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# Characterizations of Group Theory under Q-Neutrosophic Soft Environment

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**Abstract:** Neutrosophic set theory was initiated as a method to handle indeterminate uncertain data. It is identified via three independent memberships represent truth $T$, indeterminate $I$ and falsity $F$ membership degrees of an element. As a generalization of neutrosophic set theory, Q-neutrosophic set theory was established as a new hybrid model that keeps the features of Q-fuzzy soft sets which handle two-dimensional information and the features of neutrosophic soft sets in dealing with uncertainty. Different extensions of fuzzy sets have been already implemented to several algebraic structures, such as groups, symmetric groups, rings and lie algebras. Group theory is one of the most essential algebraic structures in the field of algebra. The inspiration of the current work is to broaden the idea of Q-neutrosophic soft set to group theory. In this paper the concept of Q-neutrosophic soft groups is presented. Numerous properties and basic attributes are examined. We characterize the thought of Q-level soft sets of a Q-neutrosophic soft set, which is a bridge between Q-neutrosophic soft groups and soft groups. The concept of Q-neutrosophic soft homomorphism is defined and homomorphic image and preimage of a Q-neutrosophic soft groups are investigated. Furthermore, the cartesian product of Q-neutrosophic soft groups is proposed and some relevant properties are explored.

**Keywords:** Group, Neutrosophic set, Neutrosophic group, Neutrosophic soft group, Q-neutrosophic set, Q-neutrosophic soft set, Soft group.

## 1 Introduction

Neutrosophic sets (NSs), one of the fundamental models that deal with uncertainty, first appeared in mathematics in 1998 by Smarandache [1, 2] as an extension of the concepts of the classical sets, fuzzy sets [3] and intuitionistic fuzzy sets [4]. A NS is identified via three independent membership degrees which are standard or non-standard subsets of the interval $]^{-}0, 1^{+}[$ where $^{-}0 = 0 - \delta, 1^{+} = 1 + \delta$; $\delta$ is an infinitesimal number. These memberships represent the degrees of truth $(T)$, indeterminacy $(I)$, and falsity $(F)$. This structure makes the NS an effective common framework and empowers it to deal with indeterminate information which were not considered by fuzzy and intuitionistic fuzzy sets. Molodtsov [5] raised the notion of soft sets, based on the theory of adequate parametrization, as another approach to handle uncertain data. Since its initiation, a plenty of hybrid models of soft sets have been produced, for example, soft multi set theory [6], soft expert sets [7], fuzzy soft sets [8] and neutrosophic soft sets (NSS) [9]. Recently, NSs and NSSs were studied deeply by different researchers [10]-[19].

However, none of the above models can deal with two-dimensional indeterminate, uncertain and incompatible data. This propelled researchers to amplify them to have the capacity to deal with such circumstances, for example, Q-fuzzy soft sets [20, 21], Q-neutrosophic soft sets (Q-NSSs) [22] and Q-linguistic neutrosophic variable sets [23]. A Q-NSS is an expanded model of NSSs characterized via three two-dimensional independent membership degrees to tackle two-dimensional indeterminate issues that show up in real world. It gave an appropriate parametrization notion to handle imprecise, indeterminate and inconsistent two-dimensional information. Hence, it fits the indeterminacy and two-dimensionality simultaneously. Thus, Q-NSSs were further explored by Abu Qamar and Hassan by discussing their basic operations [24], relations [22], measures of distance, similarity and entropy [25] and also extended it further to the concept of generalized Q-neutrosophic soft expert set [26].

Hybrid models of fuzzy sets and soft sets were extensively applied in different fields of mathematics, in particular they were extremely applied in classical algebraic structures. This was started by Rosenfeld in 1971 [27] when he established the idea of fuzzy subgroup, by applying fuzzy sets to the theory of groups. Since then, the theories and approaches of fuzzy soft sets on different algebraic structures developed rapidly. Mukherjee and Bhattacharya [28] studied fuzzy groups, Sharma [29] discussed intuitionistic fuzzy groups, Aktas and Cagman [30] defined soft groups and Aygunoglu and Aygun presented the concept of fuzzy soft groups [31]. Recently, many researchers have applied different hybrid models of fuzzy sets to several algebraic structures such as groups, semigroups, rings, fields and BCK/BCI-algebras [32]-[38]. NSs and NSSs have received more attention in studying the algebraic structures dealing with uncertainty. Cetkin and Aygun [39] established the concept of neutrosophic subgroups. Bera and Mahapatra introduced neutrosophic soft groups [40], neutrosophic soft rings [41], $(\alpha, \beta, \gamma)$-cut of neutrosophic soft sets and its application to neutrosophic soft groups [42] and neutrosophic normal soft groups [43]. Neutrosophic triplet groups, rings and fields and many other structures were discussed in [44, 45, 46]. Moreover, two-dimensional hybrid models of fuzzy sets and soft sets were also applied to different algebraic structures. Solairaju and Nagarajan [47] introduced the notion of Q-fuzzy groups. Thiruveni and Solairaju defined the concept of neutrosophic Q-fuzzy subgroups [48], while Rasuli [49] established Q-fuzzy and anti Q-fuzzy subrings.

