NATURAL NEUTROSOPHIC NUMBERS AND MOD NEUTROSOPHIC NUMBERS

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PREFACE

In this book authors answer the question proposed by Florentin Smarandache "Does there exist neutrosophic numbers which are such that they take values differently and behave differently from I; the indeterminate?". We have constructed a class of natural neutrosophic numbers I_0^m, I_x^m, I_y^m, I_z^m where $I_0^m \times I_0^m = I_0^m$, $I_x^m \times I_x^m = I_x^m$ and $I_y^m \times I_y^m = I_y^m$ and $I_y^m \times I_x^m = I_0^m$ and $I_z^m \times I_z^m = I_0^m$.

Here take m = 12, x = 4, y = 9 and z = 6. For more refer chapter one of this book. Thus we have defined or introduced natural neutrosophic numbers using Z_m under division.

Further there are more natural neutrosophic numbers in the MOD interval [0, m). This concept is thoroughly analysed in chapter two. Using all types of MOD planes and MOD intervals

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we have generated both natural neutrosophic numbers and MOD neutrosophic numbers.

Further the MOD intervals and MOD planes have a special type of zero divisors contributed by units in Z_n. Such type of zero divisors are termed as special pseudo zero divisors leading to the definition of special pseudo zero divisors and MOD neutrosophic numbers apart from natural (MOD) neutrosophic nilpotents, zero divisors and idempotents. Lots of open problems are suggested in this book. Certainly this paradigm of shift will give a new approach to the notion of neutrosophy.

We wish to acknowledge Dr. K Kandasamy for his sustained support and encouragement in the writing of this book.

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Chapter One

NATURAL CLASS OF NEUTROSOPHIC NUMBERS

In this book for the first time authors define a new notion called natural neutrosophic numbers. They are different from I the indeterminate or the neutrosophic number defined by Florentin [3].

As we proceed on to define them one can see how different they are from other neutrosophic numbers. Infact they naturally occur. This answers a problem by Florentin Smarandache about the existence of a natural neutrosophic number.

Throughout this book Z_n will denote the ring of modulo integers.

Clearly $\{Z_n, +, \times\}$ is a commutative finite ring of order n.

Take $Z_2 = \{0, 1\}$. $\frac{1}{1} = 1$ but $\frac{1}{0}$ is not defined and $\frac{0}{1} = 0$ and $\frac{0}{0}$ is not defined.

So if we define the operation of division clearly

$Z_2^I = \{1, 0, \dfrac{1}{0}, \dfrac{0}{0}\} = \{1, 0, I_0^2\}$ where

$I_0^2 = \dfrac{1}{0} = \dfrac{0}{0}$ that is any element divided by 0 in Z_2 is an indeterminate and is denoted by I_0^2. They will be known as natural neutrosophic numbers and I_0^n in particular the natural neutrosophic zero.

Thus $\dfrac{I_0^2}{I_0^2} = I_0^2$ (by definition).

This is just like $\dfrac{0}{n}$ ($n \neq 0$ for all $n \in Z \setminus \{0\}$ is defined as 0).

Thus $\{Z_2^I, /\}$ has the following table.

/	0	1	I_0^2
0	I_0^2	0	I_0^2
1	I_0^2	1	I_0^2
I_0^2	I_0^2	I_0^2	I_0^2

This is the way operation of division is performed on Z_2^I.

$$\dfrac{0}{I_0^2} = I_0^2, \qquad \dfrac{I_0^2}{0} = I_0^2, \qquad \dfrac{I_0^2}{I_0^2} = I_0^2,$$

$$\dfrac{1}{I_0^2} = I_0^2 \quad \text{and} \quad \dfrac{I_0^2}{1} = I_0^2.$$

Clearly / is a non commutative operation on Z_2^I.

Is / operation associative on Z_2^I?

Consider $0 / (1 / 1) = (0 / 1) = 0$ ... I

Consider $(0 / 1) / 1 = (0 / 1) = 0$ ... II

So for this triple '/' is an associative operation.

Now consider $Z_3 = \{0, 1, 2\}$, $Z_3^I = \{0, 1, 2, I_0^3\}$ is again a closed under /.

The table for Z_3^I is as follows:

/	0	1	2	I_0^3
0	I_0^3	0	0	I_0^3
1	I_0^3	1	2	I_0^3
2	I_0^3	2	1	I_0^3
I_0^3	I_0^3	I_0^3	I_0^3	I_0^3

Thus I_0^3 and I_0^2 are the natural neutrosophic elements or naturally neutrosophic elements of Z_3^I and Z_2^I respectively.

Now consider $Z_4 = \{0, 1, 2, 3\}$.

$Z_4^I = \{0, 1, 2, 3, I_0^4, I_2^4\}$ for $\dfrac{1}{2}$ is not defined so is $\dfrac{3}{2}$ also not defined.

$\dfrac{1}{0}$ is not defined and $\dfrac{2}{0}, \dfrac{3}{0}$ are all not defined and they are denoted by I_0^4.

$\dfrac{1}{2}, \dfrac{3}{2}, \dfrac{2}{2}, \dfrac{0}{2}$ are denoted by I_2^4.

$$I_2^4 \times I_2^4 = I_0^4.$$

$$I_2^4 \times I_0^4 = I_0^4 = I_0^4 \times I_2^4$$

$$I_0^4 \times I_0^4 = I_0^4.$$

Thus Z_4^I has two indeterminates I_0^4 and I_2^4.

They are natural neutrosophic elements of Z_4.

Thus if in Z_n, n is not a prime we may have more than one natural neutrosophic element.

Clearly Z_4 has two natural neutrosophic elements.

Next we find for
$$Z_5 = \{0, 1, 2, 3, 4\}$$

the natural neutrosophic elements

$$Z_5^I = \{0, 1, 2, 3, 4, I_0^5\}.$$

Thus $o\left(Z_5^I\right) = o\left(Z_4^I\right)$ but they are not isomorphic.

Consider
$$Z_6 = \{0, 1, 2, 3, 4, 5\}$$

$$Z_6^I = \{0, 1, 2, 3, 4, 5, I_0^6, I_2^6, I_4^6, I_3^6\}.$$

Clearly $3 \in Z_6$ is such that $3^2 = 3$ but also $3 \times 2 = 0$ so 3 is a zero divisor hence $\dfrac{i}{3}$; $i \in Z_6$ are all indeterminates.

Hence $|Z_6^I| = 10$ and Z_6 contributes to 4 natural neutrosophic numbers.

$$I_0^6 \times I_2^6 = I_0^6 = I_4^6 \times I_0^6 = I_3^6 \times I_0^6$$

$$I_2^6 \times I_3^6 = I_0^6, \qquad I_3^6 \times I_3^6 = I_3^6,$$

$$I_4^6 \times I_3^6 = I_0^6, \qquad I_4^6 \times I_2^6 = I_2^6,$$

$$I_2^6 \times I_2^6 = I_4^6, \qquad I_4^6 \times I_4^6 = I_4^6,$$

This is the way natural neutrosophic product is defined.

But what is $\dfrac{I_2^4}{I_0^4}$ and $\dfrac{I_2^4}{I_2^4}$ and $\dfrac{I_0^4}{I_2^4} = I_0^4$ and

$$\dfrac{I_2^4}{I_0^4} = I_2^4 \text{ and so on.}$$

Now product can be defined; addition can be made only in a very special way. Once we write Z_n^I it implies Z_n^I contains all natural neutrosophic numbers from Z_n.

Now in case of $Z_2^I = \{0, 1, I_0^2\}$ if we have to define + operation then the set

$$G = \{Z_2^I, +\} = \{0, 1, I_0^2, 1 + I_0^2\}$$

is only a semigroup under + modulo 2 as

$$I_0^2 + I_0^2 = I_0^2 \text{ (is defined)}$$

and G will be known as natural neutrosophic semigroup.

$$\{ Z_2^I, \times \} = \{0, 1, I_0^2, \times\}$$

is given by the following table.

We define $0 \times I_0^2 = I_0^2$ only and not zero.

$\times$	0	1	I_0^2
0	0	0	I_0^2
1	0	1	I_0^2
I_0^2	I_0^2	I_0^2	I_0^2

Thus $\{ Z_2^I, \times \}$ is a semigroup under $\times$, known as natural neutrosophic product semigroup.

But

$$\{ Z_2^I, +, \times \} = \{0, 1, 1 + I_0^2, I_0^2 \}$$

is a semiring.

Clearly this is not a semifield.

$\{Z_2, +, \times\}$ is a field.

Consider

$$\{ Z_3^I, + \} = \{0, 1, 2, I_0^3, 1 + I_0^3, 2 + I_0^3 \} = S.$$

Clearly S is a semigroup under +, as

$$I_0^3 + I_0^3 = I_0^3$$

so nothing will make them equal to zero for

$$I_0^3 + I_0^3 + I_0^3 = I_0^3 \neq 0.$$

The table for Z_2^I under $\times$ is as follows:

$\times$	0	1	I_0^2	$1 + I_0^2$
0	0	0	I_0^2	I_0^2
1	0	1	I_0^2	$1 + I_0^2$
I_0^2	I_0^2	I_0^2	I_0^2	I_0^2
$1 + I_0^2$	I_0^2	$1 + I_0^2$	I_0^2	$1 + I_0^2$

Thus $S = \{0, 1, I_0^2, 1 + I_0^2\}$ is only a semiring in fact a semiring of natural neutrosophic numbers and is of finite order.

This answers a very long pending question that of the existence of finite semirings of finite special characteristic n; $2 \leq n < \infty$. However it is not the classical characteristic n and semifields other than the ones got using distributive lattices L as semirings or LG the group distributive lattices or LS the semigroup distributive lattices.

Thus from this we get a class of finite semirings which are not strict semirings as $1 + 1 \equiv 0 \pmod 2$ and $1 \neq 0$.

$$Z_3^I = \{0, 1, 2, I_0^3 \ I_0^3, 1 + I_0^3, 2 + I_0^3 \mid \left(I_0^3\right)^2 = I_0^3 \text{ and}$$

$$I_0^3 + I_0^3 = I_0^3\}.$$

$$(1 + I_0^3) + 1 + I_0^3 = 2 + I_0^3.$$

$$I_0^3 + I_0^3 = I_0^3 \text{ and } \left(2 + I_0^3\right)^2 = 1 + I_0^3 \text{ and so on.}$$

Thus $\{Z_3^I, +\}$ is only a semigroup under +.

$\{Z_3^I, \times\}$ is also only a semigroup.

Thus $\{ Z_3^I, +, \times \}$ is a semiring of order 6.

Consider

Z_4^I = {0, 1, 2, 3, I_0^4, I_2^4, 1 + I_0^4, 2 + I_0^4, 3 + I_0^4, 2 + I_2^4, 1 + I_2^4, 3 + I_2^4, I_2^4 + I_0^4, 1 + I_2^4 + I_0^4, 2 + I_2^4 + I_0^4, 3 + I_2^4 + I_0^4 / I_2^4 × I_0^4 = I_0^4, I_0^4 × I_0^4 = I_0^4, I_2^4 × I_2^4 = I_0^4, I_2^4 + I_2^4 = I_2^4, I_0^4 + I_0^4 = I_0^4 }.

$\{ Z_4^I, + \}$ is a semigroup this has idempotents and $\{ Z_4^I, \times \}$ is a semigroup this has zero divisors and $\{ Z_4^I, +, \times \}$ is a semiring of finite order and it is not a semifield.

Next we study

Z_5^I = {0, 1, 2, 3, 4, I_0^5, 1 + I_0^5, 2 + I_0^5, 3 + I_0^5, 4 + I_0^5 } here

I_0^5 × I_0^5 = I_0^5, I_0^5 + I_0^5 = I_0^5; 0 × I_0^5 = I_0^5 and t I_0^5 = I_0^5 for all t ∈ Z_5. Z_5^I has 10 elements.

Infact Z_5^I is a not semifield only a semiring of order 10.

Consider Z_6^I = {0, 1, 2, 3, 4, 5, I_0^6, I_4^6, I_2^6, I_3^6, I_0^6 + I_4^6, I_0^6 + I_2^6, I_0^6 + I_3^6, I_4^6 + I_2^6, I_4^6 + I_3^6, I_2^6 + I_3^6,

$$\left(I_0^6 \right)^2 = I_0^6, \ \left(I_3^6 \right)^2 = I_3^6, \ I_4^6 \times I_3^6 = I_0^6, \ I_4^6 \times I_4^6 = I_4^6,$$

$$I_2^6 \times I_2^6 = I_4^6, \ I_3^6 \times I_2^6 = I_0^6, \ I_0^6 \times I_0^6 = I_0^6,$$

$$I_4^6 \times I_0^6 = I_0^6, \ I_3^6 \times I_0^6 = I_0^6, \ I_4^6 \times I_2^6 = I_2^6,$$

$$I_0^6 \times I_2^6 = I_0^6, \ I_0^6 + I_0^6 = I_0^6, \ I_2^6 + I_2^6 = I_2^6,$$

$$I_4^6 + I_4^6 = I_4^6, \ I_3^6 + I_3^6 = I_3^6 \}.$$

$\{ Z_6^I, \times, + \}$ is only a semiring of finite order.

Now $Z_7^I = \{0, 1, 2, 3, 4, 5, 6, I_0^7, 1 + I_0^7, 2 + I_0^7, 3 + I_0^7, 4 + I_0^7, 5 + I_0^7, 6 + I_0^7\}$ is a semiring of order 14 infact not a semifield as $4 + 3 \equiv 0 \pmod 7$.

In view of this we have the following theorem.

THEOREM 1.1: *Let* $Z_n^I = \{0, 1, 2, ..., n – 1, I_0^n, 1 + I_0^n, ..., n – 1 + I_0^n \}$; *n a prime be the natural neutrosophic set.* $\{ Z_n^I, +, \times\}$ *is not a semifield but only a strict semiring of order 2n.*

Proof follows from simple number theoretic methods.

The total number of elements generated by
$Z_6^I = \{0, 1, 2, 3, 4, 5, I_0^6, I_2^6, I_4^6, I_3^6, 1 + I_0^6, 2 + I_0^6, 4 + I_0^6, 3 + I_0^6, 1 + I_2^6, 2 + I_2^6, 4 + I_2^6, 3 + I_2^6, 1 + I_4^6, 2 + I_4^6, 3 + I_4^6, 4 + I_4^6, 1 + I_3^6, 2 + I_3^6, 3 + I_3^6, 4 + I_3^6, 5 + I_3^6, 5 + I_0^6, 5 + I_2^6, 5 + I_4^6, I_0^6 + I_2^6, I_0^6 + I_4^6, I_0^6 + I_3^6, I_2^6 + I_3^6, I_2^6 + I_4^6, I_4^6 + I_3^6, 1 + I_0^6 + I_2^6, 2 + I_0^6 + I_2^6, 3 + I_0^6 + I_2^6, 4 + I_0^6 + I_2^6, 5 + I_0^6 + I_2^6, 1 + I_2^6 + I_3^6, 2 + I_2^6 + I_3^6, 3 + I_2^6 + I_3^6, 4 + I_2^6 + I_3^6, 5 + I_2^6 + I_3^6, 1 + I_2^6 + I_4^6, 2 + I_2^6 + I_4^6, 3 + I_2^6 + I_4^6, 5 + I_2^6 + I_4^6, 4 + I_2^6 + I_4^6, 1 + I_3^6 + I_0^6, 2 + I_3^6 + I_0^6, 1 + I_0^6 + I_4^6, 3 + I_3^6 + I_0^6, 5 + I_3^6 + I_0^6, 4 + I_3^6 + I_0^6, 2 + I_0^6 + I_4^6, 4 + I_0^6 + I_4^6, 3 + I_0^6 + I_4^6, 5 + I_0^6 + I_4^6, 1 + I_3^6 + I_4^6, 2 + I_3^6 + I_4^6, 3 + I_3^6 + I_4^6, 4 + I_3^6 + I_4^6, 5 + I_3^6 + I_4^6, I_0^6 + I_2^6 + I_3^6, I_0^6 + I_2^6 + I_4^6, I_0^6 + I_3^6 + I_4^6, I_2^6 + I_3^6 + I_4^6, 1 + I_0^6 + I_2^6 + I_3^6, 3 + I_0^6 + I_2^6 + I_3^6, 2 + I_0^6 + I_2^6 + I_3^6, 4 + I_0^6 + I_2^6 + I_3^6, 5 + I_0^6 + I_2^6 + I_3^6, 1 + I_0^6 + I_2^6 + I_4^6, 2 + I_0^6 + I_2^6 + I_4^6, 4 + I_0^6 + I_2^6 + I_4^6, 3 + I_0^6 + I_2^6 + I_4^6, 5 + I_4^6 + I_2^6 + I_4^6, 1 + I_0^6 + I_3^6 + I_4^6, 2 + I_0^6 + I_3^6 + I_4^6, 3 + I_4^6 + I_3^6 + I_4^6, 4 + I_0^6 + I_3^6 + I_4^6, 5 + I_0^6 + I_3^6 + I_4^6, 1 + I_2^6 + I_3^6 + I_4^6, 2 + I_2^6 + I_3^6 + I_4^6, 3 + I_2^6 + I_3^6 + I_4^6, 5 + I_2^6 + I_3^6 + I_4^6, 4$

$+ I_2^6 + I_3^6 + I_4^6$, $I_0^6 + I_2^6 + I_3^6 + I_4^6$, $1 + I_0^6 + I_2^6 + I_3^6 + I_4^6$, $2 + I_0^6$ $+ I_2^6 + I_3^6 + I_4^6$, $3 + I_0^6 + I_2^6 + I_3^6 + I_4^6$, $4 + I_0^6 + I_2^6 + I_3^6 + I_4^6$, 5 $+ I_0^6 + I_2^6 + I_3^6 + I_4^6 \}$

The semiring $\{ \left\langle Z_6^I \right\rangle, +, \times \}$ is of order 96.

Consider $Z_8^I = \{0, 1, 2, 3, 4, 5, 6, 7, I_0^8, I_2^8, I_4^8, I_6^8, \dots,$ $1 + I_0^8 + I_2^8 + I_4^8 + I_6^8, \; 2 + I_0^8 + I_2^8 + I_4^8 + I_6^8, \dots, 7 + I_0^8 + I_2^8 +$ $I_4^8 + I_6^8 \}$ is a semiring of order 128. We make a notational default by putting Z_n^I for $\left\langle Z_n^I \right\rangle$ can be easily understood by context.

However $\{ Z_9^I, +, \times \}$ is a natural neutrosophic semiring of order 72.

Thus we cannot say with increasing n the cardinality of Z_n^I will increase.

$Z_{10}^I = \{ \langle 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, I_0^{10}, I_2^{10}, I_4^{10}, \; I_5^{10}, I_6^{10}, I_8^{10} \rangle \}$.

The number of elements in $\left\langle Z_{10}^I \right\rangle$ is of cardinality 640.

$Z_{12}^I = \{ \langle 0, 1, 2, \dots, 11, I_8^{12}, I_6^{12}, I_0^{12}, I_2^{12}, I_3^{12}, I_4^{12}, I_9^{12}, I_{10}^{12} \rangle \}$

is a semiring.

Clearly $o(Z_{12}^I) > o(Z_{10}^I)$.

In view of this we have the following theorem.

THEOREM 1.2: *Let $\{ Z_n^I, +, \times \} = S$ be a semiring.*

If n is the prime then S has only one natural indeterminate generating S.

If n is a non prime which is a product of several distinct primes then order of S is very large.

Proof is direct and hence left as an exercise to the reader.

It is to be noted that at each stage for I_t^n; $0 \le t < n$ say $I_m^n \times I_s^n$ is to be defined where $0 \le m, s < n$.

So finding the table of all product is an interesting and innovative work.

This will be first illustrated by an example or two.

Three things are important to be observed.

    i.   If on Z_n^I only product is defined.
    ii. If on Z_n^I only sum is defined.
    iii. If on Z_n^I both sum and product is defined.

In all the three cases the order of them are different.

First this will be illustrated by few examples.

Example 1.1: Let $S = \{ Z_6^I, + \}$ be the semigroup of natural neutrosophic elements.

$$Z_6^I = \{0, 1, 2, 3, 4, 5, I_0^6, I_2^6, I_3^6, I_4^6 \}$$

$S = Z_6^I = \{0, 1, 2, 3, 4, 5, I_0^6, I_2^6, I_3^6, I_4^6, 1 + I_0^6, 2 + I_0^6, 3 + I_0^6, 4 + I_0^6, 5 + I_0^6, 1 + I_2^6, 2 + I_2^6, 3 + I_2^6, 4 + I_2^6, 5 + I_2^6, 1 + I_3^6, 2 + I_3^6, 3 + I_3^6, 4 + I_3^6, 5 + I_3^6, 1 + I_4^6, 2 + I_4^6, 3 + I_4^6, 4 + I_4^6, 5 + I_4^6, I_0^6 + I_2^6, I_0^6 + I_3^6, I_0^6 + I_4^6, I_2^6 + I_3^6, I_2^6 + I_4^6, I_3^6 + I_4^6, I_0^6 + I_2^6 + I_3^6, I_0^6$

$+ I_2^6 + I_4^6$, $I_0^6 + I_3^6 + I_4^6$, $I_2^6 + I_3^6 + I_4^6$, $I_0^6 + I_2^6 + I_3^6 + I_4^6$, $1 + I_0^6$

$+ I_2^6$, $2 + I_0^6 + I_2^6$, $3 + I_0^6 + I_2^6$, $4 + I_0^6 + I_2^6$, $5 + I_0^6 + I_2^6$, $1 + I_0^6 +$

I_3^6, $2 + I_0^6 + I_3^6$, ..., $1 + I_0^6 + I_2^6 + I_4^6 + I_3^6$, ..., $5 + I_0^6 + I_2^6 + I_4^6$

$+ I_3^6$ }

Thus cardinality of S is

$6 + 4 + 4 \times 5 + 6 + 6 \times 5 + 4 + 4 \times 5 + 1 + 5 = 96.$

Now $R = \{ Z_6^I, \times \} = \{0, 1, 2, 3, 4, 5, I_0^6, I_2^6, I_4^6, I_3^6 \mid$

$I_0^6 \times I_2^6 = I_0^6$, $I_0^6 \times I_4^6 = I_0^6$, $I_0^6 \times I_3^6 = I_0^6$, $I_3^6 \times I_3^6 = I_3^6$,

$I_3^6 \times I_2^6 = I_0^6$, $I_3^6 \times I_4^6 = I_0^6$, $I_2^6 \times I_2^6 = I_4^6$,

$I_4^6 \times I_4^6 = I_4^6$, $I_2^6 \times I_4^6 = I_2^6$ }.

This is the way product operation is performed and cardinality of R is only 10 and R is a semigroup under product ×.

Let $Q = \{ Z_6^I, +, \times \}$ be the semiring; cardinality of Q is 96.

Thus $o(Q) = o(S) = 96.$

Infact we will give one more example of this situation.

Example 1.2: Let $S = \{ Z_5^I, + \}$ be a semigroup.

$S = \{0, 1, 2, 3, 4, I_0^5, 1 + I_0^5, 2 + I_0^5, 3 + I_0^5, 4 + I_0^5 \}$
and $o(S) = 10.$

Let $R = \{ Z_5^I, \times \} = \{0, 1, 2, 3, 4, I_0^5 \}.$

Clearly order of R is 6.

$Q = \{ Z_5^I, +, \times \} = \{0, 1, 2, 3, 4, I_0^5, 1 + I_0^5, 2 + I_0^5, 3 + I_0^5, 4 + I_0^5, +, \times \}$ is a semiring of order 10.

In view of this we have the following theorem.

THEOREM 1.3: *Let $\{ Z_p^I, + \} = S$ be a semigroup of natural neutrosophic numbers under +.*

$R = \{ Z_p^I, \times \}$ be the natural neutrosophic semigroup under $\times$ and $Q = \{ Z_p^I, \times, + \}$ be the natural neutrosophic semiring (p a prime). Then $o(S) = 2p$, $o(R) = p + 1$ and $o(Q) = 2p$.

Proof is left as an exercise to the reader.

Next we consider the case of Z_n^I where n is a non prime.

Example 1.3: Let $\{ Z_8^I, + \} = S$ be the natural neutrosophic semigroup under +.

$S = \{0, 1, 2, 3, 4, 5, 6, 7, I_0^8, I_2^8, I_4^8, I_6^8, 1 + I_0^8,$

$1 + I_2^8, 1 + I_4^8, 1 + I_6^8, ..., 7 + I_0^8 + I_2^8 + I_4^8 + I_6^8, + \}$

is semigroup of finite order.

However $\{Z_8, I_0^8, I_2^8, I_4^8, I_6^8, \times\}$ is a semigroup of order 12 only.

$Q = \{ Z_8^I, +, \times \}$ is a natural neutrosophic semiring of finite order.

Example 1.4: Let $\{ Z_9^I, + \} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, I_0^9, I_3^9, I_6^9, 1 + I_0^9, 1 + I_3^9, 1 + I_6^9, ..., 8 + I_0^9 + I_3^9 + I_6^9, + \}$ be a natural neutrosophic semigroup of order 72.

$$o(\{ Z_8^I, +\}) > o(\{ Z_9^I, +\}).$$

$R = \{ Z_9^I, \times\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, I_0^9, I_3^9, I_6^9, \times\}$
is a semigroup of order 12.

$Q = \{ Z_9^I, +, \times\}$ is a natural neutrosophic semiring of order 72 whereas,

$Q = \{ Z_8^I, +, \times\}$ is a natural neutrosophic semiring of order 128.

What will be the order of $\{ Z_{27}^I, \times\} = R$, the natural neutrosophic semigroup?

$$R = \{0, 1, 2, ..., 26, I_0^{27}, I_3^{27}, I_6^{27}, I_9^{27},$$
$$I_{12}^{27}, I_{15}^{27}, I_{18}^{27}, I_{21}^{27}, I_{24}^{27}, \times\}.$$

Clearly $o(Z_{3^3}^I, \times) > o(Z_{2^3}^I, \times)$.

Similarly $o(Z_{3^3}^I, +) > o(Z_{2^3}^I, \times)$.

Further $o(Z_{3^3}^I, +, \times) > o(Z_{2^3}^I, +, \times)$.

In view of this we have the following theorem.

THEOREM 1.4: *Let $\{ Z_{p^n}^I, +\} = S_1$, $S_2 = \{ Z_{q^n}^I, +\}$ (p and q are primes p > q) be natural neutrosophic semigroups.*

Then $o(S_1) > o(S_2)$.

Proof follows from simple number theoretic arguments.

THEOREM 1.5: *Let $P_1 = \{ Z_{p^n}, \times\}$ and $P_2 = \{ Z_{q^n}, \times\}$ be any two natural neutrosophic semigroups under $\times$.*

$o(P_1) > o(P_2)$ *if number of divisors of p is greater than the number of divisors of q.*

Proof follows from simple number theoretic arguments.

We will first illustrate this situation by some examples.

Example 1.5: Let $S_1 = \{ Z_{12}^I , \times \}$ and $S_2 = \{ Z_{15}^I , \times \}$ be two natural neutrosophic semigroups under $\times$.

$$o(S_1) > o(S_2) \text{ but } 15 > 12.$$

Consider the natural neutrosophic elements of Z_{12},

$$I_2^{12} , I_4^{12} , I_6^{12} , I_3^{12} , I_8^{12} , I_9^{12} , I_0^{12} .$$

The natural neutrosophic elements of Z_{15} are

$$I_0^{15} , I_3^{15} , I_5^{15} , I_6^{15} , I_9^{15} , I_{10}^{15} .$$

So $o(Z_{12}^I , \times) > o(Z_{15}^I , \times)$ however $15 > 12$ but number of divisors of 12 is 4 and that of 15 is only 2.

Z_{12} has more number of zero divisors than that of Z_{15} that is why $(Z_{12}^I , \times)$ has more number of zero divisors so is a natural neutrosophic semigroup of larger order.

Example 1.6: Let $\{ Z_{16}^I \} = \{ 0, 1, 2, 3, \ldots, 15, I_0^{16} , I_2^{16} , I_4^{16} , I_8^{16} , I_{10}^{16} , I_{12}^{16} , I_{14}^{16} , I_6^{16} \}$.

Sum of any two elements is $I_2^{16} + I_8^{16}$ and so on.

$$I_2^{16} \times I_2^{16} = I_4^{16} , \quad I_2^{16} \times I_8^{16} = I_0^{16} ,$$

$$I_4^{16} \times I_4^{16} = I_0^{16}, \; I_2^{16} \times I_{10}^{16} = I_4^{16},$$

$$I_2^{16} \times I_2^{16} = I_8^{16}, \; I_0^{16} \times I_x^{16} = I_0^{16}$$

for all $x \in \{2, 4, 6, 8, I_0, 12, 14\}$.

$$I_2^{16} \times I_{14}^{16} = I_{12}^{16}, \qquad I_2^{16} \times I_6^{16} = I_{12}^{16},$$

$$I_4^{16} \times I_{10}^{16} = I_8^{16}, \qquad I_2^{16} \times I_4^{16} = I_8^{16},$$

$$I_4^{16} \times I_8^{16} = I_0^{16}, \qquad I_4^{16} \times I_{12}^{16} = I_0^{16},$$

$$I_4^{16} \times I_{14}^{16} = I_8^{16}, \qquad I_6^{16} \times I_4^{16} = I_8^{16},$$

$$I_6^{16} \times I_6^{16} = I_4^{16}, \qquad I_6^{16} \times I_8^{16} = I_0^{16},$$

$$I_6^{16} \times I_{10}^{16} = I_{12}^{16}, \qquad I_6^{16} \times I_{12}^{16} = I_8^{16},$$

$$I_6^{16} \times I_{14}^{16} = I_4^{16}, \qquad I_8^{16} \times I_8^{16} = I_0^{16},$$

$$I_8^{16} \times I_{10}^{16} = I_0^{16}, \qquad I_8^{16} \times I_{12}^{16} = I_0^{16},$$

$$I_8^{16} \times I_{14}^{16} = I_0^{16}, \qquad I_{10}^{16} \times I_{12}^{16} = I_8^{16},$$

$$I_{10}^{16} \times I_{14}^{16} = I_{12}^{16}, \qquad I_{10}^{16} \times I_{10}^{16} = I_4^{16},$$

$$I_{12}^{16} \times I_{12}^{16} = I_0^{16}, \qquad I_{12}^{16} \times I_{14}^{16} = I_8^{16},$$

$$I_{14}^{16} \times I_{14}^{16} = I_4^{16}.$$

Thus $\{ Z_{16}^I, \times \}$ is a natural neutrosophic semigroup of order 24.

Now

$\{Z_{16}^I, +\}$ is a natural neutrosophic semigroup under +. $o(Z_{16}^I, +)$ is a very large natural neutrosophic semigroup.

Let $x = 8 + I_2^{16} + I_6^{16}$ and $y = 3 + I_8^{16} + I_6^{16} \in Z_{16}^I$,

$$x + y = 11 + I_8^{16} + 2I_6^{16} + I_2^{16}$$

Example 1.7: Let Z_{20}^I be the natural neutrosophic numbers.

$Z_{20}^I = \{0, 1, 2, \ldots, 19, I_0^{20}, I_2^{20}, I_4^{20}, I_6^{20}, I_8^{20}, I_{10}^{20}, I_{12}^{20}, I_{14}^{20},$

$I_{16}^{20}, I_{18}^{20}, I_5^{20}, I_{15}^{20}\}$. Product can be defined for

$I_{15}^{20} \times I_2^{20} = I_{10}^{20}$ and so on.

However sum of $I_2^{20} + I_5^{20}$ is taken as it is for every I_x^{20};

$x \in \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 5, 15\} \subseteq Z_{20}$.

Thus $o(Z_{20}^I, \times) = o(Z_{20}^I)$ however $o(Z_{20}^I, +) \neq o(Z_{20}^I)$.

We will illustrate this situation in case of

$Z_{10}^I = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, I_0^{10}, I_2^{10}, I_2^{10}, I_6^{10}, I_8^{10}, I_5^{10}\}$.

$\{Z_{10}^I, +\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, I_0^{10}, I_2^{10}, I_4^{10}, I_6^{10}, I_8^{10},$

$I_5^{10}, a + I_0^{10}, a + I_2^{10}, a + I_2^{10}, a + I_8^{10}, a + I_6^{10}, a + I_5^{10}, b + I_0^{10} +$

$I_2^{10}, b + I_0^{10} + I_4^{10}, \ldots, b + I_8^{10} + I_5^{10}, d + I_0^{10} + I_2^{10} + I_4^{10}, e + I_0^{10}$

$+ I_2^{10} + I_4^{10} + I_6^{10}, \ldots, d + I_8^{10} + I_6^{10} + I_5^{10}, e + I_8^{10} + I_5^{10} + I_6^{10} +$

I_4^{10}, $f + I_0^{10} + I_2^{10} + I_4^{10} + I_6^{10} + I_8^{10}$, ..., $f + I_8^{10} + I_6^{10} + I_5^{10} + I_4^{10}$,

$g + I_0^{10} + I_2^{10} + I_4^{10} + I_6^{10} + I_8^{10}$, ..., $g + I_2^{10} + I_4^{10} + I_6^{10} + I_8^{10} +$

I_5^{10}, $h + I_0^{10} + I_2^{10} + I_4^{10} + I_6^{10} + I_8^{10} + I_5^{10}$, $a \in Z_{10} \setminus \{0\}$, b, c, d,

e, f, g, h $\in Z_{10}$} is only a semigroup under +.

For $I_2^{10} + I_2^{10} = I_2^{10}$; $I_2^{10} \times I_2^{10} = I_4^{10}$ and

$I_x^{10} + I_x^{10} = I_x^{10}$ for all x $\in \{0, 2, 4\ 6, 8, 5\}$.

Infact { Z_{10}^I, +} is only a natural neutrosophic semigroup of finite order which is commutative.

Clearly the set N_I = { I_x^{10} | x $\in \{0, 2, 4, 6, 8, 5\}$} also forms a subsemigroup under + called as pure natural neutrosophic semigroup.

Infact N_I is an idempotent semigroup.

However all subsemigroups of { Z_{10}^I, +} are not idempotent subsemigroups under +.

Infact even natural neutrosophic semigroup is a Smarandache semigroup as $\{Z_n, +\} \subseteq$ { Z_n^I, +} is a group.

But { Z_n^I, $\times$} is a natural neutrosophic commutative semigroup but is a Smarandache semigroup if and only if Z_n is a S-semigroup.

Several interesting properties in this direction can be derived.

Finding ideals, zero divisors, units, idempotents of $\{ Z_n^I , \times \}$ happens to be a matter of routine.

Here we give one or two illustrative examples of them.

Example 1.8: Let $S = \{ Z_7^I , + \}$ be the natural neutrosophic semigroup. S is a S-semigroup.

$S = \{0, 1, 2, 3, 4, 5, 6, I_0, 1 + I_0, 2 + I_0, 3 + I_0, 4 + I_0, 5 + I_0, 6 + I_0, +\}$ is a semigroup under +.

Order of S is 14. Infact S is a S-semigroup.

Example 1.9: Let $S = \{ Z_{14}^I , + \} = \{0, 1, 2, \ldots, 13, I_0^{14}, I_2^{14}, I_4^{14}, I_6^{14}, I_8^{14}, I_{10}^{14}, I_{12}^{14}, I_7^{14}, a + I_x^{14}, a + I_x^{14} + I_y^{14}, \ldots, a + I_0^{14} + I_2^{14} + I_4^{14} + I_6^{14} + I_8^{14} + I_{10}^{14} + I_{12}^{14} + I_7^{14} ; a \in Z_{14}; x, y \in \{0, 2, 4, 6, 8, 10, 12, 7\}\}$ be the natural neutrosophic semigroup.

