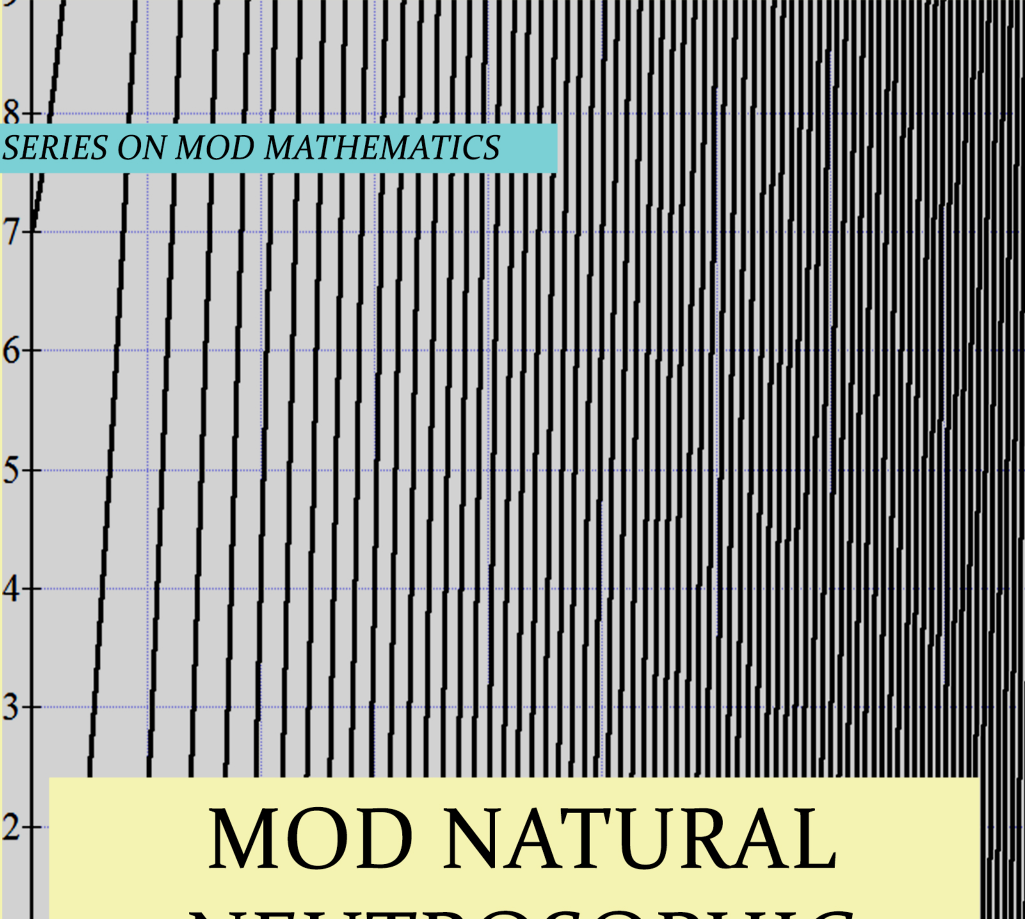


*SERIES ON MOD MATHEMATICS*

# MOD NATURAL NEUTROSOPHIC SUBSET SEMIGROUPS

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## PREFACE

In this book authors for the first time introduce the notion of MOD subsets using $Z_n$ (or $C(Z_n)$ or $\langle Z_n \cup g\rangle$ or $\langle Z_n \cup I\rangle$ or $\langle Z_n \cup h\rangle$ or $\langle Z_n \cup k\rangle$ or $Z_n^I$ or $C^I(Z_n)$ or $\langle Z_n \cup g\rangle_I$ or $\langle Z_n \cup I\rangle_I$ or $\langle Z_n \cup h\rangle_I$ or $\langle Z_n \cup k\rangle_I$. On these MOD subsets the operation '+' is defined, $S(Z_n)$ denotes the MOD subset and $\{S(Z_n), +\}$ happens to be only a Smarandache semigroup.

These S-semigroups enjoy several interesting properties. The notion of MOD universal subset and MOD absorbing subsets are defined and developed.

$\{S(Z_n), \times\}$ is also a semigroup and several properties associated with them are derived.

MOD natural neutrosophic subsets forms only a semigroup under '+'. In fact the main feature enjoyed by this structure is they have subset idempotents with respect to '+'. They are S-semigroups under '+'. These MOD natural neutrosophic subsets of all 6 types are only semigroups under '×'.

These enjoy distinct properties depending on the subset that is used. Further in these cases two types of products can be used one zero dominated product and the other MOD natural neutrosophic zero dominated product. Both the product are different. Finally using these MOD subsets matrices and polynomials are defined and developed. The new notion of MOD subset matrices are defined and these collections also under '+' (or '×') are only semigroups.

On similar lines MOD subset polynomials are defined and they also form only a semigroup under '+' (or '×').

These new notions are interesting and researchers can find lots of applications where semigroups find their applications.

Several problems at research level are also suggested for the readers.

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## Chapter One

# BASIC CONCEPTS

In this chapter we introduce the basic references which is essential for this study. Here we study the MOD subsets semigroups of various types using the six types of MOD integers as well as MOD natural neutrosophic numbers [66]. Further for subset structures refer [51]. We used dual numbers [48], special quasi dual numbers [50] and special dual like numbers [49].

For the notion of finite complex numbers refer [47]. For neutrosophic concept and structures on them refer [6-7].

The study here is significant for we are able to provide finite order semigroups under +, the addition operation, in most of the cases they are monoids and they are not groups but however all of them under + operation are Smarandache semigroups. These semigroups are of finite order.

On the other hand a study of MOD subset matrices and MOD matrix subsets are introduced. They also are only finite semigroups under '+' which are Smarandache semigroups.

Finally we introduce the notion of MOD subset polynomials subsets. Both of them under + are only semigroups. However if the polynomials are of the form $\sum_{i=0}^{\infty} a_i x^i$ then they are of infinite order.

Thus these provide examples of monoid of infinite order which are not groups. But if we use polynomials of type $\sum_{i=0}^{m} a_i x^i$, $2 \leq m < \infty$ then we get a class of MOD polynomial subsets as well as MOD subset polynomials both of which are of finite order. Still these are also only semigroups under +.

We replace the + operation by × and study the structure of the semigroup. The MOD semigroups under × have several interesting properties.

The book is endowed with several illustrative examples for that alone can give more clarity to the reader about these abstract concepts.

Finally we study the same structure in case of MOD natural neutrosophic subsets. These behave in a distinct way for we are forced to define two types of product; one the usual zero dominant product in which $0 \times I_m^t = 0$ for all $t \in \{n, c, g, h, k, I\}$ and m is a zero divisor or nilpotent or an idempotent in $Z_n$ or $C(Z_n)$ or $\langle Z_n \cup g\rangle$ or $\langle Z_n \cup h\rangle$ or $\langle Z_n \cup k\rangle$ or $\langle Z_n \cup I\rangle$ [60].

The MOD natural neutrosophic number dominant zero product is $I_0^m \times 0 = I_0^m$ where $m \in \{n, c, g, h, k, I\}$.

**Chapter Two**

# MOD SUBSET ALGEBRAIC STRUCTURES

In this chapter for the first time authors study the probable algebraic structures on $Z_n$, $C(Z_n)$, $\langle Z_n \cup I\rangle$, $\langle Z_n \cup g\rangle$, $\langle Z_n \cup h\rangle$ and $\langle Z_n \cup k\rangle$, $2 \leq n < \infty$.

Although in the 19th century set theory was introduced to make a paradigm shift in mathematics modern algebra and set topological spaces was introduced.

But unfortunately set theoretic notions was not exploited in algebraic structures they were used only in topological spaces so it is deem fit to study algebraic structures built using $Z_n$, $C(Z_n)$ and so on.

However we have defined and developed subset algebraic structures in [51].

Here for the first time we introduced algebraic structures not only on $Z_n$, $C(Z_n)$ and so on but using $Z_n^I$, $C^I(Z_n)$, $\langle Z_n \cup I\rangle_I$ and so on.

The latter theory gives many types of properties associated with them different from the existing ones.

Before we go for systematic definition first we will illustrate by some examples.

***Example 2.1:*** Let $Z_3$ be the modulo integers.

$S(Z_3)$ = {All subsets of $Z_3$ including $Z_3$} = {{0}, {1}, {2}, {1, 2}, {1, 0}, {2, 0}, {1, 2, 0}}. Clearly $o(S(Z_3)) = 7$.

***Example 2.2:*** Let $S(Z_9)$ = {{0}, {1}, …, {8}, {0, 1}, {0, 2}, …, {7,8}, {0, 1, 2}, {0, 1, 3},…, {6, 7, 8}, {0, 1, 2, 3}, …, {5, 6, 7, 8}, {0, 1, 2, 3, 4}, …, {4, 6, 7, 5, 8}, {0, 1, 2, 3, 4, 5}, …, {3, 4, 5, 6, 7, 8}, {0, 1, 2, 3, 4, 5, 6}, …, {2, 4, 5, 3, 6, 7, 8} and so on {0, 1, 2, 3, 4, 5, 6, 7, 8}} be the MOD subset collection and $o(S(Z_9)) = 2^9 - 1$.

In view of all these we prove the following theorem.

**THEOREM 2.1:** *Let $Z_n$ be the modulo integers. $S(Z_n)$ be the collection of all subsets of $Z_n$.*

$o(S(Z_n)) = 2^n - 1$.

Proof is direct and hence left as an exercise to the reader.

Throughout this book $S(Z_n)$ will denote the collection of all subsets of $Z_n$.

We will first define the operation of addition, + on $S(Z_n)$.

**DEFINITION 2.1:** *Let $Z_n$ be the ring of modulo integers $S(Z_n)$ be the collection of all subsets of $Z_n$. $S(Z_n)$ is defined as the MOD subset. Let $A, B \in S(Z_n)$, $A + B = \{a_i + b_j \ (mod\ n)$ for each $a_i \in A$ and each $b_j \in B\} \in S(Z_n)$.*

*This is the way '+' operation is defined on $S(Z_n)$.*

We will describe with examples before we proceed onto determine the properties enjoyed by this MOD subset semigroup, $\{S(Z_n), +\}$.

***Example 2.3:*** Let $S(Z_4) = \{\{0\}, \{1\}, \{2\}, \{3\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\} = Z_4\}$ be the MOD subsets of $Z_4$. Define + on $S(Z_4)$ as follows.

$\{0\} + \{1, 2, 3\} = \{1, 2, 3\}$,
$\{0\} + \{0, 3\} = \{0, 3\}$and so on.

It is left for the reader to prove $\{0\}$ acts as the additive identity of $S(Z_4)$.

It is important to note for any $X \in S(Z_4)$ we do not in general have a $Y \in S(Z_4)$ such that

$X + Y = \{0\}$.

For if $X = \{0, 2, 3\}$, $Y = \{2, 3, 1\} \in S(Z_4)$.

$$\begin{aligned} X + Y &= \{0, 2, 3\} + \{2, 3, 1\} \\ &= \{0 + 2, 0 + 3, 0 + 1, \ + 2, 2 + 3, \ 2 + 1, 3 + 2, \\ &\quad 3 + 3, 3 + 1\} \\ &= \{2, 3, 1, 0, 1, 3, 1, 2, 0\} = \{0, 1, 2, 3\} = Z_4. \end{aligned}$$

This is the way '+' operation is performed on $S(Z_4)$.

We see $\{2\} + \{0, 1, 3\} = \{2, 3, 1\} \in S(Z_4)$.

$\{2, 0\} + \{0, 1, 3\} = \{2, 3, 1, 0\} = Z_4$.

We see $Z_4 = \{2, 3, 1, 0\} \in S(Z_4)$, is such that for every $x \in S(Z_4)$, $x + Z_4 = Z_4$.

We call this $Z_4$ as the absorbing element of $S(Z_4)$.

$\{0, 2\} + \{1, 3\} = \{1, 3, 3, 1\} = \{1, 3\} \in S(Z_4)$.

We have also sets in $S(Z_4)$ such that X + Y = Y for Y in $S(Z_4)$ and for a particular X in $S(Z_4)$.

This is not possible in usual cases that in semigroups or groups or rings under +.

This situation does not follow the set theoretic results.

For {0, 2} and {1, 3} have no common factor infact a pair of disjoint sets but their sum gives the set {1, 3}. This sort of behavior in algebraic operations is very rare.

$\{0, 1\} + \{2, 3, 0\} = \{2, 3, 0, 1\} = Z_4$.

Now consider $\{0, 1, 2\} + \{3, 0, 1\} = \{3, 0, 1, 2\} = Z_4$.

However the sets {0 3 1} and {0, 1, 2} as sets are not disjoint.

***Example 2.4:*** Let $S(Z_6)$ = {(collection of all subsets of $Z_6$}. Define '+' operation on $S(Z_6)$.

Let $\{3, 2\} \in S(Z_6)$.

$\{3, 2\} + \{3, 2\} = \{0, 5, 4\}$,
$\{3, 2\} + \{0, 5, 4\} = \{3, 2, 0, 1\}$,
$\{3, 2\} + \{3, 2, 0, 1\} = \{0, 5, 2, 4, 3\}$,
$\{0, 2, 5, 4, 3\} + \{3, 2\} = \{3, 2, 1, 4, 0, 5\}$,
$\{1, 2, 3, 4, 0, 5\} + \{3, 2\} = \{4, 5, 0, 3, 2, 1\}$,
$\{1, 2, 3, 4, 5, 0\} + \{3, 2\} = \{4, 5, 0, 1, 2, 3\} = \{Z_6\}$.

After adding sets a number of times we arrive at a fixed set; such study is new and innovative.

$\{0, 3\} + \{0, 3\} = \{0, 3\}$.
$\{0, 2\} + \{0, 2\} = \{0, 2, 4\}$,
$\{0, 2, 4\} + \{0, 2\} = \{0, 2, 4\}$.

Thus these sets attains a fixed point.

That is $X + X = Y$
$Y + X = Y$.

The following operation are pertinent.

If $X = \{a_1, a_2, a_3, \ldots, a_t\} \in S(Z_n)$ is a subgroup under + of $Z_n$, then we see $X + X = X$.

$\{0, 3\} + \{0, 3\} = \{0, 3\}$,
$\{0, 2, 4\} + \{0, 4, 2\} = \{0, 4, 2\}$ and
$\{0, 1, 2, 3, 4, 5\} + \{0, 1, 2, 3, 4, 5\} = \{0, 1, 2, 3, 4, 5\}$.

Thus $\{S(Z_n), +\}$ has elements X such that $X + X = X$ so $\{S(Z_n), +\}$ is not a group under +, only a semigroup under +.

Let $X = \{5\} \in S(Z_6)$ ,
$\{5\} + \{5\} = \{4\}$,
$\{4\} + \{5\} = \{3\}$,
$\{3\} + \{5\} = \{2\}$,
$\{2\} + \{5\} = \{1\}$,
$\{5\} + \{1\} = \{0\}$.

Let $\{3, 5, 0\} = Y \in S(Z_n)$

$Y + Y = \{0, 3, 5\} + \{0, 3, 5\} = \{0, 3, 5, 2, 4\}$
$Y + \{0, 2, 3, 4, 5\} = \{0, 2, 3, 4, 5, 1\}$
$\{0, 3, 5\} + \{0, 1, 2, 3, 4, 5\} = \{0, 1, 2, 3, 4, 5\}$.

Thus we see $Z_6 = \{0, 1, 2, 3, 4, 5\}$ added with any set $X \in S(Z_6)$ gives only $Z_6$.

$X = \{0, 2, 4\}$ and $Y = \{1, 5\} \in S(Z_6)$;

$X + Y = \{0, 2, 4\} + \{1, 5\} = \{1, 5, 3\} = Z_1$.
$X + X = \{0, 2, 4\} + \{0, 2, 4\} = \{0, 2, 4\} = X$.
$Y + Y = \{1, 5\} + \{1, 5\} = \{2, 4, 0\} = V_1$.

We see X + X = Y + Y but X ≠ Y.

Further $Z_1 + Z_1$ = {1, 5, 3} + {1, 5, 3} = {2, 0, 4} = X

X + Y + X + Y = X. (X + Y ≠ X)

Y + Y = X

(X + Y) + (X + Y) = X.

X + Y + Y = {1, 5, 3} + {1, 5} = {2 0 4}

(X + Y) + X = {1, 5, 3} + {0, 2, 4} = X + Y.

Several interesting features can be analysed using the subsets.

We see the structure of $\{S(Z_6),+\}$ is only a semigroup of order $2^6 - 1$.

However $\{0\} \in S(Z_6)$ serves as the identity for any $X \in S(Z_6)$; X + {0} = X = {0} + X.

Infact $\{S(Z_6), +\}$ is a commutative monoid of finite order.

For any $X \in S(Z_6)$ we cannot in general find a Y in $S(Z_6)$ such that X + Y = {0}.

For take $\{2, 4\} \in S(Z_6)$ we do not have a $Y \in S(Z_6)$ such that Y + {2, 4} = {0}.

Thus $\{S(Z_6), +\}$ is only a monoid and not a group.

However $\{S(Z_6), +\}$ is a Smarandache MOD semigroup as $\{Z_6, +\}$ is a group under +.

In view of all these we have the following theorem.

**THEOREM 2.2:** *Let $S(Z_n)$ = {(collection of all subsets of $Z_n$)}; $2 \leq n < \infty$.*

*i) $o(S(Z_n)) = 2^n - 1$.*
*ii) $\{S(Z_n), +\}$ is only a commutative monoid.*

*iii) The set $Z_n = \{0, 1, 2, \ldots, n-1\} \in S(Z_n)$ is such that for every $X \in S(Z_n)$;*

$$X + Z_n = Z_n + X = Z_n.$$

Proof is direct and hence left as an exercise to the reader.

**DEFINITION 2.2:** *Let $\{S(Z_n), +\}$ be the MOD subsets semigroup of $Z_n$. The set $\{0, 1, 2, \ldots, n-1\} = Z_n \in S(Z_n)$ is the defined as the MOD universal subset of $S(Z_n)$.*

*We see $S(Z_n)$ has one and only one MOD universal subset. The MOD universal subset of $S(Z_n)$ will also be known as the MOD largest absorbing subset of $S(Z_n)$.*

But there are many absorbing subsets of $S(Z_n)$.

We will give some more examples before we proceed to define and develop more properties about these new MOD subset structures.

***Example 2.5:*** Let $S(Z_{12})$ = {Collection of all subsets of $Z_{12}$}; the MOD subset of $Z_{12}$, $\{S(Z_{12}), +\}$ is the MOD subset semigroup.

Let $X = \{0, 3\}$ and $Y = \{0, 6, 9, 3\} \in S(Z_{12})$
$X + Y = \{0, 3\} + \{0, 6, 9, 3\} = \{0, 3, 6, 9\} = Y$.

Let $X_1 = \{0, 2, 4\}$ and $Y = \{0, 6, 9, 3\}$ we see

$$X_1 + Y = \{0, 2, 4\} + \{0, 6, 9, 3\} = \{0, 2, 4, 6, 9, 3, 8, 11, 5, 10, 1, 7\} = Z_{12}$$

the MOD universal subset of $S(Z_{12})$ or the largest MOD subset of $S(Z_{12})$.

Let $X_2 = \{0, 8\}$ and $Y = \{0, 6, 9, 3\} \in S(Z_{12})$.
$X_2 + Y = \{0, 6, 3, 9, 8, 11, 5, 2\}$.

Clearly $X_2 + Y$ is not the MOD universal subset of $S(Z_{12})$.

Let $X_3 = \{0, 2, 8\}$ and $Y = \{0, 6, 3, 9\} \in S(Z_{12})$;

$$X_3 + Y = \{0, 2, 8\} + \{0, 6, 3, 9\}$$
$$= \{0, 6, 3, 9, 8, 2, 5, 11\} \neq Z_{12}.$$

So only $X_1 = \{0, 2, 4\}$ with Y gives the MOD largest absorbing subset of $S(Z_{12})$.

Let $X_4 = \{0, 4, 8\} \in S(Z_{12})$,
$$X_4 + Y = \{0, 4, 8\} + \{0, 6, 3, 9\}$$
$$= \{0, 6, 3, 9, 10, 7, 1, 2, 11, 4, 8, 5\} = Z_{12}.$$

Once again $X_4 = \{0, 4, 8\}$ with Y gives the MOD largest absorbing subset of $S(Z_{12})$.

Let $X_5 = \{0, 5, 7\} \in S(Z_{12})$ now we find
$$X_5 + Y = \{0, 5, 7\} + \{0, 3, 6, 9\}$$
$$= \{0, 3, 6, 9, 8, 11, 2, 5, 7, 10, 1, 4\} = Z_{12}.$$

We see $X_5 = \{0, 5, 7\}$ also yields with Y the MOD largest absorbing subset of $S(Z_{12})$.

Consider $\{0, 5, 11\} = X_6 \in S(Z_{12})$.

We find $X_6 + Y = \{0, 5, 11\} + \{0, 3, 6, 9\}$
$$= \{0, 3, 6, 9, 8, 5, 11, 2, 1\} \neq Z_{12}.$$

So $X_6$ cannot yield $Z_{12}$ when added with Y.

Let $X_7 = \{0, 7, 11\} \in S(Z_{12})$.

We find $X_7 + Y = \{0, 7, 11\} + \{0, 3, 6, 9\}$
$$= \{0, 3, 6, 9, 7, 10, 1, 4, 11, 2, 5, 8\} = Z_{12}.$$

However $X_7$ yields $Z_{12}$ the MOD largest absorbing set with Y.

Let $Z = \{0, 2, 4, 6, 8, 10\}$ and $B_1 = \{0, 1\} \in S(Z_{12})$.
$$B_1 + Z = \{0, 1\} + \{0, 2, 4, 6, 8, 10\}$$

$= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} = Z_{12}$.

So Z with $B_1$ or $B_1$ with Z give the MOD universal subset of $S(Z_{12})$.

Let $B_2 = \{0, 3\} \in S(Z_{12})$.

$B_2 + Z = \{0, 3\} + \{0, 2, 4, 6, 8, 10\}$
$= \{0, 2, 4, 6, 8, 3, 10, 5, 7, 9, 11, 1\} = Z_{12}$.

Again $B_2$ with Z yield the MOD universal subset of $S(Z_{12})$.

Let $B_3 = \{1, 3\} \in S(Z_{12})$
$B_3 + Z = \{1, 3\} + \{0, 2, 4, 6, 8, 10\} = \{1, 3, 5, 7, 9, 11\}$.

So $B_3$ cannot yield with Z the MOD largest absorbing subset or the MOD universal subset of $S(Z_{12})$.

Let $B_4 = \{0, 5\} \in S(Z_{12})$.

$B_4 + Z = \{0, 5\} + \{0, 2, 4, 6, 8, 10\}$
$= \{0, 2, 4, 6, 8, 5, 10, 7, 9, 11, 1, 3\} = Z_{12}$.

So with Z yields the MOD universal subset of $S(Z_{12})$.

Let $B_5 = \{0, 7\} \in S(Z_{12})$.

$B_5 + Z = \{0, 7\} + \{0, 2, 4, 6, 8, 10\}$
$= \{0, 2, 4, 6, 8, 10, 9, 7, 11, 1, 3, 5\} = Z_{12}$.

$B_5$ also yields with Z the MOD universal subset of $S(Z_{12})$.

Let $B_6 = \{3, 5\} \in S(Z_{12})$,
$B_6 + Z = \{3, 5\} + \{0, 2, 4, 6, 8, 10\}$
$= \{3, 5, 7, 9, 11, 1\} \neq Z_{12}$.

So $B_6$ cannot yield the MOD universal subset of $S(Z_{12})$ with Z.

Let $B_7 = \{0, 9\} \in S(Z_{12})$.
$B_7 + Z = \{0, 9\} + \{0, 2, 4, 6, 8, 10\}$
$= \{0, 2, 4, 6, 8, 9, 10, 11, 1, 3, 5, 7\} = Z_{12}$.

So $B_7$ with Z yields the MOD universal subset of $S(Z_{12})$.

Let $B_8 = \{0, 1\} \in S(Z_{12})$;
$B_8 + Z = \{0, 11\} + \{0, 2, 4, 6, 8, 10\}$
$= \{0, 2, 4, 6, 8, 10, 1, 3, 11, 5, 7, 9\} = Z_{11}$.

Thus $B_8$ with Z yields the MOD universal subset of $S(Z_{12})$.
$B_9 = \{1, 11\} \in S(Z_{12})$;
$B_9 + Z = \{1, 11\} + \{0, 2, 4, 6, 8, 10\}$
$= \{1, 3, 5, 7, 9, 11\}$.

Thus some sets yield the MOD universal subset.
Let $W = \{0, 6\} \in S(Z_{12})$ be the subset of $S(Z_{12})$.

Let $X = \{0, 5, 2, 7\} \in S(Z_{12})$
$W + X = \{0, 6\} + \{0, 2, 5, 7\}$
$= \{0, 2, 5, 7, 8, 6, 11, 1\} \neq Z_{12}$.

Thus for us to have the resultant sum to be the MOD universal subset we need atleast the product of the cardinality to be 12 and '0' should be the common element between the two sets.

***Example 2.6:*** Let $S(Z_7) = \{$(collection of all subsets of $Z_7\}$ be the MOD subset semigroup under +.

Let $X = \{0, 2\}$ and $Y = \{0, 1, 4\} \in S(Z_7)$,
$X + Y = \{0, 2, 1, 4, 3, 6\} \neq Z_7$.

Let $X = \{0, 2\}$ but $Y = \{0, 1, 6\} \in S(Z_7)$,
$X + Y = \{0, 2\} + \{0, 1, 6\} = \{0, 1, 6, 2, 3\} \neq Z_7$.

Let $X = \{0, 1, 3\}$ and $Y = \{0, 4, 5\} \in S(Z_7)$,
$X + Y = \{0, 1, 3, 4, 5, 6\} \neq Z_7$.

Let X = {0, 1, 2, 3} and Y = {0, 4, 5} ∈ $S(Z_7)$,
X + Y = {0, 1, 2, 3} + {0, 4, 5} = {0, 1, 2, 3, 4, 5, 6} = $Z_7$.
Let X = {0, 1, 2, 3} and Y = {5, 0, 6} ∈ $S(Z_7)$,
X + Y = {0, 1, 2, 3} + {5, 6, 0} = {0, 1, 2, 3, 5, 6} ≠ $Z_7$.

Let X = {0, 1, 4, 5} and Y = {0, 3} ∈ $S(Z_7)$,
X + Y = (0, 1, 4, 5} + {0, 3} = {0, 1, 4, 5, 3} ≠ $Z_7$.

So if in $Z_n$, n is a prime $S(Z_n)$ happens to behave in a very different way evident from this example.

***Example 2.7:*** Let $S(Z_4)$ = {collection of all subsets of $Z_4$} be the MOD subset semigroup under +.

Let S = {0, 2} and Y = {1, 3}
S + Y = {0, 2} + {1, 3} = {1, 3}
So S = {0, 2} leaves {1, 3} as it is even after addition.

Let X = {1, 2} and Y = {0, 3} ∈ $S(Z_4)$,
X + Y = {1, 2} + {0, 3} = {1, 2, 0} ≠ $Z_4$.

Let X = {0, 1, 2} and Y = {0, 3} ∈ $S(Z_4)$,
X + Y = {0, 1, 2} + {0, 3} = {0, 1, 2, 3} = $Z_4$.

Let X = {1, 3, 2} and Y = {0, 1} ∈ $S(Z_4)$,
X + Y = {0, 1} + {1, 3, 2} = {1, 3, 2, 0} = $Z_4$

However it is seen Y ⊆ X yet Y + X ≠ X but $Z_4$.

This is the specialty of MOD subset semigroups.

We see {0, 3} = Y and X = {1, 2, 3} ∈ $S(Z_4)$ is such that
X + Y = {0, 3} + {1, 2, 3} = {1, 2, 3, 0} = $Z_4$.

So we see if Y = {0, 3} is replaced by $Y_1$ = {0, 2} then
$Y_1$ + X = {0, 2} + {1, 2, 3} = {1, 2, 3, 0} = $Z_4$.

Let $Y_2$ = {0, 1} ∈ $S(Z_4)$.

We find $Y_2 + X = \{0, 1\} + \{1, 2, 3\} = \{0, 1, 2, 3\} = Z_4$ thus for $X = \{1, 2, 3\}$ all the three sets $\{0, 1\}$, $\{0, 2\}$ and $\{0, 3\}$ are such that their sum with X yield the MOD universal subset.

***Example 2.8:*** $S(Z_{10}) = \{$(collection of all subsets of $Z_{10}\}$ be the MOD subset semigroup under +.

Let $X = \{0, 2, 4, 6, 8\}$ and $Y = \{0, 5\} \in S(Z_{10})$.

$$X + Y = \{0, 2, 4, 6, 8\} + \{0, 5\} = \{0, 2, 4, 6, 8, 7, 9, 5, 1, 3\} = Z_{10}.$$

Thus X and Y sum lead to the MOD universal subset of $S(Z_{10})$.

Let $Y_1 = \{0, 3\}$ and $X = \{0, 2, 4, 6, 8\} \in S(Z_{10})$.

We find $Y_1 + X = \{0, 3\} + \{0, 2, 4, 6, 8\}$
$= \{0, 2, 4, 6, 8, 5, 7, 3, 9, 1\} = Z_{10}$.

Thus $Y_1$ with X also yields the MOD universal subset of $S(Z_{10})$.

Let $Y_2 = \{0, 1\} \in S(Z_{10})$;

$$Y_2 + X = \{0, 1\} + \{0, 2, 4, 6, 8\} = \{0, 2, 4, 6, 8, 3, 1, 5, 7, 9\} = Z_{10}.$$

Thus $Y_2$ with X also yields the MOD universal subset $Z_{10}$ of $S(Z_{10})$.

Let $Y_3 = \{0, 7\} \in S(Z_{10})$,

$$Y_3 + X = \{0, 7\} + \{0, 2, 4, 6, 8\} = \{0, 2, 4, 6, 8, 9, 7, 1, 3, 5\} = Z_{10}.$$

Thus $Y_3$ with X also yields the MOD universal subset of $S(Z_{10})$.

Let $Y_4 = \{0, 9\} \in S(Z_{10})$;

we find $Y_4 + X = \{0, 9\} + \{0, 2, 4, 6, 8\}$
$= \{0, 2, 4, 6, 8, 9, 1, 3, 5, 7\} = Z_{10}$.

So $Y_4$ also with X yields the MOD universal subset of $S(Z_{10})$.

These subset Y, $Y_1$, $Y_2$, $Y_3$ and $Y_4$ are the smallest order subsets of $S(Z_{10})$ which when added with X = {0, 2, 4, 6, 8} yield the MOD universal subset of $S(Z_{10})$.

***Example 2.9:*** Let $S(Z_{15})$ = {Collection of all subsets of $Z_{15}$} be the MOD subset semigroup under +.

Let X = {0, 3, 6, 9, 12} and Y = {0, 2} $\in S(Z_{15})$.

X + Y = {0, 3, 6, 9, 12} + {0, 2}
= {0, 2, 6, 9, 12, 2, 5, 8, 11, 14}.

Let Y = {0, 2, 8} $\in S(Z_{15})$.

We find X + Y = {0, 2, 8} + {0, 3, 6, 9, 12}
= {0, 3, 6, 9, 12, 2, 5, 8, 11, 14} $\neq Z_{15}$.

So Y with X does not yield the MOD universal subset of $S(Z_{15})$.

Let $Y_1$ = {0, 2, 4} $\in S(Z_{15})$.

X + $Y_1$ = {0, 2, 4} + {0, 3, 6, 9, 12}
= {0, 3, 6, 9, 12, 2, 5, 8, 11, 14, 7, 4, 10, 13, 1}
= $Z_{15}$.

We see $Y_1$ with X yields the MOD universal subset of $S(Z_{15})$.

Let $Y_2$ = {0, 8, 10} $\in S(Z_{12})$

$Y_2$ + X = {0, 8, 10} + {0, 3, 6, 9, 12} = {0, 3, 6, 9, 12, 8, 11, 14, 2, 5, 10, 13, 1, 4, 7} = $Z_{15}$, so $Y_2$ also yields when added with X the MOD universal subset of $S(Z_{15})$.

Let $Y_3$ = {0, 10, 14} $\in S(Z_{15})$.

We find

$Y_3 + X$ = {0, 10, 14} + {0, 3, 6, 9, 12}
= {0, 3, 6, 9, 12, 10, 13, 1, 4, 7, 14, 2, 5, 8, 11}
= $Z_{15}$.

Thus $Y_3$ also when added with X yields the MOD universal subset of $S(Z_{15})$.

Let $Y_4$ = {0, 8, 14} ∈ $S(Z_{15})$.

$Y_4 + X$ = {0, 8, 14} + {0, 3, 6, 9, 12}
= {0, 3, 6, 9, 12, 8, 11, 14, 2} ≠ $Z_{15}$.

So $Y_4$ cannot serve as the subset of $S(Z_{15})$ which can yield $Z_{15}$ adding with the subset X.

Let $Y_5$ = {0, 8, 12} ∈ $S(Z_{15})$.

$Y_{15} + X$= {0, 8, 12} + {0, 3, 6, 9, 12}
= {0, 3, 6, 9, 12, 8, 11, 14, 2, 5} ≠ $Z_{15}$.

Thus it is left as a open conjecture for $S(Z_n)$.

Now we propose the following open problem.

**Problem 2.1:** Let $S(Z_n)$ be the MOD subset semigroup under +; n a nonprime $2 \leq n < \infty$. Find the number of subsets X in $S(Z_n)$ which when added with the subset P of $S(Z_n)$ where the elements of P under + from a group and the sum of X + P as MOD universal subset of S(Zn). Study this question for all subsets of $Z_n$.

**DEFINITION 2.3:** *Let $S(Z_n)$ be the MOD subset semigroup.*

*Let P ∈ $S(Z_n)$ if the collection of elements in P forms a group under + then we define P to be a subgroup subset of $S(Z_n)$.*

We will illustrate these situations by some examples.

***Example 2.10:*** Let $S(Z_{24})$ be the MOD subset semigroup under +.

$P_1 = \{0, 12\}$,
$P_2 = \{0, 8, 16\}$,
$P_3 = \{0, 6, 12, 18\}$,
$P_4 = \{0, 3, 6, 9, 12, 15, 18, 21\}$,
$P_5 = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$ and
$P_6 = \{4, 8, 12, 16, 20, 0\}$ are the only subset subgroups of $S(Z_{24})$.

Finding subsets in $S(Z_{24})$ so that their sum with $P_1$ or $P_2$ or $P_3$ or $P_4$ or $P_5$ or $P_6$ yielding the subset $Z_{24}$ happens to be a matter of routine and hence left as an exercise to the reader.

We see in case of $P_3 = \{0, 6, 12, 18\}$ if we take

$X = \{0, 1, 3, 5, 7, 10\} \in S(Z_{24})$ and we calculate

$$\begin{aligned} P_3 + X &= \{0, 6, 12, 18\} + \{0, 1, 3, 5, 7, 10\} \\ &= \{0, 6, 12, 18, 1, 3, 5, 7, 11, 7, 13, 19, 9, 15, 14, \\ &\quad 21, 17, 23, 16, 10, 22, 4\} \neq Z_{24}. \end{aligned}$$

Take $B = \{0, 1, 5, 4, 20, 9\} \in S(Z_{24})$.

We find

$$\begin{aligned} P_3 + B &= \{0, 6, 12, 18\} + \{0, 1, 5, 4, 20, 9\} \\ &= \{0, 6, 12, 18, 1, 5, 9, 20, 7, 13, 19, 11, 17, 23, \\ &\quad 15, 21, 4, 3, 10, 16, 22, 2, 8, 14\} = Z_{24}. \end{aligned}$$

Thus for the MOD subgroup subset $P_3$, B in $S(Z_{24})$ serves as the subset when summed up with $P_3$ yields $Z_{24}$ the MOD universal subset of $S(Z_{24})$.

Now for the same $P_3$ consider $D = \{0, 3, 4, 5, 7, 8\}$,

$$P_3 + D = \{0, 6, 12, 18\} + \{0, 3, 4, 5, 7, 8\}$$

$= \{0, 6, 12, 18, 9, 15, 21, 3, 4, 10, 16, 22, 5, 11, 17, 23, 7, 13, 19, 1, 8, 14, 20, 2\}$.

Thus D is also another subset of $S(Z_{24})$ which yield the MOD universal subset of $S(Z_{24})$ when added with the MOD subset subgroup $P_3$ of $S(Z_{24})$.

Consider $P_2 = \{0, 8, 16\}$ of $S(Z_{24})$.

We see $X = \{0, 1, 2, 3, 4, 5, 6, 7\} \in S(Z_{24})$ is such that

$$P_2 + X = \{0, 8, 16\} + \{0, 1, 2, 3, 4, 5,6, 7\}$$
$$= \{0, 8, 16, 1, 9, 17, 2, 10, 18, 11, 19, 3, 4, 12, 20, 5, 13, 21, 6, 14, 22, 7, 15, 23\} = Z_{24},$$

the MOD universal subset of $S(Z_{24})$.

Let $P_3 = \{0, 8, 16\}$ and $Y = \{0, 9, 10, 11, 12, 13, 14, 15\} \in S(Z_{24})$ is such that

$$P_3 + Y = \{0, 8, 16\} + \{0, 9, 10, 11, 12, 13, 14, 15\}$$
$$= \{0, 8, 16, 9, 10, 11, 1, 13, 14, 15, 17, 1, 18, 2, 19, 3, 20, 4, 21, 5, 22, 6, 23, 7\} = Z_{24} \in S(Z_{24})$$

is such that $P_3 + Y$ yield the MOD universal subset of $S(Z_{24})$.

Clearly Y is not a MOD subgroup subset of $S(Z_{24})$; Y is only just a MOD subset of $S(Z_{24})$.

Consider $P_6 = \{0, 4, 8, 12, 16, 20\} \in S(Z_{24})$, we see $D = \{0, 1, 2, 3\} \in S(Z_{24})$ is such that

$$P_6 + D = \{0, 4, 8, 12, 16, 20\} + \{0, 1, 2, 3\}$$
$$= \{0, 4, 8, 12, 16, 20, 1, 2, 3, 5, 9, 13, 17, 21, 6, 10, 14, 18, 22, 7, 11, 15, 19, 23\} = Z_{24}.$$

Thus D when added with $P_6$ yields the MOD universal subset of $S(Z_{24})$.

Consider E = {0, 5, 6, 7} $\in$ S($Z_{24}$).

We find $P_6$ + E = {0, 4, 8, 12, 16, 20} + {0, 5, 6, 7}
= {0, 4, 8, 12, 16, 20, 5, 6, 7, 9, 13, 17, 21, 1, 10, 14, 18, 22, 2, 11, 15, 19, 23, 3}
= $Z_{24}$.

Thus we see for the MOD subset subgroup $P_6$ we have E $\in$ S($Z_{24}$) such that the sum of $P_6$ with E yields $Z_{24}$ the MOD universal subset of S($Z_{24}$).

Consider F = {9, 10, 11, 0} $\in$ S($Z_{24}$).

F + $P_6$ = {0, 9, 10, 11} + {0, 4, 8, 12, 16, 20}
= {0, 4, 8, 12, 16, 20, 9, 10, 11, 13, 17, 21, 1, 5, 14, 18, 22, 2, 6, 15, 19, 23, 3, 7} = $Z_{24}$.

Thus F added with $P_6$ yield the MOD universal subset of S($Z_{24}$).

Let G = {13, 14, 15, 0} $\in$ S($Z_{24}$) we find

$P_6$ + G = {0, 4, 8, 12, 16, 20} + {0, 13, 14, 15}
= {0, 4, 8, 16, 20, 12, 13, 14, 15, 17, 21, 1, 5, 9, 18, 22, 2, 6, 10, 19, 23, 3, 7, 11} = $Z_{24}$,

thus G is also a MOD subset of S($Z_{24}$) such that $P_6$ + G is the MOD universal subset of S($Z_{24}$).

Let H = {0, 17, 18, 19} $\in$ S($Z_{24}$) we find

$P_6$ + H = {0, 4, 8, 12, 16, 20} + {0, 17, 18, 19}
= {0, 4, 8, 12, 16, 20, 17, 18, 19, 21, 1, 5, 9, 13, 22, 2, 6, 10, 14, 23, 3, 7, 11, 15} = $Z_{24}$.

Hence H summed up with $P_6$ gives the MOD universal subset of S($Z_{24}$).

Let = {0, 21, 22, 23} ∈ $S(Z_{24})$, we find

$J + P_6$ = {0, 21, 22, 23} + {0, 4, 8, 12, 16, 20}
= {0, 4, 8, 12, 16, 20, 1, 5, 9, 13, 17, 2, 6, 10, 14, 18, 3, 7, 11, 15, 19, 23, 21, 22} = $Z_{24}$.

We see J summed up with $P_6$ yields the MOD universal subset $Z_{24}$ of $S(Z_{24})$.

Consider K = {0, 3, 9, 15} ∈ $S(Z_{24})$. We find

$P_6 + K$ = {0, 4, 8, 12, 16, 20} + {0, 3, 9, 15}
= {0, 4, 8, 12, 16, 20, 3, 9, 15, 7, 11, 15, 19, 23, 13, 17, 21, 1, 5} ≠ $Z_{24}$.

Take L = {0, 3, 9, 14} ∈ $S(Z_{24})$ we find

$L + P_6$ = {0, 3, 9, 14} + {0, 4, 8, 12, 16, 20}
= {0, 9, 3, 14, 4, 8, 12, 16, 20, 7, 11, 15, 19, 23, 13, 17, 21, 1, 5, 18, 22, 2, 6, 10} = $Z_{24}$.

Thus L also serves as a MOD subset which yields the MOD universal subset of $S(Z_{24})$.

Hence we have several such subsets in $S(Z_{24})$ when added with $P_6$ can yield $Z_{24}$.

What is the problem is finding means and methods to get all of them for a given MOD subgroup subset in $S(Z_n)$.

Thus we leave the following problem as open.

**Problem 2.2:** Let {$S(Z_n)$, +} be the MOD subset semigroup, $Z_n$ the MOD universal subset of $S(Z_n)$.

Let P ∈ $S(Z_n)$ be a MOD subgroup subset of $S(Z_n)$. Find all X ∈ $S(Z_n)$ such that $P + X = Z_n$.

Next we proceed onto define the notion of product on MOD subsets of $S(Z_n)$.

We will first illustrate this situation by some examples.

***Example 2.11:*** Let $S(Z_6)$ = {(collection of all subsets of $Z_6$} on $S(Z_6)$ we define product.

If X = {0, 3, 2} and Y = {1, 4} $\in S(Z_6)$ then
$X \times Y = \{0, 3, 2\} \times \{1, 4\} = \{0, 3, 2\} \in S(Z_6)$.

$X \times X = \{0, 3, 2\} \times \{0, 3, 2\} = \{0, 4, 3\}$.
$Y \times Y = \{1, 4\} \times \{1, 4\} = \{1, 4\} = Y$.

Let X = {3} and Y = {0, 2, 4} $\in S(Z_6)$ ;
$X \times Y = \{3\} \times \{0, 2, 4\} = \{0\}$

$X \times X = \{3\} \in \{3\} \times \{3\} = X$.
$Y \times Y = \{0, 2, 4\} \times \{0, 2, 4\} = \{0, 4, 2\} = Y$.

$X \times X = \{3\} \times \{3\} = \{3\} = X$.
$Y \times Y = \{0, 2, 4\} \in \{0, 2, 4\} = \{0, 4, 2\} = Y$.

Thus both X and Y are idempotents that is they are defined as MOD subset idempotents.

We call $\{S(Z_6), \times\}$ is the MOD subset semigroup.

Let B = {0, 2, 3} $\in S(Z_6)$,
$B \times B = \{0, 4, 3\} \neq B$.

Let C = {0, 4, 3} $\in S(Z_6)$,
$C \times C = \{0, 4, 3\} = C$
is a MOD subset idempotent of $S(Z_6)$.

Now {1} $\in S(Z_6)$} is such that $\{1\} \times A = A \times \{1\} = A$ for all A $\in S(Z_6)$. {1} is called as the MOD subset unit of $S(Z_6)$.

$\{0\} \in S(Z_6)$ is such that $\{0\} \times A = A \times \{0\} = \{0\}$ for all A $\in S(Z_6)$ so $\{0\}$ is defined as the MOD subset zero of $S(Z_6)$.

Clearly $S(Z_6)$ is only a MOD commutative semigroup of order $2^6 - 1$.

a = {2} and b = {3} $\in S(Z_6)$ is such that
$a \times b = \{2\} \times \{3\} = \{0\}$.

Let P = {0, 3, 2} and Q = {4, 5} $\in S(Z_6)$.

$P \times P = \{0, 3, 2\} \in \{0, 3, 2\} = \{0, 3, 4\} \neq P$.
$Q \times Q = \{4, 5\} \times \{4, 5\} = \{4, 2, 1\} \neq Q$.
$P \times Q = \{0, 3, 2\} \times \{4, 5\} = \{0, 2, 3, 4\}$.

Let T = {1, 5, 2, 3} $\in S(Z_6)$.
$T \times T = \{1, 5, 2, 3\} \times \{1, 5, 2, 3\} = \{1, 5, 2, 3, 4, 0\} = Z_6$.

Thus the MOD set T in $S(Z_6)$ is such that $T \times T$ is the MOD universal subset of $S(Z_6)$.

Let B = {1, 5} $\in S(Z_6)$.

$B \times B = \{1, 5\} \times \{1, 5\} = \{1, 5\}$ is a MOD idempotent subset of $S(Z_6)$.

Let M = {1, 5, 2} $\in S(Z_6)$,
$M \times M = \{1, 5, 2\} \in \{1, 5, 2\} = \{1, 5, 2, 4\} \neq M$.

Let Y = {1, 3, 2} and B = {1, 5} $\in S(Z_6)$
$B \times Y = \{1, 5\} \times \{1, 3, 2\} = \{1, 3, 2, 5, 3, 4\}$
$= \{1, 3, 5, 4, 2\} \neq Z_6$.

Consider C = {1, 3, 5} and Y = {1, 3, 2} $\in S(Z_6)$,

we find $C \times Y = \{1, 3, 5\} \times \{1, 3, 2\}$
$= \{1, 3, 5, 0, 4, 2\} = Z_6$.

Thus the MOD subsets C and Y in $S(Z_6)$ are such that their product yields the MOD universal subset.

Consider $C + Y = \{2, 4, 0, 3, 5, 1\} = Z_6$

Thus we see the pair of MOD subsets $\{C, Y\}$ of $S(Z_6)$ are that $C \times Y = Z_6$ and $C + Y = Z_6$.

Such MOD subsets are very unique and the study related with them will be carried out later.

***Example 2.12:*** Let $\{S(Z_{11}), \times\}$ be the MOD subset semigroup under $\times$.

Let $A = \{2, 0, 3\} \in S(Z_{11})$,
$A \times A = \{2, 0, 3\} \times \{2, 0, 3\} = \{0, 4, 6, 9\} = A^2$.

$A^2 \times A = \{0, 4, 6, 9\} \times \{0, 2, 3\} = \{0, 8, 1, 7, 5\} = A^3$.

$A^3 \times A = \{0, 1, 8, 5, 7\} \times \{0, 2, 3\} = \{0, 2, 5, 10, 3, 4\} = A^4$

$A^4 \times A = \{0, 2, 3, 5, 5, 10\} \times \{0, 2, 3\}$
$= \{0, 4, 6, 1, 10, 9, 8\} = A^5$.

$A^5 \times A = \{0, 4, 6, 1, 10, 9, 8\} \times \{0, 2, 3\}$
$= \{0, 8, 1, 2, 9, 7, 5, 3\} = A^6$.

$A^6 \times A = \{0, 1, 2, 3, 5, 7, 8, 9\} \times \{0, 2, 3\}$
$= \{0, 2, 4, 6, 10, 3, 5, 7, 9\} = A^7$.

$A^7 \times A = \{0, 2, 4, 3, 6, 10, 5, 7, 9\} \times \{0, 2, 3\}$
$= \{0, 4, 8, 6, 1, 9, 10, 3, 7, 5\} = A^8$.

$A^8 \times A = \{0, 4, 8, 6, 1, 9, 10, 3, 7, 5\} \times \{0, 2, 3\}$
$= \{0, 8, 5, 1, 2, 7, 9, 6, 3, 10, 4\} = A^9 = Z_{11}$.

So finding such A in $S(Z_{11})$ such that $A^m$ for some m is the MOD universal subset of $S(Z_{11})$.

Let B = {0, 3, 5} $\in$ S($Z_{11}$).

B $\times$ B = {0, 3, 5} $\times$ {0, 3, 5} = {0, 9, 4, 3} = $B^2$.

$B^2 \times$ B = {0, 9, 4, 3} $\times$ {0, 3, 5}
= {0, 9, 1, 5, 3} $\times$ {0, 3, 5} = {0, 9, 1, 5, 3} = $B^3$.

$B^3 \times$ B = {0, 1, 5, 9, 3} $\times$ {0, 3, 5} = {0, 3, 4, 5, 9, 1} = $B^4$.

$B^4 \times$ B = {0, 1, 3, 4, 5, 9} $\times$ {0, 3, 5}
= {0, 3, 9, 1, 4, 5} = $B^5$.

$B^5 \times B = B^6 = B^5$.

A = {0, 1, 2, 5, 6, 7, 4} $\in$ S($Z_{11}$).

A $\times$ A = {0, 1, 2, 5, 6, 7, 4} $\times$ {0, 1, 2, 5, 6, 7, 4}
= {0, 1, 2, 5, 6, 7, 4, 10, 3, 8, 9} = $A^2 = Z_{11}$.

$A^2 \times A = A^3 = A^2 = Z_{11}$.

Thus A gives after product with A we get $A^2$ to be MOD universal subset of S($Z_{11}$).

Let P = {0, 1, 5, 7, 8, 10} $\in$ S($Z_{11}$). P $\times$ P = {0, 1, 5, 7, 8, 10} $\times$ {0, 1, 5, 7, 8, 10} = {0, 1, 5, 7, 8, 10, 3, 2, 6, 4, 9} = $Z_{11}$.

Thus P is a MOD universal subset of S($Z_{11}$) as $P^2 = Z_{11}$.

B = {0, 2, 4, 6, 8, 3} $\in$ S($Z_{11}$).

B $\times$ B = {0, 2, 4, 6, 8, 3} $\times$ {0, 2, 4, 6, 8, 3}
= {0, 4, 8, 1, 5, 6, 2, 10, 3, 7, 9} = $Z_{11}$.

Similarly $B^2$ gives the MOD universal subset of S($Z_{11}$).

Let W = {0, 5, 4, 1, 3} $\in$ S($Z_{11}$).

$W \times W = \{0, 5, 4, 1, 3\} \times \{0, 5, 4, 1, 3\}$
$= \{0, 5, 4, 1, 3, 9\} = W^2$.

$W^2 \times W = \{0, 5, 4, 1, 3, 9) \times \{0, 5, 4, 1, 3\}$
$= \{0, 5, 4, 1, 3, 9\} = W^3$.

Clearly $W^2 = W^3$ so this MOD subset will not yield the MOD universal subset of $S(Z_{11})$.

Let $V = \{0, 7, 2, 1\} \in S(Z_{11})$;
$V \times V = \{0, 1, 2, 7\} \times \{0, 1, 2, 7\}$
$= \{0, 1, 2, 7, 4, 3, 5\} = V^2$.

$V^2 \times V = \{0, 1, 2, 7, 4, 3, 5\} \times \{0, 7, 1, 2\}$
$= \{0, 1, 2, 7, 4, 3, 5, 8, 10, 6\} = V^3$.

$V^3 \times V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10\} \times \{0, 7, 1, 2\}$
$= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 9\} = Z_{11}$.

Thus V is a MOD subset of $S(Z_{11})$ which gives $V^2 = Z_{11}$.

Infact if $D = \{0, 3\}$ then $D^2 = \{0, 3\} \times \{0, 3\} = \{0, 9\}$.

$D^3 = \{0, 3\} \times \{0, 9\} = \{0, 5\}$ and
$D^4 = \{0, 5\} \times \{0, 3\} = \{0, 4\}$.

$D^4 \times D = \{0, 4\} \times \{0, 3\} = \{0, 1\}$
$D^5 \times D = \{0, 1\} \times \{0, 3\} = \{0, 3\} = D$.

Thus D can never be a MOD subset of $S(Z_{11})$ which can contribute $Z_{12}$ when producted any number of times.

$D^t \neq Z_{11}$ for any $t > 1$.

Let $E = \{0, 2, 5\} \in S(Z_{11})$

$E \times E = \{0, 2, 5\} \times \{0, 2, 5\} = \{0, 4, 10, 3\} = E^2$
$E^2 \times E = \{0, 4, 10, 3\} \times \{0, 2, 5\} = \{0, 8, 9, 6, 4\} = E^3$

$E^3 \times E = \{0, 8, 9, 6, 4\} \times \{0, 2, 5\} = \{0, 5, 7, 1, 8, 9\} = E^4$

$E^4 \times E = \{0, 5, 7, 1, 8, 9\} \times \{0, 2, 5\}$
$= \{0, 10, 3, 2, 5, 7, 1\} = E^5.$

$E^5 \times E = \{0, 10, 2, 3, 5, 1, 7\} \times \{0, 2, 5\}$
$= \{0, 9, 4, 6, 10, 2, 3, 5\} = E^6.$

$E^6 \times E = \{0, 9, 4, 6, 10, 2, 3, 5\} \times \{0, 2, 5\}$
$= \{0, 7, 8, 1, 9, 4, 6, 10, 3\} = E^7.$

$E^7 \times E = \{0, 7, 8, 1, 9, 4, 6, 10, 3\} \times \{0, 2, 5\}$
$= \{0, 3, 5, 2, 7, 8, 1, 9, 6, 4\} = E^8.$

$E^8 \times E = \{0, 3, 5, 2, 7, 8, 1, 9, 6, 4\} \times \{0, 2, 5\}$
$= \{0, 6, 10, 4, 3, 5, 2, 7, 1, 8, 9\} = E^9 = Z_{11}.$

That is $E^9 = Z_{11}$.

We see if $A \in S(Z_{11})$ with $|A| \geq 3$, $|A|$ we mean the number of distinct elements in A and one of elements in A is zero then there is a $m > 1$ let $m \leq 10$ such that $A^m = Z_{11}$; then A gives the MOD universal subset after producting afinite number of times.

***Example 2.13:*** Let $\{S(Z_7), \times\}$ be the MOD subset semigroup under product.

Let $A = \{0, 3\} \in S(Z_7)$;

$A \times A = \{0, 3\} \times \{0, 3\} = \{0, 2\} = A^2$
$A^2 \times A = \{0, 2\} \times \{0, 3\} = \{0, 6\} = A^3$
$A^3 \times A = \{0, 6\} \times \{0, 3\} = \{0, 4\} = A^4$
$A^4 \times A = \{0, 4\} \times \{0, 3\} = \{0, 5\} = A^5$
$A^5 \times A = \{0, 5\} \times \{0, 3\} = \{0, 1\} = A^6$
$A^6 \times A = \{0, 1\} \times \{0, 3\} = \{0, 3\} = A.$

Thus A for no value of m can reach $Z_7$ the MOD universal subset of $S(Z_7)$.

Let B = {0, 1, 4} $\in$ S($Z_7$);

B × B = {0, 1, 4} × {0, 1, 4} = {0, 1, 4, 2} = $B^2$
$B^2$ × B = {0, 1, 4, 2} × {0, 1, 4} = {0, 1, 4, 2} = $B^3$.

Clearly $B^2 = B^3$ so we cannot get for any m, $B^m = Z_7$

Let B = {0, 2, 5} $\in$ S($Z_7$)
B × B = {0, 2, 5} × {0, 2, 5} = {0, 4, 3} = $B^2$
$B^2$ × B = {0, 2, 5} × {0, 4, 3} = {0, 1, 6} = $B^3$
$B^3$ × B = {0, 1, 6} × {0, 2, 5} = {0, 2, 5} = $B^4$ = B.

Thus $B^4$ = B so we do not have a positive integer m such that $B^m = Z_7$.

Let C = {0, 3, 5} $\in$ S($Z_7$);
C × C = {0, 3, 5} × {0, 3, 5} = {0, 2, 1, 4} = $C^2$
$C^2$ × C = {0, 2, 1, 4} × {0, 3, 5} = {0, 6, 3, 5} = $C^3$
$C^3$ × C = {0, 6, 3, 5} × {0, 3, 5} = {0, 4, 2, 1} = $C^4 = C^2$.

So this C behaves in an odd way such that $C^4 = C^2$ so
$C^3 = C^5$ and so on

Let M = {0, 4, 1, 5} $\in$ S($Z_7$)

M × M = {0, 1, 4, 5} × {0, 1, 4, 5} = {0, 1, 4, 5}
= {0, 1, 4, 5, 2, 6} = $M^2$.

$M^2$ × M= {0, 1, 4, 2, 5, 6} × {0, 1, 4, 5}
= {0, 1, 4, 2, 5, 6, 3} = $M^3 = Z_7$.

Thus $M^3 = Z_7$ so $M^3$ gives the MOD universal subset of S($Z_7$).

Let N = {0, 2, 3, 6} $\in$ S($Z_7$).

W find N × N = {0, 2, 3, 6} × {0, 2, 3, 6}
= {0, 4, 6, 5, 6, 2, 1} = $N^2 = Z_7$.

Thus $N^2$ gives the MOD universal subset of $S(Z_7)$.

Let L = {0, 1, 2, 3} $\in S(Z_7)$;

$L \times L = \{0, 1, 2, 3\} \times \{0, 2, 1, 3\} = \{0, 1, 2, 3, 4, 6\} = L^2$
$L^2 \times L = \{0, 1, 2, 3, 4, 6\} \times \{0, 1, 2, 3\}$
$= \{0, 1, 2, 3, 4, 6, 5\} = L^3 = Z_7$.

Thus $L^3$ gives the MOD universal subset of $S(Z_7)$

Let V = {0, 2, 4, 6} $\in S(Z_7)$; we find
$V \times V = \{0, 2, 4, 6\} \times \{0, 2, 4, 6\} = \{0, 4, 1, 5, 2, 3\} = V^2$

$V^2 \times V = \{0, 1, 2, 3, 4, 5\} \times \{0, 1, 2, 3, 4, 5\}$
$= \{0, 1, 2, 3, 4, 5, 6\} = Z_7$.

Thus $V^3$ gives the MOD universal subset of $S(Z_7)$.

Let S = {0, 1, 6} $\in S(Z_7)$,
$S \times S = \{0, 1, 6\} \times \{0, 1, 6\} = \{0, 1, 6\} = S$. So no power of S can give $Z_7$ and $S^2 = s$ is a MOD idempotent subset of $S(Z_7)$.

Let R = {0, 3, 5} $\in S(Z_7)$
$R \times R = \{0, 3, 5\} \times \{0, 3, 5\} = \{0, 2, 1, 4\} = R^2$
$R^2 \times R = \{0, 1, 2, 4\} \times \{0, 3, 5\} = \{0, 3, 6, 5\} = R^3$
$R^3 \times R = \{0, 3, 5, 6\} \times \{0, 3, 5\} = \{0, 2, 1, 4\} = R^4 = R^2$

So $R^4 = R^2$ can never give the MOD universal subset of $S(Z_7)$.

Let G = {0, 2, 4} $\in S(Z_7)$
$G \times G = \{0, 2, 4\} \times \{0, 2, 4\} = \{0, 4, 1, 2\} = G^2$
$G^2 \times G = \{0, 1, 2, 4\} \times \{0, 2, 4\}$
$= \{0, 2, 4, 1\} = G^3 = G^2$.

So $G^3 = G^2$ hence for no m (m a positive integer);

$G^m = Z_7$ the MOD universal subset of $S(Z_7)$.

Let T = {0, 1, 2, 3, 4, 5} $\in S(Z_7)$,
we find T × T = {0, 1, 2, 3, 4, 5} × {0, 1, 2, 4, 5}
= {0, 1, 2, 3, 4, 5, 6} = $Z_7$.

Thus T is such that $T \times T = T^2 = Z_7$ gives the MOD universal subset of $S(Z_7)$.

So for $S(Z_7)$ we see unlike $S(Z_{11})$ we do not see if $A \in S(Z_7)$ with $|A| = 3$ and $0 \in A$ then in general $A^m \neq Z_7$.

However there are subsets in $S(Z_7)$ which are such that their product taken at a stipulated number of times yields $Z_7$.

But $\{S(Z_{11}), \times\}$ behaves in a different way.

So we are forced to suggest the following problem.

**Problem 2.3:** Let $\{S(Z_p), \times\}$ be the MOD subset semigroup under product (p a prime). Characterize those MOD subset A of $S(Z_p)$ so that their power, $A^m = Z_p$, the MOD universal subset of $S(Z_p)$; $m > 1$.

Next we take $S(Z_n)$ where n is a composite number and give examples of the same.

***Example 2.14:*** Let $\{S(Z_8), \times\}$ be the MOD subset semigroup under product ×.

A = {0, 2, 4, 6} $\in S(Z_8)$,
A × A = {0, 2, 4, 6} × {0, 2, 4, 6} = {0, 4} = $A^2$
$A^2 \times A$ = {0, 2, 4, 6} × {0, 4} = {0}.
$A^3$ = {0}. We see A is a MOD nilpotent subset of $S(Z_8)$.

Let P = {0, 2, 5} $\in S(Z_8)$
P × P = {0, 2, 5} × {0, 2, 5} = {0, 4, 2, 1} = $P^2$
$P^2 \times P$ = {0, 4, 2, 1} × {0, 2, 5} = {0, 4, 2, 5, 1} = $P^3$

$P^3 \times P = \{0, 1, 2, 4, 5\} \times \{0, 2, 5\} = \{0, 2, 4, 5, 1\}$
$= P^4 = P^3$.
$P^4 = P^3$ so for no m (m > 1).
$P^m = Z_8$.

$S = \{1, 0, 5, 7\} \in S(Z_8)$;
$S \times S = \{0, 1, 5, 7\} \times \{0, 1, 5, 7\} = \{0, 1, 5, 7, 3\} = S^2$
$S^2 \times S = \{0, 1, 5, 3, 7\} \times \{0, 1, 3, 5, 7\}$
$= \{0, 1, 5, 7, 3\} = S^3 = S^2$.

Thus $S^3 = S^2$ so for no m (m a positive integer m > 1) we can get $S^m = Z_8$.

Let $T = \{0, 2, 7\} \in S(Z_8)$;

$T \times T = \{0, 2, 7\} \times \{0, 2, 7\} = \{0, 4, 6, 1\} = T^2$

$T^2 \times T = \{0, 1, 4, 6\} \times \{0, 2, 7\} = \{0, 2, 7, 1, 5, 4\} = T^3$

$T^3 \times T = \{0, 2, 7, 1, 5, 4\} \times \{0, 2, 7\}$
$= \{0, 2, 7, 4, 6, 1, 3\} = T^4$

$T^4 \times T = \{0, 2, 7, 4, 6, 1, 3\} \times \{0, 2, 7\}$
$= \{0, 4, 6, 1, 5, 2, 7\} = T^5$.

$T^t \times T = \{0, 1, 2, 4, 5, 6, 7\} \times \{0, 2, 7\}$
$= \{0, 2, 4, 1, 6, 7, 3\} = T^6 = T^4$.

Thus this will not lead to the MOD universal subset $Z_8$ $S(Z_8)$ as $T^6 = T^4$; $T^7 = T^5$ and so on.

Let $P = \{0, 1, 4\} \in S(Z_8)$
$P^2 = P \times P = \{0, 1, 4\} \times \{0, 1, 4\} = \{0, 1, 4\} = P$.

So for no m > 1 $P^m = Z_8$, so P cannot give the MOD universal subset $Z_8$ of $S(Z_8)$.

$W = \{0, 6, 1\} \in S(Z_8)$;

$W \times W = \{0, 1, 6\} \times \{1, 0, 6\} = \{0, 1, 6, 4\} = W^2$.
$W^2 \times W = \{0, 1, 4, 6\} \times \{0, 1, 6\} = \{0, 4, 6, 1\} = W^3 = W^2$

As $W^3 = W^2$ it is impossible to arrive at $W^m = Z_8$ for any $m > 1$.

***Example 2.15:*** Let $\{S(Z_{12}), \times\}$ be the MOD subset semigroup under $\times$.

Let $B = \{0, 3, 5\} \in S(Z_{12})$;
$B \times B = \{0, 3, 5\} \times \{0, 3, 5\} = \{0, 9, 3, 1\}$
$B^2 \times B = \{0, 1, 3, 9\} \times \{0, 3, 5\} = \{0, 3, 5, 9\} = B^3$
$B^3 \times B = \{0, 3, 5, 9\} \times \{0, 3, 5\} = \{0, 1, 9, 3, 7\} = B^4$
$B^4 \times B = \{0, 1, 9, 3, 7\} \times \{0, 3, 5\} = \{0, 3, 5, 9, 11\} = B^5$
$B^5 \times B = \{0, 3, 5, 9, 11\} \times \{0, 3, 5\} = \{0, 9, 3, 1, 7\} = B^6$
$B^6 \times B = \{0, 1, 3, 7, 9\} \times \{0, 3, 5\} = \{0, 3, 5, 9, 11\} = B^7$
$B^7 \times B^5$ so for no $m > 1$, $B^m = Z_{12}$, the MOD universal subset of $S(Z_{12})$.

Let $P = \{0, 3, 10\} \in S(Z_{12})$.
$P \times P = \{0, 3, 10\} \times \{0, 3, 10\} = \{0, 9, 6, 4\} = P^2$
$P^2 \times P = \{0, 6, 4, 9\} \times \{0, 10, 3\} = \{0, 4, 6, 3\} = P^3$
$P^3 \times P = \{0, 4, 6, 3\} \times \{0, 3, 10\} = \{0, 6, 9, 4\} = P^4 = P^2$.

So $P^2 = P^4$ so we see P is not a MOD subset of $S(Z_{12})$ which can contribute to the MOD universal subset of $S(Z_{12})$.

Let $V = \{0, 1, 2, 5\} \in S(Z_{12})$;
$V \times V = \{0, 1, 2, 5\} \times \{0, 1, 2, 5\} = \{0, 1, 2, 5, 4, 10\} = V^2$.
$V^2 \times V = \{0, 1, 2, 4, 5, 10\} \times \{0, 1, 2, 5\}$
$= \{0, 1, 2, 4, 5, 10, 8\} = V^3$.
$V^3 \times V = \{0, 1, 2, 4, 5, 8, 10\} \times \{0, 1, 2, 5\}$
$= \{0, 1, 2, 4, 5, 8, 10\} = V^4$.

But $V^4 = V^3$ so we see V cannot for any $(m > 1)$ give $V^m = Z_{12}$, the MOD universal subset of $S(Z_{12})$.

Let W = {0, 1, 5, 7, 2} $\in$ S($Z_{12}$);
W × W = {0, 1, 5, 2,7} × {0, 1, 2, 5, 7}
= {0, 1, 5, 2, 7, 10, 4, 11} = $W^2$.

$W^2$ × W = {0, 1, 2, 4, 5, 7, 10, 11, 8} × {0, 1, 2, 5, 7}
= {0, 1, 2, 4, 5, 7, 10, 11, 8} = $W^3$.

$W^3$ = $W^2$ so this also does not yield the MOD universal subset $Z_{12}$ of S($Z_{12}$).

Let M = {0, 1, 5, 6, 8, 9, 11} $\in$ S($Z_{12}$),

M × M = {0, 1, 5, 6, 8, 9, 11} × {0, 1, 5, 6, 8, 9, 11}
= {0, 1, 5, 6, 8, 9, 11, 4, 7, 3} = $M^2$.

$M^2$ × M = {0, 1, 5, 6, 8, 9, 11, 4,7, 3} ×
{0, 11, 5, 6, 8, 9, 11}
= {0, 1, 5, 6, 8, 9, 11, 4, 7, 3} = $M^3$.

But $M^3$ = M so M cannot yield the MOD universal subset $Z_{12}$ of S($Z_{12}$).

Thus we see in case of finding the MOD subsets which can yield MOD universal subset happens to be a very difficult problem.

In view of this we propose the following problem.

**Problem 2.4:** Let {S($Z_n$), ×} ( n a big composite number) be the MOD semigroup under product. Find all MOD subsets B of S($Z_n$) such that $B^m = Z_n$ (m > 1).

Next we proceed onto study MOD subsemigroups of S($Z_n$) by some examples.

***Example 2.16:*** Let {S($Z_6$), ×} be the MOD subset semigroup of $Z_6$ under product ×.

Let $P = \{0, 2\} \in S(Z_6)$,

$P \times P = \{0, 2\} \times \{0, 2\} = \{0, 4\} = P^2$

$P^2 \times P = \{0, 2\} \times \{0, 4\} = \{0, 2\} = P^3 = P$.

So $M = \{P, P^2, P^3 = P\}$ is a MOD subset subsemigroup of order two.

