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On The Roots of Unity in Several Complex Neutrosophic Rings

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Abstract:

Roots of unity play a basic role in the theory of algebraic extensions of fields and rings. The aim of this paper is to obtain an algorithm to find all n-th roots of unity in five different kinds of neutrosophic complex rings, where many theorems and examples will be illustrated and suggested.

Keywords: Neutrosophic root of unity, refined neutrosophic unity, n-cyclic refined neutrosophic root of unity, complex neutrosophic number

1. Introduction and preliminaries

Neutrosophic algebraic structures are considered as generalizations of classical algebraic structures. The first defined neutrosophic algebraic structure is the neutrosophic ring which was defined and studied on a wide range by Smarandache et.al [1-11].

Laterally, many other neutrosophic algebraic structures were defined such as n-cyclic refined neutrosophic rings, neutrosophic matrices, and vector spaces [12-22].

Neutrosophic complex numbers were defined as novel generalizations of classical complex numbers, in a similar way of split-complex or weak fuzzy complex numbers [23-24].

One of the most classical interesting problems in classical algebra is the extending of fields and rings by complex roots of unity. From this point of view, we study for the first time the concept of neutrosophic roots of unity, where we obtain the classification of the roots of unity in five different neutrosophic rings. In addition, many examples will be discussed and presented.

We recall some basic concepts in neutrosophic algebra.

Definition:

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n sub-indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n \ ; \ a_i \in R\}$ to be n-cyclic refined neutrosophic ring.

Operations on R_n(I) are defined as:

$$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i$$

$$= \sum_{i=0}^{n} (x_i + y_i) I_i \, , \sum_{i=0}^{n} x_i I_i$$

$$\times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j = \sum_{i,j=0}^{n} (x_i \times y_j) I_{(i+j \, mod n)}$$

× is the multiplication on the ring R.

Definition:

Let (R,+,×) be a ring, $R(I) = \{a + bI \, : \, a, b \in R\}$ is called the neutrosophic ring where I is a neutrosophic element with condition $I^2 = I$.

Definition:

Let (R,+,×) be a ring, (R(I_1, I_2) , + ,×) is called a refined neutrosophic ring generated by R ,I_1, I_2.

Definition:

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n \ ; \ a_i \in R\}$ to be n-refined neutrosophic ring.

Main concepts

Neutrosophic roots of unity.

Let $C(I)=\{X+IY;\ I^2=I\ ;X,Y\in C\}$be the complex neutrosophic ring. According to [21], we have:

$$(X+IY)^n=X^n+I[(X+Y)^n-X^n]$$

So that $X+IY$ is an n-th root of unity if and only if $(X+IY)^n=1$, hence $X^n=1$, $(X+Y)^n=1$ which is equivalent to $X,X+Y$ are two classical roots of unity.

Theorem.

n-th roots of unity in the complex neutrosophic ring $C(I)$ are:

$$U=\left\{\alpha_j+(\alpha_t-\alpha_j)I;\ \alpha_j=e^{\frac{2\pi j}{n}i},\alpha_t=e^{\frac{2\pi t}{n}i};1\le j\le n,1\le t\le n\right\}$$

Proof.

According to the previous discussion $X+IY$ is a neutrosophic n-th root of unity if and only if $X,X+Y$ are two roots of unity, thus $X=\alpha_j=e^{\frac{2\pi j}{n}i};\ 1\le j\le n\ ,X+Y=\alpha_t=e^{\frac{2\pi t}{n}i};\ 1\le t\le n.$

This implies that $X+IY=\alpha_j+(\alpha_t-\alpha_j)I.$

Theorem.

The set of n-th roots of unity in $C(I)$ is a group under multiplication. Also $U\cong Z_n\times Z_n$.

Proof.

$$\forall\, T_1=(\alpha_j)+(\alpha_t-\alpha_j)I,T_2=(\alpha_k)+(\alpha_s-\alpha_k)I;\ 1\le k,s\le n,1\le j,t\le n$$

Then:

$T_1.T_2=\alpha_j\alpha_k+(\alpha_j\alpha_s-\alpha_j\alpha_k)I+(\alpha_k\alpha_t-\alpha_j\alpha_k)I+(\alpha_t\alpha_s-\alpha_t\alpha_k-\alpha_j\alpha_s+\alpha_j\alpha_k)I$

$T_1.T_2=\alpha_j\alpha_k+(\alpha_t\alpha_s-\alpha_j\alpha_k)I=\alpha_l+(\alpha_m-\alpha_l)I\in U$

Where α_l,α_m are two roots of unity.