$\{S, +\}$ is a S-semigroup. S has subsemigroups which are idempotent subsemigroup.

$P = \{0, I_0^{14} + I_6^{14}\} \subseteq S$ is an idempotent natural pure neutrosophic subsemigroup of order two.

$$I_0^{14} + I_6^{14} + I_0^{14} + I_6^{14} = I_0^{14} + I_6^{14} \text{ as}$$

$$I_0^{14} + I_0^{14} = I_0^{14} \text{ and } I_6^{14} + I_6^{14} = I_6^{14} .$$

Consider $x = 7 + I_0^{14}$ and $y = 2 + I_7^{14} \in S$;

$$x + y = 9 + I_0^{14} + I_7^{14} .$$

However $x + x = I_0^{14}$ and $y + y = 4 + I_7^{14}$.

So all elements of S are not idempotents.

Only some of them are idempotents.

Infact S has several idempotent subsemigroups.

Clearly all subsemigroups of S are not idempotent subsemigroups.

Example 1.10: Let $S = \{ Z_{18}^I , + \}$ be the natural neutrosophic semigroup. S has subsemigroups which are idempotent subsemigroups.

S has subsemigroups which are not idempotent subsemigroups. S is a Smarandache semigroup of finite order which is commutative.

In view of this we give the following theorem.

THEOREM 1.6: *Let $\{ Z_p^I , + \} = S$ be the natural neutrosophic semigroup (p a prime).*

(i)   *o(S) = 2p.*
(ii) *S has only one natural neutrosophic element.*
(iii) *S has only two idempotent subsemigroups barring {0} subsemigroup.*
(iv) *S is a Smarandache natural neutrosophic semigroup.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 1.7: *Let $S = \{ Z_n^I , + \}$ (n a composite number) be a natural neutrosophic semigroup.*

(i)   *S is of finite order.*

(ii)   *S is a Smarandache semigroup.*
(iii) *S has subsemigroups which are idempotent*
         subsemigroups.
(iv)   *S has subsemigroups which are not idempotent*
         subsemigroups.

Proof is direct and hence left as an exercise to the reader.

We will illustrate this situation by some examples.

Example 1.11: Let $S = \{Z_7^I, +\}$ be the natural neutrosophic semigroup.

$P = Z_7 \subseteq S$ is a group so S is a S-natural neutrosophic semigroup.

$P_1 = \{0, I_0^7\} \subseteq S$ is an idempotent semigroup of S.

$P_2 = \{I_0^7\} \subseteq S$ is also an idempotent subsemigroup of S.

$P_3 = \{a + I_0^7 \mid a \in Z_7\} \subseteq S$ is not an idempotent subsemigroup of S.

Example 1.12: Let $S = \{Z_{10}^I, +\}$ be the natural neutrosophic semigroup.

S is a Smarandache semigroup as $Z_{10} \subseteq S$ is a group under +.

$\{I_x^{10}\}$ are idempotent subsemigroups $x \in \{0, 2, 4, 6, 8, 5\}$ of order one we have six such subsemigroups.

$\{0, I_x^{10}\}$ where $x \in \{0, 2, 4, 6, 8, 5\}$ are idempotent subsemigroups of order 2.

$P_1 = \{ I_0^{10} + I_2^{10}, 0 \}$ is a subsemigroup. There exists 15 idempotent subsemigroups of order two.

$R_1 = \{ I_0^{10} + I_6^{10} \} \subseteq S$ is a subsemigroup. There are 15 such subsemigroups which are idempotent subsemigroups.

$T_1 = \{ I_0^{10} + I_6^{10} + I_5^{10} \} \subseteq S$ is an idempotent subsemigroup of order one.

There exists 20 such idempotent subsemigroups.

$L = \{ I_0^{10} + I_6^{10} + I_2^{10} + I_4^{10} + I_8^{10} + I_5^{10} \}$ is again an idempotent subsemigroup.

Infact the largest idempotent subsemigroup is given by

$V = \{ 0, I_0^{10}, I_2^{10}, I_4^{10}, I_6^{10}, I_8^{10}, I_5^{10}, I_0^{10} + I_2^{10}, \ldots, I_8^{10} + I_5^{10}, I_0^{10} + I_2^{10} + I_4^{10}, \ldots, I_6^{10} + I_8^{10} + I_5^{10}, I_0^{10} + I_2^{10} + I_4^{10} + I_6^{10}, \ldots, I_4^{10} + I_8^{10} + I_6^{10} + I_5^{10}, I_0^{10} + I_2^{10} + I_6^{10} + I_8^{10} + I_4^{10}, \ldots, I_0^{10} + I_4^{10} + I_6^{10} + I_8^{10} + I_5^{10}, I_0^{10} + I_2^{10} + I_4^{10} + I_6^{10} + I_8^{10} + I_5^{10} \} \subseteq S$.

Every proper subsemigroup of V is also an idempotent subsemigroup of S.

Several interesting results can be got.

Next we proceed onto give examples of the notion of natural neutrosophic semigroup under product.

Example 1.13: Let

$S = \{ Z_{12}^I, \times \} = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, I_0^{12}, I_2^{12}, I_4^{12}, I_6^{12}, I_8^{12}, I_{10}^{12}, I_3^{12}, I_9^{12} \}$ be a natural neutrosophic semigroup.

S has several zero divisors. $4 \times 3 = 0$, $2 \times 6 = 0$, $6 \times 4 = 0$ and so on.

Certainly the indeterminates cannot lead to zeros. For product of two natural neutrosophic numbers are never zero.

However $I_0^{12} \times I_x^{12} = I_0^{12}$, $I_6^{12} \times I_6^{12} = I_0^{12}$ so these natural neutrosophic numbers are not idempotents in general that is their product is not the same.

But some of them can be; for $I_4^{12} \times I_4^{12} = I_4^{12}$,

$I_9^{12} \times I_9^{12} = I_9^{12}$, $I_8^{12} \times I_8^{12} = I_4^{12}$, $I_3^{12} \times I_3^{12} = I_9^{12}$ and so on.

S has idempotent subsemigroups.

For take $T_1 = \{ I_4^{14} \}$, T_1 is an idempotent subsemigroup of order one.

$T_2 = \{0, 4, 9\}$ is also an idempotent subsemigroup of order three.

$T_3 = \{ I_0^{12}, I_4^{12}, I_9^{12} \}$ is also an natural neutrosophic idempotent subsemigroup of order three.

$T_4 = \{ I_0^{12}, I_3^{12}, I_9^{12} \}$ is a natural neutrosophic subsemigroup which is not an idempotent subsemigroup of S.

$\{ I_0^{12}, I_6^{12} \} = T_5$ is a natural neutrosophic subsemigroup of order three but is not an idempotent subsemigroup.

$T_6 = \{0, 1, I_0^{12}, I_6^{12} \}$ is again a natural neutrosophic subsemigroup of order four and is not an idempotent subsemigroup.

Thus $\{ Z_{12}^I, \times \}$ has idempotents, units and zero divisors.

This semigroup is a S-semigroup which has idempotent subsemigroups.

Example 1.14: Let $S = \{ Z_{19}^I, \times \}$ be the natural neutrosophic semigroup.

Clearly o(S) = 38. S has no zero divisors.

S is a S-semigroup of finite order. S has only

$P = \{0, 1, I_0^{19}, 1 + I_0^{19} \}$ to be idempotents.

Infact P is an idempotent subsemigroup as

$I_0^{19} \times I_0^{19} = I_0^{19}$ and $(1 + I_0^{19}) \times (1 + I_0^{19}) = 1 + I_0^{19}$.

If S is assumed to have addition also then by o(S) = 38. If S_1 is just $\{0, 1, 2, \ldots, 19, I_0^{19} \}$ then o(S_1) = 20 and $x I_0^{19} = I_0^{19}$ and has $\{0, 1, I_0^{19} \}$ to be the set of idempotents.

All elements in $S_1 \setminus \{0, I_0^{19} \}$ are units and S_1 has no nontrivial zero divisors. S_1 has subsemigroups given by

$$P_1 = \{0, 1, I_0^{19} \},$$

$$P_2 = \{0, 1, I_0^{19}, 18\},$$

$$P_3 = \{1, I_0^{19} \},$$

$$P_4 = \{0, I_0^{19} \},$$

$$P_5 = \{1, I_0^{19}, 18\} \text{ and so on.}$$

In view of these two examples the following theorems can be proved by any interested reader.

THEOREM 1.8: *Let $S = \{Z_p^I, \times\}$ be the natural neutrosophic semigroup (p a prime). Then the following are true.*

> *(i) S is of order p + 1.*
> *(ii) S is a S-semigroup.*
> *(iii) S has no zero divisors.*
> *(iv) S has only three idempotents.*
> *(v) S has (p – 1) number of units including 1.*
> *(vi) S has idempotent subsemigroups.*
> *(viii) S has also subsemigroups which are not idempotent subsemigroups.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 1.9: *Let $S = \{Z_n^I, \times\}$ be the natural neutrosophic semigroup, n a positive composite number.*

> *i. o(S) is finite and the order of S depends on the number of zero divisors and idempotents of Z_n.*
> *ii. S is a Smarandache semigroup if and only if Z_n is a S-semigroup.*
> *iii. S has idempotents.*
> *iv. S has zero divisors.*
> *v. S has units.*
> *vi. S has subsemigroups which idempotent subsemigroups.*
> *vii. S is not an idempotent semigroup.*
> *viii. S has subsemigroups which are not idempotent subsemigroups.*

Proof is left as an exercise to the reader.

Next we proceed on to define both the operations + and $\times$ on Z_n^I.

Already examples of them are given and Z_n^I under + and $\times$ is defined as the natural neutrosophic semiring of finite order.

Now we make a formal definition.

DEFINITION 1.1: *Let $S = \langle Z_n^I, +, \times \rangle$; clearly $\{\langle Z_n^I, +\rangle\}$ is an abelian natural neutrosophic semigroup with 0 as the identity.*

$\langle Z_n^I, \times \rangle$ is a semigroup of natural neutrosophic numbers which is commutative.

Thus $\langle S, +, \times \rangle$ is defined as the natural neutrosophic semiring of finite order of special characteristic n.

We will first illustrate this situation by some examples.

Example 1.15: Let

$S = \{\langle Z_n^I, +, \times \rangle\} = \{0, 1, 2, ..., 9, I_0^{10}, I_2^{10}, I_5^{10}, I_6^{10}, I_4^{10}, I_8^{10},$

$a + I_x^{10}, a_1 + I_x^{10} + I_y^{10}, a_2 + I_x^{10} + I_y^{10} + I_z^{10}, a_3 + I_x^{10} + I_y^{10} + I_z^{10}$

$+ I_u^{10}, a_4 + I_x^{10} + I_y^{10} + I_z^{10} + I_u^{10} + I_s^{10}, a_5 + I_0^{10} + I_2^{10} + I_5^{10} + I_6^{10}$

$+ I_4^{10} + I_8^{10} \mid a \in Z_{10} \setminus \{0\}, x, y, z, u, s \in \{0, 2, 4, 6, 8, 5\}, a_1, a_2,$

$a_3, a_4, a_5 \in Z_{10}, +, \times\}$ is the natural neutrosophic semiring.

Clearly S is not a semifield as S has zero divisors.

Further a + b = 0 is not true with a ≠ 0 and b ≠ 0. So S can never be a semifield for any Z_n^I ; n any positive integer.

Let

$$x = 9 + I_8^{10} + I_5^{10} + I_6^{10}$$

and

$$y = 3 + I_2^{10} + I_4^{10} \in S$$

$$x + y \quad = 2 + I_8^{10} + I_5^{10} + I_6^{10} + I_2^{10} + I_4^{10}$$

and

$$x \times y \quad = (9 + I_8^{10} + I_5^{10} + I_6^{10}) \times (3 + I_2^{10} + I_4^{10})$$

$$= 7 + I_8^{10} + I_5^{10} + I_6^{10} + I_2^{10} + I_0^{10} + I_4^{10} \in S.$$

This is the way + and × operations are performed on S.

This natural neutrosophic semiring has zero divisors, units and idempotents.

Let

$$y = I_6^{10} + I_4^{10} + I_2^{10} \text{ and } x = (I_8^{10} + I_6^{10}) \in S,$$

$$x + y = I_6^{10} + I_4^{10} + I_2^{10} + I_8^{10}$$

$$x \times x = (I_8^{10} + I_6^{10}) \times (I_8^{10} + I_6^{10}) = I_4^{10} + I_8^{10} + I_6^{10} \neq x.$$

So this element x in S is not an idempotent.

However I_5^{10} and I_6^{10} are idempotents of S.

$$y \times y = (I_6^{10} + I_4^{10} + I_2^{10}) \times (I_6^{10} + I_4^{10} + I_2^{10})$$
$$= I_6^{10} + I_4^{10} + I_2^{10} + I_8^{10} \neq y.$$

So y ∈ S is also not an idempotent of S.

$$I_6^{10} \times I_6^{10} = I_6^{10}.$$

Let

$$s = I_6^{10} + I_4^{10} \in S$$

$$s \times s = (I_6^{10} + I_4^{10}) \times (I_6^{10} + I_4^{10}) = I_6^{10} + I_4^{10}.$$

Thus s is an idempotent of S. S has idempotents all I_x^{10};

$x \in \{0, 2, 4, 6, 8, 5\}$ are not in general idempotents.

$$I_6^{10} \times I_6^{10} = I_6^{10} \text{ and } I_5^{10} \times I_5^{10} = I_5^{10}.$$

Let

$$x = I_6^{10} + I_5^{10} \in S;$$

$$x \times x = (I_6^{10} + I_5^{10}) \times (I_6^{10} + I_5^{10})$$

$$= I_6^{10} + I_5^{10} = x \in S.$$

Thus x is an idempotent of S.

Next we can study the ideals and subsemirings of S.

Example 1.16: Let S = $\{\langle Z_{11}^{I}, +, \times \rangle\}$ be the natural neutrosophic semiring.

S = $\{0, 1, 2, 3, \ldots, 10, I_0^{11}, x + I_0^{11}; x \in Z_{11} \setminus \{0\}\}$ be the natural neutrosophic semiring.

Let

$$t = I_0^{11} \in S; t \times t = I_0^{11} \times I_0^{11} = I_0^{11} = t.$$

Let

$$x = 7 + I_0^{11} \in S;$$

$$x \times x = (7 + I_0^{11}) \times (7 + I_0^{11}) = 5 + I_0^{11} \neq x.$$

Thus x is not an idempotent.

T = {0, I_0^{11}, 1, 1 + I_0^{11} } is a collection of idempotents in S.

It is clearly verified T is not a subsemiring; however (T, ×) is an idempotent semigroup of order 4. S has no other idempotents.

We see T_1 = {0, I_0^{11} } is a subsemiring.

Example 1.17: Let S = {$\langle Z_{23}^I, +, \times \rangle$} be the natural neutrosophic semiring.

T = {0, 1, 1 + I_0^{23}, I_0^{23} } is the collection of all idempotents in S.

However T is only a subsemigroup under ×.

But T is not a subsemigroup under +. So T is not a subsemiring.

Example 1.18: Let S = {$\langle Z_{45}^I, +, \times \rangle$} be the natural neutrosophic semiring. S has subsemirings, idempotents, zero divisors and units.

{ I_3^{45}, I_0^{45}, I_6^{45}, I_9^{45}, I_{12}^{45}, I_{15}^{45}, I_{18}^{45}, I_{21}^{45}, I_{24}^{45}, I_{27}^{45}, I_{30}^{45}, I_{33}^{45}, I_{36}^{45}, I_{39}^{45}, I_{42}^{45}, I_5^{45}, I_{10}^{45}, I_{20}^{45}, I_{25}^{45}, I_{35}^{45}, I_{40}^{45} } are all natural neutrosophic numbers.

S has

$$I_{10}^{45} \times I_{10}^{45} = I_{10}^{45}, \ I_{15}^{45} \times I_{15}^{45} = I_0^{45}, \ I_{30}^{45} \times I_{30}^{45} = I_0^{45}$$
and so on.

    i.   All natural neutrosophic numbers in general are not neutrosophic idempotents.

ii. A natural neutrosophic number can be natural neutrosophic nilpotent.

We say I_x^n is natural neutrosophic nilpotent if

$$I_x^n \times I_x^n = I_0^n.$$

iii. In general $I_x^n \times I_x^n = I_y^n$; $y \neq x$ and $y \neq 0$ can occur in case of natural neutrosophic numbers.

This is evident from Example 1.18.

For

$$I_3^{45} \times I_3^{45} = I_9^{45}, \ I_5^{45} \times I_5^{45} = I_{25}^{45}, \ I_6^{45} \times I_6^{45} = I_{36}^{45}, \ I_9^{45} \times I_9^{45} = I_{36}^{45}$$

and so on.

$$I_{15}^{45} \times I_{15}^{45} = I_0^{45} \text{ and } I_{30}^{45} \times I_{30}^{45} = I_0^{45}$$

are natural neutrosophic nilpotent elements.

$$I_{10}^{45} \times I_{10}^{45} = I_{10}^{45}$$

is defined as the natural neutrosophic idempotent.

As in case of the neutrosophic number I (or indeterminacy I) we do not always have $I^2 = I$.

$$I_{36}^{45} \times I_{36}^{45} = I_{36}^{45}$$

is again a natural neutrosophic idempotent of S.

We see $P_1 = \{0, I_{36}^{45}\}$ is a subsemiring.

$P_2 = \{0, I_{10}^{45}\}$ is again an subsemiring.

$$P_3 = \{0,\ I^{45}_{36},\ I^{45}_{10},\ I^{45}_{0},\ I^{45}_{36} + I_0,\ I^{45}_{36} + I^{45}_{10},\ I^{45}_{10} + I^{45}_{0},$$

$$I^{45}_{0} +\ I^{45}_{36} + I^{45}_{10}\}$$

is again a subsemiring.

Clearly all the three subsemirings are idempotent subsemirings.

Example 1.19: Let $S = \{\langle Z^I_{2^5},\ +,\ \times\rangle\}$ be the natural neutrosophic semiring.

$$S = \{\langle 0, 1, 2, \ldots, 31,\ I^{32}_{0},\ I^{32}_{2},\ I^{32}_{4},\ I^{32}_{6},\ I^{32}_{8},\ I^{32}_{10},\ I^{32}_{12},\ I^{32}_{14},\ I^{32}_{16},$$

$$I^{32}_{18},\ I^{32}_{20},\ I^{32}_{22},\ I^{32}_{24},\ I^{32}_{26},\ I^{32}_{28},\ I^{32}_{30}\rangle\}$$

under $+$ and $\times$ operations.

$$I^{32}_{8} \times I^{32}_{8} = I^{32}_{0},\ I^{32}_{16} \times I^{32}_{16} = I^{32}_{0},\ I^{32}_{24} \times I^{32}_{24} = I^{32}_{0}.$$

This has no natural neutrosophic idempotent.

Further $I^{32}_{x} \times I^{32}_{x} = I^{32}_{4}$ or I^{32}_{16} or I^{32}_{0} only.

Clearly this semiring has no natural neutrosophic idempotents.

Example 1.20: Let

$$S = \{\langle Z^I_{27},\ +,\ \times\rangle\} = \{\langle 0, 1, 2, \ldots, 26,\ I^{27}_{0},\ I^{27}_{3},\ I^{27}_{9},$$

$$I^{27}_{6},\ I^{27}_{12},\ I^{27}_{15},\ I^{27}_{18},\ I^{27}_{21}, I^{27}_{24}\rangle,\ +,\ \times\}$$

generates the natural MOD neutrosophic semiring.

$$I^{27}_{3} \times I^{27}_{3} = I^{27}_{9},\ I^{27}_{9} \times I^{27}_{9} = I^{27}_{0},$$

$$I_6^{27} \times I_6^{27} = I_9^{27}, \ I_{12}^{27} \times I_{12}^{27} = I_9^{27},$$

$$I_{15}^{27} \times I_{15}^{27} = I_9^{27}, \ I_{18}^{27} \times I_{18}^{27} = I_0^{27},$$

$$I_{21}^{27} \times I_{21}^{27} = I_9^{27}, \ I_{24}^{27} \times I_{24}^{27} = I_9^{27}.$$

Clearly S has no natural neutrosophic idempotents. But S has natural neutrosophic nilpotents.

Example 1.21: Let

$$S = \{\langle Z_{25}^I, +, \times \rangle\} = \{\langle 0, 1, 2, \ldots, 24, \ I_0^{25}, \ I_5^{25}, \ I_{10}^{25}, \ I_{15}^{25}, \ I_{20}^{25} \rangle,$$

$+, \times\}$ be the natural neutrosophic semiring.

$$I_5^{25} \times I_5^{25} = I_0^{25},$$

$$I_{15}^{25} \times I_{15}^{25} = I_0^{25},$$

$$I_{10}^{25} \times I_{10}^{25} = I_0^{25} \text{ and}$$

$$I_{20}^{25} \times I_{20}^{25} = I_0^{25}.$$

All the natural neutrosophic elements are neutrosophic nilpotent.

However there is no neutrosophic idempotent.

Example 1.22: Let $S = \{\langle Z_{125}^I, +, \times \rangle\} = \{\langle 0, 1, 2, \ldots, 124, \ I_0^{125},$

$I_5^{125}, \ I_{10}^{125}, \ I_{15}^{125}, \ I_{20}^{125}, \ I_{25}^{125}, \ I_{30}^{125}, \ I_{35}^{125}, \ I_{40}^{125}, \ I_{45}^{125}, \ I_{50}^{125}, \ I_{55}^{125}, \ I_{60}^{125},$

$I_{65}^{125}, \ I_{70}^{125}, \ I_{75}^{125}, \ I_{80}^{125}, \ I_{85}^{125}, \ I_{90}^{125}, \ I_{95}^{125}, \ I_{100}^{125}, \ I_{105}^{125}, \ I_{110}^{125}, \ I_{115}^{125}, \ I_{120}^{125} \rangle, +,$

$\times\}$ be the natural neutrosophic semiring.

$$I_{25}^{125} \times I_{25}^{125} = I_0^{125} \; ,$$

$$I_{50}^{125} \times I_{50}^{125} = I_0^{125} \; ,$$

$$I_{75}^{125} \times I_{75}^{125} = I_0^{125} \;\; \text{and}$$

$$I_{100}^{125} \times I_{100}^{125} = I_0^{125}$$

are the natural neutrosophic nilpotents.

S has no neutrosophic idempotents.

In view of this we leave open the following conjecture.

Conjecture 1.1: Let

$S = \{\langle Z_{p^n}^I, +, \times \rangle \mid p$ is a prime and n a positive integer$\}$ be the natural neutrosophic semiring.

> (i) Can S contain natural neutrosophic idempotents?
> (ii) Find the number of natural neutrosophic elements in S.
> (iii) Find the number of natural neutrosophic nilpotents in S.

Next we proceed onto describe the result by a theorem and define the new notion of natural neutrosophic zero divisors.

THEOREM 1.10: *Let* $S = \{\langle Z_n^I, +, \times \rangle\}$ *be the natural neutrosophic semiring. S has natural neutrosophic idempotents if and only if Z_n has idempotents.*

Proof: $x \in Z_n \setminus \{0, 1\}$ such that $x^2 = x$ and $(n, x) = d \neq 0$ if and only if $I_x^n \in S$ is a natural neutrosophic idempotent of S.

DEFINITION 1.2: *Let $S = \{ Z_n^I, \times\}$ be the natural neutrosophic semigroup.*

Let I_x^n and $I_y^n \in S$; if $I_x^n \times I_y^n = I_0^n$ $(x \neq y)$ then we define this to be the natural neutrosophic zero divisor.

First we will illustrate this situation by some examples.

Example 1.23: Let $S = \{ Z_{12}^I, \times\}$ be the natural neutrosophic semigroup.

I_4^{12}, I_3^{12}, I_6^{12}, I_8^{12}, I_2^{12}, I_{10}^{12}, $I_9^{12} \in S$ contribute to natural neutrosophic zero divisors.

$$I_6^{12} \times I_2^6 = I_0^{12}, \qquad I_8^{12} \times I_6^{12} = I_0^{12},$$

$$I_6^{12} \times I_4^{12} = I_0^{12}, \qquad I_6^{12} \times I_{10}^{12} = I_0^{12},$$

$$I_2^6 \times I_4^{12} = I_0^{12}, \qquad I_3^{12} \times I_8^{12} = I_0^{12},$$

$$I_4^{12} \times I_9^{12} = I_0^{12} \quad \text{and} \quad I_8^{12} \times I_9^{12} = I_0^{12}$$

are the natural neutrosophic zero divisors of S.

Example 1.24: Let $S = \{ Z_{13}^I, \times\}$ be the natural neutrosophic semigroup. S has no natural neutrosophic zero divisors or natural neutrosophic nilpotents or natural neutrosophic idempotents.

In view of this we have the following theorem.

THEOREM 1.11: *Let $S = \{ Z_p^I, \times\}$, p a prime be the natural neutrosophic semigroup.*

    i.   *S has no natural neutrosophic nilpotents.*

ii.   S has no natural neutrosophic idempotents.

iii. S has no natural neutrosophic zero divisors.

The proof is direct and is left as an exercise to the reader.

Example 1.25: Let $S = \{Z_{17}^I, \times\}$ be the natural neutrosophic semigroup. S has no natural neutrosophic zero divisors or nilpotents of order two.

S has no natural neutrosophic idempotents.

Example 1.26: Let $S = \{Z_{24}^I, \times\}$ be the natural neutrosophic semigroup. S has natural neutrosophic zero divisors.

$$I_{12}^{24} \times I_4^{24} = I_0^{24}, \qquad I_6^{24} \times I_8^{24} = I_0^{24},$$

$$I_8^{24} \times I_3^{24} = I_0^{24}, \qquad I_{18}^{24} \times I_4^{24} = I_0^{24}$$

so on are all natural neutrosophic zero divisors.

$$I_9^{24} \times I_9^{24} = I_9^{24} \text{ and } I_{16}^{24} \times I_{16}^{24} = I_{16}^{24}$$

are natural neutrosophic idempotents.

$I_{12}^{24} \times I_{12}^{24} = I_0^{24}$ is the natural neutrosophic idempotent of S.

In view of this example we have the following theorem.

THEOREM 1.12: *Let $S = \{Z_n^I, \times\}$, n a composite number be the natural neutrosophic semigroup.*

i.   *S has natural neutrosophic idempotents if and only if Z_n has idempotents.*

ii. *S has natural neutrosophic nilpotents if and only if Z_n has nilpotents.*

iii. *S has natural neutrosophic zero divisors if and only if Z_n has zero divisors.*

Proof is direct for if x, y is a zero divisor of Z_n then I_x^n and I_y^n is a natural neutrosophic zero divisor.

If $z \in Z_n$ is an idempotent then I_z^n is the natural neutrosophic idempotent of S.

If $t \in Z_n$ is a nilpotent element of order two.

I_t^n is a natural neutrosophic nilpotent element of order two.

Hence the theorem.

Example 1.27: Let S = $\{ Z_{15}^I, \times \}$ be the natural neutrosophic semigroup.

x = $10 \in Z_{15}$ is an idempotent of Z_{15}.

I_{10}^{15} in S is a natural neutrosophic idempotent of S.

$$I_{10}^{15} \times I_{10}^{15} = I_{10}^{15}. \ x = 10 \text{ and}$$

$$y = 3 \in Z_{15} \text{ is a zero divisor in } Z_{15}.$$

Clearly I_{10}^{15}, $I_3^{15} \in S$ is such that $I_{10}^{15} \times I_3^{15} = I_0^{15}$ so is a natural neutrosophic zero divisor.

$I_6^{15} \in S$ is a natural neutrosophic idempotent, clearly 6 is an idempotent of Z_{15}.

However Z_{15} has no nilpotent so also S has no natural neutrosophic nilpotent.

Example 1.28: Let $S = \{ Z_{81}^I, \times \}$ be the natural neutrosophic semigroup.

$x = 9 \in Z_{81}$ is such that $9 \times 9 = 0$. $I_9^{81} \in S$ is a natural neutrosophic zero divisor.

$I_{18}^{81} \times I_{18}^{81} = I_0^{81}$ is again a natural neutrosophic zero divisor. S has no neutrosophic idempotents.

In view of this we have the following theorem.

THEOREM 1.13: *Let $S = \{ Z_{p^n}^I, \times \}$ p a prime be the natural neutrosophic semigroup.*

    i.   *S has natural neutrosophic zero divisors.*
    ii. *S has no natural neutrosophic idempotents.*

The proof follows from simple number theoretic techniques.

Now we proceed onto give neutrosophic natural idempotents.

Example 1.29: Let $S = \{ Z_{36}^I, \times \}$ be the natural neutrosophic semigroup.

Let $x = I_9^{36} \in S$. $I_9^{36} \times I_9^{36} = I_9^{36}$ is a natural neutrosophic idempotent $I_6^{36} \in S$ is such that

$I_6^{36} \times I_6^{36} = I_0^{36}$ so I_6^{36} is a natural neutrosophic nilpotent element of order two.

$I_9^{36} \times I_4^{36} = I_0^{36}$ is the natural neutrosophic zero divisor.

I_2^6 is a natural neutrosophic nilpotent of order two.

$I_{12}^{36} \times I_3^{36} = I_0^{36}$ is a natural neutrosophic zero divisor of S.

Thus S has natural neutrosophic zero divisors which are not natural neutrosophic nilpotents.

S has natural neutrosophic nilpotents and S has natural neutrosophic idempotents.

So in Z_n if n is a composite number and is not of the form p^m, p a prime then in general the natural neutrosophic semigroup $S = \{ Z_n^I, \times \}$ has natural neutrosophic idempotents, natural neutrosophic zero divisors which are not natural neutrosophic nilpotents and natural neutrosophic nilpotents of order two.

In view of this we can say $S = \{ Z_n^I, +, \times \}$ the natural neutrosophic semiring can have natural neutrosophic idempotents, natural neutrosophic zero divisors and natural neutrosophic nilpotents of order two.

A natural question would be can natural neutrosophic semirings have subsemirings and ideals.

Likewise can $S = \{ Z_n^I, \times \}$ have subsemigroups and ideals.

To this end we give the following examples.

Example 1.30: Let $S = \{ Z_{24}^I, +, \times \}$ be the natural neutrosophic semiring.

$M_1 = \{\langle 0, 3, 6, 9, 12, 15, 18, 21, I_0^{24}, I_3^{24}, I_6^{24}, I_9^{24}, I_{12}^{24}, I_{15}^{24},$

$I_{18}^{24}, I_{21}^{24}\rangle, +, \times\}$ is a subsemiring as well as an ideal of S.

$M_2 = \{\langle 0, 2, 4, 6, \ldots, 22, I_0^{24}, I_2^{24}, I_4^{24}, \ldots, I_{22}^{24}\rangle\}$ is an ideal of S.

$P = Z_{24} \subseteq S$ is just a subsemiring (or ring) which is not an ideal of S.

Example 1.31: Let $S = \{Z_{11}^I, +, \times\}$ be a natural neutrosophic semiring. S has no ideals. S has subsemirings.

We can derive several interesting properties about these natural neutrosophic semirings.

We suggest the following problems for this chapter.

Problems

1. Find all natural idempotents of Z_{10}.

   i.   What is the order of $\{Z_{10}^I, +\}$?
   ii. What is the order of $\{Z_{10}^I, \times\}$?
   iii. What are the algebraic structures enjoyed by $\{Z_{10}^I, \times\}$ and $\{Z_{10}^I, +\}$?

2. Can $\{Z_{12}^I, +\}$ be a group?

3. Find the order of $G = \{Z_{20}^I, +\}$?

4. Find the order of $\{Z_{20}^I, \times\}$.

5. Let $S = \{Z_{24}^I, \times\}$ be the natural neutrosophic semigroup.

      i.   Find all natural neutrosophic idempotents of S.

     ii.   Find all natural neutrosophic nilpotents of S.

   iii.   Find all zero divisors of S.

6.   Let $S = \{ Z_{42}^I , \times \}$ be the natural neutrosophic semigroup.

      i.   Can S be a Smarandache semigroup?

     ii.   What is the order of S?

   iii.   Study questions i to iii of problem 5 for this S.

   iv.   Can S have ideals?

    v.   Find all subsemigroups of S which are not ideals.

   vi.   Can S have S-ideals?

7.   Let $S = \{ Z_{15}^I , + \}$ be the natural neutrosophic semigroup.

      i.   Prove S is only a semigroup.

     ii.   Can S have ideals?

   iii.   Is S a S-semigroup?

   iv.   Find subsemigroups which are not S-ideals.

    v.   Can S have idempotents?

   vi.   Can S have S-idempotents?

   vii.   Can S have idempotents which are not S-idempotents?

8.   Let $M = \{ Z_{23}^I , + \}$ be the natural neutrosophic semigroup.

   Study questions i to vii of problem 7 for this M.

9.   Let $N = \{ Z_{24}^I , + \}$ be the natural neutrosophic semigroup.

      i.   Compare M of problem 8 with this N.

     ii.   Which of the semigroups M or N has more number of subsemigroups?

   iii.   Study questions i to vii of problem 7 for this N.

10. Let $W = \{ Z_{140}^I , + \}$ be the natural neutrosophic semigroup.
Study questions i to vii of problem 7 for this W.

11. Let $S = \{ Z_{150}^I , + \}$ be the natural neutrosophic semigroup.

    i.   Find all ideals of S.
    ii. Find all subsemigroups of S which are not ideals of S.
    iii. Can S have idempotent ideals?
    iv. Find all subsemigroups which are idempotent subsemigroups.

12. Find any interesting property associated with the semigroup $S = \{ Z_n^I , + \}$.

13. Enumerate all properties associated with the natural neutrosophic semigroup $P = \{ Z_n^I , \times \}$.

14. Compare the semigroups P and S of problem 12 and 13 for a fixed n.

15. Let $S = \{ \langle Z_{49}^I , + \rangle, \times \}$ be the semigroup under $\times$.

    i.   What is order of S?
    ii. Find all subsemigroups which are idempotent subsemigroups.
    iii. Find all idempotents of S.
    iv. Find all ideals of S which are idempotent ideals.
    v. Can S have natural neutrosophic nilpotent elements?
    vi. Find all natural neutrosophic idempotents in S.
    vii. Find all natural neutrosophic zero divisors of S.

16. Let $M = \{ \langle Z_{24}^I , + \rangle, \times \}$ be the semigroup under $\times$.

Study questions i to vii of problem 15 for this M.

17. Let $S = \{ \langle Z_{24}^I \rangle +, \times \}$ be the natural neutrosophic semiring generated under $+$ and $\times$.

    i.   Find o(S).
    ii. Is S a semifield?

    iii. Is S a S-semiring?

    iv. Find the number of natural neutrosophic idempotents in S.

    v. Find the number of natural neutrosophic elements in S.

    vi. Find the number of natural neutrosophic nilpotents of order two in S.

    vii. Find the number of natural neutrosophic zero divisors in S.

    viii. Find S-ideals if any in S.

    ix. Find all ideals of S which are not S-ideals.

    x. Can S have idempotent subsemirings?

    xi. Can idempotent subsemirings of S be S-subsemirings?

    xii. Can S have S-zero divisors?

    xiii. Can S have S-idempotents?

18. Find all the special features enjoyed by natural neutrosophic semiring $S = \{\langle Z_n^I, +, \times\rangle\}$.

19. Let $M = \{\langle Z_{98}^I, +, \times\rangle\}$ be a natural neutrosophic semiring.

    Study questions i to xiii of problem 17 for this M.