Let $B = \{0, 1, 2\} \in S(Z_6)$

$B \times B = \{0, 1, 2\} \times \{0, 2, 1\} = \{0, 1, 2, 4\} = B^2$

$B^2 \times B = \{0, 1, 2, 4\} \times \{0, 1, 2\}$

$= \{0, 1, 2, 4\} = B^3 = B^2$.

$L = \{B, B^2, B^3 = B^2\}$ is a MOD subset subsemigroup of order two in $S(Z_6)$.

Clearly L and M are not isomorphic as MOD subset subsemigroups.

Let $S = \{0, 1, 5, 2\} \in S(Z_6)$

$S \times S = \{0, 1, 5, 2\} \times \{0, 1, 2, 5\} = \{0, 1, 5, 2, 4\} = S^2$

$S^2 \times S = \{0, 1, 2, 4, 5\} \times \{0, 1, 2, 5\}$

$= \{0, 1, 2, 4, 5\} = S^3 = S^2$.

Clearly $N = \{S, S^2, S^3 = S^2\}$ is a MOD subset subsemigroup of $S(Z_6)$ and $N \cong L$ as MOD subset subsemigroups of $S(Z_6)$.

Let $W = \{0, 2, 3\} \in S(Z_6)$, we find

$W \times W = \{0, 2, 3\} \times \{0, 2, 3\} = \{0, 4, 3\} = W^2$

$W^2 \times W = \{0, 2, 3\} \times \{0, 3, 4\} = \{0, 3, 2\} = W^3 = W$.

Now $R = \{W, W^2, W^3 = W\}$ is a MOD subset subsemigroup of $S(Z_6)$.

Clearly $R \cong M$ as MOD subset subsemigroups of $S(Z_6)$.

Let $A = \{0, 3\} \in S(Z_6)$

$A \times A = \{0, 3\} \times \{0, 3\} = \{0, 3\} = A^2 = A$.

Thus B = {A, $A^2$ = A} is a MOD subset subsemigroup of order one.

Let C = {2} ∈ S($Z_6$);
C × C = {2} × {2} = {4} = $C^2$
$C^2$ × C = {4} × {2} = {2} = $C^3$ = C.

Thus S = {C, $C^2$, $C^3$ = C} is a MOD subset subsemigroup of order two.

Let T = {0, 1} and S = {0, 2} ∈ S($Z_6$).

We find the MOD subset semigroup generated by T and S.

T × T = {0, 1} × {0, 1} = {0, 1} = $T^2$ = T.
S × T = {0, 2} × {0, 2} = {0, 4} = $S^2$
$S^2$ × S = {0, 4} × {0, 2} = {0, 2} = $S^3$ = S
S × T = {0, 1} × {0, 2} = {0, 2}.

So H = {{0, 1} {0, 2}, {0, 4} / {0, 1} × {0, 1} = {0, 1}, {0, 2} × {0, 2} = {0, 4}, {0, 2} × {0, 4} = {0, 2}, {0, 2} × {0, 1} = {0, 2} and {0, 4} × {0, 1} = {0, 4}} is a MOD subset subsemigroup of order three.

It is to be noted S($Z_6$) has the elements {1} to be the identity but {0, 1} can be defined as pseudo identity of the subsets Y of S($Z_6$) where 0 ∈ Y, for in the case {0, 1} × Y = Y for all such Y ∈ S($Z_6$).

So H can be described as a MOD subset subsemigroup with the pseudo identity or specifically S is a pseudo MOD subset monoid {0, 1} × {1} = {0, 1}.

However the pseudo identity is not the identity for if Z = {2, 3, 1} ∈ S($Z_6$); {0, 1} × {2, 3, 1} = {0, 1, 2, 3} ≠ Z.

Hence our claim, but {1} × {2, 3, 1} = {1, 3, 2} = Z.

This is the major different between the identity {1} and {0, 1}.

Infact {1} serves as the identity for the pseudo identity {0, 1}.

In view of all these we give following definition.

**DEFINITION 2.4**: *Let $\{S(Z_n), \times\}$ be the MOD subset semigroup under $\times$, $\{1\} \in S(Z_n)$ is defined as the identity and $\{0, 1\}$ is defined as the pseudo identity for all subset $Y \in S(Z_n)$ such that 0 is one of the element in Y. Further $\{1\} \times \{0, 1\} = \{0, 1\}$ so $\{1\}$ serves as the identity for the pseudo identity.*

**DEFINITION 2.5**: *Let $\{S(Z_n), \times\}$ be the MOD subset semigroup. $P = \{X_1, \ldots, X_t / 2 \leq t < \infty, X_i \in S(Z_n)\} \subseteq S(Z_n)$ is defined as the pseudo MOD subset monoid if $\{0, 1\}$ acts as the pseudo identity, that is $\{0, 1\} \times X_i = X_i$ for $1 \leq i \leq t$.*

We will give more examples of this situation.

***Example 2.17:*** Let $\{S(Z_{11}), \times\}$ be the MOD subset semigroup under $\times$.

Let $A = \{6, 3, 2\} \in S(Z_{11})$
$A \times A = \{2, 3, 6\} \times \{2, 3, 6\} = \{3, 4, 6, 1, 9, 7\} = A^2$

$A^2 \times A = \{3, 1, 4, 6, 9, 7\} \times \{2, 3, 6\}$
$= \{6, 2, 8, 1, 7, 3, 9, 5, 10\} = A^3$

$A^3 \times A = \{1, 2, 3, 5, 6, 7, 8, 9, 10\} \times \{2, 3, 6\}$
$= \{2, 4, 6, 10, 1, 3, 7, 5, 9, 8\} = A^4$

$A^4 \times A = \{2, 1, 3, 4, 5, 6, 7, 8, 9, 10\} = A^5 = A^4$.

Thus $A^5 = A^4$ so A has no power to yield the MOD universal subset of $S(Z_{11})$ viz $Z_{11}$.

However if we have taken $\{0\} \cup A = \{0, 2, 3, 6\} = B$ then certainly $B^4 = Z_{11}$.

It is pertinent to observe and keep on record that in case we use in $S(Z_n)$, n a prime then if a set $A \in S(Z_n)$ such that $A^m = Z_n$ then necessarily A must contain 0.

We consider now take $B = \{0, 5, 8, 9\} \in S(Z_{11})$;

$B \times B = \{0, 5, 8, 9\} \times \{0, 5, 8, 9\} = \{0, 3, 7, 1, 9, 6, 4\} = B$
$B^2 \times B = \{0, 3, 7, 1, 9, 6, 4\} \times \{0, 5, 8, 9\}$
$= \{0, 5, 8, 9, 4, 2, 1, 6, 10\} = B^3$.

$B^3 \times B = \{0, 1, 2, 4, 5, 6, 8, 9, 10\} \times \{0, 5, 8, 9\}$
$= \{0, 5, 10, 8, 3, 7, 1, 6, 10, 4, 9\} = B^4 = Z_{11}$.

Thus B yield the MOD universal subset of $S(Z_{11})$.

Let $M = \{0, 1, 5\} \in S(Z_{11})$;
$M \times M = \{0, 1, 5\} \times \{0, 1, 5\} = \{0, 1, 5, 3\} = M^2$
$M^2 \times M = \{0, 1, 5, 3\} \times \{0, 1, 5\} = \{0, 1, 5, 3, 4\} = M^3$
$M^3 \times M = \{0, 1, 3, 4, 5\} \times \{0, 1, 5\} = \{0, 1, 3, 4, 5, 9\} = M^4$
$M^4 \times M = \{0, 1, 3, 4, 5, 9\} \times \{0, 1, 5\}$
$= \{0, 1, 3, 4, 5, 9\} = M^5$
$M^5 = M^4$ so M cannot yield the MOD universal subset $Z_{11}$ for any n, n > 1.

Now we give the following theorem.

**THEOREM 2.3:** *Let $\{S(Z_p), \times\}$, (p a prime) be the MOD subset semigroup under product. If $A \in S(Z_p)$, $A^m = Z_p$ (m > 1) only if $0 \in A$.*

Proof given $A^m = Z_p$ so $0 \in A^m$. But as p is a prime no product can yield 0 so $0 \in A$.

However if $0 \in A$ one cannot say that $A^m = Z_p$.

This is will be illustrated by the following example.

***Example 2.18:*** Let $\{S(Z_{17}), \times\}$ be the MOD subset semigroup under product $\times$.

Let $A = \{6, 0\} \in S(Z_{17})$.

$A \times A = \{0, 6\} \times \{0,6\} = \{0, 2\} = A^2$
$A^2 \times A = \{0, 2\} \times \{0, 6\} = \{0, 12\} = A^3$
$A^3 \times A = \{0, 12\} \times \{0, 6\} = \{0, 4\} = A^4$
$A^4 \times A = (0, 4\} \times \{0, 6\} = \{0, 7\} = A^5$
$A^5 \times A = \{0, 7\} \times \{0, 6\} = \{0, 8\} = A^6$
$A^6 \times A = \{0, 8\} \times \{0, 6\} = \{0, 14\} = A^7$
$A^7 \times A= \{0, 14\} \times\{0, 6\} = \{0, 6\} = A^8 = A.$

Since $A^8 = A$ for no m, $m > 1$ we can have $A^m = Z_{17}$.

But $0 \in A$. Hence the converse of the theorem is not true. That is if a set $A \in S(Z_p)$, p a prime and $0 \in A$ then in general $A^m \neq Z_p$ for some $m > 1$.

What we can say is if $A^m = Z_p$ then $0 \in A$.

Let $P = \{0, 5, 2\} \in S(Z_{17})$,

$P \times P = \{0, 2, 5\} \times \{0, 2, 5\} = \{0, 4, 10, 8\} = P^2$

$P^2 \times P = \{0, 4, 8, 10\} \times \{0, 2, 5\} = \{0, 8, 16, 3, 6\} = P^3$

$P^3 \times P = \{0, 8, 3, 6, 16\} \times \{0, 2, 5\}$
$= \{0, 16, 6, 12, 15, 13\} = P^4$

$P^4 \times P = \{0, 6, 12, 15, 16, 13\} \times \{0, 2, 5\}$
$= \{0, 12, 7, 13, 15, 9, 14\} = P^5$

$P^5 \times P = \{0, 7, 9, 12, 14, 15, 13\} \times \{0, 2, 5\}$
$= \{0, 14, 1, 7, 11, 13, 9, 2\} = P^6$

$P^6 \times P = \{0, 1, 7, 2, 9, 11, 13, 14\} \times \{0, 2, 5\}$
$= \{0, 2, 14, 4, 1, 5, 9, 11, 10\} = P^7$

$P^7 \times P = \{0, 1, 2, 4, 5, 9, 11, 10, 14\} \times \{0, 2, 5\}$
$= \{0, 2, 4, 8, 10, 1, 5, 3, 11, 16\} = P^8$

$P^8 \times P = \{0, 2, 1, 4, 8, 5, 10, 11, 3, 16\} \times \{0, 2, 5\}$
$= \{0, 4, 2, 8, 16, 10, 3, 5, 6, 15, 12\} = P^9$

$P^9 \times P = \{0, 4, 2, 8, 16, 10, 3, 5, 6, 15, 12\} \times \{0, 2, 5\}$
$= \{0, 8, 4, 16, 15, 3, 6, 10, 12, 13, 7, 9\} = P^{10}$

$P^{10} \times P = \{0, 3, 6, 4, 8, 16, 15, 12, 13, 7, 9, 10\} \times \{0, 2, 5\}$
$= \{0, 6, 12, 8, 16, 15, 7, 9, 14, 1, 3, 13, 11\} = P^{11}$

$P^{11} \times P = \{0, 1, 3, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16\} \times \{0, 2, 5\}$
$= \{0, 2, 6, 12, 14, 16, 1, 5, 7, 9, 11, 13, 15, 8\} = P^{12}$

$P^{12} \times P = \{0, 1, 2, 6, 12, 14, 16, 5, 7, 9, 11, 13, 15, 8\} \times \{0, 2, 5\}$
$= \{0, 2, 4, 12, 7, 11, 15, 10, 14, 1, 5, 9, 13, 16, 8\} = P^{13}$

$P^{13} \times P = \{0, 1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\} \times \{0, 2, 5\}$
$= \{0, 2, 4, 8, 10, 14, 16, 1, 3, 5, 7, 12, 9, 11, 13, 15\} = P^{14}$

$P^{14} \times P = \{0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\} \times \{0, 2, 5\}$
$= \{0, 2, 4, 6, 8, 10, 14, 16, 1, 3, 5, 7, 9, 11, 13, 15, 12\} = P^{15} = Z_{17}.$

Thus $P^{15} = Z_{17}$ yields the MOD universal subset of $S(Z_{17})$.

Let $B = \{0, 2, 4\} \in S(Z_{17})$,
$B \times B = \{0, 2, 4\} \times \{0, 2, 4\} = \{0, 4, 8, 16\} = B^2$
$B^2 \times B = \{0, 4, 8, 16\} \times \{0, 2, 4\} = \{0, 8, 16, 15, 13\} = B^3$
$B^3 \times B = \{0, 8, 16, 15, 13\} \times \{0, 2, 4\}$

$= \{0, 16, 15, 13, 9, 1\} = B^4$
$B^4 \times B = \{0, 1, 9, 13, 15, 16\} \times \{0, 2, 4\}$
$= \{0, 2, 1, 9, 13, 15, 4\} = B^5$
$B^5 \times B = \{0, 1, 2, 4, 9, 13, 15\} \times \{0, 2, 4\}$ and so on.

It is easily proved $B^{15} \times Z_{17}$; this task is left as an exercise to the reader.

Let $S = \{0, 4, 5\} \in S(Z_{17})$
$S \times S = \{0, 4, 5\} \times \{0, 4, 5\} = \{0, 16, 3, 8\} = S^2$
$S^2 \times S = \{0, 3, 8, 16\} \times \{0, 4, 5\} = \{0, 12, 6, 15, 13\} = S^3$
$S^3 \times S = \{0, 6, 12, 13, 15\} \times \{0, 4, 5\}$
$= \{0, 7, 14, 1, 9, 13\} = S^4$
$S^4 \times S = \{0, 1, 7, 9, 13, 14\} \times \{0, 4, 5\}$
$= \{0, 4, 11, 2, 1, 5, 14\} = S^5$ and so on.

For this S also we get $S^{15} = Z_{17}$ that S yield the MOD universal subset of $S(Z_{17})$

Let $M = \{0, 1, 3\} \in S(Z_{17})$
$M \times M = \{0, 1, 3\} \times \{0, 1, 3\} = \{0, 1, 9, 3\} = M^2$
$M^2 \times M = \{0, 1, 3, 9\} \times \{0, 1, 3\} = \{0, 1, 3, 9, 10\} = M^3$
$M^3 \times M = \{0, 1, 3, 9, 10\} \times \{0, 1, 3\}$
$= \{0, 1, 3, 9, 10, 13\} = M^4$
$M^4 \times M = \{0, 1, 3, 9, 10, 13\} \times \{0, 1, 3\}$
$= \{0, 3, 1, 9, 10, 13, 5\} = M^5$ and so on.

Thus we see after some steps say 15, $M^{15} = Z_{17}$ so M also yield the MOD universal subset $Z_{17}$ of $S(Z_{17})$.

Let $T = \{0, 1, 3, 5\} \in S(Z_{17})$
$T \times T = \{0, 1, 3, 5\} \times \{0, 1, 3, 5\}$
$= \{0, 1, 3, 5, 9, 15, 8\} = T^2$
$T^2 \times T = \{0, 1, 3, 5, 8, 9, 15\} \times \{0, 1, 3, 5\}$
$= \{0, 1, 3, 5, 8, 9, 15, 7, 10, 11\} = T^3$
$T^3 \times T = \{0, 1, 3, 5, 7, 8, 9, 10, 11, 15\} \times \{0, 1, 3, 5\}$
$= \{0, 1, 3, 5, 7, 8, 9, 10, 11, 15, 4, 13, 16, 6\} = T^4$
$T^4 \times T = \{0, 1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 16\} \times$

$\{0, 1, 3, 5\}$
$= \{0, 1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 16, 12, 14\}$
$= T^5$.

$T^5 \times T = T^6 = Z_{17}$.

Thus T is a MOD subset of $S(Z_{17})$ such that $T^6 = Z_{17}$ the MOD universal subset of $S(Z_{17})$.

Let V $= \{0, 1, 12, 10, 4\} \in S(Z_{17})$,

$V \times V = \{0, 1, 4, 10, 12\} \times \{0, 1, 4, 10, 12\}$
$= \{0, 1, 4, 10, 12, 16, 6, 14, 15, 8\} = V^2$

$V^2 \times V = \{0, 1, 4, 6, 8, 10, 12, 14, 15, 16\} \times \{0, 1, 4, 10, 12\}$
$= \{0, 1, 4, 6, 8, 10, 12, 14, 15, 16, 7, 5, 9, 13, 11\}$
$= V^3$.

$V^3 \times V = \{0, 1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$
$= \{0, 1, 4, 10, 12\}$
$= \{0, 1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 3, 2\} = V^4 = Z_{17}$.

Thus $V^4 = Z_{17}$ contributes the MOD universal subset of $S(Z_{17})$.

Let W $= \{0, 1, 2, 4, 5, 10\} \in S(Z_{17})$,

$W \times W = \{0, 1, 2, 4, 5, 10\} \times \{0, 1, 2, 4, 5, 10\}$
$= \{0, 1, 2, 4, 5, 10, 8, 3, 16, 6, 15\} = W^2$.

$W^2 \times W = \{0, 1, 2, 3, 4, 5, 6, 8, 10, 15, 16\} \times \{0, 1, 2, 4, 5, 10\}$
$= \{0, 1, 2, 3, 4, 5, 6, 8, 10, 15, 16, 12, 13, 7, 9, 14\}$
$= W^3$.

$W^3 \times W = W^4 = Z_{17}$ gives the MOD universal subset of $S(Z_{17})$

Let $A = \{0, 1, 3, 5, 7, 8\}$ and $B = \{4, 1, 10, 9\} \in S(Z_{17})$
$A \times B = \{0, 1, 3, 5, 7, 8\} \times \{4, 9, 1, 10\}$
$= \{0, 1, 3, 5, 7, 8, 4, 12, 11, 15, 9, 10, 13, 16, 2\}$
$\neq Z_{17}$.

So the product of these two sets A and B does not yield the MOD universal subset of $S(Z_{17})$.

Let $P = \{0, 1, 2, 7, 10, 16\}$ and
$Q = \{, 5, 8, 11, 4, 15,6, 9\} \in S(Z_{17})$,

$P \times Q = \{0, 1, 2, 7, 10, 16\} \times \{1, 4, 5, 6, 8, 9, 11, 15\}$
$= \{0, 1, 2, 7, 10, 16, 4, 5, 6, 8, 9, 11, 15, 13, 12, 3, 14\} = Z_{17}$.

Thus this pair of sets P and Q when their product $P \times Q$ is taken gives the MOD universal subset $Z_{17}$ of $S(Z_{17})$.

We call the pair (P, Q) as the MOD universal subset pair of $S(Z_{17})$.

Let $M = \{0, 1, 5, 7, 8, 9, 10, 15\}$ and
$N = \{1, 2, 3, 4, 11, 13\} \in S(Z_{17})$.

$M \times N = \{0, 1, 5, 7, 8, 9, 10, 15\} \times \{1, 2, 3, 4, 11, 13\}$
$= \{0, 1, 5, 7, 8, 9, 10, 15, 6, 2, 3, 4, 11, 13, 14, 16\}$
$\neq Z_{17}$.

So we cannot say always a pair of MOD subsets will contribute to the MOD universal subset of $S(Z_{17})$.

We give a few more examples before we put forth some results.

***Example 2.19:*** Let $\{S(Z_{10}), \times\}$ be the MOD subset semigroup under $\times$.

Let $A = \{0, 5, 9, 7\}$ and $B = \{0, 2, 4, 6, 8, 3\} \in S(Z_{14})$

$A \times B = \{0, 5, 9, 7\} \times \{0, 2, 4, 6, 8, 3\}$
$= \{0, 8, 4, 6, 2, 5, 7, 1\} \neq Z_{10}$.

So {A, B} is not a MOD universal subset product.

Let A = {1, 4, 6} and B = {5, 8, 7} $\in S(Z_{10})$,
$A \times B = \{1, 4, 6\} \times \{5, 8, 7\} = \{5, 8, 7, 2\} \neq Z_{10}$.

So this pair {A, B} is not a MOD universal subset pair of $S(Z_{10})$.

Let S = {0, 1, 2, 5, 8} and P = {1, 3, 7} $\in S(Z_{10})$,

$S \times P = \{0, 1, 2, 13, 5, 8\} \times \{1, 3, 7\}$
$= \{0, 1, 2, 5, 8, 3, 6, 4, 9, 7\} = Z_{10}$.

So the pair {S, P} yields the MOD universal subset $Z_{10}$ of $S(Z_{10})$. However not all MOD subsets pair can yield the MOD universal subset $Z_{10}$ of $S(Z_{10})$.

Consider A = {0, 5} and B = {0, 2, 4, 6} $\in S(Z_{10})$
$A \times B = \{0, 5\} \times \{0, 2, 4, 6\} = \{0\}$.

Thus the MOD subset pair {A, B} yields the MOD subset zero divisor of $S(Z_{10})$

Let $A \times A = \{0, 5\} \times \{0, 5\} = \{0, 5\}$ so the MOD subset A of $S(Z_{10})$ is defined as the MOD subset idempotent of $S(Z_{10})$.

Consider $B \times B = \{0, 2, 4, 6\} \times \{0, 2, 4, 6\}$
$= \{0, 4, 8, 6, 2\} \neq B$.

So it is easily observed that in general all elements of $S(Z_{10})$ are not MOD subset idempotents of $S(Z_{10})$.

Let S = {0, 2, 4, 6, 8} $\in S(Z_{10})$.
$S \times S = \{0, 2, 4, 6, 8\} \times \{0, 2, 4, 6, 8\}$

$= \{0, 4, 8, 2, 6\} = S$.

Thus S in $S(Z_{10})$ is the MOD subset idempotent of $S(Z_{10})$.

Similarly let $S = \{0, 3, 9\} \in S(Z_{10})$;

$$S \times S = \{0, 3, 9\} \times \{0, 3, 9\} = \{0, 9, 7, 1\} \neq S.$$

Now we define and develop results related with MOD subset universal pair in $S(Z_n)$.

**DEFINITION 2.6**: *Let $S(Z_n)$ be the MOD subset semigroup under product. A pair of MOD subsets $\{P, Q\}$ is MOD subset universal pair if $P \times Q = Z_n$; $Z_n$, $P, Q \in S(Z_n)$.*

*$P \neq Z_n$ and $Q \neq Z_n$.*

We have given examples of them.

However the following result is recorded.

**THEOREM 2.4**: *Let $\{S(Z_n), \times\}$ be a MOD subset semigroup under product; n a prime. There exists MOD subset universal pairs $\{P,Q\}$ such that $P \times Q = Z_n$.*

Proof is left as an exercise to the reader.

The following problem is left open.

**Problem 2.5:** Let $\{S(Z_n), \times\}$ be the MOD subset semigroup; n a composite number.

Obtain conditions on n so that $S(Z_n)$ has a MOD subset universal pair, $P, Q \in S(Z_n) \setminus Z_n$.

**DEFINITION 2.7:** *Let $\{S(Z_n), \times\}$ be the MOD subset semigroup. A pair $\{P, Q\}$, $P \neq Q \in S(Z_n) \setminus \{0\}$ is said to be a MOD zero divisor subset pair if $P \times Q = \{0\}$.*

*A MOD subset P in $S(Z_n)$ is said to be a MOD nilpotent subset if $P^m = \{0\}$ for $m > 1$.*

*A MOD subset P in $S(Z_n) \setminus \{\{0\}, \{0, 1\}\}$ is said to be a MOD idempotent subset of $S(Z_n)$ if $P \times P = P$.*

We have given examples of them.

We present only one more example before we proceed onto define $S(C(Z_n))$.

***Example 2.20:*** Let $\{S(Z_{24}), \times\}$ be the MOD subset semigroup under product.

Let $A = \{0, 6, 12\}$ and $B = \{4, 8, 10, 16\} \in S(Z_{24})$
$A \times B = \{0, 6, 12\} \times \{4, 8, 10, 16\} = \{0\}$.

Thus $\{A, B\}$ is the MOD zero divisor pair of $S(Z_{24})$.

Let $M = \{0, 12\} \in S(Z_{24})$ is such that $M \times M = \{0\}$ so M is a MOD subset nilpotent of order two in $S(Z_{24})$.

Let $N = \{0, 6\} \in S(Z_{24})$;
$N \times N = \{0, 6\} \times \{0, 6\} = \{0, 12\} \in N^2$
$N^2 \times N = \{0, 12\} \times \{0, 6\} = \{0\} = N^3$.

Thus N is a MOD subset nilpotent of order three.

Let $S = \{0, 9, 1, 6\} \in S(Z_{24})$
$S \times S = \{0, 9, 16\} \times \{0, 9, 16\} = \{0, 9, 16\} = S$.

Thus S in $S(Z_{24})$ is a MOD subset idempotent of $S(Z_{24})$.

Let $T = \{0, 10, 20, 16\}$ and $R = \{0, 6, 12, 18\} \in (Z_{24})$.
$R \times T = \{0, 10, 20, 15\} \times \{0, 6, 12, 18\} = \{0\}$.

Thus $\{R, T\}$ is a MOD subset zero divisor pair of $S(Z_{24})$.

Let $V = \{0, 12\}$ and

W = {2, 4, 6, 8, 10, 12, 14, 20, 16, 22} ∈ $S(Z_{24})$.
V × W = {0, 12} × {2, 4, 6, 8, 10, 12, 14, 20, 16, 22} = {0}.

Thus {V, W} the MOD subset pair is a MOD zero divisor. We have shown examples of MOD subset zero divisor pair, MOD subset nilpotents and MOD subset idempotents in $S(Z_{24})$.

We now illustrate properties of MOD finite complex number subsets.

Through out this book $S(C(Z_n)$ denotes the collection of all subsets of $C(Z_n)$.

***Example 2.21:*** Let $S(C(Z_3))$ = {{0}, {1}, {2},{$i_F$}, {$2i_F$}, {1 + $i_F$}, {2 + $i_F$}, {1+$2i_F$}, {2+$2i_F$}, {0, 1}, {0, 2}, {0, $i_F$}, {0, $2i_F$}, {0, 1 + $i_F$}, …, {0, 2 + $2i_F$} {1, $i_F$}, {1, $i_F$ + 2}, {1, 1 + $i_F$}, …, {1, $2i_F$ + 2}, {2, $i_F$}, {1, 2}, {2, 1 + $i_F$}, …, {2, $2i_F$ + 2} and so on {$C(Z_3)$}}.

Let A = {0, 1 + $i_F$, 2 + $i_F$, 1} and
B = {1, $2i_F$, $i_F$} ∈ $S(C(Z_3))$.

A + B = {0, 1 + $i_F$, 2 + $i_F$, 1} + {1, $i_F$, $2i_F$}
= {1, 2 + $i_F$, $i_F$, 1 + $2i_F$, 2 + $2i_F$, 2, 1 + $i_F$}

is an element of $S(C(Z_3)$.

| A | = 4, that is number of elements in A is 4.

Number of elements in B is 3. But number of elements in A + B = 7.

Thus is the way + operation is performd on $S(C(Z_3))$.

***Example 2.22:*** Let
$S(C(Z_6))$ = {collection of all subsets of the set $C(Z_6)$}. Clearly $o(S(C(Z_6))) = 2^{|C(Z_6)|} - 1$ where $|C(Z_6)\}$ denotes the number of elements in $C(Z_6)$ $S(C(Z_6))$ is defined as the MOD finite complex number subsets.

Let P = $\{3 + 2i_F, 4, 2 + 5i_F, 5i_F\}$ and Q = $\{2, 3i_F, 1 + i_F\} \in S(C(Z_6))$.

We find P + Q = $\{3 + 2i_F, 4, 2 + 5i_F, 5i_F + \{2, 3i_F, 1 + i_F\}$ = $\{5 + 2i_F, 0, 4 + 5i_F, 2 + 5i_F, 3 + 5i_F, 4 + 3i_F, 2 + 2i_F, 2i_F, 5 + i_F, 3, 1\}$.

We see o(P) = 4 and o(Q) = 3 but
o(P + Q) = 11 $\neq$ o(P) + o(Q).

Let B = $\{0, 3, 3i_F, 2\}$ and C = $\{3i_F, 3, 0\} \in S(C(Z_6))$.

$$B + C = \{0, 3, 2, 3i_F\} + \{0, 3, 3i_F\} = \{0, 3 + 3i_F, 5, 2 + 3i_F\}.$$

Clearly o(B) = 4, o(C) = 3 but o(B+C) = 4 < o(B) + o(C).

Thus we see in general o(B+C) is less than or greater than or equal o(B) + o(C).

Let X = {2} and Y = {4}

X + Y = {2} + {4} = {0}.
o(X + Y) = 1  o(X) = 1
o(Y) = 1.

Thus o(A + B) = 0 is an impossibility.

Let T = $\{0, 2, 3, 4 + i_F, 5\}$ and R = $\{i_F, 2i_F, 3i_F\} \in S(C(Z_6))$.

$$T + R = \{0, 2, 3, 4 + i_F, 5\} + \{i_F, 2i_F, 3i_F\} = \{i_F, 2i_F, 3i_F, 2 + i_F, 2 + 2i_F, 2 + 3i_F, 3 + i_F, 3 + 2i_F, 3 + 3i_F, 4 + 2i_F, 4 + 3i_F, 4 + 4i_F, 5 + i_F, 5 + 2i_F, 5 + 3i_F\}.$$

(T + R) = 15 and o(T) = 5 and o(R) = 3.
Clearly o(T + R) > o(T) + o(R).

Next we proceed define addition on $S(C(Z_n))$.

**DEFINITION 2.8:** *Let $S(C(Z_n))$ be the MOD finite complex number subsets of $C(Z_n)$.*

*Define operation + on $S(C(Z_n))$ for any A, B $\in S(C(Z_n))$ as follows.*

*A + B = {every element of A is added with every element of B}; $A + B \in S(C(Z_n))$.*

Clearly A + B = B + A, $\{0\} \in S(C(Z_n))$ is such that A + {0} = A for all $A \in S(C(Z_n))$.

$\{S(C(Z_n)), +\}$ is defined as the MOD subset finite complex number semigroup under +.

***Example 2.23:*** Let $\{S(C(Z_{10})), +\}$ be the MOD finite complex number subset semigroup.

For A = {0, 5} and B = {5} we get
A + B = {0, 5} + {5} = {0, 5} = A.
A + A = {0, 5} + {0, 5} = {0, 5}.

So A is a MOD finite complex modulo integer subset idempotent of $S((Z_{10}))$.

We see if $X \in S(C(Z_{10}))$ in general we cannot find a $Y \in S(C(Z_{10}))$ such that X + Y = {0}.

For take X = $\{0, 2, 6, 1 + i_F, 9 + i_F\}$; we have no $Y \in S(C(Z_{10}))$ such that X + Y = {0}.

Let M = $\{5 + 2i_F\}$ and N = $\{1 + 8i_F\}$ $S(C(Z_{10}))$;
M + N = $\{5 + 2i_F\} + \{1 + 8i_F\} = \{6\}$.

***Example 2.24:*** Let $\{S(C(Z_2)), +\}$ be the MOD finite complex number semigroup.

Let X = $\{0, 1 + i_F, i_F\}$ and Y = {0, 1}
X + Y = $\{0, i_F, 1 + i_F, 1\} = C(Z_2)$.

Thus X with Y yields the MOD universal finite complex number subset of $S(C(Z_2))$.

***Example 2.25:*** Let $\{S(C(Z_4)), +\}$ be the MOD universal finite complex number subset semigroup under +.

Let X = $\{0, 2, 1 + i_F, 2 + i_F, 3 + i_F, 3, 1, 1 + 2i_F, 2 + 2i_F, 3 + 2i_F\}$ and Y = $\{0, 1 + 3i_F, 2 + 3i_F, 3 + 3i_F, i_F, 2i_F, 3i_F\} \in S(C(Z_4))$.

Clearly $X + Y = C(Z_4)$.

Thus X + Y yields the MOD finite complex number universal subset of $S(C(Z_4))$.

We give the following result.

**THEOREM 2.5:** *Let $\{S(C(Z_n)), +\}$ be the MOD finite complex number subset semigroup under +.*

*i) There exists a pair {P, Q} such that $P + Q = C(Z_n)$.*
*ii) If $P + Q = C(Z_n)$ then $P \cap Q = \{0\}$.*

Proof is direct and hence left as an exerise to the reader.

We suggest the following problem.

**Problem 2.6:** Let $S(C(Z_n))$ be the MOD finite complex number subset semigroup under +.

i) Is it mandatory if {P, Q} is a MOD universal subset pair then $P \cap Q = \{0\}$?
ii) Can we find the number of such pairs?
iii) Is it possible for {P, Q} and {P, R} to be MOD universal subset pair where $R \cap Q \neq Q$ or R?

We will give some examples where we find sum of P + P; $P \in S(C(Z_n))$.

***Example 2.26:*** Let $\{S(C(Z_{13}), +\}$ be the MOD subset finite complex number semigroup under +.

Let P = $\{0, 1, 2, 3 + i_F, 5\} \in S(C(Z_{12}))$.

We find P + P = $\{0, 1, 2, 3 + i_F, 5\} + \{0, 1, 2, 3 + i_F, 5\} = \{0, 1, 2, 5, 3 + i_F, 3, 4 + i_F, 6, 4, 5 + i_F, 7, 6 + 2i_F, 8 + i_F, 10\}$ = P + P.

We see P + P ≠ P and o(P) = 5 and o(2P) = 14.

We find P + P + P = $\{0, 1, 2, 3, 4, 5, 6, 7, 10, 3 + i_F, 4 + 4i_F, 5 + i_F, 6 + i_F, 8 + i_F, 8, 11, 7 + i_F, 9 + i_F, 9, 10 + i_F, 1 + i_F, 6 + 2i_F, 7 + 2i_F, 8 + 2i_F, 9 + 2i_F, 11 + 2i_F, 11 + i_F, 2 + i_F\}$ = P + P + P; o(P + P + P) = 28 and so on.

The reader is left with the task of finding m such that $\underbrace{P + P + \ldots + P}_{m-\text{times}} = C(Z_{12})$.

So finding $\underbrace{P + P + \ldots + P}_{m-\text{times}} = C(Z_n)$ for m > 1 and P ∈ $S(C(Z_n))$ happens to be challenging and a difficult problem. However one is sure there are P ∈ $S(C(Z_n))$ such that $\underbrace{P + P + \ldots + P}_{m-\text{times}} = C(Z_n)$; but one is not in a position to know the number of such MOD finite complex number subsets P in $S(C(Z_n))$ whose sum result in the MOD finite complex number universal subset $C(Z_n)$.

**Problem 2.7:** Let $\{S(C(Z_n), +\}$ be the MOD subset finite complex number semigroup.

i) Find all P ∈ $S(C(Z_n)$ such that P + P + … + P, m times = $C(Z_n)$; m > 1.
ii) Find all MOD finite complex number subset universal pairs; {P, Q} such that P + Q = $C(Z_n)$; P, Q ∈ $S(C(Z_n) \setminus C(Z_n)$.

One can find MOD subset subsemigroups, MOD subset idempotents.

All these work is considered as a matter of routine so left as an exercise to the reader.

Next we proceed onto describe the MOD subset neutrosophic set;

$S(\langle Z_n \cup I\rangle) = \{$collection of all subsets of $\langle Z_n \cup I\rangle\}$.

***Example 2.27:*** Let $S(\langle Z_5 \cup I\rangle) = \{($collection of all subsets of $\langle Z_5 \cup I\rangle\}$ be the MOD subset neutrosophic set.

Let A = {1 + I, 3 + I, 4I, 3, 2 + I} and B = {3, 2 + 4I, 4I, 2I, I} be elements of $S(\langle Z_5 \cup I\rangle)$.

We can define the operation + on $S(\langle Z_5 \cup I\rangle)$.

Let P = {3, 2I + 1, 4I + 3, 2I} and
Q = {2, 1 + 4I, 3I, 2 + 2I} $\in S(\langle Z_5 \cup I\rangle)$.

P + Q = {3, 2I + 1, 4I + 3, 2I} + {2, 1 + 4I, 3I, 2 + 2I}
= {0, 2I + 3, 4I, 2I + 2, 4 + 4I, I, 2 + I, 3I + 4, 1 + I, 2 + 4I, 3 + 3I, 1, 3 + 2I, 2I, 4I + 3} is in $S(\langle Z_5 \cup I\rangle)$.

Clearly o(P) = 4 and o(Q) = 4, o(P + Q) = 15.

Let us now find
P + P = {3, 2I + 1, 4I + 3,2I} + {3, 2I + 1, 4I + 3, 2I}
= {1, 2I + 4, 4I + 1, 2I + 3, 2I + 4, 4I + 2, I + 4, 3I + 1, I + 3, 4I} $\in S(\langle Z_5 \cup I\rangle)$.

The following observation is mandatory.

o(P) = 4 and o(P + P) = 10.

Clearly P + P ≠ 2P; for 2P = 2{(3, 2I + 1, 4I + 3, 2I)}
= {2 × 3, 2(2I + 1), 2(4I + 3), 2 × 2I}

$= \{1, 4I + 2, 3I + 1, 4I\} \neq P + P$.

That is why we do not write P + P + P = 3P or more generally $\underbrace{P + P + \ldots + P}_{m-times} \neq mP$.

However mP is differently obtained. That is why we cannot count as in case of usual algebraic structures.

Let $L = \{2, 3, 0, 1 + I, 4I + 2\} \in S(\langle Z_5 \cup I\rangle)$.

$$\begin{aligned} L + L &= \{0, 2, 3, 1 + I, 4I + 2\} + \{0, 2, 3, 1 + I, 4I + 2\} \\ &= \{0, 2, 3, 1 + I, 4I + 2, 4, 3 + I, 4 + I, 1, 4I, 2 + 2I, \\ &\quad 4I + 4, 4I + 1, 3I + 4\} \neq 2L. \end{aligned}$$

Further $L \subseteq L + L$. Infact $L \subseteq L + L$ if and only if $0 \in L$.

Let $W = \{1, 2 + 3I, 4I, I\}$ be in $S(\langle Z_5 \cup I\rangle)$.

$$\begin{aligned} W + W &= \{1, I, 2 + 3I, 4I\} + \{1, I, 2 + 3I, 4I\} \\ &= \{2, 1 + I, 3 + 3I, 1 + 4I, 2I, 2 + 4I, 0, 3 + 4I, \\ &\quad 4 + I, 2 + 2I, 3I\}. \end{aligned}$$

Clearly $W \not\subset W + W$ as a subset.

Hence in general if $V \in S(\langle Z_5 \cup I\rangle)$ then $V \not\subseteq V + V$.

Let $R = \{1, 2, 4I + 1, 3I\}$ and
$S = \{1, 2, 4I + 1, 3I, 1 + I, 4, 2 + 3I\} \in S(\langle Z_5 \cup I\rangle)$.

We find

$$\begin{aligned} R + S &= \{1, 2, 1 + 4I, 3I\} + \{1, 2, 1 + 4I, 3I, 4, 1 + I, 2 + 3I\} \\ &= \{2, 3, 2 + 4I, 1 + 3I, 4, 3 + 4I, 2 + 3I, 1 + 2I, 3I + 2, \\ &\quad I, 0, 1, 4I, 4 + 3I, 2 + I, 3 + I, 1 + 4I, 3 + 3I, \\ &\quad 3 + 2I\}. \end{aligned}$$

However $R \not\subset R + S$, $S \not\subset R + S$ but $R \subseteq S$.

There are several of the properties enjoyed by the MOD subset neutrosophic set under + which is not a property of usual modulo integers or reals or complex or neutrosophic numbers, but true in case of MOD subsets of $S(Z_n)$) and MOD finite complex number subsets of $S(C(Z_n))$.

***Example 2.28:*** Let $S(\langle Z_{12} \cup I\rangle)$ be the collection of all MOD subsets of neutrosophic integers. Let M = {6 + 6I, 3, 3I, 10I, 4I} $\in S(\langle Z_{12} \cup I\rangle)$.

We find

$$M + M = \{6 + 6I, 3, 3I, 10I, 4I\} + \{6 + 6I, 3, 3I, 10I, 4I\}$$
$$= \{0, 9 + 6I, 6 + 9I, 6 + 4I, 6 + 10I, 6, 3 + 3I, 2I, 10I + 3, 4I + 3, 6I, I, 7I, 3 + 10I, 8I\}.$$

o(M) = 5 but order M + M = 15.

Let V = {6, 6I} and W = {6I}
W + V = {6 + 6I}.
Let X = {6 + 6I} and Y = {6, 6I}
X + Y = {6, 6I}.

Clearly {6 + 6I} = W $\in S(\langle Z_5 \cup I\rangle)$ is such that
W + W = {6 + 6I} + {6 + 6I} = {0}.

But 0 + V = V for all V $\in S(\langle Z_{12} \cup I\rangle)$.

Thus $\{S(\langle Z_{12} \cup I\rangle), +\}$ is a MOD neutrosophic subset monoid which is commutative and is of finite order.

We see there are elements in $S(\langle Z_{12} \cup I\rangle)$ such that the pair {P, Q} forms the MOD neutrosophic universal subset pair.

That is $P + Q = \langle Z_{12} \cup I\rangle$.

Also there are MOD neutrosophic subsets $P \in S(\langle Z_n \cup I\rangle)$ such that $\underbrace{P + P + \ldots + P}_{m-\text{times}} = \langle Z_n \cup I\rangle$, m > 1.

All these study is considered as a matter of routine so is left as an exercise to the reader.

We also see several properties derived in case of $S(Z_n)$ and $S(C(Z_n))$, can also be derived in case of $S(\langle Z_n \cup I\rangle)$.

Next we proceed onto describe by examples the operation of product on the MOD subsets of $S(C(Z_n))$ and $S(\langle Z_n \cup I\rangle)$.

***Example 2.29:*** Let $S(C(Z_8))$ be the collection of all MOD finite complex number subsets of $C(Z_8)$.

Let $A = \{3, 4 + 6i_F, 2, 6\}$ and
$\{5i_F, 2 + 4i_F, 7\} = B \in S(C(Z_8))$.

We find $A \times B = \{6, 2, 3, 4 + 6i\} \times \{7, 5i_F, 2 + 4i_F\}$
$= \{2, 6, 5, 4 + 2i_F, 6i_F, 2i_F, 7i_F, 4i_F + 2, 4, 6 + 4i_F, 4i_F\} \in S(C(Z_8))$.

$o(A) = 4$, $o(B) = 3$, $o(A \times B) = 11$.

$A \times A = \{6, 2, 3, 4 + 6i_F\} \times \{6, 2, 3, 4 + 6i_F\}$
$= \{4, 2, 6, 1, 4 + 2i_F, 4i_F, 4 + 2i_F\}$
$= o(A) = 4$ but $o(A \times A) = 7$.

This is the way product operation is performed on $S(C(Z_8))$.

Let $M = \{4i_F + 4, 2i_F, 6i_F + 4\}$ and
$N = \{4 + 4i_F, 4, 4i_F\} \in S(C(Z_8))$.

$M \times N = \{2i_F, 4+4i_F, 6i_F+4\} \times \{4 + 4i_F, 4, 4i_F\}$
$= \{0\}$.

Thus the MOD finite complex number subset pair {M, N} yields the MOD zero divisor subsets.

Let $B = \{4, 4 + 4i_F, 4i_F\} \in S(C(Z_8))$;
$B \times B = \{4, 4i_F, 4 + 4i_F\} \times \{4, 4i_F, 4 + 4i_F\} = \{0\}$.

Thus B is the MOD finite complex number nilpotent subset of $S(C(Z_8))$.

Infact it is left as an exercise for the reader to verify that $S(C(Z_8))$ is a finite commutative monoid.

$\{1\}$ in $S(C(Z_8))$ is such that $A \times \{1\} = A$ for all $A \in S(C(Z_8))$ serves as the MOD subset identity of $S(C(Z_8))$ under product.

Let $D = \{2, 2i_F, 4 + 4i_F, 4\} \in S(C(Z_8))$.

$D \times D = \{2, 2i_F, 4 + 4i_F, 4\} \times \{2, 2i_F, 4 + 4i_F, 4\}$

$= \{4, 4i_F\};$

$D \times D \times D = \{4, 4i_F\} \times \{2, 2i_F, 4 + 4i_F + 4\} = \{0\}.$

Thus D is the MOD finite complex number nilpotent subset of $S(C(Z_8))$.

However we see $S(C(Z_8))$ cannot have idempotents.

Thus $S(C(Z_8))$ has MOD finite complex number zero divisor subset pair, MOD finite complex number nilpotent subsets. However finding MOD finite complex number universal subset pair in $S(C(Z_8))$ happens to be challenging one as $i_F^2 = 7$ in $C(Z_8)$.

The reader is advised to work in this direction. Further find P in $S(C(Z_8))$ so that $P^m = C(Z_8)$; $m > 1$.

***Example 2.30:*** Let $\{S(C(Z_3)), \times\}$ be the MOD finite complex number subset semigroup under product where;

$C(Z_3) = \{0, 1, 2, i_F, 2i_F, 1 + i_F, 1 + 2i_F, 2 + i_F, 2 + 2i_F\}.$

Take $A = \{0, 1, 2, i_F, 2 + i_F\}$ and $B = \{1, 1 + i_F, i_F, 1 + 2i_F\} \in S(C(Z_8))$.

$A \times B = \{0, 1, 2, i_F, 2 + i_F\} \times \{1, i_F, 1 + i_F, 1 + 2i_F\}$

$= \{0, 1, 2, i_F, 2 + i_F, 1 + i_F, 1 + 2i_F, 2 + 2i_F, 2i_F\}$

$= S(C(Z_3))$.

Thus {A, B} is the MOD finite complex number subset universal pair of $S(C(Z_3))$.

Take $A = \{0, 1, 1 + i_F, i_F\} \in S(C(Z_3))$.
$A \times A = \{0, 1, i_F, 1 + i_F\} \times \{0, 1, i_F, 1 + i_F\}$
$= \{0, 1, i_F, 1 + i_F, 2, i_F + 2\} = A^2$

$A^2 \times A = \{0, 1, 2, i_F, 1 + i_F, 2 + i_F\} \times \{0, 1, i_F, 1 + i_F\}$
$= \{0, 1, 2, i_F, 1 + i_F, 2 + i_F, 2i_F, 2 + i_F, 2 + 2i_F, 1+2i_F\} = C(Z_3)$.

Thus $A \in S(C(Z_3))$ is such that $A^3 = C(Z_3)$.

Thus A is a MOD finite complex subset which gives the MOD universal subset for m = 3.

***Example 2.31:*** Let $\{S(C(Z_6)), \times\}$ be the MOD finite complex number subset semigroup under product.

Let $A = \{0, 3, 3 + 3i_F, 3i_F\}$ and
$B = \{0, 2, 2 + 2i_F, 4 + 2i_F, 2 + 4i_F, 4i_F\} \in S(C(Z_6))$.

$A \times B = \{0, 3, 3 + 3i_F, 3i_F\} \times \{0, 2, 4i_F, 2 + 2i_F, 4 + 2i_F, 2 + 4i_F\} = \{0\}$.

So the pair {A, B} is the MOD finite complex number subset zero divisor of $S(C(Z_6))$.

Further $A \times A = \{0, 3, 3 + 3i_F, 3i_F\} \times \{0, 3, 3 + 3i_F, 3i_F\}$
$= \{0, 3, 3 + 3i_F, 3i_F\} = A$.

Thus $A \in S(C(Z_6))$ is such that $A \times A = A$ so A is the MOD finite complex number subset idempotent of $S(C(Z_6))$.

Let $D = \{0, 4, 2 + 4i_F, 2, 4 + 2i_F, 4i_F, 2i_F, 2 + 2i_F, 4 + 4i_F\} \in S(C(Z_6))$.

D × D = {0, 4, 2 + $4i_F$, 4 + $2i_F$, $4i_F$ 2, $2i_F$, 2 + $2i_F$, 4 + $4i_F$} × {0, 4, 2, $4i_F$, 2 + $2i_F$, 4 + $4i_F$, 2 + $4i_F$, 4 + $2i_F$, $2i_F$} = D so D is again a MOD finite complex number of S(C($Z_6$)).

Several properties in this direction can be obtained and it is considered as a matter of routine and is left as an exercise to the reader.

Next a few examples of MOD neutrosophic subset semigroup under × is defined.

***Example 2.32:*** Let {S(C⟨$Z_{10}$ ∪ I⟩), ×} be the MOD neutrosophic subset semigroup under product ×.

Let A = {3I +5, 2I + 4, 6I + 2, 5I, 2I, 3} and
B = {0, I + 1, 4, 6I, 2I} ∈ S (⟨$Z_{10}$ ∪ I⟩).

A × B = {3I + 5, 2I + 4, 6I + 2, 5I, 2I, 3} × {0, 1+ I, 4, 1I, 2I}
= {0, 6I, 2I, 4I, I + 5, 8I + 4, 4I + 2, 3 + 3I, 6 + 2I, 8 + 4I, 8I, 2, 6I}.

This is the way product is performed on S(C⟨$Z_{10}$ ∪ I⟩).

Let A = {5 + 5I, 5I, 5} and
B = {2, 2I, 4I + 2, 6I + 4, 4I + 6, 4I + 4, 2 + 6I} ∈ S(C⟨$Z_{10}$ ∪ I⟩).

Clearly A × B = {0}.
A × A = {5 + 5I, 5I, 5} × {5 + 5I, 5I, 5}
= {5I, 5, 5 + 5I} = A.

Thus A is a MOD neutrosophic subset idempotent of S(C⟨$Z_{10}$ ∪ I⟩).

Let B = {6, 6I} ∈ S(C⟨$Z_{10}$ ∪ I⟩) is such that
B × B = {6, 6I} = B is again a MOD neutrosophic subset idempotent.

Let {5, 2 + 5I, 5I + 4} = A and
B = {0, 6I, 2I} $\in$ S($\langle Z_{10} \cup I\rangle$)

A × B = {5, 2 + 5I, 5I + 4} × {0, 6I, 2I}
= {0, 2I, 4I, 8I} $\in$ S($\langle Z_{10} \cup I\rangle$).

Let M = {0, 6 + 8I, 8 + 4I, 6I, 8I, 4, 8} and N = {5}.

M × N = {0}.

Thus the {M, N} MOD neutrosophic subset pair is a zero divisor subset of S($\langle Z_{10} \cup I\rangle$).

Let W = {0, 2, 4, 6, 8} $\in$ S($\langle Z_{10} \cup I\rangle$)

W × W = {0, 2, 4, 6, 8} × {0, 2, 4, 6, 8}
= {0, 2, 4, 6, 8} = W.

This is also a MOD neutrosophic idempotent subset of S($\langle Z_{10} \cup I\rangle$).

Let V = {0, 2I, 4I, 6I, 8I} $\in$ S($\langle Z_{10} \cup I\rangle$);
V × V = {0, 2I, 4I, 6I, 8I} × {0, 2I, 4I, 6I, 8I}
= {0, 2I, 4I, 6I, 8I} = V.

This is also a MOD neutrosophic idempotent subset of S($\langle Z_{10} \cup I\rangle$).

Let S = {collection of all subsets from the set {0, 2, 4, 8, 6}} $\subseteq$ S($\langle Z_{10} \cup I\rangle$).

Clearly S is a MOD neutrosophic subset subsemigroup of the MOD neutrosophic subset semigroup S($\langle Z_{10} \cup I\rangle$).

B = {collection of all subsets from the set {0, 2I, 4I, 8I, 6I}} be the MOD neutrosophic subset subsemigroup of the MOD neutrosophic subset semigroup S($\langle Z_{10} \cup I\rangle$).

We see B is an MOD neutrosophic subset ideal of $S(\langle Z_{10} \cup I\rangle)$ but S is not a MOD neutrosophic subset ideal of $S(\langle Z_{10} \cup I\rangle)$.

Consider F = {collection of all subsets from the set $Z_{10}$}.

F is only a MOD neutrosophic subset subsemigroup and not an ideal of $S(\langle Z_{10} \cup I\rangle)$.

Let P = {collection of all subsets from the set $Z_{10}I$}; P is a MOD neutrosophic subset subsemigroup and also a MOD neutrosophic subset ideal of $S(\langle Z_{10} \cup I\rangle)$.

In view of all these we have the following theorem.

**THEOREM 2.6:** *Let $\{S\langle Z_n \cup I\rangle), \times\}$ be the MOD neutrosophic subset semigroup under ×.*

*i) $\{S(\langle Z_n \cup I\rangle), \times\}$ has MOD neutrosophic subsemigroup which are not ideals.*
*ii) $\{S\langle Z_n \cup I\rangle, \times\}$ has MOD neutrosophic subsemigroup which is also ideal.*

Proof is direct and hence left as an exercise to the reader.

Now consider the MOD dual number subset semigroup. $S(\langle Z_n \cup g\rangle)$ = {collection of all subsets from the MOD dual number subset}; we will describe this by an example.

***Example 2.33:*** Let
$S(\langle Z_7 \cup g\rangle)$ = {collection of all MOD subset dual numbers}.

Let A = {g, 0, 2 + 5g, 1 + g} and
B = {4g, 3, 4 + 2g} $\in S(\langle Z_7 \cup I\rangle)$.

$$A \times B = \{0, 5, 1 + g, 2 + 5g\} \times \{3, 4g, 4 + 2g\}$$
$$= \{0, 3g, 6g, 4g, g, 6g + 4, 1 + 3g\}.$$

$M = \{2g, 0, 4g, 6g, 5g, g\} \in S(\langle Z_7 \cup g\rangle)$.
$M \times M = \{2g, 0, 4g, 6g, 5g, g\} \times \{2g, 0, 4g, g, 5g, 6g\}$
$= \{0\}$.

Thus M is a MOD dual number subset nilpotent of order two.

Let $B = \{1 + g, 3 + 2g, 4g + 5\} \in S(\langle Z_7 \cup g\rangle)$.

$B \times B = \{1 + g, 3 + 2g, 4g + 5\} \times \{1 + g, 3 + 2g, 4g + 5\}$
$= \{1 + 2g, 5 + 2g, 3 + 5g, 2 + 5g, 1 + g, 4 + 5g\}$
$\neq \{0\}$.

Clearly B = {collection of all MOD dual number subset from $Z_7g\} \subseteq S(\langle Z_7 \cup g\rangle)$ is a MOD dual number subsemigroup of B which is also an ideal.

C = {collection of all MOD subset from $Z_7\} \subseteq S(\langle Z_7 \cup g\rangle)$ is a MOD subset subsemigroup which is not an ideal.

So finding ideals other than B happens to be a very challenging one.

***Example 2.34:*** Let $\{S(\langle Z_{12} \cup g\rangle)\}$ be the MOD dual number subset semigroup.

We can as in case of $S(Z_n)$ $S(C(Z_n))$, $S(\langle Z_n \cup I\rangle)$ define multiplication.

Let $B = \{0, 1 + g, 4g, 8, 1 + 5g\}$ and
$C = \{6g, 4 + 6g, g + 3\} \in S(\langle Z_{12} \cup g\rangle)$

$B \times C = \{0, 4g, 8, 1 + g, 1 + 5g\} \times \{6g. g + 3, 4 + 6g\}$
$= \{0, 6g, 8g, 3 + 4g, 4 + 11g, 4g, 8, 4 + 10g, 4 + 11g\}$.

We now show how the operation of addition is performed on $S(\langle Z_{12} \cup g\rangle)$.

B + C = {0, 1 + g, 4g, 8, 1 + 5g} + {6g, 4 + 6g, g + 3} = {6g, 1 + 7g, 10g, 8 + 6g, 1 + 11g, 4 + 6g, 5 + 7g, 4 + 10g, 5 + 11g, g + 3, 4 + 2g, 5g + 3, 11 + g, 4 + 6g}.

We see o(B) = 6, o(C) = 3, o(B × C) = 8 whereas
o(B + C) = 14.

B + B = {0, 1 + g, 4g, 8, 1 + 5g} + {0, 1 + g, 4g, 8, 1 + 5g}
= {0, 1 + g, 4g, 8, 1 + 5g, 2 + 2g, 1 + 5g, 9 + 8,
2 + 6g, 8g, 8 + 4g, 1 + 9g, 4, 9 + 5g, 2 + 10g}.

o(B) = 5 but o(B + B) = 15.

B × B = {0, 1 + g, 4g, 8, 1 + 5g} × {0, 1 + g, 4g, 8, 1 + 5g}
= {0, 1 + 2g, 4g, 8 + 8g, 1 + 6g, 8, 8g, 4, 8 + 4g,
1 + 10g }.

o(B × B) = 10.

Consider D = {g, 6g, 0, 4g, 8g, 5g} and
E = {7g, 11g, 10g, 9g} ∈ S(⟨$Z_{12}$ ∪ g⟩).

D × E = {0, 5, 6g, 4g, 8g, 5g} × {7g, 11g, 10g, 9g} = {0}.

o(D) = 6, o(E) = 4, o(D × E) = 1.

D + E = {0, g, 6g, 4g, 8g, 5g} + {7g, 10g, 11g, 9g}
= {0, 7g, 8g, g, 11g, 3g, 10g, 4g, 5g, 9g, 2g}

o(D) = 6, o(E) = 4 but o(D + E) = 11.

Let M = {0, 2, 4, 6, 8, 10} ∈ S(⟨$Z_{12}$ ∪ g⟩).

M + M = {0, 2, 4, 6, 8, 10} + {0, 2, 4, 6, 8, 10}
= {0, 2, 4, 6, 8, 10} = M.

M × M = {0, 2, 4, 6, 8 10} × {0, 2, 4, 6, 8, 10}
= {0, 4, 8} ≠ M.

Thus M under + is a MOD dual number idempotent subset whereas M under × is not an MOD dual number idempotent subset.

Let G = {0, 6, 6g} ∈ S(⟨$Z_{12}$ ∪ I⟩)
G × G = {0, 6, 6g} × {0, 6, 6g} = {0}.

Thus G is a MOD dual number nilpotent subset of S(⟨$Z_{12}$ ∪ g⟩) of order two.

Infact S(⟨$Z_{12}$ ∪ g⟩) has several MOD subset dual number nilpotents of order two.

Finding properties about MOD dual number subset semigroups under + and × is left as an exercise as it is a matter of routine.

Next we proceed to describe MOD special dual like number subsets by some examples.

***Example 2.35:*** Let S(⟨$Z_{15}$ ∪ h⟩) be the MOD special dual like number subset.

Let A = {5, 5 + 3h, h, 2} and
B = {3, 0, 1, 12h} ∈ S(⟨$Z_{15}$ ∪ h⟩).

We can define the operation of addition on A.

A × A = {5, 5 + 3h, h, 2 } × {5, 5 + 3h, h, 2}
= {10, 5h, 5 + 9h, 8h, 10 + 6h, h, 2h, 4}.

o(A) = 4, o(A × A) = 7.

A + A = {2, h, 5, 5 + 3h} + {2, h, 5, 5 + 3h}
= {4, 2 + h, 7, 7 + 3h, 2h, 5 + h, 5 + 4h, 10, 10 + 3h, 10 = 6h}.

o(A + A) = 10.

Let B = {3h, h, 4h, 8h} and
D = {6h, 7h, 5h, 10h} $\in$ S($\langle Z_{15} \cup h\rangle$).

B $\times$ D = {h, 3h, 4h, 8h} $\times$ {6h, 7h, 5h, 10h}
= {6h, 3h, 9h, 7h, 13h, 11h, 5h, 0, 10h}.

o(B) = 4, o(D) = 4, o(B $\times$ D) = 9.

o(B + D) = {h, 3h, 4h, 8h} + {6h, 7h, 5h, 10h}
= {7h, 9h, 10h, 14h, 8h, 11h, 0, 6h, 13h, 3h}.

o(B + D) = 10.

This is the way operations of + and $\times$ are defined on S($\langle Z_{15} \cup h\rangle$).

Let M = {0, 5h, 10h} $\in$ S($\langle Z_{15} \cup h\rangle$).

M + M = {0, 5h, 10h} + {0, 5h, 10h} = {0, 5h, 10h} = M.

Thus the MOD special dual like number is an idempotent subset under +.

M $\times$ M = {0, 5h, 10h} $\times$ {0, 5h, 10h} = {0, 10h, 5h} = M.

Thus M is again a MOD special dual like number idempotent subset under $\times$.

Clearly M + M $\neq$ 2M but M + M = M.

P = {0, 3h, 6h, 9h, 12h} $\in$ S($\langle Z_{13} \cup h\rangle$).

P + P = {0, 3h, 6h, 0h, 12h} + {0, 3h, 6h, 9h, 12h}
= {0, 9h, 3h, 12h, 6h} = P.

Clearly P is again a MOD special dual like number idempotent subset under +.

$P \times P = \{0, 3h, 6h, 9h, 12h\} \times \{0, 9h, 3h, 6h, 12h\} = P.$

Again P is a MOD special dual like number subset idempotent under $\times$.

***Example 2.36:*** Let $\{S(\langle Z_{11} \cup h\rangle), +\}$ be the MOD special dual like number subset semigroup under +.

Let $A = \{3h, 2 + h, 6h, 0\}$ and
$B = \{8h, 9+10h, 5h, 2\} \in S(\langle Z_{11} \cup h\rangle)$.

We find

$$A + B = \{3h, 2 + h, 6h, 0\} + \{2, 5h, 10h + 9, 8h\}$$
$$= \{3h + 2, 4 + h, 2 + 6h, 2, 8h, 2 + 6h, 0, 5h, 2h + 9, 4h + 9, 10h + 9\}.$$

This is the way + operation is performed on $S(\langle Z_{11} \cup h\rangle)$.

Let $A = \{3h + 7\}$ and $B = \{8h + 4\}$ in $S(\langle Z_{11} \cup h\rangle)$ is such that $A + B = \{3h + 7\} + \{8h + 4\} = \{0\}$.

Let

$$B = \{0, h, 2h, 3h, 4h, 5h, 6h, 7h, 8h, 9h, 10h\} \in S(\langle Z_{11} \cup h\rangle).$$

Clearly B + B = B; so B is a MOD subset special dual like number idempotent of $S(\langle Z_{11} \cup h\rangle)$.

Infact $\langle B\rangle$ is a MOD subset subsemigroup under + of order one.

Next we proceed on to describe the operation of product on MOD subset special dual like number set $S(\langle Z_n \cup h\rangle)$.

***Example 2.37:*** Let $\{S(\langle Z_{20} \cup h\rangle), \times\}$ be the MOD special dual like number subset semigroup under product.

Let A = {5, 0, 10h, 15h, 10 + 5h, 5h + 15, 10h +10} and

B = {4, 4h + 8, 8h, 8h + 4, 4 + 12h, 12h + 8, 12, 12 + 4h} $\in$ S(⟨$Z_{20} \cup$ h⟩).

A × B = {0}.

Thus the MOD special dual like number pair subset is a MOD zero divisor subset pair.

Consider M = {5h, 0, 5, 16h, 16} $\in$ S(⟨$Z_{20} \cup$ h⟩).

M × M = {0, 5, 5h, 16h, 16} × {0, 5, 5h, 16, 16h}
= {0, 5, 5h, 16, 16h} = M.

Hence M is the MOD special dual like number idempotent subset of S(⟨$Z_{20} \cup$ I⟩).

Let L = {10, 10h, 10 + 10h, 0} $\in$ S(⟨$Z_{20} \cup$ h⟩).

Clearly L × L = {0, 10, 10h, 10 + 10h} × {10, 0, 10h, 10 + 10h} = {0}, so L is the MOD special dual like number nilpotent subset of order two in S(⟨$Z_{20} \cup$ h⟩).

Let B = {Collection of all subsets from the $Z_{20}$} $\subseteq$ S(⟨$Z_{20} \cup$ h⟩), B is a MOD special dual like number subset subsemigroup of S(⟨$Z_{20} \cup$ h⟩) and is not an ideal.

Consider C = {Collection of all subsets from the set $Z_{20}$h} $\subseteq$ S(⟨$Z_{20} \cup$ h⟩).

We see C is a MOD special dual like number subset subsemigroup which is also an ideal of S(⟨$Z_{20} \cup$ h⟩).

Finding MOD subset ideals. MOD subset subsemigroups happens to be a matter of routine and hence left as an exercise to the reader.

Next we briefly describe the MOD subset special quasi dual number set by some examples.

***Example 2.38:*** Let $S(\langle Z_{12} \cup k\rangle)$ = {collection of all MOD subsets from the set $\langle Z_{12} \cup k\rangle$} be the MOD subset special quasi dual number set.

Define addition on $S(\langle Z_{20} \cup k\rangle)$ as follows:

Let A = {10k, k, 0, 5 + 2k, 9} and
B = {2k, 5k, 2 + 8k, 3} $\in S(\langle Z_{12} \cup k\rangle)$.

A + B = {10k, k, 0, 5, + 2k, 9} + {2k, 5k, 2 + 8k, 3}
= {0, 3k, 2k, 5 + 4k, 6k, 5k, 5 + 7k, 2 + 6k, 2 + 9k, 2 + 8k, 7 +10k, 3 + 10k, 3 + k, 3, 8 + 2k, 9 + 2k, 9 + 5k, 11 + 8k}.

o(A) = 5, o(B) = 4 and o(A + B) = 18.

A + A = {10k, k, 0, 5 + 2k, 9} + {0, 10k, k, 5 + 2k, 9}
= {10k, 0, k, 5 + 2k, 9, 8k, 11k, 5, 9 + 10k, 2k, 3k + 5, 9 + k, 5 + 3k, 10 + 4k, 2 + 2k, 6}.

o(A) = 5 but o(A + A) = 16.

Clearly A + A ≠ 2A.

Let A = {0, 2, 4, 6, 8, 10} $\in S(\langle Z_{12} \cup k\rangle)$.
A + A = {0, 2, 4, 6, 8, 10} + {0, 2, 4, 6, 10, 8}
= {0, 2, 4, 6, 8, 10} = A.

Thus A is a MOD special quasi dual number idempotent subset of $S(\langle Z_{12} \cup k\rangle$ under +.

Let B = {0, 3k, 6k, 9k} $\in S\{\langle Z_{12} \cup k\rangle)$.
B + B = {0, 3k, 6k, 9k} + {0, 3k, 6k, 9k} = B.

Thus B is again a MOD special quasi dual number idempotent subset of $S(\langle Z_{12} \cup k\rangle)$.

We see $S(\langle Z_{12} \cup k\rangle)$ has MOD subsets such that A + A = A.

We see if A = {4, 4 + 8k, 4k, 8k, 10k + 10, 4k + 4k, 8 + 8k} and B = {6, 6k, 6 + 6k} $\in S(\langle Z_{12} \cup k\rangle)$ then

A + B = {4, 4 + 8k, 4k, 8k, 10 + 10k, 4 + 4k, 8 + 8k} + {6, 6k, 6 + 6k}

= {10, 10 + 8k, 6 + 4k, 6 + 8k, 4 + 10k, 10 + 4k, 2 + 8k, 4 + 6k, 4 + 2k, 10k, 2k, 10 + 4l. 4 + 10k, 8 + 2k, 10 + 6k, 10 + 2k, 6 + 10k, 6 + 2k, 4 + 4k, 10 + 10k, 2 + 2k}. o(A) = 7,

o(B) = 3 but o(A + B) = 21.

B + B = {6, 6k, 6 + 6k} + {6, 6k, 6 + 6k}
= {0, 6 + 6k, 6k, 6} $\neq$ B.

If T = B $\cup$ {0} = {6, 6k, 0, 6 + 6k} then T + T = T so T is again a MOD special quasi dual number subset idempotent under + of $S(\langle Z_{12} \cup k\rangle)$.

Thus $\{S\langle Z_{12} \cup k\rangle), +\}$ is only a MOD special quasi dual number subset semigroup of finite order infact a finite commutative monoid.

***Example 2.39:*** Let $S(\langle Z_6 \cup k\rangle)$ = {collection of all MOD special quasi dual number subsets} be the MOD special quasi dual number subset.

Let A = {3 + 3k, 3k, 3, 0}.

We find A × A = {3 + 3k, 3k, 3, 0} × {3 + 3k, 3k, 3, 0}
= {0, 3, 3k, 3 + 3k} = A.

Thus we get A to be a MOD special quasi dual number idempotent subset of $S(\langle Z_6 \cup k\rangle)$.

Let T = {2, 2k, 4k, 2 + 4k, 2k + 4, 4 + 4k, 0} and
S = {3, 3k, 3 + 3k, 0} $\in S(\langle Z_6 \cup k\rangle)$}.

T × S = {2, 2k, 4k, 2 + 4k, 2k + 4, 4 + 4k, 0} ×
{3, 3k, 3 + 3k, 0}
= {0}.

