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Neutrosophic Approach for Mathematical Morphology

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Abstract

The main aim of this thesis is to provide a comprehensive overview of a neutrosophic approach for mathematical morphology. The new approach is considered to be an extension of the binary mathematical morphology and the fuzzy mathematical morphology, and proposed as a new tool for binary and gray images processing and analysis. We apply the concepts of the neutrosophic crisp sets and its operations as well as the neutrosophic fuzzy sets to the classical mathematical morphological operations; introducing what we call "Neutrosophic Crisp Mathematical Morphology" and "Neutrosophic Mathematical Morphology". Several operators are to be developed, including the neutrosophic (crisp) dilation, the neutrosophic (crisp) erosion, the neutrosophic (crisp) opening and the neutrosophic (crisp) closing. Moreover, we extend the definition of some morphological filters using the neutrosophic (crisp) sets concept. For instance, we introduce the neutrosophic (crisp) boundary extraction, the neutrosophic (crisp) Top-hat and the neutrosophic (crisp) Bottom-hat filters. The idea behind the new introduced operators and filters is to act on the image in the neutrosophic (crisp) domain instead of the spatial domain. Moreover, we introduce an investigation for some algebraic properties of the introduced operations and we use some different combinations of these basic operations to produce some more advanced neutrosophic filters for boundary extraction. Explanation of the proposed operations is also provided through several examples and experimental results conducted over real life binary and grayscale images. Furthermore, we demonstrate the efficiency of the proposed operator in one of the most important image processing application. "Image threshold" the experimental results show a slight improvement when we used the new operators when comparing with the operators from both the classical and fuzzy mathematical morphology.

List of Publications

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List of Abbreviations

CSs	Crisp Sets
FSs	Fuzzy Sets
IFS	Intuitionistic Fuzzy Sets
GIFs	Generalized Intuitionistic Fuzzy Sets
NFSs	Neutrosophic Fuzzy Sets
NCSs	Neutrosophic Crisp Sets
GNS	Generalized Neutrosophic Set
INS	Intuitionistic Neutrosophic Set
MM	Mathematical Morphology
SE	Structuring Element
IAPR	International Association for Pattern Recognition
SPIE	International Society for Optical Engineering
ND	Neutrosophic Domain
NSS	Neutrosophic Soft Set

List of Symbols

$A \cap B$	Intersection between two sets
$A \cup B$	Union between two sets
coA or A^c	Complement of set
$A \subset B$	Set A is Subsets of B
A / B	Difference between two sets
$A \vee B$	Max between two sets
$A \wedge B$	Min between two sets
$\underline{1}$	Fuzzy Universe
$\underline{0}$	Fuzzy Empty
$A = \langle T_A, I_A, F_A \rangle$	Neutrosophic Fuzzy Sets
$A = \langle A^1, A^2, A^3 \rangle$	Neutrosophic Crisp Sets
1_N	Neutrosophic Fuzzy Universe
0_N	Neutrosophic Fuzzy Empty
X_N	Neutrosophic Crisp Universe
\emptyset_N	Neutrosophic Crisp Empty
$A \oplus B$	Binary Dilation of A by B
$A \ominus B$	Binary Erosion of A by B
$A \circ B$	Binary Opening of A by B
$A \bullet B$	Binary Closing of A by B
A_b	Translate of A by b
$-B$	Reflection of B
$\mu_{A \oplus B}$	Fuzzy Dilation of A by B
$\mu_{A \ominus B}$	Fuzzy Erosion of A by B
$\mu_{A \circ B}$	Fuzzy Opening of A by B
$\mu_{A \bullet B}$	Fuzzy Closing of A by B
$A \widetilde{\oplus} B$	Neutrosophic Dilation of A by B
$A \widetilde{\ominus} B$	Neutrosophic Erosion of A by B
$A \tilde{\circ} B$	Neutrosophic Opening of A by B
$A \tilde{\bullet} B$	Neutrosophic Closing of A by B
$\partial_{int}(A)$	Internal Boundary of A
$\partial_{ext}(A)$	External Boundary of A

Chapter 1

Introduction

1. Introduction

Mathematical Morphology (in short MM) has been formalized since the 1960's by Georges Matheron [47] and Jean Serra at the Centre de Morphologie Mathematique on the campus of the Paris School of Mines at Fontainebleau, France, for studying geometric and milling properties. In 1967, Matheron and Serra introduced a set formalism for analyzing binary images, which led them to work on image analysis. Their work led to the development of the theory of MM. Later Petros Maragos contributed to enrich the theory by introducing theory of lattices. Firstly the theory is purely based on set theory and operators which are defined for binary cases only, later, the theory was extended to grayscale images as well. MM gained a wide recognition after the publication of the books "Image Analysis and Mathematical Morphology" by Serra [67] and "Image Analysis and Mathematical Morphology, Theoretical Advances" edited by Serra [69]. From the mid-1970's to mid-1980's [29], MM was generalized to grayscale images and functions, this generalization yielded new operators, such as morphological gradients and hat filters. In the 1980's and 1990's, MM started to be applied to a large number of imaging problems and applications. In 1986, Serra further generalized MM [68], this time to a theoretical framework based on set theory. This generalization brought flexibility to the theory, enabling its application to a much larger number of structures, including color images, video, graphs, etc. The 1990's and 2000's also saw further theoretical advancements, including the concepts of connections and leveling, where Heink J. Heijmans gave an algebraic basis for the theory and extended the theory to Signal Processing [30, 31]. More advances in the field was presented by: the International Symposium on Mathematical Morphology (ISMM); its first six venues

were held in Barcelona (1993), Fontainebleau (1994), Atlanta (1996), Amsterdam (1998), Palo Alto (2000) and Sydney (2002). MM is now part of the basic body of techniques taught to any students of image processing courses anywhere. Far from being an academic pursuit, morphology is used in industry and businesses at many levels, for instance: quality control in industrial production, medical imaging, document processing and much more. As morphology is the study of shapes, MM mostly deals with the mathematical theory of describing shapes using set theory. MM denotes a branch of biology that deals with the forms and structures of animals and plants. It analyzes the shapes and forms of objects. In computer vision, it is used as a tool to extract image components that are useful in the representation and description of object shape. MM is a non-linear theory of image processing. Its geometry- oriented nature provides an efficient method for analyzing object shape characteristics such as size and connectivity, which are not easily accessed by linear approaches. MM has taken concepts and tools from different branches of mathematics like algebra (lattice theory), topology, discrete geometry, integral geometry, geometrical probability, partial differential equations, etc.. Early work in this discipline includes the work of Minkowski [49], Kirsch [40] and Preston [58].

Smarandache [74] introduced another concept of imprecise data called "Neutrosophic Sets". Neutrosophic set is a part of neutrosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set is a powerful general formal framework which generalizes the concept of the classic set, fuzzy set [85]. The fundamental concepts of neutrosophic set introduced by Smarandache in [74], neutrosophic theories generalizing both their classical and fuzzy counterparts. A neutrosophic linguistic variable has neutrosophic linguistic values which defined by interval neutrosophic sets characterized by three membership degrees: truth-membership, falsity-membership and indeterminacy-

membership. Each field has a neutrosophic part, i.e. that part that has indeterminacy. Thus, the neutrosophic logic was established as well as [75], neutrosophic set theory, neutrosophic probability, neutrosophic statistics, neutrosophic measure [76], neutrosophic calculus, etc. Maji, P. K. introduced the neutrosophic concept for soft sets defining "Neutrosophic Soft Set" (in short NSS) [46]. Salama, A. A [64, 65], introduced the basic properties of the concept of neutrosophic crisp set and investigated some new neutrosophic concepts. In this thesis, we are utilizing a neutrosophic approach for mathematical morphology and image processing that has become increasingly important. Neutrosophic category is the development of a crisp sets and fuzzy sets this category is more general and comprehensive. Here, we present an overview of the operations and properties of mathematical morphology, crisp sets, fuzzy sets and neutrosophic sets. In our thesis we demonstrate that neutrosophic morphological operations inherit properties and restrictions of fuzzy mathematical morphology and crisp mathematical morphology.

1.3 The thesis structure:

The remaining of this thesis consists of six chapters in addition to a list of references and structured as follows:

Chapter 1: Introduction

In this chapter we introduce a survey for both the mathematical morphology, fuzzy mathematical morphology disperse, as well as a review for the theory of neutrosophic sets.

Chapter 2: Types of Sets

The chapter was divided into three sections: the first section presents a survey of some definitions and operators from the crisp sets theory. The second section we introduces the definitions from the fuzzy sets theory. Finally, the third sections is devoted for intruding the concepts of "Neutrosophic Fuzzy Sets" and "Neutrosophic Crisp Sets".

Chapter 3: Mathematical Morphology

It's a revision for the basic definitions and properties of the binary mathematical morphology in the first section, the grayscale mathematical morphology in the second section and finally, fuzzy mathematical morphology in the third section.

Chapter 4: Neutrosophic Crisp Mathematical Morphology

The aim of this chapter is to apply the concepts of the neutrosophic crisp sets and its operations to the classical mathematical morphological operations; introducing what we call "Neutrosophic Crisp Mathematical Morphology". Several operators are to be developed, including the neutrosophic crisp dilation, the neutrosophic crisp erosion, the neutrosophic crisp opening and the neutrosophic crisp closing. Moreover, we extend the definition of some morphological filters using the neutrosophic crisp sets concept. For instance, we introduce the neutrosophic crisp boundary extraction, the neutrosophic crisp top-hat and the neutrosophic crisp bottom-hat filters.

Chapter 5: Neutrosophic Fuzzy Mathematical Morphology

In this chapter, we propose a generalization for the fuzzy morphology, as a new tool for gray image processing and analysis, using the concepts of neutrosophy. The main operations of the proposed neutrosophic morphology are introduced; namely, the neutrosophic dilation, neutrosophic erosion, neutrosophic opening and neutrosophic closing. Some algebraic properties of the introduced operations are to be investigated. Furthermore, we use different combinations of these basic operation to produce some more advanced neutrosophic boundary filters.

Chapter 6: Application

In this chapter, we experiment our operator for thresholding images, proposing in section 6.4 an algorithm for thresholding images in the neutrosophic domain instead of the spatial domain. While section 6.5 is devoted for introducing the experimental results.

Chapter 7: Conclusions

This chapter, gives conclusions and focuses on the advantage and shortcoming of our proposed techniques through this thesis. We also point out some promising directions for future research.

Chapter 2

Type of Sets

2.1 Introduction:

In many complicated problems such as, engineering problems, social, economic, computer science, medical science...etc., the data associated are not necessarily crisp, precise and deterministic because of their vague nature. Most of these problem were solved by different theories. One of these theories was the fuzzy set theory discovered by Lotfi Zadeh in 1965 [85]. In many real applications to handle uncertainty, fuzzy set is very much useful and in this one real value $\mu_A(x) \in [0,1]$ is used to represent the grade of membership of a fuzzy set A defined on the universe of X . Atanassov [3] introduced another type of fuzzy sets that is called "Intuitionistic Fuzzy Set" (in short IFS) which is more practical in real life situations. Intuitionistic fuzzy sets handle incomplete information i.e., the membership degree and non-membership degree, but not the indeterminacy and inconsistent data which exists obviously in real life systems.

In 1991, Samarandache initiated the theory of neutrosophic set as new mathematical tool for handling problems involving imprecise indeterminacy and inconsistent data [74]; where he introduced the neutrosophic components (T, I, F) which represent the membership, indeterminacy and non-membership values respectively. Later on, several researchers such as Bhowmik and Pal [6], and Salama [63, 65], studied the concept of neutrosophic crisp set. Neutrosophy introduces a new concept which represents indeterminacy with respect to some event, which can solve certain problems that cannot be solved by fuzzy logic.

The Remaining of This Chapter is Structured as Follows:

Firstly, We present a breif reversion for the concept of crisp sets with its operations and properties in §2.2.

Secondly, our goal in § 2.3, we give some definitions, for the fuzzy sets and its operations and properties.

Thirdly, Definition of the intuitionistic fuzzy sets is to be introduced in §2.4.

Finally, The purpose of § 2.5, is to explain the concepts of both neutrosophic sets and neutrosophic crisp sets.

2.2 Crisp Sets:

The concept of crisp sets is the core of most branches of mathematics, that is the concept of the group of unknown preliminary concepts. A crisp set is an unordered collection of objects, called elements or members, of the set. We write $a \in A$ to denote that a is an element of the crisp set A . The notation $a \notin A$ denotes that a is not an element of the crisp set A . The crisp set A is to be defined using the following

$$\text{characteristic function } \mu_A : X \rightarrow \{0, 1\}, \mu_A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases} \quad (2.1)$$

where $\mu_A(x)$ is the membership degree of any in the universal set X .

2.2.1 Operations on Crisp Sets:

In this section we review some basic operations which are defined on the crisp sets. To commence, we consider two crisp sets A and B , to be defined on the universe set X . Hence, we have the following operations.

The Union of Crisp Sets:

The union of two crisp sets A and B is denoted by $A \cup B$ [61]. It represents all the elements in the universe that reside in either the set A , the set B or both sets A and B ; and to be defined as the crisp set: $A \cup B = \{x | x \in A \text{ or } x \in B\}$. (2.2)

The Intersection of Crisp Sets:

The intersection of two crisp sets A and B is denoted $A \cap B$ [61]. It represents all those elements in the universe X that simultaneously reside in both sets A and B ; and to be defined as the crisp set: $A \cap B = \{x | x \in A \text{ and } x \in B\}$. (2.3)

The Complement of a Crisp Sets:

The complement of any crisp set A denoted by (coA or A^c) [61], is defined as the collection of all elements in the universe that do not reside in the crisp set A ; and to be defined as the crisp set: $A^c = \{x|x \notin A, x \in X\}$. (2.4)

The Difference between Crisp Sets:

The difference of a crisp set A with respect to B , denoted by $A - B$, is defined as the collection of all elements in the universe set that reside in A and that do not reside in B simultaneously; and to be defined as the crisp set: $A - B = \{x|x \in A \text{ and } x \notin B\}$. (2.5)

2.2.2 Properties of Crisp Sets' Operations:

For the operations defined in the previous section, the following properties are true [26, 37, 61].

The Commutativity: $A \cup B = B \cup A$, $A \cap B = B \cap A$.

The Associativity of a Crisp Sets:

$$A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C.$$

The Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

The Idempotency: $A \cup A = A$, $A \cap A = A$.

The Transitivity: if $A \subseteq B \subseteq C$, then $A \subseteq C$.

The Involution: $(A^c)^c = A$.

The Identity: $A \cup \emptyset = A$, $A \cap X = A$, $A \cap \emptyset = \emptyset$, $A \cup X = X$.

Where the symbol \subseteq means contained in or equivalent to and \subset means contained in.

2.2.3 Generalized Unions and Intersections on Crisp Sets [61]:

The Union of a collection of crisp sets is the crisp set that contains those elements that are members of at least one set in the collection.

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^n A_i = \{x: x \in A_i \exists i = 1, 2, \dots, n\}. \quad (2.6)$$

The Intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^n A_i = \{x: x \in A_i \forall i = 1, 2, \dots, n\}. \quad (2.7)$$

More generally, when I is a set, the notations $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are used to denote the intersection and union of the sets A_i for $i \in I$, respectively. Note that we have:

$$\begin{aligned} \bigcap_{i \in I} A_i &= \{x; x \in A_i, \forall i \in I\}, \\ \bigcup_{i \in I} A_i &= \{x; x \in A_i, \exists i \in I\}. \end{aligned} \quad (2.8)$$

2.3 Fuzzy Sets [85]:

In 1965, Zadeh generalized the idea of a crisp set by extending a valuation set $\{0, 1\}$ to the interval of real values $[0,1]$. The degree of membership of any particular element of a fuzzy set express the degree of compatibility of the element with a concept represented by fuzzy set. That is a fuzzy set A contains an object x to some degree $A(x)$. Fuzzy sets tend to capture vagueness exclusively via membership functions that are mappings from a given universe of discourse into the unit interval.

2.3.1 Definition [85]:

Let X be a fixed set, a fuzzy set A of X is an object having the form $A = \langle \mu_A \rangle$, where the function $\mu_A: X \rightarrow [0, 1]$ defines the degree of membership of the element $x \in X$ to the set A . The set of all fuzzy subset of X is denoted by $\mathcal{F}(X)$. The fuzzy empty set in X is denoted by $0_f = \langle \underline{0} \rangle$, where $\underline{0}: X \rightarrow [0, 1]$ and $\underline{0}(x) = 0, \forall x \in X$. Moreover, the fuzzy universe set in X is denoted by:

$$1_f = \langle \underline{1} \rangle, \text{ where } \underline{1}: X \rightarrow [0, 1] \text{ and } \underline{1}(x) = 1, \forall x \in X.$$

2.3.2 Operations on Fuzzy Sets:

Consider three fuzzy sets A, B and C in the universe X . For a given element x in the universe X , the following are the membership degrees for x under the basic fuzzy sets operations[42, 62, 85].

The Union of Fuzzy Sets:

$$(\underline{A} \cup \underline{B})(x) = \max(\underline{\mu}_A(x), \underline{\mu}_B(x)) \text{ or } (\underline{\mu}_{\underline{A} \cup \underline{B}})(x) = \underline{\mu}_A(x) \vee \underline{\mu}_B(x).$$

The Intersection of Fuzzy Sets:

$$(\underline{A} \cap \underline{B})(x) = \min(\underline{\mu}_A(x), \underline{\mu}_B(x)) \text{ or } (\underline{\mu}_{\underline{A} \cap \underline{B}})(x) = \underline{\mu}_A(x) \wedge \underline{\mu}_B(x).$$

The Complement of Fuzzy Sets: $\underline{\mu}_{\underline{A}^c}(x) = 1 - \underline{\mu}_A(x).$

The Difference of Fuzzy Sets: $(\underline{A} - \underline{B})(x) = \min(\underline{\mu}_A(x), 1 - \underline{\mu}_B(x)).$

The Containment of Fuzzy Sets: $\underline{A} \leq \underline{B} \Leftrightarrow \underline{\mu}_A(x) \leq \underline{\mu}_B(x).$

2.3.3 Properties of Fuzzy Sets' Operations:

The properties of the classical set also suits for the properties of the fuzzy sets [86]. The important properties of fuzzy set include:

The Commutativity: $\underline{A} \cup \underline{B} = \underline{B} \cup \underline{A}, \quad \underline{A} \cap \underline{B} = \underline{B} \cap \underline{A}.$

The Associativity: $\underline{A} \cup (\underline{B} \cup \underline{C}) = (\underline{A} \cup \underline{B}) \cup \underline{C}, \quad \underline{A} \cap (\underline{B} \cap \underline{C}) = (\underline{A} \cap \underline{B}) \cap \underline{C}.$

The Distributive: $\underline{A} \cup (\underline{B} \cap \underline{C}) = (\underline{A} \cup \underline{B}) \cap (\underline{A} \cup \underline{C}),$

$$\underline{A} \cap (\underline{B} \cup \underline{C}) = (\underline{A} \cap \underline{B}) \cup (\underline{A} \cap \underline{C}).$$

The Idempotency: $\underline{A} \cup \underline{A} = \underline{A}.$

The Transitivity: if $\underline{A} \subseteq \underline{B} \subseteq \underline{C}$, then $\underline{A} \subseteq \underline{C}.$

The Involution: $co(co\underline{A}) = \underline{A} \text{ or } (\underline{A}^c)^c = \underline{A}.$

The Identity: $\underline{A} \cup \underline{0} = \underline{A}, \underline{A} \cap \underline{0} = \underline{0}, \underline{A} \cup \underline{1} = \underline{1} \text{ and } \underline{A} \cap \underline{1} = \underline{A}.$

The complement: $co\underline{0} = \underline{1} \text{ and } co\underline{1} = \underline{0}.$

The Containment:

- i. If $\underline{A} \subseteq \underline{B}$ and $\underline{C} \subseteq \underline{D}$ then $\underline{A} \cup \underline{C} \subseteq \underline{B} \cup \underline{D}.$
- ii. If $\underline{A} \subseteq \underline{B}$ then $\underline{A} \cap \underline{B} = \underline{A}.$
- iii. If $\underline{A} \subseteq \underline{B}$ then $\underline{A} \cup \underline{B} = \underline{B}.$

2.3.4 Generalized Unions and Intersections on Fuzzy Sets [62]:

The Union of a collection of fuzzy sets is the set that contains those elements that are members of at least one set in the collection.

$$\cup_{i \in I} A_i: X \rightarrow [0, 1], \text{ where } x \rightarrow \sup_{i \in I} A_i(x), \forall x \in X. \quad (2.9)$$

The Intersection of a collection of Fuzzy Sets is the set that contains those elements that are members of all the sets in the collection.

$$\cap_{i \in I} A_i: X \rightarrow [0, 1], \text{ where } x \rightarrow \inf_{i \in I} A_i(x), \forall x \in X. \quad (2.10)$$

2.3.4.1 Definition:

- The set of all elements that having the degree of membership not equal zero in a fuzzy set A is said to be the support of A; and is defined as:

$$Supp(A) = \{x: x \in X \text{ and } A(x) > 0\}, \quad (2.11)$$

where $A \in \mathcal{F}(X)$ and $Supp(A) \in P(X)$.

- The all elements that having the degree of membership equal one in a fuzzy set A is said to be the support of A; and is defined as:

$$Ker(A) = \{x: x \in X \text{ and } A(x) = 1\}, \quad (2.12)$$

where $A \in \mathcal{F}(X)$ and $Ker(A) \in P(X)$.

2.3.4.2 Definition: (the weak α cut (α level))

Let A be a fuzzy set and $\alpha \in]0, 1]$ the weak α cut A_α , is defined as:

$$A_\alpha = \{x: x \in X \text{ and } A(x) \geq \alpha, \alpha \in]0, 1]\}. \quad (2.13)$$

2.3.4.3 Definition: (the strong α cut (α level))

Let A be a fuzzy set and $\alpha \in [0, 1[$ the strong α cut $A_{\bar{\alpha}}$, is defined as:

$$A_{\bar{\alpha}} = \{x: x \in X \text{ and } A(x) > \alpha, \alpha \in [0, 1[\}. \quad (2.14)$$

2.3.4.4 Definition:

For any fuzzy set $A \in \mathcal{F}(X)$, we may define the following values for the fuzzy set's height and regression.

$$\begin{aligned} \text{height } A &= \sup_{x \in X} A(x), \\ \text{plinth } A &= \inf_{x \in X} A(x). \end{aligned} \tag{2.15}$$

2.4 Intuitionistic Fuzzy Sets:

In real life, the available information is vague, inexact or insufficient, the parameters of any problem are usually defined by the decision makers in an uncertain way. Therefore, it is desirable to consider the knowledge of experts about the parameters as fuzzy data. Out of several higher order fuzzy sets, "Intuitionistic Fuzzy Sets" (in short IFS) [3, 4] have been found to be highly useful to deal with vagueness. There are situations where, due to insufficiency in the available information, the evaluation of membership values is not possible up to our satisfaction. Nevertheless, the evaluation of non-membership values is not also always possible and consequently there remains a part in deterministic on which hesitation survives. Certainly, IFS theory is more suitable to deal with such problem.

2.4.1 Definition [4]:

An intuitionistic fuzzy set, A in X , is defined to be a structure of the form:

$A = \{\langle x, \mu_A(x), \vartheta_A(x) \rangle : x \in X\}$, where the functions $\mu_A: X \rightarrow [0, 1]$ defines the degree of membership, and the functions $\vartheta_A: X \rightarrow [0, 1]$ defines the degree of non-membership of the element $x \in X$. For every element $x \in X$ in A the two degrees of membership $\langle \mu_A \rangle$ and non-membership $\langle \vartheta_A \rangle$ of x satisfy: $0 \leq \mu_A(x) + \vartheta_A(x) \leq 1$. (2.16)

When $\vartheta_A(x) = 1 - \mu_A(x)$, the set A is happen to be a fuzzy set; while A is said to be intuitionistic fuzzy set if $\vartheta_A(x) < 1 - \mu_A(x) \forall x \in X$.

Example 2.4.1:

Consider an intuitionistic fuzzy set A , with a membership function $\mu_A(x)$ and non-membership function $\vartheta_A(x)$. For some $x_0 \in X$, if we have that $\mu_A(x_0) = 0.7$ and $\vartheta_A(x_0) = 0.1$, then we can interpreted that the element x belongs to the intuitionistic

fuzzy set A by the degree 0.7; and that x_0 does not belong to the intuitionistic fuzzy set A by the degree 0.1.

2.4.2 Operations on Intuitionistic Fuzzy Set [5]:

In this section we review some basic operations which are defined on the intuitionistic fuzzy sets. To commence, we consider A , B and C to be three intuitionistic fuzzy sets defined on the universe set X . Hence, we have the following operations.

The Union of Intuitionistic Fuzzy Set:

The union of two intuitionistic fuzzy sets A and B is defined by:

$$A \cup B = \langle \max(\mu_A(x), \mu_B(x)), \min(\vartheta_A(x), \vartheta_B(x)) \rangle.$$

The Intersection of Intuitionistic Fuzzy Set:

The intersection of two intuitionistic fuzzy sets A and B is defined by:

$$A \cap B = \langle \min(\mu_A(x), \mu_B(x)), \max(\vartheta_A(x), \vartheta_B(x)) \rangle.$$

Complement of Intuitionistic Fuzzy Set:

The complement of intuitionistic fuzzy sets A is given by

$$A^c = \{ \langle x, \vartheta_A(x), \mu_A(x) \rangle : x \in X \}.$$

- $A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x) \text{ and } \vartheta_A(x) \geq \vartheta_B(x) \quad \forall x \in X,$
- $A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \text{ and } \vartheta_A(x) = \vartheta_B(x) \quad \forall x \in X.$

2.4.3 Properties of Intuitionistic Fuzzy Set [4]:

The following are the important properties of intuitionistic fuzzy set:

The Commutativity of Intuitionistic Fuzzy Set: $A \cup B = B \cup A, \quad A \cap B = B \cap A.$

Associativity of Intuitionistic Fuzzy Set:

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C.$$

Distributivity of Intuitionistic Fuzzy Set:

$$A \cup (\cap_i B_i) = \cap_i (A \cup B_i), \quad A \cap (\cup_i B_i) = \cup_i (A \cap B_i).$$

Transitivity of Intuitionistic Fuzzy Set: if $A \subset B$ and $B \subset C \Rightarrow A \subset C$.

Atanassov himself and many other authors [11, 52] studied different properties in intuitionistic fuzzy set.

2.5 Theory of Neutrosophic Set:

Several definition for the concept of neutrosophic sets were introduced by authors in literature (see for instance [74, 75]). To follow up our work we choose the following definition, which define the concepts of two neutrosophic sets; namely, the neutrosophic fuzzy sets and the neutrosophic crisp sets [65]; the two concepts are given in the following two sections §2.5.1 and §2.5.2, respectively.

2.5.1 Neutrosophic Sets:

To commence, we consider a universe of discourse X , and two neutrosophic fuzzy sets A and B of X .

The set of all neutrosophic fuzzy sets of the universe X is will be denoted by $\mathcal{N}(X)$.

2.5.1.1 Definition:

A neutrosophic fuzzy set (simply, neutrosophic set); A neutrosophic set A on $A \in \mathcal{N}(X)$. is defined as the triple structure:

$$A = \langle T_A, I_A, F_A \rangle, \text{ where } T_A, I_A, F_A : X \rightarrow [0,1]. \quad (2.17)$$

Which are the three function that define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of each element $x \in X$ to the set A . From philosophical point of view [74, 75], the neutrosophic set take the values from either real standard subset $[0, 1]$ or non-standard subset $]^{-}0,1^{+}[$ i.e.; where "1, 0" are standard part and "•" its non-standard part; "•" be such infinitesimal number; In our experiments, we will use the interval $[0,1]$ instead of the non-standard interval $]^{-}0,1^{+}[$; where the interval $]^{-}0,1^{+}[$ will be difficult to be applied in the real applications such as in scientific and engineering problems.

2.5.1.2 Definition:

- The complement of a neutrosophic set A is denoted by A^c or $(co A)$, [74] may be defined as one of the following two types: $\forall x \in X$,

$$\text{Type I: } A^c = \langle T_{A^c}, I_{A^c}, F_{A^c} \rangle, \text{ where } T_{A^c}, I_{A^c}, F_{A^c} : X \rightarrow [0,1]. \quad (2.18)$$

$$\text{Type II: } A^c = \langle F_A, I_{A^c}, T_A \rangle, \text{ where } F_A, I_{A^c}, T_A : X \rightarrow [0,1].$$

$$T_{A^c}(x) = 1 - T_A(x), I_{A^c}(x) = 1 - I_A(x) \text{ and } F_{A^c}(x) = 1 - F_A(x).$$

- The neutrosophic empty set of X , denoted by $0_{\mathcal{N}}$, may be defined as one of the following two types:

$$\text{Type I: } 0_{\mathcal{N}} = \langle \underline{0}, \underline{0}, \underline{1} \rangle, \text{ where } \underline{1}(x) = 1 \text{ and } \underline{0}(x) = 0, \forall x \in X. \quad (2.19)$$

$$\text{Type II: } 0_{\mathcal{N}} = \langle \underline{0}, \underline{1}, \underline{1} \rangle, \text{ where } \underline{1}(x) = 1 \text{ and } \underline{0}(x) = 0, \forall x \in X.$$

- The neutrosophic universe set of X , denoted by $1_{\mathcal{N}}$, may be defined as one of the following two types:

$$\text{Type I: } 1_{\mathcal{N}} = \langle \underline{1}, \underline{1}, \underline{0} \rangle, \text{ where } \underline{1}(x) = 1 \text{ and } \underline{0}(x) = 0, \forall x \in X. \quad (2.20)$$

$$\text{Type II: } 1_{\mathcal{N}} = \langle \underline{1}, \underline{0}, \underline{0} \rangle, \text{ where } \underline{1}(x) = 1 \text{ and } \underline{0}(x) = 0, \forall x \in X.$$

2.5.1.3 Definition[63]: (Containment)

A neutrosophic set A is considered to be contained in another neutrosophic set B denoted by $A \subseteq B$ according to one of the following two types.

Type I: $A \subseteq B$ if and only if;

$$T_A(x) \leq T_B(x), I_A(x) \leq I_B(x), F_A(x) \geq F_B(x), \forall x \in X. \quad (2.21)$$

Type II: $A \subseteq B$ if and only if;

$$T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x), \forall x \in X.$$

2.5.1.4 Definition: (Intersection)

The intersection of two neutrosophic sets A and B is a neutrosophic set $C = \langle T_C, I_C, F_C \rangle$, whose truth membership, indeterminacy membership and falsity membership functions are related to those of A and B by one of the following two type:

$$\begin{aligned} \text{Type I: } T_C(x) &= \min(T_A(x), T_B(x)), \quad I_C(x) = \min(I_A(x), I_B(x)), \\ F_C(x) &= \max(F_A(x), F_B(x)). \end{aligned} \tag{2.22}$$

$$\begin{aligned} \text{Type II: } T_C(x) &= \min(T_A(x), T_B(x)), \quad I_C(x) = \max(I_A(x), I_B(x)), \\ F_C(x) &= \max(F_A(x), F_B(x)). \end{aligned}$$

2.5.1.5 Definition: (Union)

Let A and B are neutrosophic sets, the union of A and B is a neutrosophic fuzzy sets, written as $C = \max(A, B)$, where $C = \langle T_C, I_C, F_C \rangle$, may be defined as two types:

$$\begin{aligned} \text{Type I: } T_C(x) &= \max(T_A(x), T_B(x)), \quad I_C(x) = \max(I_A(x), I_B(x)), \\ F_C(x) &= \min(F_A(x), F_B(x)). \end{aligned} \tag{2.23}$$

$$\begin{aligned} \text{Type II: } T_C(x) &= \max(T_A(x), T_B(x)), \quad I_C(x) = \min(I_A(x), I_B(x)), \\ F_C(x) &= \min(F_A(x), F_B(x)). \end{aligned}$$

2.5.1.6 Proposition [64]:

Let $\{A_i: i \in I\}$ be an arbitrary family of neutrosophic sets subsets in X , then:

- **Intersection** A_i may be defined as the following two types:

$$\text{Type I: } \cap_i A_i = \langle \min_{i \in I} T_i, \min_{i \in I} I_i, \max_{i \in I} F_i \rangle, \tag{2.24}$$

$$\text{Type II: } \cap_i A_i = \langle \min_{i \in I} T_i, \max_{i \in I} I_i, \max_{i \in I} F_i \rangle.$$

- **Union** A_i may be defined as the following two types:

$$\text{Type I: } \cup_i A_i = \langle \max_{i \in I} T_i, \max_{i \in I} I_i, \min_{i \in I} F_i \rangle, \tag{2.25}$$

$$\text{Type II: } \cup_i A_i = \langle \max_{i \in I} T_i, \min_{i \in I} I_i, \min_{i \in I} F_i \rangle.$$

2.5.1.7 Definition: (Strong α cut)

Let $A = \langle T_A(x), I_A(x), F_A(x) \rangle$ be a neutrosophic set of the set X . For $\alpha \in [0, 1]$, the α cut of A is fuzzy set A_α defined by as two types:

$$\text{Type I: } A_\alpha = \{x: x \in X, T_A(x) \geq \alpha, I_A(x) \geq \alpha, F_A(x) \leq 1 - \alpha\}, \tag{2.26}$$

$$\text{Type II: } A_\alpha = \{x: x \in X, T_A(x) \geq \alpha, I_A(x) \leq \alpha, F_A(x) \leq 1 - \alpha\}.$$

Condition $T_A(x) \geq \alpha$ ensures $F_A(x) \leq 1 - \alpha$ but not conversely. So α cut can be define as: $A_\alpha = \{x: x \in X, F_A(x) \leq 1 - \alpha\}$.

2.5.1.8 Definition: (Weak α -cut)

For a neutrosophic set $A = \langle T_A(x), I_A(x), F_A(x) \rangle$; For $\alpha \in [0, 1]$, the weak α -cut can be defined by as two types:

$$\text{Type I: } A_{\bar{\alpha}} = \{x: x \in X, T_A(x) > \alpha, I_A(x) > \alpha, F_A(x) < 1 - \alpha\}, \quad (2.27)$$

$$\text{Type II: } A_{\bar{\alpha}} = \{x: x \in X, T_A(x) > \alpha, I_A(x) < \alpha, F_A(x) < 1 - \alpha\}.$$

2.5.1.9 Properties of the Neutrosophic Sets [64, 75]:

One can easily prove the following properties for any neutrosophic sets $A, B, C \in \mathcal{N}(X)$.

- **Idempotency:** $A \cap A = A, A \cup A = A.$
- **Commutativity:** $A \cap B = B \cap A, A \cup B = B \cup A.$
- **Associativity:** $(A \cap B) \cap C = A \cap (B \cap C), \quad (A \cup B) \cup C = A \cup (B \cup C).$
- **Distributivity:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$
 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$
- **Absorption:** $A \cap (B \cup A) = A, \quad A \cup (B \cap A) = A.$
- **De Morgan's laws:** $(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c.$
- **Involution:** $(A^c)^c = A.$
- If $A \subseteq B$ then $B^c \subseteq A^c.$

For instance we will prove the following property: $(A \cup B)^c = A^c \cap B^c$

Let $A, B \in \mathcal{N}(X)$; $A = \langle T_A(x), I_A(x), F_A(x) \rangle$ and

$$B = \langle T_B(x), I_B(x), F_B(x) \rangle.$$

Type I: $(A \cup B)^c(x) = 1 - (A \cup B)(x)$

$$\begin{aligned} &= 1 - \langle \max(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \min(F_A(x), F_B(x)) \rangle \\ &= \langle \min(T_A^c(x), T_B^c(x)), \min(I_A^c(x), I_B^c(x)), \max(F_A^c(x), F_B^c(x)) \rangle \\ &= (A^c \cap B^c)(x). \end{aligned}$$

Type II: $(A \cup B)^c(x) = 1 - (A \cup B)(x)$

$$\begin{aligned} &= 1 - \langle \max(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \min(F_A(x), F_B(x)) \rangle \\ &= \langle \min(T_{A^c}(x), T_{B^c}(x)), \max(I_{A^c}(x), I_{B^c}(x)), \max(F_{A^c}(x), F_{B^c}(x)) \rangle \\ &= (A^c \cap B^c)(x). \end{aligned}$$

2.5.2 Neutrosophic Crisp Sets:

A neutrosophic crisp set A on the universe of discourse X , as defined in [65], is defined as a triple structure of the form: $A = \langle A^1, A^2, A^3 \rangle$, where A^1 is the set of all elements that belong in A , A^3 contains the elements that not belong in A ; while A^2 contains those elements which do not belong to neither A^1 nor A^3 .

2.5.2.1 Definition [64, 65]:

According to Salama [65], the neutrosophic crisp sets (in short NCSs) are to be categorized with respect to its components in to three different classes as follows:

- (NCSs) class I: $A^1 \cap A^2 = \varnothing, A^1 \cap A^3 = \varnothing, A^3 \cap A^2 = \varnothing$.
- (NCSs) class II: $A^1 \cap A^2 = \varnothing, A^1 \cap A^3 = \varnothing, A^3 \cap A^2 = \varnothing$ and

$$A^1 \cup A^2 \cup A^3 = X. \tag{2.28}$$
- (NCSs) class III: $A^1 \cap A^2 \cap A^3 = \varnothing, A^1 \cup A^2 \cup A^3 = X$.

2.5.2.2 Definition [65]:

Consider a universe of discourse X , the neutrosophic crisp universal set X_N and the neutrosophic crisp empty set \emptyset_N , are to be defined as follows:

- \emptyset_N may be defined as one of the following two types:
 - Type I: $\emptyset_N = \langle \emptyset, \emptyset, X \rangle,$
 - Type II: $\emptyset_N = \langle \emptyset, X, X \rangle.$
$$\tag{2.29}$$
- X_N may be defined as one of the following two types:
 - Type I: $X_N = \langle X, X, \emptyset \rangle,$
 - Type II: $X_N = \langle X, \emptyset, \emptyset \rangle.$
$$\tag{2.30}$$

2.5.2.3 Definition [65]:

Let $A = \langle A^1, A^2, A^3 \rangle$ be a NCSs in X , then the complement of the set A (A^c or (coA)) may be defined as one of the following two types:

$$\text{Type I: } coA = \langle coA^1, coA^2, coA^3 \rangle, \quad (2.31)$$

$$\text{Type II: } coA = \langle A^3, coA^2, A^1 \rangle.$$

2.5.2.1 Example: consider a universe of discourse $X = \{a, b, c, d, e, f\}$, and two neutrosophic crisp sets $A = \langle \{a, b, c, d\}, \{e\}, \{f\} \rangle$ and $B = \langle \{a, b, c\}, \emptyset, \{d, e\} \rangle$

We can deduce the following:

- The complement of A :

$$\text{Type I: } coA = \langle \{e, f\}, \{a, b, c, d, f\}, \{a, b, c, d, e\} \rangle,$$

$$\text{Type II: } coA = \langle \{a, b, c, d, e\}, \{a, b, c, d, f\}, \{e, f\} \rangle.$$

- The complement of B :

$$\text{Type I: } coB = \langle \{d, e, f\}, X, \{a, b, c, f\} \rangle,$$

$$\text{Type II: } coB = \langle \{a, b, c, f\}, X, \{d, e, f\} \rangle.$$

2.5.2.4 Definition:

For any non-empty set X , and any two NCSs A, B ; $A = \langle A^1, A^2, A^3 \rangle, B = \langle B^1, B^2, B^3 \rangle$, we may consider two possible types for containment $A \subseteq B$.

$$\text{Type I: } A \subseteq B \Leftrightarrow A^1 \subseteq B^1, A^2 \subseteq B^2 \text{ and } A^3 \supseteq B^3, \quad (2.32)$$

$$\text{Type II: } A \subseteq B \Leftrightarrow A^1 \subseteq B^1, A^2 \supseteq B^2 \text{ and } A^3 \supseteq B^3.$$

2.5.2.5 Definition [65]:

For any non-empty set X , and any two NCSs A, B ; $A = \langle A^1, A^2, A^3 \rangle, B = \langle B^1, B^2, B^3 \rangle$, we me define basic two set operations as following:

- **The Intersection** A, B may be defined as one of the following:

$$\text{Type I: } A \cap B = \langle A^1 \cap B^1, A^2 \cap B^1, A^3 \cup B^1 \rangle, \quad (2.33)$$

$$\text{Type II: } A \cap B = \langle A^1 \cap B^1, A^2 \cup B^1, A^3 \cup B^1 \rangle.$$

- **The Union** A, B may be defined as one of the following:

$$\text{Type I: } A \cup B = \langle A^1 \cup B^1, A^2 \cup B^1, A^3 \cap B^1 \rangle, \quad (2.34)$$

$$\text{Type II: } A \cup B = \langle A^1 \cup B^1, A^2 \cap B^1, A^3 \cap B^1 \rangle.$$

2.5.2.1 Proposition [65]:

Let $\{A_i: i \in I\}$ be an arbitrary family of Neutrosophic Crisp subsets in X , then:

- $\cap_i A_i$ may be defined as the following two types:

$$\text{Type I: } \cap_i A_i = \langle \cap_i A_i^1, \cap_i A_i^2, \cup_i A_i^3 \rangle, \quad (2.35)$$

$$\text{Type II: } \cap_i A_i = \langle \cap_i A_i^1, \cup_i A_i^2, \cup_i A_i^3 \rangle.$$

- $\cup_i A_i$ may be defined as the following two types:

$$\text{Type I: } \cup_i A_i = \langle \cup_i A_i^1, \cup_i A_i^2, \cap_i A_i^3 \rangle, \quad (2.36)$$

$$\text{Type II: } \cup_i A_i = \langle \cup_i A_i^1, \cap_i A_i^2, \cap_i A_i^3 \rangle.$$

2.5.2.2 Example: consider a universe of discourse $X = \{a, b, c, d, e, f\}$ and two neutrosophic crisp sets $A = \langle \{a, b, c, d\}, \{e\}, \{f\} \rangle$ and $B = \langle \{a, b, c\}, \emptyset, \{d, e\} \rangle$.

