Multiple attribute decision making method based on some normal neutrosophic Bonferroni mean operators

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Abstract: Normal neutrosophic numbers (NNNs) are a significant tool of describing the incompleteness, indeterminacy and inconsistency of the decision-making information. In this paper, we firstly propose the definition and the properties of the NNNs, and the accuracy function, the score function and the operational laws of the NNNs are developed. Then, some operators are presented, including the normal neutrosophic Bonferroni mean (NNBM) operator, the normal neutrosophic weighted Bonferroni mean (NNWBM) operator, the normal neutrosophic geometric Bonferroni mean (NNGBM) operator, the normal neutrosophic weighted geometric Bonferroni mean (NNWGBM) operator. We also study their properties and special cases. Further, we put forward a multiple attribute decision making method which is based on the NNWBM and NNWGBM operators. Finally, an illustrative example is given to verify the practicality and validity of the proposed method.

Keywords: Multiple attribute decision making; Normal neutrosophic numbers; Normal neutrosophic Bonferroni mean aggregation operator.

1. Introduction

As an important research branch of decision theory, Multiple attribute decision making (MADM) has a wide application in many areas. The multiple attribute decision making was firstly proposed and applied to select the investment policy of the enterprises by Churchman et al. [1]. However, because the fuzziness and indeterminacy of the information in real decision making are a common phenomenon, numerical values are inadequate or insufficient to model real-life decision problems. In same occasions, it can be more accurate to use the fuzzy numbers to describe the attribute values in fuzzy environment. Because of the fuzzy, indeterminate information in most of the multiple attribute decision making problems, Zadeh [2] firstly proposed the fuzzy set (FS) and used it to describe the fuzzy information. Atanassov [3] further presented the intuitionistic fuzzy set (IFS) where he added the non-membership function to the FS. Now, IFS has attracted more and more attentions in MADM and multiple attribute group decision making (MAGDM). However, due to the fact that the membership and non-membership function in IFS can only be shown by the accurate numbers, IFS is not appropriate to be used to solve the F-MADM problems. Atanassov [4], Gargov and Atanassov [5] further proposed the interval-valued intuitionistic fuzzy set (IVIFS). Liu and Zhang [6] defined triangular intuitionistic fuzzy numbers (TIFNs). Wang [7] proposed the trapezoidal intuitionistic fuzzy numbers (TrIFNs) and the interval trapezoidal intuitionistic fuzzy numbers (ITrIFNs). But, these extensions of IFS can just consider the indeterminacy.

Smarandache [8] proposed the neutrosophic set (NS) in which an independent indeterminacy-membership function was added. In NS, the truth-membership function is the same function as the membership of IFS; likewise, the falsity-membership is the same function as the non-membership of IFS. The indeterminacy-membership function is the pivotal difference between NS and IFS. The three parts, including the truth-membership, the indeterminacy membership, and the false-membership are completely independent. Based on this, Wang et al. [9] expressed the values of indeterminacy-membership, truth-membership and false-membership with the interval numbers, which has been defined the interval neutrosophic set (INS).On the basis of the Hamming and Euclidean distances, Ye [10] defined some similarity measures between INS. Then, a method about the multiple attribute decision making based on the similarity degree was proposed.

In real-life world, the normal distribution is widely applied to a lot of fields. But both the IFS and INS cannot consider the normal distribution, so the researches about the normal fuzzy information are attracting more and more attentions. Yang and Ko[11] firstly defined the normal fuzzy numbers(NFNs) to express the normal distribution phenomena. NFNs are more reasonable and realistic to express the decision-making information than FS in a random environment. Based on the NFNs and IFS, Wang and Li [12] proposed the normal intuitionistic fuzzy numbers (NIFNs) and defined its corresponding operations, the stability factor, the score function and so on. However, there have not been researches about the combination NFNs with NNs.

Now, more and more researchers pay attention to the information aggregation operators, which have become a popular research topic [26-30]. Bonferroni [13] firstly proposed the Bonferroni mean (BM) operator which can catch the interrelationship between the input arguments, BM has been applied in many application domains and attracted more and more attentions from the researchers. Yager [14] proposed some generalizations about the BM, such as the ordered weighted averaging (OWA) operator [15] and Choquet integral [16]. Yager [17] and Beliakov et al. [18] defined another generalized form of BM. Nevertheless, Zhu et al. [19] proposed the geometric Bonferroni mean (GBM) in which both the BM and geometric mean (GM) are considered.

Up to now, there is no research on the normal neutrosophic decision-making problems considering the interrelationship between the input normal neutrosophic arguments. Therefore, it is necessary to pay more attention to this issue. Because the BM operator can consider the interrelationship between the attributes, and the NNNs have the advantages of considering the normal random information and the neutrosophic variables, which can handle the incomplete, inconsistent and indeterminate information. In this paper, we extend the Bonferroni mean to aggregate the normal neutrosophic variables by combining BM aggregation operator with NNNs. We firstly propose two aggregation operators called the normal neutrosophic Bonferroni mean (NNBM) operator and the normal neutrosophic geometric Bonferroni mean (NNGBM) operator for aggregating the normal neutrosophic numbers. Then we study some properties of them and discuss their some special cases. For the situations in which the input arguments have different weight, we then develop the normal neutrosophic weighted Bonferroni mean (NNWBM) operator and the normal neutrosophic weighted geometric Bonferroni mean (NNWGBM) operator, and then, we propose two procedures for multiple attribute decision making under the environments of the NNNs based on the proposed operators.

The remainder of this paper is constructed as follows. In the next section, we introduce some basic concepts of the NNNs, some operational laws and the prominent characteristics of NNNs. In section 3, some aggregation operators on the basis of the normal neutrosophic numbers are proposed, such as the normal neutrosophic Bonferroni mean (NNBM) operator, the normal neutrosophic weighted Bonferroni mean (NNWBM) operator, the normal neutrosophic geometric Bonferroni mean (NNGBM) operator, the normal neutrosophic weighted geometric Bonferroni mean (NNWGBM) operator, and their properties are discussed. In section 4, a multiple attribute decision making method on the basis of the normal neutrosophic weighted Bonferroni mean (NNWBM) operator and the normal neutrosophic weighted geometric Bonferroni mean (NNWGBM) operator. In section 5, a numerical example is given to verify the proposed approach and to prove its effectiveness and practicality. In Section 6, we conclude the paper and give some remarks.

2. Preliminaries

2.1 The normal fuzzy set and normal intuitionistic fuzzy set

Definition 1 [17]**.** Let X be a real number set. A is denoted as $A=(a,\sigma)$. If its membership function satisfies:

$$A(x)=e^{-(\frac{x-a}{\sigma})^2} \quad (\sigma>0) \tag{1}$$

then A is called a normal fuzzy number. The set of all normal fuzzy numbers is denoted as $\tilde{N}$.

Definition 2 [18,19]**.** Suppose X is an ordinary finite non-empty set and $(a,\sigma)\in\tilde{N}$, $A=\langle(a,\sigma),\mu_A,\nu_A\rangle$ is a normal intuitionistic fuzzy number (NIFN) when its membership function is expressed as:

$$\mu_A(x)=\mu_A e^{-(\frac{x-a}{\sigma})^2}, x\in X, \tag{2}$$

and its non-membership function is expressed as:

$$\nu_A(x)=1-(1-\nu_A)e^{-(\frac{x-a}{\sigma})^2}, x\in X. \tag{3}$$

where $0\le\mu_A(x)\le1, 0\le\nu_A(x)\le1$ and $0\le\mu_A+\nu_A\le1$. When $\mu_A=1$ and $\nu_A=0$, the NIFN will become a NFN. Compared to NFNs, the NIFN adds the non-membership function, which expresses the degree of not belonging to (a,σ) . Moreover, $\pi_A(x)=1-\mu_A(x)-\nu_A(x)$ shows the degree of hesitance. The set of NIFNs is denoted by NIFNS.

