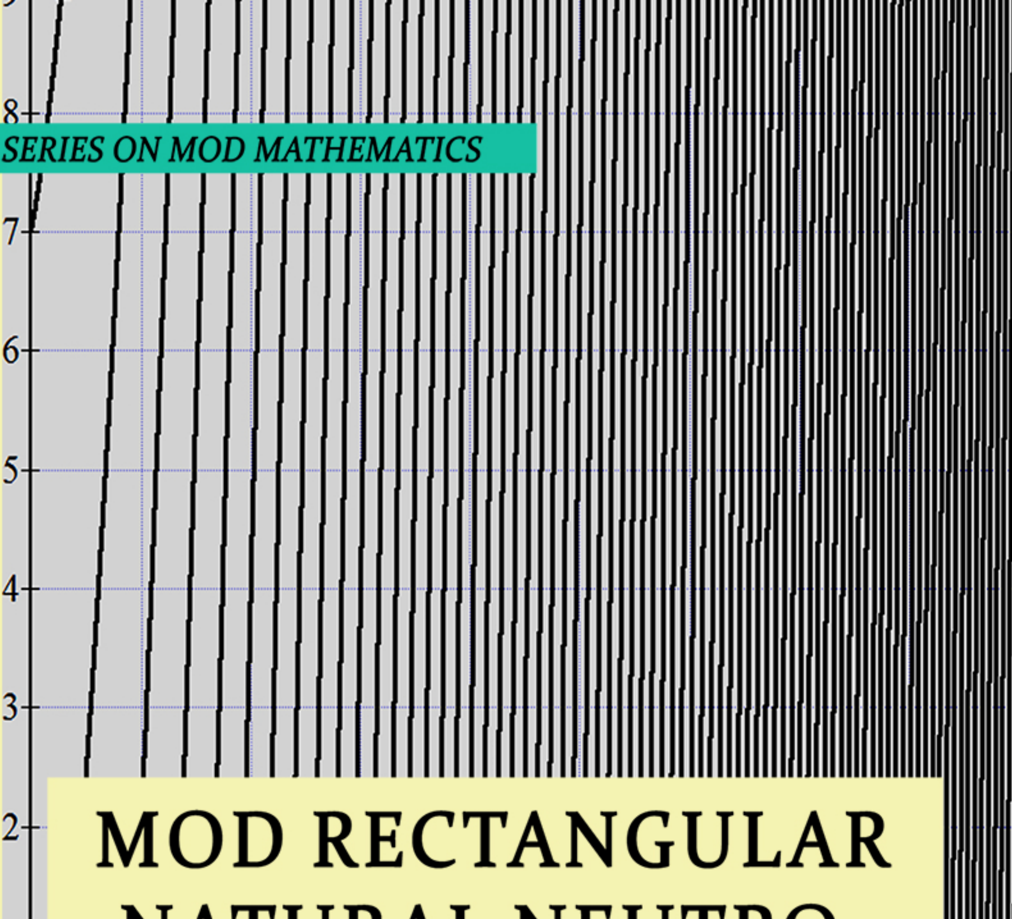


*SERIES ON MOD MATHEMATICS*

# MOD RECTANGULAR NATURAL NEUTROSOPHIC NUMBERS

W.B.VASANTHA KANDASAMY
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# CONTENTS

# PREFACE

In this book authors introduce the new notion of MOD rectangular planes. The functions on them behave very differently when compared to MOD planes (square). These are different from the usual MOD planes. Algebraic structures on these MOD rectangular planes are defined and developed.

However we have built only MOD interval natural neutrosophic products of the form ${}^{I}[0, m) \times {}^{I}[0, n)$ where $m \neq n$ and $2 \leq m, n < \infty$. These can be called as planes as one can accommodate the mod natural neutrosophic numbers in these planes. Further MOD rectangular natural neutrosophic numbers $Z_n^I \times Z_m^I$; $m \neq n$; $2 \leq m, n < \infty$ are also constructed and algebraic structures on them are defined and described. They happen to be of finite cardinality. On these MOD rectangular numbers, semigroups with respect to + and $\times$ (or $\times_0$) are defined and described. They happen to be of finite MOD rectangular natural neutrosophic sets. MOD matrix subsets are constructed and under + (or $\times$ or $\times_n$) these collections yield only semigroups.

On similar lines MOD rectangular natural neutrosophic subset coefficient polynomials are defined and under + and $\times$ or $\times_0$ they happen to be only semigroups. Study in this direction yields nice algebraic structures under a single binary operator + or $\times$ (or $\times_0$)

We wish to acknowledge Dr. K Kandasamy for his sustained support and encouragement in the writing of this book.

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**Chapter One**

# INTRODUCTION

In this book authors introduce the new notion of MOD rectangular planes $[0, n) \times [0, m)$ where $m \neq n$; $2 \leq m, n < \infty$. This concept is analogously defined to MOD rectangular modulo integers as $Z_n \times Z_m$, MOD rectangular natural neutrosophic modulo integers $Z_n^I \times Z_m^I$ and MOD rectangular natural neutrosophic interval set $^I[0, n) \times {}^I[0, m)$. However these are not planes but we choose to call them as MOD rectangular numbers. Further we cannot define MOD rectangular complex plane or dual plane or neutrosophic plane for we need $m = n$. For more about MOD structures please refer [31-41].

Here we give how the usual functions behave in the case of MOD rectangular plane $[0, n) \times [0, m)$; $(m \neq n)$. Then we proceed onto define algebraic structures on $Z_n \times Z_,$, $Z_n^I \times Z_m^I$, $^I[0, n) \times {}^I[0, m)$ and $[0, n) \times [0, m)$; $m \neq n$. We see $Z_n \times Z_m$ and $[0, n) \times [0, m)$ are groups under + modulo pair operation. For if $(a, b)$ and $(c, d) \in [0, n) \times [0, m)$ then $(a, b) + (c, d) = ((a +c) \pmod n, (b + d) \pmod m))$.

But $Z_n^I \times Z_m^I$ and ${}^I[0, n) \times {}^I[0, m)$ are only semigroups under +. So we are in a position to give semigroups of finite and infinite order which are only semigroups under +. When we extend this concept to matrices and polynomials we get matrices such that A + A = A and polynomials p(x) + p(x) = p(x) respectively.

Except for this structure we would not be in a position to get all these. $Z_n^I \times Z_m^I$ and $Z_n \times Z_m$ under product are finite order semigroups which enjoy several special features. Likewise ${}^I[0, n) \times {}^I[0, m)$ and $[0, n) \times [0, m)$ are infinite order semigroups which are also only infinite order semigroups under product.

All these give examples of non-abstract semigroups of either finite or infinite order under + or ×. Finally we see these semigroups have special and distinct features. Certainly these finite structures can find applications in several important problems related to automaton theory.

The authors also mention about matrix and polynomial MOD rectangular natural neutrosophic semigroups under + and $\times_0$ (or ×). They are always commutative monoids. In case of square matrices we can have the usual product × and get the infinite order non-commutative monoids if $[0, n) \times [0,m)$ or ${}^I[0, n) \times {}^I[0, m)$ is used and finite non commutative monoids in case of $Z_n \times Z_m$ or $Z_n^I \times Z_m^I$.

Several innovative results are obtained. For more about neutrosophic algebraic structures, refer [2-12].

## Chapter Two

# RECTANGULAR MOD PLANES

In this chapter we introduce the new notion of rectangular MOD planes. These are distinctly different from the existing real planes. Further these planes will serve a better purpose for one need not go for different types of scales only for rectangular planes. In fact rectangular MOD planes are infinitely many.

When both the x and y coordinates are real we call them as real MOD rectangular planes or MOD rectangular real planes.

We will first illustrate this by some examples.

***Example 2.1***. Let us consider the two MOD intervals [0, m) and [0, p); $p \neq m$, $2 \leq p, m < \infty$.

Now $R_n(m, p) = [0, m) \times [0, p) = \{(a, b) / a \in [0, m)$ and $b \in [0, p)\}$.

This $R_n(m, p)$ is defined as the MOD rectangular plane, which is described by the figure given in the next page.

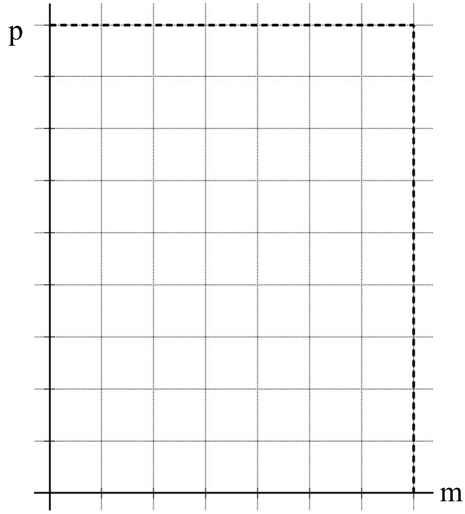


**Figure 2.1**

$R_n(m, p)$ denotes the MOD rectangular real plane. In fact we have infinitely many such rectangular MOD real planes.

***Example 2.2***. Let $R_n(5, 3) = \{(a, b) / a \in [0, 5), b \in [0, 3)\}$ be the rectangular MOD real plane clearly this is an infinite plane.

***Example 2.3***. Let $R_n(2, 4) = \{(a, b) / a \in [0, 2)$ and $b \in [0, 4)\}$ be the real rectangular MOD plane. It is evident that $R_n(2, 4)$ is an infinite real plane which has only one quadrant.

In fact we will prove that we can get appropriate transformation of $R \times R = \{(a, b) / a, b \in R\}$ into $R_n(m, p)$ and vice versa. First we describe the transformation for some particular values of m and p then for general values say of m and p. Further it is important to keep on record when m = p we get the real MOD plane $R_n(m)$ [35].

We have also described about MOD functions and transformations.

***Example 2.4.*** Let $R_n(5, 6)$ be the MOD rectangular plane.

Define map: $\eta : R \times R \to R_n(5, 6)$ by

$$\eta(a, b) = \begin{cases} (a,b) & \text{if } 0 \leq a < 5 \text{ and } 0 \leq b < 6 \\ (0,b) & \text{if } a = 5 \text{ and } 0 \leq b < 6 \\ (a,0) & \text{if } b = 6 \text{ and } 0 \leq a < 5 \\ (0,0) & \text{if } a = 5 \text{ and } b = 6 \\ (x,b) & \text{if } a > 5 \text{ and } \frac{a}{5} = n + \frac{x}{5}; 0 \leq b < 6 \\ (a,y) & \text{if } 0 \leq a < 5,\ b > 6 \text{ and } \frac{b}{6} = t + \frac{y}{6} \\ (x,y) & \text{if } a > 5 \text{ and } b > 6 \\ & \frac{a}{5} = n + \frac{x}{5} \text{ and } \frac{b}{6} = t + \frac{y}{5} \\ = (5-x, 6-y) & \text{if } a \text{ negative, } b \text{ is negative} \\ & \text{and } \frac{a}{5} = n + \frac{x}{5}; \frac{b}{6} = t + \frac{y}{6} \end{cases}$$

So if $(a, b) = (3.7, 2.01)$ then $(a, b) \in R_n(5,6)$.

If $(a, b) = (5, 0.27)$ then $\eta(a, b) = (0, 0.27)$.

If $(a, b) = (2.001, 6)$
then $\eta\ (a, b) = (2.001, 0)$.
If $(a, b) = (12.378, 10.615)$ then $\eta\ (a, b) = (2.378, 4.615)$.
If $(a, b) = (-3.21, 6.3)$ then $\eta(a, b) = (1.79, 0.3)$.
If $(a, b) = (9.32, -4.73)$, $\eta(a, b) = (4.32, 1.27)$
$\eta(-9.32, -10.41) = (0.68, 1.59)$.

This is the way the transformation from the real plane $R \times R$ is mapped onto the $R_n(5, 6)$, the MOD rectangular plane.

The MOD-rectangular plane has only one quadrant but it has the capacity to get all 4 quadrants of the $R \times R$ real plane into it.

We will leave it as an exercise for the reader to map, $\eta: R \times R \to R_n(m, p)$; this MOD transformation is akin to the MOD transformation carried out in the MOD planes. For more refer [33-35].

Next we proceed onto get the transformation from the MOD real rectangular plane to the real plane by making the following definition.

Just first we describe it by an example.

$\eta_i : R_n(9, 5) \to R \times R$, where $\eta_i$ is a map from $R_n(9,5)$ to $R \times R$ defined by the following way.

$\eta_i\ (5, 2.001) = (5, 2.001)$ or $(9n + 5, 5n + 2.001)$.

$n = 0, \pm 1, 2, \pm 3, ..., \infty$.

$\eta_i(0, 0) = (n9, m5)$; $\quad\quad n = 0, \pm 1, \pm 2, \ldots, \infty$

$m = 0, \pm 1, \pm 2, \ldots, \infty$

$\eta_i(t, s) = (9n + t, 5m + s)$ $\quad n = 0, \pm 1, \pm 2, \ldots, \infty$

$m = 0, \pm 1, \pm 2, \ldots, \infty.$

In fact $\eta_i$ does not behave like a classical map. However by the special MOD transformation we are in a position to cover the entire real plane.

Thus special MOD transformations are a specially made maps which map periodically every single element of $R_n(m, p)$ into an infinite number of elements. Thus with no difficulty we retrieve the entire real plane from the MOD rectangular real plane.

We see each point in $R_n(m, p)$ is mapped by $\eta_i$ into a real line $(-\infty, \infty)$ of a specific form.

Now our next venture is to define some sort of operations on the MOD real plane $R_n(m, p)$ or to be more specific can we have some form of well-defined algebraic structures on $R_n(m, p)$.

We will first describe this by some examples before we make the abstract definition of it.

***Example 2.5***. Let $R_n(5, 3)$ be the MOD real plane built using the intervals [0, 5) and [0, 3).

Define + on $R_n(5, 3) = \{(a, b) / a \in [0, 5), b \in [0, 3)\}$ as follows.

For (x, y) and (t, u) $\in R_n(5, 3)$ define + as

$(x, y) + (t, u) = (x + t(\text{mod } 5), y + u(\text{mod } 5))$.

In particular if (3.21, 2.114) and (4.378, 0.5667) is in $R_n(5, 3)$.

$(3.21, 2.114) + (4.378, 0.5667) = (4.588, 2.6807) \in R_n(5, 3)$.

Thus + is a closed binary operation on $R_n(5, 3)$.

Infact $(0, 0) \in R_n(5, 3)$ acts as the additive identity. If $(x, y) \in R_n(5, 3)$, $(0, 0) + (x, y) = (x, y) + (0, 0) = (x, y)$.

Thus + is a commutative binary operation on $R_n(m, p) = R_n(5, 3)$. To every $(x, y) \in R_n(m, p)$ there exists unique $(s, t) \in R_n(m, p)$ such that $(x, y) + (s, t) = (0, 0) \bmod (5, 3)$.

Infact we can say $\{R_n(5, 3), +\}$ is an additive abelian group of infinite order.

We now make the definition.

**Definition 2.1**. *Let $R_n(m, p) = \{(a, b) / a \in [0, m), b \in [0, p)\}$ be the MOD real rectangular plane. Define a binary operation + on $R_n(m, p)$ as for $(x, y)$ and $(a, b) \in R_n(m, p)$; $(x, y) + (a, b) = ((x + a) (\text{mod } m), (y + b) \text{ mod } p) \in R_n(m, p)$. Then $\{R_n(m, p), +\}$ is defined as the MOD real rectangular plane group.*

It is important to note that there exists infinitely many such MOD real rectangular plane groups as m and p can vary, $2 \leq m < \infty$ and $2 \leq p < \infty$ which is an infinite interval. This infinite group always contains subgroups of finite order as well as of infinite order.

This will be represented by the following examples.

***Example 2.6***. Let $R_n(6, 7) = \{(a, b) / a \in [0, 6)$ and $b \in [0, 7)\}$ be the MOD real rectangular plane group under +.

$B = \{(a, b) / a \in Z_6$ and $b \in Z_7\} \subseteq R_n(6, 7)$ is a subgroup of $R_n(6, 7)$ and is of finite order.

Let $S = \{(a, 0) / a \in [0, 6)\} \subseteq R_n(6, 7)$. Clearly S is a subgroup of infinite order.

Similarly $T = \{(0, b) / b \in [0, 7)\} \subseteq R_n(6, 7)$ is a subgroup of infinite order.

In view of this we have the following theorem.

**Theorem 2.1**. *Let $\{R_n(m, p), +\} = \{(a, b) / a \in [0, m), b \in [0, p), +\}$ be the MOD real rectangular plane group.*

*i) $R_n(m, p)$ always has at least 3 subgroups of finite order.*
*ii) $R_n(m, p)$ has subgroups of infinite order.*

Proof is direct and hence left as an exercise to the reader.

However examples to this effect will be provided.

***Example 2.7***. Let $R_n(12, 9) = \{(a, b) / a \in [0, 12), b \in [0, 9)\}$ be a group of MOD real rectangular plane under +.

We see $G_1=\{(a, 0) / a \in Z_{12}, +\}$,

$G_2 = \{(a, 0) / a \in \{0, 2, 4, 6, 8, 10\}, +\}$,

$G_3 = \{(0, b) / b \in Z_9, +\}$ and

$G_4 = \{(a, b) / a \in Z_{12}, b \in Z_9, +\}$ contained in $R_n(12, 9)$ are some of the finite order subgroups of $R_n(12, 9)$.

We just show how the plane looks like the usual real plane or complex plane is given by the following figures.

The real plane is as follows:

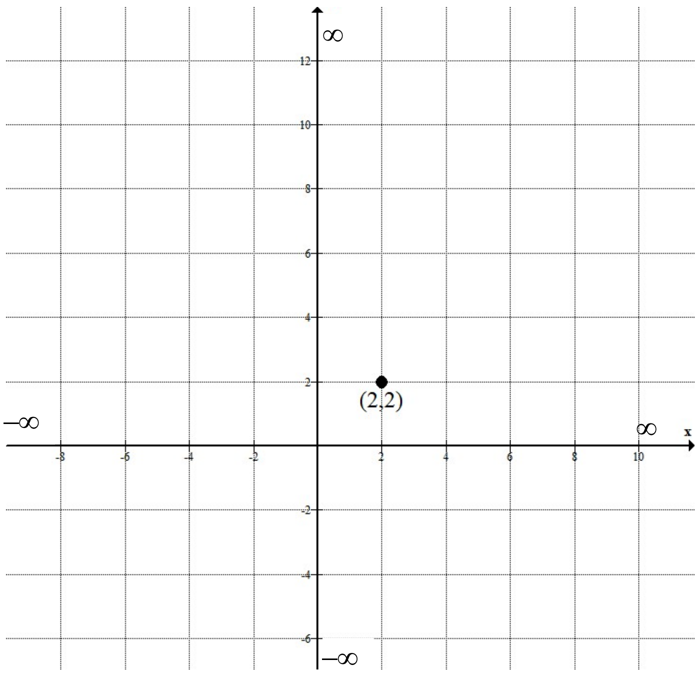


**Figure 2.2**

The complex plane is as follows:

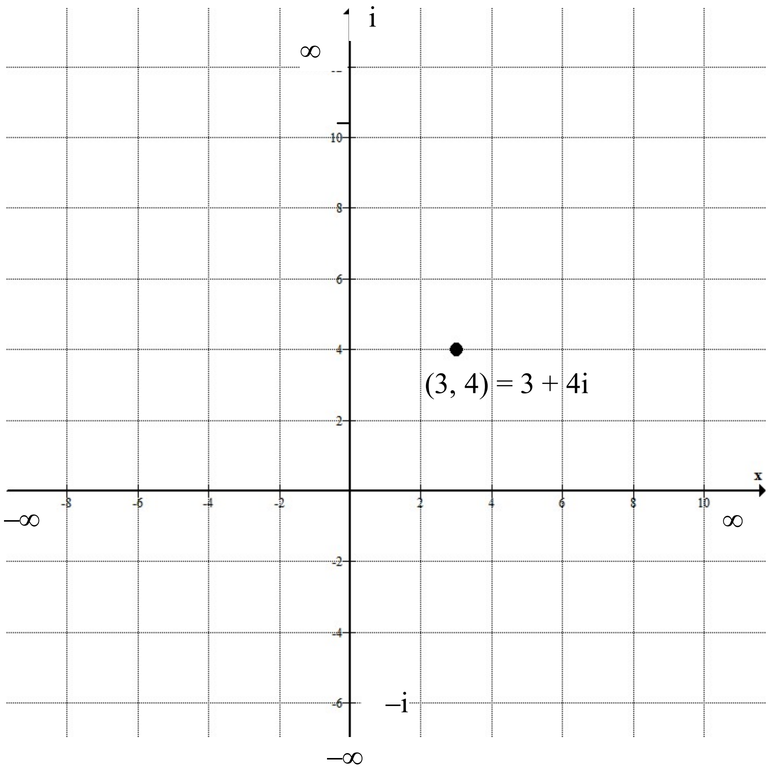


**Figure 2.3**

We see the planes on both the axis are equal and not of different sizes.

We will illustrate this situation by some examples.

***Example 2.8***. Let R(6, 3) be the MOD rectangular plane given by the following figure.

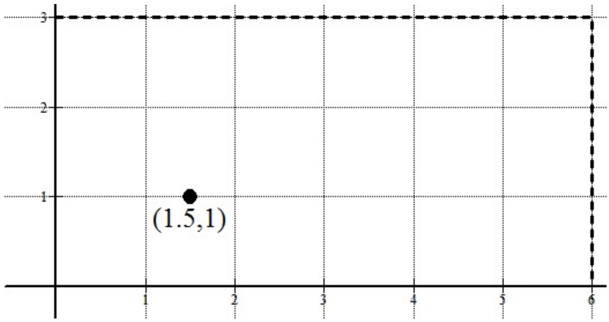


**Figure 2.4**

***Example 2.9***. Let $R_n(2, 7)$ be the MOD rectangular plane given by the following figure.

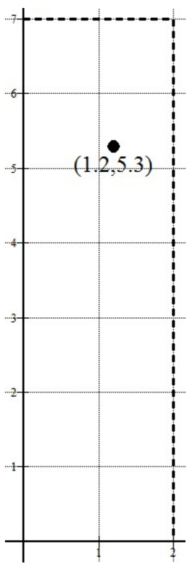


**Figure 2.5**

Let P(1.2, 5.3) be a point in the MOD rectangular plane.

We see that this MOD rectangular finite planes are much more better than the usual planes also to some extent it is better than $R_n(m)$ plane.

For even in the very simple case like storage if we choose $R_n(m, t)$ where $t < m$ then certainly it can save time and space; hence ultimately the MOD rectangular plane may be a better option in practical situations.

Next we proceed onto describe how functions can be defined in rectangular MOD planes. We will compare these functions in the real plane and MOD planes.

***Example 2.10***. $y = x$ be the function we give in the following figures; in the real plane and the MOD rectangular plane, the graph is as follows.

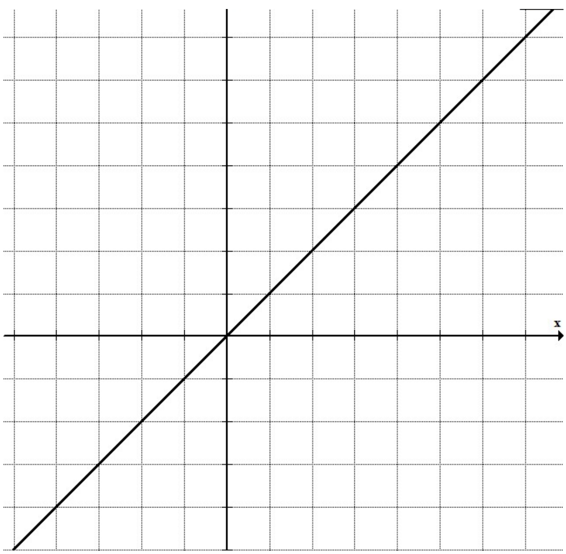


**Figure 2.6**

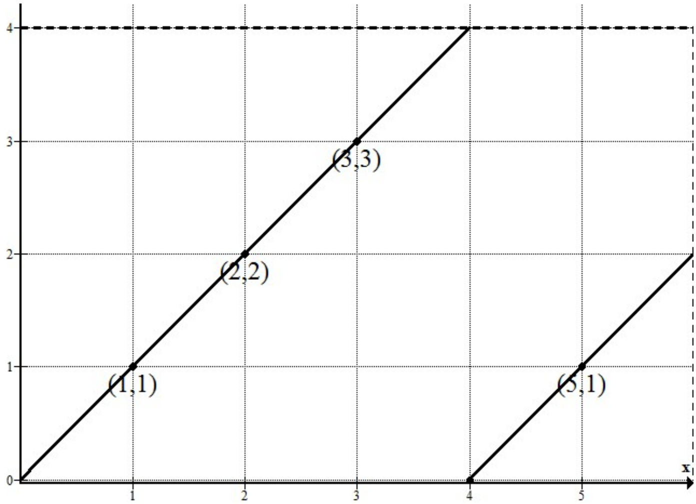


**Figure 2.7**

For when x = 4; y = 0, x = 5, y = 1 and x = 5.999, y = 1.999.

So we get in $R_n(6, 4)$ broken graphs but in the real plane the functimn y = x is a continuous line. In case of $R_n(m)$ the MOD plane we see the graph y = x is as follows.

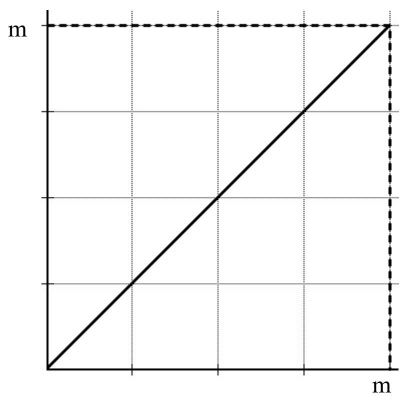


**Figure 2.8**

Let us take x = 2 and y = 8, we see the graph y = x in $R_n(2, 8)$ plane is as follows.

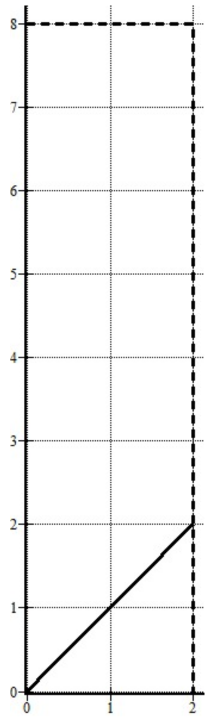


**Figure 2.9**

We see the graph is continuous.

***Example 2.11***. Let us consider the function $y = x^2$. The graph of $y = x^2$ in the real plane is as follows.

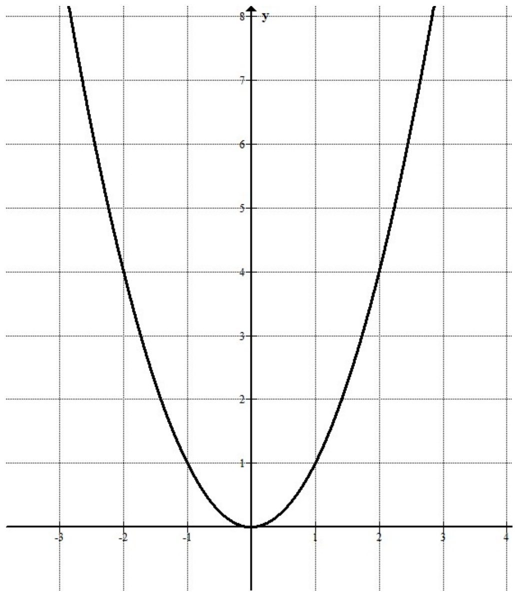

**Figure 2.10**

Next we give the graph of $y = x^2$ in the MOD real plane $R_n(5)$ by the following figure.

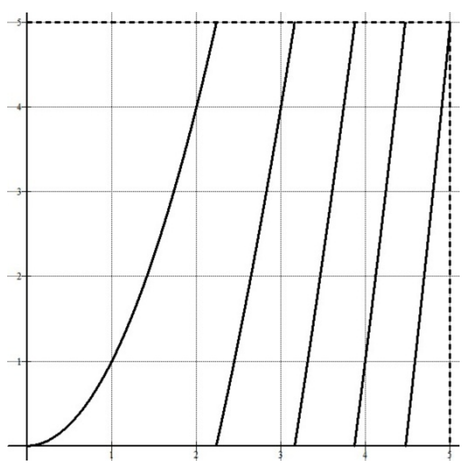

**Figure 2.11**

We can find the zeros. However in the real plane the only zero is x = 0. Now we give the graph of $y = x^2$ in the MOD rectangular plane $R_n(3, 7)$.

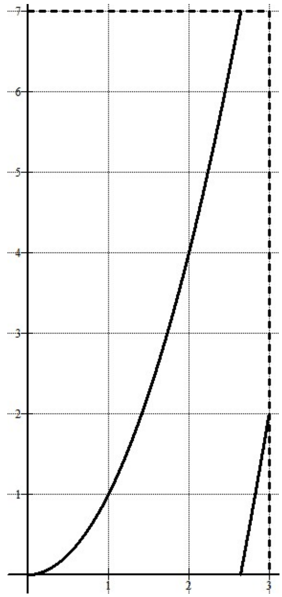


**Figure 2.12**

Consider the MOD rectangular plane $R_n(8, 4)$ given by the following figure.

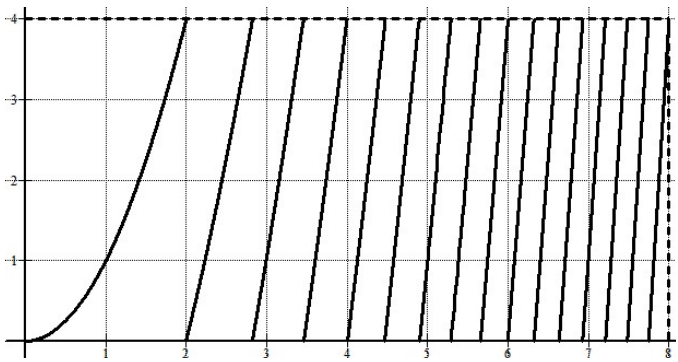


**Figure 2.13**

Finding how many branches exist happens to be a difficult problem.

Now ($n \geq 4$) if we choose to study $y = x^3$ or any $y = x^n$ ($n > 4$) finding number of branches in $R_n(m, t)$, $m \neq t$, $2 \leq m$, $t < \infty$ will continue to be a open problem.

For these curves are not continuous in $R_n(m)$ or $R_n(s, t)$ $s \neq t$; $2 \leq s, t < \infty$.

Suppose $y = ax + b$; a, b integers, we know $y = ax + b$ is defined for all values of $a, b \in R$ and the graph of the curve $y = ax + b$ can be easily plotted. But if $a, b \in Z$ we can define for the function $y = ax + b$ in $R_n(m)$ provided $1 \leq a, b < m$ and in case of $R_n(s, t)$; $y = ax + b$ can be defined only if $1 \leq a, b < s$ and t.

So using these conditions we will describe the curve $y = ax + b$ in R, $R_n(m)$ and $R_n(s, t)$; s and t taking some fixed values for a and b by examples.

***Example 2.12***. Let $y = 5x + 3$ be the function.

We trace $y = 5x + 3$ in R, $R_n(6)$, $R_n(9, 7)$ and $R_n(8,12)$ in the following. Clearly $y = 5x + 3$ is a continuous curve in the real plane.

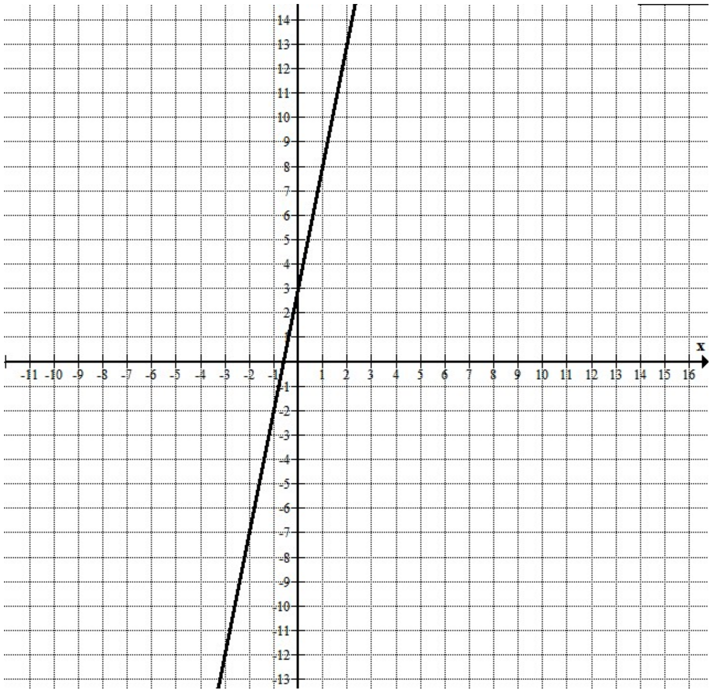


**Figure 2.14**

| x | 0 | –1 | 1 | –2 |
|---|---|---|---|---|
| y | 3 | –2 | 8 | 7 |

We find the graph of y = 5x +3 in the MOD $R_n(6)$ plane and it is given in the following figure.

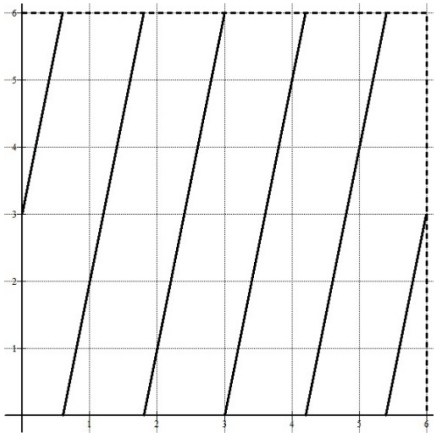

**Figure 2.15**

| x | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| y | 3 | 2 | 1 | 0 | 5 | 4 |

Consider the graph $y = 5x + 3$ in the MOD rectangular plane $R_n(7, 9)$ given by the following figure.

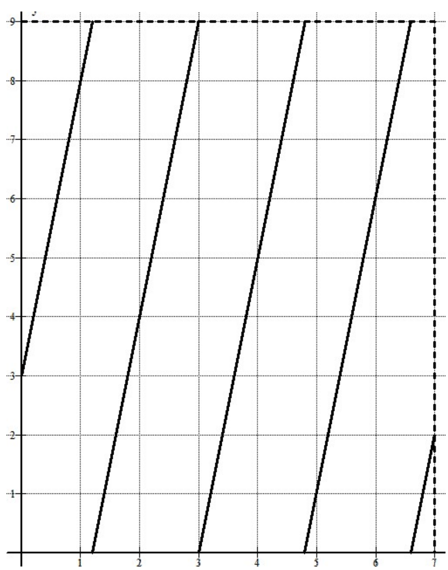

**Figure 2.16**

Next we give the graph of $y = 5x + 3$ in the MOD rectangular plane $R_n(12, 8)$ by the following figure.

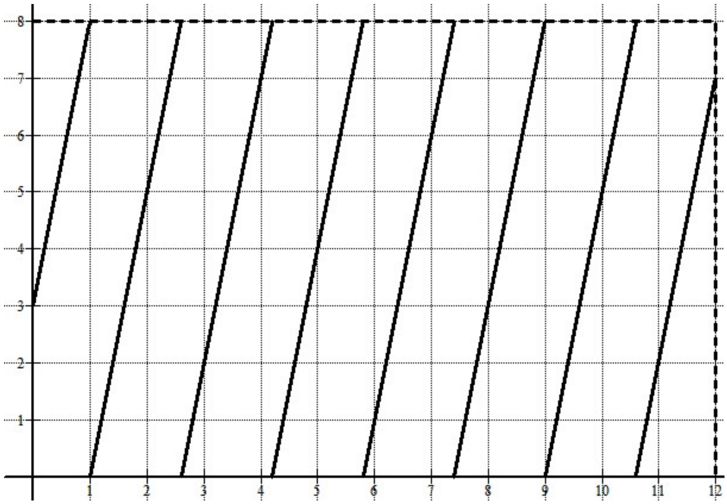


**Figure 2.17**

We see the graphs are different in these four planes.

So this study is not only innovative and interesting but it is very difficult. Several things are left as open problem / conjecture.

**Conjecture 2.1** Let $y = ax + b$, $2 \leq a, b \leq s, t < \infty$ $(s \neq t)$; be the function.

How many branches are there in $R_n(s, t)$ for the function $y = ax + b$?

i) $a / s$ and $b / s$.
ii) $a / t$ and $b / t$.
iii) Both (i) and (ii) are true.
iv) $(a, s) = 1$, $(b, s) = 1$.
v) $(b, t) = 1$ $(a, t) = 1$.

vi) (iv) and (v) are true.

Now we find the graph of y = tx in R, $R_n(m)$ and $R_n(s, p)$ ($t \geq 2$).

We will illustrate this situation by some examples.

***Example 2.13***. Let y = 2x be the given function. We will find the graph of y = 2x in the real plane R, $R_n(10)$, $R_n(5, 9)$ and $R_n(8, 6)$. The legend for y = 2x in the real plane R is as follows.

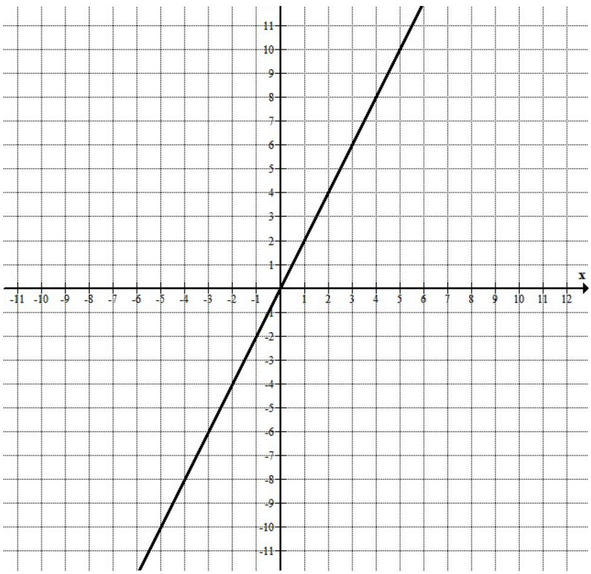


**Figure 2.18**

| x | 0 | 1 | 2 | –1 | –2 | 3 | –3 |
|---|---|---|---|---|---|---|---|
| y | 0 | 2 | 4 | –2 | –4 | 6 | –6 |

Next we proceed onto give the graph of y = 2x in the MOD plane $R_n(10)$.

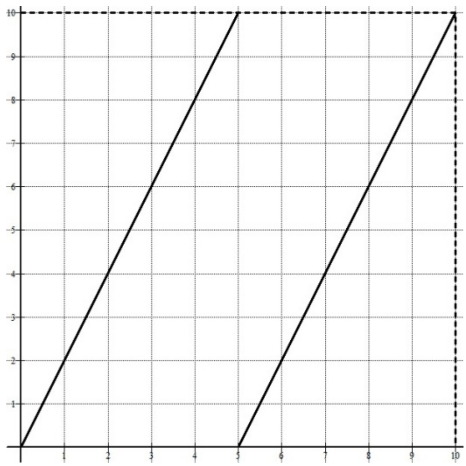

**Figure 2.19**

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| y | 0 | 2 | 4 | 4 | 8 | 0 | 2 | 4 | 6 |

The graph of y = 2x in the MOD rectangular $R_n(5, 9)$ plane is given in the following.

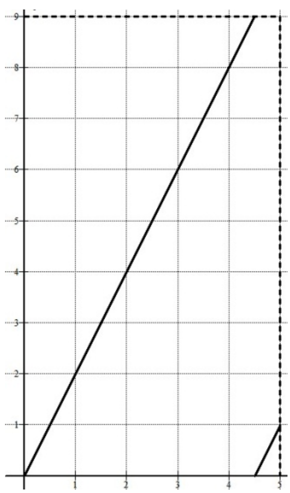

**Figure 2.20**

The table for this diagram is as follows.

| x | 0 | 1 | 2 | 3 | 4 | 4.5 | 5 |
|---|---|---|---|---|---|---|---|
| y | 0 | 2 | 4 | 6 | 8 | 0 | 1 |

Next we proceed onto describe the graph of y = 2x in the MOD rectangular plane $R_n(8, 6)$;

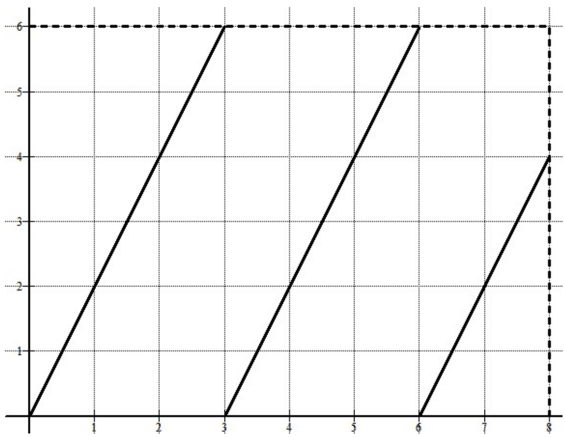

**Figure 2.21**

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| y | 0 | 2 | 4 | 6 | 0 | 2 | 4 | 6 |

We see all the graphs are distinct.

Interested reader can study for various values. However we present one more example of this situation.

***Example 2.14***. Let y = 5x be the function.

To find the graph of y = 5x in R, $R_n(7)$, $R_n(7, 8)$ and $R_n(9, 6)$.

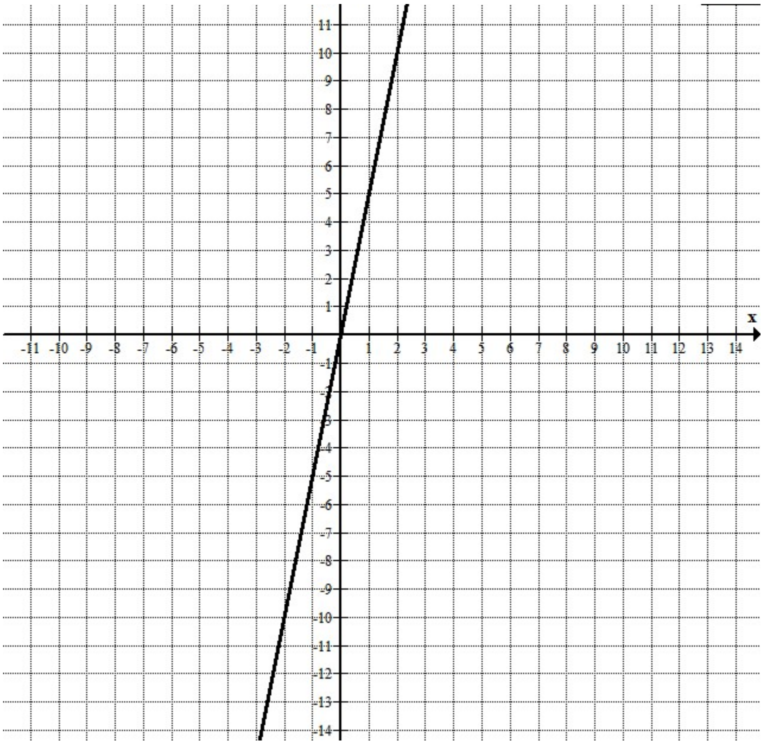


**Figure 2.22**

| x | 0 | 1 | –1 | 2 | –2 |
|---|---|---|---|---|---|
| y | 0 | 5 | –5 | 10 | –10 |

Next we proceed onto give the graph of y = 5x in the MOD plane $R_n(7)$.

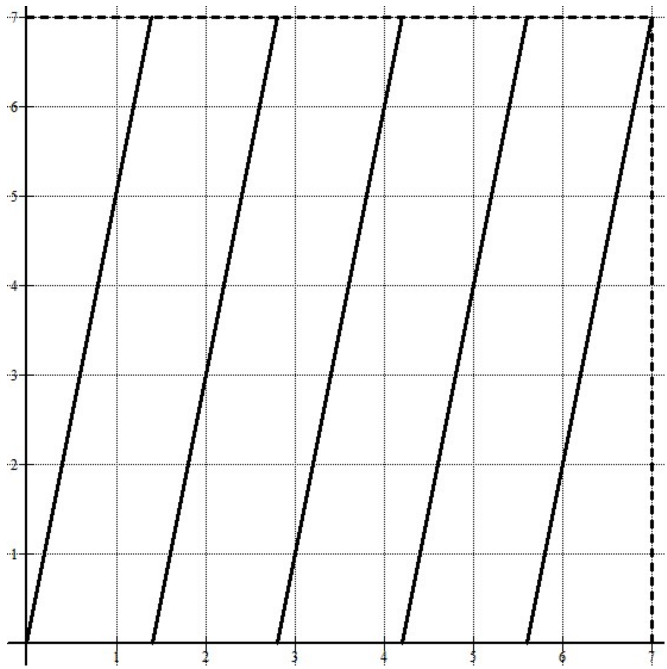


**Figure 2.23**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| x | 0 | 1 | 2 | 1.4 | 2.5 | 2.8 | 3 | 4 | 4.2 | 5 |
| y | 0 | 5 | 6 | 0 | 5.5 | 0 | 1 | 6 | 0 | 4 |
| x | 5.4 | 5.6 | 6 | 6.4 | 6.8 | | | | | |
| y | 6 | 0 | 2 | 4 | 6 | | | | | |

y = 5x has atleast 5 zeros in $R_n(7)$.

Next we proceed onto give the graph of y = 5x in the MOD rectangular plane $R_n(7, 8)$.

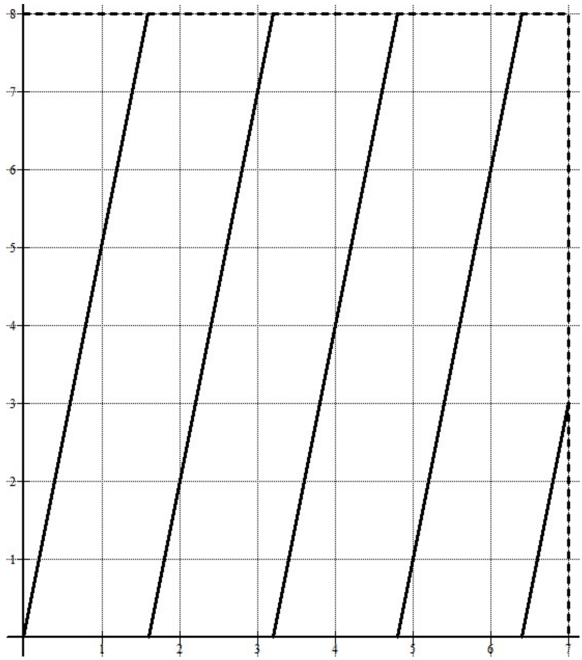


**Figure 2.24**

Next we proceed onto describe the graph of y = 5x in the MOD rectangular plane $R_n(9, 6)$ in the following.

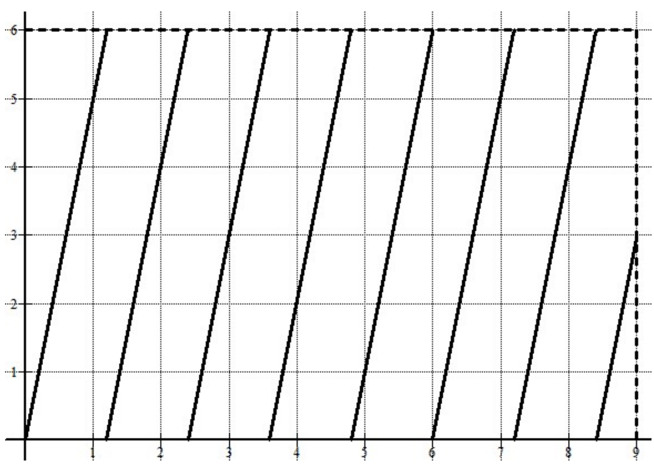


**Figure 2.25**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| x | 0 | 1 | 1.2 | 2 | 2.5 | 3 | 3.4 | 3.6 | 4 | 4.4 | 4.8 | 5 |
| y | 0 | 5 | 0 | 4 | 0 | 3 | 5 | 0 | 2 | 4 | 0 | 1 |
| x | 5.2 | 5.6 | 5.8 | 6 | 6.4 | 6.8 | 7 | 7.2 | 7.4 | 8 | 8.4 | 88 |
| y | 2 | 4 | 5 | 0 | 2 | 4 | 5 | 0 | 1 | 4 | 0 | 2 |

We see the function y = 5x has 8 zeros in the MOD rectangular plane $R_n(9, 6)$.

Further the number of zeros of y = 5x is highly dependent on the plane in which it is considered. So the distinct features enjoyed by planes is established for y = 5x and is zero only x = 0. But in case the MOD plane $R_n(7)$ and MOD rectangular planes $R_n(7, 8)$ and $R_n$ (9, 6) the function y = 5x has many zeros.