- **The Union** of A and B ,

$$\text{Type I: } A \cup B = \langle \{a, b, c, d\}, \{e\}, \emptyset \rangle,$$

$$\text{Type II: } A \cup B = \langle \{a, b, c, d\}, \emptyset, \emptyset \rangle.$$

- **The Intersection** of A and B

$$\text{Type I: } A \cap B = \langle \{a, b, c\}, \emptyset, \{d, e, f\} \rangle,$$

$$\text{Type II: } A \cap B = \langle \{a, b, c\}, \{e\}, \{d, e, f\} \rangle.$$

2.5.2.6 Properties of the Neutrosophic Crisp Sets [64, 75]:

One can easily prove the following properties for any neutrosophic crisp sets A, B and $C \in \mathcal{NC}(X)$.

- **Idempotency:** $A \cap A = A, A \cup A = A$.
- **Commutativity:** $A \cap B = B \cap A, A \cup B = B \cup A$.

- **Associativity:** $(A \cap B) \cap C = A \cap (B \cap C),$
 $(A \cup B) \cup C = A \cup (B \cup C).$
- **Distributivity:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$
 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$
- **Absorption:** $A \cap (B \cup A) = A, \quad A \cup (B \cap A) = A.$
- **De Morgan's laws:** $(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c.$
- **Involution:** $(A^c)^c = A.$
- If $A \subseteq B$ then $B^c \subseteq A^c.$

For instance we will prove the following property:

Let $A, B \in \mathcal{N}(X), \quad A = \langle A^1, A^2, A^3 \rangle$ and $B = \langle B^1, B^2, B^3 \rangle.$

$$\begin{aligned} \text{Type I: } (A \cap B)(x) &= \langle T_A(x) \cap T_B(x), I_A(x) \cap I_B(x), F_A(x) \cup F_B(x) \rangle \\ &= \langle T_B(x) \cap T_A(x), I_B(x) \cap I_A(x), F_B(x) \cup F_A(x) \rangle \\ &= (B \cap A)(x). \end{aligned}$$

$$\begin{aligned} \text{Type II: } (A \cap B)(x) &= \langle T_A(x) \cap T_B(x), I_A(x) \cup I_B(x), F_A(x) \cup F_B(x) \rangle \\ &= \langle T_B(x) \cap T_A(x), I_B(x) \cup I_A(x), F_B(x) \cup F_A(x) \rangle \\ &= (B \cap A)(x). \end{aligned}$$

2.5.2.1 Corollary [65]:

Let A, B and C be are neutrosophic sets in X , then:

- If $A \subseteq B$ and $C \subseteq D$ then $A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D.$
- If $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C.$
- If $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C.$
- If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C.$

2.6 Image as a Mathematical Object:

As a real life application for sets, we will consider the images. Mathematically, the image is considered as a set of pixels; The image as mathematical object in the Cartesian Domain, represented by an $m \times n$ matrix; $I = [g(i, j)]_{m \times n}$, with entities $g(i, j)$ corresponding to the intensity to the given pixel located at the node (i, j) . The original color image shown in Fig.2.1.

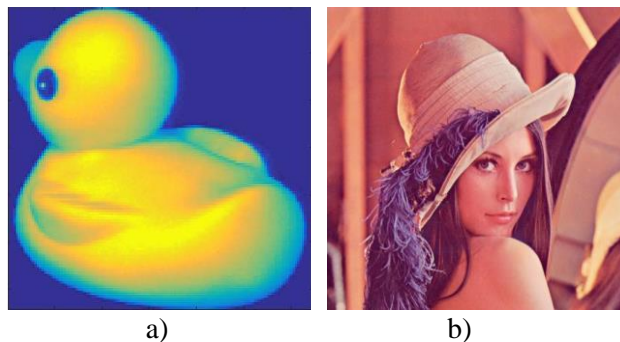


Fig. 2.1: a) Original "duck" image b)original "Lena" image

2.6.1 Binary Image[66]:

The value of a pixel of a binary image is either 1 or 0 depending on whether the pixel belongs to the foreground or to the background. In practice, images are defined over a rectangular frame called the definition domain of the image. The definition domain is often referred to as the image plane (it is actually a plane for 2-D images). Fig. 2.2, shows an example for a binary image with foreground pixels in white and background pixels in black; using the built in binary function in matlab.

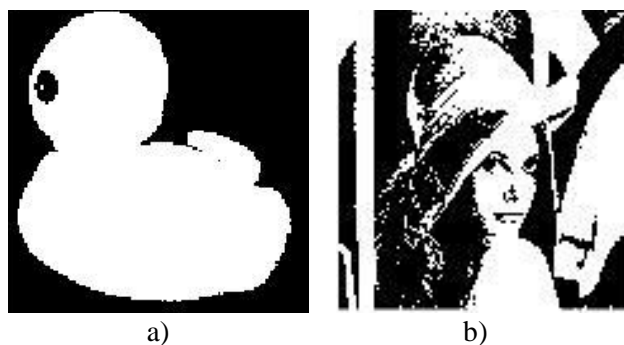


Fig. 2.2: a) Binary "Duck" image b) binary "Lena" image

2.6.2. Greyscale Image [80]:

A grayscale image is to be considered as a function $g(i, j)$, where i and j define the spatial (plane) coordinates, and the value of g at any pixel with coordinates (i, j) is called the intensity of the image at that pixel. For the grayscale image, The range of the values of the function $g(i, j)$ is not restricted to the two values, 0 and 1; as in the case of the binary image, but it is extended to a wider finite set of non-negative integers, starting from 0 to 255.

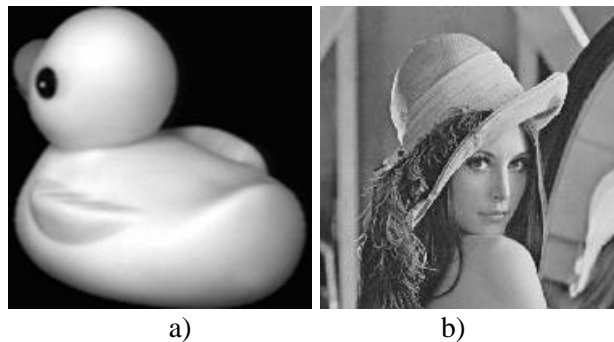


Fig. 2.3: a) Grayscale "Duck" image b) grayscale "Lena" image

2.6.3 Fuzzy Image [48]:

With the concept of fuzziness, a fuzzy image is a function that assigns to each pixel a value of membership denoting how much it belongs to the white set. Whereas, the values of the intensity of each pixel in the grayscale image ranges from 0 to 255, the membership degrees in the fuzzy image are to be expected in the interval $[0, 1]$.one of the most used methods to compute the membership degrees is to normalize the values of the intensity of each pixel. Fig.2.4, we will use the fuzzy image show in Fig.2.4(b); which is deduced form the original color image shown in Fig.2.4(a). we used the matlab to get fuzzy image.

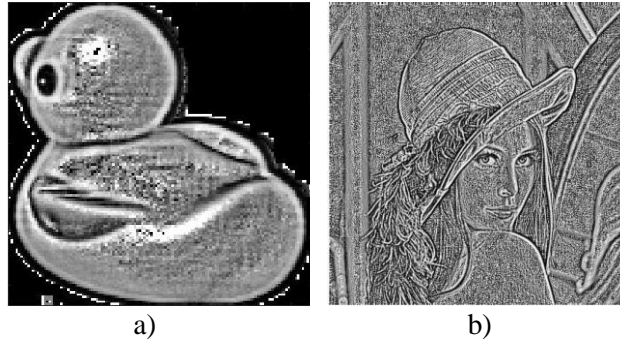


Fig. 2.4: a) Fuzzy "Duck" image b) fuzzy "Lena" image

2.6.3.1 Implementation and Experimental Results:

We use the following algorithm to transform the image into fuzzy image.

1. Read the gray scale image .
2. Compute the maximum intensity values for each pixel in image.
3. Normalize the intensity of each pixel with respect to the maximum value

$$\frac{g(i,j)}{\max(g(i,j))} ; \text{ computed in the previous step.}$$

4. Construct to the matrix $\mu(x)$.

2.6.4 Neutrosophic Image:

The image in the neutrosophic domain each pixel of the image is represented by three values.

2.6.4.1 The Image in the Neutrosophic Domain [17, 25]:

Mathematically, a gray image is represented by an $m \times n$ array $I_m = [g(i,j)]_{m \times n}$ with entities $g(i,j)$ corresponding to the intensity of the pixel located at (i,j) . In this section we are transforming the image I_m into neutrosophic domain using three membership functions T, I and F [25]. A pixel $p(i,j)$ in the image is described by a triple $(T(i,j), I(i,j), F(i,j))$. where $T(i,j)$ is the membership degree of the pixel in the white set, and $F(i,j)$ is its membership degree in the non-white (black) set; while $I(i,j)$ is how much it is neither white nor black the values of $T(i,j), I(i,j)$ and $F(i,j)$ are defined as follows:

$$\begin{aligned}
T(i, j) &= \frac{\bar{g}(i, j) - \bar{g}_{min}}{\bar{g}_{max} - \bar{g}_{min}}, \\
I(i, j) &= \frac{\delta(i, j) - \delta_{min}}{\delta_{max} - \delta_{min}}, \\
F(i, j) &= 1 - T(i, j) = \frac{\bar{g}_{max} - \bar{g}(i, j)}{\bar{g}_{max} - \bar{g}_{min}}.
\end{aligned} \tag{2.1}$$

Where $\bar{g}(i, j) = \frac{1}{\omega \times \omega} \sum_{m=i-\frac{\omega}{2}}^{m=i+\frac{\omega}{2}} \sum_{n=j-\frac{\omega}{2}}^{n=j+\frac{\omega}{2}} g(m, n)$ is the mean intensity in some

neighborhood ω of the pixel and $\delta(i, j)$ is the homogeneity value computed by the absolute value of difference between the intensity and its local mean value

$$\delta(i, j) = abs(g(i, j) - \bar{g}(i, j)), \quad \bar{g}_{max} = max \bar{g}(i, j), \quad \bar{g}_{min} = min \bar{g}(i, j),$$

$$\delta_{max} = max \delta(i, j) \text{ and } \delta_{min} = min \delta(i, j).$$

Hence, in the neutrosophic domain the image becomes a 3D matrix $Im_{ND} =$

$[T_{ij} \quad I_{ij} \quad F_{ij}]$, with dimensions $m \times n \times 3$.

2.6.4.1.a Implementation and Experimental Results:

We use the following algorithm to transform the image into neutrosophic domain.

1. Read the gray scale image.
2. Compute the local mean intensity for each pixel in image.
3. Compute the maximum and minimum values of the local. mean intensites.
4. Compute the divergence between the intensity of each. pixel and its local mean intensity.
5. Compute the maximum and minimum values of the divergence induced in the previous step.
6. Construct the matrix T,the truth valu of each pixel.
7. Construct the indeterminate matrix I.
8. Construct the falssness matrix F.

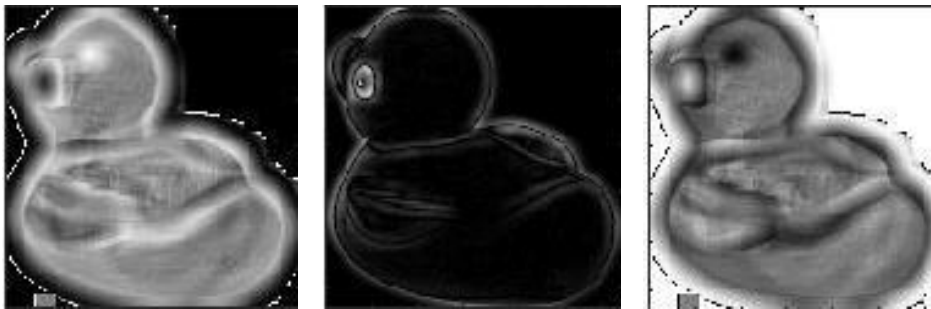


Fig.2.5(i): Neutrosophic "Duck" image $\langle T_A, I_A, F_A \rangle$ respectively

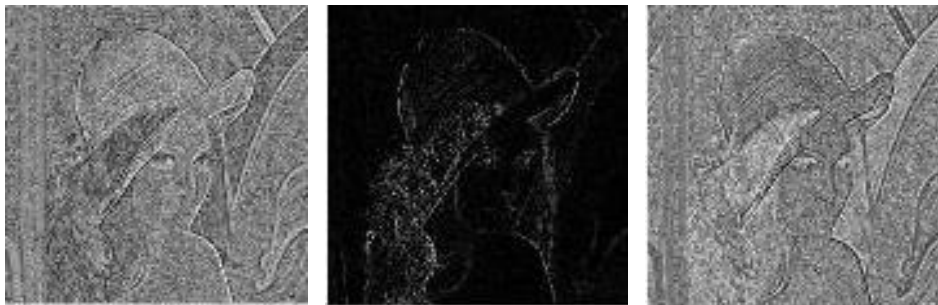


Fig.2.5(ii): Neutrosophic "Lena" image $\langle T_A, I_A, F_A \rangle$ respectively

2.6.4.2 Neutrosophic Crisp Image:

Let X be a non-empty fixed Set, a neutrosophic crisp set A , can be defined as a triple of the form $\langle A^1, A^2, A^3 \rangle$, where A^1, A^2 and A^3 are crisp subsets of X . The three components represent a classification of the elements of the space X according to some event A ; the subset A^1 contains all the elements of X that are supportive to A , A^3 contains all the elements of X that are against A , and A^2 contains all the elements of X that stand in a distance from being with or against A . Consequently, every crisp event A in X can be considered as a NCS having the form: $A = \langle A^1, A^2, A^3 \rangle$. The set of all Neutrosophic Crisp Sets of X will be denoted by $\mathcal{NC}(X)$.



Fig.2.5(i): Neutrosophic crisp "Duck" image: $\langle A^1, A^2, A^3 \rangle$ respectively



Fig.2.5(ii): Neutrosophic crisp "Lena" image: $\langle A^1, A^2, A^3 \rangle$ respectively

2.7 Conclusion:

In this chapter, the theory on crisp sets, the basic ideas of the fuzzy sets, intuitionistic sets and neutrosophic sets were discussed in detail in this chapter. The laws and properties of fuzzy sets are introduced along with that of the crisp sets. Neutrosophic operations inherit properties and restrictions of fuzzy sets. We review the definitions of the crisp sets with binary image, we will use the fuzzy image its depended on fuzzy set; where we introduce neutrosophic image and neutrosophic crisp image.

Chapter 3

Mathematical Morphology

3.1 Introduction:

In late 1960's, a relatively separate part of image analysis was developed; eventually known as "The Mathematical Morphology". Mostly, it deals with the mathematical theory of describing shapes using the concept of sets in order to extract meaningful information's from images; MM refers to a branch of nonlinear image processing and analysis developed initially by Georges Matheron and Jean Serra [47, 68], that concentrates on the geometric structure within an image. It has developed from binary morphology to grayscale morphology, in order to handle binary and grayscale images. Its basic idea is to measure corresponding shape in image using some structure element with certain shape to analyze image and recognize object. Dilating and eroding are two basic operations of MM. These two operations can make up of some compound operations, and bring some practical morphology algorithm. An image can be represented by a set of pixels. A morphological operation uses two sets of pixels, i.e., two images: the original data image to be analyzed and a structuring element which is a set of pixels constituting a specific shape.

The Remaining of This Chapter is Structured as Follows:

In § 3.2 we review the shape of the structure element introducing the basic concept of the binary morphological operators (binary dilation, binary erosion, binary opening and binary closing). The properties of several morphological filters; (different definition of boundary and hat filter) are defined in § 3.3. In § 3.4 we introduce the basic operations of grayscale morphology, namely the dilation, erosion, opening and closing. A revision of the concepts of fuzzy morphological operations, and a study of its algebraic properties, as well as some fuzzy morphological filters are to be presented in §3.5.

3.2 Structuring Element [31, 68]:

Morphological techniques probe an image with a small shape or template called a structuring element (SE); is simply a binary image, i.e. a small matrix of pixels, each with a value of 0 or 1. The matrix dimensions specify the size of SE. The SE is positioned at all possible locations in the image and it is compared with the corresponding neighborhood of pixels. Some operations test whether the element "fits" within the neighborhood, while others test whether it "hits" or intersects the neighborhood. Examples of SE: shaded square denotes a member of the SE The origins of SEs are marked by a black dot. When working with images, SE should be rectangular: append the smallest number of background elements.

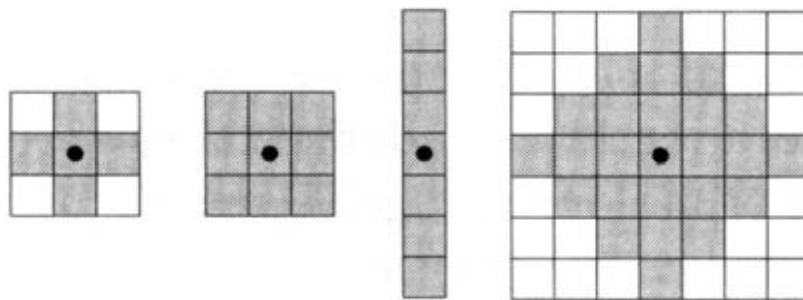


Fig 3.1: *shape of the structuring element*

3.3 Binary Morphology [72]:

In this section, we review the definitions of the classical binary morphological operators as given by Heijmans [31]; which are consistent with the original definitions of the Minkowski addition and subtraction [26]. For the purpose of visualizing the effect of these operators, we will use the binary image show in Fig.2.2.

3.3.1 Basic Binary Morphology Operations:

In this section we briefly review the basic morphological operations, the dilation, the erosion, the opening and the closing.

3.3.1.1 Binary Dilation: (Minkowski addition)

Dilation is one of the basic operations in mathematical morphology, which originally developed for binary images [49, 77]. The dilation operation uses a structuring element for exploring and expanding the shapes contained in the input image. In binary morphology, dilation is a shift-invariant (translation invariant) operator, strongly related to the Minkowski addition. For any Euclidean space E and a binary image A in E , the dilation of A by some structuring element B is defined by: $A \oplus B = \cup_{b \in B} A_b$ where A_b is the translate of the set A along the vector b , i.e., $A_b = \{a + b \in E | a \in A, b \in B\}$. The dilation is commutative and may also be given by:

$$A \oplus B = B \oplus A = \cup_{a \in A} B_a. \quad (3.1)$$

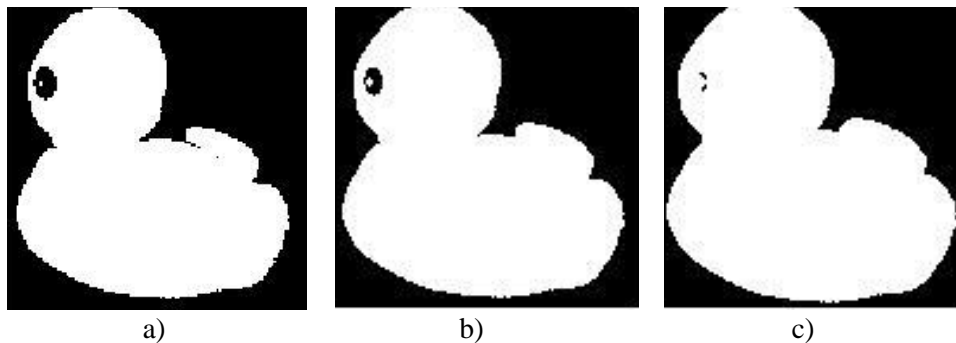


Fig. 3.2: Dilation binary image: a) Binary image b) Dilation with $SE(3)$ c) Dilation with $SE(7)$

An interpretation of the dilation of A by B can be understood as, if we put a copy of B at each pixel in A and union all of the copies, then we get $A \oplus B$. The dilation can also be obtained by: $A \oplus B = \{b \in E | (-B) \cap A \neq \emptyset\}$, where $(-B)$ denotes the reflection of B , that is, $-B = \{x \in E | -x \in B\}$. the reflection satisfies the following property: $-(A \oplus B) = (-A) \oplus (-B)$.

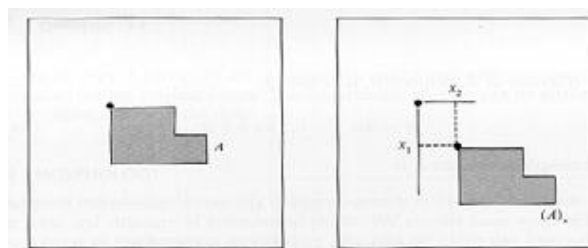


Fig. 3.3: Translation

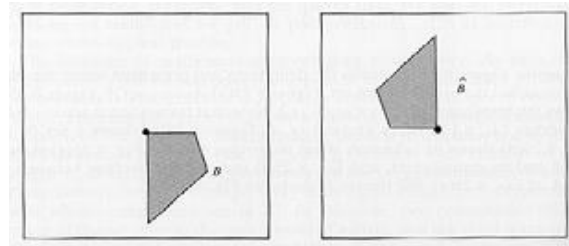
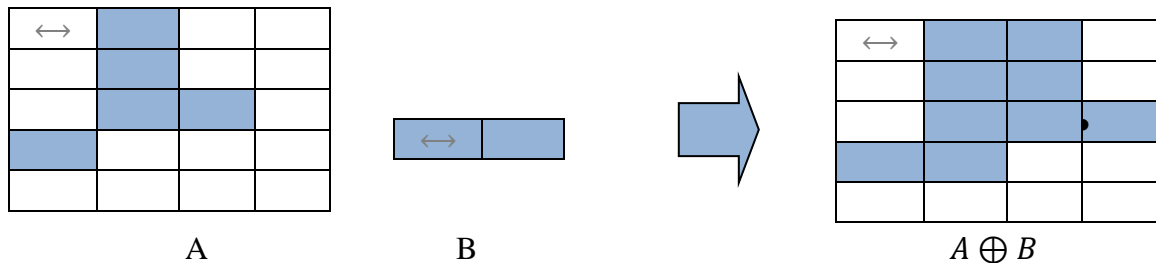


Fig. 3.4: Reflection

Example 3.1: This illustrates an instance of the dilation operation. The coordinate system we use for all the examples in the next few sections is (row, column).

$$A = \{(0, 1), (1, 1), (2, 1), (2, 2), (3, 0)\}, \quad B = \{(0, 0), (0, 1)\}$$



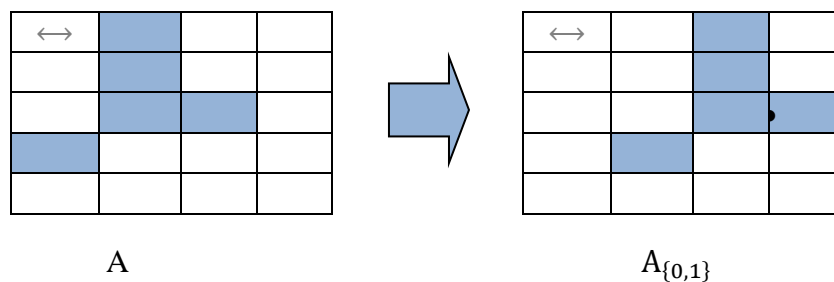
$$A \oplus B = \{(0, 1), (0, 2), (1, 1), (2, 1), (3, 1), (2, 2), (3, 0), (1, 2), (2, 3)\}.$$

In morphological dilation, the roles of the sets A and B are symmetric, that is, the dilation operation is commutative because addition is commutative [30].

3.3.1 Definition: Let A be a subset of E^N and $x \in E^N$. The translation of A by b is denoted by A_b and is defined by: $A_b = \{a + b \in E \mid a \in A, b \in B\}$. (3.3)

Example 3.2: this illustrates of translation.

$$A = \{(0, 1), (1, 1), (2, 1), (2, 2), (3, 0)\}, \quad b = (0, 1).$$



$$A_b = \{(0, 2), (1, 2), (2, 2), (2, 3), (3, 1)\}.$$

3.3.1.2 Binary Erosion: (Minkowski subtraction)

Strongly related to the Minkowski subtraction, the erosion of the binary image A by the SE B is defined by: $A \ominus B = \bigcap_{b \in B} A_{-b}$. (3.4)

Unlike dilation, erosion is not commutative, much like how addition is commutative while subtraction is not [30, 47]. An interpretation for the erosion of A by B can be understood as, if we again put a copy of B at each pixel in A , this time we count only those copies whose translated structuring elements lie entirely in A ; hence $A \ominus B$ is all pixels in A that these copies were translated to. The erosion of A by some structuring element B is defined by: $A \ominus B = \{p \in E \mid B_p \subseteq A\}$, where B_p is the translation of B by the vector p , i.e., $B_p = \{b + p \in E \mid b \in B\}$, $\forall p \in E$. (3.5)

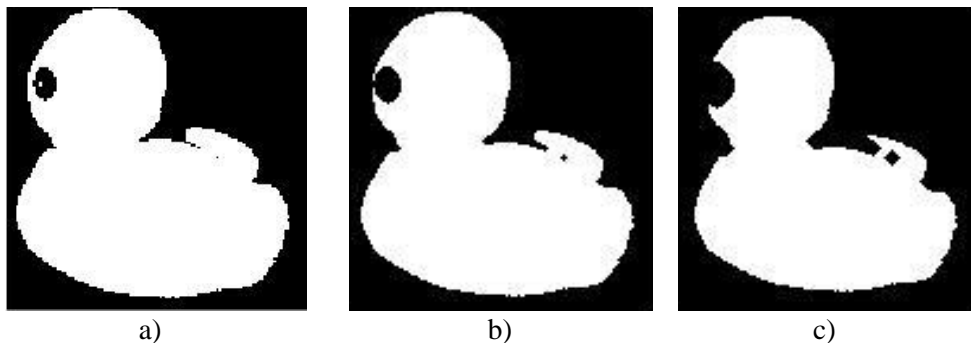


Fig. 3.5: Erosion binary image: a) Binary image b) Erosion with SE(3) c) Erosion with SE(7)

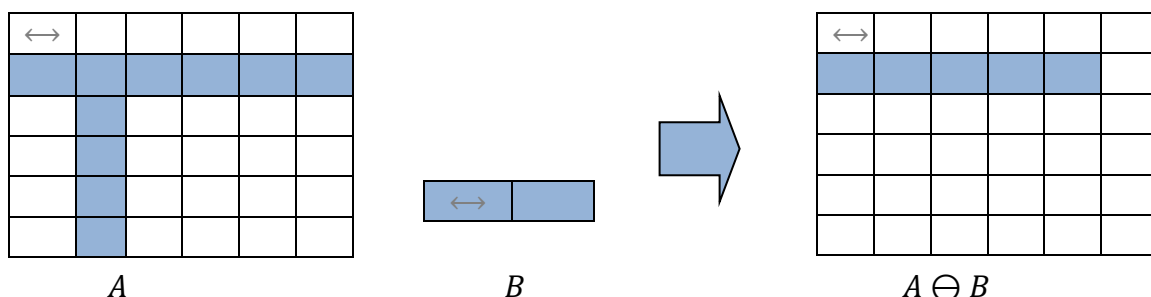
3.3.2 Definition: The erosion of A by B is denoted by $A \ominus B$ and is defined by:

$$A \ominus B = \{x \in E^N \mid x + b \in A \text{ for every } b \in B\}.$$

Example 3.3: this illustrates an instance of erosion.

$$A = \{(1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (3, 1), (4, 1), (5, 1)\},$$

$$B = \{(0,0), (0,1)\}. \quad A \ominus B = \{(1, 0), (1,1), (1,2), (1,3), (1,4)\}.$$



Erosion does not possess the commutative property. Some erosion equivalent terms are "shrink" and "reduce". Erosion of an image A by a SE B is the intersection of all translations of A by the points $-b$, where $b \in B$.

3.3.1.3 Binary Opening [49]:

The opening of A by B is obtained by the erosion of A by B , followed by dilation of the resulting image by B : $A \circ B = (A \ominus B) \oplus B$. (3.6)

The opening is also given by $A \circ B = \bigcup_{B_x \subseteq A} B_x$, which means that, an opening can be considered to be the union of all translated copies of the SE element that can fit inside the object. Generally, openings can be used to remove small objects and connections between objects.

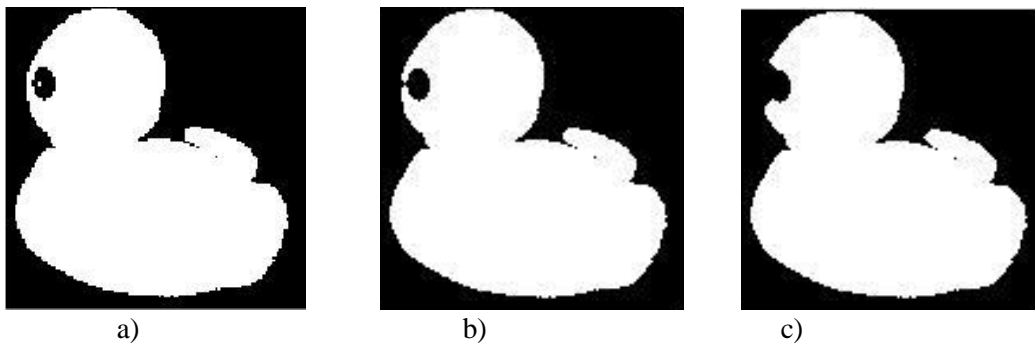


Fig.3.6: *Opening binary image: a) Binary image b) Opening image with SE(3) c) Opening image with SE(7)*

3.3.1.4 Binary Closing [28]:

The closing of A by B is obtained by the dilation of A by B , followed by erosion of the resulting structure by B : $A \bullet B = (A \oplus B) \ominus B$. (3.7)

The closing can also be obtained by $A \bullet B = (A^c \circ (-B))^c$, where A^c denotes the complement of A relative to E (that is, $A^c = \{a \in E \mid a \notin A\}$). Whereas opening removes all pixels where the SE won't fit inside the image foreground, closing fills in all places where the SE will not fit in the image background, that is opening removes some objects, while closing removes small holes.

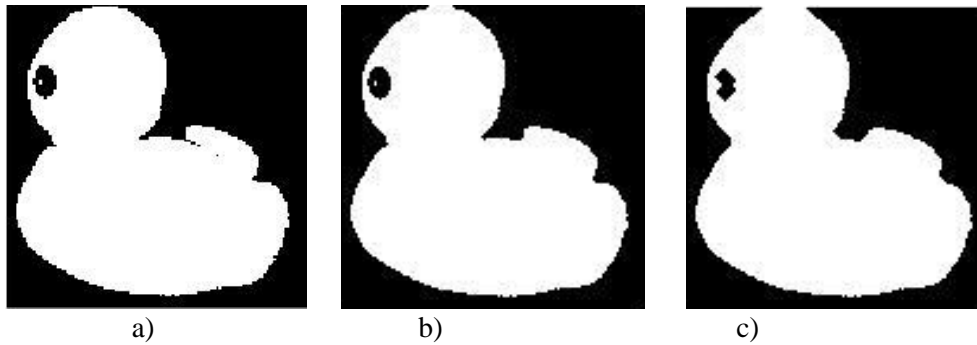


Fig. 3.7: Closing binary image: a) Binary image b) Closing image with $SE(3)$
c) Closing image with $SE(7)$

3.3.2 Properties of Binary Operations:

Here are some properties of the basic binary morphological operations (dilation, erosion, opening and closing [47]). We define the power set of X , denoted by $P(X)$, to be the set of all crisp subset of X . For all A, B and $C \in P(X)$, the following properties hold:

3.3.2.1 Properties of Binary Dilation:

- **Commutative:** $A \oplus B = B \oplus A$.
- **Associative:** $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.
- **Extensive:** $A \subseteq (A \oplus B)$ if $0 \in B$.
- **Increasing:** $A \subseteq B \Rightarrow (A \oplus C) \subseteq (B \oplus C)$.
- **Commutates with union, not with intersection:**

$$(A \cup C) \oplus B = (A \oplus B) \cup (C \oplus B).$$

- **Commutativity of dilation:** $A \oplus (C \cup B) = (A \oplus C) \cup (A \oplus B)$,
 $(A \cap C) \oplus B \subseteq (A \oplus B) \cap (C \oplus B)$,
 $A \oplus (C \cap B) \subseteq (A \oplus C) \cap (A \oplus B)$.
- **Iterativity property:** $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

3.3.2.2 Properties of Binary Erosion [72]: if B contains the origin, that is;

- **Duality of erosion and dilation with respect to complementation:**

$$A \oplus B = (A^c \ominus (-B))^c, \text{ and } A \ominus B = (A^c \oplus (-B))^c.$$

- **Anti-extensive:** $A \ominus B \subseteq A$, if B contains the origin, that is, $0 \in b$.
- **Increasing:** $A \subseteq B \Rightarrow (C \ominus A) \supseteq (C \ominus B)$.
- **Commutates with intersection, not with union:**

$$(A \cap C) \ominus B = (A \ominus B) \cap (C \ominus B),$$

$$(A \cup C) \ominus B \supseteq (A \ominus B) \cap (C \ominus B),$$

$$A \ominus (B \cap C) \supseteq (A \ominus B) \cup (A \ominus C),$$

$$A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C).$$

- **Iterativity property:** $(A \ominus B) \ominus C = A \ominus (B \oplus C)$.

3.3.2.3 Properties of Binary Opening and Closing [20]:

- $A \oplus B = (A \oplus B) \circ B = (A \bullet B) \oplus B$.
- $(A \bullet B) \bullet B = A \bullet B$.
- $A \ominus B = (A \circ B) \ominus B = (A \ominus B) \bullet B$.
- $(A \circ B) \circ B = A \circ B$.
- $A \circ B \subseteq A \subseteq A \bullet B$.
- **Increasing:** if $A \subseteq B$ then $A \circ C \subseteq B \circ C$.
- **Increasing:** if $A \subseteq B$ then $A \bullet C \subseteq B \bullet C$.
- **Opening and closing satisfy the duality that is:**

$$A \bullet B = (A^c \circ (-B))^c, \text{ and } A \circ B = (A^c \bullet (-B))^c.$$

3.3.3 Algebraic Properties Crisp Mathematical Morphology [20, 21]:

In this section, we review some of the algebraic properties of the crisp erosion and crisp dilation, as well as the crisp opening and crisp closing operator.

3.3.3.1 Properties of the Crisp Dilation [29, 31]:

Proposition 3.3.1: $A \oplus B = B \oplus A$

$$\begin{aligned} A \oplus B &= \{C \mid C = a + b \text{ for } a \in A, b \in B\} \\ &= \{C \mid C = b + a \text{ for } a \in A, b \in B\} = B \oplus A. \end{aligned}$$

Proposition 3.3.2: $A \oplus (B \oplus C) = (A \oplus B) \oplus C.$

Proof: $X \in A \oplus (B \oplus C)$ if and only if there exists $a \in A, b \in B$ and $c \in C$ such that $X = a + (b + c)$, $X \in A \oplus (B \oplus C)$, if and only if there exists $a \in A, b \in B$ and $c \in C$ such that; $X = (a + b) + c$ but $a + (b + c) = (a + b) + c$, since addition is associative, Therefore, $A \oplus (B \oplus C) = (A \oplus B) \oplus C.$ (3.8)

Proposition 3.3.3: (*Dilation is Increasing*)

$A \subseteq B$ implies $A \oplus D \subseteq B \oplus D.$

Proof: Suppose $A \subseteq B$. Let $x \in A \oplus D$. Then for some $a \in A$ and $d \in D$, $x = a + d$. Since $a \in A$ and $A \subseteq B$, $a \in B$; But $a \in B$ and $d \in D$ implies $x \in B \oplus D$.

Proposition 3.3.4:

- $(A \cap B) \oplus C \subseteq (A \oplus C) \cap (B \oplus C).$
- $A \oplus (B \cap C) \subseteq (A \oplus B) \cap (A \oplus C).$

Proof: Suppose $b \in (A \cap B) \oplus C$. Then for some $y \in A \cap B$ and $c \in C$, $a = y + c$. Now $y \in A \cap B$ implies $y \in A$ and $y \in B$. But $y \in A, c \in C$ and $a = y + c$ implies $a \in A \oplus B, y \in B, c \in C$ and $a = y + c$ implies $a \in B \oplus A$. Hence $(A \oplus C) \cap (B \oplus C), A \oplus (B \cap C) \subseteq (A \oplus B) \cap (A \oplus C).$

3.3.3.2 Properties of the Crisp Erosion :

Proposition 3.3.6: $A \ominus B = \bigcap_{b \in B} A_{-b}.$

Proof: Let $x \in A \ominus B$, Then for every $b \in B$, $x + b \in A$, But $x + b \in A$ implies $x \in A_{-b}$. Hence for every $b \in B$, $x \in A_{-b}$. This implies $x \in \bigcap_{b \in B} A_{-b}.$

Let $\bigcap_{b \in B} A_{-b}$; Then for every $b \in B$, $x \in A_{-b}$. Hence, for every $b \in B$, $x + b \in A$. Now by definition of erosion $x \in A \ominus B$.

Proposition 3.3.7: (Erosion is increasing)

$A \subseteq B$ implies $A \ominus K \subseteq B \ominus K$.

Proof: Let $x \in A \ominus K$, Then $x + K \in A$, every $k \in K$. but $A \subseteq B$. Hence, $x + k \in A$ for every $k \in K$. By definition of erosion, $x \in B \ominus K$. creasing property of erosion [61, 62]. On the other hand, if A and B are SE and B is contained in A , then the erosion of an image D by A is necessarily more severe than erosion by B , that is, D eroded by A will necessarily be contained in D eroded by B .