20. Let $N = \{\langle Z_{48}^I, +, \times\rangle\}$ be the natural neutrosophic semiring.

    Study questions i to xiii of problem 17 for this N.

21. Compare the natural neutrosophic semirings in problem 19 and 2.

    i. Which semiring N or M has more number of natural neutrosophic elements?

    ii. Which of the semiring N or M has more number of neutrosophic zero divisors?

    iii. Which of the semirings M or N has more number of natural neutrosophic nilpotents?

Chapter Two

MOD NATURAL NEUTROSOPHIC ELEMENTS IN [0,n), [0, n)g, [0, n)h AND [0, n)k

In this chapter for the first time we define MOD natural neutrosophic elements in the MOD interval or small interval [0, n). Several algebraic structures are built on the MOD natural neutrosophic elements of the MOD interval [0, n).

$^I[0, n) = \{$Collection of all elements of [0, n) together with $I_k^{[0, n)}$ where $I_k^{[0, n)}$ are MOD neutrosophic elements$\}$.

First we will represent them by examples.

Example 2.1: Let $S = \{^I[0, 3); I_{1.5}^{[0,3)}, I_0^{[0, 3)}\}$ are the one of the MOD neutrosophic zero divisors.

$$I_{1.5}^{[0,3)} \times I_{1.5}^{[0,3)} = I_{2.25}^{[0, 3)} \text{ but } 1.5 \times 2 = 3 \equiv 0 \pmod{3}.$$

The problem is can we put $I_{1.5}^{[0, 3)}$ as MOD neutrosophic element.

$I_{1.7320508075}^{[0, 3)}$ is such that

$$I_{1.7320508075}^{[0,3)} \times I_{1.7320508075}^{[0,3)} = I_0^{[0, 3)}.$$

We call $I_{1.7320508075}^{[0, 3)}$ is a MOD neutrosophic nilpotent element.

We do not accept $I_{1.5}^{[0, 3)}$ as a MOD neutrosophic element but accept it as a pseudo MOD neutrosophic element as $2 \in {}^I[0, 3)$ is such that $2 \times 2 = 1 \pmod 3$.

It is left as an open conjecture to find the number of MOD neutrosophic elements in ${}^I[0, 3)$.

Consider the MOD interval ${}^I[0, 2)$.

$x = 1.4142135625 \in [0, 2)$ is such that $x^2 = 2 \equiv 0 \pmod 2$.

Hence $I_{1.4142135625}^{[0, 2)}$ is a MOD neutrosophic nilpotent element of order 2.

It is still a open conjecture to find all MOD neutrosophic elements of ${}^I[0, 2)$.

At this stage the following are left as open conjectures.

Conjecture 2.1: Given ${}^I[0,n)$ the MOD interval to find all pseudo MOD neutrosophic elements.

Conjecture 2.2: Given ${}^I[0,n)$ to find the number of MOD neutrosophic nilpotent elements.

Conjecture 2.3: Given ${}^I[0,n)$ to find the number of MOD neutrosophic zero divisors.

Conjecture 2.4: Given ${}^I[0,n)$ to find the number of MOD neutrosophic idempotents.

Example 2.2: Let [0, 12) be the MOD interval. Clearly if [0, 12) is taken. I_6^{12} is not the MOD neutrosophic nilpotent.

Likewise I_3^{12} and I_8^{12} are not MOD neutrosophic zero divisors.

Also I_4^{12} is the natural neutrosophic idempotent but not the MOD neutrosophic idempotent element of [0, 12).

However all natural neutrosophic elements of Z_{12} are also present in [0, 12).

$x = 3.464101615 \in [0, 12)$ is such that $x \times x \equiv 12 \pmod{12} = 0$.

So $I_{3.464101615}^{[0, 12)}$ is a MOD neutrosophic nilpotent element of order two.

Consider the interval [0, 2) if we define the operation of division $^I[0, 2) = \{1, 0, 2, [0, 2), I_0^{[0, 2)}, I_{1.4142135625}^{[0, 2)}$ and so on$\}$.

If the interval [0, 3) is considered $^I[0,3) = \{1, 0, 2, [0, 3), I_0^{[0, 3)}, I_{1.5}^{[0, 3)}, I_{1.7320508075}^{[0, 3)}$ and so on$\}$.

As said earlier all these remain as open conjectures.

However for the sake of better understanding we study the MOD interval [0, 10).

Example 2.3: Let [0, 10) be the MOD interval.

We have already studied Z_{10}^I.

Now what are the special properties enjoyed by $^I[0, 10]$,

$^I[0, 10) = \{0, 1, 2, \ldots, 9, [0, 10), I_0^{[0,\,10)}, I_2^{[0,\,10)}, I_4^{[0,\,10)}, I_8^{[0,\,10)},$

$I_6^{[0,\,10)}, I_5^{[0,\,10)}, I_{2.5}^{[0,\,10)}, I_{1.25}^{[0,\,10)}, I_{8.75}^{[0,\,10)}, I_{3.125}^{[0,\,10)}, I_{7.8125}^{[0,\,10)}, I_{3.90625}^{[0,\,10)}, I_{7.5}^{[0,\,10)},$

$I_{6.25}^{[0,\,10)}, I_{1.5625}^{[0,\,10)}, I_{2.44140625}^{[0,\,10)}, I_{9.53125}^{[0,\,10)}, I_{8.125}^{[0,\,10)}, I_{5.960464478}^{[0,\,10)}$ and so on$\}$.

At this juncture we have no other option except to conjecture that $^I[0,n)$ can have infinite number of MOD neutrosophic number.

The other practical problems we face are what is the product of a MOD neutrosophic number with a pseudo neutrosophic number and so on.

This study is also left as an open conjecture.

Example 2.4: Let $^I[0, 5) = \{0, 1, 2, \ldots, 4, I_0^{[0,\,5)}, I_{1.25}^{[0,\,10)}$ and so on$\}$. Clearly $I_{1.25}^{[0,\,5)}$ is a pseudo MOD neutrosophic number $I_{1.5625}^{[0,\,5)}$ and so on. $I_{1.25}^{[0,\,5)}$ under product generates a set of pseudo MOD neutrosophic numbers; we just call them so as it is generated by a pseudo MOD neutrosophic number.

Similarly $2.5 \times 2 = 0$ so $I_{2.5}^{[0,\,5)}$ is again a pseudo MOD neutrosophic number.

The product of $I_{2.5}^{[0,\,5)} \times I_{2.5}^{[0,\,5)} = I_{1.25}^{[0,\,5)}$ is again a pseudo MOD neutrosophic number.

$$I_{2.5}^{[0,\,5)} \times I_{1.25}^{[0,\,5)} = I_{3.125}^{[0,\,5)}.$$

Now it is also left as an open conjecture to find the class of all MOD pseudo neutrosophic numbers and the MOD neutrosophic numbers and their interrelations.

As 5 is a prime Z_5^I has no natural neutrosophic number other than I_0^5.

In view of this first the following result is proved.

THEOREM 2.1: *Let $^I[0,n) = \{0, 1, \ldots, n-1, [0, n), \ldots\}$ be the MOD neutrosophic element interval. $^I[0,n)$ contains;*

    i.   *pseudo MOD neutrosophic elements.*
    ii. *natural neutrosophic elements provided; n is not a prime.*
    iii. *MOD neutrosophic elements.*

Proof: Every $a = \dfrac{n}{n-1}$ only realized as a decimal in $[0, n)$ and not a rational paves way for a pseudo MOD neutrosophic element.

For the interval $[0, 6)$; 1.2 is such that $I_{1.2}^{[0, 6)}$ is a pseudo MOD neutrosophic element of $^I[0, 6)$.

If n is a prime say p then $\dfrac{p}{2}$ realized only as a decimal in $[0, p)$; is a pseudo MOD neutrosophic element of $^I[0, p)$.

Take p = 19, then $\dfrac{p}{2} = 9.5$ is a pseudo MOD neutrosophic element as $9.5 \times 2 \equiv 0 \pmod{19}$; 2 is a unit $[0, 19)$. Hence proof of (i).

To find the proof of (ii) consider n a prime only $I_0^{[0, n)}$ is the only natural neutrosophic element.

If n is not a prime every zero divisor of Z_n paves way for a natural neutrosophic element. Hence proof of (ii). Clearly $I_x^{[0, n)}$

exists for $x = \dfrac{n}{2}$ if n is odd it will be a MOD neutrosophic element.

If n is even then $\dfrac{n}{n-1}$ is a MOD neutrosophic element.

Thus the following are left as open conjecture.

Conjecture 2.5: Let $^I[0, n)$ be the MOD neutrosophic collection.

    i.   Characterize those $^I[0, n)$ in which product of two pseudo MOD neutrosophic numbers is a pseudo MOD neutrosophic number.

    ii.   Characterize those $^I[0, n)$ in which the product of two MOD neutrosophic numbers is a MOD neutrosophic number.

    iii. Characterize those $^I[0, n)$ in which the product of MOD neutrosophic number and pseudo MOD neutrosophic number which are nilpotents.

Next we study the MOD neutrosophic elements and MOD pseudo neutrosophic elements of the MOD dual number interval.

This will be first illustrated by an example.

Example 2.5: Consider $\langle Z_5 \cup g \rangle = \{0, 1, 2, 3, 4, g, 2g, 3g, 4g, 1 + g, 1 + 2g, 1 + 3g, 1 + 4g, 2 + g, 2 + 2g, 2 + 3g, 2 + 4g, 3 + g, 3 + 2g, 3 + 4g, 3 + 3g, 4 + g, 4 + 2g, 4 + 3g, 4 + 4g\}$ be the modulo dual number $g^2 = 0$; $g \cdot 2g = 0$ and so on.

Now $\langle Z_5 \cup g \rangle_I = \{0, 1, 2, 3, 4, g, 2g, 3g, 4g, 1 + g, 2 + g, \ldots, 4 + 4g, I_g^g, I_{2g}^g, I_{3g}^g, I_{4g}^g, I_0^g$ and so on$\}$.

Thus all dual modulo integer $\langle Z_n \cup g \rangle$; $g^2 = 0$ even if n is a prime has natural neutrosophic numbers.

Several of them lead to natural neutrosophic nilpotent elements of order two.

Example 2.6: Let

$\langle Z_3 \cup g \rangle = \{0, 1, 2, g, 2g, 1 + g, 2 + g, 1 + 2g, 2 + 2g\}$.

$\langle Z_3 \cup g \rangle_I = \{0, 1, 2, I_0^g, I_g^g, I_{2g}^g \}$.

$I_g^g \times I_{2g}^g = I_0^g$. $I_g^g \times I_g^g = I_0^g$ and $I_{2g}^g \times I_{2g}^g = I_0^g$.

Example 2.7: Let $\langle Z_4 \cup g \rangle = \{0, 1, 2, 3, g, 2g, 3g, 1 + g, 1 + 2g, 1 + 3g, 2 + g, 2 + 2g, 2 + 3g, 3 + g, 3 + 2g, 3 + 3g\}$.

$\langle Z_4 \cup g \rangle_I = \{\langle Z_4 \cup g \rangle , I_0^g, I_g^g, I_{2g}^g, I_2^g, I_{3g}^g, I_{2+2g}^g, I_{2+g}^g, I_{2+3g}^g \}$ are natural neutrosophic dual numbers.

Clearly $I_g^g \times I_g^g = I_0^g$, $I_{2g}^g \times I_{2g}^g = I_0^g$,

$I_g^g \times I_{3g}^g = I_0^g$, $I_{2+2g}^g \times I_g^g = I_{2g}^g$,

$I_{2+2g}^g \times I_{2+g}^g = I_{2g}^g$ and so on.

Example 2.8: Let $\langle Z_6 \cup g \rangle = \{0, 1, 2, \ldots, 5, g, 2g, 3g, \ldots, 5g, 1 + g, 1 + 2g, \ldots, 5g + 1, \ldots, 5g + 2, \ldots, 5g + 5g\}$ be the dual number of modulo integers.

$\langle Z_6 \cup g \rangle_I = \{\langle Z_6 \cup g \rangle, I_0^g, I_g^g, I_{2g}^g, I_{3g}^g, I_3^g, I_2^g, I_4^g, I_{4g}^g, I_{5g}^g, I_{2+2g}^g, I_{3+3g}^g, I_{4+4g}^g$ and so on.$\}$.

Clearly $I^g_{2+2g} \times I^g_{3+3g} = I^g_0$, $I^g_{2+2g} \times I^g_g = I^g_{2g}$ and so on.

Now it is left as an open problem.

Problem 2.1: Let $\langle Z_n \cup g \rangle_I$ be the natural neutrosophic dual numbers.

   (i) For a given n how many such natural neutrosophic dual numbers exists?

   (ii) Find the number of natural neutrosophic dual numbers which are idempotents.

   (iii) Find the number of natural neutrosophic dual numbers which are nilpotents of order two.

This study will yield lots of interesting results in this direction.

However we will represent this situation by some examples.

Example 2.9: Let $\langle Z_{12} \cup g \rangle = \{0, 1, 2, \ldots, 11, g, \ldots, 11g, 1 + g, 1 + 2g, \ldots, 1 + 11g, 2 + g, \ldots, 2 + 11g, \ldots, 11 + 11g\}$ be the modulo dual numbers.

$\langle Z_{12} \cup g \rangle_I = \{\langle Z_{12} \cup g \rangle, I^g_2, I^g_4, I^g_8, I^g_6, I^g_{10}, I^g_g, I^g_{2g}, \ldots, I^g_{11g},$

$I^g_3, I^g_9, I^g_{2+2g}, I^g_{2+4g}, I^g_{2+6g}, I^g_{2+8g}, I^g_{2+10g}, I^g_{2+9g}, I^g_{4+2g}, I^g_{4+4g},$

$I^g_{4+6g}, \ldots, I^g_{8+9g}$ and so on$\}$.

Finding order of $\langle Z_{12} \cup g \rangle_I$ is a difficult task.

Infact $\langle Z_{12} \cup g \rangle_I$ has several neutrosophic natural dual number.

$$I^g_{2+9g} \times I^g_6 = I^g_0, \qquad I^g_{2+4g} \times I^g_6 = I^g_0$$

$$I^g_6 \times I^g_6 = I^g_0 \text{ and } \qquad I^g_9 \times I^g_9 = I^g_9$$

is a neutrosophic natural dual numbers.

$$I^g_4 \times I^g_{4g} = I^g_{4g} \text{ and so on.}$$

One of the important and interesting observations is that $\langle Z_{12} \cup g \rangle_I$ can have natural neutrosophic dual numbers which are idempotents or nilpotents of order two.

However finding all these natural neutrosophic dual numbers is a very difficult task. Further defining on them some algebraic operation like + and × happens to be still difficult.

However we are always guaranteed of more than one natural neutrosophic dual number in $\langle Z_n \cup g \rangle_I$ even if n is a prime; for when n is a prime there is one and only one natural neutrosophic number given by I^n_0.

When n is a prime $\langle Z_n \cup g \rangle_I$ has at least n number of natural neutrosophic dual numbers including I^g_0.

This is just illustrated by an example.

Example 2.10: Let $\langle Z_{13} \cup g \rangle_I$ be the natural neutrosophic dual numbers.

$\{ I^g_0, I^g_g, I^g_{2g}, \ldots, I^g_{12g} \}$ are the 13 natural neutrosophic dual numbers.

Thus the following is true.

THEOREM 2.2: *Let $\langle Z_p \cup g \rangle_I$ be the natural neutrosophic dual numbers p a prime. $\langle Z_p \cup g \rangle_I$ has at least p natural neutrosophic dual numbers.*

Proof is direct and hence left as an exercise to the reader.

Now how to define the operation + on $\langle Z_n \cup g \rangle_I$.

First we will describe this by some examples.

Example 2.11: Let $\langle Z_{10} \cup g \rangle_I$ be the natural neutrosophic dual numbers.

$\langle Z_{10} \cup g \rangle = \{0, 1, 2, \ldots, 9, g, 2g, \ldots, 9g, 1 + g, 2 + g, \ldots, 9 + g, 1 + 2g, 2 + 2g, \ldots, 9 + 2g, \ldots, 9 + 9g\}$ be the dual numbers.

$\langle Z_{10} \cup g \rangle_I = \{\langle Z_{10} \cup g \rangle_I, I_0^g, I_{2g}^g, I_{4g}^g, I_{6g}^g, I_{8g}^g, I_{5g}^g, I_{2+2g}^g,$

$I_{2+4g}^g, I_{2+6g}^g, I_{2+8g}^g, I_{4+2g}^g, I_{4+4g}^g, I_{4+6g}^g, I_{4+8g}^g, I_2^g, I_4^g, I_6^g, I_8^g$

and so on}.

It is not an easy task to find the number of natural neutrosophic dual numbers.

$$I_{2+4g}^g \times I_5^g = I_0^g;$$

$$I_{4+2g}^g \times I_{5g}^g = I_0^g \text{ and so on.}$$

It is easily verified some of the natural neutrosophic dual numbers are nilpotents and some are idempotents.

This study is both innovative and interesting.

Example 2.12: Let $\langle Z_2 \cup g \rangle_I = \{0, 1, g, 1 + g, I_0^g, I_g^g \}$.

$\langle Z_2 \cup g \rangle_I$ under + generates the following.

The set of natural neutrosophic dual numbers set under + is as follows:

$S = \{\langle Z_2 \cup g \rangle_I, +\} = \{0, 1, g, 1 + g, I_0^g, I_g^g, 1 + I_0^g, g + I_0^g, 1 + g + I_0^g, I_0^g + I_g^g, I_g^g + 1, I_g^g + g, g + I_0^g + I_g^g, 1 + I_0^g + I_g^g, 1 + g + I_g^g, 1 + g + I_g^g + I_0^g\}$.

Thus order of S is 16.

We define $I_0^g + I_0^g = I_0^g$ and $I_g^g + I_g^g = I_g^g$ and there are idempotents.

$1 + I_0^g + 1 + I_0^g = I_0^g$ and so on.

Thus this is only a semigroup as $I_0^g + I_0^g = I_0^g$ and is an idempotent so S is only defined as the natural neutrosophic dual number semigroup.

Example 2.13: Let $\langle Z_4 \cup g \rangle_I = \{1, 2, 0, 3, g, 2g, 3g, 1 + g, 1 + 2g, 1 + 3g, 2 + g, 2 + 2g, 2 + 3g, 3 + g, 3 + 2g, 3 + 3g, I_0^g, I_g^g, I_{2g}^g, I_2^g, I_{3g}^g, I_{2+2g}^g, I_{2+g}^g, I_{2+3g}^g\}$

$S = \{\langle Z_4 \cup g \rangle_I, +\} = \{\langle Z_4 \cup g \rangle_I, 1 + I_0^g, 2 + I_0^g, 3 + I_0^g, g + I_0^g, 2g + I_0^g, 3g + I_0^g, 1 + g + I_0^g, 1 + 2g + I_0^g, \dots, 3 + 3g + I_0^g + I_g^g + I_{2g}^g + I_{3g}^g + I_{2+2g}^g + I_{2+g}^g + I_{2+3g}^g\}$.

Thus S is only a natural neutrosophic dual number semigroup.

Problem 2.2: Let $S = \{\langle Z_n \cup g \rangle_I, +\}$ be the natural neutrosophic dual number semigroup.

    i.   What is the order of S if n is a prime?

    ii.   What is the order of S if n is not a prime?

We have to find the subsemigroups of S.

We will first illustrate this by some examples.

Example 2.14: Let $S = \{\langle Z_5 \cup g \rangle_I, +\}$ be the natural neutrosophic dual number semigroup.

$P_1 = \{Z_5, +\}$ is a subgroup of S so S is a Smarandache semigroup of natural neutrosophic dual numbers.

$P_2 = \{\langle Z_5 \cup g \rangle, +\}$ is also a subgroup of S.

$P_3 = \{Z_5g = \{0, g, 2g, 3g, 4g\}, +\}$ is a group under +.

$\langle Z_5 \cup g \rangle_I = \{\langle Z_5 \cup g \rangle, I_0^g, I_g^g, I_{2g}^g, I_{3g}^g, I_{4g}^g, 4 + 4g + I_0^g, 4 + 4g + I_g^g, 4 + 4g + I_{2g}^g, \ldots, 4 + 4g + I_0^g + I_g^g + I_{2g}^g + I_{3g}^g + I_{4g}^g \}$ has subgroups and subsemigroups which are not subgroups.

For $P_4 = \{Z_5, I_0^g, I_0^g + 1, I_0^g + 2, I_0^g + 3, I_0^g + 4\}$ is only a subsemigroup of S and is not a subgroup of S as $I_0^g + I_0^g = I_0^g$.

Example 2.15: Let $S = \{\langle Z_{15} \cup g \rangle_I, +\}$ be the natural neutrosophic dual number semigroup.

$N_1 = Z_{15}$, $N_2 = Z_{15}g$ and $N_3 = \langle Z_{15} \cup g \rangle$ are all subsemigroups which are subgroups.

So S is a Smarandache semigroup.

Finding order of S is a difficult job I_0^g, I_3^g, I_6^g, I_9^g, I_{12}^g, I_{10}^g and I_5^g are some of the natural neutrosophic dual numbers.

$$I_{10}^g + I_{10}^g = I_{10}^g,$$

$$I_5^g + I_5^g = I_5^g \text{ and so on.}$$

In view of all these the following theorem can be easily proved.

THEOREM 2.3: *Let* $S = \{\langle Z_n \cup g \rangle_I, +\}$ *be the natural neutrosophic dual number semigroup.*

   i.   *S is a Smarandache semigroup.*
   ii. *S has idempotents.*
   iii. *S has subsemigroups which has group structure.*
   iv. *S has subsemigroups which do not have a group structure.*

Thus we by this method get natural neutrosophic dual number semigroups which has idempotent under +.

Next we proceed onto define product operation on $\langle Z_n \cup g \rangle_I$.

There are two such semigroups of natural neutrosophic dual numbers are just generated by $\langle Z_n \cup g \rangle_I$ another S under the operation ×.

We will first illustrate this situation by some examples.

Example 2.16: Let
M = $\{\langle Z_6 \cup g \rangle_I, \times\}$ be the semigroup under ×.

M = $\{0, 1, 2, 3, 4, 5, g, 2g, 3g, 4g, 5g, I_0^g, I_2^g, I_3^g, I_4^g, I_g^g, I_{2g}^g, I_{3g}^g, I_{4g}^g, I_{5g}^g, \times\}$ and o(M) = 20.

$N_1 = \{0, 1, 2, 3, 4, 5\} \subseteq M$ is a subsemigroup of order 6 and $6 \nmid 20$.

$N_2 = \{0, 1, 2, 3, 4, 5, g, 2g, 3g, 4g, 5g\} \subseteq M$ is a subsemigroup of M.

Now $N_3 = \{1, 2, 3, 4, 5, 0, I_0^g\} \subseteq M$ is also a subsemigroup of G.

$N_4 = \{1, 5\} \subseteq M$ is a subsemigroup which is a subgroup of order 2. Thus M is a Smarandache semigroup.

This semigroup has zero divisors, idempotents and nilpotents of order two.

Example 2.17: Let $M = \{\langle Z_7 \cup g\rangle_I, \times\}$ be the natural neutrosophic dual number semigroup.

M is a S-semigroup as $Z_7 \setminus \{0\} \subseteq M$ is a group.

However M has other subsemigroups. Infact M has zero divisors.

$I_{6g}^g \times I_{6g}^g = I_0^g$ is a natural neutrosophic dual number nilpotent element of order two.

$I_{5g}^g \times I_{5g}^g = I_0^g$ so every natural neutrosophic dual number in M is a nilpotent element of order two.

Example 2.18: Let $M = \{\langle Z_{14} \cup g\rangle_I, \times\}$ be the natural neutrosophic dual number semigroup.

I_0^g, I_2^g, I_4^g, I_6^g, I_8^g, I_{10}^g, I_{12}^g, I_g^g, I_{2g}^g, I_{3g}^g, I_{4g}^g, ..., I_{13g}^g, I_{2+2g}^g, I_{2+4g}^g, I_{2+6g}^g, I_{2+8g}^g, I_{2+10g}^g, I_{2+12g}^g, I_{4+2g}^g, I_{4+4g}^g, I_{4+6g}^g, I_{4+8g}^g, I_{4+10g}^g, I_{4+12g}^g, I_{6+2g}^g, I_{6+4g}^g, ..., I_{6+12g}^g, ..., I_{12+2g}^g, I_{12+4g}^g, ...,

I_{12+12g}^g, I_{7+7g}^g, I_{7+2g}^g, …, I_{7+12g}^g, I_{2+7g}^g, …, I_{12+7g}^g and so on} are all natural neutrosophic dual numbers.

$$I_{12+7g}^g \times I_{2g}^g = I_{10g}^g$$

$$I_{7+7g}^g \times I_{2g}^g = I_0^g$$

so this semigroup has zero divisors.

In view of all these we have the following theorem.

THEOREM 2.4: *Let* $M = \{\langle Z_n \cup g \rangle_I, \times\}$ *be the natural neutrosophic dual number semigroup.*

(i)   *M is a S-semigroup if and only if Z_n is a S-semigroup.*
(ii)   *M has at least (m − 1) distinct nilpotent elements of order two.*
(iii)   *M has at least (m − 1) distinct natural neutrosophic dual numbers which are nilpotents of order two.*

Proof is direct hence left as an exercise to the reader.

Now having seen examples of natural neutrosophic dual number semigroup under product we proceed onto study natural neutrosophic dual number semigroup using $S = \{\langle Z_n \cup g \rangle_I, +\}$.

Clearly $\{S, \times\}$ is again a natural neutrosophic dual number semigroup under $\times$. Further $M \subseteq \{S, \times\}$.

We will first illustrate this situation by an example or two.

Example 2.19: Let $M = \{\{\langle Z_{10} \cup g \rangle_I, +\}, \times\}$ be the natural neutrosophic dual number semigroup.

$$I_0^g, I_2^g, I_4^g, I_6^g, I_8^g, I_g^g, I_{2g}^g, I_{3g}^g, I_{4g}^g, \ldots, I_{9g}^g,$$

$$I_{3g}^g + I_{4g}^g, I_{2g}^g + I_{9g}^g \text{ and so on.}$$

$$\left(I_g^g + I_{2g}^g + I_{4g}^g\right) \times \left(I_{8g}^g + I_{2g}^g\right) = I_0^g.$$

Thus M has zero divisors.

$$5 \times 2 = 0, \text{ for } 5, 2 \in Z_{10}$$

$$5g \times 2g = 0 \text{ for } 5, 2 \in Z_{10}$$

$$g \times 4g = 0 \text{ for } g, 4g \in \langle Z_{10} \cup g \rangle.$$

Thus M has nilpotent elements of order two.

$$x = \left(I_{4g}^g + I_{2g}^g + I_{6g}^g + I_{5g}^g\right) \text{ and}$$

$$y = \left(I_{3g}^g + I_{7g}^g + I_{9g}^g + I_g^g\right) \in M.$$

$x \times y = I_0^g$ is a neutrosophic nilpotent element of order two.

Example 2.20: Let M = $\{\langle Z_{20} \cup g \rangle_I, +, \times\}$ be the natural neutrosophic dual number semigroup.

Let $x = I_{16g}^g + I_{2g}^g + I_g^g + I_{14g}^g \in M$ is such that

$x \times x = I_0^g$ is only a natural neutrosophic dual number zero divisor. M has several nilpotent neutrosophic zero divisor.

Let $x = 10 + I_{10g}^g$ and

$$y = 2 + I_g^g \in M.$$

$$x \times y = \left(10 + I_0^g\right) . \left(2 + I_g^g\right)$$

$$= 2\,I_{10}^g + 10\,I_g^g = I_{10}^g + I_g^g \in M.$$

Example 2.21: Let $M = \{\langle\langle Z_{19} \cup g\rangle_I, +\rangle \times\}$ be the natural neutrosophic dual number semigroup.

$P_1 = Z_{19} \setminus \{0\} \subseteq M$ under $\times$ is a group.

Thus M is a S-semigroup.

$P_2 = \{Z_{19}g, \times\}$ be the zero square subsemigroup.

$P_3 = \{\langle Z_{19} \cup g\rangle, \times\}$ is the subsemigroup.

Example 2.22: Let $W = \{\langle\langle Z_{18} \cup g\rangle, +\rangle, \times\}$ be the natural neutrosophic dual number semigroup.

$P_1 = \{Z_{18}, \times\}$ be the subsemigroup.

$P_2 = \{\langle Z_{18} \cup g\rangle, \times\}$ be the subsemigroup.

I_0^g, I_2^g, I_4^g, I_6^g, I_8^g, ..., I_g^g, I_{2g}^g, ..., I_{17g}^g are some of the natural neutrosophic dual numbers.

$$x = I_{2g}^g + I_{5g}^g + I_{10g}^g + I_{11g}^g \text{ and}$$

$$y = I_{8g}^g + I_{17g}^g + I_{16g}^g \in W.$$

Clearly $x \times y = I_0^g$; thus this is a natural neutrosophic dual number zero divisor.

Also $x^2 = I_0^g$ and $y^2 = I_0^g$ are natural neutrosophic dual number nilpotent elements.

Example 2.23: Let $W = \{\langle\langle Z_{11} \cup g\rangle_I, +\rangle, \times\}$ be the natural neutrosophic dual number semigroup.

W has nilpotent elements, natural neutrosophic nilpotent elements and zero divisors and natural neutrosophic zero divisor.

Example 2.24: Let $M = \{\langle \langle Z_{12} \cup g\rangle_I, +\rangle; \times\}$ be the natural neutrosophic dual number semigroup.

x = 3 and y = 8 ∈ M is such that x × y = 0 ∈ M.

$x = I^g_{10g}$ and $y = I^g_{2g} \in M$; $x \times y = I^g_0$ is natural neutrosophic dual number zero divisor.

$I^g_{10g} \times I^g_{10g} = I^g_0$ is the natural neutrosophic dual number nilpotent element of order two. $6 \times 6 = 0 \pmod{12}$ is a nilpotent element of order two.

Next we proceed onto describe the natural neutrosophic dual number semiring.

$S = \{\langle Z_n \cup g\rangle_I, +, \times\}$ is defined as the natural neutrosophic dual number semiring.

Clearly S is of finite order so we get finite semirings a long standing problem about semirings.

For semiring of finite order of finite characteristic were got by finite distributive lattice and lattice groups taking finite groups only or lattice semigroups taking finite semigroups.

Now we will first describe this situation by some examples.

Example 2.25: Let $S = \{\langle Z_3 \cup g \rangle_I, +, \times\} = \{0, 1, 2, g, 2g, I_0^g, I_g^g,$
$I_{2g}^g, 1 + I_0^g, 2 + I_0^g, g + I_0^g, 2g + I_0^g, 1 + I_g^g, 2 + I_g^g, 2g + I_g^g, g +$
$I_g^g, 1 + I_{2g}^g, 2 + I_{2g}^g, g + I_{2g}^g, 2g + I_{2g}^g, I_0^g + I_g^g, I_0^g + I_{2g}^g, I_g^g +$
$I_{2g}^g, I_0^g + I_g^g + I_{2g}^g, 1 + I_0^g + I_g^g, 2 + I_0^g + I_g^g, g + I_0^g + I_g^g, 2g +$
$I_0^g + I_g^g, 1 + I_0^g + I_{2g}^g, 2 + I_0^g + I_{2g}^g, g + I_0^g + I_{2g}^g, 2g + I_0^g + I_{2g}^g,$
$1 + I_g^g + I_{2g}^g, 2 + I_g^g + I_{2g}^g, g + I_g^g + I_{2g}^g, 2g + I_g^g + I_{2g}^g, 1 + I_0^g +$
$I_g^g + I_{2g}^g, 2 + I_0^g + I_g^g + I_{2g}^g, g + I_0^g + I_g^g + I_{2g}^g, 2g + I_0^g + I_g^g +$
$I_{2g}^g, +, \times\}$ be the natural neutrosophic dual number semiring.

$$o(S) = 40.$$

Clearly S has zero divisors. S also has natural neutrosophic dual number zero divisors.

S has also nilpotents as well as natural neutrosophic dual number nilpotents.

$2g \times g = 0$ is a zero divisor.

$g \times g = 0$ is a nilpotent element of order two.

$x = I_g^g + I_{2g}^g$ is such that $x \times x = I_0^g$ is a natural neutrosophic dual number nilpotent of order two.

Let $x = I_g^g + I_{2g}^g$ and $y = I_0^g + I_{2g}^g \in S$.

Clearly $x \times y = I_0^g$ so is a natural neutrosophic zero divisor.

Thus S has zero divisors and natural neutrosophic dual number zero divisor.

$P_1 = \{Z_3, + \times\}$ is a field.

Can S have semifields which are not fields? The answer is no.

Several properties about them will be discussed in the following first through examples and then by proving results.

Example 2.26: Let $S = \{\langle Z_7 \cup g \rangle_I, +, \times\}$ be the natural neutrosophic dual number semiring; $o(S) < \infty$.

S has zero divisors, natural neutrosophic dual number zero divisors and nilpotents.

This semiring has subsemiring. Finding ideals in S is a difficult task.

Example 2.27: Let $S = \{\langle Z_{12} \cup g \rangle, +, \times\}$ be the natural neutrosophic dual number semiring.

S has subsemirings. Further S has zero divisors, natural neutrosophic dual number zero divisors.

S has nilpotents as well as natural neutrosophic dual number nilpotents of order two.

Several important properties can be determined.

THEOREM 2.5: *Let* $S = \{\langle Z_n \cup g \rangle, +, \times\}$ *be the natural neutrosophic dual number semiring. The following facts are true.*

   i.   *$o(S) < \infty$.*
   ii. *S has zero divisors.*
   iii. *S has natural neutrosophic dual number zero divisors.*
   iv. *S has nilpotent elements of order two.*
   v. *S has natural neutrosophic dual number nilpotent elements of order two.*

Proof is direct and hence left as an exercise to the reader.

All properties of semirings is true in case of these natural neutrosophic dual number semirings.

Next we proceed onto describe the notion of natural neutrosophic special dual like numbers.

Example 2.28: Let B = $\{\langle Z_4 \cup h \rangle_I, h^2 = h\}$ be the special dual like numbers.

P = $\{\langle Z_4 \cup h \rangle_I; h^2 = h\}$ = {0, 1, 2, 3, h, 2h, 3h, 1 + h, 2 + h, 3 + h, 1 + 2h, 2 + 2h, 3 + 3h, 1 + 3h, 3 + 2h, 2 + 3h, I_0^h, I_2^h, I_h^h, I_{2h}^h, I_{3h}^h, I_{2+2h}^h, I_{2+h}} be the natural neutrosophic special dual like numbers. This is of finite order.

Example 2.29: Let L = $\{\langle Z_{11} \cup h \rangle_I, h^2 = h\}$ = {0, 1, 2, …, 10, h, 2h, …, 10h, 1 + h, 2 + h, …, 10 + h, 1 + 2h, 2 + 2h, I_0^h, …, 10 + 2h, 1 + 10h, 2 + 10h, I_{2h}^h, …, 10 + 10h, I_{3h}^h, …, I_{10h}^h} be the natural neutrosophic special dual like numbers.

Next we proceed onto define some more operations on $\{\langle Z_n \cup h \rangle_I, h^2 = h\}$.

Example 2.30: Let S = $\{\langle Z_2 \cup h \rangle_I, +\}$ = {0, 1, h, 1 + h, I_0^h, I_h^h, I_{1+h}^h, 1 + I_0^h, 1 + I_h^h, 1 + I_{1+h}^h, I_0^h + I_h^h, I_0^h + I_{1+h}^h, I_h^h + I_{1+h}^h, I_0^h + I_h^h + I_{1+h}^h, 1 + I_0^h + I_h^h, 1 + I_0^h + I_{1+h}^h, 1 + I_h^h + I_{1+h}^h, 1 + I_0^h + I_h^h + I_{1+h}^h}.

$$\because \quad I_h^h + I_h^h = I_h^h, \quad I_0^h + I_0^h = I_0^h,$$

$I_{1+h}^h + I_{1+h}^h = I_{1+h}^h$ we see S under + is only a semigroup.