Inspired by the above discussion, in the present work we combine the idea of Q-NSS and group theory to conceptualize the notion of Q-neutrosophic soft groups (Q-NSGs) as a generalization of neutrosophic soft groups and soft groups; it is a new algebraic structure that deals with two-dimensional universal set under uncertain and indeterminate data. Some properties and basic characteristics are explored. Additionally, we define the Q-level soft set of a Q-NSS, which is a bridge between Q-NSGs and soft groups. The concept of Q-neutrosophic soft homomorphism (Q-NS hom) is defined and homomorphic image and preimage of a Q-NSG are investigated. Furthermore, the cartesian product of Q-NSGs is defined and some pertinent properties are examined. To clarify the novelty and originality of the proposed model a few contributions of numerous authors toward Q-NSGs are appeared in Table 1.

Table 1: Contributions toward Q-NSG.

| **Authors** | **Year** | **Contributions** |
|---|---|---|
| Rosenfeld [27] | 1971 | Introduction of fuzzy subgroup. |
| Aktas and Cagman [30] | 2007 | Introduction of soft group. |
| Aygunoglu and Aygun [31] | 2009 | Introduction to fuzzy soft groups. |
| Cetkin and Aygun [39] | 2015 | Introduction of neutrosophic subgroup. |
| Bera and Mahapatra [40] | 2016 | Introduction of neutrosophic soft group. |
| Solairaju and Nagarajan [47] | 2009 | Introduction of Q-fuzzy group. |
| Thiruveni and Solairaju [48] | 2018 | Introduction of neutrosophic Q-fuzzy subgroup. |
| Abu Qamar and Hassan | This paper | Introduction of Q-NSG. |

## 2 Preliminaries

We recall the elementary aspects of soft set, Q-NS and Q-NSS relevant to this study.

**Definition 2.1.** [5] A pair $(f, E)$ is a soft set over $X$ if $f$ is a mapping given by $f : E \to \mathcal{P}(X)$. That is, the soft set is a parametrized family of subsets of $X$.

**Definition 2.2.** [30] A soft set $(f, E)$ over a group $G$ is called a soft group over $G$ if $f(a)$ is a subgroup of $G$, $\forall a \in E$.

**Definition 2.3.** [31] A fuzzy soft set $(F, E)$ over a group $G$ is called a fuzzy soft group over $G$ if $\forall a \in E$, $F(a)$ is a fuzzy subgroup of $G$ in Rosenfeld's sense.

Abu Qamar and Hassan [22] proposed the notion of Q-neutrosophic set (Q-NS) in the following way.

**Definition 2.4.** [22] A Q-NS $\Gamma_Q$ in $X$ is an object of the form

$$\Gamma_Q = \Big\{ \big\langle (s,p), T_{\Gamma_Q}(s,p), I_{\Gamma_Q}(s,p), F_{\Gamma_Q}(s,p) \big\rangle : s \in X, p \in Q \Big\},$$

where $Q \neq \phi$ and $T_{\Gamma_Q}, I_{\Gamma_Q}, F_{\Gamma_Q} : X \times Q \to ]^-0, 1^+[$ are the true, indeterminacy and false membership functions, respectively with $^-0 \leq T_{\Gamma_Q} + I_{\Gamma_Q} + F_{\Gamma_Q} \leq 3^+$.

**Definition 2.5.** [22] Let $X$ be a universal set, $Q$ be a nonempty set and $A \subseteq E$ be a set of parameters. Let $\mu^l QNS(X)$ be the set of all multi Q-NSs on $X$ with dimension $l = 1$. A pair $(\Gamma_Q, A)$ is called a Q-NSS over $X$, where $\Gamma_Q : A \to \mu^l QNS(X)$ is a mapping, such that $\Gamma_Q(e) = \phi$ if $e \notin A$.

A Q-NSS can be presented as

$$(\Gamma_Q, A) = \{(e, \Gamma_Q(e)) : e \in A, \Gamma_Q \in \mu^l QNS(X)\}.$$

**Definition 2.6** ([24])**.** Let $(\Gamma_Q, A), (\Psi_Q, B) \in Q - NSS(X)$. Then, $(\Gamma_Q, A)$ is a Q-neutrosophic soft subset of $(\Psi_Q, B)$, denoted by $(\Gamma_Q, A) \subseteq (\Psi_Q, B)$, if $A \subseteq B$ and $\Gamma_Q(e) \subseteq \Psi_Q(e)$ for all $e \in A$, that is $T_{\Gamma_{Q(e)}}(s,p) \leq T_{\Psi_{Q(e)}}(s,p)$, $I_{\Gamma_{Q(e)}}(s,p) \geq I_{\Psi_{Q(e)}}(s,p)$, $F_{\Gamma_{Q(e)}}(s,p) \geq F_{\Psi_{Q(e)}}(s,p)$, for all $(s,p) \in X \times Q$.