On the other hand, we have $\alpha_j{}^{-1},\alpha_t{}^{-1}$ are two roots of unity.

So that we can put $T_3=\alpha_j{}^{-1}+(\alpha_t{}^{-1}-\alpha_j{}^{-1})I\in U.$

$$T_1.T_3=\alpha_j\alpha_j{}^{-1}+(\alpha_j\alpha_t{}^{-1}-\alpha_j\alpha_j{}^{-1}+\alpha_t\alpha_j{}^{-1}-\alpha_j\alpha_j{}^{-1}+\alpha_t\alpha_t{}^{-1}-\alpha_t\alpha_j{}^{-1}-\alpha_j\alpha_t{}^{-1}+\alpha_j\alpha_j{}^{-1})I$$

$T_1.T_3=1+(0)I=1$, thus $T_3=T_1{}^{-1}\in U$. This implies that U is a group under multiplication.

We define $f:U\to Z_n\times Z_n$ such that:

$$f\left(\alpha_j + (\alpha_t - \alpha_j)I\right) = (\alpha_j, \alpha_t)$$

f is well defined, that is because:

If $T_1 = \alpha_j + (\alpha_t - \alpha_j)I = T_2 = \alpha_k + (\alpha_s - \alpha_k)I$, then $\alpha_j = \alpha_k,\ \alpha_t = \alpha_s$, hence

$f(T_1) = (\alpha_j, \alpha_t) = (\alpha_k, \alpha_s) = f(T_2)$

f is a group homomorphism, that is because:

$T_1 \times T_2 = \alpha_j\alpha_k + (\alpha_t\alpha_s - \alpha_j\alpha_k)I$

$f(T_1 \times T_2) = (\alpha_j\alpha_k, \alpha_t\alpha_s) = (\alpha_j, \alpha_t) \times (\alpha_k, \alpha_s) = f(T_1) \times f(T_2)$.

It is clear that f is surjective. Also, f is injective that is because:

$$ker(f) = \{\alpha_j + (\alpha_t - \alpha_j)I \in U;\ (\alpha_j, \alpha_t) = (1,1)\} = \{1\}$$

Thus f is a group isomorphic, hence $U \cong Z_n \times Z_n$.

Refined neutrosophic roots of unity.

Let $C(I_1, I_2) = \{X + YI_1 + ZI_2;\ I_1I_2 = I_2I_1 = I_1, I_1^2 = I_1, I_2^2 = I_2, X, Y, Z \in C\}$ be the complex ring of refined neutrosophic numbers.

According to [ ], we have:

$(X + YI_1 + ZI_2)^n = X^n + I_1[(X + Y + Z)^n - (X + Z)^n] + I_2[(X + Z)^n - X^n]$

So that $X + YI_1 + ZI_2$ is a refined neutrosophic n-th root of unity if and only if $(X + YI_1 + ZI_2)^n = 1$, thus $X^n = (X + Y + Z)^n = (X + Z)^n = 1$, i.e, $X, X + Y + Z, X + Z$ are three classical roots of unity.

Theorem.

n-th roots of unity in the complex refined neutrosophic ring $C(I_1, I_2)$ are:

$$U = \left\{\alpha_j + (\alpha_t - \alpha_k)I_1 + (\alpha_k - \alpha_j)I_2;\ \alpha_j = e^{\frac{2\pi j}{n}i}, \alpha_t = e^{\frac{2\pi t}{n}i}, \alpha_k = e^{\frac{2\pi k}{n}i}; 1 \le j, k, t \le n\right\}$$

Proof.

According to the previous discussion $X + YI_1 + ZI_2$ is a refined neutrosophic n-th root of unity if $X, X + Y + Z, X + Z$ are three roots of unity, thus:

$X = \alpha_j, X + Z = \alpha_k, X + Y + Z = \alpha_t$ where $1 \le j, k, t \le n$ and $\alpha_j = e^{\frac{2\pi j}{n}i}, \alpha_t = e^{\frac{2\pi t}{n}i}, \alpha_k = e^{\frac{2\pi k}{n}i}$, thus:

$$\begin{cases} X = \alpha_j \\ Y = \alpha_t - \alpha_k \\ Z = \alpha_k - \alpha_j \end{cases}$$ thus $X + YI_1 + ZI_2 = \alpha_j + (\alpha_t - \alpha_k)I_1 + (\alpha_k - \alpha_j)I_2$.