**Example 2.15**. Let y = x + 1 be the function. We see the graph of y = x + 1 in R, $R_n(2)$, $R_n(5)$, $R_n(2, 7)$ and $R_n(5, 3)$.

The graph of y = x +1 in the real plane.

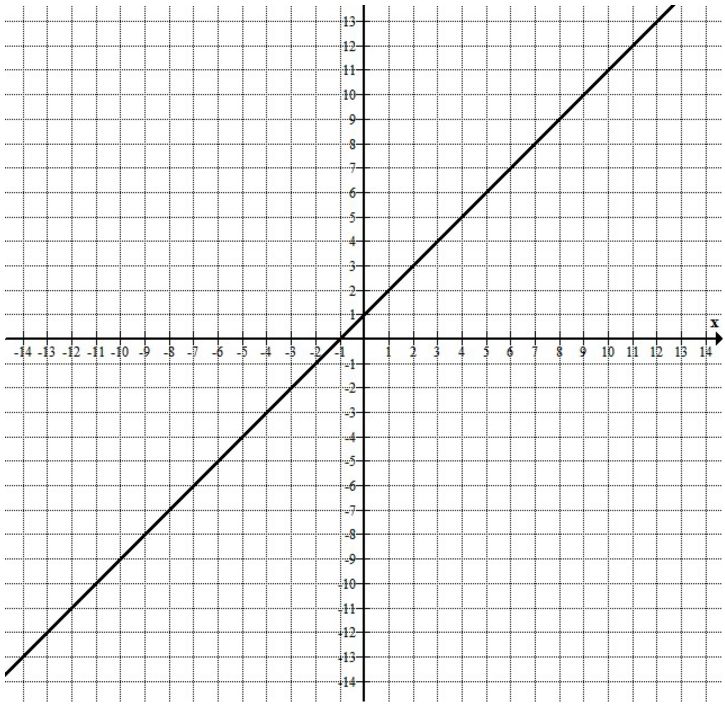


**Figure 2.26**

The graph is a straight line infact a continuous curve. We now give the graph of y = x + 1 in the MOD plane $R_n(2)$.

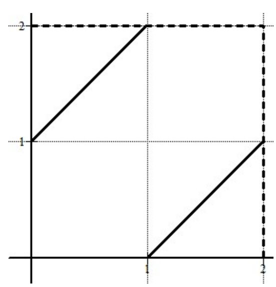

**Figure 2.27**

The MOD graph is not continuous it has two continuous branches. Similarly graph in the MOD plane $R_n(5)$ is given.

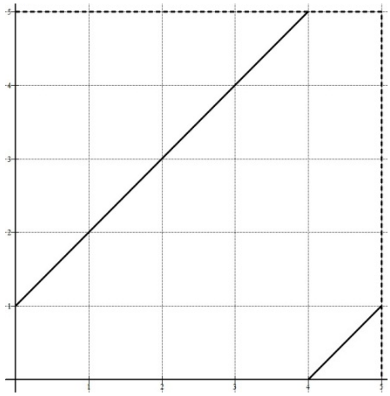

**Figure 2.28**

We see the MOD graph is not a continuous one even in the MOD plane $R_n(5)$. This also has two branches and one zero.

Now we give the graph of y = x + 1 in the MOD rectangular plane $R_n(2, 7)$.

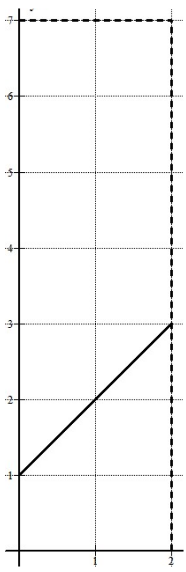

**Figure 2.29**

Next we proceed onto describe y = x + 1 in the MOD rectangular plane $R_n(5, 3)$.

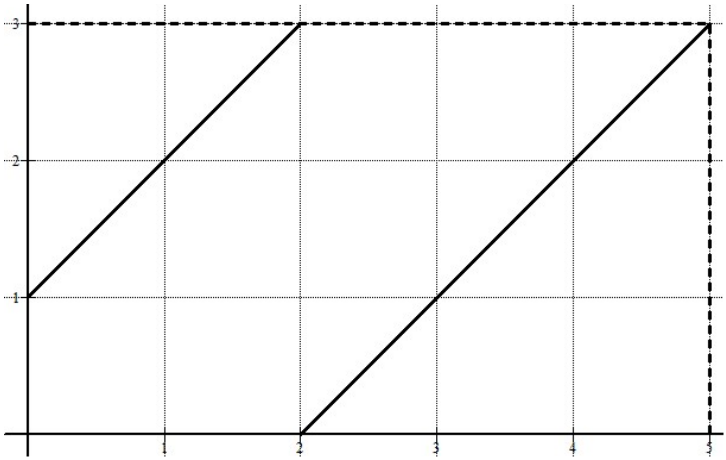


**Figure 2.30**

| x | 0 | 1 | 2 | 3 | 4 | 4.8 |
|---|---|---|---|---|---|---|
| y | 1 | 2 | 0 | 1 | 2 | 2.8 |

We see in R and $R_n(2, 7)$. The function y = x + 1 is a continuous one in the other MOD planes and MOD rectangular planes the function y = x + 1 has two branches and one zero. However both the branches are also continuous.

Next we give one more example of the function in all the three types of planes.

Let us consider $y = x^2 + 2x + 1$. This function is defined only in $Z_n$; $n \geq 3$.

***Example 2.16***. Let $y = x^2 + 2x + 1$ be the function. Let us find the graph of $y = x^2 + 2x + 1$ in R, $R_n(7)$, $R_n(8)$, $R_n(5, 8)$ and $R_n(9, 6)$.

The graph of $y = x^2 + 2x + 1$ in the real plane R is as follows.

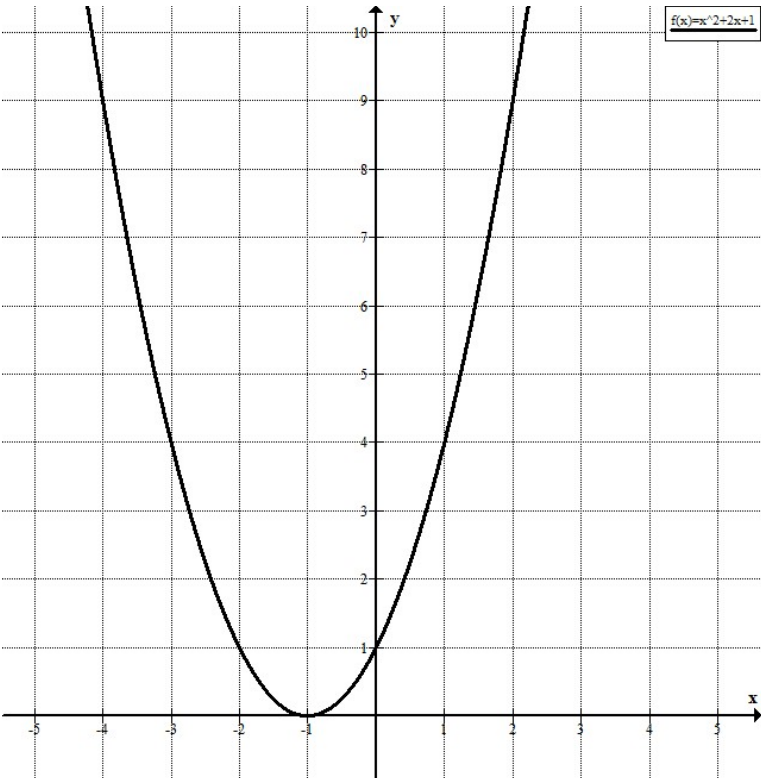


**Figure 2.31**

The graph of $y = x^2 + 2x + 1$ in the MOD real plane $R_n(7)$ is as follows.

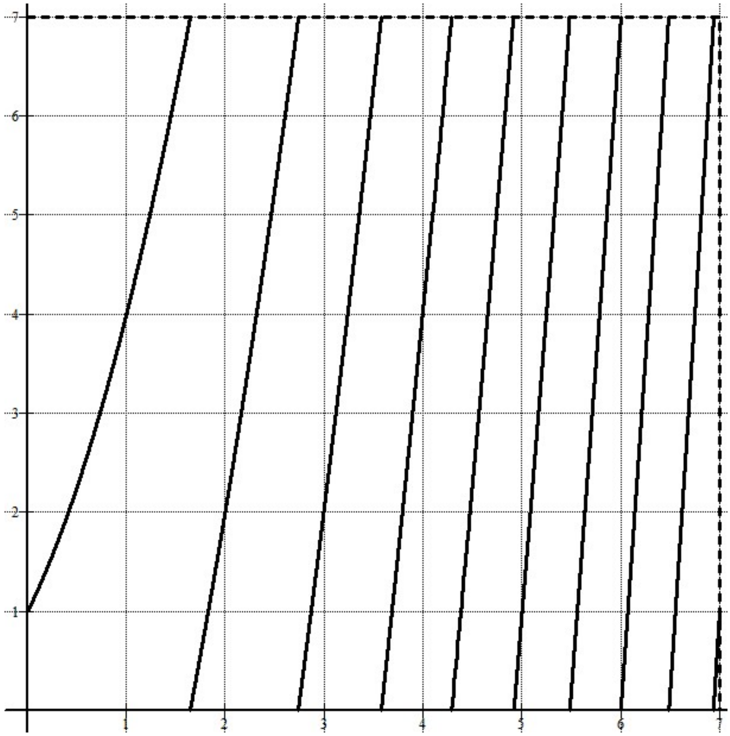


**Figure 2.32**

| x | 0 | 1 | 2 | 1.2 | 1.5 | 1.6 | 1.8 | 2 |
|---|---|---|---|---|---|---|---|---|
| y | 1 | 4 | 2 | 4.84 | 6.25 | 6.76 | 0.84 | 2 |

It is seen between 1.6 and 1.7 the zero of y lies.

| x | 2.5 | 3 | 3.4 | 3.5 | 3.6 | 3.8 | 3.9 | 4 | 4.2 | 4.5 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | 5.25 | 2 | 3.36 | 6.25 | 0.16 | 2.04 | 3.01 | 4 | 5.04 | 2.25 |

So a zero lies in between 2.5 and 3. In particular between 2.7 and 2.8. Between 3.5 and 3.6 we have again a zero.

Further between 4.2 and 4.3 there is again a zero. For at x = 4.3, y = 0.09. Now between 4.9 and 4.95 there is again a zero. Consider 5.4 and 5.5 there is a zero. Between 5.95 and 6 there is a zero. That is at x = 6, y = 0.

Between 6.4 and 6.5 there is a zero.

So there are eight zeros in the MOD plane $R_n(7)$.

We describe the graph of the function $y = x^2 + 2x + 1$ in the MOD rectangular plane $R_n(5, 8)$.

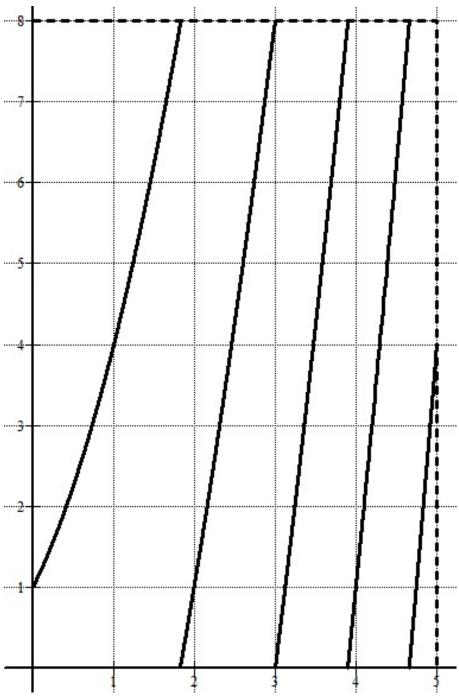


**Figure 2.33**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| x | 0 | 1 | 1.8 | 1.9 | 2 | 2.6 | 3 | 3.2 | 3.5 | 3.9 |
| y | 1 | 4 | 7.84 | 0.41 | 1 | 4.96 | 0 | 1.64 | 4.25 | 0.01 |
| x | 4 | 4.5 | 4.7 | | | | | | | |
| y | 1 | 2.25 | 0.49 | | | | | | | |

There are four zeros in the intervals (1.9, 2), 3, (3.8, 3.9) and (4.6, 4.7).

The graph of $y = x^2 + 2x + 1$ in the MOD rectangular a plane $R_n(9, 6)$ is as follows.

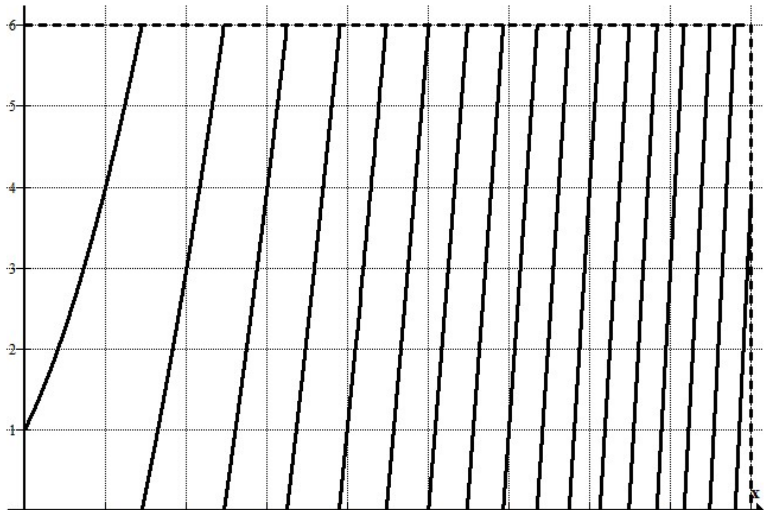


**Figure 2.34**

There is a zero between 1.4 and 1.5. There is a zero between 2.4 and 2.5.

There is another zero between 3.2 and 3.3. There is again a zero between 3.8 and 3.9.

There is a zero between 4.4 and 4.5, at x = 5, y = 0. There is a zero between 5.4 and 5.5.

Between 5.9 and 5.94 there is a zero.

There is a zero between 6.3 and 6.4. There is a zero between 6.7 and 6.8.

There is a zero between 7.1 and 7.2. Between 7.4 and 7.5 there is a zero. There is a zero between 7.8 and 7.85.

There is a zero between 8.1 and 8.2. Between 8.4 and 8.5 there is a zero. Between 8.7 and 8.8 there is a zero.

There are 16 zeros for $y = x^2 + 2x + 1$ in the MOD rectangular plane $R_n(9, 6)$.

So if in $R_n(x, y)$ if $x > y$ we have more zeros and if $x < y$ then $R_n(x, y)$ has lesser number of zeros.

It is important to record almost all properties related with functions on MOD rectangular planes happens to be different and new and this study is innovative and interesting.

We suggest a few problems in this direction.

**Problems**

1. What are the special features associated with rectangular planes $R_n(s, t)$; $(s \neq t)$?

2. Let $R_n(8, 7)$ be the MOD rectangular plane. Draw the graph of the following functions.

   a) $y = x$

   b) $y = 2x$

   c) $y = 5x$

d) $y = 6x$

e) $y = 2x + 3$

f) $y = 6x + 3$

g) $y = x^2$

h) $y = 2x^2 + 3$

i) $y = x^2 + 4x + 3$

j) $y = x^3$

k) $y = x^4 + 2x + 1$

Compare these function graphs in the plane R, $R_n(7)$ and $R_n(7, 8)$.

3. Let $y = 8x^2 + 2x + 1$ be the given function. Plot it in the following planes;

   (a) R, $R_n(9)$ and $R_n(9, 12)$.

   (b) R, $R_n(17)$ and $R_n(10,11)$.

   (c) R, $R_n(42)$ and $R_n(20, 14)$.

4. Plot the graph $y = 5x^2 + 3x + 4$ in (i) $R_n(7)$ and $R_n(7, 8)$ (ii) $R_n(14, 9)$ (iii) $R_n(15, 10)$ and (iv) $R_n(6, 9)$.

5. Let $y = 5x + 3$ be the MOD function compare the graphs in $R_n(6, 14)$ and $R_n(14, 6)$ which has more number of zeros for $y = 5x + 3$ in $R_n(6, 14)$ or in $R_n(14, 6)$.

6. Find all the zeros of $y = 3x^2 + x + 1$ in the MOD rectangular plane $R_n(5, 7)$ and $R(8, 7)$.

   Which of the plane gives more zeros?

7. Find all zeros of the function $y = x^3 + 3$ in the MOD rectangular planes $R_n(5, 7)$ and $R_n(8, 6)$.

8. Let $y = 3x^2 + 4x + 3$ be the MOD function. Find all zeros of y in the MOD rectangular planes; $R_n(8, 9)$ and $R_n(15, 8)$.

10. Let $y = 2x^2 + 3x + 4$ be the MOD function. Find all zeros in the MOD planes.

    i) $R_n(5, 7)$,

    ii) $R_n(17, 13)$,

    iii) $R_n(6, 9)$ and

    iv) $R_n(10,5)$.

11. Give an example of a function $y = f(x)$ which has no zeros in the MOD rectangular planes $R_n(8, 16)$ and $R_n(11,9)$.

Chapter Three

# ALGEBRAIC STRUCTURES USING RECTANGULAR MOD PLANES

Let $R_n(s, t)$; $s \neq t$; $2 \leq s, t < \infty$ be a MOD rectangular plane. $R_n(s, t) = \{(a, b) / a \in [0, s)$ and $b \in [0, t)\}$. We define the operation + on them.

We first give some illustrative examples.

***Example 3.1***. Let $R_n(3, 5) = \{(a, b) / a \in [0, 3)$ and $b \in [0,5)\}$ be the MOD rectangular plane. Let $x = (2.053, 3.0112)$ and $y = (1.6571, 2.43005) \in R_n(3.5)$.

We find x + y, $x + y = (2.053, 3.0112) + (1.6571, 2.43005) = ((2.053 + 1.6571) \pmod 3), (3.0112 + 2.43005) \pmod 5)) = (0.7101, 0.44125) \in R_n(3, 5)$.

Now $(0, 0) \in R_n(3,5)$ is such that $(x, y) + (0, 0) = (x, y)$ for any $(x, y) \in R_n(3, 5)$.

Let $a = (0.37, 1.325) \in R_n (3, 5)$; we have a unique $b = (2.63, 3.675)$ in $R_n (3, 5)$ such that $a + b = (0.37, 1.325) + (2.63, 3.675) = ((0.37 + 2.63) \pmod 3), (1.325 + 3.675) \pmod 5)) = (0, 0)$.

Thus in $R_n(3, 5)$ for every $a \in R_n(3, 5)$ there is a unique $b \in R_n(3, 5)$ such that $a + b = (0, 0)$. That is every element $a \in R_n(3, 5)$ there is a unique inverse element of a in $R_n(3, 5)$. So $\{R_n(3, 5), +\}$ is an abelian group of infinite order.

Clearly $\{R_n(3, 5), +\}$ has subgroups of finite order as well as infinite order.

***Example 3.2***. Let $R_n(7, 6) = \{(a, b) / a \in [0, 7), b \in [0, 6)\}$ be the MOD rectangular plane.

Let $P_1 = \{(a, b) / a \in Z_7, b \in Z_6\} \subseteq R_n(7, 6)$; $P_1$ is a subgroup of $R_n(7, 6)$ of finite order.

$\{R_n(7, 6),+\}$ is an infinite abelian group.

$W = \{(a, 0) / a \in [0, 7)\} \subseteq R_n(7, 6)$ is an infinite abelian subgroup of $R_n(7, 6)$.

$P_2 = \{(0, b) / b \in Z_6\} \subseteq R_n(7, 6)$ is a abelian subgroup of order 6.

In view of all these now we make the abstract definition.

**Definition 3.1**. *Let $G = \{R_n(s, t), +\} = \{(a, b) / a \in [0, s), b \in [0, t). s \neq t; +\}$ be the MOD rectangular group of infinite order under +.*

We enumerate a few properties associated with them.

**Theorem 3.1**. *Let $G = \{R_n(s, t), +\} = \{(a, b) / a \in [0, s), b \in [0, t); s \neq t, +\}$ be the MOD rectangular group of infinite order.*

*i) G is an abelian group.*

*ii) G has subgroups of both finite and infinite order .*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular semigroups under $\times$ by some examples.

***Example 3.3***. Let $S = \{R_n(9, 12) = \{(a, b) / a \in [0, 9)$ and $b \in [0, 12), \times\}$ be the MOD rectangular semigroup under $\times$.

Let $x = (3.271, 5.43)$ and $y = (0.32, 9.2) \in S$.

$$\begin{aligned} x \times y &= (3.271, 5.43) \times (0.32, 9.2) \\ &= (3.271 \times 0.32 \ (\text{mod } 9), 5.43 \times 9.2 \ (\text{mod } 12)) \\ &= (1.04672, 1.956) \in S. \end{aligned}$$

This is the way product operation is performed.

We see $o(S) = \infty$. Infact S has both finite order MOD subsemigroups as well as infinite order subsemigroups.

S has ideals but always ideals of S are of infinite order.

$P_1 = \{(a, b) / a \in Z_9$ and $b \in Z_{12}, \times\} \subseteq S$ is a MOD rectangular subsemigroup of S of finite order.

$P_2 = \{(a, 0) / a \in [0, 9), \times\} \subseteq S$ is a MOD rectangular subsemigroup of infinite order which is also an ideal of S.

$P_3 = \{(a, b) / a \in Z_9, b \in [0, 12), \times\}$ is a MOD rectangular subsemigroup of infinite order which is not an ideal.

Further S has infinite number of zero divisors.

$(a, 0)$ where $a \in [0, 9)$ is a zero divisor in S.

Also $(0, 0) \in S$ is such that $(0, 0) \times x = (0, 0)$ for all $x \in S$. $(1, 1) \in S$ is such that $(1, 1) \times x = x$ for all $x \in S$.

Thus S is an infinite commutative rectangular monoid.

The existence of idempotents and nilpotents are greatly dependent on s and t of the MOD rectangular plane, $R_n(s, t)$; $2 \leq s, t < \infty$, $(s \neq t)$.

We see x = (1, 4) and y = (0, 9) are MOD idempotents of S for $x \times x = x$ and $y = y \times y$.

But p = (3, 6) and q = (6, 6) in S are such that $p \times p = (0, 0)$ and $q \times q = (0, 0)$ are nilpotents of order two.

***Example 3.4***. Let $\{P, \times\} = \{R_n(11, 7) = \{(a, b) / a \in [0, 11)$ and $b \in [0, 7)\}, \times\}$ be the MOD rectangular semigroup under product.

This has no nontrivial idempotents or nilpotents as both 11 and 7 are primes. But has zero divisors.

For x = (5.5, 3.5) and y = (4, 2) $\in$ P is such that $x \times y = (5.5, 3.5) \times (4, 2) = (0, 0)$ is a non trivial zero divisor.

We see $B_1 = \{(x, y) / x \in Z_{11}$ and $y \in Z_7\} \subseteq P$ is such that $B_1$ is only a MOD rectangular subsemigroup and is not an ideal.

Infact $o(B_1) < \infty$.

$B_2 = \{(x, 0) / x \in [0, 11), \times\} \subseteq P$ is a MOD rectangular subsemigroup of infinite order which is an ideal.

$B_3 = \{(0, b) / b \in [0, 7), \times\} \subseteq P$ is also a MOD rectangular subsemigroup which is an ideal of P.

Clearly $o(B_2) = \infty$ and $o(B_3) = \infty$.

Consider $B_4 = \{(a, b) / a \in [0,11), b \in Z_7, \times\} \subseteq P$ is a MOD rectangular subsemigroup of P of infinite order which is not an ideal. Infact a submonoid of P.

***Example 3.5***. Let G = {$R_n(17, 24), \times$} = {(a, b) / a $\in$ [0,17), b $\in$ [0, 24); $\times$} be the MOD rectangular semigroup under product.

Clearly this has only partly trivial MOD idempotents like $x_1 = (0, 16)$, $x_2 = (1, 16)$, $x_3 = (0, 9)$ and $x_4 = (1, 9)$, we cannot get other types of MOD idempotents as 17 is a prime.

In view of all these we have the following theorem.

**Theorem 3.2**. *Let S = {$R_n(s, t)$ = {(a, b) / a $\in$ [0, s) and b $\in$ [0, t), $\times$} be the MOD rectangular semigroup under product* $\times$.

i) *$o(S) = \infty$.*

ii) *S is an abelian monoid.*

iii) *S has infinite number of zero divisors.*

iv) *S has MOD subsemigroups of finite order.*

v) *S has MOD rectangular subsemigroups which are ideals and all ideals are of infinite order.*

vi) *S has MOD rectangular subsemigroups of infinite order which are not ideals.*

vii) *S has nontrivial MOD idempotents and MOD nilpotents only when s and t are not primes and of the form $s = p_1^q q_1$ and $t = p_2^q q_2$ where $p_1$ and $p_2$ are primes a, b $\geq$ 2, $(p_i, q_i) = 1$, i = 1, 2 ($q \geq 1$).*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular matrix under + first by some examples.

***Example 3.6***. Let P = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \right.$ / $a_i \in R_n(8, 6)$ = {(a, b) /

a ∈ [0, 8) and b ∈ [0, 6) +}; 1 ≤ i ≤ 5, +} be the MOD rectangular matrix group under +.

Clearly o(P) = ∞.

$$\text{Let } x = \begin{bmatrix} (3, 5.1) \\ (0.25, 0.32) \\ (2.01, 3.02) \\ (6, 2.15) \\ (0.3,1.2) \end{bmatrix} \text{ and } y = \begin{bmatrix} (7.2,3) \\ (1.18,0.31) \\ (5.01,1.1) \\ (6.2,4.5) \\ (0,3.8) \end{bmatrix} \in P.$$

$$x + y = \begin{bmatrix} (3, 5.1) \\ (0.25, 0.32) \\ (2.01, 3.02) \\ (6, 2.15) \\ (0.3,1.2) \end{bmatrix} + \begin{bmatrix} (7.2,3) \\ (1.18,0.31) \\ (5.01,1.1) \\ (6.2,4.5) \\ (0,3.8) \end{bmatrix} = \begin{bmatrix} (2.2,2.1) \\ (1.43,0.63) \\ (7.04,4.12) \\ (4.2,0.65) \\ (0.3,5) \end{bmatrix} \in P.$$

For every x ∈ P we have a unique y ∈ P such that

$$x + y = (0) = \begin{bmatrix} (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \end{bmatrix}.$$

$$\text{For } x = \begin{bmatrix} (3.2, 2.01) \\ (4.07, 0.57) \\ (0.31, 1.24) \\ (3.005, 0.115) \\ (0.123, 3) \end{bmatrix} \in P.$$

We have a unique

$$y = \begin{bmatrix} (4.8, 3.99) \\ (3.93, 5.4) \\ (7.69, 4.76) \\ (4.995, 5.885) \\ (7.877, 3) \end{bmatrix} \text{ in P}$$

such that x + y = (0).

Further for

$$(a) = \begin{bmatrix} (3,5) \\ (2,4) \\ (6,2) \\ (0.31, 4.02) \\ (5.001, 2.007) \end{bmatrix}$$

$$(a) + (a) = \begin{bmatrix} (6,4) \\ (4,2) \\ (4,4) \\ (0.62, 2.04) \\ (4.002, 4.014) \end{bmatrix} \in P.$$

Can we say if a $\in$ P we will have a finite m > 2 such that $\underbrace{a + a + \ldots + a}_{m-\text{times}} = (0)$.

However the authors leave it as a exercise for the reader to prove or disprove this claim. But then in our view such m is not always possible.

This property is true if the entries of the matrix are from $Z_8$ and $Z_6$ and not from [0, 8) and [0, 6).

We can say this is a group such that the sum of a term x with itself is taken finite number of times is not (0).

***Example 3.7***. Let M = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right.$ / $a_i \in R_n(4,12) = \{(a, b)$ / $a \in [0, 4)$ and $b \in [0, 12)\}$, $1 \leq i \leq 4$, +} be the MOD rectangular group under +.

Let, $x = \begin{bmatrix} (0.3,4) & (2.12,0.3) \\ (0.7,2.7) & (0.15,2) \end{bmatrix} \in M.$

$$x + x = \begin{bmatrix} (0.3,4) & (2.12,0.3) \\ (0.7,2.7) & (0.15,2) \end{bmatrix} + \begin{bmatrix} (0.3,4) & (2.12,0.3) \\ (0.7,2.7) & (0.15,2) \end{bmatrix}$$

$$= \begin{bmatrix} (0.6,8) & (0.24,0.6) \\ (1.4,5.4) & (0.3,4) \end{bmatrix} \in M.$$

We see $y = \begin{bmatrix} (3.7,8) & (1.88,11.7) \\ (3.3,9.3) & (3.85,10) \end{bmatrix} \in M$

is such that $x + y = \begin{bmatrix} (0,0) & (0,0) \\ (0,0) & (0,0) \end{bmatrix}$.

Thus y is the inverse of x and vice versa.

Let $T_1 = \{\begin{bmatrix} (a_1,0) & (a_2,0) \\ (a_3,0) & (a_4,0) \end{bmatrix}$ / $a_i \in [0, 4)$; $1 \leq i \leq 4$, +}

$\subseteq$ M is a MOD rectangular matrix subgroup of infinite order.

$T_2 = \{\begin{bmatrix} (0,b_1) & (0,b_2) \\ (0,b_3) & (0,b_4) \end{bmatrix}$ / $b_j \in Z_{12}$; $1 \leq j \leq 4$, +} $\subseteq$ M

is a MOD rectangular matrix subgroup of finite order.

$T_3 = \{\begin{bmatrix} (a,b) & (0,0) \\ (0,0) & (0,0) \end{bmatrix}$ / a $\in$ [0, 4), b $\in$ [0,12), +} $\subseteq$ M;

is MOD rectangular matrix subgroup of infinite order.

$T_4 = \{\begin{bmatrix} (0,0) & (a,0) \\ (0,b) & (0,0) \end{bmatrix}$ / a $\in$ [0, 4) and b $\in$ [0,12), + }

$\subseteq$ M is again a MOD rectangular matrix subgroup of infinite order.

We see M has several subgroups some of finite order and some are of infinite order.

***Example 3.8***. Let W = $\{(a_1, a_2, a_3, a_4, a_5, a_6)$ / $a_i \in R_n(7, 13)$ = {(a, b) / a $\in$ [0, 7), b $\in$ [0, 13), +}, $1 \leq i \leq 6$, +} be MOD rectangular row matrix group under +. W is of infinite order.

This has MOD subgroups of both finite and infinite order.

***Example 3.9.*** Let M = { $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix}$ / $a_i \in R_n(12, 10)$ =P{a, b) /

$a \in [0, 12)$, $b \in [0, 10)$, +}; $1 \leq i \leq 12$, +} be the MOD rectangular matrix group under +.

M has several MOD subgroups of infinite order. Only a few MOD subgroups of finite order.

$P_1$ = { $\begin{bmatrix} (a,b) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ / $(a, b) \in R_n(12, 10)$, +} $\subseteq$ M

is a MOD rectangular subgroup of infinite order.

$P_2$ = { $\begin{bmatrix} (a,b) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ / $a \in Z_{12}$ and $b \in Z_{10}$, +} $\subseteq$ M

is a MOD rectangular subgroup of finite order.

$$\text{Let } P_3 = \left\{\begin{bmatrix} (a,0) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} / a \in [0, 12), +\right\} \subseteq M$$

is again a MOD rectangular subgroup of infinite order.

$$\text{Let } P_5 = \left\{\begin{bmatrix} (a,b) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} / a \in [0, 12), b \in Z_{10}, +\right\} \subseteq M$$

is also a MOD rectangular subgroup of infinite order.

Thus there are many MOD rectangular subgroups of infinite order.

Interested reader can study the properties related with it.

***Example 3.10***. Let $S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right.$ where $a_i \in R_n(23, 48)$

$= \{(a, b) / a \in [0, 23), b \in [0, 48), +\}, +; 1 \leq i \leq 9\}$ be the MOD rectangular matrix group under +. $o(S) = \infty$. S is a commutative group. S has both finite and infinite order subgroups.

Next we proceed onto describe MOD rectangular matrix semigroups by examples.

***Example 3.11***. Let V = $\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$ / $a_i \in R_n(3, 7) = \{(a, b)$ /

$a \in [0, 3)$ and $b \in [0, 7)$, $\times\}$ $\times_n$, $1 \leq i \leq 5\}$ be the MOD rectangular semigroup. o(V) = $\infty$ and V is a commutative semigroup infact a monoid.

For ({(1, 1)}) = $\begin{bmatrix} (1,1) \\ (1,1) \\ (1,1) \\ (1,1) \\ (1,1) \end{bmatrix}$ in V is the multiplicative identity.

{(0, 0)} = $\begin{bmatrix} (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \end{bmatrix}$ in V is the zero of V and

$[(0, 0)] \times A = A \times [(0,0)] = [(0, 0)]$ for all $A \in V$.

$[(1, 1)] \times A = A \times [(1, 1)] = A$ for all $A \in V$.

Let A = { $\begin{bmatrix} (0.3, 0.25) \\ (1,1) \\ (1.2, 0) \\ (0, 3.5) \\ (1, 2) \end{bmatrix}$ and

$$B = \begin{bmatrix} (0.5, 6) \\ (0.331, 6.21) \\ (1.6, 3.2213) \\ (1.5234, 2) \\ (0.1115, 2.0006) \end{bmatrix} \in V;\text{ we find}$$

$$A \times_n B = \begin{bmatrix} (0.3, 0.25) \\ (1,1) \\ (1.2, 0) \\ (0, 3.5) \\ (1, 2) \end{bmatrix} \times_n \begin{bmatrix} (0.5, 6) \\ (0.331, 6.21) \\ (1.6, 3.2213) \\ (1.5234, 2) \\ (0.1115, 2.0006) \end{bmatrix}$$

$$= \begin{bmatrix} (0.15, 1.5) \\ (0.331, 6.21) \\ (1.82, 0) \\ (0, 0) \\ (0.1115, 4.0012) \end{bmatrix} \in V.$$

Interested reader can find the MOD idempotents, nilpotents and zero divisors of V.

Next we give some more examples.

***Example 3.12***. Let W = $\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$ / $a_i \in R_n(16, 12)$ =

$\{(a, b) / a \in [0, 6), b \in [0, 12), \times\}, \times_n\}$ be the MOD rectangular matrix semigroup. o(W) = ∞.

$$P_1 = \{\begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \end{bmatrix} / a, b \in R_n (16, 12) \times_n\} \subseteq W$$

is a MOD rectangular matrix subsemigroup of W which is also an ideal of W.

$$P_2 = \{\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \end{bmatrix} / a \in R_n (16, 12) \text{ and } b = \{(x, 0) /$$

$x \in Z_{16}\}, \times_n\} \subseteq W$ is a MOD rectangular subsemigroup of infinite order but $P_2$ is not an ideal of W.

$$\text{Consider } P_3 = \{\begin{bmatrix} 0 & 0 & 0 \\ a & b & 0 \end{bmatrix} / a, b \in Z_{16} \times Z_{12} = \{(x, y) /$$

$x \in Z_{16}, y \in Z_{12}\}, \times_n\} \subseteq W$ is a MOD rectangular subsemigroup of finite order.

Clearly $P_3$ is not an ideal. It is left as an exercise to the reader to prove that if P is an ideal of W then P is of infinite order.

Let $x = \begin{bmatrix} (8,3) & (4,6) & (2,9) \\ (10,4) & (12,6) & (4,8) \end{bmatrix}$ and

$$y = \begin{bmatrix} (2,4) & (8,6) & (8,4) \\ (8,3) & (4,4) & (8,6) \end{bmatrix} \in W.$$

Clearly $x \times_n y = \begin{bmatrix} (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{bmatrix}$.

This MOD rectangular matrix semigroup has zero divisors.

Let $a = \begin{bmatrix} (8,6) & (4,0) & (12,6) \\ (0,6) & (8,6) & (4,0) \end{bmatrix} \in W$.

We see $a \times_n a = \begin{bmatrix} (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{bmatrix}$

so a in W is a MOD nilpotent matrix of order two.

This W has also MOD rectangular matrix subsemigroups which are not ideals.

$S_1 = \{\begin{bmatrix} (a,b) & (0,0) & (0,0) \\ (c,d) & (0,0) & (0,0) \end{bmatrix} / a, c \in Z_{16}, b, d \in Z_{12}, \times_n\}$

$\subseteq$ W is only MOD rectangular matrix subsemigroup which is not an ideal.

Infact $o(S_1) < \infty$ so is only a MOD rectangular finite order matrix subsemigroup.

Let $S_2 = \{\begin{bmatrix} (a,b) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} / (a, b) \in R_n(16, 12), \times_n\} \subseteq W$.

Clearly $S_2$ is a MOD rectangular matrix subsemigroup of W which is also an ideal of W.

$o(S_2) = \infty$.

It is observed that all MOD rectangular matrix ideals of W are of infinite order.

Let x = $\begin{bmatrix} (1,9) & (1,4) & (0,4) \\ (0,9) & (1,9) & (1,4) \end{bmatrix} \in W$.

We see x $\times_n$ x = x so x is a MOD rectangular idempotent matrix of W.

In view of all these we have the following theorem.

**Theorem 3.3**. *Let M = {collection of all s × t matrices with entries from $R_n$ (p, q) = {(a, b) / a ∈ [0, p) and b ∈ [0, q), ×}, $\times_n$} be the MOD rectangular matrix semigroup.*

*i)* *o(M) = ∞.*

*ii)* *M is infact a commutative monoid.*

*iii)* *M has MOD rectangular matrix subsemigroups of finite order.*

*iv)* *M has MOD rectangular matrix ideals all of which are of infinite order.*

*v)* *M has MOD rectangular matrix subsemigroups of infinite order which are not ideals.*

*vi)* *M has MOD rectangular matrix zero divisors.*

*vii)* *M has nontrivial MOD rectangular matrix idempotents and nilpotents only for appropriate p and q.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular matrix noncommutative semigroup of infinite order by examples.

***Example 3.13***. Let M = { $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ / $a_i \in R_n(6, 10)$; $1 \leq i \leq 4$, ×} be the MOD rectangular square matrix semigroup.

$$\text{Let } A = \begin{bmatrix} (3,5) & (4,2) \\ (5,1) & (1,0) \end{bmatrix} \text{ and } B = \begin{bmatrix} (0,3) & (4,8) \\ (2,2) & (3,4) \end{bmatrix} \in M$$

$$A \times B = \begin{bmatrix} (3,5) & (4,2) \\ (5,1) & (1,0) \end{bmatrix} \times \begin{bmatrix} (0,3) & (4,8) \\ (2,2) & (3,4) \end{bmatrix}$$

$$= \begin{bmatrix} (3,5)\times(0,3)+ & (3,5)\times(4,8)+ \\ (4,2)\times(2,2) & (4,2)\times(3,4) \\ (5,1)\times(0,3)+ & (5,1)\times(4,8)+ \\ (1,0)\times(2,2) & (1,0)\times(3,4) \end{bmatrix}$$

$$= \begin{bmatrix} (3,5)+(2,4) & (0,0)+(0,8) \\ (0,3)+(2,0) & (2,8)+(3,0) \end{bmatrix} = \begin{bmatrix} (5,9) & (0,8) \\ (2,3) & (5,8) \end{bmatrix} \qquad \text{(i)}$$

$$B \times A = \begin{bmatrix} (0,3) & (4,8) \\ (2,2) & (3,4) \end{bmatrix} \times \begin{bmatrix} (3,5) & (4,2) \\ (5,1) & (1,0) \end{bmatrix}$$

$$= \begin{bmatrix} (0,3)\times(3,5)+ & (0,3)\times(4,2)+ \\ (4,8)\times(5,1) & (4,8)\times(1,0) \\ (2,2)\times(3,5)+ & (2,2)\times(4,2)+ \\ (3,4)\times(5,1) & (3,4)\times(1,0) \end{bmatrix}$$

$$= \begin{bmatrix} (0,5)+(2,8) & (0,6)+(4,0) \\ (0,0)+(3,4) & (2,4)+3,0) \end{bmatrix} = \begin{bmatrix} (2,3) & (4,6) \\ (3,4) & (5,4) \end{bmatrix} \qquad \text{(ii)}$$

(i) and (ii) are distinct.

So {M, ×} is a non commutative MOD rectangular semigroup of infinite order.

Finding MOD rectangular matrix zero divisors, MOD rectangular matrix idempotents and nilpotents happens to be a difficult problem.

Similarly M has MOD rectangular matrix right ideals which are not MOD rectangular matrix left ideals.

The reader is left with the task of finding them.

***Example 3.14***. Let D = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right.$ / $a_i \in R_n(11, 7)$

= {(a, b) / a ∈ [0, 11), b ∈ [0, 7), ×}; 1 ≤ i ≤ 3, ×} be the MOD rectangular square matrix semigroup. D is a noncommutative subsemigroup of infinite order.

The reader is left with the task of finding MOD rectangular matrix right ideals, MOD rectangular matrix left ideals, MOD rectangular zero divisors (right or left) and so on.

In view of all these we have the following result.

**Theorem 3.4**. *Let S = {M = $(m_{ij})_{m\times n}$ where $m_{ij} \in R_n(s, t)$ = {(a, b) / a ∈ [0, s) and b ∈ [0, t); s ≠ t}; 1 ≤ i, j ≤ m, ×} be the MOD rectangular square matrix semigroup under ×.*

*i)* *o(S) = ∞.*

*ii)* *S has both finite and infinite order MOD rectangular square matrix subsemigroups.*

*iii)* *S has MOD rectangular square matrix right ideals which are not left ideals.*

*iv)* *All MOD rectangular square matrix right ideals or left ideals are of infinite order.*

*v)* *S has MOD rectangular square matrix ideals of infinite order.*

*vi)* *S has MOD rectangular square matrix right and left zero divisors depending on s and t.*

*vii)* *S has MOD rectangular square matrix nilpotents only for particular values of s and t.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto define and describe MOD rectangular subsets.

$S(R_n(s, t) = \{$collection of all subsets from $R_n(s, t) = \{(a, b) / a \in [0, s), b \in [0, t)\}\}$.

$o(S(R_n(s, t)) = \infty$.

***Example 3.15***. Let $S(R_n(5, 8)) = \{$collection of all subsets from $R_n(5, 8) = \{(a, b) / a \in [0, 5), b \in [0, 8)\}\}$ be the MOD rectangular subsets of $R_n (5, 8)$.

Let A = {(3, 4.002), (0.3851, 2.118), (4.0007, 2) (3, 0.00097), (0.345, 2.62), (2, 4)} $\in S(R_n(5, 8))$ and

B = {(3, 2), (1, 1), (2, 2), (1, 0), (0, 1), (0, 4), (4, 0)} $\in S(R_n(5, 8))$.

We next proceed onto describe and define operations on $S(R_n(s, t))$ by examples.

***Example 3.16***. Let T = {S($R_n$(3, 5)), +} be the MOD rectangular subset semigroup under +.

We say T is only a semigroup for if A ∈ T we may not be always in a position to find a B such that A + B = {(0, 0)}. That is why we can have only a semigroup.

Consider A = {(0, 3.2), (1, 1), (0.32, 1.5)} and B = {(1, 0.3), (2.5, 3.7)} ∈ T.

We find A + B = {(0, 3.2), (1, 1), (0.32, 1.5)} + {(1, 0.3), (2.5, 3.7)}

= {(0, 3.2) + (1, 0.3), (1, 1) + (1, 0.3), (0.32, 1.5) + (1, 0.3), (0, 3.2) + (2.5, 3.7), (1, 1) + (2.5, 3.7), (0.32, 1.5) + (2.5, 3.7)}

= {(1, 3.5) (2, 1.3), (1.32, 4.5), (2.5, 1.9), (0.5, 4.7), (2.82, (0.2)} ∈ T.

This is the way + operation is performed on T.

Clearly we see for the given A ∈ T there is no B ∈ T such that A + B = {(0, 0)}.

However it is pertinent to keep on record that for every A ∈ T, A + {(0, 0)) = {(0, 0)} + A = A.

Thus T is a MOD rectangular subset monoid of infinite order.

T has MOD rectangular subset subsemigroups of both of finite and infinite order.

$P_1$ = {S((a, 0)) / a ∈ $Z_3$)} ⊆ T is a finite MOD subset rectangular subsemigroup of finite order.

$P_1$ = {{(0, 0)}, {(2, 0), (1, 0)}, {(0, 0), (2,0)}, {(0, 0), (1, 0)}, {(0, 0), (1, 0), (2, 0)}, {(1, 0), (2, 0)} and so on}.

$P_1$ is a MOD rectangular subset subsemigroup under + of finite order.

We see for

$S_1 = \{(0, 0), (2, 0), (1, 0)\}$ and $S_2 = \{(2, 0), (0, 0)\}$ in $P_1$.

$S_1 + S_2 = \{(0, 0), (2, 0), (1, 0)\} + \{(2, 0), (0, 0)\} = \{(0, 0), (1, 0), (2, 0)\}$.

Infact $P_1$ is a MOD rectangular finite subset monoid.

Let $R_1$ = {collection of all subsets from $\{0\} \times Z_5\}$ = $S(\{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)\} = \{\{(0,0)\}, \{(0, 1)\}, \{(0, 3)\}, \ldots, \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)\}\}$. $R_1$ under + is a MOD rectangular subset submonoid of finite order.

Study in this direction is interesting and important.

Let $T_1$ = {collection of all subsets from $[0, 3) \times \{0\} = \{(a, 0) / a \in [0, 3)\}$; $T_1$ under + is a MOD rectangular subset commutative submonoid of infinite order.

If $A = \{(2.1, 0), (1.32, 0), (1, 0), (1.12, 0)\}$ and $B = \{(0.2, 0), (0.31,0)\} \in T_1$ then

$A + B = \{(2.1, 0), (1.32, 0), (1, 0), (1.12, 0)\} + \{(0.2,0), (0.31,0)\} = \{(2.2, 0), (1.52, 0), (1.2, 0), (1.32, 0), (2.41, 0), (1.63, 0), (1.31, 0), (2.43, 0)\} \in T_1$.

Thus $T_1$ is a MOD rectangular subset submonoid of infinite order.

Interested reader is left with the task of finding all related structures with these MOD rectangular subset monoids.

***Example 3.17***. Let W = {collection of all subsets from $R_n(12, 6) = \{(a, b) / a \in [0,12)$ and $b \in [0,16)\}$; +},+} be the MOD rectangular subset semigroup.

Clearly o(W) = $\infty$ and W is infact a commutative monoid as {(0, 0)} is the additive identity of W.

We see for every x $\in$ W; x + {(0, 0)} = {(0,0)} + x = x.

Let A = {(3, 2), (4.31, 2.01), (4.1, 0), (1, 0.31), (2.1, 3.2)} and B = {(0, 3), (1.201, 0), (0.1112,0)} $\in$ W.

A + B = {(3.2, (4.31, 2.01), (4.12, 0), (1, 0.31), (2.1, 3.2)} + {(0, 3), (1.201, 0), (0.1112, 0)} = {(3, 5), (4.31, 5.01), (4.12, 3), (1, 3.31), (2.1, 0.2), (4.201, 2), (5.511, 2.01), (5.321, 0), (2.201, 0.31), (3.301, 3.2), (3.112, 2), (4.4212, 2.01), (4.2312, 0), (1.1112, 0.31), (2.2112, 3.2)} is in W.

This is the way '+' operation is performed on W.

Take $P_1$ = {collection of all subsets from [0,12) $\times$ {0} = {(a, 0) / a $\in$ [0,12)}, +} $\subseteq$ W is a MOD rectangular subset sub semigroup of infinite order.

$P_2$ = {collection of all subsets from $Z_{12} \times Z_6$ = {(a, b) / b $\in$ $Z_6$, a $\in$ $Z_{12}$}, +} $\subseteq$ W is a MOD rectangular subset subsemigroup of finite order in W.

Study in this direction is interesting and important.

$P_3$ = {collection of all subsets from [0, 12) $\times$ $Z_6$ = {(a, b) where a $\in$ [0, 12), b $\in$ $Z_6$}, +} $\subseteq$ W is a MOD rectangular subset subsemigroup of infinite order.

For A = {(4.3, 2) (3.5, 5),(0.11, 4)} and B = {(2.1, 0), (6.16, 2), (9.12, 4)} $\in$ $P_3$.

A + B = {(4.3, 2), (3.5, 5), (0.11, 4)} + {(2.1, 0), (6.16, 2), (9.12, 4)} = {(6.4, 2), (5.6, 5), (2.21, 4), (10.46, 4), (9.66, 1), (6.27, 0), (1.42, 0), (0.62, 3), (9.23, 2)} $\in$ $P_3$.

This is the way the + operation is performed on $P_3$.

For $A = \{(0.334, 2), (1.963, 0), (4.321, 3), (4.0061, 1)\} \in P_3$; to have a unique $B \in P_3$ such that $A + B = \{(0, 0)\}$ is an impossibility.