Thus we see the MOD special quasi dual number subset pair of zero divisors of $S(\langle Z_6 \cup k\rangle)$.

However we do not find any MOD special quasi dual number subsets which are nilpotents in $S(\langle Z_6 \cup k\rangle)$.

So the following problem is thrown open.

**Problem 2.8:** Find condition on $\{S\langle Z_n \cup k\rangle, \times\}$ the collection of all MOD special quasi dual number subsets semigroup under product to contain MOD special quasi dual number nilpotents of order m, $m \geq 2$.

We suggest a few problems for the reader.

**Problems:**

1. Let $S(Z_{11})$ be the MOD subset semigroup under +.

   i) Find all subset in $S(Z_{11})$ such that their sum with X = {0, 2, 5, 7, 10} gives $Z_{11}$, the MOD universal subset of $S(Z_{11})$.
   ii) Let Y = {0, 1, 3, 5, 7, 9} $\in S(Z_{11})$; does there exists $P_i$ subsets in $S(Z_{11})$ so that $P_i + Y = Z_{11}$?
   iii) Let X = {0, 2, 4, 6, 8, 10} $\in S(Z_{11})$, find all Y in $S(Z_{11})$ so that $X + Y = Z_{11}$.

iv) Let B = {6, 5, 4, 7, 3, 8} $\in$ $S(Z_{11})$, does there exist sets C in $S(Z_{11})$ so that B + C = $Z_{11}$.

2. Let {$S(Z_{16})$, +} be the MOD subset semigroup.

   i) Let X = {0, 4, 8, 12} $\in$ $S(Z_{16})$, find all subsets P of $S(Z_{16})$ so that X + P = $Z_{16}$.
   ii) Find all R $\in$ S ($Z_{16}$) such that R + R = $Z_{16}$; R $\neq$ $Z_{16}$ or R $\neq$ {0}.
   iii) Find all MOD subsemigroups of $S(Z_{16})$.
   iv) Find all MOD subsemigroups of $S(Z_{16})$ which are MOD subgroups.
   v) Find all T $\in$ S ($Z_{16}$) such that T + T = T.
   vi) Can we have Y $\in$ S ($Z_{16}$) such that Y + Y = {0}?

3. Let {$S(Z_{24})$, +} be the MOD subset semigroup under +.

   i) Find all MOD subsets P in $S(Z_{24})$ so that P + P = {0}.
   ii) Find all MOD subsets R in $S(Z_{24})$ so that R + R = R.
   iii) Find all MOD subsets T in $S(Z_{24})$ so that $\underbrace{T + T + \ldots + T}_{m-\text{times}} =$ $Z_{24}$; m $\geq$ 2.
   iv) Find all MOD subsets subsemigroups which are not MOD subgroups.
   v) Find all MOD subsets subsemigroups which are MOD subgroups.
   vi) Obtain any other special striking features associated with $S(Z_{24})$.

4. Let {$S(C(Z_{10}))$, +} be the MOD finite complex subset semigroup under +.

   i) Study questions (i) to (vi) of problem 3 for this $S(C(Z_{10}))$.

ii) Obtain any other special feature associated with $S(C(Z_{10}))$.

5. Let $\{S(C(Z_{19})), +\}$ be the MOD finite complex, subset semigroup under +

   i) Study questions (i) to (vi) of problem 3 for this $S(C(Z_{19}))$.
   ii) Compare $S(C(Z_{19}))$ in problem (4) with $S(C(Z_{19}))$.

6. Let $S(C(Z_{24}))$ be the MOD finite complex number subset semigroup under +.

   i) Study questions (i) to (vi) of problem (3) for this $S(C(Z_{24}))$.
   ii) Compare $S(C(Z_{24}))$ with $S(C(Z_{10}))$ in problem 4 and $S((C(Z_{19}))$ in problem 5.

7. Let $\{S(\langle Z_{12} \cup I\rangle), +\}$ be the MOD neutrosophic subset semigroup under +.

   i) Study questions (i) to (vi) of problem (3) for this $S(\langle Z_{12} \cup I\rangle)$.
   ii) Compare the properties of $S(Z_{12})$ and $S(C(Z_{12}))$ with $S(\langle Z_{12} \cup I\rangle)$.

8. Let $\{S(\langle Z_{29} \cup I\rangle), +\}$ be the MOD neutrosophic subset semigroup.

   i) Study questions (i) to (vi) of problem (3) for this $S(\langle Z_{29} \cup I\rangle)$.
   ii) Compare $S(\langle Z_{29} \cup I\rangle)$ with $S(Z_{29})$ and $S(C(Z_{29})$ as semigroups under +.
   iii) Let $P_1 = \{S(\langle Z_{24} \cup I\rangle), +\}$ and $P_2 = \{S(\langle Z_{64} \cup I\rangle), +\}$ be MOD neutrosophic subsets with + operation.

Compare $P_1$ and $P_2$ with $S\{(\langle Z_{29} \cup I\rangle), +\}$.

9. Let $\{S(\langle Z_{18} \cup g\rangle), +\}$ be the MOD subset dual number semigroup under +.

   i) Study questions (i) to (vi) of problem (3) for this $S(\langle Z_{18} \cup g\rangle)$.
   ii) Discuss the special features associated with $S(\langle Z_{18} \cup g\rangle)$.
   iii) Compare $\{S(\langle Z_{18} \cup g\rangle), +\}$ with $\{S(Z_{18}), +\} = P_1$, $P_2 = \{S(C(Z_{18})), +\}$ and $P_3 = \{S(\langle Z_{18} \cup I\rangle), +\}$.

10. Let $\{S(\langle Z_{3^{10}} \cup g\rangle), +\}$ be the MOD subset dual number semigroup under +.

    i) Compare $\{S(\langle Z_{3^{10}} \cup g\rangle), +\}$ with $\{S(\langle Z_{3^{10}} \cup I\rangle), +\}$, $\{S(Z_{3^{10}}), +\}$, $\{S(C(Z_{48}), +\}$ and $\{S(\langle Z_{48} \cup g\rangle), +\}$.
    ii) Study questions (i) to (vi) of problem (3) for this $\{S(\langle Z_{3^{10}} \cup g\rangle), +\}$.

11. Let $M = \{S(\langle Z_{31} \cup h\rangle), +\}$ be the MOD special dual like number subset semigroup under +.

i) Study questions (i) to (vi) of problem (3) for this M.
   ii) Compare M with $N = \{S(\langle Z_{48} \cup h\rangle), +\}$.
   iii) Compare M with $P_1 = \{S(Z_{10}), +\}$, $P_2 = \{S(C(z_{47})), +\}$, $P_3 = \{S(\langle Z_{48} \cup h\rangle), +\}$, $P_4 = \{S(\langle Z_{12} \cup I\rangle), +\}$ and $P_5 = \{S(\langle Z_6 \cup g\rangle), +\}$.

12. Obtain all special and distinct features associated with MOD special dual like number subset semigroup $\{S(\langle Z_n \cup h\rangle), +\}$.

13. Let W = $\{S(\langle Z_{15} \cup k\rangle, +\}$ be the MOD special quasi dual number subset semigroup under +.

i) Study questions (i) to (vi) of problem (3) for this W.
ii) Compare this W with $P_1 = \{S(Z_{15}), +\}$, $P_2 = \{S(C(Z_{15})), +\}$, $\{P_3 = \{S(\langle Z_{15} \cup I\rangle), +\}$, $P_4 = \{S(\langle Z_{15} \cup h\rangle), +\}$ and $P_5 = \{S(\langle Z_{15} \cup g\rangle), +\}$.
iii) Compare W with $B_1 = \{S(\langle Z_{12} \cup k\rangle), +\}$, $B_2 = \{S(\langle Z_{11} \cup k\rangle), +\}$, $B_3 = \{S(\langle Z_{48} \cup k\rangle), +\}$ and $B_4 = \{S(\langle Z_{243} \cup k\rangle), +\}$.

14. Determine any other special feature associated with $\{S(\langle Z_n \cup k\rangle), +\}$ the MOD special quasi dual number subset semigroup.

15. Let V = $\{S(Z_{18}), +\}$ be the MOD subset semigroup under addition. Characterize all MOD subsets of V which contribute to MOD universal set $Z_{18}$.

16. Study problem (15) for W = $\{S(C(Z_{42})), +\}$ and M = $\{S(\langle Z_{40} \cup I\rangle), +\}$.

17. Let W = $\{S(Z_{20}), \times\}$ be the MOD subset semigroup under product ×.

i) Find all MOD subsets P of W such that $P \times Q = Z_{20}$ the MOD universal subset of $S(Z_{20})$.
ii) Find all MOD subsets P, Q such that $P \times Q = \{0\}$.
iii) Can $S(Z_{20})$ have MOD subset nilpotents?
iv) Characterize all MOD zero divisor subset pairs of $S(Z_{20})$.
v) Find all MOD subset idempotents of $S(Z_{20})$.
vi) Find all MOD subset subsemigroups which are MOD subset subgroups of $S(Z_{20})$.

vii) Find all MOD subset subsemigroups which are not MOD subset subgroups of $S(Z_{20})$.
viii) Find all MOD subset ideals of $S(Z_{20})$.
ix) Can a MOD subset ideals be a MOD subset subgroup?

18. Let $S = \{S(Z_{24}), \times\}$ be the MOD subset semigroup under $\times$. Study questions (i) to (ix) of problem (17) for this $S(Z_{48})$.

19. Let $T = \{S(Z_{37}), \times\}$ be the MOD subset semigroup under $\times$.
    i) Study questions (i) to (ix) of problem (17) for this $S(Z_{37})$.
    ii) Compare $S(Z_{37})$ with $S(Z_{48})$ and $S(Z_{20})$ in problems 18 and 17 respectively.

20. Let $Z = \{S(Z_{64}), \times\}$ be the MOD subset semigroup.

    i) Study questions (i) to (ix) of problem (17) for this $S(Z_{64})$.
    ii) Compare $S(_{64})$ with $S(Z_{20})$ of problem 18, $S(Z_{48})$ of problem 17 and $S(Z_{37})$ of problem 19.

21. Let $\{S(C(Z_{10})), \times\} = V$ be MOD finite complex number subset semigroup under $\times$.

    i) Study questions (i) to (ix) of problem (17) for this V.
    ii) Compare V with $P = \{S(Z_{10}), \times\}$.
    iii) Compare V with $S = \{S(Z_{11}), \times\}$.
    iv) Compare V with $M = \{S(Z_{81}), \times\}$.

22. Let $W = \{S(C(Z_{43})), \times\}$ be the MOD finite complex number subset semigroup under $\times$.

    i) Study questions (i) to (ix) of problem (17) for this W.
    ii) Compare W with V of problem (20).

iii) Find those P ∈ W such that there is a Q with P × Q C($Z_{43}$), the MOD universal subset of S(C($Z_{43}$)).

23. Let Z = {S(⟨$Z_{48}$ ∪ I⟩), ×} be the MOD neutrosophic subset semigroup under product.

i) Study questions (i) to (ix) of problem (16) for this Z.
ii) Compare Z with {S(C($Z_{48}$)), ×} and {S($Z_{48}$), ×}.
iii) Enumerate all special features enjoyed by Z.

24. Let M = {S(⟨$Z_{19}$ ∪ I⟩) be the MOD neutrosophic subset semigroup under product.

Study question (i) to (ix) of problem (17) for this M.

25. Let P = {S(⟨$Z_{12}$ ∪ g⟩), ×} be the MOD dual number subset semigroup under product.

i) Study questions (i) to (ix) of problem 17 for this P.
ii) Prove P has more number of MOD subset zero divisors than $S_1$ = {S($Z_{12}$), ×}, $S_2$ = {S(C($Z_{12}$)), ×}, $S_3$ = {S(⟨$Z_{12}$ ∪ I⟩), ×}.
iii) Prove P has more MOD subset ideals than $S_1$, $S_2$ and $S_3$!

26. Let B = {S(⟨$Z_{23}$ ∪ g⟩), ×} be the MOD dual number subset semigroup under ×.

i) Study questions (i) to (ix) of problem (17) for this B.
ii) Obtain all special features associated with MOD dual number subset semigroups under ×.
iii) Compare P in problem (24) with this B.

27. Let E = {S(⟨ $Z_{2^8}$ ∪ g⟩), ×} be the MOD dual number subset semigroup under ×.

    i) Study questions (i) to (ix) of problem (17) for this E.
    ii) Compare E with P and B of problem (24) and (25) respectively.
    ii) All {S⟨$Z_n$ ∪ g⟩), ×} has several MOD nilpotent subsets of order two irrespective of n, $2 \leq n < \infty$.

28. Let W = {S(⟨$Z_{15}$ ∪ h⟩), ×} be the MOD special dual like number subset semigroup under ×.

    i) Study questions (i) to (ix) of problem (17) for this W.
    ii) Compare W with$V_1$ = {S($Z_{15}$), ×}, $V_2$ = {S(C($Z_{15}$)), ×}, $V_3$ = {S(⟨$Z_{15}$ ∪ I⟩), ×} and $V_4$ = {S(⟨$Z_{15}$ ∪ g⟩), ×}.
    iii) Which of the MOD subset semigroups, $V_1$ or $V_2$ or $V_3$ or $V_4$ or $V_5$ has the maximum number of MOD subset zero divisor pairs?
    iv) Which of the MOD subset semigroups mentioned in (iii) has maximum number MOD subset nilpotents of order two?
    v) Which of the MOD subsets semigroups mentioned in (iii) has maximum number of MOD subset idempotents?
    vi) Which of the MOD subset semigroups mentioned in (iii) has maximum number of MOD subset ideals?

29. Let R = {S(⟨$Z_{29}$ ∪ h⟩), ×} be the MOD subset semigroup under product.

    i) Study questions (i) to (ix) of problem (17) for this R.
    ii) Can R have MOD subset nilpotents of order two?
    iii) Compare R with $T_1$ = {S($Z_{29}$), ×}, $T_2$ = {S(C($Z_{29}$)), ×}, $T_3$ = {S(⟨$Z_{29}$ ∪ g⟩), ×} and $T_4$ = {S(⟨$Z_{29}$ ∪ I⟩), ×}.
    iv) Obtain any other special feature associated with R.

30. Let Y = {S(⟨$Z_{42}$ ∪ k⟩), ×} be the MOD subset special quasi dual number semigroup under ×.

i) Study questions (i) to (ix) of problem (17) for this Y.
ii) Find o(Y).
iii) Compare Y with $D_1$ = {S($Z_{42}$), ×}, $D_2$ = {S(⟨$Z_{42}$ ∪ I⟩), ×}, $D_3$ = {S(C($Z_{42}$)), ×}, $D_4$ = {S(⟨$Z_{42}$ ∪ g⟩), ×} and $D_5$ = {S(⟨$Z_{42}$ ∪ I⟩), ×}.
iv) Which of the MOD subset semigroups given in (iii) will have maximum number of MOD subset idempotents?

31. Let Z = {S(⟨$Z_{19}$ ∪ k⟩), ×} be the MOD subset special quasi dual number semigroup.

i) Study questions (i) to (ix) of problem (16) for this Z.
ii) Compare Z with R in problem 29.

32. Enumerate all special features in S(⟨$Z_n$ ∪ k⟩),

i) n is a prime
ii) n is a composite number
iii) n is of the form $p^t$, t ≥ 2, p a prime.

33. Prove MOD subsets semigroups are of finite order and are commutative.

34. Can one say all MOD subsets semigroups under × are S-MOD subset semigroups?

35. Can these MOD subset semigroups under product have

i) S-MOD subset zero divisor pairs.
ii) S MOD subset idempotents.

36. Let W = {S(⟨$Z_{612}$ ∪ I⟩), ×} be the MOD subset semigroup.

i) Does W contain S-MOD subset zero divisors?
ii) Can W have MOD S-subset idempotents.

37. Characterize those MOD subset semigroups under × which has MOD subset S-zero divisors.

38. Characterize those MOD subset semigroups under × which has no MOD subset S-zero divisors.

39. Characterize those MOD subset semigroups under × which has MOD subset S-idempotents.

40. Study for Smarandache MOD zero divisors subsets and MOD S-idempotents subsets in $\{S(\langle Z_{48} \cup k\rangle), \times\}$.

## Chapter Three

# MOD NATURAL NEUTROSOPHIC SUBSET SEMIGROUPS

In this chapter we define the new notion of MOD natural neutrosophic elements subset semigroups under + and ×.

This study is very new however MOD natural neutrosophic numbers have been introduced in [60] and also a complete study of semigroups constructed using MOD natural neutrosophic numbers have been carried out in [64].

Here we study semigroups constructed using subsets of MOD natural neutrosophic numbers using $Z_n^I$ or $\langle Z_n \cup I\rangle_I$ or $\langle Z_n \cup g\rangle_I$ or $\langle Z_n \cup h\rangle_I$ or $\langle Z_n \cup k\rangle_I$ or $C^I(Z_n)$.

As continuing the notation

$S(Z_n^I)$ = {collection of all subsets from the set $Z_n^I$} which will also be known as MOD natural neutrosophic subset set.

We will describe these by the following examples.

***Example 3.1:*** Let S($Z_4^I$) = {Collection of all subsets from the set $Z_4^I$ = {0, 1, 2, 3, $I_0^4$, $I_2^4$, 1 + $I_0^4$, 2 + $I_0^4$, 3 + $I_0^4$, 1 + $I_2^4$, 2 + $I_2^4$, 3 + $I_2^4$, 1 + $I_0^4$ + $I_2^4$, 2 + $I_0^4$ + $I_2^4$, 3 + $I_2^4$ + $I_0^4$, $I_0^4$ + $I_2^4$}}.

A = {$I_0^4$, $I_2^4$ + 3, 3, 2, 1 + $I_0^4$ + $I_2^4$) ∈ S($Z_4^I$) is a MOD natural neutrosophic subset of S($Z_4^I$).

B = {0, 1, 2, 3} is a MOD natural neutrosophic subset of S($Z_4^I$).

We can define operations of + and × on S($Z_4^I$).

Thus

S($Z_n^I$) = {collection of all subsets of the set $Z_n^I$ }, $2 \leq n < \infty$.

Clearly o(S($Z_n^I$)) < ∞. However even finding order of $Z_n^I$ is a difficult task.

We will give examples of them.

***Example 3.2:*** Let {S($Z_5^I$), +} be the MOD natural neutrosophic elements subset under +, addition, and it is defined as the MOD natural neutrosophic subset semigroup under +.

If A = {4 + $I_0^5$, $I_0^5$, 3, 4, 2, 0} and

B = {$I_0^5$, 2, 4, 1 + $I_0^5$} ∈ S($Z_5^I$), then

A + B = {4 + $I_0^5$, $I_0^5$, 3, 4, 2, 0} + {$I_0^5$, 2, 4, 1 + $I_0^5$}
= {4 + $I_0^5$, $I_0^5$, 3 + $I_0^5$, 2 + $I_0^5$, 3, 1 + $I_0^5$, 0, 4, 2, 1}
= $Z_5^I$.

This is the way the operation + is performed on S($Z_5^I$).

Thus the pair of MOD natural neutrosophic elements subset is a MOD universal natural neutrosophic elements subset pair.

Let M = {0, 1, 2} and N = {0, 1 + $I_0^5$} ∈ S($Z_5^I$).

M + N = {0, 1, 2} + {0, 1 + $I_0^5$}
= {0, 2, 1 + $I_0^5$, 1, 2 + $I_0^5$, 3 + $I_0^5$}.

We find M + M = {0, 1, 2} + {0, 1, 2} = {0, 1, 2, 3, 4}.

N + N = {0, 1 + $I_0^5$} + {0, 1 + $I_0^5$} = {0, 1 + $I_0^5$, 2 + $I_0^5$}.

Let P = {0, $I_0^5$} ∈ S($Z_5^I$)
P + P = {0, $I_0^5$} + {0, $I_0^5$} = {0, $I_0^5$}

Thus P is the MOD natural neutrospohi elements idempotent subset of S($Z_5^I$).

Let M = {0, 1 + $I_0^5$, 2 + $I_0^5$, 1, 2, 3 + $I_0^5$, 4 + $I_0^5$} ∈ S($Z_5^I$).

M + M = {0, 1 + $I_0^5$, 2 + $I_0^5$, 1, 2, 3 + $I_0^5$, 4 + $I_0^5$}+
{0, 1 + $I_0^5$, 1, 2, 2 + $I_0^5$, 3 + $I_0^5$, 4 + $I_0^5$}

= {0, 1 + $I_0^5$, 2 + $I_0^5$, 1, 2, 3 + $I_0^5$, 3, 4, 4 + $I_0^5$, $I_0^5$} ≠ M.

So in general we may not have M + M = M but here M + M = $Z_5^I$.

Consider P = {collection of all subsets from {0, 1, 2, 3, 4}} ⊆ S($Z_5^I$).

P is a MOD natural neutrosophic elements subset subsemigroup as if for any X, Y ∈ P then X + Y ∈ P.

Likewise M = {collection of all subsets from {0, $I_0^5$, 1 + $I_0^5$, 2 + $I_0^5$, 3 + $I_0^5$, 4 + $I_0^5$}} ⊆ S($Z_5^I$) is a MOD natural neutrosophic elements subsets subsemigroup of S($Z_5^I$).

Let M = {0, 2 + $I_0^5$} and N = {0, 1, $I_0^5$, 3} ∈ S($Z_5^I$).

M + N = {0, 2 + $I_0^5$} + {0, 1, $I_0^5$, 3}
= {0, 2 + $I_0^5$, 1, 3 + $I_0^5$, $I_0^5$, 3}.

Next we provide one more example of MOD natural neutrosophic subset semigroup.

***Example 3.3:*** Let {S($Z_{12}^I$), +} be MOD natural neutrosophic subset semigroup.

Let A = {3, 0, 4 + $I_0^{12}$, $I_4^{12}$, $I_6^{12}$, $I_8^{12}$, $I_{10}^{12}$ + $I_9^{12}$} ∈ S($Z_{12}^I$).

A + A = {0, 3, 4 + $I_0^{12}$, $I_4^{12}$, $I_6^{12}$, $I_8^{12}$, $I_{10}^{12}$ + $I_9^{12}$} + {3, 0, $I_8^{12}$, 4 + $I_{10}^{12}$, $I_4^{12}$, $I_6^{12}$, $I_{10}^{12}$ + $I_9^{12}$}

= {0, 3, 4 + $I_0^{12}$, $I_4^{12}$, $I_6^{12}$, $I_8^{12}$, $I_{10}^{12}$ + $I_9^{12}$, 6, 7 + $I_0^{12}$, 3 + $I_4^{12}$, 3 + $I_6^{12}$, 3 + $I_8^{12}$, 3 + $I_{10}^{12}$ + $I_9^{12}$, 4 + $I_0^{12}$ + $I_8^{12}$, $I_4^{12}$ + $I_4^{12}$, $I_8^{12}$ + $I_6^{12}$, $I_8^{12}$ + $I_{10}^{12}$ + $I_9^{12}$, 8 + $I_0^{12}$, 4 + $I_0^{12}$ + $I_4^{12}$, 4 + $I_0^{12}$ + $I_6^{12}$, 4 + $I_0^{12}$ + $I_{10}^{12}$ + $I_9^{12}$, 4 + $I_4^{12}$ + $I_0^{12}$, and so on}.

This is the way + operation on S($Z_{12}^I$) is performed.

Let M = {0, 5, $I_0^{12}$ + $I_8^{12}$, 4} and N = {8, 4, $I_6^{12}$, 6} ∈ S($Z_{12}^I$).

M + N = {0, 5, $I_0^{12}$ + $I_8^{12}$} + {8, 4, 6, $I_6^{12}$}
= {8, 1, 8 + $I_0^{12}$ + $I_8^{12}$, 4, 9, 4 + $I_0^{12}$ + $I_8^{12}$, 6, 11, 6 + $I_0^{12}$ + $I_8^{12}$, $I_6^{12}$, 5 + $I_6^{12}$, $I_6^{12}$ + $I_0^{12}$ + $I_8^{12}$} ∈ S($Z_{12}^I$).

Clearly for a given A ∈ S($Z_{12}^{I}$) we may or may not be able to find a B such that A + B = $Z_{12}^{I}$, that the pair of MOD natural neutrosophic elements subsets may or may not yield the MOD natural neutrosophic elements subset universal pair.

For we can say given a pair A, B of MOD natural neutrosophic elements subset S($Z_{12}^{I}$); sure we cannot say their sum will be $Z_{12}^{I}$ but we can establish given A ∈ S($Z_{12}^{I}$) we can always find a B ∈ S($Z_{12}^{I}$) such that A + B = $Z_{12}^{I}$.

In view of this we propose the following problem.

**Problem 3.1:** Let {S($Z_n^I$), +} be the MOD natural neutrosophic elements subset semigroup under +.

For any given A ∈ S($Z_n^I$) can we find a B such that A + B = $Z_n^I$, the MOD natural neutrosophic elements subset universal pair.

ii) Given two MOD natural neutrosophic elements subsets A, B prove the pair (A, B) in general is not a MOD natural neutrosophic elements universal subset pair.

We next study MOD natural neutrosophic elements finite complex number subsets sets {S($C^I(Z_n)$)} by some examples.

***Example 3.4:*** Let S($C^I(Z_6)$) = {collection of all subsets of $C^I(Z_6)$} be the MOD finite complex number natural neutrosophic elements subset.

A = {$I_3^C$, $I_{2i_F}^C$, $I_{2+4i_F}^C$, $I_0^C$, 4 + $3i_F$} ∈ S($C^I(Z_6)$) is a MOD finite complex number natural neutrosophic elements subset of S($C^I(Z_6)$).

Let M = {$I_{3+3i_F}^C$, 2 + $3i_F$, 1 + $i_F$, 4, $5i_F$} ∈ S($C^i(Z_6)$).

We find M + M = $\{I^C_{3+3i_F}$, 2 + 2$i_F$, 4, 1 + $i_F$, 5$i_F\}$ + $\{I^C_{3+3i_F}$, 2 + 2$i_F$, 4, 1 + $i_F$, 5$i_f\}$ = $\{I^C_{3+3i_F}$, 2 + 2$i_F$ + $I^C_{3+3i_F}$, 4 + $I^C_{3+3i_F}$, 1 + $i_F$ + $I^C_{3+3i_F}$, 5$i_F$, + $I^C_{3+3i_F}$, 2 + 2$i_F$ + $I^C_{3+3i_F}$, 4 + 4$i_F$, 2$i_F$, 3 + 3$i_F$, 2 + $i_F$, 2, 5 + $i_F$, 4 + 5$i_F$, 1, 4$i_F\}$ $\in$ S($C^I(Z_6)$).

This is the way '+' operation is performed on S($C^I(Z_6)$).

We see {S($C^I(Z_6)$), +} is only a semigroup for given any A $\in$ S($C^I(Z_6)$) we may not be always in a position to find a B $\in$ S($C^I(Z_6)$) such that A + B = {0}.

However A + {0} = {0} + A = A for all A $\in$ S($C^I(Z_6)$).

Let A = {0, 2 + $I^C_3$} we cannot find a B $\in$ S($C^I(Z_6)$) such that A + B = {0}.

Infact if M = $\{I^C_4\}$ and N = $\{I^C_3\}$ then M + N = $\{I^C_4 + I^C_3\}$.

Further if S = $\{I^C_0\}$ $\in$ S($C^I(Z_6)$) then
S + S = $\{I^C_0\}$ + $\{I^C_0\}$ = $\{I^C_0\}$.

Let T + T = $\{I^C_{2+2i_F}\}$+ $\{I^C_{2+2i_F}\}$ = $\{I^C_{2+2i_F}\}$, T $\in$ S ($C^I(Z_6)$).

Thus T is a MOD finite complex natural neutrosophic subsets which are idempotents with respect to addition.

Hence {S($C^I(Z_n)$), +} is only a finite commutative semigroup with {0} as the additive identity.

***Example 3.5:*** Let {S($C^I(Z_8)$), +} be the MOD natural neutrosophic finite complex number subset semigroup under +.

Let P = {0, 2, 4, 2 + $I^C_2$, 2$i_F$ + $I^C_{4i_F}$} and
Q = {6, 2, $I^C_{2i_F}$, $I^C_{6i_F}$} $\in$ S($C^I(Z_8)$).

We find

$P + Q = \{0, 2, 4, 2 + I_2^C, 2i_F + I_{4i_F}^C\} + \{2, 6, I_{2i_F}^C, I_{6i_F}^C\} = \{2, 6, I_{2i_F}^C, I_{6i_F}^C, 4, 0, 2 + I_{2i_F}^C, 2 + I_{6i_F}^C, 4 + I_{2i_F}^C, 4 + I_{6i_F}^C, 4 + I_2^C, I_2^C, 2 + I_2^C + I_{2i_F}^C, 2 + I_2^C + I_{6i_F}^C, 2 + 2i_F + I_{4i_F}^C, 6 + 2i_F + I_{4i_F}^C, 2i_F + I_{2i_F}^C + I_{4i_F}^C, 2i_F + I_{6i_F}^C + I_{4i_F}^C\} \in S(C^I(Z_8))$.

Clearly o(P) = 5, o(Q) = 4, o(P + Q) = 18.

So, o(P + Q) = o(P) + o(Q) is not true in general.

Of course given P one can find a Q such that P + Q = $C^I(Z_8)$; that is (P, Q) is a MOD natural neutrosophic finite complex number universal subset pair of $S(C^I(Z_8)$.

We have R ∈ $S(C^I(Z_8))$ such that R + R = 0. We have also T ∈ $S(C^I(Z_8))$ such that T + T = T.

In view of all these we can prove the following theorem.

**THEOREM 3.1:** *Let {$S(Z_n^I)$ (or $S(C^I(Z_n))$, +} be the MOD natural neutrosophic finite subset (or MOD natural neutrosophic finite complex number subset) semigroup under +.*

*i) For any A ∈ $S(Z_n^I)$ (or $S(C^I(Z_n))$ we can always find a B in $S(Z_n^I)$ (or $S(C^I(Z_n))$) so that A + B = $Z_n^I$ (or ($C^I(Z_n)$). That is the MOD natural neutrosophic subset pair is a universal pair.*

*ii) Given a pair of MOD natural neutrosophic subsets then in general A + B ≠ $Z_n^I$ (or $C^I(Z_n)$).*

Proof is direct and hence left as an exercise to the reader.

***Example 3.6:*** Let {S(⟨$Z_{12}$ ∪ I⟩)} be the MOD natural neutrosophic-neutrosophic subsets with entries from $\langle Z_{12} \cup I\rangle_I$. A = {3 + I, 6 + 3I, $I_6^I$, 7I, $I_0^I$, 4I, 2, $I_4^I$, 0} and B = {0, 1 + 2I, 4I

+ 4, 9I, $I^I_{2+4I}$, $I^I_{6I+6}$, 2} are MOD natural neutrosophic-neutrosophic subsets of $S(\langle Z_{12} \cup I\rangle_I)$.

We can define the operation + on $S(\langle Z_{12} \cup I\rangle_I)$ and $\{S(\langle Z_{12} \cup I\rangle_I), +\}$ is a MOD natural neutrosophic-neutrosophic subset semigroup.

Let $A = \{I^I_{4+2I}, 0, I^I_{3I}, I^I_{3I} + I^I_{4+2I}\} \in (S(\langle Z_{12} \cup I\rangle_I)$

$$A + A = \{I^I_{4+2I}, I^I_{3I}, 0, I^I_{3I} + I^I_{4+2I}\} + \{I^I_{4+2I}, I^I_{3I}, 0, I^I_{3I} + I^I_{4+2I}\}$$

$$= \{I^I_{4+2I}, I^I_{3I}, 0, I^I_{3I} + I^I_{4+2I}\} = A.$$

Thus A is MOD natural neutrosophic-neutrosophic idempotents subset of $(S(\langle Z_{12} \cup I\rangle_I)$.

Let $A = \{0, 2, 6, 3I, 4 + I^I_{4+2I}, 3 + 8I + I^I_0\}$ and
$B = \{4, 0, 8 + I^I_{2I}\} \in S(\langle Z_{12} \cup I\rangle_I)$,

$$A + B = \{0, 2, 6, 3I, 4 + I^I_{4+2I}, I^I_0 + 3 + 8I\} + \{0, 4, 8 + I^I_{2I}\}$$

$= \{0, 2, 6, 3I, 4 + I^I_{4+4I}, I^I_0 + 3 + 8I, 4, 6, 10, 4 + 3I, 8 + I^I_{4+4I}, 7 + 8I + I^I_0, 8 + I^I_{2I}, 10 + I^I_{2I}, 2 + I^I_{2I}, 3I + 8 + I^I_{2I}, I^I_{4+4I} + I^I_{2I}, 11 + 8I + I^I_0 + I^I_{2I}\} \in S((\langle Z_{12} \cup I\rangle_I)$.

Order of A = 6, o(B) = 3 and o(A + B) = 18.

Let $A = \{0, I^I_0 + I^I_{3+4I}\} \in S((\langle Z_{12} \cup I\rangle_I)$.

$$A + A = \{0, I^I_0 + I^I_{3+4I}\} + \{0, I^I_0 + I^I_{3+4I}\}$$
$$= \{0, I^I_0 + I^I_{3+4I}\}.$$

Let B = {6 + 6I} then B + B = {6 +6I} + {6 + 6I} = {0}.

Consider P = {Collection of all subsets of $\{Z_{12} \cup I\}\} \subseteq S(\langle Z_{12} \cup I\rangle_I)$.

Clearly P is a MOD subset natural neutrosophic - neutrosophic subsemigroup of $S(\langle Z_{12} \cup I\rangle_I)$.

We can find several MOD subset natural neutrosophic-neutrosophic subsemigroups.

***Example 3.7:*** Let W = {{collection of all subsets from $\langle Z_{43} \cup I\rangle_I\} = S(\langle Z_{43} \cup I\rangle_I)$, +} be a MOD natural neutrosophic neutrosophic subsemigroup of finite order. Infact W a monoid.

There are several MOD subset natural neutrosophic neutrosophic subsemigroups given by

$B_1$ = {collection of all subset of $Z_{43}$},

$B_2$ = {collection of all subsets of $Z_{43}I$},

$B_3$ = {collection of all subsets of $\langle Z_{43} \cup I\rangle$},

$B_4 = \{0, I_0^I\}$, $B_5 = \{a + I_0^I / a \in Z_{43}\}$ and

$B_6 = \{a + I_0^I / a \in \langle Z_{43} \cup I\rangle\}$.

All these MOD subset natural neutrosophic- neutrosophic subsemigroups are of finite order.

Now for any subset $A \in S(\langle Z_{43} \cup I\rangle_I)$; we can always find a $B \in S(\langle Z_{43} \cup I\rangle_I)$ such that $A + B = \langle Z_{43} \cup I\rangle$.

***Example 3.8:*** Let {V, +} = {collection of all subsets from $\langle Z_4 \cup I\rangle_I$, +} be the MOD subset natural neutrosophic-neutrosophic semigroup under +.

Let B = $\{0, 2, 3, I_0^I, I_2^I, I_2^I + I_0^I, 3 + I_0^I\}$ and

$C = \{1, 2, 0, 3, 2I, 3I, I, I_0^I, I_I^I, I_{1+I}^I\} \in S(\langle Z_4 \cup I\rangle_I)$.

Clearly $B + C \neq \langle Z_4 \cup I\rangle_I$. However we can always find a D $\in S(\langle Z_4 \cup I\rangle_I)$ such that $D + B = \langle Z_4 \cup I\rangle_I$.

***Example 3.9:*** Let $\{S, +\} = \{S(\langle Z_2 \cup I\rangle_I), +\}$ be the MOD subset natural neutrosophic - neutrosophic semigroup under +.

$\{\langle Z_2 \cup I\rangle_I = \{0, 1, I, 1 + I, I_0^I, I_I^I, I_{1+I}^I, 1 + I_0^I, 1 + I_I^I, I + I_0^I, I + I_I^I, 1 + I_{1+I}^I, I + I_{1+I}^I, 1 + I + I_0^I, 1 + I + I_I^I, 1 + I + I_{1+I}^I, I_0^I + I_I^I, I_0^I + I_{1+I}^I, I_I^I + I_{1+I}^I, 1 + I_0^I + I_I^I, 1 + I_0^I + I_{1+I}^I, 1 + I_{1+I}^I + I_I^I, 1 + I_0^I + I_I^I, I + I_0^I + I_{1+I}^I, I + I_{1+I}^I + I_I^I, I_0^I + I_{1+I}^I + I_{1+I}^I + I_I^I, 1 + I_0^I + I_I^I + I_{1+I}^I, I + I_0^I + I_{1+I}^I + I_I^I, 1 + I + I_0^I + I_I^I + 1 + I_{I+1}^I, 1 + I + I_I^I + I_0^I, 1 + I + I_{1+I}^I + I_I^I, 1 + I + I_{1+I}^I + I_0^I, 1 + I + I_0^I + I_I^I + I_{1+I}^I\}$.

So for any A $\in$ S we can find a B $\in$ S such that $A + B = \langle Z_2 \cup I\rangle_I$.

In view of all these we have the following theorem.

**THEOREM 3.2:** *Let $G = \{S(\langle Z_n \cup I\rangle_I), +\}$ be the MOD subset natural neutrosophic - neutrosophic semigroup under +.*

*i) $o(G) < \infty$.*
*ii) For every $A \in G$ there is a $B \in G$ such that $A + B = \langle Z_n \cup I\rangle_I$.*

Proof is direct and we leave it as an exercise to the reader.

Now consider any A $\in S(\langle Z_n \cup I\rangle_I)$, we hav B such that $A + B = \langle Z_n \cup I\rangle_I$

We call the pair (A, B) to be the MOD subset natural neutrosophic-neutrosophic universal pair if $A + B = \langle Z_n \cup I\rangle_I$ and B is of least cardinality for a given A.

We leave it as a open conjecture to find all B such that $|B| = o(B)$ is the least such that for a given B,
$A + B = \langle Z_n \cup I\rangle_I$.

i) Is that B only one set or we can have more than one B?
ii) For a given A find all B in $S(\langle Z_n \cup I\rangle_I)$ so that $A + B = \langle Z_n \cup I\rangle_I$.

How many such B exist for a given A?

Further finding $o(\langle Z_n \cup I\rangle_I)$ is itself a very difficult problem.

Now we proceed onto describe MOD subset natural neutrosophic dual number semigroups under + by examples.

***Example 3.10:*** Let $S = \{S(\langle Z_{10} \cup g\rangle_I), +\}$ be the MOD natural neutrosophic subset dual number semigroup.

The reader is left with the task of finding $o(\langle Z_{10} \cup I\rangle_I)$.

Let $A = \{5, 2g + 8, I_4^g, I_5^g, I_{3g}^g, I_0^g\}$ and

$B = \{0, I_{2+2g}^g, I_{4+6g}^g, 3 + g, 5g\} \in S$; we find

$A + B = \{5, 2g + 8, I_4^g, I_5^g, I_{3g}^g, I_0^g\} + \{0, I_{2+2g}^g, I_{4+6g}^g, 3 + g, 5g\}$

$= \{5, 2g + 8, I_4^g, I_5^g, I_{3g}^g, I_0^g, I_{2+2g}^g + 5, 2g + 8 + I_{2+2g}^g, I_4^g + I_{2+2g}^g, I_5^g + I_{2+2g}^g, I_{3g}^g + I_{2+2g}^g, I_0^g + I_{2+2g}^g, 5 + I_{4+6g}^g, 2 + 8 + I_{4+6g}^g, I_4^g + I_{4+6g}^g, I_5^g + I_{4+6g}^g, I_{3g}^g + I_{4+6g}^g, I_0^g + I_{4+6g}^g, 8 + g, 1 + 3g, 3 + g + I_4^g, 3 + g + I_5^g, 3 + g + I_{3g}^g, 3 + g + I_0^g, 5 + 5g, 7g + 8, 5g + I_4^g, 5g + I_5^g, 5g + I_{3g}^g, 5g + I_0^g\} \in S(\langle Z_{10} \cup g\rangle_I)$.

This is the way '+' operation is performed on S.

However $\{0\} \in S(\langle Z_{10} \cup g\rangle_I)$ acts as the additive identity of S.

We see given $A \in S$ we in general cannot find a B in S such that $A + B = \{0\}$.

We can say only for singleton sets we can get for A, with $o(A) = 1$, and $A \in \langle Z_n \cup g\rangle$ alone is such that there is a $A_1$ such that $A + A_1 = \{0\}$.

However if $A = \{ I_2^g \} \in S$ then we have no $B \in S$ such that $A + B = \{0\}$.

$\{0, 2\} = A \in S$; $B = \{8\}$ then
$A + B = \{0, 2\} + \{8\} = \{8, 0\} \neq \{0\}$.

Thus S is only a MOD subset natural neutrosophic semigroup under + as every $A \in S$ has no additive inverse.

***Example 3.11:*** Let $B = \{S(\langle Z_{13} \cup g\rangle_I, +\}$ be the MOD subset natural neutrosophic dual number semigroup under +.

We see B has MOD subset subsemigroup, o(B) is finite.

If $A = \{0, I_{10g}^g, I_{4g}^g, I_{10g}^g + I_{4g}^g\} \in B$.

$$A + A = \{0, I_{10g}^g, I_{4g}^g, I_{10g}^g, I_{4g}^g\} + \{0, I_{10g}^g, I_{4g}^g, I_{10g}^g + I_{4g}^g\}$$
$$= \{0, I_{10g}^g, I_{4g}^g, I_{10g}^g + I_{4g}^g\} = A.$$

Thus A is an MOD subset dual number idempotent of B.

B has also MOD subset dual number idempotents.

Thus in view of all these we have the following result the proof of which is left as an exercise to the reader.

**THEOREM 3.3:** *Let $M = \{S(\langle Z_n \cup g\rangle_I), +\}$ be the MOD subset natural neutrosophic dual number semigroup.*

*i) M has MOD subset dual number idempotents with respect to +.*
*ii) M has MOD subset natural neutrosophic dual number subsemigroups.*

However it is a difficult task to find the number of MOD subset dual number idempotents.

Further finding order of $\langle Z_n \cup g\rangle_I$ and $S(\langle Z_n \cup g\rangle_I)$ is also a difficult task. Finding the number of MOD subset natural neutrosophic dual number subsemigroups is also a challenging problem.

Next we proceed onto describe by examples the notion of MOD subset natural neutrosophic special dual like number semigroups under +.

***Example 3.12:*** Let $G = \{S(\langle Z_{12} \cup h\rangle_I), +\}$ be the MOD natural neutrosophic special dual like number subset semigroup under + $o(G) < \infty$.

G has several MOD subset natural neutrosophic special dual like number subset subsemigroup.

Let $A = \{0, 6, I^h_{3+4h}, I^h_2\}$ and

$B = \{3, 5, 8h, I^h_{8h}, I^h_{9+2h}, I^h_2, I^h_0\} \in G$.

We show how the operation + is performed on G.

$A + B = \{0, 6, I^h_2, I^h_{3+4h}\} + \{3, 5, 8h, I^h_{8h}, I^h_{9+2h}, I^h_0, I^h_2\} = \{3, 5, 8h, I^h_2, I^h_{8h}, I^h_{9+2h}, 9, 3 + I^h_2, 3 + I^h_{3+4h}, 11, 5 + I^h_2, 5 + I^h_{3+4h}, 8h + 6, 8h + I^h_2, 8h + I^h_{3+4h}, 6 + I^h_{8h}, I^h_2 + I^h_{8h}, I^h_{8h} + I^h_{3+4h}, I^h_{9+2h} + 6, I^h_2 + I^h_{9+2h}, I^h_{3+4h} + I^h_{9+2h}, I^h_2 + 6, I^h_2 + I^h_{3+4h}\} \in G.$

This is the way '+' operation is performed on G.

Let A = $\{ I_{3h}^{h}, I_{4h}^{h}, I_{3h}^{h} + I_{4h}^{h}, I_{8}^{h}, I_{8}^{h} + I_{3h}^{h}, I_{4h}^{h} + I_{8}^{h}, I_{3h}^{h} + I_{4h}^{h} + I_{8}^{h} \} \in G$.

It is easily verified A + A = A.

Thus A is the MOD subset natural neutrosophic special dual like number idempotent of G.

G has several such MOD subset natural neutrosophic special dual like number subsemigroups.

Interested reader can find the number of such MOD subset natural neutrosophic special dual like number subsemigroups in G.

Even finding $o(\langle Z_{12} \cup h\rangle_I)$ is a difficult task.

Finding the number of MOD natural neutrosophic special dual like number idempotents is yet a challenging problem.

***Example 3.13:*** Let W = $\{S\langle Z_7 \cup h\rangle_I), +\}$ be the MOD natural neutrosophic special dual like number subset semigroup under +.

Let D = $\{ I_{3h}^{h}, I_{0}^{h}, 3, 4h + 1\}$ and E = $\{0, 2, I_{2h}^{h} + 4, I_{3h}^{h} + h, I_{0}^{h} + I_{4h}^{h} + 5\} \in W$. We show how D + E is calculated.

D + E = $\{ I_{3h}^{h}, I_{0}^{h}, 3, 4h + 1\} + \{0, 2, 4 + I_{2h}^{h}, h + I_{3h}^{h}, 5 + I_{0}^{h} + I_{4h}^{h} \} = \{ I_{3h}^{h}, I_{0}^{h}, 3, 4h + 1, 2 + I_{3h}^{h}, 2 + I_{0}^{h}, 5, 3 + 4h, 4 + I_{2h}^{h} + I_{3h}^{h}, I_{0}^{h} + I_{2h}^{h} + 4, I_{2h}^{h}, 4h + 5 + I_{2h}^{h}, h + I_{3h}^{h}, h + I_{0}^{h} + I_{3h}^{h}, 3 + h + I_{3h}^{h}, 5h + 1 + I_{3h}^{h}, 5 + I_{3h}^{h} + I_{0}^{h} + I_{4h}^{h}, 5 + I_{0}^{h} + I_{4h}^{h}, 1 + I_{0}^{h} + I_{3h}^{h}, 4h + 6 + I_{0}^{h} + I_{4h}^{h} \} \in W$.

This is the way + operation is performed on W.

Finding all MOD subset natural neutrosophic special dual like number idempotents is left as an exercise to the reader. Finding o(W) is also a challenging one.

The reader is expected to find the number of MOD subset natural neutrosophic special dual like number subsemigroups of W.

We have the following result.

**THEOREM 3.4:** *Let S = {S(⟨$Z_n \cup h$⟩$_I$, +} be the MOD natural neutrosophic subset special dual like number semigroup under +.*

*a) o(S) < ∞.*
*b) S has MOD subset natural neutrosophic special dual like number subsemigroups.*
*c) S has MOD subset natural neutrosophic special dual like number idempotents.*

Proof is direct and hence left as an exercise to the reader.

Next we describe by examples MOD natural neutrosophic subset special quasi dual number semigroup under +.

***Example 3.14:*** Let S = {S(⟨$Z_{15} \cup k$⟩$_I$), +} be the MOD special quasi dual number natural neutrosophic subset semigroup under +.

Let A = $\{0, 5k, I_{3k}^{k}, I_{9k+6}^{k}\}$ and

B = $\{3 + 6k, I_0^k + 3, 4k + 5 + I_{6k}^{k}\} \in S$ .

A + B = $\{0, 5k, I_{3k}^{k}, I_{9k+6}^{k}\} + \{3 + 6k, I_0^k + 3, 4k + 5 + I_{6k}^{k}\}$

$= \{3 + 6k, 11k + 3, 3 + 6k + I^k_{3k}, 6k + 3 + I^k_{9k} + 6, I^k_0 + 3, 3 + 5k + I^k_0, I^k_0 + I^k_{3k} + 3, I^k_0 + 3 + I^k_{6+9k}, 4k + 5 + I^k_{6k}, 9k + 5 + I^k_{6k}, 5 + 4k + I^k_{6k} + I^k_{3k}, 4k + 5 + I^k_{6k} + I^k_{9k+6}\} \in S.$

This is the way + operation is performed on S.

Let $A = \{I^k_0, I^k_{6k}, I^k_0 + I^k_{6k}\} \in S$;

$$A + A = \{I^k_0, I^k_{6k}, I^k_0 + I^k_{6k}\} + \{I^k_0, I^k_{6k}, I^k_0 + I^k_{6k}\}$$
$$= \{I^k_0, I^k_{6k}, I^k_0 + I^k_{6k}\} = A.$$

Thus A is the MOD subset special quasi dual number natural neutrosophic idempotent element of S.

S has several such MOD subset idempotents.

However the reader is left with the task of finding the number of such MOD natural neutrosophic special quasi dual number subset idempotents of S.

***Example 3.15:*** Let $M = \{S(\langle Z_{17} \cup k\rangle_I), +\}$ be the MOD subset natural neutrosophic special quasi dual number semigroup under +. M has MOD natural neutrosophic subset special quasi dual number subsemigroups.

$A = \{I^k_0, I^k_0 + I^k_{6k}, I^k_{6k}, 0\} \in M$ is such that A + A = A is a MOD natural neutrosophic subset special quasi dual number idempotent of M.

The reader is left with the task of finding the total number of MOD natural neutrosophic subset special quasi dual number idempotents of M.

In view of this we have the following theorem, the proof of which is left as an exercise to the reader.

**THEOREM 3.5:** *Let S = {(⟨$Z_n$ ∪ k⟩$_I$), +} be the MOD subset natural neutrosophic subset special quasi dual number semigroup under +.*

*i) o(S) < ∞.*
*ii) S has MOD subset natural neutrosophic special quasi dual number subsemigroups.*
*iii) S has MOD subset natural neutrosophic special quasi dual number idempotents.*
*iv) For every A ∈ S there exists aB, (o(B) the least) in S such that A + B = ⟨$Z_n$ ∪ k⟩$_I$.*

However we propose the following problems.

**Problem 3.2:** Let {S(⟨$Z_n$ ∪ k⟩$_I$), +} = S be the MOD subset natural neutrosophic, special quasi dual number semigroup under +.

i) For a given A ∈ S, for a fixed n; 2 ≤ n < ∞ how many B exists such that A + B = ⟨$Z_n$ ∪ k⟩$_I$?
ii) How many B with o(B) least exist for a given A in S so that A + B = ⟨$Z_n$ ∪ k⟩$_I$?
iii) Find all MOD natural neutrosophic special quasi dual number subsets B of a fixed A such that A + B = ⟨$Z_n$ ∪ k⟩$_I$.
iv) Find all MOD natural neutrosophic special quasi dual number subsets of S which are MOD idempotent subsets and study the three problems
   a. When n is a t prime, give a large value for n.
   b. When n = $p^t$ p a prime.
   c. When n is a odd composite number (large).
   d. When n is an even composite number.

Next we proceed onto describe the properties of S($Z_n^I$) under product.

Here we define two types of products.

i) $0 \times I_t^n = I_t^n$ for all appropriate t as the natural neutrosophic dominant MOD natural neutrosophic semigroups.

ii) $0 \times I_t^n = 0$ for all appropriate t as the usual zero dominant semigroup.

However even if we do not mention explicitly the structure from the way product is used we will know whether the MOD semigroup is zero dominant or MOD natural neutrosophic element dominant.

We have already seen the concept of zero dominant, this was used in [ ], for otherwise the MOD graphs edges would be very clumsy or many leaving no edge free.

We will see when it is 0 dominant what are the special properties enjoyed by it and when MOD natural neutrosophic element is dominated what are the special features associated with it, as semigroups under product.

***Example 3.16:*** Let S = $\{S(Z_{10}^I), \times\}$ be the MOD natural neutrosophic subset semigroup in which S is zero dominated.

That is $\{0\} \times A = \{0\}$ for all $A \in S$.

Let $A = \{3, 5, 7, I_5^{10}, 4 + I_2^{10}\}$ and $B = \{2, 0, 4, I_6^{10}\} \in S$. We find A × B;

$A \times B = \{3, 5, 7, I_5^{10}, 4 + I_2^{10}\} \times \{0, 2, 4, I_6^{10}\}$

$= \{0, 6, 4, I_5^{10}, 8 + I_2^{10}, 2, 8, 6 + I_2^{10}, I_2^{10}, I_6^{10}, I_6^{10} + I_2^{10}\}.$

Suppose $A = \{I_0^{10}, 5, 2 + I_8^{10}\}$ and $B = \{2, 0, I_5^{10}\} \in S$.

$A \times B = \{I_0^{10}, 5, 2 + I_8^{10}\} \times \{2, 0, I_5^{10}\}$

$= \{I_0^{10}, 0, 4 + I_8^{10}, I_0^{10}, I_5^{10}, I_5^{10} + I_0^{10}\} \in S.$

Clearly S is a MOD natural neutrosophic subset semigroup.

Infact S is a monoid as {1} ∈ S is such that

{1} × A = A × {1} = A for all A ∈ S.

Further if A = {5, 0, $I_2^{10}$} and B = {0, 2} then

A × B = {5, 0, $I_2^{10}$} × {0, 2} = {0, $I_2^{10}$}.

We see o(A) = 3, o(B) = 2 but o(A × B) = 2.

Thus these MOD subset natural neutrosophic semigroup under product behave differently.

Further if A = {0, 2, 4, 8, 6} and B = {5} ∈ S then A × B = {0, 2, 4, 8, 6} × {5} = {0}.

Thus the pair {A, B} is a MOD natural neutrosophic zero divisor pair subsets. However we can find MOD zero divisor subset if $I_t^{10}$ ∈ A or B. Only when A = {0} or B = {0}. The zero is possible. So we see if A ∈ S such that $I_t^{10}$ ∈ A then for no B ∈ S \ {0}, A × B = {0}.

The natural problem is to find for any A ∈ S a B ∈ S such that A × B = $Z_{10}^I$.

This task is really difficult and left as an exercise to the reader.

Let A = {1, 0, $I_5^{10}$} and
B = {3, 5, 4, 1, 2, 6, 7, 8, 9, $I_2^{10}$, $I_8^{10}$, $I_4^{10}$} ∈ S.

A × B = {1, 0, $I_5^{10}$} × {1, 3, 2, 4, 5, 6, 7, 8, 9, $I_2^{10}$, $I_8^{10}$, $I_4^{10}$}
= {1, 0, $I_5^{10}$, 2, 3, 4, 5, 6, 7, 8, 9, $I_{21}^{10}$, $I_8^{10}$, $I_0^{10}$, $I_4^{10}$} ≠ $Z_{10}^I$.

However if A = $\{1, 0, I_5^{10}\}$ then take B $\in Z_{10}^I \setminus \{I_5^{10}\}$ then A × B = $Z_{10}^I$.

It is an interesting problem to find the number of subsets B in $Z_{10}^I$ so that A × B = $Z_{10}^I$.

Finding those subsets B in S which has the least cardinality which gives A × B = $Z_{10}^I$. Such study is innovative and interesting so left as an exercise to the reader.

Also another natural question is how many MOD natural neutrosophic subsets in S are idempotents.

Let A = {0, 5, 1, 6} ∈S;

A × A = {0, 5, 6, 1} × {0, 6, 5, 1} = {0, 6, 5, 1} = A.

So A is a MOD natural neutrosophic idempotent subset of S.

Let B = $\{1, 6, I_6^{10}, I_0^{10}\} \in S$

$$B \times B = \{1, 6, I_6^{10}, I_0^{10}\} \times \{1, 6, I_6^{10}, I_0^{10}\} = \{1, 6, I_6^{10}, I_0^{10}\} = B.$$

Thus B is also a MOD natural neutrosophic subset idempotent of S.

Finding the total number of such MOD natural neutrosophic subset idempotents in S happens to be challenging problem.

Let D = $\{1, 6, 5, I_6^{10}, I_0^{10}, I_5^{10}, 0\} \in S$.

$$D \times D = \{1, 6, 5, 0, I_6^{10}, I_0^{10}, I_5^{10}\} \times \{1, 5, 0, I_6^{10}, I_0^{10}, I_5^{10}\} = \{0, 1, 6, 5, I_6^{10}, I_0^{10}, I_5^{10}\} = D.$$

Thus D is also a MOD natural neutrosophic subset idempotent of S.

It is yet a challenging problem to find the MOD natural neutrosophic subset idempotent M of such that o(M) is the highest cardinality.

However there are several MOD subset natural neutrosophic idempotents P of S such that o(P) = 1.

***Example 3.17:*** Let M = $\{S(Z_{11}^{I}), \times\}$ be the MOD natural neutrosophic subset semigroup which is zero dominated that is $0 \times I_0^{11} = 0$.

Let A = {0, 2, 6} and B = $\{5, 3, 1 + I_0^{11}, 7 + I_0^{11}\} \in M$.

To find A × B, A × B = $\{0, 2, 6\} \times \{3, 5, 1 + I_0^{11}, 7 + I_0^{11}\}$

$= \{0, 6, 10, 2 + I_0^{11}, 4 + I_0^{11}, 7, 8, 6 +, I_0^{11}\}$

o(A) = 3, o(B) = 4. o(A × B) = 8.

So we see o(A × B) ≠ o(A) × o(B) it may be less than o(A) × o(B) it may be less than o(A) × o(B). We cannot actually attain the equality.

Let A = $\{1, I_0^{11}, 5, 6, 8 + I_0^{11}\}$ and

B = $\{1, 0, 9, 10, 5 + I_0^{11}\} \in M$.

To find A × B; A × B = $\{1, I_0^{11}, 5, 6, 8 + I_0^{11}\} \times \{0, 1, 9, 10, 5 + I_0^{11}\} = \{0, 1, I_0^{11}, 5, 6, 8 + I_0^{11}, 9, 10, 6 + I_0^{10}, 3 + I_0^{11}, 5 + I_0^{11}, 7 + I_0^{11}\} \in M$.

Clearly o(A) = 5, o(B) = 5, o(A × B) = 12.

Now finding MOD subset natural neutrosophic idempotents and nilpotents in M is an impossibility. For if A = {0} ∈ M, then A × A = {0}. If A = {0, 1} ∈ M then A × A = {0, 1}. If B = {0, 1, $I_0^{11}$} ∈ M then

B × B = {0, 1, $I_0^{11}$}. D = {1, $I_0^{11}$} ∈ M is such that
D × D = {1, $I_0^{11}$}.

These are some of the MOD natural neutrosophic subset idempotents of M.

All these will be termed only as trivial MOD natural neutrosophic subset idempotents of M. A = {0, 4, 3, 1} ∈ M is a nontrivial MOD subset idempotent.

Now the only MOD natural neutrosophic subset nilpotents are A = {0} and B = {0, $I_0^{11}$}.

So this M has no nontrivial MOD natural neutrosophic subset nilpotents.

In view of all these we have the following theorem.

**THEOREM 3.6:** *Let S = {S($Z_n^I$), ×}, n a prime, be the MOD natural neutrosophic subset semigroup under × which is zero dominated.*

*i) S has no nontrivial MOD natural neutrosophic subset nilpotents.*
*ii) This has MOD natural neutrosophic idempotents subset given by {0, 1, a, b} where a × b = 1; a, b ∈ $Z_p$, or the set {$Z_p$} = B or the set T = {$Z_p^I$}.*

Proof is direct and hence left as an exercise to the reader.

***Example 3.18:*** Let S = {S($Z_8^I$), ×} be the MOD natural neutrosophic subset semigroup under zero dominant product that is 0 × $I_t^8$ = 0 for all t = 0, 2, 4, 6.

Let A = {2, 4, 0, 6} ∈ S.
A × A = {0, 2, 4, 6} ∈ {0, 2, 4, 6} = {0, 4} = B
A × A × A = B × A = {0, 4} × {0, 2, 6, 4} = {0}.

Thus A is a MOD nilpotent subset natural neutrosophic element of S of order three as $A^3$ = {0}.

Let M = {0, $I_0^8$} ∈ S; M × M = {0, $I_0^8$} × {0, $I_0^8$} = {0, $I_0^8$} = M is the MOD natural neutrosophic trivial idempotent subset of S.

Let P = {$I_0^8$, $I_2^8$, $I_6^8$, $I_4^8$} ∈ S;

P × P = {$I_0^8$, $I_2^8$, $I_6^8$, $I_4^8$} × {$I_0^8$, $I_2^8$, $I_6^8$, $I_4^8$} = {$I_0^8$, $I_4^8$} = $P^2$;

P × P × P = $P^2$ × P = {$I_0^8$, $I_4^8$} × {$I_0^8$, $I_2^8$, $I_6^8$, $I_4^8$} = {$I_0^8$}.

Thus P is the MOD natural neutrosophic subset which is natural neutrosophic nilpotent as $P^3$ = {$I_0^8$} the natural neutrosophic zero subset of S.

So interested reader can study this sort of MOD natural neutrosophic subset nilpotents P and MOD natural neutrosophic subset nilpotents R such that $P^t$ = {0} and $R^s$ = {$I_0^n$}; t ≥ 2 and s ≥ 2 with n = $p^m$, p a prime; m ≥ 2.

Next we proceed onto illustrate by examples MOD natural neutrosophic finite complex number subset semigroups under the zero dominant product ×.

***Example 3.19:*** Let S = {S($C^I(Z_6)$), ×} be the MOD natural neutrosophic finite complex number subset semigroup under zero dominant product that is 0 × $I_t^C$ = 0 for all relevant t; t in $C(Z_6)$.

Let A = $\{0, I_3^C, I_{2+4i_F}^C, 4\}$ and

B = $\{I_{3i_F}^8, 3 + I_0^C, 2, 3, i_F\} \in S$;

$$A \times B = \{0, I_3^C, I_{2+4i_F}^C, 4\} \times \{I_{3i_F}^8, 3 + I_0^C, 2, 3, i_F\}$$
$$= \{0, I_{3i_F}^8, I_0^C, I_3^C + I_0^C, I_{2+4i_F}^C + I_0^C, 2, 4i_F\}.$$

This is the way product operation is performed on S.

o(A) = 4 and o(B) = 5, o(A × B) = 7.

Let A = $\{0, i_F, I_0^C, 5, 1\} \in S$;

$$A \times A = \{0, i_F, I_0^C, 5, 1\} \times \{5, 1, 0, i_F, I_0^C\}$$
$$= \{0, I_0^C, 5, i_F, 1, 5i_F\} \neq A.$$

Thus finding MOD natural neutrosophic finite complex subset idempotents happens to be a difficult problem.

Let P = $\{0, 1, i_F, I_0^C, 5, 5i_F\} \in S$. We find

$$P \times P = \{0, 1, i_F, I_0^C, 5, 5i_F\} \times \{0, 1, i_F, I_0^C, 5, 5i_F\}$$
$$= \{0, 1, i_F, I_0^C, 5, 5i_F\} = P.$$

Thus P is the MOD natural neutrosophic finite complex number subset which is idempotent as $P^2 = P$.

Consider D = $\{0, 3, 4, I_0^C, 1\} \in S$.

We find $D \times D = \{0, 1, 3, 4, I_0^C\} \times \{0, 1, 3, 4, I_0^C\}$
$$= \{0, 1, 3, 4, I_0^C\} = D.$$

This D is also a MOD natural neutrosophic finite complex number idempotent subset of S.

Finding the total number of MOD natural neutrosophic finite complex number idempotent subsets happens to be a challenging problem.

***Example 3.20:*** Let S = $\{S(C^I(Z_3)), \times\}$ be the MOD natural neutrosophic finite complex number subset semigroup under the zero dominant product where $C^I(Z_3) = \{0, 1, 2, i_F, 2i_F, 1 + i_F, 2 + i_F, 1 + 2i_F, 2 + 2i_F, I_0^C, 1 + I_0^C, 2 + I_0^C, i_F + I_0^C, 2i_F + I_0^C, 1 + i_F + I_0^C, 1 + 2i_F + I_0^C, 2 + 2i_F + I_0^C\}$.

Let $A = \{1, i_F, 2, 0\}$ and $B = \{I_0^C, 1 + I_0^C, 0, 1\} \in C^I(Z_3)$.

$A \times B = \{1, i_F, 2, 0\} \times \{0, 1, I_0^C, I_0^C + 1\}$
$= \{0, 1, I_0^C, 1 + I_0^C, 2, 2 + I_0^C, i_F, i_F + I_0^C\} \in S.$

o(A) = 4, o(B) = 4 but o(A × B) = 8.

This is the way product operation is performed. Finding MOD natural neutrosophic finite complex number zero divisors happens to be difficult task.

Let us find the product of A = {0, 1, 2} ∈ S,

A × A = {0, 1, 2} × {0, 1, 2} = {0, 1, 2} = A ∈ S.

Thus A is the MOD natural neutrosophic subset finite complex number idempotent.

$B = \{0, 2, 1, i_F, 1 + i_F, 2i_F, 2 + i_F, 2 + 2i_F, 1 + 2i_F\} \in S$ is such that B × B = B. This B is again the MOD natural neutrosophic finite complex number subset idempotent of S.

Thus even if in $C^I(Z_n)$, n is prime we can have MOD natural neutrosophic finite complex number $C^I(Z_3)$ can have MOD natural neutrosophic finite complex number subset idempotents.

In view of this we have the following theorem.

**THEOREM 3.7:** *Let S = {S($C^I(Z_n)$), ×} be the MOD natural neutrosophic finite complex number subset semigroup under zero dominant product.*

*a) S has certainly MOD natural neutrosophic finite complex number subset idempotents.*
*b) S has MOD natural neutrosophic finite complex number subset subsemigroups.*
*c) S may or may not have MOD natural neutrosophic finite complex number subset nilpotents depending on M.*

Proof is left as an exercise to the reader.

We now give examples of MOD natural neutrosophic-neutrosophic subset semigroup under zero dominant product ×.

***Example 3.21:*** Let M = {S($\langle Z_4 \cup I\rangle_I$, ×} be the MOD natural neutrosophic- neutrosophic subset semigroup.

P = {0, 2, 2 + 2I, 2I} ∈ M such that

P × P = {0, 2, 2I, 2 + 2I} × {0, 2, 2I, 2 + 2I} = {0}.

So P is a MOD natural neutrosophic-neutrosophic subset nilpotent of M.

Let N = {0, I, 1} ∈ M
N × N = {0, 1, I} × {0, 1, I} = {0, 1, I} = N.

Thus N ∈ M is the MOD natural neutrosophic- neutrosophic idempotent subset of M.

***Example 3.22:*** We consider M = {S($C^I(Z_4)$), ×}.

Let D = {0, 1, 2, 3} ∈ M, to find

$D \times D = \{0, 1, 2, 3\} \times \{0, 1, 2, 3\} = \{0, 2, 3, 1\} = D$ is again a MOD natural neutrosophic - neutrosophic idempotent subset of M.

Let $Z = \{1 + i_F, i_F, 3 + i_F\}$ and
$Y = \{2i_F, 3i_F, 0\} \in M$; to find $Z \times Y$;

$$Z \times Y = \{i_F, 1 + i_F, 3 + i_F\} \times \{0, 3i_F, 2i_F\}$$
$$= \{0, 1, 3i_F + 1, i_F + 1, 2, 2i_F + 2\}.$$

Here $o(Y) = 3$, $o(Z) = 3$ but $o(Z \times Y) = 6$.

Let $T = \{0, 2, i_F, 1 + i_F, 2 + i_F, I_0^C\}$ and
$R = \{1, 2i_F, 2 + 2i_F, 1 + I_0^C, i_F + I_0^C\} \in M$.

$T \times R = \{0, 2, i_F, 1 + i_F, 2 + i_F, I_0^C\} \times \{1, 2i_F, 2 + 2i_F, 1 + I_0^C, i_F + I_0^C\} = \{0, 2, i_F, 1 + i_F, 2 + i_F, I_0^C, 2, 2i_F + 2, 2 + I_0^C, i_F + I_0^C, 1 + i_f + I_0^C, 2 + i_F + I_0^C, 2i_F + I_0^C, 2i_F + 3 + I_0^C, 3 + I_0^C, i_F + 3 + I_0^C\}$.

$o(T) = 6$, $o(R) = 5$ but $o(T \times R) = 16$.

This is the way product is obtained in M. However given $A \in M$ finding a $B \in M$ such that $A \times B = C^I(Z_4)$ happens to be a difficult task.

***Example 3.23:*** Let $S = \{S(\langle Z_5 \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic-neutrosophic subset semigroup under zero dominant product.

Let $A = \{0, I, 4 + 4I\}$ and $B = \{1, 1 + I, 3I, I_0^C\} \in S$.

$$\text{We find } A \times B = \{0, I, 4 + 4I\} \times \{1, 1 + I, 3I, I_0^I\}$$
$$= \{0, I, 4 + 4I, 3I, 4I, I_0^I, 2I, 4 + 2I\} \in S.$$

This is the way $\times$ is performed on S.

Let P = {0, 1, 2, 3, 4} ∈ S
P × P = {0, 1, 2, 3, 4} × {0, 1, 2, 3, 4} = {0, 1, 2, 3, 4} ∈ S.

Thus P is a MOD natural neutrosophic-neutrosophic subset idempotent of S.

Let T = {0, I, 2I, 3I, 4I} ∈ S.

Consider T × T = {0, I, 2I, 3I, 4I} × {0, I, 2I, 3I, 4I}
= {0, I, 2I, 3I, 4I} = T.

