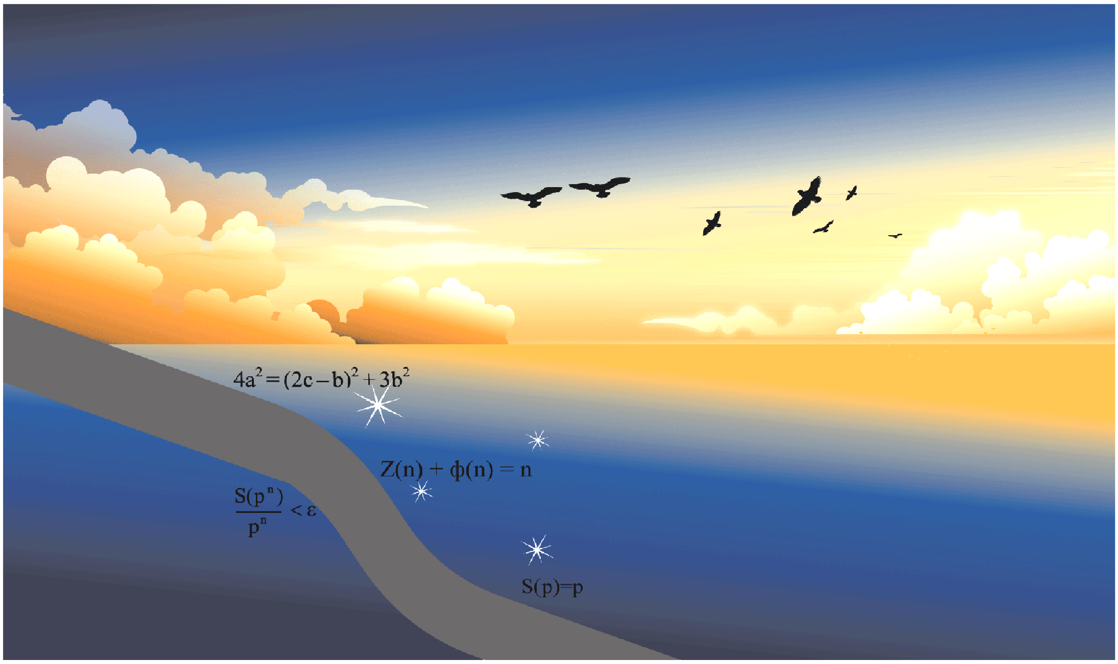
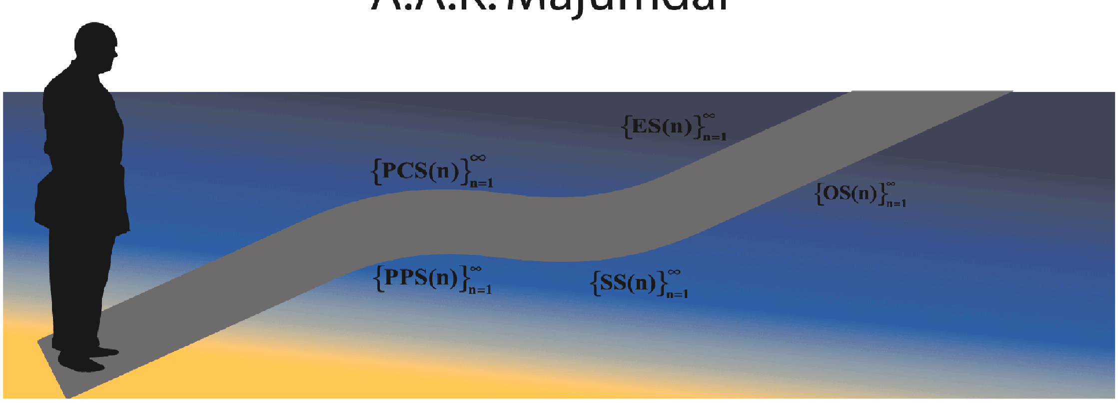


Wandering in the World of Smarandache Numbers

A.A.K. Majumdar



WANDERING

IN THE

WORLD

OF

SMARANDACHE NUMBERS

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2010

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Printed in the United States of America

ISBN : 9 – 781599 – 731247

PREFACE

God created man
And man created numbers

And F. Smarandache created, literally an infinite array of problems, known after him.

It was in mid-nineties of the last century when I received a letter from Professor Ion Patrascu of the Fratii Buzesti College, Craiova, Romania, with lots of enclosures, introducing me with this new branch of Mathematics. Though my basic undergraduate degree is in Mathematics, my research field at that time was Operations Research and Mathematical Programming.

But soon the materials sent by Professor Ion Patrascu caught my interest. And as I collected more materials from the on-line journal on Smarandache Notions, I got more and more fascinated by the simplicity of the problems. Some of the problems are simple and straight-forward, some are really challenging, and the rest are thought-provoking. Now, we have Smarandache sequences, Smarandache functions,Smarandache Fuzzy, Smarandache Anti-Geometry, Smarandache Groupoid, Quantum Smarandache Paradoxes, Smarandache Philosophy, and what not. And more importantly, mathematicians throughout the world are contributing continuously to this new field of research, enriching and diversifying it in many different directions.

At the request of Professor Ion Patrascu, I spent some time on the Smarandache Prime Product Sequence, and my first paper was published in 1998 in Smarandache Notions Journal, but the second one took a long gap till 2004. But since then, I became a regular contributor to the new journal Scientia Magna.

Then some time in 2008, Professor Ion Patrascu requested me to write a book on the Smarandache Notions. The present book is an effort to respond to the request of him.

In writing the book, the first problem I faced is to choose the contents of the book. Today the Smarandache Notions is so vast and diversified that it is indeed difficult to choose the materials. Then, I fixed on five topics, namely, Some Smarandache Sequences, Smarandache Determinant Sequences, the Smarandache Function and its generalizations, the Pseudo Smarandache Function and its generalizations, and Smarandache Number Related Triangles. Most of the results appeared before, but some results, particularly in Chapter 5, are new. In writing the book, I took the freedom of including the more recent results, found by other researchers till 2009, to keep the expositions up-to-date. At the end of Chapter 1, Chapter 3, Chapter 4 and Chapter 5, some open problems are given in the form of conjectures and questions.

I would like to take this opportunity to thank Dr. Perez for his keen interest in publishing the book.

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To

Yumiko

for her constant encouragement and enthusiasm during the preparation of the manuscript

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Chapter 0 Introduction

When we talk of Smarandache sequences, they are literally innumerable in number. For our book, we have chosen ten Smarandache sequences. They are : (1) Smarandache odd sequence, (2) Smarandache even sequence, (3) Smarandache prime product sequence, (4) Smarandache square product sequences, (5) Smarandache higher power product sequences, (6) Smarandache permutation sequence, (7) Smarandache circular sequence, (8) Smarandache reverse sequence, (9) Smarandache symmetric sequence, and (10) Smarandache pierced chain sequence. All of these sequences share the common characteristic that they are all recurrence type, that is, in each case, the n−th term can be expressed in terms of one or more of the preceding terms. Some of the properties of these ten recurrence type Smarandache sequences are studied in Chapter 1. In case of the Smarandache odd, even, circular and symmetric sequences, we show that none of these sequences satisfies the recurrence relationship for Fibonacci or Lucas numbers. We prove further that none of the Smarandache prime product and reverse sequences contains Fibonacci or Lucas numbers (in a consecutive row of three or more). For the Smarandache even, prime product, permutation and square product sequences, the question is : Are there any perfect powers in each of these sequences? We have a partial answer for the first of these sequences, while for each of the remaining sequences, we prove that no number can be expressed as a perfect power. We also prove that no number of the Smarandache higher power product sequences is square of a natural number.

In Chapter 2, we consider four Smarandache determinant sequences, namely, the Smarandache (a) cyclic determinant natural sequence, (b) cyclic arithmetic determinant sequence, (c) bisymmetric determinant natural sequence, and (d) bisymmetric arithmetic determinant sequence. Actually, the sequence (a) is a particular case of the sequence (b), and (c) is a special case of (d). In each case, we derive the explicit forms of the n−th term. We also find the sequences of the n−th partial sums for the sequences (c) and (d).

Two of the most widely studied Smarandache arithmetical functions, namely, the Smarandache function S(n) and the pseudo Smarandache function Z(n), are the subject matters of Chapter 3 and Chapter 4 respectively, where we give some elementary properties of these two functions and their generalizations.

Chapter 5 deals with the Smarandache number related triangles. Besides the Smarandache number related dissimilar Pythagorean triangles, detailed analysis on the Smarandache number related dissimilar 60^0 and 120^0 triangles is provided.

We may recall that an arithmetic function f(n) has, as its domain, the set of all positive integers n. In the classical Theory of Numbers, the most commonly encountered arithmetical functions are d(n), ϕ(n) and σ(n). Recently, the researchers are getting interested in studying equations involving S(n) or Z(n) and one or more of the classical arithmetical functions. With this in mind, we review below the relevant definitions and theorems. The classic book, "An Introduction to the Theory of Numbers", by G.H. Hardy and E.M. Wright (Clarendon Press, Oxford, UK, 5^{th} Edition (2002)), is recommended for the definitions and the corresponding formulas.

Definition 0.1 **:** Given an integer $n \geq 1$, the *divisor function* d(n) is the number of all divisors of n, including 1 and n itself.

Theorem 0.1 **:** Let

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \ldots p_s^{\alpha_s} \qquad (0.1)$$

be the representation of n in terms of its prime factors $p_1, p_2, \ldots, p_k$. Then,

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1).$$

Definition 0.2 **:** Given an integer $n \geq 1$, the *Euler phi function*, denoted by $\phi(n)$, is the number of positive integers not exceeding n, which are prime to n.

Theorem 0.2 **:** Let n have the prime factor representation of the form (0.1). Then,

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_k}\right).$$

Definition 0.3 **:** Given an integer $n \geq 1$, $\sigma(n)$ is the sum of all divisors of n (including 1 and n itself).

Theorem 0.3 **:** Let n have the prime factor representation of the form (0.1). Then,

$$\sigma(n) = \frac{p_1^{\alpha_1 + 1} - 1}{p_1 - 1} \cdot \frac{p_2^{\alpha_2 + 1} - 1}{p_2 - 1} \ldots \frac{p_k^{\alpha_k + 1} - 1}{p_k - 1}.$$

Definition 0.4 **:** An integer $n \geq 1$ is called *perfect* if and only if it is the sum of all its proper divisors (that is, all divisors including 1 but excluding n itself). Thus, n is perfect if and only if $2n = \sigma(n)$.

Theorem 0.4 **:** Any even perfect number n is of the form

$$n = 2^k(2^{k+1} - 1),$$

where the integer $k \geq 1$ is such that $2^{k+1} - 1$ is prime.

Definition 0.5 **:** An arithmetic function f(.) is called *multiplicative* if and only if

$$f(mn) = f(m)f(n) \text{ for all integers m and n with } (m, n) = 1.$$

Theorem 0.5 **:** The functions d(n), $\phi(n)$ and $\sigma(n)$ are multiplicative.

Definition 0.6 **:** An integer $n>0$ is called *f–perfect* if

$$n = \sum_{i=1}^{k} f(d_i),$$

where $d_1 \equiv 1, d_2, \ldots, d_k$ are the proper divisors of n, and f(.) is an arithmetical function.

Chapter 1, Chapter 2, Chapter 4 and Chapter 5 are based on our previous papers, published in different journals. In the meantime, some new results have been found, and those results are also included in this book. Moreover, to keep the book up-to-date, we did not hesitate to include the results of other researchers as well, very frequently with simplified proofs.

Chapter 1 Some Smarandache Sequences

For this chapter, we have chosen the following ten Smarandache sequences :

(1) Smarandache odd sequence
(2) Smarandache even sequence
(3) Smarandache prime product sequence
(4) Smarandache square product sequences
(5) Smarandache higher power product sequences
(6) Smarandache permutation sequence
(7) Smarandache circular sequence
(8) Smarandache reverse sequence
(9) Smarandache symmetric sequence
(10) Smarandache pierced chain sequence.

The above ten sequences have one property in common : They are all recurrence type.

It is conjectured that there are no Fibonacci or Lucas numbers in any of the Smarandache odd, even, circular and symmetric sequences. We confirm the conjectures by showing that none of these sequences satisfies the recurrence relationship for Fibonacci or Lucas numbers. We prove further that none of the Smarandache prime product and reverse sequences contains Fibonacci or Lucas numbers (in a consecutive row of three or more).

We recall that the sequence of Fibonacci numbers, $\{F(n)\}_{n=1}^{\infty}$, and the sequence of Lucas numbers $\{L(n)\}_{n=1}^{\infty}$, are defined through the recurrence relations satisfied by their terms :

$$F(1) = 1, \quad F(2) = 1; \quad F(n+2) = F(n+1) + F(n), \quad n \geq 1, \tag{1.1}$$

$$L(1) = 1, \quad L(2) = 3; \quad L(n+2) = L(n+1) + L(n), \quad n \geq 1. \tag{1.2}$$

For the Smarandache even, prime product, permutation and square product sequences, the question is : Are there any perfect powers in each of these sequences? We have a partial answer for the first of these sequences, while for each of the remaining sequences, we prove that no number can be expressed as a perfect power. We also prove that no number of the Smarandache higher power product sequences is square of a natural number.

For the Smarandache odd, prime product, circular, reverse and symmetric sequences, the question is : How many primes are there in each of these sequences? For the Smarandache even sequence, the question is : How many elements of the sequence are twice a prime? These questions still remain open.

After giving some preliminary results in §1.1, we study some elementary properties of the ten sequences separately in the next ten sections from §1.2 through §1.11. In each case, we give the recurrence relation satisfied by the sequence. §1.12 gives some series involving the Smarandache sequences. The chapter ends with some remarks in §1.13.

1.1 Some Preliminary Results

***Lemma 1.1.1* :** 3 divides $(10^m + 10^n + 1)$ for all integers $m,n \geq 0$.

***Proof* :** We consider the following three possible cases separately :

Case (1) : When m = n = 0. In this case, the result is clearly true.

Case (2) : When $m = 0$, $n \geq 1$. Here also, the result is seen to be true, since

$$10^m + 10^n + 1 = 10^n + 2 = (10^n - 1) + 3 = 9(1 + 10 + 10^2 + \ldots + 10^{n-1}) + 3.$$

Case (3) : When $m \geq 1$, $n \geq 1$. In this case, the result follows, since

$$10^m + 10^n + 1 = (10^m - 1) + (10^n - 1) + 3. \blacksquare$$

***Lemma 1.1.2* :** The only non–negative integer solution of the Diophantine equation $x^2 - y^2 = 1$ is $x = 1, y = 0$.

***Proof* :** The given Diophantine equation is equivalent to $(x - y)(x + y) = 1$, where both $x - y$ and $x + y$ are integers. Therefore, the only two possibilities are

(1) $x - y = 1 = x + y$, (2) $x - y = -1 = x + y$,

the first of which gives the desired non–negative solution. ■

***Corollary 1.1.1*:** Let N (>1) be a fixed number. Then,

(1) the Diophantine equation $x^2 - N = 1$ has no (positive) integer solution x;

(2) the Diophantine equation $N - y^2 = 1$ has no (positive) integer solution y.

***Lemma 1.1.3* :** Let n (≥ 2) be a fixed integer. Then, the only non–negative integer solution of the Diophantine equation $x^2 + 1 = y^n$ is $x = 0, y = 1$.

***Proof* :** By Lemma 1.1.2, it is sufficient to consider the case when $n > 2$ odd : If n is even, say, $n = 2m$ for some integer $m > 1$, then rewriting the given Diophantine equation as $(y^m)^2 - x^2 = 1$, we see that, by Lemma 1.1.2, the only non–negative integer solution is $y^m = 1$, $x = 0$, that is $x = 0, y = 1$, as required.

So, let n be odd, say, $n = 2k + 1$ for some integer $k \geq 1$. Then, the given Diophantine equation can be written as

$$x^2 = y^{2k+1} - 1 = (y - 1)(y^{2k} + y^{2k-1} + \ldots + 1). \qquad (1)$$

From (1), we see that $x = 0$ if and only if $y = 1$, since $y^{2k} + y^{2k-1} + \ldots + 1 > 0$.

Now, if $x \neq 0$, from (1), the only possibilities are the following two :

Case (1) : $y - 1 = x$, $y^{2k} + y^{2k-1} + \ldots + 1 = x$.

But then $y = x + 1$, and we are led to the equation $(x + 1)^{2k} + (x + 1)^{2k-1} + \ldots + (x + 1)^2 + 2 = 0$, which is impossible.

Case (2) : $y - 1 = 1$, $y^{2k} + y^{2k-1} + \ldots + 1 = x^2$.

Then, $y = 2$, $x^2 = 2^{2k+1} - 1$. Rewriting this equation in the following equivalent form

$$(x - 1)(x + 1) = 2(2^k - 1)(2^k + 1),$$

we see that the l.h.s. is divisible by 4, while the r.h.s. is not divisible by 4 (since both $2^k - 1$ and $2^k + 1$ are odd). And we reach to a contradiction.

The contradiction in both the cases establishes the lemma. ■

Corollary 1.1.2 **:** Let n (≥ 2) and N (>0) be two fixed integers. Then, the Diophantine equation $N^2 + 1 = y^n$ has no integer solution y.

Corollary 1.1.3 **:** Let n (≥ 2) and N (>1) be two fixed integers. Then, the Diophantine equation $x^2 + 1 = N^n$ has no (positive) integer solution x.

Lemma 1.1.4 **:** Let n (≥ 2) be a fixed integer. Then, the only non–negative integer solutions of the Diophantine equation $x^2 - y^n = 1$ are (1) x = 1, y = 0; (2) x = 3, y = 2, n = 3.

Proof **:** For n = 2, the lemma reduces to Lemma 1.1.2. So we consider the case when $n \geq 3$. From the given Diophantine equation, we see that, y = 0 if and only if $x = \pm 1$, giving the only non–negative integer solution x = 1, y = 0. To see if the given Diophantine equation has any non–zero integer solution, we assume that $x \neq 1$.

If n is even, say, n = 2m for some integer $a \geq 1$, then $x^2 - y^n \equiv x^2 - (y^m)^2 = 1$, which has no integer solution y for any $x > 1$ (by Corollary 1.1.1(2)).

Next, let n be odd, say, n = 2k + 1 for some integer $k \geq 1$. Then, $x^2 - y^{2k+1} = 1$, that is,

$$(x-1)(x+1) = y^{2k+1}.$$

We now consider the following three cases that may arise :

Case (1) : When $x - 1 = 1$, $x + 1 = y^{2k+1}$.

Here, x = 2 together with the equation $y^{2k+1} = 3$, and the latter has no integer solution y.

Case (2) : When $x - 1 = y$, $x + 1 = y^{2k}$.

Rewriting the second equation in the equivalent form $(y^k - 1)(y^k + 1) = x$, we see that $(y^k + 1) \mid x$. But this contradicts the first equation $x = y + 1$ if $k > 1$, since for $k > 1$, $y^k + 1 > y + 1 = x$.

If k = 1, then

$$(y-1)(y+1) = x \quad \Rightarrow \quad y - 1 = 1,\ y + 1 = x,$$

so that y = 2, x = 3, n = 3, which is a solution of the given Diophantine equation.

Case (3) : When $x - 1 = y^t$ for some integer t with $2 \leq t \leq k$, $k \geq 2$ (so that $x + 1 = y^{2k-t+1}$).

In this case, we have $2x = y^t[1 + y^{2(k-t)+1}]$. Since x does not divide y, it follows that

$$1 + y^{2(k-t)+1} = Cx, \text{ for some integer } C \geq 1.$$

Thus,

$$2x = y^t(Cx) \quad \Rightarrow \quad Cy^t = 2.$$

If C = 2, then y = 1, and the resulting equation $x^2 = 2$ has no integer solution. On the other hand, if $C \neq 2$, the equation $Cy^t = 2$ has no integer solution. Thus, case (3) cannot occur.

All these complete the proof of the lemma. ■

Corollary 1.1.4 **:** The only non–negative integer solution of the Diophantine equation $x^2 - y^3 = 1$ is x = 3, y = 2.

Corollary 1.1.5 **:** Let n (>3) be a fixed integer. Then, the Diophantine equation $x^2 - y^n = 1$ has x = 1, y = 0 as its only non–negative integer solution.

Corollary 1.1.6 **:** Let n (>3) and N (>0) be two fixed integers. Then, the Diophantine equation $x^2 - N^n = 1$ has no integer solution x.

Corollary 1.1.7 **:** Let n (≥ 3) and N (>1) be two fixed integers with $N \neq 3$. Then, the Diophantine equation $N^2 - y^n = 1$ has no integer solution.

1.2 Smarandache Odd Sequence $\{OS(n)\}_{n=1}^{\infty}$

The Smarandache odd sequence, denoted by $\{OS(n)\}_{n=1}^{\infty}$, is the sequence of numbers formed by repeatedly concatenating the odd positive integers (Ashbacher [1]).

The first few terms of the sequence are

1, 13, 135, 1357, 13579, 1357911, 135791113, 13579111315, … .

In general, the n–th term of the sequence is given by

$$OS(n) = \overline{135 \ldots (2n-1)},\ n \geq 1. \quad (1.2.1)$$

For any $n \geq 1$, $OS(n+1)$ can be expressed in terms of $OS(n)$ as follows : For $n \geq 1$,

$$OS(n+1) = \overline{135 \ldots (2n-1)(2n+1)}$$
$$= 10^s \times OS(n) + (2n+1) \text{ for some integer } s \geq 1. \quad (1.2.2)$$

More precisely,

$$s = \text{number of digits in } (2n+1).$$

Thus, for example,

$$OS(5) = 10 \times OS(4) + 7,$$

while,

$$OS(6) = 10^2 \times OS(5) + 11.$$

By repeated application of (1.2.2), we get

$$OS(n+3) = 10^s \times OS(n+2) + (2n+5) \text{ for some integer } s \geq 1$$
$$= 10^s[10^t \times OS(n+1) + (2n+3)] + (2n+5) \text{ for some integer } t \geq 1 \quad (1.2.3a)$$
$$= 10^{s+t}[10^u \times OS(n) + (2n+1)] + (2n+3)10^s + (2n+5) \quad (1.2.3b)$$

for some integer $u \geq 1$, so that $s \geq t \geq u \geq 1$. Therefore,

$$OS(n+3) = 10^{s+t+u} \times OS(n) + (2n+1)10^{s+t} + (2n+3)10^s + (2n+5). \quad (1.2.4)$$

From (1.2.3), we see that, for any integer $n \geq 1$,

$$OS(n+3) = 10^{s+t} \times OS(n+1) + \overline{(2n+3)(2n+5)}$$
$$= 10^{s+t+u} \times OS(n) + \overline{(2n+1)(2n+3)(2n+5)}.$$

Lemma 1.2.1 **:** 3 divides OS(n) if and only if 3 divides OS(n + 3).
Proof **:** For any s, t with $s \geq t \geq 1$, by Lemma 1.1.1,

$$3 \mid [(2n+1)10^{s+t} + (2n+3)10^s + (2n+5)] = (2n+1)(10^{s+t} + 10^s + 1) + 2(10^s + 2).$$

The result now follows by virtue of (1.2.4). ■

Lemma 1.2.2 **:** 9 divides OS(3n) for any integer $n \geq 1$.
Proof **:** First note that, the sum of digits of OS(n) is

$$1 + 3 + \ldots + (2n+1) = n^2.$$

Since the sum of digits of OS(3n) is $(3n)^2 = 9n^2$, the result follows. ■

In proving Lemma 1.2.2, we have used the fact that an integer is divisible by 9 if and only if its sum of digits is divisible by 9 (see, for example, Bernard and Child [2]).

Lemma 1.2.3 **:** 5 divides $OS(5n+3)$ for any integer $n \geq 0$.

Proof **:** From (1.2.2), for any arbitrary but fixed $n \geq 0$,

$$OS(5n+3) = 10^s \times OS(5n+2) + (10n+5) \text{ for some integer } s \geq 1.$$

The r.h.s. is clearly divisible by 5, and hence 5 divides $OS(5n+3)$. ■

Based on computer findings with all numbers from 135 through OS(2999), Ashbacher [1] conjectures that (except for the trivial cases of $n=1$, 2, for which $OS(1)=1=F(1)=F(2)=L(1)$, $OS(2)=13=F(7)$), there are no numbers in the Smarandache odd sequence that are also Fibonacci (or, Lucas) numbers. In Theorem 1.2.1 and Theorem 1.2.2, we prove the conjectures of Ashbacher in the affirmative. The proof of the theorems relies on the following results.

Lemma 1.2.4 **:** For any integer $n \geq 1$, $OS(n+1) > 10 \times OS(n)$.

Proof **:** From (1.2.2), for any $n \geq 1$,

$$OS(n+1) = 10^s \times OS(n) + (2n+1) > 10^s \times OS(n) > 10 \times OS(n),$$

where $s \geq 1$ is an integer. We thus get the desired inequality. ■

Corollary 1.2.1 **:** For any integer $n \geq 1$, $OS(n+2) - OS(n) > 9[OS(n+1) + OS(n)]$.

Proof **:** From Lemma 1.2.4,

$$OS(n+1) - OS(n) > 9 \times OS(n) \text{ for all } n \geq 1. \tag{1.2.5}$$

Now, using the inequality (1.2.5), we get

$$\begin{aligned} OS(n+2) - OS(n) &= [OS(n+2) - OS(n+1)] + [OS(n+1) - OS(n)] \\ &> 9[OS(n+1) + OS(n)], \end{aligned}$$

which establishes the result. ■

Theorem 1.2.1 **:** (Except for $n=1$, 2 for which $OS(1)=1=F(1)=F(2)$, $OS(2)=13=F(7)$) there are no numbers in the Smarandache odd sequence that are also Fibonacci numbers.

Proof **:** Using Corollary 1.2.1, we see that, for all $n \geq 1$,

$$OS(n+2) - OS(n) > 9[OS(n+1) + OS(n)] > OS(n+1).$$

Thus, no numbers of the Smarandache odd sequence satisfy the recurrence relation (1.1) satisfied by the Fibonacci numbers. ■

By similar reasoning, we have the following result.

Theorem 1.2.2 **:** (Except for $n=1$ for which $OS(1)=1=L(1)$) there are no numbers in the Smarandache odd sequence that are Lucas numbers

Thus, no numbers of the Smarandache odd sequence satisfy the recurrence relation (1.1) satisfied by the Fibonacci numbers. ■

Ashbacher [1] reports that, among the first 21 elements of the sequence, only OS(2), OS(10) and OS(16) are primes. Marimutha [3] conjectures that there are infinitely many primes in the Smarandache odd sequence, but the conjecture still remains open.

In order to search for primes in the Smarandache odd sequence, by virtue of Lemma 1.2.2 and Lemma 1.2.3, it is sufficient to check the terms of the forms $OS(3n \pm 1)$ $(n \geq 1)$, where neither $3n+1$ nor $3n-1$ is of the form $5k+3$ for some integer $k \geq 1$.

1.3 Smarandache Even Sequence $\{ES(n)\}_{n=1}^{\infty}$

The Smarandache even sequence, denoted by $\{ES(n)\}_{n=1}^{\infty}$, is the sequence of numbers formed by repeatedly concatenating the even positive integers (Ashbacher [1]).

The n–th term of the sequence is given by

$$ES(n) = \overline{24 \ldots (2n)},\ n \geq 1. \tag{1.3.1}$$

The first few terms of the sequence are

2, 24, 246, 2468, 246810, 24681012, 2468101214, …,

of which only the first is a prime number.

We note that, for any integer $n \geq 1$,

$$ES(n+1) = \overline{24 \ldots (2n)(2n+2)}$$
$$= 10^s \times ES(n) + (2n+2) \text{ for some integer } s \geq 1, \tag{1.3.2}$$

where, more precisely,

$s =$ number of digits in $(2n+2)$.

Thus, for example,

$ES(4) = 2468 = 10 \times ES(3) + 8,$

while,

$ES(5) = 246810 = 10^2 \times ES(4) + 10.$

From (1.3.2), the following result follows readily.

Lemma 1.3.1 **:** For any integer $n \geq 1$, $ES(n+1) > 10 \times ES(n)$.

Using Lemma 1.3.1, we can prove that

$$ES(n+2) - ES(n) > 9[ES(n+1) + ES(n)] \text{ for all } n \geq 1. \tag{1.3.3}$$

The poof is similar to that of Corollary 1.2.1, and is omitted here.

By repeated application of (1.3.2), we see that, for any integer $n \geq 1$,

$$ES(n+2) = 10^t \times ES(n+1) + (2n+4) \text{ for some integer } t \geq 1$$
$$= 10^t[10^u \times ES(n) + (2n+2)] + (2n+4) \text{ for some integer } u \geq 1$$
$$= 10^{u+t} \times ES(n) + (2n+2)10^t + (2n+4),$$

and

$$ES(n+3) = 10^s \times ES(n+2) + (2n+6) \text{ for some integer } s \geq 1$$
$$= 10^s[10^{u+t} \times ES(n) + (2n+2)10^t + (2n+4)] + (2n+6)$$
$$= 10^{s+t+u} \times ES(n) + (2n+2)10^{s+t} + (2n+4)10^s + (2n+6), \tag{1.3.4}$$

for some integers s, t and u with $s \geq t \geq u \geq 1$.

From (1.3.4), we see that

$$ES(n+3) = 10^{s+t} \times ES(n+1) + \overline{(2n+4)(2n+6)}$$
$$= 10^{s+t+u} \times ES(n) + \overline{(2n+2)(2n+4)(2n+6)}.$$

Using (1.3.4), together with Lemma 1.1.1, we can prove the following result.

Lemma 1.3.2 **:** 3 divides ES(n) if and only if 3 divides ES(n + 3).

Lemma 1.3.3 **:** For any integer $n \geq 1$, 3 divides ES(3n).

Proof **:** The proof is by induction on n. Since ES(3) = 246 is divisible by 3, the lemma is true for n = 1. We now assume that the result is true for some n, that is, $3 \mid ES(3n)$ for some n. Now, by Lemma 1.3.2, together with the induction hypothesis, we see that 3 divides ES(3n + 3) = ES(3(n + 1)). Thus the result is true for n + 1. ■

Corollary 1.3.1 **:** For any integer $n \geq 1$, 3 divides ES(3n − 1).

Proof **:** Let n (≥ 1) be any arbitrary but fixed integer. From (1.3.2),

$$ES(3n) = 10^s \times ES(3n-1) + 6n \text{ for some integer } s \geq 1.$$

Now, by Lemma 1.3.3, 3 divides ES(3n). Therefore, 3 must also divide ES(3n − 1). Since n is arbitrary, the lemma is proved. ■

Corollary 1.3.2 **:** For any integer $n \geq 1$, 3 does not divide ES(3n + 1).

Proof **:** Let n (≥ 1) be any arbitrary but fixed integer. From (1.3.2),

$$ES(3n+1) = 10^s \times ES(3n) + (6n+2) \text{ for some integer } s \geq 1.$$

Since 3 divides ES(3n), but 3 does not divide (6n + 2), the result follows. ■

Lemma 1.3.4 **:** 4 divides ES(2n) for any integer $n \geq 1$.

Proof **:** Since 4 divides ES(2) = 24 and 4 divides ES(4) = 2468, we see that the result is true for n = 1, 2. Now, from (1.3.2), for $n \geq 1$,

$$ES(2n) = 10^s \times ES(2n-1) + 4n,$$

where s is the number of digits in 4n. Clearly, $s \geq 2$ for all $n \geq 3$. Thus, 4 divides 10^s if $n \geq 3$, and we get the desired result. ■

Corollary 1.3.3 **:** For any integer $n \geq 0$, 4 does not divide ES(2n + 1).

Proof **:** Clearly the result is true for n = 0, since ES(1) = 2 is not divisible by 4. For $n \geq 1$, from (1.3.2),

$$ES(2n+1) = 10^s \times ES(2n) + (4n+2) \text{ for some integer } s \geq 1.$$

By Lemma 1.3.4, 4 divides ES(2n). Since 4 does not divide (4n + 2), the result follows. ■

Lemma 1.3.5 **:** For any integer $n \geq 1$, 10 divides ES(5n).

Proof **:** For any arbitrary but fixed $n \geq 1$, from (1.3.2),

$$ES(5n) = 10^s \times ES(5n-1) + 10n \text{ for some integer } s \geq 1.$$

The result is now evident from the above expression of ES(5n). ■

Corollary 1.3.4 **:** 20 divides ES(10n) for any integer $n \geq 1$.

Proof **:** follows by virtue of Lemma 1.3.4 and Lemma 1.3.5. ■

Based on the computer findings with numbers up through ES(1499) = 2468…29962998, Ashbacher [1] conjectures that (except for the case of ES(1) = 2 = F(3)) there are no numbers in the Smarandache even sequence that are also Fibonacci (or, Lucas) numbers. The following two theorems establish the validity of Ashbacher's conjectures.

Theorem 1.3.1 **:** (Except for ES(1) = 2 = F(3)) there are no numbers in the Smarandache even sequence that are Fibonacci numbers.
Proof **:** By virtue of (1.3.3), for all $n \ge 1$,

$$ES(n+2) - ES(n) > 9[ES(n+1) + ES(n)] > ES(n+1).$$

Thus, no numbers of the Smarandache odd sequence satisfy the recurrence relation (1.1) satisfied by the Fibonacci numbers. ■

By similar reasoning, we have the following result.

Theorem 1.3.2 **:** There are no numbers in the Smarandache even sequence that are Lucas numbers.

Ashbacher [1] raises the question : Are there any perfect powers in ES(n)? The following theorem gives a partial answer to the question.

Theorem 1.3.3 **:** None of the terms of the subsequence $\{ES(2n-1)\}_{n=1}^{\infty}$ is a perfect square or higher power of an integer (> 1).
Proof **:** Let, for some $n \ge 1$,

$$ES(n) = \overline{24\ \ldots\ (2n)} = x^2 \text{ for some integer } x > 1.$$

Now, since ES(n) is even for all $n \ge 1$, x must be even. Let

$$x = 2y \text{ for some integer } y \ge 1.$$

Then,

$$ES(n) = (2y)^2 = 4y^2,$$

which shows that 4 divides ES(n).
Now, if n is odd of the form $2k-1$, $k \ge 1$, by Corollary 1.3.3, ES(2k – 1) is not divisible by 4, and hence, the numbers of the form ES(2k – 1), $k \ge 1$, can not be perfect squares.

By same reasoning, none of the terms ES(2n – 1), $n \ge 1$, can be expressed as a cube or higher powers of an integer. ■

Remark 1.3.1 **:** It can be seen that, if n is of the form $k\times10^s+4$ or $k\times10^s+6$, where k ($1 \le k \le 9$) and s (≥ 1) are integers, then ES(n) cannot be a perfect square (and hence, cannot be the power of an even integer). The proof is as follows : If

$$ES(n) = x^2 \text{ for some integer } x > 1, \qquad (1)$$

then x must be an even integer. The following table gives the possible trailing digits of x and the corresponding trailing digits of x^2. Now, if n is of the form $n = k\times10^s+4$, then the trailing digit of $ES(k\times10^s+4)$ is 8 for all admissible values of k and s, and hence, it follows that the representation of $ES(k\times10^s+4)$ in the form (1) is not possible. Again, since the trailing digit of $ES(k\times10^s+6)$ is 2 for all admissible values of k and s, the numbers of the form $ES(k\times10^s+6)$ can also be not expressible in the form (1).

Trailing digit of x	Trailing digit of x^2
2	4
4	6
6	6
8	4

It thus remains to consider the cases when n is one of the three forms : (i) $n=k\times10^s$, (ii) $n=k\times10^s+2$, (iii) $n=k\times10^s+8$, (where k ($1\le k\le 9$) and s (≥ 1) are integers). Smith [4] conjectures that none of the terms of the sequence $\{ES(n)\}_{n=1}^{\infty}$ is a perfect power.

1.4 Smarandache Prime Product Sequence $\{PPS(n)\}_{n=1}^{\infty}$

Let $\{p_n\}_{n=1}^{\infty}$ be the (infinite) sequence of primes in their natural order, so that

$$p_1=2,\ p_2=3,\ p_3=5,\ p_4=7,\ p_5=11,\ p_6=13,\ \ldots .$$

The Smarandache prime product sequence, denoted by $\{PPS(n)\}_{n=1}^{\infty}$, is defined as follows (Smarandache [5]) :

$$PPS(n)=p_1p_2\ \ldots\ p_n+1,\ n\ge 1. \tag{1.4.1}$$

The following lemma gives a recurrence relation in connection with the sequence.

Lemma 1.4.1 **:** $PPS(n+1)=p_{n+1}\times PPS(n)-(p_{n+1}-1)$ for all $n\ge 1$.
Proof **:** By definition,

$$PPS(n+1)=p_1\,p_2\ \ldots\ p_n\,p_{n+1}+1=(p_1\,p_2\ \ldots\ p_n+1)p_{n+1}-p_{n+1}+1,$$

which now gives the desired relationship. ■

From Lemma 1.4.1, we get

Corollary 1.4.1 **:** $PPS(n+1)-PPS(n)=[PPS(n)-1](p_{n+1}-1)$ for all $n\ge 1$.

Lemma 1.4.2 **:** The values of PPS(n) satisfy the following inequalities :

(1) $PPS(n)<(p_n)^{n-1}$ for all $n\ge 4$,

(2) $PPS(n)<(p_n)^{n-2}$ for all $n\ge 7$,

(3) $PPS(n)<(p_n)^{n-3}$ for all $n\ge 10$,

(4) $PPS(n)<(p_{n+1})^{n-1}$ for all $n\ge 3$,

(5) $PPS(n)<(p_{n+1})^{n-2}$ for all $n\ge 6$,

(6) $PPS(n)<(p_{n+1})^{n-3}$ for all $n\ge 9$.

Proof **:** We prove parts (3) and (6) only, the proof of the other parts is similar.
To prove part (3) of the lemma, we note that the result is true for $n=10$, since

$$PPS(10)=6469693231<(p_{10})^7=29^7=17249876309.$$

Now, assuming the validity of the result for some integer k (≥ 10), and using Lemma 1.4.1, we see that,

$$PPS(k+1) = p_{k+1}\,PPS(k) - (p_{k+1} - 1) < p_{k+1}\,PPS(k)$$
$$< p_{k+1}(p_k)^{n-3} \quad \text{(by the induction hypothesis)}$$
$$< p_{k+1}(p_{k+1})^{n-3} = (p_{k+1})^{n-2},$$

where the last inequality follows from the fact that the sequence of primes, $\{p_n\}_{n=1}^{\infty}$, is strictly increasing in n (≥ 1). Thus, the result is true for $k+1$ as well.

To prove part (6) of the lemma, we note that the result is true for $n = 9$, since

$$PPS(9) = 223092871 < (p_{10})^6 = 29^6 = 594823321.$$

Now to appeal to the principle of induction, we assume that the result is true for some integer k (≥ 9). Then using Lemma 1.4.1, together with the induction hypothesis, we get

$$PPS(k+1) = p_{k+1}\,PPS(k) - (p_{k+1} - 1) < p_{k+1}\,PPS(k) < p_{k+1}(p_{k+1})^{k-3} = (p_{k+1})^{k-2}.$$

Thus the result is true for $k+1$.

All these complete the proof by induction. ■

The lemma below improves the results of Prakash [6] and Majumdar [7].

Lemma 1.4.3 **:** Each of PPS(1), PPS(2), PPS(3), PPS(4) and PPS(5) is prime, and for $n \geq 6$, PPS(n) has at most $n - 4$ prime factors, counting multiplicities.

Proof **:** Clearly, the first five terms of the sequence, namely, PPS(1) = 3, PPS(2) = 7, PPS(3) = 31, PPS(4) = 211 and PPS(5) = 2311, are all primes. Also, since

$$PPS(6) = 30031 = 59 \times 509,$$
$$PPS(7) = 510511 = 19 \times 97 \times 277,$$
$$PPS(8) = 9699691 = 347 \times 27953,$$

we see that the lemma is true for $6 \leq n \leq 8$.

Now, if p is a prime factor of PPS(n), then $p \geq p_{n+1}$. Therefore, if for some $n \geq 9$, PPS(n) has $n-3$ (or more) prime factors (counted with multiplicity), then $PPS(n) \geq (p_{n+1})^{n-3}$, and this is in contradiction with part (6) of Lemma 1.4.2.

Hence the lemma is established. ■

It has been proved that, for each $n \geq 1$, PPS(n) and PS(n+1) are relatively prime (see, Majumdar [8]). The following lemma gives the more general result.

Lemma 1.4.4 **:** For any $n \geq 1$ and $k \geq 1$, PPS(n) and PPS(n + k) can have at most $k - 1$ number of prime factors (counting multiplicities) in common.

Proof **:** For any $n \geq 1$ and $k \geq 1$,

$$PPS(n+k) - PPS(n) = p_1 p_2 \ldots p_n(p_{n+1} p_{n+2} \ldots p_{n+k} - 1). \tag{1.4.1}$$

If p is a common prime factor of PPS(n) and PPS(n + k), since $p \geq p_{n+k}$, it follows from (1.4.1) that p divides $(p_{n+1} p_{n+2} \ldots p_{n+k} - 1)$. Now if PPS(n) and PPS(n + k) have k (or more) prime factors in common, then the product of these common prime factors is greater than $(p_{n+k})^k$, which can not divide $p_{n+1} p_{n+2} \ldots p_{n+k} - 1 < (p_{n+k})^k$.

This contradiction proves the lemma. ■

Corollary 1.4.2 **:** For any integers n (≥ 1) and k (≥ 1), if all the prime factors of $p_{n+1}\,p_{n+2}\,\ldots\,p_{n+k} - 1$ are less than p_{n+k}, then PPS(n) and PPS(n + k) are relatively prime.

Proof **:** If p is any common prime factor of PPS(n) and PPS(n + k), then p divides $p_{n+1}\,p_{n+2}\,\ldots\,p_{n+k} - 1$. Also, for such p, $p > p_{n+k}$, contradicting the hypothesis of the corollary. Thus, if all the common prime factors of PPS(n) and PPS(n + k) are less than p_{n+k}, then PPS(n) and PPS(n + k) are relatively prime. ■

The following result has been proved by Prokash [6] and Majumdar [8]. Here we give a simpler proof.

Theorem 1.4.1 **:** For any $n \geq 1$, PPS(n) is never a square or higher power of an integer (>1).

Proof **:** Clearly, none of PPS(1), PPS(2), PPS(3), PPS(4) and PPS(5) can be expressed as powers of integers (by Lemma 1.4.3).

Now, if possible, let for some $n \geq 6$,

$$\mathrm{PPS}(n) = x^k \text{ for some integers } x\ (> 3) \text{ and } k\ (\geq 2). \tag{1}$$

Without loss of generality, we may assume that k is a prime (if k is a composite number, letting k = pr where p is prime, we have $\mathrm{PPS}(n) = (x^r)^p = N^p$, where $N = x^r$). By Lemma 1.4.3, $k \leq n - 4$ and so k cannot be greater than p_{n-5} ($k \geq p_{n-4} \Rightarrow k > n - 4$, since $p_n > n$ for all $n \geq 1$). Hence, k must be one of the primes $p_1, p_2, \ldots, p_{n-5}$. Also, since PPS(n) is odd, x must be odd. Let $x = 2y + 1$ for some integer $y > 0$. Then, from (1),

$$p_1 p_2 \ldots p_n = (2y + 1)^k - 1$$

$$= (2y)^k + \binom{k}{1}(2y)^{k-1} + \ldots + \binom{k}{k-1}(2y). \tag{2}$$

If k = 2, we see from (2), 4 divides $p_1 p_2 \ldots p_n$, which is absurd. On the other hand, for $k \geq 3$, since k divides $p_1 p_2 \ldots p_n$, it follows from (2) that k divides y, and consequently, k^2 divides $p_1 p_2 \ldots p_n$, which is impossible.

Hence, the representation of PPS(n) in the form (1) is not possible. ■

Using Corollary 1.4.1 and the fact that PPS(n + 1) − PPS(n) > 0, we get

$$\mathrm{PPS}(n+2) - \mathrm{PPS}(n) = [\mathrm{PPS}(n+2) - \mathrm{PPS}(n+1)] + [\mathrm{PPS}(n+1) - \mathrm{PPS}(n)]$$
$$> [\mathrm{PPS}(n+1) - 1](p_{n+2} - 1)$$
$$> 2[\mathrm{PPS}(n+1) - 1] \text{ for all } n \geq 1.$$

Hence,

$$\mathrm{PPS}(n+2) - \mathrm{PPS}(n) > \mathrm{PPS}(n+1) \text{ for all } n \geq 1. \tag{1.4.2}$$

The inequality (1.4.2) shows that no elements of the Smarandache prime product sequence satisfy the recurrence relation for Fibonacci (or, Lucas) numbers. This leads to the following theorem.

Theorem 1.4.2 **:** There are no numbers in the Smarandache prime product sequence that are Fibonacci (or Lucas) numbers (except for the trivial cases of PPS(1) = 3 = F(4) = L(2), PPS(2) = 7 = L(4)).

1.5 Smarandache Square Product Sequences $\{SPS_1(n)\}_{n=1}^{\infty}$ and $\{SPS_2(n)\}_{n=1}^{\infty}$

The Smarandache square product sequence of the first kind, denoted by $\{SPS_1(n)\}_{n=1}^{\infty}$, and the Smarandache square product sequence of the second kind, denoted by $\{SPS_2(n)\}_{n=1}^{\infty}$, are defined by (Russo [9])

$SPS_1(n) = (1^2)(2^2) \ \ldots \ (n^2) + 1, n \geq 1,$

$SPS_2(n) = (1^2)(2^2) \ \ldots \ (n^2) - 1, n \geq 1.$

Alternative expressions for $SPS_1(n)$ and $SPS_2(n)$ are

$$SPS_1(n) = (n!)^2 + 1, n \geq 1, \quad (1.5.1a)$$

$$SPS_2(n) = (n!)^2 - 1, n \geq 1. \quad (1.5.1b)$$

The first few terms of the sequence $\{SPS_1(n)\}_{n=1}^{\infty}$ are

$SPS_1(1) = 2,$

$SPS_1(2) = 5,$

$SPS_1(3) = 37,$

$SPS_1(4) = 577,$

$SPS_1(5) = 14401,$

$SPS_1(6) = 518401 = 13 \times 39877,$

$SPS_1(7) = 25401601 = 101 \times 251501,$

$SPS_1(8) = 1625702401 = 17 \times 95629553,$

$SPS_1(9) = 131681894401,$

of which the first five terms are prime numbers.

The first few terms of the sequence $\{SPS_2(n)\}_{n=1}^{\infty}$ are

$SPS_2(1) = 0,$

$SPS_2(2) = 3,$

$SPS_2(3) = 35,$

$SPS_2(4) = 575,$

$SPS_2(5) = 14399,$

$SPS_2(6) = 518399,$

$SPS_2(7) = 25401599,$

$SPS_2(8) = 1625702399,$

$SPS_2(9) = 131681894399,$

of which, disregarding the first term, the second term is prime, and the remaining terms of the sequence are all composite numbers.

Theorem 1.5.1 proves that, for any $n \geq 1$, neither of $SPS_1(n)$ and $SPS_2(n)$ is a square of an integer (> 1). Theorem 1.5.2 shows that no terms in these sequences can be cubes.

Theorem 1.5.1 **:** For any $n \geq 1$, none of $SPS_1(n)$ and $SPS_2(n)$ is a square of an integer (>1).

Proof **:** If possible, let

$$SPS_1(n) \equiv (n\,!)^2 + 1 = x^2 \text{ for some integers } n \geq 1,\ x > 1,$$

$$SPS_2(n) \equiv (n\,!)^2 - 1 = y^2 \text{ for some integers } n \geq 1,\ y > 1.$$

But these contradict Corollary 1.1.1(1) and Corollary 1.1.1(2) respectively. ■

Theorem 1.5.2 **:** For any integer $n \geq 1$, none of the $SPS_1(n)$ and $SPS_2(n)$ is a cube or higher power of an integer (>1).

Proof **:** is by contradiction. Let

$$SPS_1(n) \equiv (n!)^2 + 1 = y^m \text{ for some integers } n \geq 1,\ y > 1,\ m \geq 3,$$

$$SPS_2(n) \equiv (n!)^2 - 1 = z^s \text{ for some integer } n \geq 1,\ z \geq 1,\ s \geq 3.$$

We then have contradictions to Corollary 1.1.2 and Corollary 1.1.7 respectively. ■

Lemma 1.5.1 **:** For any integer $n \geq 1$,

(1) $SPS_1(n+1) = (n+1)^2\, SPS_1(n) - n(n+2)$,

(2) $SPS_2(n+1) = (n+1)^2\, SPS_2(n) - n(n+2)$.

Proof **:** The proof is for part (1) only, and is given below.

$$SPS_1(n+1) = [(n+1)!]^2 + 1 = (n+1)^2\,[(n!)^2 + 1] - (n+1)^2 + 1.\ ■$$

Lemma 1.5.2 **:** For any integer $n \geq 1$,

(1) $SPS_1(n+2) - SPS_1(n) > SPS_1(n+1)$,

(2) $SPS_2(n+2) - SPS_2(n) > SPS_2(n+1)$.

Proof **:** It is straightforward, using (1.5.1), to prove that

$$SPS_1(n+2) - SPS_1(n) = SPS_2(n+2) - SPS_2(n) = (n!)^2\,[(n+1)^2(n+2)^2 - 1].$$

Some algebraic manipulations give the desired inequalities. ■

Lemma 1.5.2 can be used to prove the following results.

Theorem 1.5.3 **:** (Except for the trivial cases, $SPS_1(1)=2=F(3)$, $SPS_1(2)=5=F(5)$) there are no numbers of the Smarandache square product sequence of the first kind that are Fibonacci (or Lucas) numbers.

Theorem 1.5.4 **:** (Except for the trivial case, $SPS_2(2)=3=F(4)=L(2)$) there are no numbers of the Smarandache square product sequence of the second kind that are Fibonacci (or Lucas) numbers.

The question raised by Iacobescu [10] is : How many terms of the sequence $\{SPS_1(n)\}_{n=1}^{\infty}$ are prime? The following theorem, due to Le [11], gives a partial answer to this question.

Theorem 1.5.5 **:** If n (> 2) is an even integer such that $2n+1$ is prime, then $SPS_1(n)$ is not a prime.

Russo [9], based on values of $SPS_1(n)$ and $SPS_2(n)$ for $1 \leq n \leq 20$, conjectures that each of the sequences contains only a finite number of primes.

1.6 Smarandache Higher Power Product Sequences $\{HPPS_1(n)\}_{n=1}^{\infty}$, $\{HPPS_2(n)\}_{n=1}^{\infty}$

Let m (>3) be a fixed integer. The Smarandache higher power product sequence of the first kind, denoted by $\{HPPS_1(n)\}_{n=1}^{\infty}$, and the Smarandache higher power product sequence of the second kind, denoted by $\{HPPS_2(n)\}_{n=1}^{\infty}$, are defined as follows :

$$HPPS_1(n) = (1^m)(2^m) \ldots (n^m) + 1, \; n \geq 1,$$

$$HPPS_2(n) = (1^m)(2^m) \ldots (n^m) - 1, \; n \geq 1.$$

Alternative expressions for $HPPS_1(n)$ and $HPPS_2(n)$ are

$$HPPS_1(n) = (n!)^m + 1, \; n \geq 1, \quad (1.6.1a)$$

$$HPPS_2(n) = (n!)^m - 1, \; n \geq 1. \quad (1.6.1b)$$

Lemma 1.6.1 **:** For any integer $n \geq 1$,

(1) $HPPS_1(n+1) = (n+1)^m HPPS_1(n) - [(n+1)^m + 1]$,

(2) $HPPS_2(n+1) = (n+1)^m HPPS_2(n) + [(n+1)^m + 1]$.

Theorem 1.6.1 **:** For any integer $n \geq 1$, none of $HPPS_1(n)$ and $HPPS_2(n)$ is a square of an integer (> 1).

Proof **:** If possible, let

$$HPPS_1(n) \equiv (n!)^m + 1 = x^2 \text{ for some integer } x > 1.$$

But, by Corollary 1.1.6, this permits no solution if m>3.

Again, letting, for some integer $n \geq 2$ ($HPPS_2(1) = 0$)

$$HPPS_2(n) \equiv (n!)^m - 1 = y^2 \text{ for some integer } y \geq 1,$$

we have a contradiction to Corollary 1.1.3.

Thus, we reach to a contradiction in either case, establishing the theorem. ■

The following two theorems are due to Le [12, 13].

Theorem 1.6.2 **:** If m is not a number of the form $m = 2^k$ for some integer $k \geq 1$, then the sequence $\{HPPS_1(n)\}_{n=1}^{\infty}$ contains only one prime, namely, $HPPS_1(1) = 2$.

Theorem 1.6.3 **:** If both m and $2^m - 1$ are primes, then the sequence $\{HPPS_2(n)\}_{n=1}^{\infty}$ contains only one prime, namely, $HPPS_2(2) = 2^m - 1$; otherwise, the sequence contains no prime.

Remark 1.6.1 **:** We have defined the Smarandache higher power product sequences under the restriction that m > 3, and under such restriction, as has been proved in Theorem 1.6.1, none of $HPPS_1(n)$ and $HPPS_2(n)$ is a square of an integer (>1) for any $n \geq 1$. However, if m = 3, the situation is a little bit different : For any $n \geq 1$, $HPPS_2(n) = (n!)^3 - 1$ still cannot be a perfect square of an integer (>1), by Corollary 1.1.3, but since $HPPS_1(n) = (n!)^3 + 1$, we see that $HPPS_1(2) = (2!)^3 + 1 = 3^2$, that is, $HPPS_1(2)$ is a perfect square. However, this is the only term of the sequence $\{HPPS_1(n)\}_{n=1}^{\infty}$ that can be expressed as a perfect square.

1.7 Smarandache Permutation Sequence $\{PS(n)\}_{n=1}^{\infty}$

The Smarandache permutation sequence, denoted by $\{PS(n)\}_{n=1}^{\infty}$, is defined by (Dumitrescu and Seleacu [14])

$$PS(n) = \overline{135 \ldots (2n-1)(2n)(2n-2) \ldots 42}, \ n \geq 1. \qquad (1.7.1)$$

The first few terms of the sequence are

12, 1342, 135642, 13578642, 13579108642, …,

all of which are even numbers.

For the Smarandache permutation sequence, the question raised (Dumitrescu and Seleacu [14]) is : *Is there any perfect power among these numbers*?

Smarandache conjectures that there are none. In Theorem 1.7.1, we prove the conjecture in the affirmative. To prove the theorem, we need the following results.

Lemma 1.7.1 **:** For $n \geq 2$, PS(n) is of the form $2(2k+1)$ for some integer $k > 1$.

Proof **:** Since for $n \geq 2$,

$$PS(n) = \overline{135 \ldots (2n-1)(2n)(2n-2) \ldots 42}, \qquad (1.7.2)$$

we see that PS(n) is even and after division by 2, the last digit of the quotient is 1. ■

An immediate consequence of the above lemma is the following.

Corollary 1.7.1 **:** 2^k divides PS(n) (for some integer $n \geq 2$) if and only if $k = 1$.

Theorem 1.7.1: For $n \geq 1$, PS(n) is not a perfect power.

Proof **:** The result is clearly true for n = 1, since $PS(1) = 3\times2^2$ is not a perfect power. The proof for the case $n \geq 2$ is by contradiction.

Let, for some integer $n \geq 2$,

$$PS(n) = x^k \text{ for some integers } x > 1, k \geq 2.$$

Since PS(n) is even, so is x. Let $x = 2y$ for some integer $y > 1$. Then,

$$PS(n) = (2y) = 2^k y^k,$$

which shows that 2^k divides PS(n), contradicting Corollary 1.7.1. ■

To get more insight into the numbers of the Smaradache permutation sequence, we define a new sequence, called the *reverse even sequence*, and denoted by $\{RES(n)\}_{n=1}^{\infty}$, as follows :

$$RES(n) = \overline{(2n)(2n-2) \ldots 42}, \ n \geq 1. \qquad (1.7.3)$$

The first few terms of the sequences are 2, 42, 642, 8642, 108642, 12108642, ….

We note that, for all $n \geq 1$,

$$RES(n+1) = \overline{(2n+2)(2n)(2n-2) \ldots 42}$$
$$= (2n+2)10^s + RES(n) \text{ for some integer } s \geq n, \qquad (1.7.4)$$

where, more precisely,

$$s = \text{number of digits in RES(n)}.$$

Thus, for example,

$$RES(4) = 8\times10^3 + RES(3),\ RES(5) = 10\times10^4 + RES(4),\ RES(6) = 12\times10^6 + RES(5).$$

Lemma 1.7.2 **:** For any integer $n \geq 1$, 4 divides $[RES(n+1) - RES(n)]$.

Proof **:** Since from (1.7.4),

$$RES(n+1) - RES(n) = (2n+2)10^s \text{ for some integer } s\ (\geq n \geq 1),$$

the result follows. ■

Lemma 1.7.3 **:** For $n \geq 1$, RES(n) is of the form $2(2k+1)$ for some integer $k \geq 0$.

Proof **:** The proof is by induction on n. The result is true for $n = 1$. So, we assume that the result is true for some n, that is,

$$RES(n) = 2(2k+1) \text{ for some integer } k \geq 0.$$

But, by virtue of Lemma 1.7.2,

$$RES(n+1) - RES(n) = 4k' \text{ for some integer } k' > 0,$$

which, together with the induction hypothesis, gives,

$$RES(n+1) = 4k' + RES(n) = 4(k + k') + 2.$$

Thus, the result is true for $n + 1$ as well, completing the proof. ■

Lemma 1.7.4 **:** 3 divides RES(3n) if and only if 3 divides RES(3n – 1).

Proof **:** The result follows by noting that

$$RES(3n) = (6n)\times10^s + RES(3n - 1) \text{ for some integer } s \geq n.\ ■$$

By repeated application of (1.7.4), we get successively

$$\begin{aligned} RES(n+3) &= (2n+6)10^s + RES(n+2) \text{ for some integer } s \geq n+2 \\ &= (2n+6)10^s + (2n+4)10^t + RES(n+1) \text{ for some integer } t \geq n+1 \\ &= (2n+6)10^s + (2n+4)10^t + (2n+2)10^u + RES(n) \text{ for some integer } u \geq n, \end{aligned} \quad (1.7.5)$$

so that, for some integers s, t and u with $s > t > u \geq n \geq 1$,

$$RES(n+3) - RES(n) = (2n+6)10^s + (2n+4)10^t + (2n+2)10^u. \quad (1.7.6)$$

Lemma 1.7.5 **:** 3 divides $[RES(n+3) - RES(n)]$ for any integer $n \geq 1$.

Proof **:** is evident from (1.7.6), since

$$\begin{aligned} 3 \mid\ & (2n+6)10^s + (2n+4)10^t + (2n+2)10^u \\ &= 10^u[(2n+6)(10^{s-u} + 10^{t-u} + 1) - 2(10^{s-u} + 2)].\ ■ \end{aligned}$$

Corollary 1.7.2 **:** 3 divides RES(3n) for any integer $n \geq 1$.

Proof **:** The result is true for n = 1, since RES(3) = 642 is divisible by 3. Now, assuming the validity of the result for n, so that 3 divides RES(3n), we get, from Lemma 1.7.5, 3 divides RES(3n + 3) = RES(3(n + 1)), so that the result is true for n + 1 as well.

This completes the proof by induction. ■

Corollary 1.7.3 **:** 3 divides RES(3n – 1) for any integer $n \geq 1$.

Proof **:** follows from Lemma 1.7.4, together with Corollary 1.7.2. ■

Corollary 1.7.4 **:** For any integer $n \geq 0$, 3 does not divide RES(3n + 1).

Proof **:** Clearly, the result is true for n = 0. For $n \geq 1$, from (1.7.4),

$$RES(3n+1) = (6n+2)10^s + RES(3n) \text{ for some integer } s \geq 3n.$$

Now, $3 \mid RES(3n)$ (by Corollary 1.7.2) but 3 does not divide (6n + 2). Hence the result. ■

Lemma 1.7.6 **:** RES(n + 2) − RES(n) > RES(n + 1) for any integer $n \geq 1$.

Proof **:** Using (1.7.5), we see that, for all $n \geq 1$,

$$RES(n+2) - RES(n) = (2n+4)10^t + (2n+2)10^u,$$

where t is the number of digits in RES(n + 1). Hence the lemma. ■

PS(n) can be expressed in terms of OS(n) and RES(n) as follows : For any $n \geq 1$,

$$PS(n) = 10^s \times OS(n) + RES(n) \text{ for some integer } s \geq n, \quad (1.7.7)$$

where, more precisely,

s = number of digits in RES(n).

From (1.7.7), together with Lemma 1.7.3, we observe that, for $n \geq 2$, (since 4 divides 10^s for $s \geq n \geq 2$), PS(n) is of the form 4k + 2 for some integer k > 1 (see Lemma 1.7.1).

Lemma 1.7.7 **:** 3 divides PS(n) if and only if 3 divides PS(n + 3).

Proof **:** follows by virtue of Lemma 1.2.1 and Lemma 1.7.5. ■

Lemma 1.7.8 **:** 3 divides PS(3n) for any integer $n \geq 1$.

Proof **:** follows by virtue of Lemma 1.2.2 and Corollary 1.7.2, since

$$PS(3n) = 10^s \times OS(3n) + RES(3n) \text{ for some integer } s \geq 3n. \blacksquare$$

Lemma 1.7.9 **:** 3 divides PS(3n − 2) for any integer $n \geq 1$.

Proof **:** Since 3 divides PS(1) = 12, the result is true for n = 1. To prove by induction, we assume that the result is true for some n, that is, 3 divides PS(3n − 2). But, then, by Lemma 1.7.7, 3 divides PS(3n + 1), showing that the result is true for n + 1 as well. ■

Lemma 1.7.10 **:** For any integer $n \geq 1$, PS(n + 2) − PS(n) > PS(n + 1).

Proof **:** By repeated application of (1.7.7), we get

$$PS(n+1) = 10^t \times OS(n+1) + RES(n+1) \text{ for some integer } t \geq n+1,$$

$$PS(n+2) = 10^u \times OS(n+2) + RES(n+2) \text{ for some integer } u \geq n+2,$$

where $u > t > s \geq n$. Therefore,

$$\begin{aligned} PS(n+2) - PS(n) &= [10^u \times OS(n+2) - 10^s \times OS(n)] + [RES(n+2) - RES(n)] \\ &> 10^u[OS(n+2) - OS(n)] + [RES(n+2) - RES(n)] \\ &> 10^t \times OS(n+1) + RES(n+1) \\ &= PS(n+1), \end{aligned}$$

where the last inequality follows from (1.2.6), Lemma 1.7.6, and the fact that $10^u > 10^t$. ■

Lemma 1.7.10 can be used to prove the following result (see also Le [15]).

Theorem 1.7.2 **:** There are no numbers in the Smarandache permutation sequence that are Fibonacci (or, Lucas) numbers.

1.8 Smarandache Circular Sequence $\{CS(n)\}_{n=1}^{\infty}$

The Smarandache circular (or, consecutive) sequence, denoted by $\{CS(n)\}_{n=1}^{\infty}$, is obtained by repeatedly concatenating the positive integers (Dumitrescu and Seleacu [14]).

The first few terms of the sequence are

$$1, 12, 123, 1234, 12345, 123456, \ldots.$$

The n–th tem of the circular sequence is given

$$CS(n) = \overline{123\ldots(n-1)(n)}, n \geq 1. \tag{1.8.1}$$

Since

$$CS(n+1) = \overline{123\ \ldots\ (n-1)(n)(n+1)},$$

we see that, for any integer $n \geq 1$,

$$CS(n+1) = 10^s \times CS(n) + (n+1) \text{ for some integer } s \geq 1, CS(1) = 1; \tag{1.8.2}$$

where, more precisely,

$$s = \text{number of digits in } (n+1).$$

Thus, for example,

$$CS(9) = 10 \times CS(8) + 9,$$

and

$$CS(10) = 10^2 \times CS(9) + 10.$$

From (1.8.2), we get the following result :

Lemma 1.8.1 **:** For any integer $n \geq 1$, $CS(n+1) - CS(n) > 9 \times CS(n)$.

Using Lemma 1.8.1, we get, following the proof of Corollary 1.2.1,

$$CS(n+2) - CS(n) > 9[CS(n+1) + CS(n)] \text{ for all } n \geq 1. \tag{1.8.3}$$

Thus,

$$CS(n+2) - CS(n) > CS(n+1), n \geq 1. \tag{1.8.4}$$

Based on computer search for Fibonacci (and Lucas) numbers from 12 up through $CS(2999) = 123 \ldots 29982999$, Asbacher [1] conjectures that (except for the trivial case, $CS(1)=1=F(1)=F(2)=L(1)$), there are no Fibonacci (and Lucas) numbers in the Smarandache circular sequence. The following theorem confirms the conjectures of Ashbacher.

Theorem 1.8.1 **:** There are no Fibonacci (and Lucas) numbers in the Smarandache circular sequence (except for the trivial cases of $CS(1)=1=F(1)=F(2)=L(1)$, $CS(3)=123=L(10)$).

Proof **:** is evident from (1.8.4). ■

Remark 1.8.1 **:** As has been pointed out by Ashbacher [1], CS(3) is a Lucas number. However, note that, $CS(3) \neq CS(2) + CS(1)$.

Lemma 1.8.2 **:** Let 3 divide n. Then, 3 divides CS(n) if and only if 3 divides $CS(n-1)$.

Proof **:** follows readily from (1.8.2). ■

By repeated application of (1.8.2), we get,

$$\begin{aligned} CS(n+3) &= 10^s \times CS(n+2) + (n+3) \text{ for some integer } s \geq 1 \\ &= 10^s[10^t \times CS(n+1) + (n+2)] + (n+3) \text{ for some integer } t \geq 1 \\ &= 10^{s+t}[10^u \times CS(n) + (n+1)] + (n+2)10^s + (n+3) \text{ for some integer } u \geq 1 \\ &= 10^{s+t+u} \times CS(n) + (n+1)10^{s+t} + (n+2)10^s + (n+3), \end{aligned} \tag{1.8.5}$$

where $s \geq t \geq u \geq 1$.

Lemma 1.8.3 **:** 3 divides CS(n) if and only if 3 divides CS(n + 3).

Proof **:** follows from Lemma 1.1.1, together with (1.8.5), since

$$3 \mid [(n+1)10^{s+t} + (n+2)10^s + (n+3)] = [(n+1)(10^{s+t} + 10^s + 1) + (10^s + 2)]. \blacksquare$$

Lemma 1.8.4 **:** 3 divides CS(3n) for any integer $n \geq 1$.

Proof **:** The proof is by induction on n. The result is clearly true for $n = 1$, since 3 divides CS(3) = 123. So, we assume that the result is true for some n, that is, we assume that 3 divides CS(3n) for some n. But then, by Lemma 1.8.3, 3 divides $CS(3n+3) = CS(3(n+1))$, showing that the result is true for n + 1 as well, completing induction. ■

Corollary 1.8.1 **:** 3 divides $CS(3n-1)$ for any integer $n \geq 1$.

Proof **:** From (1.8.2), for $n \geq 1$,

$$CS(3n) = 10^s \times CS(3n-1) + 3n \text{ for some integer } s \geq 1.$$

Since, by Lemma 1.8.4, 3 divides CS(3n), the result follows. ■

Corollary 1.8.2 **:** 3 does not divide CS(3n + 1) for any integer $n \geq 0$.

Proof **:** For $n = 0$, $CS(1) = 1$ is not divisible by 3.

By Lemma 1.8.4, 3 divides CS(3n) for any integer $n \geq 1$. Since (from (1.8.2)),

$$CS(3n+1) = 10^s \times CS(3n) + (3n+1),$$

and since 3 does not divide (3n + 1), we get desired the result. ■

Lemma 1.8.5 **:** For any integer $n \geq 1$, 5 divides CS(5n).

Proof **:** For $n \geq 1$, from (1.8.2),

$$CS(5n) = 10^s \times CS(5n-1) + 5n \text{ for some integer } s \geq 1.$$

Clearly, the r.h.s. is divisible by 5. Hence, 5 divides CS(5n). ■

For the Smarandache circular sequence, the question is : How many terms of the sequence are prime?

Fleuren [16] gives a table of prime factors of CS(n) for $n = 1(1)200$, which shows that none of these numbers is a prime. In the Editorial Note following the paper of Stephan [17], it is mentioned that, using a supercomputer, no prime has been found in the first 3072 terms of the Smarandache circular sequence. This gives rise to the conjecture that there is no prime in the Smarandache circular sequence. This conjecture still remains to be resolved. We note that, in order to check for prime numbers in the Smarandache circular sequence, it is sufficient to check the terms of the form CS(3n + 1) ($n \geq 1$), where 3n + 1 is odd and is not divisible by 5.

1.9 Smarandache Reverse Sequence $\{\mathbf{RS(n)}\}_{n=1}^{\infty}$

The Smarandache reverse sequence, denoted by $\{RS(n)\}_{n=1}^{\infty}$, is the sequence of numbers formed by concatenating the consecutive integers on the left side, starting with $RS(1) = 1$.

The first few terms of the sequence are (Ashbacher [1])

$$1, 21, 321, 4321, 54321, 654321, \ldots,$$

and in general, the $n-$th term is given by

$$RS(n) = \overline{n(n-1)\ldots 21}, n \geq 1. \qquad (1.9.1)$$

From (1.9.1), we see that, for all $n \geq 1$,

$$RS(n+1) = \overline{(n+1)n(n-1)\ \ldots\ 21},\ n \geq 1,$$

$$= (n+1)10^s + RS(n) \text{ for some integer } s \geq n \text{ (with } RS(1) = 1), \qquad (1.9.2)$$

where, more precisely,

$$s = \text{number of digits in } RS(n).$$

Thus, for example,

$$RS(9) = 9\times 10^8 + RS(8),\ RS(10) = 10\times 10^9 + RS(9),\ RS(11) = 11\times 10^{11} + RS(10).$$

By repeated application of (1.9.2), we get, for all $n \geq 1$,

$$RS(n+3) = (n+3)10^s + RS(n+2) \text{ for some integer } s \geq n+2$$

$$= (n+3)10^s + (n+2)10^t + RS(n+1) \text{ for some integer } t \geq n+1$$

$$= (n+3)10^s + (n+2)10^t + (n+1)10^u + RS(n) \text{ for some integer } u \geq n, \quad (1.9.3)$$

where $s > t > u \geq n$. Thus,

$$RS(n+3) = 10^u[(n+3)10^{s-u} + (n+2)10^{t-u} + (n+1)] + RS(n). \qquad (1.9.4)$$

Lemma 1.9.1 **:** For any integer $n \geq 1$,

(1) 4 divides $[RS(n+1) - RS(n)]$, **(2)** 10 divides $[RS(n+1) - RS(n)]$.

Proof **:** For all $n \geq 1$, from (1.9.2),

$$RS(n+1) - RS(n) = (n+1)10^s \text{ (with } s \geq n),$$

and the r.h.s. is divisible by both 4 and 10. ■

Corollary 1.9.1 **:** For any $n \geq 2$, $RS(n)$ is of the form $4k+1$ for some integer $k \geq 0$.

Proof **:** Since $RS(2) = 21 = 4\times 5 + 1$, the result is clearly true for $n = 2$. So, we assume the validity of the result for n, that is, we assume that $RS(n) = 4k+1$ for integer $k \geq 1$.

Now, by Lemma 1.9.1, for some integer $k' \geq 1$,

$$RS(n+1) - RS(n) = 4k' \quad \Rightarrow \quad RS(n+1) = RS(n) + 4k' = 4(k+k') + 1,$$

where the expression on the right follows by virtue of the induction hypothesis.

Thus, the result is true for $n+1$ as well, completing induction. ■

***Lemma 1.9.2* :** Let $3 \mid n$ for some n (≥ 2). Then, $3 \mid RS(n)$ if and only if $3 \mid RS(n-1)$.
***Proof* :** follows immediately from (1.9.2). ■

***Lemma 1.9.3* :** 3 divides $[RS(n+3) - RS(n)]$ for any integer $n \geq 1$.
***Proof* :** is immediate from (1.9.4), together with Lemma 1.1.1. ■

A consequence of Lemma 1.9.3 is the following.

***Corollary 1.9.2* :** 3 divides RS(n) if and only if 3 divides RS(n+3).

Using Corollary 1.9.2, the following result can be established by induction on n.

***Corollary 1.9.3* :** 3 divides RS(3n) for all $n \geq 1$.

***Corollary 1.9.4* :** 3 divides RS(3n − 1) for any integer $n \geq 1$.
***Proof* :** follows from Corollary 1.9.3, together with Lemma 1.9.2. ■

***Lemma 1.9.4* :** 3 does not divide RS(3n+1) for any integer $n \geq 0$.
***Proof* :** The result is true for $n = 0$. For $n \geq 1$, by (1.9.2),

$$RS(3n+1) = (3n+1)10^s + RS(3n).$$

This gives the desired result, since 3 divides RS(3n) but 3 does not divide (3n+1). ■

***Lemma 1.9.5* :** For any integer $n \geq 1$, $RS(n+1) > 2RS(n)$.
***Proof* :** Using (1.9.2), we see that

$$RS(n+1) = (n+1)10^s + RS(n) > 2RS(n) \text{ if and only if } RS(n) < (n+1)10^s,$$

which is true since RS(n) is an s – digit number while 10^s is an (s+1) – digit number. ■

***Corollary 1.9.5* :** For any integer $n \geq 1$, $RS(n+2) - RS(n) > RS(n+1)$.
***Proof* :** Using (1.9.3), we have (for some integers t and u with $t > u$)

$$\begin{aligned} RS(n+2) - RS(n) &= [RS(n+2) - RS(n+1)] + [RS(n+1) - RS(n)] \\ &= [(n+2)10^t - (n+1)10^u] + 2[RS(n+1) - RS(n)] \\ &> 2[RS(n+1) - RS(n)] \\ &> RS(n+1), \text{ by Lemma 1.9.5.} \end{aligned}$$

This gives the desired inequality. ■

***Theorem 1.9.1* :** There are no numbers in the Smarandache reverse sequence that are Fibonacci or Lucas numbers (except for RS(1) = 1 = F(1) = F(2) = L(1), RS(2) = 21 = F(8)).
***Proof* :** follows from Corollary 1.9.5. ■

The following expression for RS(n) is due to Alexander [18].

$$RS(n) = 1 + \sum_{i=2}^{n} i * 10^{\sum_{j=1}^{i-1}(1+[\log j])} \qquad \text{([x] denotes the integer part of x).}$$

For the Smarandache reverse sequence, the question is : How many terms of the sequence are prime? It is conjectured, based on the first 2739 terms, that RS(82) is the only prime in the sequence (see Stephan [17]). The conjecture still remains to be resolved.

1.10 Smarandache Symmetric Sequence $\{SS(n)\}_{n=1}^{\infty}$

The Smarandache symmetric sequence, denoted by $\{SS(n)\}_{n=1}^{\infty}$, is defined by (Ashbacher [1])

1, 11, 121, 12321, 1234321, 123454321, 12345654321, ….

More precisely, the n–th term of the Smarandache symmetric sequence is

$$SS(n)=\overline{12\ ...\ (n-2)(n-1)(n-2)\ ...\ 21},\ n\geq 3;\quad SS(1)=1,\ SS(2)=11. \tag{1.10.1}$$

The numbers in the Smarandache symmetric sequence can be expressed in terms of the numbers of the Smarandache circular sequence (in §1.8) and the Smarandache reverse sequence (in §1.9) as follows : For all n≥3,

$$SS(n)=10^s\times CS(n-1)\ +\ RS(n-2) \text{ for some integer } s\geq 1, \tag{1.10.2}$$

with $SS(1)=1$, $SS(2)=11$, where more precisely,

$s=$ number of digits in $RS(n-2)$.

Thus, for example, $SS(3)=10\times CS(2)+RS(1)$, $SS(4)=10^2\times CS(3)+RS(2)$.

Lemma 1.10.1 **:** 3 divides $SS(3n+1)$ for any integer $n\geq 1$.

Proof **:** Let n (≥ 1) be any arbitrary but fixed number. Then, from (1.10.2),

$$SS(3n+1)=10^s\times CS(3n)\ +\ RS(3n-1).$$

Now, by Lemma 1.8.4, 3 divides CS(3n), and by Corollary 1.9.4, 3 divides $RS(3n-1)$. Therefore, 3 divides $SS(3n+1)$. ■

Lemma 1.10.2 **:** For any integer $n\geq 1$,

(1) 3 does not divide SS(3n), **(2)** 3 does not divide $SS(3n+2)$.

Proof **:** Using (1.10.2), we see that

$$SS(3n)=10^s\times CS(3n-1)\ +\ RS(3n-2),\ n\geq 1.$$

By Corollary 1.8.1, 3 divides $CS(3n-1)$, and by Lemma 1.9.4, 3 does not divide $RS(3n-2)$. Hence, CS(3n) cannot be divisible by 3.

Again, since

$$SS(3n+2)=10^s\times CS(3n+1)\ +\ RS(3n),\ n\geq 1,$$

and since 3 does not divide $CS(3n+1)$ (by Corollary 1.8.2) and 3 divides RS(3n) (by Corollary 1.9.3), it follows that $SS(3n+2)$ is not divisible by 3. ■

Using (1.8.4) and Corollary 1.9.5, we can prove the following lemma. The proof is similar to that used in proving Lemma 1.7.10, and is omitted here.

Lemma 1.10.3 **:** For any integer $n\geq 1$, $SS(n+2)\ -\ SS(n)>SS(n+1)$.

By virtue of the inequality in Lemma 1.10.3, we have the following.

Theorem 1.10.1 **:** (Except for the trivial cases, $SS(1)=1=F(1)=L(1)$, $SS(2)=11=L(5)$), there are no members of the Smarandache symmetric sequence that are Fibonacci (or, Lucas) numbers.

The following lemma gives the expression of the difference $SS(n+1) - SS(n)$ in terms of the difference $CS(n) - CS(n-1)$.

Lemma 1.10.4 **:** $SS(n+1) - SS(n) = 10^{s+t}[CS(n) - CS(n-2)]$ for any integer $n \geq 3$, where

s = number of digits in $RS(n-2)$,

$s+t$ = number of digits in $RS(n-1)$.

Proof **:** By (1.10.2), for $n \geq 3$,

$$SS(n) = 10^s \times CS(n-1) + RS(n-2),$$

$$SS(n+1) = 10^{s+t} \times CS(n) + RS(n-1),$$

so that

$$SS(n+1) - SS(n) = 10^s[10^t \times CS(n) - CS(n-1)] + [RS(n-1) - RS(n-2)]$$

$$= 10^s[10^t \times CS(n) - CS(n-1) + (n-1)], \tag{1}$$

where the last equality follows from (1.9.2).
But,

$$t = \begin{cases} 1, & \text{if } 2 \leq n-1 \leq 9 \\ m+1, & \text{if } 10^m \leq n-1 \leq 10^{m+1} - 1 \text{ for some integer } m \geq 1 \end{cases}$$

= number of digits in $(n - 1)$.

Therefore, by (1.8.2)

$$CS(n-1) = 10^t \times CS(n-2) + (n-1),$$

so that

$$CS(n-1) - (n-1) = 10^t \times CS(n-2).$$

Now, plugging in this expression in (1), we get the desired result. ■

We observe that $SS(2) = 11$ is prime; the next eight terms of the Smarandache symmetric sequence are composite numbers and squares :

$SS(3) = 121 = 11^2$,

$SS(4) = 12321 = (3 \times 37)^2 = 111^2$,

$SS(5) = 1234321 = (11 \times 101)^2 = 1111^2$,

$SS(6) = 123454321 = (41 \times 271)^2 = 11111^{2,}$

$SS(7) = 12345654321 = (3 \times 7 \times 11 \times 13 \times 37)^2 = 111111^2$,

$SS(8) = 1234567654321 = 1111111^2$,

$SS(9) = 123456787654321 = 11111111^2$,

$SS(10) = 12345678987654321 = (9 \times 37 \times 333667)^2 = 111111111^2$.

For the Smarandache symmetric sequence, the question is : How many terms of the sequence are prime? The question still remains to be answered.