Thus we have given a finite semigroup of order 18. S has elements which satisfy a + a = a for a ∈ S.

Example 2.31: Let $S = \{\langle Z_6 \cup h\rangle_I, h^2 = h, +\} = \{0, 1, 2, 3, 4, 5,$ h, 2h, 3h, 4h, 5h, 1 + h, 2 + h, …, 5 + h, 1 + 2h, 2 + 2h, …, 5 + 2h, 1 + 3h, 2 + 3h, …, 5 + 3h, 1 + 4h, 2 + 4h, …, 5 + 4h, 5 + h, 5 + 2h, …, 5 + 5h, I_0^h, I_h^h, I_{2h}^h, I_{3h}^h, I_{4h}^h, I_{5h}^h, I_2^h, I_3^h, I_4^h, I_{2+2h}^h, I_{2+4h}^h, I_{4+2h}^h, I_{3+3h}^h, I_{5+h}^h, I_{4+2h}^h, I_{2+4h}^h, I_{5h+1}^h, I_{2+h}^h, I_{2h+1}^h, I_{1+h}^h, I_{3+h}^h, I_{3h+1}^h and so on} be the natural neutrosophic special dual like number semigroup under +.

Now $(Z_6, +) = P_1 \subseteq S$ is a subsemigroup which is a group. $\langle Z_6 \cup h\rangle = P_2$ is also a group under +.

Thus S is a Smarandache semigroup.

In view of this we have the following theorem.

THEOREM 2.6: *Let $S = \{\langle Z_n \cup h\rangle_I, h^2 = h, +\}$ be the natural neutrosophic special dual like number semigroup.*

    i.   *S has natural neutrosophic idempotents.*
    ii.   *S is a S-semigroup.*

Proof: Clearly $I_{kh}^h + I_{kh}^h = I_{kh}^h$ for all $k \in Z_n$.

Hence (i) is true.

Consider $(Z_n, +) \subseteq S$; clearly $(Z_n, +)$ is a group so S is a Smarandache semigroup.

Now we proceed onto define product on $\langle Z \cup h\rangle_h$. First we represent this by examples.

Example 2.32: Let $M = \{\langle Z_5 \cup h\rangle_I, \times\} = \{0, 1, 2, 3, 4, h, 2h, 3h,$ 4h, 1 + h, 1 + 2h, 1 + 3h, 1 + 4h, 2 + h, 2 + 2h, 2 + 3h, 2 + 4h, 3

+ h, 3 + 2h, 3 + 3h, 3 + 4h, 4 + h, 4 + 2h, 4 + 3h, 4 + 4h, I_0^h,

$I_h^h, I_{2h}^h,$ $I_{3h}^h,$ $I_{4h}^h, I_{2+3h}^h, I_{2h+3}^h, I_{h+4}^h, I_{4h+1}^h$ } be the natural

neutrosophic special dual like number semigroup.

Clearly M has zero divisors for x = 2 + 3h and y = h.

x × y = (2 + 3h)h = 0(mod 5). x = 3 + 2h and y = h is such that x × y = 0.

x = I_h^h and y = I_{2+3h}^h ∈ M; x × y = I_h^h × I_{2+3h}^h = I_0^h.

Thus M has natural neutrosophic special dual like zero divisors. P = {Z_5 \ {0}, ×} is a group so S is a Smarandache semigroup.

Example 2.33: Let
M = {⟨Z_9 ∪ h⟩$_I$, h^2 = h, ×} be the natural neutrosophic special dual like number semigroup.

Let x = 3 and y = 3h + 3 ∈ M.

x × y = 3 × 3h + 3 = 0 is a zero divisor.

Let x = 6h and y = 3 + 3h ∈ M. x × y = 0 is again a zero divisor. Consider I_{3h}^h and I_{6h}^h ∈ M.

I_{3h}^h × I_{6h}^h = I_0^h is a natural neutrosophic special dual like number zero divisor.

Clearly h × h = h is an idempotent in M.

Similarly I_h^h × I_h^h = I_h^h is a natural neutrosophic special dual like number idempotent.

Example 2.34: $M = \{\langle Z_{24} \cup h\rangle_I, \times\}$ be the natural neutrosophic special dual like number semigroup.

$M = \{\langle Z_{24} \cup h\rangle, I_0^h, I_2^h, I_3^h, I_6^h, I_4^h, I_8^h, I_{10}^h, I_{12}^h, I_{14}^h, I_{16}^h, I_{18}^h, I_{20}^h, I_{22}^h, I_9^h, I_{15}^h, I_{21}^h, I_h^h, I_{2h}^h, \ldots, I_{23h}^h, I_{2+2h}^h, I_{1+23h}^h, I_{23+h}^h, \ldots, I_{2+22h}^h, I_{22+2h}^h$ and so on$\}$ be the natural neutrosophic special dual like number semigroup.

M has zero divisors and nilpotents as well as natural neutrosophic special zero divisors and natural neutrosophic special nilpotents.

$x = 6$ and $y = 8 + 12h \in M$.

$x \times y = 6 \times 12h + 8 = 0$ is a zero divisor.

$x = 12h + h$ and $y = 12 + 8h \in M$ is such that $x \times y = 0$ is a zero divisor. $x = I_{6h}^h$ and $y = I_{12+12h}^h \in M$ is such that $x \times y = I_0^h$ is a natural neutrosophic zero divisors.

Consider $x = 12 + 12\,h \in M$; clearly $x \times x = 12 + 12h \times 12 + 12h = 0 \in M$ is a nilpotent element of order two.

$$x = I_{12+12h}^h \in M; \; x \times x = I_0^h.$$

Now having seen the special elements we proceed onto give more examples.

Example 2.35: Let

$M = \{\langle Z_{10} \cup h\rangle_I, h^2 = h, \times\} = \{0, 1, 2, \ldots, 9, h, 2h, \ldots, 9h, 1 + h, 1 + 2h, \ldots, 1 + 9h, 2 + h, 2 + 2h, \ldots, 2 + 9h, \ldots, 9 + h, 9 + 2h, 9 + 3h, \ldots, 9 + 9h, I_0^h, I_2^h, I_4^h, I_6^h, I_8^h, I_5^h, I_{1+9h}^h, I_{9+h}^h, I_{2+8h}^h, I_{8+2h}^h, I_{5+5h}^h, I_{3+7h}^h, I_{7+3h}^h, I_{4+6h}^h, I_{6+4h}^h$ and so on$\}$ be the natural neutrosophic special dual like number semigroup.

S has zero divisors and idempotents also natural neutrosophic special dual like zero divisors and idempotents.

We prove some theorems based on these examples.

THEOREM 2.7: *Let* $M = \{\langle Z_n \cup h \rangle_I, h^2 = h, \times\}$ *be the natural neutrosophic special dual like number semigroup.*

i. *M is a S-semigroup if and only if Z_n is a S-semigroup.*
ii. *M has zero divisors and special dual like number of zero divisors.*
iii. *M has idempotents as well as special dual like number idempotents.*

Proof is direct and hence left as an exercise to the reader.

Now we can define the other type of natural neutrosophic special dual like number semigroup built using the additive semigroup $\{\langle Z_n \cup h \rangle_I, +\}$.

We will first illustrate this by example.

Example 2.36: Let
$S = \{\langle\langle Z_6 \cup h \rangle_I, +\rangle, \times, h^2 = h\}$ be the natural neutrosophic special dual like semigroup under product.

Clearly $M = \{\langle Z_6 \cup h \rangle_I, h^2 = h, \times\} \subseteq S$ as a subsemigroup.

Now S contains elements of the form

$$y = I_3^h + I_2^h + I_{2h}^h \text{ and}$$

$$x = I_4^h + I_{4h}^h + I_0^h \in M.$$

$$x \times y = \left(I_4^h + I_{4h}^h + I_0^h\right) \times \left(I_3^h + I_2^h + I_{2h}^h\right)$$

$$= I_0^h + I_0^h + I_0^h + I_2^h + I_{2h}^h + I_0^h + I_{2h}^h + I_{2h}^h + I_{20}^h$$

$$= I_0^h + I_{2h}^h + I_2^h .$$

This is the way product is performed on S. This semigroup has bigger cardinality.

This semigroup has several subsemigroups.

Example 2.37: Let $S = \{\langle\langle Z_{11} \cup h\rangle_I, +\rangle, \times\}$ be the natural neutrosophic special dual like number semigroup.

S is a Smarandache semigroup as $Z_{11} \setminus \{0\} = P_1$ is a group under product.

$S = \{\langle Z_{11} \cup h\rangle, I_0^h, I_h^h, I_{2h}^h, ..., I_{10h}^h, I_{10+h}^h, I_{1+10h}^h, I_{9+2h}^h,$
$I_{2+9h}^h, I_{3+8h}^h, I_{8+3h}^h, I_{7+4h}^h, I_{4+7h}^h, I_{6+5h}^h, I_{5+6h}^h$ and so on and sums $2 + 3h + I_0^h + I_{2+9h}^h$ and so on$\}$.

Clearly $o(S) < \infty$.

$I_{2+9h}^h \times I_{5h}^h = I_0^h$ is a natural neutrosophic zero divisor.

$I_{2+9h}^h \times I_{4h}^h = I_0^h$ is again a natural neutrosophic zero divisor.

$2 + 9h \times 5h = 0$ (mod 11) and $2 + 9h \times 4h = 0$ (mod 11) are both zero divisors of S.

In view of this we have the following theorem.

THEOREM 2.8: *Let $S = \{\langle\langle Z_n \cup h\rangle_I, +\rangle, h^2 = h, \times\}$ be the natural neutrosophic special dual like number semigroup*
.

      i)   *S is a S-semigroup if and only if Z_n is a S-semigroup under $\times$.*

ii)   S has both zero divisors and natural neutrosophic
  special dual like number zero divisors.

iii) S has idempotents as well as natural neutrosophic
  idempotents.

Proof is direct and hence left as an exercise to the reader.

Next the notion of natural neutrosophic special dual number semirings are developed and described.

Example 2.38: Let S = $\{\langle Z_6 \cup h\rangle_I, +, \times\}$ be the natural neutrosophic special dual like number semiring.

Clearly S has zero divisors as well as natural neutrosophic zero divisors.

S has idempotents as well as natural neutrosophic special dual like number idempotents.

For $3 \times 3 = 3 \pmod 6$

$4 \times 4 = 4 \pmod 6$ are idempotents of S.

$$I_3^h \times I_3^h = I_3^h \qquad I_{4h}^h \times I_{4h}^h = I_{4h}^h$$

and $I_h^h \times I_h^h = I_h^h$ are natural neutrosophic idempotents of S.

Clearly $4 \times 3h = 0$, $2h \times 3h = 0$, $4 \times 3 \equiv 0$ are zero divisors of S.

$$I_{2+4h}^h \times I_{5h}^h = I_0^h, \qquad I_{h+5}^h \times I_{4h}^h = I_0^h,$$

$$I_{3h+3}^h \times I_{2h}^h = I_0^h \text{ and } \qquad I_{3h}^h \times I_4^h = I_0^h$$

are all natural neutrosophic special dual like number zero divisors.

Thus this semiring is not a semifield.

Further if $\alpha = I^h_{3+3h} + I^h_{4+2h} + I^h_{1+5h} + I^h_{5+h}$ and $\beta = I^h_{2h} + I^h_{4h} + I^h_{5h} + I^h_0 \in S$ then $\alpha \times \beta = I^h_0$ is a natural neutrosophic zero divisor of S.

$Z_6 = P_1$ is a ring in S.

$\{\langle Z_6 \cup h \rangle\} = P_2$ is again a ring in S. Thus S has also subsemirings which are rings.

Under these special conditions we define yet a new notion on semirings.

DEFINITION 2.1: *Let S be a semiring. S is said to be Smarandache Super Strong semiring (SSS-semiring) if S contains a subset P which is a field. S is said to be Smarandache Strong semiring (SS-semiring) if S contains a subset P which is a ring which is not a field.*

We first give examples of the definition.

Example 2.39: Let
$S = \{\langle Z_7 \cup g \rangle_I, g^2 = 0, +, \times\}$ be the natural neutrosophic dual number semiring.

$Z_7 \subseteq S$ is a field. So S is a SSS-semiring. $\langle Z_7 \cup g \rangle \subseteq S$ is a ring hence S is a SS-semiring.

Example 2.40: Let
$S = \{\langle Z_{15} \cup g \rangle_I, g^2 = 0, +, \times\}$ be the natural neutrosophic dual number semiring.

S is a SS-semiring as $\langle Z_{15} \cup g \rangle \subseteq S$ is a ring. $P_1 = \{0, 3, 6, 9, 12\} \subseteq S$ is a field with 6 as the multiplicative identity.

Clearly 12 is the inverse of 3 and vice versa as $3 \times 12 = 6$ the identity of P_1.

$9 \times 9 = 6$ the identity of P_1. Hence S is also a SSS-semiring.

Example 2.41: Let $S = \{\langle Z_{24} \cup h \rangle_I, h^2 = h, +, \times\}$ be the natural neutrosophic special dual like number semiring.

Clearly $P_1 \subseteq S$ is a ring so S is a SS-semiring.

$P_2 = \{0, 8, 16\} \subseteq S$ is a field with 16 as the identity with respect to product. Thus S is a SSS-semiring.

Example 2.42: Let
$M = \{(\langle Z_{17} \cup h \rangle_I), h^2 = h, +, \times\}$ be the natural neutrosophic special dual like number semiring. M is a SSS-semiring as well as SS-semiring.

So a natural neutrosophic semiring can be both a SSS-semiring as well as SS-semiring.

Next we proceed onto describe the concept of natural neutrosophic special quasi dual number sets using Z_n by some examples.

Example 2.43: Let $S = \{\langle Z_8 \cup k \rangle_I, k^2 = 7k\} = \{Z_8, k, 2k, \ldots, 7k,$ $1 + k, 2 + k, \ldots, 7 + k, 1 + 2k, 2 + 2k, \ldots, 7 + 2k, 3k + 1, 2 + 3k,$ $\ldots, 7 + 3k, \ldots, 7k + 1, 7k + 2, \ldots, 7k + 7, I_2^k, I_0^k, I_4^k, I_6^k, I_{2k}^k,$ $I_k^k, I_{3k}^k, \ldots, I_{7k}^k, I_{1+7k}^k, I_{2+5k}^k, I_{7+k}^k, \ldots\}$

S is a natural neutrosophic special quasi dual number set.

Example 2.44: Let
$S = \{\langle Z_3 \cup k \rangle_I, k^2 = 2k\} = \{0, 1, 2, 2k, k, 1 + k, 2 + k, 1 + 2k, 2 + 2k, I_{2+k}^k, I_{1+2k}^k, I_k^k, I_{2k}^k, I_0^k \}$ be the natural neutrosophic special quasi dual number set.

Example 2.45: Let
$S = \{\langle Z_2 \cup k \rangle_I, k^2 = k\} = \{0, 1, k, 1 + k, I_0^k, I_k^k, I_{1+k}^k \}$ be the natural neutrosophic special quasi dual number set.

Example 2.46: Let
$S = \{\langle Z_4 \cup k \rangle_I, k^2 = 3k\} = \{0, 1, 2, 3, k, 2k, 3k, 1 + k, 2 + k, 3 + k, 1 + 2k, 2 + 2k, 3 + 2k, 1 + 3k, 2 + 3k, 3 + 3k, I_0^k, I_2^k, I_{2k}^k, I_k^k, I_{3k}^k, I_{2+2k}^k, I_{1+3k}^k, I_{k+3}^k, I_{3+3k}^k \}$ be the natural neutrosophic special quasi dual number set.

Example 2.47: Let $S = \{\langle Z_5 \cup k \rangle_I, k^2 = 4k\} = \{0, 1, 2, 3, 4, k, 2k, 3k, 4k, 1 + k, 2 + k, 3 + k, 4 + k, 1 + 2k, 2 + 2k, 1 + 3k, 2 + 3k, 3 + 3k, 4 + 3k, 1 + 4k, 2 + 4k, 3 + 4k, 4 + 4k, 3 + 2k, 4 + 2k, I_0^k \ I_k^k, I_{2k}^k, I_{3k}^k, I_{4k}^k, I_{1+4k}^k, I_{4+k}^k, I_{2+3k}^k, I_{3+2k}^k, I_{4+4k}^k \}$ be the natural neutrosophic special quasi dual number set.

Example 2.48: Let $S = \{\langle Z_7 \cup k \rangle_I, k^2 = 6k\}$ be the natural neutrosophic special quasi dual number set.

$S = \{0, 1, 2, …, 6, k, 2k, 3k, 4k, 5k, 6k, I_0^k, I_k^k, I_{2k}^k, I_{3k}^k, I_{4k}^k, I_{5k}^k, I_{6k}^k, I_{k+6}^k, I_{6k+1}^k, I_{2k+5}^k, I_{5k+2}^k, I_{4+3k}^k, I_{3+4k}^k, I_{6+6k}^k, 1 + k, 1 + 2k, 1 + 3k, 1 + 4k, 1 + 5k, 1 + 6k, 2 + k, 2 + 2k, 2 + 3k, 2 + 4k, 2 + 5k, 2 + 6k, 3 + k, 3 + 2k, 3 + 3k, 3 + 4k, 3 + 5k, 3 + 6k, 4 + k, 4 + 2k, 4 + 3k, 4 + 4k, 4 + 5k, 4 + 6k, 5 + k, 5 + 2k,$

5 + 3k, 5 + 4k, 5 + 5k, 5 + 6k, 6 + k, 6 + 2k, 6 + 3k, 6 + 4k, 6 + 5k, 6 + 6k}

Example 2.49: Let $S = \{\langle Z_{12} \cup k\rangle_I, \ k^2 = 11k\}$ be the natural neutrosophic special quasi dual number.

$S = \{0, 1, 2, \ldots, 11k, k, 2k, \ldots, 11k, 1 + k, 2 + k, \ldots, 11 + k,$ $1 + 2k, 2 + 2k, \ldots, 11 + 2k, 1 + 3k, 2 + 3k, 3 + 3k, \ldots, 11 + 3k,$ $\ldots, 11k + 1, 11k + 2, \ldots, 11 + 11k, I_0^k, \ I_{1+11k}^k, \ I_{11+k}^k, \ldots, I_{5+7k}^k,$ $I_{7+5k}^k, \ I_{6+6k}^k\}$ be the natural neutrosophic special quasi dual number set.

Now we proceed onto give operations + and × on the set $S = \{\langle Z_n \cup k\rangle_I; \ k^2 = (n-1)k\}$ under + is described.

Example 2.50: Let

$S = \{\langle Z_{10} \cup k\rangle_I, \ k^2 = 9k, \times\} = \{0, 1, 2, \ldots, 9, k, 2k, 3k, \ldots, 9k,$ $1 + k, 1 + 2k, \ldots, 1 + 9k, 2 + k, 2 + 2k, 2 + 3k, \ldots, 2 + 9k, 3 + k,$ $3 + 2k, 3 + 3k, \ldots, 3 + 9k, 9 + k, 9 + 2k, \ldots, 9 + 9k,$ $I_0^k, I_k^k, I_{2k}^k, I_{3k}^k, \ldots, I_{9k}^k, I_{1+9k}^k, I_{9+k}^k, I_{2+8k}^k, I_{2k+8}^k, I_{3k+7}^k, I_{3+7k}^k,$ $I_{4+6k}^k, I_{6+4k}^k, \ I_{5+5k}^k, I_2^k, \ I_4^k, \ I_6^k, \ I_8^k, \ I_5^k, \ I_{2+4k}^k, \ I_{4+2k}^k, \ I_{2+6k}^k,$ $I_{6+2k}^k, \ I_{2+8k}^k, \ I_{8+2k}^k, \ I_{4k+4}^k, \ I_{2+2k}^k, \ I_{4k+6}^k, \ I_{6k+4}^k, \ I_{4k+8}^k,$ $I_{8k+4}^k, I_{6k+6}^k, \ I_{6k+8}^k, \ I_{8k+6}^k$ and so on, ×}.

S has zero divisors and S has natural neutrosophic special quasi dual zero divisors.

$5k \times 2 = 0, 2k \times 5 = 0, 5k \times 6k = 0, 2k \times 5k = 0,$

$5k \times 6 = 0, 6k \times 5 = 0, 8k \times 5 = 0, 8k \times 5k = 0.$

$8 \times 5k$ are all zero divisors of S.

$$I_{4k}^k \times I_{5+5k}^k = I_0^k \,, \qquad I_{6+2k}^k \times I_{5k}^k = I_0^k$$

are some natural neutrosophic special quasi dual number zero divisors.

This semigroup has also idempotents

$$I_{5k}^k \times I_{5k}^k = I_{5k}^k, \;\; I_{5+5k}^k \times I_{5k+5}^k = I_{5+5k}^k$$

are natural neutrosophic special quasi dual number of idempotents.

$5k \times 5k = 5k$ is an idempotent.

$$I_{6k+8}^k \times I_5^k = I_0^k \qquad I_{6k+8}^k \times I_{5k}^k = I_0^k$$

are which are natural neutrosophic special quasi dual number zero divisors.

Example 2.51: Let $S = \{\langle Z_7 \cup k \rangle_I, \, k^2 = 6k, \times\} = \{0, 1, 2, 3, 4, 6,$ 5, k, 2k, 3k, 4k, 6k, 5k, 1 + k, 1 + 2k, 1 + 3k, 1 + 4k, 1 + 5k, 1 + 6k, 2 + k, 2 + 2k, 2 + 3k, 2 + 4k, 2 + 5k, 2 + 6k, 3 + k, 3 + 2k, 3 + 3k, 3 + 4k, 3 + 5k, 3 + 6k, 4 + k, 4 + 2k, 4 + 3k, 4 + 4k, 4 + 5k, 4 + 6k, 5 + k, 5 + 2k, 5 + 3k, 5 + 4k, 5 + 5k, 5 + 6k, 6 + k, 6 + 2k, 6 + 3k, 6 + 4k, 6 + 5k, 6 + 6k, I_0^k, I_k^k, I_{2k}^k, I_{3k}^k, I_{4k}^k, I_{5k}^k, I_{6k}^k, I_{1+6k}^k, I_{6+k}^k, I_{2+5k}^k, I_{5+2k}^k, I_{4+3k}^k, I_{3+4k}^k, ..., $\times\}$ be a semigroup of finite order.

S has zero divisors and natural neutrosophic special quasi dual number zero divisors.

$(4 + 3k)k = 0$, $(3 + 4k)k = 0$,

$(6 + k)k = 0$, $(6k + 1)k = 0$ and so on are zero divisors.

$$I_k^k \times I_{6+k}^k = I_0^k,$$

$$I_{2+5k}^k \times I_k^k = I_0^k \text{ and}$$

$$I_{4+3k}^k \times I_k^k = I_0^k$$

are all natural neutrosophic special quasi dual number zero divisors.

Can S have idempotents and natural neutrosophic special quasi dual idempotents?

Example 2.52: Let $S = \{\langle Z_{18} \cup k\rangle_I, k^2 = 17k, \times\} = \{0, 1, 2, \ldots, 17, k, 2k, \ldots, 17k, I_0^k, I_k^k, I_{2k}^k, I_{3k}^k, \ldots, I_{17k}^k, I_{1+17k}^k, I_{17+k}^k, I_{2+15k}^k, I_{15+2k}^k, \ldots, I_{9+9k}^k$ and so on, $\times\}$ be the natural neutrosophic special quasi dual number semigroup.

S has idempotents and zero divisors. Further S has natural neutrosophic idempotents and zero divisors.

Example 2.53: Let $S = \{\langle Z_{20} \cup k\rangle_I, k^2 = 19k, \times\} = \{Z_{20}, kZ_{20}, I_0^k, I_k^k, I_{2k}^k, \ldots, I_{19k}^k, I_{1+19k}^k, I_{19+k}^k, I_{2+18k}^k, I_{18+2k}^k, \ldots, I_{10+10k}^k$ and so on$\}$ be the natural neutrosophic special quasi dual number semigroup.

S has zero divisors, idempotents and nilpotents. Similarly S has natural neutrosophic special quasi dual number zero divisors, idempotents and nilpotents.

We have to prove the following theorem.

THEOREM 2.9: *Let $S = \{\langle Z_n \cup k\rangle_I, k^2 = (n - 1)k, \times\}$ be the natural neutrosophic special quasi dual number semigroup.*

    i.   *S is a S-semigroup if and only if Z_n is a S-semigroup.*

    ii.   *S has zero divisors, idempotents and nilpotents if Z_n has.*

    iii. *If (ii) is true. S has natural neutrosophic special quasi dual number zero divisors, idempotents and nilpotents.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe the natural neutrosophic special quasi dual number semigroup under +.

Example 2.54: Let
$S = \{\langle Z_4 \cup k \rangle_I, k^2 = 3k, +\}$ be the natural neutrosophic special quasi dual number semigroup under addition.

$S = \{0, 1, 2, 3, k, 3k, 2k, 1 + k, 2 + k, 3 + k, 2k + 1, 2k + 2, 2k + 3, 3k + 1, 3k + 2, 3k + 3, I_0^k, I_k^k, I_{2k}^k, I_{3k}^k, I_{2+2k}^k, I_2^k,$
$I_{3+k}^k, I_{1+3k}^k, \ldots, I_0^k + I_2^k, I_{3+k}^k + I_2^k, I_{2k}^k + I_{3k}^k + I_{2+2k}^k, I_{1+3k}^k$
$+ I_{2k}^k + I_2^k + I_0^k + 3k$ and so on, +\} is not a semigroup for

$$I_{2k}^k + I_{2k}^k = I_{2k}^k, \ I_0^k + I_0^k = I_0^k, \ I_{3+k}^k + I_{k+3}^k = I_{k+3}^k$$

are all idempotents under +.

That is why S is only a semigroup under + and not a group under +.

Example 2.55: Let
$S = \{\langle Z_5 \cup k \rangle_I, k^2 = 4k, +\} = \{0, 1, 2, 3, 4, k, 2k, 4k, 3k, 1 + k, \ 1 + 3k, 1 + 2k, 1 + 4k, 2 + k, 2 + 3k, 2 + 2k, 2 + 4k, \ldots,$
$I_0^k, I_k^k, I_{2k}^k, I_{3k}^k, I_{4k}^k, I_{1+4k}^k, I_{4+k}^k, I_{3+2k}^k, I_{2+3k}^k, I_0^k + I_{1+4k}^k +$
$3k, 1 + 4k + I_{2+3k}^k + I_{2k+3}^k$ and so on\} is a semigroup under +.

Example 2.56: Let $S = \{\langle Z_{12} \cup k \rangle_I, k^2 = 11k, +\}$ is a semigroup under +. S has idempotents under +.

Next we proceed onto describe and develop the semigroup on $S = \{\langle Z_n \cup k \rangle_I, k^2 = (n-1)k, +\}$ under $\times$.

Example 2.57: Let $S = \{\langle\langle Z_9 \cup k \rangle_I, k^2 = (9-1)k, +\rangle, \times\}$ be the semigroup of natural neutrosophic special quasi dual number semigroup under $\times$.

S has nilpotents, idempotents and zero divisors as well as natural neutrosophic special quasi dual number also has nilpotents, idempotents and zero divisors.

This will represent by some elements.

$$\alpha = I_{3k}^k + I_3^k + I_{3+3k}^k \in S$$

$$\alpha \times \alpha = I_0^k \,;\, 3 \times 3 = 0$$

$\beta = I_6^k + I_{6k}^k + I_{3+6k}^k + I_{6+3k}^k$ and $\gamma = I_3^k \in S$ is such that $\beta\gamma = I_0^k$ is a natural neutrosophic special quasi dual number zero divisor.

Thus the semigroup under $\times$ gives not only more elements but more number of nilpotents, zero divisors and idempotents.

Example 2.58: Let $S = \{\langle\langle Z_{11} \cup k \rangle_I, k^2 = 10k, +\rangle, \times\}$ be the semigroup under product.

S is a natural neutrosophic special quasi dual number semigroup. S has natural neutrosophic special quasi dual number zero divisors, nilpotents and idempotents.

S is a S-semigroup for $Z_{11} \setminus \{0\}$ under product is a group of order 10.

Example 2.59: Let S = $\{\langle\langle Z_{23} \cup k\rangle_I, \; k^2 = 22k, +\rangle, \times\}$ be the natural neutrosophic special quasi dual number semigroup. S has natural neutrosophic idempotents, zero divisors and nilpotents.

Next we proceed onto describe the natural neutrosophic special quasi dual number semiring.

Example 2.60: Let S = $\{\langle Z_3 \cup k\rangle_I, +, \times\}$ = $\{0, 1, 2, k, 2k, 1 + k,$ $2 + 2k, 2 + k, 2k + 1, I_0^k, I_{2k}^k, I_k^k, I_{2k+2}^k, I_{1+k}^k, 1 + k + I_0^k +$ $I_{2k}^k, 2 + 2k + I_{1+k}^k + I_k^k, \ldots,$ and so on +, $\times\}$ be the natural neutrosophic special quasi dual number semiring.

$1 + k \in$ S is an idempotent for $1 + k \times 1 + k = 1 + k$.

$I_{1+k}^k \times I_{1+k}^k = I_{1+k}^k$. $(2k + 2)k = 0$ and $I_{2+2k}^k \times I_k^k = I_0^k$ is a natural neutrosophic zero divisor and $I_{2+4k}^k + I_{3k+1}^k +$ $I_2^k + I_3^k + I_0^k + I_{2+4k}^k + I_{3+k}^k + I_{4+2k}$ is the natural neutrosophic special quasi dual number idempotent.

Hence S is not a semifield.

Example 2.61: Let S = $\{\langle Z_6 \cup k\rangle_I, k^2 = 5k, +, \times\}$ = $\{0, 1, 2, 3, 4,$ $5, k, 2k, 3k, 4k, 5k, 1 + k, 1 + 2k, 1 + 3k, 1 + 4k, 1 + 5k, 2 + k,$ $2 + 2k, 2 + 3k, 2 + 4k, 2 + 5k, \ldots, 5 + k, 5 + 2k, 5 + 3k, 5 + 4k,$ $5 + 5k, I_0^k, I_k^k, I_{2k}^k, I_{3k}^k, I_{4k}^k, I_{5k}^k, I_{1+k}^k, I_{1+3k}^k, I_{1+4k}^k, I_{2+4k}^k,$ $I_{3+3k}^k, I_2^k, I_3^k, I_{4+2k}^k$ and sums of them as $I_{2+4k}^k + I_{3k+1}^k + I_2^k +$ $I_3^k + I_0^k + I_{2+4k}^k + I_{3+k}^k + I_{4+2k}^k, \ldots$ so on, +, $\times\}$ be the natural

neutrosophic special quasi dual number semiring. S has idempotents and zero divisors which are given in the following.

$$3k \times 4 = 0$$

$$4 + (3k + 3) = 0$$

$$(1 + k)^2 = (1 + k).$$

Clearly $I_{1+k}^k \times I_{1+k}^k = I_{1+k}^k$ and $I_{3k}^k \times I_4^k = I_0^k$ are some of the natural neutrosophic idempotents and zero divisors of the semiring S.

S has subsets which are rings like Z_6 and $\langle Z_6 \cup k \rangle$ so S is a SS-semiring.

Further S has proper subsets which are fields like

$P_1 = \{0, 2, 4\} \subseteq S$. So S is also a SSS-semiring.

In view of this we have the following theorem.

THEOREM 2.10: *Let $S = \{\langle Z_n \cup k \rangle_I, +, \times\}$ be the natural neutrosophic special dual quasi semiring.*

    i. S is a SS-semiring.

    ii. S is a SSS-semiring if and only if Z_n is a S-ring.

    iii. S has zero divisors.

    iv. S has natural neutrosophic special quasi dual zero divisors.

    v. S has idempotents.

    vi. S has natural neutrosophic special quasi dual idempotents.

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto define MOD natural neutrosophic dual number interval in the following.

Let $\langle {}^I[0, 5) \cup g \rangle$ = {0, 1, 2, 3, 4, g, 2g, 3g, 4g, 1 + 2g, 1 + g,

1 + 3g, 1 + 4g, 2 + g, 2 + 2g, x + 2g, 2 + 3g, 2 + 4g, 3 + g, x + g,

x + 4g, 3 + 2g, 3 + 3g, 3 + 4g, 4 + g, x + 3g, 4 + 2g, 4 + 3g, 4 +

4g, I_0^g, x + xg, I_g^g, I_{2g}^g, I_{3g}^g, I_{4g}^g, $I_{2.5g}$, $I_{1.25g}$, $I_{2.5}$, $I_{1.25}$, I_{xg}; x $\in$ [0,

5) and so on ...}.

Here it is important to keep on record that $\langle {}^I[0, 5) \cup g \rangle$ has infinite number of elements.

However if only [0, 5)g alone is taken, we may have that every element in [0, 5)g is a zero divisor so ${}^I[0, 5)g$ has infinite number of natural neutrosophic dual number zero divisor.

Further ${}^I[0, 5)g \subseteq (\langle [0, 5) \cup g \rangle_I)$ or ${}^I(\langle [0, 5) \cup g \rangle)$

we can use any one of the notation both notations will be used as a matter of convenience.

Consider $\langle [0, 3) \cup g \rangle_I$ = S and ${}^I[0, 3)g$ = P the collection of MOD natural neutrosophic dual numbers. Clearly the cardinality of both S and P are infinite and P $\subseteq$ S.

Further $\langle [0, n) \cup g \rangle_I$ = S and ${}^I[0, 3)_g$ = P are the infinite collection of MOD natural neutrosophic dual number.

Here it is pertinent to keep on record; n $\in$ $Z^+ \setminus \{0\}$.

It is important to record that [0, n)g is such that ever element in it contributes to a MOD natural neutrosophic number as g a dual number I_{ag}^g ; a $\in$ [0, n) is a zero divisor.

By this method one gets an infinite collection of MOD neutrosophic natural number.

Example 2.62: Let $P = \{^I[0, 4)g \mid ag; g^2 = 0, a \in [0, 4)\}$ be the collection of all MOD neutrosophic natural dual numbers.

$$P = \{[0, 4)g; I^g_{xg}; x \in [0, 4); g^2 = 0\}$$

$I^g_{xg} \times I^g_{xg} = I^g_0$ is a MOD neutrosophic nilpotent element of order two.

However $I^g_{xg} \times I^g_{tg} = I^g_0$ for all x, t $\in$ [0, 4) are MOD neutrosophic zero divisors.

Thus P has infinite number of MOD neutrosophic natural numbers.

Now we proceed onto describe some operations using them we can define mainly two operations + and $\times$.

Two types of products can be used.

One type of product is usual product other a product on the sets $\langle [0, n)g, + \rangle_I$ and $\langle \langle [0, n) \cup g \rangle_I, + \rangle$.

We describe all the three situations by examples.

Example 2.63: Let $S = \{\langle [0, 9)g \rangle_I, +\} = \{ag, I^g_{ag}, \sum_{a \in [0, 9)} I^g_{ag}$ all sums $I^g_0 + I^g_{tg}, I^g_0 + I^g_{tg} + I^g_{sg}$, s, t $\in$ [0, n) ag + I^g_0, ag + $I^g_0 + I^g_{tg}$, … so on}.

S is defined as the MOD natural neutrosophic dual number semigroup. S is a semigroup under +. S is of infinite order.