That is why we have $P_3$ to be only a MOD rectangular subset subsemigroup of infinite order.

***Example 3.18***. Let M = {collection of all subsets from the MOD rectangular plane $R_n(3, 15) = \{(a, b) / a \in [0, 3), b \in [0, 15), +\}, +\}$ be the MOD rectangular subset semigroup. $o(M) = \infty$, infact M is an infinite subset commutative monoid.

We see M has both MOD rectangular subsemigroups of finite as well as infinite order.

Let $B_1$ = {collection of all subsets from the MOD rectangular set $\{Z_3 \times \{0\}\}, +\} \subseteq M$ be a MOD rectangular subset subsemigroup of finite order.

$P = \{(2, 0), (1,0)\}$ and $Q = \{(0, 0), (1,0), (2, 0)\} \in B_1$.

We see $P + Q = \{(2,0), (0, 0), (1, 0)\} \in B_1$. This is the way '+' operation is performed on $B_1$.

Let $B_2$ = {collection of all subsets from MOD rectangular set $[0, 3) \times Z_{15} = \{(a, b) / a \in [0, 3), b \in Z_{15}, +\}, +\}$ be the MOD rectangular subset semigroup of infinite order.

We can find several MOD subset rectangular subsemigroups of infinite and finite order in M.

Next we proceed onto describe MOD subset rectangular semigroups under product operational by examples.

***Example 3.19***. Let S = {Collection of all subsets from $R_n(10,8) = \{(a, b) / a \in [0, 10), b \in [0, 8), \times\}, \times\}$ be the MOD rectangular subset semigroup under product.

Let A = {(8.2, 6.5), (2, 3.1), (4.72, 6.305), (4.05, 2.53)} and B = {(2, 4), (5, 5), (3, 3), (7, 2)} $\in$ S.

A $\times$ B = {(8.2, 6.5), (2, 3.1), (4.72, 6.305), (4.05, (2.53)} $\times$ {2, 4),(5, 5), (3, 3), (7, 2)}

= {(6.4, 2), (4, 4.4), (9.44, 1.22), (8.1, 2.12), (1, 0.5), (0, 7.5), (3.6, 7.525), (0.25, 4.65), (4.6, 3.5), (6, 1.3), (4.16, 2.915), (2.15, 7.59), (7.4, 5.0), (4, 6.2), (3.04, 2.61), (8.35, 5.06)} $\in$ S.

This is the way product operation is performed on S.

We see {(0, 0)} $\times$ A = A $\times$ {(0, 0)} = {(0, 0)} for all A $\in$ S.

Further A $\times$ {(1, 1)} = A for all A $\in$ S.

***Example 3.20***. Let B = {(collection of all subsets from $R_n(11, 17)$ = {(a, b) / a $\in$ [0, 11), b $\in$ [0, 17), $\times$}, $\times$} be the MOD subset rectangular semigroup under $\times$. o(B) = $\infty$ and B is a commutative monoid.

For every A $\in$ B, A $\times$ {(1, 1)} = A is the identity A $\times$ {(0, 0)} = {(0, 0)} for all A $\in$ B. Finding products happens to be a matter of routine so left as an exercise.

This B has MOD rectangular subset subsemigroups of finite and infinite order.

Infact all MOD rectangular subset ideals of B are of infinite order however this does not imply all MOD rectangular subsemigroups of infinite order of B are ideals.

Consider $P_1$ = {collection of all subsets from [0, 11) $\times$ $Z_{17}$ = {(0, b) / a $\in$ [0, 11), b $\in$ $Z_{17}$, $\times$}, $\times$} $\subseteq$ B, $P_1$ is a MOD rectangular subset subsemigroup of infinite order but is not an ideal of B. Hence the claim.

$P_2$ = {collection of all subsets from $Z_{11} \times Z_{17}$ = {(a, b) / a $\in Z_n$, b $\in Z_{11}$, ×}, ×} be the MOD subset rectangular subsemigroup of B. Clearly $P_2$ is not an ideal.

$P_3$ = {collection of all subsets from {0} × [0, 17) = {(0, a) / a $\in$ [0, 17), ×}, ×} $\subseteq$ B be the MOD subset rectangular subsemigroup of B. $P_3$ is a infinite order and it is also an ideal of B.

In view of all these we have the following theorem.

**Theorem 3.5.** *Let S = {collection of all subsets from $R_n(s, t)$ = {(a, b) / a $\in$ [0, s), b $\in$ [0, t), ×}, ×} be the MOD rectangular subset semigroup under ×.*

*i)* *o(S) = ∞ and S is a commutative monoid.*

*ii)* *S has MOD rectangular subsemigroups of finite or infinite order which are not ideals.*

*iii)* *S has MOD rectangular subset ideals all of which are of infinite order.*

*iv)* *S has MOD zero divisors, nilpotents and idempotents.*

*The presence of nilpotents and idempotents depends highly on s and t.*

Proof is direct and hence left as an exercise to the reader.

Now we give an example of the MOD rectangular subset zero divisors, idempotents and nilpotents.

***Example 3.21***. Let M = {collection of all subsets from $R_n(12, 24)$ = {(a, b) / a $\in$ [0, 12) and b $\in$ [0, 24), ×}, ×} be the MOD subset rectangular semigroup.

Consider P = {(0, 0.3372), (0, 9.57351), (0, 19.310006), (0, 10.033132)} and Q = {(10.3115, 0), (2.000732, 0), (5.231, 0), (1, 0)} ∈ M; clearly P × Q = {(0, 0)} so is a MOD subset zero divisor of M.

Let x = {(6, 12), (6, 0), (0, 12), (0, 0)} ∈ M; we see x × x = {(0, 0)}; so x is a MOD nilpotent rectangular subset of order two.

Let y = {(4, 16), (9, 16), (0, 9), (9, 9), (4, 9), (0, 0), (1, 9), (9, 1), (4, 1), (1, 16) (0, 16)} ∈ M. We see y × y = y so y is a MOD rectangular idempotent subset of M.

Thus the existence of MOD rectangular subset nilpotents and idempotents depends on the s and t of $R_n(s, t)$.

However A = {(0, 1), (0, 0), (0, 1), (1, 1)} is only a trivial MOD subset rectangular idempotent of $R_n(s, t)$.

Next we proceed onto describe MOD rectangular subset matrices by examples.

***Example 3.22***. Let M = {collection of all matrices A = ($a_1$, $a_2$, $a_3$, $a_4$) / $a_i$ ∈ S($R_n$(7, 9)) = {collection of all subsets from $R_n$(7, 9), +}, +; 1 ≤ i ≤ 4} be the MOD subset rectangular matrix semigroup under +.

Throughout this book S($R_n$(s, t)) = {collection of all subsets from $R_n$(s, t) = {(a, b) / a ∈ [0, s) and b ∈ [0, t)}}.

Let W = ({(0, 2.1), (1.2, 0), (1.1, 2.53)},{(0, 3), (1, 1), (0.337, 1)}, {(1, 2), (0, 3.1)}, {(4, 3), (2, 0.12), (4.3, 2.3)}}

W + W = ({(0, 2.1), (1.2, 0), (1.1, 2.53)}, {(0, 3), (1, 1), (0.337, 1)},{(1, 2), (0, 3.1)}, {(4, 3), (2, 0.12), (4.3, 2.3)}) + ({(0, 2.1), (1.2, 0), (1.1, 2.53)}, {(0, 3), (1, 1), (0.337, 1)}, {(1, 2), (0, 3.1)}, {(4.3), (2, 0.12), (4.3, 2.3)})

= {(0, 4.2), (1.2, 2.1), (1.1, 4.63), (2.4, 0), (2.3, 2.53), (2.2, 5.06)}, {(0, 6), (1, 4), (0.337, 4), (2, 2), (1.337, 2), (0.674, 2)},{(2, 4), (1, 5.1), (0, 6.2)}, {(1, 6), (6, 3.1), (1.3, 5.3), (6.3, 2.42), (4, 0.24), (1.6, 4.6)}) $\in$ M.

This is the way '+' operation is performed on M.

It is interesting to note M is an infinite semigroup which is infact a monoid. ({(0, 0)}) = ({(0, 0)}, {(0, 0)},{(0,0)},{(0, 0)} in M is such that A + ({(0, 0)}) = ({(0, 0)}) + A = A for all A $\in$ M.

We see $P_1$ = {($a_1$, $a_2$, $a_3$, $a_4$) / $a_i \in$ S($Z_7 \times Z_9$) = {collection of all subsets from $Z_7 \times Z_9$= {(a, b) / a $\in Z_7$, b $\in Z_9$}, +}; 1 $\leq$ i $\leq$ 4, +} is a MOD rectangular subset matrix subsemigroup of M.

***Example 3.23***. Let V = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right.$ / $a_i \in$ S($R_n$(3,8)) =

{collection of all subsets from $R_n$(3, 8) = {(a, b) / a $\in$[0, 3); b $\in$ [0, 8)},+}, +}, 1 $\leq$ i $\leq$ 4, +} be the MOD rectangular subset matrix semigroup under +.

o(V) = $\infty$ and V is infact a MOD rectangular subset matrix monoid which is commutative.

$$\text{Let } A = \begin{bmatrix} \{(2,3.1),(0,4.1),(0.3,2)\} \\ \{(2,5.1),(1.32,0),(1,1)\} \\ \{(2.01,0),(0,4)\} \\ \{(1,342,3.115)\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{(0,3),(0.113,2)\} \\ \{(0.31,2),(1.3,1)\} \\ \{(2,2),(0.31,4)\} \\ \{(1,2),(0.31,0.2),(0.1,5.3)\} \end{bmatrix} \in V.$$

$$A + B = \begin{bmatrix} \{(2,6.1),(0,7.1),(0.3,5), \\ (2.113,5.1),(0.113,3), \\ (0.413,4)\} \\ \{(2.31,7.1),(1.63,2),(1.31,3), \\ (0.3,6.1),(2.62,1),(2.3,2)\} \\ \{(1.01,2),(2,6),(2.32,4), \\ (0.31,0)\} \\ \{(2.342,5.115),(1.652,3.315), \\ (1.442,0.415)\} \end{bmatrix} \in V.$$

This is the way + operation is performed on V.

$$(\{(0,0)\} = \begin{bmatrix} \{(0,0)\} \\ \{(0,0)\} \\ \{(0,0)\} \\ \{(0,0)\} \end{bmatrix} \in V$$ is the identity of V with respect to +.

Consider W = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right.$ / $a_i \in S(Z_3 \times [0, 8))$ = {Collection of all subsets of $Z_3 \times [0, 8) = \{(a, b) / a \in Z_3, b \in [0, 8), +\}, +\}$, $+ 1 \leq i \leq 4\} \subseteq V$ is only a MOD rectangular subset matrix subsemigoup of infinite order.

$$Z = \{\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} / a_i \in \{ S(Z_3 \times Z_8) = \{\text{collection of all subsets}$$

from $Z_3 \times Z_8 = \{(a, b) / a \in Z_3, b \in Z_8, +\} +\} +, 1 \leq i \leq 2\} \subseteq V$ is a MOD rectangular subset matrix subsemigroup of finite order in V.

***Example 3.24***. Let $B = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ where $a_i \in S(R_n(8, 6))$

$= \{$collection of all subsets from $R_n(8, 6) = \{(a, b) / a \in [0, 8), b \in [0, 6), +\}, +\} +, 1 \leq i \leq 4\}$ is a MOD rectangular subset matrix semgroup under +.

$$o(B) = \infty.$$

$$(\{(0,0)\} = \begin{bmatrix} \{(0,0)\} & \{(0,0)\} \\ \{(0,0)\} & \{(0,0)\} \end{bmatrix}$$

in B is the additive identity of B.

It is left as an exercise to the reader to find MOD rectangular subset matrix subsemigroups of both finite and infinite order.

Infact B is a Smarandache MOD rectangular subset matrix semigroup. This task is also left as an exercise to the reader.

***Example 3.25***. Let S = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \right.$ $a_i \in S(R_n(11,15))$

= {collection of all subsets from $R_n(11, 15)$ = {(a, b) / a $\in$ [0, 11), b $\in$ [0,15), +}, +}, $1 \leq i \leq 8$, +} be the MOD rectangular subset matrix semigroup under +.

Infact S is a commutative monoid of infinite order.

Let $A = \begin{bmatrix} \{(0,0)\} & \{(2,1),(0,7)\} \\ \{(1,2)\} & \{(0.335,1.226)\} \\ \{(0.2,0.5),(7,2.1)\} & \{(1.3,2,5),(4.52,2)\} \\ \{(0,3),(1.21,0)\} & \{(1,2)\} \end{bmatrix}$ and

$$B = \begin{bmatrix} \begin{matrix}\{(2.31,0.113), \\ (1.052,3.005)\}\end{matrix} & \{(0.32,1.3)\} \\ \begin{matrix}\{(1.01,0.332), \\ (0,3.2)\}\end{matrix} & \begin{matrix}\{(2.73,1), \\ (0,2),(1,3\}\end{matrix} \\ \{(1,0.5)\} & \{(0.3,6.2)\} \\ \{(4,3.2),(1.4,1)\} & \begin{matrix}\{(4,2.1), \\ (3.115,2.55)\}\end{matrix} \end{bmatrix}$$

$$A + B = \begin{bmatrix} \{(2.31, 0.113), (1.052, 3.005)\} & \{(2.32, 2.3), (0.32, 8.3)\} \\ \{(1, 5.32), (2.01, 2.332)\} & \{(0.335, 3.226), (1.335, 4.226), (3.065, 2.226)\} \\ \{(1.2, 1), (8, 26)\} & \{(1.6, 8.7), (4.82, 8.2)\} \\ \{(4, 6.2), (1.4, 4), (5.21, 3.2), (1.61, 1)\} & \{(5, 4.1), (4.115, 4.55)\} \end{bmatrix} \in S.$$

This is the way + operation is performed on S.

Consider $W_1 = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}$ / $a_i \in$ {Collection of all subsets from $[0,11) \times \{0\} = \{(a, 0) / a \in [0,11), +\},+\}; 1 \leq i \leq 8, +\} \subseteq S$ is the MOD rectangular subset semigroup of infinite order.

Let $W_2 = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}$ / $a_i \in$ {Collection of all subsets from $Z_{11} \times Z_{15} = \{(a, b) / a \in Z_{11}, b \in Z_{15},+\}, +\}, +, 1 \leq i \leq 8\}$ be a MOD rectangular subset subsemigroup of finite order S has both subsemigroups of finite and infinite order.

In view of all these we have the following theorem.

**Theorem 3.6.** *Let S = {Collection of all t × m matrices with entries from S(R<sub>n</sub>(p, q)) = {Collection of all subsets from*

*$R_n(p, q) = \{(a, b) / a \in [0, p)$ and $b \in [0, q); 2 \leq p, q < \infty$ $(p \neq q)$ ; +}, +}, +} be the MOD rectangular subset matrix monoid.*

*i) $o(S) = \infty$ and S is commutative.*

*ii) S has both MOD rectangular subset matrix subsemigroups of both finite and infinite order.*

*iii) S is always a MOD rectangular subset matrix Smarandache semigroup.*

The proof is direct and hence left as an exercise to the reader.

We next proceed onto describe the operation of product on these MOD rectangular subset matrix semigroups under product by examples.

***Example 3.26***. Let M = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \right.$ / $a_i \in S(R_n(10,15))$ = {Collection of all subsets from $R_n(10, 15) = \{(a, b) / a \in [0, 10), b \in [0, 15),$ ×}, ×}, $\times_n$, $1 \leq i \leq 5\}$ be the MOD rectangular subset semigroup under $\times_n$.

$$\text{Let } A = \begin{bmatrix} \{(3,5.1),(2.5,0),(2,0.4)\} \\ \{(0.2,4),(0.5,6.2)\} \\ \{(1.0),(8.2,5)\} \\ \{(0,2),(6,0)\} \\ \{(4,0),(1,1)\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{(4,2),(6,8.3)\} \\ \{(1,1),(0.2,0)\} \\ \{(4,2),(0,2)\} \\ \{(1.5),(0.75)\} \\ \{(5,2.5)\} \end{bmatrix} \in M.$$

$$A \times_n B = \begin{bmatrix} \{(2,10.2),(0,0),(8,12.381), \\ (4,0.8),(5,0),(2,3.324)\} \\ \{(0.2,4),(0.5,6.2), \\ (0.04,0),(0.1,0)\} \\ \{(4,0),(0,0),(7.8,10),(0,10)\} \\ \{(0,10),(5,0),(0,0)\} \\ \{(0,0),(5,2.5)\} \end{bmatrix} \in M.$$

This is the way natural product operation $\times_n$ is performed.

$$[\{(0,0)\}] = \begin{bmatrix} \{(0,0)\} \\ \{(0,0)\} \\ \{(0,0)\} \\ \{(0,0\} \\ \{(0,0)\} \end{bmatrix} \text{ in M is such that}$$

$A \times_n [\{(0,0)\}$

$= [\{(0,0)\}] \times_n A = [\{(0,0)\}]$.

$$\text{Let } [\{(1,11)\}] = \begin{bmatrix} \{(1,1)\} \\ \{(1,1)\} \\ \{(1,1)\} \\ \{(1,1)\} \\ \{(1,1)\} \end{bmatrix} \in M \text{ is such that}$$

$[\{(1,1)\}] \times_n A = A$ for all $A \in M$, thus $[\{(1,1)\}]$ is the identity element or subset matrix of the MOD rectangular matrix semigroup M has zero divisors; for if

A =

$$\begin{bmatrix} \{(4.01,10.37),(8.34,2.1107), \\ (1.223,9.0114)\} \\ \{(0,0)\} \\ \{(5.732,0.431),(4.37,6.07), \\ (1.115,9.003)\} \\ \{(0,0)\} \\ \{(7.232,10.374),\ (5.315,3.307)\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{(0,0)\} \\ \{(4.32,7.1),(4.8,\ 0.711), \\ (0.315,12.31),(2.4,0)\} \\ \{(0,0)\} \\ \{(4.37,2.11),(7.35,0.31)\} \\ \{(0,0)\} \end{bmatrix} \in M.$$

$$A \times_n B = \begin{bmatrix} \{(0,0)\} \\ \{(0,0)\} \\ \{(0,0)\} \\ \{(0,0)\} \\ \{(0,0)\} \end{bmatrix} = [\{(0,0)\}]$$ is the MOD rectangular subset

zero divisor of M.

Let $S = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$ / $a_i \in ([0, 10) \times \{0\}$ = {Collection of all subsets

from $[0, 10) \times \{0\} = \{(a, 0) / a \in [0, 10), \times\}, \times_n, 1 \leq i \leq 5\}$ and

$R = \{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$ / $b_i \in S(\{0\} \times [0,15))$ = {collection of all subsets

from $\{0\} \times [0,15) = \{(0, b) / b \in [0,15),\times\}, \times\}, \times_n; 1 \leq i \leq 5\} \subseteq$ M be MOD rectangular subset matrix subsemigroup which is also ideal of M and they are of infinite order.

$$\text{Further } S \times R = \begin{bmatrix} \{(0,0)\} \\ \{(0,0)\} \\ \{(0,0)\} \\ \{(0,0)\} \\ \{(0,0)\} \end{bmatrix} = [\{(0,0)\}].$$

$P_1 = \{\begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ such that $a_i \in S(R_n(10,15))$ is a {collection of all subsets

from $R_n(10, 15) = \{(a, b) / a \in [0, 10), b \in [0, 15); \times\}, \times\}, \times_n\}$ $\subseteq$ M be the MOD rectangular subset matrix subsemigroup which is also an ideal of M.

We have several such ideals and this task is left as an exercise to the reader.

***Example 3.27***. Let $W = \{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} / a_i \in S(R_n(8, 12)) =$

{Collection of all subsets from $R_n(8,12) = \{(a, b) / a \in [0, 8), b \in [0,12) \times\}, \times\}; \times_n; 1 \leq i \leq 6\}$ be the MOD subset rectangular matrix semigroup under the natural product $\times_n$.

Clearly $[\{(0,0)\}] = \begin{bmatrix} \{(0,0)\} & \{(0,0)\} & \{(0,0)\} \\ \{(0,0)\} & \{(0,0)\} & \{(0,0)\} \end{bmatrix} \in W$ be

such that $A \times [\{(0,0)\}] = [\{(0,0)\}] \times A = [\{(0,0)\}]$ for all $A \in W$.

Let $[\{(1,1)\}] = \begin{bmatrix} \{(1,1)\} & \{(1,1)\} & \{(1,1)\} \\ \{(1,1)\} & \{(1,1)\} & \{(1,1)\} \end{bmatrix} \in W$ is the

MOD subset rectangular matrix identity of W;

$[\{(1, 1)\}] \times A = A \times [\{(1,1)\}] = A$ for all $A \in W$.

Infact W is a commutative MOD subset rectangular matrix monoid of infinite order.

We now give a few MOD subsets rectangular matrix ideals of W.

$$P_1 = \{\begin{bmatrix} a_1 & \{(0,0)\} & \{(0,0\} \\ \{(0,0)\} & \{(0,0)\} & \{(0,0)\} \end{bmatrix} / a_i \in S(R_n(8, 12)) =$$

{collection of all subsets from $R_n(8, 12) = \{(a, b) / a \in [0, 8), b \in [0, 12), \times\},\times\}, \times_n\}$ be the MOD rectangular subset matrix subsemigroup under natural product and infact an ideal.

Clearly $|P_1| = \infty$.

We have several such ideals.

Study in this direction is considered as the matter of routine so left as an exercise to the reader.

***Example 3.28***. Let $S = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} / a_i \in S(R_n(6,10)) =$

{Collection of all subsets from $R_n(6,10) = \{(a, b) / a \in [0, 6), b \in [0,10), \times\}, \times\}, \times;\ 1 \leq i \leq 4\}$ be the MOD subset rectangular semigroup under the usual product $\times$.

Clearly S is a non commutative semigroup.

Now $[\{(0,0)\}] = \{\begin{bmatrix} \{(0,0)\} & \{(0,0)\} \\ \{(0,0)\} & \{(0,0)\} \end{bmatrix} / a_i \in S$ is that

$A \times [\{(0,0)\}] = [\{(0,0)\}] \times A = [\{(0,0)\}]$ for all $A \in S$.

Further the multiplicative identity of S is

$$I = \begin{bmatrix} \{(1,1)\} & \{(0,0)\} \\ \{(0,0)\} & \{(1,1)\} \end{bmatrix} \in S.$$

We see A × I = I × A = A for all A × S.

$$\text{Let } A = \begin{bmatrix} \{(1,2),(0.3,5) & \{(7,1),(0.5,1), \\ (1.1, 3.01)\} & (0,.3.2)\} \\ \{(0.3, 2.1) & \{(1.3,2), \\ (2, 2), (4.5,3)\} & (0,0.5)\} \end{bmatrix} \in S.$$

We see

$$A \times I = \begin{bmatrix} \{(1,2),(0.3,5), & \{(7,1),(0.5,1), \\ (1.1,3.01)\} & (0,3.2)\} \\ \{(0.3,2.1),(2,2), & \{(1.3,2),(0,0.5)\} \\ (4.5,3)\} & \end{bmatrix}$$

$$\times \begin{bmatrix} \{(1, 1)\} & \{(0, 0)\} \\ \{(0, 0)\} & \{(1, 1)\} \end{bmatrix} = A.$$

Hence it is left as an exercise to the reader to verify I × A = A and I is the multiplicative identity with respect to the usual product.

$$\text{Let } A = \begin{bmatrix} \{(3,0.5),(4.1,2), & \{(5.5,3),(0,1) \\ (0,1.2)\} & (2,1.6)\} \\ \{(2.5,4),(4,0), & \{(3.5,2),(0,1), \\ (1,2)] & (1,2)\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{(0.5,6.5), & \{(1,2.5)\} \\ (0.3,0.4)\} & \\ \{(1.6,1.2), & \{(4,2), \\ (5,3)\} & (0,0.6)\} \end{bmatrix} \in S.$$

We first find

$$A \times B = \begin{bmatrix} \{(3,0.5),(4.1,2) & \{(5.5,3),(0,1) \\ (0,1.2)\} & (2,1.6)\} \\ \{(2.5,4),(4,0) & \{(3.5,2),(0,1) \\ (1,2)] & (1,2)\} \end{bmatrix} \times$$

$$\begin{bmatrix} \{(0.5,6.5), & \{(1,2.5)\} \\ (0.3,0.4)\} & \\ \{(1.6,1.2), & \{(4,2), \\ (5,3)\} & (0,0.5)\} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix} \{(1.5,3.25),(0.9,0.2), \\ (2.05,3),(1.23,0.8), \\ (0.736),(0,0.48)\}+ \\ \{(2.8,3.6),(3.5,9),(0,1.2), \\ (0,3),(4,4.8),(3.2,1.92)\} \end{matrix} & \begin{matrix} \{(3,1.25),(4.1,5),(0,3)\}+ \\ \{(4,6),(0,1.8),(0,2),(0,0.6), \\ (2,3.2),(0,0.96)\} \end{matrix} \\ & \\ \begin{matrix} \{(1.25,6.0),(0.75,1.6),(2,0), \\ (1.2,0),(0.5,3),(0.3,0.8)\}+ \\ \{(8.8,3.6),(3.5,9),(0,1.2) \\ (0,3),(1.6,2.4),(5,6)\} \end{matrix} & \begin{matrix} \{(2.5,0),(4,0),(1,5)\}+ \\ \{(2,4),(0,1.2),(0,2), \\ (0,0.6),\ (4,4),(0,1.2)\} \end{matrix} \end{bmatrix}.$$

The reader is given the task of simplifying them and proving in general $A \times B \neq B \times A$.

Further interested persons can prove A × B ≠ A $\times_n$ B. That is the operations × and $\times_n$ are different operations on S.

In the first place S under the usual product × is a noncommutative semigroup where as S under $\times_n$ the natural product is a commutative semigroup.

However the identities are also different. Hence it can be easily proved A × B ≠ A $\times_n$ B in general for some A, B ∈ S.

Next we give one more example.

***Example 3.29***. Let P = $\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix}$ / $a_i \in S(R_n(4, 6))$ =

{Collection of all subsets from $R_n(4, 6)$ = {(a, b) / a ∈ [0, 4), b ∈ [0,6), ×},×}, $\times_n$; 1 ≤ i ≤ 6} be the MOD subset rectangular matrix semigroup under the natural product $\times_n$.

o(P) = ∞. Infact P is a commutative monoid of infinite order.

Let $$A = \begin{bmatrix} \{(2.5,0),(0,1.6)\} & \{(3,2.5)\} \\ \{(1.1.6)\} & \{(2,0.6),(1,5)\} \\ \{(1.2,2),(2,0)\} & \{(1.6,2),(0.5,5)\} \end{bmatrix}$$ and

$$B = \begin{bmatrix} \{(3,2),(1,0)\} & \{(0.2,0.4),(2,2)\} \\ \{(0,1),(2,3)\} & \{(1,2.5)\} \\ \{(3,3)\} & \{(2,2)\} \end{bmatrix} \in P.$$

$$A \times_n B = \begin{bmatrix} \{(3.5,0),(0,3.2), & \\ (2.5,0),(0,1.6)\} & \{(0.6,0),(2.5)\} \\ \{(0,1.6),(2,4.8)\} & \{(1,2.5),(2,1.5)\} \\ \{(2,0),(3.6,6)\} & \{(3.2,4),(1,0)\} \end{bmatrix} \text{ is in P.}$$

This is the way product $\times_n$ is performed.

It is a matter of routine to find zero divisors, idempotents, nilpotents subsemigroups and ideals of P.

In view of all these we have the following theorem.

**Theorem 3.7.** *Let V = {m × q matrix with entries from $S(R_n(t, s))$ = {collection of all subsets from $R_n(t,s)$ = {(a, b) / a ∈ [0,t) and b ∈ [0, s),×}, ×}, $\times_n$} be the MOD rectangular subset matrix semigroup under the natural product $\times_n$.*

*i)* *o(V) = ∞ and V is a commutative monoid of infinite order.*

*ii)* *V has MOD rectangular subset matrix subsemigroups of finite and infinite order which are not ideals.*

*iii)* *V has MOD rectangular subset matrix ideals all of them are only of infinite order.*

*iv)* *V has nontrivial idempotents and nilpotents only for special values of t and s, 2 ≤ s, t < ∞*

*v)* *V always has zero divisors which are infinite in number.*

*vi)* *V is a S-semigroup if $Z_t$ or $Z_s$ is a S-semigroup under ×.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe the MOD rectangular matrix subset semigroups under + and $\times_n$ or $\times$ by some examples.

***Example 3.30***. Let S(M) = {collection of all subsets from

$$M = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} / a_i \in R_n(7, 6) = \{(a, b) / a \in [0, 7), b \in [0, 6)\}, +;$$

$1 \leq i \leq 4\}$, +} be the MOD rectangular matrix subsets semigroup under +.

Clearly o(S(m)) = ∞.

$$A = \{\begin{bmatrix} (2,0.5) \\ (1.5,0) \\ (1,2) \\ (0.2,0.1) \end{bmatrix}, \begin{bmatrix} (0,0) \\ (0,1) \\ (1,0.2) \\ (0,0.5) \end{bmatrix}, \begin{bmatrix} (1,2) \\ (1,1) \\ (0,0.5) \\ (0.2,0) \end{bmatrix}\} \text{ and}$$

$$B = \{\begin{bmatrix} (3,2) \\ (0,1) \\ (1,0) \\ (0.5,0) \end{bmatrix} \begin{bmatrix} (3,2) \\ (0,1) \\ (1,0) \\ (0.5,0) \end{bmatrix} \begin{bmatrix} (0,0) \\ (0,0.5) \\ (0.2,0) \\ (1,1) \end{bmatrix}\} \in S(M)$$

$$A + B = \{\begin{bmatrix} (5,2.5) \\ (1.5,1) \\ (2,2) \\ (0.7,0.1) \end{bmatrix}, \begin{bmatrix} (2,0.5) \\ (1.5,0.5) \\ (1.2,2) \\ (1.2,1.1) \end{bmatrix}, \begin{bmatrix} (3,2) \\ (0,2) \\ (1.2,0.2) \\ (1,1.5) \end{bmatrix}, \begin{bmatrix} (0,0) \\ (0,1.5) \\ (1.2,0.2) \\ (1,1.5) \end{bmatrix},$$

$$\left[\begin{matrix}(4,4)\\(1,2)\\(1,0.5)\\(0.7,0)\end{matrix}\right], \left[\begin{matrix}(1,2)\\(1,1.5)\\(0.2,0.5)\\(1.2,1)\end{matrix}\right]\} \in S(M).$$

This is the way + operation is performed on S(M).

$$\{\left[\begin{matrix}(0,0)\\(0,0)\\(0,0)\\(0,0)\end{matrix}\right]\} = \{[(0,0)]\}$$

is the zero matrix of the set S(M).

We see A + {[0,0)]} = A for all A ∈ S(M).

Let S(T) = {collection of matrix subsets from

$$T = \{\left[\begin{matrix}a_1\\a_2\\a_3\\a_4\end{matrix}\right] / a_i \in [0, 7) \times \{0\} = \{(a, 0) / a \in [0, 7)\}, +;$$

$1 \leq i \leq 4\}, +\} \subseteq$ S(M) is a MOD rectangular subset matrix subsemigroup of S(M); o(S(T)) = ∞.

Infact S(M) also has MOD rectangular subset matrix subsemigroup of finite order.

Finding them is a matter of routine so left as an exercise to the reader.

***Example 3.31***. Let S(S) = {Collection of subsets from

S = $\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix}$ where $a_i \in R_n(12, 10) = \{(a, b) / a \in [0, 12), b \in [0, 10), +\}, +, 1 \leq i \leq 8\}+\}$ be the MOD rectangular matrix subset semigroup under +. o(S(S)) = ∞ and it is easily verified S(S) has subsemigroups of finite and infinite order.

$$P = \{\begin{bmatrix} (0,3.5) & (7.2,0) & (1,3.1) & (5.6,8.01) \\ (2.3,0) & (1,1) & (0,5.71) & (2.001,0) \end{bmatrix},$$

$$\begin{bmatrix} (0.331,0) & (1,0.01) & (0.023,1) & (1,1) \\ (1,2.03) & (2.3,0.5) & (0.7,2.3) & (1,0) \end{bmatrix},$$

$$\begin{bmatrix} (0.9,0) & (6.231,1) & (1,1) & (0,2) \\ (0.115,0.112) & (1,0) & (0,3) & (7,5) \end{bmatrix}\} \in S(S).$$

This P has three matrices any A ∈ S(S) can also have infinite number of elements.

In view of this we have the following theorem.

**Theorem 3.8**. *Let S(W) = {collection of all subsets from W = {collection of all (p × q) matrices with entries from $R_n(s, t)$ = {(a, b) / a ∈ [0, s) and b ∈ [0, t)},+}, +} be the MOD rectangular matrix subset semigroup under +.*

i) *o(S(W)) = ∞ and is infact a commutative monoid.*

ii) *S(W) has MOD matrix subset subsemigroups of both finite and infinite order.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular matrix subset semigroups under $\times$.

***Example 3.32***. Let S(V) = {collection of all subsets from

$$V = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} / a_i \in R_n(6, 9) = \{(a, b) / a \in [0, 6) \text{ and } b \in [0, 9);$$

$1 \leq i \leq 3$, $\times_n\}$, $\times_n\}$ be the MOD rectangular matrix subset semigroup under the natural product $\times_n$.

$$\text{Let } A = \{\begin{bmatrix} (0,0.3) \\ (1.5,0) \\ (1,1.6) \end{bmatrix}, \begin{bmatrix} (5,2.5) \\ (3.5,2) \\ (0.16,0.12) \end{bmatrix}, \begin{bmatrix} (1,1) \\ (3,24) \\ (0.11,0.16) \end{bmatrix}\}$$

$$\text{and } B = \{\begin{bmatrix} (1,1) \\ (2,3) \\ (0.16,1.6) \end{bmatrix}, \begin{bmatrix} (0,2) \\ (0.5,0.6) \\ (0.2,0.7) \end{bmatrix}\} \in S(V).$$

$$\text{We find } A \times_n B = \{\begin{bmatrix} (0,0.3) \\ (1.5,0) \\ (1,1.6) \end{bmatrix}, \begin{bmatrix} (5,2.5) \\ (3.5,2) \\ (0.16,0.12) \end{bmatrix}, \begin{bmatrix} (1,1) \\ (3,24) \\ (0.11,0.16) \end{bmatrix}\} \times_n$$

$$\{\begin{bmatrix} (1,1) \\ (2,3) \\ (0.16,1.6) \end{bmatrix}, \begin{bmatrix} (0,2) \\ (0.5,0.6) \\ (0.2,0.7) \end{bmatrix}\}$$

$$= \{\begin{bmatrix} (0,0.3) \\ (3,0) \\ (1.6,2.56) \end{bmatrix}, \begin{bmatrix} (5,2.5) \\ (1,6) \\ (0.0256,0.192) \end{bmatrix},$$

$$\begin{bmatrix}(1,1)\\(0,7.2)\\(0.0176,0.256)\end{bmatrix}, \begin{bmatrix}(0,0.6)\\(0.75,0)\\(0.2,1.12)\end{bmatrix}, \begin{bmatrix}(0,5)\\(1.75,1.2)\\(0.032,0.084)\end{bmatrix},$$

$$\begin{bmatrix}(0,2)\\(1.5,1.44)\\(0.022,0.112)\end{bmatrix}\} \in S(V).$$

Thus we have show how the product is defined on S(V). {(0,0)}

$= \{\begin{bmatrix}(0,0)\\(0,0)\\(0,0)\end{bmatrix}\}$ in S(V) is such that for every matrix subsets A in S(V).

$$A \times_n \{\begin{bmatrix}(0,0)\\(0,0)\\(0,0)\end{bmatrix}\} = \{\begin{bmatrix}(0,0)\\(0,0)\\(0,0)\end{bmatrix}\}.$$

Further we have the matrix subset matrix $\{\begin{bmatrix}(1,1)\\(1,1)\\(1,1)\end{bmatrix}\} \in S(V)$ is

such that $A \times \{\begin{bmatrix}(1,1)\\(1,1)\\(1,1)\end{bmatrix}\} = A$ for all $A \in S(V)$.

Infact S(V) is a commutative MOD rectangular matrix subset monoid of infinite order.

S(V) also has zero divisors given by

$$A = \{\begin{bmatrix}(0.331,0)\\(3.2159,0)\\(4.032,0)\end{bmatrix}, \begin{bmatrix}(2.03,0)\\(0.36,0)\\(4.12,0)\end{bmatrix} \begin{bmatrix}(4.021,0)\\(0.3375,0)\\(0.812,0)\end{bmatrix}\} \text{ and}$$

$$B = \{\begin{bmatrix}(0,6.84)\\(0,2.006)\\(0,7.6092)\end{bmatrix}, \begin{bmatrix}(0,6.079)\\(0,2.625)\\(0,4.214)\end{bmatrix}, \begin{bmatrix}(0,2.04)\\(0,8.602)\\(0,4.704)\end{bmatrix}\} \in S(V)$$

is such that $A \times_n B = \{\begin{bmatrix}(0,0)\\(0,0)\\(0,0)\end{bmatrix}\}$.

Thus S(V) has infinite number of zero divisor.

However we see only for special values of p and q alone we can have MOD rectangular matrix subset idempotents and nilpotents. The cardinality of any $A \in S(V)$ can vary from one to infinity.

Interested reader can work with these for this work is considered as a matter of routine.

***Example 3.33***. Let S(W) = {collection of all subsets from $W = \{(a_1, a_2, a_3, a_4) / a_i \in R_n(8, 12) = \{(a, b) / a \in [0, 8)$ and $b \in [0, 12), 1 \leq i \leq 4\}, \times\}, \times\}$ be the MOD rectangular matrix subsets semigroup under product.

$o(S(W)) = \infty$ and infact S(W) is a commutative monoid for $\{(1, 1), (1, 1), (1, 1), (1, 1))\} \in S(W)$ is such

$A \times \{((1, 1), (1, 1), (1, 1), (1, 1))\} = A$ for all $A \in S(W)$.

{((0, 0), (0, 0), (0, 0), (0, 0))} $\in$ S(W) is such that A × {((0, 0), (0, 0), (0, 0), (0, 0))} = {((0, 0), (0, 0), (0, 0), (0, 0))} for all A $\in$ S(W).

Let A = {((0, 3.1), (0.356, 0), (0, 0), (0, 0)), ((0.15, 0), (1.23, 5), (0, 0), (0, 0)), ((6.752, 1.307), (1.8075, 2.555), (0, 0), (0, 0))} and

B = {((0, 0), (0, 0), (4.2, 3.75), (0.752, 0.8217)), ((0, 0) (0, 0), (1, 2.777), (0, 5.2222))} $\in$ S(W).

Clearly A × B = B × A = {((0, 0), (0, 0)), (0, 0), (0, 0)} is a zero divisor of S(W).

B = {((0, 1), (1, 4), (1, 1), (0, 9))} $\in$ S(W) is such that B × B = B is an idempotent of S(W).

D = {((0, 6), (4, 6), (4, 0), (0, 0))} $\in$ S(W) is such that D × D = {((0, 0), (0, 0), (0, 0), (0, 0))} is a nilpotent of order two.

We can find MOD rectangular matrix subsets semigroups and ideals for S(W) which is left as an exercise to the reader.

***Example 3.34***. Let S(B) = {(collection of all matrix subsets

from B = {$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ / $a_i \in R_n(6,12)$ = {(a, b) / a $\in$ [0, 6),

b $\in$[0, 12) × (or $\times_n$) 1 $\leq$ i $\leq$ 4}, × (or $\times_n$)} be the MOD rectangular matrix subset subsemigroup under × (or $\times_n$).

Under × S(B) is a non commutative MOD rectangular subset matrix monoid of infinite order where as under the product $\times_n$, S(B) is a commutative rectangular subset matrix monoid of infinite order.

In case {S(B), ×} the subset matrix $\{\begin{bmatrix} (1,1) & (0,0) \\ (0,0) & (1,1) \end{bmatrix}\}$

acts as the multiplicative identity whereas in case {(S(B), $\times_n$}

the subset matrix $\{\begin{bmatrix} (1,1) & (1,1) \\ (1,1) & (1,1) \end{bmatrix}\}$ acts as the multiplicative identity.

Clearly the identities in both the cases are distinct.

Let $A = \{\begin{pmatrix} (0,3) & (1.2,0) \\ (0.2,0.4) & (0.8,1.6) \end{pmatrix}, \begin{pmatrix} (1,1) & (2,0) \\ (0.3,1) & (0.7,0.1) \end{pmatrix}\}$

and $B = \{\begin{pmatrix} (2,1) & (0.1,0.2) \\ (0,0) & (0.4,4) \end{pmatrix}\} \in S(B)$.

$$A \times B = \{\begin{pmatrix} (0.3) & (0.48,0.6) \\ (0.4,0.4) & (0.34,6.48) \end{pmatrix}, \begin{pmatrix} (2,1) & (0.9,0.2) \\ (0.6,1) & (0.31,0.4) \end{pmatrix} \qquad \text{I}$$

$$B \times A = \{\begin{pmatrix} (2,1) & (0.1,0.2) \\ (0,0) & (0.4,4) \end{pmatrix}\} \times$$

$$\{\begin{pmatrix} (0,3) & (1.2,0) \\ (0.2,0.4) & (0.8,1.6) \end{pmatrix}, \begin{pmatrix} (1,1) & (2,0) \\ (0.3,1) & (0.7,0.1) \end{pmatrix}\} =$$

$$\{\begin{pmatrix} (0.02,3.08) & (2.48,0.32) \\ (0.08,1.6) & (0.32,6.4) \end{pmatrix}, \begin{pmatrix} (2.03,1.2) & (4.07,0.02) \\ (0.12,4) & (0.28,0.4) \end{pmatrix}\} \qquad \text{II}$$

Clearly I and II are distinct so in general A × B ≠ B × A for A, B ∈ S(B).

Consider A $\times_n$ B = $\{\begin{pmatrix} (0,3) & (0.12,0) \\ (0,0) & (0.32,6.4) \end{pmatrix}$,

$\begin{pmatrix} (2,1) & (0.2,0) \\ (0,0) & (0.28,0.4) \end{pmatrix}\} \in S(B)$.

Clearly A $\times_n$ B $\neq$ A $\times$ B or B $\times$ A. It is easily verified A $\times_n$ B = B $\times_n$ A for every A, B $\in$ S(B).

Finding substructures, nilpotents, idempotents and zero divisors of S(B) are given as exercise to the reader.

In view of all these we have the following result.

**Theorem 3.9.** *Let S(B) = {collection of all matrix subsets from B = {collection of all t $\times$ s matrices with entries from $R_n(p, q)$ = {(a, b) / a $\in$ [0, p) and b $\in$ [0, q), $\times$}, $\times_n$}, $\times_n$} be the MOD rectangular subset matrix semigroup under natural product $\times_n$.*

*i) o(S(B)) is of infinite order and S(B) is a commutative monoid.*

*ii) S(B) has both finite order subsemigroups and infinite order subsemigroups which are not ideals.*

*iii) All ideals in S(B) are of infinite order.*

*iv) S(B) has infinite number of zero divisors.*

*v) S(B) has nilpotents and idempotents only for appropriate t and s.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular subset coefficient polynomial semigroups with respect to + and $\times$ by examples.

***Example 3.35***. Let B[x] = {collection of all polynomials with coefficients as subsets from $S(R_n(3, 8))$ = {collection of all subsets from $R_n(3, 8) = \{(a, b) / a \in [0, 3)$ and $b \in [0, 8)\}, +\}$, +} be the MOD rectangular subset coefficient polynomial in the indeterminate x.

B[x] is a semigroup under addition.

$p(x) = \{(0, 2), (0.5, 0.1), (0.2, 5)\}x^2 + \{(1.2, 3), (0.8, 6)\}$ and $q(x) = \{(0.9, 0.8), (2, 5), (0.5, 0.6)\}x + \{(2, 0.2), (0.7, 1.6)\} \in B[x]$.

We find $p(x) + q(x) = \{(0, 2), (0.5, 0.1), (0.2, 5)\}x^2 + \{(1.2, 3), (0.8, 6)\} + \{(0.9, 0.8), (2, 5),(0.5, 0.6)\}x + \{(2, 0.2), (0.7, 1.6)\} = \{(0, 2), (0.5, 0.1), (0.2, 5)\}x^2 + \{(0.9, 0.8), (2, 5), (0.5, 0.6)\}x + \{(0.2, 3.2), (1.9,4.6), (2.8, 6.2), (1.5, 7.6)\} \in B[x]$.

This is the way the operation + is performed on B[x].

Clearly $o(x) = \{(0, 0)\}x^n + \{(0,0)\}x^{n-1} + \ldots + \{(0, 0)\}x + \{(0, 0)\} \in B [x]$ is such that $0(x) + p(x) = p(x)$ for all $p(x) \in B[x]$.

We can find subsemigroups under +.

Consider $P[x] = \{\sum_{i=0}^{\infty} a_i x^i$ where $a_i \in$ {collection of all subsets from $\{0\} \times [0, 8) = \{(0, a) / a \in [0, 8)\} \subseteq B[x]$ is a MOD rectangular subset coefficient polynomial subsemigroup under + of infinite order.

$R[x] = \{\sum_{i=0}^{9} a_i x^i \;/\; a_i \in$ {collection of all subsets from $Z_3 \times Z_8 = \{(a, b) / a \in Z_3, b \in Z_8\}, +\}, +\} \subseteq B[x]$ is a subsemigroup of finite order.

B[x] has several subsemigroups of finite order.

***Example 3.36***. Let S[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in$ {collection of all subsets from $R_n$(11,19) = {(a, b) / a ∈ [0, 11), b ∈ [0, 19)}, +}, +} be the MOD rectangular subset coefficient polynomial semigroup under +. S[x] has both finite and infinite order subsemigroups.

We prove the following result.

**Theorem 3.10.** *Let S[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(R_n(t, s))$ = {collection of all subsets from the MOD rectangular plane $R_n(t, s)$ = {(a, b) / a ∈ [0, t) and b ∈ [0, s)}, +}, +} be the MOD rectangular subset coefficient polynomial semigroup under +.*

*i) o(S[x]) is infinite and S[x] is infact commutative.*

*ii) S[x] has both finite and infinite order subsemigroups under +.*

The reader is left with the task of proving this theorem.

Next we describe MOD rectangular subset coefficients polynomials semigroups under × by some examples.

***Example 3.37***. Let S[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(R_n(12, 4))$ = {collection of all subsets from $R_n$(12, 4) = {(a, b) / a ∈ [0, 12), b ∈ [0, 4)}, ×}, ×} be the MOD rectangular subset coefficient polynomial semigroup under product.

Clearly o(S[x]) = ∞ and in fact S[x] is a commutative moniod with $p(x) = 1 + 0x + \ldots + 0x^n$ as the identity.

For every q(x) $\in$ S[x] we have p(x) $\times$ q(x) = p(x). Hence the claim.

Let p(x) = {(4, 2), (8, 0), (0, 2)}$x^3$ + {(4, 0), (8, 2)}x + {(4, 2), (8, 2)} and q(x) = {(6, 2), (6, 0), (0, 2)}$x^6$ + {(6, 0), (6, 2)} $\in$ S[x].

Clearly p(x) $\times$ q(x) = {(0, 0)}. This S[x] has nontrivial subset coefficient polynomial. However S[x] has no idempotent S[x] has nilpotents.

For p(x) = {(6, 2),(6, 0), (0, 2)}$x^2$ + {(0, 2), (6, 2)} $\in$ S[x] is such that p(x) $\times$ p(x) = {(0, 0)} is a nilpotent of order two.

P[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in$ S({0} $\times$ [0, 4)) = {collection of all subsets from {0} $\times$ [0, 4) = {(0, a) / a $\in$ [0, 4), $\times$}, $\times$}, $\times$} $\subseteq$ B[x] be the MOD rectangular subset coefficient polynomial subsemigroup of B[x]. Clearly P[x] is also an ideal of B[x].

Consider L[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{12} \times Z_4)$ = {collection of all subsets from $Z_{12} \times Z_4$ = {(a, b) / a $\in Z_{12}$, b $\in Z_4$, {(a, b) / a $Z_{12}$, b $\in Z_4$, $\times$}, $\times$}, $\times$} be the MOD rectangular subset coefficient polynomial subsemigroup of B[x] of infinite order but is not an ideal of B[x].

L[x] has zero divisors and nilpotents but has no idempotents.

Study in this direction is a matter of routine so left as an exercise to the reader.

In view of all these we have the following.