**Definition 2.7.** [24] The union of two Q-NSSs $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ is the Q-NSS $(\Lambda_Q, C)$ written as $(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, C)$, where $C = A \cup B$ and $\forall c \in C, (s,p) \in X \times Q$, the membership degrees of $(\Lambda_Q, C)$ are:

$$T_{\Lambda_{Q(c)}}(s,p) = \begin{cases} T_{\Gamma_{Q(c)}}(s,p) & \text{if } c \in A - B, \\ T_{\Psi_{Q(c)}}(s,p) & \text{if } c \in B - A, \\ \max\{T_{\Gamma_{Q(c)}}(s,p), T_{\Psi_{Q(c)}}(s,p)\} & \text{if } c \in A \cap B, \end{cases}$$

$$I_{\Lambda_{Q(c)}}(s,p) = \begin{cases} I_{\Gamma_{Q(c)}}(s,p) & \text{if } c \in A - B, \\ I_{\Psi_{Q(c)}}(s,p) & \text{if } c \in B - A, \\ \min\{I_{\Gamma_{Q(c)}}(s,p), I_{\Psi_{Q(c)}}(s,p)\} & \text{if } c \in A \cap B, \end{cases}$$

$$F_{\Lambda_{Q(c)}}(s,p) = \begin{cases} F_{\Gamma_{Q(c)}}(s,p) & \text{if } c \in A - B, \\ F_{\Psi_{Q(c)}}(s,p) & \text{if } c \in B - A, \\ \min\{F_{\Gamma_{Q(c)}}(s,p), F_{\Psi_{Q(c)}}(s,p)\} & \text{if } c \in A \cap B. \end{cases}$$

**Definition 2.8.** [24] The intersection of two Q-NSSs $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ is the Q-NSS $(\Xi_Q, C)$ written as $(\Gamma_Q, A) \cap (\Psi_Q, B) = (\Xi_Q, C)$, where $C = A \cap B$ and $\forall c \in C$ and $(s,p) \in X \times Q$, the membership degrees of $(\Xi_Q, C)$ are:

$$\begin{aligned} T_{\Xi_{Q(c)}}(s,p) &= \min\{T_{\Gamma_{Q(c)}}(s,p), T_{\Psi_{Q(c)}}(s,p)\}, \\ I_{\Xi_{Q(c)}}(s,p) &= \max\{I_{\Gamma_{Q(c)}}(s,p), I_{\Psi_{Q(c)}}(s,p)\}, \\ F_{\Xi_{Q(c)}}(s,p) &= \max\{F_{\Gamma_{Q(c)}}(s,p), F_{\Psi_{Q(c)}}(s,p)\}. \end{aligned}$$

**Definition 2.9.** [24] If $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ are two Q-NSSs on $X$, then $(\Gamma_Q, A)$ AND $(\Psi_Q, B)$ is the Q-NSS denoted by $(\Gamma_Q, A) \wedge (\Psi_Q, B)$ and introduced by $(\Gamma_Q, A) \wedge (\Psi_Q, B) = (\Theta_Q, A \times B)$, where $\Theta_Q(a,b) = \Gamma_Q(a) \cap \Psi_Q(b)$ $\forall (a,b) \in A \times B$ and $(s,p) \in X \times Q$, the membership degrees of $(\Theta_Q, A \times B)$ are:

$$\begin{aligned} T_{\Theta_{Q(a,b)}}(s,p) &= \min\{T_{\Gamma_{Q(a)}}(s,p), T_{\Psi_{Q(b)}}(s,p)\}, \\ I_{\Theta_{Q(a,b)}}(s,p) &= \max\{I_{\Gamma_{Q(a)}}(s,p), I_{\Psi_{Q(b)}}(s,p)\}, \\ F_{\Theta_{Q(a,b)}}(s,p) &= \max\{F_{\Gamma_{Q(a)}}(s,p), F_{\Psi_{Q(b)}}(s,p)\}. \end{aligned}$$

**Definition 2.10.** [24] If $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ are two Q-NSSs on $X$, then $(\Gamma_Q, A)$ OR $(\Psi_Q, B)$ is the Q-NSS denoted by $(\Gamma_Q, A) \vee (\Psi_Q, B)$ and introduced by $(\Gamma_Q, A) \vee (\Psi_Q, B) = (\Upsilon_Q, A \times B)$, where $\Upsilon_Q(a,b) = \Gamma_Q(a) \cup \Psi_Q(b)$ $\forall (a,b) \in A \times B$ and $(s,p) \in X \times Q$, the membership degrees of $(\Upsilon_Q, A \times B)$ are:

$$\begin{aligned} T_{\Upsilon_{Q(a,b)}}(s,p) &= \max\{T_{\Gamma_{Q(a)}}(s,p), T_{\Psi_{Q(b)}}(s,p)\}, \\ I_{\Upsilon_{Q(a,b)}}(s,p) &= \min\{I_{\Gamma_{Q(a)}}(s,p), I_{\Psi_{Q(b)}}(s,p)\}, \\ F_{\Upsilon_{Q(a,b)}}(s,p) &= \min\{F_{\Gamma_{Q(a)}}(s,p), F_{\Psi_{Q(b)}}(s,p)\}. \end{aligned}$$

**Definition 2.11.** [24] If $(\Gamma_Q, A)$ is a Q-NSS on $X$, then the necessity $\Box(\Gamma_Q, A)$ and the possibility $\Diamond(\Gamma_Q, A)$ operations of $(\Gamma_Q, A)$ are defined as: for all $e \in A$

$$\Box(\Gamma_Q, A) = \left\{ \left\langle e, [(s,p), T_{\Gamma_Q}(s,p), I_{\Gamma_Q}(s,p), 1 - T_{\Gamma_Q}(s,p)] \right\rangle : (s,p) \in X \times Q \right\}$$

and

$$\Diamond(\Gamma_Q, A) = \left\{ \left\langle e, [(s,p), 1 - F_{\Gamma_Q}(s,p), I_{\Gamma_Q}(s,p), F_{\Gamma_Q}(s,p)] \right\rangle : (s,p) \in X \times Q \right\}.$$

# 3 Q-Neutrosophic soft groups

In the current section, we propose the notion of Q-NSG and investigate some related properties. In this paper $G$ will denote a classical group.