Theorem.

Let U be the set of refined neutrosophic n-th roots of unity, then U is a group under multiplication with $U \cong Z_n \times Z_n \times Z_n$

Proof.

Let $T_1 = \alpha_j + (\alpha_t - \alpha_k)I_1 + (\alpha_k - \alpha_j)I_2$, $T_2 = \acute{\alpha}_j + (\acute{\alpha}_t - \acute{\alpha}_k)I_1 + (\acute{\alpha}_k - \acute{\alpha}_j)I_2$ be two element of U, then:

$$\begin{aligned} T_1 \times T_2 = \alpha_j\acute{\alpha}_j &+ (\alpha_j\acute{\alpha}_t - \alpha_j\acute{\alpha}_k)I_1 + (\alpha_j\acute{\alpha}_k - \alpha_j\acute{\alpha}_j)I_2 + (\acute{\alpha}_j\alpha_t - \acute{\alpha}_j\alpha_k)I_1 \\ &+ (\alpha_t\acute{\alpha}_t - \alpha_t\acute{\alpha}_k - \alpha_k\acute{\alpha}_t + \alpha_k\acute{\alpha}_k)I_1 + (\alpha_t\acute{\alpha}_k - \alpha_t\acute{\alpha}_j - \alpha_k\acute{\alpha}_k + \alpha_k\acute{\alpha}_j)I_1 \\ &+ (\acute{\alpha}_j\alpha_k - \acute{\alpha}_j\alpha_j)I_2 + (\acute{\alpha}_t\alpha_k - \acute{\alpha}_t\alpha_j - \acute{\alpha}_k\alpha_k + \acute{\alpha}_k\alpha_j)I_1 \\ &+ (\alpha_k\acute{\alpha}_k - \alpha_k\acute{\alpha}_j - \alpha_j\acute{\alpha}_k + \alpha_j\acute{\alpha}_j)I_2 \end{aligned}$$

$T_1 \times T_2 = \alpha_j\acute{\alpha}_j + (\alpha_t\acute{\alpha}_t - \alpha_k\acute{\alpha}_k)I_1 + (\alpha_k\acute{\alpha}_k - \alpha_j\acute{\alpha}_j)I_2 \in U.$

Also, ${T_1}^{-1} = {\alpha_j}^{-1} + ({\alpha_t}^{-1} - {\alpha_j}^{-1})I_1 + ({\alpha_k}^{-1} - {\alpha_j}^{-1})I_2$ is inverse of T_1, so that $(U,\times)$ is a group.

We define $f: U \to Z_n \times Z_n \times Z_n$ such that:

$$f[\alpha_j + (\alpha_t - \alpha_k)I_1 + (\alpha_k - \alpha_j)I_2] = (\alpha_j, \alpha_t, \alpha_k)$$

f is a well define one to one mapping.

f is a group homomorphism that is because:

$f(T_1 \times T_2) = (\alpha_j\acute{\alpha}_j, \alpha_t\acute{\alpha}_t, \acute{\alpha}_k\alpha_k) = (\alpha_j, \alpha_t, \alpha_k) \times (\acute{\alpha}_j, \acute{\alpha}_t, \acute{\alpha}_k) = f(T_1) \times f(T_2)$

So that $U \cong Z_n \times Z_n \times Z_n$.

2-cyclic refined neutrosophic ring.

Let $C_2(I) = \{X + YI_1 + ZI_2;\ I_1I_2 = I_2I_1 = I_1, {I_1}^2 = I_2, {I_2}^2 = I_2, X, Y, Z \in C\}$ be the 2-cyclic complex refined neutrosophic ring.

$X + YI_1 + ZI_2$ is an n-th root of unity in $C_2(I)$ if and only if $(X + YI_1 + ZI_2)^n = 1$.

Firstly, we present a formula to find the n-th power of $X + YI_1 + ZI_2$.

Theorem.

Let $X + YI_1 + ZI_2 \in C_2(I)$, then,

$$T^n = X^n + \frac{1}{2}I_1[(X + Y + Z)^n - (X - Y + Z)^n] + \frac{1}{2}I_2[(X + Y + Z)^n + (X - Y + Z)^n - 2X^n]$$

Proof.

(known befor).

Theorem.

Let $X + YI_1 + ZI_2 \in C_2(I)$, then T is n-th root of unity if and only if $X, X + Y + Z, X - Y + Z$ are three classical roots of unity.

Proof.