Proposition 3.3.8: $A \supseteq B$ implies $D \ominus A \subseteq D \ominus B$.

Proof: Let $x \in D \ominus A$. Then $x + a \in D$ for every $a \in A$. But $B \subseteq A$. Hence, $x + a \in D$ for every $a \in B$. Now by definition of erosion, $x \in D \ominus B$. The dilation and erosion transformations bear a marked similarity, in that what one does to the image foreground the other does to the image background. Indeed, their similarity can be formalized as a duality relationship. Recall that two operators are dual when the negation of a formulation employing the first operator is equal to that formulation employing operator on the negated variables. An example is De Morgan's law, illustrating the duality of union and intersection $(A \cup B)^c = A^c \cap B^c$.

3.3.1 Definition: Let $B \in E^N$. The reflection of B is denoted $(-B)$ that is $-B = \{x \in E \mid -x \in B\}$. (3.9)

The reflection occurs about the origin. Matheron [47] refers to $(-B)$ as "the symmetrical set of B with respect to the origin". Serra [69] refers to B as " B transpose". the duality of Dilation and Erosion employs both logical and geometric negation because of the different roles of the image and the SE in an expression employing these morphological operators.

Theorem 3.3.1: (*Erosion Dilation Duality*)

$$(A \ominus B)^c = A^c \oplus (-B). \quad (3.10)$$

Proof: $x \in (A \ominus B)^c$ if and only if $x \notin A \ominus B$. $x \notin A \ominus B$ if and only if there exists $b \in B$ such that $x + b \notin A$. There exists $b \in B$ such that $x + b \in A^c$. if and only if there exists $b \in B$ such that $x \in (A^c)_{-b}$. There exists $b \in B$ such that $x \in (A^c)_{-b}$ if and only if $x \in \bigcup_{b \in B} A^c_{-b}$. Now, $x \in \bigcup_{b \in B} A^c_{-b}$ if and only if $x \in \bigcup_{b \in (-B)} A^c_b$; and $x \in \bigcup_{b \in (-B)} A^c_b$ if and only if $x \in A^c \oplus (-B)$.

3.3.4 Basic Binary Morphological Filters [79]:

In image processing and analysis, it is important to extract features of objects, describe shapes, and recognize patterns. Such tasks often refer to geometric concepts, such as size, shape, and orientation. MM takes this concept from set theory, geometry, and topology and analyzes geometric structures in an image. Most essential image processing algorithms can be represented in the form of morphological operations. In this section we review several basic morphological filters, such as boundary extraction and hat filter.

3.3.4.1 Some Types of Crisp Boundary Using Dilation and Erosion:

In this section, we present some very useful words based on combinations of crisp erosions and crisp dilations and leading to the definition of morphological gradient operators [77].

3.3.4.1.a The External Crisp Boundary Filter [14]:

Boundary internal of a set A requires first the dilating of A by a SE B and then taking the set difference between its dilation and A . $B_{ext}(A) = (A \oplus B) - A$. (4.11)

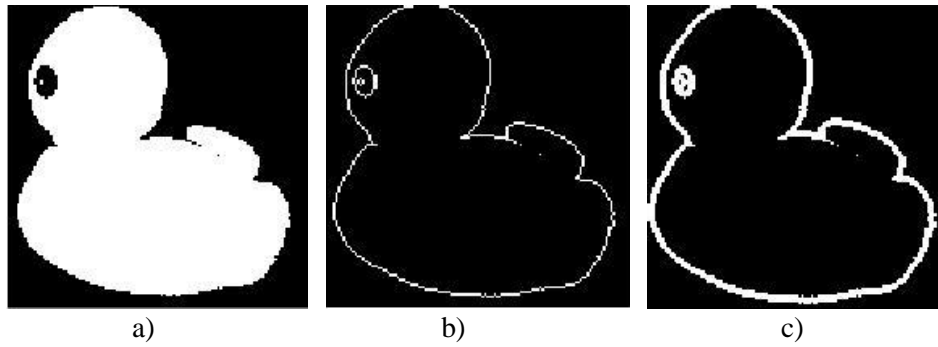


Fig. 3.8: External boundary image: a) Binary image
b) External boundary filter with SE(3) c) External boundary filter with SE(7)

This would give us all background pixels that bordered the object. Or, if we wanted all foreground pixels that bordered the background, we could use:

3.3.4.1.b The Internal Crisp Boundary Filter [51]:

Boundary internal of a set A requires first the erosion of A by a SE B and then taking the set difference between A and erosion image. That is, the boundary internal of a set A is obtained by: $B_{int}(A) = A - (A \ominus B)$. (3.12)

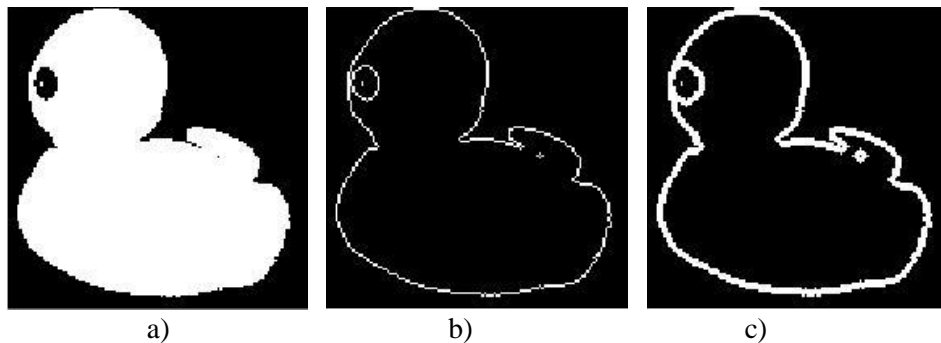


Fig. 3.9: Internal boundary image: a) Binary image
b) Internal boundary filter with SE(3) c) Internal boundary filter with SE(7)

3.3.4.1.c The Gradient Crisp Boundary Filter [44]:

Determining the gradient of an image is a fundamental image processing operation that is often used as a precursor to other, more advanced operations such as feature extraction and segmentation. The morphological gradient operator provides a simple approach to find the gradient of an image by combining the dilation and erosion operators. The morphological image gradient operator is defined as:

$$B_{gradient}(A) = (A \oplus B) - (A \ominus B). \quad (3.13)$$

The dilation thickens regions in an image and the erosion shrinks them. Therefore, their difference emphasizes the boundaries between regions. If the SE is relatively small, homogeneous areas will not be affected by dilation and erosion, so the subtraction tends to eliminate them. The net result is an image with the gradient-like effect. The effect of morphological gradient operation is shown in Fig. 3.11.

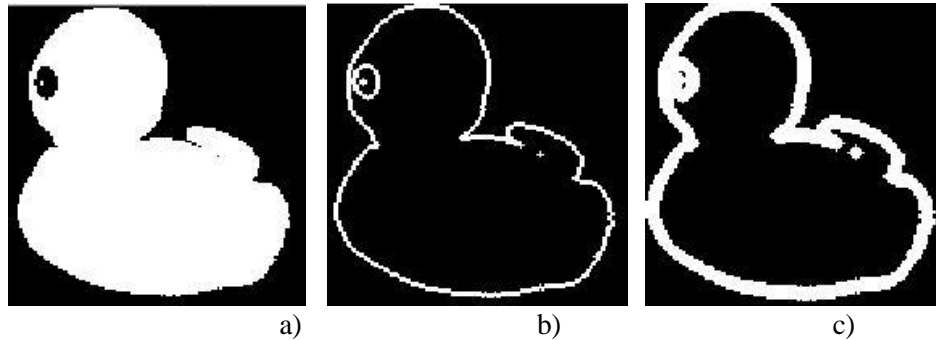


Fig. 3.10: Gradient boundary image: a) Binary image
b) Gradient boundary filter with SE(3) c) Gradient boundary filter with SE(7)

3.3.4.1.d The Outline Crisp Boundary Filter:

The outline of a binary image can be computed using erosion followed by a subtraction.

$$B_{outline}(A) = co(A \ominus B) \cap A. \quad (3.14)$$

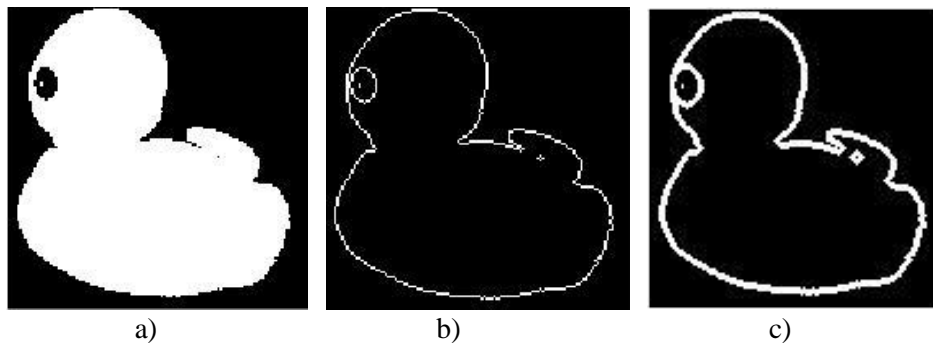


Fig. 3.11: Outline boundary image: a) Binary image
b) Outline boundary filter with SE(3) c) Outline boundary filter with SE(7)

3.3.4.2 Combination External and Internal Crisp Boundary [14, 28]:

Dilation and erosion can be used in combination with image subtraction to obtain the morphological extraction A of an image as:

$$1. B_{grad}(A) = \max[B_{ext}(A), B_{int}(A)]. \quad (3.15)$$

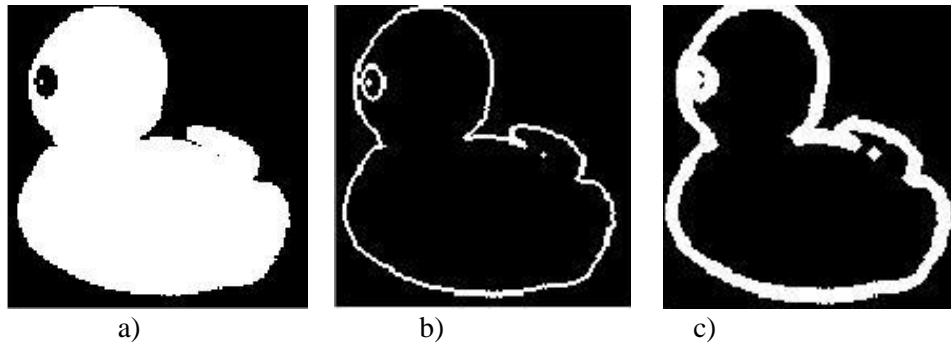


Fig. 3.12: *Grad boundary image: a) Binary image b) Grad boundary filter with SE(3) c) Grad boundary filter with SE(7)*

$$2. B_{div}(A) = [B_{ext}(A) - B_{int}(A)]. \quad (3.16)$$

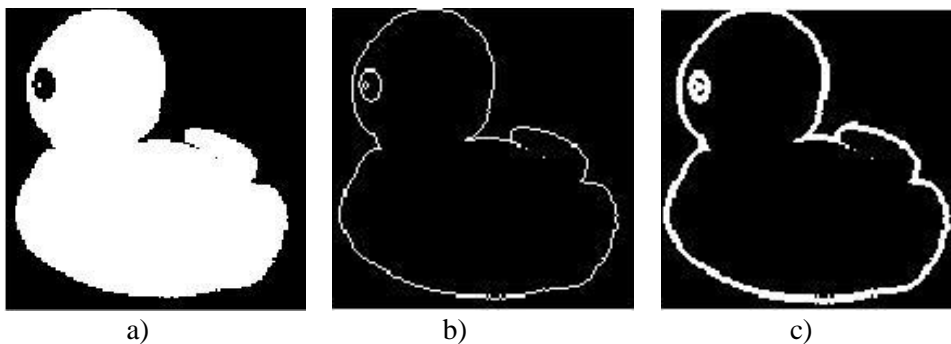


Fig. 3.13: *Div. boundary image: a) Binary image b) Div. boundary filter with SE(3) c) Div. boundary filter with SE(7)*

3.3.4.3 Hat Filter [60]:

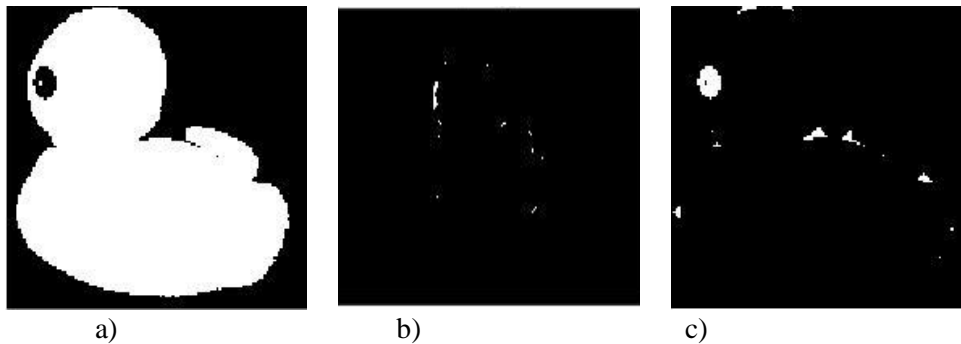
In MM and digital image processing, top-hat transform is an operation that extracts small elements and details from given images. There exist two types of hat filters: The Top-hat filter is defined as the difference between the input image and its opening by some structuring element; The Bottom-hat filter is defined dually as the difference between the closing and the input image. Top-hat filter are used for various image processing tasks, such as feature extraction, background equalization, image enhancement. If an opening removes small structures, then the difference of the original image and the opened image should bring them out. This is exactly what the white top-hat filter does, which is defined as the residue of the original and opening:

$$\textbf{Top-hat filter: } Top_{hat}(A) = A - (A \circ B). \quad (3.17)$$

The counter part of the Top-hat filter is the Bottom-hat filter which is defined as the residue of closing and the original:

$$\text{Bottom-hat filter: } Bottom_{hat}(A) = (A \bullet B) - A. \quad (3.18)$$

These filters preserve the information removed by the opening and closing operations, respectively. They are often cited as white top-hat and black top-hat. Fig. 3.16 show the results obtained when applying Top-hat and Bottom-hat filter.



a) b) c)
Fig.3.14: Hat filter image: a) Binary image b) Top-hat filter image
 c) Bottom-hat filter image

3.4 Grayscale Mathematical Morphology [80]:

Grayscale image depended on mathematics theory. It has developed from binary morphology to grayscale morphology, and it is a new method of image process. The binary morphological operations of dilation, erosion, opening and closing are all naturally extended to grayscale imagery by the use of a min or max operation. From this definition we will proceed to the representation which indicates that grayscale dilation can be computed in terms of a maximum operation and a set of addition operations. A similar plan is followed for erosion which can be evaluated in terms of a minimum operation and a set of subtraction operations. We extend the basic operations of dilation, erosion, opening, and closing to grayscale images. Assume that $f(i,j)$ is a greyscale image and b is a SE and both functions are discrete. Similarly to binary morphology, the SE are used to examine a given image for specific properties. Dilating and eroding are two basic operations of MM. These two operations can make up of some compound operations and bring some practical morphology algorithm.

3.4.1 Grayscale Dilation and Erosion [80]:

Since grayscale erosion with a flat SE computes the min intensity value of f in every neighborhood, the eroded grayscale image should be darker (bright features are reduced, dark features are thickened, background is darker). The effects of dilation are opposite.

3.4.1.1 Grayscale Dilation [77]:

The dilation of f by a flat SE b at any location (i, j) is defined as the maximum value of the image in the window outlined by $-b$ when the origin of $-b$ is at (i, j) , that is $[f \oplus b](i, j) = \max_{(s,t) \in b} \{f(i-s, j-t)\}$; where we used that $-b = b(-i, -j)$; The explanation is similar to one for erosion except for using maximum instead of minimum and that the SE is reflected about the origin.

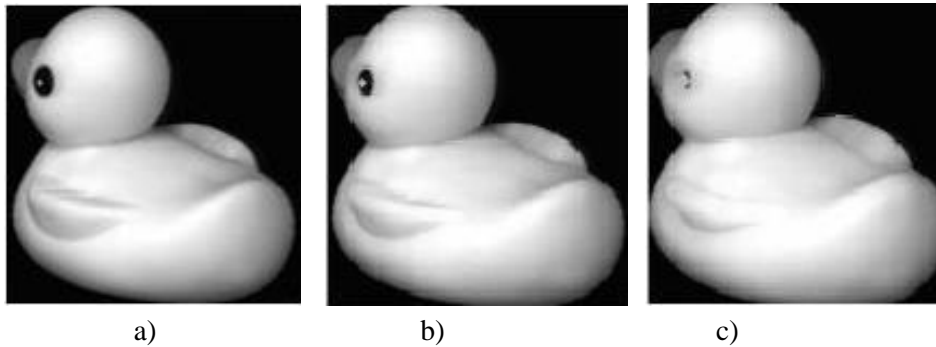


Fig. 3.15: Dilation grayscale image: a) Grayscale image
b) Dilation image with SE(3) c) Dilation image with SE(7)

3.4.1.2 Grayscale Erosion [80]:

The erosion of f by a flat SE b at any location (i, j) is defined as the minimum value of the image in the region coincident with b when the origin of b is at (i, j) . Therefore, the erosion at (i, j) of an image f by a SE b is given by:

$$[f \ominus b](i, j) = \min_{(s,t) \in b} \{f(i+s, j+t)\}. \quad (3.20)$$

where, similarly to the correlation, i and j are incremented through all values required so that the origin of visits every pixel in f . That is, to find the erosion of f by b , we place the origin of the SE at every pixel location in the image. The erosion is the minimum value of f from all values of f in the region of f coincident with b .

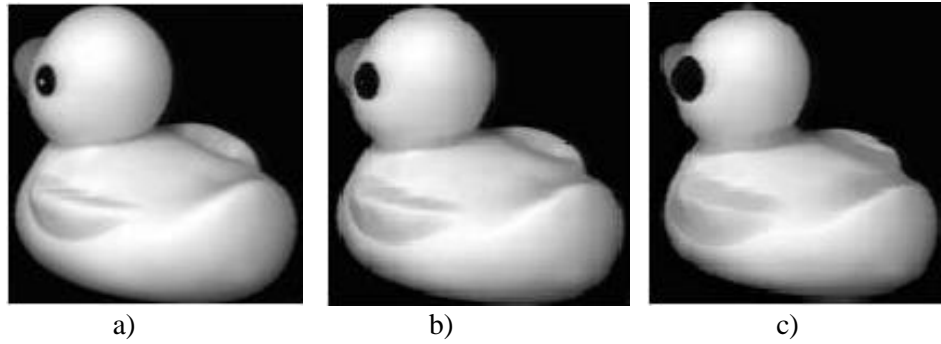


Fig. 3.16: Erosion grayscale image: a) Grayscale image
b) erosion image with SE(3) c) erosion image with SE(7)

3.4.2 Grayscale Opening and Closing [54]:

Opening and closing of images have a simple geometrical interpretation. Assume that the image $f(i, j)$ is viewed as a surface intensity values are interpreted as heights over the x_y plane. Then the opening of f by b can be interpreted as "pushing" the SE b up from below against the undersurface of f .

3.4.2.1 Grayscale Opening:

Grayscale opening operation is erosion followed by dilation, it can smooth the contour of image and remove small extrudes. $f \circ b = (f \ominus b) \oplus b$.

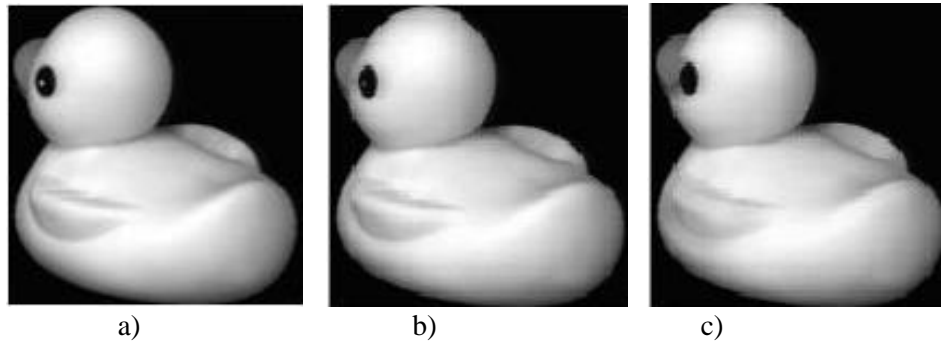


Fig. 3.17: Opening grayscale image: a) Grayscale image
b) Opening image with SE (3) c) Opening image with SE (7)

3.4.2.2 Grayscale Closing:

Grayscale closing operation is dilation followed by erosion, it can smooth the image outline and recover the holes. $f \bullet b = (f \oplus b) \ominus b$. (3.22)

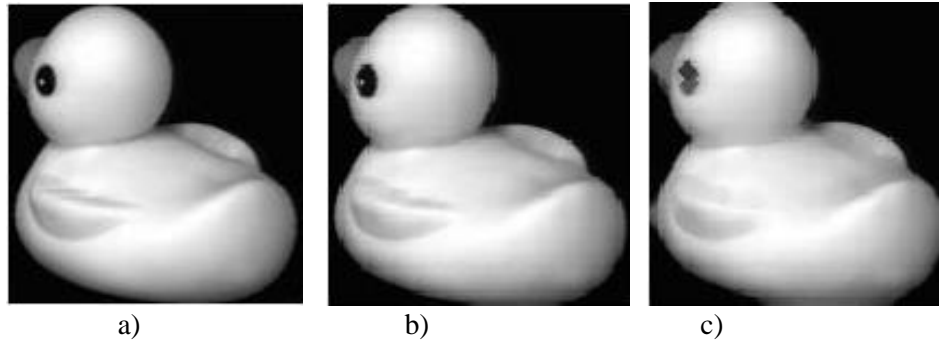


Fig. 3.18: Closing grayscale image: a) Grayscale image b) Closing image with SE(3) c) Closing image with SE(7)

3.4.3 Some Type of Grayscale Boundary Filters Using Dilation and Erosion:

The edge detection based on morphology is to do the dilation and erosion operations by using SE, $G(f)$ represents the function of the image boundary; there are some existing boundary filter based on the basic operation of morphology. there are many kinds of grayscale boundary filter (external, internal, gradient ...); the dilation thickens regions in an image and the erosion shrinks them [54,71].

3.4.3.1 Grayscale External Boundary Filter:

The dilation operator of boundary detection is

$$G_{ext}(f) = (f(i,j) \oplus b(i,j)) - f(i,j). \quad (3.23)$$

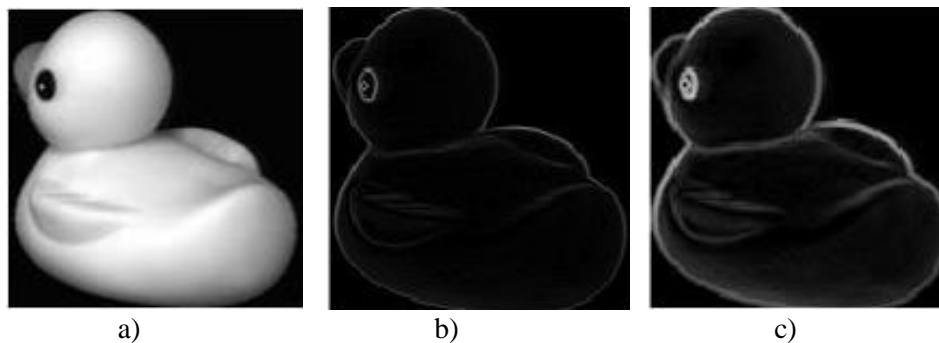


Fig. 3.19: External boundary grayscale image: a) grayscale image b) External boundary filter with SE(3) c) External boundary filter with SE(7)

3.4.3.2 Grayscale Internal Boundary Filter [51]:

The erosion operator of edge detection is

$$G_{int}(f) = f(i,j) - (f(i,j) \ominus b(i,j)) \quad (3.24)$$

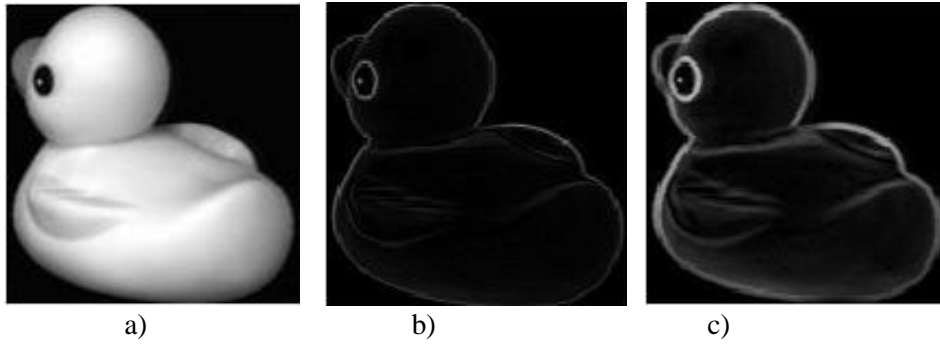


Fig. 3.20: Internal boundary grayscale image: a) Grayscale image
b) Internal boundary filter with SE(3) c) Internal boundary filter with SE(7)

3.4.3.3 Grayscale Gradient Boundary Filter [44]:

Dilation and erosion can be used in combination with image subtraction to obtain the morphological gradient G of an image as:

$$G_{gradient}(f) = f(i, j) - (f(i, j) \ominus b(i, j)). \quad (3.25)$$

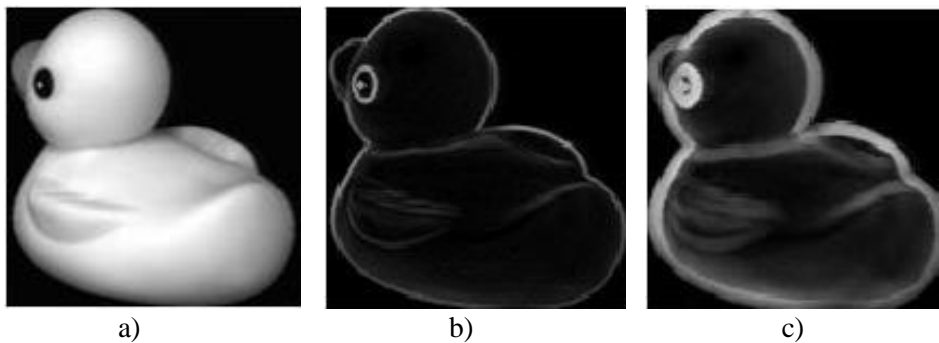


Fig. 3.21: Gradient boundary grayscale image: a) Grayscale image
b) Gradient boundary filter with SE(3) c) Gradient boundary filter with SE(7)

The above three operators implement simply and run fast.

3.4.4 Grayscale Hat Filter [31]:

Combining image subtraction with openings and closings results in top-hat and bottom-hat filter.

3.4.4.1 Grayscale Top-hat Filter:

The top-hat filter of a grayscale image f is defines as f difference its opening:

$$Top_{hat}(f) = f(i, j) - (f(i, j) \circ b(i, j)). \quad (3.26)$$

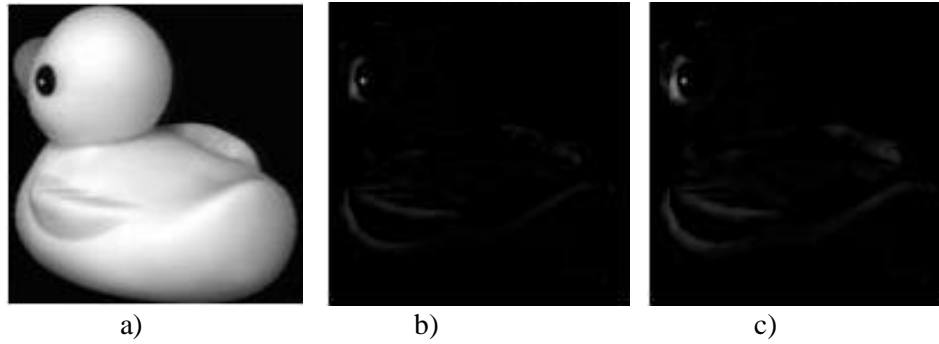


Fig. 3.22: Top-hat filter grayscale image: a) Grayscale image
b) Top-hat filter with SE (7) c) Top-hat filter with SE (9)

3.4.4.2 Grayscale Bottom-hat Filter:

Similarly, the bottom-hat filter of a grayscale image;

$$Bottom_{hat}(f) = (f(i, j) \bullet b(i, j)) - f(i, j). \quad (3.27)$$

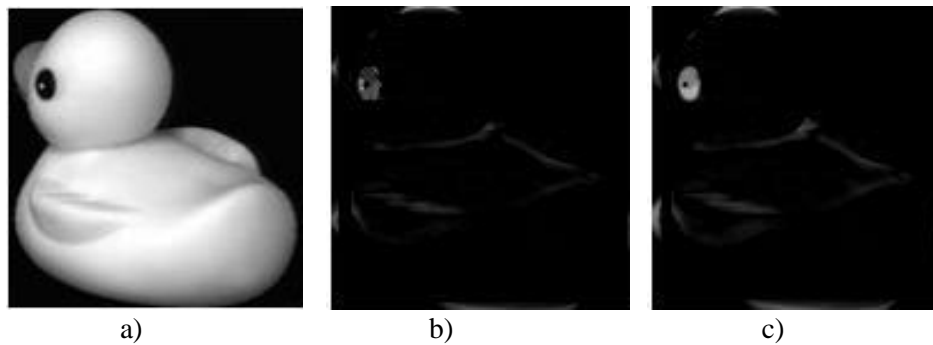


Fig. 3.23: Bottom-hat filter grayscale image: a) Grayscale image
b) Bottom-hat filter with SE(7) c) Bottom-hat filter with SE(9)

3.5 Fuzzy Mathematical Morphology:

One approach to extend mathematical morphology to grayscale images was supported by "Fuzzy Set Theory" [85], where the image is embedded into the fuzzy domain; such that each pixel value is interpreted as its membership degree to the original data set [8, 9, 12]. Hence, the fuzzy image is processed using fuzzy morphological operator, which were defined as an extension of the classical morphological operators. In this section, we review the definitions of the fuzzy morphological operators as given in [8, 9, 11]. For the purpose of visualizing the effect of these operators, we will use the fuzzy image show in Fig.2.4; which is deduced form the original color image shown in Fig.2.1. Attention will be paid here only to the four basic operations of mathematical morphology (erosion, dilation, opening and closing). Fuzzy mathematical morphology

[20, 39] has been developed to soften the classical binary morphology so as to make the operators less sensitive to image imprecision. It can also be viewed simply as an alternative grayscale morphological theory. The basic two operations of morphology are the dilation and erosion operators. The fuzzy version of the morphological dilation is used to smooth small dark regions. since all the values in the SE are positive, the output image tends to be brighter than the input. Dark elements are reduced or eliminated depending on how their shapes and sizes relate to the SE used. Whereas, the fuzzy morphological erosion. Is used to smooth small light regions. The formulae of the two basic morphological operations, as well as two different operations is ducted form them; are defined as follows:

3.5.1 Fuzzy Morphological Operations:

Mathematical morphology comprises an important toolset for analyzing spatial structures in images [39]. For binary images, the definitions of the fundamental morphological operations dilation and erosion can be related to the set theoretic Minkowski addition and subtraction. The extension of those operations to grayscale images is strongly related to ranking operations and, therefore, to the concept of ordered sets. It has been considered for a long time how to extend mathematical morphology to the case of fuzzy sets (as was done in other image processing disciplines, e.g., see [46]). Although there was a simple idea to consider grayscale images as fuzzy versions of binary images.

3.5.1.1 Fuzzy Morphological Dilation [42]:

For any grayscale image, A and any SE B (either grayscale or binary), the fuzzy dilation of A by B , (denoted by $A \oplus B$); defined as the membership function:

$$\mu_{A \oplus B} : Z^2 \rightarrow [0,1] \text{ and } \mu_{A \oplus B}(v) = \sup_{u \in Z^2} \min[\mu_A(v + u), \mu_B(u)]. \quad (3.28)$$

Where $u, v \in Z^2$ are the spatial co-ordinates of pixels in the image and SE respectively.

While μ_A, μ_B are the membership functions of the image and the SE, respectively.

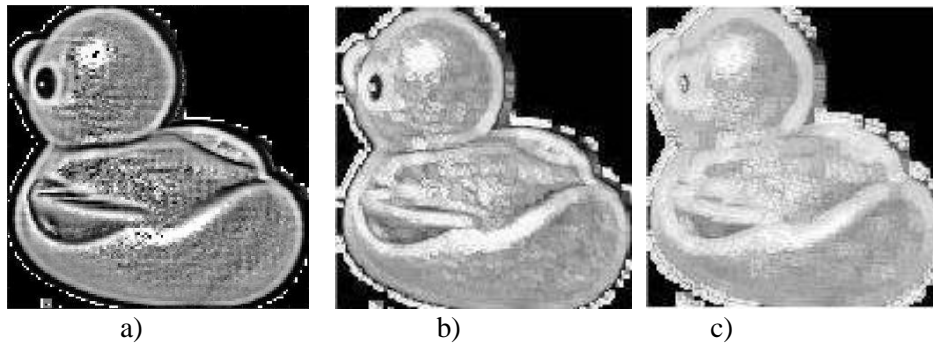


Fig.3.24: Dilation fuzzy image: a) Fuzzy image b) Fuzzy image dilated with SE(3)
c) Fuzzy image dilated with SE(5)

3.5.1.2 Fuzzy Morphological Erosion [42]:

For any grayscale image, A and any SE B (either gray-scale or binary), the fuzzy erosion of A by B , (denoted by $A \ominus B$); defined as the membership function:

$$\mu_{A \ominus B} : Z^2 \rightarrow [0,1] \text{ and } \mu_{A \ominus B}(v) = \inf_{u \in Z^2} \max[\mu_A(v + u), 1 - \mu_B(u)]. \quad (3.29)$$

Where $u, v \in Z^2$ are the spatial co-ordinates of pixels in the image and the SE, respectively. While μ_A, μ_B are the membership functions of the image and the SE respectively.

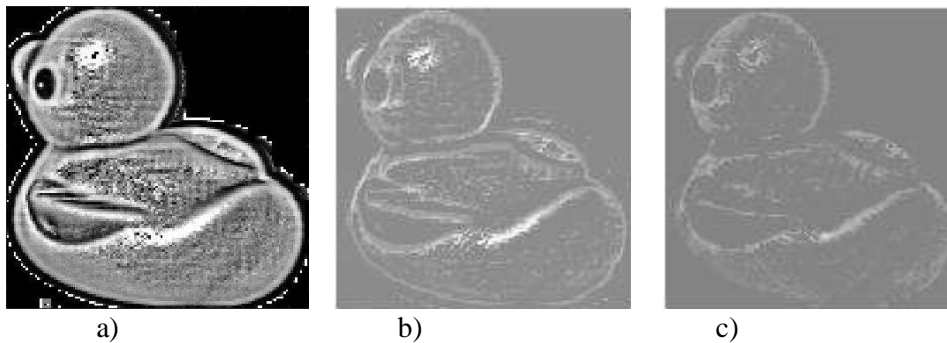


Fig.3.25: Erosion fuzzy image: a) Fuzzy image
b) Fuzzy image eroded with SE(3) c) Fuzzy image eroded with SE(5)

3.5.1.3 Fuzzy Morphological Opening [42, 47]:

Several combinations of the two basic fuzzy morphological operations, the dilation and erosion, give new operations; for instance, the fuzzy morphological opening and fuzzy morphological closing. The interpretation of the operation of fuzzy morphological opening, is that it darkens the bright regions in which the structure element does not fit, and is to be defined as the follows; For any grayscale image A , and any SE B (either

grayscale or binary), the fuzzy opening of A by B ($A \circ B$); is defined as the membership function: $\mu_{A \circ B} : Z^2 \rightarrow [0,1]$,

$$\mu_{A \circ B}(v) = \sup_{u \in Z^2} \min(\inf_{w \in Z^2} \max(\mu_A(v - u + w), 1 - \mu_B(u)), \mu_B(u)). \quad (3.30)$$

Where $u, v, w \in Z^2$ are the spatial co-ordinates of pixels in the image and μ_A, μ_B are the membership functions of the image and the SE respectively.

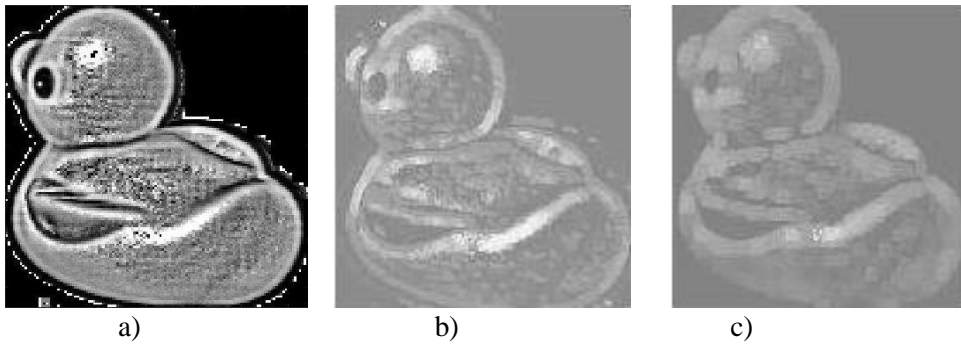


Fig.3.26: Opening fuzzy image: a) Fuzzy image b) Fuzzy image opening with SE(3)
c) Fuzzy image opening with SE(5).

3.5.1.4 Fuzzy Morphological Closing [70]:

While the fuzzy opening acts on the bright regions, the fuzzy morphological closing brighten the dark regions in which the SE does not fit, the definition goes as follows;

For any grayscale image A , and any SE B (either grayscale or binary), the fuzzy closing of A by B ($A \bullet B$); is defined as the membership function: $\mu_{A \bullet B} : Z^2 \rightarrow [0,1]$,

$$\mu_{A \bullet B}(v) = \inf_{u \in Z^2} \max(\sup_{w \in Z^2} \min(\mu_A(v - u + w), \mu_B(u)), 1 - \mu_B(u)). \quad (3.31)$$

and $u, v, w \in Z^2$ are the spatial co-ordinates of pixels in the image and μ_A, μ_B are the membership functions of the image and the SE respectively.