**Definition 3.1.** Let $(\Gamma_Q, A)$ be a Q-NSS over $G$. Then, $(\Gamma_Q, A)$ is said to be a Q-NSG over $G$ if for all $e \in A$, $\Gamma_Q(e)$ is a Q-neutrosophic subgroup of $G$, where $\Gamma_Q(e)$ is a mapping given by $\Gamma_Q(e) : G \times Q \to [0,1]^3$.

**Definition 3.2.** Let $(\Gamma_Q, A)$ be a Q-NSS over $G$. Then, $(\Gamma_Q, A)$ is said to be a Q-NSG over $G$ if for all $s, t \in G, p \in Q$ and $e \in A$ it satisfies:

1. $T_{\Gamma_{Q(e)}}(st,p) \geq \min\left\{T_{\Gamma_{Q(e)}}(s,p), T_{\Gamma_{Q(e)}}(t,p)\right\}$, $I_{\Gamma_{Q(e)}}(st,p) \leq \max\left\{I_{\Gamma_{Q(e)}}(s,p), I_{\Gamma_{Q(e)}}(t,p)\right\}$ and $F_{\Gamma_{Q(e)}}(st,p) \leq \max\left\{F_{\Gamma_{Q(e)}}(s,p), F_{\Gamma_{Q(e)}}(t,p)\right\}$.

2. $T_{\Gamma_{Q(e)}}(s^{-1}, p) \geq T_{\Gamma_{Q(e)}}(s, p), I_{\Gamma_{Q(e)}}(s^{-1}, p) \leq I_{\Gamma_{Q(e)}}(s, p)$ and $F_{\Gamma_{Q(e)}}(s^{-1}, p) \leq F_{\Gamma_{Q(e)}}(s, p)$.

**Example 3.3.** Let $G = (\mathbb{Z}, +)$ be a group and $A = 3\mathbb{Z}$ be the parametric set. Define a Q-NSS $(\Gamma_Q, A)$ as follows
for $p \in Q$ and $s, m \in \mathbb{Z}$

$$T_{\Gamma_{Q(3m)}}(s, p) = \begin{cases} 0.50 & \text{if } x = 6rm, \exists r \in \mathbb{Z}, \\ 0 & otherwise, \end{cases}$$

$$I_{\Gamma_{Q(3m)}}(s, p) = \begin{cases} 0 & \text{if } x = 6rm, \exists r \in \mathbb{Z}, \\ 0.20 & otherwise, \end{cases}$$

$$F_{\Gamma_{Q(3m)}}(s, p) = \begin{cases} 0 & \text{if } x = 6rm, \exists r \in \mathbb{Z}, \\ 0.25 & otherwise. \end{cases}$$

It is clear that $(\Gamma_Q, 3\mathbb{Z})$ is a Q-NSG over $G$.

**Theorem 3.4.** *Let $(\Gamma_Q, A)$ be a Q-NSG over G. Then, for all $s \in G$ and $p \in Q$ the following valid:*

1. $T_{\Gamma_{Q(e)}}(s^{-1}, p) = T_{\Gamma_{Q(e)}}(s, p), I_{\Gamma_{Q(e)}}(s^{-1}, p) = I_{\Gamma_{Q(e)}}(s, p)$ *and* $F_{\Gamma_{Q(e)}}(s^{-1}, p) = F_{\Gamma_{Q(e)}}(s, p)$.

2. $T_{\Gamma_{Q(e)}}(\acute{e}, p) \geq T_{\Gamma_{Q(e)}}(s, p), I_{\Gamma_{Q(e)}}(\acute{e}, p) \leq I_{\Gamma_{Q(e)}}(s, p)$ *and* $F_{\Gamma_{Q(e)}}(\acute{e}, p) \leq F_{\Gamma_{Q(e)}}(s, p)$.

*Proof.* 1. $T_{\Gamma_{Q(e)}}(s, p) = T_{\Gamma_{Q(e)}}((s^{-1})^{-1}, p) \geq T_{\Gamma_{Q(e)}}(s^{-1}, p), I_{\Gamma_{Q(e)}}(s, p) = I_{\Gamma_{Q(e)}}((s^{-1})^{-1}, p) \leq I_{\Gamma_{Q(e)}}(s^{-1}, p)$, and $F_{\Gamma_{Q(e)}}(s, p) = T_{\Gamma_{Q(e)}}((s^{-1})^{-1}, p) \leq F_{\Gamma_{Q(e)}}(s^{-1}, p)$. Now, from Definition 3.2 the result follows.