Hence once again T is a MOD natural neutrosophic - neutrosophic subset idempotent of S.

Let W = {0, 1, 2, 3, 4, I, 2I, 3I, 4I, 1 + I, 1 + 2I, 1 + 3I, 1 + 4I, 2 + I, 2 + 2I, 2 + 3I, 2 + 4I, 3 + I, 3 + 2I, 3 + 3I, 3 + 4I, 4 + I, 4 + 2I, 4 + 3I, 4 + 4I} ∈ S.

We see W × W = W, so W is again a MOD natural neutrosophic - neutrosophic idempotent subset of S.

However given A ∈ S finding B ∈ S such that A × B = $\langle Z_5 \cup I\rangle_I$ happens to be a difficult task.

Let V = {0, $I_0^I$, $I_0^I$+3, 2 + I + $I_0^I$} and

Z = {0, 1, $I_0^I$ + 3 + 2I, 4I} ∈ S.

V × Z = {0, $I_0^I$, $I_0^I$ + 3, 2 + I + $I_0^I$} × {0, 1, 4I, 3 + 2I + $I_0^I$}

= {0, $I_0^I$, $I_0^I$ + 3, 2 + I + $I_0^I$, 2I + $I_0^I$, 4 + I + $I_0^I$, 1 + 2I + $I_0^I$} ∈ S.

This is the way product operation is performed on S.

o(V) = 4, o(Z) = 4 and o(V × Z) = 7.

Let M = {0, I, 1} ∈ S
M × M = {0, 1, I} × {0, 1, I} = {0, 1, I}

Let N = {0, 1, 2, 3, 4, I, 2I, 3I, 4I} ∈ S

N × N = {0, 1, 2, 3, 4, I, 2I, 3I, 4I} × {0, 1, 2, 3, 4, I, 2I, 3I, 4I}

= {0, 1, 2, 3, 4, I, 2I, 3I, 4I} = N

Thus N is the MOD natural neutrosophic- neutrosophic idempotent subset of S.

Finding nilpotent subset of S is near to an impossibility. In view of this we have the following conjecture 3.

Conjecture 3.Let S = {S(⟨$Z_n$ ∪ I⟩$_I$)$_I$, ×}, n a prime be a zero dominant product MOD natural neutrosophic neutrosophic subset semigroup.

Prove S has no nontrivial MOD natural neutrosophic - neutrosophic nilpotent subsets.

***Example 3.24:*** Let S = {S(⟨$Z_{16}$ ∪ I⟩$_I$), ×} be the MOD natural neutrosophic - neutrosophic subset semigroup under zero dominant product.

Let B = {0, 4I, 4, 4 + 4I, 8, 8I, 8 + 8I} ∈ S;

B × B = {0, 4I, 4, 4 + 4I, 8, 8I, 8 + 8I} × {0, 4I, 4, 4 + 4I, 8, 8I, 8 + 8I} = {0}.

Thus B is the MOD natural neutrosophic - neutrosophic nilpotent subset of order two.

Let D = {0, 2I, 2 + 2I, 4I, 4, 4 + 4I} ∈ S.

D × D = {0, 2I, 4I, 2 + 2I, 4, 4 + 4I} × {0, 2I, 4I, 2 + 2I,

$4 + 4I\}$

$= \{0, 4I, 8I, 4 + 12I, 8 + 8I\} = D^2.$

$D^2 \times D = \{0, 4I, 8I, 4 + 12I, 8 + 8I\} \times \{0, 2I, 4I, 2 + 2I, 4, 4 + 4I\}$

$= \{0, 8I, 8 + 8I\} = D^3.$

$D^3 \times D = \{0, 8I, 8 + 8I\} \times \{0, 2I, 4I, 2 + 2I, 4, 4 + 4I\}$

$= \{0\} = D^4.$

Thus D is a MOD natural neutrosophic neutrosophic nilpotent subset of order four.

Let R = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15} $\in$ S.

R $\times$ R = R, so R is a MOD natural neutrosophic-neutrosophic idempotent subset of S.

Let T = {0, I, 2I, 3I, 4I, 5I, 6I, …, 10I, 11I, 12I, 13I, 14I, 15I} $\in$ S. It is easily verified T $\times$ T = T so once again T is a MOD natural neutrosophic - neutrosophic idempotent subset of S.

If B = $\{\langle Z_{16} \cup I\rangle\} \in$ S then also B $\times$ B = B is the MOD natural neutrosophic - neutrosophic idempotent subset of S.

Thus even if $n = p^t$, $t \geq 2$ p a prime still $S(\langle Z_n \cup I\rangle_I)$ has MOD natural neutrosophic - neutrosophic idempotent subsets.

In view of all these we have the following theorem.

**THEOREM 3.8:** *Let S = $\{S(\langle Z_n \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup under the zero dominated product.*

*a) S has MOD natural neutrosophic - neutrosophic idempotent subsets for all n, $2 \leq n < \infty$.*
*b) S has MOD natural neutrosophic - neutrosophic nilpotent subsets if $n = p^t$, $t \geq 2$ and p a prime.*
*c) S has MOD natural neutrosophic - neutrosophic subset subsemigroups.*

Proof is direct and hence left as an exercise to the reader.

However the task of providing given a $A \in S$ (S as in theorem 3) to find a $B \in S$ such that o(B) is the least is left as an exercise to the reader.

Next we describe by examples the MOD natural neutrosophic dual number subset semigroup under $\times$, the product is a zero dominated product.

***Example 3.25:*** Let $B = \{S(\langle Z_9 \cup g\rangle_I, \times\}$ be the MOD natural neutrosophic dual number subset semigroup under zero dominant product.

Let M = {g, 0, 3g, 3g, 6g, 2g, 4g, 8g, 7g} $\in$ B.

Clearly $M \times M = \{0\}$. So M is the MOD natural neutrosophic dual number nilpotent subset of B.

Let D = {5g, 6g, 8g, g} $\in$ M, $D \times D = \{0\}$.

Thus B has several MOD natural neutrosophic dual number subset nilpotents.

Now we describe a few MOD natural neutrosophic dual number subset idempotents A = {0, 1, 2, 3, 4, 5, 6, 7, 8} $\in$ B is such that $A \times A = A$ is a MOD natural neutrosophic dual number subset idempotent.

Let Z = {0, g, 2g, 3g, 4g, 5g, 6g, 7g, 8g} $\in$ B; $Z \times Z = \{0\}$ is the MOD natural neutrosophic dual number nilpotent subset of B.

T = {0, g, 5g} ∈ B is such that T × T = {0}.

It is important to keep on record that MOD natural neutrosophic dual number subset semigroups have many MOD natural neutrosophic dual number subsets which are nilpotent of order two.

Let W = $\{I_0^g, I_{2g}^g, I_{4g}^g, I_{3g}^g, I_{6g}^g\}$ ∈ B.

Clearly W × W = $\{I_0^g\}$ thus W is a MOD natural neutrosophic dual number natural neutrosophic zero nilpotent subset of B.

Thus in this situation only we have MOD natural neutrosophic dual number neutrosophic zero nilpotent subsets N = $\{I_0^g, I_{2g}^g, I_{4g}^g\}$ ∈ B is such that N × N = $\{I_0^g\}$.

Let A = $\{0, I_g^g, I_0^g\}$ and D = $\{I_{3g}^g, I_{4g}^g\}$ ∈ B;

A × D = $\{0, I_g^g, I_0^g\} \times \{I_{3g}^g, I_{4g}^g\} = \{0, I_0^g\}$.

Thus $\{0, I_0^g\}$ in B is defined as the mixed zero of B.

If E = $\{3, 1, I_{g+3}^g, I_{4g}^g, I_{2g}^g, 4 + 3g\}$ ∈ B, then E × $\{0, I_0^g\}$ = $\{1, 3, I_{g+3}^g, I_{4g}^g, 4 + 3g, I_{2g}^g\} \times \{0, I_0^g\} = \{0, I_0^g\}$.

Thus can we say for all x ∈ B. x × $\{0, I_0^g\} = \{0, I_0^g\}$?

***Example 3.26:*** Let S = {S(⟨$Z_7$ ∪ g⟩$_I$), ×} be the MOD natural neutrosophic subset semigroup under the zero dominant product.

Let A = $\{3, 4, 5, 2g, I_0^g\}$ and

$B = \{I_g^g, I_{3g}^g, 1, 2 + 4g\} \in S$.

$A \times B = \{3, 4, 5, 2g, I_0^g\} \times \{1, I_g^g, I_{3g}^g, 2 + 4g\}$

$= \{3, 4, 5, 2g, I_0^g, I_g^g, I_{3g}^g, 6 + 5g, 1 + 2g, 3 + 6g, 4g\}$.

This is the way product is obtained.

We find $A \times \{0, I_0^g\} = \{3, 4, 5, 2g, I_0^g\} \times \{0, I_0^g\} = \{0, I_0^g\}$. So we see for every $A \in S$; $A \times \{0, I_0^g\} = \{0, I_0^g\}$.

Further $A \times \{0\} = \{0\}$ for all A in S.

Finally $A \times \{I_0^g\} = \{I_0^g\}$ or $\{0\}$ or $\{I_0^g, 0\}$ for all $A \in S$. Thus we have three zeros and in this zero dominated MOD natural neutrosophic dual number subsets $\{0\}$ is the zero and $\{I_0^g\}$ and $\{I_0^g, 0\}$ are such that $\{I_0^g\} \times \{0\} = \{0\}$ and $\{I_0^g, 0\} \times \{0\} = \{0\}$.

Consider $M = \{0, g, 2g, 3g, 4g, 5g, 6g, tg + I_0^g, tg + I_g^g, tg + I_{2g}^g, \ldots, tg + I_{6g}^g, I_0^g, I_g^g. \ldots, I_{6g}^g, tg + I_0^g + I_g^g, tg + I_0^g + I_{2g}^g, \ldots, tg + I_{5g}^g + I_{6g}^g, tg + I_0^g + I_g^g + I_{2g}^g, \ldots, tg + I_{4g}^g + I_{5g}^g + I_{6g}^g, tg + I_0^g + I_{2g}^g + I_{3g}^g + I_{4g}^g, \ldots., tg + I_{3g}^g + I_{4g}^g + I_{5g}^g + I_{6g}^g, tg + I_0^g + I_{2g}^g + I_g^g + I_{3g}^g + I_{4g}^g, \ldots, tg + I_{2g}^g + I_{3g}^g + I_{4g}^g + I_{5g}^g + I_{6g}^g, tg + I_0^g + I_g^g + I_{2g}^g + \ldots + I_{6g}^g, t = 0, 1, 2, \ldots, 6\} \in S$ is such that if

$S(M)$ = {collection of all MOD natural neutrosophic dual number subsets from M} then $S(M)$ is a MOD natural neutrosophic dual number subsets subsemigroup of S and infact a MOD natural neutrosophic dual number subset ideal of S.

In view of all these we have the following theorem.

**THEOREM 3.9:** *Let S = {S($\langle Z_n \cup g\rangle_I$), ×} be the MOD natural neutrosophic dual number subset semigroup under zero dominant product.*

i) *S has several MOD natural neutrosophic dual number nilpotent subsets of order two.*
ii) *S has MOD natural neutrosophic subset dual number ideals.*
iii) *S has MOD natural neutrosophic subset dual number idempotents.*
iv) *S has MOD natural neutrosophic subset dual number subsemigroups which are not ideals.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe by examples MOD natural neutrosophic subset special dual like number semigroups under zero dominant product.

***Example 3.27:*** Let G = {S($\langle Z_{10} \cup h\rangle_I$), ×} be the MOD natural neutrosophic special dual like number subset semigroup under the zero dominant product.

Let B = {0, h, 1, 5, 5h, 5 + 5h} ∈ G,

$$B \times B = \{0, h, 1, 5, 5h, 5 + 5h\} \times \{0, h, 1, 5, 5h, 5 + 5h\} = \{0, 1, h, 5, 5h, 5 + 5h\} = B.$$

Thus B is the MOD natural neutrosophic special dual like number idempotent subset of G.

Let H = {0, 5, 5h, 5 + 5h, $I_0^h$, $I_5^h$} and

K = {0, 2, 2h, $I_2^h$, $I_{2h}^h$} ∈ G.

$$H \times K = \{0, 5, 5h, 5 + 5h, I_0^h, I_5^h\} \times \{0, 2, 2h, I_2^h, I_{2h}^h\} = \{0, I_0^h, I_5^h, I_2^h, I_{2h}^h\} \in G.$$

This is the way MOD natural neutrosophic special dual like number subsets product is obtained.

Let T = {0, 5, 5h, 5 + 5h, $I_0^h$} and S = {0, 2, 2h, $I_0^h$} ∈ G.

T × S = {0, 5, 5h, 5 + 5h, $I_0^h$} × {0, 2, 2h, $I_0^h$} = {0, $I_0^h$}.

Thus the product of the MOD natural neutrosophic special dual like number subsets yields the MOD natural neutrosophic special dual like number zero pair {0, $I_0^h$}.

Let M = {0, $I_{2h}^h$, $I_{4h}^h$, $I_{5h}^h$ + 5} and N = {0, $I_0^h$} ∈ G.

We find M × N = {0, $I_{2h}^h$, $I_{4h}^h$, $I_{5h}^h$ + 5} × {0, $I_0^h$} = {0, $I_0^h$}.

Thus we can prove for every Z ∈ G. Z × {0, $I_0^h$} = {0, $I_0^h$}.

Let L = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} ∈ G. Clearly L × L = L so L is a MOD natural neutrosophic special dual like number idempotent subset of G.

Let R = {0, h, 2h, 3h, 4h, …, 9h} ∈ G.

R × R = R so R is again a MOD natural neutrosophic special dual like number idempotent subset of G.

W = {⟨$Z_{10}$ ∪ h⟩} ∈ G is again a MOD natural neutrosophic special dual like number subset idempotent of G as

W × W = W.

We see G has MOD natural neutrosophic special dual like number subset idempotents and zero divisors.

However finding MOD natural neutrosophic special dual like number nilpotents happens to be a difficult task for S(⟨$Z_{10}$ ∪ h⟩$_I$) has no nontrivial nilpotent subsets.

***Example 3.28:*** Let W = {S($\langle Z_{13} \cup h\rangle_I$), ×} be the MOD natural neutrosophic special dual like number subset semigroup under the zero dominant product. Given A ∈ W the task of finding B such that A × B = $\langle Z_{13} \cup h\rangle_I$ is left as an exercise to the reader.

T = {{$Z_{13}$}} ∈ W is a MOD natural neutrosophic special dual like number subset idempotent of W.

V = {$\langle Z_{13} \cup h\rangle$} ∈ W is also a MOD natural neutrosophic special dual like number subset idempotent of W.

M = {0, h, 2h, 3h, …, 12h} ∈ W is again a MOD natural neutrosophic special dual like number subset idempotent of M as M × M = M.

Let P = {collection of all subsets from {$\langle Z_{13} \cup I_t^h\rangle$} relevant t ∈ $\langle Z_{13} \cup h\rangle$} ⊆ W is a MOD subset natural neutrosophic special dual like number ideal of W.

In view of all these we have the following theorem.

**THEOREM 3.10:** *Let S = {S($\langle Z_n \cup h\rangle_I$), ×} be the MOD natural neutrosophic special dual like number subset semigroup with zero dominant product.*

*i) S has MOD natural neutrosophic special dual like number subset idempotents, $2 \le n < \infty$.*
*ii) S has MOD natural neutrosophic special dual like number nilpotents only for $n = p^t$, $t > 2$ p a prime or n a appropriate composite number.*
*iii) S has MOD natural neutrosophic special dual like number subset zero divisors, that is A × B = {0} or A × B = {$I_0^h$} or A × B = {0, $I_0^h$} for some A, B ∈ S.*
*iv) S has MOD natural neutrosophic special dual like number subset subsemigroups which are not ideals.*
*v) S has MOD natural neutrosophic special dual like number subset subsemigroups which are ideals.*

Proof is direct and hence left as an exercise to the reader.

Next we propose the following problem.

**Problem 3.3:** Let S = {S($\langle Z_n \cup h\rangle_I$), ×} be the MOD natural neutrosophic special dual like number subset semigroup with zero dominant product.

i) Find conditions on n so that S has MOD natural neutrosophic special dual like number subset nilpotents.
ii) Given A ∈ S prove we can find always a B with minimal order (o(B) minimal) such that A × B = $\langle Z_n \cup h\rangle_I$.
iii) Is that B unique or there are more than one B mentioned in (ii) which is such that A × B = $\langle Z_n \cup h\rangle_I$?

Next we proceed onto describe by examples the MOD natural neutrosophic special quasi dual number subset semigroups under the zero dominant product.

***Example 3.29:*** Let M = {S($\langle Z_{15} \cup k\rangle_I$, ×} be the MOD natural neutrosophic special quasi dual number subset semigroup under the zero dominant product.

Let P = {$Z_5$} ∈ M, clearly P × P = P so P is a nontrivial MOD natural neutrosophic special quasi dual number subset idempotent of M.

R = {$\langle Z_{15} \cup k\rangle$} ∈ M is again a MOD natural neutrosophic special quasi dual number subset idempotent of M as R × R = R.

Infact M has several such MOD natural neutrosophic special quasi dual number subset idempotents.

Let A = {3 + k, 5, $I^k_{3k}$, $I^k_0$, 0, 1} and

$B = \{5k + 10, 10k, 0, I^k_{3+5k}\} \in M$.

We find $A \times B = \{3 + k, 5, I^k_{3k}, I^k_0, 0, 1\} \times \{5k + 10, 10k, 0, I^k_{3+5k}\}$

$= \{0, 5k + 10, 10k, I^k_{3+5k}, I^k_0, I^k_{3k}, I^k_{9k}, 10k + 5, 5k\}$.

This is the way the product operation is performed on M.

Let $A = \{5, 5k, 10k, 10\}$ and $B = \{3, 6, 9, 0\} \in M$.

$A \times B = \{5, 5k, 10k, 10\} \times \{3, 6, 9, 0\} = \{0\}$.

Thus {A, B} is the MOD natural neutrosophic special quasi dual number subset zero divisor pair.

Further $A \times \{0\} = \{0\}$ for all $A \in M$.

$A \times \{I^k_0\} = \{I^k_0\}$ or $\{I^k_0, 0\}$ or $\{0\}$ for all $A \in M$

Let $A = \{I^k_{3k}, I^k_{5k}, I^k_0, I^k_{6k+6}\}$ and $B = \{0, I^k_0\}$ then $A \times B = \{0, I^k_0\}$.

Let $A = \{I^k_{3k}, I^k_{5k}, I^k_{6k}, I^k_{6k+6}, I^k_{9k}\}$ and $B = \{I^k_0\}$ then $A \times B = \{I^k_0\}$.

Let $A = \{I^k_0, 5\}$ and $B = \{3k\}$, $A \times B = \{I^k_0, 0\}$.

Thus we see we can have any of these MOD natural neutrosophic special quasi dual number subset zero divisors.

Let N = {collection of all subsets from $Z_{15}$} $\subseteq$ M, N is a MOD natural neutrosophic special quasi dual number subsemigroup of M which is not an ideal of M.

Let S = {collection of all subsets from $Z_{15}k$} $\subseteq$ M,

S is again a MOD natural neutrosophic special quasi dual number subsemigroup of M which is not an ideal of M.

Several interesting results can be obtained in this direction.

***Example 3.30:*** Let S = {S($\langle Z_{16} \cup k\rangle_I$, ×} be the MOD natural neutrosophic special quasi dual number subset semigroup under the zero dominant product.

S has MOD natural neutrosophic special quasi dual number subset nilpotents.

Take A = {2, 4k, 8k, 1} ∈ S
$A \times A = \{2, 4k, 8k, 12\} \times \{2, 4k, 8k, 12\} = \{4, 8k, 8\} = A^2$
$A^2 \times A = \{4, 8, 8k\} \times \{2, 4k, 8k, 12\} = \{8\} = A^3$
$A^3 \times A = \{8\} \times \{2, 4k, 8k, 12\} = \{0\} = A^4$.

Thus A is the MOD natural neutrosophic special quasi dual number subset nilpotent of order four.

Let B = $\{I_2^k, I_4^k, I_8^k, I_{12k}^k\} \in S$.

$B \times B = \{I_2^k, I_4^k, I_8^k, I_{12k}^k\} \times \{I_2^k, I_4^k, I_8^k \times I_{12k}^k\}$
$= \{I_4^k, I_8^k, I_{8k}^k\} = B^2$.

$B^2 \times B = \{I_4^k, I_8^k, I_{8k}^k\} \times \{I_2^k, I_4^k, I_8^k, I_{12k}^k\} = \{I_8^k\} = B^3$
$B^3 \times B = \{I_8^k\} \times \{\{I_2^k, I_4^k, I_8^k, I_{12k}^k\} = \{I_0^k\} = B^4$.

Thus B is a MOD natural neutrosophic special quasi dual number MOD natural neutrosophic zero nilpotent subset of order 4.

Let D = $\{I_{4+4k}^k, I_0^k, 0, I_{2+4k}^k\} \in S$,

$D \times D = \{0, I_0^k, I_{4+4k}^k, I_{2+4k}^k\} \times \{0, I_0^k, I_{4+4k}^k, I_{2+4k}^k\}$

$$= \{0,\ I_0^k,\ I_{8+8k}^k\} = D^2.$$

$$D^2 \times D = \{0,\ I_0^k,\ I_{8+8k}^k\} \times \{0,\ I_0^k,\ I_{4+4k}^k,\ I_{2+4k}^k\}$$
$$= \{0,\ I_0^k\} = D^3$$

Thus D in S is the MOD natural neutrosophic special quasi dual number subset mixed nilpotent of order three as $D^3 = \{0, I_0^k\}$ which is a MOD natural neutrosophic special quasi dual number mixed zero of S.

Now we see S has MOD natural neutrosophic special quasi dual number subset zero divisors.

Let A = {8 + 4k, 0, 12} and B = {2, 0, 4 + 8k} ∈ S.

A × B = {0, 12, 8 + 4k} × {0, 2, 4 + 8k} = {0, 8, 8k} ≠ {0}.

Let A = {8 + 4k, 12} and B = {8 + 12k, 8k + 4} ∈ S.
A × B = {8 + 4k, 12} × {8 + 12k, 8k + 4} = {0}.

Thus S has MOD natural neutrosophic special quasi dual number subset pair of zero divisors.

Interested reader can work in this direction for more types of MOD natural neutrosophic special quasi dual number zero divisors and nilpotents.

However we have the following theorem.

**THEOREM 3.11:** *Let S = {S(⟨$Z_n$ ∪ k⟩$_I$), ×} be the MOD natural neutrosophic special quasi dual number subset semigroup under the zero dominant product.*

*i) S has MOD natural neutrosophic special quasi dual number subset idempotents.*
*ii) S has MOD natural neutrosophic special quasi dual number subset nilpotents only for n not a prime.*

*iii) S has MOD natural neutrosophic special quasi dual number subset subsemigroups which are not ideals.*
*iv) S has MOD natural neutrosophic special quasi dual number subset subsemigroup which are ideals.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed on describe by one or two examples MOD natural neutrosophic subsets under dominant MOD natural neutrosophic zero divisor $I_0^n$ ($I_0^k$ or $I_0^h$ or $I_0^g$ or $I_0^C$ or $I_0^I$).

That is $0 \times I_0^t = I_0^t$ for $t \in \{n, k, h, g, c, I\}$.

***Example 3.31:*** Let S = $\{S(Z_{10}^I), \times\}$ be the MOD natural neutrosophic subsets semigroup in which, MOD natural neutrosophic zero divisor dominant product is defined.

Let P = $\{5, 5 + I_2^{10}, I_5^{10}, I_0^{10}\}$ and Q = $\{2, I_5^{10}, 0, 1\} \in S$

$$P \times Q = \{5, 5 + I_2^{10}, I_5^{10}, I_0^{10}) \times \{0, 1, 2, I_5^{10}\}$$
$$= \{0, I_2^{10}, I_5^{10}, I_0^{10}, 5, 5 + I_2^{10}, I_5^{10} + I_0^{10}\}.$$

This is the way MOD subset natural neutrosophic semigroup where MOD natural neutrosophic zero divisor product is dominant is defined.

That is $0 \times I_0^{10} = I_0^{10}$.

Let B = {5, 0} and D = {2, 4, 6, 8} ∈ S,

B × D = {0, 5} × {4, 2, 6, 8} = {0}.

Thus this pair {B, D} gives the MOD natural neutrosophic zero divisor pair.

Let A = $\{I_0^{10}, I_5^{10}\}$ and B = $\{I_2^{10}, I_4^{10}, I_6^{10}, I_8^{10}\} \in S$,

$A \times B = \{I_0^{10}, I_5^{10}\} \times \{I_2^{10}, I_4^{10}, I_6^{10}, I_8^{10}\} = \{I_0^{10}\}$.

Thus the MOD natural neutrosophic subset pair gives the MOD natural neutrosophic zero divisor.

Let $E = \{I_5^{10}\ I_0^{10}, 0\}$ and $D = \{I_2^{10}, I_8^{10}, 0, I_6^{10}, I_4^{10}\} \in S$;

$E \times D = \{0, I_0^{10}, I_5^{10}\} \in \{I_2^{10}, I_8^{10}, I_6^{10}, 0, I_4^{10}\} = \{0, I_0^{10}\}$.

Thus this MOD natural neutrosophic pair gives the MOD natural neutrosophic zero divisor pair.

We see $0 \times I_2^{10} \neq 0$, $I_8^{10} \times 0 = I_8^{10}$ and so on.

We see $T = \{Z_{10}\} \in S$ is such that $T \times T = T$ is a MOD natural neutrosophic subset idempotent of S.

We see when $0 \times I_t^n = 0$ the semigroup behaves differently from the case when $0 \times I_t^n = I_t^n$.

However from the context one can easily find out how these products differ.

If $B = \{I_0^{10}, I_2^{10}, I_6^{10}\}$ and $C = \{0, I_5^{10}\} \in S$ then

$B \times C = \{I_0^{10}, I_2^{10}, I_6^{10}\} \times \{0, I_5^{10}\} = \{I_0^{10}, I_2^{10}, I_6^{10}\}$ where $0 \times I_t^{10} = I_t^{10}$.

But if we assume $I_t^{10} \times 0 = 0$ then

$B \times C = \{I_0^{10}, I_2^{10}, I_6^{10}\} \times \{0, I_5^{10}\} = \{I_0^{10}, 0\}$ is the MOD natural neutrosophic zero divisor pair.

Thus we see the very difference when $0 \times I_t^{10} = I_t^{10}$ and $I_t^{10} \times 0 = 0$.

Finding MOD natural neutrosophic subsemigroups and ideals is considered as a matter of routine and hence left as an exercise to the reader.

***Example 3.32:*** Let M = $\{S(Z_{19}^I), \times\}$ be the MOD natural neutrosophic subset semigroup under the MOD natural neutrosophic zero dominant product, that is $I_0^{19} \times 0 = I_0^{19}$.

We see if A, B $\in$ M \ {0} then A × B = {0} is an impossibility A × {0} ≠ {0} in M.

Let A = $\{4, 3 + I_0^{19}, 15, 8\}$ and B = {0} $\in$ M, then

$A \times B = \{8, 15, 4, 3 + I_0^{19}\} \times \{0\} = \{0, I_0^{19}\} \neq \{0\}$.

Let A = $\{4, 3 + I_0^{19}, 15, 8\}$ and E = $\{I_0^{19}\} \in$ S,

$A \times E = \{4, 3 + I_0^{19}, 15, 8\} \times \{I_0^{19}\} = \{I_0^{19}\}$.

Thus this is the case of MOD natural neutrosophic zero dominant product semigroup.

In case of {0} dominant MOD natural neutrosophic product semigroup.

We see $A \times E = \{4, 3 + I_0^{19}, 15, 8\} \times \{I_0^{19}\} = \{I_0^{19}\}$.

But if E = $\{I_0^{19}\}$ is replaced by F = {0}

$A \times F = \{4, 3 + I_0^{19}, 15, 8\} \times \{0\} = \{0\}$ in case

$\{0\} \times A = \{0\}$.

In case A $\times\{I_0^{19}\} = I_0^{19}$ we see

A $\times$ F = {4, 3 + $I_0^{19}$, 15, 8} $\times$ {0} = {0, $I_0^{19}$}.

So there is a difference in the results.

As per need of the situation one can use either MOD natural neutrosophic zero dominant product or just zero dominant product.

Now we will provide examples of MOD natural neutrosophic finite complex number subset under MOD natural neutrosophic finite complex number zero dominant semigroup. That is $\{I_0^C\}$ $\times$ {0} = $\{I_0^C\}$.

***Example 3.33:*** Let S = $\{S(C^I(Z_{20})), \times\}$ be the MOD natural neutrosophic finite complex number, subset, MOD natural neutrosophic finite complex number zero $I_0^C$ dominant product.

We see A = $\{I_0^C\}$ and

B = $\{5 + 3i_F, 2i + 12, I_{12}^C, I_{18}^C + I_{4i_F}^C\} \in$ S.

A $\times$ B = $\{I_0^C\} \times \{5 + 3i_F, 2i_F + 12, I_{12}^C, I_{18}^C + I_{4i_F}^C\} = \{I_0^C\}$.

Thus for all A $\in$ S we have A $\times$ $\{I_0^C\} = \{I_0^C\}$, however A $\times$ {0} $\neq$ {0} in this product.

Let P = $\{3 + i_F, I_{15+5i_F}^C, I_0^C, 4 + I_0^C, 5 + 3i_F + I_0^C\}$ and Q = $\{0, I_0^C, I_4^C, I_{5i_F}^C\} \in$ S

P $\times$ Q = $\{3 + i_F, I_{15+5i_F}^C, I_0^C, 4 + I_0^C, 5 + 3i_F + I_0^C\} \times$
$\{0, I_0^C, I_4^C, I_{5i_F}^C\}$

$= \{0, I_0^C, I_4^C, I_{5i_F}^C, I_{15i_F+15}^C, I_{10i_F}^C\}$.

This is the way product operation is performed on S.

Let A = $\{I_{10i_F}^C, I_0^C, 5 + 5i_F + I_2^C\ I_4^C + 6 + 8i_F\}$ and

B = $\{I_4^C, 5 + 3i_F, I_{8i_F}^C\} \in S$,

$A \times B = \{I_{10i_F}^C, I_0^C, 5 + 5i_F, I_2^C, I_4^C + 6 + 8i_F\} \times \{I_4^C, I_{8i_F}^C, 5 + 3i_F\}$

$= \{I_0^C, I_4^C + I_8^C, I_{16}^C + I_4^C, I_{8i_F}^C + I_{16i_F}^C, I_{12i_F}^C + I_{8i_F}^C, I_{10i_F}^C, I_2^C, I_4^C + 6\}$.

This is the way product operation is performed on S.

Clearly P = $\{Z_{20}\}$ is a MOD natural neutrosophic finite complex number subset which is an idempotent $\{C(Z_{20})\} = B \in S$ is again a MOD natural neutrosophic finite complex number subset idempotent of S.

Let M = {collection of all subsets from the set $Z_{20}\} \subseteq S$ be a MOD natural neutrosophic finite complex number subset subsemigroup which is an ideal.

N = {collection of all subsets of the set $C(Z_{20})\} \subseteq S$ is again a MOD natural neutrosophic finite complex number subset subsemigroup which is not an ideal of S.

So S has MOD natural neutrosophic finite complex number subset subsemigroups which are not ideals.

***Example 3.34:*** Let W = {collection of all subsets from $S(C^I(Z_{11}))$, ×} be the MOD natural neutrosophic finite complex number subset semigroup under '×' where × is MOD natural neutrosophic zero dominated that is $0 \times I_0^C = I_0^C$.

We see C = {collection of all subsets from $Z_{11}$} ⊆ W is a MOD natural neutrosophic finite complex number subset subsemigroup of W.

D = {collection of all subsets from $C(Z_{11})$} ⊆ W is again a MOD natural neutrosophic finite complex number subset subsemigroup of W.

Both C and D are deals of W.

Finding MOD natural neutrosophic zero divisors and nilpotents in W happens to be a difficult task.

However M = {$Z_{11}$} ∈ W is a MOD subset natural neutrosophic finite complex number idempotent of W.

N = {$C(Z_{11})$} ∈ W is also a MOD subset natural neutrosophic finite complex number idempotent of W.

Thus if in {$S(C^I(Z_p))$} p a prime finding MOD natural neutrosophic subset finite complex number zero divisors and nilpotents is left as a open conjecture.

However for p = 2 we have $S(C^I(Z_2))$ has MOD natural neutrosophic finite complex number nilpotents of order two. Also $S(C^I(Z_2))$ has MOD natural neutrosophic finite complex number subset zero divisors as A = {0, 1 +$i_F$} and B = {1 + $i_F$} gives A × B = {0}.

Further $S(C^I(Z_2))$ has MOD natural neutrosophic finite complex number MOD natural neutrosophic finite complex number mixed zero divisors.

For take A = {0, $I_0^C$, 1 + $i_F$} and

B = {$I_0^C$, 1 + $i_F$} ∈ {$S(C^I(Z_2))$, ×}.

$A \times B = \{0\ I_0^C, 1 + i_F\} \times \{I_0^C, 1 + i_F\} = \{I_0^C, 0\}$.

If $A = \{I_0^C, 1 + i_F\} \in \{S(C^I(Z_2)), \times\}$, then

$A \times A = \{I_0^C, 1 + i_F\} \times \{I_0^C\ 1 + i_F\} = \{I_0^C, 0\}$ is the MOD natural neutrosophic subset finite complex number mixed nilpotent of order two.

In view of all these we are forced to suggest in $\{S(C^I(Z_p)), \times\}$, $\{p > 2$, p a prime$\}$ finding MOD natural neutrosophic finite complex number real zero divisors $\{0\}$, or MOD natural neutrosophic finite complex number subset zero divisors $\{I_0^C\}$ or MOD natural neutrosophic finite complex number mixed zero divisors $\{I_0^C, 0\}$ happens to be a challenging problem.

However we are always guaranteed of MOD natural neutrosophic finite complex number subsets idempotents and subsemigroups in case of $(S(C^I(Z_n)), \times\}$ whatever be n.

In view of these we have the following theorem.

**THEOREM 3.12:** *Let $M = \{S(C^I(Z_n)), \times\}$ be the MOD natural neutrosophic finite complex number under MOD natural neutrosophic finite complex number zero $I_0^C$ dominating.*

a) *M has MOD natural neutrosophic finite complex number subset idempotents.*
b) *M has MOD natural neutrosophic finite complex number subset subsemigroups which are not ideals.*
c) *M has MOD natural neutrosophic finite complex number subset nilpotents and zero divisors only if n is a suitable composite number.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD natural neutrosophic dual number subset semigroup under MOD natural neutrosophic dual number zero product that is $I_0^g \times 0 = I_0^g = 0 \times I_0^g$ by some examples.

***Example 3.35:*** Let S = {S(⟨$Z_{14} \cup g$⟩$_I$), ×} by the MOD natural neutrosophic dual number subset semigroup under MOD natural neutrosophic dual number zero product that is

$I_0^g \times 0 = 9 \times I_0^g = I_0^g$

A = {2g, 3g, 5g, 8g, g} and B = {8g, 10g, 12g, 6g} ∈ S.

Clearly A × B = {0} is a MOD natural neutrosophic dual number subset zero divisor.

Let C = $\{I_0^g, I_{2g}^g, I_{4g}^g\}$ and D = $\{0, I_{3g}^g, I_{11g}^g, I_{12g}^g\}$ ∈ S;

C× D = $\{I_0^g\}$ is the MOD natural neutrosophic dual number zero divisor pair which does not give the real zero.

Thus these MOD natural neutrosophic dual number subset semigroups has pairs of zero divisors which may yield either {0} or $\{I_0^g\}$.

Infact S has several MOD natural neutrosophic dual number subsets which are real nilpotent of order two or MOD natural neutrosophic nilpotents of order two.

Let A = {5g} ∈ S, A × A = {0}.
Let B = $\{I_{4g}^g\}$ ∈ S, B × B = $\{I_0^g\}$.
Let D = {3g, 8g} ∈ S; D × D = {0}.
Let M = $\{I_{2g}^g, I_{5g}^g, I_{13g}^g\}$ ∈ S.

Clearly M × M = $\{I_0^g\}$.

Finding zero divisors of the other form happens to be a difficult task.

Let P = {collection of all subsets from $Z_{14}$} $\subseteq$ S; P is only a MOD natural neutrosophic dual number subset subsemigroup which is not an ideal of S.

B = {collection of all subsets from $\langle Z_{14} \cup g\rangle$} $\subseteq$ S; B is only a MOD natural neutrosophic dual number subset subsemigroup which is not an ideal of S.

Similarly W = {collection of all subsets from the set $Z_{14}g$} $\subseteq$ S is only a MOD natural neutrosophic dual number subset subsemigroup of S, which is not an ideal but W is also a zero square semigroup as W × W = {0}.

Infact for every a, b $\in$ W a × b = 0.

Let us give one more example of the situation before we proceed onto enumerate their related properties.

***Example 3.36:*** Let
V = {collection of all subsets from the set $S(\langle Z_{19} \cup g\rangle_I\}; \times\}$ be the MOD natural neutrosophic dual number subset semigroup.

This also has MOD natural neutrosophic dual number subset zero divisor pairs A, B $\in$ V, A × B = {0}; also has MOD natural neutrosophic dual number subset, MOD natural neutrosophic zero pairs that is A, B $\in$ V; with A × B = $\{I_0^g\}$.

Finally a mixed MOD natural neutrosophic dual number subset pair given by A × B = $\{I_0^g, 0\}$; A, B $\in$ V.

We make it clear that, even though 19 is a prime number we see V has MOD natural neutrosophic dual number zero divisors and MOD natural neutrosophic dual number nilpotents of order two.

In view of all these we have the following theorem.

**THEOREM 3.13:** *Let S = {S(⟨$Z_n$ ∪ g⟩$_I$); ×} be the MOD natural neutrosophic dual number subset semigroup in which the product is MOD natural neutrosophic dual number zero dominated that is* $I_0^g \times 0 = I_0^g = 0 \times I_0^g$.

*Then the following are true.*

i) *S has MOD natural neutrosophic dual number subset real zero divisor pairs.*
ii) *S has MOD natural neutrosophic dual number subset MOD natural neutrosophic dual number zero divisor pair; that is A × B ={*$I_0^g$*} for A, B ∈ S.*
iii) *S has MOD natural neutrosophic dual number subset subsemigroups which are not ideals.*
iv) *S has MOD natural neutrosophic dual number subset idempotents.*
v) *S has MOD natural neutrosophic dual number subset subsemigroups which are zero square semigroups.*

The proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe the MOD natural neutrosophic special dual like number subset semigroups by examples.

***Example 3.37:*** Let Z = {S(⟨$Z_{10}$ ∪ h⟩$_I$), ×} be the MOD natural neutrosophic special dual like number subset semigroup.

Let A = {5 + 2h, 5h, 3h, $I_0^h$, $I_4^h$} and

B = {3h, 2h + 4, $I_{2h}^h$, $I_0^h$, 5} ∈ Z;

$A \times B = \{5 + 2h, 3h, 5h, I_0^h, I_4^h\} \times \{3h, 2h + 4, I_{2h}^h, 5, I_0^h\}$
$= \{h, 0, 9h, 5h, I_0^h, I_4^h, 2h, 8h, I_{2h}^h\}$.

This is the way product operation is performed on Z. Here $0 \times I_0^h = I_0^h = I_0^h \times 0$.

Let A = {5h, 5, 5 + 5h, 0} and B = {2, 2h, 6h, 8h, 4h, 6h + 2, 8h + 8, 2 + 8h} ∈ Z. A × B = {0}; that is the pair A, B ∈ Z gives the MOD natural neutrosophic special dual like number subset real zero divisor.

Let $P = \{I_4^h, I_2^h, I_{2+4h}^h, I_{6h}^h\}$ and $Q = \{I_5^h, I_0^h, I_{5h}^h, I_{5+5h}^h\} \in$ Z.

$P \times Q = \{I_0^h\}$, so that pair P, Q ∈ Z contributes to the MOD natural neutrosophic special dual like number subset pair the product of which yields MOD natural neutrosophic special dual like number zero.

$T = \{Z_{10}\} \in S$ is such that $T \times T = \{Z_{10}\}$.

Thus T is a MOD natural neutrosophic special dual like number subset idempotent of Z.

Similarly $R = \{Z_{10}h\}$ is also a MOD natural neutrosophic special dual like number subset idempotent of Z.

$M = \{\langle Z_{10} \cup h\rangle\}$ is also a MOD natural neutrosophic special dual like number subset idempotent of Z.

However finding nilpotents of MOD natural neutrosophic special dual like number subset happens to be a challenging one.

Let N = {collection of all subsets from $Z_{10}$} ⊆ Z be the MOD natural neutrosophic special dual like number subset subsemigroups of Z which is not an ideal of Z.

B = {collection of all subsets from the set $Z_{10}h$} $\subseteq$ Z be the MOD natural neutrosophic special dual like number subset subsemigroup of Z which is not an ideal.

***Example 3.38:*** Let B = $\{S(\langle Z_{17} \cup h\rangle_I); \times\}$ be the MOD natural neutrosophic special dual like number subset semigroup.

This B also has MOD natural neutrosophic special dual like number zero divisors as well as MOD natural neutrosophic special dual like number idempotents.

Thus if T = $\{Z_{17}\} \in$ B. T is such that T × T = T; that is T is a MOD natural neutrosophic special dual like number idempotent subset of B.

R = $\{Z_{17}h\} \in$ B is again MOD natural neutrosophic special dual like number idempotent subsaet of B as R × R = R.

Similarly, W = $\{\langle Z_{17} \cup h\rangle\} \in$ B is again a MOD natural neutrosophic special dual like number idempotent subset of B as W × W = W.

D ={collection of all subsets from the set $Z_{17}$} $\subseteq$ B is again a MOD natural neutrosophic special dual like number subset subsemigroup of B which is not an ideal of B.

We see finding MOD natural neutrosophic special dual like number subsets of zero divisors and nilpotents happens to be a difficult problem.

**Problem 3.4:** Let M = $\{S(\langle Z_n \cup h\rangle_I), \times\}$ be the MOD natural neutrosophic special dual like number subset semigroup under the MOD natural neutrosophic special dual like number zero dominated product, that is $I_0^h \times 0 = 0 \times I_0^h = I_0^h$.

a) If n is a prime, does M contain MOD natural neutrosophic special dual like number subset pairs A, B $\in$ M such that

   i) $A \times B = \{0\}$.
   ii) $A \times B = \{I_0^h\}$.
   iii) $A \times B = \{0, I_0^h\}$.

b) If n is a prime can M contain A such that $A \times A = \{0\}$, $A \times A = \{I_0^h\}$ or $A \times A = \{0, I_0^h\}$.

Next we proceed onto describe the MOD natural neutrosophic special quasi dual number subset semigroup under MOD natural neutrosophic zero product that is
$I_0^k \times 0 = 0 \times I_0^k = I_0^k$, by examples.

***Example 3.39:*** Let S = $\{(\langle Z_{12} \cup k\rangle_I), \times\}$ be the MOD natural neutrosophic subset special quasi dual number semigroup under MOD natural neutrosophic zero dominant product,
$I_0^k \times 0 = 0 \times I_0^k = I_0^k$.

Let A = $\{6k + 6, 3k, 6k, 3, 6\}$ and
B = $\{4, 4k, 8k, 8 + 4k, 4 + 8k\} \in S$, $A \times B = \{0\}$.

Let A = $\{I_6^k, I_{6k}^k, I_0^k, I_{9+9k}^k, I_9^k\}$ and
B = $\{I_4^k, I_{4+4k}^k, I_{8k}^k, I_{8k}^k, I_{8k+4}^k\} \in S$; $A \times B = \{I_0^k\}$ is the MOD natural neutrosophic special quasi dual number, MOD natural neutrosophic special quasi dual zero.

A = $\{I_0^k, 0, I_6^k, I_{6+6k}^k\}$ and B = $\{0, I_4^k, I_{4+4k}^k, I_{8k}^k\} \in S$.

Clearly $A \times B = \{0, I_0^k\}$ is the mixed MOD natural neutrosophic special quasi dual number zero.

Let D = $\{Z_{12}\}$ be the subset of S; D × D = D is a MOD natural neutrosophic special quasi dual number subset idempotent of S.

Let E = $\{Z_{12}^{I}\}$ be the subset of S, E × E = E is a MOD natural neutrosophic special quasi dual number subset idempotent of S.

Let F = $\{\langle Z_{12} \cup k\rangle\}$ be the subset of S, F × F = F is a MOD natural neutrosophic special quasi dual number subset idempotent of S.

Let G = {collection of all subsets of $\langle Z_{12} \cup k\rangle$} ⊆ S be the MOD natural neutrosophic special quasi dual number subset subsemigroup of S and not an ideal of S.

H = {collection of all subsets of $Z_{12}k$} ⊆ S be the MOD natural neutrosophic special quasi dual number subset subsemigroup of S and not an ideal of S.

Let K = {0, 6k, 6 + 6k, 6} ∈ S; K × K = {0} is a MOD natural neutrosophic special quasi dual number nilpotent subset of S.

Let L = $\{I_0^k, I_6^k, I_{6k}^k\}$ ∈ S, L × L = $\{I_0^k\}$ is the MOD natural neutrosophic special quasi dual number subset MOD natural neutrosophic quasi dual number nilpotent.

***Example 3.40:*** Let B = {$S(\langle Z_7 \cup k\rangle_I)$ be the collection of all MOD natural neutrosophic special quasi dual number subset, ×} be the MOD natural neutrosophic special quasi dual number subset under the MOD natural neutrosophic special quasi dual number zero divisor product dominated semigroup that is $I_0^k \times 0 = 0 \times I_0^k = I_0^k$.

Let $P_1$ = {0, 1, 2, 3, 4, 5, 6} ∈ B.

Clearly $P_1 \times P_1 = P_1$ so $P_1$ is a MOD natural neutrosophic special quasi dual number idempotent subset of B.

Let $P_2 = \{0, k, 2k, 3k, 4k, 5k, 6k\} \in B$.

We see $P_2 \times P_2 = P_2$ so $P_2$ is a MOD natural neutrosophic special quasi dual number idempotent subset of B.

$P_3 = \{Z_7^I\} \in B$ is such that $P_3 \times P_3 = P_3$ is the MOD natural neutrosophic special quasi dual number idempotent subset of B.

$P_4 = \{\langle Z_7 \cup k\rangle_I\} \in B$ is a MOD natural neutrosophic special quasi dual number idempotent subset of B as $P_4 \times P_4 = P_4$.

Now finding MOD natural neutrosophic special quasi dual number nilpotent subsets or MOD natural neutrosophic special quasi dual number subset zero divisor pairs yielding $\{0\}$ or $\{I_0^k\}$ or $\{I_0^k, 0\}$ happens to be a difficult task.

Further M = {collection of all subsets of $Z_7$} $\subseteq$ B is a MOD subset natural neutrosophic special quasi dual number subsemigroup of B.

Let N = {collection of all subsets of $Z_7k$} $\subseteq$ B; N is a MOD natural neutrosophic special quasi dual number subset subsemigroup of B.

T = {collection of all subsets from the set $\langle Z_7 \cup k\rangle$} is the MOD natural neutrosophic special quasi dual number subset subsemigroup of B.

None of them are ideals of B.

In view of all these we propose the following problem.

**Problem 3.5:** Let
S = {{collection of all subsets from $\langle Z_p \cup k\rangle_I$} = $S(\langle Z_p \cup k\rangle_I)$; $\times$} be the MOD natural neutrosophic special quasi dual number

subset semigroup under MOD natural neutrosophic special quasi dual number zero dominant product operation. That is $I_0^k \times 0 = 0 \times I_0^k = I_0^k$.

i) Can S have MOD natural neutrosophic special quasi dual number nilpotent subsets?
ii) Can S have MOD natural neutrosophic special quasi dual number subset pairs A, B such that $A \times B = \{0\}$ or $A \times B = \{I_0^k\}$ or $A \times B = \{I_0^k, 0\}$?

In view of all these we have the following result.

**THEOREM 3.14:** *Let G = {collection of all subsets from the set $S(\langle Z_n \cup k\rangle_I)$, ×} be the MOD natural neutrosophic special quasi dual number subset under MOD natural neutrosophic special quasi dual number zero dominant product semigroup.*

*i) G has MOD natural neutrosophic special quasi dual number idempotent subsets for all $2 \leq n < \infty$.*
*ii) G has MOD natural neutrosophic special quasi dual number subset subsemigroup which are not ideals; $2 \leq n < \infty$.*
*iii) G has MOD natural neutrosophic special quasi dual number subsets A in G such that $A \times A = \{0\}$ or $A \times A = \{I_0^k\}$ or $A \times A = \{0, I_0^k\}$ nilpotents of order two; n an appropriate composite number.*
*iv) G has MOD natural neutrosophic special quasi dual number subset pairs A, B ∈ G such that they are MOD natural neutrosophic special quasi dual number zero divisors, that is $A \times B = \{0\}$ or $A \times B = \{I_0^k\}$ or $A \times B = \{I_0^k, 0\}$; for all composite number n; $n \geq 2$.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe by examples the MOD natural neutrosophic neutrosophic subset semigroups under product in which $I_0^I \times 0 = 0 \times I_0^I = 0$ or $I_0^I$.

***Example 3.41:*** Let M = {S($\langle Z_{16} \cup I\rangle_I$), be the collection of all subsets of $\langle Z_{16} \cup I\rangle_I$, ×} be the MOD natural neutrosophic-neutrosophic subset semigroup under the MOD natural neutrosophic-neutrosophic subset zero dominant product that is $I_0^I \times 0 = I_0^I$,

Let $A = \{1 + I, 6 + 2I, I_{4I}^I, 3 + I_{8I}^I, 0\}$ and
$B = \{I_0^I, I_{4+4I}^I, 6 + 4I, 8I\} \in M$.

$$A \times B = \{1 + I, 6 + 2I, I_{4I}^I, 3 + I_{8I}^I, 0\} \times \{I_0^I, I_{4+4I}^I, 6 + 4I, 8I\}$$

$$= \{I_0^I, I_{4+4I}^I, I_{4I}^I + I_0^I, 0, 6 + 14I, 4 + 4I, 2 + 12I + I_{8I}^I, I_{4I}^I, 8I + I_{8I}^I\} \in M,$$

this the way product operation is performed on M.

Let $A = \{4I, 4 + 4I, 8I, I_0^I\}$ and $B = \{4, 4I, 8\} \in M$.

$$A \times B = \{4I, 4 + 4I, 8I, I_0^I\} \times \{4, 4I, 8\} = \{0, I_0^I\}$$

is the MOD natural neutrosophic - neutrosophic mixed zero divisor in M.

Let $T = \{4, 4I, 4 + 4I, 8 + 8I\} \in M$.

$T \times T = \{0\}$; thus T is a MOD natural neutrosophic neutrosophic nilpotent subset of order two.

Let $P = \{I_4^I, I_{8I}^I, I_{4+4I}^I, I_{8I+8}^I, I_{4+8I}^I, I_{4+4I}^I\} \in M$; clearly

$P \times P = \{I_0^I\}$; that is P is a MOD natural neutrosophic - neutrosophic subset nilpotent of MOD natural neutrosophic - neutrosophic zero as $P \times P = \{I_0^I\}$.

Study in this direction is innovative and interesting.

B = $\{Z_{16}\}$ ∈ M is the MOD natural neutrosophic-neutrosophic subset idempotent of M as B × B = B.

Further A = $\{Z_{16}I\}$ ∈ M is again a MOD natural neutrosophic - neutrosophic subset idempotent of M as A × A = A.

Thus M has MOD natural neutrosophic - neutrosophic subset nilpotents and idempotents.

Now W = {collection of all subset of $Z_{16}$} ⊆ M is a MOD natural neutrosophic - neutrosophic subset subsemigroup of M but is not an ideal of M.

V = {collection of all subsets from the set $\langle Z_{16} \cup I\rangle_I$} ⊆ M is also a MOD natural neutrosophic - neutrosophic subset subsemigroup of M which is not an ideal.

Let D = {collection of all subsets of $\{0, I_0^I, I_t^I; t \in \langle Z_{16} \cup I\rangle_I$ is either a nilpotent or idempotent or a zero divisor in $\langle Z_{16} \cup I\rangle\}$ ⊆ M is a MOD natural neutrosophic subset subsemigroup which is an ideal of M.

Infact this ideal D of M is defined as the pure MOD natural neutrosophic - neutrosophic ideal of M.

***Example 3.42:*** Let
G = {{collection of all subsets from the set $\langle Z_{11} \cup I\rangle_I$}, ×} be the MOD natural neutrosophic - neutrosophic subset semigroup under the MOD natural neutrosophic neutrosophic zero dominant product, that is $I_0^I \times 0 = 0 \times I_0^I = I_0^I$.

Finding nontrivial MOD natural neutrosophic - neutrosophic subset nilpotents is a challenging task.

For in $\langle Z_2 \cup I\rangle_I$.

$P = \{1 + I\} \in \langle Z_2 \cup I\rangle_I$ is such that $P \times P = \{0\}$.

However $B_1 = \{Z_{11}\} \in G$ is a MOD natural neutrosophic - neutrosophic idempotent subset of G.

$B_2 = \{Z_{11}I\} \in G$ is again a MOD natural neutrosophic - neutrosophic idempotent subset of G.

$B_3 = \{\langle Z_{11} \cup I\rangle\} \in G$ is again a MOD natural neutrosophic - neutrosophic idempotent of G.

Further $P_1$ = {collection of all subsets from the set $Z_{11}$} $\subseteq G_1$ is a MOD natural neutrosophic - neutrosophic subsemigroup of G which is not an ideal of G.

$P_2$ = {collection of all subsets from the set $Z_{11}I$} $\subseteq$ G is also a MOD natural neutrosophic - neutrosophic subsemigroup of G which is not an ideal of G.

$P_3$ = {collection of all subsets from the set $\langle Z_{11} \cup I\rangle$} $\subseteq$ G is a MOD natural neutrosophic - neutrosophic subset subsemigroup of G which is not an ideal of G.

Thus we propose the following problem.

**Problem 3.6:** Let
K = {collection of all subsets from the set $\langle Z_p \cup I\rangle_I$; p a prime, ×} be the MOD natural neutrosophic - neutrosophic semigroup under product dominated by MOD natural neutrosophic - neutrosophic zero $I_0^I$; ($I_0^I \times 0 = I_0^I = 0 \times I_0^I$).

i) Can K contain non trivial MOD natural neutrosophic - neutrosophic nilpotents subsets A in K such that $A \times A = \{0\}$ or $A \times A = \{I_0^I\}$ or $A \times A = \{0, I_0^I\}$?
ii) Can K contain nontrivial MOD natural neutrosophic-neutrosophic zero divisor subset pairs A, B $\in$ K,

such that
$A \times B = \{0\}$ or $A \times B = \{I_0^I\}$ or $A \times B = \{0, I_0^I\}$?

iii) Can $P = \{I_0^I, 0, I_t^I / t \in \langle Z_p \cup I\rangle$, where t is either a nilpotent or an idempotent or a zero divisor of $\langle Z_p \cup I\rangle\}$ be an ideal such that $o(P) > p + 1$?

Now we proceed onto express the following result.

**THEOREM 3.15:** *Let $S = \{S(\langle Z_n \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic - neutrosophic subset semiring under the MOD natural neutrosophic - neutrosophic zero dominated product (i.e. $I_0^I \times 0 = 0 \times I_0^I = 0$). Then the following are true.*

*i) S has MOD natural neutrosophic - neutrosophic subset idempotents.*

*ii) S has MOD natural neutrosophic neutrosophic subsets $A \in S$ such that $A \times A = \{0\}$ or $A \times A = \{I_0^I\}$ or $A \times A = \{0, I_0^I\}$, i.e. nilpotent subsets of order two (n an appropriate composite number).*

*iii) S has MOD natural neutrosophic neutrosophic subset pair A, B in S such that $A \times B = \{0\}$ or $A \times B = \{I_0^I\}$ or $A \times B = \{0, I_0^I\}$ (For n - a composite number).*

*iv) S has MOD natural neutrosophic neutrosophic subset subsemigroups which are not ideals.*

*v) S has MOD natural neutrosophic neutrosophic subset ideal given by $P = \{I_0^I, 0, I_t^I / t \in \langle Z_n \cup I\rangle$, where t is an idempotent or nilpotent or a zero divisor or elements of the form $a^m = a$, m a positive integer.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto propose some problems for the interested reader.

**Problems**

1. Find all MOD natural neutrosophic subset subsemigroups of H = {S($Z_{24}^{I}$), +} where H is the MOD natural neutrosophic subset semigroup under +.

   i) Find o(H).
   ii) Find all MOD natural neutrosophic subset universal pairs of H.
   iii) Can H have MOD natural neutrosophic subset idempotents?

2. Let G = {S($Z_{16}^{I}$), +} be the MOD natural neutrosophic subset semigroup under +.

   i) Find o(G).
   ii) How many MOD natural neutrosophic subset pairs A, B ∈ G with A + B = $Z_{16}^{I}$ ?
   iii) Find all MOD natural neutrosophic subset idempotent of G.
   iv) Find all MOD natural neutrosophic subset subsemigroups of G.

3. Let K = {S($Z_{19}^{I}$), +} be the MOD natural neutrosophic subset semigroup under +.

   i) Study questions (i) to (iv) of problem (2) for this K.
   ii) For given A is K how many B ∈ K exists such that A + B = $Z_{19}^{I}$.

4. Enumerate all special and interesting features associated with S = {S($Z_{n}^{I}$), +} the MOD natural neutrosophic subset semigroup under +.

5. Let W ={S($C^{I}(Z_{10})$), +} be the MOD natural neutrosophic finite complex number subset semigroup under +.

i) Study questions (i) to (iv) of problem (2) for this W.
ii) Compare W with V = $\{S(Z_{10}^{I}), +\}$ as MOD natural neutrosophic subset semigroups.
iii) Prove o(V) < o(W).

6. Let M = $\{S(C^{I}(Z_{23})), +\}$ be the MOD natural neutrosophic finite complex number subset semigroup under +.

i) Study questions (i) to (iv) problem (2) for this M.
ii) Study the special and distinct features enjoyed by M.

7. Let N = $\{S(C^{I}(Z_{243})), +\}$ be the MOD natural neutrosophic subset finite complex number semigroup under +.

i) Study questions (i) to (iv) of problem (2) for this N.
ii) Compare N with T = $\{S(Z_{243}^{I}), +\}$ and derive the special features associated with N.

8. Let W = $\{S(\langle Z_{15} \cup I\rangle_I), +\}$ be the MOD natural neutrosophic- neutrosophic subset semigroup under +.

i) Study questions (i) to (iv) of problem (2) for this W.
ii) Compare W with N = $\{S(^{I}(Z_{15})), +\}$ and M = $\{S(Z_{15}^{I}), +\}$.
iii) Will o(N) > o(W) or o(W) > o(N)?

9. Let B = $\{S(\langle Z_{43} \cup I\rangle_I), +\}$ be the MOD natural neutrosophic -neutrosophic subset semigroup under +.

Study questions (i) to (iv) of problem (2) for this B.

10. Let D = $\{S(\langle Z_{64} \cup I\rangle_I), +\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup under +.

Study questions i) to (iv) of problem (2) for this D.

11. Obtain all special and distinct features enjoyed by $S = \{(\langle Z_n \cup I\rangle_I), +\}$.

12. Let $E = \{S(\langle Z_9 \cup g\rangle_I), +\}$ be the MOD natural neutrosophic dual number subset semigroup under +.

    i) Study questions (i) to (iv) of problem (2) for this E.
    ii) If $A = \{5g + 4, I_3^g, I_{5g}^g, 8 + I_{4g}^g\} \in E$ find all B in E such that $A + B = \langle Z_9 \cup g\rangle_I$.

13. Let $Z = \{S(\langle Z_{43} \cup g\rangle_I), +\}$ be the MOD natural neutrosophic subset dual number semigroup under +.

    i) Study questions (i) to (iv) of problem (2) for this Z.
    ii) Compare Z with $P = \{S(Z_{43}^I), +\}$, the MOD natural neutrosophic subset semigroup.
    iii) Compare Z with $T = \{S(\langle Z_{43} \cup I\rangle_I), +\}$ the MOD natural neutrosophic-neutrosophic subset semigroup
    iv) Compare Z with $S = \{S(C^I(Z_{43}), +\}$ the MOD natural neutrosophic finite complex number subset semigroup.
    v) Let $A = \{I_{42g}^g, I_0^g, I_{2g}^g + I_{3g}^g + 6g + I_{10g}^g, 3 + 4g, 40 + 29g, 27 + 18g, I_{23g}^g + 14g + 12\} \in Z$; how many B in Z are such that $A + B = \langle Z_{43} \cup g\rangle_I$?

14. Let $F = \{S(\langle Z_{48} \cup g\rangle_I), +\}$ be the MOD natural neutrosophic dual number subset semigroup under +.

    i) Study questions (i) to (iv) problem (2) for this F.
    ii) Can we say if in $\langle Z_n \cup g\rangle_I$ n is a composite number $S(\langle Z_n \cup g\rangle_I)$ has more number of MOD natural neutrosophic dual number subset subsemigroups?

15. Does H = $\{S(\langle Z_n \cup g\rangle_I), +\}$ enjoy any other special features than S = $\{S(\langle Z_n \cup I\rangle_I, +\}$ or M = $\{S(Z_n^I), +\}$ or N = $\{S(C^I(Z_n)), +\}$?

16. Let T = $\{S(\langle Z_8 \cup h\rangle_I), +\}$ be the MOD natural neutrosophic special dual like number subset semigroup under +.

   i) Study questions (i) to (iv) of problem (2) for this T.
   ii) Obtain all the special and distinct features enjoyed by T.
   iii) Let P = $\{7h, I^g_{2+4h}, I^h_0, I^h_{6h+2}, I^h_4, 3 + 4h + I^h_2, 2, 5h + 3, 7 + h\} \in T$.
   Find all Q in T such that P + Q = $\langle Z_8 \cup h\rangle_I$.

17. Let Y = $\{S(\langle Z_{11} \cup h\rangle_I), +\}$ be the MOD natural neutrosophic special dual like number subset semigroup under +.

   i) Study questions (i) to (iv) of problem (2) for this Y.
   ii) Compare T in problem (16) with this Y.

18. Let R = $\{S(\langle Z_{12} \cup h\rangle_I), +\}$ be the MOD natural neutrosophic special dual like number subset semigroup under +.

   i) Study questions (i) to (iv) of problem (2) for this R.
   ii) Compare this R with T and Y of problems (16) and (17) respectively.

19. Obtain all special and distinct features associated with M = $\{S(\langle Z_n \cup h\rangle_I), +\}$ the MOD natural neutrosophic special dual like number subset semigroup under +.

20. Let S = $\{S(\langle Z_4 \cup k\rangle_I), +\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under +.

i) Study questions (i) to (iv) of problem (2) for this S.
ii) Compare S with T = $\{S(Z_{14}^I), +\}$.

21. Let B = $\{S(\langle Z_{23} \cup k\rangle_I), +\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under +.

i) Study questions (i) to (iv) of problem (2) for this B.
ii) Compare this B with S in problem (20).

22. Let M = $\{S(\langle Z_{10} \cup k\rangle_I), +\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under +.

i) Study questions (i) to (iv) of problem (2) for this M.
ii) Let A = $\{5 + 5k, I_5^k, I_{2k+4}^k + 6k + 8, I_{4+4k}^k + I_8^k + I_{8k+6}^k + 4 + 5k\} \in M$, how many subsets B in M exist such that A + B = $\langle Z_{10} \cup k\rangle_I$.
iii) Find the least order of B in (ii) such that A + B = $\langle Z_{10} \cup k\rangle_I$.
iv) Does there exist more than one such B with least order?

23. Enumerate all special features associated with S = $\{S(\langle Z_n \cup k\rangle_I), +\}$ .

24. Let P = $\{S(Z_{12}^I), \times\}$ be the MOD natural neutrosophic subset semigroup under zero dominated product, that is $0 \times I_0^{12} = 0$.

i) Find o(P).
ii) How many MOD natural neutrosophic subset idempotents are in P?
iii) How many MOD natural neutrosophic subset nilpotents are in P?
iv) Find all MOD natural neutrosophic subsets zero divisor pairs of P.

v) Find all MOD natural neutrosophic subset subsemigroups of P which are not ideals of P.

vi) Find all MOD natural neutrosophic subset ideals of P.

vii) Find for any A ∈ P the total number of B in P such that A × B = $Z_{12}^{I}$; the MOD natural neutrosophic universal subset of P.

viii) Can we say for every Z ∈ P there exists at least one Y ∈ P with Z × Y = $Z_{12}^{I}$ ?

25. Let B = {S($Z_{19}^{I}$), ×} be the MOD natural neutrosophic subset semigroup which is real zero product dominated that is 0 × $I_0^{19}$ = $I_0^{19}$ × 0 = 0.

i) Study questions (i) to (viii) of problem (24) for this B.

ii) Can we have for this B MOD natural neutrosophic nilpotent subsets and MOD natural neutrosophic zero divisor subset pairs?

26. Let W = {S($Z_{64}^{I}$), ×} be the MOD natural neutrosophic subset semigroup under real zero dominated product that is $I_0^{64}$ × 0 =0.

Study questions (i) to (viii) of problem (24) for this W.

27. Let V = {S($C^I(Z_6)$), ×} be the MOD natural neutrosophic subset semigroup under zero dominated product, 0 × $I_0^C$ = 0.

i) Study questions (i) to (viii) of problem 24 for this V.

ii) Obtain all special and distinct features associated with V.

iii) Compare V with T = {S($Z_6^I$), ×} the MOD natural neutrosophic subset semigroup under ×, in which

$0 \times I_0^6 = 0$.

28. Let N = $\{S(C^I(Z_{43})), \times\}$ be the MOD natural neutrosophic finite complex number subset semigroup under real zero dominant product, that is $0 \times I_0^C = 0$.

    i) Study questions (i) to (viii) of problem (24) for this N.
    ii) Compare N with V of problem (27).

29. Enumerate all special and interesting features associated with S = $\{S(C^I(Z_n)), \times\}$ the MOD natural neutrosophic finite complex number subset semigroup under real zero dominant product, that is $0 \times I_0^C = 0$.

30. Let W = $\{S(\langle Z_{13} \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic neutrosophic subset semigroup under real zero dominant product, that is $0 \times I_0^C = 0$.

    i) Study questions (i) to (viii) of problem (24) for this W.
    ii) Obtain any other special feature associated with W.
    iii) Will P = $\{ I_0^C, 0, I_t^I$ where $t \in \langle Z_{13} \cup I\rangle$ and t is a zero divisor or a nilpotent or an idempotent, $\times\} \subseteq W$ be a MOD natural neutrosophic - neutrosophic subset ideal of W.

31. Let D = $\{S(\langle Z_{24} \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup under the real zero dominated product.

    i) Study questions (i) to (viii) of problem (24) fo this D.
    ii) Compare this D with W in problem (30).

32. Let E = $\{S(\langle Z_{5^6} \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup under the real zero dominated product.

i) Study questions (i) to (viii) of problem (24) for this E.

ii) Compare this E with D and W of problem (31) and (30) respectively.

33. Study all special and distinct features enjoyed by MOD natural neutrosophic - neutrosophic subset semigroup under real zero dominated product.

34. Let S = $\{S((\langle Z_{10} \cup g\rangle_I), \times\}$ be the MOD natural neutrosophic dual number subset semigroup under real zero dominant product, that is $0 \times I_0^g = I_0^g \times 0 = 0$.

i) Study questions (i) to (viii) of problem (25) for this S

ii) Can we say S has more number of MOD natural neutrosophic dual number subset nilpotents A such that $A \times A = \{0\}$ or $A \times A = \{I_0^g\}$ or $A \times A = \{0, I_0^g\}$?

iii) Can we conclude S will have more number of MOD natural neutrosophic dual number subsets (A, B) zero divisor pairs; that is $A \times B = \{0\}$ or $A \times B = \{I_0^g\}$ or $A \times B = \{0, I_0^g\}$?

iv) Compare S with $D = \{S(Z_{10}^I), \times\}$, $E = \{S(C^I(Z_{10})), \times\}$ and $F = \{S(\langle Z_{10} \cup g\rangle_I, \times\}$.

35. Let $R = \{S(\langle Z_{43} \cup g\rangle_I, \times\}$ be the MOD natural neutrosophic dual number subset semigroup under real zero dominant product.

i) Study questions (i) to (viii) of problem (24) for this R.

ii) Compare R with S of problem (34).

36. Enumerate all special and distinct features associated with MOD natural neutrosophic dual number subset semigroup P under real zero dominant product where
$P = \{S(\langle Z_n \cup g\rangle_I), \times\}$.