If x = 5 + I_{7g}^g + $I_{0.33g}^g$ and

y = 6 + I_0^g then x + y = 2 + I_0^g + $I_{0.33g}^g$ + I_{7g}^g ∈ S.

Thus this is only a semigroup as $I_{\alpha g}^g$ + $I_{\alpha g}^g$ = $I_{\alpha g}^g$ (by very definition). S has subsemigroups of finite order.

S has subsets which are groups say $(Z_9, +)$ is a proper subset of S which is a group. $(\langle Z_9 \cup g \rangle, +)$ is again a proper finite subset of S which is a group.

Thus S is a Smarandache semigroup of infinite order having infinite number of MOD natural neutrosophic dual number elements.

S has also infinite subsemigroups.

Example 2.64: Let S = {⟨[0, 12)g⟩$_I$, +} = {ag, I_{ag}^g ; a ∈ [0, 12),

I_0^g + I_{tg}^g + I_{sg}^g + I_{pg}^g ; t, s, p ∈ [0, 12); $\displaystyle\sum_{t \in [0, 12)}$ I_{tg} summation can

be finite or infinite} is a semigroup of infinite order.

S is a MOD natural neutrosophic dual number semigroup which has infinite number of MOD natural neutrosophic elements.

In view of this we have the following theorem.

THEOREM 2.11: *Let S = {⟨[0, n)g⟩$_I$, ×} is a MOD natural neutrosophic dual number semiring. S is a S-semigroup.*

   i. *S has infinite number of zero divisors.*
   ii. *S has infinite number of natural neutrosophic dual number elements.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe the two types of MOD neutrosophic dual number semigroups under ×.

Example 2.65: Let
$S = \{\langle[0, 6)g\rangle_I, \times\} = \{[0, 6)g, I_0^g, I_{tg}^g; t \in [0, 6), \times\}$ be a MOD natural neutrosophic dual number semigroup. S has infinite number of zero divisor, all elements in S are such that they are nilpotent elements of order two.

Infact S is a zero square semigroup. Every set with zero MOD zero I_0^g in S finite or infinite is an ideal.

These semigroups have several such special properties. S is not a S-semigroup.

Example 2.66: Let $S = \{\langle[0, 11)g\rangle_I, g^2 = 0, \times\}$ be the MOD natural neutrosophic dual number semigroup.

S is a zero square semigroup.

Every subset of S with 0 and I_0^g are ideals of S.

In view of this we have the following theorem.

THEOREM 2.12: *Let $S = \{\langle[0, n)g\rangle_I; g^2 = 0, \times\}$ be the MOD neutrosophic dual number interval semigroup.*

    i.   *S is never a S-semigroup.*
    ii.   *S is a zero square semigroup.*
    iii.   *Every proper subset with 0 and I_0^g are always ideals.*
    iv.   *S has subsemigroups both of finite and infinite order.*
    v.   *S has ideals of both finite and infinite order.*

Proof is direct and hence left as an exercise to the reader.

Now we give semigroups of MOD neutrosophic dual number semigroups got from the set

$P = \{\langle[0, n) \cup g\rangle_I, \times\}$.

Clearly $M = \{\langle[0, 3) \cup g\rangle_I, \times\}$

$= \{[0, 3), [0, 3)g, [0, 3) + [0, 3)g, I_x^g, I_{xg}^g, I_{t+sg}^g$ where x, y $\in [0, 3)$; t, s $\in [0, 3)$ and t + sg is either a zero divisor or an idempotent or a pseudo zero divisor; $\times\}$ is a MOD neutrosophic dual number semigroup of infinite order.

$S = \{\langle[0, 3)g\rangle_I, \times\} \subseteq M$. Thus M has higher cardinality. Clearly M is not a zero square semigroup.

First we will provide examples of them.

Example 2.67: Let $S = \{\langle[0, 8) \cup g\rangle_I, \times\}$ be the MOD neutrosophic dual number semigroup.

$S = \{[0, 8), [0, 8)g, I_{xg}^g; x \in [0, 8), I_{s+tg}^g$ with s + tg a nilpotent or a zero divisor or a pseudo zero divisor or an idempotent, $g^2 = 0, \times\}$.

S is not a zero square semigroup as

$$(3 + 5g) \times (2 + 3g) = 6 + 10g + 9g \pmod 8$$

$$= 6 + 3g \neq 0.$$

Thus in general S is not a zero square semigroup. S has zero divisors and units; for $\alpha = 1 + 4g$ is such that $\alpha^2 = 1$.

Study of the substructure is an interesting task.

Consider $P_1 = \{[0, 8), \times\}$ is subsemigroup of S which is not an ideal.

$P_2 = \{[0, 8)g, \times\}$ is a subsemigroup which is a zero square subsemigroup.

$P_3 = Z_8 \subseteq S$ is a finite subsemigroup of S.

$P_4 = Z_8g \subseteq S$ is a finite subsemigroup of S such that $P_4 \times P_4 = \{0\}$ so is a zero square subsemigroup of S.

Several other interesting properties of S can be derived.

Example 2.68: Let $S = \{\langle[0, 13)g\rangle_I, g^2 = 0, \times\}$ be the MOD neutrosophic dual number semigroup.

This S also has zero square subsemigroups of both finite and infinite order.

However S is not a zero square subsemigroup.

$P_1 = \{[0, 13)g, \times\}$ is a zero square subsemigroup of S as
$$P_1 \times P_1 = \{0\}.$$

$P_2 = \{Z_{13}g \mid g^2 = 0, \times\}$ is again a zero square subsemigroup
of S as $P_2 \times P_2 = \{0\}$.

Clearly S is a S-semigroup as $\{Z_{13} \setminus \{0\}, \times\}$ is a group of order 12.

$P_3 = \{\langle Z_{13} \cup g\rangle \mid g^2 = 0, \times\}$ is only a subsemigroup of S which is not a zero square subsemigroup but P_3 is a S-subsemigroup of S.

$P_4 = \{\langle[0, 13) \cup g\rangle \mid g^2 = 0, \times\}$ is a subsemigroup of S which is a S-subsemigroup of S but is not a zero square subsemigroup.

In view of all these we have the following theorem.

THEOREM 2.13: *Let S = {⟨[0, n) ∪ g⟩ₗ, g² = 0, ×} is a natural neutrosophic dual number semigroup.*

    i.   *S is a S-semigroup if and only if Z_n is a S-semigroup.*

    ii. *S is not a zero square semigroup.*

    iii. *S has subsemigroup of finite order which is a zero square subsemigroup.*

    iv. *S has subsemigroups of infinite order which is a zero square subsemigroup.*

    v. *S has nilpotent elements of order two.*

    vi. *S has zero divisors other than nilpotents of order two.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe the notion of MOD neutrosophic dual number semigroup under product got by defining product on $S = \{\langle\langle[0, n)g\rangle_I, +\rangle\}$.

We will illustrate this situation by some examples.

Example 2.69: Let
$S = \{\langle\langle[0, 6)g\rangle_I, +\rangle, \times\} = \{[0, 6)g,\ I_0^g,\ I_2^g,\ I_{ag}^g\ ;\ x \in [0, 6),\ I_{bg}^g,$
$\sum I_{a+bg}^g$ is such that it a pseudo idempotent a, b $\in [0, 6)\}$ is a MOD neutrosophic dual number semigroup. S is a S-semigroup as Z_6 is a S-semigroup.

S has infinite number of MOD neutrosophic elements some of which are MOD neutrosophic zero divisors and some are MOD neutrosophic idempotents and some of them are MOD neutrosophic pseudo zero divisors.

This semigroup is different than the earlier MOD neutrosophic dual number semigroups.

Example 2.70: Let S = {⟨[0, 10)g, +⟩, ×} be the MOD neutrosophic dual number semigroup.

$$S = \{[0, 10)g, I_0^g, I_{mg}^g, \sum_{i=0}^{t} I_{mg}^g ; t = 2, 3, 4, \ldots, \infty, \times\}$$ is a

semigroup. Every element is nilpotent of order two.

So S is a zero square semigroup and not a S-semigroup

$$x = \left(I_{0.5g}^g + I_{7.3g}^g + I_{6.332g}^g \right) \text{ and}$$

$$y = I_{9.331g}^g + I_{0.3389g}^g \in S. \ x \times y = I_0^g$$

is a MOD neutrosophic zero divisor.

In view of this we have the following theorem.

THEOREM 2.14: *Let S = {⟨[0, n)g, +⟩, g² = 0, ×} be the MOD neutrosophic dual number semigroup.*

    i.   *S is a zero square semigroup.*
    ii. *S is not a S-semigroup.*
    iii. *P = {⟨[0, g⟩ₗ, +} ⊆ S is a subsemigroup of infinite order which is also an ideal of S.*
    iv. *S has subsemigroups P of finite order which are ideals provided 0 and I_0^g are in P.*

Proof is direct and hence left as an exercise to the reader.

Example 2.71: Let S = {⟨⟨[0, 8) ∪ g⟩, +⟩ₗ, ×, g² = g} be the MOD neutrosophic dual number semigroup.

Clearly S is not a zero square semigroup. S has subsemigroups which are not ideals. S has subsemigroups of finite order which are not ideals.

Thus $\alpha = \left(I^g_{0.5g} + I^g_{7.5g} + I^g_{2g} \right)$ and

$\beta = \left(I^g_{4g} + I^g_{7g} + 3 \right) \in$ S. $\alpha \times \beta \neq I^g_0$.

So every pair does not in general contribute to zero divisor.

However $P_1 = \{[0, 8), \times\}$ is a subsemigroup of infinite order which is not an ideal.

$P_2 = \{[0, 8)g, \times\}$ is a subsemigroup of infinite order which is not an ideal by $P_2 \times P_2 = 0$.

$P_3 = \{Z_8, \times\}$ is a subsemigroup of finite order.

$P_4 = \{Z_8 g, \times\}$ is a subsemigroup of finite order.

$P_5 = \{\langle Z_8 \cup g \rangle, \times\}$ is a subsemigroup of finite order.

$P_6 = \{\langle Z_8 \cup g \rangle_I, \times\}$ is a natural neutrosophic subsemigroup of finite order.

$P_7 = \{ Z^I_8, \times\}$ is also a natural neutrosophic subsemigroup of finite order.

$P_8 = \{\langle Z^I_8 \rangle_+, \times\}$ is also a finite natural neutrosophic subsemigroup of finite order $P_7 \underset{\neq}{\subset} P_8$.

Example 2.72: Let S = $\{\langle\langle[0, 11) \cup g\rangle_I, +\rangle, g^2 = 0, \times\}$ be a MOD natural neutrosophic dual number semigroup.

S is a S-semigroup as $P_1 = \{Z_{11} \setminus \{0\}, \times\}$ is a group.

$P_2 = \{Z_{11} g, \times\}$ is a zero square subsemigroup.

$P_3 = \{\langle Z_{11} \cup g \rangle, \times\}$ is a subsemigroup.

$P_4 = \{[0, 11), \times\}$ is a subsemigroup of S of infinite order.

$P_5 = \{[0, 11)g, \times\}$ is a subsemigroup of S of infinite order which is a zero square subsemigroup.

$P_6 = \{\langle [0, 11) \cup g \rangle, \times\}$ is a subsemigroup of infinite order.

$P_7 = \{\langle [0, 11)_I, \times \rangle\}$ is a MOD neutrosophic subsemigroup of infinite order.

$P_8 = \{\langle Z_{11}^I \rangle, \times\}$ is a natural neutrosophic subsemigroup.

$P_9 = \{\langle Z_{11} \cup g \rangle_I, \times\}$ is a natural neutrosophic subsemigroup.

$P_{10} = \{ Z_{11}^I g, \times\}$ is a natural neutrosophic subsemigroup.

$P_{11} = \{\langle [0, 11)g \rangle_I\}$ is a MOD neutrosophic subsemigroup.

Thus these MOD neutrosophic semigroups has special features very much different from other semigroups.

Example 2.73: Let $S = \{\langle\langle [0, 18) \cup g \rangle_I, + \rangle, \times\}$ be the MOD natural neutrosophic dual number semigroup. S is of infinite order.

S has infinite number of zero divisors and MOD neutrosophic zero divisors.

S has finite subsemigroups as well as infinite subsemigroups. S is a S-semigroup if and only if Z_n is a S-semigroup.

THEOREM 2.15: *Let $S = \{\langle\langle [0, n) \cup g \rangle_I, + \rangle, \times\}$ be the MOD neutrosophic dual number semigroup. The following are true.*

> i. *S is a S-semigroup if and only if Z_n is a S-semigroup.*
> ii. *S has finite order subsemigroups.*
> iii. *S has infinite number of zero divisors and nilpotents of order two.*
> iv. *S has infinite number of MOD neutrosophic zero divisors and MOD neutrosophic nilpotents of order two.*
> v. *S has infinite order subsemigroup.*

Proof is direct and hence left as an exercise to the reader.

Next the notion of MOD natural neutrosophic semirings are analysed.

We will illustrate this by an example.

Example 2.74: Let $S = \{\langle [0, n) \cup g \rangle_I, +, \times\}$ be the MOD natural neutrosophic dual number semiring.

We can define $P = \{\langle [0, n)g \rangle_I, +, \times\}$ be the MOD natural neutrosophic dual number semiring.

Both S and P are semirings. Infact $P \subseteq S$ is a subsemiring.

We will develop this through examples.

Example 2.75: Let $S = \{\langle [0, 20)g \rangle_I, g^2 = 0, +, \times\}$ be a MOD neutrosophic dual number semiring.

$$I^g_{18g}, \ I^g_{19g} \in S$$

$$I^g_{18g} \times I^g_{19g} = I^g_0.$$

Thus S has MOD neutrosophic dual number zero divisors.

$I_5^g \times I_5^g = I_5^g$ is a MOD neutrosophic dual number idempotents.

Infact this semiring is a zero square semiring. Every additive subsemigroup P or subsemiring P is a zero square semiring provided P contains 0 and I_0^g is an ideal.

Thus every subsemiring is an ideal.

Example 2.76: Let S = $\{\langle[0, 23)g\rangle_I, +, \times, g^2 = 0\}$ be the MOD natural neutrosophic dual number semiring.

S is not a S-semiring. Every subsemiring which contains I_0^g is an ideal for every subsemiring contains 0.

Example 2.77: Let S = $\{\langle[0, 42)g\rangle_I, g^2 = 0, +, \times\}$ be the MOD natural neutrosophic dual number semiring. This semiring is not a S-semiring.

$Z_{42}g$ is a subsemiring but is not an ideal. If P = $(Z_{42}g \cup I_0^g)$ then P is an ideal.

In view of this the following theorem is proved.

THEOREM 2.16: *Let S = $\{\langle[0, n)g\rangle_I, g^2 = 0, +, \times\}$ be the MOD natural neutrosophic dual number semiring.*

    i.   *S is not a S-semiring.*
    ii.   *S is a zero square semiring.*
    iii. *Every subsemiring P is an ideal of S if I_0^g is in P.*

Proof is direct and hence left as an exercise to the reader.

Next we describe the MOD natural neutrosophic dual number semiring which are not zero square semiring.

Example 2.78: Let S = $\{\langle[0, 12) \cup g\rangle_I, g^2 = 0, +, \times\}$ be the MOD natural neutrosophic dual number semiring.

S is not a zero square semiring as x = 10.32 in S is such that $x \times x = x^2 \neq 0$. So S in general is not a zero square semiring.

Let $x = I_3^g$ and $y = I_{2g}^g$

$x \times y = I_{6g}^g \neq I_0^g$. Thus S is not a zero square semiring. S has subsemirings which are not ideals. For Z_{12} is a subsemiring which is not an ideal.

Likewise $\langle Z_{12} \cup g\rangle$ is a subsemiring of S which is not an ideal.

S has zero divisors MOD neutrosophic zero divisors, S has idempotents as well as MOD neutrosophic idempotents and S has nilpotents as well as MOD neutrosophic nilpotents.

For $3 \times 8 = 0$ and $I_3^g \times I_8^g = I_0^g$, $I_{10g}^g \times I_{2g}^g = I_0^g$, $I_{5g}^g \times I_{5g}^g = I_0^g$, $2g \times 11g = 0$ this accounts for some zero divisors in S.

Let $I_{6g}^g \times I_{6g}^g = I_0^g$ is a MOD neutrosophic nilpotent of order two. $I_{xg}^g \times I_{xg}^g = I_0^g$ for all $x \in [0, 12)$ thus S has infinitely many MOD neutrosophic nilpotents of order two. $6 \times 6 = 0 \pmod{12}$ is a nilpotent in [0, 12).

Also $I_4^g \times I_4^g = I_4^g$ and $I_9^g \times I_9^g = I_9^g$ are both MOD neutrosophic idempotents of S.

Clearly $4 \times 4 = 4 \pmod{12}$ and $9 \times 9 \equiv 9 \pmod{12}$ are idempotents of [0, 12).

Next P = $\{[0, 12)g, +, \times\}$ is a subsemiring which is a zero square subsemiring and is not an ideal of S.

Let $P_1 = \{\langle [0, 12)g\rangle_I, +, \times\}$ be the subsemiring which a zero square subsemiring but is not an ideal of S.

$P \subset\limits_{\neq} P_1$ both are not ideals.

In view of all these we have the following theorem.

THEOREM 2.17: *Let S = { $\langle [0, n) \cup g\rangle_I$, $g^2 = 0$, +, $\times$} be the MOD natural neutrosophic semiring.*

  i. *S has zero square subsemirings.*

  ii. *S has subrings so S is a SS-semiring.*

  iii. *S is a SSS-semiring if and only if Z_n is a S-ring.*

  iv. *S has nilpotents of order two.*

  v. *S has MOD neutrosophic elements of order two.*

  vi. *S has zero divisors.*

  vii. *S has MOD neutrosophic zero divisors.*

  viii. *S has idempotents.*

  ix. *S has MOD neutrosophic idempotents.*

  x. *S has both finite order subsemirings as well infinite order subsemirings which are not ideals.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe natural neutrosophic special quasi dual number sets and properties enjoyed by them with additional operations on defined on this set.

Example 2.79: Let
$S = \{ Z_{10}^I h; \ h^2 = h, \ 0, \ I_0^h, \ I_h^h, \ I_{2h}^h, \ \ldots, \ I_{9h}^h \}$ be a natural neutrosophic special dual like number set.

Example 2.80: Let
$S = \{ Z_{19}^I h, h^2 = h, I_0^h, I_h^h, I_{2h}^h, \ldots, I_{18h}^h, 0, h, 2h, 3h, \ldots 18h \}$ be the natural neutrosophic special dual like number set.

We can define the two basic operations $+$ and $\times$ on S.

We will first illustrate this by some examples before the properties enjoyed by them are enumerated.

Example 2.81: Let $S = \{ Z_{15}^I h, h^2 = h, 0, 1h, 2h, \ldots, 14h, I_0^h, I_h^h, I_{2h}^h, I_{3h}^h, \ldots, I_{14h}^h, \times \}$ be the natural neutrosophic special dual like number semigroup.

Clearly $5h \times 3h = 0$, $10h \times 10h = 10h$ are zero divisors.

$$I_{6h}^h \times I_{5h}^h = I_0^h \text{ and}$$

$$I_{10h}^h \times I_{6h}^h = I_0^h \text{ are neutrosophic zero divisors.}$$

$$I_{9h}^h \times I_{9h}^h = I_{16h}^h \qquad\qquad I_{4h}^h \times I_{4h}^h = I_h^h$$

$$I_{6h}^h \times I_{6h}^h = I_{6h}^h \qquad\qquad\qquad\qquad \ldots \text{ I}$$

$$I_{7h}^h \times I_{7h}^h = I_{4h}^h \qquad\qquad I_{10h}^h \times I_{10h}^h = I_{10h}^h \qquad \ldots \text{ II}$$

It is easily verified I and II are natural neutrosophic idempotents.

$P_1 = Z_{15}h \subseteq S$ is a subsemigroup of S and is not an ideal of S.

$P_2 = \{0, 3h, 6h, 9h, 12h\} \subseteq S$ is also a subsemigroup of S which is not an ideal.

$P_3 = \{0, I_{3h}^h, I_{6h}^h, I_{9h}^h, I_{12h}^h, I_0^h \} \subseteq S$ is again an ideal of S as $3h \times I_h^h = I_h^h$ for the product of any element which is not a neutrosophic element has no effect on that neutrosophic element and the neutrosophic element remain the same.

Example 2.82: Let $M = \{ Z_{24}^I h \mid h^2 = h, \times \}$ be the natural neutrosophic special dual like number semigroup.

Z_{24}, $Z_{24}h$ are subsemigroups which are not ideals.

$12h \times 4h = 0$, $12h \times 6h = 0$, $12h \times 12h = 0$, $7h \times 7h = h$, $3h \times 8h = 0$, $4h \times 12h = 0$ and so on can contribute to zero divisors.

$$I_{10h}^h \times I_{12h}^h = I_0^h \qquad I_{4h}^h \times I_{6h}^h = I_0^h \text{ and}$$

$$I_{9h}^h \times I_{8h}^h = I_0^h \text{ are neutrosophic zero divisors.}$$

$$I_{16h}^h \times I_{16h}^h = I_0^h \text{ is a neutrosophic idempotent.}$$

$$I_{9h}^h \times I_{9h}^h = I_{9h}^h, \qquad I_{20h}^h \times I_{12h}^h = I_0^h$$

$$I_{6h}^h \times I_{8h}^h = I_0^h \qquad I_{12h}^h \times I_{12h}^h = I_0^h$$

is a neutrosophic nilpotent.

Now we have the following which behaves differently from the above two examples.

Example 2.83: Let $S = \{ Z_7^I h, \times \} = \{0, h, 2h, 3h, 4h, 5h, 6h, I_0^h, I_h^h, I_{2h}^h, I_{3h}^h, I_{4h}^h, I_{5h}^h, I_{6h}^h, \times \}$ be the natural neutrosophic special dual like number semigroup.

Clearly this semigroup has no zero divisors and no neutrosophic zero divisors. This has only two subsemigroups

$P_1 = \{0, h, 2h, 3h, 4h, 5h, 6h\}$ and

$P_2 = \{ I_0^h, I_h^h, I_{2h}^h, I_{3h}^h, I_{4h}^h, I_{5h}^h, I_{6h}^h \}$.

In view of all these we have the following theorem.

THEOREM 2.18: *Let* $S = \{ Z_n^I h, h^2 = h, \times\} = \{0, h, 2h, \ldots, (n-1)h, I_0^h, I_h^h, I_{2h}^h, \ldots, I_{(n-1)h}^h, \times\}$ *be the natural neutrosophic special dual like number semigroup.*

   i.   *S has zero divisors and neutrosophic zero divisors if and only if n is not a prime.*
   ii. *S has more than two subsemigroups if and only if n is not a prime.*
   iii. *S has nilpotents of order two and neutrosophic nilpotents of order two if and only if n is not a prime.*
   iv. *S has idempotents other than h if and only if n is not a prime.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe neutrosophic special dual like number semigroup using $Z_n^I h$.

Example 2.84: Let $S = \{\langle Z_6 \cup h\rangle_I, \times\} = \{0, 1, 2, 3, 4, 5, h, 2h, 3h, 4h, 5h, 1 + h, 2 + h, 3 + h, 4 + h, 5 + h, 1 + 2h, 2 + 2h, 3 + 2h, 4 + 2h, 5 + 2h, 1 + 3h, 2 + 3h, 3 + 3h, 4 + 3h, 5 + 3h, 1 + 4h, 2 + 4h, 3 + 4h, 4 + 4h, 5 + 4h, 1 + 5h, 2 + 5h, 3 + 5h, 4 + 5h, 5 + 5h, I_0^h, I_h^h, I_{2h}^h, I_{3h}^h, I_{4h}^h, I_{5h}^h, I_2^h, I_3^h, I_{2+2h}^h, I_{3+3h}^h, I_{4+4h}^h,$

I_{2+4h}^h, I_{4+2h}^h, I_{3+h}^h, I_{1+3h}^h, I_{1+5h}^h, …, ×} is a semigroup of natural neutrosophic special dual like number.

Clearly S has zero divisors, neutrosophic zero divisors, idempotents, neutrosophic idempotents.

S has subsemigroups which are not ideals. 2h × 3h = 0, 2 × 3h = 0, 3 × 4 = 0, 3h × 4 = 0, 4h × 3 = 0, 2h × 3 = 0 and so on.

4 × 4 = 4, 3 × 3 = 3, 4h × 4h = 4h, 3h × 3h = 0, h × h = h,

$I_h^h \times I_h^h = I_h^h$, $I_{3h}^h \times I_{3h}^h = I_{3h}^h$, $I_{4h}^h \times I_{4h}^h = I_{4h}^h$,

$I_{1+5h}^h \times I_{1+5h}^h = I_{1+5h}^h$, $I_{1+3h}^h \times I_{1+3h}^h = I_{1+3h}^h$,

$I_{3h}^h \times I_4^h = I_0^h$, $I_{4h}^h \times I_3^h = I_0^h$, $I_{3+3h}^h \times I_{4h}^h = I_0^h$.

So has zero divisors and neutrosophic zero divisors. Z_6 is a subsemigroup of S. $\langle Z_6 \cup h \rangle$ is again a subsemigroup of S. Z_6^I is a natural neutrosophic subsemigroup of S.

$Z_6 h$ is also a subsemigroup of S. $Z_6^I h$ is a neutrosophic subsemigroup of S. None of these are ideals of S.

Example 2.85: Let S = {$\langle Z_5 \cup h \rangle_I$, ×} = {0, 1, 2, 3, 4, h, 2h, 3h, 4h, 1 + h, 1 + 2h, 3h + 1, 1 + 4h, 2 + h, 2 + 2h, 2 + 3h, 2 + 4h, 3 + h, 3 + 2h, 3 + 3h, 3 + 4h, 4 + h, 4 + 2h, 4 + 3h, 4 + 4h, I_0^h, I_h^h, I_{2h}^h, I_{3h}^h, I_{4h}^h, …}.

Finding zero divisors other than those

1 + 4h × h = 0, 4 + h × h = 0, 2 + 3h × h = 0,

$2h + 3 \times h = 0$, $1 + 4h \times 2h = 0$, $1 + 4h \times 3h = 0$,

$1 + 4h \times 4h = 0$, $h + 4 \times 2h = 0$, $h + 4 \times 3h = 0$,

$h + 4 \times 4h = 0$, $2 + 3h \times 2h = 0$, $3 + 2h \times 2h = 0$,

$2 + 3h \times 3h = 0$, $3 + 2h \times 3h = 0$, $2 + 3h \times 4h = 0$,

$3 + 2h \times 4h = 0$ are zero divisors of S. I^h_{1+4h}, I^h_{4+h}, I^h_{3+2h} and I^h_{2+3h} multiplied by I^h_h, I^h_{2h}, I^h_{3h} and I^h_{4h} lead to neutrosophic zero divisors. $Z_5 \setminus \{0\}$ is a group so S is S-semigroup.

Example 2.86: Let $S = \{\langle Z_{12} \cup h \rangle_I, h^2 = h, \times\} = \{0, 1, 2, …, 11, h, 2h, 3h, …, 11h, 1 + h, 2 + h, …, 11 + h, 2 + 2h, 1 + 2h, …, 2h + 11, …, 11 + 11h, I^h_0, I^h_3, I^h_4, I^h_2, I^h_6, I^h_8, I^h_{10}, I^h_9, I^h_h, I^h_{2h}, …, I^h_{11h}, I^h_{2+2h}, I^h_{4+4h}, …, I^h_{10+10h}, I^h_{1+11h}, I^h_{2+10h}, I^h_{3+9h}, …, I^h_{6+6h}, …\}$ be the natural neutrosophic special dual like number semigroup.

S has zero divisors, idempotents and nilpotents of order two as well as neutrosophic zero divisor, neutrosophic idempotents and neutrosophic nilpotents of order two. S has subsemigroups and ideals.

We prove the following theorem.

THEOREM 2.19: *Let $S = \{\langle Z_n \cup h \rangle_I, h^2 = h, \times\}$ be the natural neutrosophic special dual like number semigroup.*

    i.   *S is a S-semigroup if and only if Z_n is a S-semigroup.*

    ii.   *S has zero divisors and neutrosophic zero divisors even if n is a prime.*

    iii.   *S has idempotents and neutrosophic idempotents.*

    iv.   *S has subsemigroups which are not ideals.*

Proof is direct and hence left as an exercise to the reader.

Next we describe the operation of '+' addition on S.

Example 2.87: Let
$S = \{ Z_6^I h, + \} = \{0, 1h, 2h, 3h, 4h, 5h, I_0^h, I_h^h, I_{2h}^h, I_{3h}^h, I_{4h}^h, I_5^h,$

$h + I_0^h, 2h + I_0^h, \ldots, 3h + I_0^h + I_h^h + I_{2h}^h + I_{3h}^h + I_{4h}^h + I_{5h}^h, \ldots,$

$+\}$ be the natural neutrosophic special dual like number a semigroup. S is semigroup.

$I_h^h + I_h^h = I_h^h$, $I_{4h}^h + I_{4h}^h = I_{4h}^h$, so this is only an idempotent.

$\{Z_6 h, +\}$ is a group so S is a S-semigroup.

Example 2.88: Let $S = \{ Z_{19}^I h, + \}$ be the natural neutrosophic special dual like number semigroup. S is a S-semigroup.

S has subsemigroups. S has idempotents.

In view of this we have the following theorem.

THEOREM 2.20: *Let $S = \{ Z_n^I h \mid h^2 = h, + \}$ be the natural neutrosophic special dual like number semigroup.*

  i.   *S is a S-semigroup.*
  ii.   *S has neutrosophic idempotents.*
  iii. *S has subsemigroups.*

Proof is direct and hence left as an exercise to the reader.

Next we describe $\langle Z_n \cup h \rangle_I$ under addition by some examples.

Example 2.89: Let $S = \{\langle Z_9 \cup h\rangle_I, h^2 = h, +\}$ be the natural neutrosophic special dual like number semigroup.

S has subgroups given by Z_9 and $\langle Z_9 \cup h\rangle$ so S is a S-semigroup both Z_9 and $\langle Z_9 \cup h\rangle$ subgroups of S under +.

I_0^h, I_{3h}^h, I_3^h, I_6^h, I_{6h}^h are some of the idempotents in S;

for $I_3^h + I_3^h = I_3^h$, $I_{6h}^h + I_{6h}^h = I_{6h}^h$ and so on.

Example 2.90: Let $S = \{\langle Z_{17} \cup h\rangle_I, h^2 = h, +\}$ be the natural neutrosophic special dual like number semigroup.

S is a S-semigroup.

$I_0^h + I_0^h = I_{3h}^h + I_{3h}^h = I_{3h}^h$ and so on.

In view of this we have the following theorem.

THEOREM 2.21: *Let $S = \{\langle Z_n \cup h\rangle_I, h^2 = h, +\}$ be the natural neutrosophic special dual like number semigroup.*

    i.   S is a S-semigroup.
    ii. S has several additive idempotents.
    iii. S has subsemigroups if Z_n has proper
           subsemigroups which are not groups.

Proof is direct and hence left as an exercise to the reader.

Next we study $\{ Z_n^I h\}$ and $\{\langle Z_n \cup h\rangle_I\}$ under + and ×.

We will illustrate this situation by some examples.

Example 2.91: Let S = { Z_{12}^I h, h^2 = h, +, ×} be the neutrosophic special dual like number semiring. S has zero divisors, idempotents and nilpotents of order two.

S has also neutrosophic special dual like zero divisors, idempotents and nilpotents of order two.

3h × 4h = 0, 3 × 4h = 0, 6 × 6h = 0 are zero divisors.

6h × 4 = 0, 6h × 6h = 0, 6 × 6 = 0 nilpotent of order two.

4 × 4 = 4, 9 × 9 = 9 are idempotents.

$I_6^h \times I_6^h = I_0^h$, $I_{6h}^h \times I_{6h}^h = I_0^h$ is the neutrosophic nilpotent elements of order two.

$I_4^h \times I_{3h}^h = I_0^h$, $I_{8h}^h \times I_3^h = I_0^h$, $I_6^h \times I_4^h = I_0^h$ are some neutrosophic zero divisors of S.

Finally $I_4^h \times I_4^h = I_4^h$, $I_9^h \times I_9^h = I_9^h$, $I_{4h}^h \times I_{4h}^h = I_{4h}^h$ and

$I_{9h}^h \times I_{9h}^h = I_{9h}^h$ are neutrosophic idempotents of S.

Every subset with I_0^h and 0 is not a subsemiring.

Clearly S is SS-semiring as Z_{12} is a subring.

S   is SSS-semiring as P = {0, 4, 8} is a field. S has subsemirings which are not ideals.

Example 2.92: Let S = { Z_{19}^I h, h^2 = h, + and ×} be the natural neutrosophic semiring.

S is SSS-semiring as well as SS-semiring.

Further 15 + 4h × h = 0,

$3 + 16h \times h = 0$,

$10 + 9h \times 5h = 0$ are zero divisors of S.

$h \times h = h$ is an idempotent $I_h^h \times I_h^h = I_h^h$ is a neutrosophic idempotent and $I_{10+9h}^h \times I_{5h}^h = I_0^h$,

$I_{(4h+15)}^h \times I_{6h}^h = I_0^h$ and so on are neutrosophic zero divisors and idempotents of S.

In view of this we prove the following theorem.

THEOREM 2.22: *Let $S = \{ Z_n^I h \mid h^2 = h, +, \times \}$ be the natural neutrosophic semiring.*

    i.   *S is a SS-semiring.*

    ii.   *S is a SSS-semiring if and only if Z_n is a S-ring.*

    iii.   *S has zero divisors as well as neutrosophic zero divisors.*

    iv.   *S has idempotents and neutrosophic idempotents.*

    v.   *S has neutrosophic nilpotents if and only if Z_n has nilpotents of order two.*

Proof is direct and hence left as an exercise to the reader.

Next we illustrate this situation by some examples.

Example 2.93: Let $S = \{ \langle Z_{24} \cup h \rangle_I, h^2 = h, +, \times \}$ be the natural neutrosophic semiring.

S is a SS-semiring as Z_{24} is a ring under $+$ and $\times$. S is a SSS-semiring as Z_{24} is a S-ring.

S has zero divisors and neutrosophic zero divisors. S has idempotents as well as neutrosophic idempotents.

S has nilpotents of order two as well as neutrosophic nilpotents of order two.

$$I_{12}^h \times I_{12}^h = I_0^h \, , \, I_{12}^h \times I_{12}^h = I_0^h \, ,$$

$$I_{6h}^h \times I_4^h = I_0^h \, , \, I_6^h \times I_3^h = I_0^h$$

are some neutrosophic nilpotents and neutrosophic zero divisors.

Example 2.94: Let $S = \{\langle Z_{13} \cup h \rangle_I, \, h^2 = h, \, +, \, \times\}$ be the natural neutrosophic semiring. S has zero divisors and idempotents.

S is a SSS-semiring as Z_{13} is a field and a SS-semiring as $\langle Z_{13} \cup h \rangle$ is a ring.

$$I_{(12+h)}^h \times I_h^h = I_0^h \text{ and }$$

$$I_{(6h+7)}^h \times I_{4h}^h = I_0^h \text{ are neutrosophic zero divisors.}$$

$6h + 7 \times 4h = 0$ and $(12 + h) \times h = 0$.

THEOREM 2.23: *Let $S = \{\langle Z_n \cup h \rangle_I, \, h^2 = h, \, +, \, \times\}$ be the natural neutrosophic semiring.*

  i.   *S is a SS-semiring.*
  ii. *S is a SSS-semiring if and only if Z_n is a S-ring.*
  iii. *S has zero divisors as well as neutrosophic zero divisors.*
  iv. *S has idempotents as well as neutrosophic idempotents.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto develop MOD neutrosophic special dual like number sets and the algebraic structures which can be defined on them.