$T^n = 1$ is equivalent to:

$X^n + \frac{1}{2}I_1[(X+Y+Z)^n - (X-Y+Z)^n] + \frac{1}{2}I_2[(X+Y+Z)^n + (X-Y+Z)^n - 2X^n] = 1$,

thus:

$$\begin{cases} X^n = 1 \\ (X+Y+Z)^n - (X-Y+Z)^n = 0 \\ (X+Y+Z)^n + (X-Y+Z)^n - 2X^n = 0 \end{cases}$$

This implies that $X^n = (X+Y+Z)^n = (X-Y+Z)^n = 1$, this complete proof.

Theorem.

Let U be the set of all 2-cyclic n-th roots of unity, then:

1. $U = \left\{\alpha_j + \frac{1}{2}I_1[\alpha_t - \alpha_k] + \frac{1}{2}I_2[\alpha_t + \alpha_k - 2\alpha_j];\ \alpha_j = e^{\frac{2\pi j}{n}i}, \alpha_t = e^{\frac{2\pi t}{n}i}, \alpha_k = e^{\frac{2\pi k}{n}i}; 1 \le j,k,t \le n\right\}$.
2. $(U,\times)$ is a group.
3. $U \cong Z_n \times Z_n \times Z_n$.

Proof.

1. Assume that $T = X + YI_1 + ZI_2$ is an n-th root of unity, then $X = \alpha_j, X+Y+Z = \alpha_t, X-Y+Z = \alpha_k$ with $1 \le j,k,t \le n$ so that $Y = \frac{1}{2}[\alpha_t - \alpha_k], Z = \frac{1}{2}[\alpha_t + \alpha_k] - \alpha_j$, hence:

$$T = \alpha_j + \frac{1}{2}I_1[\alpha_t - \alpha_k] + \frac{1}{2}I_2[\alpha_t + \alpha_k - 2\alpha_j]$$

2. Let $T_1 = \alpha_j + \frac{1}{2}I_1[\alpha_t - \alpha_k] + \frac{1}{2}I_2[\alpha_t + \alpha_k - 2\alpha_j], T_2 = \acute{\alpha}_j + \frac{1}{2}I_1[\acute{\alpha}_t - \acute{\alpha}_k] + \frac{1}{2}I_2[\acute{\alpha}_t + \acute{\alpha}_k - 2\acute{\alpha}_j]$

We have $T_1 \times T_2 = \alpha_j\acute{\alpha}_j + \frac{1}{2}I_1[\alpha_t\acute{\alpha}_t - \alpha_k\acute{\alpha}_k] + \frac{1}{2}I_2[\alpha_t\acute{\alpha}_t + \alpha_k\acute{\alpha}_k - 2\alpha_j\acute{\alpha}_j] \in U$

The inverse of T_1 is $T^{-1}{}_1 = \alpha_j{}^{-1} + \frac{1}{2}I_1[\alpha_t{}^{-1} - \alpha_k{}^{-1}] + \frac{1}{2}I_2[\alpha_t{}^{-1} + \alpha_k{}^{-1} - 2\alpha_j{}^{-1}] \in U$, so that $(U,\times)$ is a group.

3. Define $f: U \to Z_n \times Z_n \times Z_n$ such that:

$$f\left[\alpha_j + \frac{1}{2}I_1[\alpha_t - \alpha_k] + \frac{1}{2}I_2[\alpha_t + \alpha_k - 2\alpha_j]\right] = (\alpha_j, \alpha_t, \alpha_k)$$

By a similar discussion of previous classification theorems, we get the proof.

Examples.

We find the 3-roots of unity in the neutrosophic complex ring $C(I)$.

In the classical case, we have three roots $\alpha_1 = 1, \alpha_2 = e^{\frac{2\pi}{3}i}, \alpha_3 = e^{\frac{4\pi}{4}i}$, thus the neutrosophic roots of unity are:

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + (\alpha_2 - \alpha_1)I, \alpha_1 + (\alpha_3 - \alpha_1)I, \alpha_2 + (\alpha_1 - \alpha_2)I, \alpha_2 + (\alpha_3 - \alpha_2)I, \alpha_3 + (\alpha_1 - \alpha_3)I, \alpha_3 + (\alpha_2 - \alpha_3)I\}$.

Example.