2.2 The neutrosophic set

Definition 3 [8]**.** Let X be a universe of discourse, with a generic element in X denoted by x. A neutrosophic number A in X is expressed as:

$$A(x)=\langle x\,|\,(T_A(x),I_A(x),F_A(x))\rangle \tag{4}$$

where, $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $\left]0^-,1^+\right[$.

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^-\le T_A(x)+I_A(x)+F_A(x)\le3^+$.

Definition 4 [9]**.** Let X be a universe of discourse, with a generic element in X denoted by x. A single valued neutrosophic number A in X is

$$A(x)=\langle x\,|\,(T_A(x),I_A(x),F_A(x))\rangle \tag{5}$$

where $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. For each point x in X, we have $T_A(x),I_A(x),F_A(x)\in[0,1]$, and $0\le T_A(x)+I_A(x)+F_A(x)\le3$.

2.3 The normal neutrosophic set

Definition 5. Suppose X is a universe of discourse, with a generic element in X denoted by x, and $(a,\sigma)\in\tilde{N}$, A normal neutrosophic number A in X is expressed as:

$$A(x)=\langle x\,|\,(a,\sigma),(T_A(x),I_A(x),F_A(x))\rangle \tag{6}$$

Where the truth-membership function $T_A(x)$ satisfies:

$$T_A(x)=T_A e^{-(\frac{x-a}{\sigma})^2}, x\in X,$$

the indeterminacy-membership function $I_A(x)$ satisfies:

$$I_A(x)=1-(1-I_A)e^{-(\frac{x-a}{\sigma})^2}, x\in X.$$

and the falsity-membership function $F_A(x)$ satisfies:

$$F_A(x)=1-(1-F_A)e^{-(\frac{x-a}{\sigma})^2}, x\in X.$$

For each point x in X, we have $T_A(x), I_A(x), F_A(x)\in[0,1]$, and $0\le T_A(x)+I_A(x)+F_A(x)\le 3$. The set of all normal neutrosophic numbers is denoted as $\tilde{R}$.

Example1. The service life of the lamp bulb obeys the normal distribution, the normal fuzzy number is $N(1000,302)$. The experts evaluate whether the service life conforms to the normal distribution. At last, the experts give the evaluation values: the degree of result in range $(1000,302)$ is 0.6; the degree of result not in range $(1000,302)$ is 0.2; and the degree of hesitance is 0.2. So, the final evaluation result about the service life of the lamp bulb is $A=\langle(1000,302),(0.6,0.2,0.2)\rangle$.

Definition 6. Let $\tilde{a}_1=\langle(a_1,\sigma_1),(T_1,I_1,F_1)\rangle$ and $\tilde{a}_2=\langle(a_2,\sigma_2),(T_2,I_2,F_2)\rangle$ be two NNNs, then the Euclidean distance between $\tilde{a}_1$ and $\tilde{a}_2$ is defined as follows:

$$d(x,y)=\frac{1}{4}\sqrt{\begin{array}{l}\left[\left(2+T_1^L-I_1^L-F_1^L\right)a_1-\left(2+T_2^L-I_2^L-F_2^L\right)a_2\right]^2\\+\frac{1}{2}\left[\left(2+T_1^L-I_1^L-F_1^L\right)\sigma_1-\left(2+T_2^L-I_2^L-F_2^L\right)\sigma_2\right]^2\end{array}} \tag{7}$$

According to the operational laws defined by Wang et al. [22], we can give the following definition.

Definition 7. Let $\tilde{a}_1=\langle(a_1,\sigma_1),(T_1,I_1,F_1)\rangle$ and $\tilde{a}_2=\langle(a_2,\sigma_2),(T_2,I_2,F_2)\rangle$ be two NNNs, then the operational rules are defined as follows.

(1) $\tilde{a}_1\oplus\tilde{a}_2=\langle(a_1+a_2,\sigma_1+\sigma_2),(T_1+T_2-T_1T_2,I_1I_2,F_1F_2)\rangle$ (8)

(2) $\tilde{a}_1\otimes\tilde{a}_2=\left\langle(a_1a_2,a_1a_2\sqrt{\frac{\sigma_1^2}{a_1^2}+\frac{\sigma_2^2}{a_2^2}}),(T_1T_2,I_1+I_2-I_1I_2,F_1+F_2-F_1F_2)\right\rangle$ (9)

(3) $\lambda\tilde{a}_1=\langle(\lambda a_1,\lambda\sigma_1),(1-(1-T_1)^\lambda,I_1^\lambda,F_1^\lambda)\rangle\quad\lambda>0$ (10)

(4) $\tilde{a}_1^\lambda=\langle(a_1^\lambda,\lambda^{1/2}a_1^{\lambda-1}\sigma_1),(T_1^\lambda,1-(1-I_1)^\lambda,1-(1-F_1)^\lambda)\rangle\ \lambda>0$ (11)

Theorem 1. Let $\tilde{a}_1 = \langle (a_1, \sigma_1), (T_1, I_1, F_1) \rangle$ and $\tilde{a}_2 = \langle (a_2, \sigma_2), (T_2, I_2, F_2) \rangle$ be two NNNs, and $\eta, \eta_1, \eta_2 > 0$, then we have

(1) $\tilde{a}_1 \oplus \tilde{a}_2 = \tilde{a}_2 \oplus \tilde{a}_1$ (12)

(2) $\tilde{a}_1 \otimes \tilde{a}_2 = \tilde{a}_2 \otimes \tilde{a}_1$ (13)

(3) $\eta(\tilde{a}_1 \oplus \tilde{a}_2) = \eta\tilde{a}_1 \oplus \eta\tilde{a}_2$ (14)

(4) $\eta_1\tilde{a}_1 \oplus \eta_2\tilde{a}_1 = (\eta_1 + \eta_2)\tilde{a}_1$ (15)

(5) $\tilde{a}_1^{\eta} \otimes \tilde{a}_2^{\eta} = (\tilde{a}_1 \otimes \tilde{a}_2)^{\eta}$ (16)

(6) $\tilde{a}_1^{\eta_1} \otimes \tilde{a}_1^{\eta_2} = \tilde{a}_1^{\eta_1 + \eta_2}$ (17)

Definition 8. Let $\tilde{a}_k = \langle (a_k, \sigma_k), (T_k, I_k, F_k) \rangle$ be a NNN, and then its score function is

$$s_1(\tilde{a}_k) = a_k(2 + T_k - I_k - F_k), \quad s_2(\tilde{a}_k) = \sigma_k(2 + T_k - I_k - F_k) \tag{18}$$

and its accuracy function is

$$h_1(\tilde{a}_k) = a_k(2 + T_k - I_k + F_k), \quad h_2(\tilde{a}_k) = \sigma_k(2 + T_k - I_k + F_k) \tag{19}$$

Definition 9. Let $\tilde{a}_1 = \langle (a_1, \sigma_1), (T_1, I_1, F_1) \rangle$ and $\tilde{a}_2 = \langle (a_2, \sigma_2), (T_2, I_2, F_2) \rangle$ be two NNNs, the values of score functions of $\tilde{a}_1$ and $\tilde{a}_2$ are $s_1(\tilde{a}_1)$, $s_2(\tilde{a}_1)$ and $s_1(\tilde{a}_2)$, $s_2(\tilde{a}_2)$, and the values of accuracy functions of $\tilde{a}_1$ and $\tilde{a}_2$ are $h_1(\tilde{a}_1)$, $h_2(\tilde{a}_1)$ and $h_1(\tilde{a}_2)$, $h_2(\tilde{a}_2)$, respectively. Then, there will be:

(1) If $s_1(\tilde{a}_1) > s_1(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$;

(2) If $s_1(\tilde{a}_1) = s_1(\tilde{a}_2)$, then

① If $h_1(\tilde{a}_1) > h_1(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$;

② If $h_1(\tilde{a}_1) = h_1(\tilde{a}_2)$, then

(i) If $s_2(\tilde{a}_1) < s_2(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$;

(ii) If $s_2(\tilde{a}_1) = s_2(\tilde{a}_2)$, then

(a) If $h_2(\tilde{a}_1) < h_2(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$;

(b)If $h_2(\tilde{a}_1) = h_2(\tilde{a}_2)$, then $\tilde{a}_1 = \tilde{a}_2$.