**Theorem 3.11.** *Let* $S[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(R_n(t, s)) =$ *{collection of all subsets from the set* $R_n(t, s) = \{(a, b) / a \in [0, t)$ *and* $b \in [0, s), \times\}, \times\}, \times\}$ *be the MOD rectangular subset polynomial coefficient semigroup under* $\times$.

i) *$o(S[x]) = \infty$ and $S[x]$ is a MOD rectangular subset coefficient polynomial monoid.*

ii) *In $S[x]$ all subsemigroups are of infinite order.*

iii) *$S[x]$ has ideals all of which are only of infinite order.*

iv) *$S[x]$ has infinite number of zero divisors what ever be $t$, and $s$ $2 \leq t, s < \infty$.*

v) *$S[x]$ has nilpotents only for special values of $t$ and $s$.*

vi) *$S[x]$ has no idempotents.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto define MOD rectangular polynomial subset semigroups under + by some examples.

***Example 3.38***. Let $S(P[x])$ = {collection of all subsets from $P[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(3,10) = \{(a, b) / a \in [0, 3)$ and $b \in [0, 10), +\}, +\}, +\}$ be the MOD rectangular polynomial subsets semigroup under +.

Let $A = \{(0, 0.5)x^3 + (2.1, 6)x + (1.2, 0.6), (2.01, 2)x^4 + (0.331, 0.5)\}$ and $B = \{(0.2, 1)x^2 + (1.6, 2), (0.32, 1)x^3 + (0.2, 0.6)\} \in S(P[x])$.

$A + B = \{(0, 0.5)x^3 + (0.2, 1)x^2 + (2.1, 6)x + (2.8, 2.6), (0.32, 1.5)x^3 + (2.1, 6)x + (1.4, 1.2), (2.01, 2)x^4 + (0.2, 1)x^2 + (1.931, 2.5), (2.01, 2)x^4 + (0.32, 1)x^3 + (1.4, 1.2)\} \in S(P[x])$.

This is the way + operation is performed on S(P[x]).

We see {(0, 0) + (0, 0)x + … + (0, 0)$x^n$} ∈ S(P[x]) acts as the additive identity of S(P[x]) so S(P[x]) is a MOD rectangular polynomial subsets monoid under + of infinite order.

It is a matter of routine to verify S(P[x]) can have both finite order MOD rectangular polynomial subset subsemigroup as well as infinite order MOD rectangular polynomial subset semigroup.

***Example 3.39***. Let S(Q[x]) = {collection of all subsets from Q[x] = {$\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(8, 12)$ = {(a, b ) / a ∈ [0, 8), b ∈ [0, 12), +}, +}, +} be the MOD rectangular polynomial subset semigroup under +. o(S(Q[x])) = ∞.

Study of properties associated with S(Q[x]) is a matter of routine so left as an exercise to the reader.

Next we proceed onto give the related result.

**Theorem 3.12.** *Let S(B[x]) = {collection of all polynomial subsets from B[x] = {$\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(s, t)$ = {(a, b) / a ∈ [0, s) and b ∈ [0, t),+}, +}, +} be the MOD rectangular polynomial subset semigroup under +.*

*i)* *o(S(B[x])) = ∞ and S(B[x]) is a MOD rectangular commutative polynomial subset monoid.*
*ii)* *S(B[x]) has subsemigroups of finite and infinite order.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed on to describe MOD rectangular subset coefficient polynomials of finite degree semigroup under + by some examples.

***Example 3.40***. Let $Q[x]_9 = \{\sum_{i=0}^{9} a_i x^i$ / $a_i \in S(R_n(7, 5))$ = {collection of all subsets from $R_n(7, 5) = \{(a, b) / a \in [0, 7), b \in [0, 5) +\}, +\}, x^{10} = 1, +\}$ be the MOD rectangular subset coefficient polynomial of finite degree semigroup under +.

Clearly $o(Q[x]_9) = \infty$ and $Q[x]_9$ is an infinite commutative MOD rectangular subset coefficient polynomial of finite degree monoid.

Let r(x) = {(0, 3.2), (0.9, 3.7), (1, 0.63), (2.3,0.8)}$x^3$ + {(1, 1.1), (2.24, 0.3), (0.37, 2)}x + {(0.5, 0.3), (3.2, 2.2)} and s(x) = {(0.3, 2), (6.32, 0) (1, 5)} + {(0.52, 1), (3.1, 4.2)}x $\in Q[x]_9$.

r(x) + s(x) = {1, 3.2), (0.9, 3.7), (1, 0.63), (2.3, 0.8)}$x^3$ + {(1.52, 2.1), (2.76, 1.3), (0.89, 3), (4.1, 5.3), (5.34, 4.5), (3.47, 6.2)}x + {(0.8, 2.3), (3.5, 4.2), (6.82, 0.3), (2.5, 2.2), (1.5, 5.3), (4.2, 7.2)} $\in Q[x]_9$.

This is the way + operation is performed on $Q[x]_9$.

Finding subsemigroups of finite and infinite order is a matter of routine so left as an exercise to the reader.

***Example 3.41***. Let $P[x]_5 = \{\sum_{i=0}^{5} a_i x^i$ / $a_i \in S(R_n(12, 18))$ = {collection of all subsets from $R_n(12, 18) = \{(a, b) / a \in [0, 12), b \in [0, 18),+\}, +\}, x^6 = 1, +\}$ be the MOD rectangular subset coefficient finite degree polynomial semigroup.

Clearly $o(P[x]_5) = \infty$. All properties associated with $P[x]_5$ is left as exercise to the reader.

In view of all these we have the following theorem.

**Theorem 3.13.** *Let $B[x]_m = \{\sum_{i=0}^{m} a_i x^i$ / $a_i \in S(R_n(t, s)) =$ {collection of all subsets with entries from $R_n(t, s) = \{(a, b)$ / $a \in [0, t), b \in [0, s), +\}, +\}, x^{m+1} = 1, +\}$ be the MOD rectangular subset coefficient finite degree polynomial semigroup under +.*

*i)* *$o(B[x]_m) = \infty$ and $B[x]_m$ is a MOD rectangular subset coefficient finite degree polynomial monoid.*

*ii)* *$B[x]_m$ has both finite and infinite order subsemigroups.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe briefly the MOD rectangular subset coefficient finite degree polynomial semigroup under × by some examples.

***Example 3.42***. Let $B[x]_{12} = \{\sum_{i=0}^{12} a_i x^i$ / $a_i \in S(R_n(10, 18) =$ {collection of all subsets from $R_n(10, 18) = \{(a ,b)$ / $a \in [0, 10)$, $b \in [0, 18), \times\}, \times\}, \times, \; x^{13} = 1\}$ be the MOD rectangular subset coefficient finite degree polynomial semigroup under ×.

Clearly $o(B[x]_{12} = \infty$.

Let $p(x) = \{(3.6, 0.72), (1.02, 5.37), (4.06, 1.32)\}x^4 + \{(0.2, 0.8), (1, 1)\}$ and $q(x) = \{(0.3, 0.5), (1.2, 0.8)\} + \{0.6, 1.2), (0.5, 5)\} x^2 \in B[x]_{12}$

$p(x) \times q(x) =[\{(3.6, 0.72), (1.02, 5.37), (4.06, 1.32)\}x^4 + \{(0.2, 0.8), (1, 1)\}] \times \{(0.3, 0.5), (1.2, 0.8)\} + \{(0.6, 1.2), (0.5, 5)\}x^2\}$

$= \{(0.108, 0.36), (0.306, 0.2685), (1.218, 0.66), (4.32, 0.576), (1.224, 4.296), (4.862, 1.056)\}x^4 + \{(0.12, 0.96), (0.6, 1.2), (0.5, 5), (0.1, 4)\}x^2 + \{(0.06, 0.4), (0.3, 0.5), (1.2, 0.8), (0.24, 0.64)\} + \{(2.16, 0.864), (0.612, 6.444), (2.436, 1.584), (1.8, 3.60), (0.51, 8.85), (2.03, 6.60)\}x^6 \in B[x]_{12}$.

This is the way product operation is performed on $B[x]_{12}$.

Clearly $B[x]_{12}$ has zero divisors, however has no idempotents and nilpotents only conditionally.

Infact $B[x]_{12}$ is a MOD rectangular subset coefficient finite degree polynomial monoid.

$B[x]_{12}$ has finite order subsemigroups. However all ideals of $B[x]_{12}$ are only of infinite order.

Reader is left with the task of studying these concepts.

***Example 3.43***. Let $R[x]_{16} = \{\sum_{i=0}^{16} a_i x^i$ / $a_i \in S(R_n(20, 9)) =$ {collection of all subsets from $R_n(20, 9) = \{(a, b) / a \in [0, 20), b \in [0, 9), \times\}, \times\}, x^{17} = 1, \times\}$ be the MOD rectangular subset coefficient finite degree polynomial semigroup under product.

$o(R[x]_{16}) = \infty$.

Let $p(x) = \{(10, 3), (0, 6), (10, 6)\}x^5 + \{(0, 3), (10, 6), (0, 6)\}x^2 + \{(10, 6), (10, 3)\} \in R[x]_{16}$.

Clearly $p(x) \times p(x) = \{(0, 0)\}$. Thus $R[x]_{16}$ has nilpotents as $Z_{20}$ and $Z_9$ have nilpotents. However $R[x]_{16}$ has infinite number of zero divisors.

For if $p(x) = \{(0, 0.35), (0, 6), (0, 6.32), (0, 8.44), (0, 5)\}x^{10} + \{(0, 3), (0, 0.6), (0, 0.8), (0, 0.7)\}$ and $q(x) = \{(5, 0), (0.39, 0), (0.489, 0), (10.3, 0) (9.25, 0)\}x^8 + \{(3, 0), (0.6, 0), (0.7, 0), (10, 0)\}x^4 + \{(8, 0), (3.5, 0), (4.25, 0)\} \in R[x]_{16}$.

Then $p(x) \times q(x) = \{(0, 0)\}$. Infact $R[x]_{16}$ has infinite number of such type of zero divisors.

$R[x]_{16}$ can never have idempotents. All ideals of $R[x]_{16}$ are of infinite order but $R[x]_{16}$ can have subsemigroups of finite order.

Let $S[x]_{16} = \{\sum_{i=0}^{16} a_i x^i$ / $a_i \in S(Z_{20} \times Z_9) = \{$collection of all subsets of $Z_{20} \times Z_9 = \{(a, b) / a \in Z_{20}$ and $b \in Z_9, \times\}, \times\}, x^{17} = 1, \times\} \subseteq R[x]_{16}$.

Clearly $S[x]_{16}$ is a MOD rectangular subset coefficient finite degree polynomial subsemigroup of finite order which is clearly not an ideal.

$B[x]_{16} = \{\sum_{i=0}^{16} a_i x^i$ / $a_i \in S(\{0\} \times [0, 9)) = \{$collection of all subsets from $\{0\} \times [0,9) = \{(0, b) / b \in [0, 9), \times\}, \times\}, x^{17} = 1, \times\} \subseteq R[x]_{16}$ is a MOD rectangular subset coefficient finite degree polynomial subsemigroup which is an ideal of $R[x]_{16}$.

Infact $o(B[x]_{16}) = \infty$.

In view of all these we have the following theorem.

**Theorem 3.14**. *Let $B[x]_m = \{\sum_{i=0}^{m} a_i x^i$ / $a_i \in S(R_n(s, t)) = \{$collection of all subsets from $R_n(s,t) = \{(a, b) / a \in [0, s), b \in [0, t), \times\}, x^{m+1} = 1, +\}, \times\}$ be the MOD rectangular subset coefficient finite degree polynomial semigroup under $\times$.*

*i)* *$oB[x]_m) = \infty$ and $B[x]_m$ is a commutative MOD rectangular subset coefficient finite degree polynomial monoid.*

*ii)* *$B[x]_m$ has both finite order and infinite order subsemigroups which are not ideals.*

*iii)* *$B[x]_m$ has ideals, all ideals are of infinite order.*

*iv)* *$B[x]_m$ has no nontrivial idempotents.*

v) *$B[x]_m$ has infinite number of zero divisors what ever be s and t.*

vi) *Only for special values of s and t, $B[x]_m$ has nilpotents.*

The proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular finite degree polynomial subset semigroup under product by examples.

***Example 3.44***. Let $S(P[x]_8)$ = {Collection of all subsets from $P[x]_8 = \{\sum_{i=0}^{8} a_i x^i$ / $a_i \in R_n(7, 10) = \{(a, b)$ / $a \in [0, 7)$ and $b \in [0, 10), +\}, +\}, x^9 = 1, +\}, +\}$ be the MOD rectangular polynomial subset semigroup under +.

$o(S(P[x]_8) = \infty$. Further $\{(0,0) + (0,0)x + \ldots + (0,0)x^n\}$ in $S(P[x]_8)$ acts as the additive identity, so $S(P[x]_8$ is a monoid.

Let A = $\{(0, 3)x^3 +(3.112, 0)x +(6.1, 2.1), (1, 2.1)x^5 + (3, 1.1)x^2 +(1, 1.2)\}$ and B = $\{(6, 8)x^3 + (3.1, 4)x + (0.334, 0.7), (0.14, 0) + (1, 1)x^5, (0.2, 1)x\} \in S(P[x])$.

We find A + B = $\{(6, 1)x^3 +(6.212, 4)x + (6.434, 2.8), (1, 2.1)x^5 + (6, 8)x^3 +(3, 1.1)x^2 +(3.1, 4)x +(1.334, 1.9), (1, 1)x^5 + (0, 3)x^3 + (3.112, 0)x + (6.24, 2.1), (2, 3.1)x^5 +(3, 1.1)x^2 + (1.14, 1.2), (0, 3)x^3 + (3.312, 1)x +(6.1, 2.1), (1, 2.1)x^5 + (3, 1.1)x^2 + (0.2, 1)x + (1, 1.2)\} \in S(P[x])$.

This is the way + operation is performed on S(P[x]). Infact S(P[x]) has both subsemigroups of finite and infinite order.

This study is considered as a matter of routine so left as an exercise to the reader.

***Example 3.45***. Let S(R[x]$_5$) = {collection of all subsets from R[x]$_5$ = { $\sum_{i=0}^{5} a_i x^i$ / $a_i \in R_n(3, 6)$ = {(a, b) / a ∈ [0, 3) and b ∈ [0, 6), +}, $x^6 = 1$, +}, +} be the MOD rectangular polynomial subset semigroup under +.

o(S[x]$_5$)) = ∞ and is a monoid with respect to +. S(R[x]$_5$) has subsemigroups of both finite and infinite order.

In view of all these we have the following theorem.

**Theorem 3.15**. *Let S(B[x]$_m$) = {collection of all subsets from B[x]$_m$ = {$\sum_{i=0}^{m} a_i x^i$ / $a_i \in R_n(s, t)$ = {(a, b) / a ∈ [0, s), b ∈ [0, t), +}, $x^{m+1}$ = 1, +},+} be the MOD rectangular finite degree polynomial subset semigroup under +.*

*i)* *o(S(B[x]$_m$) = ∞ and S(B[x]$_m$) is a commutative monoid.*

*ii)* *B[x]$_m$ has both finite order subsemigroups as well as infinite order subsemigroups.*

The proof is considered as a matter of routine so left as an exercise to the reader

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Next we proceed onto describe MOD rectangular finite degree polynomial subset semigroup under product by a few examples.

***Example 3.46***. Let S(P[x]$_4$) = {collection of all subsets from P[x]$_4$ = { $\sum_{i=0}^{4} a_i x^i$ / $a_i \in R_n(8, 6)$ = {(a, b) / a ∈ [0, 8), b ∈ [0, 6), ×}, $x^5 = 1$, ×}, ×} be the MOD rectangular subset of finite degree polynomial semigroup under ×.

$o(S(P[x]_4)) = \infty$ and $S(P[x]_4)$ is a monoid. $S(P[x]_4)$ has infinite number of zero divisors but no idempotents. The existence of nilpotents depends on the value of s, t in $R_n(s, t)$.

Consider A = {(4, 4), (4, 8), (4, 8), (0, 8)}$x^3$ + {(0, 4), (4, 0), (0, 8), (0, 12), (4, 12)} $\in$ S([P[x]$_4$).

Clearly A $\times$ A = {(0, 0)}. This is the way one can get nilpotents infact $S(P[x]_4)$ has infinite number of nilpotents of order two. However $S(P[x]_4)$ has no nontrivial idempotents.

***Example 3.47***. Let $S(V[x]_7)$ = {collection of all subsets from $V[x]_7 = \{\sum_{i=0}^{7} a_i x^i$ / $a_i \in R_n(6, 10)$ = {(a, b) / a $\in$ [0, 6), b $\in$ [0, 10)}, $x^8$ = 1, $\times$}, $\times$}; be the MOD rectangular finite degree polynomial subset semigroup under $\times$.

$S(V[x]_7)$ has no nilpotents but has infinite number of zero divisor. This is so because $Z_6$ and $Z_{10}$ have no nilpotents in them.

All other properties are realized as a matter of routine and left as an exercise to the reader.

The following results are important.

**Theorem 3.16.** *Let $S(B[x]_m)$ = {collection of all subsets from $B[x]_m = \{\sum_{i=0}^{m} a_i x^i$ / $a_i \in R_n(s ,t)$ = {(a, b) / a $\in$ [0, s) and b $\in$ [0, t), $\times$}, $x^{m+1}$ = 1, $\times$}, $\times$} be the MOD rectangular finite degree polynomial subset semigroup under $\times$.*

*i) $S(B[x]_m)$ has nilpotents if and only if $Z_s$ and $Z_t$ has nilpotents.*

*ii) If $S(B[x]_m$ has nilpotents then they are infinite in number.*

Proof is direct and hence left as an exercise to the reader.

**Theorem 3.17**. *Let $S(B[x]_m)$ = {collection of all subsets fromB$[x]_n = \{\sum_{i=0}^{m} a_i x^i$ / $a_i \in R_n(s, t) = \{(a, b) / a \in [0, s), b \in [0, t), \times\}, x^{m+1} = 1, \times\}, \times\}$ be the MOD rectangular finite degree polynomial subset semigroup under $\times$.*

*i)* *$o(S(B[x]_m)) = \infty$ and $S(B[x]_m)$ is infact a commutative monoid of infinite order.*

*ii)* *$S(B[x]_m)$ has infinite number of zero divisors.*

*iii)* *$S(B[x]_{m\_})$ has no idempotents.*

*iv)* *$S(B[x]_m)$ has nilpotents only for special values of s and t; $2 \leq s, t < \infty$.*

*v)* *$S(B[x]_m)$ has subsemigroups of finite order as well as of infinite order which are not ideals.*

*vi)* *$S(B[x]_m)$ has ideals all of which are of infinite order.*

The proof is considered as a matter of routine so left as an exercise to the reader.

Thus these MOD rectangular structures cannot be obtained in MOD complex panes or MOD dual number planes or MOD special dual like number planes or MOD special quasi dual number planes.

This is a special feature enjoyed only by MOD real planes.

Here we suggest the following problems.

## PROBLEMS

1. Let $R_n(m, q) = \{(a, b) / a \in [0, m)$ and $b \in [0, q)\}$ be the MOD rectangular plane; $m \neq q$, $2 \leq m, q < \infty$.

   i. Find all special features associated with $R_n(m, q)$.
   ii. Compare $R_n(m, q)$ with $R_n(m, m)$ and $R_n(q, q)$ in general compare it with $R_n(t, t)$; $2 \leq t < \infty$.

2. Let $S = \{R_n(5, 3)\}$ and $S_1 = \{R_n(3, 5)\}$ be the MOD rectangular planes. Compare $S_1$ and S.

3. Let $S = \{R_n(9, 8) = \{(a, b) / a \in [0, 9), b \in [0, 8), +\}$ be MOD rectangular plane group under +.

   i. Find o(S).
   ii. Find all finite MOD rectangular plane subgroup under +.
   iii. Find all infinite order MOD rectangular plane subgroup under +.
   iv. Enumerate any other special property associated with S.

4. Let $W = \{R_n(23,11) = \{(a, b) / a \in [0, 23), b \in [0, 11)\}, +\}$ be the MOD rectangular group under +.

   i. Study questions (i) to (iii) of problem (3) for this W.
   ii. Compare W with $V = \{R_n(11, 23), +\}$.

5. Let $M = \{R_n(12, 48) = \{(a, b) / a \in [0, 12), b \in [0, 48)\}, +\}$ be the MOD rectangular group under +.

   i. Study questions (i) to (iii) of problem (3) for this M.
   ii. Compare this M with W in problem (4).
   iii. Can we say M has more number finite order MOD rectangular subgroups than W in problem (4)?

6. Let S = $\{(a_1, a_2, a_3, a_4, a_5) / a_i \in \{R_n(12, 17) = \{(a, b) / a \in [0,12), b \in [0, 17); +\}, +\}$ be the MOD rectangular matrix group under +.

   i. Prove S has MOD rectangular matrix subgroup of finite order.
   ii. Prove S has MOD rectangular matrix subgroup of infinite order.
   iii. If in $S(R_n(12, 17))$ is replaced by $R_n(24, 250)$ has more numbers of MOD rectangular matrix subgroups of finite order.

7. $R = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} a_i \in R_n(15,9) = \{(a, b) / a \in [0, 15),\right.$

   $b \in [0, 9), +\}, +\}, 1 \le i \le 5\}$ is a MOD rectangular matrix group under +.

   Study questions (i) to (iii) of problem (6) for this R.

8. Let M= $\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} / a_i \in R_n(8, 19) =\right.$

   $\{(a, b) / a \in [0, 8), b \in [0, 19); +\}, 1 \le i \le 10; +\}$ be the MOD rectangular matrix group under +.

   Study questions (i) to (iii) of problem (6) for this M.

9. Let D = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right.$ / $a_i \in R_n(12, 42)$; $1 \leq i \leq$ 16, +} be the MOD rectangular matrix group under +.

Study questions (i) to (iii) of problem (6) for this D.

10. Let S = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \right.$ / $a_i \in R_n(10, 15)$; $1 \leq i \leq 10$, $\times_n$} be the MOD rectangular matrix semigroup under $\times_n$.

i. Prove o(S) = ∞ and S is a commutative monoid.
ii. Prove S has infinite number of zero divisors.
iii. Find all subsemigroups of S of finite order.
iv. Find all subsemigroups of infinite order which are not ideals.
v. Find all ideals of S of infinite order.
vi. Prove all ideals of S are of infinite order.
vii. Find all idempotents of S.
viii. Can S have nilpotents?
ix. Obtain any other special feature enjoyed by S.

11. Let M = { $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}$ / $a_i \in R_n(32, 64)$, $1 \leq i \leq$

16, $\times$ (or $\times_n$)} be the MOD rectangular matrix semigroup.

i. Study questions (i) to (ix) of problem (10) for this M under $\times$ (as well as $\times_n$).
ii. Prove M has nilpotent elements.
iii. Show M under $\times$ is a MOD rectangular matrix monoid which is noncommutative.
iv. Find all right ideals of M which are not left ideals under $\times$.
v. Find all right zero divisors of M which are not left zero divisors of M.

12. Let W = { $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix}$ / $a_i \in R_n(27, 19)$; $1 \leq i \leq$ 8,

$\times_n$} be the MOD rectangular matrix semigroup under $\times_n$.

i. Study questions (i) to (ix) of problem (10) for this W.
ii. Study W when $R_n(27, 19)$ is replaced by $R_n$ (16, 48).

13. Let P = { $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix}$ / $a_i \in R_n(120, 43)$;

$1 \leq i \leq 20$, $\times_n$} be the MOD rectangular matrix semigroup under $\times_n$.

Study questions (i) to (ix) of problem (10) for this P.

14. Let $S(M) = \{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix}$ / $a_i \in S(R_n(8, 12))$ =

{collection of all subsets from $R_n(8, 12)$ = {(a, b) / a ∈ [0, 8), b ∈ [0, 12), +}, +}, +; 1 ≤ i ≤ 15} be the MOD rectangular subset matrix semigroup under +.

i. Prove $o(S(M)) = \infty$ and S(M) is a commutative monoid.
ii. Find all subsemigroups of finite order in S(M). (Are they infinite in number).
iii. Find all subsemigroups of S(M).

15. Let S(P) = $\{(a_1, a_2, a_3, a_4, a_5)$ / $a_i \in S(R_n(43, 23))$ = {(collection of all subsets from $R_n(43, 23)$ = {(a, b) / a ∈ [0, 43), b ∈ [0, 23),+}, +},1 ≤ i ≤ 5, +} be the MOD rectangular subset matrix semigroup under +.

i. Study questions (i) to (iii) of problem (14) for this S(P).
ii. Compare with P where in P; $R_n(43, 23)$ is replaced by $R_n(24, 64)$.

16. Let $S(V) = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{bmatrix}$ / $a_i \in$

$S(R_n(8, 25))$ = {collection of all subsets from $R_n(8, 25)$ = {(a, b) / a ∈ [0, 8), b ∈{0, 25), +}, +},1 ≤ i ≤ 18, +} be the MOD rectangular subset matrix semigroup under +.

Study questions (i) to (iii) of problem (14) for this S(V).

17. Let $S(P) = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} \right.$ / $a_i \in S(R_n(7, 9) = \{$collection of all subsets from $R_n(7, 9) = \{(a, b) / a \in [0, 7), b \in [0, 9), \times, \times\}, 1 \leq i \leq 14, \times_n\}$ be the MOD rectangular subset matrix semigroup under $\times_n$, the natural product.

i. Prove $o(S(P)) = \infty$.
ii. Prove S(P) is a commutative monoid.
iii. Prove S(P) has infinite number of zero divisors whatever be s, t in $R_n(s, t)$.
iv. Prove S(P) has nontrivial idempotents and nilpotents only for special values of s and t.
v. Prove S(P) has both finite order subsemigroups as well as infinite order subsemigroups which are not ideals.
vi. Prove S(P) has ideals and all of them are of infinite order.
vii. Obtain any other special feature enjoyed by S(P).

18. Let $S(W) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \right.$ / $a_i \in$

$S(R_n(12, 24)) = \{$collection of all subsets from $R_n(12, 24) = \{(a, b) / a \in [0, 12), b \in [0, 24), \times\}, \times\}$,

$1 \leq i \leq 10, \times_n\}$ be the MOD rectangular subset matrix semigroup under $\times_n$.

Study questions (i) to (vii) of problem (17) for this S(W).

19. Let $S(B) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \right.$ / $a_i \in S(R_n(20,$

$16)) = \{$collection of all subsets from $R_n(20, 16) = \{(a, b) / a \in [0, 20), b \in [0, 16), \times\}, \times\}, 1 \leq i \leq 25, \times_n(\text{or } \times)\}$ be the MOD rectangular subset matrix semigroup under natural product $\times_n$ or usual product $\times$.

i. Study questions (i) to (vii) of problem (17) for this S(B).
ii. Prove in case usual product $\times$ on S(B) prove S(B) is a non commutative infinite semigroup.
iii. Prove S(B) has left zero divisors which are not left zero divisors.
iv. Prove S(B) has left ideals which are not right ideals.
v. Compare and contract the structure of S(B) on $\times$ and $\times_n$.

20. Let $S(P) = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right.$ / $a_i \in S(R_n(9, 27) = \{$collection of all

subsets from $R_n(9, 27) = \{(a, b) / a \in [0, 9), b \in [0,27), \times\}, \times\}, 1 \leq i \leq 5, \times_n\}$ be the MOD rectangular subset matrix semigroup under natural product $\times_n$.

Study questions (i) to (vii) of problem (17) for this S(P).

21. Let S(W) = {collection of all matrix subsets from

$$W = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} / a_i \in S(19, 11) = \{(a, b) / a \in [0, 19),$$

$b \in [0, 11), +\}, +, 1 \leq i \leq 10\}, +\}$ be the MOD rectangular matrix subset semigroup under +.

i. Prove S(W) is a commutative monoid of infinite order.
ii. Find all subsemigroups of finite order.
iii. Find all subsemigroups of infinite order.

22. Let S(M) = {collection of all matrix subsets from

$$M = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} / a_i \in$$

$R_n(24, 17), 1 \leq i \leq 14,+\},+\}$ be the MOD rectangular matrix subset semigroup under +.

Study questions (i) to (iii) of problem (21) for this S(M).

23. Let S(B) = {collection of all subsets from

$$B = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} / a_i \in R_n(23, 48),$$

$1 \leq i \leq 25, +\}, +\}$ be the MOD rectangular matrix subset semigroup under +.

Study questions (i) to (iii) of problem (21) for this S(B).

24. Let S(V) = {collection of all matrix subsets from

$$V = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} / a_i \in R_n(15, 7), \{(a, b) /$$

$a \in [0, 156), b \in [0, 7), \times\}$ $1 \leq i \leq 10\}, \times_n\}, \times_n\}$ be the MOD rectangular matrix subset semigroup under natural product $\times_n$.

i. Prove S(V) is a commutative monoid of infinite order.
ii. Prove S(V) has infinite number of zero divisors?
iii. Can S(V) have non trivial idempotents? Justify your claim.
iv. When will S(V) have nontrivial nilpotents?
v. Prove S(V) has both finite order and infinite order subsemigroups which are not ideals.
vi. Prove S(V) has ideals and all ideals are of infinite order.
vii. Obtain any other interesting feature enjoyed by S(V)

25. Let S(Q) = {collection of all matrix subsets from

$$Q = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{30} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{bmatrix} / a_i \in R_n(28, 49),$$

$= \{(a, b) / a \in [0, 28), b \in [0, 49)\ 1 \leq i \leq 36, \times_n$, (or $\leq$)}, $\times_n$ (or $\times$)} be the MOD rectangular matrix subset semigroup under natural product $\times_n$ (or usual product $\times$).

i. Study questions (i) to (vii) of problem (24) for this S(Q).
ii. Prove S(Q) under the usual product $\times$ is a non commutative monoid of infinite order.
iii. Prove S(Q) has right zero divisors which are not left zero divisors.
iv. Prove S(Q) has right ideals which are not left ideal.
v. Compare {S(Q), $\times_n$} with {S(Q), $\times$}).

26. Let S(X) = {Collection of all matrix subsets from

$$X = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} / a_i \in R_n(7, 17) = \{(a, b) / a \in [0, 7), b \in [0,1$$

7), $1 \leq i \leq 6$, $\times_n$}, $\times_n$ be the MOD rectangular matrix subset semigroup under $\times_n$.

Study questions (i) to (vii) of problem (24) for this S(X).

27. Let $P[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in [0, 8)$, b $\in [0, 12)$, +}; +} be the MOD rectangular polynomial group under +.

   i. Prove P[x] is a commutative group of infinite order.
   ii. Prove P[x] has subgroups of finite order.
   iii. Prove P[x] has subgroups of infinite order.

28. Let $V[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(43, 47) = \{(a, b)$ / a $\in$ [0, 43), b $\in$ [0, 47), +}, +} be the MOD rectangular polynomial group

   i. Study questions (i) to (iii) of problem (27) for this V[x].
   ii. Compare P[x] of problem (27) with this V[x] and show P[x] has more number of finite order subgroups.

29. Let $M[x]_5 = \{\sum_{i=0}^{5} a_i x^i$ / $a_i \in R_n(25, 16) = \{(a, b)$ / a $\in$ [0, 25), b $\in$ [0, 16), +}, $0 \leq i \leq 5$, $x^6 = 1$, +} be the MOD rectangular finite degree polynomial group under +.

   Study questions (i) to (iii) of problem (27) for this $M[x]_5$.

30. Let $V[x]_{12} = \{\sum_{i=0}^{12} a_i x^i$ / $a_i \in R_n(13, 7) = \{(a, b)$ / a $\in$ [0, 13), b $\in$ [0, 7), +}, $x^{13} = 1$, +} be the MOD rectangular finite degree polynomial group under +.

Study questions (i) to (iii) of problem (27) for this $V[x]_{12}$.

31. Let $M[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(15, 8) = \{(a, b)$ / $a \in [0, 15)$, $b \in [0, 8)$, $\times\}$, $\times\}$ be the MOD rectangular polynomial semigroup under $\times$.

    i. Prove M[x] is a commutative monoid of infinite order.
    ii. Show M[x] has infinite number of zero divisors.
    iii. Prove M[x] can never have nontrivial idempotents.
    iv. M[x] can have nilpotents only for special type of s and t in $R_n$ (s, t).
    v. Prove M[x] can never have subsemigroup of finite order.
    vi. Prove M[x] has subsemigroups of infinite order which are not ideals.
    vii. Prove all ideals of M[x] are of infinite order. Give at least three ideals of M[x].
    viii. Obtain any other special feature associated with M[x].

32. Let $B[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(17, 23) = \{(a, b)$ / $a \in [0, 17)$, $b \in [0, 23)$, $\times\}$, $\times\}$ be the MOD rectangular polynomial semigroup under $\times$.

    Study questions (i) to (viii) of problem (31) for this B[x].

33. Let $T[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(18, 49) = \{(a, b)$ / $a \in [0, 18)$, $b \in [0, 49)$, $\times\}$, $\times\}$ be the MOD rectangular polynomial semigroup under $\times$.

i. Study questions (i) to (viii) of problem (31) for this T[x].
ii. Compare this T[x] with B[x] of problem (32).

34. Let $M[x]_{20} = \{\sum_{i=0}^{20} a_i x^i$ / $a_i \in R_n(24, 48) = \{(a, b) / a \in [0, 24), b \in [0, 48), +\}$, $x^{21} = 1, +\}$ be the MOD rectangular finite degree polynomial group under +.

i. Prove $M[x]_{20}$ is a group of infinite order.
ii. Prove $M[x]_{20}$ has subgroups of finite order.
iii. Prove $M[x]_{20}$ has also subgroups of infinite order.
iv. Obtain any other special feature associated with $M[x]_{20}$.

35. Let $W[x]_7 = \{\sum_{i=0}^{7} a_i x^i$ / $a_i \in R_n(14, 19) = \{(a, b) / a \in [0, 14), b \in [0, 19), +\}$, $x^8 = 1, +\}$ be the MOD rectangular finite degree polynomial group under +.

Study questions (i) to (iv) of problem (34) for this $W[x]_7$.

36. Let $P[x]_9 = \{\sum_{i=0}^{9} a_i x^i$ / $a_i \in R_n(12, 9) = \{(a, b) / a \in [0, 12), b \in [0, 9), \times\}$, $x^{10} = 1, \times\}$ be the MOD rectangular finite degree polynomial semigroup under ×.

i. Prove $P[x]_9$ is an infinite commutative monoid.
ii. Show $P[x]_9$ has infinite number of zero divisors.
iii. Show $P[x]_9$ has no nontrivial idempotents.
iv. Prove $P[x]_9$ has nilpotents.
v. Show $P[x]_9$ can have subsemigroups of finite order and subsemigroups of infinite order which are not ideals.
vi. Prove all ideals of $P[x]x_9$ are of infinite order.
vii. Obtain any other special feature enjoyed by $P[x]_9$.

37. Let $B[x]_{12} = \{\sum_{i=0}^{12} a_i x^i$ / $a_i \in R_n(6, 241) = \{(a, b)$ / $a \in [0, 6)$, $b \in [0, 241), \times\}$, $x^{13} = 1, \times\}$ be the MOD rectangular finite degree polynomial semigroup under $\times$.

    Study questions (i) to (vii) of problem (36) for this $B[x]_{12}$.

38. Let $S[x]_9 = \{\sum_{i=0}^{9} a_i x^i$ / $a_i \in R_n(27, 81) = \{(a, b)$ / $a \in [0, 27)$, $b \in [0, 81), \times\}$, $x^{16} = 1, \times\}$ be the MOD rectangular finite degree polynomial semigroup under $\times$.

    Study questions (i) to (vii) of problem (36) for this $S[x]_9$.

39. Let $S(P[x])$ = {collection of all subsets from $P[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(3, 8) = \{(a, b)$ / $a \in [0, 3)$, $b \in [0, 8), +\}, +\}$ be the MOD rectangular polynomial subset semigroup under +.

    i. Prove S(P[x]) is a commutative monoid of infinite order.
    ii. Prove S(P[x]) can have finite or infinite order subsemigroups.
    iii. Prove S(P[x]) also has subsemigroups of infinite order.

40. Let $S(M[x])$ = {collection of all subsets from $M[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(24, 29) = \{(a, b)$ / $a \in [0, 24)$, $b \in [0, 29\}, +\}, +\}$ be the MOD rectangular polynomial subset semigroup under +.

Study questions (i) to (iii) of problem (39) for this S(M[x]).

41. Let S(V[x]) = {collection of all polynomial subsets from V[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(12, 45) = \{(a, b) / a \in [0, 12), b \in [0, 45\}, \times\}, \times\}$ be the MOD rectangular polynomial subset semigroup under ×.

i. Prove S(V[x]) is an infinite commutative monoid.
ii. Prove S(V[x]) has infinite number of zero divisors.
iii. Prove S(V[x]) has no idempotents.
iv. Can S(V[x]) has nilpotents? Justify.
v. Prove S(V[x]) has only MOD rectangular subsemigroups of infinite order which are not ideals.
vi. Obtain any other special feature associated with S(V[x]).

42. Let S(S[x]) = {collection of all subsets from S[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(3, 7) = \{(a, b) / a \in [0, 3), b \in [0, 7), \times\}, \times\}$ be the MOD rectangular polynomial subset semigroup under ×.

Study questions (i) to (vi) of problem (41) for this S(S[x]).

43. Let S(N[x]) = {collection of all subsets from N[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in R_n(64, 81) = \{(a, b) / a \in [0, 64), b \in [0, 81)\}, \times\}, \times\}$ be the MOD rectangular polynomial subset semigroup under ×.

Study questions (i) to (vii) of problem (41) for this S(N[x]).

44. Let $P[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(R_n(2, 12) = \{$collection of all subsets from $R_n$ $(2,12) = \{(a, b) / a \in [0, 2), b \in [0, 12), +\}, +\}, +\}$ be the MOD rectangular subset coefficient polynomial semigroup under +.

i. Prove P[x] is an infinite order commutative monoid.
ii. Prove P[x] has subsemigroups of finite order.
iii. Prove P[x] has subsemigroups of infinite order.
iv. Obtain all other special features associated with P[x].

45. Let $S[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(R_n(12, 81)) = \{$collection of all subsets from $R_n(12, 81) = \{(a, b) / a \in [0, 12), b \in [0, 81), +\}, +\}, +\}$ be the MOD rectangular subset coefficient polynomial semigroup.

Study questions (i) to (iv) of problem (44) for this S[x].

46. Let $V[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(R_n(26, 131)) = \{$collection of all subset from $R_n(26, 131), = \{(a, b) / a \in [0, 26), b \in [0, 131), \times\}, \times\}, \times\}$ be the MOD rectangular subset coefficient polynomial semigroup under ×.

i. Prove V[x] is an infinite commutative monoid under ×.
ii. Prove V[x] has infinite number of zero divisors.
iii. V[x] cannot have idempotents. Justify.
iv. Prove V[x] has nilpotent only for certain values of s and t of $R_n(s, t)$.
v. Does this V[x] contain nilpotents?
vi. Can V[x] have subsemigroups of finite order?
vii. Give subset subsemigroups of infinite order which are not ideals.
viii. List all ideals of V[x] and prove all ideals are of infinite order?

ix. Enumerate all special features enjoyed by V[x].

47. Let S[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(R_n(12, 48))$ = {collection of all subsets from $R_n$ (12,48) = {(a, b) / a ∈ [0, 12), b ∈ [0, 48), ×}, ×},×} be the MOD rectangular subset coefficient polynomial semigroup under ×.

Study questions (i) to (ix) of problem (46) for this S[x].

48. Let $P[x]_m = \{\sum_{i=0}^{m} a_i x^i$ / $a_i \in S(R_n(9, 12))$ = {collection of all subsets from $R_n$ (9, 12) = {(a, b) / a ∈ [0, 9), b ∈ [0, 12), ×}, ×}, $x^{m+1} = 1$} be the MOD rectangular subset coefficient polynomial semigroup under ×.

i. Prove $P[x]_m$ is a commutative monoid of infinite order.
ii. Prove $P[x]_m$ has infinite number of zero divisors.
iii. Can $P[x]_m$ have nonzero nilpotents?
iv. Prove $P[x]_m$ can have subsemigroups of finite and infinite order which are not ideals.
v. Prove $P[x]_m$ have all ideals to be of infinite order.
vi. Prove $P[x]_m$ can never have idempotents.
vii. Enumerate all special features associated with $P[x]_m$.

49. Let $B[x]_9 = \{\sum_{i=0}^{9} a_i x^i$ / $a_i \in R_n(21, 200)$ = {Collection of all subsets from $R_n(21, 200)$ = {(a, b) / a ∈ [0, 21), b ∈ [0, 200), ×}, ×}, $x^{10}$ = 1. ×} be the MOD rectangular subset coefficient polynomial semigroup under ×.

Study questions (i) to (vii) of problem (48sssss) for this $B[x]_9$.

50. Let $V[x]_{18} = \{\sum_{i=0}^{18} a_i x^i$ / $a_i \in S(R_n(27, 36)) = \{$collection of all subsets from $R_n$ (27, 36) = $\{(a, b)$ / $a \in [0, 27)$, $b \in [0, 36), \times\}, \times\}$, $x^{19} = 1, \times\}$ be the MOD rectangular subset coefficient polynomial semigroup under $\times$.

    Study questions (i) to (vii) of problem (48) for this $V[x]_{18}$.

51. Let $S(M[x]_9) = \{$collection of all subsets from $M[x]_9$ = $\{\sum_{i=0}^{9} a_i x^i$ / $a_i \in R_n(20, 19)$; $x^{10} = 1, \times\}, \times\}$ be the MOD rectangular finite degree polynomial subset semigroup under $\times$.

i. Prove $S(M[x]_9)$ is a commutative monoid of infinite order.
ii. Prove $S(M[x]_9)$ has infinite number of zero divisors.
iii. Show $S(M[x]_9)$ cannot have nontrivial idempotents.
iv. Can $S(M[x]_9)$ cannot have nontrivial idempotents?
v. Can $S(M[x]_9)$ have nontrivial nilpotents?
vi. Prove $S(M[x]_9)$ has both finite order as well infinite order subsemigroups which are not ideals.
vii. Enumerate all special features enjoyed by $S(M[x]_9)$.

52. Let = $S(T[x]_{21}) = \{$collection of all subsets from $T[x]_{21}$ = $\{\sum_{i=0}^{21} a_i x^i$ / $a_i \in R_n(81, 256) = x^{22} = 1\}, \times\}, \times\}$ be the MOD rectangular finite degree polynomial subset semigroup under $\times$.

    Study questions (i) to (vii) of problem (51) for this $S(T[x]_{21})$.

53. Let $S(P[x]_5)$ = {collection of all subsets from $P[x]_5$ = $\{\sum_{i=0}^{5} a_i x^i$ / $a_i \in R_n(22,14)$ = {(a, b) / a $\in$ [0, 22), b $\in$ [0, 14), $\times$}, $x^6 = 1\}$, $\times$} be the MOD rectangular finite degree polynomial subset semigroup under $\times$.

Study questions (i) to (vii) of problem (51) for this $S(P[x]_5)$.

**Chapter Four**

# SEMIGROUPS ON MOD RECTANGULAR NATURAL NEUTROSOPHIC NUMBERS AND THEIR PROPERTIES

In this chapter we for the first time introduce MOD rectangular natural neutrosophic numbers.

Suppose we have $Z_3 \times Z_{10}$ to be the product of two modulo integers $Z_3$ and $Z_{10}$.

We define MOD Cartesian product of natural neutrosophic number pairs of $Z_3 \times Z_{10}$ as $Z_3^I \times Z_{10}^I = \{(a, b) / a \in Z_3^I, b \in Z_{10}^I\}$, thus $Z_3^I \times Z_{10}^I$ is defined as the MOD rectangular natural neutrosophic number pairs.

We will illustrate this situation by some examples and also define the operations + and × on them.

***Example 4.1***. Let M = $Z_6^I \times Z_2^I = \{(a, b) / a \in Z_6^I, b \in Z_2^I\}$, (M, +) is only a MOD semigroup.

For $(I_0^6, I_0^2) + (I_0^6, I_0^2) = (I_0^6, I_0^2)$. Further $(I_3^6, I_0^2) + (I_3^6, I_0^2) = (I_0^6, I_0^2)$ so (M,+) is only a semigroup in fact a commutative monoid of finite order.

Let a = ($I_3^6 + I_2^6$, 1 + $I_0^2$), b = ($I_3^6 + I_0^6, I_0^2$) $\in$ M, a + b =

($I_3^6 + I_2^2$, 1 + $I_0^2$) + ($I_3^6 + I_0^6$, $I_0^2$) = ($I_3^6 + I_2^6 + I_0^6$, 1 + $I_0^2$) $\in$ M.

This is the way + operation is performed on M. In fact o(M) < ∞ and has subsemigroups of finite order.

$P_1$ = {(a, b) / a $\in$ $Z_6$, b $\in$ $Z_2$} $\subseteq$ M is a subsemigroup of M of finite order; in fact a subgroup.

$P_2$ = {(a, 0) / a $\in$ $Z_6$} $\subseteq$ M is again a subsemigroup of M of finite order in fact a subgroup.

$P_3$ = {(0, a) / a $\in$ $Z_2$} $\subseteq$ M is a subsemigroup of M of finite order which is a subgroup. $P_6$ = {(a, 0) / a $\in$ $Z_6^I$} $\subseteq$ M is a subsemigroup and is not a group under + of M.

$P_4$ = {(a, b) / a $\in$ $Z_6^I$, b $\in$ $Z_2$} $\subseteq$ M is only a subsemigroup of M and not a group.

$P_5$ = {(0, b) / b $\in$ $Z_2^I$} $\subseteq$ M is a subsemigroup which is not a subgroup of M.

Thus M is a S-MOD rectangular semigroup.

**Definition 4.1**. *Let S = {$Z_n^I \times Z_m^I$, (m ≠ n)} = {(a, b) / a $\in$ $Z_n^I$, b $\in$ $Z_m^I$}; S is defined as the MOD rectangular natural neutrosophic modulo integer set; (2 ≤ m, n < ∞).*

*(S, +) is defined as the MOD rectangular natural neutrosophic modulo integer semigroup under +. In fact (S, +) is a commutative finite semigroup.*

We have given examples of them.

***Example 4.2***. Let M = $\{ Z^I_{20} \times Z^I_7 = \{(a, b) / a \in Z^I_{20}, b \in Z^I_7\}, +\}$ be the MOD rectangular natural neutrosophic modulo integer semigroup. In fact M is a finite commutative monoid and M is a Smarandache semigroup as P = $\{(Z_{20} \times Z_7) = \{(a, b) / a \in Z_{20}, b \in Z_7\}, +\}$ is a group under +.

So M is a S–MOD rectangular natural neutrosophic modulo integer semigroup.

In view of all these we have the following theorem.

**Theorem 4.1**. *Let S = $\{ Z^I_m \times Z^I_n = \{(a, b) / a \in Z^I_m, b \in Z^I_n\}$, m $\neq$ n, $2 \leq m, n < \infty$, +} be the MOD rectangular natural neutrosophic modulo integer semigroup under +.*

*i)* *S is a commutative monoid of finite order.*

*ii)* *S is a Smarandache semigroup.*

*iii)* *S has several subsemigroups which are groups.*

*iv)* *S has several subsemigroups which are not groups.*

*v)* *S has several idempotents.*

*vi)* *S has an idempotent subsemigroup.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular natural neutrosophic finite modulo integer semigroup under × by some examples.

***Example 4.3***. Let N = $\{ Z^I_{10} \times Z^I_{12} = \{(a, b) / a \in Z^I_{10}, b \in Z^I_{12}\}, \times_0\}$ be the MOD rectangular natural neutrosophic finite modulo integer semigroup under $\times_0$.

Clearly $(1, 1) \in N$ is such that $(a, b) \times_o (1, 1) = (a, b)$ for all $(a, b) \in N$. Thus N is a commutative monoid of finite order.

We see we can define two types of products on N.

We will for sake of difference denote by $\times_0$ the zero dominated product that is $(I_t^{10}, I_s^{12}) \times_0 (0, 0) = (0, 0)$; whereas the usual product symbol $\times$ will be used to denote the natural neutrosophic number dominated product for

$(0, 0) \times (I_t^{10}, I_s^{12}) = (I_t^{10}, I_s^{12})$.

However under both products $\times$ (or $\times_0$), $(a, b) \times (I_t^{10}, I_s^{12}) = (I_t^{10}, I_s^{12}) \times (a, b) \neq (0, 0)$, i.e. $a \neq 0, b \neq 0$.

Clearly N under $\times_0$ product has zero divisors, units and idempotents.

In fact $(N, \times_0)$ has subsemigroups all of which are of finite order.

$P_1 = \{(Z_{10}^I, 0), \times_0\}$ is a subsemigroup of N.

$P_2 = \{(Z_{10}, 0), \times_0\}$ is again a subsemigroup of N.

$P_3 = \{(Z_{10}, Z_{12}), \times_0\}$ is again a subsemigroup of N and so on.

Only $P_1$ is an ideal and $P_2$ and $P_3$ are not ideals.

Consider $x = (5, 2)$ and $y = (2, 6) \in N$.

Clearly $x \times_0 y = (0, 0)$. In fact N has several zero divisors also.