1.11 Smarandache Pierced Chain Sequence $\{PCS(n)\}_{n=1}^{\infty}$

The Smarandache pierced chain sequence, denoted by $\{PCS(n)\}_{n=1}^{\infty}$, is defined by (Kashihara [19], Ashbacher [1])

$$101, 1010101, 10101010101, 101010101010101, \ldots,$$

which is obtained by successively concatenating the string 0101 to the right of the preceding terms of the sequence, starting with $SPC(1) = 101$.

We first observe that the elements of the Smarandache pierced chain sequence, $\{PCS(n)\}_{n=1}^{\infty}$, satisfy the following recurrence relation :

$$SPC(n+1) = 10^4 \times SPC(n) + 101, \ n \ge 1; \ SPC(1) = 101. \qquad (1.11.1)$$

We can now prove the following result, giving an explicit expression of SPC(n).

Lemma 1.11.1 **:** The elements of the sequence $\{PCS(n)\}_{n=1}^{\infty}$ are

$$101, 101(10^4+1), 101(10^8+10^4+1), 101(10^{12}+10^8+10^4+1), \ldots,$$

and in general,

$$SPC(n) = 101[10^{4(n-1)} + 10^{4(n-2)} + \ldots + 10^4 + 1], \ n \ge 1. \qquad (1.11.2)$$

Proof **:** The proof of (1.11.2) is by induction on n. The result is clearly true for n = 1. So, we assume that the result is true for some n.
Now, from (1.11.1), together with the induction hypothesis, we see that

$$\begin{aligned} SPC(n+1) &= 10^4 \times SPC(n) + 101 \\ &= 10^4[101(10^{4(n-1)} + 10^{4(n-2)} + \ldots + 10^4 + 1)] + 101 \\ &= 101(10^{4n} + 10^{4(n-1)} + \ldots + 10^4 + 1), \end{aligned}$$

which shows that the result is true for n + 1. ■

From Lemma 1.11.1, we see that SPC(n) is divisible by 101 for all n≥1. Thus, from the sequence $\{PCS(n)\}_{n=1}^{\infty}$, we can form another sequence, namely, the sequence $\left\{\frac{PCS(n)}{101}\right\}_{n=1}^{\infty}$, whose explicit form is given in the following corollary.

Corollary 1.11.1 **:** The elements of the sequence $\left\{\frac{PCS(n)}{101}\right\}_{n=1}^{\infty}$ are

$$1, x+1, x^2+x+1, x^3+x^2+x+1, \ldots,$$

where $x \equiv 10^4$; and in general,

$$\frac{PCS(n)}{101} = x^{n-1} + x^{n-2} + \ldots + 1, \ n \ge 1, \qquad (1.11.3)$$

The first few terms of the sequence $\left\{\dfrac{PCS(n)}{101}\right\}_{n=1}^{\infty}$ are

1, 10001, 100010001, 1000100010001, …;

and the terms of the sequence satisfy the recurrence relation, given in the following

Corollary 1.11.2 **:** For any integer $n \geq 1$,

$$\frac{PCS(n+1)}{101} = 10^{4n} + \frac{PCS(n)}{101}, \quad \frac{PCS(1)}{101} = 1.$$

Proof **:** Evident from (1.11.3). ■

From Corollary 1.11.2, we see that, for any $n \geq 1$,

$$\frac{PCS(n+2)}{101} = 10^{4(n+1)} + \frac{PCS(n+1)}{101} = 10^{4n}(10^4 + 1) + \frac{PCS(n)}{101}, \tag{1.11.4}$$

$$\frac{PCS(n+3)}{101} = 10^{4(n+2)} + \frac{PCS(n+2)}{101} = 10^{4n}(10^8 + 10^4 + 1) + \frac{PCS(n)}{101}. \tag{1.11.5}$$

Lemma 1.11.2 **:** $\dfrac{PCS(2n)}{101} = (10^{8(n-1)} + 10^{8(n-2)} + \ldots + 10^8 + 1)\dfrac{PCS(2)}{101}$, $n \geq 1$.

Proof **:** The proof is by induction on n. The result is clearly true for $n = 2$. Assuming the validity of the result for some n, we see from (1.11.4) that

$$\begin{aligned}\frac{PCS(2n+2)}{101} &= 10^{8n}(10^4 + 1) + \frac{PCS(2n)}{101} \\ &= 10^{8n}(10^4 + 1) + \left\{10^{8(n-1)} + 10^{8(n-2)} + \ldots + 10^8 + 1\right\}\frac{PCS(2)}{101} \\ &= \left\{10^{8n} + 10^{8(n-1)} + 10^{8(n-2)} + \ldots + 10^8 + 1\right\}\frac{PCS(2)}{101},\end{aligned}$$

which shows that the result is true for $n + 1$ as well. ■

Lemma 1.11.3 **:** $\dfrac{PCS(3n)}{101} = (10^{12(n-1)} + 10^{12(n-2)} + \ldots + 10^{12} + 1)\dfrac{PCS(3)}{101}$, $n \geq 1$.

Proof **:** The result is clearly true for $n = 2$. Assuming the validity of the result for some n, we see from (1.11.5) that

$$\begin{aligned}\frac{PCS(3n+3)}{101} &= 10^{12n}(10^8 + 10^4 + 1) + \frac{PCS(3n)}{101} \\ &= 10^{12n}(10^8 + 10^4 + 1) + \left\{10^{12(n-1)} + 10^{12(n-2)} + \ldots + 10^{12} + 1\right\}\frac{PCS(3)}{101} \\ &= \left\{10^{12n} + 10^{12(n-1)} + 10^{12(n-2)} + \ldots + 10^{12} + 1\right\}\frac{PCS(3)}{101},\end{aligned}$$

which shows the validity of the result for $n + 1$, thereby completing induction. ■

For the sequence $\left\{\frac{PCS(n)}{101}\right\}_{n=1}^{\infty}$, the question is : How many terms are prime?

Note that the second and third terms of the sequence are not primes, since

$$\frac{PCS(2)}{101} = 10001 = 73\times137, \quad \frac{PCS(3)}{101} = 3\times33336667.$$

We now prove the more general result.

***Theorem 1.11.1* :** For any integer $n \geq 2$, $\frac{PCS(n)}{101}$ is a composite number.

***Proof* :** The result is true for n = 2. In fact, the result is true if n is even as shown below : If n (≥ 4) is even, let $n = 2m$ for some integer m (≥ 2). Then, from (1.11.3) (where $x \equiv 10^4$),

$$\frac{PCS(2m)}{101} = x^{2m-1} + x^{2m-2} + \ldots + x + 1$$

$$= x^{2m-2}(x + 1) + \ldots + (x + 1)$$

$$= (x + 1)(x^{2m-2} + x^{2m-3} + \ldots + 1), \tag{1.11.6}$$

which shows that $\frac{PCS(2m)}{101}$ is a composite number for all m (≥ 2).

Thus, it is sufficient to consider the case when n is odd. First we consider the case when n is prime, say $n = p$, where p (≥ 3) is a prime. In this case, from (1.11.3) (with $x \equiv 10^4$),

$$\frac{PCS(p)}{101} = x^{p-1} + x^{p-2} + \ldots + 1 = \frac{x^p - 1}{x - 1}. \tag{1}$$

Let $y = 10^2$ (so that $x = y^2$). Then,

$$x - 1 = y^2 - 1 = (y + 1)(y - 1),$$

$$x^p - 1 = (y^2)^p - 1 = (y^p + 1)(y^p - 1).$$

Using the Binomial expansions of $y^p + 1$ and $y^p - 1$, we get from (1)

$$\frac{PCS(p)}{101} = \frac{x^p - 1}{x - 1} = \frac{y^{2p} - 1}{y^2 - 1} = \frac{(y^p - 1)(y^p + 1)}{(y - 1)(y + 1)}$$

$$= \frac{\{(y-1)(y^{p-1} + y^{p-2} + \ldots + 1)\}\{(y+1)(y^{p-1} - y^{p-2} + \ldots + 1)\}}{(y-1)(y+1)}$$

$$= (y^{p-1} - y^{p-2} + y^{p-3} - \ldots + 1)(y^{p-1} + y^{p-2} + y^{p-3} + \ldots + 1),$$

so that

$$\frac{PCS(p)}{101} = (y^{p-1} - y^{p-2} + y^{p-3} - \ldots + 1)(y^{p-1} + y^{p-2} + y^{p-3} + \ldots + 1). \tag{1.11.7}$$

The above expression shows that $\frac{PCS(p)}{101}$ is a composite number for each prime p (≥ 3).

Finally, we consider the case when n is odd but composite. Let $n = pr$, where p is the largest prime factor of n, and r (≥ 2) is an integer. Then,

$$\begin{aligned}\frac{PCS(n)}{101} &= \frac{PCS(pr)}{101} \\ &= x^{pr-1} + x^{pr-2} + \dots + 1 \\ &= x^{p(r-1)}(x^{p-1} + x^{p-2} + \dots + 1) + x^{p(r-2)}(x^{p-1} + x^{p-2} + \dots + 1) + \dots \\ &\quad + (x^{p-1} + x^{p-2} + \dots + 1) \\ &= (x^{p-1} + x^{p-2} + \dots + 1)[x^{p(r-1)} + x^{p(r-2)} + \dots + 1], \qquad (1.11.8)\end{aligned}$$

and hence, $\frac{PCS(n)}{101} = \frac{PCS(pr)}{101}$ is also a composite number.

All these complete the proof of the theorem. ■

Following different approaches, Kashihara [19] and Le [20] have proved by contradiction the result of Theorem 1.11.1. On the other hand, we prove the same result by actually finding out the factors of $\frac{PCS(n)}{101}$. From (1.11.6), (1.11.7) and (1.11.8), it is clear that $\frac{PCS(n)}{101}$ can not be a square for any $n \geq 2$.

Lemma 1.11.2 shows that $\frac{PCS(2n)}{101}$ can be expressed in terms of $\frac{PCS(2)}{101}$, and Lemma 1.11.3 shows that $\frac{PCS(3n)}{101}$ can be expressed in terms of $\frac{PCS(3)}{101}$. Some consequences of these results are given in the following corollaries.

Corollary 1.11.3 **:** 3 divides $\frac{PCS(3n)}{101}$ for all $n \geq 1$.

Proof **:** follows from Lemma 1.11.3. ■

Corollary 1.11.4 **:** 9 divides $\frac{PCS(9n)}{101}$ for all $n \geq 1$.

Proof **:** By virtue of Lemma 1.11.3 and Corollary 1.11.3, it is sufficient to show that

$$3 \,|\, (10^{12(3n-1)} + 10^{12(3n-2)} + \dots + 10^{12} + 1).$$

The proof is immediate, since, the number of 1 in $10^{12(3n-1)} + 10^{12(3n-2)} + \dots + 10^{12} + 1$ is 3n, and hence, it is divisible by 3. ■

Kashihara [19] raises the question : Is the sequence $\left\{\frac{PCS(n)}{101}\right\}_{n=1}^{\infty}$ square-free? Recall that an integer is called square-free if it is not divisible by the square of a prime. Corollary 1.11.4 shows that an infinite number of terms of the sequence are not square-free (see also Su Gou and Jianghua Li [21]).

1.12 Series Involving Smarandache Sequences

From the Smarandache circular sequence (given in §1.8), we can form the series

$$\sum_{n=1}^{\infty} \frac{1}{CS(n)} = 1 + \frac{1}{12} + \frac{1}{123} + \frac{1}{1234} + \ \dots \ . \tag{1.12.1}$$

The following lemma shows that the series (1.12.1) is convergent, and finds its bounds.

Lemma 1.12.1 **:** The series (1.12.1) is convergent with $1 < \sum_{n=1}^{\infty} \frac{1}{CS(n)} \le \frac{10}{9}$.

Proof **:** First note that

$$CS(n) \ge 10^{n-1} \text{ for all } n \ge 1.$$

Then,

$$\sum_{n=1}^{\infty} \frac{1}{CS(n)} \le \sum_{n=1}^{\infty} \frac{1}{10^{n-1}} = \frac{10}{9}. \ ■$$

Combining together the Smarandache circular sequence (§1.8) and the Smarandache reverse sequence (§ 1.9), we can form the series

$$\sum_{n=1}^{\infty} \frac{CS(n)}{RS(n)} = 1 + \frac{12}{21} + \frac{123}{321} + \frac{1234}{4321} + \dots \ .$$

In connection with this series, we have the following result.

Lemma 1.12.2 **:** The series $\sum_{n=1}^{\infty} \frac{CS(n)}{RS(n)}$ is divergent.

Proof **:** Let us consider the subsequence $\left\{ \frac{CS(10^n)}{RS(10^n)} \right\}_{n=1}^{\infty}$ of the given sequence. The first few terms of the subsequence are

$$\frac{\overline{12 \ \dots \ 9(10)}}{\overline{(10)9 \ \dots \ 21}}, \ \frac{\overline{12 \ \dots \ 99(100)}}{\overline{(100)99 \ \dots \ 21}}, \ \frac{\overline{12 \ \dots \ 999(1000)}}{\overline{(1000)999 \ \dots \ 21}}, \ \dots \ .$$

This shows that

$$CS(10^n) > RS(10^n) \text{ for all } n \ge 1,$$

so that $\lim_{n \to \infty} \frac{CS(10^n)}{RS(10^n)} \ne 0.$

Hence, the series $\sum_{n=1}^{\infty} \frac{CS(10^n)}{RS(10^n)}$ is divergent, and so also is $\sum_{n=1}^{\infty} \frac{CS(n)}{RS(n)}$. ■

1.13 Some Remarks

This chapter gives some elementary properties of ten Smarandache recurrence type sequences. In each case, the recurrence relation, satisfied by the sequence, is given.

In case of the Smarandache odd sequence, even sequence, prime product sequence, square product sequences of two types, permutation sequence, circular sequence and the reverse sequence, we have shown that none of these sequences satisfy the recurrence relation of the Fibonacci or Lucas numbers. This shows that, none of these sequences can contain three or more consecutive Fibonacci numbers or Lucas numbers in a row. It should be mentioned here that our theorems don't rule out the possibility of appearance of Fibonacci or Lucas numbers, scattered here and there, in any of these sequences.

In case of the Smarandache odd sequence, we find that the terms of the forms $OS(3n)$ and $OS(5n+3)$ cannot be primes; for the circular sequence, we show that the terms of the form $CS(3n)$, $CS(3n-1)$ and $CS(5n)$ cannot be primes; while, only terms of the form $SS(3n+1)$ of the symmetric sequence cannot be primes. On the other hand, it has already been settled that no terms of the pierced chain sequence is a prime, and the higher power sequences of the two kinds can each contain only one prime. Thus, the following old conjectures still remain open.

Conjecture 1.13.1 **(Marimutha [3]) :** There are infinitely many primes in the Smarandache odd sequence.

Conjecture 1.13.2 **(Russo [9]) :** The Smarandache square product sequences of two kinds each contains only a finite number of primes.

Conjecture 1.13.3 **:** The Smarandache circular sequence contains no primes.

Conjecture 1.13.4 **:** Only $RS(82)$ of the Smarandache reverse sequence is prime.

In case of the Smarandache even sequence, we find that the terms of the forms $ES(k.10^n+6)$ and $ES(k.10^n+6)$ $(1 \le k \le 9)$ cannot be perfect squares, for the Smarandache higher product sequences of two kinds, our finding is that none contains perfect squares, for the Smarandache permutation sequence, no terms can be perfect powers. So, the following question still remains to be resolved.

Question 1.13.1 **(Ashbacher [1]) :** How many terms of the Smarandache even sequence are perfect powers?

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Chapter 2 Smarandache Determinant Sequences

Murthy [1] introduced four determinant sequences, called the Smarandache cyclic determinant natural sequence (SCDNS), the Smarandache cyclic arithmetic determinant sequence (SCADS), the Smarandache bisymmetric determinant natural sequence (SBDNS), and the Smarandache bisymmetric arithmetic determinant sequence (SBADS).

In this chapter, we consider the problem of finding the n–th terms of these sequences. The expressions of the n–th terms for the Smarandache cyclic arithmetic determinant sequence and the Smarandache bisymmetric arithmetic determinant sequence are given in §2.1 and §2.2 respectively. It may be mentioned that, the SCDNS is a particular case of the SCADS, and the SBDNS is a particular case of the SBADS. In §2.3, we consider the problem of finding the sum of the first n terms of the Smarandache bisymmetric determinant natural sequence and the Smarandache bisymmetric arithmetic determinant sequence.

In §2.1 and §2.2, we would make use of the following results.

Lemma 2.1 **:** Let $D \equiv \left| d_{ij} \right|$ $(1 \le i, j \le n)$ be the determinant of order n (≥ 2) with

$$d_{ij} = \begin{cases} a, & \text{if } i = j \\ 1, & \text{otherwise} \end{cases}$$

where a is any real number. Then,

$$D \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} = (a-1)^{n-1}.$$

Proof **:** Performing the indicated column operations (where $C_i \to C_i - C_1$ indicates the column operation of subtracting the 1st column from the i–th column, $2 \le i \le n$), we get

$$D \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} \begin{matrix} = \\ C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \\ \vdots \\ C_n \to C_n - C_1 \\ \end{matrix} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a-1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & a-1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 1 & 0 & 0 & \cdots & a-1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & a-1 \end{vmatrix}$$

$$= \begin{vmatrix} a-1 & 0 & \cdots & 0 & 0 \\ 0 & a-1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & a-1 & 0 \\ 0 & 0 & \cdots & 0 & a-1 \end{vmatrix},$$

where the last determinant is diagonal of order $n-1$ whose diagonal elements are all $(a-1)$. Hence,

$$D = (a-1)^{n-1}. \blacksquare$$

Lemma 2.2 **:** Let $D^{(a)} \equiv \left| d_{ij}^{(a)} \right|$ $(1 \le i, j \le n)$ be the determinant of order n (≥ 2) with

$$d_{ij}^{(a)} = \begin{cases} a, & \text{if } i = j \\ 1, & \text{otherwise} \end{cases}$$

Then,

$$D^{(a)} \equiv \begin{vmatrix} a & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} = (a-1)^{n-1}(a+n-1).$$

Proof **:** We perform the indicated column operations (where $C_1 \to C_1 + C_2 + ... + C_n$ indicates the operation of adding all the columns from 1 through n and then replacing the 1st column by that sum, and $C_1 \to \dfrac{1}{C_1 + C_2 + ... + C_n}$ denotes the operation of taking out the common sum) to get

$$D^{(a)} \equiv \begin{vmatrix} a & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} \underset{\substack{C_1 \to C_1 + C_2 + ... + C_n \\ C_1 \to \frac{1}{C_1 + C_2 + ... + C_n}}}{=} (a+n-1) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix}$$

$$= (a+n-1)(a-1)^{n-1},$$

where the last equality follows by virtue of Lemma 2.1. ■

Lemma 2.3 **:** Let $A_n \equiv \left| a_{ij} \right|$ $(1 \le i, j \le n)$ be the determinant of order n (≥ 2) with

$$a_{ij} = \begin{cases} 1, & \text{if } i \le j \\ -1, & \text{otherwise} \end{cases}$$

Then,

$$A_n \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix} = 2^{n-1}.$$

Proof **:** The proof is by induction on n. Since

$$A_2 = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2,$$

the result is true for $n = 2$. So, we assume the validity of the result for some integer n (≥ 2).

To prove the result for $n+1$, we consider the determinant of order $n+1$, A_{n+1}, and perform the indicated column operations (where $C_1 \to C_1 + C_n$ indicates the operation of adding the n–th column to the 1st column to get the new 1st column), to get

$$A_{n+1} \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix} \underset{C_1 \to C_1 + C_n}{=} \begin{vmatrix} 2 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & 1 & 1 \\ 0 & -1 & -1 & \cdots & -1 & 1 & 1 \\ 0 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 & 1 \\ & & & & 1 & 1 \\ -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix} = 2A_n.$$

Hence, by virtue of the induction hypothesis,

$$A_{n+1} = 2\, A_n = 2^n,$$

so that the result is true for $n+1$. This completes induction. ■

Lemma 2.4 **:** For n (≥ 2), $B_n \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 \\ \vdots & & & & & & \\ 1 & 1 & -1 & \cdots & -1 & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix} = (-1)^{\left[\frac{n}{2}\right]} 2^{n-1}.$

Proof **:** We consider the following two cases depending on whether n is even or odd.

Case 1 : When n is even, say, $n = 2m$ for some integer $m \geq 1$.

In this case, starting with the determinant $B_n = B_{2m}$, we perform the indicated column operations.

$$B_n = B_{2m} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 \\ \vdots & & & & & & \\ 1 & 1 & -1 & \cdots & -1 & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix}$$

$$\underset{\substack{C_1 \leftrightarrow C_{2m} \\ C_2 \leftrightarrow C_{2m-1} \\ \vdots \\ C_m \leftrightarrow C_{m+1}}}{=} (-1)^m \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix}.$$

Here, $C_1 \leftrightarrow C_{2m}$ means that the 1st and the (2m)– th columns are interchanged, which involves $2m-1$ number of interchanges of columns, changing the sign each time. Similarly, $C_2 \leftrightarrow C_{2m-1}$ means that the 2nd and $(2m-1)$–st columns are interchanged, involving $2m-2$ number of interchanges of columns, and so on, and finally, the m–th and $(m+1)$–st columns are interchanged. Thus, the total number of interchanges of columns is

$$(2m-1) + (2m-2) + \ldots + 1 = m(2m-1),$$

which gives the number of times the sign changes. Then, by Lemma 2.3,

$$B_n = B_{2m} = (-1)^m A_{2m} = (-1)^m 2^{2m-1} = (-1)^m 2^{n-1}.$$

Case 2 : When n is odd, say, $n = 2m+1$ for some integer $m \geq 1$.

In this case, starting with $B_n = B_{2m+1}$, we perform the indicated column operations to get

$$B_n = B_{2m+1} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 \\ \vdots & & & & & & \\ 1 & 1 & -1 & \cdots & -1 & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix}$$

$$\underset{\substack{C_1 \leftrightarrow C_{2m+1} \\ C_2 \leftrightarrow C_{2m} \\ \vdots \\ C_m \leftrightarrow C_{m+2}}}{=} (-1)^m \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix}.$$

Note that, in this case, the total number of interchanges of columns is

$$(2m) + (2m-1) + \ldots + 1 = m(2m+1).$$

Therefore, by virtue of Lemma 2.3,

$$B_n = B_{2m+1} = (-1)^m A_{2m+1} = (-1)^m 2^{2m} = (-1)^m 2^{n-1}.$$

Since, in either case, $m = \left[\frac{n}{2}\right]$ ([x] being the floor function), the result is established. ■

2.1 Smarandache Cyclic Arithmetic Determinant Sequence

The Smarandache cyclic arithmetic determinant sequence is defined as follows.

Definition 2.1.1 **:** The Smarandache cyclic arithmetic determinant sequence, denoted by, {SCADS(n)}, is (where a and d are any real numbers)

$$\left\{ a, \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a \\ a+2d & a & a+d \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d & a+3d \\ a+d & a+2d & a+3d & a \\ a+2d & a+3d & a & a+d \\ a+3d & a & a+d & a+2d \end{vmatrix}, \ldots \right\}.$$

Theorem 2.1.1 **:** The n– th term of the Smarandache cyclic arithmetic determinant sequence, {SCADS(n)}, is

$$\mathrm{SCADS(n)} = \begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-2)d & a+(n-1)d \\ a+d & a+2d & a+3d & \cdots & a+(n-1)d & a \\ a+2d & a+3d & a+4d & \cdots & a & a+d \\ a+3d & a+4d & a+5d & \cdots & a+d & a+2d \\ \vdots & & & & & \\ a+(n-2)d & a+(n-1)d & a & \cdots & a+(n-4)d & a+(n-3)d \\ a+(n-1)d & a & a+d & \cdots & a+(n-3)d & a+(n-2)d \end{vmatrix}$$

$$= (-1)^{\left[\frac{n}{2}\right]} \left(a + \frac{n-1}{2} d\right) (nd)^{n-1}.$$

Proof **:** We consider separately the cases when n is even and odd respectively.

Case 1 : If $n = 2m$ for some integer $m \geq 1$.

In this case, performing the indicated column and row operations, we get successively

$$\mathrm{SCADS(n)} = \begin{vmatrix} a & a+d & a+2d & \cdots & a+(2m-3)d & a+(2m-2)d & a+(2m-1)d \\ a+d & a+2d & a+3d & \cdots & a+(2m-2)d & a+(2m-1)d & a \\ a+2d & a+3d & a+4d & \cdots & a+(2m-1)d & a & a+d \\ a+3d & a+4d & a+5d & \cdots & a & a+d & a+2d \\ \vdots & & & & & & \\ a+(2m-2)d & a+(2m-1)d & a & \cdots & a+(2m-5)d & a+(2m-4)d & a+(2m-3)d \\ a+(2m-1)d & a & a+d & \cdots & a+(2m-4)d & a+(2m-3)d & a+(2m-2)d \end{vmatrix}$$

$$\underset{\substack{C_1 \leftrightarrow C_{2m} \\ C_2 \leftrightarrow C_{2m-1} \\ \vdots \\ C_m \leftrightarrow C_{m+1}}}{=} (-1)^m \begin{vmatrix} a+(2m-1)d & a+(2m-2)d & a+(2m-3)d & \cdots & a+2d & a+d & a \\ a & a+(2m-1)d & a+(2m-2)d & \cdots & a+3d & a+2d & a+d \\ a+d & a & a+(2m-1)d & \cdots & a+4d & a+3d & a+2d \\ a+2d & a+d & a & \cdots & a+5d & a+4d & a+3d \\ \vdots & & & & & & \\ a+(2m-4)d & a+(2m-5)d & a+(2m-6)d & \cdots & a+(2m-1)d & a+(2m-2)d & a+(2m-3)d \\ a+(2m-3)d & a+(2m-4)d & a+(2m-5)d & \cdots & a & a+(2m-1)d & a+(2m-2)d \\ a+(2m-2)d & a+(2m-3)d & a+(2m-4)d & \cdots & a+d & a & a+(2m-1)d \end{vmatrix}$$

$$\underset{\substack{C_{2m} \to C_1 + C_2 + \ldots + C_{2m} \\ C_{2m} \to \frac{1}{C_1 + C_2 + \ldots + C_{2m}}}}{=} (-1)^m S_{2m} \begin{vmatrix} a+(2m-1)d & a+(2m-2)d & a+(2m-3)d & \cdots & a+2d & a+d & 1 \\ a & a+(2m-1)d & a+(2m-2)d & \cdots & a+3d & a+2d & 1 \\ a+d & a & a+(2m-1)d & \cdots & a+4d & a+3d & 1 \\ a+2d & a+d & a & \cdots & a+5d & a+4d & 1 \\ \vdots & & & & & & \\ a+(2m-4)d & a+(2m-5)d & a+(2m-6)d & \cdots & a+(2m-1)d & a+(2m-2)d & 1 \\ a+(2m-3)d & a+(2m-4)d & a+(2m-5)d & \cdots & a & a+(2m-1)d & 1 \\ a+(2m-2)d & a+(2m-3)d & a+(2m-4)d & \cdots & a+d & a & 1 \end{vmatrix}$$

$$\underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ \vdots \\ R_{2m} \to R_{2m} - R_{2m-1}}}{=} (-1)^m S_{2m} \begin{vmatrix} a+(2m-1)d & a+(2m-2)d & a+(2m-3)d & \cdots & a+2d & a+d & 1 \\ (1-2m)d & d & d & \cdots & d & d & 0 \\ d & (1-2m)d & d & \cdots & d & d & 0 \\ d & d & (1-2m)d & \cdots & d & d & 0 \\ \vdots & & & & & & \\ d & d & d & \cdots & d & d & 0 \\ d & d & d & \cdots & (1-2m)d & d & 0 \\ d & d & d & \cdots & d & (1-2m)d & 0 \end{vmatrix}$$

$$=(-1)^m\,(-1)^{2m+1}d^{2m-1}S_{2m}\begin{vmatrix}1-2m & 1 & 1 & \cdots & 1 & 1\\ 1 & 1-2m & 1 & \cdots & 1 & 1\\ 1 & 1 & 1-2m & \cdots & 1 & 1\\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & 1 & 1\\ 1 & 1 & 1 & \cdots & 1-2m & 1\\ 1 & 1 & 1 & \cdots & 1 & 1-2m\end{vmatrix},$$

where the determinant above is of order $(2m-1)$, and

$$S_{2m}=a+(a+d)+(a+2d)+\ldots+[a+(2m-1)d]=2ma+m(2m-1)d.$$

Therefore, by Lemma 2.2,

$$\text{SCADS}(n)=(-1)^{m+1}\{2ma+m(2m-1)d\}\,d^{2m-1}\{(-1)^{2m-1}(2m)^{2(m-1)}\}$$

$$=(-1)^m\left\{a+\frac{2m-1}{2}d\right\}d^{2m-1}\,(2m)^{2m-1}.$$

Case 2 : If $n=2m+1$ for some integer $m\geq 1$.

In this case, performing the indicated column and row operations, we get successively

$$\text{SCADS}(n)=\begin{vmatrix}a & a+d & a+2d & \cdots & a+(2m-2)d & a+(2m-1)d & a+2md\\ a+d & a+2d & a+3d & \cdots & a+(2m-1)d & a+2md & a\\ a+2d & a+3d & a+4d & \cdots & a+2md & a & a+d\\ a+3d & a+4d & a+5d & \cdots & a & a+d & a+2d\\ \vdots & & & & & & \\ a+(2m-1)d & a+2md & a & \cdots & a+(2m-4)d & a+(2m-3)d & a+(2m-2)d\\ a+2md & a & a+d & \cdots & a+(2m-3)d & a+(2m-2)d & a+(2m-1)d\end{vmatrix}$$

$$\underset{\substack{C_1\leftrightarrow C_{2m+1}\\ C_2\leftrightarrow C_{2m}\\ \vdots\\ C_m\leftrightarrow C_{m+2}}}{=}(-1)^m\begin{vmatrix}a+2md & a+(2m-1)d & a+(2m-2)d & \cdots & a+2d & a+d & a\\ a & a+2md & a+(2m-1)d & \cdots & a+3d & a+2d & a+d\\ a+d & a & a+2md & \cdots & a+4d & a+3d & a+2d\\ a+2d & a+d & a & \cdots & a+5d & a+4d & a+3d\\ \vdots & & & & & & \\ a+(2m-3)d & a+(2m-4)d & a+(2m-5)d & \cdots & a+2md & a+(2m-1)d & a+(2m-2)d\\ a+(2m-2)d & a+(2m-3)d & a+(2m-4)d & \cdots & a & a+2md & a+(2m-1)d\\ a+(2m-1)d & a+(2m-2)d & a+(2m-3)d & \cdots & a+d & a & a+2md\end{vmatrix}$$

$$\underset{\substack{C_{2m+1}\to C_1+C_2+\ldots+C_{2m+1}\\ C_{2m+1}\to\frac{1}{C_1+C_2+\ldots+C_{2m+1}}}}{=}(-1)^m S_{2m+1}\begin{vmatrix}a+2md & a+(2m-1)d & a+(2m-2)d & \cdots & a+2d & a+d & 1\\ a & a+2md & a+(2m-1)d & \cdots & a+3d & a+2d & 1\\ a+d & a & a+2md & \cdots & a+4d & a+3d & 1\\ a+2d & a+d & a & \cdots & a+5d & a+4d & 1\\ \vdots & & & & & & \\ a+(2m-3)d & a+(2m-4)d & a+(2m-5)d & \cdots & a+2md & a+(2m-1)d & 1\\ a+(2m-2)d & a+(2m-3)d & a+(2m-4)d & \cdots & a & a+2md & 1\\ a+(2m-1)d & a+(2m-2)d & a+(2m-3)d & \cdots & a+d & a & 1\end{vmatrix}$$

(where $S_{2m+1}=a+(a+d)+(a+2d)+\ldots+(a+2md)=(2m+1)a+m(2m+1)d=(2m+1)(a+md)$)

$$\begin{array}{l} = \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ \vdots \\ R_{2m+1} \to R_{2m+1} - R_{2m} \end{array} \quad (-1)^m S_{2m+1} \begin{vmatrix} a+2md & a+(2m-1)d & a+(2m-2)d & \cdots & a+2d & a+d & 1 \\ -2md & d & d & \cdots & d & d & 0 \\ d & -2md & d & \cdots & d & d & 0 \\ d & d & -2md & \cdots & d & d & 0 \\ \vdots & & & & & & \\ d & d & d & \cdots & d & d & 0 \\ d & d & d & \cdots & -2md & d & 0 \\ d & d & d & \cdots & d & -2md & 0 \end{vmatrix}$$

$$= (-1)^m \{(2m+1)(a+md)\} (-1)^{2m+2} d^{2m} \begin{vmatrix} -2m & 1 & 1 & \cdots & 1 & 1 \\ 1 & -2m & 1 & \cdots & 1 & 1 \\ 1 & 1 & -2m & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & -2m & 1 \\ 1 & 1 & 1 & \cdots & 1 & -2m \end{vmatrix}.$$

Now, noting that the last determinant above is of order 2m, we get, by Lemma 2.2,

$$SCADS(n) = (-1)^m (2m+1)(a+md)d^{2m} \{(-1)^{2m} (2m+1)^{2m-1}\}$$

$$= (-1)^m \left\{a + \frac{(2m-1)+1}{2} d\right\} d^{2m} (2m+1)^{2m}.$$

Now, in either case, $m = \left[\frac{n}{2}\right]$. Hence, the theorem is proved. ■

Definition 2.1.2 **:** The Smarandache cyclic determinant natural sequence, denoted by {SCDNS(n)}, is

$$\left\{1, \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}, \ldots\right\}.$$

The first few terms of the sequence {SCDNS(n)} are

1, −3, −18, 160, 1875, ….

Note that the Smarandache cyclic determinant natural sequence is a particular case of the Smarandache cyclic arithmetic determinant sequence, with $a = 1$, $d = 1$. Hence, as a consequence of Theorem 2.1.1, we have the following result.

Corollary 2.1.1 **:** The n–th term of the Smarandache cyclic determinant natural sequence is

$$SCDNS(n) = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & 1 \\ 3 & 4 & 5 & \cdots & n & 1 & 2 \\ 4 & 5 & 6 & \cdots & 1 & 2 & 3 \\ \vdots & & & & & & \\ n-1 & n & 1 & \cdots & n-4 & n-3 & n-2 \\ n & 1 & 2 & \cdots & n-3 & n-2 & n-1 \end{vmatrix} = (-1)^{\left[\frac{n}{2}\right]} \left(\frac{n+1}{2}\right) n^{n-1}.$$

2.2 Smarandache Bisymmetric Arithmetic Determinant Sequence

The Smarandache bisymmetric arithmetic determinant sequence is defined as follows.

Definition 2.2.1 **:** The Smarandache bisymmetric arithmetic determinant sequence, denoted by {SBADS(n)}, is (where a and d are any real numbers)

$$\left\{ a, \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a+d \\ a+2d & a+d & a \end{vmatrix}, \ ... \right\}.$$

Theorem 2.2.1 **:** The n–th term of the Smarandache bisymmetric arithmetic determinant sequence, {SBADS(n)}, $n \geq 5$, is

$$\text{SBADS}(n) = \begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-3)d & a+(n-2)d & a+(n-1)d \\ a+d & a+2d & a+3d & \cdots & a+(n-2)d & a+(n-1)d & a+(n-2)d \\ a+2d & a+3d & a+4d & \cdots & a+(n-1)d & a+(n-2)d & a+(n-3)d \\ \vdots & & & & & & \\ a+(n-3)d & a+(n-2)d & a+(n-1)d & \cdots & a+4d & a+3d & a+2d \\ a+(n-2)d & a+(n-1)d & a+(n-2)d & \cdots & a+3d & a+2d & a+d \\ a+(n-1)d & a+(n-2)d & a+(n-3)d & \cdots & a+2d & a+d & a \end{vmatrix}$$

$$= (-1)^{\left[\frac{n}{2}\right]} \left(a + \frac{n-1}{2} d\right) (2d)^{n-1}.$$

Proof **:** Starting with SBADS(n), we perform the indicated row and column operations.

$$\text{SBADS}(n) = \begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-3)d & a+(n-2)d & a+(n-1)d \\ a+d & a+2d & a+3d & \cdots & a+(n-2)d & a+(n-1)d & a+(n-2)d \\ a+2d & a+3d & a+4d & \cdots & a+(n-1)d & a+(n-2)d & a+(n-3)d \\ \vdots & & & & & & \\ a+(n-3)d & a+(n-2)d & a+(n-1)d & \cdots & a+4d & a+3d & a+2d \\ a+(n-2)d & a+(n-1)d & a+(n-2)d & \cdots & a+3d & a+2d & a+d \\ a+(n-1)d & a+(n-2)d & a+(n-3)d & \cdots & a+d & a+d & a \end{vmatrix}$$

$$\underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ \vdots \\ R_n \to R_n - R_{n-1}}}{=} \begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-3)d & a+(n-2)d & a+(n-1)d \\ d & d & d & \cdots & d & d & -d \\ d & d & d & \cdots & d & -d & -d \\ \vdots & & & & & & \\ d & d & d & \cdots & -d & -d & -d \\ d & d & -d & \cdots & -d & -d & -d \\ d & -d & -d & \cdots & -d & -d & -d \end{vmatrix}$$

$$\underset{C_n \to C_n + C_1}{=} d^{n-1} \begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-3)d & a+(n-2)d & 2a+(n-1)d \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & -1 & 0 \\ \vdots & & & & & & \\ 1 & 1 & 1 & \cdots & -1 & -1 & 0 \\ 1 & 1 & -1 & \cdots & -1 & -1 & 0 \\ 1 & -1 & -1 & \cdots & -1 & -1 & 0 \end{vmatrix}$$

$$= (-1)^{n+1} d^{n-1} \{2a + (n-1)d\} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -1 \\ \vdots \\ 1 & 1 & 1 & \cdots & -1 & -1 \\ 1 & 1 & -1 & \cdots & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix},$$

where the last determinant above is of order n−1. Now, appealing to Lemma 2.4, we get

$$\text{SBADS}(n) = (-1)^{n+1} d^{n-1} \{2a + (n-1)d\} \left((-1)^{\left[\frac{n-1}{2}\right]} 2^{n-2} \right).$$

Now, if n is even, say, $n = 2m$ for some integer m, then

$$(-1)^{n+1+\left[\frac{n-1}{2}\right]} = (-1)^{2m+1+(m-1)} = (-1)^m = (-1)^{\left[\frac{n}{2}\right]},$$

and if n is odd, say, $n = 2m+1$ for some integer m, then

$$(-1)^{n+1+\left[\frac{n-1}{2}\right]} = (-1)^{(2m+2)+m} = (-1)^m = (-1)^{\left[\frac{n}{2}\right]}.$$

Hence, finally, we get

$$\text{SBADS}(n) = (-1)^{\left[\frac{n}{2}\right]} d^{n-1} \{2a + (n-1)d\} 2^{n-2},$$

which we intended to prove. ■

Definition 2.2.2 **:** The Smarandache bisymmetric determinant natural sequence, denoted by{SBDNS(n)}, is

$$\left\{ 1, \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}, \ldots \right\}.$$

The first few terms of the sequence {SBDNS(n)} are

$1, -3, -8, 20, 48, -112, -256, 576, \ldots.$

Note that the Smarandache bisymmetric determinant natural sequence is a particular case of the Smarandache bisymmetric arithmetic determinant sequence, with $a = 1$, $d = 1$. Hence, we have the following result as a consequence of Theorem 2.2.1.

Corollary 2.2.1 **:** The n−th term of the Smarandache bisymmetric determinant natural sequence, {SBDNS(n)}, $n \geq 5$, is

$$\text{SBDNS}(n) = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & n-1 \\ 3 & 4 & 5 & \cdots & n & n-1 & n-2 \\ \vdots \\ n-2 & n-1 & n & \cdots & 5 & 4 & 3 \\ n-1 & n & n-1 & \cdots & 4 & 3 & 2 \\ n & n-1 & n-2 & \cdots & 3 & 2 & 1 \end{vmatrix} = (-1)^{\left[\frac{n}{2}\right]} (n+1)\, 2^{n-2}.$$

2.3 Series With Smarandache Determinant Sequences

2.3.1 Smarandache Bisymmetric Determinant Natural Sequence

We consider the Smarandache bisymmetric determinant natural sequence, $\{SBDNS(n)\}_{n=1}^{\infty}$, for which the n–th term is, by Corollary 2.2.1,

$$SBDNS(n) = (-1)^{\left[\frac{n}{2}\right]} (n+1)\, 2^{n-2}.$$

Let $\{S_n\}_{n=1}^{\infty}$ be the sequence of n–th partial sums of the sequence $\{SBDNS(n)\}_{n=1}^{\infty}$, so that

$$S_n = \sum_{k=1}^{n} SBDNS(k),\ n \geq 1.$$

In this section, we give an explicit expression for the sequence $\{S_n\}_{n=1}^{\infty}$. This is given in Theorem 2.3.1.1. To prove the theorem, we would need the following results.

Lemma 2.3.1.1 **:** For any integer $m \geq 1$, $\displaystyle\sum_{k=1,3,\ldots,(2m-1)} (2d)^{2(k-1)} = \frac{(2d)^{4m}-1}{(2d)^4-1}$.

Proof **:** Note that, the series is a geometric series with the common ratio $(2d)^4$. ■

Lemma 2.3.1.2 **:** For any integer $m \geq 1$,

$$\sum_{k=1,3,\ldots,(2m-1)} k\, y^{(k-1)/2} = \frac{2m}{y-1} y^m - \frac{(y+1)(y^m-1)}{(y-1)^2}.$$

Proof **:** Let the series on the left be denoted by s, so that

$$s = 1 + 3y + 5y^2 + \ldots + (2m-1)y^{m-1}. \quad (*)$$

Now, multiplying (*) by y and then subtracting the resulting expression from (*), we get

$$\begin{aligned}(1-y)s &= 1 + 2y + 2y^2 + \ldots + 2y^{m-1} - (2m-1)y^m \\ &= [2(1 + y + y^2 + \ldots + y^{m-1}) - 1] - (2m-1)y^m \\ &= 2\left(\frac{y^m-1}{y-1}\right) - 1 - (2m-1)y^m \\ &= \frac{(y+1)(y^m-1)}{y-1} - 2my^m,\end{aligned}$$

which now gives the desired result. ■

From Corollary 2.2.1, we see that, for any integer $k \geq 1$,

$$SBDNS(2k) + SBDNS(2k+1) = (-1)^k (6k+5) 2^{2(k-1)},$$
$$SBDNS(2k+2) + SBDNS(2k+3) = (-1)^{k+1} (6k+11) 2^{2k},$$

so that

$$\begin{aligned}&SBDNS(2k) + SBDNS(2k+1) + SBDNS(2k+2) + SBDNS(2k+3) \\ &= 3(-1)^{k+1}(6k+13)2^{2(k-1)}. \quad (1)\end{aligned}$$

Theorem 2.3.1.1 **:** For any integer $m \ge 0$,

(1) $S_{4m+1} = \frac{3}{5} m 2^{2(2m+1)} + \frac{31}{25} 2^{4m} - \frac{6}{25} = \frac{2}{25}\left\{(60m+31)2^{4m-1} - 3\right\}$,

(2) $S_{4m+2} = -\frac{1}{5} m 2^{4m+3} - \frac{11}{25} 2^{2(m+1)} - \frac{6}{25} = -\frac{2}{25}\left\{(10m+11)2^{4m+1} + 3\right\}$,

(3) $S_{4m+3} = -\frac{3}{5} m 2^{4(m+1)} - \frac{61}{25} 2^{2(2m+1)} - \frac{6}{25} = -\frac{2}{25}\left\{(60m+61)2^{4m+1} + 3\right\}$,

(4) $S_{4m+4} = \frac{1}{5} m 2^{4m+5} + \frac{1}{25} 2^{4(m+2)} - \frac{6}{25} = \frac{2}{25}\left\{(5m+8)2^{4(m+1)} - 3\right\}$.

Proof **:** The first four terms of the sequence $\{S_n\}_{n=1}^{\infty}$ are :

$S_1 = 1$, $S_2 = -2$, $S_3 = -10$, $S_4 = 10$.

(1) The sum of the first $(4m+1)$ terms of the sequence, S_{4m+1}, can be written as

$$S_{4m+1} = SBDNS(1) + SBDNS(2) + \ldots + SBDNS(4m+1)$$
$$= SBDNS(1) + \sum_{k=1,3,\ldots,(2m-1)} [SBDNS(2k) + SBDNS(2k+1) + SBDNS(2k+2) + SBDNS(2k+3)],$$

so that, by virtue of (1),

$$S_{4m+1} = SBDNS(1) + 3 \sum_{k=1,3,\ldots,(2m-1)} (-1)^{k+1}(6k+13)2^{2(k-1)}$$
$$= SBDNS(1) + 3\left\{6 \sum_{k=1,3,\ldots,(2m-1)} k 2^{2(k-1)} + 13 \sum_{k=1,3,\ldots,(2m-1)} 2^{2(k-1)}\right\}.$$

Now, appealing to Lemma 2.3.1.1 (with $d = 1$) and Lemma 2.3.1.2 (with $y = 2^4$), we get

$$S_{4m+1} = 1 + [\frac{3}{5} m 2^{4m+2} - \frac{34}{25}(2^{4m} - 1)] + \frac{13}{5}(2^{4m} - 1),$$

which now gives the desired result after some algebraic manipulations.

(2) Since $S_{4m+2} = S_{4m+1} + SBDNS(4m+2)$,

from part (1) above, together with Corollary 2.2.1, we get

$$S_{4m+2} = [\frac{3}{5} m 2^{2(2m+1)} + \frac{31}{25} 2^{4m} - \frac{6}{25}] - (4m+3)2^{4m},$$

which simplifies to the desired expression for S_{4m+2}.

(3) Since

$$S_{4m+3} = S_{4m+2} + SBDNS(4m+3) = [-\frac{1}{5} m 2^{4m+3} - \frac{11}{25} 2^{2(m+1)} - \frac{6}{25}] - (4m+4)2^{4m+1},$$

we get the desired expression for S_{4m+3} after simplifications.

(4) Since

$$S_{4m+4} = S_{4m+3} + SBDNS(4m+4) = [-\frac{3}{5} m 2^{4(m+1)} - \frac{61}{25} 2^{2(2m+1)} - \frac{6}{25}] - (4m+5)2^{4m+2},$$

the result follows after some simple algebraic simplifications.

The case when $m = 0$ can easily be verified. Hence, the proof is complete. ■

2.3.2 Smarandache Bisymmetric Arithmetic Determinant Sequence

We consider the Smarandache bisymmetric arithmetic determinant sequence, $\{SBADS(n)\}_{n=1}^{\infty}$, for which the n–th term is given by Theorem 2.2.1. Let $\{S_n\}_{n=1}^{\infty}$ be the sequence of n–th partial sums of the sequence $\{SBADS(n)\}_{n=1}^{\infty}$, so that

$$S_n = \sum_{k=1}^{n} SBADS(k),\ n \geq 1.$$

An explicit expression for the sequence $\{S_n\}_{n=1}^{\infty}$ is given below.

***Theorem 2.3.2.1* :** For any integer $m \geq 0$,

(1) $$S_{4m+1} = \frac{m(2d+1)}{4d^2+1}(2d)^{4m+2} + d\left[\frac{2a(2d+1)}{4d^2+1} - \frac{d(4d^2-4d-1)}{(4d^2+1)^2}\right](2d)^{4m} + \left[\frac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \frac{a(2d-1)}{4d^2+1}\right],$$

(2) $$S_{4m+2} = -\frac{m(2d-1)}{4d^2+1}(2d)^{4m+3} - \left[\frac{a(2d-1)}{4d^2+1} + \frac{d(4d^3+3d-1)}{(4d^2+1)^2}\right](2d)^{4m+2} + \left[\frac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \frac{a(2d-1)}{4d^2+1}\right],$$

(3) $$S_{4m+3} = -\frac{m(2d+1)}{4d^2+1}(2d)^{4(m+1)} - d\left[\frac{2a(2d+1)}{4d^2+1} + \frac{d(16d^3+4d^2+8d+3)}{(4d^2+1)^2}\right](2d)^{4m+2} + \left[\frac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \frac{a(2d-1)}{4d^2+1}\right],$$

(4) $$S_{4m+4} = -\frac{m(2d-1)}{4d^2+1}(2d)^{4m+5} + \left[\frac{a(2d-1)}{4d^2+1} + \frac{d(12d^3-4d^2+5d-2)}{(4d^2+1)^2}\right](2d)^{4(m+1)} + \left[\frac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \frac{a(2d-1)}{4d^2+1}\right].$$

***Proof* :** From Theorem 2.2.1, for any integer $k \geq 1$,

$$SBADS(2k) + SBADS(2k+1) = (-1)^k\left[a(2d+1) - \frac{d}{2} + d(2d+1)k\right](2d)^{2k-1},$$

$$SBADS(2k+2) + SBADS(2k+3) = (-1)^{k+1}\left[a(2d+1) + \frac{d}{2}(4d+1) + d(2d+1)k\right](2d)^{2k+1},$$

so that

$$SBADS(2k) + SBADS(2k+1) + SBADS(2k+2) + SBADS(2k+3)$$
$$= (-1)^{k+1} d\left[2a(2d+1)(4d^2-1) + d(16d^3+4d^2+1) + 2d(2d+1)(4d^2-1)k\right](2d)^{2(k-1)}. \qquad \text{(i)}$$

(1) Since S_{4m+1} can be written as

$$S_{4m+1} = SBADS(1) + SBADS(2) + \ldots + SBADS(4m+1)$$
$$= SBADS(1) + \sum_{k=1,3,\ldots,(2m-1)} [SBADS(2k) + SBADS(2k+1) + SBADS(2k+2) + SBADS(2k+3)],$$

by virtue of (i), Lemma 2.3.1.1 and Lemma 2.3.1.2 (with $y = (2d)^4$), we get

$$S_{4m+1} = a + d\left[2a(2d+1)(4d^2-1) + d(16d^3+4d^2+1)\right] \sum_{k=1,3,\ldots,(2m-1)} (-1)^{k+1}(2d)^{2(k-1)}$$
$$+ 2d^2(2d+1)(4d^2-1) \sum_{k=1,3,\ldots,(2m-1)} (-1)^{k+1} k(2d)^{2(k-1)}$$
$$= a + d\left[2a(2d+1)(4d^2-1) + d(16d^3+4d^2+1)\right] \frac{(2d)^{4m}-1}{(2d)^4-1}$$
$$+ 2d^2(2d+1)(4d^2-1)\left\{\frac{2m}{(2d)^4-1}(2d)^{4m} - \frac{(2d)^4+1}{[(2d)^4-1]^2}[(2d)^{4m}-1]\right\}$$
$$= a - d\left[\frac{2a(2d+1)}{4d^2+1} + \frac{d(16d^3+4d^2+1)}{(2d)^4-1}\right] + \frac{m(2d+1)}{4d^2+1}(2d)^{4m+2}$$
$$+ d\left[\frac{2a(2d+1)}{4d^2+1} + \frac{d(16d^3+4d^2+1)}{(2d)^4-1}\right](2d)^{4m} - \frac{2d^2[(2d)^4+1]}{(2d-1)(4d^2+1)^2}[(2d)^{4m}-1].$$

Now, collecting together the coefficients of $(2d)^{4m}$ and the constant terms, we get

coefficient of $(2d)^{4m}$ is $d\left[\frac{2a(2d+1)}{4d^2+1} + \frac{d(16d^3+4d^2+1)}{(2d)^4-1} - \frac{2d(16d^4+1)}{(2d-1)(4d^2+1)^2}\right]$

$$= d\left[\frac{2a(2d+1)}{4d^2+1} - \frac{d(4d^2-4d-1)}{(4d^2+1)^2}\right],$$

constant term is $a - \frac{2ad(2d+1)}{4d^2+1} - \frac{d^2(16d^3+4d^2+1)}{(2d)^4-1} + \frac{2d^2(16d^4+1)}{(2d-1)(4d^2+1)^2}$

$$= \frac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \frac{a(2d-1)}{4d^2+1}.$$

Hence, finally, we get the desired expression for S_{4m+1}.

(2) Since $S_{4m+2} = S_{4m+1} + SBADS(4m+2)$, by virtue Theorem 2.2.1,

$$S_{4m+2} = S_{4m+1} + SBADS(4m+2) = S_{4m+1} - \left(a + \frac{4m+1}{2}d\right)(2d)^{4m+1}.$$

Now, using part (1), and noting that

coefficient of $m(2d)^{4m+2}$ is $\frac{2d+1}{4d^2+1} - 1 = -2d\frac{2d-1}{4d^2+1}$,

$$\text{coefficient of } (2d)^{4m} \text{ is } d\left[\frac{2a(2d+1)}{4d^2+1}-\frac{d(4d^2-4d-1)}{(4d^2+1)^2}\right]-2d(a+\frac{d}{2})$$

$$=-4d^2\left[\frac{a(2d-1)}{4d^2+1}+\frac{d(4d^3+3d-1)}{(4d^2+1)^2}\right],$$

we get the desired expression for S_{4m+2}.

(3) Since $S_{4m+3}=S_{4m+2}+SBADS(4m+3)=S_{4m+2}-(a+\frac{4m+2}{2}d)(2d)^{4m+2}$,

using part (2), we get the desired expression for S_{4m+3}, noting that

$$\text{coefficient of } m(2d)^{4m+3} \text{ is } -\frac{2d-1}{4d^2+1}-1=-2d\frac{2d+1}{4d^2+1},$$

$$\text{coefficient of } (2d)^{4m+2} \text{ is } -\left[\frac{a(2d-1)}{4d^2+1}+\frac{d(4d^3+3d-1)}{(4d^2+1)^2}\right]-(a+d)$$

$$=-d\left[\frac{2a(2d+1)}{4d^2+1}+\frac{d(16d^3+4d^2+8d+3)}{(4d^2+1)^2}\right].$$

(4) Since $S_{4m+4}=S_{4m+3}+SBADS(4m+4)=S_{4m+3}+(a+\frac{4m+3}{2}d)(2d)^{4m+3}$, by part (3),

$$\text{coefficient of } m(2d)^{4m+4} \text{ is } -\frac{2d+1}{4d^2+1}+1=\frac{2d(2d-1)}{4d^2+1},$$

$$\text{coefficient of } (2d)^{4m+2} \text{ is } -d\left[\frac{2a(2d+1)}{4d^2+1}+\frac{d(16d^3+4d^2+8d+3)}{(4d^2+1)^2}\right]+2d(a+\frac{3}{2}d)$$

$$=4d^2\left[\frac{a(2d-1)}{4d^2+1}+\frac{d(12d^3-4d^2+5d-2)}{(4d^2+1)^2}\right],$$

and we get the desired expression for S_{4m+4}.

Now, corresponding to $m=0$, the expressions for S_1, S_2, S_3 and S_4 are given as follows :

$S_1=a$, $S_2=-a(2d-1)-d^2$, $S_3=-a(4d^2+2d-1)-d^2(4d+1)$,

$S_4=a(8d^3-4d^2-2d+1)+d^2(12d^2-4d-1)$.

All these complete the proof of the theorem. ■

References

[1] Amarnath Murthy, Smarandache Determinant Sequences, ***Smarandache Notions Journal***, **12**, 275 – 278, 2001.

[2] Majumdar, A.A.K., On Some Smarandache Determinant Sequences, ***Scientia Magna***, **4(2)**, 80 – 95, 2008.

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Chapter 3 The Smarandache Function

Possibly the most widely studied arithmetical function of Smarandache type is the celebrated Smarandache function, introduced by Smarandache [1] himself.

In this chapter, we study some of the properties of the Smarandache function, denoted by S(n), as well as its several generalizations.

We start with the formal definition of the Smarandache function. Throughout this chapter and the following ones, we shall denote by Z^+ is the set of all positive integers.

Definition 3.1 **:** For any integer $n \geq 1$, the Smarandache function, S(n), is the smallest positive integer m such that $1.2. \ldots . m \equiv m!$ is divisible by n. That is,

$$S(n) = \min\{ m : m \in Z^+, n \mid m! \}; n \geq 1.$$

Thus, for examples,

(a) $S(1) = 1$, since $1 \mid 1! = 1$,

(b) $S(4) = 4$, since 4 divides 4!, and 4 does not divide any of 1!, 2! and 3!,

(c) $S(6) = 3$, since $6 \mid 3! = 6$, and 6 divides neither 1! nor 2!.

Since the function $F(m) \equiv m!$, $m \in Z^+$, is strictly increasing in m, it follows that the function S(n) is well-defined.

By Definition 3.1, for any integer $n \geq 1$,

$$S(n) = m_0$$

if and only if the following two conditions are satisfied :

(1) n divides $m_0!$,

(2) n does not divide $(m_0 - 1)!$.

Note that, our definition of S(n) differs slightly from the traditional one in that both the domain and range of S(n) is Z^+, while the traditional definition allows the range of S(n) to include the number 0. This makes the difference in the value of S(1) : According to the traditional definition, $S(1) = 0$, whereas by Definition 3.1, $S(1) = 1$. We prefer Definition 3.1, like several other researchers, on the following points :

(1) generally, the other arithmetical functions in the Theory of Numbers, have both their domains and ranges as Z^+, and so, Definition 3.1 is in consistence with the traditional approach,

(2) the usual definition of m! is $m! = 1.2. \ldots .m$ for any integer $m \geq 1$, and we define $0! = 1$ only for the sake of convenience, particularly, in combinatorics. Moreover, by allowing the range to include 0, possibly we gain no extra advantage; rather on the contrary, some of the formulas might become complicated (see Corollary 3.1.3),

(3) according to the classical definition of S(n), though S(1) is defined, $S^k(1)$ is not defined when $k \geq 2$, where $S^k(.)$ is the k–fold composition of S(.) with itself.

Several researchers, including Ashbacher [2] and Sandor [3], have studied some of the elementary properties satisfied by S(n). These are summarized in Lemmas 3.6 – 3.12. The proofs of some of these results depend on Lemmas 3.1 – 3.5, some of which are almost trivial.

***Lemma 3.1* :** Let p be a prime. Then, p divides ab if and only if at least one of a and b is divisible by p.

***Lemma 3.2* :** If m divides a! then m divides b! for any integer $b > a$.

***Proof* :** Since b! can be written as

$$b! = a!(a+1)(a+2) \ldots (b),$$

it follows that any integer m dividing a! also divides b!. ■

***Lemma 3.3* :** For any integer $k \geq 3$, **(1)** 2k divides k!, **(2)** $4k^2$ divides (2k)!.

***Proof* :** The proof of part (1) is by induction on k. Since 6 divides $3! = 6$, we see that the result is true for $k = 3$.

(1) We assume that the result is true for some integer k, that is, we assume that

$$2k \mid k! \text{ for some integer } k > 3.$$

Then,

$$2k(k+1) \mid k!(k+1) = (k+1)! \quad \Rightarrow \quad 2(k+1) \mid (k+1)!,$$

which shows that the result is true for k + 1 as well.

(2) From part (1) above,

$$(2k).(2k) \mid k!(2k) \quad \Rightarrow \quad (4k^2) \mid (2k)!. \ ■$$

***Lemma 3.4* :** $2(n+2)$ divides $(2n+1)!$ for any integer $n \geq 1$.

***Proof* :** If $n \geq 2$, then $2n+1 > n+2$, and so the result follows by observing that

$$2(n+2) \text{ divides } (2n+1)! = (n+2)!\,(n+3) \ldots (2n).(2n+1).$$

Since 2×3 divides 3!, the result is true for $n = 1$ as well. ■

***Lemma 3.5* :** For any integers $a, b \geq 1$, a! b! divides (a + b)!.

***Proof* :** Since $\binom{a+b}{a} = \frac{(a+b)!}{a!\,b!}$, we get the desired result. ■

***Lemma 3.6* :** For any $n \in Z^+$, $S(n) \geq 1$. Moreover,

(1) $S(n) = 1$ if and only if $n = 1$,

(2) $S(n) = 2$ if and only if $n = 2$.

***Proof* :** In either case, the proof of the "if" part is trivial. So, we prove the "only if" part.

(1) Let $S(n) = 1$ for some $n \in Z^+$. Then, by definition, $n \mid 1!$, and hence, $n = 1$.

(2) Let $S(n) = 2$ for some $n \in Z^+$. Then, by definition, $n \mid 2!$, and hence, $n = 2$. ■

***Corollary 3.1* :** $S(n) \geq 3$ for any integer $n \geq 3$.

The following lemma gives an upper bound of S(n).

***Lemma 3.7* :** $S(n) \leq n$ for all $n \geq 1$.

***Proof* :** Since $n \mid n!$ for any integer $n \geq 1$, the result follows. ■

Corollary 3.2 **:** $0 < \frac{S(n)}{n} \leq 1$ for any integer $n \geq 1$.

Proof **:** Evident from Lemma 3.6 and Lemma 3.7. ■

The lower bound of S(n), given in Corollary 3.1, can be improved in the case when n is a composite number. This is given in the following

Lemma 3.8 **:** For any composite number $n \geq 4$, $S(n) \geq \max\{ S(d) : d | n \}$.

Proof **:** Let

$$S(n) = m_0 \text{ for some } m_0 \in Z^+.$$

By definition, $n | m_0!$ (and m_0 is the smallest integer with this property). Then, clearly any divisor d of n also divides $m_0!$, so that $S(d) \leq m_0$. ■

Lemma 3.9 **: (1)** $S(2n) \leq n$ for any integer $n \geq 3$; **(2)** $S(4n^2) \leq 2n$ for any integer $n \geq 2$.

Proof **:** follows immediately from Lemma 3.3 as shown below :

(1) $(2n) | n! \quad \Rightarrow \quad S(2n) \leq n$,

(2) $(4n^2) | (2n)! \quad \Rightarrow \quad S(4n^2) \leq 2n$. ■

Lemma 3.10 **:** For any integers $n_1, n_2 \geq 1$,

$$S(n_1 n_2) \leq S(n_1) + S(n_2).$$

Proof **:** Let

$$S(n_1) = m_1,\ S(n_2) = m_2 \text{ for some integers } m_1, m_2 \geq 1.$$

Then,

$$n_1 | m_1!,\ n_2 | m_2! \quad \Rightarrow \quad n_1 n_2 | m_1!\, m_2!.$$

Now, appealing to Lemma 3.5,

$$n_1 n_2 | m_1!\, m_2! \quad \Rightarrow \quad n_1 n_2 | (m_1 + m_2)!,$$

so that

$$S(n_1 n_2) \leq m_1 + m_2 = S(n_1) + S(n_2). \ ■$$

Lemma 3.11 **:** For any integers $n_1, n_2, \ldots, n_k \geq 1$,

$$S(n_1 n_2 \ldots n_k) \leq S(n_1) + S(n_2) + \ldots + S(n_k).$$

Proof **:** The proof is by induction on k. By Lemma 3.10, the result is true when $k = 2$. So, we assume that the result is true for some integer k, that is, the induction hypothesis is

$$S(n_1 n_2 \ldots n_k) \leq S(n_1) + S(n_2) + \ldots + S(n_k).$$

Then, Lemma 3.10, together with the induction hypothesis, gives

$$\begin{aligned} S((n_1 n_2 \ldots n_k) n_{k+1}) &\leq S(n_1 n_2 \ldots n_k) + S(n_{k+1}) \\ &\leq [S(n_1) + S(n_2) + \ldots + S(n_k)] + S(n_{k+1}) \\ &= S(n_1) + S(n_2) + \ldots + S(n_k) + S(n_{k+1}), \end{aligned}$$

which shows that the result is true for $k + 1$ as well. ■

Lemma 3.12 **:** For any integer $m \geq 1$, $S(m!) = m$.

Proof **:** Since $m! \mid m!$ for any integer $m \geq 1$, the result is obtained. ■

Lemma 3.12 shows that, for any integer $m_0 \geq 1$, there is at least one integer $n \geq 1$, namely, $n = m_0!$, such that $S(n) = m_0$; moreover, such an n is the maximum number satisfying the equation $S(n) = m_0$ (since any $n > m_0!$ does not divide $m_0!$).

Lemma 3.11 leads to various interesting inequalities, as has been derived by Sandor [3]. These are given in the following corollary.

Corollary 3.3 **:** The following inequalities hold :

(1) for any integers $n \geq 1$ and $k \geq 1$, $S(n^k) \leq kS(n)$;

(2) for any integers $n \geq 1$ and $r \geq 1$, $\prod_{k=1}^{r} S(n^k) \leq r![S(n)]^r$;

(3) for any integers $n \geq 1$ and $k \geq 1$, $S((n!)^k) \leq kn$;

(4) for any integers $n_1, n_2, \ldots, n_r \geq 1$, and $k_1, k_2, \ldots, k_r \geq 1$,

$$S((n_1!)^{k_1}(n_2!)^{k_2} \ldots (n_r!)^{k_r}) \leq k_1 n_1 + k_2 n_2 + \ldots + k_r n_r;$$

(5) if n_1 and n_2 are two integers such that n_2 divides n_1, then

$$S(n_1) - S(n_2) \leq S\left(\frac{n_1}{n_2}\right).$$

Proof **:** The proof is given below :

(1) Choosing $n_1 = n_2 = \ldots = n_k = n$ in Lemma 3.11, we get the desired result.

(2) From part (1) with $k = 1, 2, 3, \ldots, r$ successively, we get

$$S(n) \leq S(n),\ S(n^2) \leq 2S(n),\ S(n^3) \leq 3S(n),\ \ldots,\ S(n^r) \leq rS(n).$$

Now, multiplying all these r number of inequalities side-wise, we get the result desired.

(3) The inequality follows from part (1) above, together with Lemma 3.12 :

$$S((n!)^k) \leq kS(n!) = kn.$$

(4) From part (1), we get

$$S((n_1!)^{k_1}(n_2!)^{k_2} \ldots (n_r!)^{k_r}) \leq S((n_1!)^{k_1})S((n_2!)^{k_2}) \ldots S((n_r!)^{k_r}).$$

Now, appealing to part (3), we get the desired result.

(5) Let $n_1 = kn_2$ for some integer $k \geq 1$. Then, by Lemma 3.10,

$$S(n_1) = S(k\,n_2) \leq S(k) + S(n_2),$$

so that

$$S(n_1) - S(n_2) \leq S(k) = S\left(\frac{n_1}{n_2}\right).$$

All these complete the proof of the lemma. ■

Lemma 3.13 **:** p divides $S(p^k)$ for any prime $p \geq 2$, and any integer $k \geq 1$.

Proof **:** Let

$$S(p^k) = m \text{ for some integer } m \geq 1,$$

so that p^k divides m!. Then, p must divide m, for otherwise

$$\text{p does not divide m} \quad \Rightarrow \quad p^k \text{ divides } (m-1)! \quad \Rightarrow \quad S(p^k) \leq m-1,$$

contradicting the hypothesis. ■

Corollary 3.4 **:** For any prime $p \geq 2$, and any integer $k \geq 1$,

$$S(p^k) = \alpha p \text{ for some integer } \alpha \geq 1.$$

Proof **:** follows immediately from Lemma 3.13. ■

Lemma 3.14 **:** For any integer n and integers k, r with $k \geq r$, $S(n^k) \geq S(n^r)$.

Proof **:** Let

$$S(n^k) = m_0 \text{ for some integer } m_0 \geq 1.$$

Then,

$$n^r \mid n^k = n^r.n^{k-r},\ n^k \mid m_0! \quad \Rightarrow \quad n^r \mid m_0! \quad \Rightarrow \quad S(n^r) \leq m_0. \ ■$$

Lemma 3.15 **:** For any integers n_1 and n_2 with $n_1 \geq n_2$, $S(n_1^k) \geq S(n_2^k)$ for any $k \geq 1$.

Proof **:** Let, for any integer $k \geq 1$ fixed,

$$S(n_1^k) = m_0 \text{ for some integer } m_0 \geq 1.$$

Then,

$$n_1^k \mid m_0! \quad \Rightarrow \quad n_2^k \mid m_0! \quad \Rightarrow \quad S(n_2^k) \leq m_0. \ ■$$

Lemma 3.16 **:** For any prime $p \geq 2$, $S(p^{\alpha+\beta}) \leq S(p^\alpha)S(p^\beta)$ for any integers $\alpha, \beta \geq 1$.

Proof **:** Let,

$$S(p^{\alpha+\beta}) = m_0 \text{ for some integer } m_0 \geq 1.$$

Furthermore, let

$$S(p^\alpha) = m_1,\ S(p^\beta) = m_2 \text{ for some integer } m_1, m_2 \geq 1,$$

where, without loss of generality, we may assume that $m_2 \leq m_1$.
We want to show that $m_0 \leq m_1 m_2$. Let, on the contrary, $m_0 > m_1 m_2$ for some m_0, m_1 and m_2. Then, using Lemma 3.10, we get the following chain of inequalities :

$$m_1 m_2 < m_0 \leq m_1 + m_2 \leq 2m_1 \quad \Rightarrow \quad m_2 < 2,$$

which is a contradiction to Corollary 3.4.

This contradiction establishes the lemma. ■

In § 3.1, we give some simple explicit expressions for S(n). Two very important results in this respect are given in Theorem 3.1.1 and Theorem 3.1.2. Some generalizations of the Smarandache function are given in §3.2. § 3.3 is devoted to some miscellaneous topics. We conclude this chapter with some remarks about the properties of S(n) in the final §3.4.

3.1 Some Explicit Expressions for S(n)

In this section, we give the explicit expressions for S(n) and their consequences.

First, we state and prove the following theorem, due to Smarandache [1].

Theorem 3.1.1 **:** Let

$$n = p^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$$

be the representation of n in terms of its distinct prime factors p, p_1, p_2, …, p_k (not necessarily ordered). Then,

$$S(n) = \max\left\{ S(p^{\alpha}),\ S(p_1^{\alpha_1}),\ S(p_2^{\alpha_2}),\ \ldots,\ S(p_k^{\alpha_k}) \right\}.$$

Proof **:** For definiteness, let

$$S(p^{\alpha}) = \max\left\{ S(p^{\alpha}),\ S(p_1^{\alpha_1}),\ S(p_2^{\alpha_2}),\ \ldots,\ S(p_k^{\alpha_k}) \right\}.$$

Let

$$S(p^{\alpha}) = m_0 \text{ for some integer } m_0 \geq 1,$$

and let

$$S(p_i^{\alpha_i}) = m_i \text{ for some integer } m_i \geq 1,\ 1 \leq i \leq k.$$

Then, by definition, m_0 is the minimum integer such that

$$p^{\alpha} \mid m_0!, \qquad (1)$$

and m_i $(1 \leq i \leq k)$ is the minimum integer such that

$$p_i^{\alpha_i} \mid m_i! \quad (1 \leq i \leq k). \qquad (2)$$

Now,

$$m_0 \geq m_i \text{ for all } 1 \leq i \leq k \quad \Rightarrow \quad m_i! \mid m_0! \text{ for all } 1 \leq i \leq k.$$

Therefore, from (2),

$$p_i^{\alpha_i} \mid m_0! \quad \text{for all } 1 \leq i \leq k.$$

Now, since $(p_i^{\alpha_i}, p_j^{\alpha_j}) = 1$ for $i \neq j$, it follows that

$$p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \mid m_0!,$$

and by the same reasoning,

$$n = p^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \mid m_0!.$$

Since, by (1), m_0 is the minimum integer with this property, it follows that

$$S(n) = m_0 = S(p^{\alpha}),$$

which we intended to prove. ■

From Theorem 3.1.1, we see that, in order to calculate

$$S(n) = S(p^{\alpha}\, p_1^{\alpha_1}\, p_2^{\alpha_2} \ldots p_k^{\alpha_k}),$$

we have to know the values of

$$S(p^{\alpha}), \quad S(p_1^{\alpha_1}), \quad S(p_2^{\alpha_2}), \quad \ldots, \quad S(p_k^{\alpha_k}),$$

which are not available till now. However, we have expressions in some particular cases.

Lemma 3.1.1 **:** For any prime $p \geq 2$, $S(p) = p$.

Proof **:** Let

$$S(n) = m_0 \text{ for some } m_0 \in Z^+,$$

so that, $p \mid m_0!$. By Lemma 3.1, one of the factors of $m_0!$ must be divisible by p. Now, since $p \mid p!$ but p does not divide $(p-1)!$, the result follows. ■

Lemma 3.1.2 **:** Let $p_1, p_2, \ldots, p_k$ be k distinct primes. Then,

$$S(p_1 p_2 \ldots p_k) = \max\{p_1, p_2, \ldots, p_k\}.$$

Proof **:** By Theorem 3.1.1,

$$S(p_1 p_2 \ldots p_k) = \max\{S(p_1), S(p_2), \ldots, S(p_k)\}.$$

Now, we get the desired result by virtue of Lemma 3.1.1. ■

Lemma 3.1.3 **:** Let $n_1 \geq 1$ and $n_2 \geq 1$ be two integers with $(n_1, n_2) = 1$. Then,

$$S(n_1 n_2) = \max\{S(n_1), S(n_2)\}.$$

Proof **:** Let the representations of n_1 and n_2 in terms of their prime factors be

$$n_1 = p^{\alpha}\, p_1^{\alpha_1}\, p_2^{\alpha_2} \ldots p_r^{\alpha_r},$$

$$n_2 = q_1^{\beta_1}\, q_2^{\beta_2} \ldots q_t^{\beta_t},$$

where the primes $p, p_1, p_2, \ldots, p_r$ as well as the primes $q_1, q_2, \ldots, q_t$ are all distinct. Then,

$$n_1 n_2 = p^{\alpha}\, p_1^{\alpha_1}\, p_2^{\alpha_2} \ldots p_r^{\alpha_r}\, q_1^{\beta_1}\, q_2^{\beta_2} \ldots q_t^{\beta_t}.$$

For definiteness, let

$$S(p^{\alpha}) = \max\left\{ S(p^{\alpha}), \; S(p_1^{\alpha_1}), \; \ldots, \; S(p_r^{\alpha_r}), \; S(q_1^{\beta_1}), \; \ldots, \; S(q_t^{\beta_t}) \right\}.$$

Then, by Theorem 3.1.1,

$$S(n_1 n_2) = S(p^{\alpha}) = S(n_1). \quad ■$$

An immediate consequence of Lemma 3.1.3 is the following.

Corollary 3.1.1 **:** The function S(n) is not multiplicative.

Theorem 3.1.2 **:** Let $n_1, n_2, \ldots, n_k \geq 1$ be k integers with $(n_1, n_2, \ldots, n_k) = 1$. Then,

$$S(n_1 n_2 \ldots n_k) = \max\{S(n_1), S(n_2), \ldots, S(n_k)\}.$$

Proof **:** The proof is by induction on k. The result is true for $k = 2$, by virtue of Lemma 3.1.3. So, we assume the validity of the result for some k.

To prove the validity of the result for $k+1$, let $n_1, n_2, \ldots, n_k, n_{k+1}$ be $k+1$ integers with

$$(n_1, n_2, \ldots, n_k, n_{k+1}) = 1.$$

Then, by Lemma 3.1.3, together with the induction hypothesis, we get

$$S(n_1 n_2 \ldots n_k n_{k+1}) = \max\{S(n_1 n_2 \ldots n_k), S(n_{k+1})\}$$
$$= \max\{S(n_1), S(n_2), \ldots, S(n_k), S(n_{k+1})\}.$$

Hence the theorem. ■

From Lemma 3.1.1,

$$S(p) = p \text{ for any prime } p \geq 2. \quad (3.1.1)$$

We now want to find expressions for $S(p^k)$ for $k \geq 2$. To do so, first note that, from (3.1.1),

p divides p! but p does not divide $(p-1)!$.

Therefore,

$$p^2 \mid (2p)! = p!(p+1)(p+2) \ldots [p+(p-1)](2p), \quad (3.1.2)$$

and clearly, 2p is the minimum integer such that (2p)! is divisible by p^2.

Again,

$$p^3 \mid (3p)! = (2p)!(2p+1)(2p+2) \ldots [2p+(p-1)](3p), \quad (3.1.3)$$

and 3p is the minimum integer such that (3p)! is divisible by p^3.

Continuing the argument $p-1$ times, we see that $(p-1)p$ is the minimum integer such that

$$p^{p-1} \mid [(p-1)p]! = [(p-2)p]![(p-2)p+1][(p-2)p+2] \ldots [(p-2)p+p]$$
$$= [(p-2)p]![(p-2)p+1][(p-2)p+2] \ldots [(p-1)p]. \quad (3.1.4)$$

And continuing one more time,

$$p^p \mid (p^2)! = [(p-1)p]![(p-1)p+1][(p-1)p+2] \ldots [(p-1)p+p]$$
$$= [(p-1)p]![(p-1)p+1][(p-1)p+2] \ldots (p^2). \quad (3.1.5)$$

From (3.1.5), we see that

$$p^{p+1} \mid (p^2)!. \quad (3.1.6)$$

Summarizing the results in (3.1.2) – (3.1.6), we have the following lemma.

Lemma 3.1.4 **:** For any prime $p \geq 2$, **(1)** $S(p^k) = kp$, if $1 \leq k \leq p$, **(2)** $S(p^{p+1}) = p^2$.

By an analysis similar to that leading to Lemma 3.1.4, we have the following lemma.

Lemma 3.1.5 **:** For any prime $p \geq 2$,
(1) $S(p^{p+k+1}) = p^2 + kp$, if $1 \leq k \leq p$, **(2)** $S(p^{2(p+1)}) = 2p^2$.

Lemma 3.1.6 **:** $S(n_1 n_2) \leq S(n_1) S(n_2)$ for any integers $n_1, n_2 \geq 1$.

Proof **:** Let the representations of n_1 and n_2 in terms of their prime factors be

$$n_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r} q_1^{\beta_1} q_2^{\beta_2} \dots q_s^{\beta_s}, \quad n_2 = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_r^{\gamma_r} t_1^{\delta_1} t_2^{\delta_2} \dots t_k^{\delta_k},$$

where the primes p, $p_1, p_2, \dots, p_r$, the primes $q_1, q_2, \dots, q_s$ as well as the primes $t_1, t_2, \dots, t_k$ are all distinct, with $\alpha_1, \alpha_2, \dots, \alpha_r \geq 0$; $\beta_1, \beta_2, \dots, \beta_s \geq 0$ with at least one $\beta \geq 1$; $\gamma_1, \gamma_2, \dots, \gamma_r \geq 0$; $\delta_1, \delta_2, \dots, \delta_k \geq 0$ with at least one $\delta \geq 1$. Then,

$$n_1 n_2 = p_1^{\alpha_1 + \gamma_1} p_2^{\alpha_2 + \gamma_2} \dots p_r^{\alpha_r + \gamma_r} q_1^{\beta_1} q_2^{\beta_2} \dots q_s^{\beta_s} t_1^{\delta_1} t_2^{\delta_2} \dots t_k^{\delta_k}.$$

Therefore, by Theorem 3.1.1,

$$S(n_1 n_2) = \max\left\{ S(p_1^{\alpha_1 + \gamma_1}), \dots, S(p_r^{\alpha_r + \gamma_r}), S(q_1^{\beta_1}), \dots, S(q_s^{\beta_s})\ S(t_1^{\delta_1}), \dots, S(t_k^{\delta_k}) \right\}.$$

Now, using Lemma 3.16, we get

$$S(n_1 n_2) \leq \max\left\{ S(p_1^{\alpha_1}) S(p_1^{\gamma_1}), \dots, S(p_r^{\alpha_r}) S(p_r^{\gamma_r}), S(q_1^{\beta_1}), \dots, S(q_s^{\beta_s}), S(t_1^{\delta_1}), \dots, S(t_k^{\delta_k}) \right\}$$

$$\leq S(n_1) \ \max\left\{ S(p_1^{\gamma_1}), \dots, S(p_r^{\gamma_r}), \dots, S(t_1^{\delta_1}), \dots, S(t_k^{\delta_k}) \right\}$$

$$= S(n_1) S(n_2),$$

where we have used the facts that $S(n_1) \geq 1$, and $S(p_i^{\alpha_i}) \leq S(n_1)$ for all $1 \leq i \leq r$.

To complete the proof, note that, for any prime p, $S(np) = \max\{S(n), p\} \leq pS(n)$. ■

Corollary 3.1.2 **:** For any integers $n_1, n_2 \geq 1$, $\dfrac{S(n_1 n_2)}{n_1 n_2} \leq \min\left\{ \dfrac{S(n_1)}{n_1}, \dfrac{S(n_2)}{n_2} \right\}$.