37. Let M = $\{S(\langle Z_{14} \cup h\rangle_I), \times\}$ be the MOD natural neutrosophic special dual like number subset semigroup under real zero dominant product, $I_0^h \times 0 = 0$.

    i) Study questions (i) to (viii) of problem (24) for this M.
    ii) Compare M with P = $\{S(Z_{14}^I), \times\}$ the MOD natural neutrosophic subset semigroup.
    iii) Compare M with V = $\{S(\langle Z_{14} \cup g\rangle_I), \times\}$ the MOD natural neutrosophic dual number subset semigroup.
    iv) Compare M with W = $\{S(\langle Z_{14} \cup I\rangle_I), \times\}$ the MOD natural neutrosophic - neutrosophic subset semigroup.

38. Let F = $\{S(\langle Z_{23} \cup h\rangle_I), \times\}$ be the MOD natural neutrosophic special dual like number subset semigroup under real zero dominated product.

    i) Study questions (i) to (viii) of problem (24) for this F.
    ii) Compare F with M of problem (37).

39. Describe all the distinct features enjoyed by MOD natural neutrosophic special dual like number subset semigroup S under real zero dominant product where
S = $\{S(\langle Z_n \cup h\rangle_I), \times\}$.

40. Let T = $\{S(\langle Z_4 \cup k\rangle_I), \times\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under real zero dominant product i.e. $I_0^k \times 0 = 0$.

    i) Study questions (i) to (viii) of problem (24) for this T.
    ii) Compare this T with P = $\{S(C^I(Z_4)), \times\}$ the MOD natural neutrosophic finite complex number subset semigroup with real zero dominated product.

iii) Compare T with W = $\{S(\langle Z_4 \cup g\rangle_I), \times\}$ the MOD natural neutrosophic dual number subset semigroup under the real zero dominant product.

41. Let G = $\{S(\langle Z_{17} \cup k\rangle_I), \times\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under the real zero dominated product.

    i) Study questions (i) to (viii) of problem (24) for this G.
    ii) Compare this G with T of problem (40).

42. Let H = $\{S(\langle Z_{28} \cup k\rangle_I), \times\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under real zero dominated product.

    i) Study questions (i) to (viii) of problem (24) for this H.
    ii) Compare this H with G and T of problems (41) and (40) respectively.

43. Let J = $\{S(\langle Z_n \cup k\rangle_I), \times\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under real zero product dominated semigroup.

    Enumerate all special and distinct features enjoyed by J and compare it with H = $\{S(\langle Z_n \cup I\rangle_I), \times\}$, K = $\{S(C^I(Z_n)), \times\}$ and L = $\{S(\langle Z_n \cup g\rangle_I), \times\}$.

44. Let G = $\{S(Z_{20}^I), \times\}$ be the MOD natural neutrosophic subset semigroup under MOD natural neutrosophic zero dominated product.

    i) Study questions (i) to (viii) of problem (24) for this G.
    ii) Compare G with the MOD natural neutrosophic product semigroup which is defined under real zero product.

45. Let S = {S($Z_{47}^{I}$), ×} be the MOD natural neutrosophic subset semigroup under the MOD natural neutrosophic zero dominated product, that is $I_0^{47} \times 0 = I_0^{47}$.

i) Study questions (i) to (viii) of problem (24) for this S.

ii) If A = { $I_0^{47}$, 3 + $I_0^{47}$, 2, 5, 7, 9, 12} ∈ S find a B in S so that A × B = $Z_{47}^{I}$, the MOD universal subset of S.

a) How many such B ∈ S?
b) What is the least order of such B?
c) What is the greatest order of such B?

46. Let P = {($Z_{64}^{I}$), ×} be the MOD natural neutrosophic subset semigroup which is MOD natural neutrosophic zero dominated $I_0^{64} \times 0 = I_0^{64}$.

i) Study questions (i) to (viii) of problem (24) for this P.

ii) Find all MOD natural neutrosophic mixed zero divisors of P.

iii) Is A × {0, $I_0^{64}$} = {0, $I_0^{64}$} for A in P?

47. Let K = {S($Z_n^{I}$), ×} be the MOD natural neutrosophic subset semigroup under the MOD natural neutrosophic zero dominated product.

i) Study all special features enjoyed by K.

ii) If the product is under real zero dominated product compare both the semigroups for same S($Z_n^{I}$).

48. Let W = {S($C^I(Z_{48})$), ×} be the MOD natural neutrosophic finite complex number subset semigroup under the MOD natural neutrosophic zero dominated product.

i) Study questions (i) to (viii) of problem (24) for this W.

ii) If P = $\{I_{12}, I_0^C, I_{44}^C + 8 + 9i_F, I_{24+40i_F}^C, I_{10i_F}^C, 4, 8, 3, 39, 2, 1, I_{20}^C + I_{4i_F}^C + I_{14+12i_F}^C\} \in W$, find $Q \in W$ such that $P \times Q = C^I(Z_{48})$, the MOD natural neutrosophic finite complex number universal subset of W.

iii) For that given $P \in W$ how many such $Q \in W$ such that $P \times Q = C^I(Z_{48})$ exist?

49. Obtain all special and interesting features enjoyed by $S = \{S(C^I(Z_n)), \times\}$ the MOD natural neutrosophic finite complex number subset semigroup under MOD natural neutrosophic zero dominated product.

50. Let $G = \{S(\langle Z_{15} \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup under the MOD natural neutrosophic - neutrosophic zero dominated product, $I_0^I \times 0 = I_0^I$.

i) Study questions (i) to (viii) of problem (24) for this G.

ii) If W = $\{I_0^I, 5 + I_{5I}^I, I_{3+3I}^I + I_{9+6I}^I + 3 + 8I, 7 + 9I + I_{9I}^I, 2, 3, 5, 10, 7, 8\} \in G$ find $V \in G$ such that $W \times V = \langle Z_{15} \cup I\rangle_I$, the MOD natural neutrosophic - neutrosophic universal subset of G.

iii) How many such V's exist for this given W?

51. Let $H = \{S(\langle Z_{47} \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup under the MOD natural neutrosophic neutrosophic zero dominated product.

Study questions (i) to (viii) of problem (25) for this H.

52. Let $F = \{S(\langle Z_{5^{10}}^I \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup under the

MOD natural neutrosophic - neutrosophic zero dominated product.

i) Study questions (i) to (viii) of problem (24) for this F.
ii) Compare F with H and G of problems (51) and (50) respectively.

53. Describe all special features associated with MOD natural neutrosophic - neutrosophic subset semigroup under the MOD natural neutrosophic zero dominated product, $M = \{S(\langle Z_n \cup I\rangle_I), \times\}$.

Compare M with $P = \{S(\langle Z_n \cup I\rangle_I), \times\}$ where P enjoys the real zero dominated product, that is $0 \times I_0^I = 0$ in M.

54. Let $B = \{S(\langle Z_{18} \cup g\rangle_I), \times\}$ be the MOD natural neutrosophic dual number subset semigroup under the MOD natural neutrosophic dual number zero dominated product, that is $0 \times I_0^g = I_0^g$.

i) Study questions (i) to (viii) of problem (24) for this B.
ii) Let $T = \{I_0^g, 5g + I_{2g}^g, 9 + 8g + I_{6g+8}^g, I_{2g+4}^g + I_{6g+9}^g, I_{5g}^g, I_{7g}^g, 1\} \in B$. Find a V in B so that $T \times B = (\langle Z_{18} \cup g\rangle_I$, the MOD universal natural neutrosophic dual number subset of B.
iii) For this given T how many such B's exist.

55. Let $W = \{S(\langle Z_{128} \cup g\rangle_I), \times\}$ be the MOD natural neutrosophic dual number subset semigroup under the MOD natural neutrosophic dual number zero dominated product that is $I_0^g \times 0 = I_0^g$.

i) Study questions (i) to (viii) of problem (24) for this W.
ii) Compare the W with B in problem (54).

56. Let C = {S(⟨$Z_{53}$ ∪ g⟩$_I$), ×} be the MOD natural neutrosophic dual number subset semigroup under the MOD natural neutrosophic dual number zero dominated product, that is $I_0^g \times 0 = I_0^g$.

    i) Study questions (i) to (viii) of problem 24 for this C.
    ii) Compare C with W and B in problems 54 and 55 respectively.

57. Let S ={S(⟨$Z_n$ ∪ g⟩$_I$), ×} be the MOD natural neutrosophic dual number subset semigroup under the MOD natural neutrosophic dual zero number dominated product.

    What are the special features enjoyed by S as $g^2$ =,0 and $I_{rg}^g \times I_{sg}^g = I_0^g$ for any r, s ∈ $Z_n$

58. Let M = {S(⟨$Z_8$ ∪ h⟩$_I$), ×} be the MOD natural neutrosophic special dual like number subset semigroup under the MOD natural neutrosophic special dual like number zero dominant product that is $I_0^h \times 0 = I_0^h$.

    i) Study all special features associated with M.
    ii) Study questions (i) to (viii) of problem (24) for this M.
    iii) If H = {6 + $I_{2h}^h$, 0, 1, 7, 5, 3 + h + $I_{2+4h}^h$ } ∈ M find a K in M such that H × K = ⟨$Z_8$ ∪ h⟩$_I$), the MOD natural neutrosophic special dual like number universal subset.
    How many such K in M exists?

59. Let W ={S(⟨$Z_{20}$ ∪ h⟩$_I$), ×} be the MOD natural neutrosophic special dual like number subset semigroup under the product $I_0^h \times 0 = I_0^h$.

    Study questions (i) to (viii) of problem (24) for this W.

60. Enumerate all special and distinct features enjoyed by $S = \{S(\langle Z_n \cup h\rangle_I), \times\}$, $I_0^h \times 0 = I_0^h$, the MOD natural neutrosophic special dual like number subset semigroup.

Compare S with $R = \{S(\langle Z_n \cup g\rangle_I), \times\}$; $I_0^g \times 0 = I_0^g$, the MOD natural neutrosophic dual number subset semigroup.

61. Let $S = \{S(\langle Z_{27} \cup k\rangle_I), \times\}$ be the MOD natural neutrosophic quasi dual number subset semigroup under the MOD natural neutrosophic special quasi dual number zero dominated product, that is $I_0^k \times 0 = I_0^k$.

i) Study questions (i) to (viii) of problem (24) for this S.
ii) Compare S with $P = \{S(\langle Z_{27} \cup h\rangle_I), \times\}$ and $R = \{S(\langle Z_{27} \cup g\rangle_I), \times\}$.

62. Let $T = \{S(\langle Z_5 \cup k\rangle_I, \times\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under the product of $I_0^k \times 0 = I_0^k$.

i) Study questions (i) to (viii) of problem (24) for this T.
ii) Compare S in problem (61) with this T.
iii) If $A = \{4 + I_{2k}^k, 3 + I_0^k, I_k^k, 2, 3, 1\} \in T$ find all $B \in T$ such that $A \times B = \langle Z_5 \cup k\rangle_I$, the MOD natural neutrosophic special quasi dual number universal subset.

63. Let $V = \{S(\langle Z_{40} \cup k\rangle_I, \times\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup.

i) Study questions (i) to (viii) of problem (24) for this V.

ii) Compare V with W = $\{S(\langle Z_{40} \cup g\rangle_I), \times\}$ the MOD natural neutrosophic special quasi dual number subset semigroup.

iii) If D = $\{Z_{40}\} \in V$ does there exists a $E \in V$ such that $D \times E = \langle Z_{40} \cup k\rangle_I$ the MOD natural neutrosophic special quasi dual number universal subset of V.

64. Let S = $\{S(\langle Z_n \cup k\rangle_I), \times\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under the product $I_0^k \times 0 = I_0^k$.

i) If B = $\{Z_n k\} \in S$ find all A in S with $A \times B = \langle Z_n \cup k\rangle_I$.

ii) If T = $\{Z_n T\}$ find all $Q \in S$ with $Q \times T = \langle Z_n \cup k\rangle_I$.

iii) If V = $\langle Z_n \cup k\rangle$ find all W in S with $V \times W = \langle Z_n \cup k\rangle_I$, the MOD natural neutrosophic special quasi dual number universal subset of S.

Chapter Four

# SEMIGROUPS BUILT ON MOD SUBSET MATRICES AND MOD SUBSET POLYNOMIALS

In this chapter for the first time we introduce the new notion of MOD subset matrices of two types. Likewise we introduce the notion of MOD subset polynomials.

Let $S(Z_n)$ = {collection of all subsets of $Z_n$}.

$M = (a_{ij})_{s\times t}$ where $a_{ij} \in S(Z_n)$; $1 \leq i \leq s$, $1 \leq j \leq t\}$ denotes the collection of all MOD subset $s \times t$ matrices associated with $S(Z_n)$.

We will illustrate this situation by some examples.

***Example 4.1:*** Let

$$M = \{\begin{bmatrix} A & B \\ C & D \end{bmatrix} / A, B, C, D \in S(Z_6)$$

be the collection of all $2 \times 2$ MOD subset matrices with entries from $S(Z_6)$.

Let $P = \begin{bmatrix} \{3,2,0\} & \{1,2\} \\ \{1,2,3,4\} & \{5\} \end{bmatrix} \in M$. $R = \begin{bmatrix} \{1\} & \{0,2\} \\ \{0,2\} & \{5,3\} \end{bmatrix} \in M$.

***Example 4.2:*** Let

$$S = \{(a_1, a_2, a_3, a_4) \,/\, a_i \in S(Z_{10});\ 1 \leq i \leq 4\}$$

be the MOD subset row matrix.

$$x = (\{0, 1, 3\}, \{2, 4, 6, 5\}, \{1, 2, 6, 0, 9\}, \{1, 2\}) \in S$$
$$y = (\{1\}, \{0\}, \{Z_{10}\}, \{4, 2\}) \in S.$$

***Example 4.3:*** Let

$$T = \{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} b_j \in S(Z_{42});\ 1 \leq j \leq 4\}$$

be the MOD subset column matrix with entries from $S(Z_{42})$.

$$A = \begin{bmatrix} \{40,28,4,0\} \\ \{0,1,2,4,25\} \\ \{3,4,6,2\} \\ \{Z_{42}\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{1\} \\ \{3\} \\ \{5\} \\ \{41\} \end{bmatrix} \in T.$$

***Example 4.4:*** Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle|\ a_i \in S(Z_{19});\ 1 \leq i \leq 8\right\}$$

be the MOD subset matrix.

$$B = \begin{bmatrix} \{2,3,4\} & \{10,4,0\} \\ \{1,2,3,7\} & \{Z_{19}\} \\ \{10,8,18\} & \{1\} \\ \{1,2,15\} & \{3,16,5\} \end{bmatrix} \in W.$$

$$C = \begin{bmatrix} \{0\} & \{3\} \\ \{Z_{19}\} & \{0,2\} \\ \{1\} & \{4\} \\ \{0,6,10\} & \{0,6,4\} \end{bmatrix} \in W.$$

Now having seen examples of MOD subset matrices we proceed onto show how operations of + and × are performed on them.

We will describe these situations by examples.

***Example 4.5:*** Let

$$A = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_i \in S(Z_{10});\ 1 \le i \le 3 \right\}$$

be the MOD subset matrix collection.

$$\text{Let } x = \begin{bmatrix} \{3,2\} \\ \{5\} \\ \{7,4,2\} \end{bmatrix} \text{ and } y = \begin{bmatrix} \{0,1,5,7\} \\ \{3\} \\ \{9,2\} \end{bmatrix} \in A.$$

$$x + y = \begin{bmatrix} \{3,2\} \\ \{5\} \\ \{7,4,2\} \end{bmatrix} + \begin{bmatrix} \{0,1,5,7\} \\ \{3\} \\ \{9,2\} \end{bmatrix} = \begin{bmatrix} \{3,2,4,8,7,10,9\} \\ \{8\} \\ \{9,6,4,3,1\} \end{bmatrix} \in A.$$

This is the way '+' operation is performed on A.

Let $S = \begin{bmatrix} \{0,5\} \\ \{5,2\} \\ \{Z_{10}\} \end{bmatrix}$ and $R = \begin{bmatrix} \{5,0\} \\ \{5,8,3\} \\ \{1,7,5\} \end{bmatrix} \in A$.

$$S + R = \begin{bmatrix} \{0,5\} \\ \{5,2\} \\ \{Z_{10}\} \end{bmatrix} + \begin{bmatrix} \{5,0\} \\ \{5,8,3\} \\ \{1,7,5\} \end{bmatrix} = \begin{bmatrix} \{0,5\} \\ \{0,7,3,8,5\} \\ \{Z_{10}\} \end{bmatrix} \in A.$$

Thus we see {A, +} is only a MOD subset matrix semigroup and is not a group.

However we can prove A is a Smarandache semigroup.

In view of all these we prove the following theorem.

**THEOREM 4.1:** *Let $M = \{(m_{ij})_{t\times s} / m_{ij} \in S(Z_n);\ 1 \leq i \leq t,\ 1 \leq j \leq s, +\}$ be a MOD subset matrix semigroup under +.*

*i) $o(M) < \infty$,*
*ii) M is a Smarandache semigroup,*
*iii) M is a monoid as $(0) \in M$, and $(0) + A = A + (0) = A$.*

Proof is direct and hence left as an exercise to the reader.

We now describe MOD finite complex number subset matrix sets by some examples.

***Example 4.6:*** Let

$$B = \{(a_1, a_2, a_3, a_4, a_5, a_6) / a_i \in S\ (C(Z_{12}));\ 1 \leq i \leq 6\}$$

be the MOD finite complex number subset matrix set.

M = ({0, 1, 3, 5}, {6, 7, 10}, {11}, {5, 7, 8, 10}, {8, 9, 11}, {2}) $\in$ B is a MOD finite complex number subset of B. We can as in case of $S(Z_n)$, the MOD subset matrices define the operation + on B.

Let a = ({0, 2,1}, {7,8}, {9}, {2, 4, 3}, {5}, {$3i_F$}) and

b = ({$2i_F + 8$}, {0}, {1}, {3}, {1, 2, $4i_F$}, {$6 + i_F$, 1}) $\in$ B.

To find a + b

a + b = ({0, 1, 2}, {7, 8}, {9}, {2, 4 3}, {5}, {$3i_F$}) + ({$2i_F$ + 8}, {0}, {1}, {3}, {1, 2, $4i_F$}, {$6 + i_F$, 1})

= ({$2i_F + 8$, $2i_F + 9$, $2i_F + 9$, $2i_F + 10$}, {7, 8}, {10}, {6, 7, 5}, {6, 7, $5 + 4i_F$}, {$6 + 4i_F$, $1 + 3i_F$}).

This is the way + operation is performed on B.

Clearly {B, +} is only a finite commutative monoid as ({0}, {0}, {0}, {0}, {0}, {0}) acts as the additive identity.

***Example 4.7:*** Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in S(C(Z_7));\ 1 \leq i \leq 5 \right\}$$

be the MOD finite complex number subset column matrix.

Let x = $\begin{bmatrix} \{0, i_F + 6, 2 + 6i_F\} \\ \{1 + i_F, 4\} \\ \{2, 3, 5\} \\ \{0\} \\ \{6 + 2i_F, 5, 5i_F\} \end{bmatrix}$ and y = $\begin{bmatrix} \{2, 3\} \\ \{4 + i_F, 3\} \\ \{3\} \\ \{\{2 + i_F, 3 + 4i_F, 3\} \\ \{2 + 5i_F, 2 + 2i_F\} \end{bmatrix}$.

We now define + on S as follows:

$$x + y = \begin{bmatrix} \{0, i_F + 6, 2 + 6i_F\} \\ \{1 + i_F, 4\} \\ \{2, 3, 5\} \\ \{0\} \\ \{6 + 2i_F, 5, 5i_F\} \end{bmatrix} + \begin{bmatrix} \{2, 3\} \\ \{4 + i_F, 3\} \\ \{3\} \\ \{\{2 + i_F, 3 + 4i_F, 3\} \\ \{2 + 5i_F, 2 + 2i_F\} \end{bmatrix}$$

$$= \begin{bmatrix} \{2, 3, i_F + 1, i_F + 2, 4 + 6i_F, 5 + 6i_F\} \\ \{0, 1 + i_F, 4 + i_F, 5 + 2i_F\} \\ \{5, 6, 1\} \\ \{\{2 + i_F, 3, 3 + 4i_F\} \\ \{1, 5i_F, 2 + 3i_F, 1 + 4i_F, 2i_F, 2\} \end{bmatrix} \in S.$$

This is the way + operation is performed on S.

Thus {S, +} is a finite commutative monoid.

However finding the order of S is a challenging task for a general $C(Z_n)$; $2 \leq n < \infty$.

Next we proceed onto describe the MOD neutrosophic subset matrix sets under the + operation on them by some examples.

***Example 4.8:*** Let

$$P = \left\{\left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}\right| a_i \in S(\langle Z_4 \cup I\rangle);\ 1 \le i \le 8\right\}$$

be the MOD neutrosophic subset matrix set.

$$\text{Let } A = \begin{bmatrix} \{0,2,3\} & \{I,1+2I\} \\ \{0\} & \{I+3\} \\ \{I,2+3I\} & \{1+2I,2I\} \\ \{3\} & \{1+I,3I,2\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{1+I,2\} & \{0,3I\} \\ \{1+2I,3I\} & \{0,I,2\} \\ \{0\} & \{2\} \\ \{1+3I,1,2\} & \{2,2I\} \end{bmatrix}$$

$$A + B = \begin{bmatrix} \{0,2,3\} & \{I,1+2I\} \\ \{0\} & \{I+3\} \\ \{I,2+3I\} & \{1+2I,2I\} \\ \{3\} & \{1+I,3I,2\} \end{bmatrix} + \begin{bmatrix} \{1+I,2\} & \{0,3I\} \\ \{1+2I,3I\} & \{0,I,2\} \\ \{0\} & \{2\} \\ \{1+3I,1,2\} & \{2,2I\} \end{bmatrix}$$

$$= \begin{bmatrix} \{2,0,1+I,3+I,I,1\} & \{I,1+2I,0,1+I\} \\ \{1+2I,3I\} & \{I+3,I+1,2I+3\} \\ \{I,2+3I\} & \{2+2I,3+2I\} \\ \{0,1,3I\} & \{0,3+I,2+3I,1+3I,I,2+2I\} \end{bmatrix} \in P.$$

This is the way the sum of two MOD subset neutrosophic matrices are obtained.

$$A + A = \begin{bmatrix} \{0,2,3\} & \{I,1+2I\} \\ \{0\} & \{I+3\} \\ \{I,2+3I\} & \{1+2I,2I\} \\ \{3\} & \{1+I,3I,2\} \end{bmatrix} + \begin{bmatrix} \{0,2,3\} & \{I,1+2I\} \\ \{0\} & \{I+3\} \\ \{I,2+3I\} & \{1+2I,2I\} \\ \{3\} & \{1+I,3I,2\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,2,3,1\} & \{2I,1+3I,2\} \\ \{0\} & \{2+2I\} \\ \{2I,2\} & \{2,0,1\} \\ \{2\} & \{2+2I,3+I,2I,1,2+3I,0\} \end{bmatrix}$$

This is the way + operation is performed on P.

We can prove in general if S(M) = {Collection of all m × t matrices with entries from $S(\langle Z_n \cup I\rangle)$} then S(M) under the operation of + is a MOD neutrosophic subset matrix semigroup of finite order.

Clearly S(M) is not a group.

***Example 4.9:*** Let

$$S(N) = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in S(\langle Z_6 \cup I\rangle),\ 1 \leq i \leq 4, + \right\}$$

be the MOD neutrosophic subset matrix semigroup under +.

$$\text{Let } A = \begin{bmatrix} \{0,3,2+I,4I\} \\ \{1,2,5I+1\} \\ \{0,3+3I\} \\ \{4+2I,3+5I\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{4,2\} \\ \{3,0\} \\ \{2\} \\ \{4+4I\} \end{bmatrix} \in S(N).$$

$$A + B = \begin{bmatrix} \{0,3,2+I,4I\} \\ \{1,2,5I+1\} \\ \{0,3+3I\} \\ \{4+2I,3+5I\} \end{bmatrix} + \begin{bmatrix} \{4,2\} \\ \{3,0\} \\ \{2\} \\ \{4+4I\} \end{bmatrix}$$

$$= \begin{bmatrix} \{4,1,I,4+4I,2,5,4+I,4I+2\} \\ \{1,2,5I+1,4,5,5I+4\} \\ \{2,5+3I\} \\ \{2,1+3I\} \end{bmatrix} \in S(N).$$

This is the way '+' operation is performed on S(N).

Let $\{0\} = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix} \in S(N)$, clearly A + {0} = {0} + A = A

for all A ∈ S(N). Given A∈S(N) it is not easy to find the inverse infact inverse of A under + operation does not exist in general for every A ∈ S(N).

Let $T = \begin{bmatrix} \{3\} \\ \{2\} \\ \{5I\} \\ \{I\} \end{bmatrix} \in S(N)$, clearly $P = \begin{bmatrix} \{3\} \\ \{4\} \\ \{I\} \\ \{5I\} \end{bmatrix} \in S(N)$

is such that

$$T + P = \begin{bmatrix} \{3\} \\ \{2\} \\ \{5I\} \\ \{I\} \end{bmatrix} + \begin{bmatrix} \{3\} \\ \{4\} \\ \{I\} \\ \{5I\} \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$$

Thus P is the MOD neutrosophic subset matrix inverse of T in S(N).

All Y ∈ S(N) may not have a inverse associated with it.

For the Z = $\begin{bmatrix} \{3,0,2\} \\ \{4, I, 2I+1\} \\ \{2I+4, 4I\} \\ \{5I+1, 2+3I, 2\} \end{bmatrix}$ ∈ S(N), we cannot find a

Y in S(N) such that Z + Y = $\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$.

Hence we may not be able to find a inverse for every Y ∈ S(N).

Thus {S(N), +} is only a MOD neutrosophic subset semigroup of finite order and is commutative.

***Example 4.10:*** Let

$$S(T) = \{(a_1, a_2, a_3) / a_i \in S(\langle Z_{11} \cup I \rangle);\ 1 \leq i \leq 3, +\}$$

be the MOD neutrosophic subset semigroup under +.

({0}) = ({0}, {0}, {0}) is the additive identity in S(T) for A + ({0}) = ({0}) + A = A for all A ∈ S(T).

Thus S(T) is a finite commutative monoid.

Let P = ({0, 7, 3, 2, I, 4 + I}, {0}, {2}) and
Q = ({3, 10I + 1}, {4 + I, 3I + 4}, {3I + 4, 7}) ∈ S(T),

P + Q = ({0, 7, 3, 2, I, 4 + I}, {0}, {2}) + ({3, 1 + 10I}, {4 + I, 3I + 4}, {3I+4, 7})

= ({3, 1 + 10I, 10, 8 + 10I, 6, 4 + 10I, 5, 3 + 10I, 1, I + 3, 7 + I}, {4 + I, 3I + 4}, {2 + 3I, 6, 9}) ∈ S(T).

This is the way '+' operation is performed on S(T).

Next we proceed onto give examples of MOD dual number subset matrices and describe their properties.

***Example 4.11:*** Let

$$S(W) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in S(\langle Z_{10} \cup g \rangle);\ 1 \leq i \leq 4 \right\}$$

be the MOD dual number subset square matrices.

$$B = \begin{bmatrix} \{0, g, 4+2g, 5+9g\} & \{9+8g, 4g, 2+3g, 5g\} \\ \{1, 2g+3, 7g+9, 8g\} & \{0, 5, 1, 3g, 7g+2\} \end{bmatrix} \in S(W).$$

$$(\{0\}) = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix}$$

is defined as the MOD dual number subset identity matrix with respect to addition operation +.

$$\text{Let } M = \begin{bmatrix} \{0, 3+8g, 9g+2\} & \{4+8g, 6g+1\} \\ \{3+4g, g+1\} & \{5g+4, 2g+3\} \end{bmatrix} \text{ and}$$

$$N = \begin{bmatrix} \{4+2g, 6g+4, 3\} & \{3g+1, 5g\} \\ \{1, 2g, 3\} & \{7g+2, 6g+1\} \end{bmatrix} \in S(W),$$

$$M + N = \begin{bmatrix} \{0, 3+8g, 9g+2\} & \{4+8g, 6g+1\} \\ \{3+4g, g+1\} & \{5g+4, 2g+3\} \end{bmatrix} +$$

$$\begin{bmatrix} \{4+2g, 6g+4, 3\} & \{3g+1, 5g\} \\ \{1, 2g, 3\} & \{7g+2, 6g+1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{4+2g, 3, 6g+4, 7, 7+4g, & \{5+g, 9g+2, \\ 6+8g, g+6, 5g+6, 9g+5\} & 3g+4, g+1\} \\ \{4+4g, g+2, 3+6g, & \{2g+6, 9g+5, \\ 3g+1, 6+4g, g+4\} & g+5, 8g+4\} \end{bmatrix}$$

This is the way + operation is performed on S(W).

Thus {S(W), +} is a MOD dual number subset semigroup under +.

***Example 4.12:*** Let

$$S(V) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| \; a_i \in S(\langle Z_3 \cup g\rangle),\ 1 \leq i \leq, +\right\}$$

be the MOD dual number subset semigroup under +; S(V) is of finite order.

The task of finding the order of S(V) is left as an exercise to the reader.

$$(\{0\}) = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} \in S(V),$$

acts as the additive identity of S(V). For all G ∈ S(V).

G + {0} = {0} = {0} + G = G.

$$\text{Let } A = \begin{bmatrix} \{1+g, 2g, 2+g\} & \{0, g+2, 2g+1, 2\} \\ \{0, 2g, 2\} & \{1, g, g+1\} \\ \{1, 0, 1+2g, g\} & \{2+2g, g\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{0, 2g, 1+2g\} & \{1, 2+g\} \\ \{1, g, 2g\} & \{0, 2g+1, g\} \\ \{1+g, g, 2g\} & \{0, 1, g\} \end{bmatrix} \in S(V).$$

Consider

$$A + B = \begin{bmatrix} \{1+g, 2g, 2+g\} & \{0, g+2, 2g+1, 2\} \\ \{0, 2g, 2\} & \{1, g, g+1\} \\ \{1, 0, 1+2g, g\} & \{2+2g, g\} \end{bmatrix} + \begin{bmatrix} \{0, 2g, 1+2g\} & \{1, 2+g\} \\ \{1, g, 2g\} & \{0, 2g+1, g\} \\ \{1+g, g, 2g\} & \{0, 1, g\} \end{bmatrix}$$

$$= \begin{bmatrix} \{1+g, 2g, 2+g, 1, g, 2, 0\} & \{2+g, 1, g, 2g+2, 0, 2g+1, 1+g\} \\ \{1, g, 2, g, 0, 2+g, 2+2g, 2g+1\} & \{1, g, g+1, 2g, 2g+1, 2g+2, 2\} \\ \{2g, g, 1+g, 2g+1, 2+g, 0, 1, 2\} & \{2+2g, g, 2g, 1+g, 2\} \end{bmatrix} \in S(V).$$

This is the way '+' operation is performed on S(V).

S(V) is only a MOD dual number semigroup as every Z ∈ S(V) in general does not have an inverse.

$$B = \begin{bmatrix} \{1+g, 2+2g, 2, 1, 0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} \in S(V).$$

We cannot find a A such that $A + B = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} = (\{0\})$.

We next describe the MOD special dual like number subset matrix by some examples.

***Example 4.13:*** Let

$$S(M) = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} / a_i \in S(\langle Z_{12} \cup h\rangle);\ 1 \leq i \leq 8, +\}$$

be the MOD special dual like number subset matrix semigroup under +.

Clearly $(\{0\} = \begin{bmatrix} \{0\} & \{0\} & \{0\} & \{0\} \\ \{0\} & \{0\} & \{0\} & \{0\} \end{bmatrix} \in S(M)$

is the additive identity of S(M).

Let $A = \begin{bmatrix} \{3, 4h\} & \{0, 5+h\} & \{1, 3h\} & \{h, 4\} \\ \{2+h, 0\} & \{1, 4+3h\} & \{0, 2\} & \{1+h, 3\} \end{bmatrix}$ and

$$B = \begin{bmatrix} \{7,0\} & \{2,4+3h\} & \{1+h,0,1\} & \{7\} \\ \{10,3h\} & \{5\} & \{7+3h\} & \{9,2h\} \end{bmatrix} \in S(M).$$

$$A + B = \begin{bmatrix} \{3,4h\} & \{0,5+h\} & \{1,3h\} & \{h,4\} \\ \{2+h,0\} & \{1,4+3h\} & \{0,2\} & \{1+h,3\} \end{bmatrix} +$$

$$\begin{bmatrix} \{7,0\} & \{2,4+3h\} & \{1+h,0,1\} & \{7\} \\ \{10,3h\} & \{5\} & \{7+3h\} & \{9,2h\} \end{bmatrix}$$

$$= \begin{bmatrix} \{10,7+4h \\ 3,4h\} & \{2,4+3h,7+h, \\ 9+4h\} & \{2+h,1,2,1+4h \\ 1+3h\} & \{11,7+h\} \\ \{10,3h,h \\ 2+4h\} & \{6,9+3h\} & \{7+3h,9+3h\} & \{10+h,1+3h, \\ 0,3+2h\} \end{bmatrix}$$

$$\in S(M).$$

This is the way + operation is performed on S(M).

The reader is expected to find o(S(M)).

Clearly if

$$B = \left\{ \begin{bmatrix} a & \{0\} & \{0\} & \{0\} \\ b & \{0\} & \{0\} & \{0\} \end{bmatrix} \middle| \; a, b \in S(\langle Z_{12} \cup h \rangle), + \right\} \subseteq S(M)$$

is a MOD special dual like number subsemigroup of S(M).

S(M) has several MOD special dual like number subset matrix subsemigroups.

***Example 4.14:*** Let

$$S(P) = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in S(\langle Z_7 \cup h \rangle), 1 \leq i \leq 5, + \right\}$$

be the MOD special dual like number subset matrix semigroup under +.

$$(\{0\}) = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix} \in S(P)$$

acts as the identity element for S(P).

$$P_1 = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a \in S(\langle Z_7 \cup h \rangle), + \right\} \subseteq S(P)$$

is a MOD special dual like number subset matrix subsemigroup.

$$P_2 = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a \in S(Z_7 h), + \right\} \subseteq S(P)$$

is again a MOD special dual like number subset matrix subsemigroup.

$$P_3 = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a \in S(Z_7), + \right\} \subseteq S(P)$$

is a MOD special dual like number subset matrix semigroup.

$$P_4 = \left\{ \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a, b \in S(Z_7), + \right\}$$

is also a MOD special dual like number subset matrix subsemigroup of S(P).

$$P_5 = \left\{ \begin{bmatrix} a \\ 0 \\ b \\ 0 \\ 0 \end{bmatrix} \middle| \; a, b \in S(Z_7h), + \right\} \subseteq S(P)$$

is also a MOD special dual like number subset matrix subsemigroup of S(P).

S(P) has several MOD special dual like number subset matrix subsemigroups.

The reader is left with the task of finding the number of MOD special dual like number subset matrix subsemigroups of S(P).

Next we give some examples of MOD special quasi dual number subset matrix semigroups.

***Example 4.15:*** Let

$$S(B) = \left\{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| a_i \in S(\langle Z_9 \cup k\rangle);\ 1 \leq i \leq 6, +\right\}$$

be the MOD special quasi dual number subset matrix semigroup under +.

$$(\{0\}) = \begin{bmatrix} \{0\} & \{0\} & \{0\} \\ \{0\} & \{0\} & \{0\} \end{bmatrix}$$

is the MOD special quasi dual number subset identity matrix of S(B); for every A in S(B), A + ({0}) = A.

$$P_1 = \left\{\begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| b, a \in S(\langle Z_9 \cup k\rangle)\ +\right\} \subseteq S(B)$$

is a MOD special quasi dual number subset matrix subsemigroup of S(B).

$$P_2 = \left\{\begin{bmatrix} a & 0 & 0 \\ 0 & b & c \end{bmatrix} \middle| a, b \in S(\langle Z_9 \cup k\rangle) \text{ and }\right.$$

$$\left. C \in S(Z_9k), +\right\} \subseteq S(B)$$

is also a MOD special quasi dual number subset matrix subsemigroup of S(B).

The reader is left with the task of finding the order of $P_1$ and $P_2$.

Let

$$P_3 = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \middle| \; a, b\ c \in S(\langle Z_9 \cup k\rangle),\ d, e \in S((Z_9) \text{ and } f \in S(Z_9k), +\} \subseteq S(B) \right.$$

is a MOD special quasi dual number matrix subset subsemigroup.

***Example 4.16:*** Let

$$S(M) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in S(\langle Z_{19} \cup k\rangle);\ 1 \leq i \leq 12, +\right\}$$

be the MOD special quasi dual number subset matrix semigroup under +.

Infact S(M) is a monoid as

$$(\{0\}) = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix};$$

in S(M) is A + ({0}) = A for all A ∈ S(M).

The reader is left with the task of finding the order of S(M).

We see

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} \middle| a_i \in S(Z_{19}), 1 \leq i \leq 4, + \right\} \subseteq S(M)$$

is a MOD special quasi dual number subset matrix subsemigroup of S(M).

$$P_2 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} \middle| a_i \in S(Z_{19}k); 1 \leq i \leq 3, 1 \right\} \subseteq S(M)$$

is again a MOD special quasi dual number subset matrix subsemigroup of S(M).

$$P_3 = \left\{ \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & a_1 \\ \{0\} & \{0\} \end{bmatrix} \middle| a_i \in S(\langle Z_{19} \cup k \rangle), + \right\} \subseteq S(M)$$

is again a MOD special quasi dual number subset matrix subsemigroup of S(M).

Infact all these three MOD subset matrix subsemigroups are infact MOD subset matrix submonoids of S(M).

The reader is left with the task of finding all MOD special quasi dual number subset matrix subsemigroups of S(M).

Infact can there be MOD matrix subset matrix subsemigroups in S(M) which are not MOD matrix subset submonoids of S(M)?

In view of all these we have the following theorem.

**THEOREM 4.2:** *Let S(M) = {collection of all $t \times s$ matrices with entries from $S(Z_n)$ (or $S(C(Z_n))$ or $S(\langle Z_n \cup I\rangle)$ or $S(\langle Z_n \cup g\rangle)$ or $S(\langle Z_n \cup h\rangle)$ or $S(\langle Z_n \cup k\rangle)$); +} be the MOD subset matrix monoid (or MOD subset matrix finite complex number monoid or MOD neutrosophic subset matrix monoid or MOD dual number subset matrix monoid or MOD special dual like number subset matrix monoid or MOD special quasi dual number subset matrix monoid respectively).*

i) *$o(S(M)) < \infty$.*
ii) *S(M) has several MOD matrix subset subsemigroups.*
iii) *All MOD matrix subset subsemigroups are MOD subset matrix submonoids.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed on to describe MOD subset matrix semigroups under natural product $\times_n$ of matrices by some examples.

***Example 4.17:*** Let

$$S(M) = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in S(Z_6);\ 1 \leq i \leq 5, \times_n \right\}$$

be the MOD subset column matrix subset semigroup. We see $S(Z_6)$, the collection of all subsets of $Z_6$ under product is a semigroup.

$$\text{Let } A = \begin{bmatrix} \{0,2\} \\ \{3,4\} \\ \{5\} \\ \{1,0\} \\ \{2,1,5\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{3,1\} \\ \{5,2\} \\ \{3,4\} \\ \{2\} \\ \{1\} \end{bmatrix} \in S(M).$$

$$A \times_n B = \begin{bmatrix} \{0,2\} \\ \{3,4\} \\ \{5\} \\ \{1,0\} \\ \{2,1,5\} \end{bmatrix} \times_n \begin{bmatrix} \{3,1\} \\ \{5,2\} \\ \{3,4\} \\ \{2\} \\ \{1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,2\} \times \{3,1\} \\ \{3,4\} \times \{2,5\} \\ \{5\} \times \{3,4\} \\ \{1,0\} \times \{2\} \\ \{2,1,5\} \times \{1\} \end{bmatrix} = \begin{bmatrix} \{0,2\} \\ \{0,2,3\} \\ \{3,2\} \\ \{2,0\} \\ \{2,1,5\} \end{bmatrix} \in S(M).$$

This is the way product $\times_n$ operation is performed on S(M).

$$(\{0\}) = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix} \in S(M)$$

and for all A ∈ S(M) we have $A \times_n (\{0\}) = (\{0\}) \times_n A = (\{0\})$.

Further

$$(\{1\}) = \begin{bmatrix} \{1\} \\ \{1\} \\ \{1\} \\ \{1\} \\ \{1\} \end{bmatrix} \in S(M)$$

is defined as the MOD identity element of S(M) with respect to $\times_n$.

Clearly $(\{1\}) \times_n A = A \times_n (\{1\}) = A$ for all A ∈ S(M).

$$\text{Let } A = \begin{bmatrix} \{0,3,4\} \\ \{2,5\} \\ \{0\} \\ \{1,2,3\} \\ \{3,4,5\} \end{bmatrix} \text{ and } B \begin{bmatrix} \{0\} \\ \{0\} \\ \{1,2,3\} \\ \{0\} \\ \{0\} \end{bmatrix} \in S(M),$$

clearly $A \times_n B = \begin{bmatrix} \{0,3,4\} \\ \{2,5\} \\ \{0\} \\ \{1,2,3\} \\ \{3,4,5\} \end{bmatrix} \times_n \begin{bmatrix} \{0\} \\ \{0\} \\ \{1,2,3\} \\ \{0\} \\ \{0\} \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$.

Thus we say A is a MOD subset matrix zero divisor B and vice versa.

$$\text{Let } N = \begin{bmatrix} \{0,1,3\} \\ \{3\} \\ \{1,3\} \\ \{4\} \\ \{4,3,0\} \end{bmatrix} \in S(M).$$

$$N \times_n N = \begin{bmatrix} \{0,1,3\} \\ \{3\} \\ \{1,3\} \\ \{4\} \\ \{4,3,0\} \end{bmatrix} \times_n \begin{bmatrix} \{0,1,3\} \\ \{3\} \\ \{1,3\} \\ \{4\} \\ \{4,3,0\} \end{bmatrix} = \begin{bmatrix} \{0,1,3\} \\ \{3\} \\ \{0,3\} \\ \{4\} \\ \{4,0,3\} \end{bmatrix} = N.$$

Thus we see N in S(M) is the MOD subset idempotent matrix with respect to the product operation.

Thus S(M) is a MOD subset matrix commutative monoid of finite order which has MOD subset matrix zero divisors and MOD subset matrix idempotents in it.

However S(M) has no MOD subset matrix nilpotent elements.

***Example 4.18:*** Let S(P) = {($a_1$, $a_2$, $a_3$, $a_4$) where $a_i \in S(Z_8)$; $1 \leq i \leq 4$, ×} be the MOD subset matrix semigroup under product.

Let A = ({3, 5, 6}, {2, 7}, {1}, {0}) and

B = ({4, 3}, {5, 0} {7, 8, 3, 4}, {5, 6, 7, 0}) ∈ S(P).

A × B = ({3, 5, 6}, {2, 7}, {1}, {0}) × ({4, 3}, {5, 0}, {7, 6, 3, 4}, {5, 6, 7, 0})

= ({3, 5, 6} × {4, 3}, {2, 7} × {5, 0}, {1} × {7, 6, 3, 4}, {0} × {5, 6, 7, 0})

= ({4, 0, 1, 7, 2}, {2, 3, 0}, {7, 6, 3, 4}, {0}) ∈ S(P).

This is the way product operation is performed on the MOD matrix subset semigroup S(P).

({1}) = ({1}, {1}, {1}, {1}) ∈ S(P) acts as the multiplicative identity. For A × ({1}) = ({1}) × A = A for all A × S(P).

({0}) = ({0}, {0}, {0}, {0}) ∈ S(P) is such that A × ({0}) = ({0} × A = ({0}); for all A ⊂ S(P).

Let A = ({4, 2}, {0}, {4}, {0}) and

B = ({4}, {3, 4, 5}, {2, 4}, {7, 6, 3, 2, 1}) ∈ S(P);

A × B = ({0}, {0}, {0}, {0}) = ({0}).

So A is a MOD subset matrix zero divisor of B and vice versa.

Let $P_1$ = {($a_1$, 0, 0, 0) / $a_1 \in S(Z_8)$, ×} ⊆ S(P); clearly $P_1$ is a MOD subset matrix subsemigroup of S(P) which is also a MOD subset matrix ideal of S(P).

Let $P_2 = \{(0, a1, a_2, 0) / a_1, a_2 \in S(Z_8), \times\} \subseteq S(P)$, clearly $P_2$ is also a MOD subset matrix ideal of S(P).

Further $P_1 \cap P_2 = (\{0\})$.

Let $M = \{(a_1, a_2, a_3, a_4) / a_i \in S(\{0, 2, 4, 6\}); 1 \leq i \leq 4, \times\} \subseteq S(P)$, we see M is a MOD subset matrix ideal of S(P).

Now let $T = (\{4\}, \{4\}, \{0\}, \{2, 4\}) \in S(P)$; we find

$T \times T = (\{4\}, \{4\}, \{0\}, \{2, 4\}) \times (\{4\}, \{4\}, \{0\}, \{2, 4\})$

$= (\{4\} \times \{4\}, \{4\} \times \{4\}, \{0\} \times \{0\}, \{2, 4\} \times \{2, 4\})$

$= (\{0\}, \{0\}, \{0\}, \{4, 0\}) = T^2$

$T^2 \times T = (\{0\}, \{0\}, \{0\}, \{4, 0\}) \times (\{4\}, \{4\}, \{0\}, \{2, 4\})$

$= (\{0\}, \{0\}, \{0\}, \{0\}) = T^3 = (\{0\})$

Thus $T \in S(P)$ is a MOD subset nilpotent matrix of order three.

Hence this S(P) has MOD matrix subset nilpotents. However finding nontrivial MOD subset matrix idempotents happens to be a challenging problem.

In view of this we have the following theorem.

**THEOREM 4.3:** *Let S(B) = {collection of all m × p subset matrices with entries from $S(Z_n)$; $n = p^t$ ( $t \geq 2$); $\times_n$} be the MOD subset matrix semigroup under natural product $\times_n$.*

*i) S(B) has MOD subset matrices A such that $A^s = (\{0\})$ for some $s \geq 2$.*
*ii) S(B) has only trivial MOD subset matrix idempotents.*

Proof is direct and hence left as an exercise to the reader.

***Example 4.19:*** Let

$$S(W) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| a_i \in S(Z_{12});\ 1 \leq i \leq 6, \times_n \right\}$$

be the MOD subset matrix semigroup under the natural product $\times_n$.

$$\text{Let } A = \begin{bmatrix} \{0,6\} & \{6\} \\ \{4,2\} & \{5,3\} \\ \{1,3,2\} & \{0\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{3,7\} & \{9,3,10\} \\ \{1,5\} & \{7\} \\ \{3\} & \{8,9,10,3\} \end{bmatrix} \in S(W).$$

$$A \times_n B = \begin{bmatrix} \{0,6\} & \{6\} \\ \{4,2\} & \{5,3\} \\ \{1,3,2\} & \{0\} \end{bmatrix} \times_n \begin{bmatrix} \{3,7\} & \{9,3,10\} \\ \{1,5\} & \{7\} \\ \{3\} & \{8,9,10,3\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,6\} \times \{3,7\} & \{6\} \times \{3,910\} \\ \{4,2\} \times \{1,5\} & \{5,3\} \times \{7\} \\ \{1,2,3\} \times \{3\} & \{0\} \times \{3,9,8,10\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,6\} & \{6,0\} \\ \{4,2,8\} & \{3,5\} \\ \{3,6,9\} & \{0\} \end{bmatrix} \in S(W).$$

This is the way product operation ′n is performed on S(W).

$$(\{0\}) = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} \in S(W) \text{ is the zero of } S(W) \text{ and}$$

$$(\{1\}) = \begin{bmatrix} \{1\} & \{1\} \\ \{1\} & \{1\} \\ \{1\} & \{1\} \end{bmatrix} \in S(W)$$

is the identity element of S(W). S(W) is a commutative MOD subset matrix monoid of finite order.

The task of finding o(S(W)) is left as an exercise to the reader.

$$P = \begin{bmatrix} \{0,6\} & \{6\} \\ \{0\} & \{6\} \\ \{6,0\} & \{6,0\} \end{bmatrix} \in S(W) \text{ is such that}$$

$$P \times_n P = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} = (\{0\}).$$

Thus P is a MOD subset matrix nilpotent of order two.

Left us find some MOD subset matrix idempotents of S(W).

$$\text{Let } M = \begin{bmatrix} \{4,9\} & \{4\} \\ \{9\} & \{1,9\} \\ \{4,1,9\} & \{1,0\} \end{bmatrix} \in S(W)$$

$$M \times_n M = \begin{bmatrix} \{4,9\} & \{4\} \\ \{9\} & \{1,9\} \\ \{4,1,9\} & \{1,0\} \end{bmatrix} = M.$$

Thus M is a MOD subset matrix idempotent of S(W).

$$\text{Let } B = \begin{bmatrix} \{4,8,2\} & \{4,8\} \\ \{3,6,9\} & \{6,9\} \\ \{4,8\} & \{0,3,9\} \end{bmatrix} \text{ and}$$

$$C = \begin{bmatrix} \{6\} & \{6,3\} \\ \{4,8\} & \{4,8\} \\ \{3,6,9\} & \{4,8\} \end{bmatrix} \in S(W).$$

$$\text{Clearly } B \times_n C = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} = (\{0\}).$$

Thus B is a zero divisor of C and vice versa.

We have seen this S(W) has MOD subset matrix zero divisors, MOD subset matrix idempotents and MOD matrix subset matrix nilpotents.

So if $n = 12 = 2^2 3$ we see it has all such MOD special subset matrices.

In view of all these we put forth the following result.

**THEOREM 4.4:** *Let S(M) = {collection of all s × t subset matrices with entries from $S(Z_n)$, $\times_n$} be the MOD subset matrix semigroup under the natural product $\times_n$, where $n = p^r q$ where p is a prime and $r \geq 2$, q any composite or prime number.*

*i) Then S(M) has nontrivial MOD matrix subset idempotent.*
*ii) S(M) has nontrivial MOD matrix subset nilpotents (order of the nilpotents depends on r).*
*iii) S(M) has MOD subset matrix zero divisors.*

Proof is left as an exercise to the reader.

***Example 4.20:*** Let

$$S(V) = \left\{ \begin{bmatrix} a_1 & a_3 & a_5 & a_6 \\ a_2 & a_4 & a_7 & a_8 \end{bmatrix} \middle| a_i \in S(Z_5); 1 \leq i \leq 8, \times_n \right\}$$

be the MOD subset matrix semigroup under the natural product $\times_n$.

$$\text{Let } A = \begin{bmatrix} \{0,2\} & \{1\} & \{0,3\} & \{0\} \\ \{1,4\} & \{0\} & \{1,2\} & \{2\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{3,1\} & \{0,1,2\} & \{4,0\} & \{2,3,4\} \\ \{1,3\} & \{4,3\} & \{0\} & \{1,3\} \end{bmatrix} \in S(V).$$

$$A \times_n B = \begin{bmatrix} \{0,2\} & \{1\} & \{0,3\} & \{0\} \\ \{1,4\} & \{0\} & \{1,2\} & \{2\} \end{bmatrix}$$

$$\times_n \begin{bmatrix} \{3,1\} & \{0,1,2\} & \{4,0\} & \{2,3,4\} \\ \{1,3\} & \{4,3\} & \{0\} & \{1,3\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,2\}\times\{3,1\} & \{1\}\times\{0,1,2\} & \{0,3\}\times\{0,4\} & \{0\}\times\{2,3,4\} \\ \{1,4\}\times\{1,3\} & \{0\}\times\{4,3\} & \{1,2\}\times\{0\} & \{2\}\times\{1,3\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,1,2\} & \{0,1,2\} & \{0,2\} & \{0\} \\ \{1,4,3,2\} & \{0\} & \{0\} & \{2,1\} \end{bmatrix} \in S(V).$$

This is the way product operation $\times_n$ is performed on S(V).

We see S(V) has no nontrivial MOD subset matrix nilpotents, non trivial MOD subset matrix idempotent and nontrivial MOD matrix subset zero divisors.

In view of all these we have the following result.

**THEOREM 4.5:** *Let S(D) = {collection of all m × n subset matrices with entries from $S(Z_p)$; p a prime, $\times_n$} be the MOD subset matrix semigroup under natural product $\times_n$.*

*i) S(D) has no nontrivial MOD subset matrix idempotents.*
*ii) S(D) has no nontrivial MOD subset matrix nilpotents.*
*iii) S(D) has only MOD subset matrix zero divisors of the form $A = (a_{ij})$ and $B = (b_{ij})$; $a_{ij}, b_{ij} \in S(Z_p)$.*

*Then $A \times B = (\{0\})$ only if either $a_{ij} = \{0\}$ or $b_{ij} = \{0\}$.*

Proof is direct and hence left as an exercise to the reader.

***Example 4.21:*** Let

$$S(P) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in S(Z_{10}); 1 \leq i \leq 4 \right\}$$

be the MOD subset matrix .

We can define two types of products as the collection is a square matrix one the usual product × and the other is the natural product $\times_n$.

{S(P), ×} is a non commutative MOD subset matrix semigroup of finite order; infact a matrix subset monoid with

$$(\{1\}) = \begin{bmatrix} \{1\} & \{0\} \\ \{0\} & \{1\} \end{bmatrix}.$$

But $\{S(P), \times_n\}$ is a commutative MOD matrix subset semigroup under the natural product $\times_n$; infact a MOD subset matrix monoid with

$$(\{1\}) = \begin{bmatrix} \{1\} & \{1\} \\ \{1\} & \{1\} \end{bmatrix}$$

as the identity with respect to the natural product $\times_n$.

Both $\times$ and $\times_n$ are two distinct operations on S(P).

$$\text{Let } A = \begin{bmatrix} \{5,2,4\} & \{3,1\} \\ \{0,1\} & \{5\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{1,2\} & \{5\} \\ \{7\} & \{0,4,2\} \end{bmatrix} \in S(P).$$

$$A \times B = \begin{bmatrix} \{5,2,4\} & \{3,1\} \\ \{0,1\} & \{5\} \end{bmatrix} \times \begin{bmatrix} \{1,2\} & \{5\} \\ \{7\} & \{0,4,2\} \end{bmatrix}$$

$$= \begin{bmatrix} \{5,2,4\} \times \{1,2\} + & \{5,2,4\} \times \{5\} + \\ \{1,3\} \times \{7\} & \{3,1\} \times \{0,4,2\} \\ \{0,1\} \times \{1,2\} + & \{0,1\} \times \{5\} + \\ \{5\} \times \{7\} & \{5\} \times \{0,4,2\} \end{bmatrix}$$

$$= \begin{bmatrix} \{5,2,4,0,8\} & \{5,0\} + \\ +\{7,1\} & \{0,4,2,6\} \\ \{0,1,2\} + & \\ \{5\} & \{0,5\} + \{0\} \end{bmatrix}$$

$$= \begin{bmatrix} \{6,3,5,1,9 & \{0,4,2,6,5,9, \\ 2,7\} & 7,1\} \\ \{5,6,7\} & \{0,5\} \end{bmatrix} \qquad \text{I}$$

This is the way usual product operation × is performed on S(P).

Now we find

$$B \times A = \begin{bmatrix} \{1,2\} & \{5\} \\ \{7\} & \{0,4,2\} \end{bmatrix} \times \begin{bmatrix} \{2,4,5\} & \{3,1\} \\ \{0,1\} & \{5\} \end{bmatrix}$$

$$= \begin{bmatrix} \{1,2\} \times \{2,4,5\} & \{1,2\} \times \{3,1\} \\ +\{5\} \times \{0,1\} & +\{5\} \times \{5\} \\ \{7\} \times \{2,4,5\} & \{7\} \times \{3,1\} + \\ +\{0,4,2\} \times \{0,1\} & \{0,4,2\} \times \{5\} \end{bmatrix}$$

$$= \begin{bmatrix} \{2,4,8,5\} + & \{3,6,1,2\} + \\ \{0,5\} & \{5\} \\ \{4,8,5\} + & \{1,7\} + \\ \{0,4,2\} & \{0\} \end{bmatrix}$$

$$= \begin{bmatrix} \{7,9,3,0\} & \{8,1,6,7\} \\ \{4,8,5,2, \\ 9,6,0,7\} & \{1,7\} \end{bmatrix} \quad \text{II}$$

Clearly I and II are not equal so in general A × B = B × A for A, B ∈ S(P).

Thus {S(P), ×} is a MOD subset matrix non commutative semigroup.

Now we find

$$A \times_n B = \begin{bmatrix} \{5,2,4\} & \{3,1\} \\ \{0,1\} & \{5\} \end{bmatrix} \times_n \begin{bmatrix} \{1,2\} & \{5\} \\ \{7\} & \{0,4,2\} \end{bmatrix}$$

$$= \begin{bmatrix} \{5,2,4\} \times \{1,2\} & \{3,1\} \times \{5\} \\ \{0,1\} \times \{7\} & \{5\} \times \{0,4,2\} \end{bmatrix}$$

$$= \begin{bmatrix} \{5,2,4,0,8\} & \{5\} \\ \{0,7\} & \{0\} \end{bmatrix} \quad \text{III}$$

Clearly A $\times_n$ B = B $\times_n$ A but A × B ≠ A $\times_n$ B and B × A ≠ A $\times_n$ B evident from equations (I), (II) and (III).

Finding MOD subset matrix zero divisors in {S(P), ×} is left as a matter of routine.

Further {S(P), ×} may contain a MOD subset matrix left zero divisor which may not be a MOD subset matrix right zero divisor.

We see {S(P), ×} may have a right unit MOD subset matrix which may not be a MOD subset left unit and vice versa.

Now in view of all these we have the following result.

**THEOREM 4.6:** *Let S(M) = {collection of all m × n subset matrices with entries from $S(Z_n)$; ×} be the MOD subset square matrix semigroup under the usual product.*

*i) $o(S(M)) < \infty$ and is a non commutative monoid.*
*ii) S(M) may contain MOD subset matrix right zero divisors which may not in general be MOD subset matrix left zero divisors.*
iii) *S(M) will have MOD subset matrix right ideals which are not left ideals.*

Proof is direct and hence left as an exercise to the reader.

Next we just describe MOD finite complex number matrix subset semigroups under $\times_n$, the natural product by some examples.

***Example 4.22:*** Let S(M) = {($a_1$, …, $a_6$) be the collection of all MOD subset finite complex number matrix with entries from $S(C(Z_{10}))$; ×} be the MOD subset finite complex number matrix semigroup.

We see ({0}) = ({0}, {0}, {0}, {0}, {0}, {0}) ∈ S(M) is such that A × ({0}) = ({0}) for all A ∈ S(M).

Further ({1}) = ({1}, {1}, …, {1}) ∈ S(M) is such that A × ({1}) = A, is the identity or unit MOD subset matrix of S(M).

Thus S(M) is a MOD subset finite complex number matrix monoid of finite order.

Reader is expected to find o(S(M)).

Consider $P_1 = \{(a_1, 0, 0, 0, 0, a_2) / a_1, a_2 \in S(C(Z_{10})), \times\} \subseteq S(M)$ is a MOD finite complex number subset matrix subsemigroup or a submonoid of S(M).

Let A = ({3, $i_F$, 2 + 5$i_F$}, {2$i_F$}, {1, 2, 3 + $i_F$}, {1}, {$i_F$}, {0, 1, 2 + $i_F$}) and

B = ({0, 1, $i_F$ + 2}, {1 + 2$i_F$, 3}, {3, 4$i_F$, 2 + $i_F$}, {3}, {4$i_F$}, {0, 1, 3}) ∈ S(M).

A × B = ({3, $i_F$, 2 + 5$i_F$}, {2$i_F$}, {1, 2, 3 + $i_F$}, {1}, {$i_F$}, {0, 1, 2 + $i_F$}) × ({0, 1, $i_F$ + 2}, {1 + 2$i_F$, 3}, {3, 4$i_F$, 2 + $i_F$}, {3}, {4$i_F$}, {0, 1, 3})

= ({3, $i_F$, 2 + 5$i_F$} × {0, 1, $i_F$ + 2}, {2$i_F$} × {1 + 2$i_F$, 3}, {1, 2, 3 + $i_F$} × {3, 4$i_F$, 2 + $i_F$}, {1} × {3}, {$i_F$} × {4$i_F$}, {0, 1, 2 + $i_F$} × {0, 1, 3})

= ({0, 3, $i_F$, 2 + 5$i_F$, 3$i_F$ + 6, 9 + 2$i_F$, 9 + 2$i_F$}, {6$i_F$, 2$i_F$ + 8}, {3, 6, 9 + 3$i_F$, 4$i_F$, 8$i_F$, 12$i_F$ + 6, 2 + $i_F$, 4 + 2$i_F$, 5 + 5$i_F$}, {3}, {6}, {0, 1, 2 + $i_F$, 3, 6 + 3$i_F$}).

This is the way product operation is performed on MOD finite complex number subset matrices.

Let A = ({0}, {2,4, 6$i_F$ + 3}, {2}, {4 + 5$i_F$, 0, 3$i_F$}, {0}, {4, 2$i_F$, + 9, 3})

B = ({3 + 5$i_F$, 2, 4, 3$i_F$ + 7}, {0}, {5 + 5$i_F$, 5},{0}, {3 + 2$i_F$, 7$i_F$ + 5, 2 + 8$i_F$}, {0}) ∈ S(M).

A × B = ({0}, {0}, {0}, {0}, {0}, {0}) = ({0}).

A = ({5}, {5$i_F$}, {5 + 5$i_F$, 0}, {1, 6}, {5, 6}, {0, 1}) ∈ S(M).

Thus A × A = A is the MOD subset finite complex number matrix idempotent of S(M).

A = ({5}, {5 + 5$i_F$, 5, 5$i_F$}, {4 + 4$i_F$, 8, 8$i_F$, 2}, {0}, {4 + 4$i_F$, 6 + 2$i_F$, 6$i_F$ + 8}, {5 + 5$i_F$}) and

B = ({2, 4, 6}, {4, 6, 2, 8, 8$i_F$, 2$i_F$}, {5, 5$i_F$, 5 + 5$i_F$}, {3, 2 + 3$i_F$, 5 + 6$i_F$, 3 + 8$i_F$, 1 + 9$i_F$}, {5 + 5$i_F$, 5}, {2, 8, 6 + 6$i_F$, 8$i_F$ + 4, 4 + 4$i_F$}) $\in$ S(M) is such that

A × B = ({0}, {0}, {0}, {0}, {0}, {0}) = ({0}) is a MOD subset matrix zero divisor of S(M).

The reader is left with the task of finding the number of MOD subset matrix zero divisors of S(M).

***Example 4.23:*** Let

$$S(V) = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} \middle| \; v_i \in S(C(Z_7));\ 1 \leq i \leq 5, \times_n \right\}$$

be the MOD subset finite complex number matrix semigroup under the natural product $\times_n$.

$$\text{Let } A = \begin{bmatrix} \{3i_F, 4+2i_F, 1\} \\ \{2, 5i_F\} \\ \{0, i_F\} \\ \{2, 6i_F\} \\ \{1, 5\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{3, 4i_F\} \\ \{6+3i_F\} \\ \{2+4i_F, 3\} \\ \{1+i_F\} \\ \{3+5i_F, 3\} \end{bmatrix} \in S(V).$$

We find $A \times_n B = \begin{bmatrix} \{3i_F, 4+2i_F, 1\} \\ \{2, 5i_F\} \\ \{0, i_F\} \\ \{2, 6i_F\} \\ \{1,5\} \end{bmatrix} \times_n \begin{bmatrix} \{3, 4i_F\} \\ \{6+3i_F\} \\ \{2+4i_F, 3\} \\ \{1+i_F\} \\ \{3, 3+5i_F\} \end{bmatrix}$

$$= \begin{bmatrix} \{3i_F, 1, 4+2i_F\} \times \{3, 4i_F\} \\ \{2, 5i_F\} \times \{6+3i_F\} \\ \{0, i_F\} \times \{3, 2+4i_F\} \\ \{2, 6i_F\} \times \{1+i_F\} \\ \{1,5\} \times \{3, 3+5i_F\} \end{bmatrix} = \begin{bmatrix} \{2i_F, 3, 5+3i_F, 4i_F \\ 5i_F, 6+2i_F\} \\ \{5+6i_F, 2i_F+6\} \\ \{0, 3i_F, 2i_F+3\} \\ \{2+2i_F, 6i_F+1\} \\ \{3, 3+5i_F, 1, \\ 1+4i_F\} \end{bmatrix}$$

This is the way the natural product $\times_n$ operation is performed on S(V).

Clearly $(\{1\}) = \begin{bmatrix} \{1\} \\ \{1\} \\ \{1\} \\ \{1\} \\ \{1\} \end{bmatrix}$ acts as the multiplicative identity; for

$A \times (\{1\}) = A$ for all $A \in S(V)$.

$$(\{0\}) = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix} \in S(V)$$

is such that A × ({0}) = ({0}) for all A ∈ S(V).

We see

$$A = \begin{bmatrix} \{0\} \\ \{2, 3, 4 + i_F\} \\ \{0\} \\ \{5 + 2i_F, 3i_F\} \\ \{0\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{1 + i_F, 2i_F, 3i_F + 5, 6 + 2i_F\} \\ \{0\} \\ \{4 + i_F, 2 + 3i_F\} \\ \{0\} \\ \{1 + i_F, 2 + 5i_F, 4i_F\} \end{bmatrix} \in S(V)$$

is such that

$$A \times_n B = \begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix} = S(V)$$

has only MOD finite complex number subset matrix zeros of the following form.

It is left as an open problem to find other types of real zero divisors. For these type of zero divisors can be realized as the trivial type of MOD finite complex number subset matrix zero divisors.

However if

$$S(B) = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in S(Z_7); 1 \leq i \leq 5, \times_n \right\} \subseteq S(V)$$

is a MOD finite complex number subset matrix subsemigroup of S(V) which is not an ideal of S(V).

In view of all these the following result is put forth.

**THEOREM 4.7:** *Let S(B) = {collection of all subset matrices with entries from S(C($Z_n$)); $\times_n$} be the MOD finite complex number subset matrix semigroup.*

*i) S(B) has nontrivial MOD finite complex number subset matrix zero divisor which are nontrivial (n not a prime).*
*ii) S(B) has MOD finite complex number subset matrix subsemigroups which are not ideals.*
iii) *S(B) has MOD finite complex number subset matrix ideals.*

Proof is direct hence left as an exercise to the reader.

We leave the following as a open conjecture.

**Conjecture 4.1:** Let S(W) = {collection of all s × t MOD finite complex number subset matrix with entries from S(C($Z_p$)) p a and prime, $\times_n$}be the MOD finite complex number subset matrix semigroup under natural product.

Can S(W) have MOD finite complex number subset matrix nontrivial zero divisors?

Next we proceed onto describe MOD subset neutrosophic matrix semigroup under natural product $\times_n$ by some examples.

***Example 4.24:*** Let

$$S(W) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| \, a_i \in S(\langle Z_{12} \cup I \rangle; 1 \leq i \leq 6, \times_n \right\}$$

be the MOD neutrosophic subset matrix semigroup under natural product $\times_n$.

$$\text{Let } A = \begin{bmatrix} \{3, 4I+2, 6\} & \{2, I\} \\ \{4+6I, 2I\} & \{0\} \\ \{1, 2, 5I\} & \{3+7I, 2I+3\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{2, 0\} & \{4+5I\} \\ \{1+3I\} & \{2+5I, 4I\} \\ \{3I+8\} & \{2+4I\} \end{bmatrix} \in S(W).$$

$$A \times_n B = \begin{bmatrix} \{3, 4I+2, 6\} \times \{2, 0\} & \{2, I\} \times \{4+5I\} \\ \{4+6I, 2I\} \times \{1+3I\} & \{0\} \times \{4I, 2+5I\} \\ \{1, 2, 5I\} \times \{3I+8\} & \{3+7I, 2I+3\} \times \{2+4I\} \end{bmatrix}$$

$$= \begin{bmatrix} \{6, 8I+4, 0, 3I, 6I\} & \{8+10I, 9I\} \\ \{8I, 4+6I\} & \{0\} \\ \{3I+8, 6I+4, 7I\} & \{6+6I, 6+2I\} \end{bmatrix} \in S(W).$$

This is the way the product operation $\times_n$ is performed on S(W).

Let $A = \begin{bmatrix} \{6+6I, 3I, 3, 6+3I\} & \{4, 4+4I, 8I\} \\ \{10I+10, 10+10I, 0\} & \{6+6I, 3I\} \\ \{4+6I, 6+8I\} & \{6, 8I+8\} \end{bmatrix}$ and

$$B = \begin{bmatrix} \{34, 4I, 8, 8I, 8+8I\} & \{6, 6I, 3I, 3+6I\} \\ \{6, 6+6I, 6I\} & \{4I+4, 8, 8I\} \\ \{6, 6I, 6+6I\} & \{6, 6I+6\} \end{bmatrix} \in S(W)$$

is such that

$$A \times_n B = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} = (\{0\})$$

is the MOD neutrosophic subset matrix zero divisor of S(W).