3. Normal neutrosophic Bonferroni mean operators

3.1 NNBM and NNWBM operators

Bonferroni [13] firstly introduced the Bonferroni mean (BM) which can provide the aggregation between the max and min operators and the logical "or" and "and" operators. However the Bonferroni mean (BM) operator [13] has mostly been used in the situation where the input arguments are the non-negative real numbers. In this section, we will study the BM operator under the environments of NNNs. Based on Definition of BM [13], we define the Bonferroni mean operator of NNNs as follows:

Definition 10[13]. Suppose $p, q > 0$ and $\{\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m\}$ is a set of NNNs. The Bonferroni mean operator of NNNs is defined as

$$NNBM^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m) = \left(\frac{1}{m(m-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \tilde{a}_i^p \tilde{a}_j^q \right)^{\frac{1}{p+q}} \tag{20}$$

Theorem 2. Let $\tilde{a}_k = \left\langle \left(a_k, \sigma_k\right), \left(T_k, I_k, F_k\right) \right\rangle$ $(k = 1, 2 \cdots, m)$ be a set of NNNs, then, the result aggregated from Definition 10 will be still a NNN, and even

$$NNBM^{p,q}\left(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m\right) = \left\langle \left(\left(\frac{1}{m(\mathrm{m}-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} a_i^p a_j^q \right)^{\frac{1}{p+q}}, \frac{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{m} a_i^p a_j^q \right)^{\left(\frac{1}{p+q}-1\right)} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} a_i^{p-1} a_j^{q-1} \left(p a_j^2 \sigma_i^2 + q a_i^2 \sigma_j^2 \right)^{\frac{1}{2}}}{\sqrt{p+q}\, \sqrt[(p+q)]{m(m-1)}} \right), \right.$$

$$\left(\left(1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{m} \left(1 - T_i^p T_j^q\right) \right)^{\frac{1}{m(m-1)}} \right)^{\frac{1}{p+q}}, 1 - \left(1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{m} \left(1 - \left(1 - I_i\right)^p \left(1 - I_j\right)^q \right) \right)^{\frac{1}{m(m-1)}} \right)^{\frac{1}{p+q}}, \right. \tag{21}$$

$$\left. \left. 1 - \left(1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{m} \left(1 - \left(1 - F_i\right)^p \left(1 - F_j\right)^q \right) \right)^{\frac{1}{m(m-1)}} \right)^{\frac{1}{p+q}} \right) \right\rangle$$

Proof. By the operational rules of the NNNs, we have

$$\tilde{a}_i^p = \left(a_i^p, p^{\frac{1}{2}} a_i^{p-1} \sigma_i \right), \left(T_i^p, 1 - \left(1 - I_i\right)^p, 1 - \left(1 - F_i\right)^p \right)$$

$$\tilde{a}_j^q = \left(a_j^q, q^{\frac{1}{2}} a_j^{q-1} \sigma_j \right), \left(T_j^q, 1 - \left(1 - I_j\right)^q, 1 - \left(1 - F_j\right)^q \right)$$

and

$$\tilde{a}_i^p \otimes \tilde{a}_j^q = \left\langle \left(a_i^p a_j^q, a_i^{p-1} a_j^{q-1} \left(p a_j^2 \sigma_i^2 + q a_i^2 \sigma_j^2 \right)^{\frac{1}{2}} \right), \left(T_i^p T_j^q, 1 - \left(1 - I_i\right)^p \left(1 - I_j\right)^q, 1 - \left(1 - F_i\right)^p \left(1 - F_j\right)^q \right) \right\rangle$$

then

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{m} \tilde{a}_i^p \otimes \tilde{a}_j^q = \left\langle \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{m} a_i^p a_j^q, \sum_{\substack{i,j=1 \\ i \neq j}}^{m} a_i^{p-1} a_j^{q-1} \left(p a_j^2 \sigma_i^2 + q a_i^2 \sigma_j^2 \right)^{\frac{1}{2}} \right) \right.$$

$$\left. \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{m} \left(1 - T_i^p T_j^q\right), \prod_{\substack{i,j=1 \\ i \neq j}}^{m} 1 - \left(1 - I_i\right)^p \left(1 - I_j\right)^q, \prod_{\substack{i,j=1 \\ i \neq j}}^{m} 1 - \left(1 - F_i\right)^p \left(1 - F_j\right)^q \right) \right\rangle$$

and

$$\frac{1}{m(m-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \tilde{a}_i^p \otimes \tilde{a}_j^q = \left(\frac{1}{m(m-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} a_i^p a_j^q, \frac{1}{m(m-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} a_i^{p-1} a_j^{q-1} \left(p a_j^2 \sigma_i^2 + q a_i^2 \sigma_j^2 \right)^{\frac{1}{2}} \right),$$

$$\left(1-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-T_i^{\,p}T_j^{q}\right)\right)\right)^{1/m(m-1)},\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-I_i\right)^p\left(1-I_j\right)^q\right)\right)^{1/m(m-1)},\right.$$

$$\left.\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-F_i\right)^p\left(1-F_j\right)^q\right)\right)^{1/m(m-1)}\right)\right\rangle$$

Then

$$\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\tilde{a}_i^p\otimes\tilde{a}_j^q\right)^{1/(p+q)}=\left\langle\left(\left(\frac{1}{m(\mathrm{m}-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}a_i^pa_j^q\right)^{\frac{1}{p+q}},\frac{\left(\sum\limits_{\substack{i,j=1\\ i\neq j}}^{m}a_i^pa_j^q\right)^{\left(\frac{1}{p+q}-1\right)}\sum\limits_{\substack{i,j=1\\ i\neq j}}^{m}a_i^{p-1}a_j^{q-1}\left(pa_j^2\sigma_i^2+qa_i^2\sigma_j^2\right)^{\frac{1}{2}}}{\sqrt{p+q}^{(p+q)}\sqrt{m(m-1)}}\right),\right.$$

$$\left(\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-T_i^{\,p}T_j^{q}\right)\right)^{\frac{1}{m(m-1)}}\right)^{\frac{1}{p+q}},1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-I_i\right)^p\left(1-I_j\right)^q\right)\right)^{\frac{1}{m(m-1)}}\right)^{\frac{1}{p+q}},\right.$$

$$\left.\left.1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-F_i\right)^p\left(1-F_j\right)^q\right)\right)^{\frac{1}{m(m-1)}}\right)^{\frac{1}{p+q}}\right)\right\rangle$$

which completes the proof of the theorem 2.

In the following, we will discuss some properties of NNBM operator as follows.