Example 2.95: Let $S = \{^I[0, 3)h, \ h^2 = h\}$ be the MOD neutrosophic special dual like number set.

$S = \{ah, \ I^h_{ah} ; ah \in [0, 3)h\}$. S is of infinite cardinality.

Example 2.96: Let $P = \{^I[0, 12)h \mid h^2 = h\}$ be the natural neutrosophic special dual like number set.

Example 2.97: Let $B = \{^I[0, 14)h, \ h^2 = h\}$ be the natural neutrosophic special dual like number set.

We can define operations $+$ and $\times$ on P.

We will illustrate the product operation on this set.

Example 2.98: Let $S = \{^I[0, 10)h, \ h^2 = h, \times\}$ be the MOD neutrosophic special dual like number semigroup.

$o(S) = \infty$. S has zero divisors and MOD neutrosophic zero divisors.

$I^h_{2h} \times I^h_{5h} = I^h_0$ is a MOD zero divisor. $I^h_{5h} \times I^h_{5h} = I^h_{5h}$ is a MOD idempotent.

$I^h_h \times I^h_h = I^h_h$, $I^h_{5h} \times I^h_{5h} = I^h_0$ and so on.

Clearly this semigroup has no nilpotents.

Infact $I^h_{0.9h} \times I^h_{0.6h} = I^h_{0.54h}$; this is the way product operation is performed on MOD neutrosophic numbers.

Example 2.99: Let

S = {$^I[0, 11)h$, $\times$, $h^2 = h$} be the MOD natural neutrosophic special dual like number semigroup.

S is not a S-semigroup. S has zero divisors and MOD natural neutrosophic zero divisors.

$$I_{5.5h}^h \times I_{2h}^h = I_0^h, \qquad I_{5.5h}^h \times I_{4h}^h = I_0^h,$$

$$I_{2.75h}^h \times I_4^h = I_0^h$$

and so on are some of the MOD neutrosophic zero divisors of S.

Now we can prove the following theorem.

THEOREM 2.24: *Let* S = {$^I[0, n)h$ / $h^2 = h$, $\times$} *be the MOD natural neutrosophic special dual like number semigroup.*

  i. *S has zero divisors and MOD neutrosophic zero divisors which are infinite in number.*
  ii. *S has nilpotents if Z_n has nilpotents.*
  iii. *S has idempotents and MOD neutrosophic idempotents.*

Proof is simple and direct hence left as an exercise to the reader.

Next sum on $^I[0, n)h$ is defined and their properties are analysed.

Example 2.100: Let S = {$^I[0, 6)h$, $h^2 = h$, +} be the MOD neutrosophic special dual like number semigroup.

S is of infinite order.

S has idempotents for $I_{ah}^h + I_{ah}^h = I_{ah}^h$ for every ah $\in$ [0, 6)h.

Clearly S is a S-semigroup as $\{Z_6h, +\}$ is a group under +.

Example 2.101: Let $S = \{^I[0, 19)h; h^2 = h, +\}$ be the MOD neutrosophic special dual like number semigroup.

S is a S-semigroup. S is of infinite order and has infinite number of MOD neutrosophic idempotents;

like $I_{ah}^h + I_{ah}^h = I_{ah}^h$ for all ah $\in$ [0, 19)h.

In view of this we have the following theorem.

THEOREM 2.25: *Let $S = \{^I[0,n)h, h^2 = h, +\}$ be the MOD neutrosophic special dual like number semigroup. Then the following are true.*

    i.   S is a S-semigroup.

    ii. S has infinite number of idempotents.

Next we proceed onto study the notion of semigroup built using $S = \{^I[0, n)h, +\}$.

Example 2.102: Let

$$S = \{^I[0, 6)h, +\} = \{[0, 6)h, \sum_{ah \in [0, 6)} I_{ah} ,$$

this summation runs over 2 elements, 3 elements so on upto infinite number of terms, +} ×} be a MOD neutrosophic special dual like number semigroup.

Example 2.103: Let $S = \{^I[0, 28)h, h^2 = h, +\}, \times\}$ be a MOD neutrosophic special dual like number semigroup.

S is of infinite order. S has infinite number of zero divisors and MOD neutrosophic zero divisors.

In view of this we have the following theorem.

THEOREM 2.26: *Let* $S = \{^I[0,n)h, h^2 = h, +\}, \times\}$ *be the* MOD *neutrosophic special dual like number semigroup.*

 i. *S is a S-semigroup if and only if* Z_n *is a S-semigroup.*

 ii. *S is of infinite order.*

 iii. *S has infinite number* MOD *neutrosophic numbers.*

Proof follows directly from the definition.

Example 2.104: Let

$S = \{\langle^I[0, n)h, h^2 = h, +\rangle, \times\}$ be the MOD neutrosophic special dual like semigroup.

S is of infinite order and S has infinite number of MOD neutrosophic number.

Next we proceed onto describe MOD neutrosophic semiring $S = \{^I[0, n)h, h^2 = h, +, \times\}$.

S is an infinite semiring which has infinite number of MOD neutrosophic elements.

Example 2.105: Let $S = \{^I[0, 12)h, h^2 = h, +, \times\}$ be the MOD natural neutrosophic special quasi dual number semiring.

S has zero divisor. S is a SS-semiring as $Z_{12}h \subseteq S$ is a ring. As $P = \{0, 4h, 8h\} \subseteq S$ is a field so S is a SSS-semiring.

S has several MOD neutrosophic zero divisors.

$I^h_{1.2h} \times I^h_{10h} = I^h_0$, $I^h_{6h} \times I^h_{6h} = I^h_0$, $I^h_{8h} \times I^h_{6h} = I^h_0$ and so on.

In view of all these we have the following theorem.

THEOREM 2.27: *Let $S = \{^I[0, n)h, +, \times\}$ be the* MOD *neutrosophic special dual like number semiring.*

> i. *S is a SS-semiring.*
> ii. *S is a SSS-semiring iff $Z_n h$ is a S-ring.*
> iii. *S has zero divisors and* MOD *neutrosophic zero divisors.*
> iv. *S has idempotents and* MOD *neutrosophic idempotents.*

The proof is direct and hence left as an exercise to the reader.

Next we study the semigroup built using $\langle [0, n) \cup h \rangle_I$, $h^2 = h$.

Example 2.106: Let
$S = \{\langle [0, 10) \cup h \rangle_I, h^2 = h\}$ be the MOD neutrosophic special dual like number set.

Example 2.107: Let
$S = \{\langle [0, 15) \cup h \rangle_I, h^2 = h\}$ be the MOD neutrosophic special dual like number set.

These sets are of infinite cardinality.

Example 2.108: Let
$S = \{\langle [0, 14) \cup h \rangle_I, h^2 = h, \times\}$ be the MOD natural neutrosophic special dual like number semigroup.

$o(S) = \infty$, $7 \times 2h = 0$, $7h \times 4h = 0$, $7 \times 8 = 0$, $7h \times 2 = 0$. Thus S has zero divisors. $I_{7h}^h \times I_{2h}^h = I_0^h$, $I_7^h \times I_7^h = I_7^h$ is a MOD neutrosophic idempotent.

$I_{3.5h}^h \times I_4^h = I_0^h$ is a MOD zero divisor. S has subsemigroups which are not ideals.

Example 2.109: Let

$$S = \{\langle [0, 11) \cup h \rangle_I, h^2 = h, \times\}$$

be the MOD neutrosophic special dual like number semigroup.

S has idempotents and zero divisor as well as neutrosophic idempotents as well as zero divisors.

$$5.5h \times 2 = 0, \ 5.5 \times 2 = 0,$$

$$5.5 \times 2h = 0, \ 5.5h \times 2h = 0,$$

$$2.75 \times 4 = 0, \ 2.75 \times 4h = 0, \ 2.75h \times 4 = 0, \ 2.75 \times 8 = 0,$$

$$2.75h \times 4h = 0, \ 2.75 \times 8h = 0.$$

Further

$$I_2^h \times I_{5.5}^h = I_0^h, \ I_{2.75}^h \times I_8^h = I_0^h,$$

$$I_{2h}^h \times I_{5.5}^h = I_0^h, \ I_{8h}^h \times I_{2.75}^h = I_0^h$$

are some of the neutrosophic zero divisor.

Z_{11} is a subsemigroup. $Z_{11} \setminus \{0\}$ is a group so S is a S-semigroup.

In view of all these we have the following theorem.

THEOREM 2.28: *Let $S = \{\langle [0, n) \cup h \rangle_I, h^2 = h, \times\}$ be the MOD neutrosophic special dual like semigroup.*

    i.   *S is a S-semigroup if and only if Z_n is a S-semigroup.*

    ii. *S has idempotents and MOD neutrosophic idempotents.*

iii. S has zero divisors and MOD neutrosophic zero divisors.

The proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe + operation on

$$S = \{\langle[0, n) \cup h\rangle_i, h^2 = h\}.$$

Example 2.110: Let

$$S = \{\langle[0, 18) \cup h\rangle_I, h^2 = h, \times, +\}$$

be the MOD neutrosophic special dual like semigroup.

S has MOD neutrosophic idempotents for

$$I^h_{0.9h} + I^h_{0.9h} = I^h_{0.9h}.$$

Let $I^h_{9h} \times I^h_2 = I^h_0$, $I^h_9 \times I^h_2 = I^h_0$, $I^h_{9h} \times I^h_{2h} = I^h_0$.

Example 2.111: Let

$$V = \{\langle[0, 13) \cup h\rangle_I, h^2 = h, \times, +\}$$

be the MOD neutrosophic special dual like number semigroup.

$$I^h_{6.5h} + I^h_{6.5h} = I^h_{6.5h}, \ I^h_{6.5} \times I^h_2 = I^h_0$$

this V has pseudo MOD neutrosophic zero divisors also

$$I^h_{12+h} \times I^h_h = I^h_0, \ I^h_{10+3h} \times I^h_{2h} = I^h_0$$

and so on are all MOD neutrosophic zero divisors.

$I_{5h+8}^h \times I_{6h}^h = I_0^h$ is again a MOD neutrosophic zero divisor.

Example 2.112: Let

$$M = \{\langle [0, 12) \cup h \rangle_I, h^2 = h, \times, + \}$$

be the MOD neutrosophic special dual like semiring.

$I_6^h + I_6^h = I_6^h$, $I_6^h \times I_4^h = I_0^h$, $I_4^h \times I_{3h}^h = I_0^h$, $I_{4h}^h \times I_{3h}^h = I_0^h$ and so on. $I_{4h}^h \times I_{4h}^h = I_{4h}^h$.

THEOREM 2.29: *Let $S = \{\langle [0, n) \cup h \rangle_I, h^2 = h, + \}$ be the MOD neutrosophic special dual like semigroup.*

  i.   *$o(S) = \infty$.*
  ii. *S is S-semigroup.*
  iii. *S has no idempotents but has only MOD neutrosophic idempotents.*

Proof follows from simple techniques.

However if both + and × is defined then $S = \{\langle [0, n) \cup h \rangle_I, h^2 = h, +, \times \}$ is the MOD neutrosophic special dual like semiring.

We will illustrate this situation by some examples.

Example 2.113: Let

$$S = \{\langle [0, 20) \cup h \rangle_I, h^2 = h, \times, + \}$$

be the MOD neutrosophic special dual like number semiring. S is SSS-semiring as P = {0, 4, 8, 12, 16} $\subseteq$ S is a field with 16 as the multiplicative identity.

The table for + and × are as follows:

+	0	4	8	12	16
0	0	4	8	12	16
4	4	8	12	16	0
8	8	12	16	0	4
12	12	16	0	4	8
16	16	0	4	8	12

P is a group under +.

×	16	12	8	4
16	16	12	8	4
12	12	4	16	8
8	8	16	4	12
4	4	8	12	16

is a group.

Thus P is a field. So S is SSS-semiring. $M = Z_{20}$ is a ring so S is a SS-semiring.

S has idempotents, MOD neutrosophic idempotents and zero divisors.

Example 2.114: Let

$$S = \{\langle [0, 12) \cup h \rangle_I, h^2 = h, +, \times\}$$

be the MOD neutrosophic special dual like number semiring.

P = {0, 4, 8} ⊆ S is a field. So S is a SSS-semiring. S is also a SS-semiring.

All interesting and special elements in S can be obtained.

For $I_{3h}^h \times I_{4h}^h = I_0^h$, $I_{2h}^h \times I_{6h}^h = I_0^h$,

$I_{6h}^h \times I_{6h}^h = I_{6h}^h$ and so on.

$I_4^h \times I_4^h = I_4^h$, $I_9^h \times I_9^h = I_9^h$.

Thus S has MOD neutrosophic zero divisors, nilpotents and idempotents.

In view of all these the following result is true.

THEOREM 2.30: *Let $S = \{\langle[0, n) \cup h\rangle, h^2 = h, +, \times\}$ be the MOD neutrosophic special dual like semigroup.*

  i. *S is a SS-semiring.*
  ii. *S is a SSS-semiring if and only if Z_n is a S-ring.*
  iii. *S has infinite number of MOD neutrosophic zero divisors and zero divisors.*
  iv. *S has MOD neutrosophic idempotents.*
  v. *S has subsemiring of both finite and infinite order.*

Proof is direct and hence left as an exercise to the reader.

Next we can now study the notion of natural neutrosophic special quasi dual numbers and MOD natural neutrosophic special quasi dual number.

We first give one or two examples.

Example 2.115: Let $S = \{Z_{10}^I k, \ k^2 = 9k\}$ be the natural neutrosophic special quasi dual number set.

$S = \{Z_{10}k, \ I_k^k, \ I_0^k, \ I_{2k}^k, \ \ldots, \ I_{9k}^k, I_2^k, I_4^k, I_6^k, I_8^k, I_5^k\}$.

Example 2.116: Let
$P = \{ Z_7^I k, k^2 = 6k \} = \{0, k, 2k, \ldots, 6k, I_0^k, I_k^k, I_{2k}^k, \ldots, I_{6k}^k \}$ is the natural neutrosophic special quasi dual number set.

On $S = \{ Z_n^I k, k^2 = n - 1 \}$ we can define both the operation $+$ and $\times$.

S under $+$ is a natural neutrosophic special quasi dual number semigroup.

S under $\times$ is a natural neutrosophic special quasi dual number semigroup.

We just enumerate a few of the properties associated with them.

Example 2.117: Let $S = \{ Z_{12}^I k, k^2 = 11k, \times \}$ is the natural neutrosophic special quasi dual like number semigroup.

S has zero divisors and neutrosophic zero divisors.

$$I_{4k}^k \times I_{3k}^k = I_0^k, \qquad I_{6k}^k \times I_{4k}^k = I_0^k,$$

$$I_{6k}^k \times I_{2k}^k = I_0^k \text{ and so on.}$$

$3k \times 4k = 0 \qquad 3k \times 3k = 3k$

$4k \times 4k = 8k \qquad 6k \times 6k = 0$

$5k \times 5k = 11k \qquad 2k \times 2k = 8k \text{ and}$

$8k \times 8k = 8k.$

Thus 3k and 8k are idempotents and $I_{3k}^k \times I_{3k}^k = I_{3k}^k$ and $I_{8k}^k \times I_{8k}^k = I_{8k}^k$ are neutrosophic idempotents of S.

$I_{3k}^k \times I_{8k}^k = I_0^k$, $I_{6k}^k \times I_{2k}^k = I_0^k$ are neutrosophic zero divisor.

Now we can define on S = { Z_n^I k, $k^2 = (n-1)k$, +} the plus operation S under + operation is only a semigroup known as the natural neutrosophic special quasi dual number semigroup.

This will be illustrated just by an example.

Example 2.118: Let S = { Z_{15}^I k, $k^2 = 14k$, +} be the natural neutrosophic special quasi dual number semigroup.

$$I_{7k}^k + I_{7k}^k = I_{7k}^k, \; I_{3k}^k + I_{3k}^k = I_{3k}^k, \; x = I_{2k}^k + I_{4k}^k + I_{5k}^k$$

is the same it cannot be further reduced. Thus S has idempotents.

These examples provide a nice collection of finite semigroup under + which has idempotents.

Next the semigroup using the set S = {$\langle Z_n \cup k \rangle_I$, $k^2 = (n-1)k$}.

S under $\times$ is a semigroup. S under + is a semigroup.

We will illustrate this by examples.

Example 2.119: Let S = {$\langle Z_{10} \cup k \rangle_I$, $k^2 = 9k$, $\times$} be the natural neutrosophic semigroup.

$5k \times 3k = 5k$, $5k \times 5k = 5k$ is an idempotent and so on.

$$(3 + 4k)2k = 6k + 8 \times 9k$$
$$= 8k$$

$$(2 + 8k)k = 2k + 8 \times 9k$$
$$= 4k$$

$$(2 + 7k)k = 2k + 63k$$
$$= 5k \text{ and so on.}$$

$I_{5k}^k \times I_{5k}^k = I_{5k}^k$ is a neutrosophic idempotent of S.

$I_{5k}^k \times I_{(2+4k)}^k = I_0^k$ is a neutrosophic zero divisors.

Thus S has both zero divisors and neutrosophic zero divisors.

Example 2.120: Let $S = \{\langle Z_{12} \cup k \rangle_I, +\}$ be the natural neutrosophic special quasi dual semigroup.

$$I_{6k}^k + I_{6k}^k = I_{6k}^k$$

$I_{6k}^k + I_6^k$ is the same.

$$I_{6k}^k \times I_4^k = I_0^k, \ I_{8k}^k \times I_3^k = I_0^k.$$

Example 2.121: Let $S = \{\langle Z_7 \cup k \rangle_I, \ k^2 = 6k, \times\}$ be the natural neutrosophic special quasi dual number semigroup.

$$I_{3k}^k \times I_{3k}^k = I_{5k}^k \text{ and so on.}$$

$$I_{3.5}^k \times I_{2k}^k = I_0^k, \ I_k^k \times I_k^k = I_{6k}^k,$$

$$I_{(6+k)}^k \times I_{2k}^k = I_{3k}^k, \ I_{3.5}^k \times I_{4k}^k = I_0^k, \ I_{1.75k}^k \times I_4^k = I_0^k.$$

Thus S has neutrosophic zero divisors.

$I_{6k}^k \times I_{6k}^k = I_{6k}^k$ is a neutrosophic idempotent.

Example 2.122: Let $S = \{\langle Z_5 \cup k \rangle_I, +\}$ be the natural neutrosophic special quasi dual number semigroup.

$I_4^k + I_4^k = I_4^k$, $I_2^k + I_3^k = x$ cannot be further simplified.

$x = 3 + I_{3k}^k + I_{4k}^k$ and $y = 4 + I_2^k + I_3^k \in S$.

$x + y = 2 + I_3^k + I_4^k + I_2^k$.

This is the way + operation is performed on S.

All properties in case of natural neutrosophic special quasi dual number semigroups can be derived as in case of natural neutrosophic special dual like numbers and natural neutrosophic dual numbers.

This task is left as an exercise to the reader.

Next the concept of natural neutrosophic semirings using the set

$S = \{ Z_n^I k, +, \times\}$ and $S_1 = \{\langle Z_n \cup k\rangle_I, k^2 = (n-1)k, +, \times\}$ can be obtained as in case of semirings of natural neutrosophic special quasi dual numbers and natural neutrosophic special quasi dual numbers.

However we will just illustrate this situation by some examples.

Example 2.123: Let $S = \{\langle Z_{11} \cup k\rangle_I, k^2 = 10k, \times, +\}$ be the natural neutrosophic special quasi dual number semiring.

S is SSS-semiring. S is a SS-semiring. S has natural neutrosophic idempotents.

Example 2.124: Let $S = \{\langle Z_{20} \cup k\rangle_I, k^2 = 19k, +, \times\}$ be the natural neutrosophic special quasi dual number semiring.

S has zero divisors and neutrosophic zero divisors. S has idempotents and neutrosophic idempotents. S is a SSS-semiring and SS-semiring.

Several interesting and important results in this direction are obtained.

Almost all results on natural neutrosophic special quasi dual number semigroups and semirings can be derived as in case of natural neutrosophic special dual like numbers and natural neutrosophic dual numbers.

Next the MOD natural neutrosophic special quasi dual numbers sets and the corresponding algebraic structures can be derived as in case of earlier ones.

We will illustrate this situation by some examples.

Example 2.125: Let
$S = \{[0, 9)h \mid h^2 = 8h\}$ be the MOD special quasi dual number set.

Example 2.126: Let
$S = \{[0, 11)h \mid h^2 = 10h\}$ be the MOD special quasi dual number set.

Example 2.127: Let
$P = \{[0, 20)h \mid h^2 = 19h, \times\}$ be the MOD special quasi dual number semigroup. P has zero divisors and idempotents.

Example 2.128: Let
$W = \{[0, 19)h \mid h^2 = 18h, \times\}$ be the MOD special quasi dual number semigroup. W has several zero divisors.

For more about these structures [21].

Example 2.129: Let
$S = \{^I[0, 5)k, k^2 = 4k\}$ be the MOD neutrosophic special quasi dual number set.

Example 2.130: Let
$W = \{^I[0, 24)k, k^2 = 23k\}$ be the MOD neutrosophic special quasi dual number set.

Example 2.131: Let
$M = \{^I[0, 3)k; \ k^2 = 2k, \times\}$ be the MOD neutrosophic special quasi dual number semigroup.

$M = \{[0, 3)k, \ I_{ak}^k; \ ak \in [0, 3)k, \times\}$ be the MOD neutrosophic special quasi dual number semigroup.

M has zero divisors, MOD neutrosophic zero divisors and pseudo zero divisors and MOD neutrosophic pseudo zero divisor.

Example 2.132: Let $P = \{^I[0, 12)k, \ k^2 = 11k, \times\}$ be the MOD neutrosophic special quasi dual number semigroup.

$$I_{4k}^k \times I_{6k}^k = I_0^k \qquad\qquad 4k \times 6k = 0$$

$$I_{3k}^k \times I_{4k}^k = I_0^k \qquad 3k \times 8k = 0 \text{ and so on.}$$

$$4k \times 4k = 8k \qquad 3k \times 3k = 3k \text{ is an idempotent}$$

so $I_{3k}^k \times I_{3k}^k = I_{3k}^k$ is a MOD neutrosophic idempotent.

$I_{8k}^k \times I_{8k}^k = I_{8k}^k$ and $8k \times 8k = 8k$ are MOD neutrosophic idempotent and idempotent respectively.

Example 2.133: Let
$S = \{[0, 12)k \mid k^2 = 11k, +\}$ is a MOD special quasi dual number group of infinite order [21].

Example 2.134: Let
$M = \{[0, 15)k; \ k^2 = 14k, \times, +\}$ is only a pseudo MOD special quasi dual number ring of infinite order [21].

Example 2.135: Let
$M = \{[0, 19)k; \ k^2 = 18k, +, \times\}$ is also a pseudo MOD special quasi dual number ring of infinite order.

Example 2.136: Let

M = $\{^I[0, 10)k, k^2 = 9k, +, \times\}$ be the MOD neutrosophic special quasi dual number pseudo semiring of infinite order.

$$\text{For } I_{ak}^k \times I_{ak}^k = I_{ak}^k \text{ ; } ak \in [0, 10)k.$$

Since the distributive laws are not true. M is only a pseudo semiring.

Example 2.137: Let
S = $\{^I[0, 12)k, k^2 = 9k, +, \times\}$ be the MOD neutrosophic special quasi dual pseudo semiring which has zero divisors, MOD pseudo zero divisors, MOD neutrosophic zero divisors and pseudo zero divisors.

Next we proceed onto develop MOD neutrosophic special quasi dual number sets.

Example 2.138: Let
S = $\{\langle[0, 5) \cup k\rangle_I, k^2 = 4k\}$ be the MOD neutrosophic special quasi dual number set.

Example 2.139: Let
S = $\{\langle[0, 15) \cup k\rangle_I; k^2 = 14k\}$ be the MOD neutrosophic special quasi dual number set.

We can define the + and $\times$ operation on S and under both these operations S is only a semigroup.

Example 2.140: Let
S = $\{\langle[0, 12) \cup k\rangle_I, k^2 = 11k, \times\}$ be the MOD neutrosophic special quasi dual number semigroup. S has subsemigroups and ideals.

$$I_{3k}^k \times I_{8k}^k = I_0^k \text{ and so on.}$$

Example 2.141: Let S = $\{\langle[0, 7) \cup k\rangle_I, k^2 = 6k, +\}$ be the MOD neutrosophic special quasi dual number semigroup.

$\{[0, 7), +\}$ is a group but $I_{3k}^k \times I_{3k}^k = I_{3k}^k$ is only an idempotent so S is only a semigroup. S is always a S-semigroup.

All properties of these MOD special quasi dual number semigroups under + or × can be developed as in case of dual numbers or special dual like numbers.

Example 2.142: Let
S = $\{\langle[0, 18) \cup k\rangle_I, k^2 = 17k, +, \times\}$ be the MOD neutrosophic special quasi dual number pseudo semiring.

$$I_{6k}^k + I_{6k}^k = I_{6k}^k \qquad I_{3k}^k + I_{3k}^k = I_{3k}^k$$

$$I_{3k}^k \times I_{6k}^k = I_0^k \qquad I_{9k}^k \times I_{2k}^k = I_0^k$$

are MOD neutrosophic zero divisors.

$$I_{6k}^k \times I_{6k}^k = I_0^k \qquad I_{9k}^k \times I_{9k}^k = I_{9k}^k$$

are neutrosophic nilpotent of order two and an idempotent of order two.

All properties can be derived in case these MOD neutrosophic structures using special quasi dual numbers.

Next we proceed onto suggest problems some of which are very difficult and challenging and a few are simple.

Problems

1. Obtain any special feature associated with MOD neutrosophic elements of $I_{[0, n)}$.

2. Given [0, 19) find all pseudo zero divisors.

3. Let S = $\langle Z_7 \cup g\rangle_I$ be the natural neutrosophic set.

What are the special features enjoyed by S?

4. Define a product operation on S in problem 3 and find the algebraic enjoyed by (S ×).

5. Let $\{\langle Z_{11} \cup k \rangle_I, \times\} = S$, $k^2 = 10k$ be the natural neutrosophic special quasi dual number semigroup.

    i.   Find all ideals of S.
    ii.   Find all subsemigroups of S which are not ideals.
    iii. Find all neutrosophic idempotents.
    iv.   Find all nilpotent element of S.
    v.   Find all neutrosophic nilpotent elements of order two.
    vi.   Find all neutrosophic zero divisors.

6. Let $M = \{^I[0, 8); \times\}$ be the MOD neutrosophic semigroup.

    i.   Find all ideals of M.
    ii.   Find all MOD neutrosophic subsemigroups which are ideals.
    iii. Find all MOD neutrosophic zero divisors.
    iv.   Find all MOD neutrosophic idempotents
    v.   Find all MOD neutrosophic pseudo zero divisors.
    vi.   Find all MOD neutrosophic find subsets which are not ideals.

7. Characterize those neutrosophic semigroup $S = \{ Z_n^I \}$ which contain the neutrosophic elements of order two for various n.

8. Consider the MOD neutrosophic interval semigroup $S = {}^I[0, n)$.

    i.   Find all ideals of S.
    ii.   Find all MOD neutrosophic zero divisors.
    iii. Find the MOD neutrosophic idempotents.
    iv.   Find the MOD neutrosophic nilpotents of order two.

9. Let $S_1 = \{^I[0, 12), \times\}$ be the MOD neutrosophic semigroup. Study questions i to iv of problem 8 for this S_1.

10. Let $S_2 = \{^I[0, 23), \times\}$ be the MOD neutrosophic semigroup.

    Study questions i to iv of problem 8 for this S_2.

11. Let $M = \{\langle Z_{12} \cup g \rangle_I, g^2 = 0, \times\}$ be the neutrosophic dual number semigroup.

    i.   Find all zero divisors and natural neutrosophic zero divisors.
    ii. Find all idempotents and neutrosophic idempotents.
    iii. Find all ideals of M.
    iv. Find all subsemigroups which are not ideals.
    v.   Is M a S-semigroup?

12. Let $N = \{\langle Z_{23} \cup g \rangle_I, g^2 = 0, \times\}$ be the neutrosophic semigroup.

    Study questions i to v of problem 11 for this N.

13. Let $T = \{\langle Z_{18} \cup g \rangle_I, g^2 = 0, \times\}$ be the natural neutrosophic semigroup.

    Study questions i to v of problem 11 for this T.

14. Obtain all special features associated with natural neutrosophic dual number semigroup
    $S = \{\langle Z_n \cup g \rangle_I, g^2 = 0, \times\}$.

15. Let $L = \{\langle Z_{42} \cup h \rangle_I, h^2 = h, \times\}$ be the natural neutrosophic special dual like number semigroup.

    Study questions i to v of problem 11 for this L.

16. Let $M = \{\langle Z_{53} \cup h \rangle_I, h^2 = h, \times\}$ be the natural neutrosophic special dual like number semigroup.
    Study questions i to v of problem 11 for this M.

17. Let $Z = \{\langle Z_n \cup h\rangle_I, h^2 = h, \times\}$ be the natural neutrosophic special like number semigroup.

    Study all the special features enjoyed by Z for varying n; n odd non-prime, n even and n a prime.

18. Let $S = \{\langle Z_{121} \cup k\rangle_I, k^2 = 120k, \times\}$ be the natural neutrosophic special quasi dual number semigroup.

    Study questions i to v of problem 11 for this S.

19. Let $P = \{\langle Z_{29} \cup k\rangle_I, k^2 = 28k, \times\}$ be the natural neutrosophic special quasi dual number semigroup.

    Study questions i to v of problem 11 for this P.

20. Let $S = \{\langle [0, 20) \cup g\rangle_I, g^2 = 0, +\}$ be the MOD neutrosophic dual number semigroup.

    i.   Show S is a S-semigroup.
    ii. Prove S has neutrosophic idempotents.
    iii. Can S have idempotents?
    iv. Find all subsemigroups of S which are not ideals.
    v.   Can S have ideals?

    vi. Can S have S-ideals?
    vii. Can ideals of S be of finite order?

21. Let $P = \{\langle Z_{25} \cup g\rangle_I, g^2 = 0, \times\}$ be the natural neutrosophic dual number semigroup.

    i.   Prove P have neutrosophic idempotents.
    ii. Can P have idempotents?
    iii. Is P a S-semigroup?
    iv. Find S-ideals if any in P.
    v.   Can P have zero space subsemigroups?
    vi. Can P subsemigroups which are not ideals?

22. Let $W = \{\langle Z_{41} \cup g\rangle_I, g^2 = 0, +\}$ be the neutrosophic dual number semigroup.

    Study questions i to vi of problem 21 for this W.

23. Let $N = \{\langle Z_{48} \cup h\rangle_I, h^2 = h, +\}$ be the natural neutrosophic special dual like number semigroup.

    Study questions i to vi of problem 21 for this N.

24. Let $M = \{\langle Z_{40} \cup k\rangle_I, k^2 = 39k; +\}$ be the natural neutrosophic special quasi dual number semigroup.

    Study questions i to vi of problem 21 for this M.

25. Let $V = \{\langle Z_{29} \cup k\rangle_I, k^2 = 28k, +\}$ be the natural neutrosophic special quasi dual number semigroup.

    Study questions i to vi of problem 21 for this V.

26. Let $M = \{\langle Z_{64} \cup k\rangle, k^2 = 63k, +\}$ be the natural neutrosophic special quasi dual number.

    Study questions i to vi of problem 21 for this M.

27. Let $P = \{\langle\langle Z_{15} \cup g\rangle_I, +\rangle, g^2 = 0, \times\}$ be the natural neutrosophic dual number semigroup.

    Study all special features associated with P.

    i.   Find o(P)
    ii. Find ideals in P.
    iii. Find neutrosophic idempotents of P.
    iv. What are the special features enjoyed by elements of the form;
         $$x = m + I_{3g}^{15} + I_{3g+5}^{15} + I_{5g}^{15} + I_{3+6g}^{15} + I_{6g+9}^{15} + I_{10g}^{15}$$
         $(m \in Z_{15})$ in P.

28. Let $W = \{\langle\langle\langle Z_{10} \cup h\rangle_I, +\rangle, \times, h^2 = h\}$ be the natural neutrosophic special dual like number semigroup under $\times$.

    i.   Find all properties enjoyed by W.
    ii. Study questions i to iii of problem 27 for this W.

29. Let $S = \{\langle\langle[0, 15) \cup g\rangle_I, +\rangle, g^2 = 0, \times\}$ be the MOD neutrosophic dual number semigroup under product.

    i.   Prove S has infinite number of zero divisors.
    ii. Prove S has ideals and subsemigroups which are zero square subsemigroups.
    iii. Prove or disprove S can have ideals of finite order.
    iv. Obtain any other special feature associated with S.

30. Let $M = \{\langle[0, 23) \cup h\rangle_I, \times\}$ be the MOD neutrosophic special dual like number semigroup.

    i.   Can M have zero divisors?
    ii.   Can M have S-idempotents?
    iii. Is M a S-semigroup?
    iv.   Can M have S-zero divisors?
    v.   Can M have ideals of finite order?

    vi.   Can M have S-ideals?
    vii. Find S-subsemigroups of finite order.
    viii. Prove the number of MOD neutrosophic elements in M is infinite.
    ix.   Can M have S-MOD neutrosophic zero divisors?
    x.   Can M have MOD neutrosophic idempotents?

31. Let $V = \{\langle[0, 45) \cup k\rangle_I; k^2 = 44k, \times\}$ be the MOD neutrosophic special quasi dual number semigroup.

    Study questions i to x of problem 30 for this V.

32. Let $W = \{\langle\langle\langle[0, 14) \cup g\rangle_I, +\rangle, g^2 = 0, \times\}$ be the MOD neutrosophic dual number semigroup.

Study questions i to x of problem 30 for this W.

33. Let $Y = \{\langle\langle[0, 29) \cup k\rangle_I, +\rangle, k^2 = 28k, \times\}$ be the special quasi dual number semigroup.

   Study questions i to x of problem 30 for this Y.

34. Let $V = \{\langle\langle[0, 40) \cup h\rangle_I, +\rangle, \times; h^2 = h\}$ be the special dual like number semigroup.

   Study questions i to x of problem 30 for this V.

35. Let $X = \{\langle\langle[0, 123) \cup k\rangle_I, +\rangle, \times; k^2 = 122k\}$ be MOD neutrosophic special quasi dual number semigroup.

   Study questions i to x of problem 30 for this X.

36. Let $P = \{\langle Z_9 \cup g\rangle_I, +, g^2 = 0, \times\}$ be the natural neutrosophic dual number semiring.
   i.   Find o(P).
   ii. Can P have ideals?
   iii. Prove P is a semiring.
   iv. Find all natural neutrosophic elements of P.
   v.   Find subsemirings of P which are not ideals.
   vi. Prove there are zero square subsemirings in P.
   vii. Find any other related properties of P.

37. Let $M = \{\langle Z_{48} \cup h\rangle_I, +, \times, h^2 = h\}$ be the natural neutrosophic special dual like number semiring.

   i. Study questions i to v of problem 36 for this M.
   ii. Obtain any other special features enjoyed by these semirings.

38. Let $W = \{\langle Z_{29} \cup k\rangle_I, k^2 = 28k, +, \times\}$ be the natural neutrosophic special quasi dual number semiring.

   Study questions i to vii of problem 36 for this W.