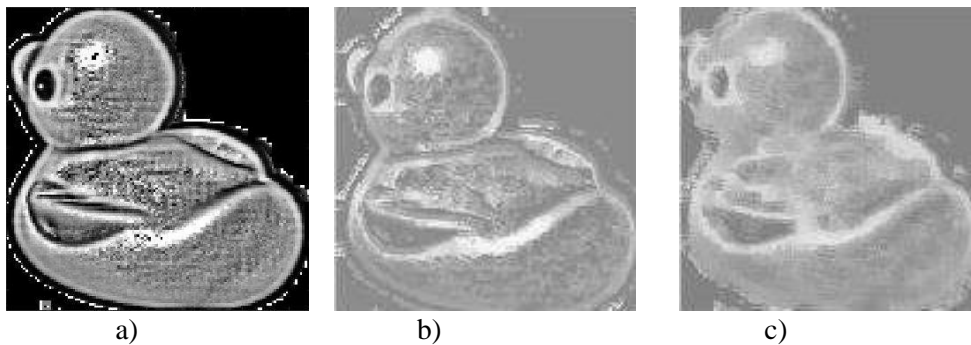


Fig.3.27: Closing fuzzy image: a) Fuzzy image b) Fuzzy image closing with SE(3)
c) Fuzzy image closing with SE(5)

3.5.2 Properties of Fuzzy Morphological Operations [71]:

Here are some properties of the basic fuzzy morphological operations (dilation, erosion, opening and closing [22, 40]). We define the power set of X , denoted by $\mathcal{F}(Z^2)$, to be the set of all fuzzy subset of X ,

- For all $A, B, C \in \mathcal{F}(Z^2)$ the following properties hold:

i. Monotonicity (increasing in both argument):

$$A \subseteq B \Rightarrow A \oplus C \subseteq B \oplus C,$$

$$A \subseteq B \Rightarrow C \oplus A \subseteq C \oplus B.$$

ii. Monotonicity (increasing in the first and decreasing in the argument):

$$A \subseteq B \Rightarrow A \ominus C \subseteq B \ominus C,$$

$$A \subseteq B \Rightarrow C \ominus A \supseteq C \ominus B.$$

iii. Monotonicity (increasing in the first argument):

$$A \subseteq B \Rightarrow A \bullet C \subseteq B \bullet C.$$

iv. Monotonicity (increasing in the first argument):

$$A \subseteq B \Rightarrow A \circ C \subseteq B \circ C.$$

- for any family $(A_i | i \in I)$ in $\mathcal{F}(Z^2)$ and $B \in \mathcal{F}(Z^2)$,

$$i. \bigcap_{i \in I} A_i \oplus B \subseteq \bigcap_{i \in I} (A_i \oplus B) \quad \text{and} \quad B \oplus \bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} (B \oplus A_i).$$

$$ii. \bigcap_{i \in I} A_i \ominus B \subseteq \bigcap_{i \in I} (A_i \ominus B) \quad \text{and} \quad B \ominus \bigcap_{i \in I} A_i \supseteq \bigcap_{i \in I} (B \ominus A_i).$$

$$iii. \bigcap_{i \in I} A_i \bullet B \subseteq \bigcap_{i \in I} (A_i \bullet B).$$

$$iv. \bigcap_{i \in I} A_i \circ B \subseteq \bigcap_{i \in I} (A_i \circ B).$$

- for any family $(A_i | i \in I)$ in $\mathcal{F}(Z^2)$ and $B \in \mathcal{F}(Z^2)$,

$$i. \bigcup_{i \in I} A_i \oplus B \supseteq \bigcup_{i \in I} (A_i \oplus B) \quad \text{and} \quad B \oplus \bigcup_{i \in I} A_i \supseteq \bigcup_{i \in I} (B \oplus A_i).$$

$$ii. \bigcup_{i \in I} A_i \ominus B \supseteq \bigcap_{i \in I} (A_i \ominus B) \quad \text{and} \quad B \ominus \bigcup_{i \in I} A_i \subseteq \bigcap_{i \in I} (B \ominus A_i).$$

$$iii. \bigcup_{i \in I} A_i \bullet B \supseteq \bigcap_{i \in I} (A_i \bullet B).$$

$$\text{iv. } \bigcup_{i \in I} A_i \circ B \supseteq \bigcap_{i \in I} (A_i \circ B).$$

3.5.3 Fuzzy Morphological Filters [81]:

In this section, we present some fuzzy morphological filters deduced from combining two or more of the four operations defined in § 3.8.1.

3.5.3.1 Some Type of Boundary Filters Using Fuzzy (Dilation and Erosion):

Both fuzzy erosion and dilation, can be combined in various ways to form several powerful morphological filters in order to extract the boundaries from some grayscale image [67].

3.5.3.1.a. Fuzzy Gradient Boundary Filter[79]:

One operation can be performed by applying the fuzzy difference over the dilation and the erosion of the two image A and structure element A . this operation is to be called the fuzzy gradient filter and is defined as follows:

$$\mu_{\partial_{\text{gradient}}(A)}(v) = \min[\mu_{(A \oplus B)}(v), 1 - \mu_{(A \ominus B)}(v)]. \quad (3.32)$$

As the dilation thickens regions in an image and the erosion shrinks them, the interpretation of the fuzzy gradient filter can be understood as emphasizing the boundaries between regions. If the SE is relatively small, the homogeneous areas will not be affected by fuzzy dilation and fuzzy erosion, then the subtraction tends to eliminate them. The effect of morphological gradient operation is shown in Fig. 3.30.

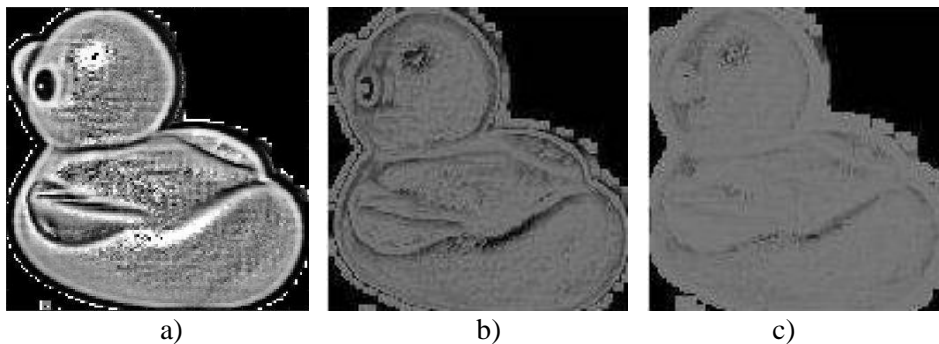


Fig.3.28: Gradient boundary fuzzy image: a) Fuzzy image
b) Gradient boundary with SE(3) c) Gradient boundary with SE(5)

Moreover there are two kinds of half gradient were deduced form the fuzzy gradient filter known as the internal gradient and the is external gradient filters.

3.5.3.1.B. Fuzzy External Boundary Filter [72]:

In this filter, a fuzzy dilation is firstly applied to the fuzzy image A by a structure element B , then the output filtered image will be the difference between fuzzy dilated image and the original fuzzy image A ; that is, the fuzzy external boundary of A is defined by:

$$\mu_{\partial_{ext}(A)}(v) = \min[\mu_{(A \oplus B)}(v), 1 - \mu_A(v)]. \quad (3.33)$$

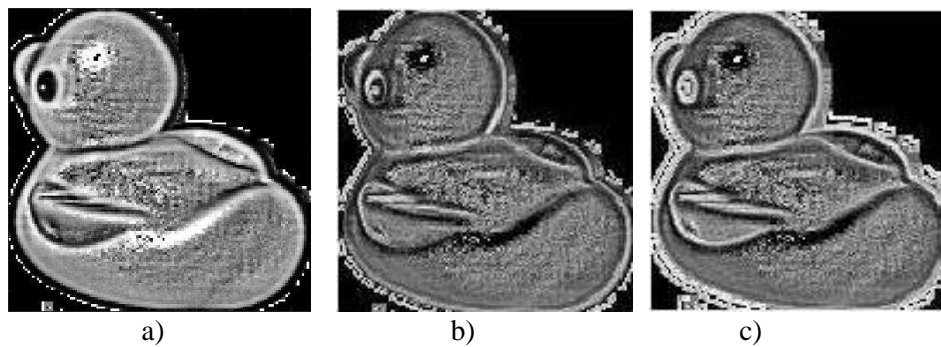


Fig.3.29: External boundary fuzzy image: a) Fuzzy image
b) External boundary filter with SE(3) c) External boundary filter with SE(5)

3.5.3.1.C. Fuzzy Internal Boundary Filter [51]:

The first step of the fuzzy internal boundary filter, is to fuzzy erode the image, hence, the output filtered image will be the difference between the original fuzzy image and the fuzzy eroded image; that is, the fuzzy internal boundary of A is defined by:

$$\mu_{\partial_{int}(A)}(v) = \min[\mu_A(v), 1 - \mu_{(A \ominus B)}(v)].$$

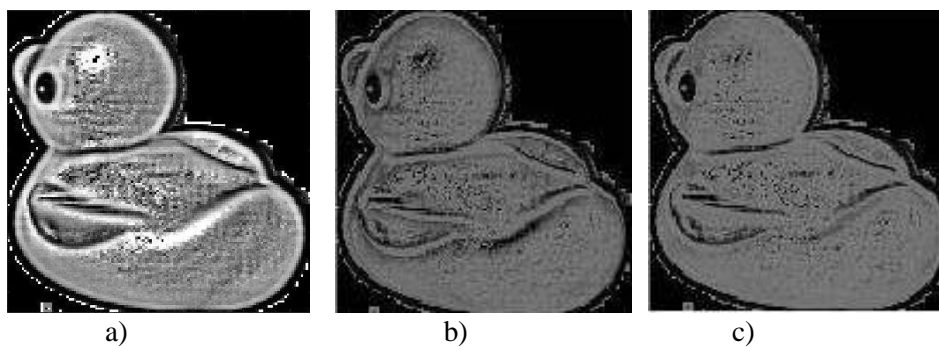


Fig.3.30: Internal boundary fuzzy image: a) Fuzzy image
b) Internal boundary filter with SE(3) b) Internal boundary filter with SE(5)

3.5.3.1.D. Fuzzy Outline Boundary Filter:

The first step of the fuzzy outline boundary filter, is the erode the image; then the complement of its erosion image hence, the output image will be the intersection between the original image and the output image that is the fuzzy outline boundary of A

$$\text{is defined by: } \mu_{\partial_{outline}(A)}(v) = \min[1 - \mu_{A \ominus B}(v), A]. \quad (3.35)$$

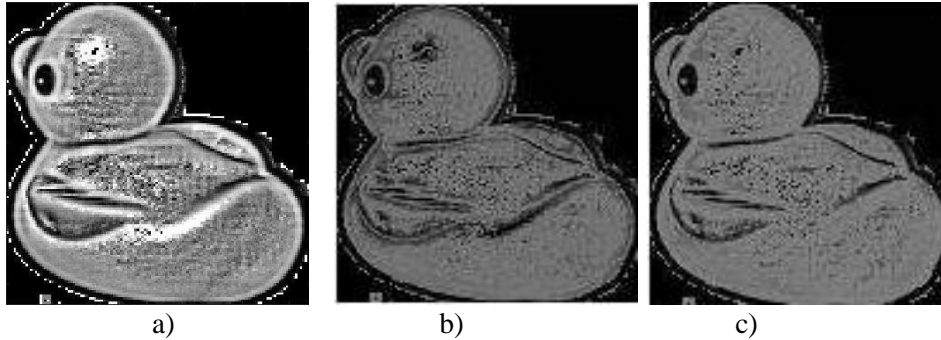


Fig.3.31: Outline boundary fuzzy image: a) fuzzy image
b) Outline boundary filter with SE(3) b) Outline boundary filter with SE(5)

3.5.3.2 Combination Fuzzy External Boundary and Fuzzy Internal Boundary:

$$1. \mu_{\partial_{sup}(A)}(v) = \max[\mu_{\partial_{int}(A)}(v), \mu_{\partial_{ext}(A)}(v)] \quad (3.36)$$

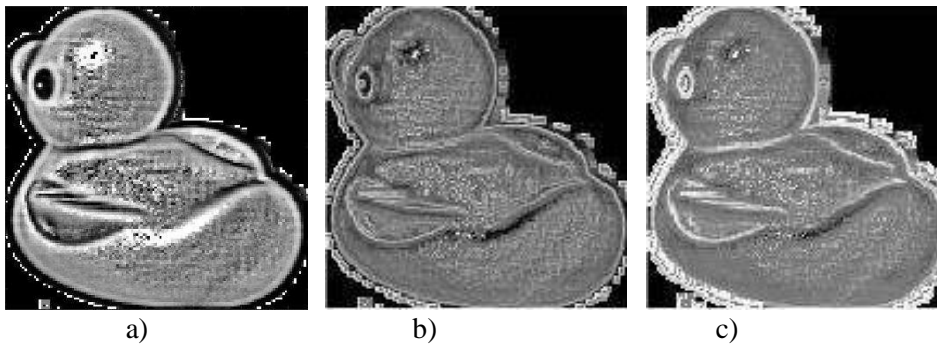


Fig.3.32: Sup. boundary fuzzy image: a) Fuzzy image
b) Sup. boundary filter with SE(3) b) Sup. boundary filter with SE(5)

$$2. \mu_{\partial_{inf}(A)}(v) = \min[\mu_{\partial_{int}(A)}(v), \mu_{\partial_{ext}(A)}(v)]. \quad (3.37)$$

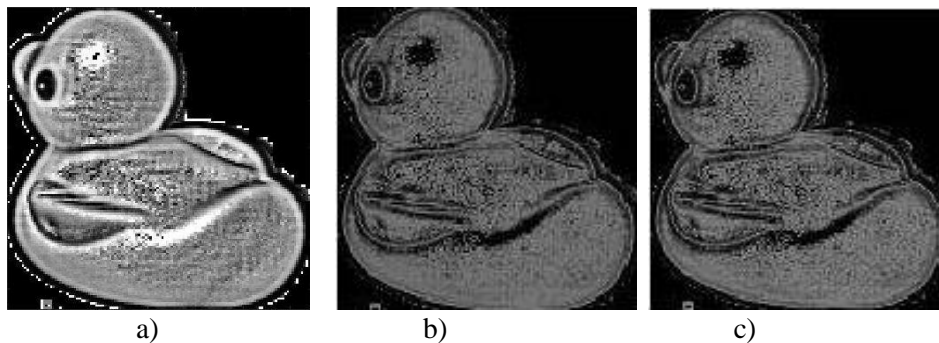


Fig.3.33: Inf. boundary fuzzy image: a) Fuzzy image
b) Inf. boundary filter with SE (3) c) Inf. boundary filter with SE (5)

$$3. \mu_{\partial_{Div}(A)}(v) = \min[\mu_{\partial_{ext}(A)}(v) - \mu_{\partial_{int}(A)}(v)]. \quad (3.38)$$

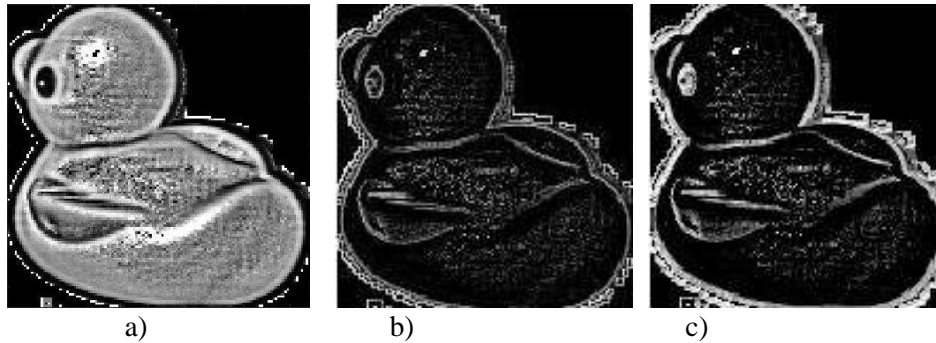


Fig.3.34: Div. boundary fuzzy image: a) fuzzy image
b) Div. boundary filter with SE (3) c) Div. boundary filter with SE(5)

3.5.3.3 Fuzzy Hat Filter [22, 40]:

The hat filters represent an important class of morphological transforms used for extracting details from signals or images. One principal application of these transforms is the removal of objects from an image.

3.5.3.3.A. Fuzzy Top-hat Filter [35]:

In mathematical morphology, top-hat filter is an operation that extracts small elements and details from given images. The top-hat filter is defined as the difference between the input image and its opening by some SE. Top-hat filter are used for various image processing tasks, such as feature extraction, background equalization, image enhancement, and others. An important use of the top-hat filter is in correcting the effects of non-uniform illumination. The fuzzy top-hat filter of A is given by:

$$\mu_{top_hat(A)}(v) = \min [\mu_A(v), 1 - \mu_{(A \circ B)}(v)].$$

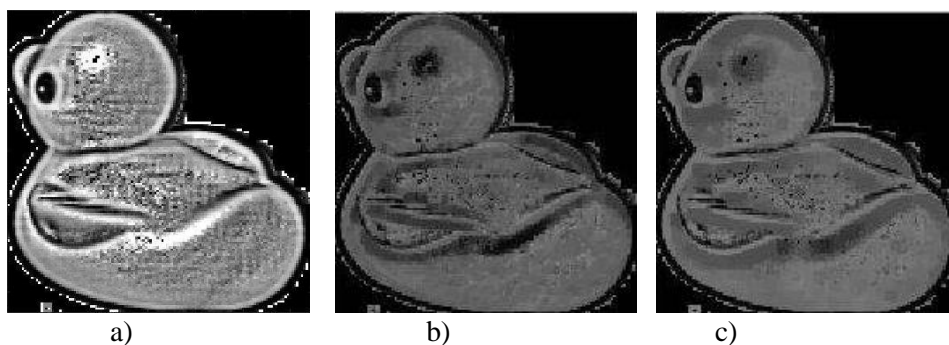


Fig.3.35: Top-hat filter fuzzy image: a) Fuzzy image b) Top-hat image with SE(3)
c) Top-hat image with SE(7)

3.5.3.3.B. Fuzzy Bottom-hat Filter [43]:

The bottom-hat morphological operator subtracts an input image from the result of morphological closing on the input image. Applied to a binary image, this filter allows getting all the pixels that were added by the closing filter but were not removed afterwards due to formed connections. The fuzzy bottom-hat filter of A is given by:

$$\mu_{Bottom_{hat}(A)}(v) = \min [\mu_{(A \bullet B)}(v), 1 - \mu_A(v)].$$

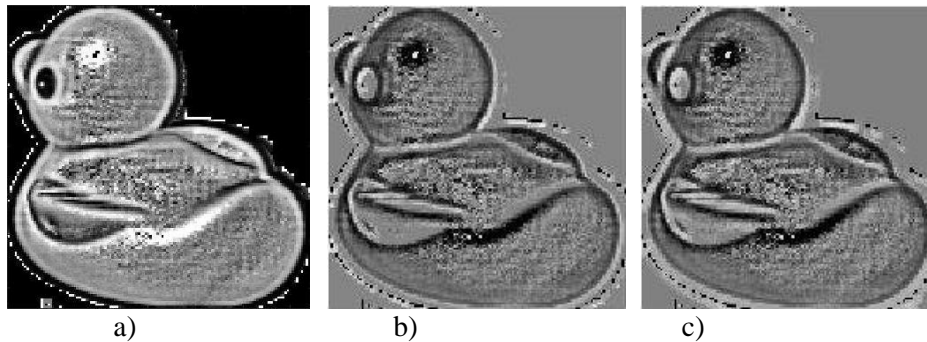


Fig.3.36: Bottom-hat filter fuzzy image: a) Fuzzy image b) Bottom-hat image with SE(3) c) Bottom-hat image with SE(7)

3.6 Conclusion:

In this chapter, we reviewed the fundamental definitions from the mathematical morphology; for both grayscale and binary images. introducing a revision for the basic morphological operators, namely, the morphological dilation, erosion, opening and closing. Some algebraic properties of the operations have been investigated. Different combinations of the defined basic operations are to be made in order to construct more advanced morphological operators and filters.

Chapter 4

Neutrosophic Crisp Mathematical Morphology

4.1 introduction:

in this chapter, we aim to apply the concepts of the neutrosophic crisp sets and its operations to the classical mathematical morphological operations; introducing what we call "Neutrosophic Crisp Mathematical Morphology". Several operators are to be developed, including the neutrosophic crisp dilation, the neutrosophic crisp erosion, the neutrosophic crisp opening and the neutrosophic crisp closing. Moreover, we extend the definition of some morphological filters using the neutrosophic crisp sets concept. For instance, we introduce the neutrosophic crisp boundary extraction, the neutrosophic crisp Top-hat and the neutrosophic crisp Bottom-hat filters. The idea behind the new introduced operators and filters is to act on the image in the neutrosophic crisp domain instead of the spatial domain.

4.2 Neutrosophic Crisp Mathematical Morphology:

As a generalization of the classical mathematical morphology, we present in this chapter the basic operations for the neutrosophic crisp mathematical morphology. To commence, we need to define the reflection and the translation of a neutrosophic set.

4.2.1 Definition:

Consider the space $X = R^n$ or Z^n ; With origin $0 = (0, \dots, 0)$ given The reflection of the SE B mirrored in its origin is defined as: $-B = \langle -B^1, -B^2, -B^3 \rangle$. (4.1)

4.2.2 Definition:

For every the $p \in A$, the translations by p is the map $p: X \rightarrow X, a \mapsto a + p$ it transforms any subset A of X into its translate by $p \in Z^2, A_p = \langle A^1_p, A^2_p, A^3_p \rangle$, where $A^1_p(u) = \{u + p: u \in A^1, p \in B^1\}$, $A^2_p(u) = \{u + p: u \in A^2, p \in B^2\}$ and

$$A^3_p(u) = \{u + p: u \in A^3, p \in B^3\}.$$

4.3 Neutrosophic Crisp Morphological Operations:

we introduce and study the mathematical morphology via neutrosophic crisp sets; The operations of neutrosophic crisp morphology dilation, erosion, opening and closing of the neutrosophic image by neutrosophic crisp structuring element.

4.3.1 Neutrosophic Crisp Dilation and Neutrosophic Crisp Erosion:

Neutrosophic mathematical morphological transformations apply to neutrosophic sets of any dimensions, those like Euclidean N-space, The two basic operations of morphological operators are dilation and erosion.

4.3.1.1 Neutrosophic Crisp Dilation Operation:

dilation "grows" or "thickens" objects in a binary image The manner and extend of this growth is image. controlled by the SE. let $A, B \in \mathcal{NC}(X)$, then we define two types of the neutrosophic crisp dilation as follows:

4.3.1.1.A. Neutrosophic Crisp Dilation of Type I:

$(A \widetilde{\oplus} B) = \langle A^1 \oplus B^1, A^2 \oplus B^2, A^3 \ominus B^3 \rangle$, where for each u and $v \in Z^2$.

$$A^1 \oplus B^1 = \bigcup_{b \in B^1} A^1_b, \quad A^2 \oplus B^2 = \bigcup_{b \in B^2} A^2_b, \quad A^3 \ominus B^3 = \bigcap_{b \in B^3} A^3_{-b}.$$

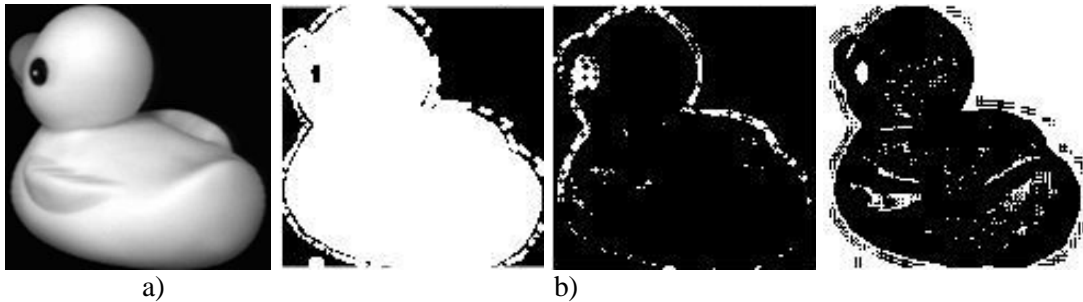


Fig.4.1(I): Neutrosophic crisp dilation in type I: a) Original image
b) Neutrosophic crisp dilation components $\langle A^1 \oplus B^1, A^2 \oplus B^2, A^3 \ominus B^3 \rangle$ respectively

4.3.1.1.B Neutrosophic Crisp Dilation of Type II:

$(A \widetilde{\oplus} B) = \langle A^1 \oplus B^1, A^2 \ominus B^2, A^3 \ominus B^3 \rangle$, where for each u and $v \in Z^2$. $A^1 \oplus B^1 =$

$$\bigcup_{b \in B^1} A^1_b, \quad A^2 \ominus B^2 = \bigcap_{b \in B^2} A^2_{-b}, \quad A^3 \ominus B^3 = \bigcap_{b \in B^3} A^3_{-b}.$$

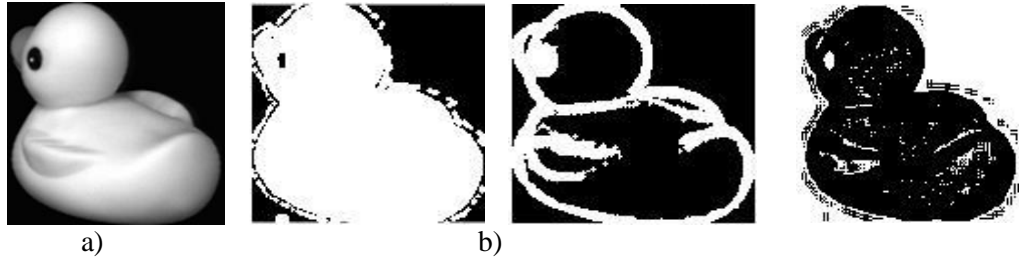


Fig.4.2(II): Neutrosophic crisp dilation in type II: a) Original image
b) Neutrosophic crisp dilation components $\langle A^1 \oplus B^1, A^2 \ominus B^2, A^3 \ominus B^3 \rangle$ respectively.

4.3.1.2 Neutrosophic Crisp Erosion Operation:

Erosion is just opposite to dilation. It is defined as the minimum value in the window.

The image after dilation will be darker than the original image. It shrinks or thins the image.

let $A, B \in \mathcal{NC}(X)$; then the neutrosophic dilation is given as two type:

4.3.1.2.A. Neutrosophic Crisp Erosion of Type I:

$$(A \widetilde{\ominus} B) = \langle A^1 \ominus B^1, A^2 \ominus B^2, A^3 \oplus B^3 \rangle, \text{ where for each } u \text{ and } v \in Z^2.$$

$$A^1 \ominus B^1 = \bigcap_{b \in B^1} A^1_{-b}, A^2 \ominus B^2 = \bigcap_{b \in B^2} A^2_{-b}, A^3 \oplus B^3 = \bigcup_{b \in B^3} A^3_b.$$

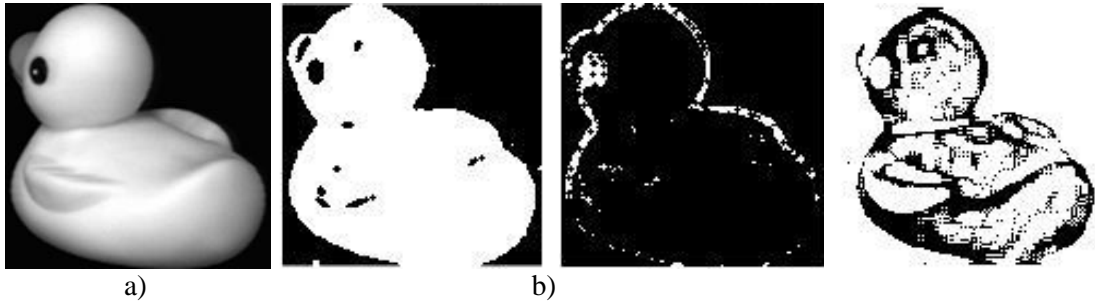


Fig.4.3(I): Neutrosophic crisp erosion in type I: a) Original image
b) Neutrosophic crisp erosion components $\langle A^1 \ominus B^1, A^2 \ominus B^2, A^3 \oplus B^3 \rangle$ respectively

4.3.1.2.B. Neutrosophic Crisp Erosion of Type II:

$$(A \widetilde{\ominus} B) = \langle A^1 \ominus B^1, A^2 \oplus B^2, A^3 \oplus B^3 \rangle, \text{ where for each } u \text{ and } v \in Z^2.$$

$$A^1 \ominus B^1 = \bigcap_{b \in B^1} A^1_{-b}, A^2 \oplus B^2 = \bigcup_{b \in B^2} A^2_b, A^3 \oplus B^3 = \bigcup_{b \in B^3} A^3_b.$$

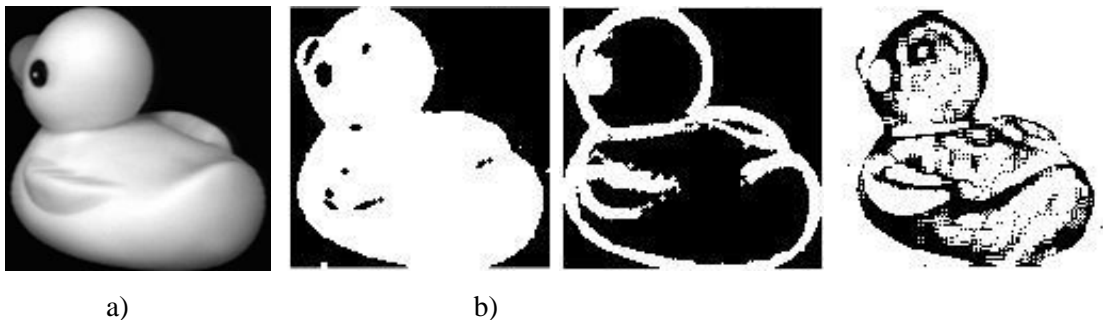


Fig.4.4(II): Neutrosophic crisp erosion in type II: a) Original image
b) Neutrosophic crisp erosion components $\langle A^1 \ominus B^1, A^2 \oplus B^2, A^3 \oplus B^3 \rangle$ respectively

4.3.2 Neutrosophic Crisp Opening and Neutrosophic Crisp Closing:

In practice, dilations and erosions are usually employed in pairs, either dilation of an image followed by the erosion of the dilated result, or image erosion followed by dilation. In either case, the result of iteratively applied dilations and erosions is an elimination of specific image detail smaller than the structuring element without the global geometric distortion of unsuppressed features.

4.3.2.1 Neutrosophic Crisp Opening Operation:

The process of "opening" an image will likely smooth the edges, remove small holes from a reference image and break narrow block connectors. The opening of an image A by a SE B ; let $A, B \in \mathcal{NC}(X)$; then we define two types of the neutrosophic crisp dilation operator as follows:

4.3.2.1.A. Neutrosophic Crisp Opening of Type I:

$$A \circ B = \langle A^1 \circ B^1, A^2 \circ B^2, A^3 \bullet B^3 \rangle,$$

$$A^1 \circ B^1 = (A^1 \ominus B^1) \oplus B^1, A^2 \circ B^2 = (A^2 \ominus B^2) \oplus B^2,$$

$$A^3 \bullet B^3 = (A^3 \oplus B^3) \ominus B^3.$$

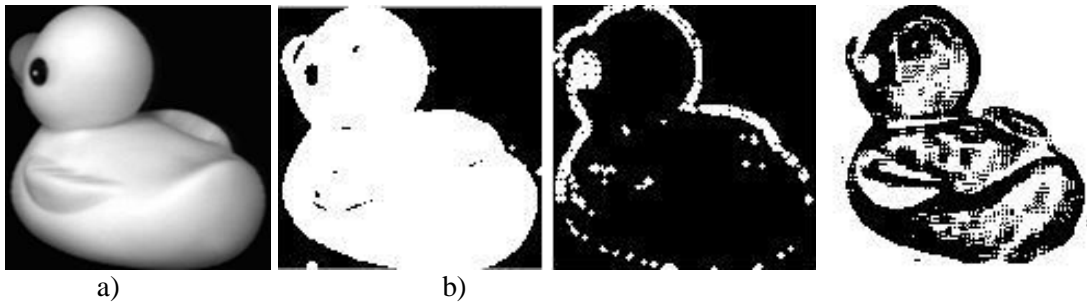


Fig.4.5(I): Neutrosophic crisp opening in type I: a) original image
b) Neutrosophic crisp opening components $\langle A^1 \circ B^1, A^2 \circ B^2, A^3 \bullet B^3 \rangle$ respectively

4.3.2.1.B. Neutrosophic Crisp Opening of Type II:

$$A \circ B = \langle A^1 \circ B^1, A^2 \bullet B^2, A^3 \bullet B^3 \rangle,$$

$$A^1 \circ B^1 = (A^1 \ominus B^1) \oplus B^1, A^2 \bullet B^2 = (A^2 \oplus B^2) \ominus B^2,$$

$$A^3 \bullet B^3 = (A^3 \oplus B^3) \ominus B^3.$$

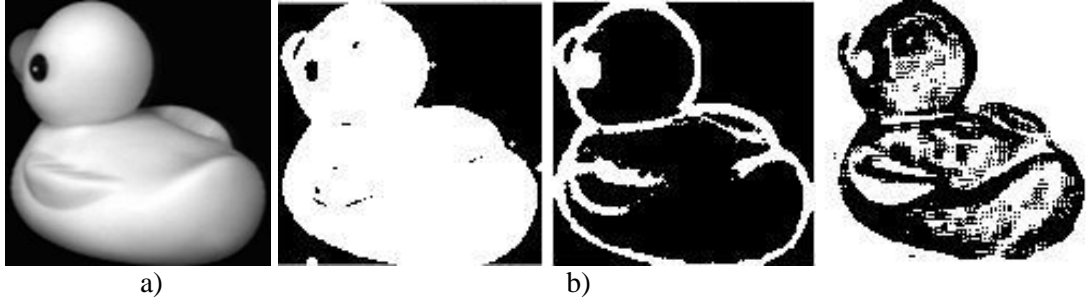


Fig.4.6(II): Neutrosophic crisp opening in type II: a) Original image
b) Neutrosophic crisp opening components $\langle A^1 \circ B^1, A^2 \bullet B^2, A^3 \bullet B^3 \rangle$ respectively

4.3.2.2 Neutrosophic Crisp Closing Operation:

Close operation can also be smoothed image of the contour. Compared with open operation, closed operation is generally used to fill the small hole and crack in the target. The main function of the connection is similar to the expansion effect, but it is also the same as the size of the target. let $A, B \in \mathcal{NC}(X)$; then the neutrosophic dilation is given as two types:

4.3.2.2.A. Neutrosophic Crisp Closing of Type I:

$$A \tilde{\circ} B = \langle A^1 \bullet B^1, A^2 \bullet B^2, A^3 \circ B^3 \rangle,$$

$$A^1 \bullet B^1 = (A^1 \ominus B^1) \oplus B^1, \quad A^2 \bullet B^2 = (A^2 \ominus B^2) \oplus B^2,$$

$$A^3 \circ B^3 = (A^3 \oplus B^3) \ominus B^3.$$

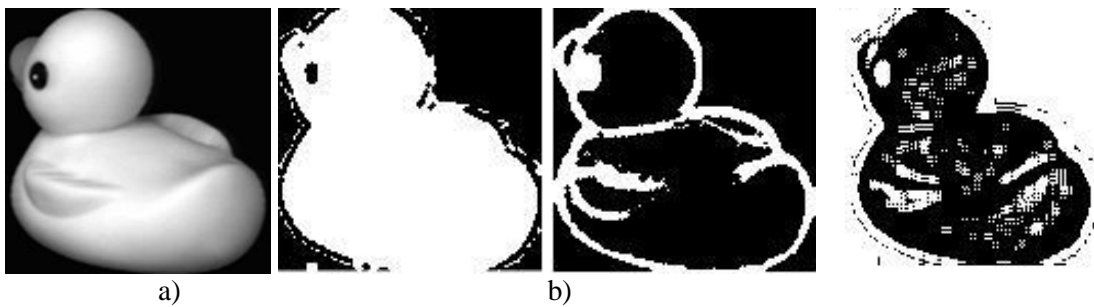


Fig.4.7(I): Neutrosophic crisp closing in type I: a) Original image
b) Neutrosophic crisp closing components $\langle A^1 \bullet B^1, A^2 \bullet B^2, A^3 \circ B^3 \rangle$ respectively

4.3.2.2.B. Neutrosophic Crisp Closing of Type II:

$$A \tilde{\circ} B = \langle A^1 \bullet B^1, A^2 \circ B^2, A^3 \circ B^3 \rangle,$$

$$A^1 \bullet B^1 = (A^1 \ominus B^1) \oplus B^1, \quad A^2 \circ B^2 = (A^2 \oplus B^2) \ominus B^2,$$

$$A^3 \circ B^3 = (A^3 \oplus B^3) \ominus B^3.$$

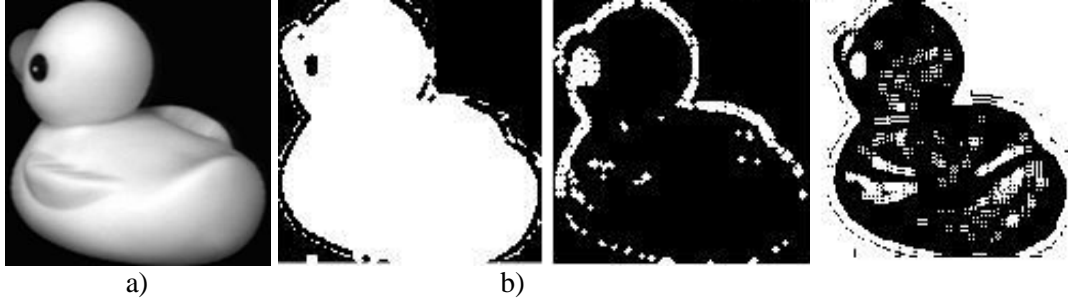


Fig.4.8(II): *Neutrosophic crisp closing in type II: a) Original image
b) Neutrosophic crisp closing components $\langle A^1 \bullet B^1, A^2 \circ B^2, A^3 \circ B^3 \rangle$ respectively*

Note: Opening and Closing remove from the image its elements (objects, noise) respectively lighter and darker than the background.

4.4 Algebraic Neutrosophic Crisp Properties:

In this section, we investigate some of the algebraic properties of the neutrosophic crisp erosion and dilation, as well as the neutrosophic crisp opening and the neutrosophic crisp closing operator.

4.4.1 Properties of the Neutrosophic Crisp Erosion:

Proposition 4.1:

The neutrosophic erosion satisfies the monotonicity for all $A, B \in \mathcal{NC}(Z^2)$. We will prove the proposition for the two types of neutrosophic crisp erosion operation as follows:

Type I:

1. $A \subseteq B \Rightarrow \langle A^1 \ominus C^1, A^2 \ominus C^2, A^3 \ominus C^3 \rangle \subseteq \langle B^1 \ominus C^1, B^2 \ominus C^2, B^3 \ominus C^3 \rangle,$
 $A^1 \ominus C^1 \subseteq B^1 \ominus C^1, A^2 \ominus C^2 \subseteq B^2 \ominus C^2$ and $A^3 \ominus C^3 \supseteq B^3 \ominus C^3.$
2. $A \subseteq B \Rightarrow \langle C^1 \ominus A^1, C^2 \ominus A^2, C^3 \ominus A^3 \rangle \subseteq \langle C^1 \ominus B^1, C^2 \ominus B^2, C^3 \ominus B^3 \rangle,$
 $C^1 \ominus A^1 \subseteq C^1 \ominus B^1, C^2 \ominus A^2 \subseteq C^2 \ominus B^2$ and $C^3 \ominus A^3 \supseteq C^3 \ominus B^3.$

Type II:

1. $A \subseteq B \Rightarrow \langle A^1 \ominus C^1, A^2 \ominus C^2, A^3 \ominus C^3 \rangle \subseteq \langle B^1 \ominus C^1, B^2 \ominus C^2, B^3 \ominus C^3 \rangle,$
 $A^1 \ominus C^1 \subseteq B^1 \ominus C^1, A^2 \ominus C^2 \supseteq B^2 \ominus C^2$ and $A^3 \ominus C^3 \supseteq B^3 \ominus C^3.$

$$2. A \subseteq B \Rightarrow \langle C^1 \ominus A^1, C^2 \ominus A^2, C^3 \ominus A^3 \rangle \subseteq \langle C^1 \ominus B^1, C^2 \ominus B^2, C^3 \ominus B^3 \rangle,$$

$$C^1 \ominus A^1 \subseteq C^1 \ominus B^1, C^2 \ominus A^2 \supseteq C^2 \ominus B^2 \text{ and } C^3 \ominus A^3 \supseteq C^3 \ominus B^3.$$

Note that: Dislike the neutrosophic crisp dilation operator, the neutrosophic crisp erosion does not satisfy commutativity and the associativity properties.