Proof **:** follows from the following chain of inequalities (using Lemma 3.7) :

$$\frac{S(n_1 n_2)}{n_1 n_2} \leq \frac{S(n_1)}{n_1} . \frac{S(n_2)}{n_2} \leq \frac{S(n_1)}{n_1}, \quad \frac{S(n_1 n_2)}{n_1 n_2} \leq \frac{S(n_1)}{n_1} . \frac{S(n_2)}{n_2} \leq \frac{S(n_2)}{n_2}. \quad ■$$

Lemma 3.1.7 **:** For any integers $a, b \geq 2$, $a b \geq a + b$.

Proof **:** Letting $a = 2 + A$, $b = 2 + B$ where $A, B \geq 0$ are integers, we see that

$$ab = (2 + A)(2 + B) \geq a + b = (2 + A)(2 + B) \text{ if and only if } A + B + AB \geq 0,$$

which is true for any integers $A, B \geq 0$, with strict equality sign only when $A = 0 = B$. ■

Corollary 3.1.3 **:** For any integers $n_1, n_2 \geq 1$,
(1) $S(n_1 n_2) \leq S(n_1) + S(n_2) \leq S(n_1) S(n_2)$, if both $n_1, n_2 \geq 2$,
(2) $S(n_1 n_2) \leq S(n_1) S(n_2) \leq S(n_1) + S(n_2)$, if $n_1 = 1$ or $n_2 = 1$.

Proof **:** follows from Lemma 3.10 and Lemma 3.1.6, together with Lemma 3.1.7. ■

3.2 Some Generalizations

Since its introduction by Smarandache [1], the Smarandache function has seen several variations. This section is devoted to two such generalizations, and are given in the following two subsections.

3.2.1 The Smarandache Dual Function $S_*(n)$

The Smarandache dual function, denoted by $S_*(n)$, introduced by Sandor [3], is defined as follows.

Definition 3.2.1.1 **:** For any integer $n \geq 1$, the Smarandache dual function, $S_*(n)$, is

$$S_*(n) = \max\left\{m \ : \ m \in Z^+, \ m! \mid n\right\}.$$

Let

$$S_*(n) = m_0 \text{ for some integer } m_0 \geq 1.$$

Then, by Definition 3.2.1.1, the following two conditions are satisfied :

(1) $m_0!$ divides n,

(2) n is not divisible by m! for any $m < m_0$.

Lemma 3.2.1.1 **:** For any integer $n \geq 1$, $1 \leq S_*(n) \leq S(n) \leq n$.

Proof **:** Let

$$S(n) = m_0, \ S_*(n) = m \text{ for some integers } m_0 \geq 1, \ m \geq 1.$$

Then, by definitions of $S_*(n)$ and $S(n)$,

$$m! \mid n, \ \ n \mid m_0! \quad \Rightarrow \quad m! \mid m_0! \quad \Rightarrow \quad m_0 \geq m.$$

Clearly,

$$1 \text{ divides any integer } n \geq 1 \quad \Rightarrow \quad S_*(n) \geq 1.$$

Finally, the right-most part of the inequality is due to Lemma 3.7.

All these complete the proof of the lemma. ■

Lemma 3.2.1.2 **:** For any prime $p \geq 2$, and any integer $n \geq p$,

$$S_*(n! + (p-1)!) = p - 1.$$

Proof **:** Writing

$$n! = (p-1)! \, p \, (p+1) \ \dots \ n,$$

we see that $(p-1)!$ divides $n! + (p-1)!$; moreover, $(p-1)!$ is the maximum integer that divides $n! + (p-1)!$, since p! divides n! but p! does not divide $(p-1)!$. ■

Corollary 3.2.1.1 **:** The equation $S_*(n) = p - 1$ has infinite number of solutions, where p is a prime.

Proof **:** By Lemma 3.2.1.2, $n = k! + (p-1)!$ is a solution for any $k \geq p$. ■

Lemma 3.2.1.3 **:** For any integers $a, b \geq 1$, $S_*(ab) \geq \max\{S_*(a), S_*(b)\}$.

Proof **:** Let

$$S_*(a) = m_0 \text{ for some integer } m_0 \geq 1.$$

Then, by definition,

$$m_0! \mid a \quad \Rightarrow \quad m_0! \mid ab \quad \Rightarrow \quad S_*(ab) \geq m_0 = S_*(a). \blacksquare$$

Lemma 3.2.1.4 **:** $S_*(n(n-1) \ldots (n-a+1)) \geq a$ for any integers n and a with $n \geq a \geq 1$.

Proof **:** follows by virtue of the fact that

$$a! \mid n(n-1) \ldots (n-a+1)! = \frac{n!}{(n-a)!} = \binom{n}{a} a!. \blacksquare$$

Lemma 3.2.1.4 **:** For any integer $n \geq 1$,

$$S_*((2n)!\,(2n+2)!) \begin{cases} = 2n+2, & \text{if } 2n+3 \text{ is a prime} \\ \geq 2n+3, & \text{if } 2n+3 \text{ is not a prime} \end{cases}$$

Proof **:** If $2n+3 = p$ is a prime, then clearly, $(2n+2)$ divides $(2n)!(2n+2)!$, but

$$2n+3 = p \text{ does not divide } (2n)!(2n+2)! = (p-3)!(p-1)!.$$

Hence, in this case, $S_*((2n)!\,(2n+2)!) = 2n+2$.

Next, let $2n+3$ be not prime, so that $n \geq 3$. We want to show that, in this case

$$2n+3 \text{ divides } (2n)!. \qquad (1)$$

If $n = 3$, then (1) is satisfied, since $2n+3 = 3^2$ divides $(2n)! = 6!$. So, let $n \geq 4$.

Case 1 : When $2n+3 = ab$ for some odd integers a and b with $b > a \geq 3$.

In this case, $b < n$, for otherwise

$$b > n \quad \Rightarrow \quad ab \geq 3b > 3n > 2n+3, \text{ a contradiction.}$$

Hence, a and b are distinct integers less than n, and hence, (1) is satisfied.

Case 2 : When $2n+3 = a^2$ for some odd integer (so that $n \geq 11$, $a \geq 5$).

Here, $a^2 - 3 > 2a$, so that

$$2n+3 = a^2 \text{ divides } a!(a+1)(a+2) \ldots (2a) \ldots (a^2-3) = (2n)!.$$

Now, by virtue of (1), $(2n+3)!$ divides $(2n)!(2n+2)!$, completing the proof. ■

Lemma 3.2.1.5 **:** For any integer $n \geq 1$, $S_*((2n+1)!\,(2n+3)!) \geq 2(n+2)$.

Proof **:** By Lemma 3.4, for any integer $n \geq 1$,

$$2(n+2) \mid (2n+1)! \quad \Rightarrow \quad [2(n+2)]! = (2n+3)!\,[2(n+2)] \mid (2n+1)!\,(2n+3)!.$$

This gives the desired result. ■

Remark 3.2.1.1 **:** As has been pointed out by Sandor [3], in certain cases, Lemma 3.2.1.5 holds true with equality signs. Some examples are given below :

(1) $S_*(3!\,5!) = 6$, (2) $S_*(5!\,7!) = 8$, (3) $S_*(7!\,9!) = 10$, (4) $S_*(9!\,11!) = 12$,

(5) $S_*(13!\,15!) = 16$, (6) $S_*(15!\,17!) = 18$, (7) $S_*(19!\,21!) = 22$. ♦

3.2.2 The Additive Analogues of S(n) and $S_*(n)$

The function, called the additive analogue of S(n), to be denoted by $S_S(n)$, has been introduced by Sandor [3].

***Definition 3.2.2.1* :** For any real number $x \in (1, \infty)$,

$$S_S(x) = \min\{m \ : \ m \in Z^+, \ x \leq m!\}.$$

The dual of $S_S(x)$, to be denoted by $S_{S^*}(x)$, is defined as follows (Sandor [3]).

***Definition 3.2.2.2* :** For any real number $x \in [1, \infty)$,

$$S_{S^*}(x) = \max\{m \ : \ m \in Z^+, \ m! \leq x\}.$$

From Definition 3.2.2.1 and Definition 3.2.2.2, it is clear that

$S_S(x) = k$ if and only if $x \in ((k-1)!, k!]$ (for any integer $k \geq 2$),

$S_{S^*}(x) = k-1$ if and only if $x \in [(k-1)!, k!)$ (for any integer $k \geq 2$).

We then have the following results, due to Sandor [3].

***Lemma 3.2.2.1* :** For any real number $x \geq 1$, and any integer $k \geq 1$,

$$S_S(x) = \begin{cases} S_{S^*}(x) + 1, & \text{if } x \in (k!, (k+1)!) \\ S_{S^*}(x), & \text{if } x = (k+1)! \end{cases}$$

***Lemma 3.2.2.2* :** $S_{S^*}(x) \sim \dfrac{\ln x}{\ln \ln x}$ as $x \to \infty$.

If the domain is restricted to the set of positive integers, Z^+, we get the function $S_S(n)$, which assume the value k (≥ 1), repeated $k! - (k-1)!$ times. Similarly, we get the function $S_{S^*}(n)$, with domain Z^+, which can take the value k (≥ 1), repeated $k! - (k-1)!$ times.

***Lemma 3.2.2.3* :** For any integer $n \geq 1$, $S(n) \geq S_{S^*}(n)$.
***Proof* :** Let

$S(n) = m_0$, $S_{S^*}(n) = m$ for some integers $m_0 \geq 1$ and $m \geq 1$.

Then, by definitions of S(n) and $S_{S^*}(n)$,

$$m_0! \geq n \text{ and } n \geq m! \quad \Rightarrow \quad m_0! \geq m!.$$

Thus, $m_0 \geq m$, which we intended to prove. ■

The lemma below is concerned with the convergence of series involving $S_{S^*}(n)$, and is due to Sandor [3].

***Lemma 3.2.2.4* :** The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n[S_{S^*}(n)]^s}$ is convergent if and only if $s > 1$.

3.3 Miscellaneous Topics

In this section, we give some properties of the Smarandache function S(n).

The following three lemmas are due to Ashbacher [2].

***Lemma 3.3.1* :** $|S(n+1) - S(n)|$ is unbounded.

***Proof* :** Let $n+1=p$, where $p \geq 7$ is a prime. Then, by Lemma 3.1.1 and Lemma 3.9,

$$S(n+1) = S(p) = p, \; S(n) = S(p-1) \leq \frac{p-1}{2}.$$

Therefore,

$$|S(n+1) - S(n)| = S(n+1) - S(n) \geq p - \frac{p-1}{2} = \frac{p+1}{2},$$

which can be made arbitrarily large (by choosing p large and larger). ■

***Lemma 3.3.2* :** For any prime $p \geq 2$ fixed, and any integer $k \geq 1$, $\frac{S(p^{k+1})}{p^{k+1}} < \frac{S(p^k)}{p^k}$,

(with equality sign only when $p=2$ and $k=1$).

***Proof* :** By Corollary 3.4,

$$S(p^k) = rp \text{ for some integer } r \geq 1.$$

Now, since p^k divides (rp)!, it follows that

$$p^{k+1} \mid (rp)!(r+1)p \;\Rightarrow\; p^{k+1} \mid [(r+1)p]! \;\Rightarrow\; S(p^{k+1}) \leq (r+1)p.$$

Therefore,

$$\frac{S(p^{k+1})}{p^{k+1}} \leq \frac{(r+1)p}{p^{k+1}} = \frac{r+1}{p^k} < \frac{S(p^k)}{p^k} = \frac{rp}{p^k}$$

if and only if $rp > r+1$, which is true for any $p \geq 2$ and any $r \geq 1$, with the exception when $p=2$ and $r=1$. In the latter case, the relationship holds with equality sign, as is verified below :

$$\frac{S(2^2)}{2^2} = \frac{4}{4} = \frac{2}{2} = \frac{S(2)}{2}.$$

Hence, the proof is complete. ■

***Lemma 3.3.3* :** For any prime $p \geq 2$ fixed, $\lim_{n\to\infty} \frac{S(p^{n+1})}{p^{n+1}} = 0.$

***Proof* :** By Lemma 3.3.2, the sequence $\left\{\frac{S(p^n)}{p^n}\right\}_{n=1}^{\infty}$ is strictly decreasing for any prime $p \geq 2$ fixed. Since the sequence is bounded (by Corollary 3.2), it follows that it is convergent. Now, for $p \geq 2$ fixed, given any $\varepsilon > 0$, however small, we can find an integer $N > 1$, such that

$$\frac{S(p^n)}{p^n} < \varepsilon \text{ whenever } n \geq N.$$

This establishes the lemma. ■

Corollary 3.3.1 **:** Given any real number $\varepsilon > 0$, however small, it is possible to find an integer $n > 1$ such that

$$\frac{S(n)}{n} < \varepsilon.$$

Proof **:** Given any real number $\varepsilon > 0$, we choose a prime $p \geq 2$. The proof of Lemma 3.3.3 shows that there is an integer such that

$$\frac{S(p^n)}{p^n} < \varepsilon \text{ whenever } n \geq N.$$

Therefore, choosing $n = p^n$, where n is an integer such that $n \geq N$, we see that the inequality in the corollary is satisfied. ■

Example 3.3.1 **:** For $p = 2$, the successive values of $\frac{S(2^n)}{2^n}$ for $n = 1, 2, \ldots$, are

$$1,\ 1,\ \frac{4}{8} = \frac{1}{2},\ \frac{6}{16} = \frac{3}{8},\ \frac{8}{32} = \frac{1}{4},\ \frac{8}{64} = \frac{1}{8},\ \frac{8}{128} = \frac{1}{16},\ \frac{10}{256} = \frac{5}{128},\ \frac{12}{512} = \frac{3}{128},\ \ldots .$$

Thus, for example, if $\varepsilon = 0.05$, then choosing $n = 2^8$, we get

$$\frac{S(n)}{n} = \frac{S(2^8)}{2^8} = \frac{5}{128} < 0.05 = \varepsilon. \ ♦$$

Example 3.3.2 **:** An alternative proof of Corollary 3.3.1 is given in Ashbacher [2]. To find n such that

$$\frac{S(n)}{n} = \varepsilon < 0.05,$$

we choose a prime q such that $\frac{1}{S(q)} = \frac{1}{q} < \varepsilon = 0.05$. The choice $q = 23$ serves the purpose. Now, let p be a prime such that $p > q$. Then, letting $n = pq$, we get

$$\frac{S(n)}{n} = \frac{S(pq)}{pq} = \frac{p}{pq} = \frac{1}{q} < \varepsilon = 0.05. \ ♦$$

The following result is due to Stuparu and Sharpe [4], who mentioned it without any proof.

Lemma 3.3.4 **:** Any solution of the equation

$$S(n) = p,$$

where $p \geq 2$ is a given prime, is of the form

$$n = kp,$$

where k is an integer such that k divides $(p-1)!$.

Proof **:** Clearly, $n = kp$, where k divides $(p-1)!$, divides $p!$. Moreover, p is the minimum integer such that $p!$ is divisible by kp. Hence,

$$S(kp) = p,$$

establishing the lemma. ■

The proof of Lemma 3.3.4 shows that, $S(n) = p$ ($p \geq 2$ is a prime) has $d((p-1)!)$ solutions.

Example 3.3.3 **:** Some particular cases of Lemma 3.3.4 are given below :

(1) The equation $S(n) = 2$ has only one solution, namely, $n = 2$.

(2) The equation $S(n) = 3$ has two solutions, namely, $n = 3, 6$.

(3) The equation $S(n) = 5$ has $4! = 8$ solutions, namely, $n = 5k$, where k is a divisor of 4!. The eight solutions are $n = 5, 10, 15, 20, 30, 40, 60, 120$. ♦

Lemma 3.3.5 **:** The only solutions of the equation $S(n) = n$ are

(1) $n = 1$, **(2)** $n = 4$, **(3)** $n = p$, where $p \geq 2$ is a prime.

Proof **:** We consider separately the cases when n is odd and n is even.

Case (1) : When n is even, say, $n = 2k$ for some integer $k \geq 1$.
In this case,

$$n = 2k \text{ does not divide } \left(\frac{n}{2}\right)! = k! \;\Rightarrow\; k = 1, 2,$$

where the last implication follows by virtue of Lemma 3.9.
Thus, $n = 2$ and $n = 4$ are the only even integers that satisfy the equation $S(n) = n$.

Case (2) : When n is odd with $n \geq 5$.
First, we consider the case when n is composite. Let

$$n = ab \text{ for some integers a and b with } b > a.$$

Then, clearly

$$n = ab \mid b! = a!(a+1)(a+2) \ldots (b) \;\Rightarrow\; S(n) \leq b < n.$$

Next, let

$$n = a^2 \text{ for some integer } a \geq 3.$$

In this case,

$$n = a^2 \mid (2a)! = a!(a+1)(a+2) \ldots (2a) \;\Rightarrow\; S(n) \leq 2a < a^2 < n.$$

Thus, no composite odd integer can be solution of the equation $S(n) = n$. And consequently, any odd integral solution of $S(n) = n$ must be an odd prime. From Lemma 3.1.1, any odd prime is a solution the equation $S(n) = n$.

All these complete the proof of the lemma. ■

Corollary 3.3.2 **:** The equation $S(kn) = n$ has an infinite number of solutions, for any integer $k \geq 1$ fixed.

Proof **:** Given the integer $k \geq 1$, let p be a prime such that $p > k$. Then,

$$S(kp) = \max\{S(k), S(p)\} = \max\{S(k), p\} = p.$$

Thus, $n = p$ is a solution of the equation $S(kn) = n$. Since there is an infinite number of primes of the prescribed form, the result follows. ■

Remark 3.3.1 **:** Since $\frac{S(n)}{n} \leq \frac{n}{n} = 1$, it follows that the equation $\frac{S(n)}{n} = k$ has no solution if $k \geq 2$.

The following lemma is due to Ashbacher [5].

Lemma 3.3.6 **:** The equation $S(n)S(n+1)=n$ has no solution.

Proof **:** Clearly, neither n nor $n+1$ can be a prime. So, let

$$n = p^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}, \qquad n+1 = q^{\beta} q_1^{\beta_1} q_2^{\beta_2} \ldots q_t^{\beta_t}$$

be the representations of n and $n+1$ in terms of their prime factors, where the primes $p, p_1, p_2, \ldots, p_r$ and the primes $q, q_1, q_2, \ldots, q_t$ are all distinct. Moreover, let for definiteness

$$S(n) = S(p^{\alpha}), \; S(n+1) = S(q^{\beta}).$$

Then, the equation $S(n)S(n+1)=n$ takes the form

$$S(p^{\alpha})S(q^{\beta}) = p^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}. \tag{1}$$

By Lemma 3.13, $S(q^{\beta})$ is divisible by q. Thus, the l.h.s. of (1) is divisible by q, but q does not divide the r.h.s. of (1). This contradiction establishes the lemma. ■

Lemma 3.3.6 proves that the equation $S(n)S(n+1)=n$ has no solution. However, the situation is different if we consider the equation

$$S(n)S(n+1)=kn, \; k \geq 2.$$

Proceeding as in the proof of Lemma 3.3.6, the above equation becomes

$$S(p^{\alpha})S(q^{\beta}) = kp^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r},$$

and the argument employed in the proof of Lemma 3.3.6 is no longer valid. For example, when $k=2$, besides the trivial solution $n=1$, another solution is $n=15$ (with $S(15)=5$, $S(16)=6$). Again, $n=2, 5, 8$ are three solutions of $S(n)S(n+1)=3n$ (with $S(8)=4$, $S(9)=6$). Corresponding to $k=4$, besides the prime solutions n=3, 7, 11, 23, a composite solution is n=2559 (with $S(2559)=853$, $S(2560)=12$). For $k=5$, in addition to the two prime solutions $n=19, 59$, four composite solutions are $n=74$, 249, 674, 1574 (with $S(74)=37$, $S(75)=10$, $S(249)=83$, $S(250)=15$, $S(674)=337$, $S(675)=10$, $S(1574)=787$, $S(1575)=10$). For $k=6$, eight prime solutions are $n=17, 47, 71, 79, 89, 179, 239, 359$, and $n=1214$ is a composite solution (with $S(1214)=607$, $S(1215)=12$). Corresponding to $k=7$, $n=13, 41, 83, 139, 167, 251$ are six prime solutions and $n=146$, 538, 734, 1322, 1371 are five prime solutions (with $S(146)=73$, $S(147)=14$, $S(538)=269$, $S(539)=14$, $S(734)=367$, $S(735)=14$, $S(1322)=661$, $S(1323)=14$, $S(1371)=457$, $S(1372)=21$). For $k=9$, $n=107, 269, 2186$ are three solutions, only the third one being composite (with $S(2186)=1093$, $S(2187)=18$). For $k=10$, in addition to the prime solutions $n=149, 199, 349$, a composite solution is $n=1874$ (with $S(1874)=937$, $S(1875)=20$). For $k=11$, six prime solutions are $n=43, 109, 131, 197, 263, 307$, and three composite solutions are $n=362$, 1814, 2661 (with $S(362)=181$, $S(363)=22$, $S(1814)=907$, $S(1815)=22$, $S(2661)=887$, $S(2662)=33$). For $k=13$, five prime solutions are $n=103, 181, 233, 311, 389$, and one composite solution is $n=1858$ (with $S(1858)=929$, $S(1859)=26$). Corresponding to $k=17$, four solutions are $n=67, 271, 373, 866$ (with $S(866)=433$, $S(867)=34$). For $k=19$, $n=37, 113, 151, 227, 379, 1082$ are solutions (with $S(1082)=541$, $S(1083)=38$), and for $k=37$, $n=73, 4106$ are solutions (with $S(4106)=2053$, $S(4107)=74$).

The problem of convergence of (infinite) series involving S(n) and its different variants has been treated by several researchers. The following two lemmas are due to Ashbacher [2].

***Lemma 3.3.7* :** The series $\sum_{n=1}^{\infty} \frac{1}{S(n)}$ is divergent.

***Proof* :** Since (by Lemma 3.7) $S(n) \leq n$ for all $n \geq 1$, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{S(n)} \geq \sum_{n=1}^{\infty} \frac{1}{n} > \infty,$$

since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. ■

***Lemma 3.3.8* :** The series $\sum_{n=1}^{\infty} \frac{1}{S(p_n)}$ is divergent, where p_n is the n–th prime.

***Proof* :** By Lemma 3.1.1,

$$\sum_{n=1}^{\infty} \frac{1}{S(p_n)} = \sum_{n=1}^{\infty} \frac{1}{p_n} > \infty,$$

since the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ is divergent (Hardy and Wright [6], Theorem 19). ■

***Lemma 3.3.9* :** The series $\sum_{n=1}^{\infty} \frac{1}{S(2n)}$ is divergent.

***Proof* :** Letting p_n be the n–th prime, by Lemma 3.1.2, $S(2p_n) = \max\{2, p_n\} = p_n$, and so

$$\sum_{n=1}^{\infty} \frac{1}{S(2n)} > \sum_{n=1}^{\infty} \frac{1}{S(2p_n)} = \sum_{n=1}^{\infty} \frac{1}{p_n} > \infty. \ ■$$

***Lemma 3.3.10* :** Both the series $\sum_{n=1}^{\infty} \frac{S(n)}{n}$ and $\sum_{n=1}^{\infty} \frac{n}{S(n)}$ are divergent.

***Proof* :** Let

$n = p_n$, $n \geq 1$ (where p_n is the n–th prime).

Then,

$$\frac{S(n)}{n} = \frac{S(p_n)}{p_n} = \frac{p_n}{p_n} = 1, \quad \frac{n}{S(n)} = \frac{p_n}{S(p_n)} = 1.$$

Therefore, both the series $\sum_{n=1}^{\infty} \frac{S(p_n)}{p_n}$ and $\sum_{n=1}^{\infty} \frac{p_n}{S(p_n)}$ are divergent. Hence,

$$\sum_{n=1}^{\infty} \frac{S(n)}{n} > \sum_{k=1}^{\infty} \frac{S(p_n)}{p_n} > \infty, \sum_{n=1}^{\infty} \frac{n}{S(n)} > \sum_{k=1}^{\infty} \frac{p_n}{S(p_n)} > \infty. \ ■$$

Next, we concentrate our attention to equations involving S(n) and other classical functions of Number Theory, such as the divisor function d(n) and the Euler phi function $\phi(n)$. These are given in Lemma 3.3.13 – Lemma 3.3.19. To prove these results, we need the results below.

Lemma 3.3.11 **:** Let $n \geq 2$ be an integer. Then,

(1) $F(\alpha) \equiv \dfrac{n^{\alpha}}{\alpha+1}$ is strictly increasing in $\alpha \geq 1$ for any $n \geq 2$ fixed, with $F(1) \geq 1$;

(2) $G(\alpha) \equiv \dfrac{n^{\alpha-1}}{\alpha(\alpha+1)}$ is strictly increasing in $\alpha \geq 2$ for any $n \geq 3$ fixed, $G(\alpha)$ is strictly increasing in $\alpha \geq 3$ when $n = 2$;

(3) $H(\alpha) \equiv \dfrac{(n+1)n^{\alpha-1}}{\alpha(\alpha+1)}$ is strictly increasing in $\alpha \geq 2$ for any $n \geq 3$ fixed, with $F(1) > 1$; $H(\alpha)$ is strictly increasing in $\alpha \geq 3$ for $n = 2$, with $F(2) = 1 = F(3)$ when $n = 2$.

Proof **:** The results corresponding to $n = 2$ in cases (2) and (3) can easily be verified.

(1) Note that

$$F(\alpha+1) \equiv \frac{n^{\alpha+1}}{\alpha+2} > \frac{n^{\alpha}}{\alpha+1} \equiv F(\alpha) \quad \text{if and only if} \quad n(\alpha+1) > \alpha+2,$$

which is true for any $\alpha \geq 1$ and any $n \geq 2$.

(2) In this case,

$$G(\alpha+1) \equiv \frac{n^{\alpha}}{(\alpha+1)(\alpha+2)} > \frac{n^{\alpha-1}}{\alpha(\alpha+1)} \equiv G(\alpha) \quad \text{if and only if} \quad n\alpha > \alpha+2$$

which is true for any $\alpha \geq 2$ and any $n \geq 3$.

(3) Since

$$H(\alpha+1) \equiv \frac{(n+1)n^{\alpha}}{(\alpha+1)(\alpha+2)} > \frac{(n+1)n^{\alpha-1}}{\alpha(\alpha+1)} \equiv H(\alpha) \quad \text{if and only if} \quad n\alpha > \alpha+2,$$

the result follows for any $\alpha \geq 2$ and any $n \geq 3$.

Hence the proof is complete. ■

Lemma 3.3.12 **:** Let $n \geq 2$ be an integer. Then,

(1) for $n \geq 2$, **(a)** $\dfrac{n^{\alpha}}{\alpha+1} \geq 2$ for $\alpha \geq 3$, **(b)** $\dfrac{n^{\alpha}}{\alpha+1} > 3$ for $\alpha \geq 4$, **(c)** $\dfrac{n^{\alpha}}{3\alpha+1} > 1$ for $\alpha \geq 4$,

(2) for $n \geq 3$, **(a)** $\dfrac{n^{\alpha}}{\alpha+1} \geq 3$ for $\alpha \geq 2$, **(b)** $\dfrac{n^{\alpha}}{4\alpha+1} > 1$ for $\alpha \geq 3$, **(c)** $\dfrac{n^{\alpha-1}}{\alpha(\alpha+1)} > 1$ for $\alpha \geq 4$,

(3) for $n \geq 5$, **(a)** $\dfrac{n^{\alpha}}{\alpha+1} > 2$ for $\alpha \geq 1$, **(b)** $\dfrac{n^{\alpha-1}}{\alpha(\alpha+1)} > 2$ for $\alpha \geq 3$,

(4) for $n \geq 7$, **(a)** $\dfrac{n^{\alpha}}{\alpha+1} > 3$ for $\alpha \geq 1$, **(b)** $\dfrac{n^{\alpha-1}}{\alpha(\alpha+1)} > 1$ for $\alpha \geq 2$,

(5) for $n \geq 2$, **(a)** $\dfrac{(n+1)n^{\alpha-1}}{\alpha(\alpha+1)} \geq 1$ for $\alpha \geq 1$, **(b)** $\dfrac{n(n+1)^{\alpha-1}}{\alpha(\alpha+1)} \geq 1$ for $\alpha \geq 1$.

We now consider the problem of finding the solutions of the equation

$$S(n) + d(n) = n.$$

The problem has been considered by Asbacher [5]. Here, we follow a different approach.

Lemma 3.3.13 **:** The only solutions of the equation

$$S(n) + d(n) = n.$$

are $n = 8, 9$.

Proof **:** Let, for some integer n,

$$S(n) + d(n) = n. \tag{3.3.1}$$

Clearly, n is not a prime, since for any prime $p \geq 2$,

$$S(n) + d(n) = p + 2 > p = n.$$

Thus, n must be a composite number.
First, we consider some particular cases.

Case 1 : When n is of the form $n = p^\alpha$, $p \geq 2$ is a prime, $(\alpha \geq 2)$.
In this case, the equation (3.3.1), together with Corollary 3.3(1), becomes

$$p^\alpha = S(p^\alpha) + d(p^\alpha) \leq p\alpha + (\alpha + 1) = \alpha(p + 1) + 1. \tag{1}$$

When $\alpha = 2$, from (1), $p^2 \leq 2p + 3$, which is true only if $p \leq 3$.

Now, let $p = 2$, so that (1) takes the form $2^\alpha \leq 3\alpha + 1$, which is absurd if $\alpha \geq 4$ (by Lemma 3.3.12).

Next, let $p = 3$ in (1) to get $3^\alpha \leq 4\alpha + 1$, which is impossible if $\alpha \geq 3$ (by Lemma 3.3.12).
Thus, in this case, the only possible candidates for (3.3.1) to hold true are $n = 2^2, 2^3, 3^2$.
Checking with these values of n, we see that

$$S(4) + d(4) = 4 + 3 = 7 > 4 = n, \; S(8) + d(8) = 4 + 4 = 8 = n,$$

$$S(9) + d(9) = 6 + 3 = 9 = n.$$

Thus, in this case, $n = 8$ and $n = 9$ are two solutions of the equation (3.3.1).

Case 2 : When n is of the form $n = 2p^\alpha$, $p \geq 3$ is a prime.
In this case, by parts (2) and (3) of Lemma 3.3.12,

$$2p^\alpha > p^\alpha + 2(\alpha + 1) \geq S(2p^\alpha) + d(2p^\alpha) \text{ if } p = 3 \text{ and } \alpha \geq 2, \text{ or if } p \geq 5 \text{ and } \alpha \geq 1.$$

Now, checking with $n = 2 \times 3 = 6$, we see that

$$S(6) + d(6) = 3 + 4 = 7 > 6 = n.$$

Thus, any n of the form $n = 2p^\alpha$ cannot be a solution of the equation (3.3.1)

Now, to deal with the general case, let

$$n = p^\alpha \; q^\beta \; p_1^{\alpha_1} \; p_2^{\alpha_2} \; \ldots \; p_k^{\alpha_k} \tag{3.3.2}$$

be the representation of n in terms of its distinct prime factors $p, q, p_1, p_2, \ldots, p_k$.

Let, for definiteness,

$$S(n) = S(p^\alpha).$$

Now, by Theorem 0.1 (in Chapter 0),

$$d(n) = (\alpha+1)(\beta+1)(\alpha_1+1)(\alpha_2+1)\ \dots\ (\alpha_k+1). \tag{2}$$

Dividing throughout of (3.3.1) by d(n), using (2) together with Lemma 3.7 (that $S(p^\alpha) \le p^\alpha$), and the fact that $\beta + 1 \ge 1$, $\alpha_i + 1 \ge 1$ for all $1 \le i \le k$, we get the following chain of inequalities :

$$\frac{p^\alpha}{\alpha+1}\cdot\frac{q^\beta}{\beta+1}\cdot\frac{p_1^{\alpha_1}}{\alpha_1+1}\cdot\frac{p_2^{\alpha_2}}{\alpha_2+1}\ \cdots\ \frac{p_k^{\alpha_k}}{\alpha_k+1} \le \frac{p^\alpha}{\alpha+1}\cdot\frac{1}{(\beta+1)(\alpha_1+1)\dots(\alpha_k+1)} + 1 \le \frac{p^\alpha}{\alpha+1} + 1,$$

so that, after rearrangement of terms, we get

$$\frac{p^\alpha}{\alpha+1}\left(\frac{q^\beta}{\beta+1}\cdot\frac{p_1^{\alpha_1}}{\alpha_1+1}\cdot\frac{p_2^{\alpha_2}}{\alpha_2+1}\ \cdots\ \frac{p_k^{\alpha_k}}{\alpha_k+1} - 1\right) \le 1. \tag{3}$$

Case 3 : When n is of the form $n = p^\alpha q^\beta$, p and q are distinct primes.

In this case, (3) takes the form

$$\frac{p^\alpha}{\alpha+1}\left(\frac{q^\beta}{\beta+1} - 1\right) \le 1. \tag{4}$$

Here also, we consider some particular cases.

First, let $p = 2$, $q \ge 3$.

Since $\frac{2^\alpha}{\alpha+1} \ge 1$ for any $\alpha \ge 1$ (by Lemma 3.3.11(1)), (4) gives $\frac{q^\beta}{\beta+1} \le 2$, which is impossible if $q = 3$ and $\beta \ge 2$, or if $q \ge 5$ and $\beta \ge 1$ (by parts (2) and (3) of Lemma 3.3.12).

For $\alpha \ge 4$, (4) with Lemma 3.3.12(1) give $3\left(\frac{q^\beta}{\beta+1} - 1\right) < 1$, which is impossible if $q \ge 3$ and $\beta \ge 1$.

Next, let $p = 3$, $q \ne 3$.

When $\alpha = 1$, so that $\frac{3^\alpha}{\alpha+1} = \frac{3}{2} > 1$, (4) gives $\frac{q^\beta}{\beta+1} < 2$, which is impossible if $q = 2$ and $\beta \ge 3$, or if $q \ge 5$ (any $\beta \ge 1$).

Since $\frac{3^\alpha}{\alpha+1} \ge 3$ for $\alpha \ge 2$ (by Lemma 3.3.12(2)), (4) gives $3\left(\frac{q^\beta}{\beta+1} - 1\right) \le 1$, which is impossible if $q = 2$ and $\beta \ge 2$ (with strict equality sign when $q = 2$, $\beta = 2$ and $\alpha = 2$), or if $q \ge 5$ and $\beta \ge 1$.

Thirdly, let $p \ge 5$, $q \ne 5$.

In this case, since $\frac{p^\alpha}{\alpha+1} > 2$ for any $\alpha \ge 1$ (by Lemma 3.3.12(3)), (4) gives $2\left(\frac{q^\beta}{\beta+1} - 1\right) < 1$, which is impossible if $q = 2$ and $\beta \ge 2$, or if $q \ge 3$ and $\beta \ge 1$.

Thus, in this case, the only possible candidates for (3.3.1) to hold true are

$$n = 3.2^2,\ 3.2^3,\ 3^2.2^2,\ 2p^\alpha.$$

The case $n = 2p^\alpha$ has been verified in Case 2. Checking with the other values of n, we see that

$$S(12)+d(12)=4+6=10<12=n,\ S(24)+d(24)=4+8=12<24=n,$$
$$S(36)+d(36)=6+9=15<36=n.$$

Hence, there is no solution of (3.3.1) of the form $n=p^{\alpha}q^{\beta}$.

Now, we confine our attention to (3) with $k\geq 1$. From (3), we get

$$\frac{q^{\beta}}{\beta+1}\cdot\frac{p_1^{\alpha_1}}{\alpha_1+1}\cdot\frac{p_2^{\alpha_2}}{\alpha_2+1}\ \cdots\ \frac{p_k^{\alpha_k}}{\alpha_k+1}<2,$$

where, without loss of generality, we may assume that $p_i\geq 5$ for all $1\leq i\leq k$. But then, by Lemma 3.3.12, we have a contradiction. ■

In the proof of Lemma 3.3.13 above, we proved more. They are summarized below.

Corollary 3.3.3 **:** Let d(n) be the divisor function. Then,

(1) the equation $S(n)+d(n)=n$ has the only solutions $n=8, 9$;

(2) the inequality $S(n)+d(n)>n$ has the solutions $n=1, 4, 6, p$;

(3) $S(n)+d(n)<n$ for any integer $n\neq 1, 4, 6, 8, 9, p$.

The following lemma is due to Jingping [7].

Lemma 3.3.14 **:** The only solutions of the equation

$$S(n)=\phi(n) \tag{3.3.3}$$

are $n=1, 8, 9, 12$.

Proof **:** Clearly, the equation (3.3.3) admits no prime solution, since for any prime $p\geq 2$,

$$S(p)=p>p-1=\phi(p).$$

Thus, any solution n of (3.3.3) must be a composite number. So, let

$$n=p^{\alpha}q^{\beta}p_1^{\alpha_1}p_2^{\alpha_2}\ \cdots\ p_k^{\alpha_k} \tag{3.3.2}$$

be the representation of n in terms of its distinct prime factors $p, q, p_1, p_2, \ldots, p_k$.

Let, for definiteness,

$$S(n)=S(p^{\alpha}).$$

Now, by Theorem 0.5 (in Chapter 0), $\phi(n)$ is multiplicative, so that (3.3.3) can be written as

$$p^{\alpha-1}(p-1)\ \phi(q^{\beta})\ \phi(p_1^{\alpha_1}p_2^{\alpha_2}\ \cdots\ p_k^{\alpha_k})\leq p^{\alpha}$$

$$\Rightarrow\ \phi(q^{\beta})\ \phi(p_1^{\alpha_1}p_2^{\alpha_2}\ \cdots\ p_k^{\alpha_k})\leq\frac{p}{p-1}\leq 2, \tag{1}$$

which is impossible if $k\geq 1$, since in such a case

$$\phi(q^{\beta})\ \phi(p_1^{\alpha_1}p_2^{\alpha_2}\ \cdots\ p_k^{\alpha_k})\geq 2^2.$$

Thus, any solution of the equation (3.3.3) can be either of the two forms $n=p^{\alpha}$ and $n=p^{\alpha}q^{\beta}$.

We now consider these two possibilities below.

Case 1 : When n is of the form $n = p^{\alpha}$ ($\alpha \geq 2$).

In this case, note that, if $p = 2$, then

$$\phi(2^{\alpha}) \equiv 2^{\alpha-1} > 2\alpha \geq S(2^{\alpha}) \text{ if } \alpha \geq 5,$$

and if $p \geq 3$, $\alpha \geq 3$, then

$$\phi(p^{\alpha}) \equiv p^{\alpha-1}(p-1) \geq 2p^{\alpha-1} > \alpha p \geq S(p^{\alpha}).$$

Thus, in this case, the only possible candidates for (3.3.3) to hold true are

$$n = 2^2, 2^3, 2^4, 3^2.$$

Checking with these values of n, we see that

$$S(4) = 4 > 2 = \phi(4),\ S(8) = 4 = \phi(8),\ S(16) = 6 < 8 = \phi(8),\ S(9) = 6 = \phi(9).$$

Thus, in this case, there are two solutions, namely, $n = 4, 9$.

Case 2 : When n is of the form $n = p^{\alpha} q^{\beta}$.

In this case, from (1), we get

$$\phi(q^{\beta}) \leq \frac{p}{p-1} \leq 2 \quad \Rightarrow \quad \phi(q^{\beta}) = 2 \quad \Rightarrow \quad \beta = 1,\ p = 2,\ q = 3.$$

With $n = 3.2^{\alpha}$, (3.3.3) takes the form

$$\phi(3.2^{\alpha}) \equiv 2^{\alpha} = S(3.2^{\alpha}) = \max\{S(2^{\alpha}), S(3)\} = S(3.2^{\alpha}),$$

with the solution $\alpha = 2$.

Hence, $n = 12$ is the only solution in this case, with

$$S(12) = 4 = \phi(12).$$

All these establish the lemma. ■

The following three lemmas are due to Yi Yuan [8].

Lemma 3.3.15 **:** The only solutions of the equation $S(n^2) = \phi(n)$ are $n = 1, 24, 50$.

Lemma 3.3.16 **:** The only solutions of the equation $S(n^3) = \phi(n)$ are $n = 1, 48, 98$.

Lemma 3.3.17 **:** The only solution of the equation $S(n^4) = \phi(n)$ is $n = 1$.

The following result has been established by Maohua Le [9].

Lemma 3.3.18 **:** The only solution of the equation $S(n + n^2 + \ldots + n^n) = \phi(n)S(1)S(2)\ldots S(n)$ is $n = 1$.

Weiguo Duan and Yanrong Xue [10] have established the following result.

Lemma 3.3.19 **:** The only solutions of the equation $S(1) + S(2) + \ldots + S(n) = \phi(\frac{n(n+1)}{2})$ are $n = 1, 10$.

***Lemma 3.3.20* :** A necessary and sufficient condition that the equation

$$[S(n)]^2 + S(n) = kn \tag{3.3.4}$$

possesses a solution for some integers $n \geq 1$ and $k \geq 1$, is that the following equation has a solution :

$$S(\frac{r(r+1)}{k}) = r \quad \text{for some integers } r \geq 1 \text{ and } k \geq 1.$$

***Proof* :** Solving the equation

$$[S(n)]^2 + S(n) - kn = 0$$

for positive S(n), we get

$$S(n) = \tfrac{1}{2}(\sqrt{1+4kn} - 1). \tag{3.3.5}$$

Thus, a necessary condition such that the equation (3.3.4) has a solution is that, k and n must be such that $1+4kn$ is a perfect square. Let

$$1 + 4kn = x^2 \text{ for some integer } x \geq 1.$$

Then,

$$4kn = x^2 - 1 = (x-1)(x+1). \tag{1}$$

Since $x-1$ and $x+1$ has the same parity, from (1), both are even. Letting $x-1=2r$, from (1),

$$4kn = (2r).(2r+2) \quad \Rightarrow \quad n = \frac{r(r+1)}{k}.$$

Finally, we get from (3.3.2),

$$S(\frac{r(r+1)}{k}) = r. \tag{3.3.6}$$

Conversely, $n = \dfrac{r(r+1)}{k}$ satisfies the equation (3.3.4), as is shown below :

$$[S(n)]^2 + S(n) = r^2 + r = r(r+1) = kn.$$

All these complete the proof of the lemma. ■

***Lemma 3.3.21* :** For any integer $k \geq 2$ fixed, the equation

$$S(\frac{n(n+1)}{k}) = n$$

has an infinite number of solutions.

***Proof* :** Given the integer $k \geq 2$, let p be a prime of the form

$$p = ka - 1, \; a \geq 1. \tag{1}$$

By Lemma 3.7, $S(\frac{p+1}{k}) \leq \frac{p+1}{k} < p$, so that by Lemma 3.1.3,

$$S(\frac{p(p+1)}{k}) = \max\{S(p),\; S(\frac{p+1}{k})\} = S(p) = p.$$

This shows that $n = p$ is a solution of the given equation. Now, since there are an infinite number of primes of the form (1), the lemma is proved. ■

Remark 3.3.2 **:** Equation (3.3.6) allows us to find the solutions of the equation (3.3.4) for any particular value of k, as given in Lemma 3.3.21.

(1) When $k=1$, the solutions of (3.3.4) are

$$n=p(p+1), \text{ where } p\geq 5 \text{ is a prime.}$$

When $p\geq 5$, $p+1\geq 6$ is a composite number, and so by Lemma 3.9,

$$S(p+1)\leq \frac{p+1}{2}<p.$$

Now, since $(p, p+1)=1$, by Lemma 3.1.3,

$$S(p(p+1))=\max\{S(p), S(p+1)\}=S(p)=p.$$

(2) When $k=2$, besides the trivial solution $n=1$, other solutions of (3.3.4) are

$$n=\frac{p(p+1)}{2}, \text{ where } p\geq 3 \text{ is a prime.}$$

In particular, any even perfect number $n=2^a(2^{a+1}-1)$ is also a solution of (3.3.4), where $a\geq 1$ is an integer such that $2^{a+1}-1$ is prime : In such a case

$$S(2^a)\leq 2^a<2^{a+1}-1 \quad \Rightarrow \quad S(n)=\max\{S(2^a), S(2^{a+1}-1)\}=2^{a+1}-1.$$

(3) When $k=5$, $r=4$ is a solution of (3.3.6), and $n=4$ is a solution of (3.3.4), besides the solutions given in Lemma 3.3.21.

Following a different approach, the above results have also been found by Rongi Chen and Maohua Le [11]

The Smarandache perfect number and the completely Smarandache perfect number are defined as follows.

Definition 3.3.1 **:** Given an integer $n\geq 1$,

(1) n is called *Smarandache perfect* if and only if

$$n=\sum_{i=1}^{k} S(d_i).$$

where $d_1\equiv 1, d_2, \ldots, d_k$ are the proper divisors of n.

(2) n is called *completely Smarandache perfect* if and only if

$$n=\sum_{d\mid n} S(d),$$

where the sum is over all divisors d of n (including n).

Remark 3.3.3 **:** The proof of Maohua Le [12], with the modification that $S(1)=0$, shows that the only Smarandache perfect number is $n=12$. Gronas [13] has shown that, under the condition $S(1)=0$, all completely Smarandache perfect numbers are $n=p$, 9, 16, 24. However, the situation is different if $S(1)=1$.

The Smarandache perfect numbers and the completely Smarandache perfect numbers are given in Lemma 3.3.22 and Lemma 3.3.23 respectively.

***Lemma 3.3.22* :** The only Smarandache perfect numbers are $n=1, 6$.

***Proof* :** Let n be an Smarandache perfect number, so that, by Definition 3.3.1,

$$n = \sum_{\substack{d|n \\ 1\le d<n}} S(d). \qquad (3.3.7)$$

Clearly, no prime is a solution of (3.3.7). Also, note that, if n is of the form $n=p^{\alpha}$ $(\alpha\ge 2)$, then the proper divisors of n are 1, p, p^2, …, $p^{\alpha-1}$, with $S(1)=1$, so that the r.h.s. of (3.3.7) is not divisible by p, while p divides the l.h.s. Thus, there is no solution of the form $n=p^{\alpha}$.

By Lemma 3.8, for any divisor d of n, $S(d)\le S(n)$, with strict inequality sign for at least one divisor of n. Therefore, from (3.3.7), we get

$$n = \sum_{\substack{d|n \\ 1\le d<n}} S(d) < [d(n)-1]S(n) \quad \Rightarrow \quad n < d(n)S(n). \qquad (1)$$

Thus, any solution n of (3.3.7) must satisfy the inequalities in (1).
We first consider some particular cases.

Case 1 : When n is of the form $n=pq$, p and q are primes with $q>p$.
In this case, $d(n)=4$, and so, from (1) we get

$$n = pq < 3q \quad \Rightarrow \quad p = 2.$$

Now, if $p=2$, then

$$\sum_{\substack{d|n \\ 1\le d<n}} S(d) = S(1)+S(p)+S(q) = 1+p+q = 3+q = n = 2q \quad \Rightarrow \quad q = 3.$$

Thus, in this case, $n=2\times 3=6$ is the only solution of the equation (3.3.7).

Case 2 : When n is of the form $n=2^{\alpha}q$, $\alpha\ge 2$, $q\ge 3$ is a prime.
First, let $S(n)=S(2^{\alpha})$. Since $d(n)=2(\alpha+1)$, from (1), we get

$$n = 2^{\alpha}q < 2(\alpha+1)S(2^{\alpha}) \le 2^2\alpha(\alpha+1) \quad \Rightarrow \quad 2^{\alpha-2}q < \alpha(\alpha+1),$$

which is impossible if (i) $q=3$ and $\alpha\ge 6$, or if (ii) $q=5$ and $\alpha\ge 4$, or if (iii) $q\ge 7$ and $\alpha\ge 2$.
Thus, the possible candidates for the equation (3.3.7) to hold true are

$$n=3.2^{\alpha},\ 2\le\alpha\le 5;\ (n=5.2^3 \text{ is to be excluded since } S(5.2^3)\ne S(2^3)).$$

But, when $n=2^2q$, $\sum_{\substack{d|n \\ 1\le d<n}} S(d) = S(1)+S(2)+S(2^2)+S(q)+S(2q) = 1+2+4+q+q = 7+2q$;

when $n=2^3q$, $\sum_{\substack{d|n \\ 1\le d<n}} S(d) = S(1)+S(2)+S(2^2)+S(2^3)+S(q)+S(2q)+S(2^2q) = 15+2q$;

for $n=2^4q$, $\sum_{\substack{d|n \\ 1\le d<n}} S(d) = S(1)+S(2)+S(2^2)+S(2^3)+S(2^4)+S(q)+S(2q)+S(2^2q)+S(2^3q) = 25+2q$;

for $n=2^5q$, $\sum_{\substack{d|n \\ 1\le d<n}} S(d) = S(1)+S(2)+S(2^2)+S(2^3)+S(2^4)+S(2^5)+S(q)+S(2q)+S(2^2q)+S(2^3q)+S(2^4q) = 39+2q$;

so that, in each case, the term on the right of (3.3.7) is odd, while n is even.

Next, let $S(n) = q$. Then, from (1), we get

$$n = 2^{\alpha} q < 2(\alpha+1)q \quad \Rightarrow \quad 2^{\alpha-1} < \alpha+1,$$

which is possible only for $\alpha = 2$, and so, $n = 2^2 q$, which is not a solution of (3.3.7) for any $q \geq 3$. Thus, there is no solution of the equation (3.3.7) of the form $n = 2^{\alpha} q$.

Case 3 : When n is of the form $n = p^{\alpha} q^{\beta}$ ($\alpha \geq 1$, $\beta \geq 1$, $\alpha\beta \geq 2$).

Let $S(n) = S(p^{\alpha})$. Now, since $d(n) = (\alpha+1)(\beta+1)$, (1) takes the form

$$n = p^{\alpha} q^{\beta} < (\alpha+1)(\beta+1) S(p^{\alpha}) \leq (\alpha+1)(\beta+1)\alpha p \quad \Rightarrow \quad \frac{p^{\alpha-1}}{\alpha(\alpha+1)} . \frac{q^{\beta}}{\beta+1} < 1. \qquad (2)$$

Now, since $\frac{p^{\alpha-1}}{\alpha(\alpha+1)} > 1$ for $p \geq 7$ and $\alpha \geq 2$, (2) is impossible for $p \geq 7$, $q \neq p$, $\alpha \geq 2$ and $\beta \geq 1$ (by part (4) of Lemma 3.3.12). Thus, it is sufficient to check for the primes $p = 2, 3, 5$ only.

First, let $p = 2$ (so that $q \geq 3$, $\alpha \geq \beta \geq 2$). Then, (2) reads as

$$\frac{2^{\alpha-1}}{\alpha(\alpha+1)} . \frac{q^{\beta}}{\beta+1} < 1. \qquad (3)$$

Since $\frac{q^{\beta}}{\beta+1} \geq 3$ if $q \geq 3$ and $\beta \geq 2$, (3) gives $\frac{3.2^{\alpha-1}}{\alpha(\alpha+1)} < 1$, which is impossible for any $\alpha \geq 1$ (by Lemma 3.3.12(5)). Thus, in this case, (3.3.7) has no solution (by Case 2 above).

Next, let $p = 3$, so that from (2)

$$\frac{3^{\alpha-1}}{\alpha(\alpha+1)} . \frac{q^{\beta}}{\beta+1} < 1. \qquad (4)$$

If $q \geq 5$ and $\beta \geq 1$ (so that $\frac{q^{\beta}}{\beta+1} > 2$), (4) gives $\frac{2.3^{\alpha-1}}{\alpha(\alpha+1)} < 1$, which is impossible for any $\alpha \geq 1$ (by Lemma 3.3.12(5)); if $q = 2$ and $\beta \geq 3$ (so that $\frac{2^{\beta}}{\beta+1} \geq 2$), from (4), $\frac{2.3^{\alpha-1}}{\alpha(\alpha+1)} < 1$, which is impossible for any $\alpha \geq 1$. Moreover, from (4), $\frac{3^{\alpha-1}}{\alpha(\alpha+1)} < 1$, which is impossible for $\alpha \geq 4$.

Thus, the only possible candidates for (3.3.7) to hold true are $n = 2.3^{\alpha}$, $2^2.3^{\alpha}$ with $2 \leq \alpha \leq 3$.

When $n = 2.3^{\alpha}$,

$$\sum_{\substack{d|n \\ 1 \leq d < n}} S(d) = S(1) + S(2) + S(3) + \ \ldots \ + S(3^{\alpha}) + S(2.3) + S(2.3^2) + \ \ldots \ + S(2.3^{\alpha-1})$$

$$= 1 + 2 + \sum_{k=1}^{\alpha} (3k) + \sum_{k=1}^{\alpha-1} (3k) = 3(\alpha^2 + 1),$$

and the equation $3(\alpha^2 + 1) = 2.3^{\alpha}$ has no solution for $\alpha \geq 2$. And when $n = 2^2.3^{\alpha}$,

$$\sum_{\substack{d|n \\ 1 \leq d < n}} S(d) = S(1) + S(2) + S(2^2) + S(3) + \ \ldots \ + S(3^{\alpha}) + S(2.3) + \ \ldots \ + S(2.3^{\alpha}) + S(2^2.3) + \ \ldots \ + S(2^2.3^{\alpha-1})$$

$$= 1 + 2 + 4 + 2\sum_{k=1}^{\alpha} (3k) + 4 + \sum_{k=2}^{\alpha-1} (3k),$$

which is not divisible by 3.

Finally, let $p=5$, so that (2) takes the form

$$\frac{5^{\alpha-1}}{\alpha(\alpha+1)}.\frac{q^{\beta}}{\beta+1}<1. \qquad (5)$$

If $q \geq 7$ and $\beta \geq 1$, then since $\frac{q^{\beta}}{\beta+1}>3$, (5) gives $\frac{3.5^{\alpha-1}}{\alpha(\alpha+1)}<1$, which is impossible for any $\alpha \geq 1$; if $q=2$ and $\beta \geq 3$ (so that $\frac{2^{\beta}}{\beta+1} \geq 2$), then from (5), $\frac{2.5^{\alpha-1}}{\alpha(\alpha+1)}<1$, which is impossible for any $\alpha \geq 1$; if $q=3$ and $\beta \geq 2$, then since $\frac{3^{\beta}}{\beta+1} \geq 3$, from (5), $\frac{3.5^{\alpha-1}}{\alpha(\alpha+1)}<1$, which is impossible for any $\alpha \geq 1$. Also, from (5), $\frac{5^{\alpha-1}}{\alpha(\alpha+1)}<1$, which is impossible for $\alpha \geq 3$.

Thus, in this case, the only possible candidates for (3.3.7) to hold true are $n=3.5^2, 2.5^2, 2^2.5^2$.

Note, however, that $n=3.5^2$ violates the inequality (5), since $\frac{5}{6}.\frac{3}{2}>1$.

Now, when $n=2.5^2$, $\sum_{\substack{d|n \\ 1\leq d<n}} S(d) = S(1)+S(2)+S(5)+S(5^2)+S(2.5) = 1+2+5+10+5 = 23$;

for $n=2^2.5^2$, $\sum_{\substack{d|n \\ 1\leq d<n}} S(d) = S(1)+S(2)+S(2^2)+S(5)+S(5^2)+S(2.5)+S(2^2.5)+S(2.5^2) = 42$.

Thus, there is no solution of (3.3.7) of the form $n=p^{\alpha}q^{\beta}$.

Now, we consider the general case. So, let

$$n = p^{\alpha} q^{\beta} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$$

be the representation of n in terms of its distinct prime factors p, q, p, $p_1, p_2, \ldots, p_k$ $(k \geq 1)$.

Let, for definiteness, $S(n)=S(p^{\alpha})$. Then, from (1), we get

$$n < d(n)S(n) \leq d(n)S(p^{\alpha})$$

$$\Rightarrow \quad \frac{p^{\alpha}}{\alpha+1}.\frac{q^{\beta}}{\beta+1}.\frac{p_1^{\alpha_1}}{\alpha_1+1}.\frac{p_2^{\alpha_2}}{\alpha_2+1} \ldots \frac{p_k^{\alpha_k}}{\alpha_k+1} < S(p^{\alpha}) \leq \alpha p. \qquad (6)$$

Now, if $p=2$, without loss of generality, $q \geq 5$, and so (6) takes the form

$$\frac{2^{\alpha}}{\alpha(\alpha+1)}.\frac{p_1^{\alpha_1}}{\alpha_1+1}.\frac{p_2^{\alpha_2}}{\alpha_2+1} \ldots \frac{p_k^{\alpha_k}}{\alpha_k+1}<1,$$

which is impossible. On the other hand, if $p=3$, with $q \geq 5$, (6) takes the form

$$\frac{2.3^{\alpha-1}}{\alpha(\alpha+1)}.\frac{p_1^{\alpha_1}}{\alpha_1+1}.\frac{p_2^{\alpha_2}}{\alpha_2+1} \ldots \frac{p_k^{\alpha_k}}{\alpha_k+1}<1,$$

which is also impossible.

All these show that $n=1, 6$ are the only solutions of (3.3.7), which we intended to prove. ■

***Lemma 3.3.23* :** The only completely Smarandache perfect numbers are $n=1, 28$.

***Proof* :** Let n be a completely Smarandache perfect number, so that, by Definition 3.3.1,

$$n = \sum_{\substack{d|n \\ 1\le d\le n}} S(d). \qquad (3.3.8)$$

Clearly, no prime is a solution of (3.3.8). Thus, any solution of (3.3.8) must be a composite number.
In this case, any solution of (3.3.8) must satisfy the following condition :

$$n = \sum_{\substack{d|n \\ 1\le d\le n}} S(d) < d(n)S(n). \qquad (1)$$

We proceed like that in the proof of Lemma 3.3.22, considering the following cases :

Case 1 : When n is of the form $n=p^{\alpha}$ $(\alpha \ge 2)$.
In this case, however, there is no solution of the equation (3.3.8).

Case 2 : When n is of the form $n=pq$ with $q \ge p$.
In this case,

$$\sum_{\substack{d|n \\ 1\le d\le n}} S(d) = S(1) + S(p) + S(q) + S(pq) = 1 + p + q + q = 1 + p + 2q,$$

which is odd only if $p=2$. But, then

$$\sum_{\substack{d|n \\ 1\le d\le n}} S(d) = 3 + 2q > 2q.$$

Thus, there is no solution in this case.

Case 3 : When n is of the form $n=2^{\alpha}q$, $q \ge 3$, $\alpha \ge 2$.
Assuming that $S(n)=S(2^{\alpha})$, the possible candidates for the equation (3.3.8) to hold true are $n=3.2^2, 3.2^3$.
But, when $n=2^2q$, the divisors are 1, 2, 2^2, q, 2q and 2^2q, so that

$$\sum_{\substack{d|n \\ 1\le d\le n}} S(d) = S(1) + S(2) + S(2^2) + S(q) + S(2q) + S(2^2q) = 1 + 2 + 4 + q + q + 4 = 11 + 2q,$$

which is odd.
Again, when $n=2^3q$, the divisors are 1, 2, 2^2, 2^3, q, 2q, 2^2q and 2^3q, so that

$$\sum_{\substack{d|n \\ 1\le d\le n}} S(d) = S(1) + S(2) + S(2^2) + S(2^3) + S(q) + S(2q) + S(2^2q) + S(2^3q)$$
$$= 1 + 2 + 4 + 4 + q + q + 4 + 4 = 19 + 2q,$$

which is odd.
Next, assuming that $S(n)=q$, so that $\alpha \ge 2$, we get from (1),

$$n = 2^{\alpha}q < 2(\alpha+1)q \quad \Rightarrow \quad 2^{\alpha-1} < \alpha + 1,$$

which is possible only for $\alpha=2$, and so, $n=2^2q$. Now, since

$$\sum_{\substack{d|n \\ 1\le d\le n}} S(d) = S(1) + S(2) + S(2^2) + S(q) + S(2q) + S(2^2 q) = 1 + 2 + 4 + q + q + q = 7 + 3q,$$

we see that

$$\sum_{\substack{d|n \\ 1\le d\le n}} S(d) = 7 + 3q = 2^2 q \quad \Rightarrow \quad q = 7.$$

Thus, in this case, $n = 2^2.7 = 28$ is the only solution of the equation (3.3.8).

Case 4 : When n is of the form $n = p^\alpha q^\beta$.

Letting $S(n) = S(p^\alpha)$ with $p = 3$, the only possible candidates for (3.3.8) to hold true are

$$n = 2.3^\alpha,\ 2^2.3^\alpha \text{ with } 2 \le \alpha \le 3.$$

When $n = 2.3^\alpha$,

$$\begin{aligned}\sum_{\substack{d|n \\ 1\le d<n}} S(d) &= S(1) + S(2) + S(3) + S(3^2) + \ \dots\ + S(3^\alpha) + S(2.3) + S(2.3^2) + \ \dots\ + S(2.3^\alpha) \\ &= 1 + 2 + 2\sum_{k=1}^{\alpha}(3k) \\ &= 1 + 2 + 2.3\frac{\alpha(\alpha+1)}{2} \\ &= 3(\alpha^2 + \alpha + 1),\end{aligned}$$

and the equation $3(\alpha^2 + \alpha + 1) = 2.3^\alpha$ has no solution for $\alpha \ge 2$.

Again, when $n = 2^2.3^\alpha$,

$$\begin{aligned}\sum_{\substack{d|n \\ 1\le d<n}} S(d) &= S(1) + S(2) + S(2^2) + S(3) + S(3^2) + \ \dots\ + S(3^\alpha) \\ &\quad + S(2.3) + S(2.3^2) + \ \dots\ + S(2.3^\alpha) + S(2^2.3) + S(2^2.3^2) + \ \dots\ + S(2^2.3^\alpha) \\ &= 1 + 2 + 4 + 2\sum_{k=1}^{\alpha}(3k) + 4 + \sum_{k=2}^{\alpha}(3k),\end{aligned}$$

which is not divisible by 3.

When $p = 5$, the only possible candidates for (1) to hold true are $n = 5.2^2,\ 2.5^2,\ 2^2.5^2$.

But, when $n = 2.5^2$, $\sum_{\substack{d|n \\ 1\le d\le n}} S(d) = S(1) + S(2) + S(5) + S(5^2) + S(2.5) + S(2.5^2) = 33;$

and when $n = 2^2.5^2$,

$$\sum_{\substack{d|n \\ 1\le d<n}} S(d) = S(1) + S(2) + S(2^2) + S(5) + S(5^2) + S(2.5) + S(2^2.5) + S(2.5^2) + S(2^2.5^2) = 52.$$

In the general case when $n = p^\alpha q^\beta p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ with $k \ge 1$, an analysis similar to that in the proof of Lemma 3.3.22 shows that the equation (3.3.8) has no solution.

All these complete the proof of the lemma. ■

The following result is due to Ashbacher [2]. Here, we give a simplified proof.

***Lemma 3.3.24* :** Given the successive values of the Smarandache function

$S(1)=1$, $S(2)=2$, $S(3)=3$, $Z(4)=4$, $S(5)=4$, $S(6)=3$, $S(7)=7$, $S(8)=4$, …,

the number r is constructed by concatenating the values in the following way :

$r = 0.12345374\ldots$.

Then, the number r is irrational.

***Proof* :** The proof is by contradiction. Let, on the contrary, r be a rational number. Then, from some point on, a group of consecutive digits must repeat infinite number of times, that is, r has the form

$$r = 0.12345374\ldots d_1d_2\ldots d_k\, d_1d_2\ldots d_k\ldots .$$

Let N be the integer such that $S(N)=d_1d_2\ldots d_i$, $1\le i\le k$. By Lemma 3.12,

$$S((d_1d_2\ldots d_i)!) = d_1d_2\ldots d_i,$$

and $(d_1d_2\ldots d_i)!$ is the maximum such number (so that, for any integer $n>(d_1d_2\ldots d_i)!$, $S(n)\ne d_1d_2\ldots d_i$). Thus, the digits $d_1d_2\ldots d_i$ in r cannot repeat infinitely often. This leads to a contradiction, and consequently, r is irrational. ■

The following lemma, giving an exact formula for the number of primes less than or equal to given integer $N\ge 4$, is due to Seagull [14].

***Lemma 3.3.25* :** Given any integer $N\ge 4$, the number of primes less than or equal to N, denoted by $\pi(N)$, is

$$\pi(N) = \sum_{n=2}^{N} \left[\frac{S(n)}{n}\right] - 1,$$

where [x] denotes the greatest integer less than or equal to x.

***Proof* :** By Lemma 3.3.5 and Lemma 3.7,

$$\left[\frac{S(n)}{n}\right] = \begin{cases} 1, & \text{if } n=4 \text{ or if } n \text{ is prime} \\ 0, & \text{otherwise} \end{cases}$$

Thus, in the above sum, a prime contributes 1, increasing the count of prime by 1. Since $n=4$ is not prime, its contribution to the sum is to be deducted. ■

The idea of Lemma 3.3.25 can be extended to find an exact formula for the number of twin primes less than or equal to a given integer $N\ge 3$. Recall that, p and $p+2$ are called twin primes if both p and $p+2$ are primes. Some of the twin prime pairs are (3, 5), (5, 7), (11, 13), (17, 19), (29, 31) and (41, 43). A formula has been put forward by Mehendale [15] to find the number of twin primes less than or equal to a given integer $N\ge 3$. In the following lemma, we reproduce the formula in a modified form.

***Lemma 3.3.26* :** Given any integer $N\ge 3$, the number of twin primes less than or equal to N, denoted by $\pi_2(N)$, is

$$\pi_2(N) = \sum_{n=3}^{N-2} \left[\frac{S(n)}{n}.\frac{S(n+2)}{n+2}\right] \text{ ([x] denotes the greatest integer not exceeding x).}$$

***Proof* :** By Lemma 3.3.5 and Lemma 3.7,

$$\left[\frac{S(n)}{n}.\frac{S(n+2)}{n+2}\right] = \begin{cases} 1, & \text{if both n and n + 2 are primes} \\ 0, & \text{otherwise} \end{cases}$$

Thus, in the sum, a contribution of a count of 1 comes only from a twin prime pair. ■

Since it is known that, the first twin prime pair in the sequence of twin primes is (3, 5), it is reasonable to start searching from $n = 3$; and this automatically excludes the case $n = 4$.

The idea is extended further in the following lemma, due to Mehendale [15].

***Lemma 3.3.27* :** Given any integer $N \geq 3$, the number of prime pairs $(p, p+2k)$ (where $k \geq 1$ is a pre–specified integer) less than or equal to N, denoted by $\pi_{2k}(N)$, is

$$\pi_{2k}(N) = \sum_{n=3}^{N-2k} \left[\frac{S(n)}{n}.\frac{S(n+2k)}{n+2k}\right],$$

(where [x] denotes the greatest integer less than or equal to x).

And we can extend still further to find a formula of the number of prime–triplets less than or equal to a given integer N. A prime–triplet is of the form $(p, p+2, p+6)$ where p, $p+2$ and $p+6$ are all primes, or of the form $(p, p+4, p+6)$ with p, $p+4$ and $p+6$ all primes. The triplets (5, 7, 11) and (11, 13, 17) are examples of prime–triplets of the first kind, while two prime–triplets of the second kind are (7, 11, 13) and (13, 17, 19). It is conjectured that, like the twin-primes, the twin–triplets (of both kinds) are infinite in number (Hardy and Wright [6], pp. 5), but the conjectures still remain open.

***Lemma 3.3.28* :** Let $N \geq 5$ be any given integer.

(1) The number of prime–triplets of the form $(p, p+2, p+6)$ less than or equal to N, denoted by $\pi_3(N)$, is

$$\pi_3(N) = \sum_{n=5}^{N-6} \left[\frac{S(n)}{n}.\frac{S(n+2)}{n+2}.\frac{S(n+6)}{n+6}\right];$$

(2) The number of prime–triplets of the form $(p, p+4, p+6)$ less than or equal to N, denoted by $\pi'_3(N)$, is

$$\pi'_3(N) = \sum_{n=7}^{N-6} \left[\frac{S(n)}{n}.\frac{S(n+4)}{n+4}.\frac{S(n+6)}{n+6}\right];$$

(where [x] denotes the greatest integer less than or equal to x).

Lemma 3.3.25 – Lemma 3.3.28 show how the Smarandache function comes into role in the formulas for determining the exact number of primes, twin primes or prime–triplets upto a given integer N. However, it should be borne in mind that, from the computational point of view, these formulas are not that useful, because the calculation of S(n) for large values of n involves much memory.

3.4 Some Observations and Remarks

In this section, some observations about some of the properties of the Smarandache function are given. Some questions are raised in the relevant cases.

***Remark 3.4.1* :** The proof of Lemma 3.3.1 shows that $S(n+1)-S(n)$ is both unbounded above and unbounded below; in fact, the proof also shows that

$$\limsup_{n\to\infty} |S(n+1)-S(n)| = +\infty.$$

The minimum value of $|S(n+1)-S(n)|$ is clearly 1, and some instances are given below :

$$|S(2)-S(1)|=1, \quad |S(3)-S(2)|=1, \quad |S(4)-S(3)|=1, \quad |S(5)-S(4)|=1,$$
$$|S(9)-S(10)|=1, \quad |S(16)-S(15)|=1, \quad |S(35)-S(36)|=1, \quad |S(64)-S(63)|=1,$$
$$|S(99)-S(100)|=1, \quad |S(176)-S(175)|=1, \quad |S(196)-S(195)|=1.$$

The question is : Is there infinitely many n with $|S(n+1)-S(n)|=1$? (see Jason Earls [16]).

***Remark 3.4.2* :** Since for any prime $p \geq 5$,

$$S(p-1) \leq \frac{p-1}{2} < p = S(p),$$

it follows that, the inequality $S(n+1) > S(n)$ holds for infinitely many n. Again, for any $p \geq 5$,

$$S(p+1) \leq \frac{p+1}{2} < p = S(p),$$

which shows that the inequality $S(n+1) < S(n)$ holds for infinitely many n.
Lemma 3.3.6 proves that the equation $S(n)S(n+1) = n$ has no solution. However, for the equation, $S(n)S(n+1) = kn$, $k \geq 2$, the situation is quite different : If n is a prime, say $n = p$, then the equation has a solution corresponding to $k = S(p+1)$. Besides this trivial case, there are solutions of the equation for particular values of k. We have examples corresponding to $k = 2, 4, 5, 6, 7, 9, 10, 11, 13, 17, 19, 37$, which shows that the equation $S(n)S(n+1) = kn$ has more than one solutions for these values of k. For these values of k, there are composite n as well. A limited search upto $n = 4800$ reveals that, there are solutions for other values of k as well. These are given below :

$k=8 : n=31, 127, 191, 223, 383,$ $\quad k=14 : n=97, 293,$
$k=22 : n=241,$ $\quad k=23 : n=137, 229, 367,$
$k=26 : n=337,$ $\quad k=41 : n=163, 409,$
$k=43 : n=257,$ $\quad k=47 : n=281,$
$k=53 : n=211, 317,$ $\quad k=59 : n=353,$
$k=67 : n=401,$ $\quad k=71 : n=283,$
$k=79 : n=157,$ $\quad k=83 : n=331,$
$k=97 : n=193,$ $\quad k=139 : n=277,$
$k=157 : n=313,$ $\quad k=199 : n=397.$

Note that, in each case, n is prime.
The question is : Is there a solution of the equation $S(n)S(n+1) = kn$ for any $k \geq 2$?

Remark 3.4.3 **:** In more detailed study of the properties of the Smarandache function S(n), the inequalities given in Lemma 3.1.6 and Corollary 3.1.3 may prove very useful. Sandor [3] derives a number of equations and inequalities involving S(n). The derivations of some of them become simpler if Lemma 3.1.6 is exploited. For example, from Lemma 3.1.6, it follows immediately that

$$S(n_1 n_2) \leq S(n_1)S(n_2) \leq n_1 S(n_2) \text{ for any integers } n_1, n_2 \geq 1.$$

As another example, let us consider the problem of finding the solution of $S(ab) = a^k S(b)$.
If $a = 1 = b$, the equation becomes trivial. So, let $a, b > 1$. Then, by Lemma 3.1.6,

$$S(ab) = a^k S(b) \leq S(a)\,S(b) \quad \Rightarrow \quad a^k \leq S(a) \leq a \quad \Rightarrow \quad k = 1,\ S(a) = a.$$

Therefore, $S(ab) = S(a)S(b)$, and hence, by Corollary 3.1.3, we must have $S(ab) = S(a) + S(b)$. Now, the proof of Lemma 3.1.7 shows that, we must have $a = 2 = b$.

Remark 3.4.4 **:** As has been proved by Sandor [3],

$$\liminf_{n\to\infty} \frac{S(n)}{n} = 0, \quad \limsup_{n\to\infty} \frac{S(n)}{n} = 1 = \max\left\{\frac{S(n)}{n} : n \in Z^+\right\}.$$

The proof is simple : In the first case, the proof follows from Lemma 3.3.3. In the second case, the proof follows from the fact that, for any prime p,

$$\frac{S(p)}{p} = 1.$$

Remark 3.4.5 **:** The following results have been established by Sandor [3] :

$$\liminf_{n\to\infty} \frac{SS(n))}{n} = 0, \quad \limsup_{n\to\infty} \frac{S(S(n))}{n} = 1 = \max\left\{\frac{S(S(n))}{n} : n \in Z^+\right\}.$$

To prove the first part, note that

$$0 < \frac{S(S(p!))}{p!} = \frac{S(p)}{p!} = \frac{p}{p!} = \frac{1}{(p-1)!} \to 0 \text{ as } p \to \infty.$$

The proof of the remaining part is as follows : For any prime p,

$$\frac{S(S(p))}{p} = \frac{S(p)}{p} = 1.$$

Remark 3.4.6 **:** The proof of Lemma 3.3.13 shows that, the inequality $S(n) + d(n) > n$ holds true for infinitely many n (namely, when $n = p$ is a prime), and the inequality $S(n) + d(n) < n$ also holds true for infinitely many n (namely, when n is of the form $n=2p^\alpha$, where $p \geq 3$ is a prime).

Remark 3.4.7 **:** In [3], Sandor conjectures that

$$S_*((2n+1)!\,(2n+3)!) = q_n - 1, \text{ where } q_n \text{ is the first prime following } 2n+3.$$

From the values of $S_*((2n+1)!\,(2n+3)!)$, given in Remark 3.2.1.1, we see that the case $n = 2$ is a violation to the above conjecture. However, for values up to $n = 10$, given in Sandor [3], $n = 2$ is the only exception to the conjecture.

The question is : Is the conjecture true for $n \neq 2$?

In this respect, we have only the result given in Lemma 3.2.1.5.

Remark 3.4.8 **:** Several researchers have considered the function

$$T_p(m) = \sum_{k=1}^{\infty} \left[\frac{m}{p^k}\right],$$

which counts the exponent of the prime p in m! (where [x] is the greatest integer less than or equal to x). Since for any prime p and any integer $\alpha \geq 1$,

$$S(p^\alpha) = \min\{m : p^\alpha \mid m!\},$$

it follows that

$$S(p^\alpha) = \min\{m : T_p(m) \geq \alpha\}.$$

Using $T_p(m)$, Farris and Mitchell [17] have proved the following result.

Lemma 3.4.1 **:** $S\left(p^{\frac{p^k - k}{p}}\right) = (p-1)p^k$ for $k \geq 1$.

Remark 3.4.9 **:** From Lemma 3.3.5, $S(n) < n$ for any composite number $n \neq 4$. To find the maximum value, $\max\left\{\frac{S(n)}{n} : n \neq 4, p\right\}$, it is sufficient to consider $\max\left\{\frac{S(p^k)}{p^k} : p \geq 2,\ k \geq 2\right\}$, where the maximum is over all primes p and all $k \geq 2$. By Lemma 3.3.2 and Lemma 3.1.4, the sequence $\left\{\frac{S(p^k)}{p^k}\right\}_{k=1}^{\infty}$ is strictly decreasing in k for any prime p fixed, with

$$\frac{S(p^k)}{p^k} = \frac{kp}{p^k} = \frac{k}{p^{k-1}} \quad \text{if } 1 \leq k \leq p.$$

Since $\frac{k}{p^{k-1}}$ is strictly deceasing in both k and p, to find $\max\left\{\frac{S(n)}{n} : n \neq 4, p\right\}$, it is sufficient to consider only the cases when $p = 2$ with $k = 3$ and $p = 3$ with $k = 2$. Since

$$\frac{S(3^2)}{3^2} = \frac{6}{9} = \frac{2}{3} > \frac{S(2^3)}{2^3} = \frac{1}{2},$$

we have the following lemma.

Lemma 3.4.2 **:** For any composite number $n > 4$, $\frac{S(n)}{n} \leq \frac{2}{3}$,

(with the equality sign for $n = 9$ only).

Ashbacher [2] raises the following question :

Question 3.4.1 **:** For what rational numbers r there exists an n such that $\frac{S(n)}{n} = r$?

From Lemma 3.4.2, we see that Question 3.4.1 is valid only when $r \in (0, \frac{2}{3}]$. Ashbacher [2] also proved that, if r is of the form $r = \frac{1}{p}$, where p is a prime, then $n = 8, 27$, and these are the only solutions.

A simpler proof is as follows : In order that $\frac{S(n)}{n}=\frac{1}{p}$, n must be of the form $n=p^{\alpha}$. From Lemma 3.1.4, Lemma 3.1.5 and Lemma 3.4.1, it is clear that $\alpha \le p+1$.

Now, when $\alpha \le p$, then

$$\frac{S(p^{\alpha})}{p^{\alpha}}=\frac{\alpha p}{p^{\alpha}}=\frac{\alpha}{p^{\alpha-1}}=\frac{1}{p} \quad \Rightarrow \quad \alpha=p,\ \alpha-2=1 \quad \Rightarrow \quad p=3=\alpha,$$

and when $\alpha=p+1$,

$$\frac{S(p^{p+1})}{p^{p+1}}=\frac{p^2}{p^{p+1}}=\frac{1}{p^{p-1}}=\frac{1}{p} \quad \Rightarrow \quad p-1=1 \quad \Rightarrow \quad p=2.$$

We conclude this chapter with the following conjecture related to S(n).

Conjecture 3.4.1 **:** For any integer $n \ge 1$, $S(n+1) \ne S(n)$.

Let, on the contrary,

$$S(n+1)=S(n)=m_0 \text{ for some integers n and } m_0. \tag{1}$$

Clearly, neither n nor $n+1$ can be prime. So, let

$$n=p^{\alpha}p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r},\quad n+1=q^{\beta}q_1^{\beta_1}q_2^{\beta_2}\dots q_t^{\beta_t}$$

be the representations of n and $n+1$ in terms of their prime factors, where the primes $p, p_1, p_2, \dots, p_r$ and the primes $q, q_1, q_2, \dots, q_t$ are all distinct. Moreover, let for definiteness

$$S(n)=S(p^{\alpha}),\ S(n+1)=S(q^{\beta}). \tag{2}$$

Thus, the problem of finding the solution of (1) reduces to finding the primes p and q, and the integers α and β such that

$$S(p^{\alpha})=S(q^{\beta}). \tag{3}$$

Without any loss of generality, we may assume that $p>q$, so that $\alpha \le \beta$. From (3), it follows that p must divide $S(q^{\beta})$ and q must divide $S(p^{\alpha})$. From Lemma 3.1.4 and Lemma 3.1.5, we see that, since $p>q$, we must have $1 \le \alpha \le p-1, \beta \ge q+1$, so that

$S(p^{\alpha})=kp$ for some integer $1 \le k \le p-1$ (p does not divide k),

$S(q^{\beta})=q^a(q+b)$ for some integers a and b (q does not divide b).

Again, since $n \mid m_0!$, $(n+1) \mid m_0!$, and since $(n, n+1)=1$, it follows that

$$n(n+1) \mid m_0! \quad \Rightarrow \quad S(n(n+1)) \le m_o \quad \Rightarrow \quad S(n(n+1))=m_0,$$

since, by Lemma 3.8, $S(n(n+1)) \ge S(n) \ge m_0$. Therefore,

$$S(n+1)=S(n) \quad \Rightarrow \quad S(n+1)=S(n)=S(n(n+1)).$$

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Chapter 4 The Pseudo Smarandache Function

Chapter 3 gives some properties of the Smarandache function S(n). The other function, widely-studied by the Smarandache number theorists, is the pseudo Smarandache function. In this chapter, the pseudo Smarandache function, denoted by Z(n), is considered.