39. Let $S = \{\langle [0, 24) \cup g \rangle_I, +, \times, g^2 = 0\}$ be the MOD neutrosophic interval dual number semigroup.

    i.   Study questions i to vii of problem 36 for this S.
    ii. Can S be a S-semiring?
    iii. Does this S enjoy any other special features?

40. Let $M = \{\langle [0, 47) \cup h \rangle_I, +, \times, h^2 = h\}$ be the MOD neutrosophic interval special dual like number semiring.

    i.   Study questions i to vii of problem 36 for this M.
    ii. Enumerate all the special features enjoyed by this M.
    iii. Can M have S-idempotents?
    iv. Can M have S-MOD neutrosophic idempotents?

41. All the special features associated with the MOD neutrosophic interval dual number semirings.

    $S = \{\langle [0, n) \cup g \rangle_I, g^2 = 0, +, \times\}$ – Study and enumerate.

42. Study all special features enjoyed by the MOD neutrosophic interval special dual like number semiring.

    $R = \{\langle [0, m) \cup h \rangle_I, h^2 = h, +, \times\}$.
    Compare S in problem 41 with this R.

43. Let $P = \{\langle [0, p) \cup k \rangle_I, k^2 = (p - 1)k, +, \times\}$ be the MOD neutrosophic interval special quasi dual number semiring.

    Compare P with S in problem 41 and compare M with P in problem 42.

44. When will P in problem 43 be a SS-semiring and SSS-semiring?

45. Let $M = \{\langle Z_n \cup g \rangle_I, g^2 = 0, +, \times\}$ be the neutrosophic dual number semiring.

    i.   Find the number of neutrosophic elements in M. (This includes elements of the form:

$$k + I_g^n + I_{5g}^n + I_{3g}^n + I_X^n \; ; \; k \in Z_n, \; x \in Z_n \text{ is either an}$$

idempotent or a zero divisor).

   ii. Can M have S-natural neutrosophic zero divisors?

  iii. Can M have natural neutrosophic idempotents?

46. Let $B = \{\langle Z_m \cup h \rangle_I, \; h^2 = h, +, \times\}$ be the natural neutrosophic interval special dual like number semiring.

    i.   Study questions i to iii of problem 45 for this B.

   ii. Compare M with B and bring out the similarities and differences.

47. Let $D = \{\langle Z_m \cup k \rangle_I, \; k^2 = (m - 1)k, \times, +\}$ be the natural neutrosophic special quasi dual number semiring.

    i.   Study questions i to iii of problem 45 for this D.

   ii. Compare D with M and B in problems 46 and 45.

48. Let $M = \{\langle [0, \; n) \cup g \rangle_I, +, \times, \; g^2 = 0\}$ be the MOD neutrosophic dual number semiring.

    i.   Prove M has infinite number of MOD neutrosophic zero divisors.

   ii. Can M have MOD neutrosophic S-zero divisors?

  iii. Can M have MOD neutrosophic idempotents?

  iv. Can M have S-MOD neutrosophic idempotents?

   v. Is it possible to find finite order ideals in M?

  vi. Can M have S-ideals?

 vii. What will be structure enjoyed by the collection of MOD neutrosophic elements?

49. Let $T = \{\langle [0, \; n) \cup h \rangle_I, \; h^2 = h, \times, +\}$ be the MOD neutrosophic special dual like number semiring.

    i.   Study questions i to vii of problem 48 for this T.

   ii. Compare T with M in problem 48.

50. Let $V = \{\langle[0, n) \cup k\rangle_I, k^2 = (n - 1)k, +, \times\}$ be the MOD neutrosophic special quasi dual number semiring.

    i.   Study questions i to vi of problem 48 for this V.
    ii. Compare T and M with V of problems 49 and 50 respectively.

51. Can there be a MOD natural neutrosophic dual number set $S = \{\langle[0, n) \cup g\rangle_I, \times\}$ which has no MOD neutrosophic idempotents for $n \in Z^+ \setminus \{1\}$.

52. Can there be a MOD natural neutrosophic special dual like number set $P = \{\langle[0, m) \cup h\rangle_I, h^2 = h, \times\}$ which has no MOD neutrosophic zero divisors for some $m \in Z^+ \setminus \{1\}$?

53. Can P in problem 52 have no MOD neutrosophic nilpotents of order greater than two?

54. Give examples of those P for which P has MOD neutrosophic nilpotent element of order greater than or equal to three.

55. Let $M = \{\langle[0, t) \cup k\rangle_I, k^2 = (t - 1)k; +, \times\}$ be the MOD neutrosophic special quasi dual number set.

    i.   Can M have MOD neutrosophic zero divisors?
    ii. Can M have MOD neutrosophic idempotents?
    iii. Can M have MOD neutrosophic nilpotents?

Chapter Three

NATURAL NEUTROSOPHIC NUMBERS IN THE FINITE COMPLEX MODULO INTEGER AND MOD NEUTROSOPHIC NUMBERS

In this chapter we for the first time study natural neutrosophic numbers in the ring $C(Z_n)$ and that of MOD neutrosophic complex numbers in $C([0, n))$.

Also natural neutrosophic numbers are introduced in $\langle Z_n \cup I \rangle$ and MOD natural neutrosophic numbers in $[0, n)I$ and $\langle [0, n) \cup I \rangle$.

These situations are described in the following.

Example 3.1: Let $B = \{C(Z_5) \mid i_F^2 = 4\} = \{0, 1, 2, 3, 4, i_F, 2i_F, 3i_F, 4i_F, 1 + i_F, 2 + i_F, \ldots, 4 + 4i_F\}$.

The natural neutrosophic complex modulo integer are $C^I(Z_5) = \{C(Z_5),\ I_0^c \}$.

We denote the natural neutrosophic complex modulo integer of $C(Z_n)$ by $C^I(Z_n)$.

Example 3.2: Let
$M = \{C(Z_4) \mid i_F^2 = 3\}$ be the finite complex modulo integers.

The natural neutrosophic complex modulo integer;

$C^I(Z_4) = \{C(Z_4),\ I_0^c,\ I_2^c,\ I_{2i_F}^c,\ I_{2+2i_F}^c,\ I_{1+i_F}^c,\ I_{1+3i_F}^c,\ I_{3+i_F}^c\}$ is the natural neutrosophic complex modulo integer set.

Example 3.3: Let
$M = \{C(Z_2) \mid i_F^2 = 1\}$ be the finite complex modulo integers.

The natural neutrosophic complex modulo integer set.

$C^I(Z_2) = \{0,\ 1,\ i_F,\ 1 + i_F,\ I_0^c,\ I_{1+i_F}^c\}$.

Clearly $o(C^I(Z_2)) = 6$.

Example 3.4: Let $P = \{C(Z_3) \mid i_F^2 = 2\}$ be the finite complex modulo integers.

$C^I(Z_3) = \{0,\ 1,\ 2,\ i_F,\ 2i_F,\ 1 + i_F,\ 2 + i_F,\ 2 + 2i_F,\ 1 + 2i_F,\ I_0^c\}$ is the natural neutrosophic finite complex modulo integer set.

Example 3.5: Let $W = \{C(Z_6) \mid i_F^2 = 5\}$ be the finite complex modulo integers.

$C^I(Z_6) = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ i_F,\ 2i_F,\ 3i_F,\ 4i_F,\ 5i_F,\ 1 + i_F,\ 1 + 2i_F,\ 1 + 3i_F,\ 1 + 4i_F,\ 1 + 5i_F,\ 2 + i_F,\ 2 + 2i_F,\ 2 + 3i_F,\ 2 + 4i_F,\ 2 + 5i_F,\ 3 + i_F,\ 3 + 2i_F,\ 3 + 3i_F,\ 3 + 4i_F,\ 3 + 5i_F,\ 4 + i_F,\ 4 + 2i_F,\ 4 + 3i_F,\ 4 + 4i_F,\ 4 + 5i_F,\ 5 + i_F,\ 5 + 2i_F,\ 5 + 3i_F,\ 5 + 4i_F,\ 5 + 5i_F,\ I_0^c,$

$I_2^c,\ I_4^c,\ I_3^c,\ I_{2i_F}^c,\ I_{4i_F}^c,\ I_{3i_F}^c,\ I_{1+i_F}^c,\ I_{1+2i_F}^c,\quad I_{1+3i_F}^c,\quad I_{1+4i_F}^c,\quad I_{1+5i_F}^c,$

$I_{2+i_F}^c,\quad I_{2+2i_F}^c,\quad I_{2+4i_F}^c,\quad I_{2+5i_F}^c,\quad I_{3+i_F}^c,\quad I_{3+3i_F}^c,\quad I_{4+i_F}^c,\quad I_{4+2i_F}^c,\qquad I_{4+4i_F}^c$

and so on} is the natural neutrosophic finite complex modulo integers.

Example 3.6: Let $C(Z_7) = \{0, 1, 2, \ldots, 6, i_F, 2i_F, \ldots, 6i_F, 1 + i_F, 1 + 2i_F, \ldots, 1 + 6i_F, 2 + i_F, \ldots, 6 + 6i_F\}$ be the finite complex modulo integers.

$C^I(Z_7) = \{C(Z_7), \ I_0^c\}$, we are yet to find some natural neutrosophic finite complex modulo integers.

Example 3.7: Let $C(Z_8) = \{0, 1, 2, \ldots, 7, i_F, 2i_F, \ldots, 7i_F, 1 + i_F, 4 + i_F, \ldots, 7 + 7i_F\}$ be the complex modulo integer.

$C^I(Z_8) = \{0, 1, 2, \ldots, 7, i_F, \ldots, 7i_F, 1 + i_F, \ldots, 5 + 6i_F, \ldots, 7 + 7i_F,$

$I_0^c, \quad I_2^c, I_4^c, I_6^c, I_{2i_F}^c, I_{4i_F}^c, \quad I_{6i_F}^c, \quad I_{2+2i_F}^c, I_{4+4i_F}^c, I_{6+6i_F}^c, \quad I_{2+6i_F}^c,$

$I_{6+2i_F}^c, I_{4+2i_F}^c, I_{2+4i_F}^c, I_{4+6i_F}^c, I_{6+4i_F}^c, I_{i_F+7}^c, I_{1+7i_F}^c, I_{3+5i_F}^c, \quad I_{5+3i_F}^c$

and so on} be the natural neutrosophic complex modulo integers.

Example 3.8: Let $B = \{C(Z_{10}) \mid i_F^2 = 9\}$ be the finite complex modulo integer.

$C^I(Z_{10}) = \{C(Z_{10}), \quad I_0^c, \quad I_2^c, I_{2i_F}^c, I_4^c, I_{4i_F}^c, \quad I_{6i_F}^c, I_6^c, I_{8i_F}^c,$

$I_8^c, I_5^c, I_{5i_F}^c, I_{1+i_F}^c, I_{2+2i_F}^c, I_{5+5i_F}^c, I_{4+4i_F}^c, I_{6+6i_F}^c, I_{8+8i_F}^c, I_{2+4i_F}^c,$

$I_{4+2i_F}^c, \quad I_{6+2i_F}^c, I_{2+6i_F}^c, I_{8+2i_F}^c, I_{2+8i_F}^c, I_{6+4i_F}^c, I_{4+6i_F}^c$ and so on} is the natural neutrosophic finite complex modulo integers.

Example 3.9: Let $S = \{C(Z_{15}), i_F^2 = 14\}$ be the finite complex modulo integers. $C^I(Z_{15}) = \{C(Z_{15}), \ I_0^c, \ I_5^c, I_{10}^c, I_3^c, I_6^c, I_9^c,$

$I_{12}^c, I_{6i_F}^c, \ I_{5i_F}^c, I_{10i_F}^c, I_{3i_F}^c, I_{6+6i_F}^c, I_{9i_F}^c, I_{6+3i_F}^c, I_{3+6i_F}^c, I_{12+9i_F}^c$ and so

on} is the natural neutrosophic finite complex modulo integer set.

Example 3.10: Let S = {C(Z$_{11}$) | i$_F^2$ = 10} be the finite complex modulo integer.

C^I(Z$_{11}$) = {C(Z$_{11}$), I$_0^c$ one has to find more elements} is the natural neutrosophic finite complex modulo integer set.

From these examples the following result is prove.

THEOREM 3.1: *Let C(Z$_n$) be the finite complex modulo integer.*

   i. *C^I(Z$_n$) is always different from C(Z$_n$).*
   ii. *C^I(Z$_n$) has more than one natural neutrosophic number (element) if n is positive integer.*

Proof: Since $0 \in$ C(Z$_n$). I$_0^c$ is always a natural neutrosophic element as $\dfrac{t}{0}$ is undefined for all x $\in$ C(Z$_n$).

Hence proof of (i) is true.

Consider C(Z$_n$) where n is a composite integer.

So Z$_n$ has p, q $\in$ Z$_n$ such that p $\times$ q = 0. Hence this paves way for I$_p^c$ and I$_q^c$ for natural neutrosophic elements.

Further I$_{pi_F}^c$ and I$_{qi_F}^c$ also are natural neutrosophic elements of C^I(Z$_n$).

Hence the result.

However the following problem is given as an open conjecture.

Conjecture 3.1: Let $S = \{C(Z_n),\ i_F^2 = n - 1\}$ be the finite complex modulo integer. If n is a prime.

    i.   Can $C^I(Z_n)$ have more than one natural neutrosophic element?

    ii.   Can $C^I(Z_n)$ have zero divisors if n is a prime?

    iii. Can $C^I(Z_n)$ have nontrivial idempotents if n is a prime?

Next we proceed onto describe some algebraic operations on $C^I(Z_n)$. Let us first describe product operation on $C^I(Z_6)$ by an example.

Example 3.11: Let $S = \{C(Z_6)\ |\ i_F^2 = 5\}$ be the finite complex modulo integer.

$$C^I(Z_6) = \{C(Z_6),\ I_0^c, I_2^c, I_4^c, I_3^c, I_{2i_F}^c,\ I_{4i_F}^c, I_{3i_F}^c, I_{2+2i_F}^c, I_{4+4i_F}^c,$$

$I_{2+4i_F}^c, I_{4+2i_F}^c, I_{3+3i_F}^c,\ I_{1+i_F}^c,\ I_{2+i_F}^c,\ I_{i_F+2}^c, I_{4+3i_F}^c$ and so on$\}$ is the natural neutrosophic complex modulo integers.

Define $\times$ operation on $C^I(Z_6)$. $S = \{C^I(Z_6), \times\}$ is defined as the natural neutrosophic complex modulo integer semigroup. S is a semigroup.

$$I_2^c \times I_2^c = I_4^c,\ I_2^c \times I_3^c = I_0^c,\ I_{2i_F}^c \times I_{4i_F}^c = I_4^c,$$

$$I_{3i_F+3}^c \times I_{1+i_F}^c = I_0^c,\ I_{4+3i_F}^c \times I_{3+3i_F}^c = I_{3+3i_F}^c \text{ and so on.}$$

This is the way product operation is performed on S.

We see S can have natural neutrosophic complex modulo integer zero divisors and idempotents.

$$I_3^c \times I_3^c = I_3^c,\ I_{3i_F}^c \times I_{3i_F}^c = I_3^c \text{ and so on.}$$

Example 3.12: Let
$S = \{C^I(Z_{12}), i_F^2 = 11, \times\}$ be the natural neutrosophic finite complex modulo integer semigroup.

S has natural neutrosophic zero divisors and idempotents.

Example 3.13: Let
$S = \{C^I(Z_{20}); i_F^2 = 19, \times\}$ be the natural neutrosophic finite complex modulo integer semigroup.

S has zero divisors and idempotents.

$$I_{2i_F}^c \times I_{10i_F}^c = I_0^c, \ I_{4i_F}^c \times I_5^c = I_0^c, \ I_5^c \times I_5^c = I_5^c.$$

S has subsemigroups which are not ideals as well as S has ideals.

Take $C(Z_{20}) \subseteq S$ is a subsemigroup of S which is not an ideal.

$P = \{0, 10, 10i_F, 10 + 10i_F, I_0^c, I_{10}^c, I_{10i_F}^c, I_{10 + 10i_F}^c \} \subseteq S$ is an ideal of S.

Hence the claim.

Example 3.14: Let
$S = \{C^I(Z_{13}), i_F^2 = 12, \times\}$ be the natural neutrosophic complex modulo integer semigroup.

S has finite subsemigroups. Finding ideals in S is a difficult problem.

In view of this we propose the following conjecture.

Conjecture 3.2: Let
$S = \{C^I(Z_p),\ i_F^2 = p - 1,\ p$ a prime, $\times\}$ be the natural neutrosophic finite complex modulo integer semigroup. Can S have ideals?

Next we proceed onto describe + operation on the natural neutrosophic finite complex modulo integers by examples.

Example 3.15: Let
$M = \{C^I(Z_{16}),\ +;\ i_F^2 = 15\}$ be the natural neutrosophic finite complex modulo integer semigroup.

M is only a semigroup as $I_8^c + I_8^c = I_8^c$, $I_{8i_F}^c + I_{4i_F}^c = I_{8i_F}^c + I_{4i_F}^c$
and $I_{2i_F}^c + I_{2i_F}^c = I_{2i_F}^c$ so only idempotents under + so S cannot be a group only a semigroup under addition.

S is of finite order.

M has subsets which are groups; viz, Z_{16} and $C(Z_{16})$ are groups hence M is a Smarandache semigroup.

So we have several such subgroups in $C^I(Z_{16})$.

Working with these structures is innovative and interesting.

Example 3.15: Let $S = \{C^I(Z_{19}),\ +\}$ be the natural neutrosophic finite complex modulo integer semigroup.

$C(Z_{19})$, Z_{19} and $Z_{19}i_F$ are the proper subsets of S which are groups under +. Thus S is a Smarandache semigroup.

Now in view of this the following result is proved.

THEOREM 3.2: *Let $S = \{C^I(Z_n),\ +\}$ be the natural neutrosophic finite complex modulo integer semigroup. S is a Smarandache semigroup.*

Proof follows from the fact $C(Z_n)$, Z_n and $Z_n i_F$ are subsemigroups of S under + which are groups. Hence the claim.

Next we proceed onto give the product structure on the additive semigroup. $S = \{C^I(Z_n), +\}$.

This will be illustrated by some examples.

Example 3.17: Let $S = \{\langle C^I(Z_4), +\rangle, \times\}$ be a natural neutrosophic finite complex modulo integer semigroup under $\times$.

Clearly

$$S = \{C(Z_4),\ I_0^c,\ I_{2i_F}^c,\ I_2^c,\ I_{2+2i_F}^c,\ I_{1+i_F}^c,\ 1 + I_0^c,\ 1 + I_{2i_F}^c,$$
$$1 + I_{2+2i_F}^c,\ 1 + I_{1+i_F}^c,\ 2 + I_{2i_F}^c,\ 2 + I_0^c,\ 2 + I_{1+i_F}^c,\ 2 + I_{2+2i_F}^c,$$
$$3 + I_0^c,\ 3 + I_{2i_F}^c,\ 3 + I_{1+i_F}^c,\ 3 + I_{2+2i_F}^c,\ I_0^c + I_{2i_F}^c,\ I_0^c + I_{1+i_F}^c,$$
$$3 + I_0^c + I_{2i_F}^c + I_2^c + I_{1+i_F}^c\ \}.$$

$I_{2i_F}^c$, $I_{1+i_F}^c$ are also known as natural neutrosophic complex numbers for $I_2^c = I_2^4$ in our usual notation,

Superfix c is used to denote the structure is from the finite complex modulo integers.

Further $2 \in C(Z_4)$ as well $2 \in Z_4$ this is only an analogous identification.

$$I_2^c \times I_{1+i_F}^c = I_{2+2i_F}^c$$

Consider

$$I_{2i_F}^c \times I_{1+i_F}^c = I_{2i_F+2}^c.$$

This is the way natural neutrosophic finite complex modulo integer products are performed.

$$I_{2i_F+2}^c \times I_{2i_F+2}^c = I_0^c.$$

There are many nilpotents of order two.

$$2 + I^c_{2i_F} + I^c_2 + I^c_{2+2i_F} = x \in S.$$

$$x^2 = I^c_0 + I^c_2 + I^c_{2i_F} + I^c_{2+2i_F}$$

For $2 I^c_{2i_F} = I^c_{2i_F} + I^c_{2i_F} = I^c_{2i_F}$.

This is the way operations are performed on S.

Example 3.18: Let
$S = \{\langle C^I(Z_7), +\rangle, \times\}$ be the natural neutrosophic finite complex modulo integer semigroup.

Finding natural neutrosophic finite complex modulo integers, zero divisors and idempotents is a difficult job.

Let us now proceed onto develop the notion of natural neutrosophic finite complex modulo integer semirings.

We will first describe this situation by some examples.

Example 3.19: Let
$S = \{\langle C^I(Z_{10})\rangle; +, \times\}$ be the natural neutrosophic finite complex modulo integer semiring.

Clearly $o(S) < \infty$ that is this is a finite semiring which has natural neutrosophic complex finite modulo integer zero divisors and idempotents.

$$\text{For } I^c_5 \times I^c_2 = I^c_0, \qquad I^c_{2i_F} \times I^c_5 = I^c_0,$$

$$I^c_{5i_F} \times I^c_{8i_F} = I^c_0, \qquad I^c_{4+4i_F} \times I^c_{5+5i_F} = I^c_0,$$

$$I^c_{8+6i_F} \times I^c_{5i_F} = I^c_0, \qquad I^c_{8i_F} \times I^c_{5+5i_F} = I^c_0,$$

$$I^c_{6i_F+4} \times I^c_5 = I^c_0$$

and so on are the natural neutrosophic complex finite modulo integer zero divisors.

$$I_5^c \times I_5^c = I_5^c \text{ and }$$

$$I_6^c \times I_6^c = I_6^c \text{ are idempotents which are in } Z_{10}^I.$$

Now $I_{6i_F}^c \times I_{6i_F}^c = I_4^c$ and

$$I_{4i_F}^c \times I_{4i_F}^c = I_4^c.$$

Can we have pure natural neutrosophic idempotents?

$$I_{5+5i_F}^c \times I_{5+5i_F}^c = I_0^c \text{ is a zero divisor.}$$

$x = I_5^c + I_{2i_F}^c + I_{5i_F}^c + I_{8+4i_F}^c$ be in S.

$x \times x = I_5^c + I_6^c + I_{8+4i_F}^c + I_0^c + I_{5i_F}^c + I_{6i_F+2}^c \in S.$

This is the way product operation is carried out in S.

Clearly Z_{10} is a subsemiring which is a ring so S is a SS-semiring.

$C(Z_{10})$ is again a subring; $B = \{0, 5\} \subseteq S$ is a field so S is a SSS-semiring.

Example 3.20: Let
$S = \{C^I(Z_{11}), i_F^2 = 10, +, \times\}$ be the natural neutrosophic finite complex modulo integer semiring.

Z_{11} is a field in S so S is a SSS-semiring. $C(Z_{11})$ is a subring in S so S is also a SS-semiring.
Finding zero divisors is a difficult job.

However for $x = 1 + I_0^c \in S$.

$x^2 = (1 + I_0^c) = 1 + I_0^c = x$ is an idempotent.

Let $x = 5 + I_0^c$ and $y = 4 + I_0^c \in S$.

$x \times y = (5 + I_0^c) \times (4 + I_0^c) = 9 + I_0^c \in S$.

Example 3.21: Let $S = \{C^I(24), i_F^2 = 23; +, \times\}$ be the natural neutrosophic finite complex modulo integer semiring. S has zero divisors, natural neutrosophic zero divisors, complex finite modulo integer zero divisors and natural neutrosophic complex modulo integer zero divisor.

We see I_{12}^c, I_6^c, I_4^c, I_3^c, I_9^c will give natural neutrosophic zero divisors.

2, 3, 4, 8, 12, 6 etc will give zero divisors. $2i_F$, $4i_F$, $3i_F$, $8i_F$, $12i_F$, $6i_F$ can give finite complex modulo integer zero divisors.

$I_{4 + 4i_F}^c$, $I_{6 + 6i_F}^c$, $I_{8i_F}^c$ and $I_{6i_F}^c$ are some of the natural neutrosophic finite complex modulo integers.

Now we have also mixed zero divisors contributed by $x = 8 + I_{8i_F}^c + I_{8 + 8i_F}^c + I_8^c$ and

$y = 6 + I_{3i_F}^c + I_{6 + 3i_F}^c + I_{6i_F}^c \in S$ is that is $x \times y = I_0^c$.

We see if n is a composite number then certainly $C^I(Z_n)$ has zero divisors.

However if n is a prime we are not always guaranteed of zero divisors.

Finding subsemirings and ideals in case of these finite natural neutrosophic complex modulo integer semirings is

considered as a matter of routine and hence is left as an exercise to the reader.

Next we proceed onto study MOD neutrosophic interval finite modulo complex numbers.

Example 3.22: Let $S = \{C^I([0, 6))\} = \{$Collection of all elements from $[0, n)$, $C([0, n))$ and those I_x^c where x is an element in $[0, n)$ such that it is a zero divisor or an idempotent or a pseudo zero divisor$\}$.

$C^I([0, 6)) = \{C([0, 6)), I_3^c, I_{3i_F}^c, I_2^c, I_{2i_F}^c, I_{2 + 2i_F}^c, I_{3 + 3i_F}^c, I_{1.2}^c, I_{1.2i_F}^c$ and so on$\}$ is the collection of all MOD neutrosophic finite complex modulo integers.

Example 3.23: Let $P = \{C^I([0, 7)), i_F^2 = 6\}$ be the MOD neutrosophic finite complex modulo integers.

$P = \{ I_0^c, I_{3.5}^c, I_{1.75}^c, I_{3.5i_F}^c, I_{3.5i_F + 3.5}^c, I_{1.75i_F}^c, C([0, 7))$ and so on$\}$.

Example 3.24: Let $M = \{C^I([0, 12)), i_F^2 = 11\}$ be the MOD neutrosophic finite complex modulo integer interval.

$M = \{C([0, 12)), I_0^c, I_6^c, I_4^c, I_3^c, I_{6i_F}^c, I_{3i_F}^c, I_{2i_F}^c, I_{4i_F}^c, I_{3 + 3i_F}^c, I_{8i_F}^c + I_8^c, I_{1.2i_F}^c, I_{1.2i_F + 1.2}^c, I_{2.4i_F + 2.4}^c$ and so on$\}$ is the MOD neutrosophic finite complex modulo integers.

Thus it is a difficult problem to find the number of MOD neutrosophic finite complex modulo integers. It may be infinite.

However the open problem is can it be for any prime p be finite?

This problem is little difficult.

Now having seen examples of MOD neutrosophic finite complex modulo integer elements we proceed onto study or employ algebraic structures on them.

Example 3.25: Let $S = \{C^I([0, 4)), \times\} = \{C([0, 4)), I_2^c, I_0^c, I_{2i_F}^c, I_{2 + 2i_F}^c, I_{3 + 3i_F}^c, I_{1 + i_F}^c$ and so on$\}$ be the MOD neutrosophic finite complex modulo integer semigroup.

This semigroup has zero divisors and MOD neutrosophic zero divisors.

However finding idempotents happens to be a difficult problem for this S.

Infact S has subsemigroups which are not ideals and of finite order.

Finding even ideals in S is a difficult task.

Example 3.26: Let $A = \{C^I([0, 5)), \times\} = \{C([0, 5)); I_0^c, I_{2.5}^c, I_{2.5i_F}^c, I_{1.25}^c, I_{1.25i_F}^c, I_{2.5 + 2.5i_F}^c, I_{1.25 + 1.25i_F}^c$, etc$\}$ be the MOD neutrosophic finite complex modulo integers.

Clearly 2.5, $2.5i_F$, 1.25, $1.25i_F$, $2.5 + 2.5i_F$, $1.25 + 1.25i_F$, $2.5 + 1.25i_F$, $2.5i_F + 1.25i_F$, 3.75, $3.75i_F$, $3.75 + 3.75i_F$, $3.75 + 2.5i_F$, $3.75i_F + 2.5$ and so on are all only pseudo zero divisors for their product with a unit 4 or 2 leads to give zero divisors hence this study is new and in these semigroups we have pseudo zero divisors which help to pave way for MOD neutrosophic pseudo divisors.

However $I_{2.5}^c \times I_{2.5}^c = I_{2.5}^c$ but it is clear none of these can lead to MOD neutrosophic zeros for both 4 and 2 are neutrosophic as $\dfrac{1}{4}$ is 4 and $\dfrac{1}{2} = 3$ so they are units in Z_5.

Such type of tricky but innovative study is left open for any interested researchers.

This sort of situation mainly prevails when one uses in the MOD interval of complex finite modulo integers $C([0, n))$, n a prime number. Hence a special type of study is needed for n a prime value.

We will just illustrate this situation by one more example before we proceed on to study other algebraic structures on $C^I([0, n))$.

Example 3.27: Let $B = \{C^I([0, n)), \times\}$ be the MOD neutrosophic finite complex modulo integer semigroup.

5.5, $5.5i_F$ are both pseudo zero divisors as

$$5.5 \times 2 = 0 \ (\text{mod } 11)$$

$$5.5i_F \times 2 = 0 \ (\text{mod } 11)$$

$$5.5 + 5.5i_F \times 2 = 0 \ (\text{mod } 11)$$

$$5.5 \times 4 \equiv 0 \ (\text{mod } 11)$$

$$5.5 \times 8 = 0 \ (\text{mod } 11)$$

$$5.5 + 5.5i_F \times 12 \equiv 0 \ (\text{mod } 11).$$

But none of the elements 2, 4, 6, 8, 10 or 12 can be neutrosophic as all of them are units.

Similarly 2.75, 2.75i_F, 2.75 + 2.75i_F, 2.75 + 5.5i_F, 2.75i_F + 5.5 are all only pseudo zero divisors.

Similarly 1.1, 2.2, 3.3, 4.4, …, 9.9 and 1.1i_F, 2.2i_F, 3.3i_F, …, 9.9i_F are all pseudo zero divisors for 10 acts on them to make them zeros.

1.375, 1.375i_F, 1.375 + 1.375i_F lead to pseudo interval zero divisors.

For 8 which is a unit in 11 makes them zero.

The study of pseudo zero divisors in MOD intervals is a challenging problem for they cannot be MOD neutrosophic zero divisors and pseudo zero divisors.

In view of all these we have the following conjecture.

Conjecture 3.3: Let S = {C^I([0, n)), n a prime, ×} be the MOD neutrosophic finite complex modulo integer semigroup.

    i.   Can S have zero divisors?

    ii.   Can S have pseudo zero divisors?

    iii.   Can S have MOD neutrosophic pseudo zero divisors? (How to develop this notion?)

    iv.   Can C^I([0, n)), n not a prime have MOD neutrosophic pseudo zero divisors?
(Clearly if n is not a prime then also [0, n) has pseudo zero divisors).

Next we define the notion of + operation on C^I([0, n)).

Let S = {C^I([0, n)), +} = {Collection of all MOD neutrosophic finite complex modulo integers under +}. S is only a semigroup and not a group as $I_x^c + I_x^c = I_x^c$ for all x ∈ C^I([0, n)); is an idempotent.

We will illustrate this situation by some examples.

Example 3.28: Let S = {$C^I([0, 6))$, +} be the MOD neutrosophic finite complex modulo integer semigroup.

$S = \{\langle C([0, 6)); \quad I_0^c, \quad I_2^c, I_3^c, I_4^c, I_{2i_F}^c, I_{4i_F}^c, \ I_{3i_F}^c, I_{2+2i_F}^c,$

$I_{3+3i_F}^c, \ldots I^c, I_{1.5}^c, I_{1.5i_F}^c, I_{1.5+1.5i_F}^c, \ I_{1.2}^c, \ I_{1.2i_F}^c, \ I_{1.2+1.2i_F}^c$ and so on$\rangle\}$

is a semigroup generated under +.

Clearly we define in general $I_{1.5i_F}^c + I_{1.5i_F}^c = I_{1.5i_F}^c$ and so on.

Hence this is an idempotent under +.

This semigroup has subsemigroups of both finite and infinite order.

There are both subsemigroups which are groups under + of infinite and finite order.

For Z_6 is a group under +. $C(Z_6)$ is again a group under +. [0, 6) is a group of infinite order under +. $[0, 6)i_F$ is also a group under + of infinite order.

$C([0, 6))$ is also a group under + is an infinite group.

Thus S is a Smarandache semigroup.

Example 3.29: Let M = {$C^I([0, 7))$, +} be the MOD neutrosophic finite complex modulo integer semigroup.

$M = \{\langle C([0, 17)), \quad I_0^c, \quad I_{8.5}^c, \quad I_{8.5i_F}^c, \quad I_{8.5+8.5i_F}^c, I_{4.25}^c,$

$I_{4.25i_F}^c, I_{4.25+4.25i_F}^c, I_{4.25}^c + I_{8.5i_F}^c + I_{4.25+4.25i_F}^c$ and so on$\rangle$, +} be the MOD neutrosophic interval semigroup. M is a S-semigroup of infinite order.

Example 3.30: Let

$S = \{\langle C^I([0, 24)), +\rangle\} = \{C^I([0, 24)), x + I_0^c, x + I_{12}^c + I_{12i_F}^c$ and so on$\}$ be the MOD neutrosophic finite complex modulo integer semigroup.

This is only a semigroup as MOD neutrosophic elements under the operation of addition is only an idempotent.

Several interesting properties can be derived which is considered as a matter of routine and left as an exercise to the reader.

Next we prove a theorem.

THEOREM 3.3: *Let $S = \{C^I([0, n)), +\}$ be the MOD neutrosophic finite complex modulo integer semigroup.*

> i.   *S is a S-semigroup.*
> ii. *S has subgroups of both finite and infinite order.*
> iii. *S has idempotents.*
> iv. *S has MOD neutrosophic idempotents.*

Proof is direct and hence left as an exercise to the reader.

Next we illustrate the semigroup under product using the additive MOD neutrosophic finite complex modulo integer semigroup under + by examples.

Example 3.31: Let $S = \{\langle C^I([0, 3)), +\rangle, \times\}$ be the MOD neutrosophic finite complex modulo integer semigroup under $\times$.

Clearly S contains elements of the form $x = 2.532 + I_0^c + I_{1.5 + 1.5i_F}^c$ and $y = 0.273 + I_{1.5}^c + I_0^c$ and so on.

$$x \times y = 2.532 \times 0.273 + I_0^c + I_{1.5 + 1.5i_F}^c + I_{1.5}^c + I_{2.25 + 2.25i_F}^c .$$

This is the way product operation is performed on S. If x = 1.5 is a pseudo zero divisor x^2, x^3, x^4, … are also assumed to be pseudo zero divisors.

Likewise I_x^c, $I_{x^2}^c$, $I_{x^3}^c$, …, $I_{x^t}^c$, … are all assumed to be pseudo zero divisors.

So we may also have infinite sums of MOD neutrosophic elements in S.

This study is new and innovative. So far there has been no research in this direction.

Example 3.32: Let
S = {⟨$C^I([0, 10))$, +⟩, ×} be the MOD neutrosophic finite complex modulo integer semigroup.

S has pseudo divisor, zero divisor, MOD neutrosophic zero divisors and pseudo zero divisors.

S has subsemigroups of both finite and infinite order.

Example 3.33: Let
S = {⟨$C^I([0, 29))$, +⟩, ×} be the MOD neutrosophic finite complex modulo integer semigroup.

S has MOD neutrosophic pseudo zero divisors. S has subsemigroups.

Next we proceed onto describe MOD neutrosophic finite complex modulo integer semirings by examples.

Example 3.34: Let
S = {$C^I([0, 16))$, +, ×, $i_F^2 = 15$} be the collection of all MOD neutrosophic finite complex modulo integer semiring.