Theorem 3. (Idempotency)**.**

Let $\{\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\}$ be a set of NNNs, if all $\tilde{a}_k$ $(k=1,2,...,m)$ are equal, i.e. $\tilde{a}_k=\tilde{a}$ $(k=1,2,...,m)$,

then

$$NNBM\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=\tilde{a}$$

Proof. since $\tilde{a}_k=\tilde{a}$ $(k=1,2,...,m)$,then according to definition 10,

$$NNBM^{\,p,q}\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m\right)=\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\tilde{a}_i^p\otimes\tilde{a}_j^q\right)^{\frac{1}{p+q}}=\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\tilde{a}^p\otimes\tilde{a}^q\right)^{\frac{1}{p+q}}$$

$$=\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\tilde{a}^{(p+q)}\right)^{\frac{1}{p+q}}=\left(\tilde{a}^{(p+q)}\right)^{\frac{1}{p+q}}=\tilde{a}$$

which completes the proof of theorem 3.

Theorem 4. (Commutativity)**.**

Let $\tilde{a}_k'$ $(k=1,2,...,m)$ is any permutation of $\tilde{a}_k$ $(k=1,2,...,m)$ .Then

$$NNBM\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=NNBM\left(\tilde{a}'_1,\tilde{a}'_2,...,\tilde{a}'_m\right)$$

Proof. Let $NNBM^{p,q}\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m\right)=\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\tilde{a}_i^p\tilde{a}_j^q\right)^{\frac{1}{p+q}}$

$$NNBM^{p,q}\left(\tilde{a}'_1,\tilde{a}'_2,\ldots,\tilde{a}'_m\right)=\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\tilde{a}_i'^p\tilde{a}_j'^q\right)^{\frac{1}{p+q}}$$

Since $\left\{\tilde{a}'_1,\tilde{a}'_2,...,\tilde{a}'_m\right\}$ is any permutation of $\left\{\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right\}$, then we have $\sum_{\substack{i,j=1\\ i\neq j}}^{m}\tilde{a}_i^p\tilde{a}_j^q=\sum_{\substack{i,j=1\\ i\neq j}}^{m}\tilde{a}_i'^p\tilde{a}_j'^q$.

Thus, $NNBM\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=NNBM\left(\tilde{a}'_1,\tilde{a}'_2,...,\tilde{a}'_m\right)$.

which completes the proof of the theorem 4.

Now we discuss some special cases of the NNBM by assigning different values to the parameters p,q:

(1) If $q=0$, then

$$NNBM^{p,0}\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m\right)=\left(\frac{1}{m}\sum_{i=1}^{m}\tilde{a}_i^p\right)^{\frac{1}{p}} \tag{22}$$

which we call it the normal neutrosophic generalized mean (NNGM) operator.

(2) If $p=1$ and $q=0$, then

$$NNBM^{1,0}\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m\right)=\frac{1}{m}\sum_{i=1}^{m}\tilde{a}_i \tag{23}$$

which we call it the normal neutrosophic mean (NNM) operator.

(3) If $p=2$ and $q=0$, then

$$NNBM^{2,0}\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m\right)=\left(\frac{1}{m}\sum_{i=1}^{m}\tilde{a}_i^2\right)^{\frac{1}{2}} \tag{24}$$

which we call it the normal neutrosophic square mean (NNSM) operator.

(4) If $p=1$ and $q=1$, then

$$NNBM^{1,1}\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m\right)=\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\tilde{a}_i\tilde{a}_j\right)^{\frac{1}{2}} \tag{25}$$

which we call it the Normal neutrosophic interrelated square mean (NNISM) operator.

The NNBM operator just considers the relationship of the aggregated arguments but ignores the importance of their weights. In the following, we will define another Bonferroni mean operator, the normal neutrosophic weighted Bonferroni mean (NNWBM) operator, to overcome the shortcoming.

Definition 11. Let $\left\{\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right\}$ be a set of NNNs. The weighted Bonferroni mean operator of NNNs is defined as

$$NNWBM^{p,q}\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m\right)=\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\left(w_i\tilde{a}_i\right)^p\otimes\left(w_j\tilde{a}_j\right)^q\right)^{\frac{1}{p+q}} \tag{26}$$

Where $w=\left(w_1,w_2,...,w_m\right)^T$ is the weight vector of NNNs. $\tilde{a}_k$ $(k=1,2,...,m)$ $0\le w_k\le 1\left(k=1,2,...,m\right)$ and $\sum_{k=1}^{m}w_k=1$.

Theorem 5. Let $\tilde{a}_k=\left\langle\left(a_k,\sigma_k\right),\left(T_k,I_k,F_k\right)\right\rangle$ $(k=1,2\cdots,m)$ be a set of the NNNs, then, the result aggregated based on the Definition11 will be still a NNN, and even

$$\begin{aligned}
&\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\left(w_i\tilde{a}_i\right)^p\otimes\left(w_j\tilde{a}_j\right)^q\right)^{\frac{1}{p+q}}=\Bigg\langle\Bigg(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(w_ia_i\right)^p\otimes\left(w_ja_j\right)^q\right)^{\frac{1}{p+q}},\\
&\frac{\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(w_ia_i\right)^p\otimes\left(w_ja_j\right)^q\right)^{\left(\frac{1}{p+q}-1\right)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(w_ia_i\right)^{p-1}\otimes\left(w_ja_j\right)^{q-1}\left(p\left(w_ja_j\right)^2\left(w_i\sigma_i\right)^2+q\left(w_ia_i\right)^2\left(w_j\sigma_j\right)^2\right)^{\frac{1}{2}}}{\sqrt{p+q}\,\sqrt[(p+q)]{n(n-1)}}\Bigg),
\end{aligned} \tag{27}$$

$$\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-T_i\right)\right)^{w_ip}\otimes\left(1-\left(1-T_j\right)\right)^{w_jq}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},$$

$$1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-I_i^{w_i}\right)^p\left(1-I_j^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-F_i^{w_i}\right)^p\left(1-F_j^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\Bigg)\Bigg\rangle$$

The NNWBM operator has the following properties:

Theorem 6. (Idempotency)**.**

Let $\left\{\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right\}$ be a collection of NNNs, if all $\tilde{a}_k$ $(k=1,2,...,m)$ are equal, i.e. $\tilde{a}_k=\tilde{a}$ $(k=1,2,...,m)$, for all k, then

$$NNWBM\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=\tilde{a}$$

The proof of the Theorem 6 can be easily completed with the same way as the Theorem 3.

Theorem 7. (Commutativity)**.**

Let $\tilde{a}'_k$ $(k=1,2,...,m)$ is any permutation of $\tilde{a}_k$ $(k=1,2,...,m)$ .Then

$$NNWBM\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=NNWBM\left(\tilde{a}'_1,\tilde{a}'_2,...,\tilde{a}'_m\right)$$

The proof of the Theorem 7 can be easily completed with the same way as the Theorem 4.