Proposition 4.2: For any family $A_i, i \in I$ in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$;

We will prove the proposition for the two types of neutrosophic crisp erosion operation as follows:

Type I: a) $\cap_{i \in I} A_i \widetilde{\ominus} B = \cap_{i \in I} (A_i \widetilde{\ominus} B)$

$$\langle \cap_{i \in I} A^1_i \ominus B^1, \cap_{i \in I} A^2_i \ominus B^2, \cap_{i \in I} A^3_i \oplus B^3 \rangle$$

$$= \langle \cap_{i \in I} (A^1_i \ominus B^1), \cap_{i \in I} (A^2_i \ominus B^2), \cap_{i \in I} (A^3_i \oplus B^3) \rangle.$$

b) $B \widetilde{\ominus} \cap_{i \in I} A_i = \cap_{i \in I} (B \widetilde{\ominus} A_i)$

$$\langle B^1 \ominus \cap_{i \in I} A^1_i, B^2 \ominus \cap_{i \in I} A^2_i, B^3 \oplus \cap_{i \in I} A^3_i \rangle$$

$$= \langle \cap_{i \in I} (B^1 \ominus A^1_i), \cap_{i \in I} (B^2 \ominus A^2_i), \cap_{i \in I} (B^3 \oplus A^3_i) \rangle.$$

Type II: a) $\cap_{i \in I} A_i \widetilde{\ominus} B = \cap_{i \in I} (A_i \widetilde{\ominus} B)$

$$\langle \cap_{i \in I} A^1_i \ominus B^1, \cap_{i \in I} A^2_i \oplus B^2, \cap_{i \in I} A^3_i \oplus B^3 \rangle$$

$$= \langle \cap_{i \in I} (A^1_i \ominus B^1), \cap_{i \in I} (A^2_i \oplus B^2), \cap_{i \in I} (A^3_i \oplus B^3) \rangle.$$

b) $B \widetilde{\ominus} \cap_{i \in I} A_i = \cap_{i \in I} (B \widetilde{\ominus} A_i)$

$$\langle B^1 \ominus \cap_{i \in I} A^1_i, B^2 \oplus \cap_{i \in I} A^2_i, B^3 \oplus \cap_{i \in I} A^3_i \rangle$$

$$= \langle \cap_{i \in I} (B^1 \ominus A^1_i), \cap_{i \in I} (B^2 \oplus A^2_i), \cap_{i \in I} (B^3 \oplus A^3_i) \rangle.$$

Proof a) in two types:

Type I:

$$\cap_{i \in I} A_i \widetilde{\ominus} B = \langle \cap_{b \in B} \left(\cap_{i \in I} A^1_{i(-b)} \right), \cap_{b \in B} \left(\cap_{i \in I} A^2_{i(-b)} \right), \cup_{b \in B} \left(\cap_{i \in I} A^3_{ib} \right) \rangle$$

$$= \langle \cap_{i \in I} \left(\cap_{b \in B} A^1_{i(-b)} \right), \cap_{i \in I} \left(\cap_{b \in B} A^2_{i(-b)} \right), \cap_{i \in I} \left(\cup_{b \in B} A^3_{ib} \right) \rangle = \cap_{i \in I} (A_i \widetilde{\ominus} B).$$

Type II: similarity, we can show that it is true in type II.

b) The proof is similar to point **a**).

Proposition 4.3: For any family $A_i, i \in I$ in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$.

We will prove the proposition for the two types of neutrosophic crisp erosion operation as follows:

Type I: a) $\cup_{i \in I} A_i \widetilde{\ominus} B = \cup_{i \in I} (A_i \widetilde{\ominus} B)$

$$\begin{aligned} & \langle \cup_{i \in I} A^1_i \ominus B^1, \cup_{i \in I} A^2_i \ominus B^2, \cup_{i \in I} A^3_i \oplus B^3 \rangle \\ &= \langle \cup_{i \in I} (A^1_i \ominus B^1), \cup_{i \in I} (A^2_i \ominus B^2), \cup_{i \in I} (A^3_i \oplus B^3) \rangle. \end{aligned}$$

b) $B \widetilde{\ominus} \cup_{i \in I} A_i = \cup_{i \in I} (B \widetilde{\ominus} A_i)$

$$\begin{aligned} & \langle B^1 \ominus \cup_{i \in I} A^1_i, B^2 \ominus \cup_{i \in I} A^2_i, B^3 \oplus \cup_{i \in I} A^3_i \rangle \\ &= \langle \cup_{i \in I} (B^1 \ominus A^1_i), \cup_{i \in I} (B^2 \ominus A^2_i), \cup_{i \in I} (B^3 \oplus A^3_i) \rangle. \end{aligned}$$

Type II: a) $\cup_{i \in I} A_i \widetilde{\ominus} B = \cup_{i \in I} (A_i \widetilde{\ominus} B)$

$$\begin{aligned} & \langle \cup_{i \in I} A^1_i \ominus B^1, \cup_{i \in I} A^2_i \oplus B^2, \cup_{i \in I} A^3_i \oplus B^3 \rangle \\ &= \langle \cup_{i \in I} (A^1_i \ominus B^1), \cup_{i \in I} (A^2_i \oplus B^2), \cup_{i \in I} (A^3_i \oplus B^3) \rangle. \end{aligned}$$

b) $B \widetilde{\ominus} \cup_{i \in I} A_i = \cup_{i \in I} (B \widetilde{\ominus} A_i)$

$$\begin{aligned} & \langle B^1 \ominus \cup_{i \in I} A^1_i, B^2 \oplus \cup_{i \in I} A^2_i, B^3 \oplus \cup_{i \in I} A^3_i \rangle \\ &= \langle \cup_{i \in I} (B^1 \ominus A^1_i), \cup_{i \in I} (B^2 \oplus A^2_i), \cup_{i \in I} (B^3 \oplus A^3_i) \rangle. \end{aligned}$$

Proof: a)

Type I:

$$\begin{aligned} \cup_{i \in I} A_i \widetilde{\ominus} B &= \langle \bigcap_{b \in B} \left(\cup_{i \in I} A^1_{i(-b)} \right), \bigcap_{b \in B} \left(\cup_{i \in I} A^2_{i(-b)} \right), \cup_{b \in B} \left(\cup_{i \in I} A^3_{ib} \right) \rangle \\ &= \langle \cup_{i \in I} \left(\bigcap_{b \in B} A^1_{i(-b)} \right), \cup_{i \in I} \left(\bigcap_{b \in B} A^2_{i(-b)} \right), \cup_{i \in I} \left(\bigcup_{b \in B} A^3_{ib} \right) \rangle = \cup_{i \in I} (A_i \oplus B). \end{aligned}$$

Type II: can be verified in a similar way as in type I. **b)** The proof is similar to point **a**).

4.4.2 Properties of the Neutrosophic Crisp Dilation:

Proposition 4.4:

The neutrosophic dilation satisfies the following properties: $\forall A, B \in \mathcal{NC}(Z^2)$.

- i) *Commutativity*: $A \widetilde{\oplus} B = B \widetilde{\oplus} A$.
- ii) *Associativity*: $(A \widetilde{\oplus} B) \widetilde{\oplus} C = A \widetilde{\oplus} (B \widetilde{\oplus} C)$.
- iii) *Monotonicity*: (increasing in both arguments):

We will prove the proposition for the two types of neutrosophic crisp dilation operation as follows:

Type I:

1. $A \subseteq B \Rightarrow \langle A^1 \oplus C^1, A^2 \oplus C^2, A^3 \oplus C^3 \rangle \subseteq \langle B^1 \oplus C^1, B^2 \oplus C^2, B^3 \oplus C^3 \rangle$
 $A^1 \oplus C^1 \subseteq B^1 \oplus C^1, A^2 \oplus C^2 \subseteq B^2 \oplus C^2$ and $A^3 \oplus C^3 \supseteq B^3 \oplus C^3$.
2. $A \subseteq B \Rightarrow \langle C^1 \oplus A^1, C^2 \oplus A^2, C^3 \oplus A^3 \rangle \subseteq \langle C^1 \oplus B^1, C^2 \oplus B^2, C^3 \oplus B^3 \rangle$
 $C^1 \oplus A^1 \subseteq C^1 \oplus B^1, C^2 \oplus A^2 \subseteq C^2 \oplus B^2$ and $C^3 \oplus A^3 \supseteq C^3 \oplus B^3$.

Type II:

1. $A \subseteq B \Rightarrow \langle A^1 \oplus C^1, A^2 \oplus C^2, A^3 \oplus C^3 \rangle \subseteq \langle B^1 \oplus C^1, B^2 \oplus C^2, B^3 \oplus C^3 \rangle$
 $A^1 \oplus C^1 \subseteq B^1 \oplus C^1, A^2 \oplus C^2 \supseteq B^2 \oplus C^2$ and $A^3 \oplus C^3 \supseteq B^3 \oplus C^3$.
2. $A \subseteq B \Rightarrow \langle C^1 \oplus A^1, C^2 \oplus A^2, C^3 \oplus A^3 \rangle \subseteq \langle C^1 \oplus B^1, C^2 \oplus B^2, C^3 \oplus B^3 \rangle$
 $C^1 \oplus A^1 \subseteq C^1 \oplus B^1, C^2 \oplus A^2 \supseteq C^2 \oplus B^2$ and $C^3 \oplus A^3 \supseteq C^3 \oplus B^3$.

Proof: i), ii), iii) Obvious in two types.

Proposition 4.5: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$.

We will prove the proposition for the two types of neutrosophic crisp dilation operation as follows:

Type I: a) $\cap_{i \in I} A_i \widetilde{\oplus} B = \cap_{i \in I} (A_i \widetilde{\oplus} B)$

$$\begin{aligned} & \langle \cap_{i \in I} A_i^1 \oplus B^1, \cap_{i \in I} A_i^2 \oplus B^2, \cap_{i \in I} A_i^3 \oplus B^3 \rangle \\ &= \langle \cap_{i \in I} (A_i^1 \oplus B^1), \cap_{i \in I} (A_i^2 \oplus B^2), \cap_{i \in I} (A_i^3 \oplus B^3) \rangle. \end{aligned}$$

$$\begin{aligned}
\text{b) } B \widetilde{\oplus} \cap_{i \in I} A_i &= \cap_{i \in I} (B \widetilde{\oplus} A_i) \\
&\langle B^1 \widetilde{\oplus} \cap_{i \in I} A^1_i, B^2 \widetilde{\oplus} \cap_{i \in I} A^2_i, B^3 \widetilde{\oplus} \cap_{i \in I} A^3_i \rangle \\
&= \langle \cap_{i \in I} (B^1 \widetilde{\oplus} A^1_i), \cap_{i \in I} (B^2 \widetilde{\oplus} A^2_i), \cap_{i \in I} (B^3 \widetilde{\oplus} A^3_i) \rangle.
\end{aligned}$$

$$\text{Type II: a) } \cap_{i \in I} A_i \widetilde{\oplus} B = \cap_{i \in I} (A_i \widetilde{\oplus} B)$$

$$\begin{aligned}
&\langle \cap_{i \in I} A^1_i \widetilde{\oplus} B^1, \cap_{i \in I} A^2_i \widetilde{\oplus} B^2, \cap_{i \in I} A^3_i \widetilde{\oplus} B^3 \rangle \\
&= \langle \cap_{i \in I} (A^1_i \widetilde{\oplus} B^1), \cap_{i \in I} (A^2_i \widetilde{\oplus} B^2), \cap_{i \in I} (A^3_i \widetilde{\oplus} B^3) \rangle.
\end{aligned}$$

$$\begin{aligned}
\text{b) } B \widetilde{\oplus} \cap_{i \in I} A_i &= \cap_{i \in I} (B \widetilde{\oplus} A_i) \\
&\langle B^1 \widetilde{\oplus} \cap_{i \in I} A^1_i, B^2 \widetilde{\oplus} \cap_{i \in I} A^2_i, B^3 \widetilde{\oplus} \cap_{i \in I} A^3_i \rangle \\
&= \langle \cap_{i \in I} (B^1 \widetilde{\oplus} A^1_i), \cap_{i \in I} (B^2 \widetilde{\oplus} A^2_i), \cap_{i \in I} (B^3 \widetilde{\oplus} A^3_i) \rangle.
\end{aligned}$$

Proof: we will prove this property for the two types of the neutrosophic crisp intersection operator:

$$\begin{aligned}
\text{Type I: } \cap_{i \in I} A_i \widetilde{\oplus} B &= \langle \cup_{b \in B} \left(\cap_{i \in I} A^1_{ib} \right), \cup_{b \in B} \left(\cap_{i \in I} A^2_{ib} \right), \cup_{b \in B} \left(\cap_{i \in I} A^3_{i(-b)} \right) \rangle \\
&= \langle \cap_{i \in I} \left(\cup_{b \in B} A^1_{ib} \right), \cap_{i \in I} \left(\cup_{b \in B} A^2_{ib} \right), \cap_{i \in I} \left(\cup_{b \in B} A^3_{i(-b)} \right) \rangle = \cap_{i \in I} (A_i \widetilde{\oplus} B).
\end{aligned}$$

Type II:

$$\begin{aligned}
\cap_{i \in I} A_i \widetilde{\oplus} B &= \langle \cup_{b \in B} \left(\cap_{i \in I} A^1_{ib} \right), \cap_{b \in B} \left(\cap_{i \in I} A^2_{i(-b)} \right), \cap_{b \in B} \left(\cap_{i \in I} A^3_{i(-b)} \right) \rangle \\
&= \langle \cap_{i \in I} \left(\cup_{b \in B} A^1_{ib} \right), \cap_{i \in I} \left(\cap_{b \in B} A^2_{i(-b)} \right), \cap_{i \in I} \left(\cap_{b \in B} A^3_{i(-b)} \right) \rangle = \cap_{i \in I} (A_i \widetilde{\oplus} B).
\end{aligned}$$

Proof: b) The proof is similar a).

Proposition 4.6: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$.

We will prove the proposition for the two types of neutrosophic crisp dilation operation as follows:

$$\text{Type I: a) } \cup_{i \in I} A_i \widetilde{\oplus} B = \cup_{i \in I} (A_i \widetilde{\oplus} B)$$

$$\begin{aligned}
&\langle \cup_{i \in I} A^1_i \widetilde{\oplus} B^1, \cup_{i \in I} A^2_i \widetilde{\oplus} B^2, \cup_{i \in I} A^3_i \widetilde{\oplus} B^3 \rangle \\
&= \langle \cup_{i \in I} (A^1_i \widetilde{\oplus} B^1), \cup_{i \in I} (A^2_i \widetilde{\oplus} B^2), \cup_{i \in I} (A^3_i \widetilde{\oplus} B^3) \rangle.
\end{aligned}$$

$$\begin{aligned}
\text{b) } B \widetilde{\oplus} \cup_{i \in I} A_i &= \cup_{i \in I} (B \widetilde{\oplus} A_i) \\
&\langle B^1 \widetilde{\oplus} \cup_{i \in I} A_i^1, B^2 \widetilde{\oplus} \cup_{i \in I} A_i^2, B^3 \widetilde{\oplus} \cup_{i \in I} A_i^3 \rangle \\
&= \langle \cup_{i \in I} (B^1 \widetilde{\oplus} A_i^1), \cup_{i \in I} (B^2 \widetilde{\oplus} A_i^2), \cup_{i \in I} (B^3 \widetilde{\oplus} A_i^3) \rangle.
\end{aligned}$$

$$\begin{aligned}
\text{Type II: a) } \cup_{i \in I} A_i \widetilde{\oplus} B &= \cup_{i \in I} (A_i \widetilde{\oplus} B) \\
&\langle \cup_{i \in I} A_i^1 \widetilde{\oplus} B^1, \cup_{i \in I} A_i^2 \widetilde{\oplus} B^2, \cup_{i \in I} A_i^3 \widetilde{\oplus} B^3 \rangle \\
&= \langle \cup_{i \in I} (A_i^1 \widetilde{\oplus} B^1), \cup_{i \in I} (A_i^2 \widetilde{\oplus} B^2), \cup_{i \in I} (A_i^3 \widetilde{\oplus} B^3) \rangle.
\end{aligned}$$

$$\begin{aligned}
\text{b) } B \widetilde{\oplus} \cup_{i \in I} A_i &= \cup_{i \in I} (B \widetilde{\oplus} A_i) \\
&\langle B^1 \widetilde{\oplus} \cup_{i \in I} A_i^1, B^2 \widetilde{\oplus} \cup_{i \in I} A_i^2, B^3 \widetilde{\oplus} \cup_{i \in I} A_i^3 \rangle \\
&= \langle \cup_{i \in I} (B^1 \widetilde{\oplus} A_i^1), \cup_{i \in I} (B^2 \widetilde{\oplus} A_i^2), \cup_{i \in I} (B^3 \widetilde{\oplus} A_i^3) \rangle.
\end{aligned}$$

Proof: a) we will prove this property for the two types of the neutrosophic crisp union operator:

Type I:

$$\begin{aligned}
\cup_{i \in I} A_i \widetilde{\oplus} B &= \langle \cup_{b \in B} \left(\cup_{i \in I} A_i^1 \right), \cup_{b \in B} \left(\cup_{i \in I} A_i^2 \right), \cap_{b \in B} \left(\cup_{i \in I} A_i^3 \right) \rangle \\
&= \langle \cup_{i \in I} \left(\cup_{b \in B} A_i^1 \right), \cup_{i \in I} \left(\cup_{b \in B} A_i^2 \right), \cup_{i \in I} \left(\cap_{b \in B} A_i^3 \right) \rangle = \cup_{i \in I} (A_i \widetilde{\oplus} B).
\end{aligned}$$

Type II:

$$\begin{aligned}
\cup_{i \in I} A_i \widetilde{\oplus} B &= \langle \cup_{b \in B} \left(\cup_{i \in I} A_i^1 \right), \cap_{b \in B} \left(\cup_{i \in I} A_i^2 \right), \cap_{b \in B} \left(\cup_{i \in I} A_i^3 \right) \rangle \\
&= \langle \cup_{i \in I} \left(\cup_{b \in B} A_i^1 \right), \cup_{i \in I} \left(\cap_{b \in B} A_i^2 \right), \cup_{i \in I} \left(\cap_{b \in B} A_i^3 \right) \rangle = \cup_{i \in I} (A_i \widetilde{\oplus} B).
\end{aligned}$$

Proof: b) The proof is similar to a).

4.4.3 Properties of the Neutrosophic Crisp Opening:

Proposition 4.7:

The neutrosophic opening satisfies the monotonicity $\forall A, B \in \mathcal{NC}(Z^2)$.

We will prove the proposition for the two types of neutrosophic crisp opening operation as follows:

Type I:

$$A \subseteq B \Rightarrow \langle A^1 \circ C^1, A^2 \circ C^2, A^3 \circ C^3 \rangle \subseteq \langle B^1 \circ C^1, B^2 \circ C^2, B^3 \circ C^3 \rangle$$

$$A^1 \circ C^1 \subseteq B^1 \circ C^1, A^2 \circ C^2 \subseteq B^2 \circ C^2 \text{ and } A^3 \circ C^3 \supseteq B^3 \circ C^3.$$

Type II:

$$A \subseteq B \Rightarrow \langle A^1 \circ C^1, A^2 \circ C^2, A^3 \circ C^3 \rangle \subseteq \langle B^1 \circ C^1, B^2 \circ C^2, B^3 \circ C^3 \rangle$$

$$A^1 \circ C^1 \subseteq B^1 \circ C^1, A^2 \circ C^2 \supseteq B^2 \circ C^2 \text{ and } A^3 \circ C^3 \supseteq B^3 \circ C^3.$$

Proposition 4.8: For any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$.

We will prove the proposition for the two types of neutrosophic crisp opening operation as follows:

Type I: $\bigcap_{i \in I} A_i \circ B = \bigcap_{i \in I} (A_i \circ B)$

$$\begin{aligned} & \langle \bigcap_{i \in I} A^1_i \circ B^1, \bigcap_{i \in I} A^2_i \circ B^2, \bigcap_{i \in I} A^3_i \circ B^3 \rangle \\ &= \langle \bigcap_{i \in I} (A^1_i \circ B^1), \bigcap_{i \in I} (A^2_i \circ B^2), \bigcap_{i \in I} (A^3_i \circ B^3) \rangle. \end{aligned}$$

Type II: $\bigcap_{i \in I} A_i \circ B = \bigcap_{i \in I} (A_i \circ B)$

$$\begin{aligned} & \langle \bigcap_{i \in I} A^1_i \circ B^1, \bigcap_{i \in I} A^2_i \circ B^2, \bigcap_{i \in I} A^3_i \circ B^3 \rangle \\ &= \langle \bigcap_{i \in I} (A^1_i \circ B^1), \bigcap_{i \in I} (A^2_i \circ B^2), \bigcap_{i \in I} (A^3_i \circ B^3) \rangle. \end{aligned}$$

Proposition 4.9: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$.

We will prove the proposition for the two types of neutrosophic crisp opening operation as follows:

Type I: $\bigcup_{i \in I} A_i \circ B = \bigcup_{i \in I} (A_i \circ B)$

$$\begin{aligned} & \langle \bigcup_{i \in I} A^1_i \circ B^1, \bigcup_{i \in I} A^2_i \circ B^2, \bigcup_{i \in I} A^3_i \circ B^3 \rangle \\ &= \langle \bigcup_{i \in I} (A^1_i \circ B^1), \bigcup_{i \in I} (A^2_i \circ B^2), \bigcup_{i \in I} (A^3_i \circ B^3) \rangle. \end{aligned}$$

Type II: $\bigcup_{i \in I} A_i \circ B = \bigcup_{i \in I} (A_i \circ B)$

$$\begin{aligned} & \langle \bigcup_{i \in I} A^1_i \circ B^1, \bigcup_{i \in I} A^2_i \circ B^2, \bigcup_{i \in I} A^3_i \circ B^3 \rangle \\ &= \langle \bigcup_{i \in I} (A^1_i \circ B^1), \bigcup_{i \in I} (A^2_i \circ B^2), \bigcup_{i \in I} (A^3_i \circ B^3) \rangle. \end{aligned}$$

Proof: Is similar to the procedure used to prove the propositions given in proposition 4.6.

4.4.4 Properties of the Neutrosophic Crisp Closing:

Proposition 4.10:

The neutrosophic closing satisfies the monotonicity $\forall A, B \in \mathcal{N}(Z^2)$. We will prove the proposition for the two types of neutrosophic crisp closing operation as follows:

Type I:

$$A \subseteq B \implies \langle A^1 \bullet C^1, A^2 \bullet C^2, A^3 \bullet C^3 \rangle \subseteq \langle B^1 \bullet C^1, B^2 \bullet C^2, B^3 \bullet C^3 \rangle$$

$$A^1 \bullet C^1 \subseteq B^1 \bullet C^1, A^2 \bullet C^2 \subseteq B^2 \bullet C^2 \text{ and } A^3 \bullet C^3 \supseteq B^3 \bullet C^3.$$

Type II:

$$A \subseteq B \implies \langle A^1 \bullet C^1, A^2 \bullet C^2, A^3 \bullet C^3 \rangle \subseteq \langle B^1 \bullet C^1, B^2 \bullet C^2, B^3 \bullet C^3 \rangle$$

$$A^1 \bullet C^1 \subseteq B^1 \bullet C^1, A^2 \bullet C^2 \supseteq B^2 \bullet C^2 \text{ and } A^3 \bullet C^3 \supseteq B^3 \bullet C^3.$$

Proposition 4.11: for any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$. We will prove the proposition for the two types of neutrosophic crisp closing operation as follows:

Type I: $\bigcap_{i \in I} A_i \tilde{\bullet} B = \bigcap_{i \in I} (A_i \tilde{\bullet} B)$

$$\langle \bigcap_{i \in I} A^1_i \bullet B^1, \bigcap_{i \in I} A^2_i \bullet B^2, \bigcap_{i \in I} A^3_i \bullet B^3 \rangle$$

$$= \langle \bigcap_{i \in I} (A^1_i \bullet B^1), \bigcap_{i \in I} (A^2_i \bullet B^2), \bigcap_{i \in I} (A^3_i \bullet B^3) \rangle.$$

Type II: $\bigcap_{i \in I} A_i \tilde{\circ} B = \bigcap_{i \in I} (A_i \tilde{\circ} B)$

$$\langle \bigcap_{i \in I} A^1_i \bullet B^1, \bigcap_{i \in I} A^2_i \bullet B^2, \bigcap_{i \in I} A^3_i \bullet B^3 \rangle$$

$$= \langle \bigcap_{i \in I} (A^1_i \bullet B^1), \bigcap_{i \in I} (A^2_i \bullet B^2), \bigcap_{i \in I} (A^3_i \bullet B^3) \rangle.$$

Proposition 4.12: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$.

Type I: $\bigcup_{i \in I} A_i \tilde{\bullet} B = \bigcup_{i \in I} (A_i \tilde{\bullet} B)$

$$\langle \bigcup_{i \in I} A^1_i \bullet B^1, \bigcup_{i \in I} A^2_i \bullet B^2, \bigcup_{i \in I} A^3_i \bullet B^3 \rangle$$

$$= \langle \bigcup_{i \in I} (A^1_i \bullet B^1), \bigcup_{i \in I} (A^2_i \bullet B^2), \bigcup_{i \in I} (A^3_i \bullet B^3) \rangle.$$

Type II: $\cup_{i \in I} A_i \tilde{\circ} B = \cup_{i \in I} (A_i \tilde{\circ} B)$

$$\begin{aligned} & \langle \cup_{i \in I} A_i^1 \bullet B^1, \cup_{i \in I} A_i^2 \circ B^2, \cup_{i \in I} A_i^3 \circ B^3 \rangle \\ & = \langle \cup_{i \in I} (A_i^1 \bullet B^1), \cup_{i \in I} (A_i^2 \circ B^2), \cup_{i \in I} (A_i^3 \circ B^3) \rangle. \end{aligned}$$

Proof: Is similar to the procedure used to prove the propositions given in proposition 4.6.

4.5 Duality of Theorem:

Erosion and dilation are duals of each other with respect to set complementation and reflection: Indicating that erosion of A by B is the complement of the dilation of the complement of A by the reflection of B and vice versa. Duality is particularly useful when the structure element is symmetric with respect to its origin, so that. Then we can obtain the erosion of an image by B simply by dilating its background (complement of A) with the same structuring element and complementing the result.

4.5.1 Duality Theorem of Neutrosophic Crisp Dilation:

let $A, B \in \mathcal{NC}(X)$; Neutrosophic Crisp Erosion and Dilation are dual operations i.e.

Type I:

$$\begin{aligned} co(\text{co}A \tilde{\oplus} B) & = \langle co(\text{co}A^1 \oplus B^1), co(\text{co}A^2 \oplus B^2), co(\text{co}A^3 \oplus B^3) \rangle \\ & = \langle A^1 \ominus B^1, A^2 \ominus B^2, A^3 \oplus B^3 \rangle = A \tilde{\ominus} B. \end{aligned}$$

Type II:

$$\begin{aligned} co(\text{co}A \tilde{\oplus} B) & = \langle co(\text{co}A^1 \oplus B^1), co(\text{co}A^2 \ominus B^2), co(\text{co}A^3 \ominus B^3) \rangle \\ & = \langle A^1 \ominus B^1, A^2 \oplus B^2, A^3 \oplus B^3 \rangle = A \tilde{\ominus} B. \end{aligned}$$

4.5.2 Duality Theorem of Neutrosophic Crisp Closing:

let $A, B \in \mathcal{NC}(X)$; Neutrosophic erosion and dilation are dual operations i.e.

Type I: $co(\text{co}A \tilde{\circ} B) = \langle co(\text{co}A^1 \bullet B^1), co(\text{co}A^2 \bullet B^2), co(\text{co}A^3 \circ B^3) \rangle$

$$= \langle A^1 \circ B^1, A^2 \circ B^2, A^3 \bullet B^3 \rangle = A \circ B.$$

$$\begin{aligned} \text{Type II: } co(coA \tilde{\circ} B) &= \langle co(coA^1 \bullet B^1), co(coA^2 \circ B^2), co(coA^3 \circ B^3) \rangle \\ &= \langle A^1 \circ B^1, A^2 \bullet B^2, A^3 \bullet B^3 \rangle = A \circ B. \end{aligned}$$

4.6 Neutrosophic Crisp Mathematical Morphological Filters:

When considering The differences of two or more of the basic neutrosophic morphological operators, given in § 4.3, yield some remarkable filters; in this section we will consider the boundary and Hat filters:

4.6.1 Some Type of Boundary Extraction Filter Using Neutrosophic Crisp Dilation and Neutrosophic Crisp Erosion:

Where A^1 is the set of all pixels that belong to the foreground of the picture, A^3 contains the pixels that belong to the background while A^2 contains those pixel which do not belong to neither A^1 nor A^3 ; let $A, B \in \mathcal{NC}(X)$, $A = \langle A^1, A^2, A^3 \rangle$, and B is some structure element of the form $B = \langle B^1, B^2, B^3 \rangle$, dilation and erosion can be used in combination with image subtraction to obtain the morphology extract boundary of an image; The dilation thickens regions in an image and the erosion shrinks them.

4.6.1.1 Neutrosophic Crisp Internal Boundary Filter:

let A and B be two neutrosophic crisp sets, $A = \langle A^1, A^2, A^3 \rangle$ and B is some structure element of the form $B = \langle B^1, B^2, B^3 \rangle$; where A^1 is the set of all pixels that belong to the foreground of the picture, A^3 contains the pixels that belong to the background while A^2 contains those pixel which do not belong to neither A^1 nor A^3 ; then the neutrosophic crisp internal boundary is defined as: $\tilde{B}_{int}(A) = B(A) \cap B^*(A)$, where;

$$B^*(A) = A^2 - [(A^3 \oplus B^3) - (A^1 \ominus B^1)], \quad B(A) = A^2 - (B_1A^1 \cup B_3A^3)$$

$$B_1A^1 = A^1 - (A^1 \ominus B^1), \quad B_3A^3 = (A^3 \oplus B^3) - A^3.$$

In the following figure (fig.4.9), we present the results obtained when applying neutrosophic crisp internal boundary filter on some grayscale image.

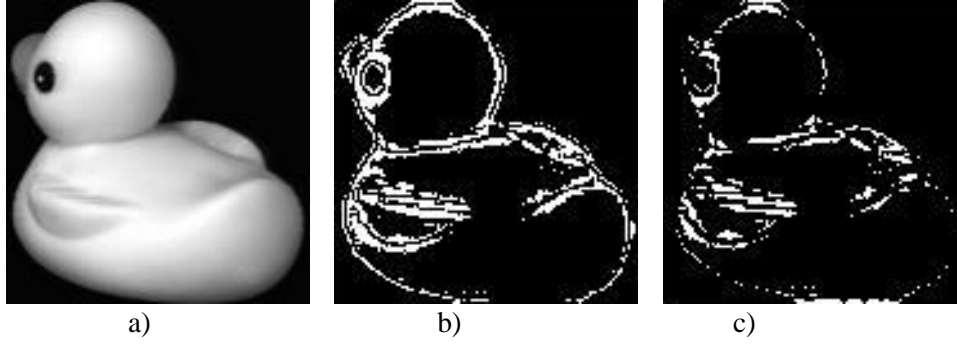


Fig. 4.9: Neutrosophic crisp internal boundary filter: a) Original image
b) Neutrosophic crisp internal boundary filter with SE(3)
c) Neutrosophic crisp internal boundary filter with SE(5)

4.6.1.2 Neutrosophic Crisp External Boundary Filter:

The simplest morphological edge detector, the dilation residue, is found by subtracting the original signal from its dilation by a small structuring element. The output is defined as: $\tilde{B}_{ext}(A) = B(A) \cap B^*(A)$, where;

$$B(A) = A^2 - (B_1A^1 \cup B_3A^3), \quad B^*(A) = A^2 - [(A^1 \oplus B^1) - (A^3 \ominus B^3)]$$

$$B_1A^1 = (A^1 \oplus B^1) - A^1, \quad B_3A^3 = A^3 - (A^3 \ominus B^3).$$

In the following figure (fig.4.10), we present the results obtained when applying neutrosophic crisp external boundary filter on some grayscale image.

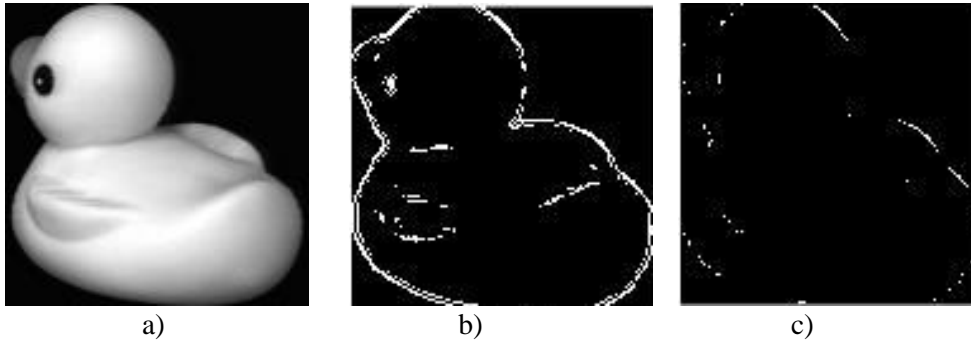


Fig. 4.10: Neutrosophic crisp external boundary filter: a) Original image
b) Neutrosophic crisp external boundary filter with SE(3)
c) Neutrosophic crisp external boundary filter with SE(5)

4.6.1.3 Neutrosophic Crisp Gradient Boundary Filter:

Neutrosophic crisp gradient boundary filter is defined as:

$$\tilde{B}_{gradient}(A) = B_1(A) \cap B_2(A), \text{ where;}$$

$$B_1(A) = A^2 - (B_1A^1 \cup B_3A^3), \quad B_2(A) = A^2 - ((A^3 \oplus B^3) - (A^1 \ominus B^1))$$

$$B_1(A^1) = (A^1 \ominus B^1) - (A^1 \oplus B^1), \quad B_3(A^3) = (A^3 \oplus B^3) - (A^3 \ominus B^3).$$

In the following figure (fig.4.11), we present the results obtained when applying neutrosophic crisp gradient boundary filter on some grayscale image.

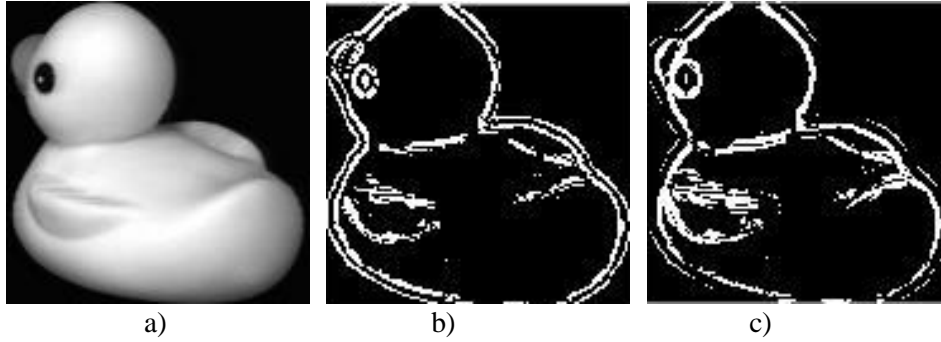


Fig.4.11: Neutrosophic crisp gradient boundary: a) Original image
b) Neutrosophic crisp gradient boundary filter with SE(3)
c) Neutrosophic crisp gradient boundary filter with SE(5)

4.6.1.4 Neutrosophic Crisp Outline Boundary Filter:

Neutrosophic crisp outline boundary filter is defined as:

$$\tilde{B}_{outline}(A) = co((B_1A^1 \cup B_3A^3) \cap A^2), \text{ where;}$$

$$B_1(A^1) = co(A^1 \ominus B^1) \cap A^1, \quad B_3(A^3) = co(A^3 \oplus B^3) \cup A^3.$$

In the following figure (fig.4.12), we present the results obtained when applying neutrosophic crisp outline boundary filter on some grayscale image.

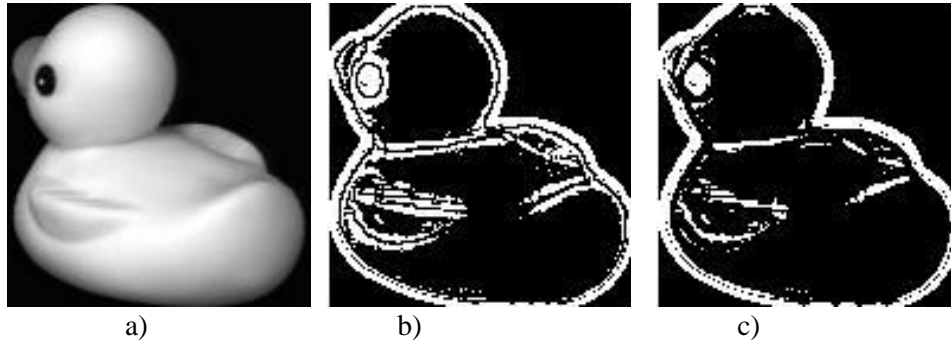


Fig. 4.12: Neutrosophic crisp outline a) Original image
b) Neutrosophic crisp outline filtered image SE(3)
c) Neutrosophic crisp outline filtered image SE(7)

4.6.2 Combinations of Neutrosophic Crisp External and Neutrosophic Crisp Internal Operators:

Dilation and erosion can be used in combination with image subtraction to obtain the morphological extraction A of an image as:

$$1. \tilde{B}_{grad}(A) = \min(B_1(A^1), A^2), \text{ where;}$$

$$B_1(A^1) = \max[\max(B_{ext}(A^1), B_{int}(A^1)), \min(B_{ext}(A^3), B_{int}(A^3))].$$

In the following figure (fig.4.13), we present the results obtained when applying neutrosophic crisp grad boundary filter on some grayscale image.

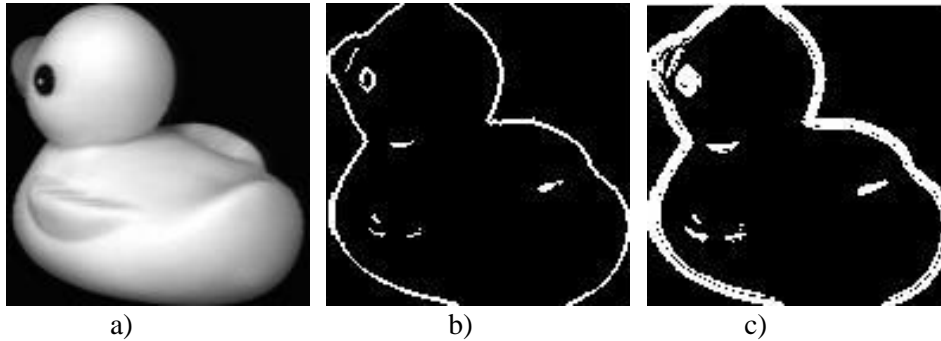


Fig. 4.13: *Neutrosophic crisp grad boundary: a) Original image
b) Neutrosophic Crisp grad Boundary filtered image with SE(3)
c) Neutrosophic Crisp grad Boundary filtered image with SE(7)*

2. $\tilde{B}_{min} = \min(B_1(A^1), A^2)$, where;

$$B_1(A^1) = \max[\min(B_{ext}(A^1), \partial_{int}(A^1)), \max(B_{ext}A^3, B_{int}(A^3))].$$

In the following figure (fig.4.14), we present the results obtained when applying neutrosophic crisp min. boundary filter on some grayscale image.