Study in this direction is realized as a matter of routine for such study has been done in earlier books [37], however only in this book the notation $\times_0$ is used but even that is not very essential as from the very context one knows what type of product is used.

However if $\times_0$ is replaced by $\times$ on N that is the neutrosophic zero dominated product we see $P_1$ is not an ideal under $\times$ only a subsemigroup. Thus it is clear that both product behave differently and they are distinct.

It is also pertinent to keep on record that $\times_0$ is most favourable to be used when we use these in MODCMs or MODRMs models for otherwise the resultant would in many cases will be a natural neutrosophic number which may not be always a feasible answer.

So the product, $\times_0$ is essential in the study of MOD mathematical models.

Further authors want to keep on record that we cannot build MOD rectangular natural neutrosophic finite complex numbers or MOD rectangular natural neutrosophic-neutrosophic finite modulo numbers or so on.

For only MOD rectangular natural neutrosophic modulo integers alone is capable of extension to MOD rectangular real plane of natural neutrosophic numbers as ${}^{I}[0, m) \times {}^{I}[0, n)$ the rest are incapable of such structures, hence the restrictions in our study.

***Example 4.4***. Let B = $\{(Z_{17}^{I} \times Z_{7}^{I}) = \{(a, b) / a \in Z_{17}^{I}, b \in Z_{7}^{I}\}$, $\times_0$ (or $\times$)} be MOD rectangular natural neutrosophic number semigroup.

Finding zero divisors of the form (x, y) $\times$ (a, b) = (0, 0) a, b, x, y none of them equal to 0 or $I_0^{17}$ or $I_0^{7}$ is an impossibility. Thus if in $Z_n^I \times Z_m^I$, m and n are primes mostly it is difficult to find zero divisors.

However B has subsemigroups and ideals depending on $\times_0$ (or $\times$).

All these are left as an exercise to the reader.

Now we give the following result.

**Theorem 4.2**. *Let B = {($Z^I_m \times Z^I_n$) = {(a, b) / a ∈ $Z^I_m$, b ∈ $Z^I_n$}, m ≠ n, 2 ≤ m, n < ∞, $\times_0$ (or ×)} be the MOD rectangular natural neutrosophic modulo integer semigroup under $\times_0$ (or ×).*

i) *B is a commutative monoid of finite order.*
ii) *B has zero divisors of the form (a, 0) × (0, b) = (0, 0) for a ∈ $Z^I_n$ and b ∈ $Z^I_m$.*
iii) *If m and n are non-primes B has more number of zero divisors.*
iv) *B has MOD subsemigroups which are not ideals.*
v) *B has ideals.*
vi) *B has nilpotents and idempotents only for special values of m and n.*
vii) *B is a S-semigroup if and only if $Z^I_n$ or $Z^I_m$ are S-semigroups under $\times_0$ or×.*

Proof is direct and hence left an exercise to the reader.

***Example 4.5***. Let M = { $Z^I_{16} \times Z^I_{24}$ = {(a, b) / a ∈ $Z^I_{16}$, b ∈ $Z^I_{24}$ } $\times_0$ (or ×)} be the MOD rectangular natural neutrosophic semigroup under $\times_0$ (or ×).

Clearly M has nilpotents, zero divisors, subsemigroups, idempotents and ideals.

Finding them is left as a task for the reader.

Clearly x = ($I^{16}_4$, $I^I_{12}$) is such that x × x = ($I^{16}_0$, $I^I_{12}$) is a MOD natural neutrosophic nilpotent.

Next we define MOD rectangular natural neutrosophic numbers.

**Definition 4.2.** *Let D = {( $^I[0, m)$, × $^I[0, n)$) = {(a, b) / a ∈ $^I[0, m)$ and b ∈ $^I[0, n)$}; m ≠ n, 2 ≤ m, n < ∞} be the MOD rectangular interval natural neutrosophic pairs.*

*Clearly $^I[0,m) \times {}^I[0, n) \neq {}^I[0, n) \times {}^I[0, m)$.*

We can using them have both + (or ×) defined on them. They are only semigroups under + (or $\times_0$ or ×).

We will illustrate this situation by some examples.

***Example 4.6***. Let S = {$^I[0, 7) \times {}^I[0, 12)$ = {(a, b) / a ∈ $^I[0, 7)$, b ∈ $^I[0, 12)$}} be the MOD rectangular natural neutrosophic interval set. Clearly o(S) = ∞.

We can define any one of the operations + or $\times_0$ or × on S and S under any one of these operations is only a commutative semigroup of infinite order.

***Example 4.7***. Let M = {$^I[0, 5) \times {}^I[0, 12)$ = {(a, b) / a ∈ $^I[0, 5)$, b ∈ [0, 12)}, +} be the MOD rectangular natural neutrosophic interval semigroup of infinite order.

$(I_0^{[0,5)}, I_0^{[0,12)}) + (0, 0) = (I_0^{[0,5)}, I_0^{[0,12)})$ ($\because I_0^{[0,5)} + 0 = I_0^{[0,5)}$ and $I_0^{[0,5)} + a = a + I_0^{[0,5)}$ for all a ∈ [0,5). Similar result holds good for $I_0^{[0,12)}$).

This sort of study is interesting and innovative $(I_0^{[0,5)}, I_6^{[0,12)}) + (I_0^{[0,5)}, I_6^{[0,12)}) = (I_0^{[0,5)}, I_0^{[0,12)})$ is an idempotent of M. Hence M is only a semigroup under + and o(M) = ∞.

Clearly $(I_0^{[0,5)}, I_0^{[0,12)})$ + (4.02, 3.001) ≠ (4.02, 3.001), in fact the sum is $(I_0^{[0,5)} + 4.02,\ I_0^{[0,12)} + 3.001)$. Thus the MOD natural neutrosophic zero does not behave like the usual zero.

In fact it adds itself with any value from [0, 5) × [0, 12).

Consider

A = {[0, 5) × 0 = {(a, 0) / a ∈ [0, 5); +} ⊆ M is only a subsemigroup of infinite order.

B = {($Z_5 \times Z_{12}$) = {(a, b) / a ∈ $Z_5$, b ∈ $Z_{12}$, +} ⊆ M is only a subsemigroup of finite order.

We see C = {($Z_5$ × {0}) = {(a, 0) / a ∈ $Z_5$, +} ⊆ M is only a subsemigroup of finite order.

In fact all the three subsemigroups are groups under +. Thus it is important to note that M is always a Smarandache semigroup under the operation +.

In fact M also has subsemigroups of finite order which are not groups.

Consider W = {( $Z_5^I$ × {0}) = {(a, 0) / a ∈ $Z_5^I$, +} ⊆ M is a MOD rectangular natural neutrosophic subsemigroup of finite order which is not a group.

Thus M has substructures which can be groups of finite or infinite order as well as subsemigroups which can be of finite or infinite order.

M also has idempotents with respect to +.

In view of all these we have the following theorem.

**Theorem 4.3**. *Let S = $^I[0, m) \times {}^I[0, n)$ = {(a, b) / a ∈ $^I[0, m)$, b ∈ $^I[0, n)$; +}; m ≠ n, 2 ≤ m, n < ∞, +} be the MOD rectangular natural neutrosophic interval semigroup under +.*

*i) S is an infinite commutative monoid with (0, 0) as the additive identity.*

*ii) S has finite subsemigroups which are subgroups.*

*iii)* *S has finite subsemigroups which are not subgroups.*

*iv)* *S has infinite subsemigroups which are subgroups.*

*v)* *S has infinite subsemigroups which are only `subsemigroups.*

*vi)* *S is a Smarandache semigroup.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular natural neutrosophic interval semigroup under $\times_0$ (or $\times$) by some examples.

***Example 4.8***. Let B = $\{{}^I[0, 6) \times {}^I[0, 42) = \{(a, b) / a \in {}^I[0, 6), b \in {}^I[0, 42)\}, \times_0\}$ be the MOD rectangular natural neutrosophic interval semigroup under the usual zero dominated product.

B has MOD zero divisors, idempotents, subsemigroups of both finite and infinite order.

B is a S-semigroup if and only if $Z_6$ and (or) $Z_{42}$ is a S-semigroup under $\times$.

***Example 4.9***. Let P = $\{{}^I[0, 17) \times {}^I[0, 53) = \{(a, b) / a \in [0, 17), b \in [0, 53)\}, \times_0\}$ be the MOD rectangular natural neutrosophic interval semigroup under usual zero dominated product $\times_0$.

Let x = (8.5, 2) and y = (2, 26.5) $\in$ P, x $\times_0$ y = (0, 0). Thus P has zero divisors.

R = $\{(a, b) / a \in Z_{17}$ and $b \in Z_{53}\} \subseteq$ P is only a subsemigroup of finite order which is not an ideal.

However P is a MOD rectangular natural neutrosophic interval Smarandache semigroup as G = $\{(a, b) / a \in Z_{17} \setminus \{0\}, b \in Z_{53} \setminus \{0\}\} \subseteq$ P is a subgroup hence the claim.

W = {(a, 0) / a ∈ $^I[0, 17)$, $\times_0$} ⊆ P is a subsemigroup which is also an ideal of P.

Studying other properties of P happens to be a matter of routine so left as an exercise to the reader.

In view of all these we have the following theorem.

**Theorem 4.4**. *Let S = {$^I[0, m) \times {}^I[0, n)$ = {(a, b) / a ∈ $^I[0, m)$, b ∈ $^I[0, n)$}, m ≠ n, 2 ≤ m, n < ∞, $\times_0$} be the MOD rectangular natural neutrosophic interval semigroup under usual zero dominant product.*

i) *S is a MOD rectangular natural neutrosophic interval commutative monoid of infinite order.*

ii) *S has subsemigroups of finite order as well as of infinite order which are not ideals.*

iii) *S has all ideals to be of infinite order.*

iv) *S is a S-semigroup if and only if ($Z_m$, $\times_0$) and or {$Z_n$,$\times_0$)} are S-semigroups.*

v) *S has zero divisors for all n.*

vi) *S has nontrivial idempotents only if $Z_m$ and $Z_n$ has idempotents.*

vii) *S has nontrivial nilpotents if and only if $Z_m$ and $Z_n$ have nilpotents.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe the MOD rectangular natural neutrosophic interval semigroup under the product in which ($I_t^m, I_s^n$) are dominant.

That is $0 \times I_0^m = I_0^m$ and $0 \times I_s^n = I_s^n$.

***Example 4.10***. Let S = {$^I[0, 9) \times {}^I[0, 8)$ = {(a, b) / a ∈ $^I[0, 9)$, b ∈ $^I[0, 8)$}, ×} be the MOD rectangular natural neutrosophic semigroup under the natural neutrosophic zero, ( $I_0^{[0,9)}$, $I_0^{[0,8)}$ ) product. S has subsemigroups of both finite and infinite order which are not ideals.

Of course P = {(a, b) / a ∈ $^I[0, 9)$ and b = $I_0^{[0,8)}$, ×} ⊆ S is the MOD rectangular natural neutrosophic interval subsemigroup which is also an ideal of S.

S has nilpotents and zero divisors.

***Example 4.11***. Let M = {$^I[0, 42) \times {}^I[0, 25)$ = {(a, b) / a ∈ $^I[0,42)$, b ∈ $^I[0, 25)$}, ×} be the MOD rectangular natural neutrosophic interval semigroup under ×.

This M has idempotents, nilpotents, zero divisors, subsemigroups of finite and infinite order and ideals which are only of infinite order.

Finding all these is considered as a matter of routine so left as an exercise to the reader.

In view of all these we prove the following theorem.

**Theorem 4.5**. *Let S = {$^I[0, m) \times {}^I[0, n)$ = {(a, b) / a ∈ $^I[0, m)$, b ∈ $^I[0, n)$}, m ≠ n, 2 ≤ m, n < ∞, ×} be the MOD rectangular natural neutrosophic interval semigroup.*

*i) S is a commutative monoid of infinite order.*

*ii) S has subsemigroups of both finite and infinite order which are not ideals.*

*iii) All ideals of S are of infinite order.*

*iv) S has MOD natural neutrosophic zero divisors as well as zero divisors.*

*v) S has idempotents and nilpotents for appropriate m and n.*

The proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular natural neutrosophic matrix subsemigroups under + or $\times_0$ or $\times$ by examples.

***Example 4.12***. Let W = $\{(a_1, a_2, a_3, a_4) / a_i \in ({}^I[0, 12) \times {}^I[0, 24)) = \{(a, b) / a \in {}^I[0, 12), b \in {}^I[0, 24)\}, 1 \leq i \leq 4, +\}$ be the MOD rectangular natural neutrosophic interval matrix semigroup under +.

$P_1 = \{(a_1, 0, 0, 0) / a_1 \in ({}^I[0, 12) \times {}^I[0, 24)) = \{(a, b) / a \in {}^I[0,12), b \in {}^I[0, 24)\}, +\}$ is a subsemigroup of infinite order.

$P_2 = \{(a_1, 0, 0, 0) / a_1 \in \{Z_{12} \times Z_{24} = (a, b) / a \in Z_{12}, b \in Z_{24}\}, +\}$ is a subsemigroup of finite order which is a group under +.

$P_3 = \{(a_1, 0, 0, 0) / a_1 = (a, 0)$ where $a_1 \in Z_{12}^I, 0 \in Z_{24}, +\}$ is also a subsemigroup of finite order.

$P_3$ is only a subsemigroup and does not have a group structure for $I_6^{12} + I_6^{12} = I_6^{12}$ in $Z_{12}^I$.

Thus we can also make a claim that W is a Smarandache semigroup. W has both finite order subsemigroups which are groups and finite order subsemigroups which are not groups. Likewise W has infinite order subsemigroups which are groups as well as infinite order subsemigroups which are not groups.

$P_1$ is an infinite order subsemigroup which is not a group.

$P_4 = \{(a_1, 0, 0, 0) / a_1 = (a, 0)$ where $a \in [0,12),+\} \subseteq W$ is a subsemigroup of W which is also a group.

***Example 4.13***. Let M = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\right.$ / $a_i \in {}^I[0,11) \times {}^I[0,43)$ = {(a, b) /

$a \in {}^I[0, 11)$ and $b \in {}^I[0, 43)$}; $1 \leq i \leq 6$, +} be the MOD rectangular interval natural neutrosophic column matrix semigroup under +.

All associated properties can be studied by any interested reader.

***Example 4.14***. Let B = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix}\right.$ / $a_i \in {}^I[0,48) \times$

${}^I[0,105)$ = {(a, b) / $a \in {}^I[0, 48)$. B $\in$ [0, 105)}, $1 \leq i \leq 15$, +} be the MOD rectangular interval natural neutrosophic matrix semigroup under +.

B has subsemigroups of both finite and infinite order and some of them are groups under +.

In view of all these we have the following result.

**Theorem 4.6**: *Let S = {collection of all t × s matrices with entries from* ${}^I[0,m) \times {}^I[0, n)$ *= {(a, b) /* $a \in {}^I[0, m)$*,* $b \in {}^I[0, n)$*},* $m \neq n$*,* $2 \leq m, n < \infty$*, +} be the MOD rectangular interval natural neutrosophic matrix semigroup under +.*

i) *S is a commutative monoid of infinite order.*

ii) *S has infinite order subsemigroups which are groups under +.*

iii) *S has finite order subsemigroups which are groups under +.*

iv) *S has finite order subsemigroups which are not groups.*

v) *S has infinite subsemigroups which are not groups.*

vi) *S is always a Smarandache semigroup immaterial of m and n.*

vii) *S has several matrices which are idempotents under +.*

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto describe MOD rectangular interval natural neutrosophic semigroups under $\times_n$ by some examples it may be 0 dominated or natural neutrosophic zero dominated product.

**Example 4.15**. Let M = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \right.$ / $a_i \in {}^I[0, 4) \times {}^I[0, 10) = \{(a, b)$ / $a \in {}^I[0, 4)$ and $b \in {}^I[0, 10)\}$, $1 \leq i \leq 5, \times\}$ be the MOD rectangular interval natural neutrosophic matrix semigroup under $\times_n$.

M has zero divisors, idempotents and subsemigroups of both finite and infinite order.

$$\begin{bmatrix} (1, 1) \\ (1, 1) \\ (1, 1) \\ (1, 1) \\ (1, 1) \end{bmatrix} \in M \text{ is the multiplicative identity, so M is a}$$

commutative monoid of infinite order.

M has ideals all of which are infinite order.

The reader is left with the task of working with them as it is a matter of routine.

***Example 4.16***. Let B = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix}\right.$ where $a_i \in {}^I[0, 7)$ $\times {}^I[0, 23) = \{(a, b) / a \in {}^I[0, 7), b \in {}^I[0, 23)\}$, $1 \leq i \leq 8$, $\times_n\}$ be the MOD rectangular interval natural neutrosophic matrix semigroup under the natural product $\times_n$.

$\begin{bmatrix} (1,1) & (1,1) & (1,1) & (1,1) \\ (1,1) & (1,1) & (1,1) & (1,1) \end{bmatrix}$ in B acts as the identity in B under $\times_n$.

B has zero divisors but finding nontrivial nilpotents or idempotents happens to be a challenging problem.

Finding substructures etc.; is considered as a matter of routine so left as an exercise to the reader.

***Example 4.17***. Let B = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right. / a_i \in {}^I[0,20) \times {}^I[0,18) =$

$\{(a, b) / a \in {}^{I}[0, 20), b \in {}^{I}[0,18)\}, 1 \leq i \leq 9; \times$ or $(\times_n)\}$ be the MOD rectangular natural neutrosophic interval matrix semigroup under usual product (or the natural product $\times_n$).

$\{B, \times\}$ is a MOD rectangular natural neutrosophic interval matrix monoid which is non commutative and is of infinite order.

$$\begin{bmatrix} (1,1) & (0,0) & (0,0) \\ (0,0) & (1,1) & (0,0) \\ (0,0) & (0,0) & (1,1) \end{bmatrix}$$ acts as the identity element of B under

the usual product $\times$.

However under the natural product $\times_n$ we see

$$\begin{bmatrix} (1,1) & (1,1) & (1,1) \\ (1,1) & (1,1) & (1,1) \\ (1,1) & (1,1) & (1,1) \end{bmatrix} \in B$$ acts as the identity.

Thus the very identity elements of $(B, \times_n)$ and $(B, \times)$ are distinct.

Further $(B, \times_n)$ is a commutative semigroup. The ideals of $(B, \times_n)$ are in general not ideals of $(B,\times)$.

B has left zero divisors which are not right zero divisors.

B has left ideals which are not right ideals.
B has subsemigroups of both finite and infinite order.

B has also ideals, right ideals and left ideals all of them are only of infinite order.

***Example 4.18***. Let M = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \right.$ / $a_i \in {}^I[0, 16) \times {}^I[0, 23)$ =

$\{(a, b) / a \in {}^I[0, 16), b \in {}^I[0, 23)\}, 1 \leq i \leq 10, \times_n\}$ be the MOD rectangular natural neutrosophic interval matrix semigroup under $\times_n$.

Clearly this has substructures and all related properties can be derived without any difficulty hence left as an exercise to the reader.

Thus we have the following result.

**Theorem 4.7**. *Let S = {collection of all t × s matrices with entries from* $^I[0, m) \times {}^I[0, n) = \{(a, b) / a \in {}^I[0, m), b \in {}^I[0, n)\}$ $m \neq n$; $2 \leq m, n < \infty$; $\times_n\}$ *be the MOD rectangular natural neutrosophic interval matrix semigroup under the natural product* $\times_n$.

*i)* *S is a commutative monoid of infinite order.*

*ii)* *S has zero divisors whatever be m and n.*

*iii)* *S has nontrivial nilpotents only for certain appropriate values of m and n.*

*iv)* *S is a S-semigroup if and only if* $Z_m$ *or* $Z_n$ *is a S-semigroup.*

*v)* *S contains idempotents comprising of MOD natural neutrosophic elements.*

*vi)* *S has subsemigroups of finite order as well as of infinite order which are not ideals of S.*

*vii)* *S has ideals and all of them are only of infinite order.*

Proof is direct and hence left as an exercise to the reader.

We now give examples of MOD natural neutrosophic rectangular interval polynomial semigroups under +.

***Example 4.19***. Let P[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^I[0, 20) \times {}^I[0, 14)$ = $\{(a, b) / a \in {}^I[0, 20), b \in {}^I[0, 14)\}, +\}$ be the MOD rectangular natural neutrosophic interval polynomial under +.

P[x] has subsemigroups of both finite and infinite order P[x] has polynomials which are idempotents.

***Example 4.20***. Let M[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^I[0, 43) \times {}^I[0, 7)$ = $\{(a, b) / a \in {}^I[0, 43), b \in {}^I[0, 7)\}, +\}$ be the MOD rectangular natural neutrosophic interval polynomial semigroup under +.

$p(x) = (3.7, 0.5)x^7 + (0.3, 4)x^2 + (6.3, 2.01)$ and

$q(x) = (25.3, 4.6)x^2 + (2.3, 1.02)x + (10.6, 5.09) \in M[x]$.

$p(x) + q(x) = (3.7, 0.5)x^7 + (25.6, 1.6)x^2 + (2.3,1.02)x + (16.9, 0.1) \in M[x]$.

This is the way '+' operation is performed on M[x].

$(0, 0) = (0, 0) + (0, 0)x + \ldots + (0, 0)x^n$ is the additive identity.

For $p(x) + (0, 0) = p(x)$ for all $p(x) \in M[x]$.

$N[x] = \{\sum_{i=0}^{10} a_i x^i$ / $a_i \in \{0\} \times Z_7 = \{(0, a) / a \in Z_7\}, +\} \subseteq$ M[x] is a subsemigroup of M[x] which is of finite order.

In fact N[x] is a group under + so M[x] is a Smarandache semigroup.

$$B[x] = \{ \sum_{i=0}^{\infty} a_i x^i \ / \ a_i \in Z_{43} \times \{0\} = \{(a, 0) / a \in Z_{43}\}, +\}$$

be the subsemigroup. Clearly B[x] is a group of infinite order

$$p[x] = \{ \sum_{i=0}^{5} a_i x^i \ / \ a_i \in \{0\} \times Z_7^I = \{(0, b) / b \in Z_7^I\}, +\} \subseteq M[x]$$

is a subsemigroup of finite order but is not a group.

Let $W[x] = \{ \sum_{i=0}^{\infty} a_i x^i / a_i \in \{0\} \times Z_7^I = \{(0, b) / b \in Z_7^I\},$ $+\} \subseteq M[x]$ is a subsemigroup of infinite order which is not a group.

Study in this direction is a matter of routine so left as an exercise to the reader.

We have the following result.

**Theorem 4.8**. *Let* $B[x] = \{\sum_{i=0}^{\infty} a_i x^i / a_i \in {}^I[0, m) \times {}^I[0, n) = \{(a, b) / a \in {}^I[0, m), b \in {}^I[0, n)\}, +\}$ *be the MOD rectangular natural neutrosophic interval polynomial semigroup under +.*

i) *S is a commutative monoid of infinite order.*
ii) *S has subsemigroups of finite order which are groups.*
iii) *S has subsemigroups of infinite order which are groups.*
iv) *S has subsemigroups of finite order which are not groups.*
v) *S has subsemigroups of infinite order which are not groups.*
vi) *S is always a Smarandache semigroup.*

vii) *S has always idempotents which are infinite in number.*

The proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular natural neutrosophic interval polynomial semigroups under product $\times_0$ (or $\times$) by some examples.

***Example 4.21***. Let B[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^I[0, 42) \times {}^I[0, 16)$ = $\{(a, b)$ / $a \in {}^I[0, 42), b \in {}^I[0, 16)\}, \times_0\}$ be the MOD rectangular natural neutrosophic interval polynomial semigroup under $\times_0$.

Clearly B[x] has zero divisors and idempotents has no nontrivial idempotent polynomials.

Of course finding nilpotent polynomials is dependent on the s and t of ${}^I[0, t) \times {}^I[0, s)$.

***Example 4.22***. Let M[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^I[0, 16) \times {}^I[0, 81)$ = $\{(a, b)$ / $a \in {}^I[0, 16), b \in {}^I[0, 81)\}; \times_0$ (or $\times$)$\}$ be the MOD rectangular natural neutrosophic interval polynomial semigroup under $\times_0$ (or $\times$).

M[x] has polynomials p(x) such that $(p(x))^t$ = (0, 0) for $t \geq 2$.

M[x] has no nontrivial idempotent polynomials.

All subsemigroups are of infinite order. Similarly all ideals are of infinite order.

M[x] has infinite number of zero divisors.

$B[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in \{0\} \times {}^I[0, 81) = \{(0, a) / a \in {}^I[0, 81)\}, \times_0$ (or $\times$)$\} \subseteq M[x]$ is an ideal of M[x].

$D[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^I[0, 16) \times \{0\} = \{(a, 0) / a \in {}^I[0, 16)\}, \times_0$ (or $\times$)$\} \subseteq M[x]$ is an ideal of M[x].

We see if $p(x) \in B[x]$ and $q(x) \in D[x]$ then $p(x) \times_0 q(x) = (0, 0)$. Further $B[x] \times D[x] = \{(0, 0)\}$.

Hence the claim, M[x] has infinite number of zero divisor polynomials.

In view of all these we have the following result.

**Theorem 4.9**. *Let* $B[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^I[0, s) \times {}^I[0, t) = \{(a, b) / a \in {}^I[0, s), b \in {}^I[0, t)\}; \times_0$ *(or* $\times$*)*$\}$ *be the MOD rectangular natural neutrosophic interval coefficient polynomial semigroup under* $\times_0$ *(or* $\times$*).*

i) *B[x] is a commutative monoid of infinite order.*

ii) *B[x] has infinite number of zero divisors.*

iii) *B[x] has no nontrivial idempotents for any s, t;* $2 \leq s, t < \infty$.

iv) *B[x] has nilpotents only for special values of s and t.*

v) *All subsemigroups of B[x] are of infinite order which are not ideals.*

vi) *B[x] has ideals all of which are of infinite order.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto develop the notion of MOD rectangular natural neutrosophic interval finite degree polynomial semigroup under + by some examples.

***Example 4.23***. Let $M[x]_9 = \{\sum_{i=0}^{9} a_i x^i$ / $a_i \in {}^I[0, 8) \times {}^I[0, 18) = \{(a, b)$ / $a \in {}^I[0, 8)$, and $b \in {}^I[0, 18); x^{10} = 1, +\}$ be the MOD rectangular natural neutrosophic interval finite degree polynomial semigroup under +.

Clearly $o(M[x]_9) = \infty$ but $M[x]_9$ has both finite order subsemigroups as well as infinite order subsemigroups.

$M[x]_9$ has several idempotents.

Finding special features associated with $M[x]_9$ is considered as a matter of routine and is left as an exercise to the reader.

***Example 4.24***. Let $M[x]_{20} = \{\sum_{i=0}^{20} a_i x^i$ / $a_i \in {}^I[0, 20) \times {}^I[0, 43) = \{(a, b)$ / $a \in {}^I[0, 20)$, and $b \in {}^I[0, 43)\}, x^{21} = 1, +\}$ be the MOD rectangular natural neutrosophic interval finite degree polynomial semigroup under +.

$M[x]_{20}$ has subsemigroups of both finite and infinite order. $M[x]_{20}$ has idempotents.

In view of all these we have the following theorem.

**Theorem 4.10**: *Let $S[x]_m = \{\sum_{i=0}^{m} a_i x^i$ / $a_i \in {}^I[0, n) \times {}^I[0, s) = \{(a, b)$ / $a \in {}^I[0, n), b \in {}^I[0, s)\}, x^{m+1} = 1; 1 \leq m, n, s < \infty, +\}$ be the MOD rectangular natural neutrosophic interval finite degree polynomial semigroup under +.*

*i)* *$S[x]_m$ is an infinite commutative monoid.*

*ii)* *$S[x]_m$ has subsemigroups of both finite and infinite order.*

*iii)* *$S[x]_m$ has nontrivial idempotents with respect to +.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular natural neutrosophic interval finite degree polynomial semigroup under $\times_0$ (or $\times$) by some examples.

***Example 4.25***. Let $P[x]_6 = \{\sum_{i=0}^{6} a_i x^i$ / $a_i \in {}^I[0, 12) \times {}^I[0, 20) = \{(a, b)$ / $a \in {}^I[0, 12), b \in {}^I[0, 20)\}, x^7 = 1, \times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval finite degree polynomial semigroup under $\times_0$ (or $\times$).

$P[x]_6$ is of infinite order. $P[x]_6$ has no nontrivial idempotents but has infinite number of zero divisors and nilpotents. $P[x]_6$ has subsemigroups of both infinite and finite order.

All these are considered as a matter of routine so left as an exercise to the reader.

***Example 4.26***. Let $M[x]_{12} = \{\sum_{i=0}^{12} a_i x^i$ / $a_i \in {}^I[0, 7) \times {}^I[0, 13) = \{(a, b)$ / $a \in {}^I[0, 7), b \in {}^I[0, 13)\}; x^{13} = 1, \times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval finite degree polynomial semigroup under ($\times_0$) or $\times$.

We see under both operation $\times_0$ (or $\times$); $M[x]_{12}$ has no nontrivial idempotents or nilpotents but has infinite number of zero divisors.

Study of substructures is considered as a matter of routine and so left as an exercise to the reader.

In view of all these we have the following result.

**Theorem 4.11**: *Let* $B[x]_m = \{\sum_{i=0}^{m} a_i x^i \; a_i x^i / a_i \in {}^I[0, s) \times {}^I[0, t) = \{(a, b) / a \in {}^I[0, s), b \in {}^I[0, t)\}; x^{m+1} = 1, 2 \leq s, t < \infty, \times_0 \text{ (or } \times)\}$ *be the MOD rectangular natural neutrosophic finite degree polynomial semigroup under* $\times_0$ *(or* $\times$*).*

- *i)* $B[x]_m$ *is a commutative monoid of infinite order.*
- *ii)* $B[x]_m$ *has infinite number of zero divisors.*
- *iii)* $B[x]_m$ *has subsemigroups of finite order.*
- *iv)* $B[x]_m$ *has subsemigroups of infinite order which are not ideals.*
- *v)* *All ideals of* $B[x]_m$ *are of infinite order.*
- *vi)* $B[x]_m$ *has no nontrivial idempotents.*
- *vii)* $B[x]_m$ *has nontrivial nilpotents only if* $Z_s$ *and* $Z_t$ *has nontrivial nilpotents.*
- *viii)* $B[x]_m$ *is a Smarandache semigroup if and only if* $Z_s$ *or* $Z_t$ *or both are S-semigroups.*

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto describe and develop algebraic structures using subsets of $Z_n^I$ and ${}^I[0, n)$ by examples.

***Example 4.27**.* $S(Z_3^I \times Z_{24}^I)$ = {collection of all subsets from $Z_3^I \times Z_{24}^I = \{(a, b) / a \in Z_3^I, b \in Z_{24}^I\}\}$ is the MOD rectangular natural neutrosophic subsets of Cartesian product of modulo integers $Z_3^I \times Z_{24}^I$.

P = {(0.5, 8 + $I_0^{24}$), (2.06, $I_0^{24}$ + $I_8^{24}$ + 3.2), ($I_0^3$ + 2.001, 1), (1.02 + $I_0^3$, $I_0^{24}$ + $I_3^{24}$ + 4.003)}.

Q = {(0.21 + $I_0^3$, 2.03), (0, 1.345 + $I_0^{24}$), (1, 2 + $I_8^{24}$)} be MOD rectangular subsets of $Z_3^I \times Z_{24}^I$.

T = {(0, $I_0^{24}$+3.889), (0, $I_6^{24}$ + 2), (0, $I_0^{24}$+ $I_3^{24}$ +6.02)} and S = {($I_0^3$ + 0.2, 0), ($I_0^3$ +2.1, 0), ($I_0^3$+1, 0), ($I_0^3$ +1.226, 0)} are also MOD rectangular subsets of $Z_3^I \times Z_{24}^I$.

We can perform either + operation or $\times_0$ or × on S($Z_3^I \times Z_{24}^I$).

It is important to keep on record that S($Z_3^I \times Z_{24}^I$) is only a semigroup under any of these operations.

**Definition 4.3.** *Let S($Z_m^I \times Z_n^I$) = {collection of all subsets from $Z_m^I \times Z_n^I$ = {(a, b) / a ∈ $Z_m^I$, b ∈ $Z_n^I$}, +} be the MOD rectangular natural neutrosophic modulo integer subset of pairs of semigroup under +.*

*Clearly (S($Z_m^I \times Z_n^I$)) is a finite monoid under +.*

We will first describe this situation by some examples.

***Example 4.28***. Let S($Z_{10}^I \times Z_{16}^I$) = {collection of all subsets from $Z_{10}^I \times Z_{16}^I$ = {(a, b) / a ∈ $Z_{10}^I$, b ∈ $Z_{16}^I$}, +} be the MOD rectangular natural neutrosophic subsets of modulo integer pairs semigroup under +.

Let A = $\{(I_5^{10} + 3, 0), (0, 4 + I_0^{16}), (I_0^{10} + 2.05, I_2^{16} + I_4^{16} + 0.5), (1, 2.32)\}$ and B = $\{(I_2^{10}, I_4^{16} + 0.7), (I_0^{10} + I_5^{10} + I_4^{10} + 2, 4)\}$ $\in S(Z_{10}^I \times Z_{16}^I)$.

A + B = $\{(I_5^{10} + I_2^{10} + 3, I_4^{16} + 0.7), (I_2^{10}, 4.7 + I_0^{16} + I_4^{16}), (I_2^{10} + I_0^{10} + 2.05, I_2^{16} + I_4^{16} + 1.2), (I_2^{10} + 1, I_4^{16} + 3.02), (I_0^{10} + I_5^{10} + I_4^{10} + 5, 4), (I_0^{10} + I_5^{10} + I_4^{16} + 2, 8 + I_0^{16}), (I_0^{10} + I_5^{10} + I_4^{10} + 4.05, I_2^{16} + I_4^{16} + 4.05), (3 + I_5^{10} + I_4^{10} + I_0^{10}, 6.32)\} \in S(Z_{10}^I \times Z_{16}^I)$.

This has subset subsemigroups of both finite order and some of them are infinite order.

Clearly $S(Z_{10}^I \times Z_{16}^I)$ has subsets which are idempotents under +.

***Example 4.29***. Let $\{S(Z_{13}^I \times Z_{48}^I), +\}$ = {Collection of all subsets from $Z_{13}^I \times Z_{48}^I$, +}, the MOD rectangular natural neutrosophic subsets of modulo finite integers under + be the semigroup. $S(Z_{13}^I \times Z_{48}^I)$ has subsemigroups of finite order as well as of infinite order.

$S(Z_{13}^I \times Z_{48}^I)$ has subsets which are idempotents.

In view of all these we have the following result.

**Theorem 4.12.** *Let B = $\{S(Z_n^I \times Z_m^I), +\}$ = {collection of all subsets from $Z_n^I \times Z_m^I$, +} be the MOD rectangular subset natural neutrosophic semigroup under +.*

*i) B is a finite commutative monoid.*

*ii) B has subset subsemigroups of finite order.*

*iii) B has subsets which are idempotents.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto discuss MOD rectangular subset natural neutrosophic finite modulo integer semigroup under $\times_0$ (or $\times$) by some examples.

***Example 4.30***. Let W = {S( $Z_{12}^I \times Z_{36}^I$ ), $\times_0$ (or $\times$)} = {collection of all subsets from $Z_{12}^I \times Z_{36}^I$ = {(a, b) / a $\in Z_{12}^I$, b $\in Z_{36}^I$}, $\times_0$ (or $\times$)} be the MOD rectangular subset natural neutrosophic finite modulo integer semigroup under $\times_0$ (or $\times$).

P = {( $I_4^{12} + I_0^{12} + 0.5, 0.2 + I_2^{36}$ ), ( $10 + I_6^{12}, 0.8 + I_0^{36}$ ), ( $I_0^{12}$ , 3)} and Q = {( $I_3^{12} + 1.2, 0.6 + I_0^{36}$ ), ( $I_6^{12} + 5, 6$ )} $\in$ W.

P $\times$ Q = {( $I_4^{12} + I_0^{12} + I_3^{12} + 0.6, 0.12 + I_2^{36} + I_0^{36}$ ), ( $I_3^{12} + I_6^{12}, I_0^{36} + 0.48$ ), ( $I_0^{12}, 1.8 + I_0^{36}$ ), ( $I_0^{12} + I_4^{12} + I_6^{12} + 2.5, 1.2 + I_2^{36}$ ), ( $I_0^{12} + I_0^{12}$ , 18), ( $I_6^{12} + I_0^{12} + 2, 4.8 + I_0^{36}$ )} $\in$ W. This is the way product operation is performed on W.

W has finite number of zero divisors. W is of finite order in fact W is a finite commutative monoid.

W has subsets which are such that A $\times$ A = A that is W has idempotent subsets.

Similarly W has subsets which are nilpotents W has subsemigroups.

***Example 4.31***. Let B = {S( $Z_{17}^I \times Z_{43}^I$ ) = {Collection of all subsets from $Z_{17}^I \times Z_{43}^I$ = {(a, b) / a $\in Z_{17}^I$, b $\in Z_{43}^I$}, $\times_0$ (or $\times$)} be the MOD rectangular subset natural neutrosophic finite modulo integer semigroup.

B has no nilpotents or idempotents B has subsemigroups. In fact B is a Smarandache semigroup.

B has zero divisors.

In view of all these we have the following theorem.

**Theorem 4.13.** *Let B = {S($Z_n^I \times Z_m^I$) = {Collection of all subsets from $Z_n^I \times Z_m^I$ = {(a, b) / a ∈ $Z_n^I$, b ∈ $Z_m^I$, m ≠ n, 2 ≤ m, n < ∞}, $\times_0$ (or ×)} be the MOD rectangular subset natural neutrosophic finite modulo integer semigroup under $\times_0$ (or ×).*

i) *B is a finite commutative monoid.*

ii) *B has subset zero divisors.*

iii) *B has subsemigroups which are not ideals.*

iv) *B is a Smarandache semigroup only if $Z_n$ or $Z_m$ is a Smarandache semigroup.*

v) *B has ideals.*

vi) *B has idempotents and nilpotents only for some appropriate values of m and n.*

The reader is left with the task of proving the above theorem.

Next we proceed onto describe MOD rectangular subset natural neutrosophic finite modulo integer matrix semigroups under + by the following examples.

***Example 4.32***. Let P = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \right.$ / $a_i \in S(Z_{12}^I \times Z_{44}^I)$ =

{Collection of all subsets from $Z_{12}^{I} \times Z_{44}^{I} = \{(a, b) / a \in Z_{12}^{I}, b \in Z_{44}^{I}\}$; $1 \leq i \leq 8$, +} be the MOD rectangular natural neutrosophic subset matrix (with subsets from $Z_{12}^{I} \times Z_{44}^{I}$) semigroup under +.

We just show how the operation + is performed on P.

Clearly ({(0, 0)}) = $\left\{\begin{bmatrix} \{(0,0)\} & \{(0,0)\} \\ \{(0,0)\} & \{(0,0)\} \\ \{(0,0)\} & \{(0,0)\} \\ \{(0,0)\} & \{(0,0)\} \end{bmatrix}\right\}$ acts the

additive identity $o(P) < \infty$, P is in fact a commutative monoid.

$$\text{Let } A = \begin{bmatrix} \{(0, I_0^{44}), (1,2) & \{(0, I_{11}^{44}), (1, 0.3) \\ (I_6^{12}, 0.5)\} & (0.32 + I_0^{12}, 1)\} \\ \{(0,0), (1,0.5) & \{(I_0^{12}, 0), (0, I_{11}^{44}) \\ \{(1,2)\} & \{(0.5, 0.77)\} \\ \{(1,2), (I_6^{12}+4, 0)\} & \{(0, I_4^{44}+3), (1, 0.3)\} \end{bmatrix}$$

and $B = \begin{bmatrix} \{(I_0^{12}, 1.2)\} & \{(0, I_8^{44}+3)\} \\ \{(1,0.3), (4, I_0^{44})\} & \{(0,0), (I_2^{12}, 0.2)\} \\ \{(4+I_0^{12}, 1+I_2^{44})\} & \{(1, 4.5)\} \\ \{(3, 4+I_3^{44})\} & \{(1, 3.2)\} \end{bmatrix}$, be elements of P.

$$A + B = \begin{bmatrix} \{(I_0^{12}, 1.2 + I_0^{44}), & \{(0, I_8^{44} + I_{11}^{44} + 3), \\ (1 + I_0^{12}, 3.2) & (1, 3.3 + I_8^{44}), (0.32 + I_0^{12}, \\ (I_0^{12} + I_6^{12}, 1.7)\} & 4 + I_8^{44})\} \\ \{(1, 0.3), (4, I_0^{44}), & \{(I_0^{12}, 0), (0, I_{11}^{44}), (I_2^{12} + I_0^{12}, \\ (2, 0.8), (5, I_0^{44} + 0.5)\} & 0.2), (I_2^{12}, I_{11}^{44} + 0.2)\} \\ \{(5 + I_0^{12}, 3 + I_2^{44})\} & \{(1.5, 5.27)\} \\ \{(4, 6 + I_3^{44}), & \{(1, 6.2 + I_4^{44}, \\ (7 + I_6^{12}, 4 + I_3^{44})\} & (2, 3.5)\} \end{bmatrix} \in P.$$

This is the way the operation + is performed on P.

The task of finding subsemigroups and idempotents is left as an exercise to the reader.

***Example 4.33***. Let S = $\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ / $a_i \in S(Z_5^I \times Z_8^I)$ = {Collection of all subsets from $Z_5^I \times Z_8^I$ = {(a, b) / a $\in Z_5^I$, b $\in Z_8^I$}, $1 \leq i \leq 3$, +} be the MOD rectangular natural neutrosophic subset matrix modulo integer semigroup under +.

Clearly o(S) < ∞ and S is a commutative monoid under +.

$(\{(0, 0)\}) = \begin{bmatrix} \{(0,0)\} \\ \{(0,0)\} \\ \{(0,0)\} \end{bmatrix} \in$ S is such that A + ({(0, 0)}) = A for all A ∈ S.

We now show how + operation is performed on S.

$$\text{Let } A = \begin{bmatrix} \{(I_0^5 + 3.2, I_2^8 + I_6^8 + 0.3), \\ (0, 6.2), (2.1, I_0^8 + 0.5)\} \\ \{0, 0.6 + I_4^8), (2.5, I_2^8 + I_0^8)\} \\ \{(4.2 + I_0^5, 2), (3, I_0^8 + 2)\} \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} \{(3.2 + I_0^5, 2 + I_0^8), (4, 2.5)\} \\ \{(I_0^5 + 2, I_6^8 + I_0^8), (1, 0)\} \\ \{(0, 0), (I_0^5, I_6^8 + I_4^8)\} \end{bmatrix} \in S.$$

$$A + B = \begin{bmatrix} \{(1.4 + I_0^5, 2.3 + I_0^8 + I_2^8 + I_6^8), (3.2 + I_0^5, 0.2 + I_0^8), \\ (0.3 + I_0^5, I_0^8 + 2.5), (2.2 + I_0^5, 2.8 + I_2^8 + I_6^8), \\ (4, 0.5), (1.1, 3 + I_0^8)\} \\ \{(I_0^5 + 2, 0.6 + I_0^8 + I_6^8 + I_4^8), (4.5 + I_0^5, \\ I_0^8 + I_6^8 + I_2^8), (1, 0.6 + I_4^8), \\ (3.5, I_2^8 + I_0^8)\} \\ \{(4.2 + I_0^5, 2), (3, I_0^8 + 2), (I_0^5 + 4.2, \\ 2 + I_6^8 + I_4^8), (3 + I_0^5, I_0^8 + I_6^8 + I_4^8 + 2)\} \end{bmatrix}.$$

This is the way + operation is performed on S.

Finding subsemigroups is considered as a matter of routine so left as exercise to the reader.

In view of all these we have the following theorem.

**Theorem 4.14.** *Let M = {collection of all t × s, (2 ≤ t, s < ∞) matrices with entries from* $S(Z_m^I \times Z_n^I)$ *= {collection of all subsets from* $Z_n^I \times Z_m^I$ *= {(a, b) / a ∈* $Z_m^I$*, b ∈* $Z_n^I$*}, m ≠ n, 2 ≤ m, n < ∞}, +} be the MOD rectangular subset natural neutrosophic finite modulo integer matrix semigroup under +.*

*i)* *o(M)< ∞ and M is a commutative monoid.*

*ii)* *M has idempotents with respect to +.*

*iii)* *M is always a Smarandache semigroup.*

*iv)* *M has subsemigroups which are subgroups as well as subsemigroups which are not subgroups.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular natural neutrosophic subset matrix semigroup under $\times_n$ natural product or usual product $\times$ for square matrices by examples.

***Example 4.34***. Let S = $\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$ where $a_i \in S\,(Z_{12}^I \times Z_8^I)$

= {collection of all subsets from $Z_{12}^I \times Z_8^I$ = {(a, b) / a $\in Z_{12}^I$, b $\in Z_8^I$},1 $\leq$ i $\leq$ 6, $\times_n$} be the MOD rectangular natural neutrosophic subset matrix semigroup under the natural product $\times_n$.

Clearly $\begin{bmatrix} \{(1,1)\} & \{(1,1)\} & \{(1,1)\} \\ \{(1,1)\} & \{(1,1)\} & \{(1,1)\} \end{bmatrix} \in$ S acts as the identity with respect to $\times_n$.

$(\{(0,0)\}) \begin{bmatrix} \{(0,0)\} & \{(0,0)\} & \{(0,0)\} \\ \{(0,0)\} & \{(0,0)\} & \{(0,0)\} \end{bmatrix} \in$ S is such that

A $\times$ ({(0,0)}) = ({(0,0)}) for all A $\in$ S.

Let $A = \begin{bmatrix} \{(I_0^{12}+1,0),(1,5)\} & \{(I_6^{12},I_4^{8})\} & \{(2,3)\} \\ \{(1,5+I_2^{8}),(I_0^{12},1)\} & \{(3,I_6^{8})\} & \{(I_8^{12},1)\} \end{bmatrix}$

$$B = \begin{bmatrix} \{(3,4)\} & \{(I_0^{12},1),(3,I_4^{8})\} & \{(I_4^{12}+4,0)\} \\ \{(5,I_6^{8})\} & \{(2,I_2^{8}),(1+I_0^{12},2)\} & \{(0,4)\} \end{bmatrix} \in S.$$

$A \times_n B =$

$$\begin{bmatrix} \{(3+I_0^{12},0),(3,2)\} & \{(I_0^{12},I_4^{8}),(I_6^{12},I_4^{8})\} & \{(8+I_4^{12},0)\} \\ \{(5,I_6^{8}+I_2^{8}),(I_0^{12},I_0^{8})\} & \{(6,I_4^{8}),(3+I_0^{12},I_6^{8})\} & \{(0,4)\} \end{bmatrix} \in S.$$

This is the way $\times_n$ is performed on S. S has matrix zero divisors, matrix nilpotents and matrix idempotents.

However the task of finding ideals and subsemigroups is left as an exercise to the reader.

***Example 4.35***. Let M = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}\right.$ / $a_i \in$ S ( $Z_{20}^I \times Z_{15}^I$ ) = {(a, b) / a $\in$ $Z_{20}^I$, b $\in$ $Z_{15}^I$ },$1 \leq i \leq 8$, $\times_n$} be the MOD rectangular natural neutrosophic subset matrix semigroup.

Find all substructures and special elements associated with M.

***Example 4.36***. Let M = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right.$ / $a_i \in$ S( $Z_{12}^I \times Z_{45}^I$ ) =

{collection of all subsets from $Z_{12}^{I} \times Z_{45}^{I} = \{(a, b) / a \in Z_{12}^{I}, b \in Z_{45}^{I}\}$, $1 \leq i \leq 4 \times$ (or $\times_n$)} be the MOD rectangular natural neutrosophic subset matrix semigroup under $\times$.

Clearly M is a non-commutative semigroup under $\times$.

We will show how $\times$ is defined.

The usual product '$\times$' is applicable only when the matrix under consideration is only a square matrix and here it is $0 \times I_s^t = I_s^t$.

Let $A = \begin{bmatrix} \{(I_6^{12}+3, I_0^{45}), (4,3)\ (0, I_5^{45})\} & \{(3, 4+I_{10}^{45})\} \\ \{(I_0^{12}+4, I_3^{45})\} & \{(4, I_0^{45}+3), (0.5, 0.1)\} \end{bmatrix}$

and $B = \begin{bmatrix} \{(0, I_3^{45}+4)\} & \{(2, I_0^{45}), (I_0^{12}+3, 6)\} \\ \{(1+I_4^{12}, 0), (1,5), (0.2, 0.3)\} & \{(0,0)\} \end{bmatrix}$.

We now find $A \times B$.