2. For the identity element $\acute{e}$ in $G$

$$\begin{aligned} T_{\Gamma_{Q(e)}}(\acute{e}, p) &= T_{\Gamma_{Q(e)}}(ss^{-1}, p) \\ &\geq \min\left\{T_{\Gamma_{Q(e)}}(s, p), T_{\Gamma_{Q(e)}}(s, p)\right\} \\ &= T_{\Gamma_{Q(e)}}(s, p), \\ I_{\Gamma_{Q(e)}}(\acute{e}, p) &= I_{\Gamma_{Q(e)}}(ss^{-1}, p) \\ &\leq \max\left\{I_{\Gamma_{Q(e)}}(s, p), I_{\Gamma_{Q(e)}}(s, p)\right\} \\ &= I_{\Gamma_{Q(e)}}(s, p) \end{aligned}$$

and

$$\begin{aligned} F_{\Gamma_{Q(e)}}(\acute{e}, p) &= F_{\Gamma_{Q(e)}}(ss^{-1}, p) \\ &\leq \max\left\{F_{\Gamma_{Q(e)}}(s, p), F_{\Gamma_{Q(e)}}(s, p)\right\} \\ &= F_{\Gamma_{Q(e)}}(s, p). \end{aligned}$$

Therefore, the result is proved. □

**Theorem 3.5.** *A Q-NSS $(\Gamma_Q, A)$ over $G$ is a Q-NSG if and only if for all $s, t \in G, p \in Q$ and $e \in A$*

1. $T_{\Gamma_Q(e)}(st^{-1}, p) \geq \min\left\{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t, p)\right\}$,
2. $I_{\Gamma_Q(e)}(st^{-1}, p) \leq \max\left\{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t, p)\right\}$ *and*
3. $F_{\Gamma_Q(e)}(st^{-1}, p) \leq \max\left\{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t, p)\right\}$.

*Proof.* Suppose that $(\Gamma_Q, A)$ is a Q-NSG over $G$. By Definition 3.2 we have

$$T_{\Gamma_Q(e)}(st^{-1}, p) \geq \min\left\{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t^{-1}, p)\right\} \geq \min\left\{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t, p)\right\},$$
$$I_{\Gamma_Q(e)}(st^{-1}, p) \leq \max\left\{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t^{-1}, p)\right\} \leq \max\left\{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t, p)\right\},$$
$$F_{\Gamma_Q(e)}(st^{-1}, p) \leq \max\left\{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t^{-1}, p)\right\} \leq \max\left\{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t, p)\right\}.$$

Thus, conditions 1,2 and 3 hold.

Conversely, suppose conditions 1,2 and 3 are satisfied. We show that for each $e \in A$ $(\Gamma_Q, A)$ is a Q-neutrosophic subgroup of $G$. From Theorem 3.4 we have $T_{\Gamma_Q(e)}(s^{-1}, p) \geq T_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(s^{-1}, p) \leq I_{\Gamma_Q(e)}(s, p)$ and $F_{\Gamma_Q(e)}(s^{-1}, p) \leq F_{\Gamma_Q(e)}(s, p)$. Next,

$$\begin{aligned} T_{\Gamma_Q(e)}(st, p) &= T_{\Gamma_Q(e)}(s(t^{-1})^{-1}, p) \\ &\geq \min\left\{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t^{-1}, p)\right\} \\ &\geq \min\left\{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t, p)\right\}, \\ I_{\Gamma_Q(e)}(st, p) &= I_{\Gamma_Q(e)}(s(t^{-1})^{-1}, p) \\ &\leq \max\left\{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t^{-1}, p)\right\} \\ &\leq \max\left\{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t, p)\right\} \end{aligned}$$

and

$$\begin{aligned} F_{\Gamma_Q(e)}(st, p) &= F_{\Gamma_Q(e)}(s(t^{-1})^{-1}, p) \\ &\leq \max\left\{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t^{-1}, p)\right\} \\ &\leq \max\left\{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t, p)\right\}. \end{aligned}$$

This completes the proof. □

**Theorem 3.6.** *Let $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ be two Q-NSGs over $G$. Then, $(\Gamma_Q, A) \wedge (\Psi_Q, B)$ and $(\Gamma_Q, A) \cap (\Psi_Q, B)$ are also Q-NSGs over $G$.*

*Proof.* We know that $(\Gamma_Q, A) \wedge (\Psi_Q, B) = (\Theta_Q, A \times B)$, where for all $(a, b) \in A \times B$ and $(s, p) \in X \times Q$

$$\begin{aligned} T_{\Theta_Q(a,b)}(s, p) &= \min\left\{T_{\Gamma_Q(a)}(s, p), T_{\Psi_Q(b)}(s, p)\right\}, \\ I_{\Theta_Q(a,b)}(s, p) &= \max\left\{I_{\Gamma_Q(a)}(s, p), I_{\Psi_Q(b)}(s, p)\right\}, \\ F_{\Theta_Q(a,b)}(s, p) &= \max\left\{F_{\Gamma_Q(a)}(s, p), F_{\Psi_Q(b)}(s, p)\right\}. \end{aligned}$$