3.2 NNGBM and NNWGBM operators

Definition 12. Suppose $p,q>0$ and $\{\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\}$ be a set of NNNs. The geometric Bonferroni mean operator of the NNNs is defined as

$$NNGBM^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m)=\frac{1}{p+q}\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(p\tilde{a}_i+q\tilde{a}_j\right)\right)^{\frac{1}{m(m-1)}} \quad (28)$$

Theorem 8. Let $\tilde{a}_k=\langle(a_k,\sigma_k),(T_k,I_k,F_k)\rangle$ $(k=1,2\cdots,m)$ be a set of the NNNs, then, the result aggregated based on the Definition 12 will be still a NNN, and even

$$NNGBM^{p,q}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m)=$$

$$\left\langle\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(pa_i+qa_j\right)^{1/m(m-1)},\frac{1}{p+q}\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\frac{\left(p\sigma_i+q\sigma_j\right)^2}{\left(pa_i+qa_j\right)^2}\right)^{1/2}\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(pa_i+qa_j\right)^{1/m(m-1)}\right),\right. \quad (29)$$

$$\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-T_i\right)^p\left(1-T_j\right)^q\right)^{1/m(m-1)}\right)^{1/p+q}\right.,$$

$$\left.\left.\left(1-\prod_{i,j=1 i\neq j}^{m}\left(1-I_i^pI_j^q\right)^{1/m(m-1)}\right)^{1/p+q},\left(1-\prod_{i,j=1 i\neq j}^{m}\left(1-F_i^pF_j^q\right)^{1/m(m-1)}\right)^{1/p+q}\right)\right\rangle$$

Proof. By the operational laws of the NNNs, we have

$$p\tilde{a}_i=\langle(pa_i,p\sigma_i),\left(1-(1-T_i)^p,I_i^p,F_i^p\right)\rangle$$

$$q\tilde{a}_j=\langle(qa_j,q\sigma_j),\left(1-(1-T_j)^q,I_j^q,F_j^q\right)\rangle$$

and

$$p\tilde{a}_i+q\tilde{a}_j=\left\langle\left(pa_i+qa_j,p\sigma_i+q\sigma_j\right),\left(1-\left(1-T_i\right)^p\left(1-T_j\right)^q,I_i^pI_j^q,F_i^pF_j^q\right)\right\rangle$$

then

$$\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(p\tilde{a}_i+q\tilde{a}_j\right)=\left\langle\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(pa_i+qa_j\right),\left(\sum_{\substack{i,j=1\\ i\neq j}}^{m}\frac{\left(p\sigma_i+q\sigma_j\right)^2}{\left(pa_i+qa_j\right)^2}\right)^{1/2}\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(pa_i+qa_j\right)\right),\right.$$

$$\left.\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-T_i\right)^p\left(1-T_j\right)^q\right),1-\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-I_i^pI_j^q\right),1-\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-F_i^pF_j^q\right)\right)\right\rangle$$

and

$$\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(p\tilde{a}_i+q\tilde{a}_j\right)\right)^{\frac{1}{m(m-1)}}=\left\langle\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(pa_i+qa_j\right)^{1/m(m-1)},\right.\right.$$

$$\left.\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\frac{\left(p\sigma_i+q\sigma_j\right)^2}{\left(pa_i+qa_j\right)^2}\right)^{1/2}\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(pa_i+qa_j\right)^{1/m(m-1)}\right),$$

$$\left.\left(\prod_{i,j=1 i\neq j}^{m}\left(1-(1-T_i)^p(1-T_j)^q\right)^{1/m(m-1)},1-\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-I_i^pI_j^q\right)^{1/m(m-1)},1-\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-F_i^pF_j^q\right)^{1/m(m-1)}\right)\right\rangle$$

then

$$\frac{1}{p+q}\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(p\tilde{a}_i+q\tilde{a}_j\right)\right)^{\frac{1}{m(m-1)}}=\left\langle\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(pa_i+qa_j\right)^{1/m(m-1)},\right.\right.$$

$$\left.\frac{1}{p+q}\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\frac{\left(p\sigma_i+q\sigma_j\right)^2}{\left(pa_i+qa_j\right)^2}\right)^{1/2}\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(pa_i+qa_j\right)^{1/m(m-1)}\right),$$

$$\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-(1-T_i)^p(1-T_j)^q\right)^{1/m(m-1)}\right)^{1/p+q},\right.$$

$$\left.\left.\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-I_i^pI_j^q\right)^{1/m(m-1)}\right)^{1/p+q},\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-F_i^pF_j^q\right)^{1/m(m-1)}\right)^{1/p+q}\right)\right\rangle$$

which completes the proof of the theorem 8.

The geometric Bonferroni mean operator of the NNNs has some properties as follows:

Theorem 9. (Idempotency)**.**

Let $\{\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\}$ be a set of the NNNs. If all $\tilde{a}_k$ $(k=1,2,...,m)$ are equal, i.e. $\tilde{a}_k=\tilde{a}$ $(k=1,2,...,m)$**,** for all k, then

$$NNGBM\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=\tilde{a}$$

The proof of the Theorem 9 can be easily completed similar to the Theorem 3.

Theorem 10. (Commutativity)**.**

Suppose $\tilde{a}_k'$ $(k=1,2,...,m)$ is any permutation of $\tilde{a}_k$ $(k=1,2,...,m)$ .Then

$$NNGBM\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=NNGBM\left(\tilde{a}_1',\tilde{a}_2',...,\tilde{a}_m'\right)$$

The proof of the Theorem 10 can be easily completed with the same way as the Theorem 4.

Now we discuss some special cases of the NNGBM by assigning different values to the parameters p,q:

(1) If $q=0$, then

$$NNGBM^{p,0}\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=\frac{1}{p}\left(\prod_{i=1}^{m}\left(p\tilde{a}_i\right)\right)^{\frac{1}{m}} \tag{30}$$

which we call it the Normal neutrosophic generalized geometric mean (NNGGM) operator.

(2) If $p=1$ and $q=0$, then

$$NNGBM^{1,0}\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=\left(\prod_{i=1}^{m}\left(\tilde{a}_i\right)\right)^{\frac{1}{m}} \tag{31}$$

which we call it the Normal neutrosophic geometric mean (NNGM) operator.

(3) If $p=2$ and $q=0$, then

$$NNGBM^{2,0}\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=\left(\prod_{i=1}^{m}\left(2\tilde{a}_i\right)\right)^{\frac{1}{m}} \tag{32}$$

which we call it the Normal neutrosophic square geometric mean (NNSGM) operator.

(4) If $p=1$ and $q=1$, then

$$NNGBM^{1,1}\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=\frac{1}{2}\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(\tilde{a}_i+\tilde{a}_j\right)\right)^{\frac{1}{m(m-1)}} \tag{33}$$

which we call it the Normal neutrosophic interrelated square geometric mean (NNISGM) operator.

Similar to the NNBM operator, the NNGBM operator also just considers the interrelationship of the input arguments and ignores their own importance. In the following, we will extend the NNGBM to the normal neutrosophic weighted Bonferroni mean (NNWGBM) operator which can not only considers the interrelationship but also takes the weights into account.

Definition 13. Let $\left\{\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right\}$ be a set of NNNs. The weighted geometric Bonferroni mean operator of the NNNs will be defined as:

$$NNWGBM^{p,q}\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=\frac{1}{p+q}\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(p\left(\tilde{a}_i\right)^{w_i}+q\left(\tilde{a}_j\right)^{w_j}\right)\right)^{\frac{1}{m(m-1)}} \tag{34}$$

Where $w=\left(w_1,w_2,...,w_m\right)^T$ is the weight vector of NNNs, $\tilde{a}_k$ $(k=1,2,...,m)$, $0\leq w_k\leq 1\left(k=1,2,...,m\right)$ and $\sum_{k=1}^{m}w_k=1$.