The pseudo Smarandache function, Z(n), introduced by Kashihara [1], is as follows :

Definition 4.1 **:** For any integer $n \geq 1$, the pseudo Smarandache function Z(n) is the smallest positive integer m such that $1+2+\ldots+m \equiv \frac{m(m+1)}{2}$ is divisible by n. Thus,

$$Z(n)=\min\{m : m \in Z^{+},\ n \mid \frac{m(m+1)}{2}\};\ n \geq 1,$$

(where Z^{+} is the set of all positive integers) :

Let T(m) be m–th the triangular number

$$T(m) \equiv \frac{m(m+1)}{2},\ m \in Z^{+}.$$

Then, T(m) is strictly increasing in m. This implies that the function Z(n) is well-defined.

By Definition 4.1, Z(n) is the smallest positive integer m such that the corresponding triangular number T(m) is divisible by n. Thus,

$$Z(n)=m_0$$

if and only if the following two conditions are satisfied :

(1) n divides $\frac{m_0(m_0+1)}{2}$,

(2) n does not divide $\frac{m(m+1)}{2}$ for any m with $1 \leq m \leq m_0-1$.

Let, for any $n \in Z^{+}$ fixed,

$$Z(n)=m_0,$$

so that

$$T(m_0) \equiv \frac{m_0(m_0+1)}{2}=kn \text{ for some } k \in Z^{+}.$$

Then, m_0 can be found by solving the above equation for the positive root. Thus,

$$m_0=\frac{1}{2}(\sqrt{1+8kn}-1),$$

where k is such that $1+8kn$ is a perfect square; moreover, the smallest m_0 is found by choosing the smallest such k. Conversely, if k is the smallest integer such that $1+8kn$ is a perfect square, then m_0 is the smallest integer such that $n \mid \frac{m_0(m_0+1)}{2}$, and hence, $Z(n)=m_0$. We thus arrive at the following equivalent definition of Z(n), as has been pointed out by Ibstedt [2].

Definition 4.2 **:** $Z(n) = \min_{k} \{\frac{1}{2}(\sqrt{1+8kn} - 1) : k \in Z^+,\ 1 + 8kn \text{ is a perfect square}\}$.

Kashihara [1], Ibstedt [2] and Ashbacher [3] studied some of the elementary properties satisfied by Z(n). Their findings are summarized in the following Lemma 4.2 – Lemma 4.6. The proofs of some of these results depend on the following lemma which is almost trivial.

Lemma 4.1 **:** Let p be a prime. Let an integer m ($>p$) be divisible by p^k for some integer k (≥ 1). Then, p^k does not divide $m+1$ (and $m-1$).

Lemma 4.2 **:** For any $n \in Z^+$, $Z(n) \geq 1$. Moreover,

(1) $Z(n) = 1$ if and only if $n = 1$,

(2) $Z(n) = 2$ if and only if $n = 3$.

Proof **:** In either case, the proof of the "if" part is trivial. So, we prove the "only if" part.

(1) Let $Z(n) = 1$ for some $n \in Z^+$. Then, by definition, $n \mid 1$, and hence, $n = 1$.

(2) Let $Z(n) = 2$ for some $n \in Z^+$. Then, by definition, $n \mid 3$, and hence, $n = 3$. ■

The following lemma gives lower and upper bounds of Z(n).

Lemma 4.3 **:** $3 \leq Z(n) \leq 2n-1$ for all $n \geq 4$.

Proof **:** $T(m) \equiv \frac{m(m+1)}{2}$ is strictly increasing in m ($m \in Z^+$) with $T(2) = 3$. Thus, $Z(n) = 2$ if and only if $n = 3$. Therefore, for $n \geq 4$, $Z(n) \geq T(2) = 3$.

Again, since $n \mid T(2n-1)$, it follows that Z(n) cannot be greater than $2n-1$. ■

The upper bound of Z(n) in Lemma 4.3 can be improved in the case when n is odd. In fact, we can prove the following result.

Lemma 4.4 **:** $Z(n) \leq n-1$ for any odd integer $n \geq 3$.

Proof **:** If $n \geq 3$ is an odd integer, then $n-1$ is even, and hence, $n \mid T(n-1) \equiv \frac{n(n-1)}{2}$. ■

The lower bound of Z(n) can also be improved. For example, since $T(4) = 10$, it follows that $Z(n) \geq 5$ for all $n \geq 11$. A better lower bound of Z(n) is given in Lemma 4.5 below for the case when n is a composite number.

Lemma 4.5 **:** For any composite number $n \geq 4$, $Z(n) \geq \max\{Z(d) : d \mid n\}$.

Proof **:** Let

$$Z(n) = m_0 \text{ for some } m_0 \in Z^+.$$

Then, $n \mid \frac{m_0(m_0+1)}{2} \equiv T(m_0)$ (and m_0 is the smallest integer with this property). Clearly, any divisor d of n also divides $T(m_0)$, so that $Z(d) \leq m_0$. ■

Lemma 4.6 **:** For any integer $n \geq 4$, $Z(n) \geq \frac{1}{2}(\sqrt{1+8n} - 1) > \sqrt{n}$.

Proof **:** From Definition 4.2, since the minimum admissible value of $k = 1$,

$$Z(n) \geq \frac{1}{2}(\sqrt{1+8n} - 1) > \sqrt{n}. \quad ■$$

This chapter gives some results related to the pseudo Smarandache function Z(n).

In §4.1, we give some preliminary results which would be needed in the following section. §4.2 gives some simple explicit expressions for Z(n). An expression for Z(pq) is given in Theorem 4.2.2, whose proof shows that the solution of Z(pq) involves the solution of two Diophantine equations in p and q. Some particular cases of Theorem 4.2.2 are given in Corollaries 4.2.3 – 4.2.18. Some generalizations of the pseudo Smarandache function are given in §4.3. §4.4 is devoted to some miscellaneous topics. We conclude this chapter with some remarks about the properties of Z(n) in the final §4.5.

4.1 Some Preliminary Results

In this section, we give some preliminary results that would be required in the next section to find explicit expressions of Z(n) in some cases.

Lemma 4.1.1 **:** $6 \mid m(m+1)(m+2)$ for any $m \in Z^+$; and $6 \mid (p^2-1)$ for any prime $p \geq 5$.

Proof **:** The first part is well-known : Since the product of two consecutive integers is divisible by 2 and the product of three consecutive integers is divisible by 3, and since $(2,3)=1$, the result follows. In particular, for any prime $p \geq 5$, $6 \mid (p-1)p(p+1)$. But since p (≥ 5) is not divisible by 6, it follows that $6 \mid (p-1)(p+1)$. ■

Lemma 4.1.2 **:** For any integer $k \geq 0$,

(1) 4 divides $3^{2k}-1$, **(2)** 4 divides $3^{2k+1}+1$.

Proof **:** In either case, the proof is trivial for k=0. So, let $k \geq 1$.

(1) Writing $3^{2k}-1=(3^k-1)(3^k+1)$, the result follows immediately.

(2) Since $3^{2k+1}+1=3(3^{2k}-1)+4$, the result follows by virtue of part (1). ■

Lemma 4.1.3 **:** For any integer $k \geq 0$,

(1) 3 divides $2^{2k}-1$, **(2)** 3 divides $2^{2k+1}+1$.

Proof **:** In either case, the proof is trivial for $k=0$. So, let $k \geq 1$.

(1) By Lemma 4.1.1, 3 divides $(2^k-1)2^k(2^k+1)$. Since 3 does not divide 2^k, 3 must divide $(2^k-1)(2^k+1)=2^{2k}-1$.

(2) Noting that $2^{2k+1}+1=2(2^{2k}-1)+3$, the result follows by virtue of part (1). ■

Lemma 4.1.4 **:** For any integer $k \geq 0$, 5 divides $2^{4k}-1$.

Proof **:** The proof is by induction on $k \geq 1$. The result is clearly true for $k=1$. So, we assume its validity for some integer k. Now, since

$$2^{4(k+1)}-1=16(2^{4k}-1)+15,$$

it follows that the result is true for $k+1$ as well, completing induction. ■

Lemma 4.1.5 **:** For any integer $k \geq 0$, 7 divides $2^{3k}-1$.

Proof **:** The result is clearly true for $k=1$. To proceed by induction, we assume the validity of the result for some integer k. Now,

$$2^{3(k+1)}-1=8(2^{3k}-1)+7,$$

showing that the result is true for $k+1$ as well, which we intended to prove. ■

***Lemma 4.1.6* :** For any integer $k \geq 0$,

(1) 11 divides $2^{5(2k+1)}+1$, **(2)** 11 divides $2^{10k}-1$.

***Proof* :** In either case, the result is true when $k=0$. So, let $k \geq 1$.

(1) The result can easily be checked to be true for $k=1$. Now, assuming its validity for some integer k, by the induction hypothesis, together with the fact that

$$2^{5(2k+3)}+1=2^{10}[2^{5(2k+1)}+1]-(2^5+1)(2^5-1),$$

it follows that the result is true for $k+1$ as well, completing induction.

(2) Since $2^{10k}-1=(2^{5k}-1)(2^{5k}+1)$, the result follows by virtue of part (1). ■

***Lemma 4.1.7* :** For any integer $k \geq 0$,

(1) 13 divides $2^{6(2k+1)}+1$, **(2)** 13 divides $2^{12k}-1$.

***Proof* :** In either case, the result is trivially true when $k=0$. So, let $k \geq 1$.

(1) The result is true for $k=1$. To proceed by induction, we assume that it is true for some integer k. Now,

$$2^{6(2k+3)}+1=2^{12}([2^{6(2k+1)}+1]-(2^6+1)(2^6-1),$$

which shows the validity of the result for $k+1$. This completes induction.

(2) Noting that $2^{12k}-1=(2^{6k}-1)(2^{6k}+1)$, the result follows by part (1) above. ■

***Lemma 4.1.8* :** For any integer $k \geq 0$,

(1) 17 divides $2^{4(2k+1)}+1$, **(2)** 17 divides $2^{8k}-1$.

***Proof* :** In either case, the result is true when $k=0$. So, let $k \geq 1$.

(1) The result is clearly true for $k=1$. To proceed by induction, we assume its validity for some integer k. Then, since

$$2^{4(2k+3)}+1=2^8([2^{4(2k+1)}+1]-(2^4+1)(2^4-1),$$

we see that the result is true for $k+1$ as well. This completes the proof by induction.

(2) Since $2^{8k}-1=(2^{4k}-1)(2^{4k}+1)$, the result follows by virtue of part (1). ■

***Lemma 4.1.9* :** For any integer $k \geq 0$,

(1) 19 divides $2^{9(2k+1)}+1$, **(2)** 19 divides $2^{18k}-1$.

***Proof* :** In either case, the result is true when $k=0$. So, let $k \geq 1$.

(1) The result is clearly true for $k=1$. To proceed by induction, we assume its validity for some integer k. Then,

$$2^{4(2k+3)}+1=2^8([2^{4(2k+1)}+1]-(2^4+1)(2^4-1),$$

showing that the result is true for $k+1$ also. This completes the proof by induction.

(2) Writing $2^{8k}-1=(2^{4k}-1)(2^{4k}+1)$, the result follows by part (1) above. ■

***Lemma 4.1.10* :** For any integer $k \geq 0$, 31 divides $2^{5k}-1$.

***Proof* :** The proof is by induction on $k \geq 1$. The result is clearly true for $k=1$. So, we assume its validity for some integer k. Now, since

$$2^{5(k+1)}-1=32(2^{4k}-1)+31,$$

it follows that the result is true for $k+1$ as well, completing induction. ■

4.2 Some Explicit Expressions for Z(n)

In this section, we first give the explicit expressions for $Z(\frac{k(k+1)}{2})$, $Z(2^k)$ and $Z(p^k)$ in Lemmas 4.2.1 – 4.2.3 (see Kashihara [1] and Ashbacher [3]). Lemmas 4.2.4 – 4.2.5 give the expressions for Z(np) and $z(np^2)$ for some particular n. We also derive the explicit forms of $Z(3.2^k)$, $Z(5.2^k)$, $Z(7.2^k)$, $Z(11.2^k)$, $Z(13.2^k)$, $Z(17.2^k)$, $Z(19.2^k)$, $Z(31.2^k)$, $Z(2p^k)$ and $Z(3p^k)$, as well as the expressions for Z(4p), Z(5p), Z(6p), Z(7p), Z(8p), Z(9p), Z(10p), Z(11p), Z(12p), Z(13p), Z(16p), Z(32p), Z(24p), Z(48) and Z(96p) in Lemmas 4.2.6 – 4.2.30 respectively. Theorems 4.2.1 – 4.2.2 give two alternative expressions for Z(pq).

Lemma 4.2.1 **:** $Z(\frac{k(k+1)}{2})=k$ for any $k\in Z^+$.

Proof **:** Noting that $k(k+1)=m(m+1)$ if and only if $k=m$, the result follows. ■

Lemma 4.2.2 **:** For any $k\in Z^+$, $Z(2^k)=2^{k+1}-1$.
Proof **:** By definition,

$$Z(2^k)=\min\{m : 2^k \mid \frac{m(m+1)}{2}\}=\min\{m : 2^{k+1} \mid m(m+1)\}. \quad (1)$$

Now, note that, only one of m and m + 1 is even, and as such, 2^{k+1} divides exactly one of m and m + 1. Hence, the minimum m in (1) is 2^{k+1}. This proves the lemma. ■

Lemma 4.2.3 **:** For any prime $p\geq 3$ and any $k\in Z^+$, $Z(p^k)=p^k-1$.
Proof **:** By definition,

$$Z(p^k)=\min\{m : p^k \mid \frac{m(m+1)}{2}\}=\min\{m : 2.p^k \mid m(m+1)\}. \quad (2)$$

If p^k divides m(m+1), then p^k must divide either m or m + 1, but not both (by Lemma 4.1). Again, note that, if p^k divides one of m and m + 1, then 2 divides the other. Thus, the minimum m in (2) may be taken as p^k-1, which establishes the lemma. ■

Lemma 4.2.4 and Lemma 4.2.5 give respectively the expressions for Z(np) and $Z(np^2)$ (p is a prime) for some particular n. Then, the expressions for Z(2p) and Z(3p) follow from Lemma 4.2.4, and are given in Corollary 4.2.1. The expressions for $Z(2p^2)$ and $Z(3p^2)$, which follow from Lemma 4.2.5, are given in Corollary 4.2.2.

Lemma 4.2.4 **:** If $p\geq 3$ is a prime and $n\geq 2$ is an integer (not divisible by p), then

$$Z(np)=\begin{cases} p-1, & \text{if } 2n \mid (p-1) \\ p, & \text{if } 2n \mid (p+1) \end{cases}$$

Proof **:** By Definition 4.1,

$$Z(np)=\min\{m : np \mid \frac{m(m+1)}{2}\}=\min\{m : 2np \mid m(m+1)\}. \quad (3)$$

Then p must divide either m or m+1, but not both (by Lemma 4.1). Thus, the minimum m in (3) is $p-1$ or p depending on whether $p-1$or $p+1$ respectively is divisible by 2n. ■

Corollary 4.2.1 **:** Let p be an odd prime. Then,

(1) for $p \geq 3$,

$$Z(2p) = \begin{cases} p-1, & \text{if } 4 \mid (p-1) \\ p, & \text{if } 4 \mid (p+1) \end{cases}$$

(2) for $p \geq 5$,

$$Z(3p) = \begin{cases} p-1, & \text{if } 3 \mid (p-1) \\ p, & \text{if } 3 \mid (p+1) \end{cases}$$

Proof **:** The above results follow immediately from Lemma 4.2.4, in Case

(1) since any prime $p \geq 3$ is either of the form 4a+1 or 4a+3 (for some integer $a \geq 1$), and in Case

(2) since any prime p is of the form $3a+1$ or $3a+2$ (for some integer $a \geq 1$), and since both $p-1$ and $p+1$ are divisible by 2. ■

Lemma 4.2.5 **:** If $p \geq 3$ is a prime and n is an integer not divisible by p, then

$$Z(np^2) = \begin{cases} p^2-1, & \text{if } 2n \mid (p^2-1) \\ p^2, & \text{if } 2n \mid (p^2+1) \end{cases}$$

Proof **:** Using the definition,

$$Z(np^2) = \min\{m : np^2 \mid \frac{m(m+1)}{2}\} = \min\{m : 2np^2 \mid m(m+1)\}. \quad (4)$$

Therefore, p^2 must divide exactly one of m and $m+1$ (by Lemma 4.1), and then 2n must divide the other. Now, if 2n divides p^2-1, then the minimum m in (4) may be taken as p^2-1; on the other hand, if p^2+1 is divisible by 2n, the minimum m is p^2+1. ■

Corollary 4.2.2 **:** Let p be an odd prime. Then,

(1) $Z(2p^2) = p^2-1$, for $p \geq 3$;

(2) $Z(3p^2) = p^2-1$, for $p \geq 5$.

Proof **:** In both the cases, the results follow from Lemma 4.2.5.

(1) Noting that 4 divides $(p-1)(p+1) = p^2-1$, the result follows immediately.

(2) By Lemma 4.1.1, 6 divides p^2-1. The result then follows. ■

Remark 4.2.1 **:** Since

$$Z(2p) \neq 3(p-1) = Z(2)\,Z(p) \text{ for any odd prime p,}$$

and

$$Z(3p^2) = p^2-1 \neq Z(2p^2) + Z(p^2) \text{ for any prime } p \geq 5,$$

it follows that Z(n) is neither multiplicative nor additive.

The expressions for $Z(3.2^k)$ and $Z(5.2^k)$, due to Ashbacher [3], are given in Lemma 4.2.6 and Lemma 4.2.7 respectively. Lemma 4.2.8 gives an expression for $Z(7.2^k)$. This result also appears in Ibstedt [4], but we follow a different approach to the proof. Note that, for any prime $p \geq 3$ fixed, the values of $Z(p.2^k)$ depends on the form of k.

Lemma 4.2.6 **:** For any integer $k \geq 1$,

$$Z(3.2^k) = \begin{cases} 2^{k+1} - 1, & \text{if k is odd} \\ 2^{k+1}, & \text{if k is even} \end{cases}$$

Proof **:** Using the definition,

$$Z(3.2^k) = \min\{m : 3.2^k \mid \frac{m(m+1)}{2}\} = \min\{m : 3.2^{k+1} \mid m(m+1)\}. \tag{5}$$

Then, 2^{k+1} must divide one of m and $m+1$ and 3 must divide the other.
Now, we consider the two possibilities separately :

Case (1) : When k is odd, say, $k = 2a+1$, for some integer $a \geq 0$.
In this case, by Lemma 4.1.3(2), 3 divides $2^{2a+1}+1 = 2^k+1$. By Lemma 4.1.3(1),

$$3 \mid (2^{k+1}-1) \quad \Rightarrow \quad 3.2^{k+1} \mid 2^{k+1}(2^{k+1}-1),$$

and hence, the minimum m in (5) may be taken as $2^{k+1}-1$.

Case (2) : When k is even, say, $k = 2a$, for some integer $a \geq 1$.
By Lemma 4.1.3(1), 3 divides $2^{2a}-1 = 2^k-1$. Also,

$$3 \mid (2^{k+1}+1) \quad \Rightarrow \quad 3.2^{k+1} \mid 2^{k+1}(2^{k+1}+1),$$

and hence, in this case, the minimum m in (5) may be taken as 2^{k+1}. ■

Lemma 4.2.7 **:** For any integer $k \geq 1$,

$$Z(5.2^k) = \begin{cases} 2^{k+2}, & \text{if } 4 \mid k \\ 2^{k+1}, & \text{if } 4 \mid (k-1) \\ 2^{k+2} - 1, & \text{if } 4 \mid (k-2) \\ 2^{k+1} - 1, & \text{if } 4 \mid (k-3) \end{cases}$$

Proof **:** Using the definition,

$$Z(5.2^k) = \min\{m : 5.2^k \mid \frac{m(m+1)}{2}\} = \min\{m : 5.2^{k+1} \mid m(m+1)\}. \tag{6}$$

Here, 2^{k+1} must divide one of m and $m+1$, and 5 must divide the other.
We have to consider separately the four possible cases :

Case (1) : When k is of the form $k = 4a$ for some integer $a \geq 1$.
In this case, by Lemma 4.1.4, 5 divides $2^{4a}-1 = 2^k-1$. Now, since

$$2^{k+2}+1 = 4(2^k-1)+5,$$

we see that

$$5 \mid (2^{k+2}+1) \quad \Rightarrow \quad 5.2^{k+1} \mid 2^{k+2}(2^{k+2}+1),$$

and hence, the minimum m in (6) may be taken as 2^{k+2}.

Case (2) : When k is of the form $k = 4a+1$ for some integer $a \geq 0$.
By Lemma 4.1.4, 5 divides $2^{4a}-1 = 2^{k-1}-1$. Now, since

$$2^{k+1}+1 = 4(2^{k-1}-1)+5 = 4(2^{4a}-1)+5,$$

we see that

$$5 \mid (2^{k+1}+1) \quad \Rightarrow \quad 5.2^{k+1} \mid 2^{k+1}(2^{k+1}+1),$$

and hence, in this case, the minimum m in (6) may be taken as 2^{k+1}.

Case (3) : When k is of the form $k=4a+2$ for some integer $a\geq 0$.
In this case, writing

$$2^{k+2}-1=16(2^{k-2}-1)+15,$$

and noting that, by Lemma 4.1.4, 5 divides $2^{4a}-1=2^{k-2}-1$, we see that

$$5 \mid (2^{k+2}-1) \quad \Rightarrow \quad 5.2^{k+1} \mid 2^{k+2}(2^{k+2}-1).$$

Hence, in this case, the minimum m in (6) may be taken as $2^{k+2}-1$.

Case (4) : When k is of the form $k=4a+3$ for some integer $a\geq 0$.
Here, since

$$2^{k+1}-1=16(2^{k-3}-1)+15,$$

and since 5 divides $2^{4a}-1=2^{k-3}-1$ (by Lemma 4.1.4), we see that

$$5 \mid (2^{k+1}-1) \quad \Rightarrow \quad 5.2^{k+1} \mid 2^{k+1}(2^{k+1}-1).$$

Hence, in this case, $Z(5.2^k)=2^{k+1}-1$.

All these complete the proof of the lemma. ■

Lemma 4.2.8 **:** For any integer $k\geq 1$,

$$Z(7.2^k)=\begin{cases} 3.2^{k+1}, & \text{if } 3 \mid k \\ 2^{k+2}-1, & \text{if } 3 \mid (k-1) \\ 2^{k+1}-1, & \text{if } 3 \mid (k-2) \end{cases}$$

Proof **:** By definition,

$$Z(7.2^k)=\min\{m : 7.2^k \mid \frac{m(m+1)}{2}\}=\min\{m : 7.2^{k+1} \mid m(m+1)\}. \quad (7)$$

Here, 2^{k+1} must divide one of m and $m+1$, and 7 must divide the other.
We now consider the three possible cases below :

Case (1) : When k is of the form $k=3a$ for some integer $a\geq 1$.
By Lemma 4.1.5, 7 divides $2^{3a}-1=2^k-1$. Now,

$$3.2^{k+1}+1=6(2^k-1)+7,$$

which shows that $3.2^{k+1}+1$ is divisible by 7. Hence, 7.2^{k+1} divides $3.2^{k+1}(3.2^{k+1}+1)$, and consequently, the minimum m in (7) can be taken as 3.2^{k+1}.

Case (2) : When k is of the form $k=3a+1$ for some integer $a\geq 0$.
Here, by virtue of Lemma 4.1.5,

$$7 \mid (2^{k+2}-1)=8(2^{k-1}-1)+7=8(2^{3a}-1)+7 \quad \Rightarrow \quad 7.2^{k+1} \mid 2^{k+2}(2^{k+2}-1),$$

and hence, in this case, the minimum m in (7) may be taken as $2^{k+2}-1$.

Case (3) : When k is of the form $k=3a+2$ for some integer $a\geq 0$.
In this case, by virtue of Lemma 4.1.5,

$$7 \mid (2^{k+1}-1)=8(2^{k-2}-1)+7=8(2^{3a}-1)+7 \quad \Rightarrow \quad 7.2^{k+1} \mid 2^{k+1}(2^{k+1}-1),$$

and hence, $Z(7.2^k)=2^{k+1}-1$.

Hence, the lemma is established. ■

In Lemmas 4.2.9 – 4.2.13 respectively, the expressions for $Z(11.2^k)$, $Z(13.2^k)$, $Z(17.2^k)$, $Z(19.2^k)$ and $Z(31.2^k)$ are given. These results would be needed later in § 4.4.

Lemma 4.2.9 **:** For any integer $k \geq 1$,

$$Z(11.2^k) = \begin{cases} 5.2^{k+1}, & \text{if } 10 \mid k \\ 3.2^{k+1} - 1, & \text{if } 10 \mid (k-1) \\ 2^{k+3}, & \text{if } 10 \mid (k-2) \\ 2^{k+2}, & \text{if } 10 \mid (k-3) \\ 2^{k+1}, & \text{if } 10 \mid (k-4) \\ 5.2^{k+1} - 1, & \text{if } 10 \mid (k-5) \\ 3.2^{k+1}, & \text{if } 10 \mid (k-6) \\ 2^{k+3} - 1, & \text{if } 10 \mid (k-7) \\ 2^{k+2} - 1, & \text{if } 10 \mid (k-8) \\ 2^{k+1} - 1, & \text{if } 10 \mid (k-9) \end{cases}$$

Proof **:** By definition,

$$Z(11.2^k) = \min\{m : 11.2^k \mid \frac{m(m+1)}{2}\} = \min\{m : 11.2^{k+1} \mid m(m+1)\}. \quad (8)$$

Here, 2^{k+1} must divide one of m and $m+1$, and 11 must divide the other.
We now consider all the possible cases below :

Case (1) : When k is of the form $k = 10a$ for some integer $a \geq 1$.
By Lemma 4.1.6(2), 11 divides $2^{10a} - 1 = 2^k - 1$. Now,

$$11 \mid (5.2^{k+1} + 1) = 10(2^k - 1) + 11 \quad \Rightarrow \quad 11.2^{k+1} \mid (5.2^{k+1}).(5.2^{k+1} + 1).$$

Therefore, the minimum m in (8) can be taken as 5.2^{k+1}.

Case (2) : When k is of the form $k = 10a + 1$ for some integer $a \geq 0$.
Here, by virtue of Lemma 4.1.6(2),

$$11 \mid (3.2^{k+1} - 1) = 12(2^{k-1} - 1) + 11 \quad \Rightarrow \quad 11.2^{k+1} \mid (3.2^{k+1}).(3.2^{k+1} - 1),$$

and hence, in this case, the minimum m in (8) may be taken as $3.2^{k+1} - 1$.

Case (3) : When k is of the form $k = 10a + 2$ for some integer $a \geq 0$.
In this case, writing $2^{k+3} = 2^{10a+5}$ in the form

$$2^{k+3} = 2^{10a+5} = (2^{10a} - 1)(2^5 + 1) + (2^5 + 1) - (2^{10a} - 1) - 1,$$

we see, by virtue of Lemma 4.1.6, that

$$11 \mid (2^{k+3} + 1) \quad \Rightarrow \quad 11.2^{k+1} \mid 2^{k+3}(2^{k+3} + 1),$$

and hence, $Z(11.2^k) = 2^{k+3}$.

Case (4) : When k is of the form $k = 10a + 3$ for some integer $a \geq 0$.
Here, since $2^{k+2} = 2^{10a+5}$, it follows that

$$11 \mid (2^{k+2} + 1) \quad \Rightarrow \quad 11.2^{k+1} \mid 2^{k+2}(2^{k+2} + 1) \quad \Rightarrow \quad Z(11.2^k) = 2^{k+2}.$$

Case (5) : When k is of the form $k = 10a + 4$ for some integer $a \geq 0$.
Here, since $2^{k+1} = 2^{10a+5}$, it follows that

$$11 \mid (2^{k+1} + 1) \quad \Rightarrow \quad 11.2^{k+1} \mid 2^{k+1}(2^{k+1} + 1) \quad \Rightarrow \quad Z(11.2^k) = 2^{k+1}.$$

Case (6) : When k is of the form $k=10a+5$ for some integer $a\geq 0$.
In this case,

$$5.2^{k+1}-1=10(2^{10a+5}+1)-11 \quad \Rightarrow \quad 11 \mid (5.2^{k+1}-1),$$

and hence, $Z(11.2^k)=5.2^{k+1}-1$.

Case (7) : When k is of the form $k=10a+6$ for some integer $a\geq 0$.
Here, the result follows from the following chain of implications :

$$3.2^{k+1}+1=12(2^{10a+5}+1)-11 \quad \Rightarrow \quad 11 \mid (3.2^{k+1}+1) \quad \Rightarrow \quad Z(11.2^k)=3.2^{k+1}.$$

Case (8) : When k is of the form $k=10a+7$ for some integer $a\geq 0$.
In this case,

$$2^{k+3}=2^{10(a+1)} \quad \Rightarrow \quad 11 \mid (2^{k+3}-1) \quad \Rightarrow \quad Z(11.2^k)=2^{k+3}-1.$$

Case (9) : When k is of the form $k=10a+8$ for some integer $a\geq 0$.
Here, the desired result is obtained as follows :

$$2^{k+2}=2^{10(a+1)} \quad \Rightarrow \quad 11 \mid (2^{k+2}-1) \quad \Rightarrow \quad Z(11.2^k)=2^{k+2}-1.$$

Case (10) : When k is of the form $k=10a+9$ for some integer $a\geq 0$.
In this case, the result follows, since

$$2^{k+1}=2^{10(a+1)} \quad \Rightarrow \quad 11 \mid (2^{k+1}-1) \quad \Rightarrow \quad Z(11.2^k)=2^{k+1}-1.$$

All these complete the proof. ■

Lemma 4.2.10 **:** For any integer $k\geq 1$,

$$Z(13.2^k)=\begin{cases} 3.2^{k+2}, & \text{if } 12 \mid k \\ 3.2^{k+1}, & \text{if } 12 \mid (k-1) \\ 5.2^{k+1}-1, & \text{if } 12 \mid (k-2) \\ 2^{k+3}, & \text{if } 12 \mid (k-3) \\ 2^{k+2}, & \text{if } 12 \mid (k-4) \\ 2^{k+1}, & \text{if } 12 \mid (k-5) \\ 3.2^{k+2}-1, & \text{if } 12 \mid (k-6) \\ 3.2^{k+1}-1, & \text{if } 12 \mid (k-7) \\ 5.2^{k+1}, & \text{if } 12 \mid (k-8) \\ 2^{k+3}-1, & \text{if } 12 \mid (k-9) \\ 2^{k+2}-1, & \text{if } 12 \mid (k-10) \\ 2^{k+1}-1, & \text{if } 12 \mid (k-11) \end{cases}$$

Proof **:** By definition,

$$Z(13.2^k)=\min\{m : 13.2^k \mid \frac{m(m+1)}{2}\}=\min\{m : 13.2^{k+1} \mid m(m+1)\}. \qquad (9)$$

Here, 2^{k+1} must divide one of m and $m+1$, and 13 must divide the other.
We now consider all the possible cases, twelve in number :

Case (1) : When k is of the form $k=12a$ for some integer $a\geq 1$.
By Lemma 4.1.7(2), 13 divides $2^{12a}-1=2^k-1$. Now,

$$3.2^{k+2}+1=12(2^k-1)+13 \quad \Rightarrow \quad 13 \mid (3.2^{k+2}+1),$$

so that, 13.2^{k+1} divides $3.2^{k+2}(3.2^{k+2}+1)$. Hence, the minimum m in (9) is 3.2^{k+2}.

Case (2) : When k is of the form $k = 12a + 1$ for some integer $a \geq 0$.
Here, by virtue of Lemma 4.1.7(2),

$$13 \mid (3.2^{k+1} + 1) = 12(2^{k-1} - 1) + 13 \quad \Rightarrow \quad 13.2^{k+1} \mid (3.2^{k+1}).(3.2^{k+1} + 1),$$

and hence, in this case, the minimum m in (9) may be taken as 3.2^{k+1}.

Case (3) : When k is of the form $k = 12a + 2$ for some integer $a \geq 0$.
In this case,

$$5.2^{k+1} - 1 = 40(2^{k-2} - 1) + 39 \quad \Rightarrow \quad 13.2^{k+1} \mid (5.2^{k+1}).(5.2^{k+1} - 1),$$

and hence, $Z(13.2^k) = 5.2^{k+1} - 1$.

Case (4) : When k is of the form $k = 12a + 3$ for some integer $a \geq 0$.
Here, $2^{k+3} = 2^{12a+6}$. Expressing this as follows :

$$2^{k+3} = 2^{12a+6} = (2^{12a} - 1)(2^6 + 1) + (2^6 + 1) - (2^{12a} - 1) - 1,$$

we see, by Lemma 4.1.7, that

$$13 \mid (2^{k+3} + 1) \quad \Rightarrow \quad 13.2^{k+1} \mid 2^{k+3}(2^{k+3} + 1) \quad \Rightarrow \quad Z(13.2^k) = 2^{k+3}.$$

Case (5) : When k is of the form $k = 12a + 4$ for some integer $a \geq 0$.
Here, since $2^{k+2} = 2^{12a+6}$, it follows that

$$13 \mid (2^{k+2} + 1) \quad \Rightarrow \quad 13.2^{k+1} \mid 2^{k+2}(2^{k+2} + 1) \quad \Rightarrow \quad Z(13.2^k) = 2^{k+2}.$$

Case (6) : When k is of the form $k = 12a + 5$ for some integer $a \geq 1$.
In this case,

$$2^{k+1} + 1 = 2^{12a+6} + 1 \quad \Rightarrow \quad 13 \mid (2^{k+1} + 1),$$

and hence, $Z(13.2^k) = 2^{k+1}$.

Case (7) : When k is of the form $k = 12a + 6$ for some integer $a \geq 0$.
Here, the result follows from the chain of implications below :

$$3.2^{k+2} - 1 = 12(2^{12a+6} + 1) - 13 \quad \Rightarrow \quad 13 \mid (3.2^{k+2} - 1) \quad \Rightarrow \quad Z(13.2^k) = 3.2^{k+2} - 1.$$

Case (8) : When k is of the form $k = 12a + 7$ for some integer $a \geq 0$.
In this case, the result is obtained as follows :

$$3.2^{k+1} - 1 = 12(2^{12a+6} + 1) - 13 \quad \Rightarrow \quad 13 \mid (3.2^{k+1} - 1) \quad \Rightarrow \quad Z(13.2^k) = 3.2^{k+1} - 1.$$

Case (9) : When k is of the form $k = 12a + 8$ for some integer $a \geq 0$.
Here,

$$5.2^{k+1} + 1 = 40(2^{12a+6} + 1) - 39 \quad \Rightarrow \quad 13 \mid (5.2^{k+1} + 1) \quad \Rightarrow \quad Z(13.2^k) = 5.2^{k+1}.$$

Case (10) : When k is of the form $k = 12a + 9$ for some integer $a \geq 0$.
In this case, by Lemma 4.1.7(2),

$$2^{k+3} - 1 = 2^{12(a+1)} - 1 \quad \Rightarrow \quad 13 \mid (2^{k+3} - 1) \quad \Rightarrow \quad Z(13.2^k) = 2^{k+3} - 1.$$

Case (11) : When k is of the form $k = 12a + 10$ for some integer $a \geq 0$.
In this case,

$$2^{k+2} - 1 = 2^{12(a+1)} - 1 \quad \Rightarrow \quad 13 \mid (2^{k+2} - 1) \quad \Rightarrow \quad Z(11.2^k) = 2^{k+2} - 1.$$

Case (12) : When k is of the form $k = 12a + 11$ for some integer $a \geq 0$.
In this case, the result follows by virtue of the fact that $13 \mid (2^{k+1} - 1)$.

Hence, the lemma is established. ■

Lemma 4.2.11 **:** For any integer $k \geq 1$,

$$Z(17.2^k) = \begin{cases} 2^{k+4}, & \text{if } 8 \mid k \\ 2^{k+3}, & \text{if } 8 \mid (k-1) \\ 2^{k+2}, & \text{if } 8 \mid (k-2) \\ 2^{k+1}, & \text{if } 8 \mid (k-3) \\ 2^{k+4}-1, & \text{if } 8 \mid (k-4) \\ 2^{k+3}-1, & \text{if } 8 \mid (k-5) \\ 2^{k+2}-1, & \text{if } 8 \mid (k-6) \\ 2^{k+1}-1, & \text{if } 8 \mid (k-7) \end{cases}$$

Proof **:** By definition,

$$Z(17.2^k) = \min\{m : 17.2^k \mid \frac{m(m+1)}{2}\} = \min\{m : 17.2^{k+1} \mid m(m+1)\}. \qquad (10)$$

Here, 2^{k+1} must divide one of m and $m+1$, and 17 must divide the other.
We now consider all the possibilities cases below :

Case (1) : When k is of the form $k = 8a$ for some integer $a \geq 1$.
By Lemma 4.1.8(2), 17 divides $2^{8a} - 1 = 2^k - 1$. Now,

$$2^{k+4} + 1 = 16(2^k - 1) + 17 \quad \Rightarrow \quad 17 \mid (2^{k+4} + 1),$$

so that 17.2^{k+4} divides $2^{k+4}(2^{k+4} + 1)$, and the minimum m in (10) can be taken as 2^{k+4}.

Case (2) : When k is of the form $k = 8a + 1$ for some integer $a \geq 0$.
Here, by virtue of Lemma 4.1.8(2),

$$17 \mid (2^{k+3} + 1) = 16(2^{8a} - 1) + 17 = 16(2^{k-1} - 1) + 17,$$

and hence, in this case, the minimum m in (10) may be taken as 2^{k+3}.

Case (3) : When k is of the form $k = 8a + 2$ for some integer $a \geq 0$.
In this case, $2^{k+1} = 2^{8a+3}$. Then, since

$$2^{k+2} = 2^{8a+4} = (2^{8a} - 1)(2^4 + 1) + (2^4 + 1) - (2^{8a} - 1) - 1,$$

by virtue of Lemma 4.1.8, it follows that

$$17 \mid (2^{k+2} + 1) \quad \Rightarrow \quad 17.2^{k+1} \mid 2^{k+2}(2^{k+2} + 1) \quad \Rightarrow \quad Z(17.2^k) = 2^{k+2}.$$

Case (4) : When k is of the form $k = 8a + 3$ for some integer $a \geq 0$.
Here, since $2^{k+1} = 2^{8a+4}$, it follows that, 17 divides $2^{k+1} + 1$. Hence, $Z(17.2^k) = 2^{k+1}$.

Case (5) : When k is of the form $k = 8a + 4$ for some integer $a \geq 0$.
In this case, $2^{k+1} = 2^{8a+5}$, so that $2^{k+4} - 1 = 16(2^{8a+4} + 1) - 17$ is divisible by 17.
Hence, $Z(17.2^k) = 2^{k+4} - 1$.

Case (6) : When k is of the form $k = 8a + 5$ for some integer $a \geq 0$.
In this case, 17 divides $2^{k+3} - 1 = 16(2^{8a+4} + 1) - 17$, and hence, $Z(17.2^k) = 2^{k+3} - 1$.

Case (7) : When k is of the form $k = 8a + 6$ for some integer $a \geq 0$.
Here, $2^{k+2} - 1 = 2^{8(a+1)} - 1$, which is divisible by 17. Thus, $Z(17.2^k) = 2^{k+2} - 1$.

Case (8) : When k is of the form $k = 8a + 7$ for some integer $a \geq 0$.
In this case, the result follows, since 17 divides $2^{k+1} - 1 = 2^{8(a+1)} - 1$. ■

The same procedure can be followed to find explicit expressions for $Z(p.2^k)$, where p is an odd prime. But, in some cases, it would be complicated as the number of possible cases to be considered becomes larger and larger. For example, to find $Z(19.2^k)$, we have to consider 18 possible cases. In the following two lemmas, we give the expressions for $Z(19.2^k)$ and $Z(31.2^k)$, with outlines of the proofs.

Lemma 4.2.12 **:** For any integer $k \geq 1$,

$$Z(19.2^k) = \begin{cases} 9.2^{k+1}, & \text{if } 19 \mid k \\ 5.2^{k+1}-1, & \text{if } 19 \mid (k-1) \\ 3.2^{k+3}-1, & \text{if } 19 \mid (k-2) \\ 3.2^{k+2}-1, & \text{if } 19 \mid (k-3) \\ 3.2^{k+1}-1, & \text{if } 19 \mid (k-4) \\ 2^{k+4}, & \text{if } 19 \mid (k-5) \\ 2^{k+3}, & \text{if } 19 \mid (k-6) \\ 2^{k+2}, & \text{if } 19 \mid (k-7) \\ 2^{k+1}, & \text{if } 19 \mid (k-8) \\ 9.2^{k+1}-1, & \text{if } 19 \mid (k-9) \\ 5.2^{k+1}, & \text{if } 19 \mid (k-10) \\ 3.2^{k+3}, & \text{if } 19 \mid (k-11) \\ 3.2^{k+2}, & \text{if } 19 \mid (k-12) \\ 3.2^{k+1}, & \text{if } 19 \mid (k-13) \\ 2^{k+4}-1, & \text{if } 19 \mid (k-14) \\ 2^{k+3}-1, & \text{if } 19 \mid (k-15) \\ 2^{k+2}-1, & \text{if } 19 \mid (k-16) \\ 2^{k+1}-1, & \text{if } 19 \mid (k-17) \end{cases}$$

Proof **:** By definition,

$$Z(19.2^k) = \min\{m : 19.2^k \mid \frac{m(m+1)}{2}\} = \min\{m : 19.2^{k+1} \mid m(m+1)\}.$$

The eighteen possible cases are given below.

Case (1) : When k is of the form $k = 18a$ for some integer $a \geq 1$.
Here, by Lemma 4.1.9(2), $19 \mid (9.2^{k+1}+1) = 18(2^{18a}-1)+19$. Thus, $Z(19.2^k) = 9.2^{k+1}$.

Case (2) : When k is of the form $k = 18a+1$ for some integer $a \geq 0$.
Since 19 divides $5.2^{k+1}-1 = 20(2^{18a}-1)+19$, we have, $Z(19.2^k) = 5.2^{k+1}-1$.

Case (3) : When k is of the form $k = 18a+2$ for some integer $a \geq 0$.
In this case, 19 divides $12.2^{k+1}-1 = 96(2^{18a}-1)+95$. Hence, $Z(19.2^k) = 3.2^{k+3}-1$.

Case (4) : When k is of the form $k = 18a+3$ for some integer $a \geq 0$.
Here, 19 divides $6.2^{k+1}-1 = 96(2^{18a}-1)+95$. Consequently, $Z(19.2^k) = 3.2^{k+2}-1$.

Case (5) : When k is of the form $k = 18a+4$ for some integer $a \geq 0$.
In this case, 19 divides $3.2^{k+1}-1 = 96(2^{18a}-1)+95$. Therefore, $Z(19.2^k) = 3.2^{k+1}-1$.

Case (6) : When k is of the form $k = 18a+5$ for some integer $a \geq 0$.
In this case, 19 divides $8.2^{k+1}+1 = 512(2^{18a}-1)+513$, and so, $Z(19.2^k) = 2^{k+4}$.

Case (7) : When k is of the form $k=18a+6$ for some integer $a\geq 0$.
Here, 19 divides $4.2^{k+1}+1=512(2^{18a}-1)+513$, and hence, $Z(19.2^k)=2^{k+3}$.

Case (8) : When k is of the form $k=18a+7$ for some integer $a\geq 0$.
In this case, 19 divides $2.2^{k+1}+1=512(2^{18a}-1)+513$. As such, $Z(19.2^k)=2^{k+2}$.

Case (9) : When k is of the form $k=18a+8$ for some integer $a\geq 0$.
Writing $2^{k+1}=2^{18a+9}=(2^{18a}-1)(2^9+1)+(2^9+1)-(2^{18a}-1)-1$, it follows, by Lemma 4.1.9(1), that 19 divides $2^{k+1}+1$. Consequently, $Z(19.2^k)=2^{k+1}$.

Case (10) : When k is of the form $k=18a+9$ for some integer $a\geq 0$.
Here, 19 divides $9.2^{k+1}-1=18(2^{18a+9}+1)-19$, so that, $Z(19.2^k)=9.2^{k+1}-1$.

Case (11) : When k is of the form $k=18a+10$ for some integer $a\geq 0$.
In this case, 19 divides $5.2^{k+1}+1=20(2^{18a+9}+1)-19$, and hence, $Z(19.2^k)=5.2^{k+1}$.

Case (12) : When k is of the form $k=18a+11$ for some integer $a\geq 0$.
Here, 19 divides $12.2^{k+1}+1=96(2^{18a+9}+1)-95$. Then, $Z(19.2^k)=3.2^{k+3}$.

Case (13) : When k is of the form $k=18a+12$ for some integer $a\geq 0$.
In this case, 19 divides $6.2^{k+1}+1=96(2^{18a+9}+1)-95$. Thus, $Z(19.2^k)=3.2^{k+2}$.

Case (14) : When k is of the form $k=18a+13$ for some integer $a\geq 0$.
Here, 19 divides $3.2^{k+1}+1=96(2^{18a+9}+1)-95$. Therefore, $Z(19.2^k)=3.2^{k+1}$.

Case (15) : When k is of the form $k=18a+14$ for some integer $a\geq 0$.
In this case, 19 divides $2^{k+4}-1=2^{18(a+1)}-1$. And consequently, $Z(19.2^k)=2^{k+4}-1$.

Case (16) : When k is of the form $k=18a+15$ for some integer $a\geq 0$.
Here, 19 divides $2^{k+3}-1=2^{18(a+1)}-1$, so that, $Z(19.2^k)=2^{k+3}-1$.

Case (17) : When k is of the form $k=18a+16$ for some integer $a\geq 0$.
In this case, 19 divides $2^{k+2}-1=2^{18(a+1)}-1$. Therefore, $Z(19.2^k)=2^{k+2}-1$.

Case (18) : When k is of the form $k=18a+17$ for some integer $a\geq 0$.
Noting that, $2^{k+1}-1=2^{18(a+1)}-1$ is divisible by 19, it follows that $Z(19.2^k)=2^{k+1}-1$.

Hence, the lemma is established. ■

Lemma 4.2.13 **:** For any integer $k\geq 1$,

$$Z(31.2^k)=\begin{cases} 15.2^{k+1}, & \text{if } 5\mid k \\ 2^{k+4}-1, & \text{if } 5\mid (k-1) \\ 2^{k+3}-1, & \text{if } 5\mid (k-2) \\ 2^{k+2}-1, & \text{if } 5\mid (k-3) \\ 2^{k+2}-1, & \text{if } 5\mid (k-3) \end{cases}$$

Proof **:** Using the definition,

$$Z(31.2^k)=\min\{m : 31.2^k \mid \frac{m(m+1)}{2}\}=\min\{m : 31.2^{k+1} \mid m(m+1)\}.$$

We now consider the five possible cases separately below :

Case (1) : When k is of the form $k=5a$ for some integer $a\geq 1$.
By Lemma 4.1.10, 31 divides $15.2^{k+1}+1=30(2^{5a}-1)+31$. Thus, $Z(31.2^k)=15.2^{k+1}$.

Case (2) : When k is of the form $k=5a+1$ for some integer $a\geq 0$.
Here, since 31 divides $8.2^{k+1}-1=32(2^{5a}-1)+31$, it follows that $Z(31.2^k)=2^{k+4}-1$.

Case (3) : When k is of the form $k = 5a + 2$ for some integer $a \geq 0$.
In this case, $2^{k+1} = 8(2^{5a} - 1) + 8$, so that 31 divides $4.2^{k+1} - 1$. Thus, $Z(31.2^k) = 2^{k+3} - 1$.

Case (4) : When k is of the form $k = 5a + 3$ for some integer $a \geq 0$.
Here, $2.2^{k+1} - 1 = 32(2^{5a} - 1) + 31$ is divisible by 31. Therefore, $Z(31.2^k) = 2^{k+2} - 1$.

Case (5) : When k is of the form $k = 5a + 4$ for some integer $a \geq 0$.
In this case, 31 divides $2^{k+1} - 1 = 2^{5(a+1)} - 1$. And so, $Z(31.2^k) = 2^{k+1} - 1$.

All these complete the proof of the lemma. ■

The expressions for $Z(2p^k)$ and $Z(3p^k)$ $(k \geq 3)$ are given in Lemmas 4.2.14 – 4.2.15.

Lemma 4.2.14 **:** If $p \geq 3$ is a prime and $k \geq 3$ is an integer, then

$$Z(2p^k) = \begin{cases} p^k, & \text{if } 4 \mid (p+1) \text{ and } k \text{ is odd} \\ p^k - 1, & \text{otherwise} \end{cases}$$

Proof **:** By definition,

$$Z(2p^k) = \min\{m : 2p^k \mid \frac{m(m+1)}{2}\} = \min\{m : 4p^k \mid m(m+1)\}.$$

We consider the two possibilities :

Case 1 : p is of the form $4a + 1$ for some integer $a \geq 1$. In this case,

$$p^k = (4a+1)^k = (4a)^k + {}_kC_1 (4a)^{k-1} + \ldots + {}_kC_{(k-1)} (4a) + 1,$$

showing that 4 divides $p^k - 1$. Hence, in this case, $Z(2p^k) = p^k - 1$.

Case 2 : p is of the form $4a + 3$ for some integer $a \geq 1$. Here,

$$p^k = (4a+3)^k = (4a)^k + {}_kC_1 (4a)^{k-1}(3) + \ldots + {}_kC_{(k-1)} (4a)3^{k-1} + 3^k.$$

(1) If $k \geq 2$ is even, then by Lemma 4.1.2, $4 \mid (3^k - 1)$, so that $4 \mid (p^k - 1)$. Thus, $Z(2p^k) = p^k - 1$.

(2) If $k \geq 3$ is odd, then by Lemma 4.1.2, $4 \mid (3^k + 1)$, and so $4 \mid (p^k + 1)$. Hence, $Z(2p^k) = p^k$.

All these complete the proof of the theorem. ■

Lemma 4.2.15 **:** If $p \geq 5$ is a prime and $k \geq 3$ is an integer, then

$$Z(3p^k) = \begin{cases} p^k, & \text{if } 3 \mid (p+1) \text{ and } k \text{ is odd} \\ p^k - 1, & \text{otherwise} \end{cases}$$

Proof **:** Using the definition,

$$Z(3p^k) = \min\{m : 3p^k \mid \frac{m(m+1)}{2}\} = \min\{m : 6p^k \mid m(m+1)\}.$$

We now consider the following two possible cases :

Case 1 : p is of the form $3a + 1$ for some integer $a \geq 1$. In this case, since

$$p^k = (3a+1)^k = (3a)^k + {}_kC_1 (3a)^{k-1} + \ldots + {}_kC_{(k-1)} (3a) + 1,$$

it follows that $3 \mid (p^k - 1)$. Thus, in this case, $Z(3p^k) = p^k - 1$.

Case 2 : p is of the form $3a + 2$ for some integer $a \geq 1$. Then,

$$p^k = (3a+2)^k = (3a)^k + {}_kC_1 (3a)^{k-1}(2) + \ldots + {}_kC_{(k-1)} (3a)2^{k-1} + 2^k.$$

(1) If $k \geq 2$ is even, then by Lemma 4.1.3, $3 \mid (2^k - 1)$, and so $3 \mid (p^k - 1)$. Thus, $Z(3p^k) = p^k - 1$.

(2) If $k \geq 3$ is odd, then by Lemma 4.1.3, $3 \mid (2^k + 1)$, so that $3 \mid (p^k + 1)$. Hence, $Z(3p^k) = p^k$. ■

Lemma 4.2.16 – Lemma 4.2.25 give the expressions for Z(4p), Z(5p), Z(6p), Z(7p), Z(8p), Z(9p), Z(10p), Z(11p), Z(12p) and Z(13p) respectively, where p is a prime.

Lemma 4.2.16 **:** If $p \geq 3$ is a prime, then

$$Z(4p) = \begin{cases} p-1, & \text{if } 8 \mid (p-1) \\ p, & \text{if } 8 \mid (p+1) \\ 3p-1, & \text{if } 8 \mid (3p-1) \\ 3p, & \text{if } 8 \mid (3p+1) \end{cases}$$

Proof **:** By definition,

$$Z(4p) = \min\{m : 4p \mid \frac{m(m+1)}{2}\} = \min\{m : 8p \mid m(m+1)\}.$$

Here, p must divide only one of m and $m+1$. We now consider separately the following four cases that may arise depending on p :

Case (1) : p is of the form $p = 8a+1$ for some integer $a \geq 1$.
In this case, 8 divides $p-1$, and hence, by Lemma 4.2.4, $Z(4p) = p-1$.

Case (2) : p is of the form $p = 8a+7$ for some integer $a \geq 0$.
Here, 8 divides $p+1$, and hence, $Z(4p) = p$ (by Lemma 4.2.4).

Case (3) : p is of the form $p = 8a+3$ for some integer $a \geq 0$.
In this case, 8 divides $3p-1 = 8(3a+1)$, and hence, $Z(4p) = 3p-1$.

Case (4) : p is of the form $p = 8a+5$ for some integer $a \geq 0$.
Here, 8 divides $3p+1 = 8(3a+2)$, and hence, $Z(4p) = 3p$.

All these complete the proof of the theorem. ■

Lemma 4.2.17 **:** If $p = 3$ or $p \geq 7$ is a prime, then

$$Z(5p) = \begin{cases} p-1, & \text{if } 10 \mid (p-1) \\ p, & \text{if } 10 \mid (p+1) \\ 2p-1, & \text{if } 5 \mid (2p-1) \\ 2p, & \text{if } 5 \mid (2p+1) \end{cases}$$

Proof **:** Using definition,

$$Z(5p) = \min\{m : 5p \mid \frac{m(m+1)}{2}\} = \min\{m : 10p \mid m(m+1)\}. \quad (11)$$

Now, p must divide one of m and $m+1$, and 5 must divide the other. By Lemma 4.2.4, the minimum m in (11) may be taken as $p-1$ or p if 10 divides $p-1$ or $p+1$ respectively.
Here, we have to consider the following possible four cases :

Case (1) : p is a prime whose last digit is 1. Here, 10 divides $p-1$, and hence, $Z(5p) = p-1$.

Case (2) : p is a prime whose last digit is 9. In this case, 10 divides $p+1$, and so, $Z(5p) = p$.

Case (3) : p is a prime whose last digit is 3. Here, $5 \mid (2p-1)$, so that $10p \mid 2p(2p-1)$.
Thus, the minimum m in (11) may be taken as $2p-1$. Hence, $Z(5p) = 2p-1$.

Case (4) : p is a prime whose last digit is 7. Here, 5 divides $2p+1$, and hence, $Z(5p) = 2p$.

Hence, the proof is complete. ■

Lemma 4.2.18 **:** If $p \geq 5$ is a prime, then

$$Z(6p) = \begin{cases} p-1, & \text{if } 12 \mid (p-1) \\ p, & \text{if } 12 \mid (p+1) \\ 3p, & \text{if } 12 \mid (p-5) \\ 3p-1, & \text{if } 12 \mid (p-7) \end{cases}$$

Proof **:** By definition,

$$Z(6p) = \min\{m : 6p \mid \frac{m(m+1)}{2}\} = \min\{m : 12p \mid m(m+1)\}.$$

Here, p must divide exactly one of m and $m+1$.
Now, the following four cases may arise depending on the nature of the prime p :

Case (1) : p is of the form $p=12a+1$ for some integer $a \geq 1$.
In this case, 12 divides $p-1$, and hence, by Lemma 4.2.4, $Z(6p)=p-1$.

Case (2) : p is of the form $p=12a+11$ for some integer $a \geq 0$.
Here, 12 divides $p+1$, and hence, $Z(6p)=p$ (by Lemma 4.2.4).

Case (3) : p is of the form $p=12a+5$ for some integer $a \geq 0$.
In this case, 4 divides $3p+1=4(9a+4)$, so that $12p \mid 3p(3p+1)$. Thus, $Z(6p)=3p$.

Case (4) : p is of the form $p=12a+7$ for some integer $a \geq 0$.
Here, 4 divides $3p-1=4(9a+5)$, and hence, $Z(6p)=3p-1$.

All these establish the lemma. ■

Lemma 4.2.19 **:** If $p \geq 3$ with $p \neq 7$ is a prime, then

$$Z(7p) = \begin{cases} p-1, & \text{if } 7 \mid (p-1) \\ p, & \text{if } 7 \mid (p+1) \\ 3p, & \text{if } 7 \mid (3p+1) \\ 3p-1, & \text{if } 7 \mid (3p-1) \\ 2p, & \text{if } 7 \mid (2p+1) \\ 2p-1, & \text{if } 7 \mid (2p-1) \end{cases}$$

Proof **:** Since

$$Z(7p) = \min\{m : 7p \mid \frac{m(m+1)}{2}\} = \min\{m : 14p \mid m(m+1)\},$$

p must divide exactly one of m or $m+1$, and 7 must divide the other.
We consider the six cases that may arise.

Case (1) : p is of the form $p=7a+1$ for some integer $a \geq 1$.
In this case, 7 divides $p-1$, and consequently, by Lemma 4.2.4, $Z(7p)=p-1$.

Case (2) : p is of the form $p=7a+6$ for some integer $a \geq 1$.
Here, 7 divides $p+1$, and so, $Z(7p)=p$ (by Lemma 4.2.4).

Case (3) : p is of the form $p=7a+2$ for some integer $a \geq 1$.
In this case, 7 divides $3p+1=7(3a+1)$, so that $7p \mid 3p(3p+1)$. Therefore, $Z(7p)=3p$.

Case (4) : p is of the form $p = 7a + 5$ for some integer $a \geq 0$.
Here, $7 | (3p-1)$. Clearly, $2 | (3p-1)$ as well. Thus, $14p | 3p(3p-1)$. Hence, $Z(7p) = 3p-1$.

Case (5) : p is of the form $p = 7a + 3$ for some integer $a \geq 0$.
In this case, 7 divides $2p+1 = 7(2a+1)$, and hence, $Z(7p) = 2p$.

Case (6) : p is of the form $p = 7a + 4$ for some integer $a \geq 1$.
Here, 7 divides $2p-1 = 7(2a+1)$, and hence, $Z(7p) = 2p-1$.

Hence, the lemma is established. ■

Lemma 4.2.20 **:** For any prime $p \geq 3$,

$$Z(8p) = \begin{cases} p-1, & \text{if } 16 | (p-1) \\ p, & \text{if } 16 | (p+1) \\ 5p, & \text{if } 16 | (5p+1) \\ 5p-1, & \text{if } 16 | (5p-1) \\ 3p, & \text{if } 16 | (3p+1) \\ 3p-1, & \text{if } 16 | (3p-1) \\ 7p-1, & \text{if } 16 | (7p-1) \\ 7p, & \text{if } 16 | (7p+1) \end{cases}$$

Proof **:** By definition,

$$Z(8p) = \min\{m : 8p | \frac{m(m+1)}{2}\} = \min\{m : 16p | m(m+1)\}.$$

Then, p must divide exactly one of m or $m+1$.
Here, we have to consider the following eight possibilities :

Case (1) : p is of the form $p = 16a+1$ for some integer $a \geq 1$.
In this case, 16 divides $p-1$, and so, by Lemma 4.2.4, $Z(8p) = p-1$.

Case (2) : p is of the form $p = 16a+15$ for some integer $a \geq 1$.
Here, 16 divides $p+1$, and hence, $Z(8p) = p$ (by Lemma 4.2.4).

Case (3) : p is of the form $p = 16a + 3$ for some integer $a \geq 0$.
In this case, 16 divides $5p+1$, so that 16p divides $5p(5p+1)$. Hence, $Z(8p) = 5p$.

Case (4) : p is of the form $p = 16a+13$ for some integer $a \geq 0$.
Here, 16 divides $5p-1 = 16(5a+4)$, and hence, $Z(8p) = 5p-1$.

Case (5) : p is of the form $p = 16a+5$ for some integer $a \geq 0$.
In this case, 16 divides $3p+1 = 16(3a+1)$, and hence, $Z(8p) = 3p$.

Case (6) : p is of the form $p = 16a+11$ for some integer $a \geq 0$.
Here, 16 divides $3p-1 = 16(3a+2)$, and hence, $Z(8p) = 3p-1$.

Case (7) : p is of the form $p = 16a+7$ for some integer $a \geq 0$.
In this case, 16 divides $7p-1 = 16(7a+3)$, and hence, $Z(8p) = 7p-1$.

Case (8) : p is of the form $p = 16a+9$ for some integer $a \geq 1$.
Here, 16 divides $7p+1 = 16(7a+4)$, and hence, $Z(8p) = 7p$.

Thus, the proof is complete. ■

Lemma 4.2.21 **:** For any prime $p \geq 5$,

$$Z(9p) = \begin{cases} p-1, & \text{if } 18 \mid (p-1) \\ p, & \text{if } 18 \mid (p+1) \\ 2p-1, & \text{if } 9 \mid (2p-1) \\ 2p, & \text{if } 9 \mid (2p+1) \\ 4p-1, & \text{if } 9 \mid (4p-1) \\ 4p, & \text{if } 9 \mid (4p+1) \end{cases}$$

Proof **:** By definition,

$$Z(9p) = \min\{m : 9p \mid \frac{m(m+1)}{2}\} = \min\{m : 18p \mid m(m+1)\}.$$

The six possible cases are considered separately below :

Case (1) : p is of the form $p = 18a + 1$ for some integer $a \geq 1$.
In this case, 18 divides $p-1$, and so, by Lemma 4.2.4, $Z(9p) = p-1$.

Case (2) : p is of the form $p = 18a + 17$ for some integer $a \geq 0$.
In this case, 18 divides $p+1$, and hence, $Z(9p) = p$ (by Lemma 4.2.4).

Case (3) : p is of the form $p = 18a + 5$ for some integer $a \geq 0$.
Here, $2p-1 = 9(4a+1)$, so that 18p divides $2p(2p-1)$. Thus, $Z(9p) = 2p-1$.

Case (4) : p is of the form $p = 18a + 13$ for some integer $a \geq 0$.
In this case, 9 divides $2p+1 = 9(4a+3)$, and consequently, $Z(9p) = 2p$.

Case (5) : p is of the form $p = 18a + 7$ for some integer $a \geq 0$.
Here, 9 divides $4p-1 = 9(8a+3)$, so that 18p divides $4p(4p-1)$. Thus, $Z(9p) = 4p-1$.

Case (6) : p is of the form $p = 18a + 11$ for some integer $a \geq 0$.
In this case, since 9 divides $4p+1 = 9(8a+5)$, it follows that, $Z(9p) = 4p$. ■

Lemma 4.2.22 **:** For any prime $p \geq 3$ with $p \neq 5$,

$$Z(10p) = \begin{cases} p-1, & \text{if } 20 \mid (p-1) \\ p, & \text{if } 20 \mid (p+1) \\ 5p, & \text{if } 20 \mid (p-3) \\ 5p-1, & \text{if } 20 \mid (p+3) \\ 3p-1, & \text{if } 20 \mid (p-7) \\ 3p, & \text{if } 20 \mid (p+7) \\ 4p-1, & \text{if } 20 \mid (p-9) \\ 4p, & \text{if } 20 \mid (p+9) \end{cases}$$

Proof **:** By definition,

$$Z(10p) = \min\{m : 10p \mid \frac{m(m+1)}{2}\} = \min\{m : 20p \mid m(m+1)\}.$$

The eight possible cases are considered below :

Case (1) : p is of the form $p = 20a + 1$ for some integer $a \geq 1$.
In this case, 20 divides $p-1$, and so, by Lemma 4.2.4, $Z(10p) = p-1$.

Case (2) : p is of the form $p = 20a + 19$ for some integer $a \geq 0$.
Here, 20 divides $p+1$, and hence, $Z(10p) = p$ (by Lemma 4.2.4).

Case (3) : p is of the form $p=20a+3$ for some integer $a \ge 0$.
Here, 4 divides $5p+1=4(25a+4)$, so that 20p divides $5p(5p+1)$. Thus, $Z(10p)=5p$.

Case (4) : p is of the form $p=20a+17$ for some integer $a \ge 0$.
In this case, 4 divides $5p-1=4(25a+21)$, and hence, $Z(10p)=5p-1$.

Case (5) : p is of the form $p=20a+7$ for some integer $a \ge 0$.
Here, 20 divides $3p-1=20(3a+1)$, and consequently, $Z(10p)=3p-1$.

Case (6) : p is of the form $p=20a+13$ for some integer $a \ge 0$.
In this case, 20 divides $3p+1=20(3a+2)$, and hence, $Z(10p)=3p$.

Case (7) : p is of the form $p=20a+9$ for some integer $a \ge 1$.
Here, 5 divides $4p-1=5(16a+7)$, so that 20p divides $4p(4p-1)$. Thus, $Z(10p)=4p-1$.

Case (8) : p is of the form $p=20a+11$ for some integer $a \ge 0$.
In this case, since 5 divides $4p+1=5(16a+9)$, it follows that, $Z(10p)=4p$. ■

***Lemma 4.2.23* :** For any prime $p \ge 3$ with $p \ne 11$,

$$Z(11p) = \begin{cases} p-1, & \text{if } 11 \mid (p-1) \\ p, & \text{if } 11 \mid (p+1) \\ 5p, & \text{if } 11 \mid (5p+1) \\ 5p-1, & \text{if } 11 \mid (5p-1) \\ 4p-1, & \text{if } 11 \mid (4p-1) \\ 4p, & \text{if } 11 \mid (4p+1) \\ 3p-1, & \text{if } 11 \mid (3p-1) \\ 3p, & \text{if } 11 \mid (3p+1) \\ 2p, & \text{if } 11 \mid (2p+1) \\ 2p-1, & \text{if } 11 \mid (2p-1) \end{cases}$$

***Proof* :** By definition,

$$Z(11p)=\min\{m : 11p \mid \frac{m(m+1)}{2}\}=\min\{m : 22p \mid m(m+1)\}.$$

Here, p must divide exactly one of m or $m+1$, and 11 must divide the other.
We have to consider ten possible cases.

Case (1) : p is of the form $p=11a+1$ for some integer $a \ge 1$.
In this case, 11 divides $p-1$, and so, by Lemma 4.2.4, $Z(11p)=p-1$.

Case (2) : p is of the form $p=11a+10$ for some integer $a \ge 1$.
Here, 11 divides $p+1$, and consequently (by Lemma 4.2.4), $Z(11p)=p$.

Case (3) : p is of the form $p=11a+2$ for some integer $a \ge 1$.
In this case, 11 divides $5p+1=11(5a+1)$, and hence, $Z(11p)=5p$.

Case (4) : p is of the form $p=11a+9$ for some integer $a \ge 1$.
Here, 11 divides $5p-1=11(5a+4)$, and therefore, $Z(11p)=5p-1$.

Case (5) : p is of the form $p=11a+3$ for some integer $a \ge 0$.
In this case, 11 divides $4p-1=11(4a+1)$, and hence, $Z(11p)=4p-1$.

Case (6) : p is of the form $p=11a+8$ for some integer $a\geq 1$.
Here, 11 divides $4p+1=11(4a+3)$, and consequently, $Z(11p)=4p$.

Case (7) : p is of the form $p=11a+4$ for some integer $a\geq 1$.
In this case, 11 divides $3p-1=11(3a+1)$, and hence, $Z(11p)=3p-1$.

Case (8) : p is of the form $p=11a+7$ for some integer $a\geq 0$.
Here, 11 divides $3p+1=11(3a+2)$, so that, $Z(11p)=3p$.

Case (9) : p is of the form $p=11a+5$ for some integer $a\geq 0$.
In this case, 11 divides $2p+1=11(2a+1)$, and hence, $Z(11p)=2p$.

Case (10) : p is of the form $p=11a+6$ for some integer $a\geq 1$.
Here, since 11 divides $2p-1=11(2a+1)$, it follows that, $Z(11p)=2p-1$. ■

Lemma 4.2.24 **:** If $p\geq 3$ is a prime, then

$$Z(12p)=\begin{cases} p-1, & \text{if } 24\,|\,(p-1) \\ p, & \text{if } 24\,|\,(p+1) \\ 3p, & \text{if } \ 8\,|\,(3p+1) \\ 3p-1, & \text{if } \ 8\,|\,(3p-1) \\ 7p-1, & \text{if } 24\,|\,(7p-1) \\ 7p, & \text{if } 24\,|\,(7p+1) \end{cases}$$

Proof **:** By definition,

$$Z(12p)=\min\{m : 12p\,|\,\frac{m(m+1)}{2}\}=\min\{m : 24p\,|\,m(m+1)\}.$$

Here, p must divide exactly one of m or $m+1$.
We consider the following eight cases that may arise depending on p :

Case (1) : p is of the form $p=24a+1$ for some integer $a\geq 1$.
In this case, 24 divides $p-1$. Therefore, by Lemma 4.2.4, $Z(12p)=p-1$.

Case (2) : p is of the form $p=24a+23$ for some integer $a\geq 0$.
Here, 24 divides $p+1$, and so, $Z(12p)=p$ (by Lemma 4.2.4).

Case (3) : p is of the form $p=24a+5$ for some integer $a\geq 0$.
In this case, 8 divides $3p+1=8(9a+2)$, so that 24p divides $3p(3p+1)$. Hence, $Z(12p)=3p$.

Case (4) : p is of the form $p=24a+19$ for some integer $a\geq 0$.
Here, 8 divides $3p-1=8(9a+7)$, and consequently, $Z(12p)=3p-1$.

Case (5) : p is of the form $p=24a+7$ for some integer $a\geq 0$.
In this case, 24 divides $7p-1=24(7a+2)$, and therefore, $Z(12p)=7p-1$.

Case (6) : p is of the form $p=24a+17$ for some integer $a\geq 0$.
Here, 24 divides $7p+1=24(7a+5)$, and hence, $Z(12p)=7p$.

Case (7) : p is of the form $p=24a+11$ for some integer $a\geq 0$.
In this case, 8 divides $3p-1=8(9a+4)$, and so, $Z(12p)=3p-1$.

Case (8) : p is of the form $p=24a+13$ for some integer $a\geq 0$.
Here, 8 divides $3p+1=8(9a+5)$, and hence, $Z(12p)=3p$.

To complete the proof of the lemma, note that $Z(36)=8$. ■

Lemma 4.2.25 **:** For any prime $p \geq 2$,

$$Z(13p) = \begin{cases} p-1, & \text{if } 13 \mid (p-1) \\ p, & \text{if } 13 \mid (p+1) \\ 6p, & \text{if } 13 \mid (6p+1) \\ 6p-1, & \text{if } 13 \mid (6p-1) \\ 4p, & \text{if } 13 \mid (4p+1) \\ 4p-1, & \text{if } 13 \mid (4p-1) \\ 3p, & \text{if } 13 \mid (3p+1) \\ 3p-1, & \text{if } 13 \mid (3p-1) \\ 5p, & \text{if } 13 \mid (5p+1) \\ 5p-1, & \text{if } 13 \mid (5p-1) \\ 2p, & \text{if } 13 \mid (2p+1) \\ 2p-1, & \text{if } 13 \mid (2p-1) \end{cases}$$

Proof **:** By definition,

$$Z(13p) = \min\{m : 13p \mid \frac{m(m+1)}{2}\} = \min\{m : 26p \mid m(m+1)\}.$$

Here, p must divide exactly one of m or $m+1$, and 13 must divide the other.
In this case, we have to consider the twelve possible cases that may arise :

Case (1) : p is of the form $p = 13a+1$ for some integer $a \geq 1$.
In this case, 13 divides $p-1$, and so, by Lemma 4.2.4, $Z(13p) = p-1$.

Case (2): p is of the form $p = 13a+12$ for some integer $a \geq 1$.
Here, 13 divides $p+1$, and hence, $Z(13p) = p$ (by Lemma 4.2.4).

Case (3) : p is of the form $p = 13a+2$ for some integer $a \geq 0$.
In this case, 13 divides $6p+1 = 13(6a+1)$, and hence, $Z(13p) = 6p$.

Case (4) : p is of the form $p = 13a+11$ for some integer $a \geq 0$.
Here, 13 divides $6p-1 = 13(6a+5)$, and hence, $Z(13p) = 6p-1$.

Case (5) : p is of the form $p = 13a+3$ for some integer $a \geq 0$.
In this case, 13 divides $4p+1 = 13(4a+1)$, and hence, $Z(13p) = 4p$.

Case (6) : p is of the form $p = 13a+10$ for some integer $a \geq 1$.
Here, 13 divides $4p-1 = 13(4a+3)$, and hence, $Z(13p) = 4p-1$.

Case (7) : p is of the form $p = 13a+4$ for some integer $a \geq 1$.
In this case, 13 divides $3p+1 = 13(3a+1)$, and hence, $Z(13p) = 3p$.

Case (8) : p is of the form $p = 13a+9$ for some integer $a \geq 1$.
Here, 13 divides $3p-1 = 13(3a+2)$, and hence, $Z(13p) = 3p-1$.

Case (9) : p is of the form $p = 13a+5$ for some integer $a \geq 0$.
In this case, 13 divides $5p+1 = 13(5a+2)$, and hence, $Z(13p) = 5p$.

Case 10 : p is of the form $p = 13a+8$ for some integer $a \geq 1$. Here, $Z(13p) = 5p-1$.

Case 11 : p is of the form $p = 13a+6$ for some integer $a \geq 1$.
In this case, 13 divides $2p+1 = 13(2a+1)$, and hence, $Z(13p) = 2p$.

Case 12 : p is of the form $p = 13a+7$ for some integer $a \geq 0$. Here, $Z(13p) = 2p-1$. ■

In Lemmas 4.2.26 – 4.2.27, we give the expressions for Z(16p) and Z(32p).

Lemma 4.2.26 **:** For any prime $p \geq 3$,

$$Z(16p) = \begin{cases} p-1, & \text{if } 32 \mid (p-1) \\ p, & \text{if } 32 \mid (p+1) \\ 11p-1, & \text{if } 32 \mid (p-3) \\ 11p, & \text{if } 32 \mid (p-29) \\ 13p-1, & \text{if } 32 \mid (p-5) \\ 13p, & \text{if } 32 \mid (p-27) \\ 9p, & \text{if } 32 \mid (p-7) \\ 9p-1, & \text{if } 32 \mid (p-25) \\ 7p, & \text{if } 32 \mid (p-9) \\ 7p-1, & \text{if } 32 \mid (p-23) \\ 3p-1, & \text{if } 32 \mid (p-11) \\ 3p, & \text{if } 32 \mid (p-21) \\ 5p-1, & \text{if } 32 \mid (p-13) \\ 5p, & \text{if } 32 \mid (p-19) \\ 15p-1, & \text{if } 32 \mid (p-15) \\ 15p, & \text{if } 32 \mid (p-17) \end{cases}$$

Proof **:** By definition,

$$Z(16p) = \min\{m : 16p \mid \frac{m(m+1)}{2}\} = \min\{m : 32p \mid m(m+1)\}.$$

Here, p must divide exactly one of m or $m+1$.

In this case, we have to consider the sixteen possible cases that may arise :

Case (1) : p is of the form $p = 32a+1$ for some integer $a \geq 1$.

In this case, 32 divides $p-1$, and so, by Lemma 4.2.4, $Z(16p) = p-1$.

Case (2) : p is of the form $p = 32a+31$ for some integer $a \geq 0$.

Here, 32 divides $p+1$, and hence, $Z(16p) = p$ (by Lemma 4.2.4).

Case (3) : p is of the form $p = 32a+3$ for some integer $a \geq 0$.

Since $11p-1 = 32(11a+1)$, it follows that 32 divides $11p-1$. Hence, $Z(16p) = 11p-1$.

Case (4) : p is of the form $p = 32a+29$ for some integer $a \geq 0$. Here, $Z(16p) = 11p$.

Case (5) : p is of the form $p = 32a+5$ for some integer $a \geq 0$.

In this case, 32 divides $13p-1 = 32(13a+2)$, and hence, $Z(16p) = 13p-1$.

Case (6) : p is of the form $p = 32a+27$ for some integer $a \geq 1$. Here, $Z(16p) = 13p$.

Case (7) : p is of the form $p = 32a+7$ for some integer $a \geq 0$.

In this case, $9p+1 = 32(9a+2)$. Therefore, 32 divides $9p+1$, and hence, $Z(16p) = 9p$.

Case (8) : p is of the form $p = 32a+25$ for some integer $a \geq 1$. Here, $Z(16p) = 9p-1$.

Case (9) : p is of the form $p = 32a+9$ for some integer $a \geq 1$.

In this case, 32 divides $7p+1 = 32(7a+2)$, and consequently, $Z(16p) = 7p$.

Case (10) : p is of the form $p = 32a+23$ for some integer $a \geq 0$. Here, $Z(16p) = 7p-1$.

Case (11) : p is of the form $p=32a+11$ for some integer $a \geq 0$.
In this case, 32 divides $3p-1=32(3a+1)$, and hence, $Z(16p)=3p-1$.

Case (12) : p is of the form $p=32a+21$ for some integer $a \geq 1$. Here, $Z(16p)=3p$.

Case (13) : p is of the form $p=32a+13$ for some integer $a \geq 0$.
Since 32 divides $5p-1=32(5a+2)$, it follows that, $Z(16p)=5p-1$.

Case (14) : p is of the form $p=32a+19$ for some integer $a \geq 0$. Here, $Z(16p)=5p$.

Case (15) : p is of the form $p=32a+15$ for some integer $a \geq 1$.
In this case, 32 divides $15p-1=32(15a+7)$, and so, $Z(16p)=15p-1$.

Case (16) : p is of the form $p=32a+17$ for some integer $a \geq 0$. Here, $Z(16p)=15p$. ■

Lemma 4.2.27 **:** For any prime $p \geq 3$,

$$Z(32p)=\begin{cases} p-1, & \text{if } 64 \mid (p-1) \\ p, & \text{if } 64 \mid (p+1) \\ 21p, & \text{if } 64 \mid (p-3) \\ 21p-1, & \text{if } 64 \mid (p-61) \\ 13p-1, & \text{if } 64 \mid (p-5) \\ 13p, & \text{if } 64 \mid (p-59) \\ 9p, & \text{if } 64 \mid (p-7) \\ 9p-1, & \text{if } 64 \mid (p-57) \\ 7p, & \text{if } 64 \mid (p-9) \\ 7p-1, & \text{if } 64 \mid (p-55) \\ 29p, & \text{if } 64 \mid (p-11) \\ 29p-1, & \text{if } 64 \mid (p-53) \\ 5p-1, & \text{if } 64 \mid (p-13) \\ 5p, & \text{if } 64 \mid (p-51) \\ 17p, & \text{if } 64 \mid (p-15) \\ 17p-1, & \text{if } 64 \mid (p-49) \\ 15p, & \text{if } 64 \mid (p-17) \\ 15p-1, & \text{if } 64 \mid (p-47) \\ 27p-1, & \text{if } 64 \mid (p-19) \\ 27p, & \text{if } 64 \mid (p-45) \\ 3p, & \text{if } 64 \mid (p-21) \\ 3p-1, & \text{if } 64 \mid (p-43) \\ 25p, & \text{if } 64 \mid (p-23) \\ 25p-1, & \text{if } 64 \mid (p-41) \\ 23p, & \text{if } 64 \mid (p-25) \\ 23p-1, & \text{if } 64 \mid (p-39) \\ 19p-1, & \text{if } 64 \mid (p-27) \\ 19p, & \text{if } 64 \mid (p-37) \\ 11p, & \text{if } 64 \mid (p-29) \\ 11p-1, & \text{if } 64 \mid (p-35) \\ 31p-1, & \text{if } 64 \mid (p-31) \\ 31p & \text{if } 64 \mid (p-33) \end{cases}$$

Proof **:** By definition,

$$Z(32p) = \min\{m : 32p \mid \frac{m(m+1)}{2}\} = \min\{m : 64p \mid m(m+1)\}.$$

Here, p must divide exactly one of m or $m+1$.

In this case, we have to consider the 32 possible cases that may arise :

Case (1) : p is of the form $p = 64a + 1$ for some integer $a \geq 1$.
In this case, 64 divides $p-1$, and so, by Lemma 4.2.4, $Z(32p) = p-1$.

Case (2) : p is of the form $p = 64a + 63$ for some integer $a \geq 1$.
Here, 64 divides $p+1$, and hence, $Z(32p) = p$ (by Lemma 4.2.4).