Now, since $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ are Q-NSGs over $G$, $\forall s, t \in G, p \in Q$ and $(a, b) \in A \times B$, we get

$$\begin{aligned}
T_{\Theta_{Q(a,b)}}(st,p) &= \min\Big\{T_{\Gamma_{Q(a)}}(st,p), T_{\Psi_{Q(b)}}(st,p)\Big\} \\
&\geq \min\Big\{\min\big\{T_{\Gamma_{Q(a)}}(s,p), T_{\Gamma_{Q(a)}}(t,p)\big\}, \min\big\{T_{\Psi_{Q(b)}}(s,p), T_{\Psi_{Q(b)}}(t,p)\big\}\Big\} \\
&= \min\Big\{\min\big\{T_{\Gamma_{Q(a)}}(s,p), T_{\Psi_{Q(b)}}(s,p)\big\}, \min\big\{T_{\Gamma_{Q(a)}}(t,p), T_{\Psi_{Q(b)}}(t,p)\big\}\Big\} \\
&= \min\Big\{T_{\Theta_{Q(a,b)}}(s,p), T_{\Theta_{Q(a,b)}}(t,p)\Big\},
\end{aligned}$$

$$\begin{aligned}
I_{\Theta_{Q(a,b)}}(st,p) &= \max\Big\{I_{\Gamma_{Q(a)}}(st,p), I_{\Psi_{Q(b)}}(st,p)\Big\} \\
&\leq \max\Big\{\max\big\{I_{\Gamma_{Q(a)}}(s,p), I_{\Gamma_{Q(a)}}(t,p)\big\}, \max\big\{I_{\Psi_{Q(b)}}(s,p), I_{\Psi_{Q(b)}}(t,p)\big\}\Big\} \\
&= \max\Big\{\max\big\{I_{\Gamma_{Q(a)}}(s,p), I_{\Psi_{Q(b)}}(s,p)\big\}, \max\big\{I_{\Gamma_{Q(a)}}(t,p), I_{\Psi_{Q(b)}}(t,p)\big\}\Big\} \\
&= \max\Big\{I_{\Theta_{Q(a,b)}}(s,p), I_{\Theta_{Q(a,b)}}(t,p)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
F_{\Theta_{Q(a,b)}}(st,p) &= \max\Big\{F_{\Gamma_{Q(a)}}(st,p), F_{\Psi_{Q(b)}}(st,p)\Big\} \\
&\leq \max\Big\{\max\big\{F_{\Gamma_{Q(a)}}(s,p), F_{\Gamma_{Q(a)}}(t,p)\big\}, \max\big\{F_{\Psi_{Q(b)}}(s,p), F_{\Psi_{Q(b)}}(t,p)\big\}\Big\} \\
&= \max\Big\{\max\big\{F_{\Gamma_{Q(a)}}(s,p), F_{\Psi_{Q(b)}}(s,p)\big\}, \max\big\{F_{\Gamma_{Q(a)}}(t,p), F_{\Psi_{Q(b)}}(t,p)\big\}\Big\} \\
&= \max\Big\{F_{\Theta_{Q(a,b)}}(s,p), F_{\Theta_{Q(a,b)}}(t,p)\Big\}.
\end{aligned}$$

Also,

$$\begin{aligned}
T_{\Theta_{Q(a,b)}}(s^{-1},p) &= \min\Big\{T_{\Gamma_{Q(a)}}(s^{-1},p), T_{\Psi_{Q(b)}}(s^{-1},p)\Big\} \\
&\geq \min\Big\{T_{\Gamma_{Q(a)}}(s,p), T_{\Psi_{Q(b)}}(s,p)\Big\} \\
&= T_{\Theta_{Q(a,b)}}(s,p),
\end{aligned}$$

$$\begin{aligned}
I_{\Theta_{Q(a,b)}}(s^{-1},p) &= \max\Big\{I_{\Gamma_{Q(a)}}(s^{-1},p), I_{\Psi_{Q(b)}}(s^{-1},p)\Big\} \\
&\leq \max\Big\{I_{\Gamma_{Q(a)}}(s,p), I_{\Psi_{Q(b)}}(s,p)\Big\} \\
&= I_{\Theta_{Q(a,b)}}(s,p),
\end{aligned}$$

$$
\begin{aligned}
F_{\Theta_Q(a,b)}(s^{-1},p) &= \max\Big\{F_{\Gamma_Q(a)}(s^{-1},p), F_{\Psi_Q(b)}(s^{-1},p)\Big\}\\
&\leq \max\Big\{F_{\Gamma_Q(a)}(s,p), F_{\Psi_Q(b)}(s,p)\Big\}\\
&= F_{\Theta_Q(a,b)}(s,p).
\end{aligned}
$$

This shows that $(\Gamma_Q, A) \wedge (\Psi_Q, B)$ is a Q-NSG. The proof of $(\Gamma_Q, A) \cap (\Psi_Q, B)$ is similar to the proof of $(\Gamma_Q, A) \wedge (\Psi_Q, B)$. $\square$

**Remark 3.7.** For two Q-NSGs $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ over $G$, $(\Gamma_Q, A) \cup (\Psi_Q, B)$ is not generally a Q-NSG over $G$.