Let

$$B = \begin{bmatrix} \{4, 4I\} & \{9, 9I\} \\ \{4, 4I\} & \{I\} \\ \{9I\} & \{9\} \end{bmatrix} \in S(W)$$

is such that

$$B \times_n B = \begin{bmatrix} \{4, 4I\} & \{9, 9I\} \\ \{4, 4I\} & \{I\} \\ \{9I\} & \{9\} \end{bmatrix} = B.$$

Thus S(W) has MOD neutrosophic matrix subset idempotents.

Infact S(W) also has nontrivial MOD neutrosophic matrix subset nilpotents given by the following matrix.

$$F = \begin{bmatrix} \{6, 6I\} & \{6I\} \\ \{6+6I\} & \{6+6I\} \\ \{0\} & \{6, 6I, 6+6I\} \end{bmatrix} \in S(W)$$

is such that

$$F \times_n F = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} = (\{0\})$$

is the MOD neutrosophic subset matrix nilpotents of order two.

In view of all these we have the following result.

**THEOREM 4.8:** *Let S(M) = {collection of all s × t subset matrices with entries from S(⟨$Z_n$ ∪ I⟩); $\times_n$} be the MOD neutrosophic subset matrix semigroup under natural product $\times_n$.*

*i) If n = $p^t q$ (t ≥ 2, p a prime of any composite number) then S(M) has MOD neutrosophic subset matrix nilpotents, the order of which is dependent on t.*
*ii) S(M) has MOD neutrosophic subset matrix idempotents.*
*iii) S(M) has MOD neutrosophic subset matrix zero divisors which are nontrivial.*
*iv) S(M) has MOD neutrosophic subset matrix ideals.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD dual number subset matrix semigroups under natural product by examples.

***Example 4.25:*** Let

$$S(P) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_9 \\ a_5 & a_6 & a_7 & a_8 & a_{10} \end{bmatrix} \middle| \; a_i \in S(\langle Z_6 \cup g\rangle); \right.$$

$1 \leq i \leq 10; \times_n\}$

be the MOD dual number subset matrix semigroup under natural product.

Let

$$A = \begin{bmatrix} \{5g, g\} & \{3g, 0\} & \{g, 2g\} & \{2g\} & \{g, 2g\} \\ \{3g, g\} & Pg, 4g\} & \{2g\} & \{4g, 0\} & \{5g, g\} \end{bmatrix} \in S(P);$$

$$\text{clearly } A \times_n A = \begin{bmatrix} \{0\} & \{0\} & \{0\} & \{0\} & \{0\} \\ \{0\} & \{0\} & \{0\} & \{0\} & \{0\} \end{bmatrix} = (\{0\}).$$

Thus A is MOD nilpotent dual number subset matrix of S(P).

Infact S(P) has several MOD nilpotent subset matrices of order two.

Infact

$$S(M) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_9 \\ a_5 & a_6 & a_7 & a_8 & a_{10} \end{bmatrix} \middle| \; a_i \in \{S(Z_6g)\}; \right.$$

$$1 \leq i \leq 10\} \subseteq S(P)$$

is a MOD dual number subset matrix subsemigroup which is also an ideal of S(P).

Infact S(M) is a zero square MOD dual number subset ideal of S(P) as S(M) × S(M) = ({0}).

We see S(P) has MOD dual number subset matrix subsemigroups which are not ideals.

For let

$$S(T) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_9 \\ a_5 & a_6 & a_7 & a_8 & a_{10} \end{bmatrix} \middle| \; a_i \in S(Z_6);\ 1 \leq i \leq 10, \times_n \right\}$$

$$\subseteq S(P)$$

is a MOD dual number subset matrix subsemigroup of S(P) which is clearly not an ideal of S(P).

$$L = \begin{bmatrix} \{4\} & \{3\} & \{0,3\} & \{4,0\} & \{0\} \\ \{0\} & \{3,0\} & \{0,4\} & \{4\} & \{3\} \end{bmatrix} \in S(P)$$

we see

$$L \times_n L = \begin{bmatrix} \{4\} & \{3\} & \{0,3\} & \{4,0\} & \{0\} \\ \{0\} & \{0,3\} & \{0,4\} & \{4\} & \{3\} \end{bmatrix} = L.$$

Thus L is a MOD dual number matrix subset idempotent of S(P).

Hence S(P) has MOD dual number matrix subset nilpotents, MOD dual numbers matrix subset idempotents and MOD dual number matrix subset zero divisors.

S(P) has MOD dual number subset ideals which are square zero divisors and MOD dual number subset subsemigroups which are not ideals.

***Example 4.26:*** Let

$$S(B) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in S(\langle Z_{11} \cup g \rangle);\ 1 \leq i \leq 4, \times \right\}$$

be the MOD subset dual number matrix semigroup under usual product. Clearly S(B) is a non commutative MOD dual number subset matrix semigroup finite order.

Let $A = \begin{bmatrix} \{3g, 4g, 10g\} & \{9g, g, 8g, 6g\} \\ \{g, 2g, 5g\} & \{2g, g, 7g\} \end{bmatrix} \in S(B)$;

clearly $A \times A = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} = (\{0\})$,

so is a MOD dual number subset matrix nilpotent of order two.

Infact S(B) has several MOD dual number subset matrix nilpotents of order two under usual product ×.

The task of finding MOD dual number subset matrix idempotents of S(B). Thus finding o(SB)) and getting MOD dual number subset matrix left zero divisors which are not MOD dual number subset matrix right zero divisors are left for the reader.

In view of all these we have the following result.

**THEOREM 4.9:** *Let S(V) = {collection of all s × t subset matrices with entries from S(⟨$Z_n$ ∪ g⟩), ×} be the MOD dual number subset matrix semigroup under usual product ×.*

*i) S(V) is a non commutative MOD subset dual number matrix semigroup of finite order.*
*ii) S(V) has several MOD subset dual number matrix nilpotents of order two.*
*iii) S(V) has MOD dual number subset matrix ideal which is a zero square subsemigroup.*
*iv) S(V) has MOD dual number subset matrix idempotents depending on $Z_n$.*

Proof is direct and hence left as an exercise to the reader.

We leave it as an exercise to the reader to show MOD dual number subset matrix semigroup under natural product $\times_n$ is different from the usual product ×.

Next we proceed onto describe MOD special dual like number subset matrix semigroup under both the natural product $\times_n$ and that of usual product $\times$ by some examples.

***Example 4.27:*** Let

$$S(D) = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| a_i \in S(\langle Z_{10} \cup h\rangle);\ 1 \leq i \leq 8, \times_n\right\}$$

be the MOD special dual like number subset matrix semigroup under the natural product $\times_n$.

$$\text{Let } A = \begin{bmatrix} \{3h, 2h+1, 5\} & \{0, 4h\} \\ \{0\} & \{2\} \\ \{h\} & \{0, 2h+1, 8h\} \\ \{5h+5, 5\} & \{8h+3, 2h\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{h, 3h\} & \{4h+2, 3\} \\ \{7h+2, 8h+3 & \{4+7h, 8h, 3h \\ 5h+8, 3h, 4\} & 7h+3, 2, 0\} \\ \{3+4h, 5h+1\} & \{2\} \\ \{3\} & \{h, 2+3h\} \end{bmatrix} \in S(D).$$

We find

$$A \times_n B = \begin{bmatrix} \{3h, 2h+1, 5\} & \{0, 4h\} \\ \{0\} & \{2\} \\ \{h\} & \{0, 2h+1, 8h\} \\ \{5h+5, 5\} & \{8h+3, 2h\} \end{bmatrix} \times_n$$

$$\begin{bmatrix} \{h, 3h\} & \{4h+2, 3\} \\ \{7h+2, 8h+3 & \{4+7h, 8h, 3h \\ 5h+8, 3h, 4\} & 7h+3, 2, 0\} \\ \{3+4h, 5h+1\} & \{2\} \\ \{3\} & \{h, 2+3h\} \end{bmatrix}$$

$$= \begin{bmatrix} \{3h, 2h+1, 5\} \times & \{0, 4h\} \times \\ \{h, 3h\} & \{3, 4h+2\} \\ \{0\} \times \{7h+2, 8h+3, & \{2\} \times \{4+7h, 8h, \\ 5h+8, 3h, 4\} & 3h, 0, , 7h+2\} \\ \{h\} \times \{3+4h, & \{0, 8h, 2h+1\} \times \\ 5h+1\} & \{2\} \\ \{5, 5h+5\} \times \{3\} & \{2h, 8h+3\} \times \{h, 2+3h\} \end{bmatrix}$$

$$= \begin{bmatrix} \{3h, 7h, 5h, 9h\} & \{0, 2h, 4h\} \\ \{0\} & \{8+4h, 6h, 0, 4, 4h+6\} \\ \{7h, 6h\} & \{0, 6h, 4h+2\} \\ \{5, 5h+5\} & \{2h, h, 0, 6+9h\} \end{bmatrix}.$$

This is the way natural product is performed on S(D).

Let $A = \begin{bmatrix} \{6\} & \{6h\} \\ \{5+5h\} & \{5h\} \\ \{5, 6\} & \{5h, 6h\} \\ \{0, 6, 6h\} & \{0, 5, 5h\} \end{bmatrix} \in S(D).$

$A \times_n A = A$ so A is the MOD special dual like number subset matrix idempotent of S(D).

$$\text{Let } M = \begin{bmatrix} \{2, 4h, 6h\} & \{5, 5h, 5h+5\} \\ \{0, 5\} & \{2+4h, 8h+2\} \\ \{6h+6, 8+6h, 4+6h, 4h\} & \{5h, 0\} \\ \{2+2h, 8+6h\} & \{8h+4, 2h+8\} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \{5, 5h, 5+5h\} & \{2, 2+4h, 4h, 8h, 6h, 6+2h\} \\ \{2, 2h, 4h+6, 8h+2, 4+6h, 4, 8h\} & \{5, 0\} \\ \{4, 5h\} & \{2, 2+8h, 6h+4, 8+2h\} \\ \{5, 5h, 0, 5+5h\} & \{5, 0\} \end{bmatrix}$$

$$M \times_n B = \begin{bmatrix} \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \\ \{0\} & \{0\} \end{bmatrix} = (\{0\}).$$

Thus M is a MOD special dual like number matrix subset zero divisor.

Let

$$S(P) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \; a_i \in S(Z_{10}h);\ 1 \leq i \leq 8, \times_n \right\} \subseteq S(D)$$

be the MOD special dual like number matrix subset subsemigroup of S(D).

Clearly S(P) is also a MOD special dual like number matrix subset ideal of S(D). We give one more example.

***Example 4.28:*** Let

$$S(W) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in S(\langle Z_{11} \cup h \rangle);\ 1 \leq i \leq 4,\ \times_n \right\}$$

be the MOD special dual like number subset matrix semigroup under the natural product $\times_n$.

$$\text{Let } A = \begin{bmatrix} \{3h+5, 3h, 2h, 0, 10\} & \{1+h, h, 4+3h, 2+h\} \\ \{0, 1, h, 5h, 6h+6\} & \{h, 0, 2h+1, 6h, 9\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \begin{matrix}\{0, h, 4h+2, \\ 3h+6\}\end{matrix} & \{6h+4, 0, 1\} \\ \{1, 7h, 5h, 10h, 10\} & \{0, 1, 10, 5h, 5\} \end{bmatrix} \in S(W);$$

$$A \times_n B = \begin{bmatrix} \{3h+5, 3h, 2h, 0, 10\} & \{1+h, h, 4+3h, 2+h\} \\ \{0, 1, h, 5h, 6h+6\} & \{h, 0, 2h+1, 6h, 9\} \end{bmatrix}$$

$$\times_n \begin{bmatrix} \begin{matrix}\{0, h, 4h+2, \\ 3h+6\}\end{matrix} & \{6h+4, 0, 1\} \\ \{1, 7h, 5h, 10h, 10\} & \{0, 1, 10, 5h, 5\} \end{bmatrix}$$

$$= \begin{bmatrix} \{3h+5,3h,2h,0,10\}\times & \{h,1+h,4+3h,2+h\}\times \\ \{4h+2,3h+6\} & \{0,1,6h+4\} \\ \{0,1,h,5h,6+6h\} & \{0,h,2h+1,6h,9\}\times \\ \times\{1,7h,5h,10h,10\} & \{0,1,10,5,5h\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,8h,3h,2h,10h,7h+9, & \{0,h,1+h,4+3h,2+h, \\ 5+8h,h,7h,10+5h,8+9h\} & 10h,+5h,5+10h,8\} \\ \{0,1,h,5h,6+6h,7h,2h,7h & \{0,h,2h+1,6h,9,5h,10h+5 \\ 3h,10h,6h,10,5+5h\} & 8h,1,10h,10+9h,4h,h\} \end{bmatrix}$$

This is the way natural product $\times_n$ is obtained in S(W).

Let

$$S(V) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in S(Z_{11}h);\ 1 \le i \le 4, \times_n \right\} \subseteq S(W)$$

is the MOD subset special dual like number matrix subsemigroup which is also an ideal of S(W).

Now if we replace in S(W) the natural product $\times_n$ by $\times$ then S(W) will be a non commutative MOD subset special dual like number matrix non commutative semigroup under the usual product $\times$.

This two operations are distinct we will just show by some an illustration.

$$\text{Let } A = \begin{bmatrix} \{h,2h,1+h,0\} & \{10h,10,5h\} \\ \{5+10h,10+5h, & \{10+2h,0,1, \\ h,5h,1,0\} & h+2\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{3h, 0, 10\} & \{4h, 2+6h, 8h\} \\ \{0, 10+2h, 6h, 4h, 4, 8\} & \{8, 7, 5h, 10, h, 1\} \end{bmatrix} \in S(W);$$

we now define the usual product × in S(W).

$$A \times B = \begin{bmatrix} \{h, 2h, 1+h, 0\} & \{10h, 10, 5h\} \\ \{5+10h, 10+5h, & \{10+2h, 0, 1, \\ h, 5h, 1, 0\} & h+2\} \end{bmatrix} \times$$

$$\begin{bmatrix} \{3h, 0, 10\} & \{4h, 2+6h, 8h\} \\ \{0, 10+2h, 6h, 4h, 4, 8\} & \{8, 7, 5h, 10, h, 1\} \end{bmatrix}$$

=

$$\begin{bmatrix} \{0,10h,9h,10+10h,3h,6h & \{0,8h,4h,5h,2+3h\} \\ +\{0,10h,1+9h,5h,8h, & +\{3,3h,4,4h,7h,6h, \\ 9h,7h,7,3h\} & 2h,h,1,5h,10,10h\} \\ \{h,4h,3h,6+h,1+6h, & \{5h,9h,4h,10h,7h,8h,2+6h, \\ 6h,10h,0\}+\{0,1,10+2h,9+3h, & 9+h,10\}+\{5h+3,8,8h+5,4+3h, \\ 6h,7h,4h,h,7+8h, & 7,7h+3,h,3h,10+2h,1,h+2,1+9h,10, \\ 4,4h+8,3+5h,8,8h+5\} & 2h,10h+9,5h\} \end{bmatrix}$$

=

=

$$\begin{bmatrix} \{0,10h,9h,10+10h,3h,6h,8h,10+9h, \\ 2h,5h,7h,1+8h,1+7h,1+h,1+4h, \\ 4h,10+4h,6h,10+8h,h,10+6h, \\ 10h+7,9h+7,10h+6,7+3h,6h+7, \\ 10+2h,10h+3,9h+3,2+10h,3+6h\} \\ \{h,4h,3h,6+h,1+6h,6h,10h,0,8,h+1 \\ 4h+1,3h+1,7+h,2+6h,5h,5,10h+1,1, \\ 10+3h,8h+5,10+6h,10+5h,5+3h, \\ 8h,9h,7h+5,10+8h,10+h,10+2h, \\ 9+4h,3h+5,9+7h,9+6h,4+4h,10+9h, \\ 6+3h,8+h,9+9h,9+3h,2h,8+4h, \\ 10h,9h,6+7h,7h,8+3h,6+8h,1+2h, \\ 6+2h,1+7h,3+h,5h+8,8h+8,7h+8, \\ 3+5h,9+6h,9,10h+8,3h+8,4h+8, \\ 8+6h,3+6h,3+9h,3+8h, \\ 9h+5,h+5,4,3,3+4h,3+5h\} \end{bmatrix}$$

$$\begin{bmatrix} \{3,3h,4,4h,7h,6h,2h,h,1,5h,10,10h, \\ 8h+3,0,4+8h,9h,1+8h,10+8h, \\ 8h,3+4h,4+4h,10+4h,5h+10,5h+2, \\ 2+4h,6+3h,2+7h,2+9h,3+3h, \\ 2+8h,1+3h,2+h\} \\ \{10h+3,3h+3,9h+3,4h+3,h+3, \\ 2h+3,h,5,1+6h,9h,10h,10+7h, \\ 10,10+5h,7+8h,2+5h,8+5h, \\ 8+9h,6,7h,9+6h,8+4h,8+10h, \\ 8+7h,6h,5+h,2h,4+8h,8+8h, \\ 8+6h,6+h,10h,4+h,8h,7, \\ 2h+5,6h+5,5h,4+7h,h+5,7h+5, \\ 4h+5,4+2h,0,7+3h,3+9h,4+10h, \\ 6+9h,8h+4,7+4h,7+1h,4h+3, \\ 3+3h,7+5h,7+9h,h+3,5h+3, \\ 3,6h+3,3h+3,4h+3,5+2h, \\ 2+5h,2+7h,2+7h, \\ 9+2h,10+h\} \end{bmatrix}$$

This is the way the usual product operation is performed on S(D).

If we replace the usual product operation by natural product $\times_n$ certainly S(D) is commutative and both operations are distinct.

Further we see under both the operations $\times$ as well as $\times_n$ the set

$$S(V) = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in S(Z_{11}h);\ 1 \leq i \leq 4;\ \times \text{ or } \times_n\right\}$$

is a MOD special dual like number subset matrix subsemigroup which is an ideal.

This is the only common feature enjoyed by S(D), under both $\times$ and $\times_n$.

However MOD special dual like number zero divisor subset matrices say A, B $\in$ S(D) need not in general imply $A \times B = (\{0\})$ then $A \times_n B = (\{0\})$ and vice versa.

Next we supply yet another example of this set up.

***Example 4.29:*** Let

$$S(W) = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| a_i \in S(\langle Z_4 \cup h\rangle);\ 1 \leq i \leq 4,\ \times_n\right\}$$

be the MOD special dual like number subset matrix semigroup under the natural product $\times_n$.

$$\text{Let } A = \begin{bmatrix} \{3,2,0,h\} \\ \{2+2h,2\} \\ \{3,1+h\} \\ \{2h,2,0\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{3h,1\} \\ \{2,2h\} \\ \{1+2h\} \\ \{3+2h\} \end{bmatrix} \in S(W);$$

$$A \times_n B = \begin{bmatrix} \{3,2,0,h\} \\ \{2,2+2h\} \\ \{3,1+h\} \\ \{0,2,2h\} \end{bmatrix} \times_n \begin{bmatrix} \{1,3h\} \\ \{2,2h\} \\ \{1+2h\} \\ \{3+2h\} \end{bmatrix}$$

$$= \begin{bmatrix} \{3,2,0,h\} \times \{1,3h\} \\ \{2,2+2h\} \times \{2,2h\} \\ \{3,1+h\} \times \{1+2h\} \\ \{0,2,2h\} \times \{3+2h\} \end{bmatrix} = \begin{bmatrix} \{3,2,0,h,2h,3h\} \\ \{0\} \\ \{3+2h,1+2h\} \\ \{0,2,2h\} \end{bmatrix} \in S(W).$$

This is the way natural product operation is performed on S(W).

$$P_1 = \left\{ \begin{bmatrix} a \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix} \middle| \, a \in S(Z_4h); \times_n \right\} \subseteq S(W)$$

is a MOD special dual like number subset matrix ideal of S(W).

$$\text{Let } P_2 = \left\{ \begin{bmatrix} a \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix} \middle| \, a \in S(Z_4), \times_n \right\} \subseteq S(W)$$

is a MOD special dual like number subset matrix subsemigroup of S(W) which is not an ideal of S(W)

$$\text{Let } P_3 = \left\{ \begin{bmatrix} a \\ \{0\} \\ \{0\} \\ \{0\} \end{bmatrix} \middle| \; a \in S(\langle Z_4 \cup h \rangle); \times_n \right\} \subseteq S(W)$$

is again a MOD subset special dual like number matrix ideal of S(W).

Infact S(W) has several MOD special dual like number subset matrices subsemigroup which are not ideal.

In view of all these we have the following theorem.

**THEOREM 4.10:** *Let S(M) = {collection of all s × t matrices with entries from subsets of S(⟨$Z_n$ ∪ h⟩); $\times_n$} be the MOD special dual like number subset matrix semigroup under the natural product $\times_n$.*

i) *S(M) has MOD special dual like number nilpotents, idempotents and zero divisors if n is an appropriate composite number of the form $n = p^m q$ $m \geq 2$, p a prime and q a composite or another prime.*
ii) *S(M) has always MOD special dual like number subset matrix ideals if the entries are from S($Z_n h$) no matter whatever be n.*

Proof is direct and hence left as an exercise for the reader.

Next we proceed onto describe the MOD special quasi dual number subset matrix semigroups under $\times_n$ and also under usual product × provided the matrices under consideration are square matrices by examples.

***Example 4.30:*** Let

$$S(B) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in S(\langle Z_6 \cup k\rangle);\ 1 \leq i \leq 9,\ \times_n \right\}$$

be the MOD special quasi dual number subset matrix semigroup under the natural product $\times_n$.

$$\text{Let } A = \begin{bmatrix} \{3,4,0\} & \{1,4\} & \{1\} \\ \{0,3\} & \{3k,3\} & \{3+3k\} \\ \{4,3k\} & \{4,0,3k\} & \{1,3k\} \end{bmatrix} \in S(B).$$

We see $A \times_n A = A$, that is A is the MOD special quasi dual number subset matrix idempotent.

Suppose we impose the usual product that is we find $A \times A$, then

$$A \times A = \begin{bmatrix} \{0,3,4\} & \{1,4\} & \{1\} \\ \{0,3\} & \{3,3k\} & \{3+3k\} \\ \{4,3k\} & \{0,4,3k\} & \{1,3k\} \end{bmatrix} \times$$

$$\begin{bmatrix} \{0,3,4\} & \{1,4\} & \{1\} \\ \{0,3\} & \{3,3k\} & \{3+3k\} \\ \{4,3k\} & \{0,4,3k\} & \{1,3k\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,3,4\}\times\{0,3,4\} & \{3,0,4\}\times\{1,4\} & \{0,3,4\}\times\{1\}+ \\ +\{0,3\}\times\{1,4\}+ & +\{1,4\}\times\{3,3k\} & \{1,4\}\times\{3+3k\} \\ \{1\}\times\{4,3k\} & +\{1\}\times\{0,4,3k\} & +\{1\}\times\{1,3k\} \\ \{0,3\}\times\{0,3,4\} & \{0,3\}\times\{1,4\}+ & \{0,3\}\times\{1\}+ \\ +\{3,3k\}\times\{0,3\}+ & \{3,3k\}\times\{3,3k\} & \{3,3k\}\times\{3+3k\}+ \\ \{3+3k\}\times\{4,3k\} & +\{3+3k\}\times\{0,4,3k\} & \{3+3k\}\times\{1,3k\} \\ \{4,3k\}\times\{0,3,4\} & \{4,3k\}\times\{1,4\}+ & \{4,3k\}\times\{1\}+ \\ +\{0,4,3k\}\times\{0,3\} & \{3,3k\}\times\{0,4,3k\} & \{0,4,3k\}\times\{3+3k\}+ \\ +\{1,3k\}\times\{4,3k\} & +\{1,3k\}\times\{0,4,3k\} & \{1+3k\}\times\{1,3k\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,3,4\}+\{0,3\} +\{4,3k\} & \{3,0,4\}+\{3,3k\} +\{0,4,3k\} & \{0,3,4\}+ \{3+3k\}, \{1,3k\} \\ \{0,3\}+\{3,3k,0\} +\{0\} & \{0,3\}+\{3,3k\} +\{0\} & \{0,3\}+ \{3+3k,0\} +\{3+3k,0\} \\ \{0,3k,4\}+ \{0,3k\}+ \{4,3k\} & \{3k,4\}+ \{0,3k\}+ \{0,4,3k\} & \{4,3k\}+\{0\}+ \{1,3k\} \end{bmatrix}$$

$$= \begin{bmatrix} \{4,1,2,5,3k, 3+3k,3k+4, 1+3k\} & \{0,1,3,3+k,3k, 3k+4,5,1+3k, 3+3k,2+3k,4\} & \{4+3k,1+3k, 2+3k,0, 1,3\} \\ \{0,3,3k,3+3k\} & \{3+3k,3,0,3k\} & \{0,3,3k,3+3k\} \\ \{0,4,3k, 2+3k,4+3k\} & \{0,4,3k,2, 2+3k,4+3k\} & \{5,3k+1 0,4+3k\} \end{bmatrix} \neq A.$$

Thus we see the two operations behave in a very distinct way.

Let $A = \begin{bmatrix} \{0,3k\} & \{0,3+3k\} & \{0,3,3k\} \\ \{2\} & \{4\} & \{2+4k\} \\ \{4k+4\} & \{2+2k,4\} & \{3+3k\} \end{bmatrix}$ and

$B = \begin{bmatrix} \{4\} & \{4,4k\} & \{2\} \\ \{3,3k\} & \{3+3k\} & \{3,3k\} \\ \{0,3\} & \{3,3+3k\} & \{4,2\} \end{bmatrix}$ be elements of S(B).

$$\text{Clearly } A \times_n B = \begin{bmatrix} \{3,3k\} & \{0,3+3k\} & \{0,3,3k\} \\ \{2\} & \{4\} & \{2+4k\} \\ \{4+4k\} & \{4,2+2k\} & \{3+3k\} \end{bmatrix} \times_n$$

$$\begin{bmatrix} \{4\} & \{4,4k\} & \{2\} \\ \{3,3k\} & \{3+3k\} & \{3,3k\} \\ \{0,3k\} & \{3,3+3k\} & \{4,2\} \end{bmatrix}$$

$$= \begin{bmatrix} \{3,3k\}\times\{4\} & \{0,3+3k\}\times\{4,4k\} & \{\{0,3,3k\}\times\{2\} \\ \{2\}\times\{3,3k\} & \{4\}\times\{3+3k\} & \{2+4k\}\times\{3,3k\} \\ \{4+4k\}\times\{0,3\} & \{4,2+2k\}\times\{3,3+3k\} & \{3+3k\}\times\{4,2\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0\} & \{0\} & \{0\} \\ \{0\} & \{0\} & \{0\} \\ \{0\} & \{0\} & \{0\} \end{bmatrix} = (\{0\})$$

Thus under the natural product $\times_n$. A, B is a zero divisor pair. Now we find A × B

$$= \begin{bmatrix} \{3,3k\} & \{0,3+3k\} & \{0,3,3k\} \\ \{2\} & \{4\} & \{2+4k\} \\ \{4k+4\} & \{4,2+2k\} & \{3+3k\} \end{bmatrix} \times$$

$$\begin{bmatrix} \{4\} & \{4,4k\} & \{2\} \\ \{3,3k\} & \{3+3k\} & \{3,3k\} \\ \{0,3\} & \{3,3+3k\} & \{2,4\} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix}\{3,3k\}\times\{4\} \\ +\{0,3+3k\}\times\{3,3k\} \\ +\{0,3,3k\}\times\{0,3\}\end{matrix} & \begin{matrix}\{3,3k\}\times\{4,4k\} \\ +\{0,3+3k\}\times\{3+3k\} \\ +\{0,3,3k\}\times\{3,3+3k\}\end{matrix} & \begin{matrix}\{3,3k\}\times\{2\}+ \\ \{0,3+3k\}\times\{3,3k\}+ \\ \{3,3k,0\}\times\{2,4\}\end{matrix} \\ \begin{matrix}\{2\}\times\{4\}+\{4\}\times\{3,3k\} \\ +\{2+4k\}\times\{0,3\}\end{matrix} & \begin{matrix}\{2\}\times\{4,4k\}+\{4\} \\ \times\{3+3k\}+\{2+4k\} \\ \times\{3,3+3k\}\end{matrix} & \begin{matrix}\{2\}\times\{2\}+\{4\}\times \\ \{3,3k\}+\{2+4k\} \\ \times\{2,4\}\end{matrix} \\ \begin{matrix}\{4k+4\}\times\{4\}+ \\ \{4,2+2k\}\times\{3,3k\} \\ +\{3+3k\}\times\{0,3\}\end{matrix} & \begin{matrix}\{4,4k\}\times\{4+4k\}+ \\ \{3+3k\}\times\{4,2+2k\} \\ +\{3+3k\}\times\{3,3+3k\}\end{matrix} & \begin{matrix}\{4k+4\}\times\{2\}+ \\ \{4,2+2k\}\times\{3,3k\} \\ +\{3+3k\}\times\{2,4\}\end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,3+3k,3,3k\} & \{0,3+3k,3k,3\} & \{0,3+3k\} \\ \{2\} & \{2,2k\} & \{2k+2k,4k\} \\ \{4+4k,k+1\} & \{3+3k,1+k\} & \{2+2k\} \end{bmatrix}.$$

Clearly the pair A, B in S(B) does not yield a MOD zero divisor of special quasi dual numbers under the usual product ×.

However

$$S(P) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in S(Z_6k);\ 1 \le i \le 9, \times_n \right\}$$

as well {S(P), ×} are both MOD special quasi dual number subset matrix ideals of S(D).

This property alone is common to both the product × and $\times_n$.

In view of all these we give the following result.

**THEOREM 4.11:** *Let S(M) = {collection of all s × t matrices with entries from subsets $S(\langle Z_n \cup k\rangle)$; $\times_n$} be the MOD special quasi dual number subset matrix semigroup under natural product.*

*i) S(M) has MOD special quasi dual number subset matrix idempotents, nilpotents and zero divisors for $n = p^m q$, $m \geq 2$ p is a prime, q may be prime or non prime.*
*ii) S(W) = {collection of all s × t matrices with subset entries from $S(Z_n k)$ under natural product $\times_n$} is a MOD special quasi dual number subset matrix ideal of S(M).*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto study MOD subset coefficient polynomial semigroup under product by some example.

***Example 4.31:*** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_{12}); \times \right\}$$

be the MOD subset coefficient polynomial semigroup under product.

Clearly o(S[x]) = ∞.

Let $p(x) = \{0, 3, 6\} + \{2, 6\}x + \{5, 1, 8\}x^2$ and

$q(x) = \{4, 8, 1\} + \{4, 6, 3\} x^3 \in S[x]$

$p(x) \times q(x) = (\{0, 3, 6\} + \{2, 6\}x + \{5, 1, 8\}x^2) \times (\{4, 8, 1\} + \{4, 6, 3\}x^3)$

$= (\{0, 3, 6\} \times \{1, 4, 8\}) + (\{2, 6\} \times \{4, 8, 1\})x + (\{5, 1, 8\} \times \{1, 4, 8\})x^2 + (\{0, 3, 6\} \times \{4, 6, 3\})x^3 + (\{2, 6\} \times \{4, 6, 3\})x^4 + (\{5, 1, 8\} \times \{4, 6, 3\})x^5$

$= \{0, 3, 6\} + \{8, 4, 0, 2, 6\}x + \{1, 5, 8, 2, 4\}x^2 + \{0, 9,6\}x^3 + \{8, 6, 0\}x^4 + \{2, 4, 6, 0\}x^5 \in S[x]$.

This is the way product of two polynomials is made in S[x].

$(0) = 0 + 0x + 0x^2 + \ldots + 0x^n \in S[x]$ is such that $p(x) \times (0) = (0)$, S[x] is a commutative semigroup of infinite order.

Let $p(x) = \{0, 4\} + \{0, 8\} x^2 + \{0, 8, 4\} x^4$ and

$q(x) = \{0, 3, 6\} + \{6, 3, 9\}x^3 \in S[x]$.

$p(x) \times q(x) = (\{0, 4\} + \{0, 8\}x^2 + \{0, 4, 8\}x^4) \times (\{0, 3, 6\} + \{3, 6, 9\}x^3)$

$=\{0\} + \{0\}x^2 + \{0\}x^4 + \{0\}x^3 + \{0\}x^5 + \{0\}x^7 = (0)$.

This we see S[x] can have pairs of polynomials such that $p(x) \times q(x) = (\{0\})$ is a zero divisor in S[x].

We can also find MOD subset coefficient polynomials subsemigroups under × as well as ideals of S[x].

Let

$$P[x] = \left\{ \sum_{i \in 2Z^+ \cup \{0\}}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_{12}), \times \right\} \subseteq S[x]$$

is a MOD subset coefficient polynomial subsemigroup of S[x] of infinite order which is clearly not an ideal of S[x].

For p(x) = {0, 5, 10, 2} x + {8, 4, 0, 1} ∈ S[x] and
q(x) = {0, 5, 8} + {0, 9, 8, 2} $x^6$ ∈ P[x]

Consider p(x) × q(x) = ({8, 4, 0, 1} + {0, 2, 5, 10} x) × ({0, 5, 8} + {0, 2, 8, 9})$x^6$

= ({8, 4, 0, 1} × {0, 5, 8}) + ({0, 2, 5, 10} × {0, 5, 8})x + ({8, 4, 0, 1} × {0, 2, 8, 9})$x^6$ + ({0, 2, 5, 10} × {0, 2, 8, 9})$x^7$

= {0, 5, 2, 4, 8} + {0, 10, 1, 2, 4, 8}x + {0, 2, 8, 4, 9}$x^6$ + {0, 4, 10, 2, 8, 6, 9}$x^7$ ∉ P[x].

Hence our claim P[x] is only MOD subset polynomial coefficient subsemigroup.

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\{0, 2, 4, 6, 8, 10\}), \times \right\} \subseteq S[x]$$

is a MOD subset coefficient polynomial subsemigroup which is also an ideal of S[x].

The reader is left with the task of finding MOD subset coefficient polynomial subsemigroups, ideals and zero divisors. However it is kept on record that S[x] no idempotents.

***Example 4.32:*** Let

$$T[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_{19}); \times \right\}$$

be the MOD subset coefficient polynomial semigroup under ×.

Clearly T[x] has no zero divisors, however T[x] has no ideals but has subsemigorups of infinite order.

In view of all these we have the following theorem.

**THEOREM 4.12:** *Let W[x] =* $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_n), \times\right\}$ *be the MOD subset coefficient polynomial semigroup.*

*i) W[x] has no idempotent MOD subset polynomials.*
*ii) W[x] has MOD subset coefficient polynomials which are nilpotents and zero divisors only for some appropriate n, n =* $p^t q$ *(t ≥ 2 p a prime and q any number other than p).*
*iii) W[x] has MOD subset coefficient polynomial subsemigroup for all n.*
*iv) W[x] has no MOD subset coefficient polynomial ideals for n a prime.*

Proof is left as an exercise to the reader.

Next we describe by example MOD finite complex number subset coefficient polynomial semigroups under ×.

***Example 4.33:*** Let

$$D[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(C(Z_6)); \times\right\}$$

be the MOD subset finite complex number coefficient polynomial semigroup under product.

$$P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_6), \times\right\} \subseteq D[x]$$

is a MOD subset finite complex coefficient polynomial subsemigroup under × of D[x], however P[x] is not a MOD subset finite complex number coefficient polynomial ideal of D[x].

Find MOD subset finite complex number coefficient polynomial ideals of D[x].

Clearly D[x] has MOD subset finite complex number coefficient zero divisor polynomials.

However D[x] has no nontrivial MOD subset finite complex number coefficient nilpotent or idempotent polynomials.

It is left for the reader to find under what conditions MOD finite complex number subset coefficient polynomials has ideals.

For finding ideals happens to be yet another challenging problem.

***Example 4.34:*** Let

$$S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(C(Z_{13})), \times\right\}$$

be the MOD finite complex number subset coefficient polynomial semigroup under ×.

S[x] has no MOD finite complex number coefficient subset zero divisors or idempotents or nilpotents. However S[x] has MOD finite complex number coefficient subset polynomial subsemigroups but no ideals.

In view of this we propose the following problem.

**Problem 4.1:** Let $S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(C(Z_p)) \text{ p a prime}, \times\right\}$

be the MOD finite complex number subset coefficient polynomial semigroup under ×.

Can S[x] have MOD finite complex number subset coefficient polynomial ideal?

**THEOREM 4.13:** *Let*

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_j \in S(C(Z_n)); \times \right\}$$

*be the MOD finite complex number subset coefficient polynomial semigroup under ×.*

i) *For appropriate n, S[x] has zero divisors, nilpotents and ideals.*
ii) *For n a prime S[x] has no zero divisors and nilpotents.*
iii) *S[x] has no nontrivial MOD finite complex number subset coefficient polynomial idempotent.*

The reader is left with the task of proving this theorem.

Next we proceed onto describe MOD dual number subset coefficient polynomial semigroups under × by example.

***Example 4.35:*** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{10} \cup g \rangle); \times \right\}$$

be the MOD dual number subset coefficient polynomial semigroup under product.

S[x] has infinite number of MOD subset dual number coefficient polynomials such that they are zero divisors.

S[x] has infinite number of MOD subset dual number coefficient nilpotent polynomials of order two.

Let p(x) = {3g, 2g, 5g, 6g} $x^3$ + {g, 2g, 9g, 8g}x + {g, 0, 8g, 7g} ∈ S[x].

Consider p(x) × p(x) = ({3g, 2g, 5g, 6g}$x^3$ + {g, 2g, 9g, 8g}x + {0, g, 8g, 7g}) × ({3g, 2g, 5g, 6g}$x^3$ + {g, 2g, 9g, 8g}x + {0, g, 7g, 8g})

= ({3g, 2g, 5g, 6g} × {3g, 2g, 5g, 6g})$x^6$ + ({3g, 2g, 5g, 6g} × {g, 2g, 9g, 8g})$x^4$ + ({0, g, 7g, 8g} × {2g, 3g, 5g, 6g})$x^3$ + ({g, 2g, 8g, 9g} × {3g, 2g, 5g, 6g})$x^4$ + ({g, 2g, 8g, 9g} × {g, 2g, 8g, 9g})$x^2$ + ({g, 2g, 8g, 9g} × {0, g, 7g, 8g})x + ({0, g, 7g, 8g} × {3g, 2g, 5g, 6g}) $x^3$ + ({0, g, 7g, 8g} × {g, 2g, 9g, 8g}) x + ({0, g, 7g, 8g} × {0, g, 7g, 8g}) = {0} $x^6$ + {0}$x^4$ + {0}$x^3$ + {0}$x^4$ + {0}$x^2$ + {0}x + {0}$x^3$ + {0}x + {0} = ({0}).

Thus p(x) is a MOD subset dual number coefficient polynomial which is nilpotent of order two in S[x].

Infact it is pertinent to keep on record that S[x] has infinite number of MOD dual number subset coefficient nilpotent polynomials of order two.

Let p(x) = {3g, g, 0} $x^3$ + {2g, g, 4g, 7g} x + {5g, 6g, 9g, 8g, 0, 2g} and

q(x) = {4g, 5g, 8g, 9g, 0} $x^2$ + {3g, 2g, g, 6g, 5g, 3g} ∈ S[x].

Clearly p(x) × q(x) = ({0}).

Thus p(x) is a MOD dual number subset coefficient polynomial zero divisor of S[x].

We see S[x] has infinite number of MOD subset dual number coefficient polynomial pairs which are zero divisors.

This is the marked difference between the MOD subset coefficient polynomial semigroup and MOD dual number coefficient subset polynomial semigroup under ×.

Further

$$B[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_{10}g); \times\right\} \subseteq S[x]$$

is a MOD dual number subset coefficient polynomial subsemigroup which is also a MOD dual number subset coefficient polynomial ideal of S[x].

Infact B[x] is a zero divisor subsemigroup as for any p(x), q(x) ∈ B[x]; p(x) × q(x) = ({0}), that is B[x] × B[x] = ([0}).

However in S[x] we do not have MOD dual number subset coefficient polynomials which are idempotents.

***Example 4.36:*** Let

$$S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_7 \cup g\rangle); \times\right\}$$

be the MOD dual number subset coefficient polynomial semigroup under product.

Let $p(x) = \{3g, 2g, 0, g\} + \{4g, 5g, 6g\}x^2 + \{6g, g, 0\}x^3$ and

$q(x) = \{4g, 0\} + \{2g, 3g, g\}x + \{5g, 6g, 0, g\}x^5 \in S[x]$.

p(x) × q(x) = ({0}). Thus we see even though $Z_7$ is a prime field yet we have MOD dual number subset coefficient polynomial pairs which are zero divisors.

Also p(x) × p(x) = ({0}) and q(x) × q(x) = ({0}) are both MOD dual number subset coefficient polynomials which are nilpotents of order two.

$$D[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_7); \times \right\} \subseteq S[x]$$

is a MOD subset dual number coefficient polynomial subsemigroup which has no nontrivial MOD dual number subset coefficient polynomial zero divisors or nilpotents.

D[x] is not an ideal of S[x].

$$B[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_7 g); \times \right\} \subseteq S[x]$$

is a MOD dual number subset coefficient polynomial subsemigroup which is also an ideal.

This B[x] has nontrivial MOD subset dual number coefficient polynomial pairs which are zero divisors as well as nilpotents of order two.

However B[x], D[x] and S[x] has no MOD subset dual number coefficient polynomial idempotents which are nontrivial.

For a = {0, 1} ∈ S[x] then, we have a × a = {0, 1} × {0, 1} = {0, 1} = a is a trivial MOD subset dual number coefficient constant polynomial idempotent.

Inview of all these we have the following theorem.

**THEOREM 4.14:** *Let* $S[x] = \{\sum a_i x^i \mid a_i \in S(\langle Z_n \cup g\rangle); \times\}$ *be MOD dual number subset coefficient polynomial semigroup under* ×.

*i) S[x] has no nontrivial MOD dual number subset coefficient polynomial idempotents.*

*ii) S[x] has nontrivial MOD dual number subset coefficient polynomial zero divisors and nilpotents whatever be n.*

*iii) S[x] has nontrivial MOD dual number coefficient polynomial subsemigroups which are not ideals.*

*iv) S[x] has nontrivial MOD dual number coefficient polynomials subsemigroups D[x] which are not ideals but D[x] × D[x] = ({0})*

*v) S[x] has nontrivial MOD dual number coefficient polynomial subsemigroups* $T[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S(Z_n g); \times \right\} \subseteq S[x]$ *to be an ideal with T[x] × T[x] = ({0}).*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD neutrosophic subset coefficient polynomial semigroups by examples.

***Example 4.37:*** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S(\langle Z_{17} \cup I\rangle); \times \right\}$$

be the MOD neutrosophic subset coefficient polynomial semigroup under ×.

Clearly S[x] has no nontrivial neutrosophic subset coefficient idempotent polynomials or nilpotent polynomials or zero divisor polynomials as in $Z_{17}$, 17 is a prime.

However if $p(x) = \{3I, 4 + 2I, 7\} + \{0, 4I, 2 + I, 3\}x^2$ and

$q(x) = \{0, I, 4, 3I\} + \{2 + 4I, 3, 1\} x^2 \in S[x]$; then

$p(x) \times q(x) = (\{3I, 4 + 2I, 7\} + \{0, 4I, 2+ I, 3\}x^2) \times (\{0, I, 4, 3I\} \times \{1, 3, 2 + 4I\}x^2)$

= ({3I, 4 + 2I, 7} × {0, I, 4, 3I}) + ({0, 4I, 2 + I, 3} × {0, I, 4, 3I})$x^2$ + ({3I, 7, 4 + 2I} × {1, 3, 2 + 4I})$x^2$ + ({0, 4I, 2 + I, 3} × {1, 3, 2 + 4I})$x^4$

= {0, 3I, 6I, 7I, 12I, 16 + 8I, 11, 9I, I, 4I} + {0, 4I, 3I, 16I, 8 + 4I, 12, 12I, 9I}$x^2$ + {3I, 7, 4 + 2I, 9I, 4, 12 + 6I, I, 14 + 11I, 8 + 11I}$x^2$ + {0, 4I, 2 + I, 3, 12I, 9I, 9, 7I, 4 + 14I, 6 + 12I}$x^4$

= {0, 3I, 6I, 7I, 12I, 6 + 8I, 11, 9I, I, 4I} +{3I, 7, 4 + 2I, 9I, 4, 12, 6I, I, 14 + 11I, 8 + 11I, 7I, 7 + 4I, 4 + 6I, 13I, 0, 4 + 16I, 4 + 4I, 12 + 10I, 5I, 14 + 15I, 12 + 5I, 8 +15I, 6I, 7 + 3I, 4 + 5I, 14 + 10I, 8 + 10I, 12I, 4 + 3I, 12 + 9I, 4I, 14 + 14I, 8 + 14I, 2I, 16I + 7, 4 + I, 8 + 7I, 15 + 4I, 12 + 6I, 8 + 13I, 12 + 4I, 3 +10I, 8 + 5I, 5 + 15I, 16 + 15I, 12 + 3I, 2, 16 + 2I, 12 + 9I, 16, 7 + 6I, 12 + I, 9 + 11I, 20 + 11I, 15I, 12I + 7, 4 + 14I, 4I, 4 + 12I, 12 + I, 13I, 14 + 6I, 8 + 6I, 12I, 7 + 9I, 4 + 11I, I, 4 + iI, 12 + 15I, 10I, 14 + 3I, 8 + 14I} $x^2$ + {0, 4I, 2 + I, 3, 12I, 9I, 9, 7I, 4 + 14I, 6 + 12I}$x^4$.

This is the way product operation is performed on S[x].

$$B[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S\{Z_{17}I); \times\} \subseteq S[x] \right.$$

is a MOD neutrosophic subset coefficient polynomial subsemigroup which is also an ideal of S[x].

This S[x] has no nontrivial idempotents.

The trivial idempotent MOD neutrosophic subsets are {0}, {1}, {I}, {0, 1}, {0, I}, {1, I} and {0, 1, I}.

They can only yield to constant MOD subset neutrosophic polynomials.

***Example 4.38:*** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in (\langle Z_{12} \cup I\rangle); \times \right\}$$

be the MOD neutrosophic subset coefficient polynomial semigroup under product operation.

Let p(x) = {0, 4, 2} $x^3$ + {0, 8, 8I, 4I} and q(x) = {0, 6, 6I, 6 + 6I} + {0, 3, 3 + 3I, 6I, 6 + 3I}x ∈ S[x].

p(x) × q(x) = ({0, 2, 4}$x^3$ + {0, 8, 8I, 4I}) × ({0, 6, 6I, 6 + 6I} + {0, 3, 3 + 3I, 6I, 6 + 3I}x) = ({0}).

Thus S[x] has MOD neutrosophic subset coefficient polynomial pair which are zero divisors.

However S[x] has no idempotents.

Let p(x) = {0, 6} + {0, 6I, 6, 6 + 6I}$x^3$ ∈ S[x];

clearly p(x) × p(x) = ({0}) so p(x) is a MOD neutrosophic subset coefficient polynomial nilpotent of S[x].

S[x] has trivial idempotents given by {0, 4, 9, 1}, {0, 4, 9}, {0, 4, 1}, {0, 9, 1}, {0, 4}, {1, 4}, {1, 9} and {0, 9}.

Clearly

$$M[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_{12}I), \times \right\} \subseteq S[x]$$

is a MOD neutrosophic subset coefficient polynomial subsemigroup which is also an ideal of S[x].

But

$$T[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_{12}), \times\right\} \subseteq S[x]$$

is a MOD neutrosophic subset coefficient polynomial subsemigroup which is not an ideal.

In view of all these we have the following result.

**THEOREM 4.15:** *Let*

$$S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_n \cup I\rangle); \times\right\}$$

*be the MOD neutrosophic subset coefficient polynomial semigroup under product.*

*i) S[x] has no nontrivial MOD neutrosophic subset coefficient polynomial idempotents.*
*ii) S[x] has MOD neutrosophic subset coefficient polynomial pairs which can contribute to zero divisors provided n is an appropriate composite number.*
*iii) S[x] has MOD neutrosophic subset coefficient polynomial nilpotents if $n = p^t q$, $t \geq 2$, p a prime and q a composite number*
*iv) $D[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_n I), \times\right\}$ is the MOD neutrosophic subset coefficient polynomial subsemigroup which is an ideal of S[x].*
*v) S[x] has MOD neutrosophic subset coefficient polynomial subsemigroups which are not ideals.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe by examples MOD subset special dual like number coefficient polynomial semigroups under product.

***Example 4.39:*** Let

$$B[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{13} \cup h \rangle), \times \right\}$$

be the MOD subset special dual like number coefficient polynomial semigroup under product.

We see B[x] has no nontrivial idempotents. B[x] has no nontrivial nilpotents as well as zero divisor polynomials.

However B[x] has MOD subset special dual like number coefficient polynomial subsemigroups which are ideals.

Let p(x) = {3, 5h, 2 + h, 1} + {0, 4h, 3}x + {0, 1, 3h}$x^2$ and

q(x) = {0, 4h, 2}$x^4$ + {1, 2, 3, 0} ∈ B[x].

We find p(x) × q(x) = ({3, 5h, 2 + h, 1} + {0, 4h, 3}x + {0, 1, 3, h})$x^2$ × ({0, 1, 2, 3} + {0, 2, 4h}$x^4$ )

= {3, 5h, 1, 2 + h} × {0, 1, 2, 3} + ({3, 5h, 1, 2 + h} × {0, 2, 4h})$x^4$ + ({0, 4h, 3} × {0, 1, 2, 3})x + ({0, 3, 4h} × {0, 2, 4h}) $x^5$ + ({0, 1, 3h} × {0, 1, 2, 3})$x^2$ + ({0, 1, 3h} × {0, 2, 4h})$x^6$

= {0, 3, 5h, 1, 2 + h, 6, 10h, 2, 4 + 2h, 9, 2h, 6 + 3h} + {0, 6, 10h, 2, 8 + 4h, 4 + 2h, 12, 7h, 4h}$x^4$ + {0, 4h, 3, 8h, 6, 12h, 9}x + {0, 6, 8h, 12h, 3h}$x^5$ + {0, 1, 3h, 2, 6h, 3, 9h}$x^2$ + {0, 2, 6h, 4h, 12h}$x^6$ ∈ B[x].

This is the way product operation is performed in B[x]. B[x] has no MOD idempotents or zero divisors or nilpotents.

***Example 4.40:*** Let

$$M[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{15} \cup h \rangle); \times \right\}$$

be the MOD subset special dual like number coefficient polynomial semigroup under the product. M[x] has zero divisors has no nontrivial nilpotents or idempotents.

p(x) = {5, 5h, 10} + {10, 10 + 10h, 5 + 10h, 0} $x^2$ and

q(x) = {3, 0, 3h} + {3 + 3h, 6}$x^2$ + {3h, 6h, 6 + 3h}$x^4$ ∈ M[x].

p(x) × q(x) = ({5, 5h, 10} + {10, 10 + 10h, 5 + 10h, 0})$x^2$ × ({0, 3, 3h} + {3 + 3h, 6h})$x^2$ + {6h, 3h, 6 + 3h} $x^4$)

= {5, 5h, 10} × {0, 3, 3h} + ({5, 5h, 10} × {3 + 3h, 6h})$x^2$ + ({5, 5h, 10} × {6h, 3h, 6 + 3h})$x^4$ + ({10, 10 + 10h, 5 + 10h, 0} × {0, 3, 3h}) $x^2$ + ({10, 10 + 10h, 5 + 10h, 0} × {6 + 3h, 3h, 6h}) $x^6$ + ({10, 10 + 10h, 5 + 10h, 0} × {6h, 3+ 3h})$x^4$ = ({0}).

Thus this pair of MOD special dual like number subset coefficient polynomials is a zero divisor subset polynomial pair.

We can find infinite number of MOD special dual like number subset coefficient polynomial zero divisor pairs.

However M[x] has no nilpotents only trivial idempotents of the form {0}, {1}, {6}, {10}, {6, 10, 0}, {1, 6}, {0, 6}, {1, 10}, {0, 10}, {0, 1, 6, 10} as $6^2 = 6$ (mod 15) and 10 × 10 = 10 (mod 15).

This has no nontrivial nilpotent subset only {0} is the trivial nilpotent subset.

In view of all these we put forth the following theorem.

**THEOREM 4.16:** *Let* $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S(\langle Z_n \cup h \rangle);\ \times \right\}$ *be the MOD special dual like number subset coefficient polynomial semigroup under product.*

*Then*

i) *S[x] has MOD special dual like number subset coefficient polynomial subsemigroup which is not an ideal (what ever be n).*
ii) *S[x] has MOD special dual like number subset coefficient polynomial semigroups which is also an ideal.*
iii) *S[x] has MOD special dual like number subset coefficient polynomial nilpotents and zero divisors only for appropriate n; $n = p^t s$ ($t \geq 2$, p a prime s a composite or a prime different from p).*
iv) *S[x] has no MOD special dual like number subset coefficient polynomial which is an idempotent polynomial.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD special quasi dual number subset coefficient polynomial semigroup under product operation by some examples.

***Example 4.41:*** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{20} \cup k \rangle); \times \right\}$$

be the MOD special quasi dual number subset coefficient polynomial semigroup under product.

Let p(x) = {0, k, 2k + 1, 4k} + {1, 5k, 2 + 3k} x + {1, 0, 2k, 8k, 4}$x^2$ and

q(x) = {0, 4k, 10k} + {10, 10 + 10k, 1, 0, k}$x^4$ ∈ S[x].

p(x) × q(x) = ({0, k, 4k, 2k + 1} + {1, 5k, 2+ 2k}x + {1, 0, 4, 8k, 2k}$x^2$) × ({0, 4k, 10k} + {10, 10 + 10k, 1, 0, k}$x^4$)

= {0, k, 4k, 2k + 1} × {0, 4k, 10k} + ({0, k, 4k, 2k + 1} × {10, 10k + 10, 1, 0, k})$x^4$ + ({1, 5k, 2 + 3k} × {0, 4k, 10l})x + ({1, 5k, 2 + 3k} × {10, 10 + 10k, 1, 0, k})$x^5$ + ({0, 1, 4, 2k, 8k} × {0, 4k, 10k})$x^6$ = {0, 16k, 4k, 16k, 10k} + {0, 10k, 10, 10k + 10, k, 4k, 2k + 1, 19k, 16k, 19k}$x^4$ + {10, 10k, 10 + 10k, 1, 5k, 2 + 3k, 0, k, 15k, 19k}$x^5$ + {0, 4k, 6k, 10k} x + {0, 4k, 16k, 12k,18k, 10k}$x^2$ + {0, 10, 10 + 10k, 1, 4, 8k, 2k, k, 4k, 12k, 18k} $x^6$.

This is the way product operation is performed on S[x].

Consider p(x) = {10, 10k} + {0, 10, 10k + 10} $x^2$ + {10 + 10k, 10k}$x^4$ ∈ S[x].

Clearly p(x) × p(x) = ({0}). Thus p(x) is a nilpotent MOD subset coefficient polynomial.

Thus S[x] has both MOD special quasi dual number subset coefficient polynomial zero divisors and nilpotents, however has no idempotents.

***Example 4.42:*** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_5 \cup k \rangle); \times \right\}$$

be the MOD special quasi dual number subset coefficient polynomial semigroup under product.

Let p(x) = {k, 1, 0} + {1 + k, 4, 2}x + {1, 3k, 3}$x^2$ and

q(x) = {0, k} + {1, 2, 3, k, 4} $x^4$ ∈ S[x].

p(x) × q(x) = ({0, 1, k} + {2, 4, 1 + k} x + {1, 3k, 3}$x^2$) × ({0 k} + {1, 2, 3, k, 4}$x^4$)

= {0, 1, k}× {0, k} + ({2, 4, 1 + k} × {0, k}) x + ({1, 3k, 3} × {0, k}) $x^2$ + ({0, 1, k} × {1, 2, 3, k, 4})$x^4$ + ({2, 4, 1 + k}×{1, 2, 3, 4, k}) $x^5$ + ({1, 3, 3k} × {1, 2, 3, 4, k})$x^6$

= {0, k, 4k} + {0, 2k, 4k}x + {0, k, 3k, 2k}$x^5$ {0, 1, k, 2, 2k, 3, 3k, k, 4k, 4} $x^4$ + {2, 4, 1 + k, 2, 3, 2 + 2k, 1, 3 + 3k, 4 + 4k, 2k, 4k, 0}$x^5$ + {1, 2, 3, 4, k, 3k, 4k, 2k}$x^6$.

This is the way product operation is performed on S[x].

S[x] has no MOD special quasi dual number subset coefficient polynomial idempotents or zero divisors or nilpotents.

In view of all these we have the following result.

**THEOREM 4.17:** *Let S[x] be the MOD special quasi dual number subset coefficient polynomial semigroup under product,*

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_n \cup k\rangle); \times \right\}.$$

*i) S[x] has no MOD nilpotents or idempotents or zero divisors if n = p, a prime.*
*ii) S[x] for all n has a MOD special quasi dual number subset coefficient subsemigroups which are not ideals.*
*iii)* $P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_n k); \times \right\} \subseteq S[x]$ *is a MOD special quasi dual number subset coefficient polynomial ideal of S[x].*
*iv) S[x] has MOD nilpotents and zero divisors if n = $p^t q$, t ≥ 2 p a prime and q a composite number of a prime different from p.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto give by simple illustrations MOD subset coefficient polynomial semigroup under product of finite order very briefly.

***Example 4.43:*** Let

$$S[x]_8 = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_6);\ x^9 = 1, \times \right\}$$

be the MOD subset coefficient polynomial semigroup under product. $S[x]_8$ is of finite order and is commutative.

Let $p(x) = \{3, 2, 4\} + \{0, 5, 1\}x^5$ and

$q(x) = \{1, 0, 3\} + \{2, 4, 5, 1\}\, x^4 \in S[x]_8$.

$p(x) \times q(x) = (\{3, 2, 4\} + \{0, 5, 1\}\, x^5) \times (\{1, 0, 3\} + \{1, 2, 4, 5\}\, x^4)$

$= \{3, 2, 4\} \times \{1, 0, 3\} + \{3, 2, 4\} \times \{1, 2, 4, 5\}\, x^4\, (\{0, 7, 5\} \times \{1, 0, 3\})\, x^5 + (\{0, 1, 5\} \times \{1, 2, 4, 5\})x^9$

$= \{3, 2, 4, 0\} + \{3, 2, 4, 0\}x^4 + \{0, 1, 5, 3\}\, x^5 + \{0, 1, 5, 2, 4\}$ (as $x^9 = 1$}

$= (\{3, 2, 4, 0\} + \{0, 1, 5, 2, 4\}) + \{3, 2, 4, 0\}\, x^4 + \{0, 1, 5, 3\}\, x^5$

$= \{3, 2, 4, 0, 1, 5\} + \{3, 2, 4, 0\}x^4 + \{0, 1, 5, 3\}\, x^5$.

This is the way product operation is performed on $S[x]_8$. The reader is left with the task of finding the order of $S[x]_8$.

We see if $p(x) = \{0, 3\}\, x^2 + \{3\}$ and $q(x) = \{0, 2\} + \{0, 2, 4\}x^6 \in S[x]_8$, then $p(x) \times q(x) = [\{3\} + \{0, 3\}x^2\,] \times [\{0, 2\} + \{0, 2, 4\}\, x^6\}$ $= \{0\} + \{0\}\, x^2 + \{0\}x^6 + \{0\}x^8 = (\{0\})$.

Thus $S[x]_8$ has zero divisors.

However $S[x]_8$ has no idempotents or nilpotents.

***Example 4.44:*** Let

$$S[x]_4 = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_7);\ x^5 = 1, \times \right\}$$

be the MOD subset coefficient polynomial semigroup under product operation.

Let $p(x) = \{0, 3, 2\} + \{6, 5, 1\}\, x^4$ and

$q(x) = \{1, 2\} + \{3, 4, 2\}\, x^3 \in S[x]_4$,

$p(x) \times q(x) = (\{0, 3, 2\} + \{6, 5, 1\}\, x^4) \times (\{1, 2\} + \{3, 4, 2\}\, x^3)$

$= \{0, 3, 2\} \times \{1, 2\} + \{0, 3, 2\} \times \{3, 4, 2\}x^3 + \{6, 5, 1\} \times \{1, 2\}\, x^4 + \{6, 5, 1\} \times \{3, 4, 2\}x^3 = \{0, 3, 2, 4, 6\} + \{0, 2, 6, 5, 1, 4\}x^3 + \{6, 5, 1, 2, 3\}\, x^4 + \{4, 1, 3, 6, 2, 5\}x^2 \in S[x]_4$.

This is the way the product operation is performed on $S[x]_4$. Clearly $S[x]_4$ has no zero divisors or nilpotents or idempotents.

But $S[x]_4$ is of finite order and the reader is expected with the task of finding the order of $S[x]_4$.

In view of all these we have the following result.

**THEOREM 4.18:** *Let* $S[x]_m = \left\{ \sum_{i=0}^{m} a_i x^i \;\middle|\; a_i \in S(Z_n);\ x^{m+1} = 1, \times \right\}$ *be the MOD subset coefficient polynomial semigroup under product.*

*i)* $o(S[x]_m) < \infty$.

*ii) $S[x]_m$ has zero divisors and nilpotents for appropriate n.*
*iii) $S[x]_m$ has no zero divisors and nilpotents when n is a prime.*

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto describe by example the MOD finite complex number subset coefficient polynomial semigroup of finite order by examples.

***Example 4.45:*** Let

$$M[x]_6 = \left\{ \sum_{i=0}^{6} a_i x^i \;\middle|\; a_i \in S\,(C(Z_{10}); x^7 = 1, \times \right\}$$

be the MOD finite complex number subset coefficient polynomial semigroup.

The reader is left with the task of finding the order of $M[x]_6$. $M[x]_6$ has MOD zero divisors but no nilpotents zero divisors but no nilpotents.

Infact $M[x]_6$ has ideals.

$$P[x]_6 = \left\{ \sum_{i=0}^{6} a_i x^i \;\middle|\; a_i \in S\,(\{0, 2, 4, 6, 8\}); x^2 = 1, \times \right\} \subseteq M[x]_6$$

is a MOD subset finite complex number coefficient polynomial subsemigroup which is not an ideal of $M[x]_6$ is a MOD subset finite complex number coefficient polynomial subsemigroup which is not an ideal of $M[x]_6$.

Let $p(x) = \{0, 5\} + \{0, 5, 5i_F\}\, x^3$ and

$q(x) = \{0, 2\} + \{0, 2i_F, 4\}\, x + \{0, 4 + 4i_F, 8, 8i_F\}\, x^4 \in M[x]_6$.

Clearly p(x) × q(x) = ({0}).

Consider

$$T[x]_6 = \left\{ \sum_{i=0}^{6} a_i x^i \;\middle|\; a_i \in S(\{0, 5, 5i_F, 5 + 5i_F\}); x^7 = 1, \times \right\} \subseteq M[x]_6$$

is again a MOD finite complex number subset subsemigroup which is also an ideal of $M[x]_6$.

However $M[x]_6$ has no nilpotents.

***Example 4.46:*** Let

$$B[x]_7 = \left\{ \sum_{i=0}^{7} a_i x^i \;\middle|\; a_i \in S(C(Z_7)); x^8 = 1, \times \right\}$$

be the MOD finite complex number subset coefficient polynomial semigroup under product of finite order, $B[x]_7$ has no idempotents, zero divisors or nilpotents.

Even finding MOD finite complex number coefficient subsemigroups and ideals happens to be a difficult task.

Inview of all these we propose the following problem.

**Problem 4.2:** Let

$$S[x]_m = \left\{ \sum_{i=0}^{m} a_i x^i \;\middle|\; a_i \in S(C(Z_p)); x^{m+1} = 1, \times \right\}$$

be the MOD finite complex number subset coefficient polynomial semigroup (p a prime).

i) Can $S[x]_m$ have MOD finite complex number subset coefficient polynomial ideals?

ii) Can $S[x]_m$ have MOD zero divisor subset coefficient polynomial pairs?
iii) Can $S[x]_m$ have MOD nilpotent subset coefficient polynomials?

Next we describe one or two examples of MOD neutrosophic subset coefficient polynomial semigroup of finite order.

***Example 4.47:*** Let

$$W[x]_{10} = \left\{ \sum_{i=0}^{10} a_i x^i \;\middle|\; a_i \in S\{(\langle Z_{12} \cup I\rangle),\ x^{11} = 1, \times\} \right.$$

be the MOD neutrosophic subset coefficient polynomial semigroup under ×.

$W[x]_{10}$ has MOD neutrosophic subset coefficient polynomial zero divisors and nilpotents.

Also $W[x]_{10}$ has ideals.

***Example 4.48:*** Let

$$V[x]_{12} = \{ \sum_{i=0}^{12} a_i x^i \mid x^{13} = 1, a_i \in S\{(\langle Z_{13} \cup I\rangle), \times\}$$

be the MOD neutrosophic subset coefficient polynomial semigroup under ×.

$V[x]_{12}$ has no MOD neutrosophic subset coefficient polynomial idempotent or nilpotent or zero divisors.

$V[x]_{12}$ has nontrivial MOD neutrosophic subset coefficient polynomials ideals.

The task of finding the order of $V[x]_{12}$ is left as an exercise to the reader.

Next we proceed onto describe MOD dual number subset coefficient polynomial semigroups under product × by some examples.

***Example 4.49:*** Let

$$P[x]_{15} = \left\{ \sum_{i=0}^{15} a_i x^i \;\middle|\; a_i \in S\{(\langle Z_{14} \cup g\rangle),\ x^{16} = 1, \times\} \right.$$

be the MOD dual number subset coefficient polynomial semigroup under ×.

Clearly $P[x]_{15}$ has MOD dual number subset coefficient polynomials which are nilpotents of order two. Also $P[x]_{15}$ has MOD dual number subset coefficient polynomials which are zero divisors.

Let $p(x) = \{3g, 10g, 6g, g, 9g\} + \{4g, 0, 8g\}x^3 + \{10g, 13g, 2, 0\}x^5 \in P[x]_{15}$.

Clearly $p(x) \times p(x) = (\{0\})$.

Let $q(x) = \{3g, 10g\} + \{5g, 12g, g\}x^3 + \{g, 2g, 5g, 7g, 9g, 11g\}x^5 + \{3g, 6g, 9g, 12g\}\ x^8 \in P[x]_{15}$.

We see $p(x) \times q(x) = (\{0\})$. This $P[x]_5$ has MOD nilpotent polynomials as well as MOD zero divisor polynomial pairs.

Further

$$W[x]_{15} = \left\{ \sum_{i=0}^{15} a_i x^i \;\middle|\; x^{16} = 1,\ a_i \in S(Z_{14}g),\ \times \right\} \subseteq P[x]_{15}$$

is a MOD subset dual number coefficient subset polynomial ideal of $P[x]_{15}$.

$$\text{Let } B[x]_{15} = \left\{\sum_{i=0}^{15} a_i x^i \;\middle|\; a_i \in S(Z_{14}) \times\right\} \subseteq P[x]_{15};$$

$B[x]_{15}$ is only a MOD dual number coefficient subset polynomial subsemigroup of $P[x]_{15}$ which is not an ideal of $P[x]_5$.

In view of all these we have the following theorem.

**THEOREM 4.19:** *Let* $S[x]_m = \left\{\sum_{i=0}^{m} a_i x^i \;\middle|\; a_i \in S\{(\langle Z_n \cup g\rangle),\; x^{11} = 1, \times\right\}$ *be the MOD dual number coefficient subset polynomial semigroup under ×.*

*i) $S[x]_m$ has several MOD dual number coefficient subset polynomial zero divisors and nilpotents even if n is a prime.*
*ii) $S[x]_m$ has MOD dual number subset coefficient polynomial subsemigroups which are not ideals even if n is prime.*
*iii) $S[x]_m$ has MOD dual number subset coefficient polynomial ideal even if n is a prime.*
*iv) $S[x]_m$ has no nontrivial MOD dual number subset coefficient polynomial idempotent.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD special dual like number subset coefficient polynomial semigroups of finite order by some examples.

***Example 4.50:*** Let

$$W[x]_8 = \left\{\sum_{i=0}^{8} a_i x^i \;\middle|\; a_i \in S\{\langle Z_{12} \cup h\rangle); \times \;\; x^9 = 1\right\}$$

be the MOD special dual like number subset coefficient polynomial semigroup under ×.

Clearly if p(x) = {0, 4, 8} + {0, 8h, 4h+4}$x^2$ + {4 + 8h, 8h + 8, 0}$x^4$ and

q(x) = {0, 3h} + {0, 3h + 3, 6h}$x^2$ + {0, 6 + 6h, 6h, 6 + 3h}$x^5$ ∈ $W[x]_8$ then clearly

p(x) × q(x) = ({0}); that is $W[x]_8$ has non trivial MOD special dual like number coefficient subset zero divisor pairs.