The 2-nd roots of unity in $C(I_1, I_2)$ are:

$$U = \{\alpha_1, \alpha_2, \alpha_1 + (\alpha_2 - \alpha_1)I_1, \alpha_1 + (\alpha_1 - \alpha_2)I_1 + (\alpha_2 - \alpha_1)I_2, \alpha_2 + (\alpha_1 - \alpha_2)I_2, \alpha_2 + (\alpha_2 - \alpha_1)I_1 + (\alpha_1 - \alpha_2)I_2, \alpha_1 + (\alpha_2 - \alpha_1)I_2, \alpha_2 + (\alpha_1 - \alpha_2)I_1\}\ ;\ \alpha_1 = 1, \alpha_2 = -1$$

Thus

$U = \{1, -1, 1 - 2I_1, 1 + 2I_1 - 2I_2, -1 + 2I_2, -1 - 2I_1 + 2I_2, 1 - 2I_2, -1 + 2I_1\}$.

3- Refined and 4-Refined Neutrosophic roots Of Unity

Definition.

Let C be the complex field, the 3-refined neutrosophic complex ring is defined as follows:

$C_3(I) = \{a + bI_1 + cI_2 + dI_3\ ; a, b, c, d \in C\}$, with $I_i.I_j = I_{min(i,j)}, {I_i}^2 = I_i; 1 \le i \le 3$.

The 4-refined neutrosophic complex ring is defined:

$C_4(I) = \{a + bI_1 + cI_2 + dI_3 + eI_4\ ; a, b, c, d, e \in C\}$, with $I_i.I_j = I_{min(i,j)}, {I_i}^2 = I_i; 1 \le i \le 4$.

Definition.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 \in C_3(I)$, then X is called the n-th root of unity if and only if $X^n = 1$.

X is called the 3-refined neutrosophic root of unity.

Definition.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 \in C_4(I)$, then X is called the n-th root of unity if and only if $X^n = 1$.

X is called the 4-refined neutrosophic root of unity.

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 \in C_3(I), n \in N$, then:

$$X^n = {x_0}^n + [(x_0 + x_1 + x_2 + x_3)^n - (x_0 + x_2 + x_3)^n]I_1 + [(x_0 + x_2 + x_3)^n - (x_0 + x_3)^n]I_2 + [(x_0 + x_3)^n - {x_0}^n]I_3$$

For the proof see [ ].

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 \in C_3(I)$, then X is a 3-refined neutrosophic root of unity if and only if $x_0, x_0 + x_3, x_0 + x_2 + x_3x_0 + x_1 + x_2 + x_3$ are roots of unity.

Proof.

$X^n = 1 \Leftrightarrow {x_0}^n = 1, (x_0 + x_3)^n = 1, (x_0 + x_2 + x_3)^n = 1, (x_0 + x_1 + x_2 + x_3)^n = 1$, thus the proof is complete.

Now, we find the 3-refined neutrosophic roots of unity.

Let $U = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be the set of classical n-th roots of unity.

If X is 3-refined neutrosophic roots of unity, then $x_0 \in U, x_0 + x_3 \in U, x_0 + x_2 + x_3 \in U, x_0 + x_1 + x_2 + x_3 \in U$.

If $x_0 = \alpha_i, x_0 + x_3 = \alpha_j, x_0 + x_2 + x_3 = \alpha_t, x_0 + x_1 + x_2 + x_3 = \alpha_s$, where $i, j, t, s \in \{1, \dots, n\}$, thus

$x_0 = \alpha_i, x_3 = \alpha_j - \alpha_i, x_2 = \alpha_t - \alpha_j, x_1 = \alpha_s - \alpha_t$.

Remark.

For n, there exists n^4 root of unity in $C_3(I)$.

Example.

For $n = 3$, we have $U = \{1, \alpha_1, \alpha_2\}$, with $\alpha_1 = e^{i\frac{2\pi}{3}}, \alpha_2 = e^{i\frac{4\pi}{3}}$, hence the 3-refined neutrosophic cubic roots of unity are:

$X = t_0 + (t_1 - t_2)I_1 + (t_2 - t_3)I_2 + (t_3 - t_0)I_3$, where $t_i \in U$.

We show some of them:

$X = 1 + (\alpha_1 - \alpha_2)I_1 + (\alpha_2 - 1)I_2 + (1 - \alpha_2)I_3$, $(t_0 = 1, t_1 = \alpha_1, t_2 = \alpha_2, t_3 = \alpha_1)$.

$Y = \alpha_2 + (\alpha_2 - \alpha_1)I_1 + (\alpha_2 - 1)I_2 + (1 - \alpha_2)I_3$, $(t_0 = \alpha_2, t_1 = \alpha_1, t_2 = \alpha_2, t_3 = 1)$.