For example, let $G = (\mathbb{Z}, +)$ and $E = 2\mathbb{Z}$. Define the two Q-NSGs $(\Gamma_Q, E)$ and $(\Psi_Q, E)$ over $G$ as the following for $s, m \in \mathbb{Z}, p \in Q$

$$
T_{\Gamma_Q(2m)}(s,p) = \begin{cases} 0.50 & \text{if } x = 4rm, \exists r \in \mathbb{Z},\\ 0 & otherwise, \end{cases}
$$

$$
I_{\Gamma_Q(2m)}(s,p) = \begin{cases} 0 & \text{if } x = 4rm, \exists r \in \mathbb{Z},\\ 0.25 & otherwise, \end{cases}
$$

$$
F_{\Gamma_Q(2m)}(s,p) = \begin{cases} 0 & \text{if } x = 4rm, \exists r \in \mathbb{Z},\\ 0.10 & otherwise, \end{cases}
$$

and

$$
T_{\Psi_Q(2m)}(s,p) = \begin{cases} 0.67 & \text{if } x = 6rm, \exists r \in \mathbb{Z},\\ 0 & otherwise, \end{cases}
$$

$$
I_{\Psi_Q(3m)}(s,p) = \begin{cases} 0 & \text{if } x = 6rm, \exists r \in \mathbb{Z},\\ 0.20 & otherwise, \end{cases}
$$

$$
F_{\Psi_Q(3m)}(s,p) = \begin{cases} 0 & \text{if } x = 6rm, \exists r \in \mathbb{Z},\\ 0.17 & otherwise. \end{cases}
$$

Let $(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, E)$. For $m = 3, s = 12, t = 18$ we have

$$
T_{\Lambda_Q(6)}(12.18^{-1},p) = T_{\Lambda_Q(6)}(-6,p) = \max\Big\{T_{\Gamma_Q(6)}(-6,p), T_{\Psi_Q(6)}(-6,p)\Big\} = \max\{0,0\} = 0
$$

and

$$
\begin{aligned}
\min\Big\{&T_{\Lambda_Q(6)}(12,p), T_{\Lambda_Q(6)}(18,p)\Big\}\\
&= \min\Big\{\max\big\{T_{\Gamma_Q(6)}(12,p), T_{\Psi_Q(6)}(12,p)\big\}, \max\big\{T_{\Gamma_Q(6)}(18,p), T_{\Psi_Q(6)}(18,p)\big\}\Big\}\\
&= \min\Big\{\max\big\{0.50, 0.67\big\}, \max\big\{0, 0.67\big\}\Big\}\\
&= \min\Big\{0.67, 0.67\Big\} = 0.67.
\end{aligned}
$$

Hence, $T_{\Lambda_Q(6)}(12.18^{-1}, p) = 0 < \min\Big\{T_{\Lambda_Q(6)}(12, p), T_{\Lambda_Q(6)}(18, p)\Big\} = 0.67$; i.e. $(\Lambda_Q, E) = (\Gamma_Q, A) \cup (\Psi_Q, B)$ is not a Q-NSG.

**Theorem 3.8.** *If $(\Gamma_Q, A)$ is a Q-NSG over $G$, then $\square(\Gamma_Q, A)$ and $\lozenge(\Gamma_Q, A)$ are Q-NSGs over $G$.*

*Proof.* Let $(\Gamma_Q, A)$ be a Q-NSG over $G$. Then, for each $e \in A$, $s, t \in G$ and $p \in Q$ we have

$$\begin{aligned}
F_{\square\Gamma_Q(e)}(st^{-1}, p) &= 1 - T_{\Gamma_Q(e)}(st^{-1}, p)\\
&\leq 1 - \min\Big\{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t, p)\Big\}\\
&= \max\Big\{1 - T_{\Gamma_Q(e)}(s, p), 1 - T_{\Gamma_Q(e)}(t, p)\Big\}\\
&= \max\Big\{F_{\square\Gamma_Q(e)}(s, p), F_{\square\Gamma_Q(e)}(t, p)\Big\}.
\end{aligned}$$

Hence, $\square(\Gamma_Q, A)$ is a Q-NSG. Similarly, we can prove the second part. $\square$

**Definition 3.9.** Let $(\Gamma_Q, A)$ be a Q-NSG over $G$. Let $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma \leq 3$. Then $(\Gamma_Q, A)_{(\alpha,\beta,\gamma)}$ is a Q-level soft set of $(\Gamma_Q, A)$ defined by

$$(\Gamma_Q, A)_{(\alpha,\beta,\gamma)} = \Big\{s \in G, p \in Q : T_{\Gamma_Q(e)}(s, p) \geq \alpha, I_{\Gamma_Q(e)}(s, p) \leq \beta, F_{\Gamma_Q(e)}(s, p) \leq \gamma\Big\}$$

for all $e \in A$.

The next theorem provides a bridge between Q-NSG and soft group.