Let p(x) = {0, 6} + {0, 6h}x +{0, 6 + 6h}$x^2$ ∈ $W[x]_8$.

We see p(x) × p(x) = ({0}) is a MOD special dual like number subset coefficient polynomial nilpotent of order two.

However $W[x]_8$ has no nontrivial MOD subset special quasi dual number coefficient polynomial idempotents.

We see $V[x]_8 = \left\{\sum_{i=0}^{8} a_i x^i \;\middle|\; a_i \in S(Z_{12}\, h);\; x^9 = 1, \times\right\} \subseteq W[x]_8$

is a MOD special dual like number subset coefficient polynomial subsemigroup of $W[x]_8$ whicih is also an ideal of $w[x]_8$.

***Example 4.51:*** Let

$$S[x]_{10} = \left\{\sum_{i=0}^{10} a_i x^i \;\middle|\; a_i \in S\{(\langle Z_{13} \cup h\rangle),\; x^{11} = 1, \times\right\}$$

be the MOD special dual like number subset coefficient polynomial semigroup under ×.

Clearly $P[x]_{10} = \left\{\sum_{i=0}^{10} a_i x^i \;\middle|\; a_i \in S(Z_{13});\; x^{11} = 1, \times\right\} \subseteq S[x]_{10}$

is the MOD special dual like number subset coefficient polynomial subsemigroup of $S[x]_{10}$ which is not an ideal of $S[x]_{10}$.

Consider

$$R[x]_{10} = \left\{\sum_{i=0}^{10} a_i x^i \;\middle|\; a_i \in S(Z_{13}h),\ x^{11} = 1, \times\right\} \subseteq S[x]_{10}$$

is a MOD special dual like number subset coefficient polynomial subsemigroup of $S[x]_{10}$ which is also an ideal of $S[x]_{10}$.

However finding nontrivial MOD special dual like number subset coefficient polynomial zero divisor pairs or nilpotents or idempotents in $S[x]_{10}$ is a different task.

In view of all these we have the following theorem.

**THEOREM 4.20:** *Let* $Q[x]_m = \left\{\sum_{i=0}^{m} a_i x^i \;\middle|\; a_i \in S\{(\langle Z_n \cup h\rangle);\ x^{m+1} = 1;\ \times\right\}$ *be the MOD special dual like number subset coefficient polynomial semigroup under* $\times$.

*i)* $o(Q[x]) < \infty$.

*ii)* $Q[x]_m$ *has MOD special dual like number subset coefficient polynomial subsemigroups which are not ideals* $(2 \leq n < \infty)$

*iii)* $Q[x]_m$ *has MOD special dual like number subset coefficient polynomial subsemigroups which are ideals of* $Q[x]_m$.

*iv) If n is a composite number of the form* $n = p^t q$; $t \geq 2$, *p a prime and q a distinct prime from p or a composite number then* $Q[x]_m$ *has nontrivial MOD special dual like number subset coefficient polynomials which are nilpotents or zero divisor pairs.*

*v) What ever be n,* $Q[x]_m$ *has no nontrivial idempotent.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD special quasi dual number subset coefficient polynomials semigroups under × by some examples.

***Example 4.52:*** Let

$$P[x]_6 = \left\{ \sum_{i=0}^{6} a_i x^i \;\middle|\; a_i \in S\{(\langle Z_{20} \cup k\rangle),\ x^7 = 1, \times\} \right.$$

be the MOD special quasi dual number subset coefficient polynomial semigroup under ×.

Let p(x) = {5, 5 + 5k, 10} + {5 + 10k, 10k, 10 + 5l} $x^2$ + {0, 15k + 5, 15k} $x^4$ and

q(x) = {4, 8k} + {4, 8 + 8k, 12}x + {12 + 12k, 8 + 16k, 16 + 4k} $x^3$ ∈ $P[x]_6$.

Clearly p(x) × q(x) = ({0}).

Thus there are MOD special quasi dual number subset coefficient polynomial zero divisor pairs.

We can have only nilpotents of the form
{0, 10} + {10, 10k, 0}$x^2$ + {0, 10 + 10k}$x^3$ + {0, 10k}$x^4$ = p(x) such that p(x) × p(x) = ({0}).

However we do not have nontrivial idempotents.

Only a very few nilpotents arising from subset coefficient {0, 10, 10k, 10 + 10}.

But

$$D[x]_6 = \left\{\sum_{i=0}^{6} a_i x^i \;\middle|\; a_i \in S(Z_{20}),\ x^7 = 1, \times\right\} \subseteq P[x]_6$$

is a MOD special quasi dual number subset coefficient polynomial subsemigroup which is not an ideal of $P[x]_6$.

Consider

$$E[x]_6 = \left\{\sum_{i=0}^{6} a_i x^i \;\middle|\; a_i \in S(Z_{20}k);\ x^7 = 1, \times\right\} \subseteq P[x]_6;$$

clearly $E[x]_6$ is a MOD special quasi dual number subset coefficient polynomial subsemigroup which is also an ideal of $P[x]_6$.

***Example 4.53:*** Let

$$G[x]_9 = \left\{\sum_{i=0}^{9} a_i x^i \;\middle|\; a_i \in S\{(\langle Z_5 \cup k\rangle),\ x^{10} = 1, \times\right\}$$

be the MOD special quasi dual number subset coefficient polynomial semigroup under product ×.

$G[x]_9$ has no nontrivial nilpotents, zero divisors or idempotents. However $G[x]_9$ has both MOD special quasi dual number subset coefficient polynomial ideals and subsemigroups which are not ideals.

In view of all these we hav the following theorem.

**THEOREM 4.21:** *Let* $H[x]_m = \left\{\sum_{i=0}^{m} a_i x^i \;\middle|\; a_i \in S\{(\langle Z_n \cup k\rangle),\ x^{m+1} = 1, \times\right\}$ *be the MOD special quasi dual number subset coefficient polynomial semigroup under ×.*

i) *$H[x]_m$ has both MOD special quasi dual number subset coefficient polynomial subsemigroup which is not an ideal and a MOD special quasi dual number subset polynomial subsemigroup which is an ideal for all $2 \leq n < \infty$.*
ii) *$H[x]_m$ has no nontrivial MOD special quasi dual number subset coefficient polynomial idempotents for $2 \leq n < \infty$.*
iii) *$H[x]_m$ has nontrivial MOD special quasi dual number subset coefficient polynomial zero divisors and nilpotents in n if n is of the type, $n = p^t q$ ($t \geq 2$, p a prime and $p \times q$ and a composite or some other prime).*

Prime is direct and hence left as an exercise to the reader.

In almost all cases these MOD subset coefficient polynomial semigroups are Smarandache semigroups.

In view of this we have the following result.

**THEOREM 4.22:** *Let S[x] (or $S[x]_m$) be the MOD subset (finite complex or dual number or special quasi dual number of special dual like number or special neutrosophic) coefficient polynomial semigroup under ×, infinite semigroup (or finite semigroup) respectively. S[x] (or $S[x]_m$ ) is a Smarandache MOD semigroup if and only if $Z_n$ in $S(Z_n)$ or $Z_n$ in $S(C(Z_n))$ or $S(\langle Z_n \cup g\rangle)$ or $S(\langle Z_n \cup k\rangle)$ or $S(\langle Z_n \cup h\rangle)$ or $S(\langle Z_n \cup I\rangle)$ is a Smarandache semigroup.*

**Proof :** Let $S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S(Z_n)\right.$, (or $S(C(Z_n))$ or $S(\langle Z_n \cup g\rangle)$ or $S(\langle Z_n \cup h)$ or $S(\langle Z_n \cup k\rangle)$, $S(\langle Z_n \cup I\rangle))$ ×} ($S[x]_m$ = $\{\sum_{i=0}^{m} a_i x^i$ / $a_i \in S(Z_n)$ (or $S(C(Z_n)$ or $S(\langle Z_n \cup I\rangle)$ or $S(\langle Z_n \cup g\rangle)$ or $S(\langle Z_n \cup h\rangle)$ or $S(\langle Z_n \cup k\rangle))$, $x^{m+1} = 1$, ×} be the MOD subset coefficient polynomial semigroup under product.

If $S(Z_n)$ is a Smarandache semigroup then n is a prime or $Z_n$ has a subsemigroup $H \subseteq Z_n$ such that H is a group under product.

Hence B = {{1}, {2}, {3}, {4}, {5}, {6}, …, {p}} (or if h = $\{h_1, \ldots, h_t\}$ is a group under product $H \subseteq Z_n$ then M = $\{\{h_1\}, \{h_2\}, \ldots, \{h_t\}\}$) under product is a subset group under × if and only if $S(Z_n)$ is a S-semigroup under product. Thus the result.

We will illustrate this situation by an example or two.

***Example 4.54:*** Let

$$S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_{13});\; \times\right\}$$

be the MOD subset coefficient polynomial semigroup under product ×.

B = {{1}, {2}, {3}, {4}, {5}, {6}, …, {12}} ⊆ S[x] is a MOD subset subsemigroup under × which is in fact a group. Hence S[x] is a Smarandache MOD - subset coefficient polynomial semigroup under ×.

***Example 4.55:*** Let

$$S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(C(Z_{12}));\; \times\right\}$$

be the MOD subset finite complex coefficient polynomial semigroup $P_1$ = {{1}, {5}}, $P_2$ = {{1}, {11}} and $P_3$ = {{1}, {7}} are all subgroups under × and they are cyclic groups of order two. Hence S[x] is a MOD subset finite complex number polynomial coefficient Smarandache semigroup.

***Example 4.56:*** Let

$$B[x]_9 = \left\{ \sum_{i=0}^{9} a_i x^i \;\middle|\; x^{10} = 1,\ a_i \in S(\langle Z_{17} \cup h \rangle), \times \right\}$$

be the MOD subset special dual like number coefficient polynomial semigroup under product.

$B[x]_9$ is a Smarandache MOD subset special dual like number coefficient polynomial semigroup as T = {{1}, {2}, {3}, {4}, …, {16}} $\subseteq$ $B[x]_9$ is a group under $\times$.

Interested reader can get more properties about these Smarandache notions.

Infact we can have the following example.

***Example 4.57:*** Let

$$S[x]_{10} = \left\{ \sum_{i=0}^{10} a_i x^i \;\middle|\; a_i \in S(\langle Z_{24} \cup g \rangle);\ x^{11} = 1, \times \right\}$$

be the MOD dual number coefficient subset polynomial semigroup under product.

$S[x]_{10}$ is a S-semigroup as {{1}, {7}} = $B_1$ and $B_2$ = {{1}, {23}} are subset groups under product.

Table for $B_1$

| x | {1} | {7} |
|---|---|---|
| {1} | {1} | {7} |
| {7{ | {7} | {1} |

Clearly $B_1$ is a cyclic subset group of order two.

Table for $B_2$

| x | {1} | {23} |
|---|---|---|
| {1} | {1} | {23} |
| {23{ | {23} | {1} |

Is a MOD subset subgroup of $S[x]_{10}$.

Hence our claim $S[x]_{10}$ is a Smarandache MOD subset dual number coefficient polynomial semigroup of finite order.

Infact we can say under × all MOD subset semigroups are Smarandache MOD subset semigroups what ever be n of $Z_n$.

Next we want to briefly describe MOD subset matrices that is collection of subset matrices where matrices take entries from $Z_n$ or $C(Z_n)$ or $\langle Z_n \cup g\rangle$ or $\langle Z_n \cup I\rangle$ or $\langle Z_n \cup k\rangle$ or $\langle Z_n \cup h\rangle$.

We describe them by examples. Secondly we also build MOD subset polynomials with coefficients from $Z_n$ or $C(Z_n)$ or $\langle Z_n \cup I\rangle$ and so on.

We will describe both the situations by some examples.

We give a definition of both these notions.

**DEFINITION 4.1:** *Let M = {collection of all m × t matrices with entries from $Z_n$ or $C(Z_n)$ or $\langle Z_n \cup g\rangle$ or $\langle Z_n \cup h\rangle$ or $\langle Z_n \cup I\rangle$ or $\langle Z_n \cup k\rangle$ under $\times_n$ natural product operation}.*

*S(M) = {collection of all subset matrices with elements from M}. We can define $\times_n$ on S(M) and {S(M), $\times_n$} is defined as the MOD subset matrix semigroup.*

**DEFINITION 4.2:** *Let* $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_n \text{ or } \langle Z_n \cup g\rangle \text{ or } \right.$ *$\langle Z_n \cup k\rangle$ or $C(Z_n)$; ×} be the MOD polynomials under ×.*

*Let S(S[x]) = {collection of all subsets from S[x]}.*

*{S(S[x]), ×} is defined as the MOD subset polynomials semigroup under × of infinite order.*

If S[x] is replaced by $S[x]_m$ then $\{S(S[x]_m), \times\}$ is defined as the MOD subset polynomial semigroup under ×. Clearly $S(S[x]_m)$ is of finite order and is infact a commutative monoid.

Next we proceed onto describe the MOD subset matrix sets by some examples.

***Example 4.58:*** Let M = {collection of all 1 × 5 row matrices with entries from $Z_9$} = $\{\{a_1, a_2, a_3\}, \{b_1, b_2, b_3, b_4, b_5\}, \ldots, \{t_1, t_2, \ldots, t_p\}\}$ where $a_i$, $b_i$ and $t_j$ are 1 × 5 matrices with entries from $Z_9$.

Let P = {(0, 1, 0 ,0, 2), (2, 3, 4, 0, 0), (0, 8, 0, 0, 0)} and

Q = {(2, 1, 0, 1, 2), (1, 1, 1, 1 ,1), (0, 0, 0, 1, 1), (2, 5, 0,0,6)} ∈ M.

P × Q = {(0, 1, 0, 0, 2), (2, 3, 4, 0, 0), (0, 8, 0, 0, 0)} × {(2,1, 0, 1, 2), (1, 1, 1, 1, 1), {(0, 0, 0, 1, 1), (2, 5, 0, 0, 6)}

= {(0 1 0 0 4), (4 3 0 0 0), (0 8 0 0 0), (0 1 0 0 0), (2 3 4 0 0), (0 8 0 0 0), (0 0 0 0 2), (0 0 0 0 0), (0 5 0 0 3), (4 6 0 0 0) ,(0 4 0 0 0)}.

This is the way product operation is performed on M.

Clearly {(0 0 0 0 0)} ∈ M is such that

A × {(0 0 0 0 0)} = {(0 0 0 0 0)} × A = {(0 0 0 0 0)} for all A ∈ M.

B = {(1 1 1 1 1)} ∈ M is such that A × B = B × A = A for all A ∈ M.

***Example 4.59:*** Let S(P) = {collection of all subset matrices from the set of matrices

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_i \in Z_{12};\ 1 \leq i \leq 3, \times_n \}, \times_n \right\}$$

be the MOD subset matrix semigroup under natural product $\times_n$.

$$A = \{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} \} \text{ and } B = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix} \} \in S(P)$$

$$A \times_n B = \{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} \} \times_n \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix} \}$$

$$= \{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \times_n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} \times_n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \times_n \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \times_n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

$$\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \times_n \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \times_n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} \times_n \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix} \}$$

$$= \{ \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \} \in S(P).$$

This is the way product operation is performed on S(P). Clearly S(P) has MOD subset matrix zero divisors, idempotents and nilpotents given by

$$A = \{\begin{bmatrix}0\\9\\4\end{bmatrix}, \begin{bmatrix}0\\4\\9\end{bmatrix} \begin{bmatrix}0\\0\\0\end{bmatrix}\} \in S(P)$$

is such that $A \times_n A = \{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\9\\4\end{bmatrix}, \begin{bmatrix}0\\4\\9\end{bmatrix}\}$.

Thus A is a MOD subset idempotent matrix of S(P).

$$\text{Let } B = \{\begin{bmatrix}0\\4\\8\end{bmatrix}, \begin{bmatrix}4\\8\\0\end{bmatrix} \begin{bmatrix}8\\8\\8\end{bmatrix}\} \text{ and}$$

$$A = \{\begin{bmatrix}6\\3\\6\end{bmatrix}, \begin{bmatrix}3\\0\\6\end{bmatrix} \begin{bmatrix}0\\6\\6\end{bmatrix}\} \in S(P).$$

$$\text{Clearly } A \times_n B = \{\begin{bmatrix}0\\0\\0\end{bmatrix}\}.$$

Thus A, B is a MOD zero divisor subset pair of S(P).

$$\text{Let } B = \{\begin{bmatrix}6\\6\\6\end{bmatrix}, \begin{bmatrix}0\\6\\0\end{bmatrix} \begin{bmatrix}0\\0\\6\end{bmatrix} \begin{bmatrix}0\\0\\0\end{bmatrix}\} \in S(P).$$

$$\text{We see } B \times_n B = \{\begin{bmatrix}0\\0\\0\end{bmatrix}\}.$$

Thus B is a MOD subset nilpotent matrix of order two in S(P).

Let S(M) = {collection of all subsets of matrices from

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_i \in \{0, 2, 4, 6, 8, 10\}, 1 \leq i \leq 3\}, \times_n \right\}$$

be the MOD subset matrix subsemigroup of S(P) which is also a MOD subset matrix ideal of S(P).

Let S(W) = {collection of all subsets from

$$W = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \end{bmatrix} \middle| \; a_1, a_2 \in Z_{12}, \times_n\}, \times_n \right\} \subseteq S(P)$$

be a MOD subset matrix ideal of S(P).

Thus S(P) in this situation has MOD subset subsemigroups, ideals, idempotents zero divisors and nilpotents.

***Example 4.60:*** Let S(W) = {collection of all subsets from

$$W = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| \; a_i \in C(Z_5); \times_n; 1 \leq i \leq 6\}, \times_n \right\}$$

be the MOD subset finite complex matrix semigroup under the natural product.

S(W) has MOD finite complex number subset subsemigroups as well as ideals.

However finding nontrivial zero divisors is a challenging one.

For we say if A = { $\begin{pmatrix} 0 & 0 & 0 \\ a_2 & a_3 & 0 \end{pmatrix}$ ) and

$$B = \{\begin{pmatrix} b_1 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}\} \in S(W) \text{ then}$$

$$A \times_n B = \{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\};$$

such type of MOD matrix zero divisors will only be termed as trivial MOD subset matrix zero divisors.

S(W) has no nontrivial MOD subset finite complex number idempotents or nilpotents or zero divisors.

S(M) = {collection of all subset matrices from

$$M = \left\{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| \; a_i \in Z_5, 1 \leq i \leq 6, \times_n\}, \times_n\} \subseteq S(W)$$

is only a MOD subset finite complex number matrix subsemigroup which is not an ideal of S(W).

Consider S(N) = {collection of all subset finite complex number matrices from

$$N = \{\begin{pmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix} / a_1, a_2 \in C(Z_5), \times_n\}, \times_n\}\} \subseteq S(W)$$

is a MOD finite complex number subset matrix subsemigroup which is also an ideal of S(W).

***Example 4.61:*** Let S(Y) = {collection of all MOD subsets dual number matrices from

$$Y = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} / a_i \in \langle Z_6 \cup g\rangle; 1 \leq i \leq 6, \times_n\}, \times_n\}$$

be the MOD subset dual number matrix semigroup under the natural product $\times_n$.

$$\text{Let } A = \{\begin{bmatrix} 0 & 3g \\ g & 4g \\ 5g & 0 \\ 2g & g \end{bmatrix}, \begin{bmatrix} 0 & g \\ 0 & 2g \\ 0 & 3g \\ 0 & 4g \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ g & 2g \\ 3g & 4g \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ g & 2g \\ g & 3g \\ 4g & 0 \end{bmatrix}\} \text{ and}$$

$$B = \{\begin{bmatrix} g & 5g \\ 2g & 2g \\ 4g & 3g \\ 5g & g \end{bmatrix}, \begin{bmatrix} 0 & g \\ 0 & g \\ 0 & g \\ 0 & 3g \end{bmatrix}, \begin{bmatrix} 0 & g \\ g & 0 \\ g & 2g \\ 0 & 5g \end{bmatrix}\} \in S(Y);$$

$$\text{clearly } A \times_n B = \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\} \text{ and}$$

$$A \times_n A = \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\} \text{ and } B \times_n B = \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\}.$$

Thus S(Y) has MOD subset dual number matrix zero divisors and nilpotents.

$$M = \{\begin{bmatrix} 0 & 0 \\ 0 & 4 \\ 3 & 0 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 4 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\} \in S(Y)$$

is such that $M \times_n M = M$ so M is a MOD subset dual number matrix idempotent of S(Y).

This S(Y) has MOD subset dual number matrix ideals as well as subsemigroups which are not ideals.

This task is left as an exercise to the reader as it is considered as a matter of routine.

Here we proceed onto give examples of MOD subset matrix sets and their properties under natural product $\times_n$.

***Example 4.62:*** Let S(P) = {collection of all subsets from the set of MOD special quasi dual number 6 × 1 matrices

$$\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in \langle Z_{12} \cup k \rangle \;\; 1 \leq i \leq 6, \times_n\}\right\}$$

be the MOD subset matrices set.

$$\text{Let } A = \{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1+k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 9+k \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 \\ 0 \\ 0 \\ 1 \\ 0 \\ 6+k \end{bmatrix}\},$$

$$B = \{\begin{bmatrix} 9k+3 \\ 0 \\ 6+k \\ 0 \\ 0 \\ 6k+2 \end{bmatrix}\},$$

$$\text{and } C = \{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1+3k \\ 1 \\ 5+k \end{bmatrix}, \begin{bmatrix} 10k \\ 10 \\ 10 \\ 6 \\ 6+5k \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 3+2k \\ 4 \\ 3 \\ 2+k \\ 0 \end{bmatrix}\};$$

clearly it is left as an exercise to verify that

$(A \times_n B) \times_n C = A \times_n (B \times_n C)$.

$$\text{Clearly } \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} = \{(0)\}$$

acts as $\{(0)\} \times_n A = A \times_n \{(0)\} = \{(0)\}$ for all $A \in S(P)$.

Further

$$\{(1)\} = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\} \in S(P) \text{ is such that}$$

$$A \times \{(1)\} = \{(1)\} \times A = A$$

for all $A \in S(P)$.

It is easily verified S(P) has MOD subset special quasi dual number matrix zero divisors, nilpotents and idempotents.

Further S(P) has MOD subset special quasi dual number matrix subsemigroups which are not ideals as well as subsemigroups which are ideals.

***Example 4.63:*** Let

$$B = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \ a_i \in \langle Z_{11} \cup I\rangle;\ 1 \leq i \leq 4, \times \text{ (or } \times_n)\right\};$$

S(B) = {collection of all subsets from B, × or ($\times_n$)} be the MOD subset neutrosophic matrix semigroup under × (or $\times_n$) {S(B), ×} be a non commutative MOD subset neutrosophic matrix semigroup; whereas {S(B), $\times_n$} is a MOD subset neutrosophic commutative matrix semigroup.

S(T) = {collection of all subset matrices from

$$T = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in Z_{11}I,\ 1 \le i \le 4\right\} \times_n (\text{or } \times)\} \subseteq S(B)$$

is a MOD subset neutrosophic matrix ideal of S(B).

S(V) = {collection of all subset matrices from

$$V = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in Z_{11};\ 1 \le i \le 4\right\}, \times (\text{or } \times_n)\} \subseteq S(B)$$

is a MOD subset matrix neutrosophic subsemigroup of S(B) which is not an ideal of S(B).

Now having seen MOD subset matrix semigroups under product we give a few related results.

**THEOREM 4.23:** *Let S(M) = {collection of all subsets from M = {m × t matrices with entries from $Z_n$ (or $C(Z_n)$ or $\langle Z_n \cup I\rangle$ or $\langle Z_n \cup h\rangle$ or $\langle Z_n \cup g\rangle$ or $\langle Z_n \cup k\rangle$; $\times_n$ ($m \neq t$)}, $\times_n$} be the MOD subset matrix semigroup.*

*i) $o(S(M)) < \infty$.*
*ii) S(M) is a MOD subset matrix commutative monoid.*
*iii) S(M) has MOD subset subsemigroups which are not ideals.*
*iv) S(M) has MOD subset subsemigroups which are ideals.*
*v) S(M) has MOD zero divisors, nilpotents and idempotents for appropriate n.*

Proof is direct and left as an exercise to the reader.

All properties (i) to (v) hold good if $Z_n$ is replaced by $C(Z_n)$ or $\langle Z_n \cup I\rangle$ or $\langle Z_n \cup h\rangle$ or $\langle Z_n \cup k\rangle$ or $\langle Z_n \cup g\rangle$.

**Corollary 4.2:** If in the above theorem m = t then all results of theorem hold good under the usual product $\times_n$.

**Corollary 4.3:** If in the above theorem m = t and the natural product $\times_n$ is replaced by $\times$ the usual product the MOD subset semigroup has right ideals which are not left ideals, subsemigroups which are not ideals, right zero divisors which are not left zero divisors.

The proof of the above two corollaries is left as an exception to the reader.

Next we proceed onto describe by examples MOD subset polynomial semigroups.

***Example 4.64:*** Let S(M[x]) = {collection of all subset from the MOD semigroups of polynomials

$$M[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{10}, \times\right\}, \times\}$$

be the MOD subset polynomial semigroup.

Clearly $o(S(M[x])) = \infty$.

Let $A = \{x^3 + 1, 3x + 4, 5\}$ and $B = \{2x + 5, 2, 4x^4 + 8x^2 + 2\} \in S(M[x])$.

$A \times B = \{x^3 + 1, 3x + 4, 5\} \times \{2x + 5, 2, 4x^4 + 8x^2 + 2\} = \{5, 0, 2x + 5 + 2x^4 + 5x^3, 2x^3 + 2, 4x^4 + 8x^2 + 2 + 4x^7 + 8x^5 + 2x^3, 6x + 8, 8 + 2x^2 + 6x^4 + 6x + 2x^5 + 6x^5\} \in S(M[x])$.

This is the way product operation is performed on S(M[x]). Let $\{(0)\} = \{(0 + 0x + 0x^2 + \ldots + 0x^n)\}$ be the MOD subset polynomial.

Clearly {(0)} × p(x) = {(0)} for all {p(x)} ∈ S(M[x]); P[x] = {collection of some polynomials from M[x]}.

Likewise if {(1)} = {(1 + 0x + … + $0x^n$)} ∈ S(M[x]); then P[x] × {(1)} = P[x].

Let T[x] = {5x + 5, $5x^3$, 0, $5x^9 + 5x^{12}$, 5} and S[x] = {2x + 4, $4x^2 + 8x + 2$, $8x^6 + 4x + 2$} ∈ S(M[x]); T[x] × S[x] = {(0)}.

However S(M[x]) has no MOD subset polynomial idempotents whatever be n.

Consider (P[x]) = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 2, 4, 6, 8\}, \times\right\}$, let S(P[x]) = {collection of all MOD subset polynomials from P[x], ×} ⊆ S(M[x]) is a MOD subset polynomial subsemigroup which is also a MOD subset polynomial ideal of S(M[x]).

The task of finding MOD subset polynomial structures is left as an exercise to the reader.

***Example 4.65:*** Let S(W[x]) = {collection of all MOD subset polynomials from the set

$$W[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in (\langle Z_{17} \cup g\rangle), \times\right\}$$

be the MOD dual number subset polynomial semigroup.

Let p(x) = {$3gx^4 + 4gx^2 + 5g$, $15g + 8g\,x^8$, 10g + 2gx} and q(x) = {$14g\,x^4 + 10gx^2 + 11g + 7g$, $2gx^8 + 15gx^4$ +,10g, 0, $12g\,x^{18}$, $10gx^{10} + 12g$} ∈ S(W[x]).

Clearly p(x) × q(x) = {(0)}, p(x) × q(x) = {(0)} and q(x) × q(x) = {(0)}, thus S(W[x]) has MOD subset dual number polynomials which are MOD subset dual number zero divisor polynomial pairs and MOD subset dual number nilpotent polynomials of order two.

Infact S(W[x]) has MOD subset dual number polynomial subsemigroups which are not ideals as well as subsemigroups which are ideals.

This task is left as an exercise to the reader.

***Example 4.66:*** Let

$$V[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in C(Z_{12}), \times \right\}$$

be the MOD finite complex number coefficient polynomial semigroup under ×.

S(V[x]) = {collection of all subset polynomials from V[x], ×} be the MOD complex coefficient polynomial subsets semigroup.

It is left as an exercise for the reader to verify that S(V[x]) has MOD subset zero divisors and MOD subset nilpotents.

Infact S(V[x]) has MOD subset subsemigroups as well as ideals. Both ideals and subsemigroups of S(V[x]) are of infinite order.

***Example 4.67:*** Let S(M[x]) = {collection of all subsets from the MOD neutrosophic coefficient polynomials from

$$M[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \langle Z_{19} \cup I \rangle, \times \right\}, \times\}$$

be the MOD subset neutrosophic coefficient polynomials semigroup under product operation ×.

Reader is left with the task of finding MOD subset neutrosophic coefficient ideals and subsemigroups.

Finding zero divisors, and nilpotents happens to be a difficult job has no nontrivial idempotent.

The reader is left with the task of studying MOD semigroups by replacing $\langle Z_{19} \cup I\rangle$ in example 4.68 by $\langle Z_n \cup h\rangle$ and later by $\langle Z_n \cup k\rangle$ and obtain similar results.

As this study is a matter of routine the reader is left at this task. However we give the following theorem.

**THEOREM 4.24:** *Let S(P[x]) = {collection of all subsets from the MOD polynomials*

$$P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_n \;\; \times\right\} \text{ (or}$$

$$P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in C(Z_n), \times\right\} \text{ or}$$

$$P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \langle Z_n \cup g\rangle, \times\right\} \text{ or}$$

$$P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \langle Z_n \cup I\rangle, \times\right\} \text{ or}$$

$$P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \langle Z_n \cup h\rangle, \times\right\} \text{ or}$$

$$P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \langle Z_n \cup k\rangle, \times\right\}\text{) and so on.}$$

*i) S(P[x]) has zero divisors and nilpotents for appropriate n.*
*ii) S([x]) has MOD subset polynomial subsemigroups and ideals for all n in case of $\langle Z_n \cup h\rangle$ and $\langle Z_n \cup k\rangle$.*

*iii) S(P[x]) has MOD subset polynomial subsemigroups and ideals for appropriate n when $C(Z_n)$ or $Z_n$ is used.*

Proof is direct and hence left as an exercise to the reader.

We now briefly illustrate $S(P[x]_m)$ by some examples.

***Example 4.68:*** Let $S(P[x]_9)$ = {collection of all subsets from

$$P[x]_9 = \left\{ \sum_{i=0}^{9} a_i x^i \;\middle|\; a_i \in \langle Z_{12} \cup I \rangle, x^{10} = 1, \times \right\}, \times\}$$

be the MOD neutrosophic polynomial subset semigroup under product ×.

$S(P[x]_9)$ is of finite order.

Let $A = \{10Ix^3 + 3x + 4I, (2I+4)\, x^2 + 2I\}$ and

$B = \{5Ix^3 + 7\} \in S[p[x]_9)$.

We can calculate A × B as a matter of routine.

Infact $S(P[x]_9)$ has MOD zero divisors and nilpotents of course only trivial idempotents.

***Example 4.69:*** Let $S(M[x]_3)$ = {collection of all subsets from

$$M[x]_3 = \left\{ \sum_{i=0}^{3} a_i x^i \;\middle|\; x^4 = 1, a_i \in C(Z_7) \times \right\}, \times\}$$

be the MOD finite complex number coefficient subset polynomial semigroup.

The reader is expected to find substructures and special elements like nilpotents and idempotents if it exists.

In view of all these we can have the following result.

**THEOREM 4.25:** *Let $S(M[x]_m)$ = {collection of all subsets from MOD polynomial semigroup*

$$M[x]_m = \left\{\sum_{i=0}^{m} a_i x^i \,\middle|\, a_i \in Z_n,\ x^{m+1} = 1, \times\right\},$$

$$(\text{or } M[x]_m = \left\{\sum_{i=0}^{m} a_i x^i \,\middle|\, a_i \in C(Z_n),\ x^{m+1} = 1, \times\right\} \text{ or}$$

$$M[x]_m = \left\{\sum_{i=0}^{m} a_i x^i \,\middle|\, a_i \in \langle Z_n \cup I\rangle,\ x^{m+1} = 1, \times\right\}, \text{ or}$$

$$M[x]_m = \left\{\sum_{i=0}^{m} a_i x^i \,\middle|\, a_i \in \langle Z_n \cup g\rangle,\ x^{m+1} = 1, \times\right\} \text{ or}$$

$$M[x]_m = \left\{\sum_{i=0}^{m} a_i x^i \,\middle|\, a_i \in \langle Z_n \cup h\rangle,\ x^{m+1} = 1, \times\right\}, \text{ or}$$

$$M[x]_m = \left\{\sum_{i=0}^{m} a_i x^i \,\middle|\, a_i \in \langle Z_n \cup k\rangle,\ x^{m+1} = 1,x\ \}\ \times\right\}$$

*be the MOD subset polynomial semigroup (or MOD) subset finite complex number polynomial semigroup or MOD subset neutrosophic polynomial semigroup or MOD subset dual number polynomial semigroup or MOD subset special dual like number subset polynomial semigroup and MOD subset special quasi dual number polynomial semigroup respectively.*

*i) $S(M[x]_m)$ has MOD subset polynomial subsemirings and ideals for $C(Z_n)$, $\langle Z_n \cup I\rangle$, $\langle Z_n \cup g\rangle$, $\langle Z_n \cup h\rangle$ and $\langle Z_n \cup k\rangle$.*
*ii) $o(S(M[x])) < \infty$ for all sets, $Z_n$, $C(Z_n)$ and so on.*
*iii) Clearly $S(M[x]_m)$ has zero divisors and nilpotents.*

*iv) The MOD dual number polynomial subset $S(M[x]_m)$ using $\langle Z_n \cup g\rangle$ behaves distinctly different from other MOD semigroups.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD matrix subset collection under + operation by examples.

***Example 4.70:*** Let P(M) = {collection of all subsets from the MOD matrix semigroup

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle|\ a_i \in Z_8;\ 1 \leq i \leq 4, +\}, +\right\}$$

be the MOD matrix subset semigroup under addition.

$$\text{Let } A = \{\begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}\} \text{ and}$$

$$B = \{\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\} \in P(M).$$

$$A + B = \{\begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}\} + \{\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\}$$

$$= \{ \begin{bmatrix}0\\0\\2\\3\end{bmatrix} + \begin{bmatrix}3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\\3\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\end{bmatrix} + \begin{bmatrix}3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\0\end{bmatrix} +$$

$$\begin{bmatrix}3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\\0\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\} = \{ \begin{bmatrix}5\\1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\3\\3\end{bmatrix}, \begin{bmatrix}1\\1\\3\\4\end{bmatrix}, \begin{bmatrix}3\\2\\1\\1\end{bmatrix}, \begin{bmatrix}4\\0\\3\\0\end{bmatrix}, \begin{bmatrix}2\\1\\3\\1\end{bmatrix}\}$$

$$\in S(M).$$

This is the way + operation is performed on S(M).

S(M) is a MOD subset matrix commutative monoid of finite order as

$$\{(0)\} = \{ \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \} \in S(M)$$

is such that A + {(0)} = A for all A ∈ S(M).

Clearly S(M) has MOD subsemigroup.

***Example 4.71:*** Let

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| \; a_i \in \langle Z_{13} \cup k \rangle, 1 \leq i \leq 6, + \right\}$$

be the MOD special quasi dual number matrix semigroup.

S(M) = {collection of all subsets matrices from M under +} be the MOD special quasi dual number subset matrix semigroup under +.

Let $A = \left\{\begin{pmatrix} 0 & 0 & 2 \\ 0 & 5 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 5 \end{pmatrix}\right\}$ and

$$B = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 2 \\ 5 & 7 & 8 \end{pmatrix}\right\} \in S(M).$$

$$A + B + \left\{\begin{pmatrix} 0 & 0 & 2 \\ 0 & 5 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & k \\ 1 & 1 & 1+2k \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2k \\ 0 & 0 & 5 \end{pmatrix}\right\} +$$

$$\left\{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 2 \\ k+5 & 7 & 8 \end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix} 1 & 2 & 5 \\ 0 & 5 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 3k \\ 1 & 1 & 1+2k \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3+2k \\ 0 & 0 & 5 \end{pmatrix},\right.$$

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2k \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & k \\ 1 & 1 & 1+2k \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 4 \\ k+5 & 12 & 8 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 2+k \\ k+6 & 8 & 9+2k \end{pmatrix},$$

$$\left.\begin{pmatrix} 3 & 1 & 2+2k \\ k+5 & 7 & 8 \end{pmatrix}\right\} \in S(M).$$

This is the way + operation is performed on S(M).

The reader is left with the task of finding MOD special dual like number subset matrix subsemigroup.

Next we describe + operation on S(P[x]), the MOD polynomial subsets by some examples.

***Example 4.72:*** Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \langle Z_9 \cup g \rangle; + \right\}$$

be the MOD dual number coefficient polynomial semigroup under +.

Let S(P[x]) = {collection of all subsets from P[x], +} be the MOD dual number coefficient polynomial subset semigroup under +.

Let A = $\{(3 + g)\, x^8 + (4 + 8g),\ 5gx^3 + 8\}$ and

B = $\{(4 + 3g)\, x^2 + 7x + (8 + 2g)\} \in S(P[x])$.

A + B = $\{(3 + g)\, x^8 + (4 + 8g),\ 5gx^3 + 8\} + \{(4 + 3g)\, x^2 + 7x + 8 + 2g\} = (3 + g)\, x^8 + (4 + 3g)x^2 + 7x + (3 + g),\ 5gx^3 + (4 + 3g)x^2 + 7x + 7 + 2g\} \in S(P[x])$.

This is the way + operation is performed on S(P[x]). The reader is left with the task of proving (S(P[x]), +) is a MOD dual number coefficient polynomial monoid under + of infinite order.

Clearly $\{(0)\} = \{0 + 0.x + 0.x^2 + \ldots + 0.x^n\} \in S(P[x])$ is such that for every A ∈ S(P[x]); A + {(0)} = A.

Hence S(P[x]) is an infinite order monoid.

The reader is left with the task of finding subsemigroups and submonoids of S(P[x]).

We give yet another example.

***Example 4.73:*** Let S(P[x]) = {collection of all MOD finite complex number coefficient polynomial subsets with entries from

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in C(Z_{15}); +\right\}, +\}$$

be the MOD finite complex number coefficient polynomials subsets semigroup under + operation. P[x] has subsemigroups of both finite and infinite order.

The reader is left with the task of finding such subsemigroups.

***Example 4.74:*** Let S(M[x]) = {collection of all subsets from the MOD neutrosophic coefficient polynomial semigroup

$$M[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \langle Z_{11} \cup I\rangle, \times\right\}, \times\}$$

be the MOD neutrosophic coefficient polynomial subset semigroup under +.

o(S(M[x]) = ∞ and S(M[x]) is and S(M[x]) is an infinite monoid.

Let S(P[x]) = {collection of all MOD neutrosophic polynomial coefficient from the semigroup;

$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{11}I, +\right\}, \times\} \subseteq S(M[x])$ be the MOD neutrosophic polynomial coefficient subset semigroup.

Let $M = \left\{ \sum_{i=0}^{10} a_i x^i \;\middle|\; a_i \in < Z_{11} \cup I>, +\right\}$ and

S(M) = {collection of all subsets from M, +} is a MOD neutrosophic polynomial coefficient subset subsemigroup of finite order.

Thus S(M[x]) can have several MOD neutrosophic coefficient polynomial subset subsemigroup of finite order under +.

Let S(W) = {collection of all MOD neutrosophic polynomial coefficient subsets from

$$W = \left\{ \sum_{i=0}^{7} a_i x^i \;\middle|\; a_i \in \langle Z_{11} \cup I \rangle; +\right\}, +\}$$

be the MOD neutrosophic polynomial coefficient subset subsemigroup of finite.

The reader is left with the task of finding o(S(M)) and o(S(W)).

***Example 4.75:*** Let $S(M[x]_9)$ = {collection of all MOD dual number coefficient polynomial subsets from

$$M[x]_9 = \left\{ \sum_{i=0}^{9} a_i x^i \;\middle|\; a_i \in \langle Z_{12} \cup g \rangle, x^{10} = 1, +\right\}, +\}$$

be the MOD dual number coefficient polynomial subset semigroup under +. $o(S(M[x]_9)$ is finite and has finite order MOD dual number coefficient polynomial subsemigroups.

***Example 4.76:*** Let $S(V[x]_{12})$ = {collection of all subsets from

$$V[x]_{12} = \left\{ \sum_{i=0}^{12} a_i x^i \;\middle|\; a_i \in \langle Z_{10} \cup I \rangle, x^{13} = 1, +\right\}, +\}$$

be the MOD neutrosophic coefficient polynomial subset semigroup of finite order.

This has subsemigroups all of which are finite order.

$S(P[x]_{12})$ = {collection of all subsets from

$$P[x]_{12} = \left\{\sum_{i=0}^{12} a_i x^i \;\middle|\; a_i \in Z_{12}I;\ x^{13} = 1, +\right\}, +\}$$ is a MOD neutrosophic coefficient polynomial subset subsemigroup of finite order.

Let $A = \{5Ix^3 + 6Ix + 8I, 10I + 9Ix^6\}$ and

$B = \{5I + 6Ix^3, 10Ix^7 + 3I\} \in S(P[x]_{12})$.

$A + B = \{5Ix^3 + 6Ix + 8I, 10I + 9I + x^6\} + \{5I + 6Ix^3, 10Ix, 3I\} = \{5Ix^3, 6Ix + 8I + 5I + 6Ix^3, 10I + 9Ix^6 + 5I + 6Ix^3, 5Ix^3 + 6Ix + 8I + 10I + 9I x^6, 10I + 9Ix^6 + 10Ix\} = \{11Ix^3 + 6Ix + I, 9Ix^6 + 6Ix^3 + 3I, 9Ix^6 + 5Ix^3 + 6Ix + 6I, 9Ix^6 + 10Ix^7 + I\} \in S(P[x]_{12})$.

This is the way the '+' operation is performed on $S(P[x]_{12})$. Interested reader can analyse the related properties of $S(P[x]_{12})$.

***Example 4.77:*** Let $S(W[x]_{18})$ = {collection of all MOD polynomials subsets from the MOD polynomial semigroup.

$$W[x]_{18} = \left\{\sum_{i=0}^{18} a_i x^i \;\middle|\; a_i \in Z_{17}, x^{19} = 1, +\right\}, +\}$$

be the MOD polynomial subset semigroup under +.

Let $A = \{x^{10}, 16x^3 + 11, 10, 3x^5 + 2\}$ and
$B = \{3x^7 + 1, 10x^4 + 1, 0, 8\} \in S(W[x]_{18})$.

$A + B = \{x^{10}, 6x^3 + 11, 10, 3x^5 + 2\} + \{3x^7 + 1, 10x^4 + 1, 0, 8\}$

$= \{x^{10} + 3x^7 + 1, 3x^7 + 6x^3 + 12, 3x^7 + 11, 3x^7 + 3x^5 + 3, 10x^4 + 1 + x^{10}, 6x^3 + 12 + 10x^4, 10x^4 + 11, 3x^5 + 10x^4 + 3, x^{10}, 6x^3 + 11, 10, 3x^5 + 2, x^{10} + 88, 6x^3 + 2, 1, 3x^4 + 10\} \in S(W[x]_{18})$.

The reader is left with the task of finding MOD polynomial subset subsemigroups. Thus we see one can have two operations MOD polynomial subsets. Under both of them they behave differently.

Further we can have likewise two types of operations on MOD subset matrices they are also different and distinct.

It is left as an exercise to work with these structures.

Infact we suggest a set of problems some of which are challenging some of them easy just routine exercises.

**Problems:**

1. Let $M = \{(a_1, a_2, a_3, a_4, a_5) / a_i \in S(Z_9), 1 \leq i \leq 5, +\}$ be the MOD subset matrix semigroup under +.

   i) Find order of M.
   ii) How many MOD subset subsemigroups of M are there?
   iii) Is M a Smarandache MOD subset matrix semigroup?

2. Let

$$S = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} / a_i \in S(Z_{43}), 1 \leq i \leq 8, +\}$$

   be the MOD subset matrix semigroup under +.

i) Study questions (i) to (iii) of problem (1) for this S.
ii) Compare S with M is problems (1).

3. Let

$$A = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} a_i \in S(Z_{480}), 1 \leq i \leq 15, +\}$$

be the MOD subset matrix semigroup.

i) Study questions (i) to (iii) of problem (1) for this A.
ii) Compare A with S and M of problems (2) and (1) respectively.

4. Obtain any other special and interesting features associated with $M = \{(a_{ij})_{s\times t} / a_{ij} \in S(Z_n); 1 \leq i \leq s \text{ and } 1 \leq j \leq t, +\}$.

5. Let

$$B = \{\begin{bmatrix} a_1 & a_6 & a_{11} \\ a_2 & a_7 & a_{12} \\ a_3 & a_8 & a_{13} \\ a_4 & a_9 & a_{14} \\ a_5 & a_{10} & a_{15} \end{bmatrix} / a_i \in S(C(Z_{10})); 1 \leq i \leq 15, +\}$$

be the MOD subset finite complex number matrix semigroup.

Study questions (i) to (iii) of problem (1) for this B.

6. Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} / a_i \in S(Z_{41}); 1 \leq i \leq 9, + \right\}$$

be the MOD finite complex number matrix semigroup.

Study questions (i) to (iii) of problem (1) for this T.

7. Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| a_i \in S(\langle Z_7 \cup I \rangle; 1 \leq i \leq 15, \right.$$

+} be the MOD neutrosophic subset matrix semigroup.

i) Study questions (i) to (iii) of problem (1) for this W.
ii) Compare W with T of problem 6.

8. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} / a_i \in S(\langle Z_{12} \cup g \rangle; 1 \leq i \leq 15, + \right\}$$

be the MOD subset dual number matrix semigroup.

i) Study questions (i) to (iii) of problem (1) for this V.
ii) Compare V with W in problem 7.

9. Obtain all special and striking features enjoyed by MOD subset dual number matrix semigroups.

10. Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \middle/ a_i \in S(\langle Z_{12} \cup h\rangle; \; 1 \leq i \leq 10, +\right\}$$

be the MOD special dual like number matrix semigroup.

Study questions (i) to (iii) of problem (1) for this S.

11. Let

$$D = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle/ a_i \in S(\langle Z_{15} \cup k\rangle; \; 1 \leq i \leq 16, +\right\}$$

be the MOD special quasi dual number matrix semigroup.

Study questions (i) to (iii) problem (1) for this D.

12. Enumerate all special and distinct features associates with MOD subset special quasi dual number matrix semigroup.

13. Let M = $\{(a_1, a_2, a_3, a_4, a_5) / a_i \in S(Z_6); 1 \leq i \leq 6, \times\}$ be the MOD subset matrix semigroup.

    i) Find o(M).
    ii) Find all MOD subset matrix subsemigroups which are not ideals.
    iii) Find all MOD subset matrix subsemigroups which are ideals.
    iv) Find all MOD subset zero divisors and S-zero divisors if any.
    v) Find all MOD subset nilpotent and idempotents of M.
    vi) Obtain any other special feature enjoyed by M and compare some M under +.

14. Let

$$S = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} / a_i \in S(C(Z_{12})); 1 \leq i \leq 15, \times_n\}$$

be the MOD subset finite complex number semigroup under $\times$.

Study questions (i) to (v) of problem (13) for this S.

15. Let

$$P = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} / a_i \in S(\langle Z_{10} \cup I\rangle; 1 \leq i \leq 12, \times_n\}$$

be the MOD subset neutrosophic semigroup under $\times$.

Study questions (i) to (v) of problem (13) for this P.

16. Let

$$V = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} / a_i \in S(\langle Z_{23} \cup g\rangle; 1 \leq i \leq 16, \times_n\}$$

be the MOD subset dual number matrix semigroup under product.

Study questions (i) to (v) of problem (13) for this V.

17. Obtain all special features enjoyed by MOD subset dual number matrix semigroups and compare it with MOD subset neutrosophic matrix semigroups both under ×.

18. Let

$$N = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \middle/ a_i \in S(\langle Z_{15} \cup k \rangle); \; 1 \leq i \leq 15, \times_n \right\}$$

be the MOD subset special quasi dual number semigroup.

Study questions (i) to (v) of problem (13) for this N.

19. Let

$$x = \left\{ \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} \middle/ a_i \in S(\langle Z_{19} \cup I \rangle); \; 1 \leq i \leq 9, \times_n \right\}$$

be the MOD subset neutrosophic semigroup under natural product $\times_n$.

i) Study questions (i) to (v) of problem (13) for this $\times_n$.

ii) Replace the operation $\times_n$ by × and study and distinguish all the properties under $\times_n$ and under ×.

20. Let

$$S[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_{12}); +\right\}$$

be the MOD subset polynomial coefficient semigroup under +.

i) Find all MOD subset subsemigroups of S[x]
ii) Show S[x] can have finite order subset subsemigroups of S[x].
iii) Obtain any other special feature associated by S[x].

21. Let

$$P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(C(Z_{17})), +\right\}$$

be the MOD subset finite complex number polynomial coefficient semigroup.

Study questions (i) to (iii) of problem (20) for this P[x].

22. Let $D[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{18} \cup I\rangle), +\right\}$ be the MOD neutrosophic subset coefficient polynomial semigroup.

Study questions (i) to (iii) of problem (20) for this D[x].

23. Let $W[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{23} \cup g\rangle), +\right\}$ be the MOD dual number subset coefficient polynomial semigroup.

Study questions (i) to (iii) of problem (20) for this W[x].

24. Let

$$B[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{15} \cup h\rangle), +\right\}$$

be the MOD special dual like number subset coefficient polynomial semigroup.

Study questions (i) to (iii) of problem (20) for this B[x].

25. Let

$$L[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{18} \cup k\rangle), +\right\}$$

be the MOD special quasi dual number subset coefficient semigroup.

Study questions (i) to (iii) of problem (20) for this L[x].

26. Obtain all special and distinct features enjoyed by the 6 different MOD subset coefficient polynomials semigroup under +.

    Compare and construct them with each other by enumerating all special features.

27. Let

$$T[x]_9 = \left\{\sum_{i=0}^{9} a_i x^i \;\middle|\; a_i \in S(Z_{48}); +, x^{10} = 1\right\}$$

be the MOD subset coefficient polynomial subsemigroups of finite order under +,

i) Find $o(T[x]_9)$.
ii) Find all MOD subset coefficient polynomial subsemigroups.
iii) Can we say using S ($Z_{48}$) will yield more MOD subset coefficient polynomial subsemigroups than by using

$S(Z_{47})$?

28. Let

$$M[x]_{20} = \left\{ \sum_{i=0}^{20} a_i x^i \;\middle|\; a_i \in S(C(Z_{28})),\ x^{21} = 1, + \right\}$$

be the MOD subset finite complex number coefficient polynomial semigroup under +.

i) Study questions (i) to (ii) of problem (27) for this $M[x]_{20}$.
ii) Enumerate all special features associated with $M[x]_{20}$ and compare it with $T[x]_9$ in problem (27)

29. Let

$$W[x]_5 = \{ \sum_{i=0}^{5} a_i x^i \,/\, a_i \in S\langle Z_{10} \cup I\rangle,\ x^6 = 1, + \}$$

be the MOD subset finite neutrosophic coefficient polynomial semigroup.

i) Study questions (i) to (ii) of problem (27) for this $W[x]_5$.
ii) Compare $W[x]_5$ with $M[x]_{20}$ and $T[x]_9$ of problem 28 and 27 respectively.

30. Let

$$V[x]_8 = \{ \sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in S(Z_{20} \cup k\rangle,\ x^9 = 1, + \}$$

be the MOD subset special quasi dual number coefficient polynomial semigroup under +.

Study questions (i) to (ii) of problem (27) for this $V[x]_8$.

31. What are the special features associated with MOD subset special quasi dual number subset coefficient polynomial semigroups under +.

32. Let

$$F[x]_8 = \{\sum_{i=0}^{18} a_i x^i \,/\, a_i \in S(\langle Z_{17} \cup g\rangle, x^{19} = 1, +\}$$

be the MOD subset special dual number coefficient polynomial semigroup under +.

Study questions (i) to (ii) of problem (27) for this $F[x]_{18}$.

33. Let

$$P[x]_8 = \{\sum_{i=0}^{8} a_i x^i \,/\, a_i \in S(\langle Z_{27} \cup h\rangle); x^9 = 1, +\}$$

be the MOD subset special dual like number coefficient polynomial semigroup under +.

i) Study questions (i) to (ii) of problem (27) for this $P[x]_8$.
ii) Study all special and distinct features associated with MOD subset special dual like number coefficient polynomial semigroups.

34. Let

$$S[x] = \{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in S(Z_{12}) \times\}$$

be the MOD subset coefficient polynomial semigroup under ×.

i) Find all MOD subset coefficient polynomial subsemigroup of under ×.
ii) Find all MOD subset coefficient polynomial ideals of S[x] under ×.
iii) Find all MOD zero divisors of S[x].
iv) Find all MOD nilpotents of S[x].

v) Prove S[x] cannot have MOD idempotents.
vi) Can S[x] of finite order MOD subsemigroup?

35. Let P[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(\langle Z_{18} \cup g\rangle \times\}$ be the MOD dual number subset coefficient polynomial semigroup under ×.

    Study questions (i) to (vi) of problem (34) for this P[x].

36. Let

$$T[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S\langle C(Z_{19})\rangle;\ \times \right\}$$

    be the MOD subset finite complex number coefficient polynomial semigroup under ×.

    i) Study questions (i) to (vi) of problem (34) for this T[x].
    ii) Find all special feature enjoyed by MOD subset finite complex number coefficient polynomial under ×.

37. Let

$$W[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S\langle (Z_{92} \cup k\rangle; \times \right\}$$

    be the MOD special quasi dual number subset coefficient polynomial semigroup.

    i) Study questions (i) to (vi) of problem (33) for this M[x]
    ii) Find all special features enjoyed by the MOD subset special dual like number coefficient polynomial semigroup.

38. Let P[x] = $\{\sum_{i=0}^{9} a_i x^i$ / $a_i \in S(Z_{24})$; $x^{10} = 1$, ×} be the MOD subset polynomial semigroup under ×.

    i) Find o(P[x])

ii) Find all MOD subset polynomial subsemigroups which are not ideals.
iii) Find all MOD subset polynomial subsemigroups which are ideals.
iv) Find all MOD zero divisors and MOD nilpotents of P[x].
v) Give all special features associated P[x].

39. Let $M[x] = \{\sum_{i=0}^{6} a_ix^i / a_i \in S\langle(Z_{12} \cup g\rangle; x^7 = 1, \times\}$ be the MOD subset dual number polynomial semigroup.

i) Study questions (i) to (v) of problem 38.
ii) Distinguish M[x] and P[x] of problem 38.
iii) Prove M[x] enjoys special features entirely different from other MOD subset coefficient polynomial semigroup.

40. Let

$$V[x] = \{\sum_{i=0}^{\infty} a_ix^i / a_i \in S(C(Z_{18})); \times\}$$

be the MOD special quasi dual number subset coefficient polynomial semigroup.

Study questions (i) to (vi) of problem (34) for this V[x].

41. Let

$$G[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{10} \cup k\rangle), \times\right\}$$

be the MOD subset special quasi dual like number coefficient polynomial semigroup.

i) Study questions (i) to (vi) of problem (34) for this G[x].
ii) Compare G[x] with V[x] of problem (40).

42. Let

$$H[x]_{12} = =\{\sum_{i=0}^{12} a_i x^i \,/\, a_i \in S(\langle Z_{10} \cup h\rangle);\ x^{13} = 1,\ \times\}$$

be the MOD subset special dual like number coefficient polynomial semigroup.

i) Study questions (i) to (v) of problem (38) for this $H[x]_{12}$

43. Let S(M) = {collection of matrix subsets from

$$M = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} a_i \in Z_{18},\ 1 \le i \le 8,\ +\},\ +\}$$

be the MOD matrix subset semigroup under +.

i) Find o(M).
ii) Find all MOD subsemigroups of M.
iii) Can M have MOD idempotents under +.
iv) Enumerate all special features associated with S(M).

44. Let S(T) = {collection of all subsets from

$$T = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} a_i \in C(Z_8);\ 1 \le i \le 5,\ +\}$$

be MOD finite complex number subset matrices, semigroup under +.

Study questions (i) to (iv) of problem (43).

45. Let W(S) = {collection of all subsets from the matrix set with entries from $\langle Z_{10} \cup I \rangle$.

$$S = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_i \in \langle Z_{10} \cup I \rangle, 1 \leq i \leq 4, +\}, \_\}$$

be the MOD neutrosophic matrix subset semigroup under +. Study questions (i) to (iv) of problem (43) for this W(S).

46. Let H(M) = {collection of all subset matrix from

$$M = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \mid a_i \in \langle Z_{23} \cup g \rangle, 1 \leq i \leq 10, +\}, +\}$$

be the MOD dual number subset matrix semigroup under +.

Study questions (i) to (iv) of problem (43).

47. Let V(N) = {collection of all subset matrices from

$$N = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} / a_i \in Z_{48}; 1 \leq i \leq 12, \times_n\}, \times_n\}$$

be the MOD subset matrix semigroup under $\in$.

i) Find 0(V(N))
ii) Find all MOD subset matrix zero divisors and nilpotents of V(N).
iii) Find all MOD subset matrix subsemigroups which are not ideals of V(N).
iv) Find all MOD subset matrix subsemigroups which are

ideals of V(N).

v) Prove / disprove V(N) is a Smarandache MOD matrix subset semigroup.

vi) Obtain all special features associated with V(N).

48. Let W(M) = {collection of all subset matrix from

$$M = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} / a_i \in \langle Z_{13} \cup g\rangle, 1 \leq i \leq 6, \times_n\}, \times_n\}$$

be the MOD subset dual number matrix semigroup under $\times_n$, the natural product.

Study questions (i) to (vi) of problem (47) for this W(M).

49. Let G(P) = {collection of matrix subsets from

$$P = \{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} / a_i \in C(Z_{49}); 1 \leq i \leq 9, + (\text{or } \times_n) \times (\text{or } \times_n)\}$$

be the MOD subset matrix finite complex number semigroup under × (or $\times_n$).

i) Study questions (i) to (vi) of problem (47) for this G(P).
ii) If x is used prove G(P) is non commutative.
iii) Find in G(P) a MOD left zero divisor which is not a right zero divisor and vice versa under the usual product ×.
iv) Find all MOD left ideals which are not MOD right ideals in G(P) under the usual product.

50. Let T(H) = {collection of all of matrix subsets from

$$H = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} / a_i \in \langle Z_{24} \cup g\rangle, \times_n,$$

$$1 \leq i \leq 10\}, \times_n\}$$

be the MOD dual number subset matrix semigroup.

i) Study questions (i) to (vi) of problem (47) for this T(H).
ii) Prove T(H) has more number of MOD zero divisors and nilpotents even if n is a prime.

51. Let S(W) = {collection of all subsets from the MOD special quasi dual number matrix semigroup

$$W = \{\begin{bmatrix} a_1 & a_8 & a_{15} \\ a_2 & a_9 & a_{16} \\ a_3 & a_{10} & a_{17} \\ a_4 & a_{11} & a_{18} \\ a_5 & a_{12} & a_{19} \\ a_6 & a_{13} & a_{20} \\ a_{21} & a_{14} & a_{21} \end{bmatrix} / a_i \in \langle Z_9 \cup I\rangle, 1 \leq i \leq 21, \times_n\}, \times_n\}$$

be the MOD neutrosophic subset matrix semigroup under natural product $\times_n$

i) Study questions (i) to (vi) of problem (47) for this S(W).
ii) Distinguish T(H) of problem (50) from S(W).

52. Let S(M) = {collection of all subsets from the MOD matrix special dual like number semigroup.

$$M = \left\{ \begin{bmatrix} a_1 & a_6 & a_{11} \\ a_2 & a_7 & a_{12} \\ a_3 & a_8 & a_{13} \\ a_4 & a_9 & a_{14} \\ a_5 & a_{10} & a_{15} \end{bmatrix} \Big/ a_i \in \langle Z_{13} \cup h \rangle, 1 \leq i \leq 15, \times_n \right\}, \times_n\}$$

be the MOD subset special dual like number semigroup under $\times_n$.

Study questions (i) to (vi) of problem (47) for this S(M).

53. Let S(P[x]) = {collection of all subsets from the set P[x] where

$$a_i\ P[x] = \{\sum_{i=0}^{\infty} d_i x^i \,/\, d_i \in Z_{48}\}, \times\}$$

be the MOD subset polynomial semigroup under product.

i) Prove S(P[x]) is a commutative monoid of infinite order.
ii) Find all MOD ideals of S(P[x]).
iii) Find all MOD subsemigroups of S(P[x]) which are not ideals.
iv) Find all MOD subset zero divisors and nilpotents of S(P[x]).
v) Prove S(P[x]) cannot have idempotents.

54. Let S(W[x]) = {collection of all subsets from the MOD semigroup.

$$W[x] = \{\sum_{i=0}^{\infty} d_i x^i \,/\, d_i \in (Z_{47}), \times\}, \times\}$$

be the MOD subset of finite complex number polynomials semigroup.

Study questions (i) to (v) of problem (53) for this W[x].

55. Let S(Z[x]) = {collection of all subsets from the MOD special dual like number polynomial semigroup

$$z[x] = \{\sum_{i=0}^{\infty} a_i x^i / a_i \in \langle Z_{40} \cup h\rangle, \times\}, \times\}$$

be the MOD special dual like number polynomials subsets semigroup.

Study questions (i) to (v) of problem (53) for this S(Z[x]).

56. Let S(B[x]) = {collection of all subsets from the MOD dual number coefficient polynomial semigroup

$$B[x] = \{\sum_{i=0}^{\infty} a_i x^i / a_i \in \langle Z_{20} \cup g\rangle; \times\}, \times\}$$

be the MOD subset polynomial dual number semigroup.

i) Study questions (i) to (v) of problem (53) for this S(B[x]).
ii) Prove S(B[x]) has more number of MOD zero divisors and MOD nilpotents.

57. Let $S(P[x]_8)$ = {collection of all subsets polynomials from MOD polynomial semigroup.

$$P[x]_8 = \{\sum_{i=0}^{\infty} a_i x^i / a_i \in Z_{26}, x^9 = 1, \times\}, \times\}$$

be the MOD subset polynomial semigroup under ×.

i) Find $o(SP[x]_8)$.
ii) Find all MOD zero divisors and nilpotents.
iii) Find all MOD subset ideals.
iv) Find all MOD subset subsemigroups which are not

ideals.

v) Prove $S(P[x]_8)$ has no idempotents.

58. Let $S(Q[x]_6)$ = {collection of all subset polynomials from

$$Q[x]_6 = \{\sum_{i=0}^{6} a_i x^i \,/\, a_i \in \langle Z_{10} \cup I\rangle, x^7 = 1, \times\}, \times\}$$

be the MOD neutrosophic subset polynomials semigroup.

Study questions (i) to (v) of problem (57) for this $S(Q[x]_6)$.

59. Let $S(T[x]_{10})$ = {collection of all subset polynomials from

$$T[x]_{10} = \{\sum_{i=0}^{10} a_i x^i \,/\, x^{11} = 1, a_i \in \langle Z_{19} \cup g\rangle; \times\}, \times\}$$

be the MOD dual number subset polynomial semigroup.

i) Study questions (i) to (v) of problem (59) for this $T[x]_{10}$.

60. Let $S(V[x]_{18})$ = {collection of all polynomial subsets from

$$V[x]_{18} = \{\sum_{i=0}^{18} a_i x^i \,/\, x^{19} = 1, a_i \in \langle Z_{12} \cup k\rangle, \times\}, \times\}$$

be the MOD polynomial special quasi dual number subset semigroup.

i) Study questions (i) to (v) of problem (57) for this $S(V[x]_{18})$.
ii) Compare $S(V[x]_{18})$ when $\langle Z_{12} \cup k\rangle$ is replaced by $\langle Z_{12} \cup h\rangle$ or $\langle Z_{12} \cup g\rangle$ or $\langle Z_{12} \cup I\rangle$, or $C(Z_{12})$ and $Z_{12}$.

61. Let $S(P[x])$ = {collection of all MOD subset polynomials from

$$P[x] = \{\sum_{i=0}^{18} a_i x^i / a_i \in Z_{483}, +\}, +\}$$

be the MOD subset polynomial semigroup.

i) Prove S(P[x]) has MOD subset semigroups of finite order.
ii) Find all MOD subset subsemigroups of infinite order.
iii) Prove o(S(P[x]) = ∞.

62. Let S (G[x]) = {collection of all subset polynomials from

$$G[x] = \{\sum_{i=0}^{18} a_i x^i / a_i \in C(Z_{11}), +\}, +\}$$

be the MOD finite number complex subset polynomial semigroup under +.

Study questions (i) to (iii) of problem (61) for this S(G[x]).

63. Let S(V[x]) = {collection of all subset polynomial from

$$V[x] = \{\sum_{i=0}^{18} a_i x^i / a_i \in \langle Z_{12} \cup g\rangle\}, +\}, +\}$$

be the MOD finite dual number subset polynomial semigroup under +.

i) Study questions (i) to (iii) of problem (61) for this S(V[x]).
ii) Compare S(V[x]) with S(G[x]) of problem (62).

64. Let $S(B[x]_{10})$ = {collection of all subset polynomials from

$$B[x]_{10} = \{\sum_{i=0}^{10} a_i x^i / x^{11} = 1, a_i \in Z_{48}, +\}, +\}$$

be the MOD subset polynomial semigroup under +.

i) Find $o(S(B[x]_{10}))$.
ii) Find all MOD subset polynomial subsemigroups.
iii) Can $S(B[x]_{10})$ have nontrivial idempotents?

65. Let $S(V[x]_6)$ = {collection of all polynomial subset from

$$V[x]_6 = \{\sum_{i=0}^{6} a_i x^i \,/\, x^7 = 1, a_i \in C(Z_{27}), +\}, +\}$$

be the MOD finite complex number polynomial subset semigroup under +.

Study questions (i) to (iii) of problem (64) for this $S(V[x]_6)$.

66. Let $S(G[x]_{12})$ = {collection of all polynomial subsets from

$$G[x]_{12} = \{\sum_{i=0}^{12} a_i x^i \,/\, x^{13} = 1, a_i \in \langle Z_{43} \cup g\rangle, +\}, +\}$$

be the MOD dual number coefficient polynomial subset semigroup under +.

Study questions (i) to (iii) of problem (64) for this $S(G[x]_{12})$.

67. Let $S(M[x]_{19})$ = {collection of all polynomial subsets from

$$M[x]_{19} = \{\sum_{i=0}^{19} a_i x^i \,/\, x^{20} = 1, a_i \in \langle Z_{10} \cup k\rangle, +\}, +\}$$

be the MOD special quasi dual number polynomial subset semigroup under +.

Study questions (i) to (iii) of problem (64) for this $S(M[x]_{14})$.

68. Differentiate all special features associated with $S(M[x]_{19})$ in problem 67 with $S(M[x]_{19})$ under product operation.

69. Compare $S(M[x]_{19})$ on problem 67 with $S(G[x]_{12})$ in problem 66.

70. Enumerate all special features enjoyed by the following MOD semigroups. $S(Q[x]_m)$ with entries from $Z_n$ or $C(Z_n)$ or $\langle Z_n \cup g\rangle$, $\langle Z_n \cup h\rangle$, $\langle Z_n \cup k\rangle$ and $\langle Z_n \cup I\rangle$; $2 \leq m < \infty$, $x^{m+1} = 1$.

    Compare each of them.

Chapter Five

# MOD SUBSET NATURAL NEUTROSOPHIC SEMIGROUPS

In this chapter we just describe MOD natural neutrosophic number semigroups of different types built using subsets from $Z_n^I$ or $C^I(Z_n)$ or $\langle Z_n \cup I\rangle_I$ or $\langle Z_n \cup h\rangle_I$ or $\langle Z_n \cup g\rangle_I$ or $\langle Z_n \cup k\rangle_I$.

We describe by examples and derive several important and interesting properties associated with them.

For properties of MOD natural neutrosophic numbers please refer [60].

We first describe the notions by examples.

$S(Z_n^I)$ = {Collection of all subsets from $Z_n^I$}.

***Example 5.1:*** $S(Z_6^I)$ = {Collection of all subsets from $Z_6^I$}.

A = {$I_0^6, I_3^6 + I_2^6$, 5 + $I_4^6$, 3 + $I_0^6$ + $I_2^6$} and

B = {1, 2, 0, $I_0^6$ + 5, 4 + $I_2^6$ + $I_4^6$, $I_3^6$ + $I_2^6$+1} are MOD subset natural neutrosophic numbers in $S(Z_6^I)$.

***Example 5.2:*** S $(C^I(Z_8))$ = {Collection of all subsets from MOD natural neutrosophic finite complex numbers of $C^I(Z_8)$}.