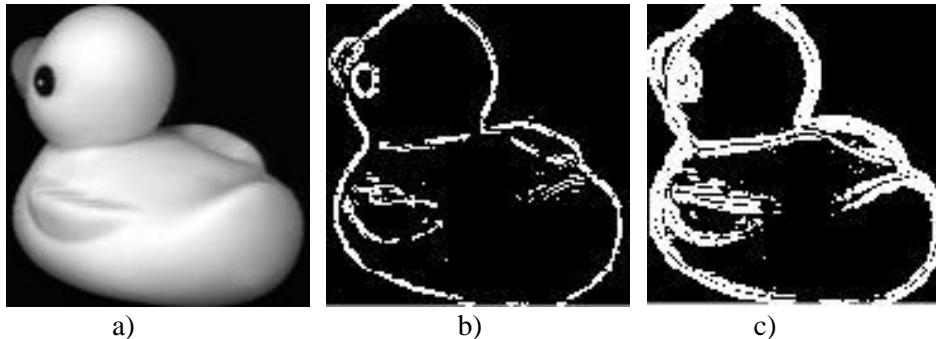


Fig. 4.14: *Neutrosophic crisp min. boundary: a) Original image
b) Neutrosophic Crisp min Boundary filter image with SE(3)
c) Neutrosophic Crisp min Boundary filter image with SE(7)*

3. $\tilde{B}_{div} = \min(B_1(A^1), A^2)$, where;

$$B_1(A^1) = \max[(B_{ext}(A^1) - B_{int}(A^1)), (B_{ext}(A^3) - B_{int}(A^3))].$$

In the following figure (fig.4.15), we present the results obtained when applying neutrosophic crisp Div. boundary filter on some grayscale image.

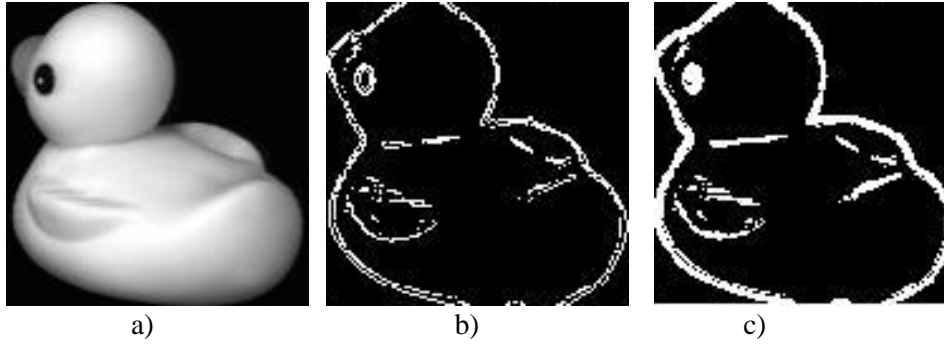


Fig. 4.15: *Neutrosophic crisp div. boundary: a) Original image
b) Neutrosophic Crisp Div. Boundary filtered image with SE(3)
c) Neutrosophic Crisp Div. Boundary filtered image with SE(7)*

4.6.3 Neutrosophic Crisp Hat Filters:

Filters described above remove image objects or noise of certain kind. Sometimes, however, instead of removing, one needs to detect objects of particular characteristics. The descriptions "white" and "black" indicates types of objects which are detected by a particular operator lighter or darker than the background. The mentioned above, main property of top-hat filter can be applied to contrast enhancement. Indeed, by combining the original image with images with detected objects, the contrast improves. This combination is performed by adding to the original image the result of white top-hat and by subtracting the result of a black top-hat:

- **Neutrosophic Crisp Top-hat Filter:**

$$\widetilde{Top}_{hat}(A) = B(A) \cap B^*(A), \text{ where;}$$

$$B_1A^1 = A^1 - (A^1 \circ B^1), \quad B_3A^3 = (A^3 \bullet B^3) - A^3$$

$$B(A) = A^2 - (\partial_1A^1 \cup \partial_3A^3), \quad B^*(A) = A^2 - [(A^1 \circ B^1) - (A^3 \bullet B^3)].$$

In the following figure (fig.4.16), we present the results obtained when applying neutrosophic crisp top-hat boundary filter on some grayscale image.

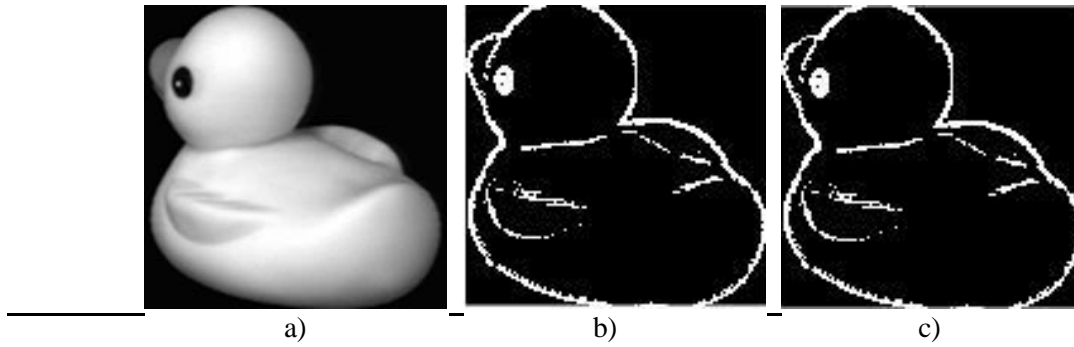


Fig. 4.16: *Neutrosophic crisp top-hat filter: a)Original image
b) Neutrosophic Crisp top-hat Boundary filtered with SE(3)
c)Neutrosophic Crisp top-hat Boundary filtered with SE(7)*

- **Neutrosophic Crisp Bottom-hat Filter:**

$\widetilde{Bottom}_{hat}(A) = B(A) \cap B^*(A)$, where;

$$B(A) = A^2 - (\partial_1(A^1) \cup \partial_3(A^3)), \quad B^*(A) = A^2 - [(A^1 \bullet B^1) - (A^3 \circ B^3)]$$

$$B_1(A^1) = (A^1 \bullet B^1) - A^1, \quad B_3(A^3) = A^3 - (A^3 \circ B^3).$$

In the following figure (fig.4.17), we present the results obtained when applying neutrosophic crisp bottom-hat boundary filter on some grayscale image.

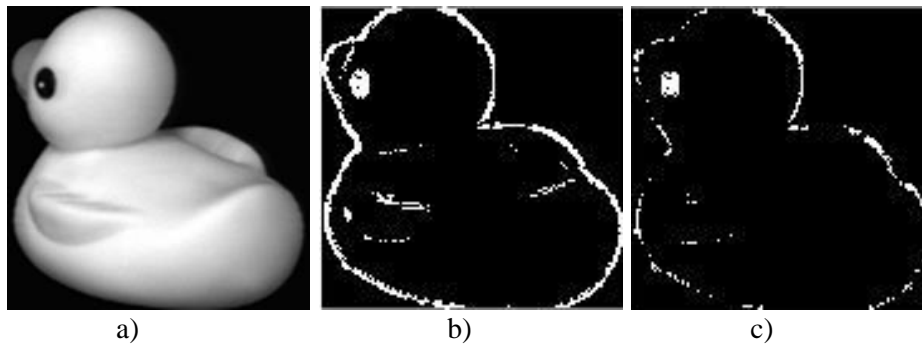


Fig. 4.17: *Neutrosophic crisp bottom-hat filter: a)Original image
b) Neutrosophic crisp bottom-hat Boundary filtered with SE(3)
c)neutrosophic crisp bottom-hat Boundary filtered with SE(5)*

4.7 Conclusion:

In this chapter we established a foundation for what we called "Neutrosophic Crisp Mathematical Morphology". Our aim was to generalize the concepts of the classical mathematical morphology. For this purpose, we developed several neutrosophic crisp morphological operators; namely, the neutrosophic crisp dilation, the neutrosophic crisp erosion, the neutrosophic crisp opening and the neutrosophic crisp closing operators.

These operators were presented in two different types, each type is determined according to the behavior of the second component of the triple structure of the operator. Furthermore, we developed three neutrosophic crisp morphological filters; namely, the neutrosophic crisp boundary, the neutrosophic crisp Top-hat and the neutrosophic crisp Bottom-hat filters. Some promising experimental results were presented to visualize the effect of the new introduced operators and filters on the image in the neutrosophic domain instead of the spatial domain.

Chapter 5

Neutrosophic Mathematical Morphology

5.1 Neutrosophic Mathematical Morphology:

The aim of this chapter is to introduce a new approach to mathematical morphology based on neutrosophic set theory. In order to propose "The Neutrosophic Mathematical Morphology", the concept of neutrosophic morphology based on the fact that the basic morphological operators make use of fuzzy set operators. Hence, such expressions can easily be extended using the context of neutrosophic sets. Basic definitions for neutrosophic morphological operations are extracted and a study of its algebraic properties is presented. In our work we demonstrate that neutrosophic morphological operations inherit properties and restrictions of fuzzy mathematical morphology. The operations of neutrosophic dilation, neutrosophic erosion, neutrosophic opening and neutrosophic closing of the neutrosophic image by neutrosophic structuring element, are defined in terms of their membership, in determinant and non-membership functions; which is defined for the first time as far as we know.

Definition 5.1:

The reflection of the SE B mirrored in its origin is defined as:

- $-B = \langle -T_B, -I_B, -F_B \rangle$, where;

$$-T_B(u) = T_B(-u), -I_B(u) = I_B(-u) \quad \text{and} \quad -F_B(u) = F_B(-u).$$

- For every p in E , Translation of A by $p \in Z^2$ is

$$A_p = \langle T_{A_p}, I_{A_p}, F_{A_p} \rangle, \text{ where;}$$

$$T_{A_p}(u) = T_{A_p}(u + p), I_{A_p}(u) = I_{A_p}(u + p) \quad \text{and} \quad F_{A_p}(u) = F_{A_p}(u + p).$$

most morphological operations on neutrosophic can be obtained by combining neutrosophic set theoretical operations with two basic operations, dilation and erosion.

5.2 Neutrosophic Morphological Operations:

The neutrosophy concept is introduced to morphology by a triple degree to which the structuring element fits into the image in the three levels of trueness, indeterminacy, and falseness. The operations of neutrosophic erosion, dilation, opening and closing of the neutrosophic image by neutrosophic SE, are defined in terms of their membership, indeterminacy and non-membership functions; which is defined for the first time as far as we know.

5.2.1 Neutrosophic Dilation and Neutrosophic Erosion:

The two basic operations for the construction of morphological operators, namely, neutrosophic dilation and neutrosophic erosion. are based on the two Minkowski set operations, the Minkowski addition and subtraction of two neutrosophic sets; respectively. we may define the follows:

5.2.1.1 Neutrosophic Dilation Operation:

Let A and B , be two neutrosophic sets, the neutrosophic dilation of a neutrosophic set B to a neutrosophic set A is defined by:

$(A \widetilde{\oplus} B) = \langle T_{A \widetilde{\oplus} B}, I_{A \widetilde{\oplus} B}, F_{A \widetilde{\oplus} B} \rangle$, where for each $u, v \in Z^2$. The three components, $T_{A \widetilde{\oplus} B}$, $I_{A \widetilde{\oplus} B}$ and $F_{A \widetilde{\oplus} B}$ are to be defined in to different types as follows:

5.2.1.1.A. Neutrosophic Dilation of Type I:

$$T_{A \widetilde{\oplus} B}(v) = \sup_{u \in Z^2} \min(T_A(v + u), T_B(u)),$$

$$I_{A \widetilde{\oplus} B}(v) = \sup_{u \in Z^2} \min(I_A(v + u), I_B(u)),$$

$$F_{A \widetilde{\oplus} B}(v) = \inf_{u \in Z^2} \max(F_A(v + u), 1 - F_B(u)).$$

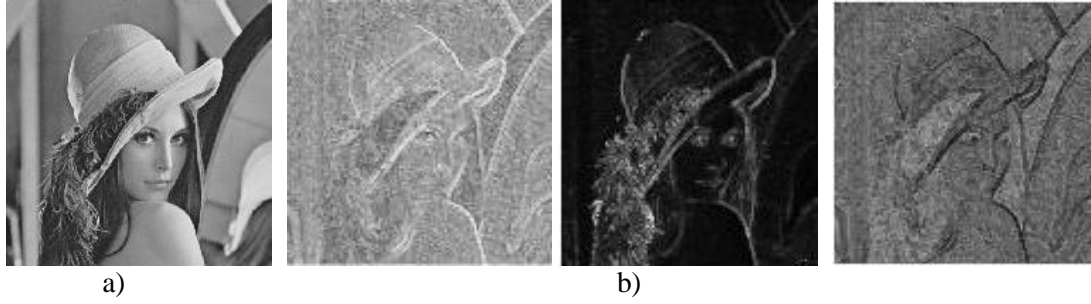


Fig.5.1(I): Applying the neutrosophic dilation operator: a) Original image
b) Neutrosophic component of the dilated image in type I $\langle T_{A\oplus B}, I_{A\oplus B}, F_{A\oplus B} \rangle$ respectively

5.2.1.1.B. Neutrosophic Dilation of Type II:

$$T_{A\oplus B}(v) = \sup_{u \in Z^2} \min(T_A(v+u), T_B(u)),$$

$$I_{A\oplus B}(v) = \inf_{u \in Z^2} \max(I_A(v+u), 1 - I_B(u)),$$

$$F_{A\oplus B}(v) = \inf_{u \in Z^2} \max(F_A(v+u), 1 - F_B(u)).$$

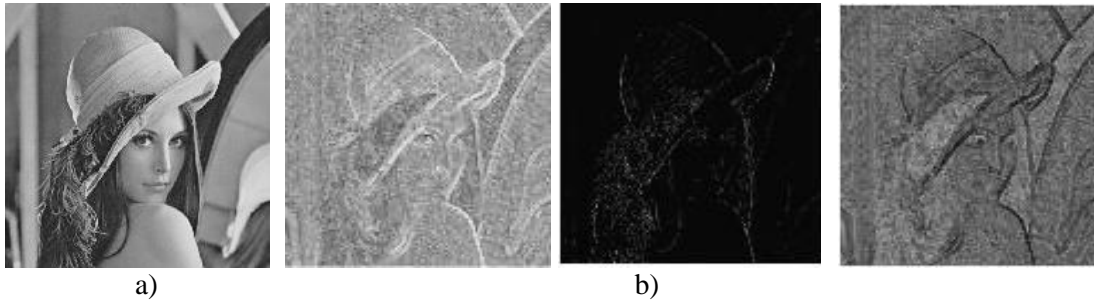


Fig.5.2(II): Applying the neutrosophic dilation operator: a) Original image
b) Neutrosophic component of the dilated image in type II $\langle T_{A\oplus B}, I_{A\oplus B}, F_{A\oplus B} \rangle$ respectively

5.2.1.2 Neutrosophic Erosion Operation:

let A and B, be two neutrosophic sets, The neutrosophic erosion of a neutrosophic set B from a neutrosophic set A is defined as: $(A \ominus B) = \langle T_{A\ominus B}, I_{A\ominus B}, F_{A\ominus B} \rangle$; where for each $u, v \in Z^2$. The three components, $T_{A\ominus B}$, $I_{A\ominus B}$ and $F_{A\ominus B}$ are to be defined in to different types as follows:

5.2.1.2.A. Neutrosophic Erosion of Type I:

$$T_{A\ominus B}(v) = \inf_{u \in Z^2} \max(T_A(v+u), 1 - T_B(u)),$$

$$I_{A\ominus B}(v) = \inf_{u \in Z^2} \max(I_A(v+u), 1 - I_B(u)),$$

$$F_{A\ominus B}(v) = \sup_{u \in Z^2} \min(F_A(v+u), F_B(u)).$$

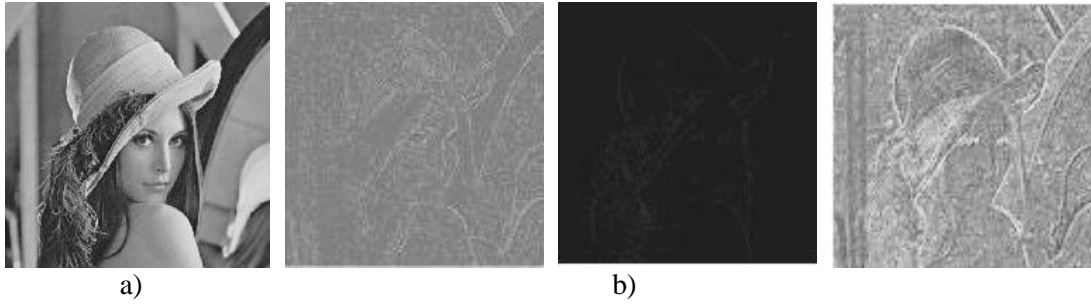


Fig.5.3(I): Applying the neutrosophic erosion operator: a)original image b)neutrosophic components of the eroded in type I $\langle T_{A\tilde{\ominus}B}, I_{A\tilde{\ominus}B}, F_{A\tilde{\ominus}B} \rangle$ respectively

5.2.1.2.B. Neutrosophic Erosion of Type II:

$$T_{A\tilde{\ominus}B}(v) = \inf_{u \in Z^2} \max(T_A(v+u), 1 - T_B(u)),$$

$$I_{A\tilde{\ominus}B}(v) = \sup_{u \in Z^2} \min(I_A(v+u), I_B(u)),$$

$$F_{A\tilde{\ominus}B}(v) = \sup_{u \in Z^2} \min(F_A(v+u), F_B(u)).$$

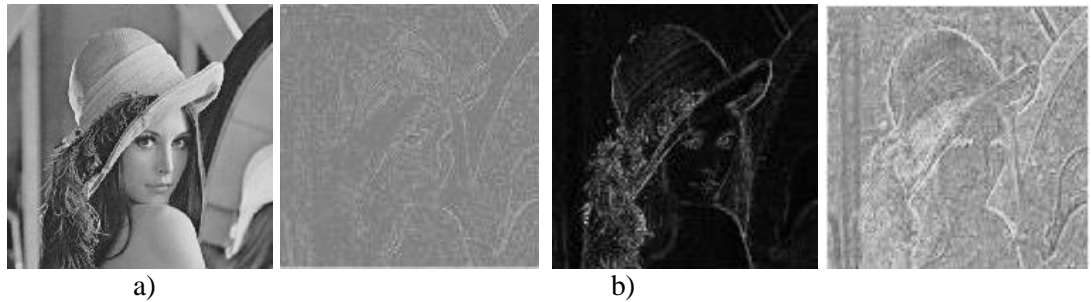


Fig.5.4(II): Applying the neutrosophic erosion operator: a)Original image b)neutrosophic components of the eroded in type II $\langle T_{A\tilde{\ominus}B}, I_{A\tilde{\ominus}B}, F_{A\tilde{\ominus}B} \rangle$ respectively

5.2.2 Neutrosophic Opening and Neutrosophic Closing:

The combination of the two main operations, neutrosophic dilation and neutrosophic erosion, can produce more complex sequences. Neutrosophic opening and neutrosophic closing are the most useful of these for morphological filtering.

5.2.2.1 Neutrosophic Opening Operation:

A neutrosophic opened image, is the result of eroding a neutrosophic image A by a neutrosophic SE B ; followed by a neutrosophic dilation operation by the same element B , and to be defined as the triple structure: $(A \tilde{\circ} B) = \langle T_{A\tilde{\circ}B}, I_{A\tilde{\circ}B}, F_{A\tilde{\circ}B} \rangle$, where $u, v, w \in Z^2$. The three components, $T_{A\tilde{\circ}B}$, $I_{A\tilde{\circ}B}$ and $F_{A\tilde{\circ}B}$ are to be defined in two different types as follows:

5.2.2.1.A. Neutrosophic Opening of Type I:

$$T_{A\tilde{\circ}B}(v) = \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max(T_A(v - u + w), 1 - T_B(w)), T_B(u) \right],$$

$$I_{A\tilde{\circ}B}(v) = \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max(I_A(v - u + w), 1 - I_B(w)), I_B(u) \right],$$

$$F_{A\tilde{\circ}B}(v) = \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min(F_A(v - u + w), F_B(w)), 1 - F_B(u) \right].$$

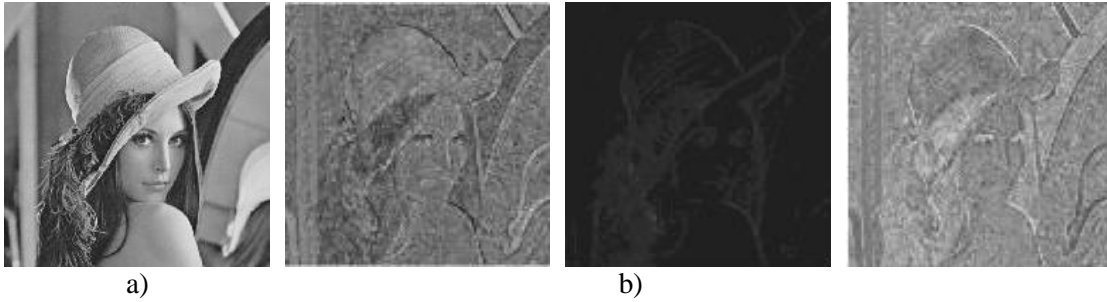


Fig.5.5(I): Applying the neutrosophic opening operator: a)Original image
b) neutrosophic opening components in type I $\langle T_{A\tilde{\circ}B}, I_{A\tilde{\circ}B}, F_{A\tilde{\circ}B} \rangle$ respectively

5.2.2.1.B. Neutrosophic Opening of Type II:

$$T_{A\tilde{\circ}B}(v) = \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max(T_A(v - u + w), 1 - T_B(w)), T_B(u) \right],$$

$$I_{A\tilde{\circ}B}(v) = \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max(I_A(v - u + w), I_B(w)), 1 - I_B(u) \right],$$

$$F_{A\tilde{\circ}B}(v) = \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min(F_A(v - u + w), F_B(w)), 1 - F_B(u) \right].$$

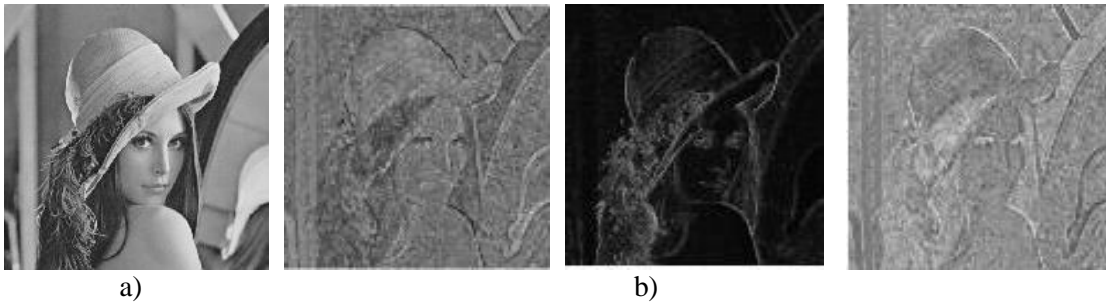


Fig.5.6(II): Applying the neutrosophic opening operator: a)Original image
b) Neutrosophic opening components in type II $\langle T_{A\tilde{\circ}B}, I_{A\tilde{\circ}B}, F_{A\tilde{\circ}B} \rangle$ respectively

5.2.2.2 Neutrosophic Closing Operation:

A neutrosophic closed image, is the result of dilation a neutrosophic image A by a neutrosophic structure element B; followed by a neutrosophic erosion operation by the same element B, and to be defined as the triple structure: $(A \tilde{\bullet} B) = \langle T_{A\tilde{\bullet}B}, I_{A\tilde{\bullet}B}, F_{A\tilde{\bullet}B} \rangle$,

where The three components, $T_{A\tilde{B}}$, $I_{A\tilde{B}}$ and $F_{A\tilde{B}}$ are to be defined in two different types as follows: for each $u, v, w \in Z^2$.

5.2.2.2.A. Neutrosophic Closing Type I:

$$T_{A\tilde{B}}(v) = \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min(T_A(v - u + w), T_B(w)), 1 - T_B(u) \right],$$

$$I_{A\tilde{B}}(v) = \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min(I_A(v - u + w), I_B(w)), 1 - I_B(u) \right],$$

$$F_{A\tilde{B}}(v) = \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max(F_A(v - u + w), 1 - F_B(w)), F_B(u) \right].$$

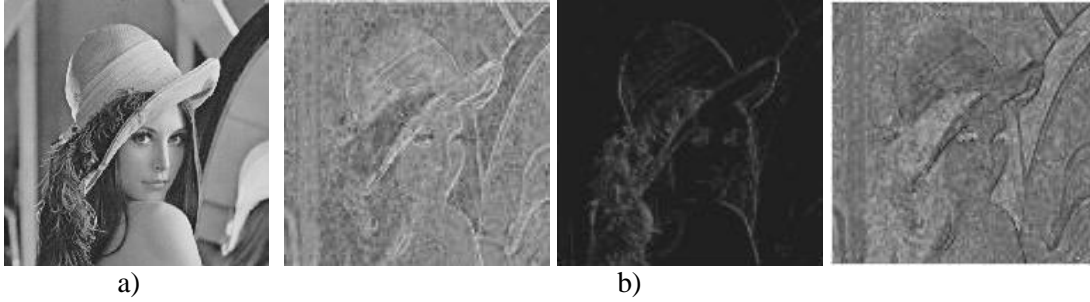


Fig.5.7(I): Applying the neutrosophic closing operator: a)Original image b)Neutrosophic closing components in type I $\langle T_{A\tilde{B}}, I_{A\tilde{B}}, F_{A\tilde{B}} \rangle$ respectively

5.2.2.2.B. Neutrosophic Closing Type II:

$$T_{A\tilde{B}}(v) = \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min(T_A(v - u + w), T_B(w)), 1 - T_B(u) \right],$$

$$I_{A\tilde{B}}(v) = \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max(I_A(v - u + w), 1 - I_B(w)), I_B(u) \right],$$

$$F_{A\tilde{B}}(v) = \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max(F_A(v - u + w), 1 - F_B(w)), F_B(u) \right].$$

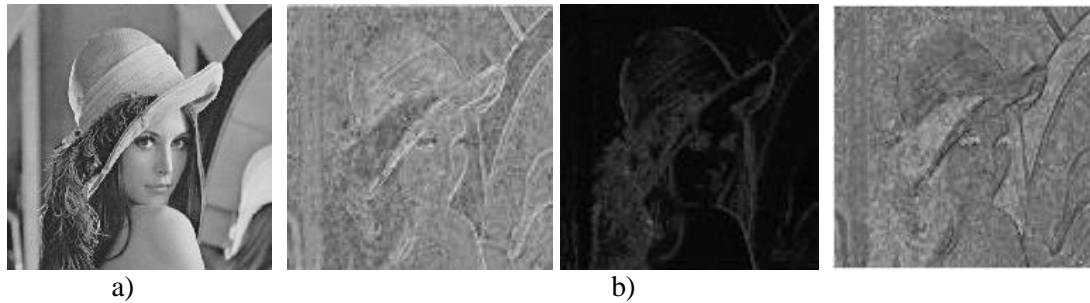


Fig.5.8(II): Applying the neutrosophic closing operator: a)Original image b)Neutrosophic closing components in type II $\langle T_{A\tilde{B}}, I_{A\tilde{B}}, F_{A\tilde{B}} \rangle$ respectively

5.3 Algebraic Properties of Neutrosophic Morphological Operations:

In this section we investigate some of the algebraic properties of the neutrosophic morphological operation; neutrosophic dilation, neutrosophic erosion, neutrosophic opening and neutrosophic closing. The algebraic properties for neutrosophic

mathematical morphology erosion and dilation, as well as for neutrosophic opening and closing operations are now considered.

5.3.1 Properties of the Neutrosophic Erosion Operation:

Proposition 5.1: for any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$.

$$\langle T_{\bigcap_{i \in I} A_i \bar{\ominus} B}, I_{\bigcap_{i \in I} A_i \bar{\ominus} B}, F_{\bigcap_{i \in I} A_i \bar{\ominus} B} \rangle \subseteq \langle T_{\bigcap_{i \in I} (A_i \bar{\ominus} B)}, I_{\bigcap_{i \in I} (A_i \bar{\ominus} B)}, F_{\bigcap_{i \in I} (A_i \bar{\ominus} B)} \rangle.$$

We will prove the proposition for the two types of neutrosophic erosion operation as follows:

Type I:

$$\begin{aligned} T_{\bigcap_{i \in I} A_i \bar{\ominus} B}(v) &= \inf_{u \in Z^2} \max \left(T_{\bigcap_{i \in I} A_i}(v+u), 1 - T_B(u) \right) \\ &= \inf_{u \in Z^2} \max \left(\inf_{i \in I} T_{A_i}(v+u), 1 - T_B(u) \right) \\ &\subseteq \inf_{u \in Z^2} \inf_{i \in I} (\max T_{A_i}(v+u), 1 - T_B(u)) && \subseteq T_{\bigcap_{i \in I} (A_i \bar{\ominus} B)}(v). \end{aligned}$$

$$\begin{aligned} I_{\bigcap_{i \in I} A_i \bar{\ominus} B}(v) &= \inf_{u \in Z^2} \max (I_{\bigcap_{i \in I} A_i}(v+u), 1 - I_B(u)) \\ &= \inf_{u \in Z^2} \max \left(\inf_{i \in I} I_{A_i}(v+u), 1 - I_B(u) \right) \\ &\subseteq \inf_{u \in Z^2} \inf_{i \in I} (\max I_{A_i}(v+u), 1 - I_B(u)) && \subseteq I_{\bigcap_{i \in I} (A_i \bar{\ominus} B)}(v). \end{aligned}$$

$$\begin{aligned} F_{\bigcap_{i \in I} A_i \bar{\ominus} B}(v) &= \sup_{u \in Z^2} \min (F_{\bigcap_{i \in I} A_i}(v+u), F_B(u)) \\ &= \sup_{u \in Z^2} \min \left(\inf_{i \in I} F_{A_i}(v+u), F_B(u) \right) \\ &= \sup_{u \in Z^2} \inf_{i \in I} (\min F_{A_i}(v+u), F_B(u)) && \subseteq F_{\bigcap_{i \in I} (A_i \bar{\ominus} B)}(v). \end{aligned}$$

Type II:

$$\begin{aligned} T_{\bigcap_{i \in I} A_i \bar{\ominus} B}(v) &= \inf_{u \in Z^2} \max \left(T_{\bigcap_{i \in I} A_i}(v+u), 1 - T_B(u) \right) \\ &= \inf_{u \in Z^2} \max \left(\inf_{i \in I} T_{A_i}(v+u), 1 - T_B(u) \right) \\ &\subseteq \inf_{u \in Z^2} \inf_{i \in I} (\max T_{A_i}(v+u), 1 - T_B(u)) && \subseteq T_{\bigcap_{i \in I} (A_i \bar{\ominus} B)}(v). \end{aligned}$$

$$\begin{aligned}
I_{\bigcap_{i \in I} A_i \widetilde{\ominus} B}(v) &= \sup_{u \in Z^2} \min \left(I_{\bigcap_{i \in I} A_i}(v+u), I_B(u) \right) \\
&= \sup_{u \in Z^2} \min \left(\inf_{i \in I} I_{A_i}(v+u), I_B(u) \right) \\
&= \sup_{u \in Z^2} \inf_{i \in I} \left(\min I_{A_i}(v+u), I_B(u) \right) \quad \subseteq I_{\bigcap_{i \in I} (A_i \widetilde{\ominus} B)}(v).
\end{aligned}$$

$$\begin{aligned}
F_{\bigcap_{i \in I} A_i \widetilde{\ominus} B}(v) &= \sup_{u \in Z^2} \min \left(F_{\bigcap_{i \in I} A_i}(v+u), F_B(u) \right) \\
&= \sup_{u \in Z^2} \min \left(\inf_{i \in I} F_{A_i}(v+u), F_B(u) \right) \\
&= \sup_{u \in Z^2} \inf_{i \in I} \left(\min F_{A_i}(v+u), F_B(u) \right) \quad \subseteq F_{\bigcap_{i \in I} (A_i \widetilde{\ominus} B)}(v).
\end{aligned}$$

Proposition 5.2: for any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$.

$$\langle T_{\bigcup_{i \in I} A_i \widetilde{\ominus} B}, I_{\bigcup_{i \in I} A_i \widetilde{\ominus} B}, F_{\bigcup_{i \in I} A_i \widetilde{\ominus} B} \rangle \supseteq \langle T_{\bigcup_{i \in I} (A_i \widetilde{\ominus} B)}, I_{\bigcup_{i \in I} (A_i \widetilde{\ominus} B)}, F_{\bigcup_{i \in I} (A_i \widetilde{\ominus} B)} \rangle.$$

We will prove the proposition for the two types of neutrosophic erosion operation as follows:

Type I:

$$\begin{aligned}
T_{\bigcup_{i \in I} A_i \widetilde{\ominus} B}(v) &= \inf_{u \in Z^2} \max \left(T_{\bigcup_{i \in I} A_i}(v+u), 1 - T_B(u) \right) \\
&= \inf_{u \in Z^2} \max \left(\sup_{i \in I} T_{A_i}(v+u), 1 - T_B(u) \right) \\
&= \inf_{u \in Z^2} \sup_{i \in I} \left(\max T_{A_i}(v+u), 1 - T_B(u) \right) \quad \supseteq T_{\bigcup_{i \in I} (A_i \widetilde{\ominus} B)}(v).
\end{aligned}$$

$$\begin{aligned}
I_{\bigcup_{i \in I} A_i \widetilde{\ominus} B}(v) &= \inf_{u \in Z^2} \max \left(I_{\bigcup_{i \in I} A_i}(v+u), 1 - I_B(u) \right) \\
&= \inf_{u \in Z^2} \max \left(\sup_{i \in I} I_{A_i}(v+u), 1 - I_B(u) \right) \\
&= \inf_{u \in Z^2} \sup_{i \in I} \left(\max I_{A_i}(v+u), 1 - I_B(u) \right) \quad \supseteq I_{\bigcup_{i \in I} (A_i \widetilde{\ominus} B)}(v).
\end{aligned}$$

$$\begin{aligned}
F_{\bigcup_{i \in I} A_i \widetilde{\ominus} B}(v) &= \sup_{u \in Z^2} \min \left(F_{\bigcup_{i \in I} A_i}(v+u), F_B(u) \right) \\
&= \sup_{u \in Z^2} \min \left(\sup_{i \in I} F_{A_i}(v+u), F_B(u) \right)
\end{aligned}$$

$$\sup_{u \in Z^2} \sup_{i \in I} (\min F_{A_i}(v+u), F_B(u)) \quad \supseteq F_{\bigcup_{i \in I} (A_i \oplus B)}(v).$$

The **proof** of type II is similar to type I.