Theorem 11. Let $\tilde{a}_k=\left\langle\left(a_k,\sigma_k\right),\left(T_k,I_k,F_k\right)\right\rangle$ $(k=1,2\cdots,m)$ be a set of the NNNs, then, the result aggregated based on the Definition 13 will be still a NNN, and even

$$NNWGBM^{p,q}\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=\left\langle\left(\frac{1}{p+q}\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(pa_i^{w_i}+qa_j^{w_j}\right)^{\frac{1}{m(m-1)}},\right.\right.$$

$$\left.\frac{1}{p+q}\left(\frac{1}{m(m-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\frac{\left(pw_i^{1/2}a_i^{w_i-1}\sigma_i+qw_j^{1/2}a_j^{w_j-1}\sigma_j\right)^2}{\left(pa_i^{w_i}+qa_j^{w_j}\right)^2}\right)^{1/2}\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(pa_i^{w_i}+qa_j^{w_j}\right)^{\frac{1}{m(m-1)}}\right), \tag{35}$$

$$\left(1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{m}\left(1-\left(1-T_i^{w_i}\right)^p\left(1-T_j^{w_j}\right)^q\right)^{1/m(m-1)}\right)^{\frac{1}{p+q}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{m}\left(1-\left(1-\left(1-I_i\right)^{w_i}\right)^p\left(1-\left(1-I_j\right)^{w_j}\right)^q\right)^{1/m(m-1)}\right)^{\frac{1}{p+q}},\right.$$

$$\left.\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{m}\left(1-\left(1-\left(1-F_i\right)^{w_i}\right)^p\otimes\left(1-\left(1-F_j\right)^{w_j}\right)^q\right)^{1/m(m-1)}\right)^{\frac{1}{p+q}}\right)\Bigg\rangle$$

The weighted geometric Bonferroni mean of the NNNs has some properties as follow.

Theorem 12. (Idempotency)**.**

Let $\{\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\}$ be a set of NNNs, if all $\tilde{a}_k$ $(k=1,2,...,m)$ are equal, i.e. $\tilde{a}_k=\tilde{a}$ $(k=1,2,...,m)$, for all k, then

$$NNWPG\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=\tilde{a}$$

The proof of the Theorem12 can be easily completed with the same way as the Theorem 3.

Theorem 13 (Commutativity)**.**

Let $\tilde{a}'_k$ $(k=1,2,...,m)$ be any permutation of $\tilde{a}_k$ $(k=1,2,...,m)$. Then

$$NNGBM\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right)=NNGBM\left(\tilde{a}'_1,\tilde{a}'_2,...,\tilde{a}'_m\right)$$

The proof of the Theorem 13 can be easily completed similar to Theorem 4.

4. A multiple attribute decision making method on the basis of NNWBM and NNWGBM operator

In this section, we will apply the normal neutrosophic weighted geometric Bonferroni mean(NNWBM) operator (or NNWGBM) to solve the multiple attribute decision making problems on the basis of the NNNs.

For a multiple attribute decision making problem, suppose $A=\{A_1,A_2,\cdots,A_m\}$ is the set of the alternatives, and $C=\{C_1,C_2,\cdots,C_n\}$ is the set of the attributes. Suppose each attributes are independent, and the evaluation value of the alternative A_i on the condition of the attribute C_i is $\bar{a}_{ij}=\left\langle(a_{ij},\sigma_{ij}),(T_{ij},I_{ij},F_{ij})\right\rangle$, which is presented by the form of the NNN, where $T_{ij},I_{ij},F_{ij}\in[0,1]$ and $T_{ij}+I_{ij}+F_{ij}\leq 3$. The weight vector of the attribute is $w=(w_1,w_2,\cdots,w_n)$, which $w_j\in[0,1],\sum_{j=1}^{n}w_j=1$.

Then, we use the normal neutrosophic weighted geometric Bonferroni mean(NNWBM) operator (or NNWGBM) to develop a method to deal with the multiple attribute decision making problems as follows.

Step 1. Normalize the decision matrix.

Because there are two types of attribute, i.e., the benefit type and the cost type, we firstly convert the different types to the same one. So, the decision matrix of normal neutrosophic variables $D=(\bar{a}_{ij})_{m\times n}$ will be converted to the standardized matrix $D=(\tilde{a}_{ij})_{m\times n}$

For the benefit type: $\tilde{a}_{ij} = \left\langle \left(\frac{a_{ij}}{\max_i (a_{ij})}, \frac{\sigma_{ij}}{\max_i (a_{ij})} \frac{\sigma_{ij}}{a_{ij}} \right), \left(T_{ij}, I_{ij}, F_{ij} \right) \right\rangle$ (36)

For the cost type: $\tilde{a}_{ij} = \left\langle \left(\frac{\min_i (a_{ij})}{a_{ij}}, \frac{\sigma_{ij}}{\max_i (a_{ij})} \frac{\sigma_{ij}}{a_{ij}} \right), \left(F_{ij}, 1 - I_{ij}, T_{ij} \right) \right\rangle$

(37)

Step 2. Calculate the comprehensive evaluation values of the alternatives based on the NNWBM operator (or NNWGBM). (generally, we can take $p = q = 1$)

$$\tilde{a}_i = NNWBM^{p,q}\left(\tilde{a}_{i1}, \tilde{a}_{i2}, \ldots, \tilde{a}_{in}\right) = \left\langle \left(\left(\frac{1}{n(n-1)} \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \left(w_j a_{ij} \right)^p \otimes \left(w_k a_{ik} \right)^q \right)^{\frac{1}{p+q}}, \right.\right.$$

$$\frac{\left(\sum_{\substack{j,k=1 \\ j \neq k}}^{n} \left(w_j a_{ij} \right)^p \otimes \left(w_k a_{ik} \right)^q \right)^{\left(\frac{1}{p+q} - 1 \right)} \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \left(w_j a_{ij} \right)^{p-1} \otimes \left(w_k a_{ik} \right)^{q-1} \left(p\left(w_k a_{ik} \right)^2 \left(w_j \sigma_{ij} \right)^2 + q\left(w_j a_{ij} \right)^2 \left(w_k \sigma_{ik} \right)^2 \right)^{\frac{1}{2}}}{\sqrt{p+q}\, \sqrt[(p+q)]{n(n-1)}} \quad (38)$$

$$\left(1 - \left(\prod_{\substack{j,k=1 \\ j \neq k}}^{n} \left(1 - \left(1 - \left(1 - T_{ij} \right) \right)^{w_j p} \otimes \left(1 - \left(1 - T_{ik} \right) \right)^{w_k q} \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}},$$

$$\left. 1 - \left(1 - \left(\prod_{\substack{j,k=1 \\ j \neq k}}^{n} \left(1 - \left(1 - I_{ij}^{w_j} \right)^p \left(1 - I_{ik}^{w_k} \right)^q \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, 1 - \left(1 - \left(\prod_{\substack{j,k=1 \\ j \neq k}}^{n} \left(1 - \left(1 - F_{ij}^{w_j} \right)^p \left(1 - F_{ik}^{w_k} \right)^q \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right) \right\rangle$$

or

$$\tilde{a}_i = NNWGBM^{p,q}\left(\tilde{a}_{i1}, \tilde{a}_{i2}, \ldots, \tilde{a}_{in}\right) = \left\langle \left(\frac{1}{p+q} \prod_{\substack{j,k=1 \\ j \neq k}}^{n} \left(p a_{ij}^{w_j} + q a_{ik}^{w_k} \right)^{\frac{1}{n(n-1)}}, \right.\right.$$

$$\frac{1}{p+q} \left(\frac{1}{n(n-1)} \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \frac{\left(p w_j^{1/2} a_{ij}^{w_j - 1} \sigma_{ij} + q w_k^{1/2} a_{ik}^{w_k - 1} \sigma_{ik} \right)^2}{\left(p a_{ij}^{w_j} + q a_{ik}^{w_k} \right)^2} \right)^{1/2} \prod_{\substack{j,k=1 \\ j \neq k}}^{n} \left(p a_{ij}^{w_j} + q a_{ik}^{w_k} \right)^{\frac{1}{n(n-1)}} \Bigg),$$