Case (3) : p is of the form $p = 64a + 3$ for some integer $a \geq 0$.
In this case, $21p + 1 = 64(21a + 1)$, so that 64 divides $21p+1$. Hence, $Z(32p) = 21p$.

Case (4) : p is of the form $p = 64a + 61$ for some integer $a \geq 0$.
Here, $21p - 1 = 64(21a + 20)$. Thus, $32 \mid (11p+1)$, and consequently, $Z(32p) = 21p - 1$.

Case (5) : p is of the form $p = 64a + 5$ for some integer $a \geq 0$.
Since 64 divides $13p - 1 = 64(13a + 1)$, it follows that, $Z(32p) = 13p - 1$.

Case (6) : p is of the form $p = 64a + 59$ for some integer $a \geq 0$.
Here, 64 divides $13p + 1 = 64(13a + 12)$. Therefore, $Z(32p) = 13p$.

Case (7) : p is of the form $p = 64a + 7$ for some integer $a \geq 0$.
In this case, $9p + 1 = 64(9a + 1)$. Thus, 64 divides $9p + 1$, and hence, $Z(32p) = 9p$.

Case (8) : p is of the form $p = 64a + 57$ for some integer $a \geq 1$.
Here, 64 divides $9p - 1 = 64(9a + 8)$, and as such, $Z(32p) = 9p - 1$.

Case (9) : p is of the form $p = 64a + 9$ for some integer $a \geq 1$.
Since 64 divides $7p + 1 = 64(7a + 1)$, it follows that, $Z(32p) = 7p$.

Case (10) : p is of the form $p = 64a + 55$ for some integer $a \geq 1$.
In this case, 64 divides $7p - 1 = 64(7a + 6)$, and hence, $Z(32p) = 7p - 1$.

Case (11) : p is of the form $p = 64a + 11$ for some integer $a \geq 0$.
Here, $29p + 1 = 64(29a + 5)$, so that 64 divides $29p + 1$. Thus, $Z(32p) = 29p$.

Case (12) : p is of the form $p = 64a + 53$ for some integer $a \geq 0$.
In this case, 64 divides $29p - 1 = 64(29a + 24)$, and so, $Z(32p) = 29p - 1$.

Case (13) : p is of the form $p = 64a + 13$ for some integer $a \geq 0$.
Here, 64 divides $5p - 1 = 64(5a + 1)$, and consequently, $Z(32p) = 5p - 1$.

Case (14) : p is of the form $p = 64a + 51$ for some integer $a \geq 1$.
Since 64 divides $5p + 1 = 64(5a + 4)$, it follows that, $Z(32p) = 5p$.

Case (15) : p is of the form $p = 64a + 15$ for some integer $a \geq 1$.
In this case, 64 divides $17p + 1 = 64(17a + 4)$, and hence, $Z(32p) = 17p$.

Case (16) : p is of the form $p = 64a + 49$ for some integer $a \geq 1$.
Here, $17p - 1 = 64(17a + 13)$. Thus, 64 divides $17p - 1$, and so, $Z(32p) = 17p - 1$.

Case (17) : p is of the form $p = 64a + 17$ for some integer $a \geq 0$.
In this case, 64 divides $15p + 1 = 64(15a + 4)$, and consequently, $Z(32p) = 15p$.

Case (18) : p is of the form $p = 64a + 47$ for some integer $a \geq 0$.
Here, 64 divides $15p-1 = 64(15a+11)$, and hence, $Z(32p) = 15p-1$.

Case (19) : p is of the form $p = 64a + 19$ for some integer $a \geq 0$.
In this case, $27p-1 = 64(27a+8)$, so that 64 divides $27p-1$. Hence, $Z(32p) = 27p-1$.

Case (20) : p is of the form $p = 64a + 45$ for some integer $a \geq 1$.
Here, $27p+1 = 64(27a+19)$. Thus, 64 divides $27p+1$, and so, $Z(32p) = 27p$.

Case (21) : p is of the form $p = 64a + 21$ for some integer $a \geq 1$.
In this case, $3p+1 = 64(a+1)$, so that 64p divides $3p(3p+1)$. Thus, $Z(32p) = 3p$.

Case (22) : p is of the form $p = 64a + 43$ for some integer $a \geq 0$.
Here, $3p-1 = 64(3a+2)$. Thus, 64p divides $3p(3p-1)$, and so, $Z(32p) = 3p-1$.

Case (23) : p is of the form $p = 64a + 23$ for some integer $a \geq 0$.
In this case, $25p+1 = 64(25a+9)$. Therefore, 64 divides $25p+1$, and hence, $Z(32p) = 25p$.

Case (24) : p is of the form $p = 64a + 41$ for some integer $a \geq 0$.
Here, 64 divides $25p-1 = 64(25a+16)$, and hence, $Z(32p) = 25p-1$.

Case (25) : p is of the form $p = 64a + 25$ for some integer $a \geq 1$.
In this case, 64 divides $23p+1 = 64(23a+9)$, and hence, $Z(32p) = 23p$.

Case (26) : p is of the form $p = 64a + 39$ for some integer $a \geq 1$.
Here, 64 divides $23p-1 = 64(23a+14)$, and hence, $Z(32p) = 23p-1$.

Case (27) : p is of the form $p = 64a + 27$ for some integer $a \geq 1$.
In this case, $19p-1 = 64(19a+8)$, so that 64 divides $19p-1$. Thus, $Z(32p) = 19p-1$.

Case (28) : p is of the form $p = 64a + 37$ for some integer $a \geq 0$.
Here, 64 divides $19p+1 = 64(19a+11)$, and hence, $Z(32p) = 19p$.

Case (29) : p is of the form $p = 64a + 29$ for some integer $a \geq 0$.
Since $11p+1 = 64(11a+5)$, it follows that 64p divides $11p(11p+1)$, and so, $Z(32p) = 11p$.

Case (30) : p is of the form $p = 64a + 35$ for some integer $a \geq 1$.
Here, $64 \mid (11p-1) = 64(11a+6)$, so that 64p divides $11p(11p-1)$. Thus, $Z(32p) = 11p-1$.

Case (31) : p is of the form $p = 64a + 31$ for some integer $a \geq 0$.
In this case, 64 divides $31p-1 = 64(31a+15)$, and hence, $Z(32p) = 31p-1$.

Case (32) : p is of the form $p = 64a + 33$ for some integer $a \geq 1$.
Here, $31p+1 = 64(31a+16)$. Thus, 64 divides $31p+1$, and consequently, $Z(32p) = 31p$.

All these complete the proof of the lemma. ■

We now give the expressions for Z(24p), Z(48p) and Z(96p) in the following three lemmas. We shall make use of these results in proving Lemma 4.4.18 and Lemma 4.4.21 in §4.4 in connection with equations involving Z(n) and the arithmetic functions like d(n) (the divisor function) and $\phi(n)$ (Euler's phi function).

***Lemma 4.2.28* :** For any prime $p \geq 5$,

$$Z(24p) = \begin{cases} p-1, & \text{if } 48 \mid (p-1) \\ p, & \text{if } 48 \mid (p+1) \\ 3p, & \text{if } 16 \mid (3p+1) \\ 3p-1, & \text{if } 16 \mid (3p-1) \\ 7p-1, & \text{if } 48 \mid (7p-1) \\ 7p, & \text{if } 48 \mid (7p+1) \\ 11p, & \text{if } 48 \mid (11p+1) \\ 11p-1, & \text{if } 48 \mid (11p-1) \\ 15p, & \text{if } 48 \mid (15p+1) \\ 15p-1, & \text{if } 48 \mid (15p-1) \\ 5p, & \text{if } 8 \mid (5p+1) \\ 5p-1, & \text{if } 8 \mid (5p-1) \\ 9p, & \text{if } 16 \mid (9p+1) \\ 9p-1, & \text{if } 16 \mid (9p-1) \end{cases}$$

***Proof* :** By definition,

$$Z(24p) = \min\{m : 24p \mid \frac{m(m+1)}{2}\} = \min\{m : 48p \mid m(m+1)\}.$$

Here, p must divide exactly one of m or $m+1$.

In this case, we have to consider the sixteen possible cases that may arise :

Case (1) : p is of the form $p = 48a + 1$ for some integer $a \geq 1$.

In this case, 48 divides $p-1$, and so, by Lemma 4.2.4, $Z(24p) = p-1$.

Case (2) : p is of the form $p = 48a + 47$ for some integer $a \geq 0$.

Here, 48 divides $p+1$, and hence, $Z(24p) = p$ (by Lemma 4.2.4).

Case (3) : p is of the form $p = 48a + 5$ for some integer $a \geq 0$.

In this case, $3p+1 = 16(9a+1)$, so that 48p divides $3p(3p+1)$. Hence, $Z(24p) = 3p$.

Case (4) : p is of the form $p = 48a + 43$ for some integer $a \geq 0$.

Here, $3p-1 = 16(9a+8)$. Thus, 48p divides $3p(3p-1)$, and consequently, $Z(24p) = 3p-1$.

Case (5) : p is of the form $p = 48a + 7$ for some integer $a \geq 0$.

In this case, 48 divides $7p - 1 = 48(7a+1)$, and hence, $Z(24p) = 7p-1$.

Case (6) : p is of the form $p = 48a + 41$ for some integer $a \geq 0$.

Here, 48 divides $7p+1 = 48(7a+6)$, and consequently, $Z(24p) = 7p$.

Case (7) : p is of the form $p = 48a + 11$ for some integer $a \geq 0$.

In this case, $3p-1 = 16(9a+2)$, and so, 48p divides $3p(3p-1)$. Hence, $Z(24p) = 3p-1$.

Case (8) : p is of the form $p = 48a + 37$ for some integer $a \geq 0$.

Here, 16 divides $3p+1 = 16(9a+7)$, so that 48p divides $3p(3p+1)$. Thus, $Z(24p) = 3p$.

Case (9) : p is of the form $p = 48a + 13$ for some integer $a \geq 0$.

Since, 48 divides $11p+1 = 48(11a+3)$, we see that, $Z(24p) = 11p$.

Case (10) : p is of the form $p = 48a + 35$ for some integer $a \geq 1$.

In this case, 48 divides $11p-1=48(11a+8)$, and hence, $Z(24p)=11p-1$.

Case (11) : p is of the form $p=48a+17$ for some integer $a \geq 0$.

Here, $15p+1=16(45a+16)$ so that 48p divides $15p(15p+1)$, and hence, $Z(24p)=15p$.

Case (12) : p is of the form $p=48a+31$ for some integer $a \geq 0$.

Note that, $15p-1=16(45a+25)$. Thus, 48p divides $15p(15p-1)$, and so, $Z(24p)=15p-1$.

Case (13) : p is of the form $p=48a+19$ for some integer $a \geq 0$.

In this case, 48 divides $5p+1=48(5a+2)$, and hence, $Z(24p)=5p$.

Case (14) : p is of the form $p=48a+29$ for some integer $a \geq 0$.

Here, 48 divides $5p-1=48(5a+3)$, and hence, $Z(24p)=5p-1$.

Case (15) : p is of the form $p=48a+23$ for some integer $a \geq 0$.

Since $9p+1=16(27a+13)$, it follows that 48p divides $9p(9p+1)$, and so, $Z(24p)=9p$.

Case (16) : p is of the form $p=48a+25$ for some integer $a \geq 1$.

Here, $9p-1=16(27a+14)$. Thus, 48p divides $9p(9p-1)$, and hence, $Z(24p)=9p-1$.

All these establish the lemma. ■

Lemma 4.2.29 **:** For any prime $p \geq 5$,

$$Z(48p)=\begin{cases} p-1, & \text{if } 96 \mid (p-1) \\ p, & \text{if } 96 \mid (p+1) \\ 19p, & \text{if } 96 \mid (19p+1) \\ 19p-1, & \text{if } 96 \mid (19p-1) \\ 9p, & \text{if } 32 \mid (9p+1) \\ 9p-1, & \text{if } 32 \mid (9p-1) \\ 3p-1, & \text{if } 32 \mid (3p+1) \\ 3p, & \text{if } 32 \mid (3p-1) \\ 27p, & \text{if } 32 \mid (27p+1) \\ 27p-1, & \text{if } 32 \mid (27p-1) \\ 15p, & \text{if } 32 \mid (15p+1) \\ 15p-1, & \text{if } 32 \mid (15p-1) \\ 5p-1, & \text{if } 96 \mid (5p+1) \\ 5p, & \text{if } 96 \mid (5p-1) \\ 25p, & \text{if } 96 \mid (25p+1) \\ 25p-1, & \text{if } 96 \mid (25p-1) \\ 21p-1, & \text{if } 32 \mid (21p-1) \\ 21p, & \text{if } 32 \mid (21p+1) \\ 31p-1, & \text{if } 96 \mid (31p-1) \\ 31p, & \text{if } 96 \mid (31p+1) \\ 11p-1, & \text{if } 96 \mid (11p-1) \\ 11p, & \text{if } 96 \mid (11p+1) \\ 13p-1, & \text{if } 96 \mid (13p-1) \\ 13p, & \text{if } 96 \mid (13p+1) \\ 7p, & \text{if } 32 \mid (7p+1) \\ 7p-1, & \text{if } 32 \mid (7p-1) \end{cases}$$

Proof **:** By definition, $Z(48p) = \min\{m : 96p \mid m(m+1)\}$.

Here, p must divide exactly one of m or $m+1$.

In this case, we have to consider the 32 possible cases that may arise :

Case (1) : p is of the form $p = 96a+1$ for some integer $a \geq 1$. In this case, $Z(48p) = p-1$.

Case (2) : p is of the form $p = 96a+95$ for some integer $a \geq 1$. Here, $Z(48p) = p$.

Case (3) : p is of the form $p = 96a+5$ for some integer $a \geq 0$.

In this case, $19p+1 = 96(19a+1)$, so that 96p divides $19p+1$. Hence, $Z(48p) = 19p$.

Case (4) : p is of the form $p = 96a+91$ for some integer $a \geq 1$. Here, $Z(48p) = 19p-1$.

Case (5) : p is of the form $p = 96a+7$ for some integer $a \geq 0$.

In this case, $9p+1=32(27a+2)$, so that 96p divides $9p(9p+1)$. Hence, $Z(48p) = 9p$.

Case (6) : p is of the form $p = 96a+89$ for some integer $a \geq 0$. In this case, $Z(48p) = 9p-1$.

Case (7) : p is of the form $p = 96a+11$ for some integer $a \geq 0$.

Here, 32 divides $3p-1 = 32(9a+1)$, and so, 96p divides $3p(3p-1)$. Thus, $Z(48p) = 3p-1$.

Case (8) : p is of the form $p = 96a+85$ for some integer $a \geq 1$. In this case, $Z(48p) = 3p$.

Case (9) : p is of the form $p = 96a+13$ for some integer $a \geq 0$.

Since 32 divides $27p+1 = 32(81a+11)$, we see that, $Z(48p) = 27p$.

Case (10) : p is of the form $p = 96a+83$ for some integer $a \geq 0$. Here, $Z(48p) = 27p-1$.

Case (11) : p is of the form $p = 96a+17$ for some integer $a \geq 0$.

Here, 32 divides $15p+1 = 32(45a+8)$, so that 96p divides $15p(15p+1)$. Thus, $Z(48p) = 15p$.

Case (12) : p is of the form $p = 96a+79$ for some integer $a \geq 0$. Here, $Z(48p) = 15p-1$.

Case (13) : p is of the form $p = 96a+19$ for some integer $a \geq 0$.

In this case, 96 divides $5p+1 = 96(5a+1)$, and hence, $Z(48p) = 5p$.

Case (14) : p is of the form $p = 96a+77$ for some integer $a \geq 1$. Here, $Z(48p) = 5p-1$.

Case (15) : p is of the form $p = 96a+23$ for some integer $a \geq 0$.

Since $25p+1 = 96(25a+6)$, it follows that 96 divides $25p+1$, and so, $Z(48p) = 25p$.

Case (16) : p is of the form $p = 96a+73$ for some integer $a \geq 0$.

Here, $25p-1 = 96(25a+19)$. Thus, 96 divides $25p-1$, and hence, $Z(48p) = 25p-1$.

Case (17) : p is of the form $p = 96a+25$ for some integer $a \geq 1$. Here, $Z(48p) = 9p-1$.

Case (18) : p is of the form $p = 96a+71$ for some integer $a \geq 0$. In this case, $Z(48p) = 9p$.

Case (19) : p is of the form $p = 96a+29$ for some integer $a \geq 0$.

Here, 32 divides $21p-1 = 32(63a+19)$, so that 96p divides $21p(21p-1)$. Thus, $Z(48p) = 21p-1$.

Case (20) : p is of the form $p = 96a+67$ for some integer $a \geq 0$. In this case, $Z(48p) = 21p$.

Case (21) : p is of the form $p = 96a+31$ for some integer $a \geq 0$.

In this case, 96 divides $31p-1 = 96(31a+10)$, and hence, $Z(48p) = 31p-1$.

Case (22) : p is of the form $p = 96a+65$ for some integer $a \geq 1$. In this case, $Z(48p) = 31p$.

Case (23) : p is of the form $p = 96a+35$ for some integer $a \geq 1$.

Here, 96 divides $11p-1 = 96(11a+4)$, and so, $Z(48p) = 11p-1$.

Case (24) : p is of the form $p=96a+61$ for some integer $a\geq 0$. Here, $Z(48p)=11p$.

Case (25) : p is of the form $p=96a+37$ for some integer $a\geq 0$.
Here, 96 divides $13p-1=96(13a+5)$, and so, $Z(48p)=13p-1$.

Case (26) : p is of the form $p=96a+59$ for some integer $a\geq 0$. In this case, $Z(48p)=13p$.

Case (27) : p is of the form $p=96a+41$ for some integer $a\geq 0$.
Here, 96 divides $7p+1=96(7a+3)$, so that , $Z(48p)=7p$.

Case (28) : p is of the form $p=96a+55$ for some integer $a\geq 1$. Here, $Z(48p)=7p-1$.

Case (29) : p is of the form $p=96a+43$ for some integer $a\geq 0$.
Here, 32 divides $3p-1=32(9a+4)$, so that, 96p divides $3p(3p-1)$. Hence, $Z(48p)=3p-1$.

Case (30) : p is of the form $p=96a+53$ for some integer $a\geq 0$. In this case, $Z(48p)=3p$.

Case (31) : p is of the form $p=96a+47$ for some integer $a\geq 0$.
Here, 32 divides $15p-1=32(45a+22)$, and so, 96p divides $15p(15p-1)$. Thus, $Z(48p)=15p-1$.

Case (32) : p is of the form $p=96a+49$ for some integer $a\geq 0$. Here, $Z(48p)=15p$.

Hence the lemma is established. ■

Lemma 4.2.30 **:** For any prime $p\geq 5$, the values of Z(96p) are as follows :

Form of p	**Z(96p)**	**Form of p**	**Z(96p)**	**Form of p**	**Z(96p)**
192a+1	p–1	192a+35	11p–1	192a+67	21p
192a+191	p	192a+157	11p	192a+125	21p–1
192a+5	51p	192a+37	45p–1	192a+71	9p
192a+187	51p–1	192a+155	45p	192a+121	9p–1
192a+7	9p	192a+41	39p	192a+73	57p–1
192a+185	9p–1	192a+151	39p–1	192a+119	57p
192a+11	35p–1	192a+43	3p–1	192a+77	5p–1
192a+181	35p	192a+149	3p	192a+115	5p
192a+13	59p	192a+47	15p–1	192a+79	17p
192a+179	59p–1	192a+145	15p	192a+113	17p–1
192a+17	15p	192a+49	47p	192a+83	27p–1
192a+175	15p–1	192a+143	47p–1	192a+109	27p
192a+19	27p–1	192a+53	29p–1	192a+85	3p
192a+173	27p	192a+139	29p	192a+107	3p–1
192a+23	25p	192a+55	7p–1	192a+89	41p–1
192a+169	25p–1	192a+137	7p	192a+103	41p
192a+25	23p	192a+59	13p	192a+91	19p–1
192a+167	23p–1	192a+133	13p–1	192a+101	19p
192a+29	53p–1	192a+61	21p–1	192a+95	33p
192a+163	53p	192a+131	21p	192a+97	33p–1
192a+31	31p–1	192a+65	63p		
192a+161	31p	192a+127	63p–1		

Proof **:** By definition, $Z(96p) = \min\{m : 192p \mid m(m+1)\}$.

Here, p must divide exactly one of m or $m+1$.

In this case, we have to consider the 64 possible cases that may arise :

Case (1) : p is of the form $p = 192a + 1$ for some integer $a \geq 1$. In this case, $Z(96p) = p - 1$.

Case (2) : p is of the form $p = 192a + 191$ for some integer $a \geq 0$. Here, $Z(96p) = p$.

Case 3 : p is of the form $p = 192a + 5$ for some integer $a \geq 0$.

In this case, $51p + 1 = 64(153a + 4)$, so that 192p divides $51p(51p + 1)$. Hence, $Z(96p) = 51p$.

Case (4) : p is of the form $p = 192a + 187$ for some integer $a \geq 1$.

Here, 192p divides $51p(51p - 1) = 192(17p).(153a + 149)$, and consequently, $Z(96p) = 51p - 1$.

Case (5) : p is of the form $p = 192a + 7$ for some integer $a \geq 0$.

In this case, $9p + 1 = 64(27a + 1)$, so that 192p divides $9p(9p + 1)$. Hence, $Z(96p) = 9p$.

Case (6) : p is of the form $p = 192a + 185$ for some integer $a \geq 1$.

Here, 192 divides $9p(9p - 1) = 192(3p).(27a + 26)$, and consequently, $Z(96p) = 9p - 1$.

Case (7) : p is of the form $p = 192a + 11$ for some integer $a \geq 0$.

In this case, 192 divides $35p - 1 = 192(35a + 2)$, and so, $Z(96p) = 35p - 1$.

Case (8) : p is of the form $p = 192a + 181$ for some integer $a \geq 0$.

Here, 192 divides $35p + 1 = 192(35a + 33)$. And consequently, $Z(96p) = 35p$.

Case (9) : p is of the form $p = 192a + 13$ for some integer $a \geq 0$.

Since, 192 divides $59p + 1 = 192(59a + 4)$, we see that, $Z(96p) = 59p$.

Case (10) : p is of the form $p = 192a + 179$ for some integer $a \geq 0$.

In this case, 192 divides $59p - 1 = 192(59a + 55)$, and hence, $Z(96p) = 59p - 1$.

Case (11) : p is of the form $p = 192a + 17$ for some integer $a \geq 0$.

Here, 64 divides $15p + 1 = 64(45a + 4)$, so that 192p divides $15p(15p + 1)$. Thus, $Z(96p) = 15p$.

Case (12) : p is of the form $p = 192a + 175$ for some integer $a \geq 1$.

In this case, 192p divides $15p(15p - 1) = 192(5p).(45a + 41)$, and so, $Z(96p) = 15p - 1$.

Case (13) : p is of the form $p = 192a + 19$ for some integer $a \geq 0$.

In this case, 64 divides $27p - 1 = 64(81a + 8)$ and 3p divides 27p, and hence, $Z(96p) = 27p - 1$.

Case (14) : p is of the form $p = 192a + 173$ for some integer $a \geq 0$.

Here, 64 divides $27p + 1 = 64(81a + 73)$, and hence, $Z(96p) = 27p$.

Case (15) : p is of the form $p = 192a + 23$ for some integer $a \geq 0$.

Since $25p + 1 = 192(25a + 3)$, it follows that 192 divides $25p + 1$, and so, $Z(96p) = 25p$.

Case (16) : p is of the form $p = 192a + 169$ for some integer $a \geq 1$.

Here, $25p - 1 = 192(25a + 22)$. Thus, 192 divides $25p - 1$, and hence, $Z(96p) = 25p - 1$.

Case (17) : p is of the form $p = 192a + 25$ for some integer $a \geq 1$.

Here, $23p + 1 = 192(23a + 3)$, so that, 192 divides $23p + 1$. Therefore, $Z(96p) = 23p$.

Case (18) : p is of the form $p = 192a + 167$ for some integer $a \geq 0$.

In this case, $23p - 1 = 192(23a + 20)$, and so, 192 divides $23p - 1$. Thus, $Z(96p) = 23p - 1$.

Case (19) : p is of the form $p=192a+29$ for some integer $a\geq 0$.
Here, 192 divides $53p-1=192(53a+8)$, so that, $Z(96p)=53p-1$.

Case (20) : p is of the form $p=192a+163$ for some integer $a\geq 0$.
Since, 192 divides $53p+1=192(53a+45)$, it follows that, $Z(96p)=53p$.

Case (21) : p is of the form $p=192a+31$ for some integer $a\geq 0$.
In this case, 192 divides $31p-1=192(31a+5)$, and hence, $Z(96p)=31p-1$.

Case (22) : p is of the form $p=192a+161$ for some integer $a\geq 1$.
Since, 192 divides $31p+1=192(31a+26)$, it follows that, $Z(96p)=31p$.

Case (23) : p is of the form $p=192a+35$ for some integer $a\geq 1$.
Here, 192 divides $11p-1=192(11a+2)$, and so, $Z(96p)=11p-1$.

Case (24) : p is of the form $p=192a+157$ for some integer $a\geq 0$.
Since, 192 divides $11p+1=192(11a+9)$, it follows that, $Z(96p)=11p$.

Case (25) : p is of the form $p=192a+37$ for some integer $a\geq 0$.
Here, 64 divides $45p-1=64(135a+26)$, and so, $Z(96p)=45p-1$.

Case (26) : p is of the form $p=192a+155$ for some integer $a\geq 1$.
In this case, 64 divides $45p+1=64(13a+109)$, and hence, $Z(96p)=45p$.

Case (27) : p is of the form $p=192a+41$ for some integer $a\geq 0$.
Here, 64 divides $39p+1=64(117a+25)$, so that , $Z(96p)=39p$.

Case (28) : p is of the form $p=192a+151$ for some integer $a\geq 1$.
Since 64 divides $39p-1=64(117a+92)$, it follows that, $Z(96p)=39p-1$.

Case (29) : p is of the form $p=192a+43$ for some integer $a\geq 0$.
Here, 64 divides $3p-1=64(9a+2)$, so that 192p divides $3p(3p-1)$. Hence, $Z(96p)=3p-1$.

Case (30) : p is of the form $p=192a+149$ for some integer $a\geq 0$. Then, $Z(96p)=3p$.

Case (31) : p is of the form $p=192a+47$ for some integer $a\geq 0$.
Here, 64 divides $15p-1=64(45a+11)$, and 192p divides $15p(15p-1)$. Thus, $Z(96p)=15p-1$.

Case (32) : p is of the form $p=192a+145$ for some integer $a\geq 1$.
In this case, 64 divides $15p+1=64(45a+34)$. Consequently, $Z(96p)=15p$.

Case (33) : p is of the form $p=192a+49$ for some integer $a\geq 1$.
In this case, 192 divides $47p+1=192(47a+12)$, and consequently, $Z(96p)=47p$.

Case (34) : p is of the form $p=192a+143$ for some integer $a\geq 1$. Here, $Z(96p)=47p-1$.

Case (35) : p is of the form $p=192a+53$ for some integer $a\geq 0$.
In this case, 192 divides $29p-1=192(29a+8)$, so that, $Z(96p)=29p-1$.

Case (36) : p is of the form $p=192a+139$ for some integer $a\geq 0$. Here, $Z(96p)=29p$.

Case (37) : p is of the form $p=192a+55$ for some integer $a\geq 1$.
In this case, 192 divides $7p-1=192(7a+2)$. Hence, $Z(96p)=7p-1$.

Case (38) : p is of the form $p=192a+137$ for some integer $a\geq 0$.
Here, 192 divides $7p+1=192(7a+5)$, and consequently, $Z(96p)=7p$.

Case (39) : p is of the form $p=192a+59$ for some integer $a \geq 0$.
In this case, 192 divides $13p+1=192(13a+4)$, and so, $Z(96p)=13p$.

Case (40) : p is of the form $p=192a+133$ for some integer $a \geq 1$.
Here, 192 divides $13p-1=192(13a+9)$. And consequently, $Z(96p)=13p-1$.

Case (41) : p is of the form $p=192a+61$ for some integer $a \geq 0$.
In this case, 64 divides $21p-1=64(63a+20)$. Therefore, $Z(96p)=21p-1$.

Case (42) : p is of the form $p=192a+131$ for some integer $a \geq 0$.
Since, 64 divides $21p+1=64(63a+43)$, it follows that, $Z(96p)=21p$.

Case (43) : p is of the form $p=192a+65$ for some integer $a \geq 1$.
Here, 64 divides $63p+1=64(189a+64)$, and so, 192p divides $63p(63p+1)$. Thus, $Z(96p)=63p$.

Case (44) : p is of the form $p=192a+127$ for some integer $a \geq 0$.
In this case, 192p divides $63p(63p-1)$, and so, $Z(63p)=63p-1$.

Case (45) : p is of the form $p=192a+67$ for some integer $a \geq 0$.
In this case, 64 divides $21p+1=64(63a+22)$, and hence, $Z(96p)=21p$.

Case (46) : p is of the form $p=192a+125$ for some integer $a \geq 1$.
Here, 64 divides $21p-1=64(63a+41)$, and hence, $Z(96p)=21p-1$.

Case (47) : p is of the form $p=192a+71$ for some integer $a \geq 0$.
Since $9p+1=64(27a+10)$, it follows that 192p divides $9p(9p+1)$, and so, $Z(96p)=9p$.

Case (48) : p is of the form $p=192a+121$ for some integer $a \geq 1$.
Here, $9p-1=64(27a+17)$. Thus, 192p divides $9p(9p-1)$, and hence, $Z(96p)=9p-1$.

Case (49) : p is of the form $p=192a+73$ for some integer $a \geq 0$.
In this case, $57p-1=64(171a+65)$, and 3p divides 57p. Therefore, $Z(96p)=57p-1$.

Case (50) : p is of the form $p=192a+119$ for some integer $a \geq 1$.
Here, $57p+1=64(171a+106)$. Therefore, $Z(96p)=57p$.

Case (51) : p is of the form $p=192a+77$ for some integer $a \geq 1$.
In this case, 192 divides $5p-1=192(5a+2)$, and hence, $Z(96p)=5p-1$.

Case (52) : p is of the form $p=192a+115$ for some integer $a \geq 1$. Here, $Z(96p)=5p$.

Case (53) : p is of the form $p=192a+79$ for some integer $a \geq 0$.
Here, 192 divides $17p+1=192(17a+7)$, and so, $Z(96p)=17p$.

Case (54) : p is of the form $p=192a+113$ for some integer $a \geq 0$.
Since, 192 divides $17p-1=192(17a+10)$, it follows that, $Z(96p)=17p-1$.

Case (55) : p is of the form $p=192a+83$ for some integer $a \geq 0$.
Here, $64 \mid (27p-1)=64(81a+35)$, so that 192p divides $27p(27p-1)$. Thus, $Z(96p)=27p-1$.

Case (56) : p is of the form $p=192a+109$ for some integer $a \geq 0$. Here, $Z(96p)=27p$.

Case (57) : p is of the form $p=192a+85$ for some integer $a \geq 1$.
Here, 64 divides $3p+1=64(9a+4)$, so that 192p divides $3p(3p+1)$. Thus, $Z(96p)=3p$.

Case (58) : p is of the form $p=192a+107$ for some integer $a \geq 0$. Here, $Z(96p)=3p-1$.

Case (59) : p is of the form $p = 192a + 89$ for some integer $a \geq 0$.
Here, 192 divides $41p - 1 = 192(41a + 19)$, so that, $Z(96p) = 41p - 1$.

Case (60) : p is of the form $p = 192a + 103$ for some integer $a \geq 0$. Here, $Z(96p) = 41p$.

Case (61) : p is of the form $p = 192a + 91$ for some integer $a \geq 1$.
Here, 192 divides $19p - 1 = 192(19a + 9)$. Hence, $Z(96p) = 19p - 1$.

Case (62) : p is of the form $p = 192a + 101$ for some integer $a \geq 0$. Here, $Z(96p) = 19p$.

Case (63) : p is of the form $p = 192a + 95$ for some integer $a \geq 1$.
Here, 64 divides $33p + 1 = 64(99a + 49)$, and so, $192p$ divides $33p(33p + 1)$. Hence, $Z(96p) = 33p$.

Case (64) : p is of the form $p = 192a + 97$ for some integer $a \geq 0$. Here, $Z(96p) = 33p - 1$.

All these complete the proof of the lemma. ■

When n is a composite number, we observe the following :
Let

$$Z(n) = m_o \text{ for some integer } m_o \geq 1 \text{ (so that n divides } \frac{m_0(m_0 + 1)}{2}\text{)}.$$

We now consider the following two cases that may arise :

Case 1 : m_o is even (so that m_o+1 is odd).
In this case, n does not divide $\frac{m_0}{2}$, for otherwise,

$$n \mid \frac{m_0}{2} \quad \Rightarrow \quad n \mid \frac{m_0(m_0 - 1)}{2} \quad \Rightarrow \quad Z(n) \leq m_o - 1.$$

Case 2 : m_o is odd (so that m_o+1 is even).
If n is even, then, n does not divide m_o. If n is odd, n does not divide m_o, for

$$n \mid m_o \quad \Rightarrow \quad n \mid \frac{m_0(m_0 - 1)}{2} \quad \Rightarrow \quad Z(n) \leq m_o - 1.$$

Thus, if n is a composite number, n does not divide m_o.
Now, let the representation of n in terms of its distinct prime factors 2, $p_1, p_2, \ldots, p_s$, be

$$n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \ldots p_s^{\alpha_s} \quad (\alpha \geq 0,\ \alpha_i \geq 0 \text{ for all } 1 \leq i \leq s),$$

where $p_1, p_2, \ldots, p_s$ are not necessarily ordered. Then, one of m_o and m_o+1 is divisible by

$$2^{\alpha+1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i} \quad \text{for some } 1 \leq i < s;$$

and the other one is divisible by

$$p_j^{\alpha_j} p_{j+1}^{\alpha_{j+1}} \ldots p_s^{\alpha_s} \quad \text{for } i + 1 \leq j \leq s.$$

We now proceed to find an expression for Z(pq), where p and q are two distinct primes. The following result is due to Ibstedt [2].

Theorem 4.2.1 **:** Let p and q be two primes with $q > p$. Let $g = q - p$. Then,

$$Z(pq) = \min\{\frac{p(qk+1)}{g}, \frac{q(pk-1)}{g}\},$$

where both $pk+1$ and $pk-1$ are divisible by g.

Proof **:** By definition, $Z(pq) = \min\{m : pq \mid \frac{m(m+1)}{2}\}$.

Now, we consider the three possible cases separately.

Case (1) : When p divides m and q divides $(m+1)$. In this case,

$$m = px, \ m+1 = qy \text{ for some integers } x \geq 1, \ y \geq 1,$$

leading to the equation $p(x-y) = gy - 1$. This shows that p divides $gy - 1$. Now,

$$gy - 1 = pk \quad \Rightarrow \quad y = \frac{pk+1}{g}, \ x = \frac{qk+1}{g}, \ m = \frac{p(qk+1)}{g}.$$

Case (2) : When p divides $(m+1)$ and q divides m. In this case,

$$m + 1 = px, \ m = qy \text{ for some integers } x \geq 1, \ y \geq 1.$$

Thus, we get the equation $p(x-y) = gy + 1$. This shows that p divides $gy + 1$. Now,

$$gy + 1 = pk \quad \Rightarrow \quad y = \frac{pk-1}{g}, \ m = \frac{p(qk+1)}{g}.$$

Case (3) : When pq divides $(m+1)$.

This case cannot occur (see Case 3 in the proof of Theorem 4.2.2).

Hence, the theorem is proved. ■

In Theorem 4.2.2, we give an alternative expression for Z(pq). In this connection, we state the following lemma (see, for example, Theorem 2.12.2 of Gioia [5]).

Lemma 4.2.31 **:** Let p and q be two distinct primes. Then, the Diophantine equation

$$qy - px = 1$$

has an infinite number of solutions. Moreover, if (x_o, y_o) is a solution of the Diophantine equation, then any solution is of the form

$$x = x_o + qt, \ y = y_o + pt, \text{ where } t \geq 0 \text{ is an integer.}$$

Theorem 4.2.2 **:** Let p and q be two primes with $q > p \geq 5$. Then,

$$Z(pq) = \min\{qy_o - 1, px_o - 1\},$$

where

$$y_o = \min\{y : x, y \in Z^+, \ qy - px = 1\},$$

$$x_o = \min\{x : x, y \in Z^+, \ px - qy = 1\}.$$

Proof **:** Since

$$Z(pq) = \min\{m : pq \mid \frac{m(m+1)}{2}\}, \tag{A}$$

it follows that we have to consider the three cases below that may arise :

Case 1 : When p divides m and q divides $(m+1)$. In this case,

$m = px$ for some integer $x \geq 1$,

$m+1 = qy$ for some integer $y \geq 1$.

From these two equations, we get the Diophantine equation

$qy - px = 1.$

By Lemma 4.2.31, the above Diophantine equation has infinite number of solutions. Let

$y_o = \min\{y : x, y \in Z^+, qy - px = 1\}.$

For this y_o, the corresponding x_o is given by the equation $q_o y - p_o x = 1$. Note that y_o and x_o cannot be both odd or both even. Then, the minimum m in (A) is given by

$m+1 = qy_o \quad \Rightarrow \quad m = qy_o - 1.$

Case 2 : When p divides $(m+1)$ and q divides m. Here,

$m+1 = px$ for some integer $x \geq 1$,

$m = qy$ for some integer $y \geq 1$.

These two equations lead to the Diophantine equation.

$px - qy = 1.$

Let

$x_o = \min\{x : x, y \in Z^+, px - qy = 1\}.$

For this x_o, the corresponding y_o is given by $y_o = \dfrac{px_0 - 1}{q}$. Here also, x_o and y_o both cannot be odd or even simultaneously. The minimum m in (A) is given by

$m+1 = px_o \quad \Rightarrow \quad m = px_o - 1.$

Case 3 : When pq divides $(m+1)$. In this case, $m = pq - 1$. But then, by Case 1 and Case 2 above, this does not give the minimum m. Thus, this case cannot occur.

The proof of the theorem now follows by virtue of Case 1 and Case 2. ■

***Remark 4.2.2* :** Let p and q be two primes with $q > p \geq 5$. Let

$q = kp + \ell$ for some integers k and ℓ with $k \geq 1$ and $1 \leq \ell \leq p-1$.

We now consider the two cases given in Theorem 4.2.2 :

Case 1 : When p divides m and q divides $(m+1)$. In this case,

$m = px$ for some integer $x \geq 1$,

$m+1 = qy = (kp + \ell)y$ for some integer $y \geq 1$.

From the above two equations, we get

$$\ell y - (x - ky)p = 1. \tag{4.2.1}$$

Case 2 : When p divides $(m+1)$ and q divides m. Here,

$m+1 = px$ for some integer $x \geq 1$,

$m = (kp + \ell)y$ for some integer $y \geq 1$.

These two equations lead to

$$(x - ky)p - \ell y = 1. \tag{4.2.2}$$

Some particular cases are given in Corollary 4.2.3 – Corollary 4.2.18 below.

Corollary 4.2.3 **:** Let p and q be two primes with $q>p\geq 5$. Let

$$q=kp+1 \text{ for some integer } k\geq 2.$$

Then,

$$Z(pq)=q-1.$$

Proof **:** From (4.2.1) with $\ell=1$, we get

$$y-(x-ky)p=1,$$

the minimum solution of which is $y=1$, $x=ky=k$. Then, the minimum m in (A) is given by

$$m+1=qy=q \quad \Rightarrow \quad m=q-1.$$

Note that, from (4.2.2) with $\ell=1$, we have $(x-ky)p-y=1$, with the least possible solution

$$y=p-1 \text{ (and } x-ky=1). \blacksquare$$

Corollary 4.2.4 **:** Let p and q be two primes with $q>p\geq 5$. Let

$$q=(k+1)p-1 \text{ for some integer } k\geq 1.$$

Then,

$$Z(pq)=q.$$

Proof **:** From (4.2.2) with $\ell=p-1$, we have,

$$y-[(k+1)y-x]p=1,$$

whose minimum solution is $y=1$, $x=(k+1)y=k+1$. Then, the minimum m in (A) is given by

$$m=qy=q.$$

Note that, from (4.2.1) with $\ell=p-1$, we have $[(k+1)y-x]p-y=1$, with the least possible solution

$$y=p-1 \text{ (and } (k+1)y-x=1). \blacksquare$$

Corollary 4.2.5 **:** Let p and q be two primes with $q>p\geq 5$. Let

$$q=kp+2 \text{ for some integer } k\geq 1.$$

Then,

$$Z(pq)=\frac{q(p-1)}{2}.$$

Proof **:** With $\ell=2$ in (4.2.2), we have

$$(x-ky)p-2y=1,$$

with the minimum solution

$$y=\frac{p-1}{2} \quad \text{(and } x-ky=1).$$

This gives $m=qy=\frac{q(p-1)}{2}$ as one possible solution of (A).

Now, (4.2.1) with $\ell=2$ gives

$$2y-(x-ky)p=1,$$

with the minimum solution

$$y=\frac{p+1}{2} \quad \text{(and } x=ky+1\text{)}.$$

This gives

$$m=qy-1=\frac{q(p+1)}{2}-1$$

as another possible solution of (A). Now, since

$$\frac{q(p+1)}{2}-1>\frac{q(p-1)}{2},$$

it follows that

$$Z(pq)=\frac{q(p-1)}{2},$$

which we intended to prove. ■

***Corollary 4.2.6* :** Let p and q be two primes with $q>p\geq 5$. Let

$$q=(k+1)p-2 \text{ for some integer } k\geq 1.$$

Then,

$$Z(pq)=\frac{q(p-1)}{2}-1.$$

***Proof* :** From (4.2.1) with $\ell=p-2$, we get,

$$[(k+1)y-x]p-2y=1,$$

whose minimum solution is

$$y=\frac{p-1}{2} \quad \text{(and } x=(k+1)y-1\text{)}.$$

This gives $m=qy-1=\frac{q(p-1)}{2}-1$ as one possible solution of (A).

Note that, (4.2.2) with $\ell=p-2$ gives

$$2y-[(k+1)y-x]p=1,$$

with the minimum solution

$$y=\frac{p+1}{2} \quad \text{(and } x=(k+1)y-1\text{)}.$$

Corresponding to this case, we get

$$m=qy=\frac{q(p+1)}{2}$$

as another possible solution of (A). But since

$$\frac{q(p+1)}{2}>\frac{q(p-1)}{2}-1,$$

it follows that

$$Z(pq)=\frac{q(p-1)}{2}-1,$$

establishing the desired result. ■

***Corollary 4.2.7* :** Let p and q be two primes with $q>p\geq 7$. Let

$$q=kp+3 \text{ for some integer } k\geq 1.$$

Then,

$$Z(pq)=\begin{cases}\frac{q(p-1)}{3}, & \text{if } 3\,|\,(p-1)\\ \frac{q(p+1)}{3}-1, & \text{if } 3\,|\,(p+1)\end{cases}$$

***Proof* :** From (4.2.1) and (4.2.2) with $\ell=3$, we have respectively

$$3y-(x-ky)p=1, \quad \text{(i)}$$

$$(x-ky)p-3y=1. \quad \text{(ii)}$$

We now consider the following two possible cases :

Case 1 : When 3 divides $(p-1)$.

In this case, the minimum solution is obtained from (ii), which is

$$y=\frac{p-1}{3} \quad (\text{and } x-ky=1).$$

Also, $p-1$ is divisible by 2 as well. Therefore, the minimum m in (A) is

$$m=qy=\frac{q(p-1)}{3}.$$

Case 2 : When 3 divides $(p+1)$.

In this case, (i) gives the minimum solution, which is

$$y=\frac{p+1}{3} \quad (\text{and } x-ky=1).$$

Note that, 2 divides $p+1$. Therefore, the minimum m in (A) is

$$m=qy-1=\frac{q(p+1)}{3}-1.$$

Thus, the result is established. ■

***Corollary 4.2.8* :** Let p and q be two primes with $q>p\geq 7$. Let

$$q=(k+1)p-3 \text{ for some integer } k\geq 1.$$

Then,

$$Z(pq)=\begin{cases}\frac{q(p+1)}{3}, & \text{if } 3\,|\,(p+1)\\ \frac{q(p-1)}{3}-1, & \text{if } 3\,|\,(p-1)\end{cases}$$

***Proof* :** The equations (4.2.1) and (4.2.2) with $\ell=p-3$ give respectively

$$[(k+1)y-x]p-3y=1, \quad \text{(i)}$$

$$3y-[(k+1)y-x]p=1. \quad \text{(ii)}$$

We now consider the following two cases :

Case 1 : When 3 divides $(p+1)$.

In this case, the minimum solution, obtained from (ii), is

$$y = \frac{p+1}{3} \quad (\text{and } x = (k+1)y - 1).$$

Moreover, 2 divides $p+1$. Therefore, the minimum m in (A) is

$$m = qy = \frac{q(p+1)}{3}.$$

Case 2 : When 3 divides $(p-1)$.

In this case, the minimum solution, obtained from (i), is

$$y = \frac{p-1}{3} \quad (\text{and } x = (k+1)y - 1).$$

Moreover, 2 divides $p-1$. Therefore, the minimum m in (A) is

$$m = qy = \frac{q(p-1)}{3} - 1. \blacksquare$$

***Corollary 4.2.9* :** Let p and q be two primes with $q > p \geq 7$. Let

$q = kp + 4$ for some integer $k \geq 1$.

Then,

$$Z(pq) = \begin{cases} \dfrac{q(p-1)}{4}, & \text{if } 4 \mid (p-1) \\ \dfrac{q(p+1)}{4} - 1, & \text{if } 4 \mid (p+1) \end{cases}$$

***Proof* :** With $\ell = 4$, (4.2.1) and (4.2.2) take the respective forms

$$4y - (x - ky)p = 1, \quad \text{(i)}$$

$$(x - ky)p - 4y = 1. \quad \text{(ii)}$$

Now, for any prime $p \geq 7$, exactly one of the following two cases can occur : Either $p-1$ is divisible by 4, or $p+1$ is divisible by 4.

We consider these two possibilities separately below :

Case 1 : When 4 divides $(p-1)$.

In this case, the minimum solution is obtained from (ii), which is

$$y = \frac{p-1}{4} \quad (\text{and } x = ky + 1).$$

Therefore, the minimum m in (A) is

$$m = qy = \frac{q(p-1)}{4}.$$

Case 2 : When 4 divides $(p+1)$.

In this case, (i) gives the minimum solution, which is

$$y = \frac{p+1}{4} \quad (\text{and } x = ky + 1).$$

Therefore, the minimum m in (A) is

$$m = qy - 1 = \frac{q(p+1)}{4} - 1. \blacksquare$$

***Corollary 4.2.10* :** Let p and q be two primes with $q>p\geq 7$. Let

$q=(k+1)p-4$ for some integer $k\geq 1$.

Then,

$$Z(pq)=\begin{cases}\dfrac{q(p+1)}{4}, & \text{if } 4\,|\,(p+1)\\[2mm] \dfrac{q(p-1)}{4}-1, & \text{if } 4\,|\,(p+1)\end{cases}$$

***Proof* :** From (4.2.1) and (4.2.2) with $\ell=p-4$, we have respectively

$$[(k+1)y-x]p-4y=1, \qquad \text{(i)}$$

$$4y-[(k+1)y-x]p=1. \qquad \text{(ii)}$$

We now consider the following two cases which are the only possibilities (as noted in the proof of Corollary 4.2.9) :

Case 1 : When 4 divides $(p+1)$.

In this case, the minimum solution obtained from (ii) is

$$y=\frac{p+1}{4} \quad (\text{and } x=(k+1)y-1).$$

Therefore, the minimum m in (A) is $m=qy=\dfrac{q(p+1)}{4}$.

Case 2 : When 4 divides $(p-1)$.

In this case, the minimum solution, obtained from (i), is

$$y=\frac{p-1}{4} \quad (\text{and } x=(k+1)y-1).$$

Therefore, the minimum m in (A) is $m=qy=\dfrac{q(p-1)}{4}-1$. ■

***Corollary 4.2.11* :** Let p and q be two primes with $q>p\geq 11$. Let

$q=kp+5$ for some integer $k\geq 1$.

Then,

$$Z(pq)=\begin{cases}\dfrac{q(p-1)}{5}, & \text{if } 5\,|\,(p-1)\\[2mm] q(2a+1)-1, & \text{if } p=5a+2\\ q(2a+1), & \text{if } p=5a+3\\ \dfrac{q(p+1)}{5}-1, & \text{if } 5\,|\,(p+1)\end{cases}=\begin{cases}\dfrac{q(p-1)}{5}, & \text{if } 5\,|\,(p-1)\\[2mm] \dfrac{q(2p+1)}{5}-1, & \text{if } 5\,|\,(p-2)\\[2mm] \dfrac{q(2p-1)}{5}, & \text{if } 5\,|\,(p-3)\\[2mm] \dfrac{q(p+1)}{5}-1, & \text{if } 5\,|\,(p+1)\end{cases}$$

***Proof* :** The equations (4.2.1) and (4.2.2) with $\ell=5$ are respectively

$$5y-(x-ky)p=1, \qquad \text{(i)}$$

$$(x-ky)p-5y=1. \qquad \text{(ii)}$$

Now, for any prime $p\geq 7$, exactly one of the following four cases occur :

Case 1 : When p is of the form $p=5a+1$ for some integer $a\geq 2$.
In this case, 5 divides $(p-1)$. Then, the minimum solution is obtained from (ii), which is $y=\frac{p-1}{5}$ (and $x-ky=1$). Therefore, the minimum m in (A) is $m=qy=\frac{q(p-1)}{5}$.

Case 2 : When p is of the form $p=5a+2$ for some integer $a\geq 2$.
In this case, from (i) and (ii), we get respectively

$$1=5y-(x-ky)(5a+2)=5[y-(x-ky)a]-2(x-ky), \qquad \text{(iii)}$$

$$1=(x-ky)(5a+2)-5y=2(x-ky)-5[y-(x-ky)a]. \qquad \text{(iv)}$$

Clearly, the minimum solution is obtained from (iii), which is

$$y-(x-ky)a=1,\ x-ky=2 \quad \Rightarrow \quad y=2a+1 \text{ (and } x=k(2a+1)+2).$$

Hence, in this case, the minimum m in (A) is $m=qy-1=q(2a+1)-1$.

Case 3 : When p is of the form $p=5a+3$ for some integer $a\geq 2$. From (i) and (ii), we get

$$1=5y-(x-ky)(5a+3)=5[y-(x-ky)a]-3(x-ky), \qquad \text{(v)}$$

$$1=(x-ky)(5a+3)-5y=3(x-ky)-5[y-(x-ky)a]. \qquad \text{(vi)}$$

The minimum solution is obtained from (vi) as follows :

$$y-(x-ky)a=1,\ x-ky=2 \quad \Rightarrow \quad y=2a+1 \text{ (and } x=k(2a+1)+2).$$

Hence, in this case, the minimum m in (A) is $m=qy=q(2a+1)$.

Case 4 : When p is of the form $p=5a+4$ for some integer $a\geq 2$.
In this case, 5 divides $(p+1)$. Then, the minimum solution, obtained from (i), is

$$y=\frac{p+1}{5} \text{ (and } x-ky=1).$$

Therefore, the minimum m in (A) is $m=qy-1=\frac{q(p+1)}{5}-1$. ■

Corollary 4.2.12 **:** Let p and q be two primes with $q>p\geq 11$. Let

$$q=(k+1)p-5 \text{ for some integer } k\geq 1.$$

Then,

$$Z(pq)=\begin{cases}\frac{q(p-1)}{5}-1, & \text{if } 5\,|\,(p-1)\\ q(2a+1), & \text{if } p=5a+2\\ q(2a+1)-1, & \text{if } p=5a+3\\ \frac{q(p+1)}{5}, & \text{if } 5\,|\,(p+1)\end{cases} =\begin{cases}\frac{q(p-1)}{5}-1, & \text{if } 5\,|\,(p-1)\\ \frac{q(2p+1)}{5}, & \text{if } 5\,|\,(p-2)\\ \frac{q(2p-1)}{5}-1, & \text{if } 5\,|\,(p-3)\\ \frac{q(p+1)}{5}, & \text{if } 5\,|\,(p+1)\end{cases}$$

Proof **:** From (4.2.1) and (4.2.2) with $\ell=p-5$, we have respectively

$$[(k+1)y-x]p-5y=1, \qquad \text{(i)}$$

$$5y-[(k+1)y-x]p=1. \qquad \text{(ii)}$$

As in the proof of Corollary 4.2.11, we consider the following four possibilities :

Case 1 : When p is of the form $p=5a+1$ for some integer $a \geq 2$.

In this case, 5 divides $(p-1)$. Then, the minimum solution is obtained from (i), which is

$$y=\frac{p-1}{5} \quad (\text{and } x=(k+1)y-1).$$

Therefore, the minimum m in (A) is

$$m=qy-1=\frac{q(p-1)}{5}-1.$$

Case 2 : When p is of the form $p=5a+2$ for some integer $a \geq 2$.

In this case, from (i) and (ii), we get respectively

$$1=[(k+1)y-x](5a+2)-5y=2[(k+1)y-x]-5[y-a\{(k+1)y-x\}], \qquad \text{(iii)}$$

$$1=5y-[(k+1)y-x](5a+2)=5[y-a\{(k+1)y-x\}]-2[(k+1)y-x]. \qquad \text{(iv)}$$

Clearly, the minimum solution is obtained from (iv), which is

$$y-a\{(k+1)y-x\}=1,\ (k+1)y-x=2 \quad \Rightarrow \quad y=2a+1 \ (\text{and } x=(k+1)(2a+1)-2).$$

Hence, in this case, the minimum m in (A) is $m=qy=q(2a+1)$.

Case 3 : When p is of the form $p=5a+3$ for some integer $a \geq 2$.

In this case, from (i) and (ii), we get respectively

$$1=[(k+1)y-x](5a+3)-5y=3[(k+1)y-x]-5[y-a\{(k+1)y-x\}], \qquad \text{(v)}$$

$$1=5y-[(k+1)y-x](5a+3)=5[y-a\{(k+1)y-x\}]-3[(k+1)y-x]. \qquad \text{(vi)}$$

The minimum solution is obtained from (v) as follows :

$$y-a\{(k+1)y-x\}=1,\ (k+1)y-x=2 \quad \Rightarrow \quad y=2a+1 \ (\text{and } x=(k+1)(2a+1)-2).$$

Hence, in this case, the minimum m in (A) is $m=qy-1=q(2a+1)-1$.

Case 4 : When p is of the form $p=5a+4$ for some integer $a \geq 2$.

In this case, 5 divides $(p+1)$. Then, the minimum solution is obtained from (ii). The minimum y and the minimum m in (A) are given by

$$y=\frac{p+1}{5} \quad (\text{and } x=(k+1)y-1),\ m=qy=\frac{q(p+1)}{5}. \ \blacksquare$$

***Corollary 4.2.13* :** Let p and q be two primes with $q>p\geq 13$. Let

$$q=kp+6 \text{ for some integer } k\geq 1.$$

Then,

$$Z(pq)=\begin{cases}\dfrac{q(p-1)}{6}, & \text{if } 6\,|\,(p-1)\\[2mm] \dfrac{q(p+1)}{6}-1, & \text{if } 6\,|\,(p+1)\end{cases}$$

***Proof* :** The equations (4.2.1) and (4.2.2) with $\ell=6$ become respectively

$$6y-(x-ky)p=1, \qquad \text{(i)}$$

$$(x-ky)p-6y=1. \qquad \text{(ii)}$$

Now, for any prime $p\geq 13$, exactly one of the following two cases can occur : Either $p-1$ is divisible by 6, or $p+1$ is divisible by 6.

We thus consider the two possibilities separately below :

Case 1 : When 6 divides $(p-1)$.
In this case, the minimum solution is obtained from (ii), which is

$$y=\frac{p-1}{6} \quad \text{(and } x=ky+1\text{)}.$$

Therefore, the minimum m in (A) is

$$m=qy=\frac{q(p-1)}{6}.$$

Case 2 : When 6 divides $(p+1)$.
In this case, (i) gives the minimum solution, which is

$$y=\frac{p+1}{6} \quad \text{(and } x=ky+1\text{)}.$$

Therefore, the minimum m in (A) is

$$m=qy-1=\frac{q(p+1)}{6}-1. \blacksquare$$

***Corollary 4.2.14* :** Let p and q be two primes with $q>p\geq 13$. Let $q=(k+1)p-6$ for some integer $k\geq 1$.
Then,

$$Z(pq)=\begin{cases}\dfrac{q(p+1)}{6}, & \text{if } 6\,|\,(p+1)\\[2ex] \dfrac{q(p-1)}{6}-1, & \text{if } 6\,|\,(p-1)\end{cases}$$

***Proof* :** From (4.2.1) and (4.2.2) with $\ell=p-6$, we have respectively

$$[(k+1)y-x]p-6y=1, \tag{i}$$

$$6y-[(k+1)y-x]p=1. \tag{ii}$$

We now consider the following two cases which are the only possibilities (as noted in the proof of Corollary 4.2.13) :

Case 1 : When 6 divides $(p+1)$.
In this case, the minimum solution, obtained from (ii), is

$$y=\frac{p+1}{6} \quad \text{(and } x=(k+1)y-1\text{)}.$$

Therefore, the minimum m in (A) is

$$m=qy=\frac{q(p+1)}{6}.$$

Case 2 : When 6 divides $(p-1)$.
Here, the minimum solution is obtained from (i), which is

$$y=\frac{p-1}{6} \quad \text{(and } x=(k+1)y-1\text{)}.$$

Therefore, the minimum m in (A) is

$$m=qy=\frac{q(p-1)}{6}-1. \blacksquare$$

Corollary 4.2.15 **:** Let p and q be two primes with $q > p \geq 13$. Let

$q = kp + 7$ for some integer $k \geq 1$.

Then,

$$Z(pq) = \begin{cases} \dfrac{q(p-1)}{7}, & \text{if } 7 \mid (p-1) \\ q(3a+1)-1, & \text{if } p = 7a+2 \\ q(2a+1)-1, & \text{if } p = 7a+3 \\ q(2a+1), & \text{if } p = 7a+4 \\ q(3a+2), & \text{if } p = 7a+5 \\ \dfrac{q(p+1)}{7}-1, & \text{if } 7 \mid (p+1) \end{cases} = \begin{cases} \frac{q(p-1)}{7}, & \text{if } 7 \mid (p-1) \\ \frac{q(3p+1)}{7}-1, & \text{if } 7 \mid (p-2) \\ \frac{q(2p+1)}{7}-1, & \text{if } 7 \mid (p-3) \\ \frac{q(2p-1)}{7}, & \text{if } 7 \mid (p-4) \\ \frac{q(3p-1)}{7}, & \text{if } 7 \mid (p-5) \\ \frac{q(p+1)}{7}-1, & \text{if } 7 \mid (p+1) \end{cases}$$

Proof **:** The equations (4.2.1) and (4.2.2) with $\ell = 7$ are respectively

$$7y-(x-ky)p=1, \tag{i}$$

$$(x-ky)p-7y=1. \tag{ii}$$

Now, for any prime $p \geq 11$, exactly one of the following six cases occur :

Case 1 : When p is of the form $p=7a+1$ for some integer $a \geq 2$.

In this case, 7 divides $(p-1)$. Then, the minimum solution is obtained from (ii), which is $y = \dfrac{p-1}{7}$ (and $x-ky=1$). Therefore, the minimum m in (A) is

$$m = qy = \frac{q(p-1)}{7}.$$

Case 2 : When p is of the form $p=7a+2$ for some integer $a \geq 2$.

In this case, from (i) and (ii), we get respectively

$$1 = 7y-(x-ky)(7a+2) = 7[y-(x-ky)a]-2(x-ky), \tag{iii}$$

$$1 = (x-ky)(7a+2)-7y = 2(x-ky)-7[y-(x-ky)a]. \tag{iv}$$

Clearly, the minimum solution is obtained from (iii), which is

$$y-(x-ky)a = 1,\ x-ky = 3 \quad \Rightarrow \quad y = 3a+1 \text{ (and } x = k(3a+1)+3).$$

Hence, in this case, the minimum m in (A) is $m = qy-1 = q(3a+1)-1$.

Case 3 : When p is of the form $p=7a+3$ for some integer $a \geq 2$.

Here, from (i) and (ii),

$$1 = 7y-(x-ky)(7a+3) = 7[y-(x-ky)a]-3(x-ky), \tag{v}$$

$$1 = (x-ky)(7a+3)-7y = 3(x-ky)-7[y-(x-ky)a]. \tag{vi}$$

The minimum solution is obtained from (v) as follows :

$$y-(x-ky)a = 1,\ x-ky = 2 \quad \Rightarrow \quad y = 2a+1 \text{ (and } x = k(2a+1)+2).$$

Hence, in this case, the minimum m in (A) is $m = qy = q(2a+1)-1$.

Case 4 : When p is of the form $p=7a+4$ for some integer $a \geq 2$.

In this case, from (i) and (ii), we get respectively

$$1 = 7y-(x-ky)(7a+4) = 7[y-(x-ky)a]-4(x-ky), \tag{vii}$$

$$1 = (x-ky)(7a+4)-7y = 4(x-ky)-7[y-(x-ky)a]. \tag{viii}$$

Clearly, the minimum solution is obtained from (viii), which is

$$y-(x-ky)a=1,\ x-ky=2 \quad \Rightarrow \quad y=2a+1 \text{ (and } x=k(2a+1)+2).$$

Hence, in this case, the minimum m in (A) is $m=qy=q(2a+1)$.

Case 5 : When p is of the form $p=7a+5$ for some integer $a\geq 2$.

From (i) and (ii), we have

$$1=7y-(x-ky)(7a+5)=7[y-(x-ky)a]-5(x-ky), \tag{ix}$$

$$1=(x-ky)(7a+3)-7y=5(x-ky)-7[y-(x-ky)a]. \tag{x}$$

The minimum solution is obtained from (x), which is

$$y-(x-ky)a=2,\ x-ky=3 \quad \Rightarrow \quad y=3a+2 \text{ (and } x=k(3a+2)+3).$$

Hence, in this case, the minimum m in (A) is $m=qy=q(3a+2)$.

Case 6 : When p is of the form $p=7a+6$ for some integer $a\geq 2$.

In this case, 7 divides $(p+1)$. Then, the minimum solution is obtained from (i), which is

$$y=\frac{p+1}{7} \quad \text{(and } x-ky=1).$$

Therefore, the minimum m in (A) is

$$m=qy=\frac{q(p+1)}{7}-1. \ \blacklozenge$$

Corollary 4.2.16 **:** Let p and q be two primes with $q>p\geq 13$. Let

$$q=(k+1)p-7 \text{ for some integer } k\geq 1.$$

Then,

$$Z(pq)=\begin{cases}\dfrac{q(p-1)}{7}-1, & \text{if } 7\,|\,(p-1)\\ q(3a+1), & \text{if } p=7a+2\\ q(2a+1), & \text{if } p=7a+3\\ q(2a+1)-1, & \text{if } p=7a+4\\ q(3a+2)-1, & \text{if } p=7a+5\\ \dfrac{q(p+1)}{7}, & \text{if } 7\,|\,(p+1)\end{cases} = \begin{cases}\frac{q(p-1)}{7}-1, & \text{if } 7\,|\,(p-1)\\ \frac{q(3p+1)}{7}, & \text{if } 7\,|\,(p-2)\\ \frac{q(2p+1)}{7}, & \text{if } 7\,|\,(p-3)\\ \frac{q(2p-1)}{7}-1, & \text{if } 7\,|\,(p-4)\\ \frac{q(3p-1)}{7}-1, & \text{if } 7\,|\,(p-5)\\ \frac{q(p+1)}{7}, & \text{if } 7\,|\,(p+1)\end{cases}$$

Proof **:** From (4.2.1) and (4.2.2) with $\ell=p-7$, we have respectively

$$[(k+1)y-x]p-7y=1, \tag{i}$$

$$7y-[(k+1)y-x]p=1. \tag{ii}$$

We now consider the following six possibilities :

Case 1 : When p is of the form $p=7a+1$ for some integer $a\geq 2$.

In this case, 7 divides $(p-1)$. Then, the minimum solution is obtained from (i), which is

$$y=\frac{p-1}{7} \quad \text{(and } x=(k+1)y-1).$$

Therefore, the minimum m in (A) is

$$m=qy-1=\frac{q(p-1)}{7}-1.$$

Case 2 : When p is of the form $p=7a+2$ for some integer $a\geq 2$.
In this case, from (i) and (ii), we get respectively

$$1=[(k+1)y-x](7a+2)-7y=2[(k+1)y-x]-7[y-a\{(k+1)y-x\}], \quad \text{(iii)}$$

$$1=7y-[(k+1)y-x](7a+2)=7[y-a\{(k+1)y-x\}]-2[(k+1)y-x]. \quad \text{(iv)}$$

Clearly, the minimum solution is obtained from (iv), which is

$$y-a\{(k+1)y-x\}=1,\ (k+1)y-x=3 \quad \Rightarrow \quad y=3a+1 \text{ (and } x=(k+1)(3a+1)-3).$$

Hence, in this case, the minimum m in (A) is $m=qy=q(3a+1)$.

Case 3 : When p is of the form $p=7a+3$ for some integer $a\geq 2$.
Here, from (i) and (ii),

$$1=[(k+1)y-x](7a+3)-7y=3[(k+1)y-x]-7[y-a\{(k+1)y-x\}], \quad \text{(v)}$$

$$1=7y-[(k+1)y-x](7a+3)=7[y-a\{(k+1)y-x\}]-3[(k+1)y-x]. \quad \text{(vi)}$$

Then, (vi) gives the minimum solution, which is

$$y-a\{(k+1)y-x\}=1,\ (k+1)y-x=2 \quad \Rightarrow \quad y=2a+1 \text{ (and } x=(k+1)(2a+1)-2).$$

Hence, in this case, the minimum m in (A) is $m=qy=q(2a+1)$.

Case 4 : When p is of the form $p=7a+4$ for some integer $a\geq 2$.
Here, from (i) and (ii),

$$1=[(k+1)y-x](7a+4)-7y=4[(k+1)y-x]-7[y-a\{(k+1)y-x\}], \quad \text{(vii)}$$

$$1=7y-[(k+1)y-x](7a+4)=7[y-a\{(k+1)y-x\}]-4[(k+1)y-x]. \quad \text{(viii)}$$

Clearly, the minimum solution is obtained from (vii) as follows :

$$y-a\{(k+1)y-x\}=1,\ (k+1)y-x=2 \quad \Rightarrow \quad y=2a+1 \text{ (and } x=(k+1)(2a+1)-2).$$

Hence, in this case, the minimum m in (A) is $m=qy-1=q(2a+1)-1$.

Case 5 : When p is of the form $p=7a+5$ for some integer $a\geq 2$.
In this case, from (i) and (ii), we get respectively

$$1=[(k+1)y-x](7a+5)-7y=5[(k+1)y-x]-7[y-a\{(k+1)y-x\}], \quad \text{(ix)}$$

$$1=7y-[(k+1)y-x](7a+5)=7[y-a\{(k+1)y-x\}]-5[(k+1)y-x]. \quad \text{(x)}$$

Then, (ix) gives the following minimum solution :

$$y-a\{(k+1)y-x\}=2,\ (k+1)y-x=3 \quad \Rightarrow \quad y=3a+2 \text{ (and } x=(k+1)(3a+2)-3).$$

Hence, in this case, the minimum m in (A) is $m=qy-1=q(3a+2)-1$.

Case 6 : When p is of the form $p=7a+6$ for some integer $a\geq 2$.
In this case, 7 divides $(p+1)$. Then, the minimum solution is obtained from (ii), which is

$$y=\frac{p+1}{7} \quad \text{(and } x=(k+1)y-1).$$

Therefore, the minimum m in (A) is

$$m=qy=\frac{q(p+1)}{7}. \ \blacksquare$$

***Corollary 4.2.17* :** Let p and q be two primes with $q>p\geq 13$. Let $q=kp+8$ for some integer $k\geq 1$.

Then,

$$Z(pq)=\begin{cases}\dfrac{q(p-1)}{8}, & \text{if } 8\,|\,(p-1)\\ q(3a+1), & \text{if } p=8a+3\\ q(3a+2)-1, & \text{if } p=8a+5\\ \dfrac{q(p+1)}{8}-1, & \text{if } 8\,|\,(p+1)\end{cases}=\begin{cases}\frac{q(p-1)}{8}, & \text{if } 8\,|\,(p-1)\\ \frac{q(3p-1)}{8}, & \text{if } 8\,|\,(p-3)\\ \frac{q(3p+1)}{8}-1, & \text{if } 8\,|\,(p-5)\\ \frac{q(p+1)}{8}-1, & \text{if } 8\,|\,(p+1)\end{cases}$$

***Proof* :** With $\ell=8$ in (4.2.1) and (4.2.2), we get respectively

$$8y-(x-ky)p=1, \tag{i}$$

$$(x-ky)p-8y=1. \tag{ii}$$

Now, for any prime $p\geq 13$, exactly one of the following four cases occur :

Case 1 : When p is of the form $p=8a+1$ for some integer $a\geq 2$.

In this case, 8 divides $(p-1)$. Then, the minimum solution is obtained from (ii), which is $y=\dfrac{p-1}{8}$ (and $x-ky=1$). Therefore, the minimum m in (A) is $m=qy=\dfrac{q(p-1)}{8}$.

Case 2 : When p is of the form $p=8a+3$ for some integer $a\geq 2$.

In this case, from (i) and (ii), we get respectively

$$1=8y-(x-ky)(8a+3)=8[y-(x-ky)a]-3(x-ky), \tag{iii}$$

$$1=(x-ky)(8a+3)-8y=3(x-ky)-8[y-(x-ky)a]. \tag{iv}$$

Clearly, the minimum solution is obtained from (iv), which is

$$y-(x-ky)a=1,\ x-ky=3 \quad\Rightarrow\quad y=3a+1 \text{ (and } x=k(3a+1)+3).$$

Hence, in this case, the minimum m in (A) is $m=qy=q(3a+1)$.

Case 3 : When p is of the form $p=8a+5$ for some integer $a\geq 1$.

From (i) and (ii), we get

$$1=8y-(x-ky)(8a+5)=8[y-(x-ky)a]-5(x-ky), \tag{iv}$$

$$1=(x-ky)(8a+5)-8y=5(x-ky)-8[y-(x-ky)a]. \tag{v}$$

The minimum solution is obtained from (iv) as follows :

$$y-(x-ky)a=2,\ x-ky=3 \quad\Rightarrow\quad y=3a+2 \text{ (and } x=k(3a+2)+3).$$

Hence, in this case, the minimum m in (A) is $m=qy-1=q(3a+2)-1$.

Case 4 : When p is of the form $p=8a+7$ for some integer $a\geq 1$.

In this case, 8 divides $(p+1)$. Then, the minimum solution is obtained from (i), which is

$$y=\frac{p+1}{8} \text{ (and } x-ky=1).$$

Therefore, the minimum m in (A) is

$$m=qy-1=\frac{q(p+1)}{8}-1. \ \blacksquare$$

Corollary 4.2.18 **:** Let p and q be two primes with $q > p \geq 13$. Let

$q = (k+1)p - 8$ for some integer $k \geq 1$.

Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{8} - 1, & \text{if } 8 \mid (p-1) \\ q(3a+1) - 1, & \text{if } p = 8a+3 \\ q(3a+2), & \text{if } p = 8a+5 \\ \frac{q(p+1)}{8}, & \text{if } 8 \mid (p+1) \end{cases} = \begin{cases} \frac{q(p-1)}{8} - 1, & \text{if } 8 \mid (p-1) \\ \frac{q(3p-1)}{8} - 1, & \text{if } 8 \mid (p-3) \\ \frac{q(3p+1)}{8}, & \text{if } 8 \mid (p-5) \\ \frac{q(p+1)}{8}, & \text{if } 8 \mid (p+1) \end{cases}$$

Proof **:** From (4.2.1) and (4.2.2) with $\ell = p - 8$, we have respectively

$$[(k+1)y - x]p - 8y = 1, \qquad \text{(i)}$$

$$8y - [(k+1)y - x]p = 1. \qquad \text{(ii)}$$

We consider the four possibilities that may arise :

Case 1 : When p is of the form $p = 8a+1$ for some integer $a \geq 2$.

In this case, 8 divides $(p-1)$. Then, the minimum solution is obtained from (i), which is

$$y = \frac{p-1}{8} \quad (\text{and } x = (k+1)y - 1).$$

Therefore, the minimum m in (A) is $m = qy - 1 = \frac{q(p-1)}{8} - 1$.

Case 2 : When p is of the form $p = 8a+3$ for some integer $a \geq 2$.

In this case, from (i) and (ii), we get respectively

$$1 = [(k+1)y - x](8a+3) - 8y = 3[(k+1)y - x] - 8[y - a\{(k+1)y - x\}], \qquad \text{(iii)}$$

$$1 = 8y - [(k+1)y - x](8a+3) = 8[y - a\{(k+1)y - x\}] - 3[(k+1)y - x]. \qquad \text{(iv)}$$

Clearly, the minimum solution is obtained from (iii), which is

$$y - a\{(k+1)y - x\} = 1,\ (k+1)y - x = 3 \quad \Rightarrow \quad y = 3a+1 \ (\text{and } x = (k+1)(3a+1) - 3).$$

Hence, in this case, the minimum m in (A) is $m = qy - 1 = q(3a+1) - 1$.

Case 3 : When p is of the form $p = 8a+5$ for some integer $a \geq 1$.

In this case, from (i) and (ii), we get respectively

$$1 = [(k+1)y - x](8a+5) - 8y = 5[(k+1)y - x] - 8[y - a\{(k+1)y - x\}], \qquad \text{(v)}$$

$$1 = 8y - [(k+1)y - x](8a+5) = 8[y - a\{(k+1)y - x\}] - 5[(k+1)y - x]. \qquad \text{(vi)}$$

The minimum solution is obtained from (vi) as follows :

$$y - a\{(k+1)y - x\} = 2,\ (k+1)y - x = 3 \quad \Rightarrow \quad y = 3a+2 \ (\text{and } x = (k+1)(3a+2) - 3).$$

Hence, in this case, the minimum m in (A) is $m = qy = q(3a+2)$.

Case 4 : When p is of the form $p = 8a+7$ for some integer $a \geq 2$.

In this case, 8 divides $(p+1)$. Then, the minimum solution is obtained from (ii), which is

$$y = \frac{p+1}{8} \quad (\text{and } x = (k+1)y - 1).$$

Therefore, the minimum m in (A) is

$$m = qy = \frac{q(p+1)}{8}. \ \blacksquare$$

4.3 Some Generalizations

Since its introduction by Kashihara [1], the pseudo Smarandache function has seen several variations. This section is devoted to some of these generalizations, given in the following two subsections.

4.3.1 The Pseudo Smarandache Dual Function $Z_*(n)$

The pseudo Smarandache dual function, denoted by $Z_*(n)$, introduced by Sandor [6], is defined as follows.

Definition 4.3.1.1 **:** For any integer $n \geq 1$, the pseudo Smarandache dual function, $Z_*(n)$, is

$$Z_*(n) = \max\left\{m \ : \ m \in Z^+, \ \frac{m(m+1)}{2} \mid n\right\}.$$

Let

$$Z_*(n) = m_0 \text{ for some integer } m_0 \geq 1.$$

Then, by Definition 4.3.1.1, the following two conditions are satisfied :

(1) $\frac{m_0(m_0+1)}{2}$ divides n,

(2) n is not divisible by $\frac{m(m+1)}{2}$ for any $m > m_0$.

Lemma 4.3.1.1 **:** For any integer $n \geq 1$, $1 \leq Z_*(n) \leq \frac{1}{2}(\sqrt{1+8n}-1)$.

Proof **:** Let

$$Z_*(n) = m_0 \text{ for some integer } m_0 \geq 1,$$

so that, by definition,

$$n = k.\frac{m_0(m_0+1)}{2} \quad \text{for some integer } k \geq 1.$$

Then,

$$n \geq \frac{m_0(m_0+1)}{2},$$

which gives the r.h.s. inequality. The other part of the inequality is obvious. ■

Lemma 4.3.1.2 **:** $Z_*(\frac{k(k+1)}{2}) = k$ for any integer $k \geq 1$.

Proof **:** By definition,

$$Z_*(\frac{k(k+1)}{2}) = \max\left\{m : \frac{m(m+1)}{2} \mid \frac{k(k+1)}{2}\right\}. \quad (1)$$

Now, since $m(m+1) = k(k+1)$ if and only if $m = k$, the maximum m in (1) is k. ■

Corollary 4.3.1.1 **:** Let p and $q=2p-1$ be two primes. Then, $Z_*(pq)=q$.

Proof **:** Since $pq=(\frac{q+1}{2}).q$, the result follows from Lemma 4.3.1.2. ■

Lemma 4.3.1.3 **:** For any integers $a, b \geq 1$, $Z_*(ab) \geq \max\{Z_*(a), Z_*(b)\}$.

Proof **:** Let

$Z_*(a) = m_0$ for some integer $m_0 \geq 1$.

Then, by definition,

$$\frac{m_0(m_0+1)}{2} \mid a \quad \Rightarrow \quad \frac{m_0(m_0+1)}{2} \mid ab \quad \Rightarrow \quad Z_*(ab) \geq m_0 = Z_*(a). \ ■$$

Lemma 4.3.1.4 **:** Let $p \geq 3$ be a prime. Then, for any integer $k \geq 1$,

$$Z_*(p^k) = \begin{cases} 2, \text{ if } p=3 \\ 1, \text{ if } p \neq 3 \end{cases}$$

Proof **:** By definition, $Z_*(p^k) = \max\left\{m \ : \ m \in Z^+, \ \frac{m(m+1)}{2} \mid p^k\right\}$.

If one of m and $m+1$ is p, then the other one must be 2. Now,

$m+1=p, m=2 \quad \Rightarrow \quad p=3,$

$m+1=2, m=p \quad \Rightarrow \quad p=1.$

Thus, in the second case when $p \neq 3$, we must have $m=1$. ■

Lemma 4.3.1.5 **:** For any integer $n \geq 1$, $Z(n) \geq Z_*(n)$.

Proof **:** Let

$Z(n)=m_0$, $Z_*(n)=m$ for some integers $m_0 \geq 1$ and $m \geq 1$.

Then, by definitions of $Z_*(n)$ and $Z(n)$ respectively,

$$\frac{m(m+1)}{2} \mid n, \quad n \mid \frac{m_0(m_0+1)}{2} \quad \Rightarrow \quad \frac{m(m+1)}{2} \mid \frac{m_0(m_0+1)}{2}.$$

Thus, $m_0 \geq m$, which we intended to prove. ■

Lemma 4.3.1.6 **:** Any solution of the equation $Z(n)=Z_*(n)$ is of the form

$$n=\frac{k(k+1)}{2}, \ k \geq 1.$$

Proof **:** Let

$Z(n)=k=Z_*(n)$ for some integer $k \geq 1$.

Then,

$$\frac{k(k+1)}{2} \mid n, \quad n \mid \frac{k(k+1)}{2} \quad \Rightarrow \quad n=\frac{k(k+1)}{2}. \ ■$$

Corollary 4.3.1.2 **:** There is an infinite number of n satisfying the equation $Z(n)=Z_*(n)$.

4.3.2 The Additive Analogues of Z(n) and $Z_*(n)$

The function, called the additive analogue of Z(n), to be denoted by $Z_S(n)$, has been introduced by Sandor [7].

Definition 4.3.2.1 **:** For any real number $x \in (0, \infty)$,

$$Z_S(x) = \min\left\{m \ : \ m \in Z^+, \ x \leq \frac{m(m+1)}{2}\right\}.$$

The dual of $Z_S(x)$, to be denoted by $Z_{S^*}(x)$, is defined as follows (Sandor [7]).

Definition 4.3.2.2 **:** For any real number $x \in [1, \infty)$,

$$Z_{S^*}(x) = \max\left\{m \ : \ m \in Z^+, \ \frac{m(m+1)}{2} \leq x\right\}.$$

From Definition 4.3.2.1 and Definition 4.3.2.2, it is clear that, for any integer $k \geq 1$,

$$Z_S(x) = k \text{ if and only if } x \in (\frac{(k-1)k}{2}, \frac{k(k+1)}{2}],$$

$$Z_{S^*}(x) = k \text{ if and only if } x \in [\frac{k(k+1)}{2}, \frac{(k+1)(k+2)}{2}).$$

We then have the following result.