5.3.2 Properties of the Neutrosophic Dilation Operation:

Proposition 5.3: for any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$.

$$\langle T_{\bigcap_{i \in I} A_i \oplus B}, I_{\bigcap_{i \in I} A_i \oplus B}, F_{\bigcap_{i \in I} A_i \oplus B} \rangle \subseteq \langle T_{\bigcap_{i \in I} (A_i \oplus B)}, I_{\bigcap_{i \in I} (A_i \oplus B)}, F_{\bigcap_{i \in I} (A_i \oplus B)} \rangle.$$

We will prove the proposition for the two types of neutrosophic dilation operation as follows: We will prove the proposition for the two types of neutrosophic dilation operation as follows:

Type I:

$$\begin{aligned} T_{\bigcap_{i \in I} A_i \oplus B}(v) &= \sup_{u \in Z^2} \min \left(T_{\bigcap_{i \in I} A_i}(v+u), T_B(u) \right) \\ &= \sup_{u \in Z^2} \min \left(\inf_{i \in I} T_{A_i}(v+u), T_B(u) \right) = \sup_{u \in Z^2} \inf_{i \in I} (\min T_{A_i}(v+u), T_B(u)) \\ &\subseteq \inf_{i \in I} \sup_{u \in Z^2} (\min T_{A_i}(v+u), T_B(u)) \subseteq T_{\bigcap_{i \in I} (A_i \oplus B)}(v). \end{aligned}$$

$$\begin{aligned} I_{\bigcap_{i \in I} A_i \oplus B}(v) &= \sup_{u \in Z^2} \min (I_{\bigcap_{i \in I} A_i}(v+u), I_B(u)) \\ &= \sup_{u \in Z^2} \min \left(\inf_{i \in I} I_{A_i}(v+u), I_B(u) \right) \subseteq I_{\bigcap_{i \in I} (A_i \oplus B)}(v). \end{aligned}$$

$$\begin{aligned} F_{\bigcap_{i \in I} A_i \oplus B}(v) &= \inf_{u \in Z^2} \max \left(F_{\bigcap_{i \in I} A_i}(v+u), 1 - F_B(u) \right) \\ &= \inf_{u \in Z^2} \max \left(\inf_{i \in I} F_{A_i}(v+u), 1 - F_B(u) \right) \\ &\subseteq \inf_{u \in Z^2} \inf_{i \in I} (\max F_{A_i}(v+u), 1 - F_B(u)) \subseteq F_{\bigcap_{i \in I} (A_i \oplus B)}(v) \end{aligned}$$

Type II:

$$\begin{aligned} T_{\bigcap_{i \in I} A_i \oplus B}(v) &= \sup_{u \in Z^2} \min \left(T_{\bigcap_{i \in I} A_i}(v+u), T_B(u) \right) \\ &= \sup_{u \in Z^2} \min \left(\inf_{i \in I} T_{A_i}(v+u), T_B(u) \right) = \sup_{u \in Z^2} \inf_{i \in I} (\min T_{A_i}(v+u), T_B(u)) \end{aligned}$$

$$\subseteq \inf_{i \in I} \sup_{u \in Z^2} (\min T_{A_i}(v+u), T_B(u)) \quad \subseteq T_{\bigcap_{i \in I} (A_i \oplus B)}(v).$$

$$\begin{aligned} I_{\bigcap_{i \in I} A_i \oplus B}(v) &= \inf_{u \in Z^2} \max(I_{\bigcap_{i \in I} A_i}(v+u), 1 - I_B(u)) \\ &= \inf_{u \in Z^2} \max\left(\inf_{i \in I} I_{A_i}(v+u), 1 - I_B(u)\right) \\ &\subseteq \inf_{u \in Z^2} \inf_{i \in I} (\max I_{A_i}(v+u), 1 - I_B(u)) \quad \subseteq I_{\bigcap_{i \in I} (A_i \oplus B)}(v). \end{aligned}$$

$$\begin{aligned} F_{\bigcap_{i \in I} A_i \oplus B}(v) &= \inf_{u \in Z^2} \max(F_{\bigcap_{i \in I} A_i}(v+u), 1 - F_B(u)) \\ &= \inf_{u \in Z^2} \max\left(\inf_{i \in I} F_{A_i}(v+u), 1 - F_B(u)\right) \\ &\subseteq \inf_{u \in Z^2} \inf_{i \in I} (\max F_{A_i}(v+u), 1 - F_B(u)) \quad \subseteq F_{\bigcap_{i \in I} (A_i \oplus B)}(v). \end{aligned}$$

Proposition 5.4: for any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$.

$$\langle T_{\bigcup_{i \in I} A_i \oplus B}, I_{\bigcup_{i \in I} A_i \oplus B}, F_{\bigcup_{i \in I} A_i \oplus B} \rangle \supseteq \langle T_{\bigcup_{i \in I} (A_i \oplus B)}, I_{\bigcup_{i \in I} (A_i \oplus B)}, F_{\bigcup_{i \in I} (A_i \oplus B)} \rangle.$$

We will prove the proposition for the two types of neutrosophic dilation operation as follows:

Type I:

$$\begin{aligned} T_{\bigcup_{i \in I} A_i \oplus B}(v) &= \sup_{u \in Z^2} \min\left(T_{\bigcup_{i \in I} A_i}(v+u), T_B(u)\right) \\ &= \sup_{u \in Z^2} \min\left(\sup_{i \in I} T_{A_i}(v+u), T_B(u)\right) \\ &\supseteq \sup_{u \in Z^2} \left(\sup_{i \in I} \min T_{A_i}(v+u), T_B(u)\right) \quad \supseteq T_{\bigcup_{i \in I} (A_i \oplus B)}(v). \end{aligned}$$

$$\begin{aligned} I_{B \oplus \bigcup_{i \in I} A_i}(v) &= \sup_{u \in Z^2} \min\left(I_{\bigcup_{i \in I} A_i}(v+u), I_B(u)\right) \\ &= \sup_{u \in Z^2} \min\left(\sup_{i \in I} I_{A_i}(v+u), I_B(u)\right) \\ &\supseteq \sup_{u \in Z^2} \left(\sup_{i \in I} \min I_{A_i}(v+u), I_B(u)\right) \quad \supseteq I_{\bigcup_{i \in I} (A_i \oplus B)}(v). \end{aligned}$$

$$\begin{aligned}
F_{\bigcup_{i \in I} A_i \oplus B} (v) &= \inf_{u \in Z^2} \max \left(F_{\bigcup_{i \in I} A_i} (v + u), 1 - F_B(u) \right) \\
&= \inf_{u \in Z^2} \max \left(\sup_{i \in I} F_{A_i} (v + u), 1 - F_B(u) \right) \\
&= \inf_{u \in Z^2} \left(\sup_{i \in I} \max F_{A_i} (v + u), 1 - F_B(u) \right) \supseteq F_{\bigcup_{i \in I} (A_i \oplus B)} (v).
\end{aligned}$$

Type II:

$$\begin{aligned}
T_{\bigcup_{i \in I} A_i \oplus B} (v) &= \sup_{u \in Z^2} \min \left(T_{\bigcup_{i \in I} A_i} (v + u), T_B(u) \right) \\
&= \sup_{u \in Z^2} \min \left(\sup_{i \in I} T_{A_i} (v + u), T_B(u) \right) \\
&\supseteq \sup_{u \in Z^2} \left(\sup_{i \in I} \min T_{A_i} (v + u), T_B(u) \right) \supseteq T_{\bigcup_{i \in I} (A_i \oplus B)} (v).
\end{aligned}$$

$$\begin{aligned}
I_{B \oplus \bigcup_{i \in I} A_i} (v) &= \inf_{u \in Z^2} \max \left(I_{\bigcup_{i \in I} A_i} (v + u), 1 - I_B(u) \right) \\
&= \inf_{u \in Z^2} \max \left(\sup_{i \in I} I_{A_i} (v + u), 1 - I_B(u) \right) \\
&= \inf_{u \in Z^2} \left(\sup_{i \in I} \max I_{A_i} (v + u), 1 - I_B(u) \right) \supseteq I_{\bigcup_{i \in I} (A_i \oplus B)} (v).
\end{aligned}$$

$$\begin{aligned}
F_{\bigcup_{i \in I} A_i \oplus B} (v) &= \inf_{u \in Z^2} \max \left(F_{\bigcup_{i \in I} A_i} (v + u), 1 - F_B(u) \right) \\
&= \inf_{u \in Z^2} \max \left(\sup_{i \in I} F_{A_i} (v + u), 1 - F_B(u) \right) \\
&= \inf_{u \in Z^2} \left(\sup_{i \in I} \max F_{A_i} (v + u), 1 - F_B(u) \right) \supseteq F_{\bigcup_{i \in I} (A_i \oplus B)} (v).
\end{aligned}$$

5.3.3 Properties of the Neutrosophic Closing Operation:

Proposition 5.5: for any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$.

$$\langle T_{\bigcap_{i \in I} A_i \circ B}, I_{\bigcap_{i \in I} A_i \circ B}, F_{\bigcap_{i \in I} A_i \circ B} \rangle \subseteq \langle T_{\bigcap_{i \in I} (A_i \circ B)}, I_{\bigcap_{i \in I} (A_i \circ B)}, F_{\bigcap_{i \in I} (A_i \circ B)} \rangle.$$

We will prove the proposition for the two types of neutrosophic closing operation as follows:

Type I:

$$\begin{aligned}
T_{\bigcap_{i \in I} A_i \tilde{*} B}(v) &= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(T_{\bigcap_{i \in I} A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(\inf_{i \in I} T_{A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&\subseteq \inf_{u \in Z^2} \max \left[\inf_{i \in I} \sup_{w \in Z^2} \min \left(T_{A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&\subseteq \inf_{i \in I} \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(T_{A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&\subseteq T_{\bigcap_{i \in I} (A_i \tilde{*} B)}(v).
\end{aligned}$$

$$\begin{aligned}
I_{\bigcap_{i \in I} A_i \tilde{*} B}(v) &= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(I_{\bigcap_{i \in I} A_i}(v - u + w), I_B(w) \right), 1 - I_B(u) \right] \\
&= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(\inf_{i \in I} I_{A_i}(v - u + w), I_B(w) \right), 1 - I_B(u) \right] \\
&\subseteq \inf_{u \in Z^2} \max \left[\inf_{i \in I} \sup_{w \in Z^2} \min \left(I_{A_i}(v - u + w), I_B(w) \right), 1 - I_B(u) \right] \\
&\subseteq \inf_{i \in I} \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(I_{A_i}(v - u + w), I_B(w) \right), 1 - I_B(u) \right] \\
&\subseteq I_{\bigcap_{i \in I} (A_i \tilde{*} B)}(v).
\end{aligned}$$

$$\begin{aligned}
F_{\bigcap_{i \in I} A_i \tilde{*} B}(v) &= \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(F_{\bigcap_{i \in I} A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&= \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(\inf_{i \in I} F_{A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&\subseteq \sup_{u \in Z^2} \min \left[\inf_{i \in I} \inf_{w \in Z^2} \max \left(F_{A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&\subseteq \inf_{i \in I} \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(F_{A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&\subseteq F_{\bigcap_{i \in I} (A_i \tilde{*} B)}(v).
\end{aligned}$$

Type II:

$$\begin{aligned}
T_{\cap_{i \in I} A_i \tilde{*} B}(v) &= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(T_{\cap_{i \in I} A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(\inf_{i \in I} T_{A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&\subseteq \inf_{u \in Z^2} \max \left[\inf_{i \in I} \sup_{w \in Z^2} \min \left(T_{A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&\subseteq \inf_{i \in I} \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(T_{A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&\subseteq T_{\cap_{i \in I} (A_i \tilde{*} B)}(v).
\end{aligned}$$

$$\begin{aligned}
I_{\cap_{i \in I} A_i \tilde{*} B}(v) &= \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(I_{\cap_{i \in I} A_i}(v - u + w), 1 - I_B(w) \right), I_B(u) \right] \\
&= \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(\inf_{i \in I} I_{A_i}(v - u + w), 1 - I_B(w) \right), I_B(u) \right] \\
&\subseteq \sup_{u \in Z^2} \min \left[\inf_{i \in I} \inf_{w \in Z^2} \max \left(I_{A_i}(v - u + w), 1 - I_B(w) \right), I_B(u) \right] \\
&\subseteq \inf_{i \in I} \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(I_{A_i}(v - u + w), 1 - I_B(w) \right), I_B(u) \right] \\
&\subseteq I_{\cap_{i \in I} (A_i \tilde{*} B)}(v).
\end{aligned}$$

$$\begin{aligned}
F_{\cap_{i \in I} A_i \tilde{*} B}(v) &= \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(F_{\cap_{i \in I} A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&= \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(\inf_{i \in I} F_{A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&\subseteq \sup_{u \in Z^2} \min \left[\inf_{i \in I} \inf_{w \in Z^2} \max \left(F_{A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&\subseteq \inf_{i \in I} \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(F_{A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&\subseteq F_{\cap_{i \in I} (A_i \tilde{*} B)}(v).
\end{aligned}$$

Proposition 5.6: for any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$.

$$\langle T_{\cup_{i \in I} A_i \tilde{*} B}, I_{\cup_{i \in I} A_i \tilde{*} B}, F_{\cup_{i \in I} A_i \tilde{*} B} \rangle \supseteq \langle T_{\cup_{i \in I} (A_i \tilde{*} B)}, I_{\cup_{i \in I} (A_i \tilde{*} B)}, F_{\cup_{i \in I} (A_i \tilde{*} B)} \rangle.$$

We will prove the proposition for the two types of neutrosophic closing operation as follows:

Type I:

$$\begin{aligned}
T_{\cup_{i \in I} A_i \tilde{*} B}(v) &= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(T_{\cup_{i \in I} A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(\sup_{i \in I} T_{A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&\supseteq \inf_{u \in Z^2} \max \left[\sup_{i \in I} \sup_{w \in Z^2} \min \left(T_{A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&\supseteq \inf_{i \in I} \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(T_{A_i}(v - u + w), T_B(w) \right), 1 - T_B(u) \right] \\
&\supseteq T_{\cup_{i \in I} (A_i \tilde{*} B)}(v).
\end{aligned}$$

$$\begin{aligned}
I_{\cup_{i \in I} A_i \tilde{*} B}(v) &= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(I_{\cup_{i \in I} A_i}(v - u + w), I_B(w) \right), 1 - I_B(u) \right] \\
&= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(\sup_{i \in I} I_{A_i}(v - u + w), I_B(w) \right), 1 - I_B(u) \right] \\
&\supseteq \inf_{u \in Z^2} \max \left[\sup_{i \in I} \sup_{w \in Z^2} \min \left(I_{A_i}(v - u + w), I_B(w) \right), 1 - I_B(u) \right] \\
&\supseteq \inf_{i \in I} \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min \left(I_{A_i}(v - u + w), I_B(w) \right), 1 - I_B(u) \right] \\
&\supseteq I_{\cup_{i \in I} (A_i \tilde{*} B)}(v).
\end{aligned}$$

$$\begin{aligned}
F_{\cup_{i \in I} A_i \tilde{*} B}(v) &= \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(F_{\cup_{i \in I} A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&= \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(\sup_{i \in I} F_{A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&\supseteq \sup_{u \in Z^2} \min \left[\sup_{i \in I} \inf_{w \in Z^2} \max \left(F_{A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&\supseteq \sup_{i \in I} \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max \left(F_{A_i}(v - u + w), 1 - F_B(w) \right), F_B(u) \right] \\
&\supseteq F_{\cup_{i \in I} (A_i \tilde{*} B)}(v).
\end{aligned}$$

The **proof** type II is similar to type I.

5.3.4 Properties of the Neutrosophic Opening Operation:

The neutrosophic opening satisfies the following properties:

Proposition 5.7: for any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$.

$$\langle T_{\bigcap_{i \in I} A_i \ominus B}, I_{\bigcap_{i \in I} A_i \ominus B}, F_{\bigcap_{i \in I} A_i \ominus B} \rangle \subseteq \langle T_{\bigcap_{i \in I} (A_i \ominus B)}, I_{\bigcap_{i \in I} (A_i \ominus B)}, F_{\bigcap_{i \in I} (A_i \ominus B)} \rangle.$$

Proposition 5.8: for any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$.

$$\langle T_{\bigcup_{i \in I} A_i \ominus B}, I_{\bigcup_{i \in I} A_i \ominus B}, F_{\bigcup_{i \in I} A_i \ominus B} \rangle \supseteq \langle T_{\bigcup_{i \in I} (A_i \ominus B)}, I_{\bigcup_{i \in I} (A_i \ominus B)}, F_{\bigcup_{i \in I} (A_i \ominus B)} \rangle.$$

Proof: Is similar to the procedure used to prove the propositions given in § 5.3.3.

5.4 Duality Theorem:

5.4.1 Duality Theorem of Neutrosophic Dilation:

let A and B are two neutrosophic sets. Neutrosophic erosion and dilation are dual

operations i.e. $(A^c \widetilde{\oplus} B)^c = \langle T_{(A^c \widetilde{\oplus} B)^c}, I_{(A^c \widetilde{\oplus} B)^c}, F_{(A^c \widetilde{\oplus} B)^c} \rangle$.

where for each $u, v \in Z^2$. We will prove the proposition for the two types of neutrosophic dilation operation as follows:

Type I:

$$\begin{aligned} T_{(A^c \widetilde{\oplus} B)^c}(v) &= 1 - T_{(A^c \widetilde{\oplus} B)}(v) \\ &= 1 - \sup_{u \in Z^2} \min(T_{A^c}(v+u), T_B(u)) = \inf_{u \in Z^2} [1 - \min(T_{A^c}(v+u), T_B(u))] \\ &= \inf_{u \in Z^2} [\max(1 - T_{A^c}(v+u), 1 - T_B(u))] \\ &= \inf_{u \in Z^2} [\max(T_A(v+u), 1 - T_B(u))] = T_{A \ominus B}(v). \end{aligned}$$

$$\begin{aligned} I_{(A^c \widetilde{\oplus} B)^c}(v) &= 1 - I_{(A^c \widetilde{\oplus} B)}(v) \\ &= 1 - \sup_{x \in R^n} \min(I_{A^c}(v+u), I_B(u)) = \inf_{u \in Z^2} [1 - \min(I_{A^c}(v+u), I_B(x))] \\ &= \inf_{u \in Z^2} [\max(1 - I_{A^c}(v+u), 1 - I_B(u))] \\ &= \inf_{u \in Z^2} [\max(I_A(v+u), 1 - I_B(u))] = I_{A \ominus B}(v). \end{aligned}$$

$$\begin{aligned}
F_{(A^c \oplus B)^c}(v) &= 1 - F_{(A^c \oplus B)}(v) \\
&= 1 - [\inf_{x \in R^n} \max(F_{A^c}(v + u), 1 - F_B(u))] \\
&= \sup_{u \in Z^2} [1 - \max(F_{A^c}(v + u), 1 - F_B(u))] \\
&= \sup_{u \in Z^2} [\min(F_A(v + u), F_B(u))] = F_{A \ominus B}(v). \\
\langle T_{(A^c \oplus B)^c}, I_{(A^c \oplus B)^c}, F_{(A^c \oplus B)^c} \rangle &= \langle T_{A \ominus B}, I_{A \ominus B}, F_{A \ominus B} \rangle.
\end{aligned}$$

Type II:

$$\begin{aligned}
T_{(A^c \oplus B)^c}(v) &= 1 - T_{(A^c \oplus B)}(v) \\
&= 1 - \sup_{u \in Z^2} \min(T_{A^c}(v + u), T_B(u)) \\
&= \inf_{u \in Z^2} [1 - \min(T_{A^c}(v + u), T_B(u))] = \inf_{u \in Z^2} [\max(1 - T_{A^c}(v + u), 1 - T_B(u))] \\
&= \inf_{u \in Z^2} [\max(T_A(v + u), 1 - T_B(u))] = T_{A \ominus B}(v).
\end{aligned}$$

$$\begin{aligned}
I_{(A^c \oplus B)^c}(v) &= 1 - I_{(A^c \oplus B)}(v). \\
&= 1 - [\inf_{x \in R^n} \max(I_{A^c}(v + u), 1 - I_B(u))] \\
&= \sup_{u \in Z^2} [1 - \max(I_{A^c}(v + u), 1 - I_B(u))] \\
&= \sup_{u \in Z^2} [\min(I_A(v + u), I_B(u))] = I_{A \ominus B}(v).
\end{aligned}$$

$$\begin{aligned}
F_{(A^c \oplus B)^c}(v) &= 1 - F_{(A^c \oplus B)}(v) \\
&= 1 - [\inf_{u \in Z^2} \max(F_{A^c}(v + u), 1 - F_B(u))] \\
&= \sup_{u \in Z^2} [1 - \max(F_{A^c}(v + u), 1 - F_B(u))] \\
&= \sup_{u \in Z^2} [\min(F_A(v + u), F_B(u))] = F_{A \ominus B}(v).
\end{aligned}$$

$$\langle T_{(A^c \oplus B)^c}, I_{(A^c \oplus B)^c}, F_{(A^c \oplus B)^c} \rangle = \langle T_{A \ominus B}, I_{A \ominus B}, F_{A \ominus B} \rangle.$$

5.4.2 Duality Theorem of Neutrosophic Closing:

let A and B are two neutrosophic sets, neutrosophic opening and neutrosophic closing are also dual operation i.e.

$(A^c \tilde{\circ} B)^c = \langle T_{(A^c \tilde{\circ} B)^c}, I_{(A^c \tilde{\circ} B)^c}, F_{(A^c \tilde{\circ} B)^c} \rangle$, where for all $x \in X$. We will prove the proposition for the two neutrosophic types follows:

Type I:

$$\begin{aligned}
T_{(A^c \tilde{\circ} B)^c}(v) &= 1 - T_{A^c \tilde{\circ} B}(v) \\
&= 1 - \inf_{u \in Z^2} \max[\sup_{z \in R^n} \min(T_{A^c}(v - u + w), T_{B(w)}), 1 - T_B(u)] \\
&= \sup_{u \in Z^2} \min \left[1 - \sup_{w \in Z^2} \min(T_{A^c}(v - u + w), T_{B(w)}), T_B(u) \right] \\
&= \sup_{u \in Z^2} \min[\inf_{w \in Z^2} \max(1 - T_{A^c}(v - u + w), 1 - T_{B(w)}), T_B(u)] \\
&= \sup_{u \in Z^2} \min[\inf_{w \in Z^2} \max(T_A(v - u + w), 1 - T_{B(w)}), T_B(u)] = T_{A \tilde{\circ} B}(v).
\end{aligned}$$

$$\begin{aligned}
I_{(A^c \tilde{\circ} B)^c}(v) &= 1 - I_{A^c \tilde{\circ} B}(v) \\
&= 1 - \inf_{u \in Z^2} \max[\sup_{w \in Z^2} \min(I_{A^c}(v - u + w), B(w)), 1 - I_B(u)] \\
&= \sup_{u \in Z^2} \min \left[1 - \sup_{w \in Z^2} \min(I_{A^c}(v - u + w), B(w)), I_B(u) \right] \\
&= \sup_{u \in Z^2} \min[\inf_{w \in Z^2} \max(1 - I_{A^c}(v - u + w), 1 - B(w)), I_B(u)] \\
&= \sup_{u \in Z^2} \min[\inf_{w \in Z^2} \max(I_A(v - u + w), 1 - B(w)), I_B(u)] = I_{A \tilde{\circ} B}(v).
\end{aligned}$$

$$\begin{aligned}
F_{(A^c \tilde{\circ} B)^c}(v) &= 1 - F_{A^c \tilde{\circ} B}(v) \\
&= 1 - \sup_{u \in Z^2} \min[\inf_{w \in Z^2} \max(F_{A^c}(v - u + w), 1 - F_B(w)), F_B(u)] \\
&= \inf_{u \in Z^2} \max[1 - \inf_{w \in Z^2} \max(F_{A^c}(v - u + w), 1 - F_B(w)), 1 - F_B(u)] \\
&= \inf_{u \in Z^2} \max[\sup_{w \in Z^2} \min(F_A(v - u + w), F_B(w)), 1 - F_B(u)] = F_{A \tilde{\circ} B}(v).
\end{aligned}$$

$$\langle T_{(A^c \tilde{\circ} B)^c}, I_{(A^c \tilde{\circ} B)^c}, F_{(A^c \tilde{\circ} B)^c} \rangle = \langle T_{A \tilde{\circ} B}, I_{A \tilde{\circ} B}, F_{A \tilde{\circ} B} \rangle.$$

Type II:

$$\begin{aligned}
T_{(A^c \tilde{\circ} B)^c}(v) &= 1 - T_{A^c \tilde{\circ} B}(v) \\
&= 1 - \inf_{u \in Z^2} \max[\sup_{w \in Z^2} \min(T_{A^c}(v - u + w), T_{B(w)}), 1 - T_B(u)]
\end{aligned}$$

$$\begin{aligned}
&= \sup_{u \in Z^2} \min \left[1 - \sup_{w \in Z^2} \min(T_{A^c}(v - u + w), T_{B(w)}), T_B(u) \right] \\
&= \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max(1 - T_{A^c}(v - u + w), 1 - T_{B(w)}), T_B(u) \right] \\
&= \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max(T_A(v - u + w), 1 - T_{B(w)}), T_B(u) \right] = T_{A \tilde{\circ} B}(v).
\end{aligned}$$

$$\begin{aligned}
I_{(A^c \tilde{\circ} B)^c}(v) &= 1 - I_{A^c \tilde{\circ} B}(v) \\
&= 1 - \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max(I_{A^c}(v - u + w), 1 - I_B(w)), I_B(u) \right] \\
&= \inf_{u \in Z^2} \max \left[1 - \inf_{w \in Z^2} \max(I_{A^c}(v - u + w), 1 - I_B(w)), 1 - I_B(u) \right] \\
&= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min(I_A(v - u + w), I_B(w)), 1 - I_B(u) \right] = I_{A \tilde{\circ} B}(v).
\end{aligned}$$

$$\begin{aligned}
F_{(A^c \tilde{\circ} B)^c}(v) &= 1 - F_{A^c \tilde{\circ} B}(v) \\
&= 1 - \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max(F_{A^c}(v - u + w), 1 - F_B(w)), F_B(u) \right] \\
&= \inf_{u \in Z^2} \max \left[1 - \inf_{w \in Z^2} \max(F_{A^c}(v - u + w), 1 - F_B(w)), 1 - F_B(u) \right] \\
&= \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min(F_A(v - u + w), F_B(w)), 1 - F_B(u) \right] = F_{A \tilde{\circ} B}(v).
\end{aligned}$$

$$\langle T_{(A^c \tilde{\circ} B)^c}, I_{(A^c \tilde{\circ} B)^c}, F_{(A^c \tilde{\circ} B)^c} \rangle = \langle T_{A \tilde{\circ} B}, I_{A \tilde{\circ} B}, F_{A \tilde{\circ} B} \rangle$$

5.5 Neutrosophic Mathematical Morphological Filters:

When considering The differences of two or more of the basic neutrosophic morphological operators, given in § 5.2, yield some remarkable filters; in this section we will consider the boundary and Hat filters:

5.5.1 Some Type of Boundary Extraction Filter Using Neutrosophic Dilation and

Neutrosophic Erosion:

As the neutrosophic dilation thickens regions in the true level of image, and the neutrosophic erosion shrinks them, the neutrosophic differences between the image and either its neutrosophic dilation or erosion may emphasize the boundaries between regions included in the image. Therefore, several boundary filters may be obtained as follows:

5.5.1.1 Neutrosophic Gradient Boundary:

To commence, we will investigate the neutrosophic gradient filter which is the mean value of the three components of the neutrosophic difference between the neutrosophic dilation of some image and its neutrosophic erosion. We get the neutrosophic gradient of the image by applying the mean of these boundaries. If the structure element is relatively small, the homogeneous areas will not be affected by neutrosophic dilation and neutrosophic erosion, then the subtraction tends to eliminate them. The effect of neutrosophic morphological gradient operation is shown in Fig. 5.9 and to be defined in two different types as follows:

Type I:

$$\tilde{d}_{gradient} = \left(\frac{1}{3}\right) * \left[\begin{array}{c} \min \left(T_{A\tilde{\oplus}B}(v), 1 - T_{A\tilde{\ominus}B}(v) \right), \min \left(I_{A\tilde{\oplus}B}(v), 1 - I_{A\tilde{\ominus}B}(v) \right) \\ \max \left(F_{A\tilde{\oplus}B}(v), 1 - F_{A\tilde{\ominus}B}(v) \right) \end{array} \right].$$

In the following figure (fig.5.9 (I)), we present the results obtained when applying neutrosophic gradient boundary filter on some grayscale image.



Fig. 5.9(I): Applying the neutrosophic gradient boundary: a) Original image
b) Neutrosophic gradient boundary filtered

Type II:

$$\tilde{d}_{gradient} =$$

$$\left(\frac{1}{3}\right) * \left[\begin{array}{c} \min \left(T_{A\tilde{\oplus}B}(v), 1 - T_{A\tilde{\ominus}B}(v) \right), \max \left(I_{A\tilde{\oplus}B}(v), 1 - T_{A\tilde{\ominus}B}(v) \right) \\ \max \left(F_{A\tilde{\oplus}B}(v), 1 - T_{A\tilde{\ominus}B}(v) \right) \end{array} \right].$$

In the following figure (fig.5.10 (II)), we present the results obtained when applying neutrosophic gradient boundary filter on some grayscale image.



Fig.5.10 (II): Applying the neutrosophic gradient boundary: a) original image
b) Neutrosophic gradient boundary filtered

5.5.1.2 Neutrosophic External Boundary:

In this filter, a neutrosophic dilation is firstly applied to the neutrosophic image A by some neutrosophic structure element B; hence, the output filtered image will be the neutrosophic difference between neutrosophic dilated image and the neutrosophic image A. That is, the neutrosophic external boundary of A is to be defined in two different types as follows:

Type I:

$$\tilde{\delta}_{ext} = (1/3) * \left[\begin{array}{c} \min(T_{A\oplus B}(v), 1 - T_A(v)), \min(I_{A\oplus B}(v), 1 - I_A(v)), \\ \max(F_{A\oplus B}(v), 1 - F_A(v)) \end{array} \right].$$

In the following figure (fig.5.11 (I)), we present the results obtained when applying neutrosophic external boundary filter on some grayscale image.

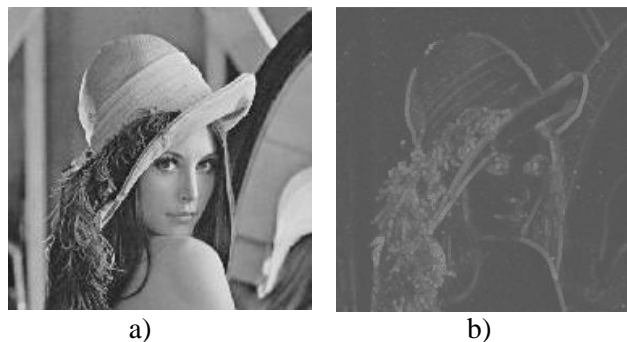


Fig.5.11(I): Applying the neutrosophic external boundary: a) original image
b) Neutrosophic external boundary filtered image

Type II:

$$\tilde{\delta}_{ext} = (1/3) * \left[\begin{array}{c} \min(T_{A\oplus B}(v), 1 - T_A(v)), \max(I_{A\oplus B}(v), 1 - I_A(v)), \\ \max(F_{A\oplus B}(v), 1 - F_A(v)) \end{array} \right].$$

In the following figure (fig.5.12 (II)), we present the results obtained when applying neutrosophic external boundary filter on some grayscale image.



Fig.5.12(II): Applying the neutrosophic external boundary: a) Original image
b) Neutrosophic external boundary filtered image

5.5.1.3 Neutrosophic Internal Boundary:

The main step of the neutrosophic internal boundary filter, is to get the neutrosophic erosion of the neutrosophic image, hence, the output filtered image will be the neutrosophic difference between the original image in neutrosophic domain and the neutrosophic eroded image that is the neutrosophic internal boundary of the neutrosophic image A is to be defined in two different types as follows:

Type I:

$$\tilde{\delta}_{int} = (1/3) * \left[\begin{array}{c} \min \left(T_A(v), 1 - \left(T_{A\ominus B}(v) \right) \right), \min \left(I_A(v), 1 - I_{A\ominus B}(v) \right), \\ \max \left(F_A(v), 1 - F_{A\ominus B}(v) \right) \end{array} \right].$$

In the following figure (fig.5.13 (I)), we present the results obtained when applying neutrosophic internal boundary filter on some grayscale image.



Fig. 5.13(I): Applying the neutrosophic internal boundary: a) Original image
b) Neutrosophic internal boundary filtered image

Type II:

$$\tilde{d}_{int} = (1/3) * \left[\begin{array}{c} \min \left(T_A(v), 1 - \left(T_{A\tilde{\ominus}B}(v) \right) \right), \min \left(I_A(v), 1 - I_{A\tilde{\ominus}B}(v) \right), \\ \max \left(F_A(v), 1 - F_{A\tilde{\ominus}B}(v) \right) \end{array} \right].$$

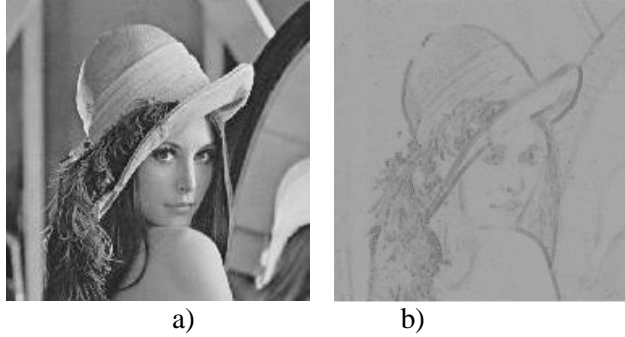


Fig.5.14(II): Applying the neutrosophic internal boundary: a) original image
b) Neutrosophic internal boundary filtered image

5.5.1.4 Neutrosophic Outline Boundary:

The main step of the neutrosophic outline boundary filter, is to get the complement of the neutrosophic erosion of the neutrosophic image, hence, the output filtered image will be the neutrosophic difference between the original image in neutrosophic domain and the neutrosophic eroded image that is the neutrosophic outline boundary of the neutrosophic image A is to be defined as follows: $\tilde{d}_{outline}(A) = (\partial_1 A^1 \cup \partial_3 A^3) \cap A^2$, where; $\partial_1(A^1) = co(A^1 \ominus B^1) \cap A^1$, $\partial_3(A^3) = co(A^3 \oplus B^3) \cup A^3$. In the following figure (fig.5.15), we present the results obtained when applying neutrosophic outline boundary filter on some grayscale image.

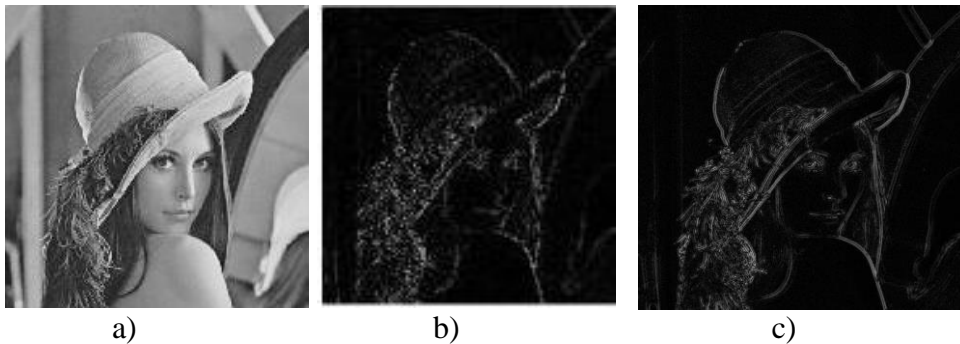


Fig. 5.15: Neutrosophic outline boundary: a) Original image
b) Neutrosophic outline Boundary filtered image with SE(3)
c) Neutrosophic outline Boundary filtered image with SE (7)

5.5.2 Some Combination Neutrosophic External and Internal Boundary Filters:

1. $\tilde{\partial}_{sup} = \left(\frac{1}{3}\right) * [T_{max} + I_{max} + F_{min}]$, where;

$$T_{max} = \max(\partial_{ext}(T), \partial_{int}(T)), \quad I_{max} = \max(\partial_{ext}(I), \partial_{int}(I)),$$

$$F_{min} = \min(\partial_{ext}(F), \partial_{int}(F)).$$

In the following figure (fig.5.16), we present the results obtained when applying neutrosophic sup. boundary filter on some grayscale image.

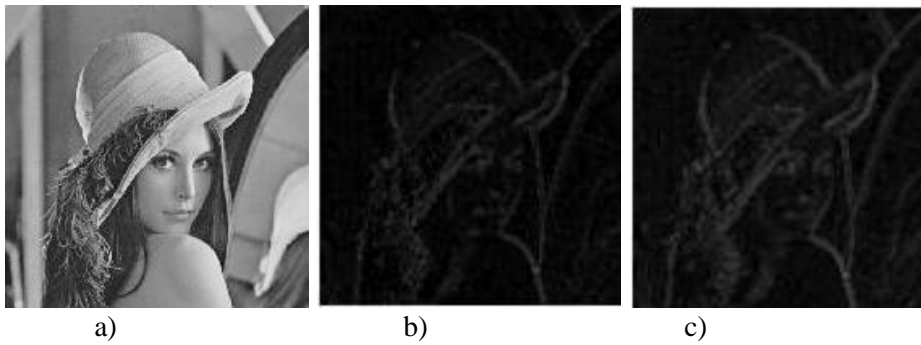


Fig.5.16: Neutrosophic sup. boundary: a) original image
b) Neutrosophic sup. Boundary filtered image with SE(5)
c) Neutrosophic sup Boundary filtered image SE (7)

2. $\tilde{\partial}_{grad} = (1/3) * [T_{sum} + I_{sum} + F_{sum}]$, where;

$$T_{sum} = \partial_{ext}(T) + \partial_{int}(T), \quad F_{sum} = \partial_{ext}(F) + \partial_{int}(F),$$

$$I_{sum} = \partial_{ext}(I) + \partial_{int}(I).$$

In the following figure (fig.5.17), we present the results obtained when applying neutrosophic grad boundary filter on some grayscale image.

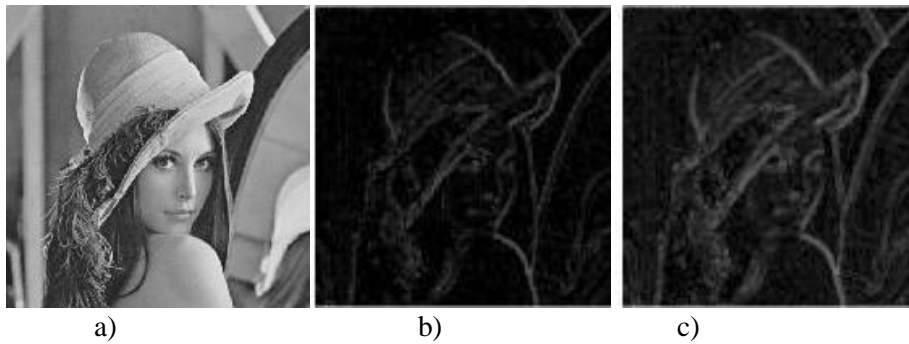


Fig. 5.17: Neutrosophic grad boundary: a)Original image
b) Neutrosophic grad boundary filtered image with SE(5)
c) Neutrosophic grad boundary filtered image with SE (7)

5.5.4 Neutrosophic Hat Filters:

The main function of the hat filters is to extract small elements and details from given image. In this section we will introduce two forms of the neutrosophic hat filters; namely, the neutrosophic top-hat and the neutrosophic bottom-hat filters. In the classical mathematical morphology, the top-hat filters plays a very important rule in several tasks of processing disciplines; such as: feature extraction, back ground equalization, feature extraction, background equalization, image enhancement,...etc.

5.5.4.1 Neutrosophic Top-hat Filter:

In the classical mathematical morphology, the top-hat filters plays a very important rule in several tasks of processing disciplines; such as: feature extraction, back ground equalization, feature extraction, background equalization, image enhancement,...etc. In this section we will generalize the concept of the top-hat filter using the neutrosophy concepts; that is the neutrosophic top-hat filter is to be defined as the neutrosophic difference between the neutrosophic image and its neutrosophic opening image. The neutrosophic top-hat filter of the neutrosophic image A is to be defined in two different types as follows:

Type I:

$$\widetilde{Top}_{hat} = (1/3) * \left[\begin{array}{c} \min(T_A(v), 1 - T_{A\delta B}(v)), \min(I_A(v), 1 - I_{A\delta B}(v)) \\ \max(F_A(v), 1 - F_{A\delta B}(v)) \end{array} \right].$$

In the following figure (fig.5.18 (I) (II)), we present the results obtained when applying neutrosophic top-hat filter boundary filter on some grayscale image.



Fig.5.18(I): Applying the neutrosophic top-hat filter: a)Original image
b) Neutrosophic top-hat boundary filtered in type I

Type II:

$$\widetilde{Top}_{hat} = (1/3) * \left[\begin{array}{c} \min(T_A(v), 1 - T_{A\tilde{\delta}B}(v)), \max(I_A(v), 1 - I_{A\tilde{\delta}B}(v)), \\ \max(F_A(v), 1 - F_{A\tilde{\delta}B}(v)) \end{array} \right].$$



Fig.5.19(II): Applying the neutrosophic top-hat filter: a) Original image
b) Neutrosophic top-hat boundary filtered in type II

5.5.3.2 Neutrosophic Bottom-hat Filter:

In the classical mathematical morphology, the bottom-hat filters plays a very important rule in several tasks of processing disciplines; such as: feature extraction, back ground equalization, feature extraction, background equalization, image enhancement,...etc. In this section we will generalize the concept of the bottom-hat filter using the neutrosophy concepts; that is the neutrosophic bottom-hat filter is to be defined as the neutrosophic difference between neutrosophic closing and image the neutrosophic image. The neutrosophic bottom-hat filter of the neutrosophic image A is to be defined as follows: in two different types as follows:

Type I:

$$\widetilde{Bottom}_{hat} = (1/3) * \left[\begin{array}{c} \min(T_{A\tilde{*}B}(v), 1 - T_A(v)), \min(I_{A\tilde{*}B}(v), 1 - I_A(v)), \\ \max(F_A(v), 1 - F_{A\tilde{*}B}(v)) \end{array} \right].$$

In the following figure (fig.5.20 (I) (II)), we present the results obtained when applying neutrosophic bottom-hat filter Boundary filter on some grayscale image.

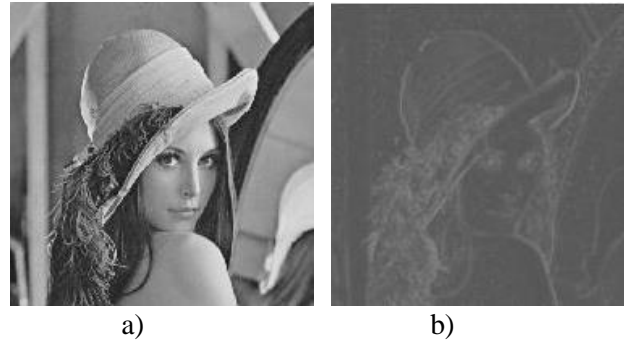


Fig.5.20(I): Applying the neutrosophic bottom-hat filter: a) Original image
b) Neutrosophic bottom-hat boundary filtered in type II

Type II:

$$\widetilde{\text{Bottom}}_{\text{hat}} = (1/3) * \begin{bmatrix} \min(T_{A\tilde{*}B}(v), 1 - T_A(v)), \min(I_{A\tilde{*}B}(v), 1 - I_A(v)), \\ \max(F_A(v), 1 - F_{A\tilde{*}B}(v)) \end{bmatrix}$$

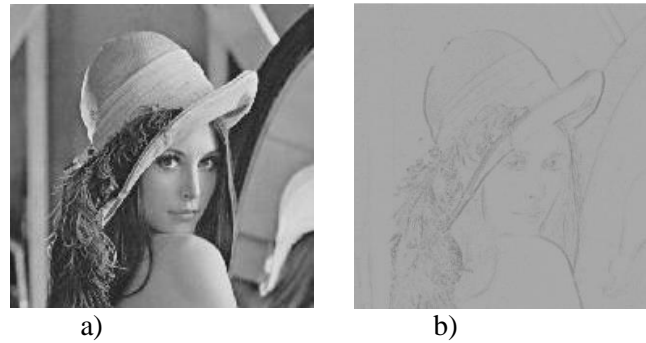


Fig.5.21 (II): Applying the neutrosophic bottom-hat filter: a) Original image
b) Neutrosophic bottom-hat boundary filtered in type II

5.6 Conclusion:

In this paper, we have proposed a new technique for analyzing and processing images; either grayscale or binary. The technique is a generalization for the fuzzy mathematical morphology; it handles the image in the neutrosophic domain. In such domain the image analyzed into three different layers; the first layer describes how much each pixel belongs to the white set, the third layer describes how much each pixel belongs to the non-white (black) set, while the second layer describes how much the pixel is neither white nor black. The properties of each layer were used to define the basic operations for what we called "Neutrosophic Mathematical Morphology". mainly, we introduced four basic operations; namely, the neutrosophic dilation, the neutrosophic erosion, the neutrosophic closing and the neutrosophic opening. The algebraic properties of the proposed operation were discussed. Furthermore, we introduced some advanced

neutrosophic filters using different combinations of the basic operators. Hence, we experimented the new introduced operators and filters using "Lena" image to investigate the effect of each over the image.