$A \times B =$

$$\begin{bmatrix} \{(I_6^{12}, I_0^{45}), (0, 12 + I_3^{45}), & \{(6 + I_6^{12}, I_0^{45}), (I_6^{12} + I_0^{12} + \\ (0, I_{15}^{45} + I_5^{45})\} + \{(3 + I_4^{12}, 0), & 6, I_0^{45}), (8, I_0^{45}), (I_0^{12}, 18), (0, I_0^{45}), \\ (3,\ 20 + I_{10}^{45}),\ (0.6, 1.2 + I_0^{45})\} & (I_0^{12}, I_5^{45})\} + \{(0,0)\} \\ \{(I_0^{12}, I_9^{45} + I_3^{45}), (4 + I_4^{12}, I_0^{45}), & \{(I_0^{12} + 8, I_0^{45}), \\ (4, 15 + I_0^{45}), (0.8, 0.9 + I_0^{45}), & (I_0^{12}, I_3^{45})\} \\ (4, 15 + I_0^{45}), (0.5, 0.5), (0.8, & +\{(0, I_0^{45}) + \\ 0.9 + I_0^{45}), (0.1, 0.03)\} & (0,0)\} \end{bmatrix}.$$

The formal form of A × B is obtained after +.

It is left for the reader to verify that in general A × B ≠ B × A.

Further A $\times_n$ B ≠ A × B as both the products behave in a very different way.

In view of all these we have the following theorem.

**Theorem 4.15.** *Let S = {Collection of all s × t matrices with entries from* $S(Z_m^I \times Z_n^I)$ *= {Collection of all subsets from* $Z_m^I \times Z_n^I$ *= {(a, b) / a* $\in Z_m^I$*, b* $\in Z_n^I$*, m ≠ n, 2 ≤ m, n < ∞},* $\times_n$*} be the MOD rectangular natural neutrosophic subset matrix semigroup under* $\times_n$*.*

*i) o(S) < ∞ and is a commutative monoid.*

*ii) S has zero divisors.*

*iii) S has nontrivial idempotents and nilpotents only for special values of m and n.*

*iv) S has subsemigroups which are not ideals.*

*v) S has ideals.*

The proof is left as an exercise to the reader.

Next we introduce the new notion of MOD rectangular natural neutrosophic matrix subset semigroups under + by some examples.

***Example 4.37***. Let S(M) = {Collection of all subsets from

$$M = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} / a_i \in Z_{10}^I \times Z_{16}^I = \{(a, b) / a \in Z_{10}^I, b \in Z_{16}^I, 1 \leq i$$

$\leq$ 4, +}, +} be the MOD rectangular natural neutrosophic matrix subset semigroup under +.

$$\text{Let } A = \{\begin{bmatrix} (0,0.3) \\ (I_5^{10},1) \\ (0.5,0.8+I_8^{16}) \\ (1,5) \end{bmatrix} \begin{bmatrix} (1,0.5) \\ (1,I_2^{16}) \\ (0.3,I_0^{16}) \\ (I_2^{10},I_2^{16}) \end{bmatrix} \begin{bmatrix} (0,0) \\ (1,1) \\ (2,0.3) \\ (0.5,0.3) \end{bmatrix}\} \text{ and}$$

$$B = \{\begin{bmatrix} (1,I_0^{16}) \\ (0,1) \\ (1,0) \\ (I_0^{10},0) \end{bmatrix} \begin{bmatrix} (0,0.3) \\ (I_5^{10},I_8^{16}) \\ (0,2) \\ (1,I_8^{16}) \end{bmatrix}\} \in M.$$

$$A + B = \{\begin{bmatrix} (1,I_0^{16}+0.3) \\ (I_5^{10},2) \\ (1.5,0.8+I_8^{16}) \\ (1+I_0^{10},5) \end{bmatrix}, \begin{bmatrix} (0,0.6) \\ (I_5^{10},1+I_8^{16}) \\ (0.5,2.8+I_8^{16}) \\ (2,5+I_8^{16}) \end{bmatrix}, \begin{bmatrix} (2,0.5+I_0^{16}) \\ (1,1+I_2^{16}) \\ (1.3,I_0^{16}) \\ (I_0^{10}+I_2^{10},I_2^{16}) \end{bmatrix}$$

$$\begin{bmatrix} (1,0.8) \\ (1+I_5^{10}, I_8^{16}+I_2^{16}) \\ (0.3, 2+I_0^{16}) \\ (I_2^{16}+1, I_2^{16}+I_8^{16}) \end{bmatrix}, \begin{bmatrix} (1,I_0^{16}) \\ (1,2) \\ (3,0.3) \\ (0.5+I_0^{16}, 0.3) \end{bmatrix}, \begin{bmatrix} (0,0.3) \\ (1+I_5^{10}, 1+I_8^{16}) \\ (2,2.3) \\ (1.5, 0.3+I_8^{16}) \end{bmatrix}\} \in M.$$

This is the way + operation is performed on M. M is a commutative monoid of finite order.

In fact M is a Smarandache semigroup.

M has several subsemigroup some of which are group.

***Example 4.38***. Let S(W) = {(collection of all matrix subsets

from W = $\{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$ / $a_i \in Z_{40}^I \times Z_{12}^I$ = {(a, b) / a $\in Z_{40}^I$, b

$\in Z_{12}^I$, 1 ≤ i ≤ 6, +}, +} be the MOD rectangular natural neutrosophic matrix subset semigroup under +.

$\{\begin{pmatrix} (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{pmatrix}\}$ = {(0, 0)} in S(W) is such that

A + {(0, 0)} = A for all A ∈ S(W).

We see if A = $\{\begin{pmatrix} (0, I_8^{12}+1) & (I_8^{40}+5, 0.3) & (0,2) \\ (1, I_6^{12}+3) & (0,5,0.8) & (1.32, I_6^{12}) \end{pmatrix}$,

$\begin{pmatrix} (1,1) & (3,2) & (0.4+I_6^{12}, I_5^{40}+I_{10}^{40}+1) \\ (0,3) & (0,0) & (1+I_3^{12}, 5+I_8^{40}+I_0^{40}) \end{pmatrix}\}$ and

B = $\{\begin{pmatrix} (1,2.5) & (I_5^{40}+3, I_0^{12}) & (5,7) \\ (0,1) & (5, I_6^{12}+I_0^{12}) & (1,3) \end{pmatrix}\} \in$ S(W).

$$A + B = \{\begin{pmatrix} (1,3.5+I_8^{12}) & (8+I_8^{40}+I_5^{40},0.3+I_0^{12}) & (5,9) \\ (1,4+I_6^{12}) & (5.5,\ 0.8+I_6^{12}+I_0^{12}) & (2.32,3+I_6^{12}) \end{pmatrix},$$

$$\begin{pmatrix} (2,3.5) & (6+I_5^{40},2+I_0^{12}) & (5.4+I_6^{12},8+I_5^{40}+I_{10}^{40}) \\ (0,4) & (5,I_6^{12}+I_0^{12}) & (2+I_3^{12},8+I_8^{40}+I_0^{40}) \end{pmatrix} \in S(W).$$

This is the way '+' operation is performed on S(W).

The reader is expected to find substructures of S(W).

In view of all these we have the following theorem.

**Theorem 4.16.** *Let S(B) = {collection of matrix subsets from B = {collection of all s × t matrices with entries from $Z_m^I \times Z_n^I$ = {(a, b) / a ∈ [0, m) and b ∈ [0, n), +, 2 ≤ s, t < ∞, m ≠ n; 2 ≤ m, n < ∞}, +} be the MOD rectangular natural neutrosophic matrix subset semigroup under +.*

- *i) S(B) is a finite commutative monoid.*
- *ii) S(B) has idempotents.*
- *iii) S(B) has subsemigroups which are groups.*
- *iv) S(B) has subsemigroups which are not groups.*
- *v) S(B) is a Smarandache semigroup.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular natural neutrosophic matrix subset semigroup under $\times_n$ (or ×) by some examples.

***Example 4.39***. Let S(M) = {Collection of all matrix subsets from M = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}\right.$ / $a_i \in Z_{10}^I \times Z_6^I$ = {(a, b) / a $\in Z_{10}^I$, b $\in Z_6^I$, 1 ≤ i ≤ 4, $\times_n$}, $\times_n$} be the MOD rectangular natural neutrosophic matrix subset semigroup under the natural product $\times_n$.

Clearly $\left\{\begin{bmatrix} (1,1) \\ (1,1) \\ (1,1) \\ (1,1) \end{bmatrix}\right\}$ = {((1, 1))} in S(M) acts as the multiplicative identity so for all A ∈ S(M). A × {[(1,1)]} = A.

$$\text{Let } P = \left\{\begin{bmatrix} (I_2^{10},5) \\ (0, I_3^6+2) \\ (2, I_2^6+I_0^6) \\ (0,1) \end{bmatrix}, \begin{bmatrix} (1,1) \\ (2,2) \\ (I_5^{10}, I_0^6) \\ (0,1) \end{bmatrix} \begin{bmatrix} (1, I_0^6) \\ (0,0) \\ (1,1) \\ (I_0^{10},0) \end{bmatrix}\right\}$$

$$\text{and } Q = \left\{\begin{bmatrix} (3+I_2^{10}, 4+I_2^6) \\ (0,1) \\ (4+I_5^{10}, 0.3+I_4^6) \\ (1,0) \end{bmatrix}, \begin{bmatrix} (1,0.2) \\ (3,0.5) \\ (0.6, I_2^6) \\ (I_2^{10}+0.2,1) \end{bmatrix}\right\} \in S(M).$$

$$P \times_n Q = \left\{\begin{bmatrix} (I_2^{10}+I_4^{10}, 2+I_2^6) \\ (0, I_3^6+2) \\ (8+I_5^{10}, I_2^6+I_0^6+I_8^6) \\ (0,0) \end{bmatrix}, \begin{bmatrix} (3+I_2^{10}, 4+I_2^6) \\ (0,2) \\ (I_5^{10}, I_0^6) \\ (0,0) \end{bmatrix},\right.$$

$$\begin{bmatrix} (3+I_2^{10}, I_0^6) \\ (0,0) \\ (4+I_5^{10}, 0.3+I_4^6) \\ (I_0^{10}, 0) \end{bmatrix}, \begin{bmatrix} (I_2^{10}, 1) \\ (0, 1+I_3^6 \\ (1.2, I_4^6+I_0^6) \\ (I_2^{10}, 1) \end{bmatrix} \begin{bmatrix} (1, 0.2) \\ (6,1) \\ (I_5^{10}, I_0^6) \\ (I_2^{10}, 1) \end{bmatrix}, \begin{bmatrix} (1, I_0^6) \\ (0,0) \\ (0.6, I_2^6) \\ (I_0^{10}, 0) \end{bmatrix} \} \in$$

S(M).

This is the way product operation is performed on S(M). Further S(M) has zero divisor matrix subsets.

S(M) also has idempotent matrix subsets.

However in this case finding nontrivial nilpotent matrix subsets happens to be a very difficult problem.

The task of finding subsemigroups and ideals is left as an exercise to the reader.

***Example 4.40***. Let S(B) = {Collection of all matrix subsets

from B = $\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix}$ / $a_i \in Z_{15}^I \times Z_{24}^I$ = {(a, b) / a $\in Z_{15}^I$,

b $\in Z_{24}^I$ = {(a, b) / a $\in Z_{15}^I \times Z_{24}^I$ = {(a, b) / a $\in Z_{15}^I$, b $\in Z_{24}^I$, $1 \leq i \leq 8$}, $\times_n$} be the MOD rectangular natural neutrosophic matrix subset semigroup under the natural product $\times_n$.

Clearly o(S(B) < ∞ and S(B) has matrix subset zero divisors and matrix subset idempotents.

The task of finding substructures like ideals and subsemigroups is left as an exercise to the reader.

***Example 4.41***. Let S(V) = {Collection of matrix subsets from

$$V = \{\begin{pmatrix} a_1 & a_2 & a_3 \\ (0,0) & a_4 & a_5 \\ (0,0) & (0,0) & a_6 \end{pmatrix} / a_i \in Z_4^I \times Z_7^I = \{(a, b) / a \in Z_4^I,$$

$b \in Z_7^I$, $1 \leq i \leq 6$, $\times\}$, $\times\}$ be the MOD rectangular natural neutrosophic matrix subset semigroup under the usual product of $\times$.

Clearly S(V) is a non-commutative monoid of finite order.

The reader is left with the task of finding right zero divisors which are not left zero divisors and vice versa.

Further the reader is expected to find right ideal which are not left ideals and vice versa.

Will S(V) be a Smarandache semigroup?

In view of all these we have the following theorem.

**Theorem 4.17**. *Let S(M) = {Collection of all matrix subsets from M = {Collection of all s × t matrices ($2 \leq s, t < \infty$) with entries from $Z_n^I \times Z_m^I = \{(a, b) / a \in Z_n^I$ and $b \in Z_m^I$; $m \neq n$, $2 \leq m, n < \infty\}$; $\times_n$ (or $\times$)}, $\times_n$ (or $\times$)} be the MOD rectangular natural neutrosophic matrix subset semigroup under $\times_n$ (or $\times$) (the usual product, $\times$ is defined only if t = s).*

*i)* *$o(S(M)) < \infty$ and is a commutative monoid.*

*ii)* *S(M) has subset zero divisors for all m, n, $2 \leq m, n < \infty$.*

*iii)* *S(M) has nontrivial nilpotents and idempotents only for appropriate m and n.*

*iv)* *S(M) has subsemigroups which are not ideals.*

*v)* *S(M) has ideals.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular natural neutrosophic polynomial subsets semigroup under addition +, by some examples.

***Example 4.42***. Let S(R[x]) = {collection of all polynomial subsets from R[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in (Z_{10}^I \times Z_{23}^I)$ = {(a, b) / a $\in Z_{10}^I$, b $\in Z_{23}^I$, +}, +} be the MOD rectangular natural neutrosophic polynomial subset semigroup under $\times_0$ (or $\times$).

Let A = {(0.3, $I_0^{23}$ + 4)$x^7$ + (5, 3)$x^4$ + ($I_2^{10}$ + $I_5^{10}$ + 3, 0), (6.3 + $I_2^{10}$, $I_0^{23}$) + (0, 2)$x^2$} and B = {($I_5^{10}$, 2.4)$x^2$ + (1, 0.5 + $I_0^{23}$)} $\in$ S(R[x]).

A + B = {(0.3, $I_0^{23}$+4)$x^7$ + (5, 3)$x^4$ + ($I_5^{10}$, 2.4) $x^2$ + ($I_2^{10}$ + $I_5^{10}$+4, 0.5+$I_0^{23}$), ($I_5^{10}$, 4.4)$x^2$ + (7.3 + $I_2^{10}$, 0.5 + $I_0^{23}$)} $\in$ S(R[x]). This is the way the operation + is performed on S(R[x]).

S(R[x]) has subsemigroups.

In fact S(R[x]) has idempotent polynomial subsets under +.

Let {(0, 0)}= {(0,0) + (0,0)x + …+ (0, 0)$x^n$} acts as the additive identity of every polynomial subset A of S(R[x]). Several properties associated with S(R[x]) is a matter of routine so left as an exercise to the reader.

***Example 4.43***. Let S(W[x]) = {Collection of all polynomial subsets from W[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_{19}^I \times Z_{43}^I)$ = {(a, b) / a $\in Z_{19}^I$, b $\in Z_{43}^I$, +}, +} be the MOD rectangular polynomial subset natural neutrosophic semigroup under +.

o(S(W[x]) = ∞. In fact S(W[x]) has subsemigroups of finite order as well as of infinite order.

S(P[x]) = {collection of all polynomial subsets from P[x] = $\{\sum_{i=0}^{5} a_i x^i$ / $a_i \in (Z_{10}^I \times Z_{23}^I)$ = {(a, b) / a $\in Z_{10}^I$, b $\in Z_{23}^I$, +}, +} ⊆ S(W[x]) is a MOD rectangular natural neutrosophic polynomial subset subsemigroup of finite order.

In fact S(W[x]) has infinitely many subsemigroups of finite order.

Let S(B[x]) = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in \{0\} \times Z_{43}^I$, +}, +} ⊆ S(W[x]) is a subsemigroup of infinite order. S(W[x]) has several such subsemigroups of infinite order.

In view of all these we have the following theorem.

**Theorem 4.18**. *Let S(B[x]) = {collection of all polynomial subsets from B[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_m^I \times Z_n^I)$ = {(a, b) / a $\in Z_m^I$, b $\in Z_n^I$, m ≠ n, 2 ≤ m, n < ∞, +}, +} be the MOD rectangular natural neutrosophic polynomial subset semigroup under +.*

i) *S(B[x]) is an infinite commutative monoid under +.*

ii) *S(B[x]) has subsets which are idempotents under +.*

*iii)* *S(B[x]) has subsemigroups of finite order which are infinite in number.*

*iv)* *S(B[x]) has subsemigroups of infinite order.*

Proof is left as an exercise to the reader.

Next we proceed onto describe MOD rectangular natural neutrosophic polynomial subset semigroups under × operation by some examples.

***Example 4.44.*** Let S(V[x]) = {Collection of all polynomial subsets from V[x] = { $\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_4^I \times Z_9^I$ = {(a, b) / a $\in Z_4^I$, b $\in Z_9^I$, +}, +} be the MOD rectangular natural neutrosophic polynomial subset semigroup under ×. S(V[x]) has infinite number of zero divisors.

No subsemigroup is of finite order. All subsemigroups and ideals are of infinite order.

Finding nontrivial idempotents is an impossibility.

However S(V[x]) has nontrivial nilpotents. Finding all these is a matter of routine so left as an exercise to the reader.

Let A= {(0, 3)$x^3$ + ( $I_2^4 + I_0^4$, $I_0^9$), (0, 6.8) + (0.3, 0)$x^2$ + ( $I_0^4$, $I_0^9$ + 1)x}, +} and B = {(0, 1) + ( $I_0^4$, 2)$x^2$, (1, 0)x + ( $I_2^4$ + 2, $I_3^9$)} $\in$ S(V[x]).

A × B = {(0, 3)$x^3$ + ( $I_0^4 + I_2^4$, $I_0^9$) + ( $I_0^4$, 6)$x^5$ + ( $I_0^4$, $I_0^9$)$x^2$, (0, 6.8) + ( $I_0^4$, $I_0^9$+1) x + ( $I_0^4$, 4.6)$x^2$ + ( $I_0^4$, 0)$x^4$ + ( $I_0^4$, $I_0^9$+2)$x^3$, (0, 0)$x^4$ + ( $I_0^4$ + $I_0^4$, $I_0^9$+ 2)$x^3$, (0, 0)$x^4$ + ( $I_0^4$ + $I_0^4$, $I_0^9$)x + ( $I_2^4$, $I_3^9$)$x^3$ + ( $I_0^4 + I_2^4$, $I_0^9$), (0, 0)x + (0.3, 0)$x^3$ + ( $I_2^4$, $I_3^9$) + ( $I_2^4$ + 0.6, $I_3^9$)$x^2$+ ( $I_0^4$, $I_0^9$)$x^2$ + ( $I_0^4$, $I_3^9 + I_0^9$)x}.

This is the way product operation is performed on S(V[x]).

It is left as an exercise for the reader to prove there are infinite number of zero divisors and all subsemigroups are of infinite order.

***Example 4.45***. Let S(B[x]) = {collection of all polynomial subsets from B[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_{12}^I \times Z_7^I$ = {(a, b) / a $\in Z_{12}^I$, b $\in Z_7^I$ = {(a, b) / a $\in Z_{12}^I$, b $\in Z_7^I$, ×}, ×} be the MOD rectangular natural neutrosophic subset polynomial semigroup under product ×. S(B[x]) has infinite number of polynomial subset zero divisors.

Further S(B[x]) has no finite order subsemigroups all subsemigroups of S(B[x]) are of infinite order. In fact S(B[x]) has no idempotents.

Finding special elements and substructures of S(B[x]) is a matter of routine so left as an exercise to the reader.

In view of all these we have the following theorem.

**Theorem 4.19**. *Let S(B[x]) = {collection of all polynomial subsets from B[x] =* $\{\sum_{i=0}^{\infty} a_i x^i$ */* $a_i \in Z_m^I \times Z_n^I$ *= {(a, b) / a* $\in Z_m^I$*, b* $\in Z_n^I$*,* $m \neq n$*,* $2 \leq m, n < \infty$*, ×}, ×} be the MOD rectangular natural neutrosophic polynomial subset semigroup under ×.*

- *i) S(B[x]) is an infinite commutative monoid.*
- *ii) S(B[x]) has infinite number of zero divisors.*
- *iii) S(B[x]) has nilpotents only for appropriate m and n.*
- *iv) SB([x]) has no idempotents.*

*v)* *All subsemigroups of S(B[x]) are of infinite order.*

*vi)* *S(B[x]) has ideals all of which are of infinite order.*

Proof is direct and hence left as an exercise to the reader.

Next consider the MOD rectangular natural neutrosophic finite degree polynomial subset semigroup under + given by the following example.

***Example 4.46.*** Let $S(P[x]_9)$ = {collection of all polynomial subsets from $P[x]_9 = \{\sum_{i=0}^{9} a_i x^i / a_i \in Z^I_{12} \times Z^I_{45} = \{(a, b) / a \in Z^I_{12}$ , $b \in Z^I_{45}$, $x^{10} = 1, \times\}, \times\}$ be the MOD rectangular natural neutrosophic finite degree polynomial subset semigroup under +.

$o(S(P[x]_9)) < \infty$ and is a commutative monoid.

$A = \{(3, 0.5 + I^{45}_0)x^6 + (2.5 + I^{12}_6, 0), (5 + I^{12}_0 + I^{12}_3, 1)x^2 + (0.3, 4.5)\}$ and $B = \{(2.5 + I^{12}_8, 3 + I^{45}_5)x^6 + (0.73, 2)x^2 + (0.89 + I^{12}_2, 0.5 + I^{45}_{40})\} \in S(P[x]_9)$. We find A + B,

$A + B = \{(5.5 + I^{12}_8, 3.5 + I^{45}_0 + I^{45}_5)x^6 + (0.73, 2) x^2 + (3.39 + I^{12}_6 + I^{12}_2, 0.5 + I^{45}_{40}), (2.5 + I^{12}_8, 3 + I^{45}_5)x^6 + (5.73 + I^{12}_0 + I^{12}_3, 3)x^2 + (1.19 + I^{12}_2, 5 + I^{45}_{40})\} \in S(P[x]_9)$.

This is the way '+' operation is performed on $S(P[x]_9)$.

In fact $S(P[x]_9)$ has idempotents with respect to +.

Further $S(P[x]_9)$ has subsemigroups all of which are of finite order.

Finding these are left as an exercise to the reader.

***Example 4.47***. Let S(V[x]$_{17}$) = {collection of all polynomial subsets from V[x]$_{17}$ = { $\sum_{i=0}^{17} a_i x^i$ / $a_i \in Z_{11}^I \times Z_3^I$ = {(a, b) / a $\in Z_{11}^I$, b $\in Z_3^I$, $x^{18}$ = 1, ×}, ×} be the MOD rectangular natural neutrosophic finite degree polynomial subset semigroup under +.

o(S(V[x]$_{19}$)) < ∞ and S(V[x]$_{19}$) has subsemigroups and idempotents with respect to +.

All these is a matter of routine so left as an exercise to the reader.

In view of all these we have the following result.

**Theorem 4.20**. *Let S(B[x]$_q$) = {collection of all polynomial subsets from B[x]$_q$ = {$\sum_{i=0}^{q} a_i x^i$ / 0 ≤ q < ∞, $x^{q+1}$ = 1, $a_i \in Z_m^I \times Z_n^I$ = {(a, b) / a $\in Z_m^I$, b $\in Z_n^I$, m ≠ n, 2 ≤ m, n < ∞, ×}, +} be MOD rectangular natural neutrosophic finite degree polynomial subset semigroup under +.*

*i) o(S(B[x]$_q$)) < ∞ and S(B[x]$_q$) is a finite commutative monoid.*

*ii) S(B[x]$_q$) has idempotents with respect to +.*

*iii) S(B[x]$_q$) has subsemigroups.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular finite degree polynomial natural neutrosophic subset semigroup under product by some examples.

***Example 4.48***. Let $S(R[x]_9)$ = {collection of all polynomial subsets from $R[x]_9 = \{\sum_{i=0}^{9} a_i x^i \,/\, a_i \in Z_8^I \times Z_{16}^I = \{(a, b) \,/\, a \in Z_8^I$, $b \in Z_{16}^I$, $x^{10} = 1, \times\}, \times\}$ be the MOD rectangular natural neutrosophic finite polynomial subset semigroup under ×.

Clearly $o(S(R[x]_9) < \infty$ is a commutative monoid $S(R[x]_9$ has several zero divisors. In fact $S(R[x]_9)$ has no idempotents but has nilpotents which are nontrivial.

$S(R[x]_9)$ has subsemigroups which are not ideals. But $S(R[x]_9)$ also has ideals.

Study in this direction is a matter of routine so left as an exercise to the reader.

***Example 4.49***. Let $S(M[x]_4)$ = {collection of all subsets from $M[x]_4 = \{\sum_{i=0}^{4} a_i x^i \,/\, a_i \in (Z_{10}^I \times Z_{18}^I) = \{(a, b) \,/\, a \in Z_{10}^I$, $b \in Z_{18}^I$, $x^5 = 1, \times\}, \times\}$ be the MOD rectangular natural neutrosophic finite degree polynomial subset semigroup under ×. This has several zero divisors, no nontrivial idempotents or nilpotents.

$S(M[x]_4)$ has both subsemigroups which are not ideals as well as which are ideals.

This study is also a matter of routine so left as an exercise to the reader.

We have the following result.

**Theorem 4.21**. *Let $S(B[x]_q)$ = {collection of all finite degree polynomial subsets from $B[x]_q = \{\sum_{i=0}^{q} a_i x^i \,/\, a_i \in (Z_m^I \times Z_n^I) = \{(a, b) \,/\, a \in Z_m^I$, $b \in Z_n^I$, $m \neq n$, $2 \leq m, n < \infty$, $x^{q+1} = 1, \times\}, \times\}$*

*be the MOD rectangular natural neutrosophic finite degree polynomial subset semigroup under ×.*

i) *$S(B[x]_q)$ is a finite commutative monoid.*

ii) *$S(B[x]_q$ has zero divisors.*

iii) *$S(B[x]_q)$ has no nontrivial idempotents.*

iv) *$S(B[x]_q)$ has nontrivial nilpotent subsets only for appropriate m and n.*

v) *$S(B[x]_q)$ has subsemigroups which are not ideals.*

vi) *$S(B[x]_q)$ has ideals.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under + by some examples.

***Example 4.50.*** Let $P[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{10}^I \times Z_{24}^I)$ = {Collection of all subsets from $Z_{10}^I \times Z_{24}^I$, = {(a, b) / a ∈ $Z_{10}^I$, b ∈ $Z_{24}^I$, +}, +} be the MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under +.

$p(x) = \{(0, 2),(I_5^{10}, 2.5), (1, I_{12}^{24})\} x^2 + \{(1, 0), (2, I_0^{24})\}$ and $q(x) = \{(I_2^{10}, 1), (0.5, 5.2)\}x^2 + \{(I_5^{10} + 4, 3 + I_2^{24})\ (3, 10)\} \in P[x]$.

$p(x) + q(x) = \{(I_2^{10}, 3), (I_5^{10} + I_2^{10}, 3.5), (1 + I_2^{10}, 1 + I_{12}^{24}), (0.5, 7.2), (0.5 + I_5^{10}, 7.7), (1.5, I_{12}^{24} + 5.2)\}x^2 + \{(5 + I_5^{10}, 3 + I_2^{24}), (6 + I_5^{10}, I_5^{24} + I_2^{24} + 3), (4,10), (5,10 + I_0^{24})\} \in P[x]$. This is the way + operation is performed on P[x].

$\{(0, 0)\} + \{(0, 0)\}x + \ldots + \{(0, 0)\}x^n = \{(0, 0)\}$ is the additive identity in P[x].

In fact P[x] has idempotents under + also has subsemigroups of both finite and infinite order.

***Example 4.51***. Let R[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{20}^I \times Z_5^I)$ = {Collection of all subsets from $Z_{20}^I \times Z_5^I$, = {(a, b) / a $\in Z_{20}^I$, b $\in Z_5^I$, +}, +} be the MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under +.

R[x] has subsemigroups of both finite and infinite order. R[x] has idempotents under +.

In view of all these we have the following theorem.

**Theorem 4.22**. *Let V[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_m^I \times Z_n^I)$ = {Collection of all subsets from $Z_m^I \times Z_n^I$, = {(a, b) / a $\in Z_m^I$, b $\in Z_n^I$, +; $m \neq n$; $2 \leq m, n < \infty$}, +} be the MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under +.*

i) *V[x] is a commutative monoid of infinite order.*

ii) *V[x] has nontrivial idempotents.*

iii) *V[x] has subsemigroups of finite order which are infinite in number.*

iv) *V[x] has subsemigroups of infinite order.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto describe by examples MOD rectangular natural neutrosophic subset coefficient polynomial semigroups under product $\times_0$ (or $\times$) in the following.

**Example 4.52**. Let S[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{12}^I \times Z_7^I)$ = {Collection of all subsets from $Z_{12}^I \times Z_7^I$, = {(a, b) / a $\in Z_{12}^I$, b $\in Z_7^I$, $\times$}, $\times$} be the MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under product.

S[x] has infinite number of zero divisors. S[x] has subsemigroups which are not ideals.

All subsemigroups and ideals of S[x] are of infinite order. S[x] has no idempotent.

In this S[x] has nilpotents of the form p(x) = {(6, 0)}$x^n$ + {(6, 0)} and nothing other than these. All coefficients are only the subsets from {(6, 0)}.

M[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(\{0\} \times Z_7^I)$ = {Collection of all subsets from {0} $\times$ $Z_7^I$, = {(0, b) / b $\in Z_7^I$}, $\times$}, $\times$} $\subseteq$ S[x] is a subsemigroup which is also an ideal of S[x].

Clearly o(M[x]) = $\infty$.

Let T[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{12} \times Z_7)$ = {Collection of all subsets from $Z_{12} \times Z_7$ = {(a, b) / a $\in Z_{12}$, b $\in Z_7$, $\times$}, $\times$}, $\times$} be the MOD rectangular subset coefficient polynomial subsemigroup of S[x].

Clearly T[x] is only a subsemigroup and not an ideal of S[x].

***Example 4.53***. Let M[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{40}^I \times Z_{21}^I)$ = {Collection of all subsets from $Z_{40}^I \times Z_{21}^I$, = {(a, b) / a $\in Z_{40}^I$, b $\in Z_{21}^I$, ×}, ×}, ×} be the MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under product.

This has infinite number of zero divisors no idempotents. However both ideals and subsemigroups of M[x] are of infinite order only.

In view of all these we have the following result.

**Theorem 4.23**. *Let B[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_n^I \times Z_m^I)$ = {Collection of all subsets from $Z_n^I \times Z_m^I$, = {(a, b) / a $\in Z_n^I$, b $\in Z_m^I$, $m \neq n$, $2 \leq m$, $n < \infty$}, ×}} be the MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under ×.*

i) *B[x] is an infinite commutative monoid.*
ii) *B[x] has infinite number of zero divisors.*
iii) *B[x] has no idempotents.*
iv) *Both ideals and subsemigroups which are not ideals are of infinite order.*
v) *Only for specific values of m and n B[x] has nilpotents.*

The proof is left as an exercise to the reader.

Next we proceed onto describe by examples MOD rectangular natural neutrosophic subset coefficient finite degree polynomials semigroup under +.

***Example 4.54***. Let $B[x]_7 = \{\sum_{i=0}^{7} a_i x^i$ / $a_i \in S(Z_9^I \times Z_{12}^I) =$ {Collection of all subsets from $Z_9^I \times Z_{12}^I$, = {(a, b) / a $\in Z_9^I$, b $\in Z_{12}^I$, +}, +} be the MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroup under +.

Let p(x) = $\{(I_3^9, 0.5), (0, I_3^{12} + 4.2), (1, 4.2)\}x^2 + \{(0, 1), (1, 25)\}$ and q(x) = $\{(4, I_8^{12}), (3 + I_6^9, 0)\}x^2 + \{(2, 4 + I_0^{12}), (0.3, 0.72)\}x + \{(4 + I_3^9 + I_0^9, 3 + I_0^{12} + I_{10}^{12}), (2, 0.3)\} \in B[x]_7$.

We find p(x) + q(x), p(x) + q(x) = $\{(4 + I_3^9, 0.5 + I_8^{12}), (3 + I_6^9, I_3^9, 0.5), (4, I_3^{12} + I_8^{12} + 4), (3 + I_6^9, I_3^{12} + 4), (5, 4.2 + I_8^{12}), (4 + I_6^9, 4.2)\}x^2 + \{(2, 4 + I_0^{12}), (0.3, 0.72)\}x + \{(4 + I_3^9 + I_0^9, 4 + I_2^{12} + I_{10}^{12}), (5 + I_3^9 + I_0^9, 5.5 + I_2^{12} + I_{10}^{12}), (2, 1.3), (3, 2.8)\} \in B[x]_7$.

This is the way + operation is performed on $B[x]_7$.

In fact $B[x]_7$ is a finite commutative monoid under +.

Further $B[x]_7$ is a Smarandache semigroup, $B[x]_7$ has idempotents with respect to +.

**Example 4.55**. Let $F[x]_{12} = \{\sum_{i=0}^{12} a_i x^i$ / $a_i \in S(Z_{17}^I \times Z_{43}^I) =$ {Collection of all subsets from $Z_{17}^I \times Z_{43}^I$, = {(a, b) / a $\in Z_{17}^I$, b $\in Z_{43}^I$, +}, +}, +; $x^{13} = 1$} be the MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroup under +.

The task of finding subsemigroups idempotent polynomials are left as an exercise to the reader.

Find $o(F[x]_{12})$.

It is left as an open conjecture of finding $F[x]_m$.

**Conjecture 4.1** *Let* $F[x]_m = \{\sum_{i=0}^{m} a_i x^i$ / $a_i \in S(Z_s^I \times Z_t^I)$ = *{Collection of all subsets from* $Z_s^I \times Z_t^I$, = $\{(a, b)$ / $a \in Z_s^I$, $b \in Z_t^I$ }, $x^{m+1} = 1$, $1 \leq m < \infty$, $s \neq t$, $2 \leq s, t < \infty\}$ *be the MOD rectangular natural neutrosophic subset coefficient finite degree polynomial set.*

*What is the* $o(F[x]_m)$?

We have the following result.

**Theorem 4.24**. *Let* $B[x]_s = \{\sum_{i=0}^{s} a_i x^i$ / $a_i \in S(Z_m^I \times Z_n^I)$ = *{Collection of all subsets from* $Z_m^I \times Z_n^I$, = $\{(a, b)$ / $a \in Z_m^I$, $b \in Z_n^I$, +}, +}, $x^{s+1} = 1$, $1 \leq s < \infty$, $m \neq n$, $2 \leq m, n < \infty$, +} *be the MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroup under +.*

i) *$B[x]_s$ is a commutative monoid of finite order*

ii) *$B[x]_s$ has idempotent subset polynomials.*

iii) *$B[x]_s$ is a Smarandache semigroup.*

iv) *$B[x]_s$ has subsemigroups some of which are groups.*

v) *$B[x]_s$ has subsemigroups which are not groups.*

Next we proceed onto describe MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroup under product $\times_0$ (or $\times$) by some examples.

***Example 4.56***. Let $T[x]_8 = \{\sum_{i=0}^{8} a_i x^i$ / $a_i \in S(Z_{10}^I \times Z_{14}^I) =$ {Collection of all subsets from $Z_{10}^I \times Z_{14}^I$, = {(a, b) / a $\in Z_{10}^I$, b $\in Z_{14}^I$, ×}, ×}, $x^9 = 1$, ×} be the MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroup under product.

$T[x]_8$ is a finite commutative monoid.

This has zero divisors and has no nontrivial idempotents.

$o[T[x]_8) < \infty$. Further $T[x]_8$ has subsemigroups which are ideals.

$T[x]_8$ also has subsemigroups which are not ideals.

However finding nontrivial subgroups under × in $T[x]_8$ is an impossibility.

Interested reader can work with substructures and special elements in them.

***Example 4.57***. Let $B[x]_{18} = \{\sum_{i=0}^{18} a_i x^i$ / $a_i \in S(Z_{27}^I \times Z_{32}^I) =$ {Collection of all subsets from $Z_{27}^I \times Z_{32}^I$, = {(a, b) / a $\in Z_{27}^I$, b $\in Z_{32}^I$}, ×}, $x^{19} = 1$, ×} be the MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroup under ×.

This has infinite number of zero divisors.

$P[x]_{18} = \{\sum_{i=0}^{18} a_i x^i$ / $a_i \in S(\{0\} \times Z_{32}^I) =$ {Collection of all subsets from $\{0\} \times Z_{32}^I$, = {(0, b) / b $\in Z_{32}^I$, ×}, ×}, $\subseteq B[x]_{18}$ is S-subsemigroup of $B[x]_{18}$.

$$R[x]_{18} = \{\sum_{i=0}^{18} a_i x^i \,/\, a_i \in S(Z_{27}^I \times \{0\}) = \{(a, 0) \,/\, a \in Z_{27}^I,$$

$\times\}$, $\subseteq B[x]_{18}$ is also a subsemigroup of $B[x]_{18}$.

Let $p(x) \in P[x]_{18}$ and $q(x) \in R[x]_{18}$;

clearly $p(x) \times q(x) = \{(0, 0)\}$ is a zero divisor.

In fact $P[x]_{18} \times R[x]_{18} = \{(0,0)\}$.

$B[x]_{18}$ has several zero divisors. If the product is a usual zero dominated semigroup then $P[x]_{18}$ and $R[x]_{18}$ are ideals.

Otherwise they are not ideal under the natural neutrosophic zero dominated product.

Study in this direction is important and interesting.

In view of all these we have the following the theorem.

**Theorem 4.25**. *Let $S[x]_m = \{\sum_{i=0}^{m} a_i x^i \,/\, a_i \in S(Z_s^I \times Z_t^I) =$ {Collection of all subsets from $Z_s^I \times Z_t^I$, $= \{(a, b) \,/\, a \in Z_s^I,\ b \in Z_t^I\}$, $t \neq s$, $2 \leq t$, $s < \infty$, $\times\}$, $x^{m+1} = 1$, $1 \leq m < \infty$, $\times\}$ be the MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroup under product $\times_0$ (or $\times$)*

.

i) *$S[x]_m$ is a finite order commutative monoid.*

ii) *$S[x]_m$ has zero divisors.*

iii) *$S[x]_m$ has no idempotents.*

iv) *$S[x]_m$ has nilpotents for appropriate values of s and t.*

v) *$S[x]_m$ has subsemigroups which are not ideals.*

vi) *$S[x]_m$ has subsemigroups which are ideals.*

The proof is left as an exercise to the reader.

Next we proceed onto study algebraic structures using the MOD rectangular interval of natural neutrosophic elements.

${}^{I}[0, m) \times {}^{I}[0, n) = \{(a, b) / a \in {}^{I}[0, m), b \in {}^{I}[0, n); m \neq n, 2 \leq m, n < \infty\}$ by giving some examples.

***Example 4.58***. Let $S = \{{}^{I}[0, 8) \times {}^{I}[0, 15) = \{(a, b) / a \in {}^{I}[0, 8), b \in {}^{I}[0, 15)\},+\}$ be the MOD rectangular natural neutrosophic interval semigroup under +.

Clearly S is an infinite monoid. S has subsemigroups of both finite and infinite order.

$P = \{Z_8 \times Z_{15}, +\}$ is a subsemigroup of finite order which is in fact a group under +.

Thus P is a Smarandache semigroup. It is left as an exercise for the reader to find idempotents of P.

***Example 4.59***. Let $S = \{{}^{I}[0, 7) \times {}^{I}[0, 17) = \{(a, b) / a \in {}^{I}[0, 7), b \in {}^{I}[0, 17)\}, +\}$ be the MOD rectangular natural neutrosophic interval semigroup under +.

S is a Smarandache semigroup of infinite order.

However S has only very few idempotents under +.

***Example 4.60***. Let $M = \{{}^{I}[0, 28) \times {}^{I}[0, 29) = \{(a, b) / a \in {}^{I}[0, 28), b \in {}^{I}[0, 29)\}, \times_0 \text{ (or } \times)\}$ be the MOD rectangular natural neutrosophic interval semigroup under the product $\times_0$ (or $\times$).

M is of infinite order.

M has infinitely many zero divisors. Idempotents and nilpotents are dependent on m and n of ${}^{I}[0, m)$ and ${}^{I}[0, n)$. S has subsemigroups of both finite and infinite order. But all ideals of M are of infinite order.

This also has infinite number of natural neutrosophic zero divisors.

So study in this direction is a matter of routine so left as an exercise to the reader.

***Example 4.61***. Let P = $\{{}^I[0, 24) \times {}^I[0, 412) = \{(a, b) / a \in {}^I[0, 24), b \in {}^I[0, 412)\}, \times_0 \text{ (or } \times)\}$ be the MOD rectangular natural neutrosophic interval semigroup under $\times_0$ (or $\times$).

P is of infinite order and P has subsemigroups of both finite and infinite order. All ideals of P are of infinite order.

P has infinite number of zero divisors and natural neutrosophic zero divisors.

However existence of idempotents and nilpotents are dependent on 24 and 412 only.

***Example 4.62***. Let W = $\{{}^I[0, 17) \times {}^I[0, 47) = \{(a, b) / a \in {}^I[0, 17), b \in {}^I[0, 47)\}, \times_0 \text{ (or } \times)\}$ be the MOD rectangular natural neutrosophic interval semigroup under product $\times_0$ (or $\times$).

W has zero divisors which are infinite in number, however finding idempotents and nilpotents happens to be a difficult or nontrivial idempotents and nilpotents do not exist.

This has both finite and infinite subsemigroups and all ideals are of infinite order.

Next we proceed onto describe MOD rectangular natural neutrosophic interval matrix semigroups under + and $\times_n$ by some examples.

***Example 4.63***. Let W = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right.$ / $a_i \in {}^{I}[0, 20) \times {}^{I}[0, 32)$ =

{(a, b) / a ∈ $^{I}[0, 20) \times {}^{I}[0, 32)$ = {(a, b) / a ∈ $^{I}[0, 20)$, b ∈ $^{I}[0, 32)$} $1 \leq i \leq 6$, +} be the MOD rectangular natural neutrosophic interval matrix semigroup.

o(W) = ∞ W has subsemigroups of both finite and infinite order. W has matrix idempotent under +.

***Example 4.64***. Let M = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right.$ where $a_i \in {}^{I}[0,23) \times$

$^{I}[0, 47)$ = {(a, b) / a ∈ $^{I}[0, 23)$, b ∈ $^{I}[0, 47)$} $1 \leq i \leq 9$, +} be the MOD rectangular natural neutrosophic interval semigroup under +. o(M) = ∞.

M has idempotents with respect to +.

M is a S-semigroup.

M has finite order subsemigroups which are groups. M also has infinite order subsemigroups which are groups. These semigroups enjoy several special features. In fact these are only semigroups of infinite order naturally defined and not abstractly.

For we have $I_t^m + I_t^m = I_t^m$ for appropriate t in [0, m), $2 \leq m < \infty$. Thus this new class of infinite order semigroups under + happens to be a very different non abstract structure.

We have the following result.

***Theorem 4.26**. Let S = {Collection of all s × t matrices with entries from $^{I}[0, m) \times {}^{I}[0, n) = \{(a, b) / a \in [0, m), b \in [0, n)\}$, +} be the MOD rectangular natural neutrosophic interval matrix semigroup under +.*

- *i) S is an infinite order commutative monoid.*
- *ii) S has subsemigroups of finite order which are groups.*
- *iii) S has subsemigroups of finite order which are not groups.*
- *iv) S has infinite order subsemigroups which are groups.*
- *v) S has infinite order subsemigroups which are not groups.*
- *vi) S is a S-semigroup under +.*
- *vii) S has idempotent matrices.*

Proof is direct and hence left as an exercise to the reader.

Next we describe by examples the MOD rectangular natural neutrosophic interval semigroups under $\times_0$ and $\times$.

***Example 4.65***. Let M = $\{{}^{I}[0,10) \times {}^{I}[0, 48) = \{(a, b) / a \in {}^{I}[0,10), b \in {}^{I}[0, 48)\}, \times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval semigroup under $\times_0$ (or $\times$).

o(M) = ∞, M is a commutative monoid. M has infinite number of zero divisors as well infinite number of natural neutrosophic zero divisors under ×.

M has subsemigroups of finite order. M is a S-semigroup if and only if one of $Z_m$ or $Z_n$ is a S-semigroup.

Finding nilpotents is also dependent on m and n. All these are a matter of routine so left as an exercise to the reader. M has ideals but all ideals of M are of infinite order.

***Example 4.66***. Let B = $\{{}^{I}[0, 19) \times {}^{I}[0, 43) = \{(a, b) / a \in {}^{I}[0,19), b \in {}^{I}[0, 43)\}, \times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval semigroup under $\times_0$ (or $\times$).

Finding nontrivial idempotents and nilpotents is an impossibility. However B has infinite number of zero divisors. B is S-semigroups B has subsemigroups of finite order which are group.

Finding substructures is a matter of routine so left as an exercise to the reader.

***Example 4.67***. Let S = $\{{}^{I}[0, 24) \times {}^{I}[0, 7) = \{(a, b) / a \in {}^{I}[0, 24), b \in {}^{I}[0, 7)\}, \times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval semigroup under $\times_0$ (or $\times$).

Finding substructures and special elements is left as an exercise to the reader. However S is a S-semigroup.

In view of all these we have the following theorem.

**Theorem 4.27**. *Let S = $\{{}^{I}[0, m) \times {}^{I}[0, n) = \{(a, b) / a \in {}^{I}[0, m), b \in {}^{I}[0,n)\}, m \neq n, 2 \leq m < \infty , \times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval semigroup under $\times_0$ (or $\times$).*

i) *S is an infinite commutative monoid.*

ii) *S has subsemigroups of both finite and infinite order which are not ideals.*

iii) *All ideals of S are of infinite order.*

iv) *S has infinite number of zero divisors.*

*v)* *S has nontrivial nilpotents and idempotents only for special values of m and n.*

*vi)* *S is a S-semigroup if and only if $Z_n$ or $Z_m$ is a S-semigroup.*

Proof is direct and hence left as an exercise to the reader.

Next we describe by examples MOD rectangular natural neutrosophic interval matrix semigroups under $\times_n$ or $\times$ the natural product or the usual product.

***Example 4.68***. Let W = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right.$ / $a_i \in {}^I[0,20) \times {}^I[0,16) = \{(a, b)$ / $a \in {}^I[0,20)$, $b \in {}^I[0, 16)\}$, $1 \leq i \leq 6$, $\times_n\}$ be the MOD rectangular natural neutrosophic interval matrix semigroup. o(W) = ∞ and W is a commutative monoid.

W has infinite number of zero divisor matrices.

However nontrivial idempotent matrices and nilpotent matrices are possible only when $Z_{20}$ and $Z_{16}$ have such elements. Such study and conclusions are a matter of routine.

W has both finite order as well as infinite order subsemigroups. All ideals of W are of infinite order.

Study in this direction is a matter of routine so left as an exercise to the reader.

***Example 4.69***. Let S = $\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix}$ $a_i \in {}^I[0, 47) \times {}^I[0, 23) = \{(a, b) / a \in {}^I[0, 47), b \in {}^I[0, 23)\}, 1 \leq i \leq 8, \times_n\}$ be the MOD rectangular natural neutrosophic interval matrix semigroup.

S has finite and infinite order subsemigroups. S is as S-semigroups.

S has infinite number of zero divisors no nontrivial idempotents or nilpotents.

***Example 4.70***. Let P = $\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}$ where $a_i \in {}^I[0, 48) \times {}^I[0, 35) = \{(a, b) / a \in {}^I[0, 48), b \in {}^I[0, 35)\}, 1 \leq i \leq 16, \times$ (or $\times_n$)$\}$ be the MOD rectangular natural neutrosophic interval matrix semigroup under the usual product $\times$ or the natural product $\times_n$.

$\{P, \times\}$ is a noncommutative monoid of infinite order.

The specialty about this structure is $\{P, \times\}$ can have right zero divisors which are not left zero divisors and vice versa.

Similarity $(P, \times)$ has right ideals which are not left ideals and vice versa.

However $\{P, \times_n\}$ is a commutative monoid of infinite order so such things do not occur.

So we can have idempotents in {P, ×} which are little difficult to be obtained.

In view of all these we can prove results related to {P, ×} and {P, $\times_n$} which we view as a matter of routine so leave it as an exercise to the reader.

Next we proceed onto describe MOD rectangular natural neutrosophic subset interval semigroup under + and product $\times_0$ (or ×) by some examples.

***Example 4.71***. Let S = {Collection of all subsets from $^I[0, 3) \times {}^I[0, 16)$ = {(a, b) / a ∈ $^I[0, 3)$, b ∈ $^I[0, 16)$}, +} be a MOD rectangular natural neutrosophic interval subset semigroup under +.