And so on.

Remark.

Since $(U,\times)$ is a group with order n(cyclic group), the corresponding set of 3-refined neutrosophic roots of unity is an abelian group with order n^4.

Also, it is isomorphic to $U_n \times U_n \times U_n \times U_n$.

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3, Y = y_0 + y_1I_1 + y_2I_2 + y_3I_3 \in C_3(I)$, then:

$X^Y = {x_0}^{y_0} + [(x_0 + x_1 + x_2 + x_3)^{y_0+y_1+y_2+y_3} - (x_0 + x_2 + x_3)^{y_0+y_2+y_3}]I_1 + [(x_0 + x_2 + x_3)^{y_0+y_2+y_3} - (x_0 + x_3)^{y_0+y_3}]I_2 + [(x_0 + x_3)^{y_0+y_3} - {x_0}^{y_0}]I_3$.

Check [ ].

Definition.

We define the unity duplet (X, Y) as follows:

(X, Y) is a unity duplet if and only if $X^Y = 1$, where $X \in C_3(I), Y \in C_3(I)$.

Theorem.

Let (X, Y) be a unity duplet, this equivalents:

$x_0{}^{y_0} = (x_0 + x_3)^{y_0+y_3} = (x_0 + x_2 + x_3)^{y_0+y_2+y_3} = (x_0 + x_1 + x_2 + x_3)^{y_0+y_1+y_2+y_3} = 1.$

The proof is clear.

Example.

Take $X = 1 + \left(e^{i\frac{2\pi}{3}} - i\right) I_1 + \left(i - e^{i\frac{\pi}{4}}\right) I_2 + \left(e^{i\frac{\pi}{4}} - 1\right) I_3, Y = 2 - I_1 - 4I_2 + 6I_3.$

We have $X^Y = 1$, hence (X, Y) is a unity duplet.

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 \in C_4(I), n \in N$, then:

$$\begin{aligned} X^n = x_0{}^n &+ [(x_0 + x_1 + x_2 + x_3 + x_4)^n - (x_0 + x_2 + x_3 + x_4)^n]I_1 \\ &+ [(x_0 + x_2 + x_3 + x_4)^n - (x_0 + x_3 + x_4)^n]I_2 \\ &+ [(x_0 + x_3 + x_4)^n - (x_0 + x_4)^n]I_3 + [(x_0 + x_4)^n - x_0{}^n]I_4 \end{aligned}$$

Proof.

The proof can be checked easily by induction.

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 \in C_4(I)$, then X is an n-th root of unity if and only if: $x_0, x_0 + x_4, x_0 + x_3 + x_4, x_0 + x_2 + x_3 + x_4, x_0 + x_1 + x_2 + x_3 + x_4$ are classical n-th roots of unity.

The proof is clear.

Remark.

If $U = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is the set of n-th roots of unity, the corresponding 4-refined neutrosophic roots of unity are:

$\{t_0 + t_1I_1 + t_2I_2 + t_3I_3 + t_4I_4; t_0 = \alpha_i, t_4 = \alpha_j - \alpha_i, t_3 = \alpha_k - \alpha_j, t_2 = \alpha_s - \alpha_k, t_1 = \alpha_l - \alpha_s\}$

where $k, j, i, s, l \in \{1, \dots, n\}$.

Example.

For $n = 4$, we have $U = \{1, -1, i, -i\}$.

The 4-refined neutrosophic roots of unity for $n = 4$ are:

$\{X = t_0 + t_1 I_1 + t_2 I_2 + t_3 I_3 + t_4 I_4\}$, with $t_0 \in U, t_0 + t_4 \in U, t_0 + t_3 + t_4 \in U, t_0 + t_2 + t_3 + t_4 \in U, t_0 + t_1 + t_2 + t_3 + t_4 \in U$.

For example $X = i + (-2i)I_2 + (-1 + i)I_3 + (1 - i)I_4$.

$(t_0 = i, t_4 = 1 - i, t_3 = -1 + i, t_2 = -2i, t_1 = 0)$.

Conclusion

In this paper, we have studied the roots of unity of five neutrosophic different kinds of rings, where the roots of unity in neutrosophic rings, refined neutrosophic rings, 3-refined, 4-refined neutrosophic rings, and 2-cyclic refined neutrosophic rings are obtained and classified as direct products of well known classical finite groups. Many related examples were presented and discussed to clarify the validity of our work.

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