$$\left(1 - \left(1 - \prod_{\substack{j,k=1 \\ j \neq k}}^{n} \left(1 - \left(1 - T_{ij}^{w_j} \right)^p \left(1 - T_{ik}^{w_k} \right)^q \right)^{1/n(n-1)} \right)^{\frac{1}{p+q}}, \quad (39)$$

$$\left(1 - \prod_{\substack{j,k=1 \\ j \neq k}}^{n} \left(1 - \left(1 - \left(1 - I_{ij} \right)^{w_j} \right)^p \otimes \left(1 - \left(1 - I_{ik} \right)^{w_k} \right)^q \right)^{1/n(n-1)} \right)^{\frac{1}{p+q}},$$

$$\left(1-\prod_{\substack{j,k=1\\j\neq k}}^{n}\left(1-\left(1-\left(1-F_{ij}\right)^{w_j}\right)^{p}\otimes\left(1-\left(1-F_{ik}\right)^{w_k}\right)^{q}\right)^{1/n(n-1)}\right)^{\frac{1}{p+q}}\Bigg)\Bigg\rangle$$

where $i=1,2,\cdots,m$.

Step 3. Calculate the score value of each comprehensive evaluation value by equation (18).

Step 4. Rank all the alternatives $\{A_1, A_2, \cdots, A_m\}$ and select the most desirable one(s) according to the definition 9.

Step 5. End

5. The numerical example

In this section, based on NNWBM operator (NNWGBM), a numerical example is given to verify the proposed approach.

There is a company which is planning to invest some money to an industry (cited from [10]). There are four alternative companies to be chosen, including: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. There are three evaluation attributes, including: (1) C_1 is the risk; (2) C_2 is the growth; (3) C_3 is the environment. We can know the attributes C_1 and C_2 are benefit criteria, and the type of C_3 is cost. The weight vector of the attributes is $\omega=(0.35,0.25,0.4)$. The final evaluation outcomes are expressed by the NNNs, and shown in table 1.

Table 1 The evaluation values of four alternatives with respect to the three attributes

	C1	C2	C3
A1	<(3,0.4),(0.4,0.2,0.3)>	<(7,0.6),(0.4,0.1,0.2)>	<(5,0.4),(0.7,0.2,0.4)>
A2	<(4,0.2),(0.6,0.1,0.2)>	<(8,0.4),(0.6,0.1,0.2)>	<(6,0.7),(0.3,0.5,0.8)>
A3	<(3.5,0.3),(0.3,0.2,0.3)>	<(6,0.2),(0.5,0.2,0.3)>	<(5.5,0.6),(0.4,0.2,0.7)>
A4	<(5,0.5),(0.7,0.1,0.2)>	<(7,0.5),(0.6,0.1,0.1)>	<(4.5,0.5),(0.6,0.3,0.8)>

5.1 Procedure of decision making method based on the NNWBM operator.

(1) Normalize the decision matrix

Since C1 and C2 are benefit attributes, and C3 is a cost attribute, we utilize the formulas (36) and (37) to obtain the standardized decision matrix, which is shown in Table 2.

Table 2. The standardized decision matrix

	C1	C2	C3
A1	<(0.6,0.1067),(0.4,0.2,0.3)>	<(0.875,0.0875),(0.4,0.1,0.2)>	<(0.9,0.0475),(0.4,0.8,0.7)>
A2	<(0.8,0.02),(0.6,0.1,0.2)>	<(1,0.0333),(0.6,0.1,0.2)>	<(0.75,0.1167),(0.8,0.5,0.3)>
A3	<(0.7,0.0514),(0.3,0.2,0.3)>	<(0.75,0.0111),(0.5,0.2,0.3)>	<(0.818,0.0935),(0.7,0.8,0.4)>
A4	<(1,0.1),(0.7,0.1,0.2)>	<(0.875,0.0595),(0.6,0.1,0.1)>	<(1,0.0794),(0.8,0.7,0.6)>

(2) Calculate the comprehensive evaluation value of each alternative by formula (38).(suppose p=q=1).

$$\tilde{a}_1=\langle(0.1827,0.0208),(0.5704,0.7848,0.8133)\rangle$$

$$\tilde{a}_2=\langle(0.1954,0.0169),(0.6111,0.7084,0.7258)\rangle$$

$$\tilde{a}_3=\langle(0.1761,0.0143),(0.6232,0.8102,0.7834)\rangle$$

$$\tilde{a}_4 = \langle (0.2251, 0.0190), (0.5770, 0.7419, 0.7535) \rangle$$

(3) Calculate the score function by formula (18).

$$s_1(\tilde{a}_1) = 0.1776, s_1(\tilde{a}_2) = 0.2299, s_1(\tilde{a}_3) = 0.1813, s_1(\tilde{a}_4) = 0.2435$$

(4) Rank all of the alternatives and choose the most desirable one by the score function.

According to the score function $s_1(\tilde{a}_i)$, the ranking is $A_4 \succ A_2 \succ A_3 \succ A_1$.

Thus, the best alternative is A_4.

5.2 Procedure of decision making method based on the NNWGBM operator.

(1)Normalize the decision matrix

Since C1 and C2 are benefit attributes, and C3 is a cost criterion, we use the formulas (36) and (37) to get the standardized decision matrix, which can be shown in Table 3.

Table 3. The standardized decision matrix

	C1	C2	C3
A1	<(0.6,0.1067),(0.4,0.2,0.3)>	<(0.875,0.0875),(0.4,0.1,0.2)>	<(0.9,0.0475),(0.4,0.8,0.7)>
A2	<(0.8,0.02),(0.6,0.1,0.2)>	<(1,0.0333),(0.6,0.1,0.2)>	<(0.75,0.1167),(0.8,0.5,0.3)>
A3	<(0.7,0.0514),(0.3,0.2,0.3)>	<(0.75,0.0111),(0.5,0.2,0.3)>	<(0.818,0.0935),(0.7,0.8,0.4)>
A4	<(1,0.1),(0.7,0.1,0.2)>	<(0.875,0.0595),(0.6,0.1,0.1)>	<(1,0.0794),(0.8,0.7,0.6)>

(2) Calculate the comprehensive evaluation value of each alternative by formula (39). (suppose p=q=1).

$$\tilde{a}_1 = \langle (0.6783, 0.0302), (0.8143, 0.0917, 0.1101) \rangle$$

$$\tilde{a}_2 = \langle (0.6850, 0.0224), (0.8556, 0.0517, 0.0596) \rangle$$

$$\tilde{a}_3 = \langle (0.6748, 0.0207), (0.8567, 0.1050, 0.0892) \rangle$$

$$\tilde{a}_4 = \langle (0.7032, 0.0240), (0.8372, 0.0643, 0.0744) \rangle$$

(3) Calculate the score function by formula (18).

$$s_1(\tilde{a}_1) = 1.7721, s_1(\tilde{a}_2) = 1.8798, s_1(\tilde{a}_3) = 1.7968, s_1(\tilde{a}_4) = 1.8977$$

(4) Rank all of the alternatives and choose the most desirable one by the score function.

According to the score function $s_1(\tilde{a}_i)$, the ranking is $A_4 \succ A_2 \succ A_3 \succ A_1$.

Thus, the best alternative is A_4.