Here S = {S($^I[0, 3) \times {}^I[0, 16)$) = {collection of all subsets from $^I[0, 3) \times {}^I[0,16)$ = {(a, b) / a ∈ $^I[0, 3)$, b ∈ $^I[0, 16)$}, +}.

Clearly S is an infinite commutative monoid.

S has finite order subsemigroups which are groups.

S has also infinite order subsemigroups which are groups.

Let A = {$(0.2 + I_0^3, 2), (1, I_4^{16}), (0.34, I_4^{16} + 0.5)$} and B = {$(0.7 + I_0^3, 0.3), (I_0^3, 4 + I_0^{16})$} ∈ S.

We find A + B; A + B = {$(0.9 + I_0^3, 2.3), (1.7 + I_0^3, 0.3 + I_4^{16}), (1.04 + I_0^3, 0.8 + I_4^{16}), (0.2 + I_0^3, 6 + I_0^{16}), (1 + I_0^3, 4 + I_4^{16} + I_0^{16}), (I_0^3 + 0.34, 4.5 + I_4^{16} + I_0^{16})$} ∈ S.

This is the way + operation is performed on S.

Clearly $\{(0, 0)\} \in S$ is such that $A + \{(0, 0)\} = A$ for all $A \in S$. We can have subsets A in S such that $A + A = A$.

Hence S can have idempotents subsets.

***Example 4.72***. Let $B = \{S({}^{I}[0, 24) \times {}^{I}[0, 45)) = \{$collection of all subsets from ${}^{I}[0, 24) \times {}^{I}[0, 45) = \{(a, b) / a \in {}^{I}[0, 24), b \in {}^{I}[0, 45)\}, +\}$ be the MOD rectangular natural neutrosophic interval subset semigroup under +.

B has nontrivial idempotent subsets and has subsemigroups of finite and infinite order.

$A = \{(I_0^{24},\ I_0^{45}),\ (0,\ 0),\ (I_2^{24}, I_3^{45}),\ (I_0^{24} + I_3^{24},\ I_0^{45} + I_5^{45})$ $(I_3^{24}, I_5^{45}),\ (I_0^{24} + I_2^{24}, I_0^{45} + I_3^{45}),\ (I_0^{24} + I_2^{24} + I_3^{24}, I_0^{45} + I_3^{45} + I_5^{45})\} \in B$ is such that $A + A = A$. Hence our claim.

Study in this direction is a matter of routine so left as an exercise to the reader.

Next we describe by an example the notion of MOD rectangular natural neutrosophic interval semigroup under $\times_0$ (or $\times$).

***Example 4.73***. Let $M = \{S({}^{I}[0,14) \times {}^{I}[0, 48)) = \{(a, b) / a \in {}^{I}[0, 14), b \in {}^{I}[0, 48)\}, \times_0$ (or $\times$)$\}$ be the MOD rectangular natural neutrosophic interval subset semigroup under $\times_0$ (or $\times$).

M has idempotent subsets and has infinite number of zero divisors.

However nontrivial nilpotents cannot be found except nilpotents got from $\{(0, 12), (0, 6), (0, 0)\} = A$.

In view of all these we have the usual results in case of these semigroups also; this task is left as an exercise to the reader.

Next we describe MOD rectangular natural neutrosophic interval subset matrix semigroups under + and product by the following examples.

***Example 4.74***. Let M = $\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$ / $a_i \in S(^I[0,42) \times {}^I[0,24))$ =

{collection of all subsets from $^I[0, 42) \times {}^I[0, 24)$ = {(a, b) / a $\in$ $^I[0, 42)$, b $\in$ $^I[0, 24)$}, $1 \leq i \leq 5$, $\times_n$} be the MOD rectangular natural neutrosophic subset interval matrix semigroup and $\times_n$.

M has infinite number of zero divisors, finite number of nontrivial idempotents and nilpotents.

M has both finite order subsemigroups and infinite order subsemigroups.

M has ideals all of which are of infinite order.

These are all left as exercise to the reader.

***Example 4.75***. Let V = $\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix}$ / $a_i \in S(^I[0, 7) \times$

$^I[0, 41))$ = {collection of all subsets from $^I[0, 7) \times {}^I[0, 41)$ = {(a, b) / a $\in$ $^I[0, 7)$, b $\in$ $^I[0, 41)$}, $1 \leq i \leq 8$, $\times_n$} be the MOD rectangular natural neutrosophic interval subset matrix semigroup under natural product.

V has no nontrivial nilpotents or idempotents, however V has infinite number of zero divisors.

***Example 4.76***. Let W = $\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix}$ / $a_i \in$

S (${}^{I}[0, 48) \times {}^{I}[0,324)$) = {collection of all subsets from ${}^{I}[0,48) \times {}^{I}[0, 324)$ = {(a, b) / a $\in {}^{I}[0, 48)$, b $\in {}^{I}[0, 324)$}, $1 \leq i \leq 5$, ×} be the MOD rectangular natural neutrosophic interval subset matrix semigroup under usual product.

Clearly W is a non-commutative monoid of infinite order.

W has both left zero divisors which are not right zero divisors and vice versa. Also W has right ideals which are not left ideals and vice versa.

Study in this direction is a matter of routine so left as an exercise to the reader.

Next we describe MOD rectangular natural neutrosophic interval matrix subsets using + (and ×) by some examples.

***Example 4.77***. Let S(P) = {Collection of all matrix subsets from

P = $\{\begin{bmatrix} a_1 & a_6 \\ a_2 & a_7 \\ a_3 & a_8 \\ a_4 & a_9 \\ a_5 & a_{10} \end{bmatrix}$ / $a_i \in {}^{I}[0, 14) \times {}^{I}[0, 9)$ = {(a, b) / a $\in {}^{I}[0, 14)$, b $\in {}^{I}[0, 9)$}, $1 \leq i \leq 70$, +} be the MOD rectangular natural neutrosophic interval matrix subset semigroup under }.

S(P) is of infinite order has subsemigroups of finite order which are groups also subsemigroups of finite and infinite order which are not groups.

All ideals of S(P) are of infinite order S(P) has infinite number of zero divisors.

***Example 4.78***. Let S(B) = {collection all matrix subsets from B = {(a, b, c) / a, b, c $\in {}^{I}[0, 29) \times {}^{I}[0, 13)$ = {(x, y) / x $\in {}^{I}[0, 29)$, y $\in {}^{I}[0, 13)$}, +} be the MOD rectangular natural neutrosophic interval matrix subsets semigroup under +.

Let A = {((0.5, 1 + $I_0^{13}$), (4.3 + $I_0^{29}$, 2), (1, 0)), (0.36 + $I_0^{29}$, 5), (2, 0), (0, 0.331))} and

D = {((0, 4.3+ $I_0^{13}$), ($I_0^{29}$, 2), ($I_0^{29}$, $I_0^{13}$))} $\in$ S(B).

A + D = {((0.5, 5.3 + $I_0^{13}$), (4.3 + $I_0^{29}$, 4), (1 + $I_0^{29}$, $I_0^{13}$)), ((0.36 + $I_0^{29}$, 9.3 + $I_0^{13}$), (2 + $I_0^{29}$,2), ($I_0^{29}$, 0.331+ $I_0^{13}$))} $\in$ S(B).

This is the way + operation is performed on S(B). S(B) is in fact an infinite commutative monoid under +.

All properties in this case can be derived which is considered as a matter of routine so left as an exercise to the reader.

***Example 4.79***. Let S(M) = {collection of all matrix subsets

from M = { $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ / $a_i \in {}^{I}[0, 12) \times {}^{I}[0, 15)$ = {(a, b) / a $\in {}^{I}[0, 12)$,

$b \in {}^{I}[0,15))$, $1 \leq i \leq 4$, $\times_n\}$, $\times_n\}$ be the MOD rectangular natural neutrosophic interval matrix subset semigroup under natural product $\times_n$.

S(M) is an infinite commutative monoid S(M) has infinite number of zero divisors. All ideals of S(M) are of infinite order.

$$A = \{\begin{bmatrix} (0, 3) \\ (I_4^{12}, 0.5) \\ (I_6^{12}, I_0^{15}) \\ (1, 0) \end{bmatrix}, \begin{bmatrix} (I_0^{12}, 0) \\ (0, I_0^{15}) \\ (1, I_5^{15}) \\ (0, 0) \end{bmatrix}\} \text{ and}$$

$$B = \{\begin{bmatrix} (I_8^{12}, 4 + I_5^{15}) \\ (I_0^{12}, I_0^{15}) \\ (2, 0) \\ (1, I_6^{15}) \end{bmatrix}, \begin{bmatrix} (1, 2) \\ (0.3, 1) \\ (I_8^{12}, 0) \\ (0, I_0^{15}) \end{bmatrix}\} \in S(M). \text{ We find } A \times_n B;$$

$$A \times_n B = \{\begin{bmatrix} (I_8^{12}, 12 + I_5^{15}) \\ (I_0^{12}, I_0^{15}) \\ (I_6^{12}, I_0^{15}) \\ (1, I_6^{15}) \end{bmatrix}, \begin{bmatrix} (I_0^{12}, I_5^{15}) \\ (I_0^{12}, I_0^{15}) \\ (2, I_5^{15}) \\ (0, I_6^{15}) \end{bmatrix}, \begin{bmatrix} (0, 6) \\ (I_4^{12}, 0.5) \\ (I_0^{12}, I_0^{15}) \\ (0, I_{10}^{15}) \end{bmatrix}, \begin{bmatrix} (I_0^{12}, 0) \\ (0, I_0^{15}) \\ (I_8^{12}, I_5^{15}) \\ (0, I_{10}^{15}) \end{bmatrix}$$

$\in$ S(M). This is the way $\times_n$ operation is performed on S(M).

S(M) is an infinite commutative monoid. All ideals of S(M) are of infinite order.

However S(M) has subsets subsemigroups of finite order.

For S(P) = {collection of all matrix subsets from

P = {collection of all matrix subsets from P = { $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ /

$a_i \in Z_{12} \times Z_{15} = \{(a, b) / a \in Z_{12}, b \in Z_{15}\}, 1 \leq i \leq 4, \times_n\}, \times_n\} \subseteq$ S(M) is a subset subsemigroup of S(M) which is of finite order.

As study in this direction is a matter of routine we leave this task to the reader.

Further all results can be derived with appropriate modifications.

***Example 4.80***. Let S(M) = {collection of all matrix subsets

from M = $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix}$ / $a_i \in {}^I[0, 48) \times {}^I[0, 105) =$

$\{(a, b) / a \in {}^I[0, 48), b \in {}^I[0, 105)\}, 1 \leq i \leq 25, \times$ (or $\times_n$)}, $\times$ (or $\times_n$)} be the MOD rectangular natural neutrosophic matrix subset semigroup under usual product $\times$ or the natural product $\times_n$.

{(S(M), $\times$} is a MOD rectangular natural neutrosophic matrix subset noncommutative monoid of infinite order.

{S(M), $\times$} has right zero divisors which are not left zero divisors and vice versa. {S(M), $\times$} has right ideals which are not left ideals and vice versa.

However {(S(M), ×} also has MOD rectangular matrix subsemigroups of finite order, but all ideals of {S(M), ×} are only of infinite order.

Clearly {S(M), $\times_n$} is a MOD rectangular natural neutrosophic interval matrix subset commutative monoid of infinite order.

All ideals of {S(M), $\times_n$} are of infinite order but {S(M), $\times_n$} has both finite order as well as infinite order subsemigroups.

Interested reader can study in this direction.

Next we proceed onto describe MOD rectangular natural neutrosophic interval coefficient polynomial semigroups under + and then under product × by some examples.

***Example 4.81***. Let P[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^I[0,4) \times {}^I[0, 35)$ = {(a, b) / a $\in {}^I[0, 4)$, b $\in {}^I[0, 35)$}, +} be the MOD rectangular natural neutrosophic interval coefficient polynomial semigroup under +.

P[x] has finite order subsemigroups.

For R[x] = $\{\sum_{i=0}^{20} a_i x^i$ / $a_i \in Z_4 \times Z_{35}$ = {(a, b) / a $\in Z_4$, b $\in Z_{35}$}, +} is a finite subsemigroup of P[x]. In fact P[x] has infinite number of such finite order subsemigroups, however there are also infinite order subsemigroups.

Further P[x] has idempotent polynomials under +.

P[x] has subsemigroups which are groups under +. So P[x] is a S-semigroup.

In view of all these we have the following result.

**Theorem 4.28**. *Let* $S[x] = \{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in {}^{I}[0,m) \times {}^{I}[0, n) = \{(a, b) \,/\, a \in {}^{I}[0, m),\ b \in {}^{I}[0, n),\ m \neq n;\ 2 \leq m, n < \infty\}, +\}$ *be the MOD rectangular natural neutrosophic interval coefficient polynomial semigroup under +.*

*Then the following are true.*

i) *S[x] is a commutative monoid of infinite order.*

ii) *S[x] has idempotent polynomials under +.*

iii) *S[x] has finite order subsemigroups which are not groups.*

iv) *S[x] is a S-semigroup.*

v) *S[x] has infinite order subsemigroups which are groups.*

vi) *S[x] has finite order subsemigroups which are groups.*

The proof is direct with appropriate modifications so left as an exercise to the reader.

Next we describe MOD rectangular natural neutrosophic interval coefficient polynomial semigroups under product $\times_0$ (or $\times$) by examples.

***Example 4.82***. Let $W[x] = \{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in {}^{I}[0, 10) \times {}^{I}[0, 19) = \{(a, b) \,/\, a \in {}^{I}[0, 10),\ b \in {}^{I}[0, 19)\}, \times_0 \text{ (or } \times)\}$ be the MOD rectangular natural neutrosophic interval coefficient polynomial semigroup under $\times_0$ (or $\times$).

W[x] is of infinite order W[x] has infinite number of zero divisors and has no idempotents. Further all subsemigroups and ideals of W[x] are only of infinite order.

Study of this can be done as in case of other semigroups with appropriate modifications.

***Example 4.83***. Let B[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^I[0, 43) \times {}^I[0, 53)$ = $\{(a, b)$ / $a \in {}^I[0, 43)$, $b \in {}^I[0, 53)\}$, $\times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval coefficient semigroup under $\times_0$ (or $\times$).

Clearly B[x] has infinite number of zero divisors but no idempotents or nilpotents. All subsemigroups of B[x] are of infinite order.

B[x] also has ideals.

In view of all these we have the following theorem the proof of which is left as an exercise to the reader.

**Theorem 4.29**. *Let V[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^I[0, m) \times {}^I[0, n)$ = $\{(a, b)$ / $a \in {}^I[0, m)$, $b \in {}^I[0, n)\}$, $m \neq n$, $2 \leq m, n < \infty\}$, $\times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval coefficient semigroup under $\times_0$ (or $\times$).*

*Then*

- *i) V[x] is an infinite commutative monoid.*
- *ii) V[x] has infinite number of zero divisors.*
- *iii) V[x] has no idempotents.*
- *iv) V[x] has nontrivial nilpotents only for appropriate values of n and m.*
- *v) All ideals and subsemigroups of V[x] are of infinite order.*
- *vi) V[x] is S-semigroup if and only if one of $Z_n$ or $Z_m$ is a S-semigroup.*

Proof is direct and hence left as an exercise to the reader.

Next we describe MOD rectangular natural neutrosophic interval coefficient polynomials of finite degree semigroups under + by some examples.

***Example 4.84***. Let $P[x]_5 = \{\sum_{i=0}^{5} a_i x^i$ / $a_i \in {}^I[0, 44) \times {}^I[0, 24) = \{(a, b)$ / $a \in {}^I[0, 44)$ and $b \in {}^I[0, 24)\}, +\}$ be the MOD rectangular natural neutrosophic interval coefficient finite degree polynomial semigroup under +.

$P[x]_5$ is an infinite commutative monoid. $P[x]_5$ has both finite order as well as infinite order subsemigroups.

$P[x]_5$ also has subsemigroups which are groups.

$P[x]_5$ has nontrivial idempotents under +.

Study is this direction is innovative and interesting and this task is left as an exercise to the reading.

***Example 4.85***. Let $M[x]_{10} = \{\sum_{i=0}^{10} a_i x^i$ / $a_i \in {}^I[0, 15) \times {}^I[0, 17) = \{(a, b)$ / $a \in {}^I[0, 15), b \in {}^I[0, 17)\}, x^{11} = 1, \times_0$ (or $\times$)$\}$ be the MOD rectangular natural neutrosophic interval coefficient finite degree polynomial semigroup under $\times_0$ (or $\times$).

$M[x]_{10}$ has finite order subsemigroups as well as infinite order subsemigroups. $M[x]_{10}$ has infinite number of zero divisors. Nilpotents are possible depending on m and n.

Clearly $M[x]_{10}$ has no nontrivial idempotents.

Such study is interesting and left as an exercise to the reader.

All related results can be obtained with appropriate modifications.

Next we proceed onto describe by examples MOD rectangular natural neutrosophic interval subset coefficient polynomial semigroups under + by examples.

***Example 4.86***. Let M[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in$ S $({}^{I}[0, 6) \times {}^{I}[0, 11))$ = {Collection of all interval subsets from ${}^{I}[0, 16) \times {}^{I}[0, 11)$ = (a, b) / a $\in {}^{I}[0, 16)$, b $\in {}^{I}[0, 11)$}, +} be the MOD rectangular natural neutrosophic interval subset coefficient polynomial semigroup under +.

M[x] is a commutative monoid of infinite order.

M[x] has subsemigroups of finite order as well as infinite order.

Let A = $\{(0, I_0^{11}), (I_2^{16} + 1, 0), (1, 5)\} + \{(4, I_0^{11}+3), (0.331, 2)\}x^2 + \{(3, 2),(1, I_0^{11} + 0.5)\}x^3$ and

B = $\{(5, 0.3), (I_4^{16}, 0.5 + I_0^{11})\} + \{(3.3, 2.1)\ (1.3 + I_8^{16}, 0)\}x + \{(2 + I_2^{16} + I_{10}^{16}, 0.32), (1 + I_0^{16}, 0.33 + I_0^{11})\}x^2 \in$ M[x].

A + B = $\{(5, 0.3 + I_0^{11}), (6 + I_2^{16}, 0.3)\ (6, 5.3), (I_4^{16}, 0.5 + I_0^{11}), (2 + I_2^{16} + I_4^{16}, 0.5 + I_0^{11})\} + \{(3.3, 2.1)\ (1.3 + I_8^{16}, 0)\}x + \{(4, I_0^{11}+3), (0.331, 2)\}x^2 + \{(3, 2), (1, I_0^{11}+0.5)\}x^3 + \{(6 + I_2^{16} + I_{10}^{16}, 3.32 + I_0^{11}), (5 + I_0^{16}, 3.33 + I_0^{11}), (2.331 + I_2^{16} + I_{10}^{16}, 2.32), (1.331 + I_0^{16}, 2.33 + I_0^{11})\} \in$ M[x].

This is the way + operation is performed on M[x] has nontrivial idempotents with respect to +.

***Example 4.87***. Let B[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S\,({}^{I}[0, 48) \times {}^{I}[0, 30))$ = {Collection of all interval subsets from ${}^{I}[0, 48) \times {}^{I}[0, 30)$ = (a, b) / a $\in {}^{I}[0, 48)$, b $\in {}^{I}[0, 30)\}$, $\times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval subset coefficient polynomial semigroup under $\times_0$ (or $\times$).

B[x] is an infinite commutative monoid. All subsemigroups and ideals of B[x] are of infinite order. B[x] has nontrivial idempotents but has infinite number of zero divisors or MOD natural neutrosophic zero divisors. Has nilpotents.

Study in this direction is innovative and interesting and left as an exercise to the reader.

Next we proceed on to give examples of MOD rectangular natural neutrosophic interval subset coefficient finite degree polynomial semigroups under +.

***Example 4.88***. Let $B[x]_7 = \{\sum_{i=0}^{7} a_i x^i$ / $a_i \in S\,({}^{I}[0, 12) \times {}^{I}[0, 8))$ = {Collection of all subsets from ${}^{I}[0, 12) \times {}^{I}[0, 8)$ = (a, b) / a $\in {}^{I}[0,12)$, b $\in {}^{I}[0,8)\}$, $x^8 = 1$, +} be the MOD rectangular natural neutrosophic interval subset coefficient finite degree polynomial semigroup under +.

$B[x]_7$ is an infinite order commutative monoid. $B[x]_7$ has subsemigroups of finite order.

$B[x]_7$ has infinite subsemigroups also. Further $B[x]_7$ has idempotents.

***Example 4.89***. Let $P[x]_{12} = \{\sum_{i=0}^{12} a_i x^i$ / $a_i \in S\,({}^{I}[0, 49) \times {}^{I}[0, 125))$ = {Collection of all subsets from ${}^{I}[0, 49) \times {}^{I}[0, 125)$ = {(a, b) / a $\in {}^{I}[0, 49)$, b $\in {}^{I}[0, 125)\}$, $x^{13} = 1$, $\times_0$ (or $\times$)} be the

MOD rectangular natural neutrosophic interval subset coefficient finite degree polynomial semigroup under $\times_0$ (or $\times$).

$P[x]_{12}$ is an infinite commutative monoid. $P[x]_{12}$ has infinite number of zero divisors (or MOD natural neutrosophic zero divisors).

However $P[x]_1$ has no nontrivial idempotents. $P[x]_{12}$ has both finite and infinite order subsemigroups. $P[x]_{12}$ has ideals all of which are of infinite order.

This new structure has very many nice properties and the reader is expected to analyze them.

***Example 4.90***. Let S(B[x]) = {collection of all polynomial subsets from B[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^I[0, 40) \times {}^I[0, 24)$ = {(a, b) / $a \in {}^I[0, 40)$, $b \in {}^I[0, 24)\}$, +} be the MOD rectangular natural neutrosophic interval polynomial subset semigroup under +. S(B[x]) is an infinite commutative monoid.

S(B[x]) has infinite number of finite order subsemigroups. S(B[x]) also has infinite order subsemigroups S(B[x]) has no idempotents.

Study in this direction is a matter of routine.

***Example 4.91***. Let S(M[x]) = {collection of all polynomial subsets from M[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^I[0, 14) \times {}^I[0, 41)$ = {(a, b) / $a \in {}^I[0, 14)$, $b \in {}^I[0, 41)\}$, $\times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval polynomial subset semigroup under $\times_0$ (or $\times$). S(M[x]) is commutative monoid of infinite order.

All subsemigroups and ideals of S(M[x]) are of infinite order S(M[x]) has infinite number of zero divisors.

Let A = $\{(0.3, I_0^{41})x^2 + (I_7^{14}, 0.5), (I_0^{14}+0.3, 2)x^3 + (2, 3.7)x + (0, 1)\}$ and B = $\{(0.6, I_0^{41} + 3)x^2 + (0.3 + I_0^{14}, 0)x + (0.3, 2), (4, 5)x + (3, 2 + I_0^{41})\} \in (M[x])$.

$A \times B = \{(0.18, I_0^{41})\, x^4 + (I_7^{14}, I_0^{41} + 1.5)x^2 + (0.09 + I_0^{14}, I_0^{41})x^3 + (I_7^{14} + I_0^{14}, 0)x + (0.09, I_0^{41})x^2 + (I_7^{14}, 1), (1.2, I_0^{41})x^3 + (I_7^{14}, 2.5)x + (0.9, I_0^{41})x^2 + (I_7^{14}, 1 + I_0^{41}), (I_0^{14}+0.18, 6 + I_0^{41})x^5 + (1.2, I_0^{41} + 11.1)x^3, (0, I_0^{41} + 3)x^2 + (I_0^{14}+0.09, 0)x^4 + (0.6 + I_0^{14}, 0)x^2 + (I_0^{14}, 0)x + (I_0^{14} + 0.09, 4)x^3 + (0.6, 7.4)x + (0, 2), (I_0^{14} + 1.2, 10)x^4 + (8, 18.5)x^2 + (0, 5)x + (I_0^{14} + 0.9, 4 + I_0^{41})x^3 + (6, 7.4 + I_0^{41})x + (0, 2 + I_0^{41})\} \in S(M[x])$. This is the way × operation is performed on S(M[x]).

Interested reader can find the product. $A \times_0 B$ and show $A \times B \neq A \times_0 B$.

If P = $\{(0, 3)x^3 + (0, I_0^{41})x^2 + (0,9), (0, 21.5 + I_0^{41})x^7 + (0, I_0^{41} + 14.2)\}$ and Q = $\{(3, 0)x^2 + (7, 0)x + (4.327, 0), (I_2^{14}, 0)\, x^9 + (I_2^{14} + I_6^{14} + 3, 0)\} \in S(M[x])$, then $P \times_0 Q = \{(0, 0)\}$ but $P \times Q \neq \{(0,0)\}$.

This task of verifying the above conditions is left as an exercise to the reader.

***Example 4.92***. Let $S(B[x]_9)$ = {Collection of all polynomial subsets from $B[x]_9 = \{\sum_{i=0}^{9} a_i x^i \,/\, a_i \in {}^I[0, 16) \times {}^I[0, 12) = \{(a, b) \,/\, a \in {}^I[0, 16), b \in {}^I[0, 12)\}, x^{10} = 1, +\}, +\}$ be the MOD rectangular natural neutrosophic interval polynomial subsets semigroup under +.

$S(B[x]_9)$ is an infinite order commutative monoid.

Let A = {( $I_2^{16}$, $I_4^{12}$ )$x^3$ + (0, 4 + $I_6^{12}$ ), ( $I_8^{16} + I_0^{16}$, 0)$x^4$ + (0, $I_6^{12} + I_8^{12}$ )} and B = {( $I_0^{16}$, $I_6^{12}$ + 3)$x^3$ + (6, $I_6^{12}$ )x + (4 + $I_2^{16}$, $I_0^{12}$ ), (6 + $I_8^{16}$, $I_0^{12}$ ) + (3, 6 $I_8^{12}$ )$x^5$} ∈ S(B[x]$_9$).

We now find A + B, A + B = {( $I_2^{16}$ + $I_0^{16}$, $I_4^{12}$ + $I_6^{12}$ +3)$x^3$ + (6, $I_6^{12}$ )x + (4 + $I_2^{16}$, 4 + $I_6^{12}$ + $I_0^{12}$ ), ( $I_8^{16} + I_0^{16}$, 0)$x^4$ + ( $I_0^{16}$, 3+ $I_6^{12}$ )$x^3$ + (6, $I_6^{12}$ )x + (4 + $I_2^{16}$, $I_6^{12}$ + $I_8^{12} + I_0^{12}$ ), ( $I_2^{16}$, $I_4^{12}$ )$x^3$ + (3, 6 + $I_8^{12}$ )$x^5$ + (6 + $I_8^{16}$, $I_0^{12}$ + $I_6^{12}$ + 4), (3, 6 + $I_8^{12}$ )$x^5$ + ( $I_8^{16} + I_0^{16}$ ,0)$x^4$ + (6 + $I_8^{16}$, $I_0^{12}$ + $I_6^{12} + I_8^{12}$ )} ∈ S(B[x]$_9$).

This is the way + operation is performed on S(B[x]$_9$). S(B[x]$_9$) has subsemigroups of both finite and infinite order.

The reader is left with the task of finding idempotents polynomial subsets of S(B[x]$_9$).

***Example 4.93***. Let S(M[x]$_{12}$) = {collection of all polynomial subsets from M[x]$_{12}$ = { $\sum_{i=0}^{12} a_i x^i$ / $a_i$ ∈ $^I$[0,15) × $^I$[0, 24) = {(a, b) / a ∈ $^I$[0, 5), b ∈ $^I$[0, 24)}, $x^{13}$ = 1, $\times_0$ (or ×) +} $\times_0$ (or ×)} be the MOD rectangular natural neutrosophic interval polynomial subsets semigroup under $\times_0$ (or ×).

Clearly S(M[x]$_{12}$) is a commutative monoid of infinite order S(M[x]$_{12}$) has infinite number of zero divisors and MOD natural neutrosophic zero divisors.

S(M[x]$_{12}$) has both finite and infinite order subsemigroups, but all ideals of S(M[x]$_{12}$) are of infinite order. S(M[x]$_{12}$) has no nontrivial idempotents.

Let A = $\{(0, 4)x^3 + (3 + I_0^{15}, I_2^{24}), (4, 2 + I_2^{24})x^3 + (4 + I_0^{15}, 0)x + (2, 3 + I_0^{24})\}$ and B = $\{(2, 3 + I_0^{24})x^2 + (4, 5), (0.3 + I_0^{15}, I_8^{24})x^3 + (3 + I_5^{15}, 0)\} \in S(M[x]_{12})$.

A × B = $\{(0, 12 + I_0^{24})x^5 + (6 + I_5^{15}, I_2^{24} + I_0^{24})x^2, + (0, 20)x^3 + (12 + I_5^{15}, I_2^{24}), (8, 6 + I_0^{24} + I_{12}^{24})x^5 + (1, 10 + I_{12}^{24})x^3 + (8 + I_0^{15}, 0)x^3 + (4, 9 + I_0^{24})x^2 + (8, 15 + I_0^{24}) + (1 + I_0^{15}, 0)x, (0, I_8^{24})x^6 + (0.9 + I_5^{15} + I_0^{15}, I_{16}^{24})x^3 + (12 + I_5^{15}, 0)x^3 + (9 + I_{10}^{15} + I_5^{15}, 0), (1.2 + I_0^{15}, I_8^{24} + I_{12}^{24})x^6 + (1.2 + I_0^{15}, 0)x^4 + (0.6 + I_0^{15}, I_8^{24} + I_0^{24})x^3 + (12 + I_5^{15}, 0)x^3 + (12 + I_5^{15} + I_0^{15}, 0)x + (6 + I_5^{15}, 0)\} \in S(M[x]_{12})$.

This is the way × operation is performed on $S(M[x]_{12})$.

The reader is left with the task of finding A $\times_0$ B $\in$ $M[x]_{12}$) and show A × B ≠ A $\times_0$ B.

It is pertinent to keep on record that the usual zero divisor dominated product $\times_0$ and that of × the MOD natural neutrosophic zero dominated product are different, so the zero divisors under $\times_0$ are different from zero divisors under ×.

A $\times_0$ B = {(0, 0)} need not in general imply A × B = $\{I_0^n, I_0^n\}$ and vice versa.

This can be easily established by examples.

Further we cannot have nontrivial idempotents in case of all four types of polynomial semigroups.

Finally the nontrivial nilpotents under $\times_0$ is different from × and vice versa.

Study in this direction can be carried out as in case of polynomial subset coefficients from S($Z_m^I \times Z_n^I$), m ≠ n, (2 ≤ m, n < ∞).

Only major difference in using S($^I$[0, m) × $^I$[0, n)); m ≠ n, (2 ≤ m, n < ∞) instead of S($Z_m^I \times Z_n^I$) at times when we use finite degree subset polynomial or polynomial subset the semigroups are of finite order.

Further we see in this we have other types of zero divisors which makes the structure over intervals interesting. These are infinite in structure.

However it is recorded we in $^I$[0, m) use only those neutrosophic elements arising from $Z_m$ and not from [0, m).

This has be elaborately in discussed in books on MOD structure [32-37].

We proceed on to suggested problems for the reader.

**PROBLEMS**

1. Let W = {$Z_n^I \times Z_m^I$ = {(a, b) / a ∈ $Z_n^I$, b ∈ $Z_m^I$}, 2 ≤ m, n < ∞, m ≠ n} be the MOD rectangular natural neutrosophic finite modulo number set.

   Find the order of W.

2. Let S = {$Z_{10}^I \times Z_{16}^I$ = {(a, b) / a ∈ $Z_{10}^I$, b ∈ $Z_{16}^I$}, +} be the MOD rectangular natural neutrosophic finite modulo number semigroup under +.

   i) Find o(S).

   ii) Find all idempotents of S under +.

iii) Prove S is a Smarandache semigroup.

iv) Find the number of subsemigroups of S which are subgroups of S under +.

v) Find the number of subsemigroups of S which are not subgroups of S.

vi) Find all special features associated with S.

3. Let B = { $Z_{19}^{I} \times Z_{47}^{I}$ = {(a, b) / a $\in Z_{19}^{I}$, b $\in Z_{47}^{I}$, +} be the MOD rectangular natural neutrosophic finite modulo integer semigroup under +.

i) Study questions (i) to (vi) of problem (2) for this B.

ii) Compare S of problem (2) with this B.

iii) Which of the semigroups (S, +) in problem (2) or (B, +) has more number of idempotents?

4. Let T = { $Z_{24}^{I} \times Z_{60}^{I}$ = {(a, b) / a $\in Z_{24}^{I}$, b $\in Z_{60}^{I}$ },+} be the MOD rectangular natural neutrosophic finite modulo integer semigroup under +.

i) Study questions (i) to (vi) of problem (2) for this T.

ii) Compare T with B and S of problems (3) and (2) respectively.

iii) Find which of the semigroups T or S or B has maximum number of idempotents under +.

5. Let W = $\{Z_8^I \times Z_{26}^I = \{(a, b) / a \in Z_8^I, b \in Z_{26}^I\}, \times\}$ be the MOD rectangular natural neutrosophic modulo integer semigroup under $\times$.

   i) Prove W is a finite commutative monoid.
   ii) Find all zero divisors of W.
   iii) Find all nontrivial nilpotents of W.
   iv) Does W contain nontrivial idempotents?
   v) Find all subsemigroups of W which are not ideals.
   vi) Find all ideals of W.
   vii) Enumerate all special features associated with W.
   viii) Study these questions under the zero dominated product and the natural neutrosophic zero dominated product and compare W under these two products.

6. Let V = $\{Z_{43}^I \times Z_{29}^I = \{(a, b) / a \in Z_{43}^I, b \in Z_{29}^I\}, \times\}$ be the MOD rectangular natural neutrosophic finite modulo integer semigroup under $\times_0$ (or $\times$).

   i) Study questions (i) to (viii) of problem (5) for this V.
   ii) Compare this V with W of problem (5).

7. Let B = $\{Z_{20}^I \times Z_{12}^I = \{(a, b) / a \in Z_{20}^I, b \in Z_{12}^I\}, \times_0, (\times)\}$ be the MOD rectangular natural neutrosophic modulo integer semigroup under $\times_0$ (or $\times$).

i) Study questions (i) to (viii) of problem (5) for this B.

ii) Compare this B with V of problem 6.

8. Let M = {Collection of all subsets from $Z_{20}^{I} \times Z_{42}^{I}$, +} be the MOD rectangular natural neutrosophic subset semigroup under +.

i) Find o(M).

ii) Prove M is a commutative monoid.

iii) Is M a S-semigroup?

iv) Find all idempotents of M.

v) Find all subsemigroups of M.

vi) Enumerate all special features enjoyed by M.

9. Let W = {Collection of all subsets from $Z_{23}^{I} \times Z_{53}^{I}$ = {(a, b) / a $\in Z_{23}^{I}$, b $\in Z_{53}^{I}$,+}, +} be the MOD rectangular natural neutrosophic subset semigroup under +.

i) Study questions (i) to (vi) of problem (8) for this W.

ii) Compare W with M of problem 8.

10. Let P = {Collection of all subsets from $Z_{29}^{I} \times Z_{48}^{I}$ = {(a, b) / a $\in Z_{29}^{I}$, b $\in Z_{48}^{I}$, +}, +} be the MOD rectangular natural neutrosophic subset semigroup under +.

i) Study questions (i) to (vi) of problem (8) for this D.

ii) Compare this P with W of problem 9.

11. Let M = {Collection of all subsets from $Z_{24}^{I} \times Z_{7}^{I} = \{(a, b) / a \in Z_{24}^{I}, b \in Z_{7}^{I}\}, \times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic subset semigroup under $\times_0$ (or $\times$).

i) Show M is a finite commutative monoid.

ii) Find o(M).

iii) Prove M has always zero divisors in case of $\times_0$ and natural neutrosophic zero divisor in case of $\times$ product.

iv) Prove M can have nilpotent or natural neutrosophic nilpotents for $\times_0$ (or $\times$) respectively.

v) Is M a S-semigroup?

vi) Can M have nontrivial idempotents?

vii) Find all subsemigroups which are not ideals of M.

viii) Find all ideals of M.

ix) Obtain all special features enjoyed by M.

12. Let B = {Collection of all subsets from $Z_{12}^{I} \times Z_{63}^{I} = \{(a, b) / a \in Z_{12}^{I}, b \in Z_{63}^{I}, \times_0$ (or $\times$)} $\times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic subset semigroup under $\times_0$ (or $\times$).

Study questions (i) to (ix) of problem (11) for this B.

13. Let R = {collection of all subsets from $Z_3^I \times Z_{13}^I$ = {(a, b) where a ∈ $Z_3^I$, b ∈ $Z_{13}^I$, $\times_0$ (or ×)} $\times_0$ (or ×)} be the MOD rectangular natural neutrosophic subset semigroup under $\times_0$ (or ×).

i) Study questions (i) to (ix) of problem (11) for this R.

ii) Compare B of problem 12 with this R.

14. Let A = {($a_1$, $a_2$, $a_3$, $a_4$) / $a_i$ ∈ $Z_{10}^I \times Z_{48}^I$ = {(a, b) / a ∈ $Z_{10}^I$, b ∈ $Z_{48}^I$};1 ≤ i ≤ 4, +} be the MOD rectangular natural neutrosophic finite modulo integer matrix semigroup under +.

i) Prove A is a commutative monoid.

ii) Find o(A).

iii) Prove A is a S-semigroup.

iv) Find all idempotent matrices of A.

v) Find all subsemigroups of A.

vi) Find all subsemigroups of A which are groups.

iv) Obtain any other special feature associated with A.

15. Let B = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right.$ / $a_i$ ∈ $Z_{19}^I \times Z_{43}^I$ = {(a, b) / a ∈ $Z_{19}^I$,

$b \in Z_{43}^{I}$, $1 \leq i \leq 5$, +} be the MOD rectangular natural neutrosophic finite modulo integer matrix semigroup under +.

Study questions (i) to (vii) of problem (14) for this B.

16. Let L = { $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}$ / $a_i \in Z_{12}^{I} \times Z_{135}^{I}$ = {(a, b) / $a \in Z_{12}^{I}$, $b \in Z_{135}^{I}$, $1 \leq i \leq 12$, +} be the MOD rectangular natural neutrosophic finite modulo integer matrix semigroup under +.

Study questions (i) to (vii) of problem (14) for this L.

17. Let W = { $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix}$ / $a_i \in Z_{24}^{I} \times Z_{47}^{I}$ = {(a, b) / $a \in Z_{24}^{I}$, $b \in Z_{47}^{I}$, $1 \leq i \leq 14$, $\times_n$}; be the MOD rectangular natural neutrosophic finite modulo integer matrix semigroup under natural product with usual zero domination or the MOD natural neutrosophic zero domination.

i) Find o(W).

ii) Prove W is a commutative monoid.

iii) Find all zero divisors of W.

iv) Can W have nontrivial nilpotents?

v) Can W have nontrivial idempotents?

vi) Is W a S-semigroup?

vii) Find all subsemigroups of W which are not ideals.

viii) Find all ideals of W.

ix) Obtain all special features associated with W.

18. Let B = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right.$ / $a_i \in Z_{48}^I \times Z_3^I = \{(a, b)$ /

$a \in Z_{48}^I$, if $\in Z_3^I$, $1 \leq i \leq 16$, $\times_n$ or $\times\}$ be the MOD rectangular natural neutrosophic matrix semigroup under $\times_n$ the natural product or usual product $\times$.

i) Prove $\{B, \times\}$ is a non-commutative monoid of finite order.

ii) Prove $\{B, \times_n\}$ is a commutative monoid.

iii) Prove $\{B, \times_n\}$ is different from $\{B, \times\}$.

iv) Find all zero divisors of $\{B, \times_n\}$.

v) Find all left and right zero divisors of $\{B, \times\}$.

vi) Show a zero divisor in $\{B, \times_n\}$ need not in general be a zero divisor in $\{B, \times\}$ and vice versa.

vii) Find all nilpotents in $\{B, \times_n\}$ and $\{B, \times\}$.

viii) Find all idempotents of $\{B, \times_n\}$ and $\{B, \times\}$.

ix) Find all subsemigroups of $\{B, \times\}$ and $\{B, \times_n\}$.

x) Does their exist semigroups P such that $\{P, \times\} \subseteq \{B, \times\}$ and $\{P, \times_n\} \subseteq \{B, \times_n\}$ are same?

xi) Find all right ideals and left ideals of $\{B, \times\}$.

xii) Find all ideals of $\{B, \times_n\}$.

xiii) Distinguish between $\{B, \times_n\}$ and $\{B, \times\}$.

19. Let M = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{30} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{bmatrix} \right. /$

$a_i \in Z^I_{48} \times Z^I_{250} = \{(a, b) / a \in Z^I_{48}, b \in Z^I_{250}, 1 \leq i \leq 36,$ $\times_n$ (or $\times$)$\}$ be the MOD rectangular natural neutrosophic matrix semigroup under natural product $\times_n$ or the usual product $\times$.

Study questions (i) to (xii) of problems (18) for this M.

20. Let W = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right. / a_i \in Z^I_{10} \times Z^I_5 = \{(a, b) /$

$a_i \in Z_{10}^I$, $b \in Z_5^I$, $1 \leq i \leq 18$, $\times_n\}$, $\times_n\}$ be the MOD rectangular natural neutrosophic matrix semigroup under natural product $\times_n$.

Study questions (i) to (xii) of problem (18) for this W.

21. Let S(B) = {Collection of all matrix subsets from the set

$$B = \{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} / a_i \in Z_{25}^I \times Z_{38}^I = \{(a, b) / a \in Z_{25}^I \times Z_{38}^I =$$

$\{(a, b) / a \in Z_{25}^I$, $b \in Z_{38}^I$; $1 \leq i \leq 5, +\}$ $\times\}$ be the MOD rectangular natural neutrosophic matrix subset semigroup under +.

i) Find o(S(B)).
ii) Prove S(B) is a finite commutative monoid.
iii) Find all idempotents of S(B).
iv) Find all subsemigroups of S(B) which are groups.
v) Prove S(B) is a S-semigroup.
vi) Find all subsemigroups of S(B) which are not groups.
vii) Determine any other special feature associated with S(B).

22. Let S(P) = {collection of all matrix subsets from

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{bmatrix} \right. / a_i \in Z_{45}^{I} \times Z_{23}^{I} =$$

$\{(a, b) / a \in Z_{45}^{I}, b \in Z_{23}^{I}\}$ $1 \leq i \leq 18, +\}$ be the MOD rectangular natural neutrosophic matrix subsets semigroup under +.

Study questions (i) to (vii) of problem (21) for this S(P).

23. Let S(B) = {collection of all matrix subsets from

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \right. / a_i \in Z_{12}^{I} \times Z_{47}^{I} =$$

$\{(a, b) / a_i \in Z_{12}^{I}, b \in Z_{47}^{I}\}$ $1 \leq i \leq 25, +\}$ be the MOD rectangular matrix subset semigroup under +.

Study questions (i) to (vii) of problem (21) for this S(B).

24. Let S(W) = {Collection of all matrix subsets from

$$W = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} \text{ where } a_i \in Z_{20}^I \times Z_{47}^I = \{(a, b) /$$

$a \in Z_{20}^I$, $b \in Z_{47}^I$, $1 \leq i \leq 14, \times_n\}$ $\times_n\}$ be the MOD rectangular natural neutrosophic matrix subset semigroup under natural product $\times_n$.

i) Find o(S(W)).

ii) Prove S(W) is a commutative monoid.

iii) Find all zero divisor matrix subsets of S(W).

iv) Can S(W) contain idempotent matrix subsets?

v) Can S(W) have nontrivial nilpotent matrix subsets?

vi) Find all subsemigroups which are not ideals of S(W).

vii) Find all subsemigroups which are ideals of S(W).

viii) Is S(W) a S-semigroup?

ix) Enumerate any of the special feature associated with S(W).

25. Let S(E) = {Collection of all matrix subsets from

$$E = \{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} / a_i \in \{Z_{40}^I \times Z_{37}^I = \{(a, b) / a \in Z_{40}^I,$$

$b \in Z_{37}^I\}$, $1 \leq i \leq 6$, $\times_n\}, \times_n\}$ be the MOD rectangular natural neutrosophic matrix subsets semigroups under natural product $\times_n$.

Study questions (i) to (ix) of problem (24) for this S(E).

26. Let S(P) = {Collection of all matrix subsets from P = $\{(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) / a_i \in Z_{13}^I \times Z_{47}^I = \{(a, b) / a \in Z_{13}^I, b \in Z_{47}^I\}, 1 \leq i \leq 8, \times\}, \times\}$ be the MOD rectangular natural neutrosophic matrix subset semigroup under product $\times$.

Study questions (i) to (ix) of problem (24) for this S(P).

Compare S(P) with S(E) of problem (25).

27. Let S(B) = {Collection of all subsets from

$$B = \{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \text{ where } a_i \in Z_{24}^I \times Z_{45}^I = \{(a, b) /$$

$a \in Z_{24}^I$, $b \in Z_{45}^I\}$, $1 \leq i \leq 9$, $\times_n$ (or $\times$)} be the MOD rectangular natural neutrosophic matrix subset semigroup under natural product $\times_n$ (or usual product $\times$).

i) Find o(S(B)).

ii) Prove {S(B), ×} is a non-commutative monoid.

iii) Show {S(B), ×} has left matrix subset zero divisors which are not in general right matrix subset zero divisors.

iv) Prove {S(B), ×} has matrix subset left ideals which are not right ideals.

v) Find all ideals of {S(B), ×}.

vi) Can {S(B), ×} have nontrivial nilpotents?

vii) Can {S(B), ×} have nontrivial idempotents?

viii) Compare {S(B),×} with {S(B), $\times_n$}.

ix) Study questions (i) to (ix) of problem 24 for {S(B), $\times_n$}.

28. Let S(S) = {Collection of all matrix subsets from

$$S = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} / a_i \in Z_{42}^I \times Z_{56}^I =$$

{(a, b) / a $\in Z_{42}^I \times Z_{56}^I$ = {(a, b) / a $\in Z_{42}^I$, b $\in Z_{56}^I$}, 1 $\leq$ i $\leq$ 25, × (or $\times_n$)}, × (or $\times_n$)} be the MOD rectangular natural neutrosophic matrix subset semigroup under × (or $\times_n$).

i) Study questions (i) to (ix) of problem (27) for this S(S).

ii) Compare this S(S) with S(B) in problem 27.

29. Let $P = \{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} / a_i \in S(Z_{12}^I \times Z_{40}^I) =$

{Collection of all subsets from $Z_{12}^I \times Z_{40}^I = \{(a, b) / a \in Z_{12}^I, b \in Z_{40}^I\}\}$, $1 \leq i \leq 18, +\}$ be the MOD rectangular natural neutrosophic subset matrix semigroup under +.

i) Find o(P).

ii) Prove P is a commutative monoid under +.

iii) Find all nontrivial idempotents of P.

iv) Find all subsemigroups of P which are groups.

v) Find all subsemigroups of P which are not groups.

vi) Prove P is a S-semigroup.

viii) Obtain any other special feature associated with P.

30. Let S = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \right.$ / $a_i \in S(Z_{19}^I \times Z_{23}^I)$ = {Collection of all subsets from $S(Z_{19}^I \times Z_{23}^I)$ = {Collection of all subsets from $Z_{19}^I \times Z_{23}^I$ = {(a, b) / a $\in Z_{19}^I$, b $\in Z_{23}^I$}, $1 \leq i \leq 9$, +} be the MOD rectangular natural neutrosophic subset matrix semigroup under +.

i) Study questions (i) to (vii) of problem (29) for this S.

ii) Compare this S with P of problem (29).

31. Let M = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix} \right.$ / $a_i \in S(Z_{40}^I \times Z_{28}^I)$ = {Collection of all subsets from $Z_{40}^I \times Z_{28}^I$ = {(a, b) / a $\in Z_{40}^I$, b $\in Z_{28}^I$}}, $1 \leq i \leq 20$, +} be the MOD rectangular natural neutrosophic subset matrix semigroup under +.

Study questions (i) to (vii) of problem (29) for this M.

32. Let L = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\right.$ / $a_i \in S(Z_{15}^I \times Z_9^I)$ = {Collection of all subsets from $Z_{15}^I \times Z_9^I$ = {(a, b) / a $\in Z_{15}^I$, b $\in Z_9^I$}, $1 \leq i \leq 6$, $\times_n$} be the MOD rectangular natural neutrosophic subset matrix semigroup under natural product $\times_n$.

i) Find o(L).

ii) Prove L is a commutative monoid.

iii) Find all subset matrix zero divisors of L.

iv) Does L contain nontrivial subset matrix idempotents?

v) Does L contain nontrivial subset matrix nilpotents?

vi) Is L a S-semigroup?

vii) Can L have subsemigroups which are groups?

viii) Find all subset matrix subsemigroups which are not ideals.

ix) Find all subset matrix subsemigroups which are ideals.

x) Can L contain S-ideals?

xi) Can L contain S-zero divisors?

xii) Can L contain S-idempotent?