Lemma 4.3.2.1 **:** For any real number $x \geq 1$, and any integer $k \geq 1$,

$$Z_S(x) = \begin{cases} Z_{S^*}(x) + 1, & \text{if } x \in (\frac{k(k+1)}{2}, \frac{(k+1)(k+2)}{2}) \\ Z_{S^*}(x), & \text{if } x = \frac{k(k+1)}{2} \end{cases}$$

Lemma 4.3.2.2 **:** For any fixed real number $x \geq 1$,

$$\tfrac{1}{2}(\sqrt{8x+1} - 3) < Z_{S^*}(x) \leq \tfrac{1}{2}(\sqrt{8x+1} - 1). \tag{4.3.2.1}$$

Proof **:** Let, for some real number $x \geq 1$ fixed,

$Z_{S^*}(x) = k$ for some integer $k \geq 1$.

Then, by definition,

$$\frac{k(k+1)}{2} \leq x < \frac{(k+1)(k+2)}{2}, \tag{1}$$

where k is the largest integer satisfying the above inequality. Now, (1) is satisfied

if and only if $k^2 + k - 2x \leq 0 < k^2 + 3k - 2(x-1)$

if and only if $\frac{1}{2}(\sqrt{8x+1} - 3) < k \leq \frac{1}{2}(\sqrt{8x+1} - 1)$,

which now gives the desired result. ■

Corollary 4.3.2.1 **:** For any fixed real number $x \geq 1$,

$Z_{S^*}(x) = [\frac{1}{2}(\sqrt{8x+1} - 1]$ ([y] denotes the integer part of y).

***Proof* :** The result follows from Lemma 4.3.2.2, noting that

$$\tfrac{1}{2}(\sqrt{8x+1}-3)-\tfrac{1}{2}(\sqrt{8x+1}-1)=1,$$

k is an integer, and further k is the maximum integer satisfying the inequality (4.3.2.1). ■

***Corollary 4.3.2.2* :** $Z_{S*}(x) \sim \tfrac{1}{2}(\sqrt{8x+1}-1)$ as $x \to \infty$.

***Proof* :** follows from Corollary 4.3.2.1 above. ■

If the domain is restricted to the set of positive integers, Z^+, we get the function $Z_S(n)$. The first few terms of $Z_S(n)$ are

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, …,

where the value k (≥ 1) is repeated $\frac{k(k+1)}{2} - \frac{(k-1)k}{2} = k$ times. Similarly, we get the function $Z_{S*}(n)$, with domain Z^+, which assumes the successive values

1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, …,

where the value k (≥ 1) is repeated $\frac{(k+1)(k+2)}{2} - \frac{k(k+1)}{2} = k+1$ times.

***Lemma 4.3.2.3* :** For any integer $n \geq 1$, $Z(n) \geq Z_{S*}(n)$.

***Proof* :** Let

$Z(n) = m_0$, $Z_{S*}(n) = m$ for some integers $m_0 \geq 1$ and $m \geq 1$.

Then, by definitions of $Z(n)$ and $Z_{S*}(n)$,

$$n \mid \frac{m_0(m_0+1)}{2}, \quad n \geq \frac{m(m+1)}{2} \quad \Rightarrow \quad \frac{m_0(m_0+1)}{2} \geq n \geq \frac{m(m+1)}{2}.$$

Thus, $m_0 \geq m$, which we intended to prove. ■

***Lemma 4.3.2.4* :** Any solution of the equation $Z(n) = Z_{S*}(n)$ is of the form

$$n = \frac{k(k+1)}{2}, \ k \geq 1.$$

***Proof* :** Let

$Z(n) = k = Z_{S*}(n)$ for some integer $k \geq 1$.

Then,

$$n \mid \frac{k(k+1)}{2}, \quad n \geq \frac{k(k+1)}{2} \quad \Rightarrow \quad n = \frac{k(k+1)}{2}. \ ■$$

***Corollary 4.3.2.3* :** Any solution of the equation $Z(n) = Z_S(n)$ is of the form

$$n = \frac{k(k+1)}{2}, \ k \geq 1.$$

***Proof* :** follows from Lemma 4.3.2.4 and Lemma 4.3.2.1. ■

4.4 Miscellaneous Topics

In this section, we give some properties of the pseudo Smarandache function Z(n).

One major difference in the properties of S(n) and Z(n) is that, though the equation $S(n)=n$ has an infinite number of solutions, the equation $Z(n)=n$ does not permit any solution, as is proved in Lemma 4.4.1. The next lemma, Lemma 4.4.2, is due to Ashbacher [3].

Lemma 4.4.1 **:** The equation $Z(n)=n$ has no solution.

Proof **:** Let, for some integer $n\geq 1$, $Z(n)=n$, so that,

$$n \mid \frac{n(n+1)}{2} \quad \Rightarrow \quad n \mid \frac{(n-1)n}{2} \quad \Rightarrow \quad Z(n)\leq n-1,$$

which is a contradiction to the assumption. ■

Lemma 4.4.2 **:** $|Z(n+1)-Z(n)|$ is unbounded.

Proof **:** Let $n+1=2^k$ for some integer $k\geq 1$. Then, by Lemma 4.2.2 and Lemma 4.4,

$$Z(n+1)=Z(2^k)=2^{k+1}-1,\ Z(n)\leq n-1=2^k-2.$$

Therefore,

$$|Z(n+1)-Z(n)|=Z(n+1)-Z(n)\geq (2^{k+1}-1)-(2^k-2)=2^k+1,$$

which can be made arbitrarily large (by choosing k large and larger). ■

The following lemma gives a set of conditions such that $Z(n+1)$ divides Z(n).

Lemma 4.4.3 **:** Let p be a prime of the form $p=4a+1$, $a\geq 1$ is an integer.
Let q be another prime such that $q=2p-1=8a+1$.
Then, Z(2p) divides $Z(q)=Z(2p-1)$.

Proof **:** By Corollary 4.2.1(1), $Z(2p)=p-1$.

By construction, q is a prime, so that, by Lemma 4.2.3, $Z(q)=Z(2p-1)=2(p-1)$.

Hence, $Z(2p)=p-1$ divides $Z(q)=Z(2p-1)=2(p-1)$. ■

Lemma 4.4.4 **:** Given any integer $k\geq 2$, the equation $Z(kn)=n$ has an infinite number of solutions.

Proof **:** Given the integer $k\geq 2$, we choose a prime p such that $2k \mid (p+1)$. Then,

$Z(kp)=p$ (by Lemma 4.2.4).

Now, given k, there is an infinite number of primes of the desired form.

All these complete the proof of the lemma. ■

The following three results, given in Lemma 4.4.5 – Lemma 4.4.7, are due to Pinch [8], though the proofs given here are modified ones.

Lemma 4.4.5 **:** For any integer $k\geq 2$ fixed, the equation $n=kZ(n)$ has an infinite number of solutions.

Proof **:** The equation $n=kZ(n)$ is equivalent to $Z(kn)=n$. ■

Lemma 4.4.6 **:** The ratio $\frac{Z(2n)}{Z(n)}$ is unbounded above.

Proof **:** Given the integer $k \geq 1$, we choose a prime p such that $2^k \mid (p+1)$. Let

$$n = \frac{p(p+1)}{2} \equiv T(p) \text{ (so that } Z(n) = p).$$

Let

$$Z(2n) = m.$$

Now,

$$2^k p \mid p(p+1) = 2n,\ 2n \mid \tfrac{m(m+1)}{2} \ \Rightarrow\ 2^k p \mid \tfrac{m(m+1)}{2} \ \Rightarrow\ 2^{k+1} p \mid m(m+1).$$

We now consider the following two cases :

Case (1) : When p divides m and 2^{k+1} divides $(m+1)$.

In this case,

$$m = pt \text{ for some integer } t > 1.$$

Note that, $t \neq 1$, for otherwise

$$t = 1 \quad \Rightarrow \quad m = p \quad \Rightarrow \quad T(m) = T(p) = n \quad \Rightarrow \quad n = T(m) \geq 2n.$$

Now,

$$2^k \mid (p+1) \quad \Rightarrow \quad 2^k \mid (m+t),$$

and since $2^k \mid (m+1)$, it follows that $2^k \mid (t-1)$, so that $t \geq 2^k + 1$. Hence,

$$\frac{Z(2n)}{Z(n)} = \frac{m}{p} = t > 2^k,$$

which can be made arbitrarily large (by choosing k sufficiently large).

Case (2) : When 2^{k+1} divides m and p divides $(m+1)$.

In this case,

$$m = pt - 1 \text{ for some integer } t > 1.$$

In this case, $t \neq 1$, for otherwise

$$t = 1 \quad \Rightarrow \quad m = p-1 \quad \Rightarrow \quad T(m) = T(p-1) \quad \Rightarrow \quad n = T(p) > T(p-1) \geq 2n.$$

Now,

$$2^k \mid (p+1) \quad \Rightarrow \quad 2^k \mid (m+t+1),$$

and since $2^k \mid m$, it follows that $2^k \mid (t+1)$, so that $t \geq 2^k - 1$. Hence,

$$\frac{Z(2n)}{Z(n)} = \frac{m}{p} = t - \frac{1}{p} \geq 2^k - \frac{1}{p} - 1,$$

which can be made arbitrarily large (by choosing k sufficiently large). ■

Example 4.4.1 **:** To find n such that the ratio $\frac{Z(2n)}{Z(n)}$ is greater than 30, we have to choose a prime p such that $32 = 2^k \mid (p+1)$. Choosing $p = 31$, so that $n = \frac{31 \times 32}{2} = 496$, we get

$$\frac{Z(2n)}{Z(n)} = \frac{Z(992)}{Z(496)} = \frac{960}{31} = 30.97 > 30. \ ♦$$

***Lemma 4.4.7* :** Given any real number $L > 0$, the inequalities

$$\frac{Z(n+1)}{Z(n)} > L, \quad \frac{Z(n-1)}{Z(n)} > L$$

each has infinite number of solutions.

***Proof* :** Given the number $L > 0$, we choose an integer k as follows :

$k = \max\{4, [2L]\}$ ([x] denotes the least integer not less than x).

Let

$$n = \frac{k(k+1)}{2}(1+kt) \equiv T(k)(1+kt),$$

where $t \geq 1$ is an integer such that

$$n+1 = T(k)(1+kt)+1 \tag{1}$$

is prime. Let

$$m = k[1+(k+1)t].$$

Note that

$$m(m+1) = k(k+1)(1+kt)[1+(k+1)t] = 2n[1+(k+1)t],$$

so that

$$n \mid \frac{m(m+1)}{2} \quad \Rightarrow \quad Z(n) \leq m.$$

Therefore,

$$\frac{Z(n+1)}{Z(n)} \geq \frac{n}{m} = \frac{T(k)(1+kt)}{k+2tT(k)}.$$

Now,

$$f(t) \equiv \frac{T(k)(1+kt)}{k+2tT(k)} > \frac{T(k)[1+k(t+1)]}{k+2T(k)(t+1)} \equiv f(t+1)$$

if and only if $(1+kt)[k+2T(k)(t+1)] > [1+k(t+1)][k+2tT(k)]$,

that is, if and only if $2T(k) > k^2$,

which is true.

This shows that the function $f(t) \equiv \dfrac{T(k)(1+kt)}{k+2tT(k)}$ is strictly decreasing in t. Now, since

$$\lim_{t\to\infty} f(t) \equiv \frac{k}{2},$$

it follows that

$$\frac{Z(n+1)}{Z(n)} > \frac{k}{2} \geq L.$$

Next, for $k \geq 4$, let $n = T(k)(1+kt)$,

where $t \geq 1$ is an integer such that

$$n-1 = T(k)(1+kt)-1 \tag{2}$$

is prime. Then,

$$\frac{Z(n-1)}{Z(n)} \geq \frac{n-2}{m} = \frac{T(k)(1+kt)-2}{k+2tT(k)}.$$

Now,

$$g(t) \equiv \frac{T(k)(1+kt)-2}{k+2tT(k)} > \frac{T(k)[1+k(t+1)]-2}{k+2T(k)(t+1)} \equiv g(t+1)$$

if and only if

$$[T(k)(1+kt)-2][k+2T(k)(t+1)] > [T(k)\{1+k(t+1)\}-2][k+2tT(k)]$$

that is, if and only if $2[T(k)]^2 - 4T(k) > k^2\,T(k)$

that is, if and only if $k(k+1)-4 > k^2$,

which is true.

This shows that the function $g(t) \equiv \dfrac{T(k)(1+kt)-2}{k+2tT(k)}$ is strictly decreasing in t. Now, since

$$\lim_{t\to\infty} g(t) \equiv \frac{k}{2},$$

it follows that

$$\frac{Z(n-1)}{Z(n)} > \frac{k}{2} \geq L. \tag{3}$$

Since there is an infinite number of primes of the forms (1) or (2), the result follows. ■

Remark 4.4.1 **:** In Lemma 4.4.7, k can be any integer greater than 1. But we choose $k \geq 4$ simply for the convenience in proving the inequality (3).

Example 4.4.2 **:** To find the sequences $\left\{\dfrac{Z(n+1)}{Z(n)}\right\}$ and $\left\{\dfrac{Z(n-1)}{Z(n)}\right\}$, each term of which is greater than $L = 2$, we choose $k = 4$. Then, $T(k) = 10$.

(1) We have to choose the integer $t \geq 0$ such that

$$n+1 = T(k)(1+kt)+1 = 10(1+4t)+1 = 11+40t$$

is prime. We then get successively the primes

11, 131, 211, 251, 331, 491, …

with

$$\frac{Z(11)}{Z(10)} = \frac{10}{4} = 2.5, \qquad \frac{Z(131)}{Z(130)} = \frac{130}{39} = 3.33,$$

$$\frac{Z(211)}{Z(210)} = \frac{210}{20} = 10.5, \qquad \frac{Z(251)}{Z(250)} = \frac{250}{124} = 2.016,$$

$$\frac{Z(331)}{Z(330)} = \frac{330}{44} = 7.5, \qquad \frac{Z(491)}{Z(490)} = \frac{490}{195} = 2.51, \ldots.$$

(2) We have to choose $t \geq 0$ such that

$$n-1 = 10(1+4t)-1 = 9+40t$$

is prime. The successive primes are

89, 409, 449, 769, …

with

$$\frac{Z(89)}{Z(90)} = \frac{88}{35} = 2.51, \qquad \frac{Z(409)}{Z(410)} = \frac{408}{40} = 10.2,$$

$$\frac{Z(449)}{Z(450)} = \frac{448}{99} = 4.52, \quad \frac{Z(769)}{Z(770)} = \frac{768}{55} = 13.96, \ldots . \ \blacklozenge$$

Lemma 4.4.8 **:** Given any even integer $k \geq 2$, the equation

$$\frac{Z(n+1)}{Z(n)} = k$$

has solutions.

Proof **:** Given the even integer $k \geq 2$, we choose a prime p such that $2k \mid (p+1)$. Let q be another prime of the form

$$q = kp+1.$$

Now, letting

$$n = kp \text{ (so that } n+1 = q),$$

by virtue of Lemma 4.2.4, $Z(kp) = p$, so that

$$\frac{Z(n+1)}{Z(n)} = \frac{Z(q)}{Z(kp)} = \frac{Z(kp+1)}{Z(kp)} = \frac{kp}{p} = k. \quad \blacksquare$$

Example 4.4.3 **:** Corresponding to $k = 2$ in Lemma 4.4.8, some of the (p, q) pairs are

$$(p, q) = (3, 7), (11, 23), (23, 47),$$

with

$$\frac{Z(7)}{Z(6)} = \frac{6}{3} = 2, \ \frac{Z(23)}{Z(22)} = \frac{22}{11} = 2, \ \frac{Z(47)}{Z(46)} = \frac{46}{23} = 2.$$

Corresponding to $k = 4$, we have

$$(p, q) = (7, 29), (79, 317), (127, 509),$$

with

$$\frac{Z(29)}{Z(28)} = \frac{28}{7} = 4, \ \frac{Z(317)}{Z(316)} = \frac{316}{79} = 2, \ \frac{Z(509)}{Z(508)} = \frac{508}{127} = 4. \ \blacklozenge$$

Remark 4.4.2 **:** Since $\frac{Z(n)}{n} \leq \frac{2n-1}{n} < 2$, it follows that the equation

$$\frac{Z(n)}{n} = k$$

has no solution if $k \geq 2$. The case $k = 1$ is forbidden by Lemma 4.4.1.

The problem of convergence of (infinite) series involving Z(n) and its different variants has been treated by several researchers. The following two lemmas are due to Kashihara [1].

***Lemma 4.4.9* :** The series $\sum_{n=1}^{\infty} \frac{1}{Z(n)}$ is divergent.

***Proof* :** The proof follows by observing that

$$\sum_{n=1}^{\infty} \frac{1}{Z(n)} > \sum_{n=1,3,\ ...} \frac{1}{Z(n)} \geq \sum_{n=1}^{\infty} \frac{1}{2n-1} > \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} > \infty,$$

where we have used the fact that the harmonic series is divergent. ■

***Lemma 4.4.10* :** The series $\sum_{n=1}^{\infty} \frac{Z(n)}{n}$ is divergent.

***Proof* :** First note that the series $\sum_{k=1}^{\infty} \frac{2^{k+1}-1}{2^k}$ is divergent, since $\lim_{k\to\infty} \frac{2^{k+1}-1}{2^k} \neq 0.$

Therefore,

$$\sum_{n=1}^{\infty} \frac{Z(n)}{n} > \sum_{k=1}^{\infty} \frac{2^{k+1}-1}{2^k} > \infty. \ ■$$

Pinch [8] has proved the more general result, given below.

***Lemma 4.4.11* :** The series $\sum_{n=1}^{\infty} \frac{1}{[Z(n)]^s}$ is convergent if and only if $s > 1$.

The following results are due to Sandor [9].

***Lemma 4.4.12* :** $\liminf_{n\to\infty} \frac{Z_*(n)}{Z(n)} = 0, \ \limsup_{n\to\infty} \frac{Z_*(n)}{Z(n)} = 1.$

***Proof* :** With $n = p$, together with Lemma 4.2.3 and Lemma 4.3.1.4, we have

$$\frac{Z_*(n)}{Z(n)} = \frac{Z_*(p)}{Z(p)} = \begin{cases} \frac{2}{p-1}, & \text{if } p = 3 \\ \frac{1}{p-1}, & \text{if } p \neq 3 \end{cases}$$

from which the first limiting value is obtained.

Next, let $n = \frac{k(k+1)}{2}$ for any integer $k \geq 1$. Since by Lemma 4.2.1 and Lemma 4.3.1.2,

$$Z(\frac{k(k+1)}{2}) = k = Z_*(\frac{k(k+1)}{2}),$$

the other part of the lemma is established. ■

***Corollary 4.4.1* :** The series $\sum_{n=1}^{\infty} \frac{Z_*(n)}{Z(n)}$ is divergent.

***Lemma 4.4.13* :** $\liminf_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} = 0, \ \limsup_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} = \sqrt{2}.$

***Proof* :** With $n = p$, together with Lemma 4.2.3 and Lemma 4.3.1.4, we get the first limiting value (as in the proof of Lemma 4.4.12 above).

To get the remaining part, first note that, for any integer $k \geq 1$,

$$\lim_{k\to\infty} \frac{Z_*(\frac{k(k+1)}{2})}{\sqrt{\frac{k(k+1)}{2}}} = \lim_{k\to\infty} \frac{k}{\sqrt{\frac{k(k+1)}{2}}} = \sqrt{2}\left(\lim_{k\to\infty} \sqrt{\frac{k}{k+1}}\right) = \sqrt{2}.$$

Now, from Lemma 4.3.1.1,

$$\frac{Z_*(n)}{\sqrt{n}} \leq \sqrt{2}\left(\sqrt{1+\frac{1}{8n}} - \sqrt{\frac{1}{8n}}\right) \leq \sqrt{2}. \ \blacksquare$$

***Corollary 4.4.2* :** The series $\sum_{n=1}^{\infty} \frac{Z_*(n)}{\sqrt{n}}$ is divergent.

***Lemma 4.4.14* :** Each of the series $\sum_{n=1}^{\infty} \frac{Z_*(n)}{n!}$ and $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{Z_*(n)}{n!}$ converges to an irrational number.

The next three lemmas are concerned with the convergence of series involving $Z_S(n)$ and $Z_{S^*}(n)$. The first one is due to Sandor [7], and the rest are due to Jiangli Gao [10].

***Lemma 4.4.15* :** The series $\sum_{n=1}^{\infty} \frac{1}{n[Z_{S^*}(n)]^s}$ is convergent for $s > 0$.

***Lemma 4.4.16* :** For any $s > 2$,

(1) $\sum_{n=1}^{\infty} \frac{1}{[Z_S(n)]^s} = \zeta(s-1)$, **(2)** $\sum_{n=1}^{\infty} \frac{1}{[Z_{S^*}(n)]^s} = \zeta(s-1) + \zeta(s)$.

where $\zeta(s)$ is the Riemann zeta function.

***Proof* :** Let

$$I_k \equiv (\frac{(k-1)k}{2}, \frac{k(k+1)}{2}], \ J_k \equiv [\frac{k(k+1)}{2}, \frac{(k+1)(k+2)}{2}) .$$

(1) From the definition, $Z_S(n) = k$ if $n \in I_k$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{[Z_S(n)]^s} = \sum_{k=1}^{\infty} \sum_{n \in I_k} \frac{1}{[Z_S(n)]^s} = \sum_{k=1}^{\infty} \frac{k}{k^s} = \zeta(s-1).$$

(2) Since $Z_{S^*}(n) = k$ if $n \in J_k$, we get,

$$\sum_{n=1}^{\infty} \frac{1}{[Z_{S^*}(n)]^s} = \sum_{k=1}^{\infty} \sum_{n \in J_k} \frac{1}{[Z_{S^*}(n)]^s} = \sum_{k=1}^{\infty} \frac{k+1}{k^s} = \zeta(s-1) + \zeta(s). \ \blacksquare$$

Lemma 4.4.17 **:** For any $s > 1$,

(1) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{[Z_S(n)]^s} = (1-\frac{1}{2^s})\zeta(s) - \frac{2}{4^s}\zeta(s, \tfrac{1}{4}),$$

(2) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{[Z_{S*}(n)]^s} = -(1-\frac{1}{2^{s-1}})\zeta(s).$$

where $\zeta(s,\alpha)$ is the Hurwitz zeta function.

Proof **:** Let I_k and J_k be as in the proof of Lemma 4.4.16.

(1) First note that

$$\sum_{n \in I_k} \frac{(-1)^n}{[Z_S(n)]^s} = \begin{cases} -\frac{1}{k^s}, & \text{if } k = 1, 5, 9, \dots \\ 0, & \text{if } k \text{ is even} \\ \frac{1}{k^s}, & \text{if } k = 3, 7, 11, \dots \end{cases}$$

In the above sum, the sign of the first element of I_k is $(-1)^{[1+2+\dots+(k-1)]+1}$.

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{[Z_S(n)]^s} = -\sum_{k=1,5,\dots} \frac{1}{k^s} + \sum_{k=3,7,\dots} \frac{1}{k^s} = \sum_{k=1,3,5,\dots} \frac{1}{k^s} - 2\sum_{k=1,5,\dots} \frac{1}{k^s},$$

which now gives the desired result, since

$$\sum_{k=1,3,5,\dots} \frac{1}{k^s} = (1-\frac{1}{2^s})\zeta(s), \quad \sum_{k=1,5,\dots} \frac{1}{k^s} = \sum_{k=0}^{\infty} \frac{1}{(4k+1)^s} = \frac{1}{4^s}\zeta(s, \tfrac{1}{4}).$$

(2) In this case,

$$\sum_{n \in J_k} \frac{(-1)^n}{[Z_S(n)]^s} = \begin{cases} -\frac{1}{k^s}, & \text{if } k = 2, 6, 10, \dots \\ 0, & \text{if } k \text{ is odd} \\ \frac{1}{k^s}, & \text{if } k = 4, 8, 12, \dots \end{cases}$$

with the first element of J_k having the sign $(-1)^{1+2+\dots+k}$.

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{[Z_{S*}(n)]^s} = -\sum_{k=2,6,\dots} \frac{1}{k^s} + \sum_{k=4,8,\dots} \frac{1}{k^s} = -\frac{1}{2^s}\sum_{k=1,3,5,\dots} \frac{1}{k^s} + \frac{1}{4^s}\sum_{k=1,2,3,\dots} \frac{1}{k^s}$$

$$= -\frac{1}{2^s}(1-\frac{1}{2^s})\sum_{k=1,2,3,\dots} \frac{1}{k^s} + \frac{1}{4^s}\sum_{k=1,2,3,\dots} \frac{1}{k^s},$$

which now gives the desired result after simplification. ■

We now consider the problem of finding the solutions of the equation

$$Z(n) = d(n).$$

The problem has been considered by Zhong Li [11], and later by Maohua Le [12]. Lemma 4.4.18 gives the result in respect to the above equation. We follow the same line of approach as that adopted by Maohua Le, filling in the gaps in his proof.

Lemma 4.4.18 **:** $Z(n) > d(n)$ for all integers $n > 120$.

Proof **:** If $n = p$, where $p \geq 5$ is a prime, then

$$Z(p) = p - 1 > 2 = d(n),$$

so that, the result is true if n is a prime. Thus, it is sufficient to consider the case when n is a composite number.

Let there be an integer n such that

$$Z(n) \leq d(n). \tag{1}$$

Let n have the prime factor representation of the form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \ldots p_s^{\alpha_s} \tag{4.4.1}$$

so that (by Theorem 0.1 in Chapter 0)

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1).$$

Let the function $f(p^\alpha)$ be defined as follows :

$$f(p^\alpha) = \frac{p^{\alpha/2}}{\alpha + 1}. \tag{2}$$

Then, from (4.4.1) and (2),

$$\begin{aligned} f(p_1^{\alpha_1}) f(p_2^{\alpha_2}) \ldots f(p_k^{\alpha_k}) &= \frac{p_1^{\alpha_1/2}}{\alpha_1 + 1} \cdot \frac{p_2^{\alpha_2/2}}{\alpha_2 + 1} \cdots \frac{p_k^{\alpha_k/2}}{\alpha_k + 1} \\ &= \frac{\sqrt{n}}{d(n)} \\ &\leq \frac{\sqrt{n}}{Z(n)} \\ &< 1. \end{aligned} \tag{3}$$

Now,

$$f(p^\alpha) \begin{cases} > 1, & \text{if } p = 2 \text{ and } \alpha \geq 6 \\ \geq 1, & \text{if } p = 3 \text{ and } \alpha \geq 2 \\ \geq \dfrac{\sqrt{5}}{2}, & \text{if } p = 5 \text{ and } \alpha \geq 1 \\ > \dfrac{\sqrt{5}}{2}, & \text{if } p \geq 7 \text{ and } \alpha \geq 1 \end{cases} \tag{4.4.2}$$

To verify (4.4.2), first note that

$$f(p^{\alpha}) \equiv \frac{p^{\alpha/2}}{\alpha+1} \begin{cases} \geq 1, & \text{if and only if } p^{\alpha} \geq (\alpha+1)^2 \\ \geq \frac{\sqrt{5}}{2}, & \text{if and only if } p^{\alpha} \geq \frac{5}{4}(\alpha+1)^2 \end{cases} \tag{4.4.3}$$

from which the last two inequalities of (4.4.2) follow immediately. Again, since

$$2^5 < 6^2 \text{ but } 2^6 > 7^2, \text{ and } 3 < 2^2 \text{ but } 3^2 \geq 3^2,$$

the first two inequalities in (4.4.2) are established.

But, (4.4.2) contradicts (3) except for the following five cases :

$$\left.\begin{array}{l} \text{(1) When } n = 2^{\alpha},\ 1 \leq \alpha \leq 5, \\ \text{(2) When } n = 3p, \text{ where } p \geq 5 \text{ is a prime,} \\ \text{(3) When } n = 2^{\alpha}p,\ 1 \leq \alpha \leq 5 \text{ and } p \geq 3 \text{ is a prime,} \\ \text{(4) When } n = 2^{\alpha}.3p,\ 1 \leq \alpha \leq 5 \text{ and } p \geq 5 \text{ is a prime,} \\ \text{(5) When } n = 2^{\alpha}5^2,\ 1 \leq \alpha \leq 5. \end{array}\right\} \tag{4.4.4}$$

Thus, except for the numbers of the forms listed above, the result in Lemma 4.4.18 is true.

To complete the proof, we have to consider the case when n is of the forms (1) – (5) in (4.4.4). This is done below.

In Case (1), when $n = 2^{\alpha}$, by Lemma 4.2.2,

$$Z(n) = Z(2^{\alpha}) = 2^{\alpha+1} - 1 > \alpha + 1 = d(2^{\alpha}) = d(n) \text{ for } 1 \leq \alpha \leq 5.$$

In Case (2), when $n = 3p$, by Corollary 4.2.1(2),

$$Z(n) = Z(3p) \geq p - 1 \geq 4 = d(3p) = d(n) \text{ for any prime } p \geq 5.$$

In Case (3), when $n = 2^{\alpha}p$, $1 \leq \alpha \leq 5$, $d(n) = 2(\alpha+1)$.

Now, from Corollary 4.2.1(1),

$$Z(2p) \geq p - 1 \geq 4 = d(2p) \text{ for } p \geq 5,$$

(with $Z(2\times3) = 3 < 4 = d(6)$, $Z(2\times5) = 4 = d(10)$),

from Lemma 4.2.16,

$$Z(2^2 p) \geq p - 1 \geq 6 = d(2^2 p) \text{ for } p \geq 7 \text{ (with strict inequality when } p = 7),$$

$$Z(2^2 p) \geq 3p - 1 > 6 = d(2^2 p) \text{ for } p=3, 5,$$

from Lemma 4.2.20,

$$Z(2^3 p) \geq p - 1 > 8 = d(2^3 p) \text{ for } p \geq 11,$$

$$Z(2^3 p) \geq 3p > 8 = d(2^3 p) \text{ for } p = 3, 5, 7,$$

from Lemma 4.2.26,

$$Z(2^4 p) \geq p - 1 \geq 10 = d(2^4 p) \text{ for } p \geq 11 \text{ (with strict inequality when } p = 11),$$

$$Z(2^4 p) \geq 9p \geq 10 = d(2^4 p) \text{ for } p=3, 5, 7,$$

from Lemma 4.2.27,

$$Z(2^5 p) \geq p - 1 \geq 12 = d(2^5 p) \text{ for } p \geq 13 \text{ (with strict inequality sign for } p = 13),$$

$$Z(2^5 p) \geq 9p > 12 = d(2^5 p) \text{ for } p = 3, 5, 7, 11.$$

In Case (4), when $n = 2^{\alpha}.3p$, $1 \le \alpha \le 5$ and $p \ge 5$ is a prime, $d(n) = 4(\alpha + 1)$.
Now, from Lemma 4.2.18,

$$Z(6p) \ge p - 1 > 8 = d(6p) \text{ for } p \ge 11,$$

$$Z(6p) \ge 3p - 1 > 8 = d(6p) \text{ for } p = 5, 7,$$

(with $Z(6\times3) = 8 > 6 = d(18)$),
from Lemma 4.2.24,

$$Z(12p) \ge p - 1 \ge 12 = d(12p) \text{ for } p \ge 13 \text{ (with strict inequality when } p = 13),$$

$$Z(12p) \ge 3p - 1 > 12 = d(12p) \text{ for } p = 5, 7, 11,$$

(with $Z(12\times3) = 8 < 9 = d(36)$),
from Lemma 4.2.28,

$$Z(24p) \ge p - 1 \ge 16 = d(24p) \text{ for } p \ge 17 \text{ (with strict inequality when } p = 17),$$

$$Z(24p) \ge 3p - 1 > 16 = d(24p) \text{ for } p = 7, 11, 13,$$

(with $Z(24\times3) = 63 > 12 = d(72)$).
from Lemma 4.2.29,

$$Z(48p) \ge p - 1 \ge 20 = d(48p) \text{ for } p \ge 23 \text{ (with strict inequality when } p = 23),$$

$$Z(48p) \ge 3p - 1 \ge 20 = d(48p) \text{ for } p = 7, 11, 13, 17, 19,$$

(with strict inequality for $p = 7$; $Z(48\times3) = 63 > 15 = d(144)$, $Z(48\times5) = 95 > 15 = d(240)$).
To check for $Z(96p)$, we shall make use of Lemma 4.2.29 and Lemma 4.5.
Now, since $Z(96p) \ge Z(48p)$, we see that

$$Z(96p) \ge p - 1 > 24 = d(96p) \text{ for } p \ge 29,$$

$$Z(96p) \ge 3p - 1 > 24 = d(96p) \text{ for } p = 11, 13, 17, 19, 23,$$

(with $Z(96\times3) = 63 > 18 = d(288)$, $Z(96\times5) = 255 > 24 = d(480)$, $Z(96\times7) = 63 > 24 = d(672)$).
In Case (5), when $n = 2^{\alpha}5^2$, $1 \le \alpha \le 5$, note that $d(2^{\alpha}5^2) = d(2^{\alpha}5^2) = 3(\alpha + 1)$.
But, by Lemma 4.5,

$$Z(2^{\alpha}5^2) \ge Z(5^2) = 24 > 3(\alpha + 1) = d(2^{\alpha}5^2) \text{ for all } 1 \le \alpha \le 5.$$

All these complete the proof of the lemma. ■

In the proof of Lemma 4.4.18 above, we proved more. They are summarized below.

Corollary 4.4.3 **:** Let $d(n)$ be the divisor function. Then,
(4) the equation $Z(n) = d(n)$ has the only solutions $n = 1, 3, 10$;
(5) the inequality $Z(n) < d(n)$ has the only solutions $n = 6, 36, 120$;
(6) $Z(n) > d(n)$ for any integer $n \ne 1, 3, 6, 10, 36, 120$.

It is interesting to note here that, all the numbers for which the result fails to hold true (that is, the numbers 1, 3, 6, 10, 36 and 120) are triangular numbers.

Corollary 4.4.4 **:** $Z(n) + \phi(n) > d(n)$ for any integer $n \geq 1$.

Proof **:** From Corollary 4.4.3, the result follows, except possibly for the numbers 6, 36 and 120. Now, by Theorem 0.2 (in Chapter 0),

$$\phi(6) = \phi(2 \times 3) = 2,\ \phi(36) = \phi(2^2.3^2) = 12,\ \phi(120) = \phi(2^3.3.5) = 32.$$

Checking with these numbers, we get the desired result. ■

An immediate consequence of Corollary 4.4.4 is the following corollary. The same problem has also been treated by Zhong Li and Maohua Le [13].

Corollary 4.4.5 **:** The equation $Z(n) + \phi(n) = d(n)$ has no solution.

The following two results are due to Ashbacher [3].

Lemma 4.4.19 **:** The equation $Z(n) = \phi(n)$ has an infinite number of solutions.

Proof **:** The proof is simple : For any prime $p \geq 3$,

$$Z(p) = p - 1 = \phi(p). \ ■$$

Lemma 4.4.20 **:** The equation $Z(n) + \phi(n) = n$ has an infinite number of solutions.

Proof **:** Let

$$n = 3.2^{2k}, \text{ where } k \geq 1 \text{ is an integer.}$$

Then, from Lemma 4.2.6 and Theorem 0.2 (in Chapter 0),

$$Z(n) = Z(3.2^{2k}) = 2^{2k+1},\ \phi(n) = \phi(3.2^{2k}) = (3.2^{2k})(1 - \tfrac{1}{3})(1 - \tfrac{1}{2}) = 2^{2k}.$$

For such an n,

$$Z(n) + \phi(n) = Z(3.2^{2k}) + \phi(3.2^{2k}) = 2^{2k+1} + 2^{2k} = 3.2^{2k} = n. \ ■$$

Another related problem of interest is the following, due to Zhong Li [11].

Lemma 4.4.21 **:** The equation $Z(n) + d(n) = n$ has the only solution $n = 56$.

Proof **:** Let, for some integer $n \geq 1$,

$$Z(n) + d(n) = n. \tag{1}$$

By definition,

$$n \mid \frac{Z(n)[Z(n)+1)]}{2} = \frac{[n - d(n)][n - \{d(n) - 1\}]}{2}$$

$$= \frac{n(n+1) + d(n)[d(n) - 1]}{2} - nd(n). \tag{2}$$

Now, if n is odd, from (2), we see that

$$n \mid \frac{d(n)[d(n) - 1]}{2} \quad \Rightarrow \quad n \mid d(n)[d(n) - 1].$$

On the other hand, if n is even, from (2),

$$d(n)[d(n) - 1] = (2k - 1)n \text{ for some integer } k \geq 1.$$

Thus, in either case,

$$d(n)[d(n)-1] \geq n \quad \Rightarrow \quad [d(n)]^2 > n \quad \Rightarrow \quad \frac{\sqrt{n}}{d(n)} < 1. \qquad (3)$$

But, this contradicts (4.4.2) except for the five cases given in (4.4.4). Thus, the equation (1) has no solution, except for the case of numbers of the forms of (4.4.4).

To complete the proof, we now consider the five cases in (4.4.4) separately.

In Case (1), when $n = 2^\alpha$, by Lemma 4.2.2,

$$Z(n) + d(n) = Z(2^\alpha) + d(2^\alpha) = 2^{\alpha+1} + \alpha > 2^\alpha = n \text{ for any } \alpha \geq 1.$$

In Case (2), when $n = 3p$, by Corollary 4.2.1(2),

$$Z(n) + d(n) = Z(3p) + d(3p) \leq p + 3 < 3p = n \text{ for any prime } p \geq 5.$$

In Case (3), when $n = 2^\alpha p$, $1 \leq \alpha \leq 5$, $d(n) = 2(\alpha + 1)$.

Now, from Corollary 4.2.1(1),

$$Z(2p) + d(2p) \leq p + 3 < 2p \text{ for } p \geq 5 \text{ (with } Z(2\times3) + d(2\times3) = 3 + 4 > 6),$$

from Lemma 4.2.16,

$$Z(2^2 p) + d(2^2 p) \leq 3p + 5 < 4p \text{ for } p \geq 7,$$

$(Z(2^2.3) + d(2^2.3) = 8 + 6 > 12,\ Z(2^2.5) + d(2^2.5) = 15 + 6 > 20)$,

from Lemma 4.2.20,

$$Z(2^3 p) + d(2^3 p) \leq 7p + 7 < 8p \text{ for } p \geq 11,$$

$(Z(24) + d(24) = 15 + 8 < 24,\ Z(40) + d(40) = 15 + 8 < 40,\ Z(56) + d(56) = 48 + 8 = 56)$,

from Lemma 4.2.26,

$$Z(2^4 p) + d(2^4 p) \leq 15p + 9 < 16p \text{ for } p \geq 11,$$

$(Z(48) + d(48) = 32 + 10 < 48,\ Z(80) + d(80) = 64 + 10 < 80,\ Z(112) + d(112) = 63 + 10 < 112)$,

from Lemma 4.2.27,

$$Z(2^5 p) + d(2^5 p) \leq 31p + 11 < 32p \text{ for } p \geq 11,$$

(with strict inequality for $p = 11$, and $Z(96) + d(96) = 63 + 12 < 96$,

$Z(160) + d(160) = 64 + 12 < 160,\ Z(224) + d(224) = 63 + 12 < 224)$.

In Case (4), when $n = 2^\alpha.3p$, $1 \leq \alpha \leq 5$ and $p \geq 5$ is a prime, $d(n) = 4(\alpha + 1)$.

Now, from Lemma 4.2.18,

$$Z(6p) + d(6p) \leq 3p + 7 < 6p \text{ for } p \geq 5,$$

(with $Z(6\times3) + d(6\times3) = 8 + 6 = 14 < 18$),

from Lemma 4.2.24,

$$Z(12p) + d(12p) \leq 7p + 11 < 12p \text{ for } p \geq 5,$$

(with $Z(12\times3) + d(12\times3) = 8 + 9 = 17 < 36$),

from Lemma 4.2.28,

$$Z(24p) + d(24p) \le 15p + 15 < 24p \text{ for } p \ge 5,$$

(with $Z(24\times3) + d(24\times3) = 63 + 12 = 75 > 72$),

from Lemma 4.2.29,

$$Z(48p) + d(48p) \le 31p + 19 < 48p \text{ for } p \ge 3,$$

(with $Z(48\times3) + d(48\times3) = 63 + 15 = 78 < 144$),

from Lemma 4.2.30,

$$Z(96p) + d(96p) \le 59p + 23 < 96p \text{ for } p \ge 3.$$

(with $Z(96\times3) + d(96\times3) = 63 + 18 = 81 < 288$).

In Case (5), when $n = 2^\alpha 5^2$, $1 \le \alpha \le 5$, we verify the five cases directly :

$$Z(50) + d(50) = 24 + 6 < 50, \; Z(100) + d(100) = 24 + 9 < 100,$$

$$Z(200) + d(200) = 175 + 12 < 200, \; Z(400) + d(400) = 224 + 15 < 400,$$

$$Z(800) + d(800) = 575 + 18 < 800.$$

All these complete the proof of the lemma. ■

The proof of Lemma 4.4.21 shows that $n = 56$ is the only solution of the equation $Z(n) + d(n) = n$, and in other cases, either $Z(n) + d(n) > n$ or $Z(n) + d(n) < n$.

The following result is due to Maohua Le [12], whose proof relies on a result of Ibstedt [4] which is not complete. See, however, Conjecture 4.5.2 in §4.5.

Lemma 4.4.22 **:** The only solutions of the equation $Z(n) = \sigma(n)$ are $n = 2^k$, $k \ge 1$.

It is straight forward to verify that $Z(2^k) = 2^{k+1} - 1 = \sigma(2^k)$.

Definition 4.4.1 **:** An integer $n \ge 1$ is called pseudo Smarandache perfect if and only if

$$n = \sum_{i=1}^{k} Z(d_i).$$

where $d_1 \equiv 1, d_2, \ldots, d_k$ are the proper divisors of n.

In [14], Ashbacher reports that the only pseudo Smarandache perfect numbers less than 10^6 are n=4, 6, 471544. However, since n=471544 is of the form n = 8p with p = 58943 (a prime), its only proper divisors are 1, 2, 4, 8, p, 2p and 4p. Since 8 divides p + 1 = 58944, it follows from Lemma 4.2.4 that

$$Z(p) = p - 1, \; Z(2p) = p, \; Z(4p) = p,$$

so that

$$n = 471544 > \sum_{i=1}^{k} Z(d_i).$$

Thus, n = 471544 is not pseudo Smarandache perfect. Note that, the number 6 is also perfect.

Lemma 4.4.23 **:** Any solution of the equation

$$[Z(n)]^2 + Z(n) = 2n \qquad (1)$$

is of the form $n = \frac{k(k+1)}{2}$, $k \geq 1$.

Proof **:** Solving the equation $[Z(n)]^2 + Z(n) - 2n = 0$ for positive Z(n), we get

$$Z(n) = \tfrac{1}{2}(\sqrt{8n+1} - 1).$$

This shows that (1) has an integer solution only if $8n+1$ is a perfect square. Let

$$8n + 1 = x^2 \text{ for some integer } x \geq 3.$$

Then,

$$8n = (x-1)(x+1). \qquad (2)$$

Since $x-1$ and $x+1$ have the same parity, from (2), we see that both must be even. Let

$$x - 1 = 2k \text{ for some integer } k \geq 1.$$

Then, from (2),

$$8n = 2k(2k+2) \quad \Rightarrow \quad n = \frac{k(k+1)}{2}.$$

Hence, the lemma. ■

We now look at some equations and inequalities involving both S(n) and Z(n).

Lemma 4.4.24 **:** The solution of the equation $S(n) = Z(n)$ is

$$n = tp,$$

where t is such that $t \mid (p-1)!$ and $Z(tp) = p$.

Proof **:** Let, for some integer $n \geq 1$,

$$S(n) = Z(n) = m \text{ for some integer } m \geq 1.$$

Furthermore, let

$$n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_j^{\alpha_j} q_{j+1}^{\alpha_{j+1}} \ldots q_k^{\alpha_k}$$

be the representation of n in terms of its prime factors $p_1, p_2, \ldots, p_k$ (not necessarily ordered).

Now, m cannot be even, for otherwise, (since $n \mid \frac{m(m+1)}{2}$) for some $1 \leq j \leq k$,

$$q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_j^{\alpha_j} \,\Big|\, \frac{m}{2} \quad \text{and} \quad q_{j+1}^{\alpha_{j+1}} q_{j+2}^{\alpha_{j+2}} \ldots q_k^{\alpha_k} \,\Big|\, (m+1)\,.$$

Again, since $n \mid m!$, it follows that

$$q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_j^{\alpha_j} \,\Big|\, \left(\frac{m}{2}\right)! \quad \text{and} \quad q_{j+1}^{\alpha_{j+1}} q_{j+2}^{\alpha_{j+2}} \ldots q_k^{\alpha_k} \,\Big|\, (m-1)!\,.$$

But then

$$n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_j^{\alpha_j} q_{j+1}^{\alpha_{j+1}} \ldots q_k^{\alpha_k} \,\Big|\, (m-1)! \quad \Rightarrow \quad S(n) \leq m-1,$$

and we reach to a contradiction. Thus, m must be odd, and

$$q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_j^{\alpha_j} \mid m,$$

$$q_{j+1}^{\alpha_{j+1}} q_{j+2}^{\alpha_{j+2}} \ldots q_k^{\alpha_k} \mid \frac{m+1}{2} \quad \text{and} \quad q_{j+1}^{\alpha_{j+1}} q_{j+2}^{\alpha_{j+2}} \ldots q_k^{\alpha_k} \mid (m-1)!\,.$$

Now, we show that m must be a prime. The proof is by contradiction. So, let

$$m = p^{\beta} p_1^{\beta_1} p_2^{\beta_2} \ldots p_r^{\beta_r}$$

with

$$p^{\beta} = \max\left\{ p^{\beta},\ p_1^{\beta_1},\ \ldots,\ p_r^{\beta_r} \right\}.$$

But then, for some $1 \le j \le k$,

$$q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_j^{\alpha_j} \mid \left(p^{\beta}\right)! \quad \text{and} \quad q_{j+1}^{\alpha_{j+1}} q_{j+2}^{\alpha_{j+2}} \ldots q_k^{\alpha_k} \mid (m-1)!$$

$$\Rightarrow \quad n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_j^{\alpha_j} q_{j+1}^{\alpha_{j+1}} \ldots q_k^{\alpha_k} \mid (m-1)!,$$

a contradiction. Again, if $m = p^{\beta}$ with $\beta \ge 2$, then n does not divide $(m-1)!$, but

$$n \mid (p^{\beta})! = 1.2 \ldots (2p) \ldots p^2 \ldots (2p^2) \ldots (p^{\beta-1}) \ldots (2p^{\beta-1}) \ldots p^{\beta},$$

so that the power of p in n is more than $\beta(\beta-1)$. But then, n does not divide $\frac{p^{\beta}(p^{\beta}+1)}{2}$.

Thus, m must be a prime, say, $m = p$. Therefore, the problem of finding the solution of $S(n) = Z(n)$ reduces to the problem of finding the solution to the following equations :

$$S(n) = p,\ Z(n) = p.$$

The lemma now follows by virtue of Lemma 3.3.4 (in Chapter 3). ■

Example 4.4.4 **:** Let p be such that $(p+1)$ is divisible by 4. Then,

$$Z(2p) = p = S(2p).$$

Hence, $n = 2p$, where p is a prime such that $4 \mid (p+1)$, is a solution of $S(n) = Z(n)$.
This shows that the equation $S(n) = Z(n)$ possesses an infinite number of solutions. ♦

Example 4.4.5 **:** Let $p \ge 5$ be a prime such that $3 \mid (p+1)$. Then, $S(3p) = p = Z(3p)$. ♦

Example 4.4.6 **:** An even perfect number $n = 2^k(2^{k+1}-1)$ is a solution of $S(n) = Z(n)$, since, by Lemma 4.2.1 and Remark 3.3.2 (in Chapter 3),

$$Z(2^k p) = Z\left(\frac{2^{k+1}(2^{k+1}-1)}{2}\right) = 2^{k+1} - 1 = S(2^k p). \ ♦$$

Since $Z(p) = p-1 < p = S(p)$, it follows that $Z(n) < S(n)$ infinitely often. Again, since $Z(2^k) = 2^{k+1} - 1 > 2^k \ge S(2^k)$, $Z(n) > S(n)$ infinitely often.

***Lemma 4.4.25* :** The equation $Z(S(n)) = Z(n)$ has an infinite number of solutions.

***Proof* :** Clearly, any solution of $S(n) = n$ is a solution of $Z(S(n)) = Z(n)$. Besides the trivial solution $n = p$ (where $p \ge 3$ is a prime), another set of solution is

$$n = \frac{(p-1)p}{2}, \ p \ge 3 \text{ is prime,}$$

since, in such a case, $\frac{p-1}{2} < p$, and by Lemma 4.2.1,

$$Z(\frac{(p-1)p}{2}) = p-1, \ S(\frac{(p-1)p}{2}) = p,$$

Hence, for such an n,

$$Z(S(n)) = Z(S(\frac{(p-1)p}{2})) = Z(p) = p-1 = S(\frac{(p-1)p}{2}) = S(n). \ \blacksquare$$

***Lemma 4.4.26* :** The equation $S(Z(n)) = S(n)$ has an infinite number of solutions.

***Proof* :** Let $n = 2p$, where $p \ge 3$ is a prime such that 4 divides $(p+1)$. Then,

$$S(Z(n)) = S(Z(2p)) = S(p) = p = S(2p) = S(n). \ \blacksquare$$

***Lemma 4.4.27* :** The equation $S(Z(n)) = Z(n)$ has an infinite number of solutions.

***Proof* :** Let $n = 2p$, where $p \ge 3$ is a prime such that 4 divides $(p+1)$. Then,

$$S(Z(n)) = S(p) = p = Z(2p) = Z(n). \ \blacksquare$$

***Lemma 4.4.28* :** The equation $Z(S(n)) = S(n)$ has no solution.

***Proof* :** follows from Lemma 4.4.1. ■

The following two results are due to Maohua Le [15].

***Lemma 4.4.29* :** There exists infinitely many n satisfying the inequality $S(Z(n)) > Z(S(n))$.

***Proof* :** Let $n = 2p$, where $p \ge 3$ is a prime such that 4 divides $(p+1)$. Then,

$$S(Z(n)) = S(p) = p > p-1 = Z(p) = Z(S(p)). \ \blacksquare$$

***Lemma 4.4.30* :** There exists infinitely many n satisfying the inequality $S(Z(n)) < Z(S(n))$.

***Proof* :** Let $n = p$, where $p \ge 3$ is a prime. For such an n,

$$S(Z(n)) = S(Z(p)) = S(p-1) < p-1 = Z(p) = Z(S(p)). \ \blacksquare$$

Ashbacher [3] conjectures that, there is an infinite number of integers n such that $Z(n)S(n)$ $(Z(n) \ne S(n))$ is a perfect square. The following lemma confirms the conjecture.

***Lemma 4.4.31* :** Let p be a prime of the form

$$p = 8a + 3 \tag{1}$$

for some integer $a \ge 1$, such that $3a + 1 = x^2$ is a perfect square for some integer $x \ge 2$.
Let q be another prime of the form

$$q = kp + 8 \text{ for some integer } k \ge 1. \tag{2}$$

Then, $Z(pq)S(pq)$ is a perfect square.

Proof **:** Let p and q be two primes of the forms (1) and (2) respectively. Then, by Corollary 4.2.17, $Z(pq) = qx^2$. Now, since $S(pq) = q$, it follows that

$$Z(pq)S(pq) = (qx^2).(q) = (qx)^2. \blacksquare$$

Corollary 4.4.6 **:** There exists an infinite number of n such that $Z(n) \neq S(n)$ but $Z(n)S(n)$ is a perfect square.

Proof **:** Let q be a prime of the form

$$q = 11a + 8 \text{ for some integer } a \geq 1. \qquad (*)$$

Then, by Corollary 4.2.17, $Z(11q) = 4q$, so that, $Z(11q)S(11q) = (4q).(q) = (2q)^2$.

Now, since there is an infinite number of primes of the form (*), the result is proved. ■

Example 4.4.7 **:** Using Corollary 4.4.6, we get successively the primes

$$q = 19, 41, 107, 151, 173, 239, 283, 349, \ldots,$$

with

$$Z(11\times19)\,S(11\times19) = (2\times19)^2, \quad Z(11\times41)\,S(11\times41) = (2\times41)^2,$$
$$Z(11\times107)\,S(11\times107) = (2\times107)^2, \quad Z(11\times151)\,S(11\times151) = (2\times151)^2,$$
$$Z(11\times173)\,S(11\times173) = (2\times173)^2, \quad Z(11\times239)\,S(11\times239) = (2\times239)^2,$$
$$Z(11\times283)\,S(11\times283) = (2\times283)^2, \quad Z(11\times349)\,S(11\times349) = (2\times349)^2. \blacklozenge$$

Lemma 4.4.32 **:** $\liminf\limits_{n\to\infty} \dfrac{n}{Z(n)} = 0.$

Proof **:** Let p and q be two primes with $q = kp - 1$ for some integer $k \geq 1$.

Let $n = pq$. Then, by Corollary 4.2.4,

$$\frac{n}{Z(n)} = \frac{pq}{Z(pq)} = \frac{pq}{q} = \frac{1}{p},$$

which can be made arbitrarily small for infinitely many primes q. ■

The following result is due to Ashbacher [3]. Here, we give a simplified proof.

Lemma 4.4.33 **:** Given the successive values of the pseudo Smarandache function

$$Z(1) = 1, Z(2) = 3, Z(3) = 2, Z(4) = 7, Z(5) = 4, \ldots,$$

the number r is constructed by concatenating the values in the following way :

$$r = 0.13274\ldots.$$

Then, the number r is irrational.

Proof **:** The proof is by contradiction. Let, on the contrary, r be rational. Then, from some point on, a group of consecutive digits must repeat infinite number of times. Let $N = t_1 t_2 \ldots t_i$ in the recurring group correspond to n (so that $Z(n) = N$). But, by Lemma 4.2.1,

$$Z(k) \neq N \text{ for any integer } k > \frac{N(N+1)}{2},$$

so that the group of integers N cannot repeat infinitely often. ■

4.5 Some Observations and Remarks

Some observations about the pseudo Smarandache function are given below :

***Remark 4.5.1* :** The following questions are raised by Kashihara [1]) :

(1) Is there any integer n such that $Z(n) > Z(n+1) > Z(n+2) > Z(n+3)$?

(2) Is there any integer n such that $Z(n) < Z(n+1) < Z(n+2) < Z(n+3)$?

The following examples answer the questions in the affirmative :

(1) $Z(256) = 511 > 256 = Z(257) > Z(258) = 128 > 111 = Z(259) > Z(260) = 39$,

(2) $Z(159) = 53 < 64 = Z(160) < Z(161) = 69 < 80 = Z(162) < Z(163) = 162$.

Ashbacher [3] provides lists of seven consecutive increasing and decreasing terms of the sequence $\{Z(n)\}$, with $n = 18886$ in the first case and $n = 7561$ in the second case.

***Remark 4.5.2* :** The proof of Lemma 4.4.2 shows that $Z(n+1) - Z(n)$ is both unbounded above and unbounded below. The minimum value of $|Z(n+1) - Z(n)|$ is 1, and looking at the values of Z(n), it seems safe to conjecture that $|Z(n+1) - Z(n)| = 1$ if and only if $n = 2, 5$.

***Remark 4.5.3* :** In [3], Ashbacher raises the question : Is there a value of k where there are only finite number of solutions to the equation $Z(kn) = n$? Lemma 4.4.4 answers to this question in the negative.

***Remark 4.5.4* :** Lemma 4.4.19 shows that the equation $Z(n) = \phi(n)$ has an infinite number of solutions, namely, $n = p$, where $p \geq 3$ is a prime. A second set of solution, provided by Ashbacher [3], is

$$n = 2p, \text{ where p is a prime such that 4 divides } p - 1.$$

Then, Ashbacher poses the question : Are there any other infinite family of solutions to the equation $Z(n) = \phi(n)$?

To answer the above question, we first note that, by Lemma 4.2.4,

$$Z(kp) = p - 1 \text{ for any prime p such that 2k divides } p - 1.$$

Now, since (by virtue of the multiplicative property of $\phi(n)$)

$$\phi(kp) = \phi(k)\,\phi(p),$$

it follows that

$$Z(kp) = \phi(kp) \text{ if and only if } \phi(k) = 1.$$

But

$$\phi(k) = 1 \text{ if and only if } k = 1, 2.$$

Thus, for any solution of the equation $Z(n) = \phi(n)$ of the form $n = kp$, we must have $k = 1, 2$.

It may be mentioned that, by Corollary 4.2.1(1), for any prime p such that 4 divides $p - 1$,

$$Z(2p) = p - 1,$$

so that, by Theorem 0.2 (in Chapter 0),

$$\phi(2p) = 2p\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{p}\right) = p - 1 = Z(2p).$$

Remark 4.5.5 **:** From Lemma 4.4.6, we see that $\frac{Z(2n)}{Z(n)}$ can be made arbitrarily large. It can be shown that, the equation $\frac{Z(2n)}{Z(n)} = 1$ has infinite number of solutions. The proof is as follows : Let p be a prime of the form $p = 4a + 1$ for some integer $a \geq 1$. Then, choosing $n = p^k$, where $k \geq 1$ is any integer, by Lemma 4.2.3 and Lemma 4.2.14,

$$\frac{Z(2n)}{Z(n)} = \frac{Z(2p^k)}{Z(p^k)} = \frac{p^k - 1}{p^k - 1} = 1.$$

Thus, for $p = 5$, we get $n = 5^2, 5^3, 5^4$, with

$$\frac{Z(2.5^2)}{Z(5^2)} = \frac{24}{24} = 1, \quad \frac{Z(2.5^3)}{Z(5^3)} = \frac{124}{124} = 1, \quad \frac{Z(2.5^4)}{Z(5^4)} = \frac{624}{624} = 1,$$

while, corresponding to $p = 13$, we get $n = 13^2, 13^3$, with

$$\frac{Z(2.13^2)}{Z(13^2)} = \frac{168}{168} = 1, \quad \frac{Z(2.13^3)}{Z(13^3)} = \frac{2196}{2196} = 1.$$

Again, keeping $k = 1$, we get $n = 5^2, 13^2, 17^2, 29^2, 37^2, 41^2$, with

$$\frac{Z(2.17^2)}{Z(17^2)} = \frac{288}{288} = 1, \quad \frac{Z(2.29^2)}{Z(29^2)} = \frac{840}{840} = 1, \quad \frac{Z(2.37^2)}{Z(37^2)} = \frac{1368}{1368} = 1, \quad \frac{Z(2.41^2)}{Z(41^2)} = \frac{1680}{1680} = 1.$$

Another infinite family of solutions to $Z(2n) = Z(n)$ can be constructed as follows : Let p be a prime of the form $p = 4a - 1$ for some integer $a \geq 1$. Let $n = p^k$, where $k \geq 2$ is any even integer. Then, by Lemma 4.2.3 and Lemma 4.2.14,

$$\frac{Z(2n)}{Z(n)} = \frac{Z(2p^k)}{Z(p^k)} = \frac{p^k - 1}{p^k - 1} = 1.$$

For example, for $p = 3$, we have

$$\frac{Z(2.3^2)}{Z(3^2)} = \frac{8}{8} = 1, \quad \frac{Z(2.3^4)}{Z(3^4)} = \frac{80}{80} = 1, \quad \frac{Z(2.3^6)}{Z(3^6)} = \frac{728}{728} = 1,$$

and for $p = 7$, we get

$$\frac{Z(2.7^2)}{Z(7^2)} = \frac{48}{48} = 1, \quad \frac{Z(2.7^4)}{Z(7^4)} = \frac{2400}{2400} = 1.$$

Remark 4.5.6 **:** The equation $\frac{Z(3n)}{Z(n)} = 1$ has infinite number of solutions. To prove this, let $n = p^2$, where $p \geq 3$ is a prime. Then, by Lemma 4.2.3 and Corollary 4.2.2(2),

$$\frac{Z(3n)}{Z(n)} = \frac{Z(3p^2)}{Z(p^2)} = \frac{p^2 - 1}{p^2 - 1} = 1.$$

We now concentrate on some conjectures related to Z(n).

Conjecture 4.5.1 **:** $Z(n) = 2n - 1$ if and only if $n = 2^k$ for some integer $k \geq 0$.

Let, for some integer $n \geq 1$,

$$Z(n) = m_o, \text{ where } m_o = 2n - 1 \text{ (so that } n \mid \frac{m_0(m_0+1)}{2}).$$

Note that the conjecture is true for $n = 1$ (with $k = 0$). Also, note that n must be composite. Let

$$Z(2n) = m_1.$$

We want to show that $m_1 = 2m_o + 1$.

Since $n \mid \frac{m_0+1}{2}$, it follows that

$$2n \mid \frac{2(m_0+1)}{2} = \frac{(2m_0+1)+1}{2};$$

moreover,

$$2n \text{ does not divide } \frac{2(m+1)}{2} = \frac{(2m+1)+1}{2} \quad \text{for all } 1 \leq m \leq m_o - 1.$$

Thus,

$$m_1 = 2m_o + 1 = 2(2n-1) + 1 = 2^2 n - 1.$$

All these show that

$$Z(n) = 2n - 1 \quad \Rightarrow \quad Z(2n) = 2^2 n - 1.$$

Continuing this argument, we see that

$$Z(n) = 2n - 1 \quad \Rightarrow \quad Z(2^k n) = 2^{k+1} n - 1 \text{ for any integer } k \geq 1.$$

Since $Z(1) = 1$, it then follows that $Z(2^k) = 2^{k+1} - 1$.

Conjecture 4.5.2 **:** If n is not of the form 2^k for some integer $k \geq 0$, then $Z(n) < n$.

By Lemma 4.4.1, we can exclude the possibility that $Z(n) = n$. So, let

$$Z(n) = m_o \text{ with } m_o > n.$$

Note that, n must be a composite number, not of the form p^k ($p \geq 3$ is prime, $k \geq 0$). Let

$$m_o = an + b \text{ for some integers } a \geq 1,\ 1 \leq b \leq n - 1.$$

Then,

$$m_o(m_o + 1) = (an + b)(an + b + 1) = n(a^2 n + 2ab + a) + b(b + 1).$$

Therefore,

$$n \mid m_o(m_o + 1) \text{ if and only if } b + 1 = n.$$

But, by Conjecture 4.5.1, $b + 1 = n$ leads to the case when n is of the form 2^k. Thus, the assertion is proved.

***Conjecture 4.5.3* :** $Z(n) = n - 1$ if and only if $n = p^k$ for some prime $p \geq 3$, $k \geq 1$.

Let, for some integer $n \geq 2$,

$$Z(n) = m_o, \text{ where } m_o = n - 1.$$

Then, $2 \mid m_o$ and $n \mid (m_o + 1)$; moreover, n does not divide $m + 1$ for any $1 \leq m \leq m_o - 1$.
Let

$$Z(n^2) = m_1.$$

Since $n \mid (m_o + 1)$, it follows that

$$n^2 \mid (m_o + 1)^2 = (m_o^2 + 2m_o) + 1;$$

moreover,

$$n^2 \text{ does not divide } (m + 1)^2 = (m^2 + 2m) + 1 \text{ for all } 1 \leq m \leq m_o - 1.$$

Thus,

$$m_1 = m_o^2 + 2m_o = (n - 1)^2 + 2(n - 1) = n^2 - 1,$$

so that (since $2 \mid m_o \;\; \Rightarrow \;\; 2 \mid m_1$)

$$Z(n) = n - 1 \quad \Rightarrow \quad Z(n^2) = n^2 - 1.$$

Continuing this argument, we see that

$$Z(n) = n - 1 \quad \Rightarrow \quad Z(n^{2k}) = n^{2k} - 1 \text{ for any integer } k \geq 1.$$

Next, let

$$Z(n^{2k+1}) = m_2 \text{ for some integer } k \geq 1.$$

Since $n \mid (m_o + 1)$, it follows that

$$n^{2k+1} \mid (m_o + 1)^{2k+1} = [(m_o + 1)^{2k+1} - 1] + 1;$$

moreover,

$$n^{2k+1} \text{ does not divide } (m + 1)^{2k+1} = [(m + 1)^{2k+1} - 1] + 1 \text{ for any } 1 \leq m \leq m_o - 1.$$

Thus,

$$m_2 = (m_o + 1)^{2k+1} - 1 = n^{2k+1} - 1,$$

so that (since $2 \mid m_o \;\; \Rightarrow \;\; 2 \mid m_2$)

$$Z(n) = n - 1 \quad \Rightarrow \quad Z(n^{2k+1}) = n^{2k+1} - 1 \text{ for any integer } k \geq 1.$$

All these show that

$$Z(n) = n - 1 \quad \Rightarrow \quad Z(n^k) = n^k - 1 \text{ for any integer } k \geq 1.$$

Finally, since

$$Z(p) = p - 1 \text{ for any prime } p \geq 3,$$

it follows that

$$Z(p^k) = p^k - 1.$$

***Remark 4.5.7* :** Kashihara [1] proposes to find all the values of n such that $Z(n + 1) = Z(n)$.
In this respect, we make the following conjecture :

Conjecture 4.5.4 **:** For any integer $n \geq 1$, $Z(n+1) \neq Z(n)$.

Let

$$Z(n+1) = Z(n) = m_o \text{ for some } n \in Z^+,\ m_o \geq 1.$$

Then,

$$n \mid \frac{m_0(m_0+1)}{2},\ (n+1) \mid \frac{m_0(m_0+1)}{2} \ \Rightarrow\ n(n+1) \mid \frac{m_0(m_0+1)}{2},$$

since $(n, n+1) = 1$. Therefore,

$$Z(n(n+1)) \leq m_o \quad \Rightarrow \quad Z(n(n+1)) = m_0,$$

since, by Lemma 4.5, $Z(n(n+1)) \geq Z(n) \geq m_0$. Therefore,

$$n(n+1) \mid \frac{m_0(m_0+1)}{2} \ \Rightarrow\ \frac{n(n+1)}{2} \mid \frac{m_0(m_0+1)}{2} \ \Rightarrow\ Z(\frac{n(n+1)}{2}) \leq m_o.$$

Thus, by virtue of Lemma 4.2.1,

$$Z(\frac{n(n+1)}{2}) = n \leq m_o = Z(n).$$

But, the above inequality contradicts Conjecture 4.5.2.

Remark 4.5.8 **:** Kashihara [1] raises the following question : Given any integer $m_o \geq 1$, how many n are there such that $Z(n) = m_o$?

Given any integer $m_o \geq 1$, let

$$Z^{-1}(m_o) = \{n : n \in Z^+,\ Z(n) = m_o\}. \tag{4.5.1}$$

By Lemma 4.2.1, $Z^{-1}(m_o)$ is defined for all integers $m_0 \in Z^+$; moreover,

$$n_{max} \equiv \frac{m_0(m_0+1)}{2} \in Z^{-1}(m_o).$$

This shows that the set $Z^{-1}(m_o)$ is non-empty with n_{max} its largest element.

Note that, $n \in Z^{-1}(m_o)$ if and only if the following two conditions are satisfied

(1) n divides $\frac{m_0(m_0+1)}{2}$,

(2) n does not divide $\frac{m(m+1)}{2}$ for any m with $3 \leq m \leq m_o - 1$.

Thus, for examples,

$$Z^{-1}(1) = \{1\},\ Z^{-1}(2) = \{3\},\ Z^{-1}(8) = \{8, 12, 18, 36\}.$$

We can look at (4.5.1) from another point of view : On the set Z^+, we define the relation $\mathfrak{R}$ as follows :

$$\text{For any } n_1, n_2 \in Z^+,\ n_1 \mathfrak{R} n_2 \text{ if and only if } Z(n_1) = Z(n_2). \tag{4.5.2}$$

It is an easy exercise to verify that $\mathfrak{R}$ is an equivalence relation on Z^+. The sets $Z^{-1}(m)$, $m \in Z^+$, are then the equivalence classes induced by the equivalence relation $\mathfrak{R}$ on Z^+ with the following two properties :

(1) $\sum_{m=1}^{\infty} Z^{-1}(m) = Z^+$,

(2) $Z^{-1}(m_1) \cap Z^{-1}(m_2) = \varnothing$, if $m_1 \neq m_2$.

Thus, for any $n \in Z^+$, there is one and only one $m \in Z^+$ such that $n \in Z^{-1}(m)$.

Given any integer $m_o \geq 1$, let $C(m_o)$ be the number of integers n such that $Z(n) = m_o$, that is, $C(m_o)$ denotes the number of elements of $Z^{-1}(m_o)$. Then, for $m_o \geq 1$,

$$1 \leq C(m_0) \leq \begin{cases} d(m_0)\left[d(\frac{m_0+1)}{2} - 1\right], \text{ if } m_0 \text{ is odd} \\ d(\frac{m_0}{2})[d(m_0+1)-1], \text{ if } m_0 \text{ is even} \end{cases}$$

where, for any integer n, d(n) denotes the number of divisors of n including 1 and n. Note, however, that the above upper bound is not tight.

Now, let $p \geq 3$ be a prime. By Lemma 4.2.3, $Z(p) = p - 1$, so that

$$p \in Z^{-1}(p-1) \text{ for all } p \geq 3.$$

Let $n \in Z^{-1}(p-1)$, so that, n divides $\frac{p(p-1)}{2}$. Then, n must divide p, for otherwise

$$n \mid \frac{p-1}{2} \quad \Rightarrow \quad n \mid \frac{(p-1)(p-2)}{2} \quad \Rightarrow \quad Z(n) \leq p-2,$$

contradicting the assumption. Thus, any element of $Z^{-1}(p-1)$ is a multiple of p. In particular, p is the minimum element of $Z^{-1}(p-1)$. Thus, if $p \geq 5$ is a prime, then $Z^{-1}(p-1)$ contains at least two elements, namely, p and $\frac{p(p-1)}{2}$.

Next, let p be a prime factor of $\frac{m_0(m_0+1)}{2}$. Since $Z(p) = p - 1$, we see that

$$p \in Z^{-1}(m_o) \text{ if and only if } p - 1 \geq m_o,$$

that is, if and only if $p \geq m_o + 1$.

In Lemma 4.4.3, we actually proved that the equation $\frac{Z(n)}{Z(n+1)} = 2$ has solutions. However, the number of solutions is rather scarce. A limited search upto $n \leq 5000$ revealed only 21 solutions :

n = 73, 193, 673, 1153, 1201, 1321, 1657, 1753, 2017, 2137, 2401,
2473, 2593, 3217, 3313, 3361, 3481, 3697, 3721, 4057, 4177.

In this connection, the following questions may be posed :

Question 4.5.1 **:** Does the equation $\frac{Z(n)}{Z(n+1)} = k \ (k \geq 2)$

(1) have an infinite number of solutions when $k = 2$? **(2)** have solutions for $k \neq 2$?

Lemma 4.4.5 shows that the equation $\frac{n}{Z(n)} = k \ (k \geq 2)$ has an infinite number of solutions. When $k = 2$, we can find more solutions. For example, letting

$n = 2p^a$, where p is a prime such that 4 divides $(p+1)$, a is an odd integer,

we have, by virtue of Lemma 4.2.14,

$$\frac{n}{Z(n)} = \frac{2p^a}{Z(2p^a)} = \frac{2p^a}{p^a} = 2.$$

Again,

$n = 3p^a$, where p is a prime such that 3 divides $(p+1)$, a is an odd integer,

is a solution corresponding to $k = 3$, since by Lemma 4.2.15,

$$\frac{n}{Z(n)} = \frac{3p^a}{Z(3p^a)} = \frac{3p^a}{p^a} = 3.$$

This suggests the following question :

Question 4.5.2 **:** Does the equation $\frac{n}{Z(n)} = k \ (k \geq 2)$ have other solutions? Does it have solutions for $0 < k < 1$.

The following question is related to Lemma 4.4.8.

Question 4.5.3 **:** Does the equation $\frac{Z(n+1)}{Z(n)} = k$

(1) have solutions when $k \geq 3$ is odd?

(2) have an infinite number of solutions when k is even? when k is odd?

The following question has been raised by Ashbacher [3] :

Question 4.5.4 **:** What is the minimum value of s_0 such that the series $\sum_{n=1}^{\infty} \frac{1}{[S(n)+Z(n)]^s}$ is convergent for all $s > s_0$?

Ashbacher [3] has proved that the series $\sum_{n=1}^{\infty} \frac{1}{S(n)+Z(n)}$ is divergent. The proof is simple : Let p_n be the n–th prime. Then,

$$\sum_{n=1}^{\infty} \frac{1}{S(n)+Z(n)} > \sum_{n=1}^{\infty} \frac{1}{S(p_n)+Z(p_n)} = \sum_{n=1}^{\infty} \frac{1}{p_n + (p_n - 1)} > \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{1}{p_n} > \infty.$$

We close this chapter with the following questions and problems, old but still open.

Question 4.5.5 **:** Are there solutions to the following equations?

(1) $Z(n+2)=Z(n+1)+Z(n)$?

(2) $Z(n)=Z(n+1)+Z(n+2)$?

(3) $Z(n+2)=Z(n+1)Z(n)$?

(4) $Z(n)=Z(n+1)Z(n+2)$?

(5) $Z(n+2)+Z(n)=2Z(n+1)$?

(6) $Z(n+2)Z(n)=[Z(n+1)]^2$?

Question 4.5.6 **:** Are there infinitely many instances of 3 consecutive increasing or decreasing terms in the sequence $\{Z(n)\}_{n=1}^{\infty}$?

Problem 4.5.1 **:** Find all solutions of the equation $Z(n)+1=S(n)$.

Note that, for any prime $p \geq 3$,

$$Z(p)=p-1,\ S(p)=p,$$

so that $n=p$ is a solution of the equation $Z(n)+1=S(n)$. So, the question is : Are there other solutions to the equation $Z(n)+1=S(n)$?

Question 4.5.7 **:** Given a palindromic number n, what is the largest value of k such that $Z^r(n)$ are palindromic for all $1 \leq r \leq k$, where $Z^r(n)$ is the r–fold composition of Z(n) with itself?

Recall that, a number n is called palindromic if it reads the same forwards and backwards. Given a palindromic prime, it is possible, in some cases to find palindromic number n such that Z(n) is also a palindrome, using Lemma 4.2.4. Thus, starting with the palindromic prime 101, we can get $n=303, 404, 606, 707, 808, 909, 1111$ such that Z(n) is a palindrome in each case :

$$Z(303)=101,\ Z(404)=303=Z(606)=Z(808),$$
$$Z(707)=202,\ Z(909)=404,\ Z(1111)=505.$$

As has been pointed out by Ashbacher [3],

$$Z^3(909)=Z^2(404)=Z(303)=101.$$

The prime 131 gives rise to three palindromes : $Z(262)=131=Z(1441)=Z(2882)$.

Clearly, the palindromic numbers formed in this way are also palindromic with respect to the Smarandache function S(n) as well, since, for example,

$$S(101n)=101 \text{ for any } 1 \leq n \leq 9.$$

Given below are other examples where both n and Z(n) are palindromes :

$$Z(444)=111=Z(777)=Z(888) \text{ (but 111 is not prime).}$$

$$Z(1001)=77=Z(3003),\ Z(2002)=363 \text{ (but 1001 is not prime).}$$

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Chapter 5 Smarandache Number Related Triangles

This chapter is devoted to Smarandache function related and pseudo Smarandache function related triangles, defined formally below. The concept of the Smarandache function related triangles was first introduced by Sastry [1]. Later, it was extended by Ashbacher [2] to include the pseudo Smarandache function related triangles as well.

Let ΔABC be a triangle with sides $a = BC$, $b = AC$, $c = AB$. Following Sastry [1], we denote by $T(a, b, c)$ the triangle ΔABC. Let $T(a', b', c')$ be a second triangle with sides of lengths a', b', c'. Recall that two triangles $T(a, b, c)$ and $T(a', b', c')$ are similar if and only if their three sides are proportional (in any order). Thus, for example, if

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} \quad \text{(or if } \frac{a}{c'} = \frac{b}{a'} = \frac{c}{b'}\text{)},$$

then the two triangles $T(a, b, c)$ and $T(a', b', c')$ are similar.

The following two definitions are due to Sastry [1] and Ashbacher [2] respectively.

Definition 5.1 **:** Two triangles $T(a, b, c)$ and $T(a', b', c')$ (where a, b, c and a', b', c' are all positive integers) are said to be Smarandache function related (or, S−related) if

$$S(a) = S(a'),\ S(b) = S(b'),\ S(c) = S(c'),$$

(where S(.) is the Smarandache function, defined in Chapter 3).

Definition 5.2 **:** Two triangles $T(a, b, c)$ and $T(a', b', c')$ are said to be pseudo Smarandache function related (or, Z−related) if

$$Z(a) = Z(a'),\ Z(b) = Z(b'),\ Z(c) = Z(c'),$$

(where Z(.) is the pseudo Smarandache function, treated in Chapter 4).

A different way of relating two triangles has been proposed by Sastry [1] : The triangle ΔABC, with angles α, β and γ (α, β and γ being positive integers), can be denoted by $T(\alpha, \beta, \gamma)$. Then, we have the following definition, due to Sastry [1] and Ashbacher [2].

Definition 5.3 **:** Given two triangles, $T(\alpha, \beta, \gamma)$ and $T(\alpha', \beta', \gamma')$, with

$$\alpha + \beta + \gamma = 180 = \alpha' + \beta' + \gamma', \tag{5.1}$$

(1) they are said to be Smarandache function related (or, S−related) if

$$S(\alpha) = S(\alpha'),\ S(\beta) = S(\beta'),\ S(\gamma) = S(\gamma');$$

(2) they are said to be pseudo Smarandache function related (or, Z−related) if

$$Z(\alpha) = Z(\alpha'),\ Z(\beta) = Z(\beta'),\ Z(\gamma) = Z(\gamma').$$

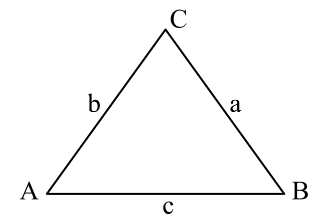
In Definition 5.1 − Definition 5.2, the sides of the pair of triangles are S−related/Z−related, while their angles, measured in degrees, are S−related/Z−related in Definition 5.3. Note that the condition (5.1) merely states the fact the sum of the three angles of a triangle is 180 degrees.

§5.1 gives some preliminary results, which are used in §5.2 and §5.3 for the 60 degrees and 120 degrees triangles respectively. The Pythagorean case is considered in §5.4. §5.5 treats the case given in Definition 5.3. Some remarks are given in the final §5.6.