Let A = { $I_4^C$, $I_2^C + I_0^C$ +5, $I_6^C + I_4^C + I_{2i_F}^C$ + 5$i_F$ + 6} and

B = { $I_{4i_F}^C$ + 0, $I_0^C$, $I_{6i_F+4}^C$, $I_{2+4i_F}^C$, 2 + 6$i_F$ + $I_{4+2i_F}^C$ } ∈ S$(C^I(Z_8))$.

S($\langle Z_n \cup I\rangle_I$) = {Collection of all subsets from the MOD natural neutrosophic - neutrosophic set $\langle Z_n \cup I\rangle_I$}.

***Example 5.3:*** S($\langle Z_{10} \cup I\rangle_I$) = {Collection of all MOD natural neutrosophic-neutrosophic subsets from $\langle Z_{10} \cup I\rangle_I$}.

Let B = { $I_0^I$, $I_{5+5I}^I$ +3+2I, 3 + $I_{6+6I}^I + I_2^I$, $I_{8I}^I$ + $I_{3I}^I$ + 4+5I} and

C = { $I_{2I}^I$, $I_{8I+2}^I$, $I_{6I}^I$ + $I_{4I}^I$ + $I_{2I+2}^I$ +4 + 7I} ∈ S($\langle Z_{10} \cup I\rangle_I$) are MOD natural neutrosophic - neutrosophic subsets of S($\langle Z_{10} \cup I\rangle_I$).

***Example 5.4:*** S($\langle Z_{18} \cup g\rangle_I$) = {Collection of all MOD natural neutrosophic dual number subsets from S($\langle Z_{18} \cup g\rangle_I$}.

Let S = { $I_{3g}^g$, $I_{7g}^g$, $I_{2g}^g$, $I_{9+9g}^g$ + $I_{6g+3}^g$ +5g + 9} and

T = {3 + 4g + $I_0^g, I_9^g$, $I_{6+8g}^g$ + $I_g^g$, $I_8^g$ + $I_2^g$ + $I_3^g$+1} ∈ S($\langle Z_{18} \cup g\rangle_I$).

Thus S($\langle Z_n \cup g\rangle_I$) = {Collection of all subsets from the MOD natural neutrosophic dual number set $\langle Z_n \cup g\rangle_I$}.

Let S($\langle Z_n \cup h\rangle_I$) = {Collection of all subsets from the MOD natural neutrosophic special dual like number set with entries from $\langle Z_n \cup h\rangle_I$}.

***Example 5.5:*** Let S(⟨$Z_{12}$ ∪ h⟩$_I$) be the MOD natural neutrosophic special dual like number subset.

Let A = { $I_0^h$ , $I_{6h+2}^h$ + $I_{3h+4}^h$ + $I_{10h}^h$ + $I_6^h$ + 3 + 6h, $I_{10h+2}^h$ + $I_4^h$ + h + 9} and

B = { $I_3^h$ + $I_4^h$ + $I_0^h$ + 8, 4 + 3h, 10, 11+h, $I_{10h}^h$ + $I_{2h}^h$ + 7, $I_8^h$ } ∈ S(⟨$Z_{12}$ ∪ h⟩$_I$) are MOD natural neutrosophic special dual like number subset of S(⟨$Z_{12}$ ∪ h⟩$_I$).

S(⟨$Z_n$ ∪ k⟩$_I$) = {Collection of all MOD natural neutrosophic special quasi dual number subsets of ⟨$Z_n$ ∪ k⟩$_I$}.

***Example 5.6:*** Let S(⟨$Z_{16}$ ∪ k⟩$_I$) = {Collection of all MOD natural neutrosophic quasi dual number subsets of ⟨$Z_{16}$ ∪ k⟩$_I$}.

Let P = { $I_4^k$ , $I_{2k}^k$ , $I_{8+2k}^k$ , 3 +9k + $I_{10k}^k$ } and

Q = { $I_0^k$ , $I_{10+8k}^k$ , $I_8^k$ + $I_{6+4k}^k$ + $I_{2k+10}^k$ , 8 + 9k + $I_4^k$ , 10, 7k, $I_{8+8k}^k$ } ∈ S(⟨$Z_{16}$ ∪ k⟩$_I$) are MOD natural neutrosophic special quasi dual number subsets of S(⟨$Z_{16}$ ∪ k⟩$_I$).

We see on S( $Z_n^I$ ) or S($C^I$($Z_n$)) or S(⟨$Z_n$ ∪ I⟩$_I$) or S(⟨$Z_n$ ∪ h⟩$_I$) or S(⟨$Z_n$ ∪ g⟩$_I$) and S(⟨$Z_n$ ∪ k⟩$_I$) we can define algebraic operations + or × and under each of these operations all these six MOD sets are only MOD semigroups.

We will illustrate this situation by some examples.

***Example 5.7:*** Let H = {S( $Z_6^I$ ), +} be the MOD natural neutrosophic subset semigroup under +.

Let A = { $I_2^6$ + $I_3^6$ , 3 + $I_0^6$ } and

B = { $I_4^6$ , $I_2^6$ + 4, $I_0^6$ + $I_4^6$ , 3} ∈ H.

$$A + B = \{I_2^6 + I_3^6, I_0^6 + 3\} + \{3, I_4^6, I_2^6 + 4, I_0^6 + I_4^6\}$$

$$= \{I_2^6 + I_3^6 + 3, I_0^6, I_2^6 + I_3^6 + I_4^6, 3 + I_0^6 + I_4^6, 4 + I_2^6 + I_2^6, 1 + I_0^6 + I_2^6, I_0^6 + I_4^6 + I_3^6 + I_2^6, I_0^6 + I_2^6 + 3\}.$$

Thus is the way + operation is performed on H.

H is infact a monoid as $\{0\} \in H$ is such that $\{0\} + A = A$ for all $A \in H$.

***Example 5.8:*** Let $M = \{S(\langle Z_{10} \cup g\rangle_I), +\}$ be the MOD natural neutrosophic dual number subset monoid under +.

Let $A = \{6g + I_0^g, 5 + 3g, I_{5g+4}^g\}$ and

$B = \{I_{4g}^g, 2, I_{6g+8}^g\} \in M$.

$A + B = \{6g + I_0^g, 5 + 3g, I_{5g+4}^g\} + \{I_{4g}^g, 2, I_{6g+8}^g\}$

$= \{6g + 2 + I_0^g, 7+3g, 2+I_{5g+4}^g, 6 + I_0^g + I_{6g+8}^g, I_{6g+8}^g + 5 + 3g, I_{5g+4}^g + I_{6g+8}^g, I_{4g}^g + I_0^g + 6g, 5 + 3g + I_{4g}^g, I_{5g+4}^g + I_{4g}^g\} \in M$.

The reader is expected to find MOD natural neutrosophic dual number subset monoid of finite order.

***Example 5.9:*** Let $V = \{S(\langle Z_{15} \cup I\rangle_I), +\}$ be the MOD natural neutrosophic - neutrosophic number subset monoid of finite order.

Let $A = \{I_{3I}^I, I_{6+3I}^I, I_0^I + I + 8\}$ and

$B = \{I_{8I}^I, I_{6I+4}^I, I_6^I, I_9^I + 7 + 3I\} \in V$.

The reader is left with the task of finding A + B.

***Example 5.10:*** Let G = $S(C^I(Z_9), +\}$ be the MOD natural neutrosophic finite complex number subset monoid of finite order.

Let P = $\{ I_3^C, I_{6i_F}^C, I_0^C + 4 + i_F\}$ and

Q = $\{ I_{3i_F+6}^C, I_{6i_F+3}^C, 3, I_0^C + I_6^C + I_{3i_F}^C + 2 + i_F\} \in G$.

The reader is expected to find P + Q.

Likewise we can find $\{S(\langle Z_{19} \cup k\rangle_I), +\}$ and $\{S(\langle Z_{20} \cup k\rangle_I), +\}$ and this is left as an exercise to the reader as it is considered as a matter of routine.

**THEOREM 5.1:** *Let {H, +} = {$S(Z_n^I)$ (or $S(C^I(Z_n))$ or $S(\langle Z_n \cup g\rangle_I)$ or $S(\langle Z_n \cup h\rangle_I)$ or $S(\langle Z_n \cup k\rangle_I)$ or $S(\langle Z_n \cup I\rangle_I)$), +} be the MOD natural neutrosophic subset (or finite complex number subset or dual number subset or special dual like number subset or special quasi dual number subset or neutrosophic subset) semigroup under +.*

*(i) $o(H) < \infty$.*
*(ii) H has MOD natural neutrosophic subsemigroups.*

The proof is left as an exercise as it is considered as a matter of routine.

We now give examples of MOD natural neutrosophic subset monoid under × by some examples.

***Example 5.11:*** Let $\{S, \times\} = \{S(Z_{12}^I), \times\}$ be the MOD natural neutrosophic subset semigroup.

Let A = $\{ I_0^{12}, I_8^{12} + 4, I_6^{12} \}$ and

B = $\{ I_{10}^{12} + I_2^{12}, 6, 0, 6 + I_4^{12} \} \in \{S, \times\}$.

$A \times B = \{I_0^{12},\ I_8^{12} + 4,\ I_6^{12}\} \times \{I_{10}^{12} + I_2^{12}, 0, 6, 6 + I_4^{12}\}$

$= \{I_0^{12},\ I_{10}^{12} + I_2^{12} + I_4^{12} + I_8^{12},\ I_6^{12} + I_0^{12},\ I_8^{12},\ I_6^{12},\ I_4^{12} + I_8^{12}\}.$

This is the way product operation is performed on B.

Thus B has MOD natural neutrosophic zeros also.

We can on B define another type of operation in which $I_0^{12} \times 0 = I_0^{12}$ and $I_0^{12} \times 0 = 0$. By the context we can understand what product is used.

If $I_0^{12} \times 0 = I_0^{12}$ then we say the product is natural neutrosophic zero dominated product. That is what we have used in A × B.

The other product $I_0^{12} \times 0 = 0$, then we say the product is natural zero dominated product.

Now we find A × B under natural zero dominated product.

$A \times B = \{0,\ I_6^{12},\ I_0^{12},\ I_8^{12},\ I_4^{12} + I_0^{12} + I_8^{12} + I_{10}^{12},\ I_6^{12} + I_0^{12},$
$I_4^{12} + I_8^{12}\}.$

The reader is left with the task of finding both types of products.

However under both types of product they attain only a monoid structure and in one the dominated usual zero will contribute to zero divisors in another type the MOD natural neutrosophic dominant zero will contribute to more MOD natural neutrosophic zero divisors.

Further the reader is left with the task of finding MOD natural neutrosophic subset ideals and subsemigroups which are not ideals.

Further subset nilpotents of two types and subset idempotents.

We will give some more examples.

***Example 5.12:*** Let $\{V, \times\} = \{S(\langle Z_{18} \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup.

$A = \{6 + 3I + I_{3I}^I, I_0^I, 0, 2 + I_{9I}^I\}$ and

$B = \{0, I_{6I+3}^I, I_{2I}^I + 4\} \in V.$

We find the MOD natural neutrosophic - neutrosophic subsets of V. We find the usual zero dominated product first.

$A \times B = \{0, I_{6I+3}^I + I_{9I}^I, I_0^I, 0, I_{6I+3}^I + I_{9I}^I, 6 + I_{2I}^I + 12I + I_{6I}^I + I_{3I}^I, 8 + I_{9I}^I + I_{2I}^I + I_0^I\}$ --- I

Now we find the MOD natural neutrosophic zero dominated product that is $I_0^I \times 0 = I_0^I$.

$A \times B = \{I_{3I}^I, I_0^I, I_{9I}^I, 0, 6 + 12I + I_{2I}^I + I_{3I}^I + I_{6I}^I, I_{6I+3}^I + I_{9I}^I, I_{6I+3}^I, 8 + I_0^I, I_{9I}^I + I_{2I}^I\}$ ---- II

Clearly I and II are distinct thus the product defined two distinct MOD natural neutrosophic - neutrosophic subset monoid which is commutative and is of finite order.

There are several MOD natural neutrosophic subset zero divisors as well as usual zero divisors.

Study in this direction is a matter of routine with only change which is to be observed is whether usual zero dominant product is taken or MOD natural neutrosophic dominant zero is taken.

Such study is similar to one done in chapter I.

***Example 5.13:*** Let W = $\{S(\langle Z_{20} \cup g\rangle_I), \times\}$ be the MOD natural neutrosophic dual number subset semigroup under product. This has many subset nilpotents and zero divisors.

$P_1 = \{S(Z_{20}), \times\} \subseteq W$ is a subset subsemigroup which is not an ideal.

$P_2 = \{S(Z_{20}g), \times\} \subseteq W$ is again subset subsemigroup which is not an ideal.

$P_3 = \{S(\langle Z_{20} \cup g\rangle_I), \times\} \subseteq W$ is again a subset subsemigroup which is not an ideal.

Study in this direction is interesting and innovative.

$P_2$ is a zero square semigroup.

***Example 5.14:*** Let M = $\{S(\langle Z_{13} \cup k\rangle_I), \times\}$ be the MOD natural neutrosophic subset special quasi dual number semigroup.

$P_1 = \{S(Z_{13}), \times\}$ is a MOD subset subsemigroup which is not an ideal.

$P_2 = \{S(Z_{13}k), \times\}$ is again a MOD subset subsemigroup which is not a subset ideal.

$P_3 = \{S(\langle Z_{13} \cup k\rangle), \times\}$ is again a MOD subset subsemigroup which is not a subset ideal. We see finding MOD subset zero divisors and nilpotents happens to be a problem.

Further finding MOD subset idempotents is also a challenging one as $Z_{13}$ is used in this example.

We see $P_4 = \{S(\langle I_0^k, I_t^k \rangle$ | t is an idempotent or zero divisor in $Z_{13}), \times\}$ is again a MOD natural neutrosophic subset subsemigroups which is also an ideal in the MOD natural neutrosophic zero dominated product.

In view of all these we have the following theorem.

**THEOREM 5.2:** *Let $P_1 = \{S(Z_n^I), \times\}$ (or $P_2 = \{S(C^I(Z_n), \times\}$ or $\{P_3 = S(\langle Z_n \cup I\rangle_I), \times\}$ or $\{P_4 = S(\langle Z_n \cup g\rangle_I), \times\}$ or $\{P_5 = S(\langle Z_n \cup h\rangle_I), \times\}$ or $\{P_6 = S(\langle Z_n \cup k\rangle_I), \times\}$) be MOD natural neutrosophic subset semigroup under MOD natural zero dominated product.*

*Then the following are true.*

*(i) $o(S(Z_n^I))$ (and all other such subset MOD natural neutrosophic semigroups under $\times$) is of finite order.*

*(ii) $\{S(Z_n^I), \times\}$ has MOD natural neutrosophic subset zero divisors and subset nilpotent if n is an appropriate number.*

*(iii) Whatever be n, prime or otherwise $P_1, P_2, \ldots, P_6$ has MOD natural neutrosophic subset subsemigroups which are not ideals.*

*(iv) $T = S(\{\langle I_0^t, I_m^t\rangle \mid m$ is a zero divisor or idempotent or nilpotent in $Z_n$ (or $C(Z_n)$ or $\langle Z_n \cup I\rangle$ or $\langle Z_n \cup g\rangle$ or $\langle Z_n \cup h\rangle$ or $\langle Z_n \cup k\rangle)\} \subseteq P_i$, $i = 1, 2, \ldots, 6$ are MOD subset ideals of $P_i$, $i = 1, 2, \ldots, 6$.*

*(v) $P_3, P_4, P_5$ and $P_6$ has ideals of the form $\{S(\langle Z_nI \cup I_t^I\rangle\}$, $\{S(\langle Z_ng \cup I_t^g\rangle\}$, $\{S(\langle Z_nh \cup I_t^h\rangle\}$, $\{S(\langle Z_nk \cup I_t^k\rangle\}$ in $P_3, P_4, P_5$ and $P_6$ respectively.*

Proof is direct and hence left as an exercise to the reader.

In case of product in theorem 5 is replaced by n MOD natural neutrosophic usual zero dominated product then with appropriate changes;

this theorem can be proved.

Next we briefly describe MOD natural neutrosophic matrix subset semigroups under + and two types of products by examples.

***Example 5.15:*** Let P = {($a_1$, $a_2$, $a_3$) | $a_i \in S(Z_6^I)$, $1 \leq i \leq 3$, +} be the MOD natural neutrosophic subset matrix semigroup under +.

Let A = ({0, 3 + $I_0^6$, $I_2^6$}, {1, 4, $I_4^6$}, {$I_4^6 + I_2^6$}) and

B = ({2, 1, $I_0^6 + I_4^6$}, {$I_0^6, I_2^6$}, {$I_4^6, I_3^6$, 0}) $\in$ P.

We find A + B = ({0, 3 + $I_0^6$, $I_2^6$}, {1, 4, $I_4^6$}, {$I_4^6 + I_2^6$}) + ({2, 1, $I_0^6 + I_4^6$}, {$I_0^6, I_2^6$}, {$I_4^6, I_3^6$, 0})

= ({0, 3 + $I_0^6$, $I_2^6$} + {2, 1, $I_0^6 + I_4^6$}, {1, 4, $I_4^6$} + {$I_0^6, I_2^6$}, {$I_4^6 + I_2^6$} + {$I_4^6, I_3^6$, 0})

= ({2, 5 + $I_0^6$, 2 + $I_2^6$, 1, 4 + $I_0^6$, 1 + $I_2^6$, $I_0^6$ + $I_4^6$, $I_0^6$ + $I_4^6$ + 3, $I_0^6$ + $I_4^6$ + $I_2^6$}, {$I_0^6$ + 1, $I_0^6$ + 4, $I_0^6$ + $I_4^6$, $I_2^6$+1, $I_2^6$+4, $I_4^6 + I_2^6$}, {$I_4^6 + I_2^6$, $I_3^6$+ $I_2^6$ + $I_4^6$}). This is the way + operation is performed on P.

The reader is left with the task of finding MOD natural neutrosophic subset matrix subsemigroups.

***Example 5.16:*** Let

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in \langle Z_4 \cup I \rangle_I); \; 1 \leq i \leq 4, + \right\}$$

be the MOD natural neutrosophic - neutrosophic subset matrix semigroup under +.

This has MOD natural neutrosophic - neutrosophic subset matrix subsemigroup under +.

As this is considered as a matter of routine the reader is left with the task of finding them.

Also if A, B $\in$ W then one can find A + B.

***Example 5.17:*** Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in S(\langle Z_{12} \cup k\rangle_I);\; 1 \leq i \leq 9, + \right\}$$

be the MOD natural neutrosophic special quasi dual number subset matrix semigroup under +.

P has MOD natural neutrosophic subset matrix subsemigroup, this is left as an exercise to the reader.

In view of all these we give the following result.

**THEOREM 5.3:** *Let P = {Collection of all m × t subset matrices with entries from $S(Z_n^I)$ or $S(C^I(Z_n))$ or $S(\langle Z_n \cup I\rangle_I)$ or $(S(\langle Z_n \cup h\rangle_I)$ or $S(\langle Z_n \cup k\rangle_I)$ or $S(\langle Z_n \cup g\rangle_I))$, +} be the MOD natural neutrosophic subset matrix semigroup under +.*

*(i) $o(P) < \infty$.*

*(ii) P has MOD natural neutrosophic subset matrix subsemigroups under +.*

*(iii) P has MOD natural neutrosophic matrix subsets which are idempotents under +.*

The reader is left with the task of giving the proof.

Next we proceed onto describe both types of MOD natural neutrosophic subset matrix semigroups under the product by some more examples.

***Example 5.18:*** Let

$$P = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| a_i \in S(Z_{12}^{I});\ 1 \le i \le 6, \times_n\right\}$$

be the MOD natural neutrosophic subset matrix semigroup under usual zero dominated product that is $0 \times I_t^{12} = 0$.

$$\text{Let } A = \begin{pmatrix} \{3, 4, I_3^{12}\} & \{0, I_6^{12}\} & \{I_6^{12}, 6\} \\ \{0, I_6^{12} + I_8^{12}\} & \{I_9^{12}\} & \{0, I_0^{12}\} \end{pmatrix}$$

$$\text{and } B = \begin{pmatrix} \{0\} & \{1\} & \{I_0^{12}, 2\} \\ \{6\} & \{I_{10}^{12}, I_4^{12}\} & \{I_9^{12} + I_8^{12}\} \end{pmatrix} \in S.$$

$$\text{We find } A \times_n B = \begin{pmatrix} \{3, 4, I_3^{12}\} & \{0, I_6^{12}\} & \{I_6^{12}, 6\} \\ \{0, I_6^{12} + I_8^{12}\} & \{I_9^{12}\} & \{0, I_0^{12}\} \end{pmatrix}$$

$$\times_n \begin{pmatrix} \{0\} & \{1\} & \{I_0^{12}, 2\} \\ \{6\} & \{I_{10}^{12}, I_4^{12}\} & \{I_9^{12} + I_8^{12}\} \end{pmatrix}$$

$$= \begin{pmatrix} \{0\} & \{0, I_6^{12}\} & \{0, I_6^{12}, I_0^{12}\} \\ \{0, I_6^{12} + I_8^{12}\} & \{I_6^{12}, I_0^{12}\} & \{0, I_0^{12}\} \end{pmatrix}.$$

This is the way product operation is performed on S.

We see S has MOD natural neutrosophic matrix subset zero divisors, nilpotents and idempotents.

This task is left as an exercise to the reader.

Further this S has MOD natural neutrosophic matrix subset subsemigroups which are not ideals as well as subsemigroups which are ideals.

***Example 5.19:*** Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} \middle| \; a_i \in S(\langle Z_{10} \cup g\rangle_I);\ 1 \leq i \leq 14, \times_n \right\}$$

be the MOD natural neutrosophic dual number subset matrix semigroup under the MOD natural neutrosophic zero dominant product $I_0^g \times 0 = I_0^g$.

The task of finding MOD natural neutrosophic dual number subset matrix zero divisors nilpotents, idempotents ideals and subsemigroups are left as an exercise to the reader.

In view of all these we have the following result.

**THEOREM 5.4:** *Let D = {Collection of all m × t subset matrices with entries from $S(Z_n^I)$ or $(S(C^I(Z_n))$ or $S(\langle Z_n \cup I\rangle_I)$ or $(S(\langle Z_n \cup g\rangle_I)$ or $S(\langle Z_n \cup k\rangle_I)$ or $S(\langle Z_n \cup h\rangle_I))$, $\times_n$} be the MOD natural neutrosophic subset matrix semigroup under the natural zero dominated product or MOD natural neutrosophic zero dominated product.*

*Then (i) D has MOD natural neutrosophic subset matrix zero divisors, nilpotents and idempotents only if $n = p^s q$ ($s \geq 2$, p a prime) and for all n is case of $S(\langle Z_n \cup g\rangle_I)$ barring only MOD natural neutrosophic matrix subset idempotents.*

*(ii) D has MOD natural neutrosophic subset matrix subsemigroups which are not ideals.*

*(iii) D has MOD natural neutrosophic subset matrix subsemigroups which are ideals.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe with examples MOD natural neutrosophic matrix subset semigroups under + and ×.

***Example 5.20:*** Let S(M) = {Collection of all matrix subsets from

$$M = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right| a_i \in (Z_{10}^I);\ 1 \leq i \leq 3, +\}, +\right\}$$

be the MOD natural neutrosophic matrix subset semigroup under +.

$$\text{Let } A = \left\{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} I_0^{10} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ I_2^{10} + I_5^{10} \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ I_6^{10} \end{bmatrix}\right\} \text{ and}$$

$$B = \left\{\begin{bmatrix} 4 + I_2^{10} \\ 0 \\ I_8^{10} \end{bmatrix}, \begin{bmatrix} 4 \\ 6 + I_2^{10} \\ 0 \end{bmatrix}\right\} \in S(M).$$

$$A + B = \left\{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} I_0^{10} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ I_2^{10} + I_5^{10} \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ I_6^{10} \end{bmatrix}\right\} +$$

$$\left\{\begin{bmatrix} 4 + I_2^{10} \\ 0 \\ I_8^{10} \end{bmatrix}, \begin{bmatrix} 4 \\ 6 + I_2^{10} \\ 0 \end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 + I_2^{10} \\ 0 \\ I_8^{10} \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 + I_2^{10} \\ 0 \end{bmatrix}, \begin{bmatrix} I_0^{10} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 + I_2^{10} \\ 0 \\ I_8^{10} \end{bmatrix},\right.$$

$$\begin{bmatrix} I_0^{10} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 + I_2^{10} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ I_2^{10} + I_5^{10} \end{bmatrix} + \begin{bmatrix} 4 + I_2^{10} \\ 0 \\ I_8^{10} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ I_2^{10} + I_5^{10} \end{bmatrix} +$$

$$\left.\begin{bmatrix} 4 \\ 6 + I_2^{10} \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ I_6^{10} \end{bmatrix} + \begin{bmatrix} 4 + I_2^{10} \\ 0 \\ I_8^{10} \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ I_6^{10} \end{bmatrix} + \begin{bmatrix} 4 \\ 6 + I_2^{10} \\ 0 \end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix} 7 + I_2^{10} \\ 0 \\ 1 + I_8^{10} \end{bmatrix}, \begin{bmatrix} 7 \\ 6 + I_2^{10} \\ 1 \end{bmatrix}, \begin{bmatrix} 4 + I_0^{10} + I_2^{10} \\ 0 \\ I_8^{10} \end{bmatrix},\right.$$

$$\begin{bmatrix} 4 + I_0^{10} \\ 6 + I_2^{10} \\ 0 \end{bmatrix}, \begin{bmatrix} 4 + I_2^{10} \\ 0 \\ I_2^{10} + I_5^{10} + I_8^{10} \end{bmatrix}, \begin{bmatrix} 4 \\ 6 + I_2^{10} \\ I_2^{10} + I_5^{10} \end{bmatrix},$$

$$\left.\begin{bmatrix} 9+I_2^{10} \\ 2 \\ I_6^{10}+I_8^{10} \end{bmatrix}, \begin{bmatrix} 9 \\ 8+I_2^{10} \\ I_6^{10} \end{bmatrix}\right\}.$$

This is the way + operation is performed on S(M).

The reader is left with the task of finding MOD natural neutrosophic matrix subset subsemigroups.

***Example 5.21:*** Let S(P) = {Collection of all matrix subsets from

$$P = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_i \in S(\langle Z_9 \cup g\rangle_I);\ 1 \leq i \leq 6, +\}, +\right\}$$

be the MOD natural neutrosophic dual number matrix subsets semigroup under +.

$$\text{Let } A = \left\{\begin{pmatrix} 3 & I_0^g & 4 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} I_3^g & I_{3g}^g & 0 \\ 5+I_0^g & 0 & 7 \end{pmatrix}, \begin{pmatrix} I_0^g & I_{3g+6}^g & I_{2g}^g \\ 0 & 2 & 4 \end{pmatrix}\right\}$$

and

$$B = \left\{\begin{pmatrix} I_g^g & 0 & 7 \\ 7g+3 & 1 & I_{2g}^g+I_{8g}^g \end{pmatrix}, \begin{pmatrix} 0 & 1 & 5 \\ I_0^g & I_{5g}^g & I_{8g}^g+3 \end{pmatrix}, \begin{pmatrix} I_{7g}^g & 1 & 1 \\ 1 & I_{6g}^g & 0 \end{pmatrix}\right\}$$

$\in$ S(P).

The reader is expected to find A +B. Further prove;

$$\{(0)\} = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

acts as the additive identity of S(P); that is A + {(0)} = A for all A ∈ S(P).

Find atleast 3 MOD natural neutrosophic subset matrix subsemigroups of S(P).

***Example 5.22:*** Let S(V) = {Collection of all matrix subsets from

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| \; a_i \in C^I(Z_{15});\ 1 \leq i \leq 10, +\}, +\right\}$$

be the MOD natural neutrosophic subset matrix finite complex number semigroup under +.

Find MOD natural neutrosophic finite complex number subset matrix subsemigroups.

Find o(S(V)).

In view of all these we have the following theorem.

**THEOREM 5.5:** *Let S(W) = {Collection of all matrix subsets from* *$W = \{M = (m_{ij})_{s\times t} \mid m_{ij} \in Z_n^I$ or $(C^I(Z_n))$ or $\langle Z_n \cup I\rangle_I$ or $\langle Z_n \cup g\rangle_I)$ or $\langle Z_n \cup k\rangle_I$ or $\langle Z_n \cup h\rangle_I$, $1 \leq i \leq s$, $1 \leq j \leq t$; $+\}$ be the MOD natural neutrosophic subset matrix semigroup under +.*

*(i) $o(S(W)) < \infty$.*
*(ii) S(W) has several MOD natural neutrosophic matrix subset subsemigroups.*

Next we describe by an example or two the notion of MOD matrix subset natural neutrosophic semigroups under zero dominant product as well as MOD natural neutrosophic zero dominant product.

***Example 5.23:*** Let S(B) = {Collection of all matrix subsets from

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \middle| \, a_i \in Z^I_{14} ; 1 \leq i \leq 8, \times_n \right\}, \times_n \}$$

be the MOD natural neutrosophic subset MOD natural neutrosophic matrix subset semigroup under zero dominant product.

$$\text{Let } P = \left\{ \begin{pmatrix} 7 & 0 & I_0^{14} & I_2^{14} \\ 0 & I_7^{14} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 \\ I_2^{14} & I_6^{14} & 0 & I_8^{14} + I_6^{14} \end{pmatrix}, \begin{pmatrix} 0 & 0 & I_0^{14} & I_2^{14} \\ 6 & 7 & 0 & I_8^{14} \end{pmatrix} \right\} \text{ and}$$

$$R = \left\{ \begin{pmatrix} 13 & I_8^{14} + I_6^{14} & 0 & 7 \\ I_6^{14} & I_2^{14} + 5 & I_0^{14} & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 & I_6^{14} + I_0^{14} + I_8^{14} & 0 \\ I_2^{14} & 0 & 0 & I_6^{14} \end{pmatrix} \right\} \in S(B).$$

The reader is assigned the task of finding A $\times_n$ A and B $\times_n$ A.

Further find MOD natural neutrosphic matrix subset zero divisors and idempotents.

Find also MOD natural neutrosophic matrix subset ideals and subsemigroups.

***Example 5.24:*** Let S(M) = {Collection of all subset matrix from

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in \langle Z_{10} \cup h \rangle_I,\ 1 \leq i \leq 6,\ \times_n \right\}\ \times_n \}$$

be the MOD natural neutrosophic special dual like number subset matrix semigroup under the MOD natural neutrosphic zero dominant product $\times_n$.

Find MOD subset matrix natural zero divisors and idempotents of S(M).

In view of all these we have the following result.

**THEOREM 5.6:** *Let S(W) = {Collection of all matrix subsets from W = {Collection of all s × t matrices with entries from $Z_n^I$ or $C^I(Z_n)$ or $\langle Z_n \cup I \rangle_I$ or $\langle Z_n \cup g \rangle_I$ or $\langle Z_n \cup k \rangle_I$ or $\langle Z_n \cup h \rangle_I$ ; $\times_n$}, $\times_n$} be the MOD natural neutrosophic subset matrix or MOD natural neutrosophic finite complex number subset matrix or MOD natural neutrosophic - neutrosophic subset matrix or MOD natural neutrosophic dual number subset matrix or MOD natural neutrosophic special dual like number subset matrix or MOD natural neutrosophic special quasi dual number subset matrix semigroup under zero dominated product or MOD dominated semigroup respectively.*

*(i) Then o(S(W)) < ∞.*

*(ii) S(W) has MOD subset matrix subsemigroups which are not ideals as well as subsemigroups which are ideals.*

*(iii) S(W) has MOD subset matrix zero divisor, nilpotents and idempotents for appropriate n.*

The proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe briefly the concept of MOD natural neutrosophic number subset coefficient polynomial semigroups under + and two types of products by examples.

***Example 5.25:*** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_{10}^{I}), + \right\}$$

be the MOD natural neutrosophic subset coefficient polynomial semigroup under +.

Let $p(x) = \{0, 4, 5 + I_5^{10}\}x^3 + \{I_2^{10} + 4, 2\}$

and $q(x) = \{6, 4, 3, I_2^{10} + I_6^{10} + 5\}x^2 + \{4, 6+ I_6^{10}\} \in S[x]$.

$$p(x) + q(x) = \{0, 4, 5, I_5^{10}\}x^3 + \{I_2^{10} + 4, 2\} + \{6, 4, 3, I_2^{10} + I_6^{10} + 5\}x^2 + \{4, 6 + I_6^{10}\}$$

$= \{0, 4, 5, I_5^{10}\}x^3 + \{6, 4, 3, I_2^{10} + I_6^{10} + 5\}x^2 + \{6, 8 + I_2^{10}, 8 + I_6^{10}, I_2^{10} + I_6^{10}\} \in S[x]$.

This is the way + operation is performed on S[x].

The reader is left with the task of finding MOD natural neutrosophic subset subsemigroups of finite order.

Further the reader is expected to prove all ideals of S[x] have infinite cardinality.

We prove S[x] has no nilpotents or zero divisors or non trivial idempotents.

Let p(x) = {0, 5}$x^3$ + {5} and

q(x) = {2, 4, 0, 6}$x^4$ + {6, 8, 2}$x^2$ + {0, 2, 4, 6, 8} ∈ S[x].

Clearly p(x) + q(x) ∈ S[x].

***Example 5.26:*** Let W[x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{20} \cup g\rangle_I);\ + \right\}$ be the MOD natural neutrosophic dual number subset coefficient polynomial semigroup under +.

Enumerate all properties enjoyed by W[x].

We give the following result.

**THEOREM 5.7:** *Let S[x] =* $\left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_n^I) \text{ or } S(C^I(Z_n)) \text{ or } S(\langle Z_n \cup I\rangle_I) \text{ or } S(\langle Z_n \cup g\rangle_I) \text{ or } S(\langle Z_n \cup h\rangle_I) \text{ or } S(\langle Z_n \cup k\rangle_I);\ + \right\}$ *be the MOD natural neutrosophic subset coefficient polynomial semigroup or MOD natural neutrosophic finite complex number subset coefficient polynomial semigroup or MOD natural neutrosophic - neutrosophic subset coefficient polynomial semigroup or MOD natural neutrosophic dual number coefficient polynomial semigroup or MOD natural neutrosophic special dual like number subset coefficient polynomial semigroup or MOD natural neutrosophic special quasi dual number subset coefficient polynomial semigroup respectively.*

*(i) o(S[x]) =∞.*
*(ii) S[x] has MOD natural neutrosophic subset coefficient polynomial subsemigroups of finite order.*

Next we proceed onto describe MOD natural neutrosophic subset polynomial semigroups under both the products, zero dominated product as well as MOD natural neutrosophic zero dominated product by some examples.

***Example 5.27:*** Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(C^I(Z_{12})); \times \right\}$$

be the MOD natural neutrosophic finite complex number coefficient polynomial subset semigroup under zero dominated product. S[x] has MOD natural neutrosophic finite complex number subset coefficient polynomial zero divisors and nilpotents but has no nontrivial idempotents.

However all MOD natural neutrosophic subset polynomial subsemigroups and ideals are infinite order only. All these are left as an exercise for the reader to verify.

***Example 5.28:*** Let $P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{13} \cup g\rangle_I); \times \right\}$ be the MOD natural neutrosophic dual number subset coefficient polynomial semigroup under MOD natural neutrosophic zero dominant product.

P[x] has infinite number of MOD natural neutrosophic subset polynomial zero divisors and nilpotents.

However P[x] has non trivial idempotents.

In view of this we prove the following result.

**THEOREM 5.8:** *Let* $B[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_n^I) \text{ or } S(C^I(Z_n)) \right.$ *or* $S(\langle Z_n \cup I\rangle_I)$ *or* $S(\langle Z_n \cup g\rangle_I)$ *or* $S(\langle Z_n \cup h\rangle_I)$ *or* $\left. S(\langle Z_n \cup k\rangle_I); \times \right\}$ *be the MOD natural neutrosophic subset coefficient polynomial semigroup of any one of the coefficient subsets semigroup under* *the zero dominant product or MOD natural neutrosophic dominant product.*

*(i) B[x] is of infinite order.*
*(ii) B[x] has no nontrivial idempotents.*
*(iii) B[x] has MOD natural neutrosophic subset coefficient nilpotents or zero divisors which mainly depend on n.*
*(iv) B[x] has all MOD natural neutrosophic subset coefficient subsemigroup or ideals are only of infinite order.*

Proof is direct and hence left as an exercise to the reader.

**Corollary 5.1:** If in theorem the set $S(\langle Z_n \cup g\rangle_I)$ is taken then the MOD natural neutrosophic dual number coefficient polynomials has more number of nilpotents and zero divisors.

The proof is also left as an exercise to the reader.

Next we will briefly describe MOD natural neutrosophic subset polynomials and define the + and two types of product operations by examples.

***Example 5.29:*** Let S(P[x]) = {Collection of all subsets from $P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_{16}^I, +\right\}, +\}$ be the MOD natural neutrosophic polynomial subset semigroup.

Let

$$A = \{3x^3 + (I_0^{16} + I_2^{16})x + I_4^{16},\ I_0^{16} + I_8^{16}x^2,\ 5x^2 + 10x + 3\} \text{ and}$$

$$B = \{5x^7 + (10 + I_6^{16})x+4,\ 8 + 5x + I_8^{16}x^2\} \in S(P[x]).$$

We can find A + B, which is a matter of routine and this task is left as an exercise to the reader.

Infact S(P[x]) is of infinite order.

This has subsemigroup both finite and infinite order.

***Example 5.30***: Let S(R[x]) = {Collection of all subsets from

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \langle Z_{12} \cup h\rangle_I), +\}; +\}$$

be the MOD natural neutrosophic special dual like number polynomial subset semigroup under +.

All properties can be derived about S(R[x]) as a matter of routine.

In view of all these we have the following result.

**THEOREM 5.9:** *Let S(B[x]) = {Collection of all subset polynomial from*

$$B[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_n^I \text{ or } C^I(Z_n) \text{ or } \langle Z_n \cup I\rangle_I \text{ or } \langle Z_n \cup g\rangle_I \text{ or } \right.$$

$$\langle Z_n \cup h\rangle_I) \text{ or } \langle Z_n \cup k\rangle_I;\ +\}\ +\}$$

*be the MOD natural neutrosophic subset polynomial semigroup of any one of the six types.*

*S(B[x]) has MOD natural neutrosophic subset polynomial subsemigroups of both finite and infinite order.*

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto give examples of MOD natural neutrosophic polynomials subset semigroups under the product in which 0 is dominated and another product is which the MOD natural neutrosophic zero is dominated by the following example.

***Example 5.31:*** Let S(P) = {Collection of all polynomial subsets from

$$P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \langle Z_{10} \cup k\rangle_I; \times\}, \times\right\}$$

be the MOD natural neutrosophic special quasi dual number polynomial subset semigorup under the usual zero dominated product.

o(S(P[x])) = ∞. Further all MOD natural neutrosophic special quasi dual number polynomial subset subsemigroups or ideals are of infinite order.

This S(P[x]) has MOD natural neutrosophic special quasi dual number polynomial subset zero divisors. However S(P[x]) has no idempotents.

Let A = $\{5x^3 + (I_5^k + I_0^k)x, (6 + I_0^k) + (4 + I_2^k)x^2, I_0^k x^3\}$ and B = $\{(I_0^k + I_{5k}^k)x^3, (I_{5k+2}^k + 3) + (I_{2+5k}^k + I_{4+8k}^k + I_0^k + I_6^k)x^2]$ ∈ S(P[x]).

It is a matter of routine to find A × B, so is left as an exercise for the reader.

***Example 5.32:*** Let S(T[x]) = {Collection of all polynomial subsets from

$$T[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in C^I(Z_{27}), \times\}, \times\right\}$$

be the MOD natural neutrosophic finite complex number subset polynomial semigroup under × with usual zero dominated product.

The reader is left with the task of finding all related properties of S(T[x]).

In view of all these we have the following theorem the proof of which is left as an exercise to the reader.

**THEOREM 5.10:** *Let S(P[x]) = {Collection of all polynomial subsets from P[x] =* $\left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in Z_n^I \text{ or } C^I(Z_n)\ \langle Z_n \cup I\rangle_I \text{ or } \langle Z_n \cup g\rangle_I \text{ or } \langle Z_n \cup h\rangle_I \text{ or } \langle Z_n \cup k\rangle_I\right.$ *under zero dominated product × or the MOD natural neutrosophic zero dominated product}, ×} be the MOD natural neutrosophic subset polynomial semigroup under × defined over any of the six sets (* $Z_n^I$ *or* $C^I(Z_n)$ *and so on) then*

*(i) o(S(P[x])) = ∞.*
*(ii) S(P[x]) is a commutative monoid.*
*(iii) S(P[x]) has MOD natural neutrosophic subset polynomial subsemigroups as well as ideals all of them are only of infinite order.*
*(iv) S(P[x]) has MOD natural neutrosophic zero divisors and nilpotents for appropriate n.*

Next we proceed onto describe MOD natural neutrosophic subset polynomial semigroup of finite order by examples.

***Example 5.33:*** Let $S(P[x]_8)$ = {Collection of all polynomial subsets from $P[x]_8 = \left\{\sum_{i=0}^{8} a_i x^i \;\middle|\; a_i \in \langle Z_{10} \cup g\rangle_I);\ x^9 = 1, +\right\}, +\}$ be the MOD natural neutrosophic dual number polynomial subset semigroup under + of finite order.

$o(S(P[x]_8)) < \infty$. $S(P[x]_8)$ has MOD natural neutrosophic dual number polynomial subset subsemigroup. Infact $S(P[x]_8)$ is a commutative finite monoid.

***Example 5.34:*** Let $S(R[x]_3)$ = {Collection of all polynomial subset from $R[x]_3 = \left\{\sum_{i=0}^{3} a_i x^i \;\middle|\; a_i \in C^I(Z_{12});\ x^4 = 1, +\right\}, +\}$ be the

MOD natural neutrosophic finite complex number polynomial subset semigroup under + of finite order.

Find all subsemigroups of $S(R[x]_3)$.

In view of all these we have the following result.

**THEOREM 5.11:** *Let $S(P[x]_m)$ = {Collection of all polynomial subsets from*

$$P[x]_m = \left\{ \sum_{i=0}^{m} a_i x^i \;\middle|\; a_i \in Z_n^I \text{ or } C^I(Z_n) \text{ or } \langle Z_n \cup I\rangle_I \text{ or } \langle Z_n \cup g\rangle_I \text{ or } \right.$$

$$\left. \langle Z_n \cup h\rangle_I \text{ or } \langle Z_n \cup k\rangle_I \; x^{m+1} = 1, +\right\}, +\}$$

*be the MOD natural neutrosophic polynomial subsets semigroup of any one (taking entries from $Z_n^I$ or $C^I Z_n$ or so on $\langle Z_n \cup k\rangle_I$) under +,*

*(i) $o(S(P[x]_m) < \infty$ and is a finite commutative monoid.*
*(ii) $S(P[x]_m)$ has several MOD natural neutrosophic polynomial subset subsemigroups.*

Proof is direct and hence left as an exercise to the reader.

Next we describe MOD natural neutrosophic polynomial subset semigroup of finite order under both the types of operations by examples.

***Example 5.35:*** Let $S(P[x]_8)$ = {Collection of all polynomial subset from the set $P[x]_8 = \left\{ \sum_{i=0}^{8} a_i x^i \;\middle|\; a_i \in Z_{18}^I,\ x^9 = 1, \times \right\} \times\}$ be the MOD natural neutrosophic polynomial subset semigroup of finite order under zero dominant product.

$S(P[x]_8)$ has MOD natural neutrosophic polynomial subset zero divisors and nilpotents.

This has ideals and subsemigroups which are not ideals.

The reader is left with the task of finding them.

***Example 5.36:*** Let $S(B[x]_5)$ = {Collection of all polynomial subsets from

$$B[x]_5 = \left\{\sum_{i=0}^{6} a_i x^i \,\middle|\, a_i \in \langle Z_{10} \cup g\rangle_I,\ x^6 = 1, \times\right\}, \times\}$$

be the MOD natural neutrosophic dual number polynomial subset semigroup under MOD natural neutrosophic dual number zero product.

Find all MOD natural neutrosophic dual number subsemigroups and ideals.

This has lots of MOD subset zero divisors.

Inview of this we have the following results.

**THEOREM 5.12:** *Let $S(M[x]_m)$ = {Collection of all polynomial subsets from*

$$M[x]_m = \left\{\sum_{i=0}^{m} a_i x^i \,\middle|\, a_i \in Z_n^I,\ x^{m+1} = 1 \text{ or } C^I(Z_n) \text{ or } \langle Z_n \cup I\rangle_I \text{ or } \langle Z_n \cup h\rangle_I \text{ or } \langle Z_n \cup k\rangle_I \text{ or } \langle Z_n \cup g\rangle_I, \times\right\}, \times\}$$

*be the MOD natural neutrosophic polynomial subsets semigroup using any of $Z_n^I$ (or $C^I(Z_n)$ or $\langle Z_n \cup I\rangle_I$ and so on) under zero dominant product or under MOD natural neutrosophic zero dominant product.*

*(i) $o(S(M[x]_m)) < \infty$.*
*(ii) This $S(M[x]_m)$ has MOD ideals and subsemigroups.*

*(iii) $S(M[x]_m)$ has MOD natural neutrosophic polynomial subset zero divisors and nilpotents.*

Proof is direct and hence left as an exercise to the reader.

Thus we have given only a few examples and results however the reader is left with the task of finding more properties of them.

All these are suggested by the following problems some of which are difficult, some are at research level and some of them are simple exercises.

**Problems:**

1. Let B = $\{S(Z_{60}^{I}), +\}$ be the MOD natural neutrosophic subset semigroup under +.

   (i) Find $o(S(Z_{60}^{I}))$.
   (ii) Find all MOD natural neutrosophic subset idempotents under + of B.
   (iii) Find all MOD natural neutrosophic subset subsemigroups of B.

2. Let A = $\{S(Z_{47}^{I}), +\}$ be the MOD natural neutrosophic subset semigroup under +.

   Study questions (i) to (iii) of problem (1) for this A.

3. Let D = $\{S(Z_{128}^{I}), +\}$ be the MOD natural neutrosophic subset semigroup under +.

   Study questions (i) to (iii) of problem (1) for this D.

4. Let M = $\{S(C^{I}(Z_{12})), +\}$ be the MOD natural neutrosophic finite complex number subset semigroup under +.

(i) Study questions (i) to (iii) of problem (1) for this M.
(ii) Compare M with P = $\{S(Z_{12}^{I}), +\}$.

5. Let N = $\{S(C^{I}(Z_{43})), +\}$ be the MOD natural neutrosophic finite complex number subset semigroup under +.

   (i) Study questions (i) to (iii) of problem (1) for this N.
   (ii) Compare this N with M of problem 4.

6. Let P = $S(\langle Z_{10} \cup I\rangle_I), +\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup under +.

   Study questions (i) to (iii) of problem (1) for this P.

7. Let V = $S(\langle Z_{23} \cup I\rangle_I), +\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup under +.

   (i) Study questions (i) to (iii) of problem (1) for this V.
   (ii) Compare V with this P of problem 6.

8. Let E = $S(\langle Z_{19} \cup g\rangle_I), +\}$ be the MOD natural neutrosophic dual number subset semigroup under +.

   Study questions (i) to (iii) of problem (1) for this E.

9. Let F = $S(\langle Z_{48} \cup g\rangle_I), +\}$ be the MOD natural neutrosophic dual number subset semigroup.

   (i) Study questions (i) to (iii) of problem (1) for this F.
   (ii) Compare E of problem 8 for this F.

10. Let G = $S(\langle Z_{244} \cup k\rangle_I), +\}$ be the MOD natural neutrosophic quasi dual number subset semigroup under +.

    Study questions (i) to (iii) of problem (1) for this G.

11. Let H = $S(\langle Z_{53} \cup k\rangle_I), +\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under +.

(i) Study questions (i) to (iii) of problem (1) for this H.
(ii) Compare H with G of problem 10.

12. Let J = S(⟨$Z_{113}$ ∪ h⟩$_I$), +} be the MOD subset natural neutrosophic special dual like number subset semigroup under +.

    Study questions (i) to (iii) of problem (1) for this J.

13. Let K = {S(⟨$Z_{48}$ ∪ h⟩$_I$), +} be the MOD subset natural neutrosophic special dual like number subset semigroup under +.

    (i) Study questions (i) to (iii) of problem (1) for this K.
    (ii) Compare J of problem (12) with this K.

14. Enumerate all special features associated with MOD natural neutrosophic subset semigroups under +.

15. Let T = {S($Z_{24}^I$), ×} where × is zero dominated product MOD natural neutrosophic subset semigroup.

    (i) Find o(T).
    (ii) Enumerate all MOD subset zero divisors.
    (iii) Prove T have ideals find all MOD subset natural neutrosophic ideals of T.
    (iv) Find all MOD natural neutrosophic subsemigroup of T which are not ideals of T.
    (v) List out all MOD natural neutrosophic nilpotents subsets of T.
    (vi) List out all MOD natural idempotent subsets of T (if any find out).

16. Let P = {S($Z_{29}^I$), ×} be the MOD natural neutrosophic zero dominated product MOD natural neutrosophic semigroup.

    Study questions (i) to (vi) of problem (15) for this P.

17. Let M = $\{S(C^I(Z_{40})), \times\}$ be the MOD natural neutrosophic finite complex number zero dominated product semigroup.

    Study questions (i) to (vi) of problem (15) for this M.

18. Let W = $\{S(C^I(Z_{59})), \times\}$ be the MOD natural neutrosophic finite complex number zero dominated product semigroup.

    Study questions (i) to (vi) of problem (15) for this W.

19. Let S = $\{S(C^I(Z_{42})), \times\}$ be the MOD natural neutrosophic finite complex number zero dominated product semigroup.

    (i) Study questions (i) to (vi) of problem (15) for this S.
    (ii) Compare S with M and W of problem 17 and 18 respectively.

20. Let W = $\{S(\langle Z_{42} \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic neutrosophic subset semigroup under usual zero dominated product.

    Study questions (i) to (vi) of problem (15) for this W.

21. Let T = $\{S(\langle Z_{193} \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup under usual zero dominated product.

    (i) Study questions (i) to (vi) of problem (15) for this T.
    (ii) Compare T with W of problem 20.

22. Let E = $\{S(\langle Z_{26} \cup g\rangle_I), \times\}$ be the MOD natural neutrosophic dual number subset semigroup which is usual zero dominated.

    Study questions (i) to (vi) of problem (15) for this E.

23. Let F = $\{S(\langle Z_{53} \cup g\rangle_I), \times\}$ be the MOD natural neutrosophic dual number MOD natural neutrosophic zero dominated subset semigroup.

(i) Study questions (i) to (vi) of problem (15) for this F.
(ii) Compare F with E in problem 22.

24. Let Z = $\{S(\langle Z_{24} \cup h\rangle_I), \times\}$ be the MOD natural neutrosophic special dual like number subset semigroup under the usual zero dominated product.

(i) Study questions (i) to (vi) of problem (15) for this Z.
(ii) If in Z the zero dominated product is replaced by MOD natural neutrosophic zero dominated product compare them.

25. Let K = $\{S(\langle Z_{19} \cup k\rangle_I), \times\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under usual zero divisor dominated product.

Study questions (i) to (vi) of problem (15) for this K.

26. Let V = $\{S(\langle Z_{424} \cup k\rangle_I), \times\}$ be the MOD natural neutrosophic special quasi dual number subset semigroup under MOD natural neutrosophic zero dominated product.

(i) Study questions (i) to (vi) of problem (15) for this V.
(ii) Compare K of problem 25 with this V.

27. Let N = $\{S(\langle Z_n \cup I\rangle_I), \times\}$ be the MOD natural neutrosophic - neutrosophic subset semigroup.

Enumerate all properties associated with N.

28. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i \in S(Z_{24}^I); 1 \leq i \leq 12; +\right\}$

be the MOD natural neutrosophic subset matrix semigroup under +.

(i) Study the special features of this semigroup.
(ii) Find all subsemigroup of M.
(iii) Find o(M).

29. Let $Y = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| a_i \in S(C^I(Z_{47})). + \right\}$

be the MOD natural neutrosophic finite complex number subset semigroup under +.

Study questions (i) to (iii) of problem (28) for this Y.

30. Let $Y = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \middle| a_i \in S(\langle Z_{16} \cup I\rangle_I), 1 \leq i \leq 10; + \right\}$

be the MOD natural neutrosophic - neutrosophic matrix subset semigroup under +.

Study questions (i) to (iii) of problem (28) for this C.

31. Enumerate all special properties enjoyed by MOD natural neutrosophic - neutrosophic matrix subset semigroups under +.

32. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle| a_i \in S(\langle Z_{10} \cup k\rangle_I), 1 \leq i \leq 12; + \right\}$

be the MOD natural neutrosophic special quasi dual number subset matrix semigroup under +.

Study questions (i) to (iii) of problem (28) for this M.

33. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i \in S(\langle Z_{42} \cup g\rangle_I), 1 \leq i \leq 12; +\right\}$

be the MOD natural neutrosophic dual number matrix subset semigroup under +.

Study questions (i) to (iii) of problem (28) for this T.

34. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix} \middle| a_i \in S(\langle Z_{28} \cup h\rangle_I), 1 \leq i \leq 20; +\right\}$

be the MOD natural neutrosophic special dual like number matrix semigroup.

Study questions (i) to (iii) of problem (28) for this B.

35. Let D = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in S(Z_{48}^I); 1 \leq i \leq 5; \times\}$

be the MOD natural neutrosophic subset matrix semigroup under zero dominated product.

(i) Find o(P).
(ii) Find all MOD natural neutrosophic subset matrix subsemigroups which are not ideals.

(iii) Find all MOD natural neutrosophic subset matrix subsemigroup which are ideals.
(iv) Find all MOD zero divisors of D.
(v) Find all nilpotents of D (if any).
(vi) Find all idempotents of D (if any).
(vii) Find any other special features associated with D.

36. Let $E = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in S(C^I(Z_{47})); 1 \leq i \leq 6; \times_n \right\}$

be the MOD natural neutrosophic subset matrix MOD natural neutrosophic zero dominated finite complex number semigroup under natural product $\times_n$.

Study questions (i) to (vii) of problem (35) for this E.

37. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| a_i \in S(\langle Z_{216} \cup g\rangle_I), 1 \leq i \leq 16; \times_n \right\}$

be the MOD natural neutrosophic dual number matrix subset semigroup under any of the product.

(i) Study questions (i) to (vii) of problem (35) for this B.
(ii) Compare B with both the products.

38. Let $P = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix} \middle| a_i \in S(\langle Z_{12} \cup h\rangle_I), 1 \leq i \leq 15; \times_n \right\}$

be the MOD natural neutrosophic special dual like number subset matrix semigroup.

Study questions (i) to (vii) of problem (35) for this P.

39. Let W = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} \middle| a_i \in S(\langle Z_{144} \cup k\rangle_I);\ 1 \leq i \leq 14;\ \times_n \right\}$

be the MOD natural neutrosophic special quasi dual number subset matrix semigroup under natural product $\times_n$.

Study questions (i) to (vii) of problem (35) for this W.

40. Let S(M) = {Collection of all subsets from

M = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in Z_{24}^I;\ 1 \leq i \leq 5;\ + \right\}, +\}$

be the MOD natural neutrosophic matrix subsets (from M) semigroup under +.

(i) Find o(S(M)).
(ii) Find all MOD natural neutrosophic subset subsemigroup.
(iii) Find all MOD natural neutrosophic subset elements which are idempotent subsets under +.

41. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i \in (C^I(Z_{20}));\right.$

    $1 \leq i \leq 12; +\}$ be the MOD natural neutrosophic semigroup under +.

    Define S(V) = {Collection of all matrix subsets from V; +} the MOD natural neutrosophic finite complex number subset matrix semigroup under +.

    Study questions (i) to (iii) of problem (40) for this S(V).

42. Let S(W) = {Collection of all subsets from

    $W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i \in S\langle Z_{20} \cup I\rangle_I; 1 \leq i \leq 4; +\right\}, +\}$ be the

    MOD natural neutrosophic - neutrosophic matrix subset semigroup under +.

    Study questions (i) to (iii) of problem (40) for this S(W).

43. Let S(B) = {Collection of all subsets from

    $B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \middle| a_i \in \langle Z_{13} \cup g\rangle_I; 1 \leq i \leq 8; +\right\} +\}$

    be the MOD natural neutrosophic - neutrosophic matrix subset semigroup under +.

    Study questions (i) to (iii) of problem (40) for this S(B).

44. Let T = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{bmatrix} \middle| \; a_i \in \langle Z_{24} \cup k\rangle_I; 1 \leq i \leq 21; +\right\}$

be the MOD natural neutrosophic special quasi dual number semigroup under +.

S(T) = {Collection of all subset matrices from T, +} be the MOD natural neutrosophic special quasi dual number subset matrix semigroup under +.

Study questions (i) to (iii) of problem (40) for this T.

45. Let S(M) = {Collection of all matrix subsets from

M = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in Z^I_{12}; 1 \leq i \leq 12; \times_n\right\}; \times_n\}$

be the MOD natural neutrosophic matrix subsets semigroup under $\times_n$.

(i) Find o(S(M)).
(ii) Find all MOD natural neutrosophic zero divisors.
(iii) Find all MOD natural neutrosophic nilpotents.
(iv) Prove this has MOD natural neutrosophic idempotents.
(v) How many MOD natural neutrosophic subsemigroup which are not ideals?

(vi) How many MOD natural neutrosophic ideals exist in S(M)?
(vii) Obtain any other special property associated with S(M).

46. Let S(D) = {Collection of matrix subset from

$$D = \left\{ \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \right| a_i \in \langle Z_{27} \cup g \rangle_I;\ 1 \leq i \leq 24; \times_n \right\}; \times_n \}$$

be the MOD natural neutrosophic dual number matrix subset semigroup under $\times_n$.

Study questions (i) to (vii) of problem (45) for this S(D).

47. Let S(D) = {Collection of all matrix subsets from

$$P = \left\{ \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right| a_i \in \langle Z_{12} \cup I \rangle_I;\ 1 \leq i \leq 6; \times_n \right\}; \times_n \}$$

be the MOD natural neutrosophic - neutrosophic subset matrix semigroup.

Study questions (i) to (vii) of problem (45) for this S(P).

48. Let S(Z) = {Collection of all matrix subsets from

$$Z = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in \langle Z_5 \cup k\rangle_I;\; 1 \leq i \leq 4; \times_n \text{ (or } \times)\right\};$$

$\times_n$(or $\times$)}

be the MOD natural neutrosophic special quasi dual number matrix subsets semigroup under natural product or usual product.

(i) Study questions (i) to (vii) of problem (45) for this S(Z) under both the natural product or usual product.
(ii) Prove {S(Z), $\times$} is a non commutative finite monoid.
(iii) Show {S(Z), $\times$} has rights zero divisors which are not left zero divisors and vice versa.
(iv) Find in {S(Z), $\times$} right ideals which are not left ideals and vice versa.

49. Let $S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_{20}^I);\; + \right\}$

be the MOD natural neutrosophic subset coefficient polynomial semigroup under +.

(i) Prove S[x] has MOD natural neutrosophic subset coefficient polynomial subsemigroups of finite order.
(ii) Find idempotents under + in S[x].

50. Let $P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(C^I(Z_{44}));\; + \right\}$

be the MOD natural neutrosophic finite complex number subset coefficient polynomial semigroup under +.

Study questions (i) to (ii) of problem (49) for this P[x].

51. Let $M[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_9 \cup g\rangle_I); +\right\}$

be the MOD natural neutrosophic dual number subset coefficient polynomial semigroup.

Study questions (i) to (ii) of problem (49) for this M[x].

52. Let $V[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{43} \cup I\rangle_I); +\right\}$

be the MOD natural neutrosophic - neutrosophic subset coefficient polynomial semigroup under +.

Study questions (i) to (ii) of problem (49) for this V[x].

53. Let $P[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(Z_{48}^I); \times\right\}$

be the MOD natural neutrosophic subset coefficient polynomial semigroup under zero dominant product of MOD natural neutrosophic zero dominant product.

(i) Prove all MOD natural neutrosophic subset ideals and subsemigroups of P[x] are of infinite order.
(ii) Does P[x] have MOD natural neutrosophic subset coefficient polynomial zero divisors?
(iii) Does P[x] have MOD natural neutrosophic subset coefficient polynomial idempotent?
(iv) Can P[x] have MOD natural neutrosophic subset polynomial coefficient nilpotents?
(v) Enumerate all special features enjoyed by P[x].

54. Let $M[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(\langle Z_{93} \cup g\rangle_I); \times\right\}$

be the MOD natural neutrosophic dual number subset coefficient polynomial semigroup.

Study questions (i) to (v) of problem (53) for this M[x].

55. Let $B[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S(\langle Z_{14} \cup I\rangle_I); \times \right\}$

be the MOD natural neutrosophic - neutrosophic subset coefficient polynomial semigroup.

Study questions (i) to (v) of problem (53) for this B[x].

56. Let $T[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S(\langle Z_{43} \cup k\rangle_I); \times \right\}$

be the MOD natural neutrosophic subset coefficient polynomial semigroup of special quasi dual numbers.

Study questions (i) to (v) of problem (53) for this T[x].

57. Let S(P[x]) = {Collection of all subset polynomials from

$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S(C^I(Z_{284}), +\}; \; +\right\}$

be the MOD natural neutrosophic finite complex number subset polynomials semigroup under +.

(i) Find all MOD natural neutrosophic finite complex number subset polynomial subsemigroup of S(P[x]).
(ii) Can S(P[x]) have MOD natural neutrosophic idempotents under +?

58. Let S(R[x]) = {Collection of all subset polynomials from

$$R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle Z_{24} \cup k \rangle_I; +\right\}; \ +\}$$

be the MOD natural neutrosophic special quasi dual number coefficient polynomials subsets semigroup under +.

Study questions (i) to (ii) of problem (57) for this R[x].

59. Let S(W[x]) = {Collection of all subsets from

$$W[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle Z_9 \cup I \rangle_I; +\right\}; \ +\}$$

be the MOD natural neutrosophic - neutrosophic polynomials subsets semigroup under +.

Study questions (i) to (ii) of problem (57) for this W[x].

60. Let $P[x]_8 = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S(Z_{87}^I); x^9 = 1; +\right\}$

be the MOD natural neutrosophic subset coefficient polynomials under + of finite order.

(i) Find $o(P[x]_8)$.
(ii) Find all idempotents in $P[x]_8$.
(iii) Find all MOD natural neutrosophic subset coefficient polynomial subsemigroup of $P[x]_8$.

61. Let $S[x]_{11} = \left\{ \sum_{i=0}^{11} a_i x^i \,\middle|\, a_i \in S(C^I(Z_{24})); x^{12} = 1; +\right\}$

be the MOD natural neutrosophic finite complex number subset coefficient polynomial semigroup.

Study questions (i) to (iii) of problem (60) for this $S[x]_{11}$.

62. Let $W[x]_9 = \left\{\sum_{i=0}^{9} a_i x^i \,\middle|\, a_i \in \langle Z_{12} \cup g\rangle_I;; x^{10} = 1; +\right\}$

be the MOD natural neutrosophic dual number subset coefficient polynomial semigroup under +.

Study questions (i) to (iii) of problem (60) for this $W[x]_9$.

63. Let $M[x]_3 = \left\{\sum_{i=0}^{3} a_i x^i \,\middle|\, a_i \in S(\langle Z_{17} \cup k\rangle_I; x^4 = 1; +\right\}$

be the MOD natural neutrosophic special quasi dual number subset coefficient polynomial semigroup under +.
Study questions (i) to (iii) of problem (60) for this $M[x]_3$.

64. Let $B[x]_8 = \left\{\sum_{i=0}^{8} a_i x^i \,\middle|\, a_i \in S(C^I(Z_{42}); x^9 = 1; \times\right\}$

be the MOD natural neutrosophic finite complex number subset coefficient semigroup under product (can be usual zero dominated product or MOD natural neutrosophic zero dominated product).

(i) Find $o(B[x]_8)$.
(ii) Find all zero divisors in $B[x]_8$.
(iii) Prove $B[x]_8$ cannot have idempotents.
(iv) Prove $B[x]_8$ has nilpotents.
(v) Find all MOD natural neutrosophic subset subsemigroups which are not ideals of $B[x]_8$.
(vi) Find all MOD natural neutrosophic ideals of $B[x]_8$.
(vii) Enumerate any of the special features associated with it.

65. Let $C[x]_5 = \left\{\sum_{i=0}^{5} a_i x^i \;\middle|\; a_i \in S(\langle Z_{24} \cup I\rangle_I;\ x^6 = 1; \times\right\}$

be the MOD natural neutrosophic - neutrosophic subset coefficient polynomial semigroup.

Study questions (i) to (vii) of problem (64) for this $C[x]_5$.

66. Let $V[x]_{15} = \left\{\sum_{i=0}^{15} a_i x^i \;\middle|\; a_i \in S(\langle Z_{12} \cup g\rangle_I;\ x^{16} = 1; \times\right\}$

be the MOD natural neutrosophic dual number subset coefficient polynomial semigroup.

Study questions (i) to (vii) of problem (64) for this $V[x]_{15}$.

67. Let $S(P[x]_7)$ = {Collection of all subsets from

$P[x]_7 = \left\{\sum_{i=0}^{7} a_i x^i \;\middle|\; a_i \in Z_{43}^{I};\ x^8 = 1; +\right\}$

be the MOD natural neutrosophic polynomial subsets semigroup under +.

(i) Find $o(S(P[x])_7$.
(ii) Find all MOD polynomial subset subsemigroups of $S(P[x]_7)$.
(iii) Find idempotents if any in $S(P[x]_7)$.

68. Let $S(B[x]_{18})$ = {Collection of all polynomial subsets from

$B[x]_{18} = \left\{\sum_{i=0}^{18} a_i x^i \;\middle|\; a_i \in \langle Z_{12} \cup I\rangle_I;\ x^{19} = 1; +\right\}; +\}$

be the MOD natural neutrosophic - neutrosophic coefficient polynomial subsets semigroup under +.

Study questions (i) to (iii) of problem (67) for this $S(B[x]_{18})$.

69. Let $S(S[x]_{10})$ = {Collection of all polynomial subsets from

$$S[x]_{10} = \left\{ \sum_{i=0}^{11} a_i x^i \,\middle|\, a_i \in \langle Z_9 \cup k\rangle_I;\ x^{12} = 1; +\right\}; +\}$$

be the MOD natural neutrosophic special quasi dual number coefficient polynomial subsets semigroup under +.

Study questions (i) to (iii) of problem (67) for this $S(S[x]_{10})$.

70. Let $S(W[x]_{10})$ = {Collection of all subsets from

$$W[x]_{10} = \left\{ \sum_{i=0}^{10} a_i x^i \,\middle|\, a_i \in Z_{25}^I;\ x^{11} = 1; \times\right\}; \times\}$$

be the MOD natural neutrosophic polynomial neutrosophic subsets (from $W[x]_{10}$) semigroup under usual zero dominant product or MOD natural neutrosophic zero dominant product.

(i) Find $o(S(W[x]_{10}))$.
(ii) Can $S(W[x]_{10})$ divisors?
(iii) Does $S(W[x]_{10})$ contain non trivial idempotents?
(iv) Can $S(W[x]_{10})$ have non trivial nilpotents?
(v) Find all MOD subset subsemigroups in $S(W[x]_{10})$ which are not ideals of $S(W[x]_{10})$.
(vi) Find all ideals of $S(W[x]_{10})$.
(vii) Obtain any other special feature enjoyed by $S(W[x]_{10})$.

71. Let $S(P[x]_{18})$ = {Collection of all subsets from

$$P[x]_{18} = \left\{\sum_{i=0}^{18} a_i x^i \,\middle|\, a_i \in C^I(Z_{12});\ x^{19} = 1; \times\right\}; \times\}$$

be the MOD natural neutrosophic finite complex number coefficient polynomials subsets semigroup under ×.

Study questions (i) to (vii) of problem (70) for this $S(P[x]_{18})$.

72. Study problem (70) when $C^I(Z_{12})$ is replaced by $\langle Z_{27} \cup I\rangle_I$.

73. Let $S(B[x]_{27})$ = {Collection of all subsets from

$$B[x]_{27} = \left\{\sum_{i=0}^{27} a_i x^i \,\middle|\, a_i \in \langle Z_{18} \cup \rangle_I;\ x^{28} = 1; \times\right\}; \times\}$$

be the MOD natural neutrosophic special dual like number coefficient polynomial subset semigroup under ×.

Study questions (i) to (vii) of problem (70) for this $S(B[x]_{27})$.

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# INDEX

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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In this book authors have introduced the notion of MOD natural neutrosophic subset semigroups. They enjoy very many special properties. They are only semigroups even under addition. This book will provide several open problems to a semigroup theorist.

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