Chapter 6

Neutrosophic Morphology Threshold

6.1 Introduction:

An important technique in image segmentation, the image thresholding is also an important step towards pattern detection and recognition, which determines the quality of many image analysis tasks. It is used to extract the meaningful objects from the image, Thresholding is one of the most simplest and pre-processing step for image applications. Basically, the classical thresholding generates some binary image in which the pixels with zero value belonging to the background while the pixels belonging to the foreground have the value 1. The process of partitioning the image into mutually exclusive regions (background and foreground) needs to choose an appropriate gray level in the original image to be used for classifying the pixels whether it is inside or outside some specific range; such process is not easy. A number of excellent investigations on various thresholding techniques have been reported in the literature; for instance, Kapur et al [38], Li and Lee [45] used the concept of entropy; while Brink and Pendcock [13] Abutaleb [1] used two-dimensional entropy to threshold an image. Otsu in [56], suggested the threshold detection by maximizing the class separability. Whatmough [82] used the exponential hull method, which is a variation of convex hull for concavity analysis. Kittler and Illingworth [41] minimized the classification error probability based on the condition that a mixture of Gaussian densities governs the histograms. Several researchers have investigated fuzzy based thresholding techniques. Pal and Rosenfeld [57] optimized the fuzzy compactness using the Zadeh S-function for the membership evaluation for image thresholding. Huang and Wang [32] used Shannon and Yagers measure for fuzzy thresholding. Ramar et al. [59] used the neural network for selecting the best threshold using various fuzzy measures. Fuzzy

homogeneity vectors and fuzzy co-occurrence matrix was reported by Cheng and Chen [16] for image thresholding. Cheng et al. [18] integrated neutrosophic set with a modified fuzzy c-means algorithm for segmentation. Some mean operation was utilized to eliminate the indeterminacy [41]. An improved clustering method IFCM was presented using the neutrosophic values after applying a-mean operation. Hanbay and Talu [27] proposed a thresholding algorithm for synthetic aperture radar image using neutrosophic sets. As a framework to deal with uncertain cases, neutrosophic sets can be used to describe the image having uncertain information and has been applied to image processing techniques, such as image thresholding, de noising and segmentation.

In this chapter, a new image threshold technique based on neutrosophic sets is presented; the chapter is organized as follows: In §2 definitions of thresholding, in §2 the theory of neutrosophy whereas, the concepts of fuzzy morphology are introduced in §3..

6.2 Image Thresholding

The classical thresholding creates binary images from grey-level ones by turning all pixels below some threshold to "0" and all pixels about that threshold to "1". Binary images are popular, but images are normally acquired as grayscale images. Ideally, objects in the image should appear consistently brighter (or darker) than the background. Under such conditions, which transform an image into a binary image (first choosing a grey level (Thr) in the original image) by transforming each pixel according to whether it is inside or outside a specified range. If $g(x, y)$ is a threshold version of $f(x, y)$ at some global threshold (Thr), it can be defined as [55],

$$g(x, y) = \begin{cases} 0 & \text{if } f(x, y) < Thr \\ 1 & \text{if } f(x, y) \geq Thr \end{cases} \quad (6.1)$$

Thresholding operation is defined as: $Thr = Thr[x, y, p(x, y), f(x, y)]$. In this equation, Thr stands for the threshold value; $f(x, y)$ is the gray level of point (x, y) and $p(x, y)$

denotes some local properties of this point, such as the average gray level of a neighborhood. Based on this, there are two types of image thresholding techniques available: global and local.

Global thresholding: When Thr depends only on $f(x, y)$ (in other words, only on gray-level values) and the value of Thr solely relates to the character of pixels, this thresholding technique is called global thresholding. A histogram of the input image intensity should reveal two peaks, corresponding respectively to the signals from the background and the object. (Note: It is a core assumption of the current version of the 3DMA software that the input data set consists of 2 phases, a phase comprising the object of interest and a single other background phase). Global thresholding consists of setting an intensity value (threshold) such that all pixels having intensity value below the threshold belong to one phase, the remainder belong to the other. Global thresholding is as good as the degree of intensity separation between the two peaks in the image. It is an unsophisticated segmentation choice. The global thresholding option in 3DMA allows the user to pick a single global threshold for a 3D image or separate thresholds for each 2D slice in the image. Some experimental options has also been provided to provide automatic choice of threshold by performing a binormal fit to the two-peak histogram and setting a threshold at the inerpeak minimum as determined by the normal fits. The thresholding option outputs the segmented image slice wise, in a packed bit (0,1) format. All voxels having intensity below the threshold value are set to "0"; the rest are set to "1".

Local thresholding: If threshold Thr depends on $f(x, y)$ and $p(x, y)$, this thresholding is called local thresholding. This method divides an original image into several sub regions, and chooses various thresholds T for each sub region reasonably [83, 84]. Local thresholding method is superior to the global ones for poorly and unevenly illuminated images. Niblack proposes a local thresholding technique based on the local

mean and local standard deviation [55]. The drawback of this algorithm is to determine the size of the neighborhood that is set by the user and it depends on the information available in the images. The window size should be small enough to preserve the local details and at the same time, it should be large enough to suppress noise. One of the well-known local thresholding methods is to fit a plane or biquadratic function to match the background gray-level variations [34] for unevenly illuminated images. A more advanced way is to generate a threshold surface where the threshold level changes dynamically over the image pixel to pixel [19]. Milgram et al. use gradient or edge information to segment images and assumed that different objects may have different thresholds, but each object has a fixed threshold with respect to its background [50].

Otsu method [56]: it is one of the most successful methods for image thresholding because of its simple calculation. Otsu is an automatic threshold selection region based segmentation method. It is a type of global thresholding in which it depend only on some gray value of the image. Otsu method was proposed by Scholar Otsu in 1979, which is widely used because it is simple and effective [50]. The Otsu method requires computing a gray-level histogram before running. However, because of the one-dimensional which only consider the gray-level information, it does not give better segmentation result. So, for that two dimensional Otsu algorithms was proposed which works on both gray-level threshold of each pixel as well as its Spatial correlation information within the neighborhood. So Otsu algorithm can obtain satisfactory segmentation results when it is applied to the noisy images [15].

6.3 Neutrosophic Image Entropy:

For a gray image, the entropy is utilized to evaluate the distribution of the gray levels. If the entropy is maximum, the intensities have equal probability. If the entropy is small, the intensity distribution is non-uniform. Neutrosophic entropy of an image is defined as

the summation of the entropies of three subsets T , I and F which is employed to evaluate the distribution of the elements in neutrosophic domain [25]:

$$En_{NS} = En_T + En_I + En_F. \quad (6.2)$$

$$En_T = - \sum_{i,j=\min(T)}^{\max(T)} P_T(i,j) \ln P_T(i,j),$$

$$En_I = - \sum_{i,j=\min(I)}^{\max(I)} P_I(i,j) \ln P_I(i,j),$$

$$En_F = - \sum_{i,j=\min(F)}^{\max(F)} P_F(i,j) \ln P_F(i,j).$$

Where; En_T , En_I and En_F are the entropies of sets T , I and F respectively. $P_T(i,j)$, $P_I(i,j)$ and $P_F(i,j)$ are the probabilities of elements in T , I and F respectively, whose values equal to i .

6.4 The Proposed Algorithm:

In this section, we suggest some algorithm for image thresholding as follows:

A. Transform input image into Neutrosophic Domain:

A Neutrosophic image P_{NS} is represented by three memberships sets T , I and F . A pixel P in the image is described as $P(t, i, f)$ and belongs to bright pixel set, W in the following way: it is $t\%$ true, $i\%$ indeterminate, and $f\%$ false as bright pixel, where t varies in T , i varies in I , and f varies in F .

B. Comparing Neutrosophic Image with Threshold:

If we directly fed the input image to the neutrosophic domain, then the image obtained are not clear. There is an uncertainty in the assignment of pixels as a pixel may belong to more than one pixel. For each pixels, indeterminacy value is generally greater.

C. *Neutrosophic Image Entropy:*

The entropy is computed for the image in its three layers, the truthness, the indeterminacy, and falseness. It evaluates the distribution of the intensity of each pixel in each layer.

D. *Apply Mathematical Morphology on the Neutrosophic Image:*

After transforming the image into a neutrosophic image, we experimenting with the basic morphological operations (neutrosophic dilation, neutrosophic erosion, neutrosophic opening, neutrosophic closing); results showed that we get better thresholding when using the operation.

E. *Neutrosophic Morphologic Image Entropy:*

In this step we compute the entropy for each component of the image resulted after applying the neutrosophic morphological opening operation.

F. *The Neutrosophic Thresholding Value:*

Now, we use the entropy computed in the previous step to deduce a neutrosophic value for thresholding the neutrosophic image.

G. *The Neutrosophic Image Segmentation:*

Finally, we use the deduced neutrosophic threshold value in order to segment the image under consideration this proposed algorithm may be explained by the following steps:

6.4.1 The Proposed Algorithm Steps:

Step 1: Convert the image to Neutrosophic Domain using equation 2.1.

Step 2: Compute the entropy on neutrosophic image using equation 6.2.

Step 3: Apply morphology operation on the neutrosophic image.

Step 4: Compute the neutrosophic entropy for the output of step 3, using equation 6.2.

Step 5: Compute the threshold value; $Thr = \frac{En_I(i+1) - En_I(i)}{En_I(i)}$.

Step 6: Segment the image according to the value induced in step 5 by OTSU's method [56]. The flowchart of the proposed algorithm is shown in Fig. 6.1.

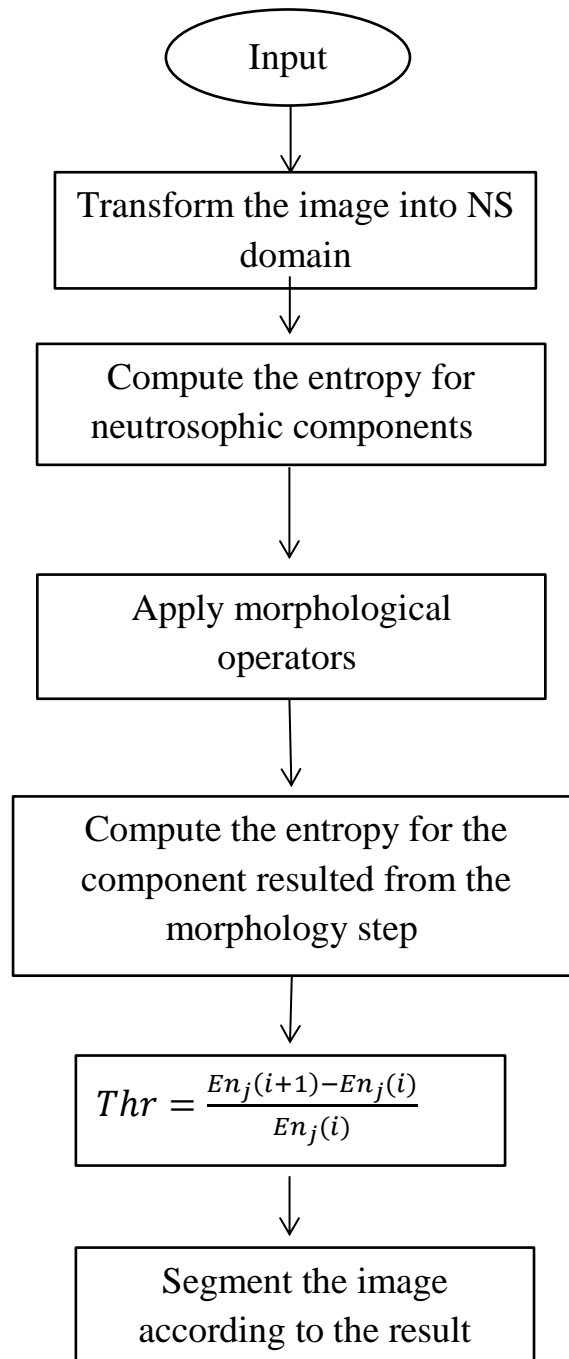
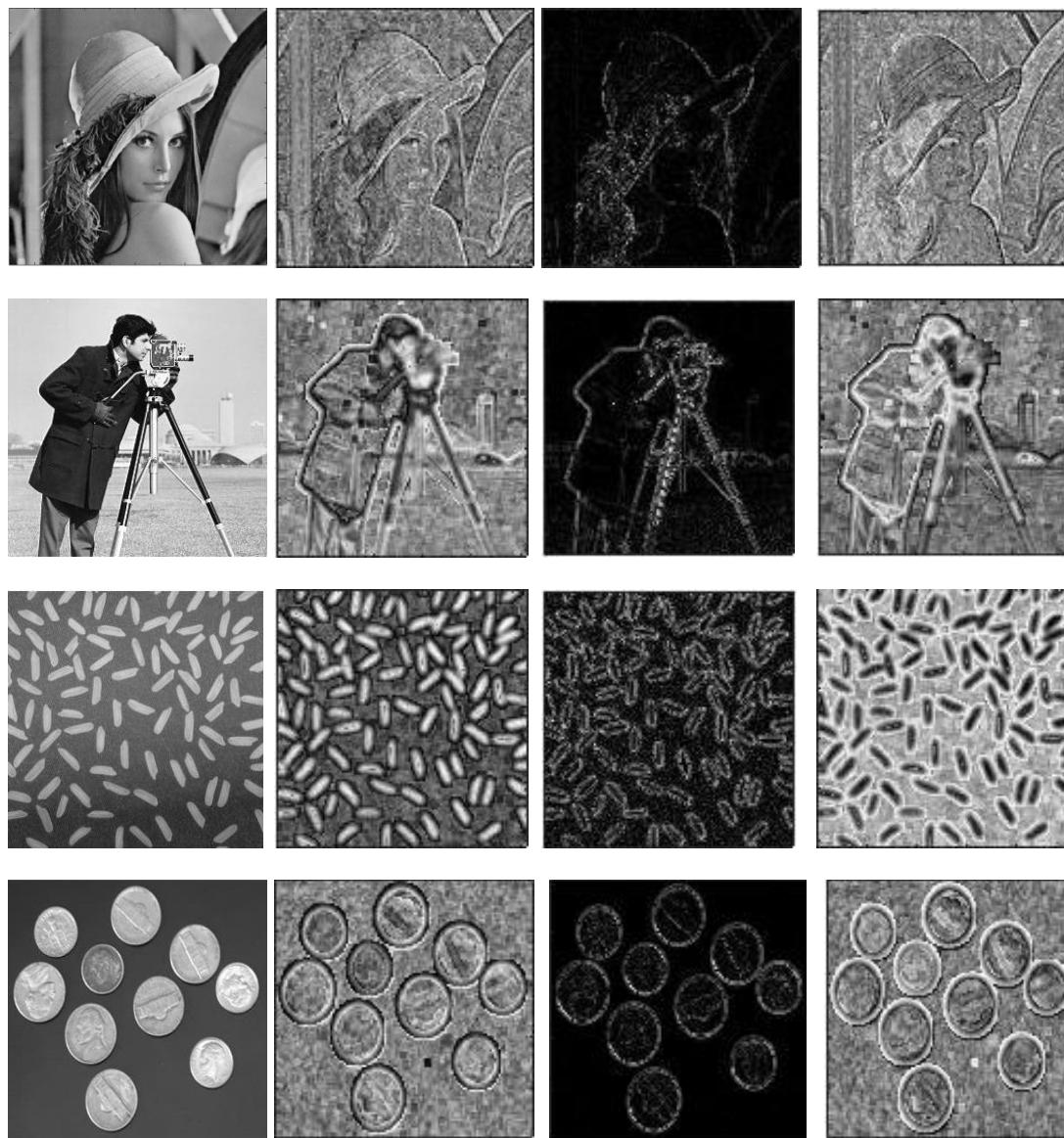


Fig. 6.1: flowchart of the proposed algorithm

6.5 Experiments:

In this chapter, We use several images for our experimental work; the images "Lena, Camera man, Rice and Coin". The steps of our experiment is as follows:

First step is to transform the image from the spatial domain into the neutrosophic domain. The results of this step for each image are shown in the following figure (fig.6.2):



a) **Fig.6.2:** a) gray image b) Neutrosophic image $\langle T_A, I_A, F_A \rangle$ respectively

Second step is to apply the neutrosophic morphology operation for the neutrosophic components

- Using neutrosophic opening operation the result are shown in the following figure (fig.6.3)

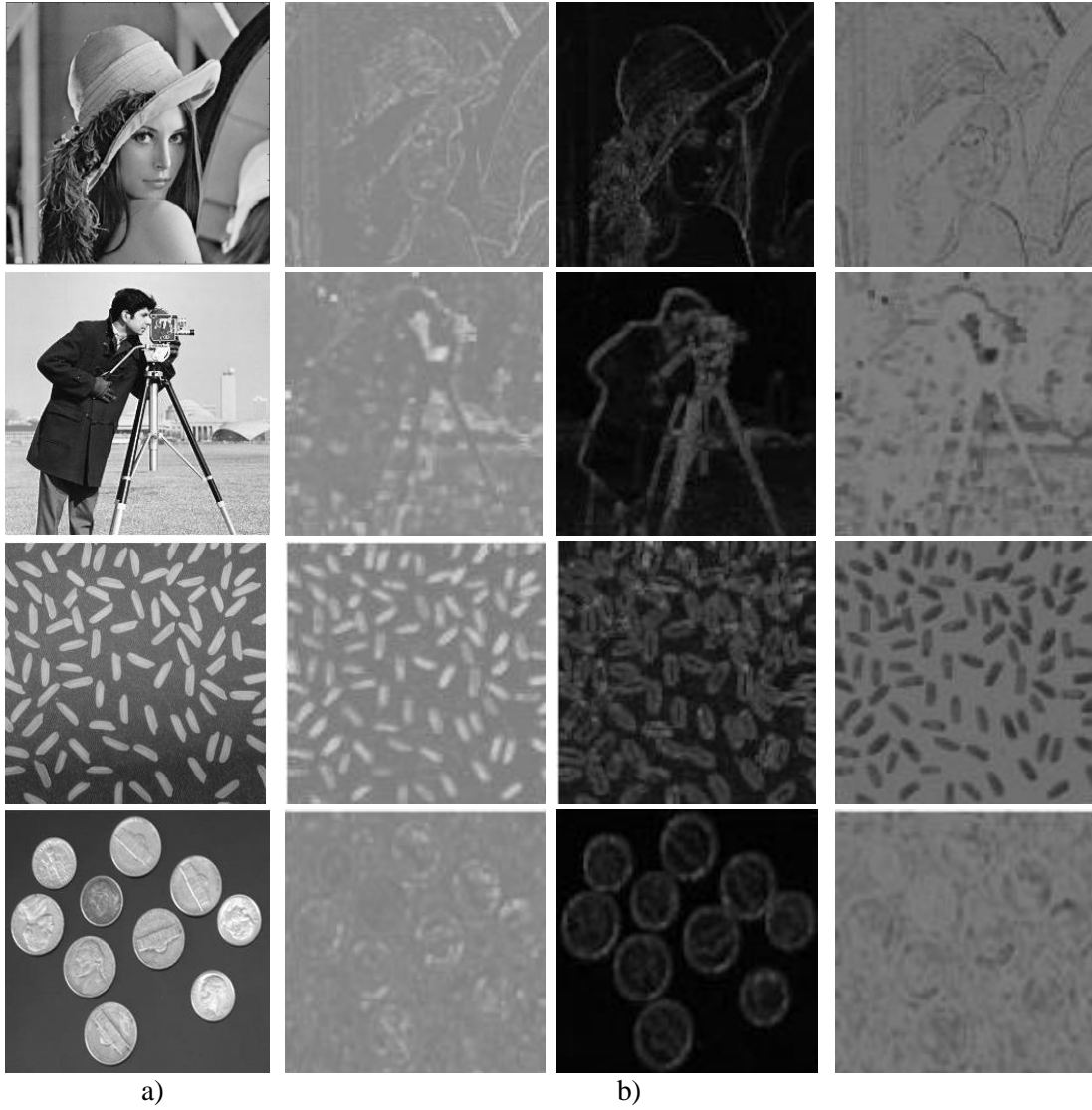


Fig.6.3: a) gray image b) Neutrosophic opening image $\langle T_{A\tilde{\sigma}B}, I_{A\tilde{\sigma}B}, F_{A\tilde{\sigma}B} \rangle$ respectively

Third step: compute *Threshold* value by using the relation,

$$Thr = \frac{En_j(i+1) - En_j(i)}{En_j(i)}$$

The following table show the thresholding value for each image.

Image \ <i>Thr</i>	Lenna	Camera man	Rice	Coin
Threshold Value (OTSU)	0.0301	0.0332	0.0386	0.0338
Threshold Value (NMM)	0.033	0.0403	0.0330	0.0464

Table 1: the thresholding values for the images under consideration

Forth step apply OTSU's method [56], using Thr value determined in the previous step. The following figure (fig.6.4) shows the results obtained.

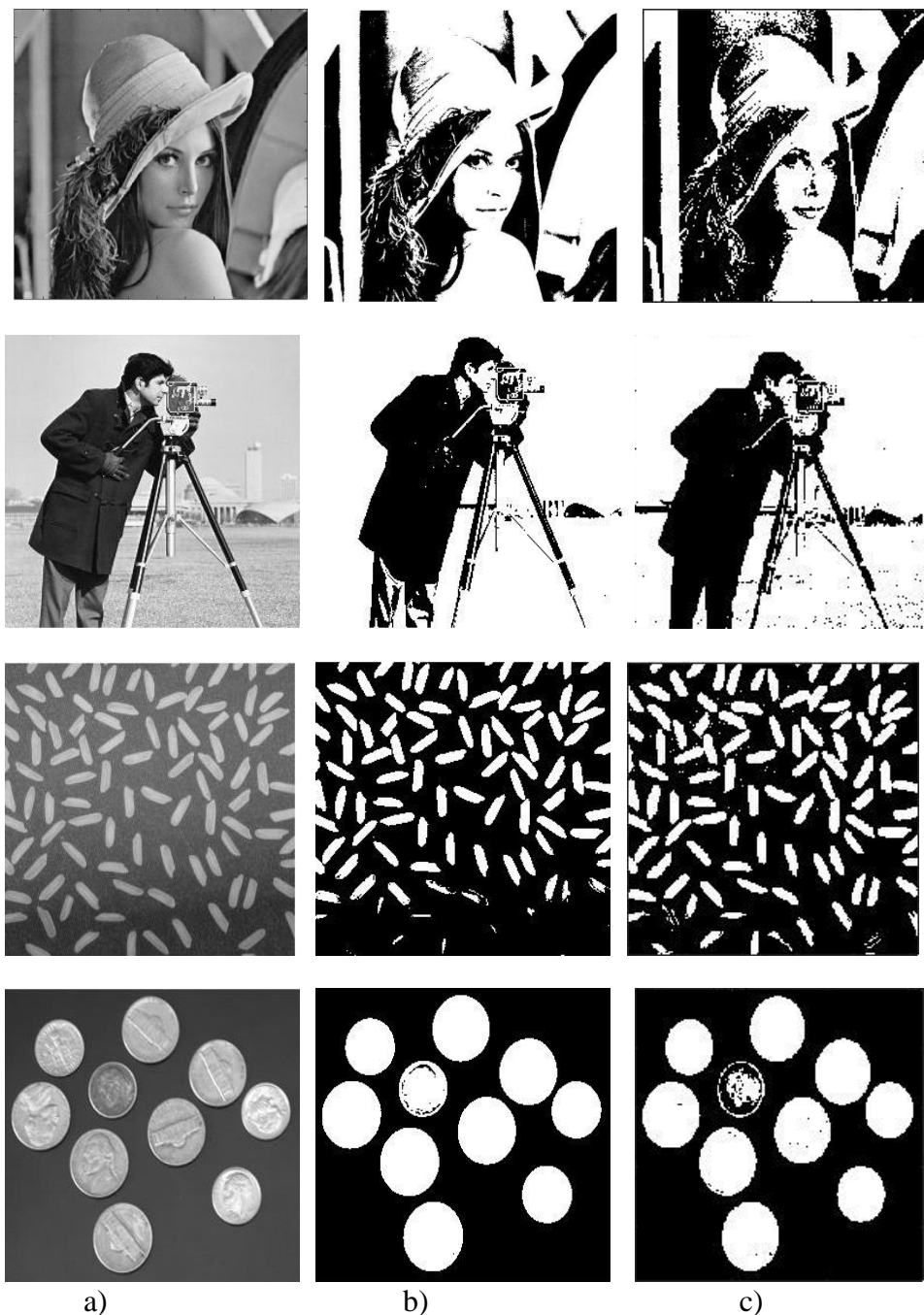


Fig.6.4: a) gray image b) OTSU's method image threshold c) Neutrosophic morphological method image threshold

6.6 Neutrosophic Morphological Method Image Threshold:

To measure the accuracy of our algorithm, we compare between the misclassified pixels between the ideally segmented image and actually segmented image by in our experiments. The quality of the resulting images can be described in terms of signal to

noise ratio (SNR):

$$SNR = 10 \log_{10} \left[\frac{\sum_{r=0}^{H-1} \sum_{c=0}^{W-1} I^2(r, c)}{\sum_{r=0}^{H-1} \sum_{c=0}^{W-1} (I(r, c) - I_n(r, c))^2} \right].$$

Where; $I(r, c)$ and $I_n(r, c)$ represent the intensities of pixel (r, c) in the ideally segmented and actually segmented images, respectively. The following table shows the SNR values computed for each image.

Image \ SNR	Lenna	Camera man	Rice	Coin
SNR (OTSU)	45.1762	46.559	45.9378	46.3514
SNR (NMM)	48.1308	48.1308	48.13	48.1307

Table 6.2: the SNR values for the images under consideration

Finally, we experimented our algorithm on several images in "tif" format. The images under consideration with their neutrosophic components were given in (fig.6.2). namely, the images are, "Lena, Camera man, Rice and Coin". In the fig.6.4, the results when applying the neutrosophic opening operation followed by calculating the entropy to produce the thresholding value were given. From the resulting threshold images, it has been observed that results using the neutrosophic operations measures gives good results when compared to gray thresh (Otsu's method).

6.7 Conclusion and Discussion:

This chapter proposes an image segmentation method using neutrosophic mathematical morphology. An algorithm for image thresholding has been proposed. Finally, the image in neutrosophic domain is segmented. The experimental results show that the proposed method cannot only perform better on synthesis images.

Chapter 7

Conclusion and Future Work

This chapter concludes thesis activities and results presented through the thesis and presenting future work can be conducted. Thesis introduced the overview of a new technique for analyzing and processing images; either binary or grayscale. The technique is a generalization for the fuzzy mathematical morphology; it handles the image in the neutrosophic domain. In such domain the image analyzed into three different layers; the first layer describes how much each pixel belongs to the white set, the third layer describes how much each pixel belongs to the non-white (black) set, while the second layer describes how much the pixel is neither white nor black. The properties of each layer were used to define the basic operations for what we called "Neutrosophic Mathematical Morphology". mainly, we introduced four basic operations; namely, the neutrosophic dilation, the neutrosophic erosion, the neutrosophic closing and the neutrosophic opening. The algebraic properties of the proposed operation were discussed. Furthermore, we introduced some advanced neutrosophic filters using different combinations of the basic operators. Some promising experimental results were presented to visualize the effect of the new introduced operators and filters on the image in the neutrosophic domain instead of the spatial domain. We used "Lena" and "duck" images to investigate the effect of each of the new operators over the image. A literature review for the types of sets was presented in chapter 2, as well as a brief revision for the basic definitions and operations of crisp sets, fuzzy sets and neutrosophic sets and their properties.

In chapter 3, we discussed the theory of both the classical and fuzzy mathematical morphology and their various operators for binary and grayscale images. Also some algebraic properties of the basic operators dilation and erosion were discussed. Besides

the two primary operations of erosion and dilation, there are two secondary operations that play key roles in morphological image processing, these being opening and its dual, closing. Which possesses more geometric formulation in terms of the structuring element.

In chapter 4, we established a foundation for what we called, "Neutrosophic Crisp Mathematical Morphology". It is a new approach to mathematical morphology based on neutrosophic set theory. In addition, we were able to prove that neutrosophic morphological operations inherited some properties and restrictions from fuzzy mathematical morphology. Furthermore, we developed three neutrosophic crisp morphological filters; namely, the neutrosophic crisp external boundary, the neutrosophic crisp internal boundary, the neutrosophic crisp gradient boundary, the neutrosophic crisp Top-hat and the neutrosophic crisp Bottom-hat filters.

Chapter 5 generalized the concepts of the classical mathematical morphology into the neutrosophic domain. For this purpose, we developed several neutrosophic morphological operators inherit properties and restrictions of fuzzy mathematical morphology; namely, the neutrosophic dilation, the neutrosophic erosion, the neutrosophic opening and the neutrosophic closing operators. These operators were presented in two different types, each type is determined according to the behavior of the second component of the triple structure of the operator. Furthermore, we developed three neutrosophic morphological filters; namely, the neutrosophic external boundary, the neutrosophic internal boundary, the neutrosophic gradient boundary, the neutrosophic Top-hat and the neutrosophic Bottom-hat filters. Some promising experimental results were presented to visualize the effect of the new introduced operators and filters on the image in the neutrosophic domain instead of the spatial domain.

Finally, in chapter 6 we applied the neutrosophic mathematical morphological operators proposed in this thesis to one of the most important image's processing application, namely, the image thresholding. The chapter also gave a promising results showing an improvement comparing with the exciting thresholding techniques.

In future, we plan to apply the introduced concepts to more image processing applications. For instance, Image Smoothing, Enhancement, Retrieval. We also plan to examine the neutrosophic morphological operators with medical imaging.

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كلية العلوم
قسم الرياضيات وعلوم الحاسب



جامعة بورسعيد

مدخل نيتروسوفكي للمورفولوجي الرياضي

رسالة مقدمة من

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ملخص الرسالة

ظهرت البداية الحقيقية لعلم المورفولوجي الرياضي (توصيف الشكل- التشكل الرياضي) عام 1960 علي يد العالم (Serra)، وأصبح موضوع خصب في مجال معالجة الصور وتحليلها، حيث بدأ ظهوره في المواقع الهندسية والمناجم بغرض معرفة خواص المواد والخامات عن طريق دراسة الشكل الخارجي من خلال صور تلك المواد. حيث يمكن إعتبار الصورة رياضياً كفتة من النقاط في المستوي، فإعتمد هذا العلم في الأساس علي إستخدام مبادئ نظرية الفئات وعلم التوبولوجي، حيث أنه يقوم بدراسة البناء الهندسي للأشكال الموجودة داخل الصور. كما يقدم الحلول للكثير من التطبيقات في مجال معالجة الصور، كالإستشعار عن بعد والتعرف الضوئي على الحروف في مجال الرادار وكذلك معالجة الصور الطبية. ورغم أن المورفولوجي الرياضي يطبق بشكل أساسي على الصور الرقمية، إلا أنه يمكن إستخدامه أيضاً على بناءات أخرى كالأشكال (Graphs) وشبكات السطوح (Surface meshes) والمجسمات (Solid). وقد تم تقديم المفاهيم الأساسية في المورفولوجي الرياضي من الهندسة والتوبولوجي. وقد أصبح المورفولوجي الرياضي هو أساس في معالجة الصور ويعتمد علي مجموعة من العمليات التي تقوم بتغيير الصورة تبعاً لهذه المفاهيم والعمليات، والعمليتان الأساسيتان هما التآكل (erosion) والتوسيع (dilation)، في كلا العمليتين يؤخذ في الإعتبار جزئين من البيانات كمدخلين: الأول يكون الصورة المراد تآكلها أو إتساعها، والثاني يطلق عليه اسم العنصر الهيكلي. وقد تم تعريف عمليات أخرى مشتقة من هاتين العمليتين مثل عمليتي الفتح (opening) والإغلاق (closing) واللتين تعتبران مؤثرين هامين في علم المورفولوجي الرياضي. وفي البداية أُستخدم المورفولوجي الرياضي في عمليات معالجة وتحليل الصور الثنائية (binary images) وذلك من خلال إستخدام عمليتي التآكل والتوسيع لإستخراج حدود الأشكال الموجودة في الصورة. ثم تم تعميم هذا التطبيق وإستخدامه في معالجة وتحليل الصور الرمادية (grayscale images). ونظراً لإعتماد المورفولوجي الرياضي علي نظرية الفئات فقد بدأ في التطور تبعاً لتطور مفهوم

الفئة رياضياً. فمع ظهور مفهوم الفئات الفازية -الضبابية- (fuzzy sets) علي يد العالم (لطفي زاده) عام (1965)[85]، التي أعطت معني جديداً وهاماً أسهم في التقليل من درجة العشوائية في البيانات وساعدنا في الوصول إلي نتائج عالية الدقة. وتم إدخال المفهوم الجديد للفئات علي تعريف العمليات الأساسية للمورفولوجي الرياضي مما أُطلق عليه المورفولوجي الرياضي الفازي (fuzzy mathematical morphology) وذلك من خلال عمل العالم (Kerre) [39]. وفي العقد الاخير من القرن الماضي، قُدم مفهوم جديد للفئة النيتروسوفكية والذي يُعد تعميم للفئة الفازية علي يد العالم (Smarandache) عام (1999)[74]، وكان ذلك دافعاً لوضع مبادئ لمنطق جديد غير كلاسيكي هو المنطق النيتروسوفكي (neutrosophic logic) كتعميم للمنطق الفازي (fuzzy logic). وقدّم A. A. Salama [62] نظرية الفئات النيتروسوفكية العادية كتعميم لنظرية الفئات الكلاسيكية. كما ساهم في تطوير وإدخال وصياغه ودراسة مفاهيم جديدة في مجالات التوبولوجي والإحصاء وعلوم الحاسب ونظم المعلومات الكلاسيكية عن طريق فئات النيتروسوفك. ويعتبر المنطق النيتروسوفكي فرع جديد يدرس أصل وطبيعة اللاتحديد بالإضافة إلى تفاعل كل الأطياف والفروض المختلفة التي يضعها الإنسان في قضية ما، بحيث يأخذ هذا المنطق بعين الاعتبار كل فكرة مع مضادها - نقيضها - مع اللاتحديد، والفكرة الرئيسية للمنطق النيتروسوفكي هي إمكانية التمييز بين ثلاث ابعاد هي الإلتناء (T) وعدم الإلتناء (F) واللاتحديد (I) حيث يُعطي لكلاً منهم درجة عضوية والتي يُعبر عنها بالشكل $\langle T, I, F \rangle$ فإن ذلك يعطي وصفاً أكثر دقة لبيانات الظاهرة المطلوب دراستها والوصول إلي نتائج عالية الدقة تساهم في إتخاذ القرارات المناسبة لدي متخذي القرار. ومن هنا جاء دور المنطق النيتروسوفكي الذي قدم لنا نوعين من الفئات النيتروسوفكية التي تعمم المفهوم الفازي والمفهوم الكلاسيكي للفئات. ومن هذا المنطلق قدمنا في هذا العمل مقدمة لإدخال المفهوم الجديد للفئات النيتروسوفكية علي العمليات الأساسية لعلم المورفولوجي الرياضي (mathematical morphology). وقد بدأت دراستنا علي الفئات العادية فقدمنا المورفولوجي الرياضي

النيتروسوفكي العادي (neutrosophic crisp mathematical morphology) وتطبيقه علي الصور الثنائية. ثم قدمنا المفهوم النيتروسوفكي علي الفئات الفازية وهو ما أطلقنا عليه المورفولوجي الرياضي النيتروسوفكي (neutrosophic mathematical morphology). وقمنا بتطبيق المفاهيم النيتروسوفكية الجديدة علي الصور الرمادية. وتم إستنتاج بعض العلاقات الرياضية والتوبولوجية وكيفية إستخدام مثل هذه العلاقات في معالجة وتحليل الصور وكأحد التطبيقات الهامة في مجال معالجة الصور، فقد إختارنا (threshold image) وذلك لإختبار المفاهيم الجديدة التي قدمناها عمليا.

تشمل الرسالة سبعة فصول:

الفصل الاول: هذا الفصل هو مقدمة لإعطاء فكرة عامة عن المورفولوجي الرياضي وكيفية تطوره من مورفولوجي رياضي عادي إلى المورفولوجي الفازي وكذلك تطور الفئات من فئات عادية إلى فئات فازية ثم الفئات النيتروسوفكية التي تعتبر أعم وأشمل، كما إستعرضنا في هذا الفصل بعض الدراسات السابقة والشرح للمفاهيم الرئيسية المتعلقة بهدف الرسالة.

الفصل الثاني: يستعرض هذا الفصل بعض الدراسات والنظريات المتعلقة بتطور مفهوم أنواع الفئات، كما يعرض بالتفصيل الفئات النيتروسوفكية، حيث أنها تقسم إلي جزئيين وهما:

- 1- الفئات النيتروسوفكية المعتمدة علي الفئات الفازية وأهم التعريفات والنظريات.
- 2- الفئات النيتروسوفكية المعتمدة علي الفئات العادية وكيفية توليدها من فئات النيتروسوفك الفازية وعرض أهم التعريفات والنظريات وإعطاء أمثلة عليها.

وقدمنا في هذا الفصل الأنواع المختلفة للصور وكيفية تطورها من الصور الثنائية الى الصور الرمادية ثم الى الصور الفازية.

الفصل الثالث: يعرض تغطية شاملة للمفاهيم المستخدمة في الرسالة:

- 1- المورفولوجي الرياضي العادي وعرض العمليات الأساسية وكذلك عرض وافي لبعض العلاقات الرياضية المعرفة علي هذه العمليات وتوضيحها بأمثله وأيضا صور توضيحية.
- 2- المورفولوجي الرياضي الرمادي و إعطاء فكرة عامة عن أهم العمليات وخواصها.
- 3- المورفولوجي الرياضي الفازي حيث قدمنا شرح وافي لجميع العمليات والعلاقات الرياضية وتطبيقها في معالجة الصور.

الفصل الرابع: يهتم هذا الفصل بتقديم مفهوماً جديداً والذي يعتبر تعميماً للمورفولوجي الرياضي الكلاسيكي وإمتداداً لمفهوم الفئة النيتروسوفكيه العادية وأطلقنا عليه المورفولوجي الرياضي النيتروسوفكي العادي وقد خصصنا هذا الفصل لدراسة العمليات الأساسية والمؤثرات الهامة التي قمنا بتقديمها لتكون نواه للمورفولوجي الرياضي النيتروسوفكي العادي، كما قمنا بدراسة وإستنتاج بعض العلاقات والخواص المورفولوجيه المعروفة علي هذه المفاهيم. هذا بالإضافة لإجراء بعض التطبيقات العملية في مجال تحليل ومعالجة الصور الثنائية.

الفصل الخامس: في هذا الفصل قمنا بإستخدام مفهوم الفئات النيتروسوفكيه الفازية وتعميمها لتقديم ما أطلقنا عليه المورفولوجي الرياضي النيتروسوفكي. كوسيلة جديده يمكن إستخدامها في مجال تحليل ومعالجة الصور الرمادية. كما قدمنا لبعض العمليات الهامة والتي يمكن إستخدامها في مجال إستخراج حدود الصورة.

الفصل السادس: نعرض في هذا الفصل النتائج التي حصلنا عليها عند إجراء التجارب العملية علي بعض نماذج من الصور وذلك عند إستخدام العمليات الجديدة التي قدمت في هذه الرسالة و تطبيقها في مجال (Image Thresholding)، كما قمنا بإجراء مقارنه بين هذه النتائج مع نتائج سابقه.

الفصل السابع: نعرض في هذا الفصل تحليلاً للنتائج التي حصلنا عليها، كما نقدم بعض الإقتراحات للعمل المستقبلي.