5.1 Some Preliminary Results

Let $T(a, b, c)$ be a triangle with sides a, b and c, and angles $\angle A$, $\angle B$ and $\angle C$, as shown in the figure. Then,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad (5.1.1)$$



where

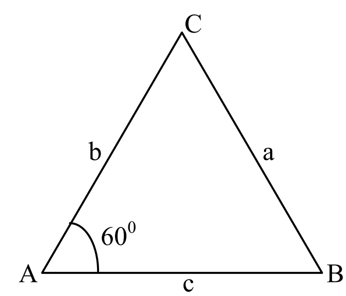
$$\begin{aligned} \sin C &= \sin(180 - (A + B)) \\ &= \sin(A + B) \\ &= \sin A \cos B + \cos A \sin B. \end{aligned}$$

Lemma 5.1.1 **:** Let $T(a, b, c)$ be a triangle with sides a, b and c, whose $\angle A = 60^0$ (as shown in the figure). Then,

$$4a^2 = (2c - b)^2 + 3b^2. \qquad (5.1.2)$$

Proof **:** If $\angle A = 60^0$, then

$$\sin C = \frac{1}{2}(\sqrt{3}\cos B + \sin B).$$



Therefore, (5.1.1) reads as

$$\frac{2a}{\sqrt{3}} = \frac{b}{\sin B} = \frac{2c}{\sqrt{3}\cos B + \sin B}. \qquad (1)$$

From the RHS of (1),

$$\sqrt{3}\, b \cos B = (2c - b)\sin B.$$

Squaring both sides, we get

$$3b^2(1 - \sin^2 B) = (2c - b)^2 \sin^2 B,$$

that is,

$$[(2c - b)^2 + 3b^2]\sin^2 B = 3b^2. \qquad (2)$$

Again, from the LHS of (1),

$$\sin B = \frac{\sqrt{3}b}{2a}. \qquad (3)$$

Now, eliminating sinB from (2) and (3), we finally get,

$$\frac{3b^2}{(2c - b)^2 + 3b^2} = \left(\frac{\sqrt{3}b}{2a}\right)^2,$$

which gives the desired result after simplifications. ■

Lemma 5.1.2 **:** Let $T(a, b, c)$ be a triangle with sides a, b and c, whose $\angle A = 120^0$ (as shown in the figure). Then,

$$4a^2 = (2c + b)^2 + 3b^2. \qquad (5.1.3)$$

Proof **:** If $\angle A = 120^0$, then

$$\sin C = \frac{1}{2}(\sqrt{3}\cos B - \sin B).$$

Therefore, (5.1.1) takes the form

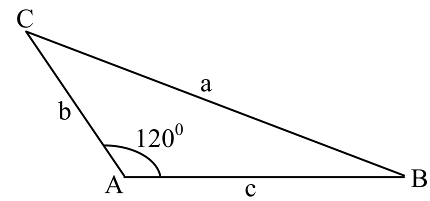
$$\frac{2a}{\sqrt{3}} = \frac{b}{\sin B} = \frac{2c}{\sqrt{3}\cos B - \sin B}. \qquad (i)$$

The LHS of (i) gives

$$\sin B = \frac{\sqrt{3}b}{2a}, \qquad \text{(ii)}$$

while the RHS of (i) gives

$$\frac{2a}{\sqrt{3}} = \frac{b}{\sin B} = \frac{2c}{\sqrt{3}\cos B - \sin B}. \qquad \text{(iii)}$$



Now, eliminating sinB from (iii), using (ii), we get the desired result. ■

Note that, when $\angle A = 60^0$, then either $b \le a \le c$ or else, $c \le a \le b$ (that is, a is in between the smallest and the largest sides of the triangle $T(a, b, c)$); and if $\angle A = 120^0$, then a is the largest side of the triangle. Also, note that, for our problem, a, b, and c are positive integer-valued.

Lemma 5.1.3 **:** If (a_0, b_0, c_0) is a non–trivial solution of the Diophantine equation

$$4a^2 = (2c - b)^2 + 3b^2, \qquad (5.1.2)$$

then one of b_0 and c_0 is greater than a_0 and the other one is less than a_0.

Proof **:** We write (5.1.2) in the equivalent form

$$a^2 = b^2 - bc + c^2. \qquad (5.1.2a)$$

Let (a_0, b_0, c_0) be a solution of (5.1.2), so that

$$a_0^2 = b_0^2 - b_0 c_0 + c_0^2. \qquad (1)$$

We now consider the following two cases :

Case (1) : Let $b_0 > a_0$, $c_0 > a_0$.

Let

$$b_0 = a_0 + m, \ c_0 = a_0 + n \text{ for some integers } m > 0, \ n > 0, \text{ with } m \ne n.$$

Then, from (1), we get

$$a_0^2 = (a_0 + m)^2 - (a_0 + m)(a_0 + n) + (a_0 + n)^2$$

that is, $\quad 0 = a_0(m + n) + m^2 + n^2 - mn$

that is, $\quad mn = a_0(m + n) + m^2 + n^2. \qquad (2)$

Now, since $m^2 + n^2 > 2mn$, from (2), we get

$$mn > m^2 + n^2 > 2mn,$$

leading to a contradiction.

Case (2) : Let $b_0 < a_0$, $c_0 < a_0$.

Without loss of generality, we may assume that $c_0 > b_0$. Let

$$a_0 = b_0 + c, \ c_0 = b_0 + d \text{ for some integers } c > 0, \ d > 0, \text{ where } c > d.$$

Then, from (1),

$$(b_0 + c)^2 = b_0^2 - b_0(b_0 + d) + (b_0 + d)^2$$

that is, $\quad c(2b_0 + c) = d(b_0 + d). \qquad (3)$

But, this leads to a contradiction, since

$$c > d \quad \Rightarrow \quad c(2b_0 + c) > d(2b_0 + c) > d(2b_0 + d) > d(b_0 + d).$$

Thus, neither Case (1) nor Case (2) can happen.

All these establish the lemma. ■

By Lemma 5.1.3, if (a_0, b_0, c_0) is a (non–trivial) solution of the Diophantine equation $4a^2 = (2c - b)^2 + 3b^2$, then, without loss of generality,

$$b_0 < a_0 < c_0. \qquad (5.1.4)$$

If one solution of the Diophantine equation (5.1.2) is known, then we can find a second independent solution. This is given in the following lemma.

Lemma 5.1.4 **:** If (a_0, b_0, c_0) is a solution of the Diophantine equation

$$4a^2 = (2c - b)^2 + 3b^2, \qquad (5.1.2)$$

then $(a_0, c_0 - b_0, c_0)$ is also a solution of (5.1.2).

Proof **:** If (a_0, b_0, c_0) is a solution of (5.1.2), then (since (5.1.2) is equivalent to (5.1.2a))

$$a_0^2 = b_0^2 - b_0 c_0 + c_0^2. \qquad (1)$$

Rewriting (1) in the equivalent form

$$a_0^2 = (c_0 - b_0)^2 - (c_0 - b_0)c_0 + c_0^2,$$

we see that $(a_0, c_0 - b_0, c_0)$ is also a solution of $a^2 = b^2 - bc + c^2$, or equivalently, of (5.1.2). ■

Lemma 5.1.4 shows that the Diophantine equation $4a^2 = (2c - b)^2 + 3b^2$ possesses (positive integer) solutions in pairs, namely, (a_0, b_0, c_0) and $(a_0, c_0 - b_0, c_0)$, which are independent. Note that, by symmetry, (a_0, c_0, b_0) and $(a_0, c_0, c_0 - b_0)$ are also solutions of the Diophantine equation.

The following table gives the solutions of $4a^2 = (2c - b)^2 + 3b^2$ when $1 \le a \le 100$.

Table 5.1.1 : Solutions of $4a^2 = (2c - b)^2 + 3b^2$ ***for*** $1 \le a \le 100$

a	b	c	a	b	c	a	b	c
7	3	8	49	16	55	91	11	96
	5	8		39	55		85	96
13	7	15		21	56		19	99
	8	15		35	56		80	99
19	5	21	61	9	65		39	104
	16	21		56	65		65	104
31	11	35	67	32	77		49	105
	24	35		45	77		56	105
37	7	40	73	17	80	97	55	112
	33	40		63	80		57	112
43	13	48	79	40	91			
	35	48		51	91			

The limited search for the solutions of the Diophantine equation $4a^2 = (2c - b)^2 + 3b^2$ for $1 \le a \le 100$ shows that, it possesses solutions only for certain primes (and their multiples), namely, when $a = 7$, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97. In each case, there are two independent solutions (that are relevant to our problem of interest).

In Table 5.1.1 above, only the solutions of $4a^2 = (2c - b)^2 + 3b^2$ for primes a are shown, with the exceptions when $a = 49 = 7^2$ and $a = 91 = 3 \times 13$; when $a = 49$, there are four independent solutions, while for $a = 91$, there are eight independent solutions to the Diophantine equation.

Next, we confine our attention to the Diophantine equation

$$4a^2 = (2c + b)^2 + 3b^2.$$

In this case, we have the following result.

Lemma 5.1.5 **:** If (a_0, b_0, c_0) is a solution of the Diophantine equation

$$4a^2 = (2c + b)^2 + 3b^2, \qquad (5.1.3)$$

then $a_0 > \max\{b_0, c_0\}$.

Proof **:** We write (5.1.3) in the equivalent form

$$a^2 = b^2 + bc + c^2. \qquad (5.1.3a)$$

If (a_0, b_0, c_0) be a solution of (5.1.3), then

$$a_0^2 = b_0^2 + b_0 c_0 + c_0^2. \qquad (1)$$

If $b_0 > a_0$, then from (1),

$$a_0^2 > a_0^2 + a_0 c_0 + c_0^2 \quad \Rightarrow \quad 0 > a_0 c_0 + c_0^2,$$

which is impossible. Hence, $b_0 < a_0$. Again,

$$c_0 > a_0 \quad \Rightarrow \quad a_0^2 > b_0^2 + a_0 b_0 + a_0^2 \quad \Rightarrow \quad 0 > b_0^2 + a_0 b_0,$$

leading to a contradiction. Hence, $c_0 < a_0$. ■

The following lemma gives the relationship between the solutions of (5.1.2) and (5.1.3).

Lemma 5.1.6 **:** If (a_0, b_0, c_0) is a solution of the Diophantine equation

$$4a^2 = (2c - b)^2 + 3b^2, \qquad (5.1.2)$$

then $(a_0, b_0, c_0 - b_0)$ is a solution of the Diophantine equation

$$4a^2 = (2c + b)^2 + 3b^2. \qquad (5.1.3)$$

Proof **:** If (a_0, b_0, c_0) is a solution of the Diophantine equation (5.1.2), then

$$a_0^2 = b_0^2 - b_0 c_0 + c_0^2. \qquad (i)$$

Now, rewriting (i) as

$$a_0^2 = b_0^2 + b_0(c_0 - b_0) + (c_0 - b_0)^2,$$

we see that $(a_0, b_0, c_0 - b_0)$ is a solution of $a^2 = b^2 + bc + c^2$, which is equivalent to (5.1.3). ■

The following table gives the solutions of $4a^2 = (2c + b)^2 + 3b^2$ for $1 \le a \le 100$.

Table 5.1.2 : Solutions of $4a^2 = (2c + b)^2 + 3b^2$ for $1 \le a \le 100$

a	b	c	a	b	c	a	b	c
7	3	5	49	16	39	91	11	85
13	7	8		21	35		19	80
19	5	16	61	9	56		39	65
31	11	24	67	32	45		49	56
37	7	33	73	17	63	97	55	57
43	13	35	79	40	51			

The above Table 5.1.2 shows that, there are solutions of the Diophantine equation $4a^2 = (2c + b)^2 + 3b^2$ when $a = 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97$ (and multiples of these primes), and in each case, there is only one solution; on the other hand, there are two independent solutions when $a = 49$, while $a = 91$ admits of four independent solutions.

5.2 Families of 60 Degrees Smarandache Function Related and Pseudo Smarandache Function Related Triangles

We start with the following result related to the 60 degrees triangles.

Lemma 5.2.1 **:** Let $T(a, b, c)$ be a triangle with $\angle A = 60^0$, and integer–valued sides a, b and c. Then,

(1) a, b and c satisfy the Diophantine equation

$$4a^2 = (2c - b)^2 + 3b^2, \tag{5.2.1}$$

(2) the Diophantine equation (5.2.1) has an infinite number of solutions.

Proof **:** Part (1) of the lemma is a restatement of Lemma 5.1.1.

Now, if (a_0, b_0, c_0) is a solution of the Diophantine equation (5.2.1), then (ka_0, kb_0, kc_0) is also its solution for any integer $k \geq 2$, and hence, (5.2.1) possesses an infinite number of solutions. Now, by inspection, we see that

$$a_1 = 7, b_1 = 3, c_1 = 8, \tag{5.2.2}$$

is a solution of (5.2.1). ■

Note that,

$$a_2 = 7, b_2 = 5, c_2 = 8, \tag{5.2.3}$$

is a second independent solution of the Diophantine equation (5.2.1), as can readily be verified. Recall that, two solutions are dependent if one is a (non-zero) constant multiple of the other; otherwise, they are independent.

Theorem 5.2.1 **:** There is an infinite number of pairs of dissimilar 60 degrees triangles that are S−related.

Proof **:** We consider the following pair of dissimilar triangles

$$T(7p, 3p, 8p) \text{ and } T(7p, 5p, 8p),$$

where $p \geq 11$ is a prime. By Lemma 5.1.1, each is a 60 degrees triangle.

Now, since for any prime $p \geq 7$,

$$S(3p) = p = S(5p),$$

it follows that the (dissimilar) triangles $T(7p, 3p, 8p)$ and $T(7p, 5p, 8p)$ are S−related. ■

Theorem 5.2.2 **:** There is an infinite number of pairs of dissimilar 60 degrees triangles that are Z−related.

Proof **:** We consider the following pair of dissimilar 60 degrees triangles

$$T(7p, 3p, 8p) \text{ and } T(7p, 5p, 8p),$$

where p is a prime of the form

$$p = (3 \times 10)n - 1 = 30n - 1, n \in Z^+. \tag{5.2.4}$$

Since $6 | (p + 1)$, $10 | (p + 1)$, by Lemma 4.2.4 (in Chapter 4),

$$Z(3p) = p = Z(5p).$$

Thus, the pair of triangles $T(7p, 3p, 8p)$ and $T(7p, 5p, 8p)$ are Z−related.

Now, since there are an infinite number of primes of the form (5.2.4) (by Dirichlet's Theorem, see, for example, Hardy and Wright [3], Theorem 15, pp. 13), we have the desired result. ■

It may be mentioned here that, if p is a prime of the form

$$p = (3 \times 10)n + 1 = 30n + 1,\ n \in Z^+, \tag{5.2.5}$$

then the pair of dissimilar 60 degrees triangles T(7p, 3p, 8p) and T(7p, 5p, 8p) forms a second infinite family of pairs of triangles that are Z–related, since for such p, both 6 and 10 divide (p – 1), and by Lemma 4.2.4,

$$Z(3p) = p - 1 = Z(5p).$$

We now prove the following theorem.

Theorem 5.2.3 **:** There exists an infinite family of pairs of 60 degrees triangles that are both S–related and Z–related.

Proof **:** From Theorem 5.2.1 and Theorem 5.2.2, we see that the pair of 60 degrees triangles

$$T(7p, 3p, 8p) \text{ and } T(7p, 5p, 8p),$$

where p is a prime of the form

$$p = (3 \times 10)n - 1 = 30n - 1,\ n \in Z^+.$$

are both S–related and Z–related, since for such p, S(3p) = p = S(5p). ■

5.3 Families of 120 Degrees Smarandache Function Related and Pseudo Smarandache Function Related Triangles

We start with the following lemma that characterizes 120 degrees triangles.

Lemma 5.3.1 **:** Let T(a, b, c) be a triangle with $\angle A = 120^0$, and integer–valued sides a, b and c. Then,

(1) a, b and c satisfy the Diophantine equation

$$4a^2 = (2c + b)^2 + 3b^2, \tag{5.3.1}$$

(2) the Diophantine equation (5.3.1) has an infinite number of solutions.

Proof **:** Part (1) follows from Lemma 5.1.2.

Again, since $a_0 = 7$, $b_0 = 3$, $c_0 = 5$ is a solution of the Diophantine equation (5.3.1), it has infinite number of solutions of the form (7k, 3k, 5k), $k \geq 1$. This proves part (2) of the lemma.

Hence, the proof is complete. ■

Theorem 5.3.1 **:** There is an infinite number of pairs of dissimilar 120 degrees triangles that are S–related.

Proof **:** We consider the pair of dissimilar triangles

$$T(49p, 16p, 39p) \text{ and } T(49p, 21p, 35p),$$

each of which is a 120 degrees triangle, by Lemma 5.1.2.

Now, for any prime $p \geq 17$,

$$S(16p) = p = S(39p) = S(21p) = S(35p).$$

Thus, for any prime $p \geq 17$, the (dissimilar) triangles T(49p, 16p, 39p) and T(49p, 21p, 35p) are S–related.

This establishes the theorem. ■

In proving Theorem 5.3.1 above, we have made use of the fact that (see Table 5.1.2)

(1) $a_1 = 49, b_1 = 16, c_1 = 39,$ (5.3.2)

(2) $a_2 = 49, b_2 = 21, c_2 = 35,$ (5.3.3)

are two independent solutions of the Diophantine equation $4a^2 = (2c+b)^2 + 3b^2$.

Theorem 5.3.2 **:** There is an infinite number of pairs of dissimilar 120 degrees triangles that are Z–related.

Proof **:** We start with the pair of dissimilar 120 degrees triangles

$$T(49p, 16p, 39p) \text{ and } T(49p, 21p, 35p),$$

where p is a prime of the form

$$p = (2^5 \times 3 \times 5 \times 7^2 \times 13)n - 1 = 305760n - 1, \; n \in Z^+. \quad (5.3.4)$$

Since

$$2 \times 16 \,|\, (p+1), \; 2 \times 39 \,|\, (p+1), \; 2 \times 21 \,|\, (p+1), \; 2 \times 35 \,|\, (p+1),$$

by Lemma 4.2.4, for any prime p of the form (5.3.4),

$$Z(16p) = Z(39p) = Z(21p) = Z(35p) = p.$$

Therefore, for any p of the form (5.3.4), the family of pairs of (dissimilar) 120 degrees triangles T(49p, 16p, 39p) and T(49p, 21p, 35p) are Z–related.

All these complete the proof of the theorem. ■

A second infinite family of (dissimilar) 120 degrees triangles that are Z–related, can be formed as follows : Let p be a prime of the form

$$p = (2^5 \times 3 \times 5 \times 7^2 \times 13)n + 1 = 305760n + 1, \; n \in Z^+. \quad (5.3.5)$$

By Lemma 4.2.4, for such p,

$$Z(16p) = Z(39p) = Z(21p) = Z(35p) = p - 1,$$

since

$$2 \times 16 \,|\, (p-1), \; 2 \times 39 \,|\, (p-1), \; 2 \times 21 \,|\, (p-1), \; 2 \times 35 \,|\, (p-1).$$

It then follows that the pair of triangles T(49p, 16p, 39p) and T(49p, 21p, 35p) are Z–related for any prime p of the form (5.3.5).

Theorem 5.3.3 **:** There is an infinite number of pairs of dissimilar 120 degrees triangles that are both S–related and Z–related.

Proof **:** From Theorem 5.3.1 and Theorem 5.3.2, we see that the pair of triangles

$$T(49p, 16p, 39p) \text{ and } T(49p, 21p, 35p),$$

where p is a prime of the form

$$p = (2^5 \times 3 \times 5 \times 7^2 \times 13)n + 1 = 305760n + 1, \; n \in Z^+,$$

are both S–related and Z–related, since for such p,

$$S(16p) = S(39p) = S(21p) = S(35p) = p.$$

All these complete the proof of the theorem. ■

5.4 Families of Smarandache Function Related and Pseudo Smarandache Function Related Pythagorean Triangles

A triangle is called Pythagorean (or, right–angled) if one of its angles is 90 degrees.

Recall that a triangle $T(a, b, c)$ (with sides of length a, b and c) is Pythagorean, with $\angle C = 90^0$, if and only if $a^2 + b^2 = c^2$. Thus, the problem of finding the S−related/Z−related Pythagorean triangles is related to the solutions of the Diophantine equation

$$a^2 + b^2 = c^2.$$

In what follows, we shall make use of the following two independent solutions of the Diophantine equation $a^2 + b^2 = c^2$:

(1) $a_0 = 3$, $b_0 = 4$, $c_0 = 5$, (2) $a_0 = 5$, $b_0 = 12$, $c_0 = 13$.

The following result is due to Ashbacher [2].

Theorem 5.4.1 **:** There are an infinite family of pairs of dissimilar Pythagorean triangles that are S−related.

Proof **:** For any prime $p \ge 17$, the two families of dissimilar Pythagorean triangles

$$T(3p, 4p, 5p) \text{ and } T(5p, 12p, 13p)$$

are S−related, since

$$S(3p) = S(4p) = S(5p) = S(12p) = S(13p) = p. \blacksquare$$

Theorem 5.4.2 **:** There are an infinite number of pairs of dissimilar Pythagorean triangles that are Z−related.

Proof **:** We consider the pair of dissimilar Pythagorean triangles

$$T(3p, 4p, 5p) \text{ and } T(5p, 12p, 13p), \quad (5.4.1)$$

where p is a prime of the form

$$p = (2^3 \times 3 \times 5 \times 13)n + 1 = 1560n + 1, \; n \in Z^+. \quad (5.4.2)$$

By Lemma 4.2.4,

$$Z(3p) = Z(4p) = Z(5p) = Z(12p) = Z(13p) = p - 1,$$

so that the triangles $T(3p, 4p, 5p)$ and $T(5p, 12p, 13p)$ are Z−related.
Since there is an infinite number of primes of the form (5.4.2), we get the desired infinite number of pairs of dissimilar Z−related Pythagorean triangles. ■

Theorem 5.4.3 **:** There is an infinite number of pairs of dissimilar Pythagorean triangles that are both S−related and Z−related.

Proof **:** Follows from Theorem 5.4.1 and Theorem 5.4.2, since the pair of triangles

$$T(3p, 4p, 5p) \text{ and } T(5p, 12p, 13p),$$

where p is a prime of the form

$$p = (2^3 \times 3 \times 5 \times 13)n + 1 = 1560n + 1, \; n \in Z^+,$$

are both S−related and Z−related. ■

Theorem 5.4.4 **:** There exists a family of infinite number of Pythagorean triangles which are S−related, pair–wise, to the triangles of a second family of infinite number of 60 degrees triangles.

Proof **:** We consider the Pythagorean triangle T(3p, 4p, 5p) and the 60 degrees triangle T(7p, 3p, 8p). Now, for any prime $p \geq 11$,

$$S(3p) = S(4p) = S(5p) = S(7p) = S(8p) = p.$$

Thus, the triangles T(3p, 4p, 5p) and T(7p, 3p, 8p) are S−related. ■

Theorem 5.4.5 **:** There exists a family of infinite number of Pythagorean triangles which are Z−related, pair–wise, to the triangles of a second infinite family of 60 degrees triangles.

Proof **:** The Pythagorean triangle T(3p, 4p, 5p) and the 60 degrees triangle T(7p, 3p, 8p) are Z−related for any prime p of the form

$$p = (2^4 \times 3 \times 5 \times 7)n - 1 = 1680n - 1, \; n \in Z^+, \tag{5.4.3}$$

since, for such p,

$$Z(3p) = Z(4p) = Z(5p) = Z(7p) = Z(8p) = p. \; ■$$

Note that, if p is a prime of the form

$$p = (2^4 \times 3 \times 5 \times 7)n + 1 = 1680n + 1, \; n \in Z^+, \tag{5.4.4}$$

then, for such p, $Z(3p) = Z(4p) = Z(5p) = Z(7p) = Z(8p) = p - 1$.

Thus, the Pythagorean triangle T(3p, 4p, 5p) and the 60 degrees triangle T(7p, 3p, 8p) are Z−related for any prime p of the form (5.4.4).

Theorem 5.4.6 **:** There exists a family of infinite number of Pythagorean triangles which are both S−related and Z−related, pair–wise, to the triangles of a second family of infinite number of 60 degrees triangles.

Proof **:** We consider the Pythagorean triangle T(3p, 4p, 5p) and the 60 degrees triangle T(7p, 8p, 5p), where p is a prime of the form (5.4.3). Then, by Theorem 5.4.4 and Theorem 5.4.5, they are both S−related and Z−related. ■

Theorem 5.4.7 **:** There exists a family of infinite number of Pythagorean triangles which are S−related, pair–wise, to the triangles of a second infinite family of 120 degrees triangles.

Proof **:** We consider the Pythagorean triangle T(3p, 4p, 5p) and the 120 degrees triangle T(7p, 3p, 5p), where $p \geq 11$ is a prime. Since, for such p,

$$S(3p) = S(4p) = S(5p) = S(7p) = p,$$

the triangles T(3p, 4p, 5p) and T(7p, 3p, 5p) are S−related. ■

Theorem 5.4.8 **:** There exists a family of infinite number of Pythagorean triangles which are Z−related, pair–wise, to the triangles of a second infinite family of 120 degrees triangles.

Proof **:** For any prime p of the form

$$p = (2^3 \times 3 \times 5)n - 1 = 120n - 1, \; n \in Z^+, \tag{5.4.5}$$

the Pythagorean triangle T(3p, 4p, 5p) and the 120 degrees triangle T(7p, 3p, 5p) are Z–related, since, for such p,

$$Z(3p) = Z(4p) = Z(5p) = Z(7p) = p. \; ■$$

It may be mentioned here that, if p is a prime of the form

$$p = (2^3 \times 3 \times 5)n + 1 = 120n + 1, \ n \in Z^+, \qquad (5.4.6)$$

then, for such p, $Z(3p) = Z(4p) = Z(5p) = Z(7p) = Z(3p) = p - 1$,

and hence, the Pythagorean triangle $T(3p, 4p, 5p)$ and the 120 degrees triangle $T(7p, 3p, 5p)$ are Z−related.

Theorem 5.4.9 **:** There exists a family of infinite number of Pythagorean triangles which are both S−related and Z−related, pair−wise, to the triangles of a second family of infinite number of 120 degrees triangles.

Proof **:** By Theorem 5.4.7 and Theorem 5.4.8, the Pythagorean triangle $T(3p, 4p, 5p)$ and the 120 degrees triangle $T(7p, 3p, 5p)$ are both S−related and Z−related, where p is a prime of the form (5.4.5). ■

In passing, we state and prove the following three theorems concerning the 60 degrees and 120 degrees S−related/Z−related triangles.

Theorem 5.4.10 **:** There exists a family of infinite number of 60 degrees triangles which are S−related, pair−wise, to the triangles of a second infinite family 120 degrees triangles.

Proof **:** The 60 degrees triangle $T(7p, 3p, 8p)$ and the 120 degrees triangle $T(7p, 3p, 5p)$ are S−related for any prime $p \geq 11$, since, in such a case,

$$S(7p) = S(3p) = S(8p) = S(5p) = p. \ ■$$

Theorem 5.4.11 **:** There exists a family of infinite number of 60 degrees triangles which are Z−related, pair−wise, to the triangles of a second infinite family of 120 degrees triangles.

Proof **:** If p is a prime of the form

$$p = (2^4 \times 3 \times 5 \times 7)n - 1 = 1680n - 1, \ n \in Z^+, \qquad (5.4.7)$$

then the 60 degrees triangle T(7p, 3p, 8p) and the 120 degrees triangle T(7p, 3p, 5p) are Z−related, since, for such p,

$$Z(7p) = Z(3p) = Z(8p) = Z(5p) = p. \ ■$$

Note that, if p is a prime of the form

$$p = (2^4 \times 3 \times 5 \times 7)n + 1 = 1680n + 1, \ n \in Z^+, \qquad (5.4.8)$$

then

$$Z(7p) = Z(3p) = Z(8p) = Z(5p) = p - 1.$$

With such a value of p, we get a second pair of 60 degrees triangle $T(7p, 3p, 8p)$ and the 120 degrees triangle $T(7p, 3p, 5p)$ which are Z−related.

Theorem 5.4.12 **:** There exists a family of infinite number of 60 degrees triangles which are both S−related and Z−related, pair−wise, to the triangles of a second family of infinite number of 120 degrees triangles.

Proof **:** If p is a prime of the form (5.4.7), by Theorems 5.4.10 and Theorem 5.4.11, the 60 degrees triangle $T(7p, 3p, 8p)$ and the 120 degrees triangle $T(7p, 3p, 5p)$ are both S−related and Z−related. ■

5.5 Smarandache Function Related and Pseudo Smarandache Function Related Triangles in Terms of Their Angles

In this section, we consider the problem of finding the S–related and Z–related triangles in terms of their angles, that is, in the sense of Definition 5.3.

We consider the case of S–related triangles and Z–related triangles separately in §5.5.1 and §5.5.2 respectively.

5.5.1 Smarandache Function Related Triangles

Using a computer program, Ashbacher [2] searched for S–related pairs of triangles (in the sense of Definition 5.3). He reports three such pairs.

However, in this case, the introduction of $S^{-1}(m)$ may be helpful. The Smarandache function S(n) is clearly not bijective. However, we can define the inverse $S^{-1}(m)$ as follows :

$$S^{-1}(m) = \{n \in Z^+ \ : \ S(n) = m\} \text{ for any integer } m \geq 1, \tag{5.5.1a}$$

with

$$S^{-1}(1) = 1. \tag{5.5.1b}$$

Then, for any $m \in Z^+ \equiv \{1, 2, 3, 4, \ldots\}$, the set $S^{-1}(m)$ is non–empty and bounded with m! as its largest element. Clearly, $n \in S^{-1}(m)$ if and only if the following two conditions are satisfied :

(1) n divides m!,

(2) n does not divide $\ell!$ for any ℓ with $1 \leq \ell \leq m-1$.

We can look at (5.5.1) from a different point of view : On the set Z^+, we define the relation $\Re$ as follows :

$$\text{For any } n_1, n_2 \in Z^+, \ n_1 \Re n_2 \text{ if and only if } S(n_1) = S(n_2). \tag{5.5.2}$$

It is then straightforward to verify that $\Re$ is an equivalence relation on Z^+. It is well–known that an equivalence relation induces a partition (on the set Z^+) (see, for example, Gioia [4], Theorem 11.2, pp. 32). The sets $S^{-1}(m)$, $m \in Z^+$, are, in fact, the equivalence classes induced by the equivalence relation $\Re$ on Z^+, and possess the following two properties :

(1) $\sum_{m=1}^{\infty} S^{-1}(m) = Z^+$,

(2) $S^{-1}(m_1) \cap S^{-1}(m_2) = \varnothing$, if $m_1 \neq m_2$.

Thus, for any $n \in Z^+$, there is one and only one $m \in Z^+$ such that $n \in S^{-1}(m)$.

To deal with the condition (5.1) (that is, the condition $\alpha + \beta + \gamma = 180 = \alpha' + \beta' + \gamma'$), we consider the restricted sets $S^{-1}(m \mid \pi)$, defined as follows :

$$S^{-1}(m \mid \pi) = \{n \in Z^+ \ : S(n) = m, \ 1 \leq m \leq 180\}. \tag{5.5.3}$$

The following table gives such restricted sets related to our problem.

Table 5.5.1 : Values of $S^{-1}(m|\pi) = \{n \in Z^+ : S(n) = m,\ 1 \le n \le 180\}$

m	$S^{-1}(m\|\pi)$	m	$S^{-1}(m\|\pi)$
1	{1}	59	{59, 118, 177}
2	{2}	61	{61, 122}
3	{3, 6}	67	{67, 134}
4	{4, 8, 12, 24}	71	{71, 142}
5	{5, 10, 15, 20, 30, 40, 60, 120}	73	{73, 146}
6	{9, 16, 18, 36, 45, 48, 72, 80, 90, 144, 180}	79	{79, 158}
7	{7, 14, 21, 28, 35, 42, 56, 63, 70, 84, 105, 112, 126, 140, 168}	83	{83, 166}
8	{32, 64, 96, 128, 160}	89	{89, 178}
9	{27, 54, 81, 108, 135, 162}	97	{97}
10	{25, 50, 75, 100, 150, 175}	101	{101}
11	{11, 22, 33, 44, 55, 66, 77, 88, 99, 110, 132, 154, 165, 176}	103	{103}
13	{13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, 156}	107	{107}
14	{49, 98, 147}	109	{109}
15	{125}	113	{113}
17	{17, 34, 51, 68, 85, 102, 119, 136, 153, 170}	127	{127}
19	{19, 38, 57, 76, 95, 114, 133, 152, 171}	131	{131}
22	{121}	137	{137}
23	{23, 46, 69, 92, 115, 138, 161}	139	{139}
26	{169}	149	{149}
29	{29, 58, 87, 116, 145, 174}	151	{151}
31	{31, 62, 93, 124, 155}	157	{157}
37	{37, 74, 111, 148}	163	{163}
41	{41, 82, 123, 164}	167	{167}
43	{43, 86, 129, 172}	173	{173}
47	{47, 94, 141}	179	{179}
53	{53, 106, 159}		

Clearly, two S–related triangles $T(\alpha, \beta, \gamma)$ and $T(\alpha', \beta', \gamma')$ must satisfy the following condition :

$$\alpha, \alpha' \in S^{-1}(m_1 | \pi)\ ; \quad \beta, \beta' \in S^{-1}(m_2 | \pi)\ ; \quad \gamma, \gamma' \in S^{-1}(m_3 | \pi) \ \text{ for some } m_1, m_2, m_3 \in Z^+ .$$

Thus, for example, choosing

$$\alpha, \alpha' \in \{1\} = S^{-1}(1 | \pi)\ ;$$

$$\beta, \beta' \in \{11, 22, 33, 44, 55, 66, 77, 88, 99, 110, 132, 154, 165, 176\} = S^{-1}(11 | \pi)\ ;$$

$$\gamma, \gamma' \in \{7, 14, 21, 28, 35, 42, 56, 63, 70, 84, 105, 112, 126, 140, 168\} = S^{-1}(7 | \pi)\ ,$$

we can construct the S−related triangles T(1, 11, 168) and T(1, 165, 14). Note that, since $S^{-1}(1\,|\,\pi) = \{1\}$ is a singleton set, if we choose $\alpha = 1$, then we must also choose $\alpha' = 1$. Again, choosing $\alpha, \alpha', \beta, \beta' \in \{25, 50, 75, 100, 150, 175\} = S^{-1}(10\,|\,\pi)$, we can form the pair of S−related triangles T(25, 150, 5) and T(75, 100, 5) with the characteristic that

$$S(\alpha) = S(\alpha') = 10 = S(\beta) = S(\beta') \text{ (and } S(\gamma) = S(\gamma') = 5).$$

Choosing $\alpha, \alpha', \beta, \beta' \in \{5, 10, 15, 20, 30, 40, 60, 120\} = S^{-1}(5\,|\,\pi)$, we see that the equilateral triangle T(60, 60, 60) is S−related to each of the two triangles T(30, 30, 120) and T(20, 40, 120)! It may be mentioned here that the pair T(60, 60, 60) and T(20, 40, 120) are Z−related as well.

Ashbacher [2] cites the triangles T(3, 7, 170) and T(6, 21, 153) as an example of a S−related pair where all the six angles are different. We found several by trial–and–error from Table 5.5.1, some of which are given below :

(1) T(5, 25, 150) and T(30, 50, 100) (S(5) = S(30) = 5, S(25) = S(50) = 10 = S(150) = S(100)),
(2) T(20, 60, 100) and T(10, 120, 50) (S(20) = S(10) = 5 = S(60) = S(120), S(100) = S(50) = 10),
(3) T(5, 45, 130) and T(60, 16, 104) (S(5) = S(60) = 5, S(45) = S(16) = 6, S(130) = S(104) = 13),
(4) T(5, 85, 90) and T(60, 102, 18) (S(5) = S(60) = 5, S(85) = S(102) = 17, S(90) = S(18) = 6),
(5) T(37, 50, 93) and T(74, 75, 31) (S(37) = S(74) = 37, S(50) = S(75) = 10, S(93) = S(31) = 31),
(6) T(40, 50, 90) and T(60, 75, 45) (S(40) = S(60) = 5, S(50) = S(75) = 10, S(90) = S(45) = 6).
(7) T(20, 70, 90) and T(60, 84, 36) (S(20) = S(60) = 5, S(70) = S(84) = S(90) = S(36) = 6).

The last two are examples where all the six angles are acute.

An exhaustive computer search by Ashbacher [2] for pairs of all dissimilar S−related triangles $T(\alpha, \beta, \gamma)$ and $T(\alpha', \beta', \gamma')$ $(\alpha + \beta + \gamma = 180 = \alpha' + \beta' + \gamma')$ with values of α in the range $1 \le \alpha \le 178$, revealed that α cannot take the following values in any triplet pairs :

$$\left.\begin{array}{l} 83, 97, 107, 113, 121, 127, 137, 139, 149, 151, \\ 163, 166, 167, 169, 172, 173, 174, 175, 176, 177, 178. \end{array}\right\} \tag{5.5.4}$$

A closer look at the values in (5.5.4) and those in Table 5.5.1 reveals the following facts :

(1) Generally, the numbers belonging to singleton sets in $S^{-1}(m\,|\,\pi)$ do not appear in any 180 degrees pairs of triplets, with the exceptions of $\alpha = 1, 2, 101, 103, 109, 125, 131, 157$. For the last six cases, we can form the following examples :

(a) T(101, 9, 70) and T(101, 16, 63) (with S(9) = 6 = S(16), S(70) = 7 = S(63)),

(b) With $\alpha = 103$, we have a set of four triangles,

T(103, 7, 70), T(103, 14, 63), T(103, 21, 56) and T(103, 35, 42)

(with S(7) = S(14) = S(21) = S(35) = 7 = S(70) = S(63) = S(56) = S(42)),

which are pair–wise S−related, and a second set of three triangles,

T(103,11,66), T(103, 22, 55) and T(103,33,44)

(with S(11)=S(22)=S(33)=11, S(66)=S(55)=S(44) = 11),

which are pair–wise S−related,

(c) T(109, 11, 60) and T(109, 66, 5) (with S(11) = 11 = S(66), S(60) = 5 = S(5)),

(d) T(125, 5, 50) and T(125, 30, 25) (with S(5) = 5 = S(30), S(50) = 10 = S(25)), as well as the pair T(125, 11, 44) and T(125, 22, 33) (with S(11) = S(22) = 11 = S(44) = S(33)),

(e) For α=131, we have three triangles, T(131, 7, 42), T(131, 14, 35) and T(131, 21, 28) (with S(7) = S(14) = S(21) = 7 = S(42) = S(35) = S(28)), any two of which are S–related,

(f) T(157, 9, 14) and T(157, 16, 7) (with S(9) = 6 = S(16), S(14) = 7 = S(7)).

(2) None of $83, 166 \in S^{-1}(83 \mid \pi)$ appears in any triplet.

(3) Both $59, 118 \in S^{-1}(59 \mid \pi)$ can appear in one or the other triplet pair, but $177 \in S^{-1}(59 \mid \pi)$ cannot appear in any triplet pair. The reason is as follows : If $\alpha = 177$, then the only option for β and γ are β = 1, γ = 2. Noting that both $S^{-1}(1 \mid \pi) = \{1\}$ and $S^{-1}(2 \mid \pi) = \{2\}$ are singleton sets, we cannot find a second dissimilar triangle S–related to T(177, 1, 2). The same is true when $\alpha = 173 \in S^{-1}(173 \mid \pi)$ (which is a singleton set), $\alpha = 175 \in S^{-1}(10 \mid \pi)$ and $\alpha = 176 \in S^{-1}(11 \mid \pi)$. Again, $43, 86, 129 \in S^{-1}(43 \mid \pi)$ can appear in one or the other triplet, but $172 \in S^{-1}(59 \mid \pi)$ cannot appear in any triplet. The detail is shown below.

	$\alpha = 172$	$\alpha' = 172$	$\alpha' = 129$	$\alpha' = 86$	$\alpha' = 43$	Remark
(β,γ)	(1, 7)	Similar Triangle	Not Possible	Not Possible	Not Possible	–
	(2, 6)	Similar Triangle	Not Possible	Not Possible	Not Possible	$\beta' + \gamma' \le 8$
	(3, 5)	Similar Triangle	Not Possible	Not Possible	Not Possible	–
	(4, 4)	Similar Triangle	Not Possible	Not Possible	Not Possible	$\beta' + \gamma' \le 48$

On the other hand, if $\alpha = 171$, we can get the S–related pair of triangles, T(171, 4, 5) and T(152, 12, 10) (with S(171) = S(152) = 19, S(4) = S(12) = 4, S(5) = S(10) = 5), and corresponding to α = 170, we have two pairs, namely, the pair T(170, 3, 7) and T(153, 6, 21) (with S(170) = S(153) = 17, S(3) = S(6) = 3, S(7) = S(21) = 7), and the pair of triangles T(170, 4, 6) and T(153, 24, 3) (with S(4) = S(24) = 4).

We looked for 60 degrees and 120 degrees S–related pairs of triangles, where a 60 (120) degrees triangle is one whose one angle is 60 (120) degrees, in the sense of Definition 5.3. We got only one pair of the first type, namely, the pair of triangles T(60, 30, 90) and T(60, 40, 80) (with S(30) = S(40) = 5, S(90) = S(80) = 6), and two pairs of the second type, namely, the pair T(120, 24, 36) and T(120, 12, 48) (with S(24) = S(12) = 4, S(36) = S(48) = 6), as well as the pair T(120, 20, 40), T(120, 30, 30) (S(20) = S(30) = S(40) = 6). We also found three pairs of Pythagorean triangles that are S–related; they are

(i) T(90, 40, 50) and T(90, 15, 75) (with S(40) = S(15) = 5, S(50) = S(75) = 10),

(ii) T(90, 36, 54) and T(90, 9, 81) (with S(36) = S(9) = 6, S(54) = S(81) = 9),

(iii) T(90, 18, 72) and T(90, 45, 45) (with S(18) = S(45) = 6 = S(72)).

In (5.5.3), the set $S^{-1}(m)$ has been defined for any integer $m \in Z^+$. It might be a problem of interest to study the characteristics of the sets $S^{-1}(m)$. Of particular interest is the set $S^{-1}(p)$, where p is an odd prime. Some elementary properties of $S^{-1}(m)$ are given (without proof) in Vasile Seleacu [5].

We first prove the following lemma.

***Lemma 5.5.1* :** p divides all the elements of $S^{-1}(p)$.

***Proof* :** Let $n \in S^{-1}(p)$, so that $S(n) = p$.

Now,

$$n \mid p! \quad \Rightarrow \quad p! = na \text{ for some integer } a \ge 1.$$

Clearly, p does not divide a, for otherwise,

$$p \mid a \quad \Rightarrow \quad a = bp \text{ for some integer } b \ge 1$$
$$\Rightarrow p! = nbp$$
$$\Rightarrow nb = (p-1)!$$
$$\Rightarrow S(n) \le (p-1)!,$$

contradicting the definition of S(n).

Therefore, p does not divide a, and this, in turn, proves the lemma. ■

Now, we observe the following facts about $S^{-1}(p)$, where $p \ge 3$ is a prime :

(1) $p, p! \in S^{-1}(p)$, p being the smallest element and p! being the largest element of $S^{-1}(p)$,

(2) $kp \in S^{-1}(p)$ for all $1 \le k \le p-1$,

(3) $(p+2k-1)p \in S^{-1}(p)$ for all $1 \le k \le \frac{p-1}{2}$,

(4) if $n \in S^{-1}(p)$ then $p \mid n$,

(5) $S^{-1}(p)$ contains $d((p-1)!)$ number of elements, where $d((p-1)!)$ denotes the number of (positive) divisors of $(p-1)!$ (see Stuparu and Sharpe [6]).

Note that, if $S(n) = m$, then m is prime if n is prime. It then follows that, if m is a positive composite number, then n must also be composite too. Thus, if m is composite, then all the elements of $S^{-1}(m)$ are also composite numbers. Again, for any fixed prime $p \ge 3$, $S^{-1}(p)$ contains exactly one prime (namely, p itself). Thus, the sequence of primes p_1, p_2, … are contained in the respective sets $S^{-1}(p_1)$, $S^{-1}(p_2)$, … (which are pair-wise disjoint). The number of elements of the set $S^{-1}(m)$ is $d(n!) - d((n-1)!)$ (see Vasile Seleacu [5]).

A formula for the least element of $S^{-1}(m)$, denoted by $S^{-1}_{\min}(m)$, is given in Vasile Seleacu [5] : If m is a composite number of the form

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k},$$

where $p_1, p_2, \ldots, p_k$ are the prime factors of m with $p_1 < p_2 < \ldots < p_k$, then

$$S^{-1}_{\min}(m) = p_k^{\ell_k(m!) - \alpha_k + 1},$$

where $\ell_k(m!)$ is the exponent of p_k in the prime factors decomposition of m!. Note that,

$$S(S^{-1}_{\min}(m)) = m \quad \text{for any } m \in Z^+;$$

however, $S^{-1}_{\min}(S(8)) = S^{-1}_{\min}(4) = 4$.

5.5.2 Pseudo Smarandache Function Related Triangles

Using a computer program, Ashbacher [2] searched for Z–related pairs of triangles (in the sense of Definition 5.3). He reports three such pairs.

However, in this case also, like the S–related case in §5.5.1, the introduction of $Z^{-1}(m)$ may be helpful. Recall from §4.5 in Chapter 4 that $Z^{-1}(m)$ is defined as follows :

$$Z^{-1}(m) = \{n \in Z^+ \;:\; Z(n) = m\} \text{ for any integer } m \ge 1, \tag{5.5.5}$$

where, $n \in Z^{-1}(m)$ if and only if the following two conditions are satisfied :

(1) n divides $\dfrac{m(m+1)}{2}$,

(2) n does not divide $\dfrac{\ell(\ell+1)}{2}$ for any ℓ with $1 \le \ell \le m-1$.

The condition (5.1) can be dealt with by considering the restricted sets $Z^{-1}(m \mid \pi)$:

$$Z^{-1}(m \mid \pi) = \{n \in Z^+ \;:\; Z(n) = m,\; 1 \le n \le 178\}. \tag{5.5.6}$$

Table 5.5.2 gives such restricted sets related to our problem.

Clearly, two Z–related triangles T(α, β, γ) and T(α′, β′, γ′) must satisfy the following condition :

$$\alpha, \alpha' \in Z^{-1}(m_1 \mid \pi)\,;\; \beta, \beta' \in Z^{-1}(m_2 \mid \pi),\; \gamma, \gamma' \in Z^{-1}(m_3 \mid \pi) \text{ for some } m_1, m_2, m_3 \in Z^+.$$

Thus, for example, choosing

$$\alpha, \alpha' \in \{2, 6\} = Z^{-1}(3 \mid \pi)\,;\; \beta, \beta' \in \{44, 48, 88, 132, 176\} = Z^{-1}(32 \mid \pi),$$

we can construct the Z–related triangles T(2, 48, 130) and T(6, 44, 130). Again, choosing $\alpha, \alpha', \beta, \beta' \in \{25, 50, 75, 100, 150\} = Z^{-1}(24 \mid \pi)$, we can form the pair of Z–related triangles T(25, 150, 5) and T(75, 100, 5) with the characteristic that

$$Z(\alpha) = Z(\alpha') = 24 = Z(\beta) = Z(\beta') \text{ (and } Z(\gamma) = Z(\gamma') = 4).$$

Choosing $\alpha, \alpha', \beta, \beta' \in \{8, 20, 24, 30, 40, 60, 120\} = Z^{-1}(15 \mid \pi)$, we see that the equilateral triangle T(60, 60, 60) is Z–related to the triangle T(20, 40, 120)!

The triangles T(4, 16, 160) and T(14, 62, 104), found by Ashbacher [2] by computer search, are an example of a Z–related pair where all the six angles are different. We have found three more, given below :

(1) T(8, 12, 160) and T(40, 36, 104),
(with Z(8) = Z(40) = 15, Z(12) = Z(36) = 8, Z(160) = Z(104) = 64),

(2) T(16, 20, 144) and T(124, 24, 32),
(with Z(16) = Z(124) = 31, Z(20) = Z(24) = 15, Z(144) = Z(32) = 63),

(3) T(37, 50, 93) and T(74, 75, 31)
(with Z(37) = Z(74) = 36, Z(50) = Z(75) = 24, Z(93) = Z(31) = 30).

Table 5.5.2 : Values of $Z^{-1}(m|\pi) = \{n \in \mathbb{Z}^+ : Z(n) = m,\ 1 \le n \le 180\}$

m	$Z^{-1}(m\|\pi)$	m	$Z^{-1}(m\|\pi)$	m	$Z^{-1}(m\|\pi)$
1	{1}	35	{90}	80	{81, 108, 162, 180}
2	{3}	36	{37, 74, 111}	82	{83}
3	{2, 6}	39	{52, 130, 156}	83	{166}
4	{5, 10}	40	{41, 82, 164}	84	{170}
5	{15}	41	{123}	87	{116, 174}
6	{7, 21}	42	{43, 129}	88	{89, 178}
7	{4, 14, 28}	43	{86}	95	{152}
8	{9, 12, 18, 36}	44	{99, 110, 165}	96	{97}
9	{45}	45	{115}	100	{101}
10	{11, 55}	46	{47}	102	{103}
11	{22, 33, 66}	47	{94, 141}	106	{107}
12	{13, 26, 39, 78}	48	{49, 56, 84, 98, 147, 168}	108	{109}
13	{91}	49	{175}	111	{148}
14	{35, 105}	51	{102}	112	{113}
15	{8, 20, 24, 30, 40, 60, 120}	52	{53, 106}	120	{121}
16	{17, 34, 68, 136}	53	{159}	124	{125}
17	{51, 153}	54	{135}	126	{127}
18	{19, 57, 171}	55	{140, 154}	127	{64}
19	{38, 95}	56	{76, 114, 133}	128	{172}
20	{42, 70}	58	{59}	130	{131}
21	{77}	59	{118, 177}	136	{137}
22	{23}	60	{61, 122}	138	{139}
23	{46, 69, 92, 138}	63	{32, 72, 96, 112, 144}	148	{149}
24	{25, 50, 75, 100, 150}	64	{80, 104, 160}	150	{151}
25	{65}	65	{143}	156	{157}
26	{27, 117}	66	{67}	162	{163}
27	{54, 63, 126}	67	{134}	166	{167}
28	{29, 58}	69	{161}	168	{169}
29	{87, 145}	70	{71}	172	{173}
30	{31, 93, 155}	71	{142}	178	{179}
31	{16, 62, 124}	72	{73, 146}	255	{128}
32	{44, 48, 88, 132, 176}	78	{79}		
34	{85, 119}	79	{158}		

Ashbacher [2], based on an exhaustive computer search for pairs of all dissimilar Z–related triangles $T(\alpha, \beta, \gamma)$ and $T(\alpha', \beta', \gamma')$ ($\alpha + \beta + \gamma = 180 = \alpha' + \beta' + \gamma'$) with values of α in the range $1 \le \alpha \le 178$, reports that α cannot take the following values in any pair of triplets :

$$\left.\begin{array}{l} 1, 15, 23, 35, 41, 45, 51, 59, 65, 67, 71, 73, 77, 79, 82, 83, 86, 87, 89, \\ 90, 91, 97, 101, 102, 105, 107, 109, 113, 115, 116, 118, 121, 123, 125, 126, 127, \\ 131, 134, 135, 137, 139, 141, 142, 143, 148, 149, 151, 152, 153, 157, 158, 159, \\ 161, 163, 164, 166, 167, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178. \end{array}\right\} \quad (5.5.7)$$

Can the table of the sets $Z^{-1}(m\,|\,\pi)$, Table 5.5.2, be utilized in explaining this observation? Very large value of α often forces the two triangles to be similar. For example, if we take $\alpha = 174 \in Z^{-1}(87\,|\,\pi)$, then the three possible pairs of values of (β, γ) are $(1, 5)$, $(2, 4)$ and $(3, 3)$. Note that the two sets $Z^{-1}(1\,|\,\pi) = \{1\}$ and $Z^{-1}(2\,|\,\pi) = \{3\}$ are each singleton. Thus, if $\alpha' = 174$, then the two triangles $T(\alpha, \beta, \gamma)$ and $T(\alpha', \beta', \gamma')$ must be similar. On the other hand, if $\alpha' = 116 \in Z^{-1}(87\,|\,\pi)$, then we cannot find β', γ' with $\beta' + \gamma' = 64$. Now, we consider the case when $\alpha = 164$. This case is summarized in the tabular form below :

	α=164	α'=164	α'=82	α'=41	Remark
(β, γ)	(1,15)	Similar Triangle	Not Possible	Not Possible	Both belong to singleton sets
	(2,14)	Similar Triangle	Not Possible	Not Possible	$\beta' + \gamma' \le 34$
	(3,13)	Similar Triangle	Not Possible	Not Possible	3 belongs to a singleton set
	(4,12)	Similar Triangle	Not Possible	Not Possible	$\beta' + \gamma' \le 64$
	(5,11)	Similar Triangle	Not Possible	Not Possible	$\beta' + \gamma' \le 65$
	(6,10)	Similar Triangle	Not Possible	Not Possible	$\beta' + \gamma' \le 16$
	(7,9)	Similar Triangle	Not Possible	Not Possible	$\beta' + \gamma' \le 57$
	(8,8)	Similar Triangle	Not Possible	Not Possible	

However, if $\alpha = 165$, we can get two dissimilar Z–related triangles, namely, $T(165,7,8)$ and $T(81, 21, 60)$.

A closer look at the values in (5.5.7) and Table 5.5.2 reveals the following facts :

(1) The numbers not appearing in any 180 degrees pair of triplets are all belong to singleton sets, with the exception of 3, 47, 64, and 103. For the last three cases, we have the following examples :

(a) $T(47, 76, 57)$ and $T(47, 114, 19)$ (with $Z(76) = 56 = Z(114)$, $Z(57) = 18 = Z(19)$),

(b) $T(64, 12, 104)$ and $T(64, 36, 80)$ (with $Z(12) = 8 = Z(36)$, $Z(104) = 64 = Z(80)$),

(c) $T(103, 11, 66)$ and $T(103, 55, 22)$ (with $Z(11) = 10 = Z(55)$, $Z(66) = 11 = Z(22)$).

(2) In most of the cases, if a value does not appear in the triplet pair, all other values of the corresponding $Z^{-1}(m\,|\,\pi)$ also do not appear in other triplet pair, with the exception of $145 \in Z^{-1}(29\,|\,\pi)$ ($87 \in Z^{-1}(29\,|\,\pi)$ does not appear in any triplet pair), and $146 \in Z^{-1}(72\,|\,\pi)$ ($73 \in Z^{-1}(72\,|\,\pi)$ does not appear in any triplet pair). In this connection, we may mention the following pair of triplets :

(a) $T(145, 14, 21)$ and $T(145, 28, 7)$ (with $Z(14) = 7 = Z(28)$, $Z(21) = 6 = Z(7)$),

(b) $T(146, 4, 30)$ and $T(146, 14, 20)$ (with $Z(4) = 7 = Z(14)$, $Z(30) = 15 = Z(20)$).

(3) The only number in $Z^{-1}(27\,|\,\pi)$ that cannot appear in any triplet pair is 126.

We searched for 60 degrees, 120 degrees and Pythagorean Z–related pairs of triangles, in the sense of Definition 5.3 (where a 60 (120) degrees triangle is one whose one angle is 60 (120) degrees). We have found the following two pairs of Z–related 60 degrees triangles :

(1) $T(8, 60, 112)$ and $T(24, 60, 96)$ (with $Z(8) = Z(24) = 15$, $Z(112) = Z(96) = 63$),

(2) $T(32, 60, 88)$ and $T(72, 60, 48)$ (with $Z(32) = Z(72) = 63$, $Z(88) = Z(48) = 32$).

Note that, in the second example, all the six angles are acute.

As for the 120 degrees Z–related triangles, we have found only one pair, namely, the triangles $T(20, 40, 120)$ and $T(30, 30, 120)$ (with $(Z(20) = Z(30) = Z(40) = Z(120) = 15)$, while our search for a pair of Z–related Pythagorean triangles went in vain.

We also looked for dissimilar pairs of Z–related triangles where all the six angles are acute. In addition to the second pair listed above, we have the following pairs :

(i) $T(72, 20, 88)$, $T(72, 60, 48)$ (with $Z(20) = 15 = Z(60)$, $Z(88) = 32 = Z(48)$),

(ii) $T(88, 20, 72)$, $T(88, 60, 32)$ (with $Z(20) = 15 = Z(60)$, $Z(72) = 63 = Z(32)$),

(iii) $T(54, 42, 84)$, $T(54, 70, 56)$ (with $Z(42) = 20 = Z(70)$, $Z(84) = 48 = Z(56)$),

(iv) $T(70, 22, 88)$, $T(70, 66, 44)$ (with $Z(22) = 11 = Z(66)$, $Z(88) = 32 = Z(44)$),

(v) $T(80, 25, 75)$, $T(80, 50, 50)$ (with $Z(25) = 24 = Z(50) = Z(75)$),

(vi) $T(81, 11, 88)$, $T(81, 55, 44)$ (with $(Z(11) = 10 = Z(55)$, $Z(88) = 32 = Z(44)$).

It may be mentioned here that the last four pairs are S–related as well, since

$$S(42) = 7 = S(70) = S(84) = S(56),\ S(22) = 11 = S(66) = S(88) = S(44),$$
$$S(25) = 10 = S(50) = S(75),\ S(11) = 11 = S(55) = S(88) = S(44).$$

Using computer search, we looked for Z–related pairs of triangles under the additional condition that $\alpha = \alpha'$. Our search resulted in 59 such pairs, of which 25 pairs are S–related as well. Our search shows that $\alpha = \alpha'$ cannot take the following values in any pair :

$$\left.\begin{array}{l} 1, 2, 3, 6, 7, 8, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24, 29, 31, 34, 35, 38, 39, 40, \\ 41, 43, 45, 46, 49, 50, 51, 52, 53, 56, 57, 58, 59, 61, 62, 65, 66, 67, 69, 73, 74, \\ 76, 77, 78, 79, 82, 83, 84, 85, 86, 87, 89, 90, 91, 92, 93, 95, 96, 97, 98, 99, 101, \\ 102, 104, 105, 106, 107, 109, 110, 111, 113, 114, 115, 116, 117, 118, 119, 121, \\ 122, 123, 124, 125, 126, 127, 133, 134, 135, 141, 142, 143, 144, 147, 148, \\ 149, 150, 151, 152, 153, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, \\ 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178. \end{array}\right\}$$

Similar search for S–related pairs of triangles (under the same condition that $\alpha = \alpha'$) found 1072 pairs, with the maximum number of 46 pairs when $\alpha = \alpha' = 37$. The search shows further that $\alpha = \alpha'$ cannot take the following values in any pair :

$$\left.\begin{array}{l} 58, 71, 73, 77, 83, 86, 91, 97, 107, 111, 113, 118, 121, 123, 127, 132, 137, \\ 139, 141, 142, 144, 147, 148, 149, 151, 153, 154, 156, 158, 159, 161, 162, \\ 163, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178. \end{array}\right\}$$

5.6 Some Observations and Remarks

Theorem 5.2.1 and Theorem 5.2.2 in §5.2 prove respectively the existence of pairs of infinite families of 60 degrees triangles that are S−related and Z−related, the corresponding results in the case of the 120 degrees triangles are given in Theorem 5.3.1 and Theorem 5.3.2 respectively in §5.3, while Theorem 5.4.1 and Theorem 5.4.2 respectively in §5.4 settles the cases with the Pythagorean triangles. However, note that, the values of a, b and c, as well as the values a′, b′, and c′ in the relevant pairs of triangles $T(a, b, c)$ and $T(a', b', c')$ are quite large, particularly for the case of Z–related triangles. This is unavoidable if we want to prove the infinitude of the families of pairs of dissimilar 60 degrees, 120 degrees and Pythagorean triangles that are S−related or Z−related.

It may be mentioned that, it is possible to find pairs of dissimilar 60 degrees (and 120 degrees) triangles $T(a, b, c)$ and $T(a', b', c')$ that are S−related or Z–related, with smaller values of a, b, c, and a′, b′, c′. To find such pairs of triangles manually (without using any computer program), we have to look at the solutions of the Diophantine equations (5.2.1) and (5.3.1) more closely.

First, we consider the Diophantine equation

$$4a^2 = (2c-b)^2 + 3b^2. \tag{5.2.1}$$

By Lemma 5.2.1, the problem of finding the pair of dissimilar 60 degrees triangles is equivalent to the problem of finding (non−trivial) independent solutions of the Diophantine equation (5.2.1). Lemma 5.1.4 guarantees that the solutions of the equation (5.2.1) occur in independent pairs. Once we can find a pair of independent solutions of (5.2.1), we can manipulate with them to find a pair of triangles that are S−related or Z−related. We illustrate it with the help of an example below.

***Example 5.6.1* :** Two independent solutions of the Diophantine equation (5.2.1) are

(1) $a_1 = 7, b_1 = 3, c_1 = 8,$ (5.2.2)

(2) $a_2 = 7, b_2 = 5, c_2 = 8.$ (5.2.3)

Using these two solutions, we can form the family of (dissimilar) pairs of 60 degrees triangles, $T(7k, 3k, 8k)$ and $T(7k, 5k, 8k)$, where $k \geq 1$ is an integer. Now, if we choose k such that

$$Z(3k) = Z(5k), \tag{1}$$

then the corresponding pair of triangles becomes Z−related. Choosing $k = 8$ (so that $Z(3k) = Z(24) = 15 = Z(40) = Z(5k)$), we get the pair of 60 degrees triangles $T(56, 24, 64)$ and $T(56, 40, 64)$, which are Z−related. The choice $k = 14$ gives a second pair of triangles $T(98, 42, 112)$ and $T(98, 70, 112)$ (since $Z(42) = 20 = Z(70)$). And $k = 33$ gives the third pair (since $Z(99) = 44 = Z(165)$). We can still get more, besides those corresponding to $k = 29, 31$ (from (5.2.4) and (5.2.5) with $n = 1$), for example, those corresponding to $k = 36, 52$.

Thus, for this particular pair of independent solutions of the Diophantine equation (5.2.1), the problem of finding pair(s) of (dissimilar) 60 degrees Z−related triangles reduces to the problem of finding the solutions of (1). So, the question arises : How to find the solutions of (1)?

Note that, since $S(42)=7=S(70)$, the pair of triangles $T(98,42,112)$ and $T(98,70,112)$ is S–related as well. That is, $T(98,42,112)$ and $T(98,70,112)$ are both S–related and Z–related.
With the solutions (5.2.2) and (5.2.3) of (5.2.1), we can think of other possibilities as well :

Case (1) : Are there integers k and k_1, with $k \neq k_1$ $(k, k_1 \geq 1)$, such that

$$S(7k)=S(7k_1),\ S(3k)=S(3k_1),\ S(8k)=S(8k_1)?$$

Case (2) : Are there integers k and k_1, with $k \neq k_1$ $(k, k_1 \geq 1)$, such that

$$S(7k)=S(7k_1),\ S(5k)=S(5k_1),\ S(8k)=S(8k_1)?$$

Case (3) : Are there integers k and k_1 $(k, k_1 \geq 1)$, such that

$$S(7k)=S(7k_1),\ S(3k)=S(5k_1),\ S(8k)=S(8k_1)?$$

In Case (2), choosing $k=2$ and $k_1=2\times3=6$, we get the pair of triangles $T(7, 5, 8)$ and $T(21, 15, 24)$, which are S–related (with $S(7)=7=S(21)$, $S(5)=5=S(15)$, $S(8)=4=S(24)$).

In Case (3), choosing $k=2\times5=10$ and $k_1=2$, we get the pair of triangles $T(70, 30, 80)$ and $T(14, 10, 16)$, which are S–related (with $S(70)=7=S(14)$, $S(30)=5=S(10)$, $S(80)=6=S(16)$).

In the case of the Z–related triangles, the three possibilities are

Case (i) : Are there integers k and k_1, with $k \neq k_1$ $(k, k_1 \geq 1)$, such that

$$Z(7k)=Z(7k_1),\ Z(3k)=Z(3k_1),\ Z(8k)=Z(8k_1)?$$

Case (ii) : Are there integers k and k_1, with $k \neq k_1$ $(k, k_1 \geq 1)$, such that

$$Z(7k)=Z(7k_1),\ Z(5k)=Z(5k_1),\ Z(8k)=Z(8k_1)?$$

Case (iii) : Are there integers k and k_1 $(k, k_1 \geq 1)$, such that

$$Z(7k)=Z(7k_1),\ Z(3k)=Z(5k_1),\ Z(8k)=Z(8k_1)?$$

These cases are more difficult to deal with. Case (iii) with $k=14$, $k_1=42$, gives the pair of Z–related triangles $T(98,42,112)$ and $T(294,210,336)$, since

$$Z(98)=48=Z(294),\ Z(42)=20=Z(210),\ Z(112)=63=Z(336).\ \blacklozenge$$

Example 5.6.2 **:** Since

(1) $a_3=13,\ b_3=7,\ c_3=15,$ **(2)** $a_4=13,\ b_4=8,\ c_4=15.$

are two independent solutions of the Diophantine equation (5.2.1) (as can easily be verified), the two triangles $T(13k,7k,15k)$ and $T(13k,8k,15k)$ are 60 degrees Z–related if k is such that $Z(7k)=Z(8k)$. Since, for $k=7$, $Z(49)=48=Z(56)$, we get the desired pair of triangles $T(91,49,105)$ and $T(91,56,105)$. Choosing $k=11$, we get the pair of triangles $T(143,77,165)$ and $T(143,88,165)$, which are S–related. ♦

Example 5.6.3 **:** We now consider the following pair :

(1) $a_1=7,\ b_1=3,\ c_1=8,$ **(2)** $a_4=13,\ b_4=7,\ c_4=15.$

By Example 5.6.1 and Example 5.6.2, each is a solution of the Diophantine equation (5.2.1). Now, using these two solutions, we can form pairs of 60 degrees triangles that are S–related. One such pair is $T(119,51,136)$ and $T(221,119,255)$, since

$$S(119)=S(7\times17)=17=S(13\times17)=S(221),$$
$$S(51)=S(3\times17)=17=S(7\times17)=S(119),$$
$$S(136)=S(8\times17)=17=S(15\times17)=S(255).\ \blacklozenge$$

Next, we confine our attention to the Diophantine equation

$$4a^2 = (2c + b)^2 + 3b^2. \quad (5.1.2)$$

A look at Table 5.1.2 shows that, it is more difficult to form pairs of 120 degrees triangles that are S−related/Z−related, even with the two independent solutions of the Diophantine equation (5.1.2) corresponding to $a = 49$. From Theorem 5.3.1 with $p = 17$, we get the pair of 120 degrees triangles T(833, 272, 663) and T(833, 357, 595) that are S−related. The following example illustrates how to construct pairs of 120 degrees S–related triangles, using the solutions of (5.1.2).

Example 5.6.4 **:** Two independent solutions of the Diophantine equation (5.1.2) are

(1) $a_1 = 7, b_1 = 3, c_1 = 5$, **(2)** $a_2 = 13, b_2 = 7, c_2 = 8$.

Using these two solutions, we can form a pair of (dissimilar) 120 degrees triangles, $T(7k, 3k, 5k)$ and $T(13k_1, 7k_1, 8k_1)$, where $k \geq 1$ and $k_1 \geq 1$ are integers. Now, if we choose k and k_1 such that

$$S(7k) = S(13k_1),\ S(3k) = S(7k_1),\ S(5k) = S(8k_1),$$

then the corresponding pair of triangles becomes S−related. Choosing $k = 17$, $k_1 = 17$ (so that $S(7k) = S(119) = 17 = S(221) = S(13k_1)$, $S(3k) = S(51) = 17 = S(119) = S(7k_1)$, $S(5k) = S(85) = 7$, $17 = S(136) = S(8k_1)$), we get the pair of 120 degrees triangles T(119, 51, 85) and T(221, 119, 136), which are S−related. We can now form more; in fact, each member of the family of similar triangles T(119m, 51m, 85m), $1 \leq m \leq 6$, is S−related to each of the family of similar triangles T(221n, 119n, 136n), $1 \leq n \leq 6$. ♦

Remark 5.6.1 **:** To construct a pair of (dissimilar) 120 degrees S−related triangles, using the (independent) solutions (13, 7, 8) and (49, 16, 39) of the Diophantine equation (5.1.2), we consider the triangles T(13k, 7k, 8k) and $T(49k_1, 16k_1, 39k_1)$, where $k \geq 1$ and $k_1 \geq 1$ are integers such that $S(13k) = S(49k_1)$, $S(7k) = S(16k_1)$, $S(8k) = S(39k_1)$. Choosing $k = k_1 = 17$, we get the desired pair T(221, 119, 136) and T(833, 272, 663). However, if we agree that the order of the sides (of the two triangles) can be disregarded (that is, the triangles T(a, b, c) and T(a′, b′, c′) are S−related if S(a) equals any one of S(a′), S(b′) and S(c′), S(b) equals any one of the remaining two, and S(c) equals the third one), then we can get another pair of S−related triangles from T(13k, 7k, 8k) and $T(49k_1, 16k_1, 39k_1)$ as follows : We choose $k = 2^2 \times 7$ and $k_1 = 2$. The resulting pair, also found by Ashbacher [2] by computer search, is T(364, 196, 224) and T(98, 32, 78), where

$$S(a) = S(364) = 13 = S(78) = S(c'),\ S(b) = S(196) = 14 = S(98) = S(a'),$$
$$S(c) = S(224) = 8 = S(32) = S(b').$$

The construction of a pair of 120 degrees Z−related triangles, using the solutions of the Diophantine equation (5.1.2), is much more difficult, particularly because only very little is known about the values of Z(n). We illustrate below how this pair, also found by Ashbacher [2], can be obtained from the solutions of the Diophantine equation (5.1.2).

Example 5.6.5 **:** The pair of 120 degrees triangles T(13k, 7k, 8k) and $T(49k_1, 16k_1, 39k_1)$ (considered in Remark 5.6.1) are Z−related if we choose $k = 24$ and $k_1 = 12$. The resulting pair is T(312, 168, 192) and T(588, 192, 468), with $Z(312) = 143 = Z(468)$, $Z(168) = 48 = Z(588)$. Note that, in this case, the order of the sides of the two triangles is disregarded. ♦

We close this last chapter of the book with the following open problems and conjectures.

Question 5.6.1 **:** Is it possible to find a general explicit formula for all the solutions of the Diophantine equation $4a^2 = (2c - b)^2 + 3b^2$?

Table 5.1.1 shows that, for $1 \le a \le 100$, the solutions of the equation $4a^2 = (2c - b)^2 + 3b^2$ exists when a is a prime of the form $3n + 1$. Note that, if a solution of $4a^2 = (2c - b)^2 + 3b^2$ is known, then a second independent solution of the equation is also known, by Lemma 5.1.4; moreover, this enables us to find a solution of the Diophantine equation $4a^2 = (2c + b)^2 + 3b^2$ as well, by virtue of Lemma 5.1.6.

Question 5.6.2 **:** Is it possible to find a good upper bound to the number of pairs of dissimilar triangles that are S−related/Z−related, in the sense of Definition 5.1?

Question 5.6.3 **:** Is it possible to find a good upper bound to the number of pairs of dissimilar 60 degrees triangles that are S−related/Z−related, in the sense of Definition 5.1?

Question 5.6.4 **:** Is it possible to find better upper bounds to the number of elements of the set $S^{-1}(m \mid \pi)$?

Conjecture 5.6.1 **:** There is no pair of dissimilar 90 degrees triangles that are Z−related, in the sense of Definition 5.3.

Question 5.6.5 **:** Given a triangle $T(a, b, c)$, what is the condition that a similar triangle $T(a', b', c')$ exists such that $T(a, b, c)$ and $T(a', b', c')$ are S–related/Z–related?

Given the right-angled triangle $T(3, 4, 5)$, we can find a similar triangle, namely, $T(6, 8, 10)$, such that $T(3, 4, 5)$ and $T(6, 8, 10)$ are S–related; but, there is no triangle which is similar and Z–related to $T(3, 4, 5)$. However, the right–angled triangles $T(18, 80, 82)$ and $T(36, 160, 164)$ are Z–related. Note that, the triangles $T(18, 80, 82)$ and $T(36, 160, 164)$ are not S–related.