S has zero divisors and MOD neutrosophic zero divisors. S is not a semifield. S is only a pseudo semiring as + and × are not distributive.

Let us consider $x = I_0^c + I_{4i_F}^c + I_{2i_F}^c$ and $y = I_0^c + I_8^c + I_{8i_F}^c + I_{8+8i_F}^c \in S$. $x \times y = I_0^c$ is a MOD neutrosophic finite complex modulo integer zero divisor.

Several properties of infinite semirings can be adopted to these pseudo infinite semiring with appropriate changes.

Example 3.35: Let $M = \{C^I([0, 10)), +, \times\}$ be the MOD neutrosophic finite complex modulo integer pseudo semirings.

Let $x = I_5^c + I_{5i_F}^c + I_{5+5i_F}^c \in M$.

$$x \times x = \left(I_5^c + I_{5i_F}^c + I_{5+5i_F}^c \right)$$

$$= I_5^c + I_{5i_F}^c + I_{5+5i_F}^c + I_0^c$$
$$\neq x.$$

But $x + x = x$.

This is the way product and + operation is performed on M.

M has subsemirings which are rings so M is SS-semiring. Also M has a subset which is a field so M is SSS-semifield.

Finding ideals in these pseudo semirings is a difficult problem.

Example 3.36: Let $S = \{C^I([0, 29)), +, \times\}$ be the MOD neutrosophic finite complex modulo integer pseudo semiring.

S has pseudo zero divisors 14.5, $14.5i_F$, $14.5 + 14.5i_F$, 7.25, $7.25i_F$, $7.25 + 7.25i_F$, 2.9, 5.8, 7.7, $2.9i_F$, $5.8i_F$, $7.7i_F$, $11.6i_F$ and so on contribute to pseudo zero divisors.

Hence $I^c_{14.5}$, $I^c_{2.9}$, $I^c_{5.8}$, $I^c_{2.9i_F}$ and so on are MOD neutrosophic pseudo zero divisors.

Study in this direction is innovative and interesting.

Since this study of pseudo semirings using MOD neutrosophic finite complex modulo integers considered as a matter of routine.

This work is left as an exercise to the reader.

Next the concept of natural neutrosophic finite neutrosophic modulo analogue using the MOD intervals $\langle [0, n) \cup I \rangle_I$ is developed and described in the following.

Example 3.37: Let S = $\{\langle Z_9 \cup I \rangle_I\}$ = {Collection of all natural neutrosophic numbers in $\langle Z_9 \cup I \rangle$ got by adopting division in $\langle Z_9 \cup I \rangle$} = $\{\langle Z_9 \cup I \rangle$, I^I_0, I^I_3, I^I_6, I^I_I, I^I_{2I}, I^I_{3I}, I^I_{4I}, …, I^I_{8I}, I^I_{3+3I}, I^I_{6+6I} and so on}.

Finding order of S is a difficult job.

$$I^I_{3+3I} \times I^I_{6+6I} = I^I_0 \qquad\qquad I^I_{3+3I} \times I^I_{3+3I} = I^I_0$$

$$I^I_{6+6I} \times I^I_{6+6I} = I^I_0 \qquad\qquad I^I_{3I} \times I^I_{3+3I} = I^I_0 \text{ and so on.}$$

Thus S has zero divisor and natural neutrosophic zero divisors.

Study in this direction is interesting and innovative.

Example 3.38: Let
W = $\{\langle Z_{43} \cup I\rangle_I, \times\}$ be the natural neutrosophic finite neutrosophic modulo integer semigroup.

W = $\{\langle Z_{43} \cup I\rangle, I_0^I, I_{nI}^I; n \in Z_{43}$ and so on, $\times\}$ is a semigroup for $I^2 = I$ and I is an invertible neutrosophic number which is also assumed to be an indeterminate.

Example 3.39: Let
M = $\{\langle Z_{12} \cup I\rangle_I, \times\}$ be the natural neutrosophic finite neutrosophic modulo integer semigroup.

M has both zero divisors, idempotents and natural neutrosophic idempotents and zero divisors.

Next we give on $\langle Z_n \cup I\rangle$, the operation of addition.

Example 3.40: Let S = $\{\langle Z_3 \cup I\rangle_I, +\} = \{0, 1, 2, 0, I, 2I, 1 + I, 2 + I, 2 + 2I, 1 + 2I, I_0^I, I_I^I, I_{2I}^I$ and so on, $+\}$.

$(I_0^I + I_I^I + I_{2I}^I = x$ we see $x + x = x)$ be the natural neutrosophic finite neutrosophic modulo integers semigroup under +.

Example 3.41: Let M = $\{\langle Z_4 \cup I\rangle_I; +\} = \{\langle Z_4 \cup I\rangle, I_0^I, I_2^I, I_{2I}^I, I_{2 + 2I}^I, I_{1 + 2I}^I, I_{2 + I}^I, I_{1 + 3I}^I, I_{3 + I}^I$ and so on, $+\}$ be the natural neutrosophic finite neutrosophic modulo integer semigroup.

$$x = I_{1 + 3I}^I + I_2^I, y = I_0^I + I_{2 + 2I}^I \in M$$

$$y + y = I_0^I + I_{2 + 2I}^I \in M$$

$$x + y = I_0^I + I_{2 + 2I}^I + I_2^I + I_{1 + 3I}^I \in M.$$

Example 3.42: Let P = {$\langle Z_8 \cup I \rangle_I$, +} = {$\langle Z_8 \cup I \rangle$, I_I^I, I_{2I}^I, I_{3I}^I, I_{4I}^I, ..., I_{7I}^I, I_{2I+4I}^I, I_{2+6I}^I, I_2^I, I_4^I, I_6^I and so on, +} be the MOD neutrosophic finite neutrosophic modulo integer semigroup.

$$I_6^I + I_6^I = I_6^I,$$

$$I_{6+6I}^I + I_{4I+4}^I + I_{2I}^I \text{ and so on are elements of P.}$$

Example 3.43: Let M = {$\langle Z_{12} \cup I \rangle_I$, +} = {$\langle Z_{12} \cup I \rangle$, I_0^I, I_I^I, ..., I_{11I}^I, I_2^I, I_3^I, I_4^I, I_6^I, I_8^I, I_{10}^I, I_9^I, $I_{11I}^I + I_8^I$, $I_{10}^I + I_I^I$ and so on, +} be the natural neutrosophic finite neutrosophic modulo integer semigroup.

We see M is a S-semigroup.

P_1 = {Z_{12}, +} is a group under +.

$P_2 = \langle Z_{12} \cup I \rangle$ is a S-semigroup. Thus M is a S-semigroup.

The following theorems are proved.

THEOREM 3.4: *Let S = {$\langle Z_n \cup I \rangle_I$, $\times$} be a natural neutrosophic finite neutrosophic integers semigroup.*

    i.   *S is a S-semigroup if and only if Z_n is a S-semigroup.*
    ii.   *If S has zero divisors then S has natural neutrosophic zero divisors.*
    iii. *S has subsemigroups.*
    iv. *S has idempotents and natural neutrosophic idempotents.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 3.5: *Let S = {$\langle Z_n \cup I \rangle_I$, +} be the natural neutrosophic finite neutrosophic integers semigroup.*

    i.   *S is always a S-semigroup.*
    ii.   *o(S) < ∞.*
    iii.   *S has neutrosophic idempotents.*

Proof is direct and is left as an exercise to the reader.

Next we proceed onto define a new product on the additive semigroup built using
$S = \langle\langle Z_n \cup I\rangle, +\rangle$. $o(S) < \infty$ but $o(S) > o(\{\langle Z_n \cup I\rangle_I, \times\})$.

We will first describe this situation by an example.

Example 3.44: Let $P = \{\langle\langle Z_8 \cup I\rangle_I, +\rangle, \times\}$ be the natural neutrosophic finite integer neutrosophic semigroup.

S has idempotents zero divisors and neutrosophic zero divisors and neutrosophic idempotents.

For $x = I_0^I + I_4^I + I_6^I \in P$ is such that

$$x \times x \quad = (I_0^I + I_4^I + I_6^I) \times (I_0^I + I_4^I + I_6^I)$$

$$= I_0^I + I_0^I + I_4^I + I_4^I$$
$$= I_0^I + I_4^I \in P.$$

This is the way product operation is performed on P.

$y = I_0^I + I_4^I$ is a natural neutrosophic zero divisors.

Thus x is a natural neutrosophic nilpotent element of order three.

Example 3.45: Let $W = \{\langle\langle Z_{12} \cup I\rangle_I, +\rangle, \times\}$ be the natural neutrosophic finite neutrosophic semigroup.

Let $x = I_{3I}^I + I_{4I}^I + I_{6I}^I + I_{2I}^I$ and $y = I_8^I + I_{4+4I}^I \in W$.

$x \times y = \left(I_{3I}^I + I_{4I}^I + I_{6I}^I + I_{2I}^I \right) \times \left(I_8^I + I_{4+4I}^I \right)$

$\qquad = I_0^I + I_{8I}^I + I_{4I}^I + I_{4I}^I + I_{8I}^I$

$\qquad = I_0^I + I_{4I}^I + I_{8I}^I \in W$.

W is a S-semigroup.

This is the way product is performed on W.

Example 3.46: Let $V = \{\langle\langle Z_{11} \cup I\rangle_I, +\rangle \times\}$ be the natural neutrosophic finite neutrosophic integer semigroup.

V has neutrosophic idempotents. Thus V has subsemigroups.

Infact V is a S-semigroup.

Next we proceed on to develop the neutrosophic interval semirings.

All properties associated with natural neutrosophic semigroups can be derived as matter of routine so left as an exercise to the reader.

Example 3.47: Let $S = \{\langle Z_9 \cup I\rangle_I, +, \times\}$ be the natural neutrosophic finite neutrosophic modulo integer semiring.

o(S) is finite and this is the first time naturally finite semirings are constructed.

These semirings solves the open conjectures; does there exist semirings of finite characteristic.

The answer is yes we can have semirings of finite characteristic some of them are of infinite order and some are of finite order.

Example 3.48: Let S = $\{\langle Z_{12} \cup I \rangle_I, +, \times\}$ be the natural neutrosophic finite neutrosophic modulo integer semiring.

Clearly S is of characteristic 12. We see S has both zero divisors and natural neutrosophic zero divisors.

Several properties associated with them can be derived. S has also natural neutrosophic idempotents.

For x = I_{4I}^I, y = I_{9I}^I, z = $I_{4I}^I + I_{9I}^I + I_0^I$ are all natural neutrosophic idempotents in S.

For $x^2 = \left(I_0^I + I_{4I}^I + I_{9I}^I\right) \times \left(I_0^I + I_{4I}^I + I_{9I}^I\right)$

$$= I_0^I + I_{4I}^I + I_{9I}^I = x.$$

Hence z is a natural neutrosophic idempotent.

S contains a subring so S is a SS-semiring.

Several properties regarding natural neutrosophic finite neutrosophic modulo integer semirings can be derived and it is considered as a matter of routine and is left as an exercise to the reader.

Next we proceed onto describe MOD neutrosophic interval semigroup under + and × and the pseudo semirings.

Example 3.49: Let B = $\{\langle [0, 6) \cup I \rangle_I, \times\}$ be the MOD natural neutrosophic interval semigroup.

B has zero divisors, idempotents, MOD neutrosophic zero divisors and idempotents.

$$I_3^I \times I_3^I = I_3^I, \qquad I_{3I}^I \times I_{3I}^I = I_{3I}^I,$$

$$I_{4I}^I \times I_{4I}^I = I_{4I}^I, \quad I_4^I \times I_4^I = I_4^I$$

and so on are MOD neutrosophic idempotents in B.

$$\text{Consider } I_3^I \times I_{4I}^I = I_0^I, \qquad I_{3I}^I \times I_{2I}^I = I_0^I,$$

$$I_{4I}^I \times I_3^I = I_0^I$$

and so on are MOD neutrosophic zero divisors in B.

Example 3.50: Let M = $\{\langle [0, 7) \cup I \rangle_I, \times\}$ be the MOD neutrosophic interval neutrosophic semigroup of infinite order.

$I_{6I}^I, I_0^I, I_{4I}^I, I_I^I$ are in M and some of them are idempotents.

However pseudo zero divisor and MOD neutrosophic zero divisors of M are given.

$I_{3.5I}^I, I_{3.5}^I, I_{3.5+3.5I}^I, I_{1.75}^I, I_{1.75I}^I, I_{1.75+1.75I}^I$ and so on.

Study of existence MOD neutrosophic idempotents and zero divisors happens to be an interesting and a difficult problem, when the MOD intervals [0, n), where n a prime is used.

Example 3.51: Let S = $\{\langle [0, 18) \cup I \rangle_I, \times\}$ be the MOD neutrosophic interval neutrosophic semigroup.

S has zero divisors, pseudo zero divisors as well as MOD neutrosophic zero divisors and pseudo zero divisors.

$I_0^I, I_9^I, I_2^I, I_3^I, I_{9I}^I, I_{6I}^I, I_{4I}^I, \ldots, I_{12I}^I, I_{4.5}^I, I_{4.5I}^I, I_{12I+6}^I$ and so on.

$$I_{12I}^I \times I_{12I}^I = I_0^I \qquad\qquad I_{6I}^I \times I_{6I}^I = I_0^I$$

and so on are MOD neutrosophic zero divisors of S.

$$I_9^I \times I_9^I = I_9^I \text{ and } \qquad I_{9I}^I \times I_{9I}^I = I_{9I}^I$$

$$I_{9+9I}^I \times I_{9+9I}^I = I_{9+9I}^I$$

are some of the MOD neutrosophic idempotents of S.

In view of this the following theorem proved.

THEOREM 3.6: *Let* $B = \{\langle [0, n) \cup I \rangle_I, \times\}$ *be the* MOD *neutrosophic interval neutrosophic semigroup.*

  i.   *B has* MOD *neutrosophic zero divisors if* Z_n *has zero divisors.*
  ii. *B has* MOD *neutrosophic idempotents if* Z_n *has idempotents.*
  iii. *B has* MOD *neutrosophic pseudo zero divisors if* $[0, n)$ *has pseudo zero divisors.*

Proof is direct and hence left as an exercise to the reader.

Next we generate MOD neutrosophic interval neutrosophic semigroup under the operation +.

This is illustrated by some examples.

Example 3.52: Let $S = \{\langle\langle [0, 6) \cup I \rangle_I, + \rangle\}$ be the MOD neutrosophic interval neutrosophic semigroup as $I_{3I}^I + I_{3I}^I = I_{3I}^I$ and

$$I_{2I+2}^I + I_{2I+2}^I = I_{2I+2}^I \text{ are idempotents under sum.}$$

Further as $P_1 = \langle [0, 6) \cup I \rangle \subseteq S$ and P_1 is a group.

Likewise $P_2 = [0, 6)$ is also a group under +.

$P_3 = \langle Z_6 \cup I \rangle$ is again a group under +.

$P_4 = Z_6$ is again a group under +.

Thus $P_4 \subseteq P_3 \subseteq P_1 \subseteq S$ is chain of group and hence S is a S-semigroup.

Several interesting properties in this direction can be derived and it is considered as a matter of routine and left as an exercise to the reader.

Example 3.53: Let $W = \{\langle\langle[0, 7) \cup I\rangle_I, +\}$ be the MOD neutrosophic interval neutrosophic semigroup.

$I_0^I, I_{3.5}^I, I_{3.5I}^I, I_{3.5 + 3.5I}^I, I_{1.75}^I, I_{1.75I}^I, I_{1.75 + 1.75I}^I$ are all some of the MOD neutrosophic idempotents of W.

In view of all these we have the following result.

THEOREM 3.7: *Let*
$S = \{\langle\langle[0, n) \cup I\rangle_I, +\rangle\}$ be the MOD neutrosophic interval neutrosophic semigroup generated under +.

  i.   *S is a S-semigroup.*
  ii. *S has MOD neutrosophic idempotents..*
  iii. *S has subsemigroups of infinite order which are not ideals.*

The proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD neutrosophic interval neutrosophic semigroups under $\times$ on the semigroup $\langle\langle[0, n) \cup I\rangle_I, +\rangle$ by examples.

Example 3.54: Let $M = \{\langle\langle[0, 10) \cup I\rangle_I, +\rangle, \times\}$ be the MOD neutrosophic interval neutrosophic semigroup under product.

M is of infinite order. M has zero divisors, idempotents, pseudo zero divisors, neutrosophic zero divisors, idempotents and neutrosophic pseudo zero divisors.

Example 3.55: Let B = $\{\langle\langle[0,\ 11)\ \cup\ I\rangle_I,\ +\rangle,\ \times\}$ be the MOD neutrosophic interval neutrosophic semigroup under $\times$.

B is a S-semigroup.

We define $I \times I_p^I = I_p^I$ as we give more importance to the natural neutrosophic numbers only so $I \times I_{pI}^I = I_{pI}^I, p \in [0, 11)$.

Study in this direction is interesting and innovative and is left for the reader as it is considered as a matter of routine.

Next we proceed onto give examples of MOD neutrosophic interval neutrosophic semirings.

Example 3.56: Let S = $\{\langle[0,\ 9)\ \cup\ I\rangle_I,\ +,\ \times\}$ be the MOD neutrosophic interval neutrosophic pseudo semiring.

S has MOD neutrosophic zero divisors and MOD neutrosophic idempotents. S is only a pseudo semiring as the distributive laws are not true in general.

$I_0^I, I_3^I, I_6^I, I_{3I}^I, I_{6I}^I, I_{3+3I}^I, I_{3+6I}^I, I_{3I+6}^I, I_{6I+3}^I$ are some of the MOD neutrosophic elements of S.

We see $I_3^I \times I_{6I}^I = I_0^I$, $I_{6I}^I \times I_{6I}^I = I_0^I$ and so on.

Example 3.57: Let S = $\{\langle[0,\ 14)\ \cup\ I\rangle_I,\ +,\ \times\}$ be the MOD neutrosophic interval neutrosophic pseudo semiring.

S is not a semifield. S has zero divisors and MOD neutrosophic zero divisors.

$I_2^I \times I_{7I}^I = I_0^I, I_{4I}^I \times I_{7I}^I = I_0^I$ are neutrosophic zero divisors.

$$I_{7I}^I \times I_{7I}^I = I_{7I}^I, \qquad I_7^I \times I_7^I = I_7^I \text{ and}$$

$$I_{7I+7}^I \times I_{7+7I}^I = I_{7+7I}^I$$

are MOD neutrosophic idempotents. $Z_{14} \subseteq S$ is a ring so S is a SS-ring. Likewise S can be proved to be SSS-semiring.

Thus study in this direction is interesting and innovative and it is left as an exercise to the reader.

Example 3.58: Let M = $\{\langle [0,\ 12) \cup I \rangle_I,\ +,\ \times \}$ be the MOD neutrosophic interval neutrosophic pseudo semiring.

Let x = $I_3^I + I_{6I}^I + I_{9+9I}^I + I_0^I$ and

y = $I_4^I + I_{4+4I}^I + I_{8I}^I \in M;$

$x \times y = I_0^I$

thus M has MOD neutrosophic zero divisors. So M is not a pseudo semifield. Now $Z_{12} \subseteq M$ is a subring of M so M is a SS-semiring.

Clearly if x = $I_4^I + I_{4I}^I \in M.$

$$x \times x = \left(I_4^I + I_{4I}^I \right) \times \left(I_4^I + I_{4I}^I \right)$$

$$= I_4^I + I_{4I}^I + I_{4I}^I + I_{4I}^I$$

$$= I_4^I + I_{4I}^I = x \in M.$$

Thus M has MOD neutrosophic idempotents.

Also $I_4^I + I_4^I = I_4^I$ is a MOD neutrosophic idempotent with respect to + of M.

Let $x = I_9^I + I_{4I}^I + I_{9I}^I + I_0^I + I_4^I \in M$.

$x \times x = I_9^I + I_{4I}^I + I_{9I}^I + I_4^I + I_0^I = x$ is also an idempotent which is MOD neutrosophic.

Let $P = [0, 12)I \subseteq M$; clearly P is a pseudo ideal of M.

$R = [0, 12)$ is only a pseudo subsemiring of M.

Thus M has ideals of infinite order as well as subsemirings of infinite order which are not ideals.

In view of all these we have the following theorem.

THEOREM 3.8: *Let* $S = \{\langle [0, n) \cup I \rangle_I, +, \times\}$ *be the MOD neutrosophic interval neutrosophic pseudo semiring.*

    i.   *$o(S) = \infty$.*
    ii.   *S has MOD neutrosophic zero divisors if [0, n) has zero divisors.*
    iii.   *S has MOD neutrosophic idempotents.*
    iv.   *S has [0, n)I = P $\subseteq$ S to be the MOD neutrosophic pseudo ideal.*
    v.   *R = [0, n) $\subseteq$ S is a MOD neutrosophic pseudo subsemiring.*
    vi.   *S is a SS-pseudo semiring.*
    vii. *S is a SSS-pseudo semiring if and only if Z_n is a S-ring.*

Proof is direct and hence left as an exercise to the reader.

We suggest the following problems.

Problems

1. Give some interesting properties associated with $C^I(Z_n)$ the natural neutrosophic finite complex modulo integers.

2. Let $S = \{C^I(Z_{40}), \times\}$ be the natural neutrosophic finite complex modulo integer semigroup.

    i.   Is S a S-semigroup?
    ii.   Find o(S).
    iii. Can S have S-ideals?
    iv.   Can S have S-zero divisors?
    v.   Can S have natural neutrosophic S zero divisors?
    vi.   Can S have idempotents?
    vii. Can S have natural neutrosophic S-idempotents?
    viii. Obtain any other special property associated with S.

3. Let $S = \{C^I(Z_{27}), \times\}$ be the natural neutrosophic finite complex modulo integer semigroup.

    Study question i to viii of problem 2 for this S.

4. Let $W = \{C^I(Z_{47}), \times\}$ be the natural neutrosophic finite complex modulo integer semigroup.

    Study questions i to viii of problem 2 for this W.

5. Let $M = \{C^I(Z_n), \times\}$ be the natural neutrosophic finite complex modulo integer semigroup.

    Obtain all special features enjoyed by M when;
    i.   n is a prime
    ii.   n is a composite number
    iii. $n = p^t$; p a prime t > 0.

6. Let $S = \{C^I(Z_{40}), +\}$ be the natural neutrosophic finite complex modulo integer semigroup under +.

    i.   Find o(S).

ii. Find S-ideals if any in S.
iii. Is S a S-semigroup?
iv. Find subsemigroups which are not ideals.
v.   Find idempotents of S.

7. Let $P = \{C^I(Z_{29}), +\}$ be the natural neutrosophic complex modulo integer semigroup.

    Study questions i to v of problem 6 for this P.

8. Let $S = \{C^I(Z_n), +\}$ be the natural neutrosophic finite complex modulo integer semigroup.

    Study all the special features enjoyed by S when
    i.   n is a prime.
    ii. n is a composite number.
    iii. $n = p^t$; p a prime $t \geq 2$.

9. Let $V = \{\langle C^I(Z_{10}), +\rangle, \times\}$ be the natural neutrosophic finite complex modulo integer semigroup under $\times$.

    i.   Find o(V).
    ii.   Find all natural neutrosophic elements of V.
    iii. Find all natural neutrosophic zero divisors of V.
    iv.   Find all natural neutrosophic idempotents of V.
    v.   Is V a S-semigroups?
    vi.   Can V have S-ideals?
    vii. Can V have S-zero divisors?
    viii. Obtain any other property enjoyed by V.

10. Let $W = \{\langle C^I(17), +\rangle, \times\}$ be the natural neutrosophic finite complex modulo integer semigroup under $\times$.

    Study questions i to vii of problem 9 for this W.

11. Let $X = \{\langle C^I(64), +\rangle, \times\}$ be the natural neutrosophic finite complex modulo integer semigroup under $\times$.

    Study questions i to vii of problem 9 for this X.

12. Let $S = \{C^I(Z_{15}), +, \times\}$ be the natural neutrosophic finite complex modulo integer semiring.

    i.   Find o(S).
    ii.   Show S is a semiring of finite characteristic 15.
    iii. Find all idempotents of S.
    iv.   Is S a S-semiring?
    v.   Can S have zero divisors?
    vi.   Prove or disprove S has ideals.
    vii. Can S have subsemirings which are not ideals?
    viii. Find any other special feature enjoyed by S.
    ix.   Is S a SS-semiring?
    x.   Is S a SSS-semiring?

13. Let $M = \{C^I(Z_{47}), +, \times\}$ be the natural neutrosophic finite complex modulo integer semiring.

    i.   Study questions i to x of problem 12 for this M.
    ii. Obtain any other special or distinct feature enjoyed by M.

14. Let $W = \{C^I(Z_{243}), +, \times\}$ be the natural neutrosophic finite complex modulo integer semiring.

    Study questions i to x of problem 12 for this W.

15. Let $S = \{C([0, 20))_I\}$ be the MOD neutrosophic finite complex modulo integer set.

    i.   Study all the special features enjoyed by S.
    ii. If 20 is replaced by 29 study the special features and compare them.

16. Let $B = \{C^I([0, 24)), \times\}$ be the MOD neutrosophic finite complex modulo integer semigroup.

    i.   Prove o(B) = ∞.
    ii. Can ideals of B be of finite order?

iii. Is B a S-semigroup?
iv. Find all MOD neutrosophic zero divisors.
   Is it a finite collection or an infinite collection?
v. Find all MOD neutrosophic idempotents. (Is it a finite collection?)
vi. Can B have S-MOD neutrosophic zero divisors?
vii. Can B have S MOD neutrosophic idempotents?
viii. Obtain any other special feature enjoyed by B.

17. Let $M = \{C^I([0, 19)), \times\}$ be the MOD neutrosophic finite complex modulo integer semigroup.

    Study questions i to viii of problem 16 for this M.

18. Let $W = \{C^I([0, 64)), \times\}$ be the MOD neutrosophic finite complex modulo integer semigroup.

    Study questions i to viii of problem 16 for this W.

19. Let $V = \{C^I([0, 64)), +\}$ be the MOD neutrosophic finite complex modulo integer semigroup.

    Study questions i to viii of problem 16 for this V.

20. Let $S = \{\langle C([0, 45)), +\rangle_I\}$ be the MOD neutrosophic finite complex modulo integer additive semigroup.

    i.   Find all MOD neutrosophic idempotents of S.
    ii. Can S have ideals?
    iii. Prove S is always a S-semigroup.
    iv. Can S have any other special feature enjoyed by it?
    v.   Prove S has also finite order subsemigroups.

21. Let $W = \{\langle C([0, 37)), +\rangle_I\}$ be the MOD neutrosophic finite complex modulo integer semigroup.

    Study question from i to v of problem 20 for this W.

22. Let A = {⟨C([0, 128)), +⟩_I} be the MOD neutrosophic finite complex modulo integer semigroup.

    Study question from i to v of problem 20 for this A.

23. Let V = {⟨C([0, 48)), +⟩_I, ×} be the MOD neutrosophic finite complex modulo integer semigroup under ×.

    i.   Show o(V) = ∞.

    ii. Show V has infinite number of MOD neutrosophic zero divisors.
    iii. Show V has infinite order ideals.
    iv. Can V have finite order ideals?
    v.   Can V have S-subsemigroups?
    vi. Can V have S MOD neutrosophic idempotents?
    vii. Can V have S-MOD neutrosophic zero divisors?
    viii.Can S have nilpotents of order two?
    ix. Obtain any other special feature enjoyed by S.

24. Let W = {⟨C([0, 23), +⟩_I, ×} be the MOD neutrosophic finite complex modulo integer semigroup.

    Study questions i to ix of problem 23 for this W.

25. Let V = {⟨C([0, 625), +⟩_I, ×} be the MOD neutrosophic finite complex modulo integer semigroup.

    Study questions i to ix of problem 23 for this V.

26. Let D = {C^I([0, 28), +, ×} be the MOD neutrosophic finite complex modulo integer pseudo semiring.

    i.   Find o(D).
    ii. Can D be a S-semiring?
    iii. Is D a SS-semiring?
    iv. Can D be a SSS-semiring?
    v.   Can D have MOD neutrosophic idempotents?
    vi. Can D have MOD neutrosophic zero divisors?

vii. Can D have S-MOD neutrosophic zero divisors?

viii.Can D have pseudo S-ideals?

ix. Can S have pseudo ideals of finite order?

x. Obtain any other special feature enjoyed by S.

27. Let $V = \{C^I([0, 37)), +, \times\}$ be the MOD neutrosophic finite complex modulo integer pseudo semiring.

Study questions i to x of problem 26 for this V.

28. Let $M = \{C^I([0, 256)), +, \times\}$ be the MOD neutrosophic finite complex modulo integer pseudo semiring.

Study questions i to x of problem 26 for this M.

29. Let $S = \{\langle Z_8 \cup I\rangle_I\}$ be the natural neutrosophic modulo integer set.

i. Study all properties enjoyed by S.

ii. Find o(S).

30. Let $Z = \{\langle Z_{23} \cup I\rangle_I\}$ be the natural neutrosophic finite neutrosophic modulo integer set.

Study questions i and ii of problem 29 for this Z.

31. Let $Y = \{\langle Z_{48} \cup I\rangle_I\}$ be the natural neutrosophic finite neutrosophic modulo integer set.

Study questions i and ii of problem 29 for this Y.

32. Let $L = \{\langle Z_{256} \cup I\rangle_I\}$ be the natural neutrosophic finite neutrosophic modulo integer set.

Study questions i and ii of problem 29 for this L.

33. Let $M = \{\langle Z_{47} \cup I\rangle_I, \times\}$ be the natural neutrosophic finite natural neutrosophic modulo integer semigroup.

i. Find o(M).

    ii. Can M have S-ideals?

    iii. Is M a S-semigroup?

    iv. Can M have natural neutrosophic zero divisors?

    v. Can M have natural neutrosophic S-idempotents?

    vi. Find any other special feature enjoyed by M.

    vii. Can M have ideals which are not S-ideals?

    viii.Obtain any other special feature enjoyed by M.

34. Let $N = \{\langle Z_{24} \cup I\rangle_I, \times\}$ be the natural neutrosophic modulo integer semigroup.

    Study questions i and viii of problem 33 for this N.

35. Let $P = \{\langle Z_{256} \cup I\rangle_I, \times\}$ be the natural neutrosophic modulo integer neutrosophic semigroup.

    Study questions i and viii of problem 33 for this P.

36. Let $L = \{\langle\langle Z_{25} \cup I\rangle, +\rangle_I\}$ be the natural neutrosophic modulo neutrosophic integer semigroup under +.

    i. Prove L is a S-semigroup.

    ii. Show L is not a group as it contains idempotents under +.

    iii. $o(L) < \infty$ prove.

    iv. Can L have ideals?

    v. Can L have finite order subsemigroups?

37. Let $Q = \{\langle\langle Z_{28} \cup I\rangle +\rangle_I\}$ be the natural neutrosophic finite neutrosophic modulo integer semigroup.

    Study questions i and v of problem 36 for this Q.

38. Study questions of problem 36 for Z_{25} replaced by Z_{53}.

39. Let $W = \{Q, \times\}$ (Q as in problem 37) be the natural neutrosophic finite natural modulo integer neutrosophic semigroup under $\times$.

    i.   Find o(W).
    ii. Is W a S-semigroup?
    iii. Can W have S-ideals?
    iv. Can W have S-idempotents?
    v.   Find all natural neutrosophic elements of W.
    vi. Find subsemigroups which are not ideals.
    vii. Can W have natural neutrosophic zero divisors?
    viii.Can W have S-natural neutrosophic zero divisors?

40. $M = \{\langle\langle\langle Z_{45} \cup I\rangle_I, +\rangle, \times\}$ be the natural neutrosophic finite neutrosophic modulo integer semigroup under $\times$.

    Study questions i to viii of problem 39 for this M.

41. Study question i to viii of problem 39 by replacing Z_{45} by Z_{13} in problem 40.

42. Let $M = \{\langle Z_{40} \cup I\rangle_I, +, \times\}$ be the natural neutrosophic finite modulo integer neutrosophic semiring.

    i.   Prove o(M) is finite.
    ii. Prove characteristics of M is 40.
    iii. Show M is never a semifield.
    iv. Prove M has additive idempotents.
    v.   Prove M has idempotents under product.
    vi. Prove M has zero divisors.
    vii. Prove M have S-zero divisors.
    viii.Can M have ideals?
    ix. Can M have S-ideals?

43. Let $N = \{\langle Z_{217} \cup I\rangle_I, +, \times\}$ be the natural neutrosophic finite neutrosophic modulo integer semiring.

    Study questions i and ix of problem 42 for this N.

44. Let $T = \{\langle Z_{14} \cup I\rangle_I, +, \times\}$ be the natural neutrosophic finite modulo integer semiring.

    Study questions i and ix of problem 42 for this T.

45. Let M = {⟨([0, n) ∪ I⟩ᵢ} be the MOD neutrosophic interval modulo integer set.

    i.   Study all properties associated with M.
    ii. Study if n is replaced by 4, 5, 81 and 48.

46. Let B = {⟨[0, 20) ∪ I⟩ᵢ, ×} be the MOD neutrosophic interval semigroup.

    i.   Prove B is of infinite order.
    ii.   Can B be a S-semigroup?
    iii. Can B have MOD neutrosophic zero divisors?
    iv.   Can B have MOD neutrosophic idempotents?
    v.   Can ideals of B be of finite order?
    vi.   Can B have S-MOD neutrosophic zero divisors?
    vii. Can B have S-MOD neutrosophic idempotents?

47. In problem (46) in B, [0, 20) is replaced by [0, 43), study questions i to vii of problem 46 for that B.

48. Let M = {⟨[0, 49) ∪ I⟩ᵢ, ×} be the MOD neutrosophic interval semigroup.

    Study questions i to vii of problem 46 for this M.

49. Let L = {⟨⟨[0, n) ∪ I⟩ᵢ, +⟩} be the MOD neutrosophic interval semigroup under +.

    i.   Prove L is always a S-semigroup.
    ii.   Prove L has idempotents.
    iii. Obtain all special features enjoyed by L.
    iv.   Can L have ideals?
    v.   Can L have subsemigroups of infinite order which are not ideals?
    vi.   Obtain all special features enjoyed by L.

50. Let S = {⟨[0, 9) ∪ I⟩ᵢ, +, ×} be the MOD neutrosophic interval pseudo semiring.

i.   Prove o(S) = ∞.
ii.   Can S have ideals of finite order?
iii. Can S have S-idempotents and S-MOD neutrosophic idempotents?
iv.   Can S have S-ideals?
v.   Can S have S-zero divisors and S-MOD neutrosophic zero divisors?
vi.   Prove S is a SS-semiring.
vii. When will S be a SSS-semiring?
viii. Obtain any other special features enjoyed by S.

51. Let $V = \{\langle [0, 48) \cup I \rangle_I, +, \times\}$ be the MOD neutrosophic interval pseudo semiring.

    Study questions i to viii of problem 50 for this V.

52. Let $W = \{\langle [0, 143) \cup I \rangle_I, +, \times\}$ be the MOD neutrosophic interval pseudo semiring.

    Study questions i to viii of problem 50 for this W.

53. Let $M = \{\langle [0, 144) \cup I \rangle_I, +, \times\}$ be the MOD neutrosophic interval pseudo semiring.

    Study questions i to viii of problem 50 for this M.

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.
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The authors in this book
introduce a new class
of natural neutrosophic
numbers using MOD
intervals.

These natural MOD
neutrosophic numbers
behave in a different
way for the product
of two natural
neutrosophic numbers
can be neutrosophic
zero divisors or
idempotents or nilpotents.
Several open problems
are suggested in this book.