33. Let M = $\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix}$ / $a_i \in S\ (Z_{12}^I \times Z_{25}^I)$ =

{Collection of all subsets from $Z_{12}^I \times Z_{25}^I = \{(a, b)$ / $a \in Z_{12}^I$, $b \in Z_{25}^I\}\}$, $1 \leq i \leq 8$, $\times_n\}$ be the MOD rectangular natural neutrosophic subset matrix semigroup under $\times_n$.

Study questions (i) to (xii) of problem (32) for this M.

34. Let V = $\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix}$ / $a_i \in S\ (Z_{36}^I \times Z_{49}^I)$

= {Collection of all subsets from $Z_{26}^I \times Z_{49}^I = \{(a, b)$ / $a \in Z_{26}^I$, $b \in Z_{49}^I\}$ = (a, b) / $a \in Z_{26}^I$, $b \in Z_{49}^I\}$, $1 \leq i \leq 25$, $\times$ (or $\times_n$) be the MOD rectangular natural neutrosophic subset matrix semigroup under $\times$ (or $\times_n$).

i) Can questions (i) to (xii) of problem (32) be analysed for this V.

ii) Study only those questions (i) to (xii) of problem (32) which are relevant to (V, $\times$).

iii) Find all right ideals of (V, $\times$) which are not left ideals of (V,$\times$) and vice versa.

iv) Find all right zero divisors of (V, $\times$) which are not left zero divisors and vice versa.

v) Compare (V, $\times_n$) with (V,$\times$).

35. Let P[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_{18}^I \times Z_{42}^I$ = {(a, b) / a $\in Z_{18}^I$, b $\in Z_{42}^I$ +}, +} be the MOD rectangular natural neutrosophic coefficient polynomial semigroup under +.

i) Prove P[x] is a commutative monoid of infinite order.

ii) Prove P[x] has subsemigroups of finite order which are infinite in number.

iii) Prove P[x] also has subsemigroups of infinite order.

iv) Prove P[x] has idempotents.

v) Enumerate any other special feature associated with P[x].

36. Let S[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_{29}^I \times Z_5^I$ = {(a, b) / a $\in Z_{29}^I$, b $\in Z_5^I$ }, +} be the MOD rectangular natural neutrosophic coefficient polynomial semigroup under +.

i) Study questions (i) to (v) of problem (35) for this S[x].

ii) Compare S[x] with P[x] of problem 35.

37. Let $B[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in \{Z_{12}^I \times Z_{17}^I = \{(a, b) / a \in Z_{12}^I$, $b \in Z_{17}^I\}$, $+\}$ be the MOD rectangular natural neutrosophic coefficient polynomial semigroup under +.

Study questions (i) to (v) of problem (35) for this B[x].

38. Let $W[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_{48}^I \times Z_{75}^I\}$, $\times\}$ be the MOD rectangular natural neutrosophic coefficient polynomial semigroup under ×.

i) Find all zero divisors of W[x].

ii) Is W[x] a S-semigroup?

iii) Can W[x] have nilpotent polynomials?

iv) Prove W[x] has no nontrivial idempotent polynomials.

v) Prove W[x] is a commutative monoid of infinite order.

vi) Find all subsemigroups which are not ideals of W[x].

vii) Find all ideals of W[x].

viii) Find all S-ideals of W[x].

ix) Prove all subsemigroups and ideals are only of infinite order.

x) Obtain any other special feature associated with W[x].

39. Let $V[x] = \{ \sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_7^I \times Z_{43}^I = \{(a, b) / a \in Z_7^I$, $b \in Z_{43}^I\}$, $\times\}$ be the MOD rectangular natural neutrosophic coefficient polynomial semigroup.

i) Study questions (i) to (x) of problem (38) for this V[x].

ii) Compare this V[x] with W[x] of problem 38.

40. Let $B[x] = \{ \sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_{48}^I \times Z_{29}^I = \{(a, b) / a \in Z_{48}^I$, $b \in Z_{29}^I\}$, $\times\}$ be the MOD rectangular natural neutrosophic coefficient polynomial semigroup under product.

Study questions (i) to (x) of problem (38) for this B[x].

41. Let $M[x]_9 = \{ \sum_{i=0}^{9} a_i x^i$ / $a_i \in Z_{12}^I \times Z_{43}^I = \{(a, b) / a \in Z_{12}^I$, $b \in Z_{43}^I\}$, $x^{10} = 1, +\}$ be the MOD rectangular natural neutrosophic coefficient finite degree polynomial semigroup under +.

i) Find $o(M[x]_9)$.

ii) Prove $M[x]_9$ is commutative monoid.

iii) Prove $M[x]_9$ is a S-semigroup.

iv) Find all subsemigroups which are groups.

v) Find all subsemigroups which are not groups.

vi) Find all idempotents in $M[x]_9$.

vii) Enumerate any other special feature associated with $M[x]_9$.

42. Let $V[x]_{18}$ = = $\{\sum_{i=0}^{18} a_i x^i$ / $a_i \in Z_{23}^I \times Z_{11}^I$ = {(a, b) / a $\in$ $Z_{23}^I$, b $\in$ $Z_{11}^I$}, $x^{19}$ = 1, +} be the MOD rectangular natural neutrosophic coefficient finite degree polynomial semigroup under +.

i) Study questions (i) to (vii) of problem (41) for this $V[x]_{18}$.

ii) Compare $V[x]_{18}$ with $M[x]_9$ in problem 41.

43. Let $S[x]_{27}$ = $\{\sum_{i=0}^{27} a_i x^i$ / $a_i \in Z_{48}^I \times Z_{210}^I$ = {(a, b) / a $\in$ $Z_{48}^I$, b $\in$ $Z_{210}^I$ }, $x^{28}$ = 1, +} be the MOD rectangular natural neutrosophic coefficient finite degree polynomial semigroup under +.

Study questions (i) to (vii) of problem (41) for this $S[x]_{27}$.

44. Let $M[x]_7$ = $\{\sum_{i=0}^{7} a_i x^i$ / $a_i \in Z_{12}^I \times Z_{105}^I$ = {(a, b) / a $\in$ $Z_{12}^I$, b $\in$ $Z_{105}^I$ }, $x^8$ = 1, ×} be the MOD rectangular natural neutrosophic coefficient finite degree polynomial semigroup under ×.

i) Find $o(M[x]_7)$.
ii) Prove $M[x]_7$ is a commutative monoid.
iii) Show $M[x]_7$ has zero divisors.
iv) Prove $M[x]_7$ cannot have nontrivial idempotents.
v) Prove $M[x]_7$ has nontrivial nilpotents.
vi) Is $M[x]_7$ a S-semigroup?
vii) Prove $M[x]_7$ has ideals.
viii) Prove $M[x]_7$ has subsemigroups which are not ideals.
ix) Can $M[x]_7$ has S–zero divisors?

x) Can $M[x]_7$ have S–ideals?

xi) Enumerate any special feature associated with $M[x]_7$.

45. Let $B[x]_{126} = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_{47}^I \times Z_{23}^I = \{(a, b)$ / $a \in Z_{47}^I$, $b \in Z_{23}^I\}$, $x^{127} = 1, \times\}$ be the MOD rectangular natural neutrosophic coefficient finite degree polynomial semigroup under $\times$.

Study questions (i) to (x) of problem 44 for this $B[x]_{126}$.

46. Let $S[x]_{189} = \{\sum_{i=0}^{189} a_i x^i$ / $a_i \in Z_{48}^I \times Z_{39}^I = \{(a, b)$ / $a \in Z_{48}^I$, $b \in Z_{39}^I = \{(a, b)$ / $a \in Z_{48}^I$, $b \in Z_{39}^I\}$, $x^{190} = 1, \times\}$ be the MOD rectangular natural neutrosophic coefficient polynomial of finite degree semigroup under $\times$.

Study questions (i) to (x) of problem (44) for this $S[x]_{189}$.

47. Let $B[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{10}^I \times Z_{19}^I) = \{$Collection of all subsets from $Z_{10}^I \times Z_{19}^I = \{(a, b)$ / $a \in Z_{10}^I$, $b \in Z_{19}^I\}$, $+\},+\}$ be the MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under $+$.

i) Prove B[x] is a commutative monoid of infinite order.

ii) Show B[x] idempotents with respect to +.

iii) Prove B[x] has subsemigroups of finite order.

iv) Is B[x] a S-semigroup?

v) Prove B[x] also has subsemigroups of infinite order.

48. Let S[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{24}^I \times Z_{43}^I$ = {Collection of all subsets from $Z_{24}^I \times Z_{43}^I = \{(a, b) / a \in Z_{24}^I, b \in Z_{43}^I$ = $\{(a, b) / a \in Z_{24}^I, b \in Z_{43}^I\}$, +},+} be the MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under +.

Study questions (i) to (v) of problem (47) for this S[x].

49. Let W[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{20}^I \times Z_{47}^I)$ = {Collection of all subsets from $Z_{20}^I \times Z_{47}^I = \{(a, b) / a \in Z_{20}^I, b \in Z_{47}^I\}$, ×}, ×} be the MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under ×.

i) Prove W[x] is an infinite commutative monoid.

ii) Prove W[x] has infinite number of zero divisors.

iii) Prove W[x] cannot have nontrivial idempotents.

iv) Can W[x] have nontrivial nilpotents?

v) Is W[x] a S-semigroup?

vi) Can W[x] have S-zero divisors?

vii) Can W[x] have S-ideals?

viii) Prove all ideals and subsemigroups are of infinite order.

ix) Can W[x] have a S-subsemigroup which is not a S-ideal?

x) Obtain any other special feature associated with W[x].

50. Let $V[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{19}^I \times Z_{53}^I)$ = {Collection of all subsets from $Z_{19}^I \times Z_{53}^I$ = {(a, b) / a $\in Z_{19}^I$, b $\in Z_{53}^I$}, ×}, ×} be the MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under ×.

i) Study questions (i) to (x) of problem (49) for this V[x].

ii) Comparative W[x] of problem 49 with this V[x].

51. Let $B[x] = \{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{24}^I \times Z_{45}^I)$ = {Collection of all subsets from $Z_{24}^I \times Z_{45}^I$ = {(a, b) / a $\in Z_{24}^I$, b $\in Z_{45}^I$}, ×}, ×} be the MOD rectangular natural neutrosophic subset coefficient polynomial semigroup under ×.

i) Study questions (i) to (x) of problem (49) for this B[x].

ii) Compare this B[x] with V[x] and W[x] of problems 50 and 49 respectively.

52. Let $M[x] = \{\sum_{i=0}^{8} a_i x^i$ / $a_i \in S(Z_7^I \times Z_{48}^I)$ = {Collection of all subsets from $Z_7^I \times Z_{48}^I$ = {(a, b) / a $\in Z_7^I$, b $\in Z_{48}^I$}, $x^9$ =1; +},+} be MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroup under +.

i) Find $o(M[x]_8)$.

ii) Prove $M[x]_8$ is a commutative monoid.

iii) Is $M[x]_8$ a S-semigroup?

iv) Prove $M[x]_8$ has idempotents.

v) Can $M[x]_8$ have S-idempotents?

vi) Find all subsemigroups of $M[x]_8$.

vii) Can $M[x]_8$ have subsemigroups which are not ideals? Justify your claim.

viii) Enumerate any other special feature associated with $M[x]_8$.

53. Let $B[x]_{18} = \{\sum_{i=0}^{18} a_i x^i$ / $a_i \in S(Z_{18}^I \times Z_{42}^I)$ = {Collection of all subsets from $Z_{18}^I \times Z_{42}^I$ = {(a, b) / a $\in Z_{18}^I$, b $\in Z_{42}^I$}, +} $x^{19}$ = 1, +} be the MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroup under +.

Study questions (i) to (vii) problem (52) for this $B[x]_{18}$.

54. Let $S[x]_7 = \{\sum_{i=0}^{27} a_i x^i$ / $a_i \in S(Z_{11}^I \times Z_3^I)$ = {Collection of all subsets from $Z_{11}^I \times Z_3^I$ = {(a, b) / a $\in Z_{11}^I$, b $\in Z_3^I$}, +}, $x^{28}$ = 1, +} be the MOD rectangular natural

neutrosophic subset coefficient polynomials of finite degree semigroup under +.

i) Study questions (i) to (viii) of problem (52) for this $S[x]_{27}$.

ii) Compare $S[x]_{27}$ with $B[x]_{18}$ in problem 53.

55. Let $S[x]_{14} = \{ \sum_{i=0}^{14} a_i x^i$ / $a_i \in S(Z^I_{22} \times Z^I_{26}) = \{$Collection of all subsets from $Z^I_{22} \times Z^I_{26} = \{(a, b) / a \in Z^I_{22}$, $b \in Z^I_{26}\}$, $\times\}$, $x^{15} = 1, \times\}$ be the MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroups under $\times$.

i) Find the order of $S[x]_{14}$.

ii) Prove $S[x]_{14}$ is a commutative monoid.

iii) Prove $S[x]_{14}$ has both zero divisors as well as natural neutrosophic zero divisors.

iv) Can $S[x]_{14}$ have S-zero divisors and (or) S-natural neutrosophic zero divisors?

v) Can $S[x]_{14}$ have S-ideals?

vi) Is $S[x]_{14}$ a S-semigroups?

vii) Find all S-subsemigroups of $S[x]_{14}$ which are not S-ideals.

viii) How many ideals are in $S[x]_{14}$ which are not S-ideals?

ix) Enumerate any other special feature enjoyed by $S[x]_{14}$.

56. Let $M[x]_{27} = \{\sum_{i=0}^{27} a_i x^i$ / $a_i \in S(Z^I_{28} \times Z^I_{43})$ = {Collection of all subsets from $Z^I_{28} \times Z^I_{43} = \{(a, b)$ / $a \in Z^I_{28}$, $b \in Z^I_{43}\} \times\}$ $x^{28} = 1, \times\}$ be the MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroup under product.

Study questions (i) to (ix) of problem (55) for this $M[x]_{27}$.

57. Let $S[x]_{32} = \{\sum_{i=0}^{32} a_i x^i$ / $a_i \in S(Z^I_{27} \times Z^I_{90})$ = {Collection of all subsets from $Z^I_{27} \times Z^I_{90} = \{(a, b)$ / $a \in Z^I_{27}$, $b \in Z^I_{90} = \{(a, b)$ / $a \in Z^I_{27}$, $b \in Z^I_{90}\}, \times\}$, $x^{33} = 1, \times\}$ be the MOD rectangular natural neutrosophic subset coefficient finite degree polynomial semigroup under $\times$.

Study questions (i) to (ix) of problem 55 for this $S[x]_{32}$.

58. Let S(M[x]) = {collection of all subsets from M[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z^I_{24} \times Z^I_{49} = \{(a, b)$ / $a \in Z^I_{24}$, $b \in Z^I_{49}$, $+\}, +\}$ be the MOD rectangular natural neutrosophic polynomial subset semigroup under +.

i) Show S(M[x]) is a commutative monoid of infinite order.

ii) Prove (M[x]) can have both finite order subsemigroups and infinite order subsemigroups.

iii) Prove S(M[x]) has idempotents.

iv) Obtain all special features enjoyed by S(M[x]).

vi) Is S(M[x]) a S-semigroup?

59. Let S(B[x]) = {collection of all subsets from B[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S(Z_{24}^I \times Z_{43}^I) =$ {Collection of all subsets from $Z_{43}^I \times Z_3^I = \{(a, b) / a \in Z_{43}^I, b \in Z_3^I\}$, +},+} be the MOD rectangular natural neutrosophic polynomial subset semigroup under +.

i) Study questions (i) to (v) of problem (58) for this S(B[x]).

ii) Compare this S(B[x]) with S(M[x]) of problem 58.

iii) Let S(S[x]) = {collection of all polynomial subsets from S[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_{44}^I \times Z_{13}^I =$ $\{(a, b) / a \in Z_{44}^I, b \in Z_{13}^I$ ,+}, +} be the MOD rectangular natural neutrosophic polynomial subset semigroup under +.

Study questions (i) to (v) of problem (58) for this S(S[x]).

60. Let S(P[x]) = {Collection of all polynomial subsets from P[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_{42}^I \times Z_{280}^I = \{(a, b) / a \in Z_{42}^I, b \in Z_{280}^I\}$, ×}, ×} be the MOD rectangular natural neutrosophic polynomial subset semigroup under ×.

i) Show S(P[x]) is a commutative monoid of infinite order.

ii) Prove S(P[x]) has infinite number of zero divisors.

iii) Can S(P[x]) have S-zero divisors?
iv) Is S(P[x]) a S-semigroup?
v) Can S(P[x]) have S-ideals?
vi) Prove S(P[x]) has no idempotents.
vii) Can S(P[x]) have nilpotents?
viii) Prove S(P[x]) cannot have subsemigroups of finite order.
ix) Can S(P[x]) have subsemigroups which are not S-subsemigroups?
x) Obtain all special features associated with S(P[x]).

61. Let S(X[x]) = {Collection of all polynomial subsets from W[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in (Z_{43}^I \times Z_7^I)$ = {(a, b) / a $\in$ $Z_{43}^I$, b $\in$ $Z_7^I$}, ×}, ×} be the MOD rectangular natural neutrosophic polynomial subset semigroup under ×.

Study questions (i) to (x) of problem (60) for this S(W[x]).

62. Let S(V[x]) = {collection of all polynomial subsets from V[x] = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in Z_{27}^I \times Z_4^I$ = {(a, b) / a $\in$ $Z_{27}^I$, b $\in$ $Z_4^I$}, ×}, ×} be the MOD rectangular natural neutrosophic polynomial subset semigroup under ×.

Study questions (i) to (x) of problem (60) for this S(V[x]).

63. Let $S(B[x]_9)$ = {collection of all finite degree polynomial subsets from $B[x]_9$ = $\{\sum_{i=0}^{9} a_i x^i$ / $a_i \in Z_{12}^I \times$

$Z_2^I = \{(a, b) / a \in Z_{12}^I, b \in Z_2^I\}, x^{10} = 1, +\},+\}$ be the MOD rectangular finite degree polynomial subset semigroup under +.

i) Find o(S(B[x]$_9$).

ii) Prove S(B[x]$_9$) is a commutative monoid under +.

iii) Find all idempotents of S(B[x]$_9$).

iv) Is S(B[x]$_9$) a S-semigroup?

v) Find all subsemigroups of S(B[x]$_9$) which are groups of S(B[x]$_9$) and those which are not groups.

vi) Find all subsemigroups of S(B[x]$_9$) which are not groups.

vii) Find all S-subsemigroups of S(B[x]$_9$).

viii) Find all subsemigroups which are not S-subsemigroups.

64. Let S(M[x]$_{23}$) = {collection of all finite degree polynomial subsets from M[x]$_{23}$ = $\{\sum_{i=0}^{23} a_i x^i$ / $a_i \in Z_{43}^I \times Z_{23}^I = \{(a, b) / a \in Z_{43}^I, b \in Z_{23}^I\}, x^{24} = 1, +\}$ be the MOD rectangular natural neutrosophic polynomial subset semigroup.

Study questions (i) to (viii) of problem (63) for this S(M[x]$_{23}$).

65. Let S(B[x]$_{10}$) = {collection of all finite degree polynomial subsets from B[x$_{10}$] = $\{\sum_{i=0}^{10} a_i x^i$ / $a_i \in Z_{43}^I \times Z_{16}^I = \{(a, b) / a \in Z_{43}^I, b \in Z_{16}^I\}, x^{10} = 1\}, +\}$ be the

MOD rectangular natural neutrosophic finite degree polynomial subset semigroup under +.

i) Study questions (i) to (viii) of problem (63) for this $S(B[x]_{10})$.

ii) Compare $S(B[x]_{10})$ with $(M[x]_{23})$ of problem 64.

66. Let $S(M[x]_{45})$ = {collection of all polynomial subsets from $M[x]_{45} = \{\sum_{i=0}^{45} a_i x^i$ / $a_i \in Z_{12}^I \times Z_{45}^I = \{(a, b)$ / $a \in Z_{12}^I$, $b \in Z_{45}^I\}$, $x^{46} = 1$, ×} be the MOD rectangular natural neutrosophic finite degree polynomial subset semigroup under ×.

i) Find $o(S(M[x]_{45}))$.

ii) Prove $S(M[x]_{45})$ is a commutative monoid.

iii) Find all zero divisors of $S(M[x]_{45})$.

iv) Can $S(M[x]_{45})$ have S-zero divisors?

v) Prove $S(M[x]_{45})$ cannot have nontrivial idempotents,

vi) Show $S(M[x]_{45})$ has subsemigroups which are not ideals.

vii) Is $S(M[x]_{45})$ a S-semigroup?

viii) Find all ideals of $S(M[x]_{45})$.

ix) Can $S(M[x]_{45})$ have S-ideals?

x) Can $S(M[x]_{45})$ have nontrivial nilpotents?

xi) Can $S(M[x]_{45})$ have S-subsemigroups which are not S-ideals?

xii) Enumerate all special features associated with $S(M[x]_{45})$.

67. Let $S(B[x]_{27})$ = {collection of all polynomial subsets from $B[x]_{27} = \{\sum_{i=0}^{27} a_i x^i / a_i \in Z_{19}^I \times Z_{43}^I = \{(a, b) / a \in Z_{19}^I, b \in Z_{43}^I\}, x^{28} = 1, \times\}, \times\}$ be the MOD rectangular natural neutrosophic finite degree polynomial subset semigroup under ×.

i) Study questions (i) to (xii) of problem 66 for this $S(B[x]_{27})$.

ii) Compare $S(B[x]_{27})$ with $S(M[x]_{45})$ of problem.

68. Let $S(S[x]_{45})$ = {Collection of all finite degree polynomial subsets from $S[x]_{45} = \{\sum_{i=0}^{45} a_i x^i / a_i \in Z_{16}^I \times Z_{625}^I = \{(a, b) / a \in Z_{16}^I, b \in Z_{625}^I\}, \times\}, x^{46} = 1\}, \times\}$ be the MOD rectangular natural neutrosophic finite degree polynomial subset semigroup under ×.

Study questions (i) to (xii) of problem (66) for this $(S[x]_{45}))$.

69. Let S = {collection of all elements from ${}^I[0,8) \times {}^I[0, 24)$ $= \{(a, b) / a \in {}^I[0, 8), b \in {}^I[0, 24)\}, +\}$ be the MOD rectangular natural neutrosophic interval semigroup under +.

i) Prove S is only a commutative monoid of infinite order.

ii) Prove S has subsemigroups which are subgroups of both finite or infinite order.

iii) Prove S is a S-semigroup.

iv) Find all idempotents of S.

v) Enumerate any other special feature associated with S.

70. Let W= $\{{}^{I}[0, 19) \times {}^{I}[0, 43) = \{(a, b) / a \in {}^{I}[0, 19), b \in {}^{I}[0,43)\}, +\}$ be the MOD rectangular natural neutrosophic interval semigroup under +.

Study questions (i) to (v) of problem 69 for this W.

71. Let P = $\{{}^{I}[0,41) \times {}^{I}[0, 37) = \{\{a, b) / a \in {}^{I}[0, 41), b \in {}^{I}[0, 37)\}, \times_0$ (or $\times)\}$ be the MOD rectangular natural neutrosophic interval semigroup under $\times_0$ (or $\times$).

i) Prove P is a commutative monoid of infinite order.

ii) Prove P has infinite number of zero divisors.

iii) Prove P has subsemigroups of finite and infinite order which are not ideals.

iv) Prove all ideals of P are of infinite order.

v) Can P have nontrivial idempotents? Justify.

vi) Can P have nontrivial nilpotents? Justify.

vii) Is P a S-semigroup?

viii) Can P have S-zero divisors?

ix) Can P have S-ideals?

72. Let M = $\{{}^{I}[0,48) \times {}^{I}[0,30) = \{(a, b) / a \in {}^{I}[0, 48), b\ {}^{I}[0,30)\}\ \times$ (or $\times_0)\}$ be the MOD rectangular natural neutrosophic interval semigroup under $\times$ (or $\times_0$).

Study questions (i) to (ix) of problem (71) for this M.

73. Let S = {S($^I[0,48) \times {}^I[0,25)$) = {(collection of all subsets from $^I[0,48) \times {}^I[0,25)$ = {(a, b) / a ∈ $^I[0,48)$, b ∈ $^I[0,25)$}, +}, +} be the MOD rectangular natural neutrosophic interval subset semigroup under +.

Study questions (i) to (v) of problem (69) for this S.

74. Let B = {S ($^I[0,42) \times {}^I[0,20)$) = {collection of all subsets from $^I[0,42) \times {}^I[0,20)$ = {(a, b) / a ∈ $^I[0, 42)$, b ∈ $^I[0,20)$}, $\times_0$ (or ×)}, be the MOD rectangular natural neutrosophic subset interval semigroup under $\times_0$ (or ×).

Study questions (i) to (ix) of problem (71) for this B.

75. Let M = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \right.$ / $a_i \in {}^I[0, 47) \times {}^I[0, 31)$ = {(a, b) /

a ∈ $^I[0, 47)$, b ∈ $^I[0, 31)$, + ; $1 \leq i \leq 12$} be the MOD rectangular natural neutrosophic interval matrix semigroup under +.

i) Prove M is a commutative monoid of infinite order.

ii) Can M has matrix idempotents?

iii) Prove M has matrix idempotents.

iv) Is M a S-matrix semigroup?

v) Find all subgroups of M which are of finite order.

vi) Find all infinite order subsemigroups of M.

vii) Enumerate any other special feature enjoyed by M.

76. Let B = $\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix}$ / $a_i \in {}^I[0, 48) \times {}^I[0,16)$

= $\{(a, b) / a \in {}^I[0,48), b \in {}^I[0,16)\}$, +; $1 \leq i \leq 10\}$ be the MOD rectangular natural neutrosophic interval matrix semigroup under +.

i) Study questions (i) to (vii) of problem (75) for this B.

ii) Compare this B with M of problem (75).

77. Let W = $\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix}$ / $a_i \in {}^I[0, 44) \times {}^I[0,23)$ =

$\{(a, b) / a \in {}^I[0,44), b \in {}^I[0,23)\}$, $1 \leq i \leq 20$, $\times_n\}$ be the MOD rectangular natural neutrosophic interval matrix semigroup under natural product $\times_n$.

i) Prove W is an infinite order commutative monoid.

ii) Prove W has infinite number of zero divisors.

iii) Can W be a S-semigroup?

iv) Can W have S-zero divisors?

v) Can W have matrix idempotents?

vi) Can W have nontrivial matrix nilpotents?

vii) Can W have matrix S-idempotents?

viii) Prove all ideals of W are of infinite order.

ix) Can W have S-ideals?

x) Can W have units and S-units?

xi) Obtain any other special features associated with W.

78. Let $B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} / a_i \in {}^I[0, 48) \times {}^I[0,38) = \{(a, b) / a \in\right.$ ${}^I[0,48)$, $b \in {}^I[0,38)\}, +; 1 \leq i \leq 9, \times_n\}$ be the MOD rectangular natural neutrosophic interval matrix semigroup under natural product $\times_n$.

Study questions (i) to (xi) of problem 77 for this B.

79. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \right. / a_i \in {}^I[0, 49) \times$

${}^I[0,28) = \{(a, b) / a \in {}^I[0,49), b \in {}^I[0,28)\}$, $1 \leq i \leq 25$, $\times$ (or $\times_n$)} be the MOD rectangular natural neutrosophic interval matrix semigroup under the usual product $\times$ or the natural product $\times_n$.

i) Study questions (i) to (xi) of problem 77 in case of $(S,\times)$ and $(S, \times_n)$.

ii) Prove $(S,\times)$ is a non-commutative monoid of infinite order.

iii) Find all right ideals which are not left ideals and vice versa in $(S,\times)$.

iv) Can $(S,\times)$ contain right zero divisors which are not left zero divisors and vice versa?

v) Enumerate all special features enjoyed by $(S,\times)$.

vi) Compare $(S,\times)$ with $(S, \times_n)$.

80. Let $B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right. / a_i \in {}^I[0, 37) \times {}^I[0,29) = \{(a, b)$

$/ a \in {}^I[0,37), b \in {}^I[0,29)\}$, $1 \leq i \leq 9$, $\times$ (or $\times_n$)} be the MOD rectangular natural neutrosophic interval matrix semigroup.

Study questions (i) to (vi) of problem (79) for this $(W,\times)$ and $(W, \times_n)$.

81. Let $P = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$ / $a_i \in S({}^I[0, 7) \times {}^I[0,48))$ = {collection of all subsets from ${}^I[0,7) \times {}^I[0,48) = \{(a, b) / a \in {}^I[0,7), b \in {}^I[0,48)\}$, +; $1 \leq i \leq 5$, +} be the MOD rectangular natural neutrosophic interval subset matrix semigroup under +.

Study questions (i) to (vii) of problem (75) for this P.

82. Let $B = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$ / $a_i \in S ({}^I[0, 40) \times {}^I[0,28)$ = {collection of all subsets from ${}^I[0,40) \times {}^I[0,28) = \{(a, b) / a \in {}^I[0,40), b \in {}^I[0,28), +\}$, $1 \leq i \leq 24$, +}be the MOD rectangular natural neutrosophic interval matrix subset semigroup under +.

i) Study questions (i) to (vii) of problem 75 for this B.

ii) Compare with this B the P of problem 81.

83. Let T = { $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{26} & a_{27} & a_{28} & a_{29} & a_{30} \end{bmatrix}$ / $a_i \in S({}^I[0, 144) \times$ ${}^I[0,126)$ = {Collection of all subsets from ${}^I[0,144) \times {}^I[0,126)$ = {(a, b) / a ∈ ${}^I[0,144)$, b ∈ ${}^I[0,126)$} $\times_n$}, $\times_n$, $1 \leq i \leq 13$} be the MOD rectangular natural neutrosophic subset matrix semigroup under the natural product.

i) Study questions (i) to (xi) of problem 79 for this T.

ii) Obtain any other special feature associated with T.

84. Let B = { $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}$ / $a_i \in S\,({}^I[0, 9) \times {}^I[0,420)$ = {(a, b) / a ∈ ${}^I[0,9)$, b ∈ ${}^I[0,420)$}, ×} $1 \leq i \leq 16$, × (or $\times_n$)} be the MOD rectangular natural neutrosophic interval subset matrix semigroup under usual product × or the natural product $\times_n$.

Study questions (i) to (vi) of problem (79) for this (B, ×) and (B, $\times_n$).

85. Let S(B) = {Collection of all matrix subsets with entries

$$\text{from } B = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} / a_i \in {}^{I}[0, 4) \times {}^{I}[0,21) = \{(a, b) /$$

$a \in {}^{I}[0,4)$, $b \in {}^{I}[0,21)\}$, $1 \leq i \leq 5$, $+\}$ be the MOD rectangular natural neutrosophic interval matrix subset semigroup under +.

Study questions (i) to (vii) of problem (75) for this S(B).

86. Let S(M) = {Collection of all matrix subsets from

$$M = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \text{ where } a_i \in {}^{I}[0, 19) \times {}^{I}[0, 325) = \{(a, b) /$$

$a \in {}^{I}[0, 19)$, $b \in {}^{I}[0, 325)\}$, $1 \leq i \leq 7$, $\times_n\}$, $\times_n\}$ be the MOD rectangular natural neutrosophic interval matrix subset semigroup and the natural product $\times_n$.

i) Study questions (i) to (xi) of problem (77) for this S(M).

ii) Obtain all special features enjoyed by S(M).

87. Let S(W) = {Collection of all matrix subsets from

$$W = \{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} / a_i \in {}^{I}[0, 42) \times {}^{I}[0, 12) =$$

$\{(a, b) / a \in {}^{I}[0, 42), b \in {}^{I}[0, 12)\}$, $1 \leq i \leq 16$, $\times$ (or $\times_n$)}, $\times$ (or $\times_n$)} (where $\times$ is the usual product on S(W)) be the MOD rectangular natural neutrosophic interval matrix subset semigroup under $\times$ (or $\times_n$).

Study questions (i) to (vi) of problem (79) for this {S(W), $\times$} or (S(W), $\times_n$).

88. Let $P[x] = \{\sum_{i=0}^{\infty} a_i x^i / a_i \in {}^{I}[0, 10) \times {}^{I}[0, 48) = \{(a, b) / a \in {}^{I}[0, 10), b \in {}^{I}[0, 48)\}, +\}$ be the MOD rectangular natural neutrosophic interval coefficient polynomial semigroup under +.

i) Prove P[x] is an infinite order commutative monoid.

ii) Prove (P[x], +) has idempotent polynomial.

iii) Prove (P[x],+) has infinite number finite order subsemigroups under +.

iv) Is P[x] a S-semigroup?

v) Find all subsemigroups of infinite order in P[x].

vi) Give any other special feature enjoyed by {P[x], +}.

89. Let $V[x] = \{\sum_{i=0}^{\infty} a_i x^i / a_i \in {}^I[0, 43) \times {}^I[0, 23) = \{(a, b) / a \in {}^I[0, 43), b \in {}^I[0, 23)\}, +\}$ be the MOD rectangular natural neutrosophic interval coefficient semigroup under +.

    Study questions (i) to (vi) of problem (88) for this V[x].

90. Let $S[x] = \{\sum_{i=0}^{\infty} a_i x^i / a_i \in {}^I[0, 40) \times {}^I[0, 126) = \{(a, b) / a \in {}^I[0, 40), b \in {}^I[0, 126)\} \times_0 (\text{or } \times)\}$ be the MOD rectangular natural neutrosophic interval coefficient polynomial semigroup under $\times_0$ (or $\times$).

    i) Prove S[x] is an infinite commutative monoid.
    ii) Prove S[x] has infinite number of zero divisors.
    iii) Can S[x] have S–zero divisors?
    iv) Prove all subsemigroups and ideals of S[x] are of infinite order.
    v) Prove S[x] has no polynomial idempotents.
    vi) Can S[x] have nontrivial nilpotents?
    vii) Can S[x] have S-ideals?
    viii) Obtain any other special feature enjoyed by S[x].

91. Let $B[x] = \{\sum_{i=0}^{\infty} a_i x^i / a_i \in {}^I[0, 19) \times {}^I[0, 31)) = \{(a, b) / a \in {}^I[0, 19), b \in {}^I[0, 31)\}, \times_0 (\text{or } \times)\}$ be the MOD rectangular natural neutrosophic interval coefficient polynomial semigroup under $\times_0$ (or $\times$).

Study questions (i) to (viii) of problem (90) for this B[x].

92. Let S(B[x]) = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S\ ({}^{I}[0, 40) \times {}^{I}[0, 13))$ = {Collection of all subsets from ${}^{I}[0, 40) \times {}^{I}[0, 13)$ = (a, b) / a $\in {}^{I}[0, 40)$, b $\in {}^{I}[0, 13)$}, +}, +} be the MOD rectangular natural neutrosophic interval subset coefficient polynomial semigroup under +.

Study questions (i) to (vi) of problem (88) for this S(B[x]).

93. Let S(W[x]) = $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in S\ ({}^{I}[0, 28) \times {}^{I}[0, 35))$ = {Collection of all subsets from ${}^{I}[0, 28) \times {}^{I}[0, 35)$}, $\times_0$ (or $\times$)} $\times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval, subset coefficient polynomial semigroup under $\times_0$ (or $\times$).

Study questions (i) to (viii) of problem (90) for this S(W[x]).

94. Let S(V[x]) = {Collection of all polynomial subsets from V[x] $\{\sum_{i=0}^{\infty} a_i x^i$ / $a_i \in {}^{I}[0,43) \times {}^{I}[0, 24)$ = {(a, b) / a $\in {}^{I}[0,43)$, b $\in {}^{I}[0,24)$}, +},+} be the MOD rectangular natural neutrosophic interval polynomial subset semigroup under +.

Study questions (i) to (vi) of problem (88) for this S(V[x]).

95. Let S(P[x]) = {collection of all polynomial subsets from $P[x] = \{\sum_{i=0}^{\infty} a_i x^i / a_i \in {}^I[0,25) \times {}^I[0, 40) = \{(a, b) / a \in {}^I[0,25), b \in {}^I[0,40)\}, \times_0$ (or $\times$)},$\times_0$ (or $\times$)} be the MOD rectangular natural neutrosophic interval polynomial subset semigroup under $\times_0$ (or $\times$).

Study questions (i) to (viii) of problem (90) for this S(P[x]).

96. Let $P[x]_{20} = \{\sum_{i=0}^{9} a_i x^i / a_i \in {}^I[0, 3) \times {}^I[0, 40) = \{(a, b) / a \in {}^I[0, 3), b \in {}^I[0, 40)\}, +\}, +\}$ be the MOD rectangular natural neutrosophic interval coefficient of polynomial of degree less than or equal to 9.

Study questions (i) to (vi) of problem (88) for this $P[x]_9$.

97. If in problem (96) '+' is replaced by $\times_0$ (or $\times$) then for $\{P[x]_9, \times_0$ (or $\times$)}.

Study questions (i) to (viii) of problem 90.

98. Let $W[x]_{19} = \{\sum_{i=0}^{19} a_i x^i / a_i \in S\,({}^I[0, 42) \times {}^I[0, 89)) =$ {Collection of all subsets from ${}^I[0, 42) \times {}^I[0, 89)$ = (a, b) where $a \in {}^I[0, 42), b \in {}^I[0, 89)\}, +\}$ $x^{20} = 1, +\}$ be the MOD rectangular natural neutrosophic interval subset coefficient polynomials of degree less than or equal to 19 semigroup under +.

i) Study questions (i) to (vi) of problem (88) for this $W[x]_{19}$.

ii) Study questions (i) to (viii) of problem (96) for $\{W[x]_{19}, \times_0$(or $\times$)}.

99. Let $S(B[x]_{18})$ = {Collection of all polynomial subsets from $B[x]_{18} = \{\sum_{i=0}^{18} a_i x^i$ / $a_i \in {}^{I}[0, 23) \times {}^{I}[0, 40) = \{(a, b)$ / $a \in {}^{I}[0, 23)$, $b \in {}^{I}[0, 40)\}$, $x^{19} = 1\}$ be the MOD rectangular natural neutrosophic interval polynomial subset collection.

i) Study for $\{S(B[x]_{18}), +\}$ questions (i) to (vi) of problem (88).

ii) Study questions (i) to (viii) of problem (96) for $\{S(B[x]_{18}), \times_0(\text{or } \times)\}$.

# FURTHER READING


1. Lang, S., Algebra, Addison Wesley, (1967).
2. Salama, A.A. and Smarandache, F., Neutrosophic Crisp Set Theory, Educational Publisher, Columbus, Ohio, 2015.
3. Smarandache, F. (editor), Proceedings of the First International Conference on Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics, Univ. of New Mexico – Gallup, 2001. http://www.gallup.unm.edu/~smarandache/NeutrosophicProceedings.pdf
4. Smarandache, F. Neutrosophic Logic - Generalization of the Intuitionistic Fuzzy Logic, presented at the Special Session on Intuitionistic Fuzzy Sets and Related Concepts, of International EUSFLAT Conference, Zittau, Germany, 10-12 September 2003. http://lanl.arxiv.org/ftp/math/papers/0303/0303009.pdf
5. Smarandache, F., (editor), Multispace and Multistructure. Neutrosophic Transdisciplinarity (100 Collected Papers of Sciences), Vol. IV, North-European Scientific Publishers, Hanko, Finland, 800 p., 2010.
6. Smarandache, F., (editor), Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics, University of New Mexico, Gallup Campus, Xiquan, Phoenix, Arizona, 147 p., 2002.
7. Smarandache, F., Collected Papers III, Editura Abaddaba, Oradea, 2000. http://www.gallup.unm.edu/~smarandache/CP3.pdf

8. Smarandache, F., H. Wang, Y.-Q. Zhang, R. Sunderra man, Interval Neutrosophic Sets and Logic: Theory and Applications in Computing, Hexis, Phoenix, Arizona, 87 p., 2005.
9. Smarandache, F., Neutrosophic Theory and its Applications, Collected Papers, Vol. I, EuropaNova, Brussels, Belgium, 480 p., 2014.
10. Smarandache, Florentin, Definitions Derived from Neutrosophics, In Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics, University of New Mexico, Gallup, 1-3 December (2001).
11. Smarandache, Florentin, Neutrosophic Logic—Generalization of the Intuitionistic Fuzzy Logic, Special Session on Intuitionistic Fuzzy Sets and Related Concepts, International EUSFLAT Conference, Zittau, Germany, 10-12 September 2003.
12. Smarandache, F., Neutrosophy. Neutrosophic Probability, Set, and Logic, American Research Press, Rehoboth, 105 p., 1998. - Republished in 2000, 2003, 2005, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics (second, third, and respectively fourth edition), American Research Press, 156 p.; - Chinese translation by F. Liu, Xiquan Chinese Branch, 121 p., 2003; Сущность нейтрософии, Russian partial translation by D. Rabounski, Hexis, Phoenix, Arizona, 32 p., 2006. II. Neutrosophic Algebraic Structures - Edited Books
13. Vasantha Kandasamy, W.B., and Smarandache, F., Algebraic Structures using Subsets, Educational Publisher Inc, Ohio, (2013).
14. Vasantha Kandasamy, W.B., and Smarandache, F., Algebraic Structures using [0, n), Educational Publisher Inc, Ohio, (2013).
15. Vasantha Kandasamy, W.B., and Smarandache, F., Algebraic Structures on the fuzzy interval [0, 1), Educational Publisher Inc, Ohio, (2014).
16. Vasantha Kandasamy, W.B., and Smarandache, F., Algebraic Structures on Fuzzy Unit squares and Neutrosophic unit square, Educational Publisher Inc, Ohio, (2014).

17. Vasantha Kandasamy, W.B., and Smarandache, F., Algebraic Structures on Real and Neutrosophic square, Educational Publisher Inc, Ohio, (2014).
18. Vasantha Kandasamy, W.B., and Smarandache, F., Algebraic Structures on Real and Neutrosophic Semi Open Squares, Education Publisher, Columbus, Ohio, 2014.
19. Vasantha Kandasamy, W.B., and Smarandache, F., Basic Neutrosophic algebraic structures and their applications to fuzzy and Neutrosophic models, Hexis, Church Rock, 2004
20. Vasantha Kandasamy, W.B., and Smarandache, F., Dual Numbers, Zip Publishing, Ohio, (2012).
21. Vasantha Kandasamy, W.B., and Smarandache, F., Finite Neutrosophic Complex Numbers, Zip Publishing, Ohio, (2011).
22. Vasantha Kandasamy, W.B., and Smarandache, F., Fuzzy Interval Matrices, Neutrosophic Interval Matrices and their Applications, Hexis, Phoenix, (2006).
23. Vasantha Kandasamy, W.B., and Smarandache, F., Interval Semigroups, Kappa and Omega, Glendale, (2011).
24. Vasantha Kandasamy, W.B., and Smarandache, F., N-algebraic structures and S-N-algebraic structures, Hexis, Phoenix, Arizona, (2005).
25. Vasantha Kandasamy, W.B., and Smarandache, F., Natural Product on Matrices, Zip Publishing Inc, Ohio, (2012).
26. Vasantha Kandasamy, W.B., and Smarandache, F., Neutrosophic algebraic structures and neutrosophic N-algebraic structures, Hexis, Phoenix, Arizona, (2006).
27. Vasantha Kandasamy, W.B., and Smarandache, F., Neutrosophic Rings, Hexis, Phoenix, Arizona, 2006.
28. Vasantha Kandasamy, W.B., and Smarandache, F., Smarandache Neutrosophic algebraic structures, Hexis, Phoenix, Arizona, (2006).
29. Vasantha Kandasamy, W.B., and Smarandache, F., Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures, Hexis, Phoenix, Arizona, 2006.
30. Vasantha Kandasamy, W.B., and Smarandache, F., Special dual like numbers and lattices, Zip Publishing, Ohio, (2012).
31. Vasantha Kandasamy, W.B., Smarandache, F. and Ilanthenral, K., MOD Cognitive Maps models and MOD natural neutrosophic Cognitive Maps models, EuropaNova, (2016).

32. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Algebraic Structures on MOD planes, EuropaNova, (2015).
33. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., MOD planes, EuropaNova, (2015).
34. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., MOD Functions, EuropaNova, (2015).
35. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Multidimensional MOD planes, EuropaNova, (2015).
36. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., MOD Pseudo Linear Algebras, EuropaNova, (2015).
37. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Natural Neutrosophic numbers and MOD Neutrosophic numbers, EuropaNova, (2015).
38. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Problems in MOD Structures, EuropaNova, (2015).
39. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Semigroups on MOD Natural Neutrosophic Elements, EuropaNova, (2016).
40. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Special Type of Fixed Points of MOD Matrix Operators, EuropaNova, (2016).
41. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Special Type of Fixed Points pairs of MOD Rectangular Matrix Operators, EuropaNova, (2016).
42. Vasantha Kandasamy, W.B., A note on semigroup rings which are Boolean rings, Scientist of Phys. Sci., 3 no.1 (1991) 67-68.
43. Vasantha Kandasamy, W.B., A note on semigroup rings which are pre-p-rings, Kyungpook Math. J., 34 no.2 (1994) 223-225. ISSN: 1225-6951
44. Vasantha Kandasamy, W.B., A note on the modular semigroup ring of a finite idempotent semigroup, J. Natural Phys. Sci., 8 (1994) 91-94.
45. Vasantha Kandasamy, W.B., Filial semigroups and semigroup rings, Libertas Mathematica, 12 (1992) 35-37.
46. Vasantha Kandasamy, W.B., f-semigroup rings, Math. Education, 28 no.3 (1994) 162-164.
47. Vasantha Kandasamy, W.B., I*- Semigroups, Math. Education, 28 (1994) 30-31.

48. Vasantha Kandasamy, W.B., J-semigroups and J-semigroup rings, Math. Education, 28(1994) 84-86.
49. Vasantha Kandasamy, W.B., Modularity and supermodularity of subsemigroups in finite semigroups, Proc. Pak. Acad. Sci., 30 no.1 (1993) 45-47.
50. Vasantha Kandasamy, W.B., On group semigroup rings, Octogon Math. Mag., 3 no.2 (1995) 44-46.
51. Vasantha Kandasamy, W.B., On null semigroups, Math. Education, 24 no.3 (1992) 186-187.
52. Vasantha Kandasamy, W.B., On quasi semi group rings, Math. Education, 35 no.4 (2001) 192-195.
53. Vasantha Kandasamy, W.B., On semigroup rings in which (xy)n = xy, J. Bihar Math. Soc., 16 (1995) 47-50.
54. Vasantha Kandasamy, W.B., On semigroup rings which are Marot rings, Revista Intergracion Temas De Mathematics, 9 no.1 (1991) 59-62.
55. Vasantha Kandasamy, W.B., On semigroup superboolean Algebras, Analele Stiintifice Ale Universtatii Al. I. Cuza Iasi., 4 (1995) 33-36.
56. Vasantha Kandasamy, W.B., On SPI semigroups, Proc. Pak. Acad. Sci., 31 no.1 (1994) 49-50.
57. Vasantha Kandasamy, W.B., Semi group rings which are chinese rings, J. of Math. Res. Exposition, 13 No.3 (1993) 375-376.
58. Vasantha Kandasamy, W.B., Semigroup rings of ordered semigroups which are reduced rings, J. Math. Res. Expo., 10 no.4 (1990) 493-494.
59. Vasantha Kandasamy, W.B., Semigroup rings that are pre-Boolean rings, J. Fuzhou Univ., 20 no.1 (1992) 6-8.
60. Vasantha Kandasamy, W.B., Semigroup rings which are p-rings, Bull. Cal. Math. Soc., 82 (1990) 191-192.
61. Vasantha Kandasamy, W.B., Semigroup rings which are zero square rings, News Bull. Cal. Math. Soc., 12 no.4 (1989) 8-10.
62. Vasantha Kandasamy, W.B., Semi-idempotents in semigroup rings, J. Guizhou Inst. of Tech., 18 no.4 (1989) 73-74.
63. Vasantha Kandasamy, W.B., Smarandache Automaton, Smarandache Notions Journal, (2002). http://www.gallup.unm.edu/~smarandache/Automaton.pdf

64. Vasantha Kandasamy, W.B., Smarandache Idempotents, Smarandache Notions Journal, (2002). http://www.gallup.unm.edu/~smarandache/Idempotents.pdf

65. Vasantha Kandasamy, W.B., Smarandache Semigroups, American Research Press, (2002).

66. Vasantha Kandasamy, W.B., The units of semigroup seminear-rings, Opuscula Mathematica, 15 (1995) 113-114. ISSN: 1232-9274

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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In this book authors introduce the new notion of MOD rectangular numbers and MOD rectangular planes. This concept is not possible in real planes. Further the most innovative part of this book is introduction of MOD rectangular natural neutrosophic numbers and the study of algebraic structures on them. Several open problems are suggested.

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