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APPENDIX A.1 : Values of Z(n) for n = 1(1)5000

Table A.1.1 : Values of Z(n) for n = 1(1)352

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1	1	45	9	89	88	133	56	177	59	221	51	265	105	309	102
2	3	46	23	90	35	134	67	178	88	222	36	266	56	310	124
3	2	47	46	91	13	135	54	179	178	223	222	267	89	311	310
4	7	48	32	92	23	136	16	180	80	224	63	268	200	312	143
5	4	49	48	93	30	137	136	181	180	225	99	269	268	313	312
6	3	50	24	94	47	138	23	182	91	226	112	270	80	314	156
7	6	51	17	95	19	139	138	183	60	227	226	271	270	315	35
8	15	52	39	96	63	140	55	184	160	228	56	272	255	316	79
9	8	53	52	97	96	141	47	185	74	229	228	273	77	317	316
10	4	54	27	98	48	142	71	186	92	230	115	274	136	318	159
11	10	55	10	99	44	143	65	187	33	231	21	275	99	319	87
12	8	56	48	100	24	144	63	188	47	232	144	276	23	320	255
13	12	57	18	101	100	145	29	189	27	233	232	277	276	321	107
14	7	58	28	102	51	146	72	190	19	234	116	278	139	322	91
15	5	59	58	103	102	147	48	191	190	235	94	279	62	323	152
16	31	60	15	104	64	148	111	192	128	236	176	280	160	324	80
17	16	61	60	105	14	149	148	193	192	237	78	281	280	325	25
18	8	62	31	106	52	150	24	194	96	238	84	282	47	326	163
19	18	63	27	107	106	151	150	195	39	239	238	283	282	327	108
20	15	64	127	108	80	152	95	196	48	240	95	284	71	328	287
21	6	65	25	109	108	153	17	197	196	241	240	285	75	329	140
22	11	66	11	110	44	154	55	198	44	242	120	286	143	330	44
23	22	67	66	111	36	155	30	199	198	243	242	287	41	331	330
24	15	68	16	112	63	156	39	200	175	244	183	288	63	332	248
25	24	69	23	113	112	157	156	201	66	245	49	289	288	333	36
26	12	70	20	114	56	158	79	202	100	246	123	290	115	334	167
27	26	71	70	115	45	159	53	203	28	247	38	291	96	335	134
28	7	72	63	116	87	160	64	204	119	248	31	292	72	336	63
29	28	73	72	117	26	161	69	205	40	249	83	293	292	337	336
30	15	74	36	118	59	162	80	206	103	250	124	294	48	338	168
31	30	75	24	119	34	163	162	207	45	251	250	295	59	339	113
32	63	76	56	120	15	164	40	208	64	252	63	296	111	340	119
33	11	77	21	121	120	165	44	209	76	253	22	297	54	341	154
34	16	78	12	122	60	166	83	210	20	254	127	298	148	342	152
35	14	79	78	123	41	167	166	211	210	255	50	299	91	343	342
36	8	80	64	124	31	168	48	212	159	256	511	300	24	344	128
37	36	81	80	125	124	169	168	213	71	257	256	301	42	345	45
38	19	82	40	126	27	170	84	214	107	258	128	302	151	346	172
39	12	83	82	127	126	171	18	215	85	259	111	303	101	347	346
40	15	84	48	128	255	172	128	216	80	260	39	304	95	348	87
41	40	85	34	129	42	173	172	217	62	261	116	305	60	349	348
42	20	86	43	130	39	174	87	218	108	262	131	306	135	350	175
43	42	87	29	131	130	175	49	219	72	263	262	307	306	351	26
44	32	88	32	132	32	176	32	220	55	264	32	308	55	352	319

Table A.1.2 : Values of Z(n) for n = 353(1)720

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
353	352	399	56	445	89	491	490	537	179	583	264	629	221	675	324
354	59	400	224	446	223	492	287	538	268	584	511	630	35	676	168
355	70	401	400	447	149	493	203	539	98	585	90	631	630	677	676
356	88	402	200	448	384	494	208	540	80	586	292	632	79	678	339
357	84	403	155	449	448	495	44	541	540	587	586	633	210	679	97
358	179	404	303	450	99	496	31	542	271	588	48	634	316	680	255
359	358	405	80	451	164	497	70	543	180	589	247	635	254	681	227
360	80	406	28	452	112	498	83	544	255	590	59	636	159	682	340
361	360	407	110	453	150	499	498	545	109	591	197	637	195	683	682
362	180	408	255	454	227	500	375	546	104	592	480	638	87	684	152
363	120	409	408	455	90	501	167	547	546	593	592	639	71	685	274
364	104	410	40	456	95	502	251	548	136	594	296	640	255	686	343
365	145	411	137	457	456	503	502	549	243	595	34	641	640	687	228
366	60	412	103	458	228	504	63	550	99	596	447	642	107	688	128
367	366	413	118	459	135	505	100	551	57	597	198	643	642	689	52
368	160	414	207	460	160	506	252	552	207	598	91	644	160	690	275
369	81	415	165	461	460	507	168	553	237	599	598	645	129	691	690
370	184	416	64	462	132	508	127	554	276	600	224	646	152	692	519
371	105	417	138	463	462	509	508	555	74	601	600	647	646	693	98
372	216	418	76	464	319	510	84	556	416	602	300	648	80	694	347
373	372	419	418	465	30	511	146	557	556	603	134	649	176	695	139
374	187	420	104	466	232	512	1023	558	216	604	151	650	299	696	144
375	125	421	420	467	466	513	189	559	129	605	120	651	62	697	204
376	47	422	211	468	143	514	256	560	160	606	303	652	488	698	348
377	116	423	188	469	133	515	205	561	33	607	606	653	652	699	233
378	27	424	159	470	140	516	128	562	280	608	512	654	108	700	175
379	378	425	50	471	156	517	187	563	562	609	174	655	130	701	700
380	95	426	71	472	176	518	111	564	47	610	60	656	287	702	324
381	126	427	182	473	43	519	173	565	225	611	234	657	72	703	37
382	191	428	320	474	236	520	64	566	283	612	135	658	140	704	384
383	382	429	65	475	75	521	520	567	161	613	612	659	658	705	140
384	255	430	215	476	119	522	116	568	496	614	307	660	120	706	352
385	55	431	430	477	53	523	522	569	568	615	164	661	660	707	202
386	192	432	351	478	239	524	392	570	75	616	175	662	331	708	176
387	171	433	432	479	478	525	125	571	570	617	616	663	51	709	708
388	96	434	216	480	255	526	263	572	143	618	308	664	415	710	284
389	388	435	29	481	221	527	186	573	191	619	618	665	189	711	315
390	39	436	327	482	240	528	32	574	287	620	279	666	36	712	623
391	68	437	114	483	69	529	528	575	275	621	161	667	115	713	92
392	48	438	72	484	120	530	159	576	512	622	311	668	167	714	84
393	131	439	438	485	194	531	117	577	576	623	266	669	222	715	65
394	196	440	175	486	243	532	56	578	288	624	351	670	200	716	536
395	79	441	98	487	486	533	246	579	192	625	624	671	121	717	239
396	143	442	51	488	304	534	267	580	144	626	312	672	63	718	359
397	396	443	442	489	162	535	214	581	83	627	132	673	672	719	718
398	199	444	111	490	195	536	335	582	96	628	471	674	336	720	224

Table A.1.3 : Values of Z(n) for n = 721(1)1088

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
721	308	767	117	813	270	859	858	905	180	951	317	997	996	1043	447
722	360	768	512	814	296	860	215	906	452	952	272	998	499	1044	144
723	240	769	768	815	325	861	41	907	906	953	952	999	296	1045	209
724	543	770	55	816	255	862	431	908	680	954	423	1000	624	1046	523
725	174	771	257	817	171	863	862	909	404	955	190	1001	77	1047	348
726	120	772	192	818	408	864	512	910	104	956	239	1002	167	1048	655
727	726	773	772	819	90	865	345	911	910	957	87	1003	118	1049	1048
728	272	774	171	820	40	866	432	912	95	958	479	1004	752	1050	224
729	728	775	124	821	820	867	288	913	165	959	273	1005	134	1051	1050
730	219	776	96	822	411	868	216	914	456	960	255	1006	503	1052	263
731	85	777	111	823	822	869	395	915	60	961	960	1007	265	1053	324
732	183	778	388	824	720	870	144	916	687	962	259	1008	63	1054	340
733	732	779	246	825	99	871	402	917	392	963	107	1009	1008	1055	210
734	367	780	39	826	412	872	544	918	135	964	240	1010	100	1056	384
735	195	781	142	827	826	873	387	919	918	965	385	1011	336	1057	301
736	575	782	68	828	207	874	436	920	160	966	252	1012	528	1058	528
737	66	783	377	829	828	875	125	921	306	967	966	1013	1012	1059	353
738	287	784	735	830	415	876	72	922	460	968	847	1014	168	1060	159
739	738	785	314	831	276	877	876	923	142	969	152	1015	174	1061	1060
740	184	786	131	832	767	878	439	924	231	970	484	1016	127	1062	531
741	38	787	786	833	391	879	293	925	74	971	970	1017	225	1063	1062
742	371	788	591	834	416	880	319	926	463	972	728	1018	508	1064	399
743	742	789	263	835	334	881	880	927	206	973	139	1019	1018	1065	284
744	464	790	79	836	208	882	440	928	319	974	487	1020	119	1066	532
745	149	791	112	837	216	883	882	929	928	975	299	1021	1020	1067	484
746	372	792	143	838	419	884	272	930	155	976	671	1022	364	1068	623
747	332	793	182	839	838	885	59	931	342	977	976	1023	186	1069	1068
748	407	794	396	840	224	886	443	932	232	978	488	1024	2047	1070	320
749	321	795	105	841	840	887	886	933	311	979	88	1025	450	1071	153
750	375	796	199	842	420	888	111	934	467	980	440	1026	323	1072	736
751	750	797	796	843	281	889	126	935	220	981	108	1027	78	1073	406
752	704	798	56	844	632	890	355	936	143	982	491	1028	256	1074	179
753	251	799	187	845	169	891	242	937	936	983	982	1029	342	1075	300
754	116	800	575	846	188	892	223	938	335	984	287	1030	515	1076	807
755	150	801	89	847	363	893	94	939	312	985	394	1031	1030	1077	359
756	216	802	400	848	159	894	447	940	375	986	203	1032	128	1078	440
757	756	803	219	849	282	895	179	941	940	987	140	1033	1032	1079	415
758	379	804	200	850	424	896	511	942	156	988	208	1034	187	1080	80
759	230	805	69	851	184	897	207	943	368	989	344	1035	45	1081	46
760	95	806	155	852	71	898	448	944	767	990	44	1036	111	1082	540
761	760	807	269	853	852	899	434	945	189	991	990	1037	305	1083	360
762	380	808	303	854	244	900	224	946	43	992	960	1038	519	1084	271
763	217	809	808	855	170	901	424	947	946	993	330	1039	1038	1085	154
764	191	810	80	856	320	902	164	948	552	994	496	1040	64	1086	180
765	135	811	810	857	856	903	42	949	364	995	199	1041	347	1087	1086
766	383	812	231	858	143	904	112	950	75	996	248	1042	520	1088	255

Table A.1.4 : Values of Z(n) for n = 1089(1)1456

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1089	242	1135	454	1181	1180	1227	408	1273	133	1319	1318	1365	90	1411	663
1090	435	1136	639	1182	591	1228	920	1274	195	1320	384	1366	683	1412	352
1091	1090	1137	378	1183	168	1229	1228	1275	50	1321	1320	1367	1366	1413	314
1092	104	1138	568	1184	703	1230	164	1276	87	1322	660	1368	512	1414	504
1093	1092	1139	67	1185	315	1231	1230	1277	1276	1323	539	1369	1368	1415	565
1094	547	1140	95	1186	592	1232	384	1278	71	1324	992	1370	684	1416	176
1095	219	1141	489	1187	1186	1233	548	1279	1278	1325	424	1371	456	1417	545
1096	959	1142	571	1188	296	1234	616	1280	1024	1326	51	1372	343	1418	708
1097	1096	1143	126	1189	492	1235	285	1281	182	1327	1326	1373	1372	1419	429
1098	243	1144	143	1190	84	1236	720	1282	640	1328	415	1374	228	1420	639
1099	314	1145	229	1191	396	1237	1236	1283	1282	1329	443	1375	374	1421	637
1100	175	1146	191	1192	447	1238	619	1284	320	1330	399	1376	128	1422	315
1101	366	1147	185	1193	1192	1239	294	1285	514	1331	1330	1377	323	1423	1422
1102	551	1148	287	1194	596	1240	464	1286	643	1332	296	1378	52	1424	800
1103	1102	1149	383	1195	239	1241	510	1287	143	1333	558	1379	196	1425	75
1104	575	1150	275	1196	207	1242	459	1288	160	1334	115	1380	344	1426	92
1105	169	1151	1150	1197	189	1243	451	1289	1288	1335	89	1381	1380	1427	1426
1106	315	1152	512	1198	599	1244	311	1290	215	1336	1168	1382	691	1428	119
1107	81	1153	1152	1199	109	1245	165	1291	1290	1337	573	1383	461	1429	1428
1108	831	1154	576	1200	224	1246	356	1292	152	1338	668	1384	864	1430	220
1109	1108	1155	209	1201	1200	1247	86	1293	431	1339	103	1385	554	1431	53
1110	444	1156	288	1202	600	1248	767	1294	647	1340	200	1386	252	1432	895
1111	505	1157	533	1203	401	1249	1248	1295	259	1341	297	1387	437	1433	1432
1112	416	1158	192	1204	343	1250	624	1296	1215	1342	671	1388	1040	1434	239
1113	105	1159	304	1205	240	1251	278	1297	1296	1343	237	1389	462	1435	245
1114	556	1160	144	1206	468	1252	312	1298	176	1344	384	1390	139	1436	359
1115	445	1161	215	1207	425	1253	357	1299	432	1345	269	1391	428	1437	479
1116	216	1162	83	1208	1056	1254	132	1300	624	1346	672	1392	608	1438	719
1117	1116	1163	1162	1209	155	1255	250	1301	1300	1347	449	1393	398	1439	1438
1118	559	1164	96	1210	120	1256	784	1302	216	1348	336	1394	204	1440	575
1119	372	1165	465	1211	518	1257	419	1303	1302	1349	284	1395	279	1441	131
1120	384	1166	264	1212	303	1258	407	1304	815	1350	324	1396	1047	1442	308
1121	531	1167	389	1213	1212	1259	1258	1305	144	1351	385	1397	253	1443	221
1122	407	1168	511	1214	607	1260	224	1306	652	1352	1183	1398	699	1444	360
1123	1122	1169	167	1215	485	1261	194	1307	1306	1353	164	1399	1398	1445	289
1124	280	1170	260	1216	512	1262	631	1308	327	1354	676	1400	175	1446	240
1125	125	1171	1170	1217	1216	1263	420	1309	153	1355	270	1401	467	1447	1446
1126	563	1172	879	1218	203	1264	1184	1310	524	1356	791	1402	700	1448	543
1127	391	1173	68	1219	529	1265	230	1311	114	1357	413	1403	183	1449	161
1128	47	1174	587	1220	304	1266	632	1312	1024	1358	679	1404	351	1450	724
1129	1128	1175	375	1221	110	1267	181	1313	403	1359	603	1405	280	1451	1450
1130	339	1176	48	1222	376	1268	951	1314	72	1360	255	1406	703	1452	120
1131	116	1177	428	1223	1222	1269	188	1315	525	1361	1360	1407	335	1453	1452
1132	848	1178	247	1224	288	1270	380	1316	328	1362	227	1408	1023	1454	727
1133	308	1179	261	1225	49	1271	123	1317	438	1363	376	1409	1408	1455	194
1134	567	1180	295	1226	612	1272	159	1318	659	1364	495	1410	140	1456	832

Table A.1.5 : Values of Z(n) for n = 1457(1)1824

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1457	93	1503	333	1549	1548	1595	319	1641	546	1687	482	1733	1732	1779	593
1458	728	1504	704	1550	124	1596	56	1642	820	1688	1055	1734	288	1780	800
1459	1458	1505	300	1551	329	1597	1596	1643	371	1689	563	1735	694	1781	273
1460	584	1506	251	1552	96	1598	187	1644	959	1690	675	1736	496	1782	648
1461	486	1507	274	1553	1552	1599	246	1645	140	1691	266	1737	386	1783	1782
1462	731	1508	376	1554	111	1600	1024	1646	823	1692	423	1738	395	1784	223
1463	76	1509	503	1555	310	1601	1600	1647	243	1693	1692	1739	517	1785	84
1464	671	1510	604	1556	1167	1602	711	1648	927	1694	363	1740	144	1786	892
1465	585	1511	1510	1557	692	1603	686	1649	679	1695	225	1741	1740	1787	1786
1466	732	1512	783	1558	532	1604	400	1650	99	1696	1536	1742	468	1788	447
1467	162	1513	356	1559	1558	1605	320	1651	507	1697	1696	1743	83	1789	1788
1468	367	1514	756	1560	480	1606	219	1652	944	1698	848	1744	544	1790	179
1469	338	1515	404	1561	223	1607	1606	1653	57	1699	1698	1745	349	1791	198
1470	195	1516	1136	1562	780	1608	335	1654	827	1700	424	1746	387	1792	511
1471	1470	1517	369	1563	521	1609	1608	1655	330	1701	728	1747	1746	1793	814
1472	896	1518	252	1564	391	1610	160	1656	207	1702	184	1748	551	1794	207
1473	491	1519	588	1565	625	1611	179	1657	1656	1703	130	1749	264	1795	359
1474	736	1520	95	1566	783	1612	247	1658	828	1704	639	1750	875	1796	448
1475	649	1521	675	1567	1566	1613	1612	1659	237	1705	154	1751	102	1797	599
1476	287	1522	760	1568	832	1614	807	1660	415	1706	852	1752	656	1798	464
1477	210	1523	1522	1569	522	1615	170	1661	604	1707	569	1753	1752	1799	770
1478	739	1524	888	1570	784	1616	1312	1662	276	1708	671	1754	876	1800	224
1479	203	1525	549	1571	1570	1617	98	1663	1662	1709	1708	1755	324	1801	1800
1480	480	1526	763	1572	392	1618	808	1664	767	1710	360	1756	439	1802	424
1481	1480	1527	509	1573	363	1619	1618	1665	369	1711	58	1757	251	1803	600
1482	455	1528	191	1574	787	1620	80	1666	391	1712	320	1758	879	1804	615
1483	1482	1529	417	1575	125	1621	1620	1667	1666	1713	570	1759	1758	1805	360
1484	847	1530	135	1576	591	1622	811	1668	416	1714	856	1760	319	1806	300
1485	54	1531	1530	1577	664	1623	540	1669	1668	1715	685	1761	587	1807	416
1486	743	1532	383	1578	263	1624	608	1670	500	1716	143	1762	880	1808	1695
1487	1486	1533	146	1579	1578	1625	624	1671	557	1717	101	1763	860	1809	134
1488	960	1534	767	1580	79	1626	812	1672	208	1718	859	1764	440	1810	180
1489	1488	1535	614	1581	186	1627	1626	1673	238	1719	764	1765	705	1811	1810
1490	595	1536	1023	1582	112	1628	296	1674	216	1720	559	1766	883	1812	1056
1491	426	1537	318	1583	1582	1629	180	1675	200	1721	1720	1767	341	1813	147
1492	1119	1538	768	1584	351	1630	815	1676	1256	1722	287	1768	272	1814	907
1493	1492	1539	323	1585	634	1631	699	1677	129	1723	1722	1769	609	1815	120
1494	332	1540	55	1586	792	1632	255	1678	839	1724	431	1770	59	1816	1135
1495	299	1541	736	1587	528	1633	781	1679	437	1725	275	1771	230	1817	552
1496	527	1542	771	1588	1191	1634	171	1680	224	1726	863	1772	1328	1818	404
1497	498	1543	1542	1589	454	1635	435	1681	1680	1727	627	1773	197	1819	748
1498	427	1544	192	1590	159	1636	408	1682	840	1728	512	1774	887	1820	104
1499	1498	1545	309	1591	258	1637	1636	1683	153	1729	455	1775	425	1821	606
1500	375	1546	772	1592	1392	1638	468	1684	1263	1730	519	1776	480	1822	911
1501	474	1547	272	1593	648	1639	297	1685	674	1731	576	1777	1776	1823	1822
1502	751	1548	215	1594	796	1640	655	1686	843	1732	432	1778	888	1824	512

Table A.1.6 : Values of Z(n) for n = 1825(1)2192

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1825	875	1871	1870	1917	567	1963	754	2009	245	2055	410	2101	572	2147	113
1826	747	1872	351	1918	959	1964	1472	2010	200	2056	256	2102	1051	2148	536
1827	377	1873	1872	1919	303	1965	524	2011	2010	2057	968	2103	701	2149	307
1828	456	1874	936	1920	255	1966	983	2012	503	2058	1028	2104	1840	2150	300
1829	589	1875	624	1921	339	1967	280	2013	549	2059	638	2105	420	2151	477
1830	60	1876	335	1922	960	1968	287	2014	1007	2060	720	2106	324	2152	1344
1831	1830	1877	1876	1923	641	1969	715	2015	155	2061	458	2107	343	2153	2152
1832	687	1878	312	1924	480	1970	984	2016	63	2062	1031	2108	527	2154	359
1833	234	1879	1878	1925	175	1971	729	2017	2016	2063	2062	2109	665	2155	430
1834	392	1880	704	1926	107	1972	696	2018	1008	2064	128	2110	844	2156	440
1835	734	1881	341	1927	328	1973	1972	2019	672	2065	294	2111	2110	2157	719
1836	135	1882	940	1928	240	1974	140	2020	504	2066	1032	2112	384	2158	415
1837	835	1883	538	1929	642	1975	474	2021	516	2067	636	2113	2112	2159	254
1838	919	1884	471	1930	579	1976	208	2022	336	2068	704	2114	755	2160	864
1839	612	1885	260	1931	1930	1977	659	2023	867	2069	2068	2115	234	2161	2160
1840	160	1886	368	1932	552	1978	344	2024	528	2070	459	2116	528	2162	1080
1841	525	1887	221	1933	1932	1979	1978	2025	324	2071	436	2117	145	2163	308
1842	920	1888	767	1934	967	1980	440	2026	1012	2072	111	2118	1059	2164	1623
1843	873	1889	1888	1935	215	1981	566	2027	2026	2073	690	2119	325	2165	865
1844	1383	1890	539	1936	1088	1982	991	2028	168	2074	731	2120	159	2166	360
1845	369	1891	61	1937	298	1983	660	2029	2028	2075	249	2121	504	2167	197
1846	780	1892	472	1938	152	1984	1023	2030	580	2076	519	2122	1060	2168	271
1847	1846	1893	630	1939	553	1985	794	2031	677	2077	402	2123	385	2169	963
1848	384	1894	947	1940	679	1986	992	2032	127	2078	1039	2124	648	2170	279
1849	1848	1895	379	1941	647	1987	1986	2033	855	2079	539	2125	374	2171	1001
1850	924	1896	1184	1942	971	1988	496	2034	791	2080	64	2126	1063	2172	543
1851	617	1897	812	1943	870	1989	441	2035	110	2081	2080	2127	708	2173	901
1852	463	1898	364	1944	1215	1990	199	2036	1527	2082	347	2128	608	2174	1087
1853	544	1899	422	1945	389	1991	362	2037	581	2083	2082	2129	2128	2175	174
1854	720	1900	399	1946	139	1992	912	2038	1019	2084	520	2130	284	2176	255
1855	105	1901	1900	1947	176	1993	1992	2039	2038	2085	555	2131	2130	2177	622
1856	1536	1902	951	1948	487	1994	996	2040	255	2086	447	2132	1312	2178	1088
1857	618	1903	692	1949	1948	1995	189	2041	156	2087	2086	2133	1026	2179	2178
1858	928	1904	832	1950	299	1996	1496	2042	1020	2088	144	2134	484	2180	544
1859	506	1905	254	1951	1950	1997	1996	2043	908	2089	2088	2135	244	2181	726
1860	279	1906	952	1952	1280	1998	296	2044	511	2090	759	2136	623	2182	1091
1861	1860	1907	1906	1953	62	1999	1998	2045	409	2091	204	2137	2136	2183	295
1862	588	1908	423	1954	976	2000	1375	2046	495	2092	1568	2138	1068	2184	272
1863	161	1909	414	1955	390	2001	551	2047	712	2093	91	2139	92	2185	114
1864	1631	1910	764	1956	488	2002	363	2048	4095	2094	348	2140	320	2186	1092
1865	745	1911	195	1957	721	2003	2002	2049	683	2095	419	2141	2140	2187	2186
1866	311	1912	239	1958	88	2004	167	2050	1024	2096	1440	2142	476	2188	1640
1867	1866	1913	1912	1959	653	2005	400	2051	293	2097	233	2143	2142	2189	198
1868	1400	1914	87	1960	735	2006	884	2052	512	2098	1048	2144	1407	2190	219
1869	266	1915	765	1961	370	2007	891	2053	2052	2099	2098	2145	65	2191	938
1870	220	1916	479	1962	108	2008	752	2054	948	2100	224	2146	1072	2192	959

Table A.1.7 : Values of Z(n) for n = 2193(1)2558

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
2193	645	2239	2238	2285	914	2331	629	2377	2376	2423	2422	2469	822	2515	1005
2194	1096	2240	384	2286	1016	2332	264	2378	492	2424	303	2470	455	2516	407
2195	439	2241	1079	2287	2286	2333	2332	2379	182	2425	775	2471	706	2517	839
2196	792	2242	531	2288	351	2334	1167	2380	119	2426	1212	2472	720	2518	1259
2197	2196	2243	2242	2289	545	2335	934	2381	2380	2427	809	2473	2472	2519	1144
2198	784	2244	407	2290	915	2336	511	2382	396	2428	607	2474	1236	2520	224
2199	732	2245	449	2291	869	2337	246	2383	2382	2429	693	2475	99	2521	2520
2200	175	2246	1123	2292	191	2338	167	2384	447	2430	1215	2476	1856	2522	1260
2201	496	2247	321	2293	2292	2339	2338	2385	530	2431	220	2477	2476	2523	840
2202	1100	2248	1967	2294	1147	2340	584	2386	1192	2432	512	2478	531	2524	631
2203	2202	2249	519	2295	135	2341	2340	2387	154	2433	810	2479	1072	2525	100
2204	551	2250	999	2296	287	2342	1171	2388	1392	2434	1216	2480	960	2526	420
2205	440	2251	2250	2297	2296	2343	638	2389	2388	2435	974	2481	827	2527	721
2206	1103	2252	1688	2298	383	2344	879	2390	239	2436	231	2482	1240	2528	1343
2207	2206	2253	750	2299	968	2345	335	2391	797	2437	2436	2483	572	2529	1124
2208	575	2254	391	2300	575	2346	68	2392	207	2438	1219	2484	1080	2530	275
2209	2208	2255	164	2301	117	2347	2346	2393	2392	2439	270	2485	70	2531	2530
2210	220	2256	704	2302	1151	2348	1760	2394	531	2440	304	2486	451	2532	632
2211	66	2257	1036	2303	1127	2349	405	2395	479	2441	2440	2487	828	2533	595
2212	552	2258	1128	2304	512	2350	375	2396	599	2442	296	2488	2176	2534	1267
2213	2212	2259	251	2305	460	2351	2350	2397	611	2443	349	2489	1178	2535	675
2214	1107	2260	904	2306	1152	2352	735	2398	1199	2444	376	2490	995	2536	1584
2215	885	2261	322	2307	768	2353	181	2399	2398	2445	489	2491	423	2537	472
2216	831	2262	116	2308	576	2354	428	2400	575	2446	1223	2492	623	2538	188
2217	738	2263	1022	2309	2308	2355	314	2401	2400	2447	2446	2493	1107	2539	2538
2218	1108	2264	848	2310	384	2356	247	2402	1200	2448	288	2494	1160	2540	1015
2219	951	2265	150	2311	2310	2357	2356	2403	890	2449	868	2495	499	2541	363
2220	480	2266	308	2312	288	2358	1179	2404	600	2450	1175	2496	767	2542	123
2221	2220	2267	2266	2313	513	2359	336	2405	259	2451	171	2497	681	2543	2542
2222	1111	2268	567	2314	623	2360	944	2406	1203	2452	1839	2498	1248	2544	159
2223	494	2269	2268	2315	925	2361	786	2407	580	2453	891	2499	441	2545	509
2224	416	2270	680	2316	192	2362	1180	2408	559	2454	408	2500	624	2546	1139
2225	800	2271	756	2317	993	2363	833	2409	219	2455	490	2501	122	2547	566
2226	371	2272	639	2318	304	2364	591	2410	240	2456	1535	2502	972	2548	832
2227	917	2273	2272	2319	773	2365	429	2411	2410	2457	350	2503	2502	2549	2548
2228	1671	2274	1136	2320	319	2366	168	2412	1071	2458	1228	2504	2191	2550	900
2229	743	2275	350	2321	1055	2367	1052	2413	380	2459	2458	2505	500	2551	2550
2230	1115	2276	568	2322	215	2368	1664	2414	1207	2460	615	2506	895	2552	319
2231	873	2277	252	2323	505	2369	206	2415	69	2461	321	2507	436	2553	666
2232	495	2278	67	2324	664	2370	315	2416	1056	2462	1231	2508	759	2554	1276
2233	231	2279	688	2325	650	2371	2370	2417	2416	2463	821	2509	1157	2555	364
2234	1116	2280	95	2326	1163	2372	592	2418	155	2464	384	2510	1004	2556	71
2235	149	2281	2280	2327	715	2373	678	2419	943	2465	289	2511	1053	2557	2556
2236	559	2282	651	2328	96	2374	1187	2420	120	2466	548	2512	1727	2558	1279
2237	2236	2283	761	2329	136	2375	874	2421	269	2467	2466	2513	1077	2559	852
2238	372	2284	1712	2330	699	2376	351	2422	692	2468	616	2514	419	2560	1024

Table A.1.8 : Values of Z(n) for n = 2561(1)2928

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
2561	1182	2607	395	2653	378	2699	2698	2745	305	2791	2790	2837	2836	2883	960
2562	671	2608	1792	2654	1327	2700	999	2746	1372	2792	1744	2838	516	2884	720
2563	1165	2609	2608	2655	530	2701	73	2747	737	2793	342	2839	1002	2885	1154
2564	640	2610	144	2656	2240	2702	1351	2748	687	2794	1143	2840	639	2886	480
2565	189	2611	1119	2657	2656	2703	476	2749	2748	2795	129	2841	947	2887	2886
2566	1283	2612	1959	2658	443	2704	1183	2750	1000	2796	1631	2842	783	2888	2527
2567	1207	2613	402	2659	2658	2705	540	2751	392	2797	2796	2843	2842	2889	107
2568	320	2614	1307	2660	399	2706	164	2752	128	2798	1399	2844	711	2890	1155
2569	734	2615	1045	2661	887	2707	2706	2753	2752	2799	621	2845	569	2891	294
2570	1284	2616	1199	2662	1331	2708	2031	2754	323	2800	224	2846	1423	2892	240
2571	857	2617	2616	2663	2662	2709	602	2755	550	2801	2800	2847	584	2893	263
2572	1928	2618	560	2664	1295	2710	1084	2756	688	2802	467	2848	2047	2894	1447
2573	248	2619	485	2665	779	2711	2710	2757	918	2803	2802	2849	406	2895	579
2574	143	2620	655	2666	1332	2712	1695	2758	196	2804	2103	2850	75	2896	543
2575	824	2621	2620	2667	126	2713	2712	2759	712	2805	374	2851	2850	2897	2896
2576	160	2622	551	2668	551	2714	943	2760	575	2806	183	2852	712	2898	252
2577	858	2623	731	2669	628	2715	180	2761	1254	2807	1203	2853	1268	2899	1338
2578	1288	2624	1024	2670	444	2716	679	2762	1380	2808	351	2854	1427	2900	1624
2579	2578	2625	125	2671	2670	2717	208	2763	306	2809	2808	2855	570	2901	966
2580	215	2626	403	2672	1503	2718	603	2764	2072	2810	280	2856	272	2902	1451
2581	1246	2627	851	2673	242	2719	2718	2765	315	2811	936	2857	2856	2903	2902
2582	1291	2628	72	2674	763	2720	255	2766	1383	2812	703	2858	1428	2904	1088
2583	287	2629	956	2675	749	2721	906	2767	2766	2813	290	2859	953	2905	580
2584	816	2630	1315	2676	1560	2722	1360	2768	864	2814	335	2860	935	2906	1452
2585	329	2631	876	2677	2676	2723	777	2769	780	2815	1125	2861	2860	2907	152
2586	431	2632	1456	2678	103	2724	680	2770	1384	2816	1023	2862	1431	2908	727
2587	597	2633	2632	2679	798	2725	1199	2771	815	2817	1251	2863	818	2909	2908
2588	647	2634	1316	2680	335	2726	376	2772	440	2818	1408	2864	895	2910	1164
2589	863	2635	340	2681	1148	2727	404	2773	235	2819	2818	2865	764	2911	1065
2590	259	2636	1976	2682	1043	2728	495	2774	1387	2820	375	2866	1432	2912	832
2591	2590	2637	585	2683	2682	2729	2728	2775	74	2821	650	2867	610	2913	971
2592	1215	2638	1319	2684	671	2730	104	2776	1040	2822	663	2868	239	2914	1363
2593	2592	2639	377	2685	179	2731	2730	2777	2776	2823	941	2869	151	2915	264
2594	1296	2640	384	2686	1343	2732	2048	2778	1388	2824	352	2870	615	2916	728
2595	345	2641	417	2687	2686	2733	911	2779	1190	2825	225	2871	638	2917	2916
2596	176	2642	1320	2688	1280	2734	1367	2780	695	2826	1412	2872	2512	2918	1459
2597	636	2643	881	2689	2688	2735	1094	2781	1133	2827	770	2873	169	2919	833
2598	432	2644	1983	2690	1075	2736	512	2782	428	2828	504	2874	479	2920	1679
2599	1242	2645	529	2691	207	2737	322	2783	483	2829	368	2875	874	2921	253
2600	624	2646	539	2692	672	2738	1368	2784	1536	2830	1415	2876	719	2922	1460
2601	288	2647	2646	2693	2692	2739	165	2785	1114	2831	893	2877	273	2923	1184
2602	1300	2648	992	2694	1347	2740	959	2786	1392	2832	767	2878	1439	2924	816
2603	684	2649	882	2695	440	2741	2740	2787	929	2833	2832	2879	2878	2925	324
2604	216	2650	424	2696	336	2742	456	2788	696	2834	871	2880	639	2926	76
2605	520	2651	241	2697	434	2743	844	2789	2788	2835	405	2881	602	2927	2926
2606	1303	2652	272	2698	284	2744	2400	2790	279	2836	2127	2882	131	2928	671

Table A.1.9 : Values of Z(n) for n = 2929(1)3296

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
2929	202	2975	475	3021	741	3067	3066	3113	1132	3159	728	3205	640	3251	3250
2930	879	2976	960	3022	1511	3068	767	3114	692	3160	79	3206	916	3252	1896
2931	977	2977	1144	3023	3022	3069	341	3115	889	3161	435	3207	1068	3253	3252
2932	2199	2978	1488	3024	1728	3070	920	3116	1311	3162	527	3208	400	3254	1627
2933	419	2979	1323	3025	725	3071	332	3117	1038	3163	3162	3209	3208	3255	279
2934	1304	2980	744	3026	356	3072	2048	3118	1559	3164	112	3210	320	3256	703
2935	1174	2981	813	3027	1008	3073	1316	3119	3118	3165	210	3211	1520	3257	3256
2936	367	2982	567	3028	2271	3074	1536	3120	480	3166	1583	3212	583	3258	180
2937	890	2983	627	3029	233	3075	450	3121	3120	3167	3166	3213	594	3259	3258
2938	1468	2984	1119	3030	404	3076	768	3122	223	3168	1088	3214	1607	3260	815
2939	2938	2985	795	3031	433	3077	543	3123	693	3169	3168	3215	1285	3261	1086
2940	440	2986	1492	3032	1136	3078	323	3124	1704	3170	1584	3216	1407	3262	699
2941	1037	2987	927	3033	674	3079	3078	3125	3124	3171	755	3217	3216	3263	753
2942	1471	2988	1079	3034	1147	3080	175	3126	1563	3172	792	3218	1608	3264	255
2943	108	2989	244	3035	1214	3081	78	3127	530	3173	835	3219	666	3265	1305
2944	2047	2990	299	3036	528	3082	736	3128	1104	3174	528	3220	160	3266	851
2945	589	2991	996	3037	3036	3083	3082	3129	447	3175	1524	3221	3220	3267	242
2946	491	2992	1088	3038	588	3084	1799	3130	939	3176	1984	3222	179	3268	816
2947	420	2993	656	3039	1013	3085	1234	3131	403	3177	1412	3223	879	3269	1400
2948	736	2994	1496	3040	1215	3086	1543	3132	783	3178	1588	3224	1456	3270	435
2949	983	2995	599	3041	3040	3087	342	3133	481	3179	1155	3225	300	3271	3270
2950	1475	2996	1176	3042	675	3088	192	3134	1567	3180	159	3226	1612	3272	2863
2951	454	2997	1295	3043	357	3089	3088	3135	209	3181	3180	3227	461	3273	1091
2952	287	2998	1499	3044	760	3090	515	3136	2303	3182	1332	3228	807	3274	1636
2953	2952	2999	2998	3045	174	3091	561	3137	3136	3183	1061	3229	3228	3275	524
2954	1476	3000	624	3046	1523	3092	2319	3138	1568	3184	1791	3230	475	3276	728
2955	590	3001	3000	3047	1385	3093	1031	3139	730	3185	195	3231	359	3277	1130
2956	2216	3002	1500	3048	1904	3094	272	3140	784	3186	648	3232	1919	3278	1639
2957	2956	3003	77	3049	3048	3095	619	3141	1395	3187	3186	3233	1219	3279	1092
2958	203	3004	751	3050	975	3096	1376	3142	1571	3188	2391	3234	440	3280	1024
2959	538	3005	600	3051	1242	3097	1140	3143	448	3189	1062	3235	1294	3281	578
2960	480	3006	1503	3052	1743	3098	1548	3144	1440	3190	319	3236	808	3282	1640
2961	188	3007	775	3053	1419	3099	1032	3145	629	3191	3190	3237	663	3283	1273
2962	1480	3008	2303	3054	1527	3100	775	3146	363	3192	399	3238	1619	3284	2463
2963	2962	3009	884	3055	234	3101	1329	3147	1049	3193	309	3239	1106	3285	584
2964	455	3010	300	3056	191	3102	516	3148	2360	3194	1596	3240	80	3286	371
2965	1185	3011	3010	3057	1019	3103	1391	3149	469	3195	639	3241	462	3287	1557
2966	1483	3012	752	3058	1111	3104	3007	3150	224	3196	799	3242	1620	3288	959
2967	344	3013	1310	3059	322	3105	459	3151	137	3197	138	3243	1034	3289	506
2968	847	3014	1232	3060	135	3106	1552	3152	2560	3198	779	3244	2432	3290	140
2969	2968	3015	134	3061	3060	3107	1195	3153	1050	3199	1371	3245	649	3291	1097
2970	539	3016	1247	3062	1531	3108	111	3154	664	3200	1024	3246	540	3292	823
2971	2970	3017	861	3063	1020	3109	3108	3155	630	3201	582	3247	764	3293	444
2972	743	3018	503	3064	383	3110	1244	3156	263	3202	1600	3248	608	3294	243
2973	990	3019	3018	3065	1225	3111	305	3157	286	3203	3202	3249	360	3295	659
2974	1487	3020	1359	3066	875	3112	1167	3158	1579	3204	711	3250	624	3296	2368

Table A.1.10 : Values of Z(n) for n = 3297(1)3664

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
3297	314	3343	3342	3389	3388	3435	915	3481	3480	3527	3526	3573	396	3619	329
3298	679	3344	703	3390	339	3436	2576	3482	1740	3528	783	3574	1787	3620	904
3299	3298	3345	669	3391	3390	3437	490	3483	1376	3529	3528	3575	649	3621	425
3300	824	3346	1672	3392	1536	3438	764	3484	871	3530	1059	3576	447	3622	1811
3301	3300	3347	3346	3393	116	3439	361	3485	204	3531	428	3577	146	3623	3622
3302	507	3348	216	3394	1696	3440	1375	3486	83	3532	2648	3578	1788	3624	1056
3303	1467	3349	985	3395	679	3441	185	3487	1584	3533	3532	3579	1193	3625	1624
3304	944	3350	200	3396	848	3442	1720	3488	2943	3534	588	3580	895	3626	147
3305	660	3351	1116	3397	473	3443	626	3489	1163	3535	504	3581	3580	3627	558
3306	551	3352	2095	3398	1699	3444	287	3490	1395	3536	832	3582	1592	3628	2720
3307	3306	3353	958	3399	308	3445	689	3491	3490	3537	917	3583	3582	3629	190
3308	2480	3354	1247	3400	799	3446	1723	3492	872	3538	1159	3584	3072	3630	120
3309	1103	3355	549	3401	1253	3447	765	3493	1497	3539	3538	3585	239	3631	3630
3310	1324	3356	839	3402	728	3448	431	3494	1747	3540	944	3586	1792	3632	2496
3311	902	3357	746	3403	82	3449	3448	3495	465	3541	3540	3587	1054	3633	518
3312	575	3358	1679	3404	184	3450	275	3496	1311	3542	252	3588	207	3634	552
3313	3312	3359	3358	3405	680	3451	203	3497	806	3543	1181	3589	776	3635	1454
3314	1656	3360	384	3406	1572	3452	863	3498	264	3544	1328	3590	359	3636	504
3315	390	3361	3360	3407	3406	3453	1151	3499	3498	3545	709	3591	189	3637	3636
3316	2487	3362	1680	3408	639	3454	627	3500	1000	3546	1575	3592	448	3638	748
3317	1177	3363	531	3409	973	3455	690	3501	1556	3547	3546	3593	3592	3639	1212
3318	315	3364	840	3410	340	3456	512	3502	1648	3548	887	3594	599	3640	559
3319	3318	3365	1345	3411	378	3457	3456	3503	1581	3549	168	3595	719	3641	330
3320	415	3366	747	3412	2559	3458	455	3504	1824	3550	1775	3596	464	3642	1820
3321	81	3367	259	3413	3412	3459	1152	3505	700	3551	1272	3597	1089	3643	3642
3322	604	3368	1263	3414	1707	3460	519	3506	1752	3552	1664	3598	1028	3644	911
3323	3322	3369	1122	3415	1365	3461	3460	3507	167	3553	968	3599	1769	3645	729
3324	831	3370	1684	3416	671	3462	576	3508	2631	3554	1776	3600	224	3646	1823
3325	399	3371	3370	3417	1071	3463	3462	3509	725	3555	315	3601	831	3647	1042
3326	1663	3372	1967	3418	1708	3464	432	3510	324	3556	888	3602	1800	3648	512
3327	1109	3373	3372	3419	1052	3465	440	3511	3510	3557	3556	3603	1200	3649	533
3328	2560	3374	1204	3420	360	3466	1732	3512	3072	3558	1779	3604	424	3650	875
3329	3328	3375	999	3421	1243	3467	3466	3513	1170	3559	3558	3605	720	3651	1217
3330	999	3376	1055	3422	1652	3468	288	3514	251	3560	800	3606	600	3652	912
3331	3330	3377	307	3423	489	3469	3468	3515	665	3561	1187	3607	3606	3653	1404
3332	391	3378	563	3424	320	3470	1040	3516	879	3562	1507	3608	1968	3654	783
3333	605	3379	217	3425	274	3471	533	3517	3516	3563	1526	3609	801	3655	85
3334	1667	3380	1520	3426	1712	3472	1952	3518	1759	3564	648	3610	360	3656	3199
3335	115	3381	735	3427	298	3473	1057	3519	459	3565	620	3611	942	3657	689
3336	416	3382	1424	3428	856	3474	1736	3520	384	3566	1783	3612	903	3658	1239
3337	141	3383	1393	3429	1269	3475	1250	3521	503	3567	492	3613	3612	3659	3658
3338	1668	3384	1503	3430	1715	3476	1264	3522	587	3568	223	3614	416	3660	975
3339	476	3385	1354	3431	657	3477	854	3523	1625	3569	1161	3615	240	3661	1568
3340	1335	3386	1692	3432	143	3478	1739	3524	880	3570	84	3616	1920	3662	1831
3341	1027	3387	1128	3433	3432	3479	1420	3525	375	3571	3570	3617	3616	3663	296
3342	1671	3388	847	3434	1615	3480	144	3526	860	3572	1880	3618	1808	3664	2976

Table A.1.11 : Values of Z(n) for n = 3665(1)4032

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
3665	1465	3711	1236	3757	1156	3803	3802	3849	1283	3895	779	3941	1126	3987	1772
3666	611	3712	1536	3758	1879	3804	951	3850	175	3896	3408	3942	1971	3988	2991
3667	1158	3713	1738	3759	357	3805	760	3851	3850	3897	432	3943	3942	3989	3988
3668	392	3714	1856	3760	704	3806	692	3852	855	3898	1948	3944	2175	3990	399
3669	1223	3715	1485	3761	3760	3807	1457	3853	3852	3899	1113	3945	525	3991	1534
3670	1100	3716	928	3762	836	3808	832	3854	328	3900	624	3946	1972	3992	2495
3671	3670	3717	413	3763	212	3809	585	3855	770	3901	1410	3947	3946	3993	1331
3672	1376	3718	1352	3764	2823	3810	380	3856	3615	3902	1951	3948	1127	3994	1996
3673	3672	3719	3718	3765	1004	3811	1442	3857	608	3903	1301	3949	1077	3995	799
3674	835	3720	464	3766	1344	3812	952	3858	1928	3904	1280	3950	1500	3996	296
3675	1175	3721	3720	3767	3766	3813	123	3859	680	3905	780	3951	1755	3997	1141
3676	919	3722	1860	3768	1727	3814	1907	3860	1544	3906	216	3952	1728	3998	1999
3677	3676	3723	510	3769	3768	3815	545	3861	351	3907	3906	3953	1474	3999	558
3678	612	3724	1519	3770	260	3816	1007	3862	1931	3908	976	3954	659	4000	2624
3679	1131	3725	149	3771	837	3817	693	3863	3862	3909	1302	3955	790	4001	4000
3680	575	3726	1863	3772	368	3818	1908	3864	735	3910	459	3956	344	4002	551
3681	818	3727	3726	3773	1715	3819	1139	3865	1545	3911	3910	3957	1319	4003	4002
3682	1315	3728	1631	3774	407	3820	1719	3866	1932	3912	815	3958	1979	4004	1000
3683	1015	3729	791	3775	150	3821	3820	3867	1289	3913	559	3959	962	4005	89
3684	920	3730	1119	3776	767	3822	195	3868	967	3914	1235	3960	495	4006	2003
3685	670	3731	286	3777	1259	3823	3822	3869	583	3915	405	3961	1631	4007	4006
3686	1843	3732	311	3778	1888	3824	3584	3870	215	3916	88	3962	1980	4008	1503
3687	1229	3733	3732	3779	3778	3825	900	3871	1421	3917	3916	3963	1320	4009	1899
3688	2304	3734	1867	3780	944	3826	1912	3872	1088	3918	1959	3964	991	4010	400
3689	713	3735	414	3781	398	3827	1246	3873	1290	3919	3918	3965	610	4011	573
3690	819	3736	2335	3782	1891	3828	87	3874	1936	3920	735	3966	660	4012	1887
3691	3690	3737	1110	3783	194	3829	546	3875	124	3921	1307	3967	3966	4013	4012
3692	1703	3738	356	3784	1375	3830	1915	3876	152	3922	1960	3968	1023	4014	891
3693	1230	3739	3738	3785	1514	3831	1277	3877	3876	3923	3922	3969	1862	4015	219
3694	1847	3740	560	3786	1892	3832	479	3878	1939	3924	872	3970	1984	4016	3263
3695	739	3741	86	3787	1623	3833	3832	3879	431	3925	1099	3971	1804	4017	1235
3696	384	3742	1871	3788	2840	3834	567	3880	1455	3926	1208	3972	992	4018	1763
3697	3696	3743	1576	3789	1683	3835	649	3881	3880	3927	153	3973	1507	4019	4018
3698	1848	3744	1664	3790	379	3836	959	3882	647	3928	1472	3974	1987	4020	200
3699	1781	3745	749	3791	1784	3837	1278	3883	352	3929	3928	3975	900	4021	4020
3700	999	3746	1872	3792	1184	3838	303	3884	2912	3930	524	3976	496	4022	2011
3701	3700	3747	1248	3793	3792	3839	1396	3885	629	3931	3930	3977	1066	4023	297
3702	1851	3748	936	3794	812	3840	1535	3886	1072	3932	983	3978	611	4024	3520
3703	1057	3749	1793	3795	230	3841	667	3887	506	3933	873	3979	345	4025	574
3704	463	3750	624	3796	584	3842	339	3888	1215	3934	280	3980	199	4026	671
3705	285	3751	1209	3797	3796	3843	854	3889	3888	3935	1574	3981	1326	4027	4026
3706	544	3752	335	3798	1476	3844	960	3890	1555	3936	1599	3982	1628	4028	1007
3707	1011	3753	972	3799	261	3845	769	3891	1296	3937	1270	3983	1707	4029	237
3708	720	3754	1876	3800	399	3846	1923	3892	1112	3938	715	3984	2240	4030	155
3709	3708	3755	750	3801	1085	3847	3846	3893	458	3939	909	3985	1594	4031	695
3710	1484	3756	312	3802	1900	3848	480	3894	176	3940	984	3986	1992	4032	1791

Table A.1.12 : Values of Z(n) for n = 4033(1)4400

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
4033	1961	4079	4078	4125	374	4171	387	4217	4216	4263	783	4309	278	4355	870
4034	2016	4080	255	4126	2063	4172	447	4218	740	4264	1312	4310	1724	4356	1088
4035	269	4081	847	4127	4126	4173	428	4219	4218	4265	1705	4311	1916	4357	4356
4036	1008	4082	156	4128	128	4174	2087	4220	1055	4266	2132	4312	1616	4358	2179
4037	1100	4083	1361	4129	4128	4175	500	4221	468	4267	1003	4313	227	4359	1452
4038	672	4084	3063	4130	944	4176	927	4222	2111	4268	1551	4314	719	4360	544
4039	1154	4085	645	4131	1700	4177	4176	4223	205	4269	1422	4315	1725	4361	979
4040	1615	4086	908	4132	1032	4178	2088	4224	1023	4270	244	4316	415	4362	2180
4041	449	4087	670	4133	4132	4179	398	4225	675	4271	4270	4317	1439	4363	4362
4042	516	4088	511	4134	636	4180	759	4226	2112	4272	800	4318	1904	4364	3272
4043	311	4089	986	4135	1654	4181	2034	4227	1409	4273	4272	4319	616	4365	485
4044	336	4090	1635	4136	704	4182	204	4228	1056	4274	2136	4320	1215	4366	295
4045	809	4091	4090	4137	1182	4183	1691	4229	4228	4275	1025	4321	1043	4367	396
4046	867	4092	495	4138	2068	4184	1568	4230	1080	4276	3207	4322	2160	4368	896
4047	284	4093	4092	4139	4138	4185	620	4231	4230	4277	987	4323	131	4369	2056
4048	736	4094	712	4140	575	4186	91	4232	528	4278	92	4324	1080	4370	759
4049	4048	4095	90	4141	1312	4187	158	4233	663	4279	1166	4325	2075	4371	93
4050	324	4096	8191	4142	436	4188	1047	4234	1971	4280	320	4326	308	4372	3279
4051	4050	4097	1445	4143	1380	4189	354	4235	1210	4281	1427	4327	4326	4373	4372
4052	3039	4098	683	4144	1183	4190	419	4236	2471	4282	2140	4328	2704	4374	2187
4053	965	4099	4098	4145	829	4191	1143	4237	892	4283	4282	4329	702	4375	1875
4054	2027	4100	1024	4146	2072	4192	2751	4238	2119	4284	1071	4330	1299	4376	2735
4055	810	4101	1367	4147	637	4193	1197	4239	1727	4285	1714	4331	426	4377	1458
4056	1520	4102	2051	4148	1768	4194	1863	4240	159	4286	2143	4332	360	4378	2188
4057	4056	4103	373	4149	1844	4195	839	4241	4240	4287	1428	4333	1238	4379	754
4058	2028	4104	512	4150	2075	4196	1048	4242	504	4288	1407	4334	2167	4380	584
4059	450	4105	820	4151	1778	4197	1398	4243	4242	4289	4288	4335	1155	4381	337
4060	840	4106	2052	4152	864	4198	2099	4244	3183	4290	780	4336	4064	4382	1252
4061	1178	4107	1368	4153	4152	4199	493	4245	849	4291	1225	4337	4336	4383	486
4062	2031	4108	1975	4154	2076	4200	224	4246	2123	4292	1072	4338	963	4384	959
4063	238	4109	587	4155	554	4201	4200	4247	1643	4293	1377	4339	4338	4385	1754
4064	127	4110	684	4156	1039	4202	572	4248	944	4294	2147	4340	279	4386	731
4065	270	4111	4110	4157	4156	4203	467	4249	1820	4295	859	4341	1446	4387	1926
4066	855	4112	256	4158	539	4204	3152	4250	2124	4296	1968	4342	2171	4388	1096
4067	1078	4113	1827	4159	4158	4205	840	4251	545	4297	4296	4343	2020	4389	132
4068	791	4114	968	4160	1664	4206	2103	4252	1063	4298	307	4344	543	4390	439
4069	312	4115	1645	4161	437	4207	601	4253	4252	4299	1433	4345	395	4391	4390
4070	924	4116	2400	4162	2080	4208	2367	4254	708	4300	1375	4346	1271	4392	1647
4071	413	4117	1610	4163	1448	4209	183	4255	184	4301	781	4347	161	4393	1909
4072	2544	4118	1420	4164	1040	4210	420	4256	1728	4302	2151	4348	1087	4394	2196
4073	4072	4119	1373	4165	1224	4211	4210	4257	945	4303	662	4349	4348	4395	585
4074	776	4120	720	4166	2083	4212	728	4258	2128	4304	1344	4350	899	4396	784
4075	325	4121	1585	4167	926	4213	1914	4259	4258	4305	245	4351	228	4397	4396
4076	3056	4122	2060	4168	3647	4214	343	4260	639	4306	2152	4352	4096	4398	732
4077	755	4123	588	4169	758	4215	1124	4261	4260	4307	1533	4353	1451	4399	1908
4078	2039	4124	1031	4170	555	4216	527	4262	2131	4308	359	4354	2176	4400	1375

Table A.1.13 : Values of Z(n) for n = 4401(1)4768

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
4401	162	4447	4446	4493	4492	4539	356	4585	524	4631	1683	4677	1559	4723	4722
4402	496	4448	4031	4494	1176	4540	680	4586	2292	4632	192	4678	2339	4724	3543
4403	629	4449	1482	4495	434	4541	1672	4587	417	4633	451	4679	4678	4725	350
4404	2568	4450	800	4496	2528	4542	756	4588	2479	4634	1323	4680	1520	4726	2363
4405	880	4451	4450	4497	1499	4543	825	4589	2118	4635	720	4681	1208	4727	1304
4406	2203	4452	1007	4498	519	4544	639	4590	135	4636	304	4682	2340	4728	591
4407	338	4453	365	4499	2045	4545	404	4591	4590	4637	4636	4683	1337	4729	4728
4408	608	4454	2227	4500	999	4546	2272	4592	287	4638	2319	4684	3512	4730	1375
4409	4408	4455	890	4501	643	4547	4546	4593	1530	4639	4638	4685	1874	4731	912
4410	440	4456	2784	4502	2251	4548	1136	4594	2296	4640	319	4686	780	4732	168
4411	802	4457	4456	4503	474	4549	4548	4595	919	4641	272	4687	1634	4733	4732
4412	1103	4458	743	4504	2815	4550	1000	4596	383	4642	1055	4688	3808	4734	1052
4413	1470	4459	1715	4505	424	4551	369	4597	4596	4643	4642	4689	521	4735	1894
4414	2207	4460	1560	4506	2252	4552	3983	4598	968	4644	215	4690	335	4736	3071
4415	1765	4461	1487	4507	4506	4553	1884	4599	657	4645	929	4691	4690	4737	1578
4416	896	4462	2231	4508	391	4554	252	4600	575	4646	2323	4692	1104	4738	2368
4417	630	4463	4462	4509	1836	4555	910	4601	214	4647	1548	4693	1443	4739	2030
4418	2208	4464	1952	4510	164	4556	1071	4602	767	4648	1743	4694	2347	4740	1184
4419	981	4465	94	4511	1040	4557	588	4603	4602	4649	4648	4695	939	4741	2155
4420	935	4466	231	4512	704	4558	688	4604	1151	4650	899	4696	1760	4742	2371
4421	4420	4467	1488	4513	4512	4559	1551	4605	614	4651	4650	4697	671	4743	1053
4422	803	4468	3351	4514	1036	4560	95	4606	1127	4652	3488	4698	1943	4744	592
4423	4422	4469	327	4515	300	4561	4560	4607	271	4653	846	4699	888	4745	364
4424	1343	4470	744	4516	1128	4562	2280	4608	4095	4654	715	4700	375	4746	791
4425	825	4471	526	4517	4516	4563	675	4609	418	4655	930	4701	1566	4747	2020
4426	2212	4472	559	4518	251	4564	1792	4610	460	4656	96	4702	2351	4748	3560
4427	931	4473	567	4519	4518	4565	165	4611	318	4657	4656	4703	4702	4749	1583
4428	2295	4474	2236	4520	1695	4566	2283	4612	1152	4658	136	4704	2303	4750	1500
4429	515	4475	1074	4521	1232	4567	4566	4613	658	4659	1553	4705	940	4751	4750
4430	2215	4476	1119	4522	475	4568	1712	4614	768	4660	1864	4706	2171	4752	351
4431	210	4477	1331	4523	4522	4569	1523	4615	780	4661	236	4707	522	4753	97
4432	831	4478	2239	4524	1247	4570	2284	4616	576	4662	1035	4708	1176	4754	2376
4433	1209	4479	1493	4525	724	4571	1959	4617	1215	4663	4662	4709	1938	4755	950
4434	2216	4480	1280	4526	1240	4572	1016	4618	2308	4664	847	4710	1884	4756	696
4435	1774	4481	4480	4527	503	4573	1614	4619	744	4665	1244	4711	672	4757	1206
4436	3327	4482	1079	4528	3679	4574	2287	4620	384	4666	2332	4712	1519	4758	792
4437	782	4483	4482	4529	1294	4575	549	4621	4620	4667	1794	4713	1571	4759	4758
4438	951	4484	1120	4530	755	4576	2431	4622	2311	4668	1167	4714	2356	4760	560
4439	965	4485	299	4531	1379	4577	597	4623	804	4669	1218	4715	574	4761	2115
4440	480	4486	2243	4532	824	4578	980	4624	288	4670	1400	4716	1440	4762	2380
4441	4440	4487	1281	4533	1511	4579	722	4625	999	4671	864	4717	2225	4763	1298
4442	2220	4488	527	4534	2267	4580	1144	4626	1799	4672	511	4718	336	4764	1191
4443	1481	4489	4488	4535	1814	4581	1017	4627	1322	4673	4672	4719	363	4765	1905
4444	1111	4490	1795	4536	1295	4582	2291	4628	623	4674	779	4720	1120	4766	2383
4445	889	4491	998	4537	2093	4583	4582	4629	1542	4675	374	4721	4720	4767	1134
4446	1196	4492	3368	4538	2268	4584	191	4630	2315	4676	167	4722	2360	4768	447

Table A.1.14 : Values of Z(n) for n = 4769(1)5000

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
4769	1254	4798	2399	4827	1608	4856	607	4885	1954	4914	728	4943	4942	4972	791
4770	900	4799	4798	4828	1207	4857	1619	4886	2443	4915	1965	4944	927	4973	4972
4771	1468	4800	2175	4829	439	4858	1735	4887	1809	4916	3687	4945	344	4974	828
4772	1192	4801	4800	4830	644	4859	903	4888	1456	4917	297	4946	2472	4975	199
4773	258	4802	2400	4831	4830	4860	1215	4889	4888	4918	2459	4947	969	4976	2176
4774	868	4803	1601	4832	3775	4861	4860	4890	815	4919	4818	4948	3711	4977	315
4775	2100	4804	1200	4833	1431	4862	220	4891	803	4920	1599	4949	1616	4978	2488
4776	1392	4805	960	4834	2416	4863	1620	4892	1223	4921	665	4950	99	4979	766
4777	561	4806	1512	4835	1934	4864	512	4893	699	4922	2139	4951	4950	4980	1079
4778	2388	4807	759	4836	1208	4865	139	4894	2447	4923	2187	4952	1856	4981	1172
4779	648	4808	4207	4837	2072	4866	2432	4895	890	4924	1231	4953	507	4982	423
4780	239	4809	686	4838	943	4867	2355	4896	1088	4925	1575	4954	2476	4983	1056
4781	1365	4810	259	4839	1613	4868	1216	4897	2241	4926	2463	4955	990	4984	623
4782	2391	4811	849	4840	1935	4869	540	4898	868	4927	2274	4956	944	4985	1994
4783	4782	4812	2807	4841	2162	4870	1460	4899	851	4928	384	4957	4956	4986	1107
4784	896	4813	4812	4842	2151	4871	4870	4900	1175	4929	371	4958	1072	4987	4986
4785	725	4814	580	4843	667	4872	608	4901	1014	4930	1275	4959	1044	4988	1160
4786	2392	4815	855	4844	1903	4873	1771	4902	171	4931	4930	4960	960	4989	1662
4787	4786	4816	1504	4845	170	4874	2436	4903	4902	4932	1232	4961	2419	4990	499
4788	1007	4817	4816	4846	2423	4875	624	4904	1839	4933	4932	4962	827	4991	713
4789	4788	4818	219	4847	1702	4876	2967	4905	980	4934	2467	4963	2127	4992	767
4790	479	4819	1342	4848	1919	4877	4876	4906	891	4935	140	4964	1240	4993	4992
4791	1596	4820	240	4849	1118	4878	2168	4907	700	4936	4319	4965	330	4994	1815
4792	4192	4821	1607	4850	775	4879	1189	4908	408	4937	4936	4966	572	4995	999
4793	4792	4822	2411	4851	98	4880	1280	4909	4908	4938	2468	4967	4966	4996	1248
4794	611	4823	636	4852	3639	4881	1626	4910	1964	4939	2244	4968	2943	4997	1577
4795	685	4824	1071	4853	1265	4882	2440	4911	1637	4940	455	4969	4968	4998	1224
4796	1199	4825	1350	4854	2427	4883	513	4912	1535	4941	243	4970	1064	4999	4998
4797	819	4826	380	4855	970	4884	296	4913	4912	4942	1764	4971	1656	5000	624

APPENDIX A.2 : Values of Z(np); n = 6, 7, 8, 10, 11, 12, 13, 16, 24, 32, 48, 96

Table 2.1 : Values of Z(6p), Z(7p), Z(8p), Z(10p), Z(11p), Z(12p), Z(13p), Z(16p), Z(24p), Z(32p), Z(48p) and Z(96p) for $2 \leq p \leq 181$

p	Z(6p)	Z(7p)	Z(8p)	Z(10p)	Z(11p)	Z(12p)	Z(13p)	Z(16p)	Z(24p)	Z(32p)	Z(48p)	Z(96p)
2	8	7	31	15	11	15	12	63	32	127	63	128
3	8	6	15	15	11	8	12	32	63	63	63	63
5	15	14	15	24	10	15	25	64	15	64	95	255
7	20	48	48	20	21	48	13	63	48	63	63	63
11	11	21	32	44	120	32	65	32	32	319	32	384
13	12	13	64	39	65	39	168	64	143	64	351	767
17	51	34	16	84	33	119	51	255	255	255	255	255
19	56	56	95	19	76	56	38	95	95	512	95	512
23	23	69	160	115	22	23	91	160	207	575	575	575
29	87	28	144	115	87	87	116	319	144	319	608	1536
31	92	62	31	124	154	216	155	31	464	960	960	960
37	36	111	111	184	110	111	221	480	111	703	480	1664
41	123	41	287	40	164	287	246	287	287	1024	287	1599
43	128	42	128	215	43	128	129	128	128	128	128	128
47	47	140	47	140	187	47	234	704	47	704	704	704
53	159	105	159	159	264	159	52	159	159	1536	159	1536
59	59	118	176	59	176	176	117	767	176	767	767	767
61	60	182	304	60	121	183	182	671	671	1280	671	1280
67	200	133	335	200	66	200	402	736	335	1407	1407	1407
71	71	70	496	284	142	71	142	639	639	639	639	639
73	72	146	511	219	219	72	364	511	656	511	1824	4160
79	236	237	79	79	395	552	78	1184	1184	1343	1184	1343
83	83	83	415	415	165	248	415	415	912	2240	2240	2240
89	267	266	623	355	88	623	533	800	623	2047	800	3648
97	96	97	96	484	484	96	194	96	96	3007	96	3200
101	303	202	303	100	505	303	403	1312	303	1919	1919	1919
103	308	308	720	515	308	720	103	927	720	2368	927	4223
107	107	321	320	320	428	320	428	320	320	320	320	320
109	108	217	544	435	109	327	545	544	1199	2943	2943	2943
113	339	112	112	339	451	791	338	1695	1695	1920	1695	1920
127	380	126	127	380	253	888	507	127	1904	127	3936	8000
131	131	392	655	524	131	392	130	1440	1440	2751	1440	2751
137	411	273	959	684	274	959	273	959	959	959	959	959
139	416	139	416	139	417	416	416	416	416	4031	416	4031
149	447	447	447	595	297	447	298	447	447	447	447	447
151	452	301	1056	604	604	1056	754	1056	1056	3775	1056	5888
157	156	314	784	784	627	471	156	1727	1727	1727	1727	1727
163	488	489	815	815	814	488	325	1792	815	1792	3423	8639
167	167	167	1168	500	835	167	1001	1503	1503	3840	1503	3840
173	519	518	864	519	692	519	519	864	864	4671	864	4671
179	179	357	895	179	715	536	715	895	1968	895	4832	10560
181	180	181	543	180	362	543	181	543	543	5248	543	6335

Table 2.2 : Values of Z(6p), Z(7p), Z(8p), Z(10p), Z(11p), Z(12p), Z(13p), Z(16p), Z(24p), Z(32p), Z(48p) and Z(96p) for $191 \le p \le 449$

p	Z(6p)	Z(7p)	Z(8p)	Z(10p)	Z(11p)	Z(12p)	Z(13p)	Z(16p)	Z(24p)	Z(32p)	Z(48p)	Z(96p)
191	191	573	191	764	572	191	572	191	191	191	191	191
193	192	385	192	579	385	192	1157	192	192	192	192	192
197	591	196	591	984	197	591	1182	2560	591	2560	3743	10047
199	596	398	1392	199	198	1392	597	1791	1392	1791	1791	1791
211	632	210	1055	844	1055	632	844	1055	1055	5696	1055	5696
223	668	223	223	1115	891	1560	1338	223	3344	6912	6912	6912
227	227	454	1135	680	681	680	454	2496	2496	2496	2496	2496
229	228	686	687	915	1144	687	1144	2976	687	4351	2976	10304
233	699	699	1631	699	1165	1631	233	1631	1631	5824	1631	9087
239	239	238	239	239	956	239	1195	3584	239	3584	3584	3584
241	240	482	240	240	241	240	481	3615	240	4096	3615	11327
251	251	251	752	1004	1254	752	753	3263	752	3263	3263	3263
257	771	770	256	1284	770	1799	1027	256	3855	256	7967	16191
263	263	525	1840	1315	263	263	1052	2367	2367	2367	2367	2367
269	807	538	1344	1075	538	807	806	1344	1344	1344	1344	1344
271	812	812	271	1084	813	1896	1625	4064	4064	4607	4064	4607
277	276	553	831	1384	1385	831	831	831	831	831	831	831
281	843	280	1967	280	561	1967	1404	2528	1967	6463	2528	11520
283	848	566	848	1415	1132	848	1131	3679	848	5376	5376	5376
293	879	293	879	879	879	879	585	3808	879	5567	5567	5567
307	920	307	1535	920	307	920	1534	1535	1535	1535	1535	1535
311	311	622	2176	1244	1243	311	311	2176	2799	2176	7775	17727
313	312	938	2191	939	626	312	312	2816	2816	2816	2816	2816
317	951	951	1584	1584	1584	951	1585	3487	1584	6656	6656	6656
331	992	993	992	1324	330	992	662	992	992	9599	992	9599
337	336	336	336	1684	1011	336	337	5055	336	5055	5055	5055
347	347	693	1040	1040	693	1040	1040	4511	1040	6592	4511	15615
349	348	349	1744	1395	1396	1047	2093	3839	3839	3839	3839	3839
353	1059	706	352	1059	352	2471	2118	352	5295	10943	10943	10943
359	359	1077	2512	359	1077	359	1794	3231	3231	8256	3231	8256
367	1100	734	367	1100	1100	2568	1468	5504	5504	5504	5504	5504
373	372	1119	1119	1119	373	1119	1118	1119	1119	10816	1119	13055
379	1136	378	1136	379	758	1136	2274	4927	1136	4927	7200	19328
383	383	1148	383	1915	1914	383	766	383	383	383	383	383
389	1167	777	1167	1555	1166	1167	389	5056	1167	5056	7391	19839
397	396	1190	1984	1984	396	1191	793	1984	4367	1984	10719	23423
401	1203	1203	400	400	802	2807	2405	6015	6015	6015	6015	6015
409	408	818	2863	1635	2045	408	818	3680	3680	9407	3680	9407
419	419	419	2095	419	418	1256	1676	4608	4608	4608	4608	4608
421	420	420	1263	420	1683	1263	2105	5472	1263	7999	5472	18944
431	431	861	431	1724	2155	431	2586	6464	431	6464	6464	6464
433	432	433	432	1299	1298	432	1299	6495	432	7360	6495	20351
439	1316	1316	3072	439	439	3072	1755	3072	3072	3072	3072	3072
443	443	1329	1328	2215	1771	1328	442	5759	1328	5759	5759	5759
449	1347	448	448	1795	2244	3143	897	448	6735	448	13919	28287

Table 2.3 : Values of Z(6p), Z(7p), Z(8p), Z(10p), Z(11p), Z(12p), Z(13p), Z(16p), Z(24p), Z(32p), Z(48p) and Z(96p) for $457 \le p \le 743$

p	Z(6p)	Z(7p)	Z(8p)	Z(10p)	Z(11p)	Z(12p)	Z(13p)	Z(16p)	Z(24p)	Z(32p)	Z(48p)	Z(96p)
457	456	1371	3199	2284	913	456	2742	3199	4112	3199	11424	26048
461	1383	461	2304	460	461	1383	922	2304	2304	2304	2304	2304
463	1388	462	463	2315	462	3240	2314	6944	6944	7871	6944	7871
467	467	1400	2335	1400	934	1400	467	2335	5136	12608	12608	12608
479	479	958	479	479	957	479	2873	479	479	14848	479	15807
487	1460	973	3408	1460	1947	3408	974	4383	3408	11200	4383	19967
491	491	490	1472	1964	1473	1472	1963	1472	1472	1472	1472	1472
499	1496	1497	2495	499	1496	1496	2495	2495	2495	2495	2495	2495
503	503	503	3520	2515	2012	503	1508	3520	4527	3520	12575	28671
509	1527	1526	2544	2035	2035	1527	3054	5599	2544	10688	10688	10688
521	1563	1042	3647	520	1562	3647	520	3647	3647	3647	3647	3647
523	1568	1568	1568	2615	1045	1568	2092	1568	1568	15167	1568	15167
541	540	1623	2704	540	2705	1623	2704	5951	5951	5951	5951	5951
547	1640	546	2735	1640	2188	1640	546	6016	2735	6016	11487	28991
557	1671	1113	2784	2784	1671	1671	3341	2784	2784	15039	2784	15039
563	563	1126	2815	2815	2815	1688	1689	2815	6192	2815	15200	33216
569	1707	1707	3983	2275	2276	3983	2275	5120	3983	5120	5120	5120
571	1712	1141	1712	2284	571	1712	571	7423	1712	7423	10848	29120
577	576	1154	576	2884	1154	576	2885	576	576	576	576	576
587	587	587	1760	1760	1760	1760	3522	1760	1760	17023	1760	20544
593	1779	1778	592	1779	593	4151	2964	8895	8895	8895	8895	8895
599	599	1197	4192	599	1198	599	598	4192	5391	14975	14975	14975
601	600	601	4207	600	1803	600	2404	5408	5408	13823	5408	13823
607	1820	1820	607	1820	3035	4248	1820	607	9104	18816	18816	18816
613	612	1225	1839	1839	2452	1839	3678	7968	1839	11647	7968	27584
617	1851	616	4319	3084	616	4319	1234	4319	4319	15424	4319	24063
619	1856	1238	1856	619	2475	1856	3094	1856	1856	1856	1856	1856
631	1892	630	4416	2524	1892	4416	1261	4416	4416	4416	4416	4416
641	1923	1281	640	640	2563	4487	1923	640	9615	640	19871	40383
643	1928	643	3215	3215	1286	1928	1286	7072	3215	13503	13503	13503
647	647	1294	4528	1940	3234	647	2587	5823	5823	5823	5823	5823
653	1959	1959	3264	1959	1958	1959	2612	3264	3264	3264	3264	3264
659	659	658	3295	659	659	1976	1976	3295	7248	17792	17792	17792
661	660	1322	1983	660	660	1983	3965	1983	1983	1983	1983	1983
673	672	672	672	2019	3365	672	2691	672	672	20863	672	22208
677	2031	2030	2031	3384	1353	2031	676	8800	2031	12863	12863	12863
683	683	1365	2048	3415	682	2048	1365	2048	2048	2048	2048	2048
691	2072	2072	3455	2764	3454	2072	4146	3455	3455	3455	3455	3455
701	2103	700	3504	700	2804	2103	701	7711	3504	14720	14720	14720
709	708	2127	2127	2835	1418	2127	1417	9216	2127	9216	9216	9216
719	719	2156	719	719	2156	719	2157	10784	719	12223	10784	33792
727	2180	727	5088	2180	726	5088	727	5088	5088	18175	5088	28352
733	732	2198	3664	2199	2199	2199	3665	8063	8063	8063	8063	8063
739	2216	1477	3695	739	3695	2216	4433	8128	3695	8128	15519	39167
743	743	742	5200	3715	1485	743	4458	6687	6687	17088	6687	17088

Table 2.4 : Values of Z(6p), Z(7p), Z(8p), Z(10p), Z(11p), Z(12p), Z(13p), Z(16p), Z(24p), Z(32p), Z(48p) and Z(96p) for 751 ≤ p ≤ 1051

p	Z(6p)	Z(7p)	Z(8p)	Z(10p)	Z(11p)	Z(12p)	Z(13p)	Z(16p)	Z(24p)	Z(32p)	Z(48p)	Z(96p)
751	2252	2253	751	3004	3003	5256	3003	11264	11264	11264	11264	11264
757	756	756	2271	3784	3784	2271	3028	2271	2271	21952	2271	26495
761	2283	2282	5327	760	3805	5327	1521	6848	5327	6848	6848	6848
769	768	769	768	3075	769	768	4614	768	768	768	768	768
773	2319	1546	2319	2319	3091	2319	1546	10048	2319	10048	14687	39423
787	2360	1574	3935	2360	1573	2360	1573	3935	3935	21248	3935	21248
797	2391	797	3984	3984	1594	2391	2391	8767	3984	8767	16736	42240
809	2427	1617	5663	3235	1617	5663	3236	5663	5663	20224	5663	31551
811	2432	811	2432	3244	3244	2432	4055	2432	2432	2432	2432	2432
821	2463	2463	2463	820	2463	2463	4926	2463	2463	23808	2463	23808
823	2468	1645	5760	4115	4114	5760	2469	5760	5760	5760	5760	5760
827	827	826	2480	2480	4135	2480	4134	10751	2480	10751	10751	10751
829	828	1658	4144	3315	2486	2487	3315	9119	9119	17408	9119	17408
839	839	839	5872	839	3355	839	1677	7551	7551	7551	7551	7551
853	852	853	2559	2559	1705	2559	4264	2559	2559	2559	2559	2559
857	2571	1714	5999	4284	857	5999	857	7712	5999	19711	7712	35136
859	2576	2576	2576	859	858	2576	858	11167	2576	16320	16320	16320
863	863	2589	863	4315	1726	863	4315	863	863	26752	863	28479
877	876	2631	4384	4384	3508	2631	1754	4384	9647	23679	23679	23679
881	2643	881	880	880	880	6167	3523	13215	13215	14976	13215	14976
883	2648	882	4415	4415	3531	2648	883	4415	4415	4415	4415	4415
887	887	2660	6208	2660	2661	887	3548	6208	7983	6208	22175	50559
907	2720	1813	2720	2720	1814	2720	3627	2720	2720	26303	2720	26303
911	911	910	911	3644	4554	911	910	13664	911	15487	13664	42816
919	2756	2757	6432	919	1837	6432	2756	6432	6432	22975	6432	35840
929	2787	2786	928	3715	1858	6503	1858	928	13935	28799	28799	28799
937	936	937	6559	4684	4685	936	936	6559	8432	23424	23424	23424
941	2823	1882	4704	940	1881	2823	4705	4704	4704	25407	4704	25407
947	947	2841	4735	2840	946	2840	5681	4735	10416	4735	25568	55872
953	2859	952	6671	2859	2859	6671	2859	8576	6671	8576	8576	8576
967	2900	966	6768	2900	967	6768	4835	8703	6768	8703	8703	8703
971	971	2912	2912	3884	3883	2912	2912	2912	2912	28159	2912	33984
977	2931	1953	976	4884	4884	6839	5862	14655	14655	14655	14655	14655
983	983	1966	6880	4915	2948	983	4914	6880	8847	24575	24575	24575
991	2972	1981	991	3964	990	6936	3964	991	14864	30720	30720	30720
997	996	1994	2991	4984	2991	2991	2990	12960	2991	18943	12960	44864
1009	1008	1008	1008	4035	4036	1008	5044	15135	1008	17152	15135	47423
1013	3039	3038	3039	3039	1012	3039	1013	3039	3039	29376	3039	29376
1019	1019	2037	3056	1019	3057	3056	5095	13247	3056	13247	13247	13247
1021	1020	1021	5104	1020	5104	3063	2041	11231	11231	21440	11231	21440
1031	1031	3093	7216	4124	4124	1031	3093	9279	9279	9279	9279	9279
1033	1032	2065	7231	3099	1033	1032	2066	7231	9296	7231	25824	58880
1039	3116	2078	1039	1039	2078	7272	1039	15584	15584	17663	15584	17663
1049	3147	1049	7343	4195	3146	7343	3146	9440	7343	24127	9440	43008
1051	3152	1050	3152	4204	2101	3152	6305	13663	3152	19968	19968	19968

Table 2.5 : Values of Z(6p), Z(7p), Z(8p), Z(10p), Z(11p), Z(12p), Z(13p), Z(16p), Z(24p), Z(32p), Z(48p) and Z(96p) for 1061 ≤ p ≤ 1399

p	Z(6p)	Z(7p)	Z(8p)	Z(10p)	Z(11p)	Z(12p)	Z(13p)	Z(16p)	Z(24p)	Z(32p)	Z(48p)	Z(96p)
1061	3183	2121	3183	1060	2122	3183	5304	13792	3183	20159	20159	20159
1063	3188	1063	7440	5315	3189	7440	4251	9567	7440	24448	9567	43583
1069	1068	3206	5344	4275	5345	3207	4276	5344	11759	28863	28863	28863
1087	3260	3261	1087	3260	5434	7608	5434	1087	16304	1087	33696	68480
1091	1091	1091	5455	4364	5455	3272	1091	12000	12000	22911	12000	22911
1093	1092	1092	3279	3279	3278	3279	1092	14208	3279	14208	14208	14208
1097	3291	3290	7679	5484	4388	7679	5485	7679	7679	7679	7679	7679
1103	1103	2205	1103	5515	4411	1103	6617	16544	1103	18751	16544	51840
1109	3327	2218	3327	4435	5544	3327	3327	3327	3327	3327	3327	3327
1117	1116	2233	5584	5584	2233	3351	1117	12287	12287	12287	12287	12287
1123	3368	2246	5615	5615	1122	3368	5615	12352	5615	12352	23583	59519
1129	1128	3387	7903	4515	3387	1128	6773	7903	10160	28224	28224	28224
1151	1151	2302	1151	4604	3453	1151	2301	1151	1151	1151	1151	1151
1153	1152	3458	1152	3459	5764	1152	3458	1152	1152	1152	1152	1152
1163	1163	1162	3488	5815	4652	3488	2326	3488	3488	33727	3488	40704
1171	3512	3513	5855	4684	2342	3512	1170	5855	5855	31616	5855	31616
1181	3543	3542	5904	1180	3542	3543	7085	12991	5904	12991	24800	62592
1187	1187	2373	5935	3560	1187	3560	3561	13056	13056	13056	13056	13056
1193	3579	2386	8351	3579	2386	8351	4771	8351	8351	29824	8351	46527
1201	1200	2401	1200	1200	6005	1200	6005	18015	1200	20416	18015	56447
1213	1212	3639	6064	3639	4851	3639	3639	13343	13343	25472	13343	25472
1217	3651	1217	1216	6084	3651	8519	6084	1216	18255	1216	37727	76671
1223	1223	3668	8560	6115	6115	1223	1222	11007	11007	11007	11007	11007
1229	3687	2457	6144	4915	4916	3687	2457	6144	6144	6144	6144	6144
1231	3692	1231	1231	4924	1231	8616	3692	18464	18464	20927	18464	20927
1237	1236	3710	3711	6184	2474	3711	7422	3711	3711	3711	3711	3711
1249	1248	2498	1248	4995	2497	1248	1248	1248	1248	38719	1248	41216
1259	1259	1259	3776	1259	2518	3776	7553	3776	3776	3776	3776	3776
1277	3831	2554	6384	6384	1276	3831	5108	14047	6384	26816	26816	26816
1279	3836	3836	1279	1279	5115	8952	6395	1279	19184	1279	39648	80576
1283	1283	3849	6415	6415	3849	3848	3848	14112	14112	26943	14112	26943
1289	3867	1288	9023	5155	6445	9023	7734	9023	9023	9023	9023	9023
1291	3872	2582	3872	5164	3872	3872	3873	3872	3872	37439	3872	37439
1297	1296	3891	1296	6484	1297	1296	5187	19455	1296	19455	19455	19455
1301	3903	1301	3903	1300	5203	3903	1300	3903	3903	3903	3903	3903
1303	3908	1302	9120	6515	2606	9120	5212	9120	9120	32575	9120	50816
1307	1307	3920	3920	3920	6534	3920	2613	16991	3920	24832	16991	58815
1319	1319	2638	9232	1319	1319	1319	2638	11871	11871	30336	11871	30336
1321	1320	3962	9247	1320	1320	1320	6604	9247	11888	33024	33024	33024
1327	3980	2653	1327	3980	3981	9288	1326	19904	19904	19904	19904	19904
1361	4083	2722	1360	1360	5444	9527	4082	20415	20415	20415	20415	20415
1367	1367	4101	9568	4100	5467	1367	8202	9568	12303	34175	34175	34175
1373	4119	1372	6864	4119	6864	4119	6864	15103	6864	15103	28832	72768
1381	1380	4143	4143	1380	2761	4143	5524	17952	4143	26239	17952	62144
1399	4196	1399	9792	1399	6995	9792	6994	9792	9792	9792	9792	9792

Table 2.6 : Values of Z(6p), Z(7p), Z(8p), Z(10p), Z(11p), Z(12p), Z(13p), Z(16p), Z(24p), Z(32p), Z(48p) and Z(96p) for $1409 \le p \le 1511$

p	Z(6p)	Z(7p)	Z(8p)	Z(10p)	Z(11p)	Z(12p)	Z(13p)	Z(16p)	Z(24p)	Z(32p)	Z(48p)	Z(96p)
1409	4227	4227	1408	5635	1408	9863	7045	1408	21135	1408	43679	88767
1423	4268	4269	1423	7115	4268	9960	2846	21344	21344	24191	21344	24191
1427	1427	1427	7135	4280	5708	4280	5707	7135	15696	38528	38528	38528
1429	1428	1428	4287	5715	1429	4287	1429	4287	4287	4287	4287	4287
1433	4299	4298	10031	4299	5731	10031	5732	12896	10031	32959	12896	58752
1439	1439	2877	1439	1439	7194	1439	4316	1439	1439	44608	1439	47487
1447	4340	4340	10128	4340	2893	10128	4341	13023	10128	33280	13023	59327
1451	1451	4353	4352	5804	1451	4352	7254	4352	4352	4352	4352	4352
1453	1452	2905	7264	4359	1452	4359	5811	7264	15983	39231	39231	39231
1459	4376	2918	7295	1459	4377	4376	5836	7295	7295	7295	7295	7295
1471	4412	1470	1471	5884	5884	10296	8826	1471	22064	1471	45600	92672
1481	4443	2961	10367	1480	4443	10367	1481	10367	10367	10367	10367	10367
1483	4448	1483	4448	7415	7414	4448	1482	4448	4448	43007	4448	43007
1487	1487	2974	1487	4460	7435	1487	7435	22304	1487	25279	22304	69888
1489	1488	4466	1488	5955	4466	1488	2977	22335	1488	22335	22335	22335
1493	4479	4479	4479	4479	5972	4479	8957	4479	4479	4479	4479	4479
1499	1499	1498	4496	1499	5995	4496	4497	19487	4496	28480	19487	67455
1511	1511	1511	10576	6044	4532	1511	6044	13599	13599	34752	13599	34752

APPENDIX A.3 : Values of $Z(p.2^k)$ for p = 11, 13, 17, 19, 23, 29, 31

Table A.3.1 : Values of $Z(11.2^k)$, $Z(13.2^k)$, $Z(17.2^k)$, $Z(19.2^k)$, $Z(23.2^k)$, $Z(29.2^k)$, $Z(31.2^k)$

k	$Z(11.2^k)$	$Z(13.2^k)$	$Z(17.2^k)$	$Z(19.2^k)$	$Z(23.2^k)$	$Z(29.2^k)$	$Z(31.2^k)$
1	11	12	16	19	23	28	31
2	32	39	16	56	23	87	31
3	32	64	16	95	160	144	31
4	32	64	255	95	160	319	31
5	319	64	255	512	575	319	960
6	384	767	255	512	896	1536	1023
7	1023	767	255	512	2047	1536	1023
8	1023	2560	4096	512	2047	1536	1023
9	1023	4095	4096	9215	2047	13311	1023
10	10240	4095	4096	10420	2047	16384	30720
11	12287	4095	4096	28671	45056	16384	32767
12	32768	49152	65535	49152	49151	16384	32767
13	32768	49152	65535	49152	49151	16384	32767
14	32768	163839	65535	262143	327680	458751	32767
15	327679	262144	65535	262143	327680	458751	983040
16	303216	262144	1048576	262143	1179647	1441792	1048575
17	1048575	262144	1048576	262143	1835008	2359295	1048575
18	1048575	3145727	1048576	4718592	4194303	5242880	1048575
19	1048575	3145727	1048576	5242879	4194303	5242880	1048575
20	10485760	10485760	16777215	14680064	4194303	24165823	31457280
21	12582911	16777215	16777215	25165823	4194303	24165823	33554431
22	33554432	16777215	16777215	25165823	92274688	24165823	33554431
23	33554432	16777215	16777215	134217728	100663295	167772161	33554431
24	33554432	16777215	234881025	134217728	100663295	167772161	33554431
25	268435454	201326592	268435456	134217728	268435458	239886426	251077228
26	268435454	268435454	268435456	134217728	268435458	239886426	251077228
27	268435454	268435454	268435456	298888120	268435458	239886426	251077228

Acknowledgements

Part of the work was done during the Academic Development Leave (ADL) for six months from September 2010 to March 2011 from the Ritsumeikan Asis-Pacific University, Japan. The author acknowledges with thanks the Ritsumeikan APU for granting the ADL.

Thanks are also due to the Department of Mathematics, Jahangirnagar University, Bangladesh, for allowing their research resources during the author's ADL period.

Last, but not the least, my colleague at APU, Professor Dr. Hary Gunarto, deserves my thanks for his computer works in the preparation of the tables in the Appendices at the end of the book.

Khan Md. Anwarus Salam, a graduate student at the University of Electro-Communications, Japan, volunteered for the design of the cover page with the theme in accordance with the title of the book.

SUBJECT INDEX

This book covers only a part of the wide and diverse field of the Smarandache Notions, and contains some of the materials that I gathered as I wandered in the world of Smarandache. Most of the materials are already published in different journals, but some materials are new and appear for the first time in this book. All the results are provided with proofs.

- Chapter 1 gives eleven recursive type Smarandache sequences, namely, the Smarandache Odd, Even, Prime Product, Square Product (of two types), Higher Power Product (of two types), Permutation, Circular, Reverse, Symmetric and Pierced Chain sequences
- Chapter 2 deals with the Smarandache Cyclic Arithmetic Determinant and Bisymmetric Arithmetic Determinant sequences, and series involving the terms of the Smarandache bisymmetric determinant natural and bisymmetric arithmetic determinant sequences
- Chapter 3 treats the Smarandache function S(n)
- Chapter 4 considers, in rather more detail, the pseudo Smarandache function Z(n)
- And the Smarandache S-related and Z-related triangles are the subject matter of Chapter 5.

To make the book self-contained, some well-known results of the classical Number Theory are given in Chapter 0. In order to make the book up-to-date, the major results of other researchers are also included in the book.

At the end of each chapter, several open problems are given.

ISBN 9781599731247

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