5.3 Analysis the effect of the factor p, q

In order to demonstrate the influence of the parameter p, q on decision making results of this example, we use the different values p, q in NNWBM or NNWGBM operator in step 4 to rank the alternatives. The ranking results are shown in Table 4 and Table 5.

Table 4 Ordering of the alternatives by utilizing the different p, q in NNWBM operator

p, q	Score values $s_1(\tilde{a}_i)$	Ranking

$p=0,q=1$	$s_1(\tilde{a}_1)=0.1292, s_1(\tilde{a}_2)=0.1457,$ $s_1(\tilde{a}_3)=0.1235, s_1(\tilde{a}_4)=0.1648$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$p=0,q=2$	$s_1(\tilde{a}_1)=0.2357, s_1(\tilde{a}_2)=0.2517,$ $s_1(\tilde{a}_3)=0.2206, s_1(\tilde{a}_4)=0.3003$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$p=0,q=10$	$s_1(\tilde{a}_1)=0.4494, s_1(\tilde{a}_2)=0.4448,$ $s_1(\tilde{a}_3)=0.4163, s_1(\tilde{a}_4)=0.5902$	$A_4 \succ A_1 \succ A_2 \succ A_3$
$p=1,q=0$	$s_1(\tilde{a}_1)=0.0879, s_1(\tilde{a}_2)=0.1338,$ $s_1(\tilde{a}_3)=0.1002, s_1(\tilde{a}_4)=0.1486$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=2,q=0$	$s_1(\tilde{a}_1)=0.1569, s_1(\tilde{a}_2)=0.2394,$ $s_1(\tilde{a}_3)=0.1756, s_1(\tilde{a}_4)=0.2679$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=10,q=0$	$s_1(\tilde{a}_1)=0.2772, s_1(\tilde{a}_2)=0.4181,$ $s_1(\tilde{a}_3)=0.3158, s_1(\tilde{a}_4)=0.5166$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=2,q=1$	$s_1(\tilde{a}_1)=0.2087, s_1(\tilde{a}_2)=0.2824,$ $s_1(\tilde{a}_3)=0.2179, s_1(\tilde{a}_4)=0.3061$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=10,q=1$	$s_1(\tilde{a}_1)=0.2853, s_1(\tilde{a}_2)=0.4185$ $s_1(\tilde{a}_3)=0.3181, s_1(\tilde{a}_4)=0.5075$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1,q=1$	$s_1(\tilde{a}_1)=0.1776, s_1(\tilde{a}_2)=0.2299$ $s_1(\tilde{a}_3)=0.1813, s_1(\tilde{a}_4)=0.2435$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1,q=2$	$s_1(\tilde{a}_1)=0.2408, s_1(\tilde{a}_2)=0.2848$ $s_1(\tilde{a}_3)=0.2350, s_1(\tilde{a}_4)=0.3161$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$p=1,q=10$	$s_1(\tilde{a}_1)=0.4270, s_1(\tilde{a}_2)=0.4328,$ $s_1(\tilde{a}_3)=0.3942, s_1(\tilde{a}_4)=0.5552$	$A_4 \succ A_2 \succ A_1 \succ A_3$

Table 5 Ordering of the alternatives by utilizing the different p,q in NNWGBM operator

p,q	Score values $s_1(\tilde{a}_i)$	Ranking
$p=0,q=1$	$s_1(\tilde{a}_1)=2.5272, s_1(\tilde{a}_2)=2.6809,$ $s_1(\tilde{a}_3)=2.5721, s_1(\tilde{a}_4)=2.6587$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$p=0,q=2$	$s_1(\tilde{a}_1)=1.6499, s_1(\tilde{a}_2)=1.8015,$ $s_1(\tilde{a}_3)=1.7067 s_1(\tilde{a}_4)=1.7247$	$A_2 \succ A_4 \succ A_3 \succ A_1$

$p=0,q=10$	$s_1(\tilde{a}_1)=0.6120, s_1(\tilde{a}_2)=0.7106,$ $s_1(\tilde{a}_3)=0.6521, s_1(\tilde{a}_4)=0.6168$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$p=1,q=0$	$s_1(\tilde{a}_1)=2.5272, s_1(\tilde{a}_2)=2.6809,$ $s_1(\tilde{a}_3)=2.5721, s_1(\tilde{a}_4)=2.6587$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$p=2,q=0$	$s_1(\tilde{a}_1)=1.6499, s_1(\tilde{a}_2)=1.8015,$ $s_1(\tilde{a}_3)=1.7067, s_1(\tilde{a}_4)=1.7247$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$p=10,q=0$	$s_1(\tilde{a}_1)=0.6121, s_1(\tilde{a}_2)=0.7106,$ $s_1(\tilde{a}_3)=0.6521, s_1(\tilde{a}_4)=0.6168$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$p=2,q=1$	$s_1(\tilde{a}_1)=1.3853, s_1(\tilde{a}_2)=1.5031,$ $s_1(\tilde{a}_3)=1.4232, s_1(\tilde{a}_4)=1.4891$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$p=10,q=1$	$s_1(\tilde{a}_1)=0.6081, s_1(\tilde{a}_2)=0.7005,$ $s_1(\tilde{a}_3)=0.6433, s_1(\tilde{a}_4)=0.6295$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$p=1,q=1$	$s_1(\tilde{a}_1)=1.7721, s_1(\tilde{a}_2)=1.8798$ $s_1(\tilde{a}_3)=1.7968, s_1(\tilde{a}_4)=1.8977$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1,q=2$	$s_1(\tilde{a}_1)=1.4110, s_1(\tilde{a}_2)=1.4988$ $s_1(\tilde{a}_3)=1.4304, s_1(\tilde{a}_4)=1.5024$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1,q=10$	$s_1(\tilde{a}_1)=0.6236, s_1(\tilde{a}_2)=0.6972,$ $s_1(\tilde{a}_3)=0.6486, s_1(\tilde{a}_4)=0.6370$	$A_2 \succ A_3 \succ A_4 \succ A_1$

As we can see from Table 4, the ordering of the alternatives may be different for the different values of p, q in NNWBA operator. But the best alternative is the same one A_4. In the Table 5, the ordering of the alternatives also may be different for the different values of p, q. The best alternative is A_2 or A_4.In practical applications, we generally adopt the values of the two parameters as $p=q=1$, which are not only easy and intuitive but also fully capture the correlations between criteria.

6. Conclusions

The multiple attribute decision making method on the basis of normal neutrosophic variables has a wider application in many domains. The normal neutrosophic set (NNS) will be more appropriate to deal with the incompleteness, indeterminacy and inconsistency of the decision-making information, and the Bonferroni mean (BM) operator can consider the interrelationships between the input arguments. So, in this paper, we proposed two aggregation operators called the normal neutrosophic Bonferroni mean (NNBM) operator and the normal neutrosophic geometric Bonferroni mean (NNGBM) operator for aggregating the information expressed by the normal neutrosophic numbers.

We studied some properties of them and discuss their some special cases. For the situations in which the input arguments have different weight, we then developed the normal neutrosophic weighted Bonferroni mean (NNWBM) operator and the normal neutrosophic weighted geometric Bonferroni mean (NNWGBM) operator, on the basis of which we propose two procedures for multiple attribute decision making under the environments where the information is expressed by the NNNs. Moreover, we use the NNWBM operator and NNWGBM operator to aggregate the evaluation information of alternatives, so the decision makers can get the desirable alternative according with their interest and the practical need by changing the values of p, q, which makes the results of the proposed multiple attribute decision making method more flexible and reliable. In the further research, the study about the applications of the new decision making method is necessary and significative because the applications of the normal distribution is widely distributed in many domain in the uncertain environment.

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