**Theorem 3.10.** *Let $(\Gamma_Q, A)$ be a Q-NSS over $G$. Then, $(\Gamma_Q, A)$ is a Q-NSG over $G$ if and only if for all $\alpha, \beta, \gamma \in [0, 1]$ the Q-level soft set $(\Gamma_Q, A)_{(\alpha,\beta,\gamma)} \neq \phi$ is a soft group over $G$.*

*Proof.* Let $(\Gamma_Q, A)$ be a Q-NSG over $G$, $s, t \in (\Gamma_Q(e))_{(\alpha,\beta,\gamma)}$ and $p \in Q$, for arbitrary $\alpha, \beta, \gamma \in [0, 1]$ and $e \in A$.
Then we have $T_{\Gamma_Q(e)}(s, p) \geq \alpha, I_{\Gamma_Q(e)}(s, p) \leq \beta, F_{\Gamma_Q(e)}(s, p) \leq \gamma$. Since $(\Gamma_Q, A)$ is a Q-NSG over $G$, then we have

$$\begin{aligned}
T_{\Gamma_Q(e)}(st, p) &\geq \min\Big\{T_{\Gamma_Q(e)}(s, p), T_{\Gamma_Q(e)}(t, p)\Big\} \geq \Big\{\alpha, \alpha\Big\} = \alpha,\\
I_{\Gamma_Q(e)}(st, p) &\leq \max\Big\{I_{\Gamma_Q(e)}(s, p), I_{\Gamma_Q(e)}(t, p)\Big\} \leq \Big\{\beta, \beta\Big\} = \beta,\\
F_{\Gamma_Q(e)}(st, p) &\leq \max\Big\{F_{\Gamma_Q(e)}(s, p), F_{\Gamma_Q(e)}(t, p)\Big\} \leq \Big\{\gamma, \gamma\Big\} = \gamma.
\end{aligned}$$

Therefore, $st \in (\Gamma_Q(e))_{(\alpha,\beta,\gamma)}$. Furthermore $T_{\Gamma_Q(e)}(s^{-1}, p) \geq \alpha, I_{\Gamma_Q(e)}(s^{-1}, p) \leq \beta, F_{\Gamma_Q(e)}(s^{-1}, p) \leq \gamma$. So, $s^{-1} \in (\Gamma_Q(e))_{(\alpha,\beta,\gamma)}$. Hence $(\Gamma_Q(e))_{(\alpha,\beta,\gamma)}$ is a subgroup over $G$, $\forall e \in A$.

Conversely, suppose $(\Gamma_Q, A)$ is not a Q-NSG over $G$. Then, there exists $e \in A$ such that $\Gamma_Q(e)$ is not a Q-neutrosophic subgroup of $G$. Then, there exist $s_1, t_1 \in G$ and $p \in Q$ such that

$$\begin{aligned}
T_{\Gamma_Q(e)}(s_1 t_1^{-1}, p) &< \min\Big\{T_{\Gamma_Q(e)}(s_1, p), T_{\Gamma_Q(e)}(t_1, p)\Big\},\\
I_{\Gamma_Q(e)}(s_1 t_1^{-1}, p) &> \max\Big\{I_{\Gamma_Q(e)}(s_1, p), I_{\Gamma_Q(e)}(t_1, p)\Big\}
\end{aligned}$$

and

$$F_{\Gamma_Q(e)}(s_1t_1^{-1},p) > \max\Big\{F_{\Gamma_Q(e)}(s_1,p), F_{\Gamma_Q(e)}(t_1,p)\Big\}.$$

Let us assume that, $T_{\Gamma_Q(e)}(s_1t_1^{-1},p) < \min\Big\{T_{\Gamma_Q(e)}(s_1,p), T_{\Gamma_Q(e)}(t_1,p)\Big\}$. Let $T_{\Gamma_Q(e)}(s_1,p) = \alpha_1$, $T_{\Gamma_Q(e)}(t_1,p) = \alpha_2$ and $T_{\Gamma_Q(e)}(s_1t_1^{-1},p) = \alpha_3$. If we take $\alpha = \min\{\alpha_1,\alpha_2\}$, then $s_1t_1^{-1} \notin (\Gamma_Q(e))_{(\alpha,\beta,\gamma)}$. But, since

$$T_{\Gamma_Q(e)}(s_1,p) = \alpha_1 \geq \min\{\alpha_1,\alpha_2\} = \alpha$$

and

$$T_{\Gamma_Q(e)}(t_1,p) = \alpha_2 \geq \min\{\alpha_1,\alpha_2\} = \alpha.$$

For $I_{\Gamma_Q(e)}(s_1,p) \leq \beta, I_{\Gamma_Q(e)}(t_1,p) \leq \beta, F_{\Gamma_Q(e)}(s_1,p) \leq \gamma, F_{\Gamma_Q(e)}(t_1,p) \leq \gamma$, we have $s_1, t_1 \in (\Gamma_Q(e))_{(\alpha,\beta,\gamma)}$. This contradicts with the fact that $(\Gamma_Q, A)_{(\alpha,\beta,\gamma)}$ is a soft group over $G$.
Similarly, we can show that $I_{\Gamma_Q(e)}(s_1t_1^{-1},p) > \max\Big\{I_{\Gamma_Q(e)}(s_1,p), I_{\Gamma_Q(e)}(t_1,p)\Big\}$ and $F_{\Gamma_Q(e)}(s_1t_1^{-1},p) > \max\Big\{F_{\Gamma_Q(e)}(s_1,p), F_{\Gamma_Q(e)}(t_1,p)\Big\}$. □

# 4 Homomorphism of Q-neutrosophic soft groups

In the following, we define the Q-neutrosophic soft function (Q-NS fn), and then define the image and preimage of a Q-NSS under Q-NS fn. Moreover, we define the Q-neutrosophic soft homomorphism (Q-NS hom) and prove that the homomorphic image and pre-image of a Q-NSG are also Q-NSGs.

**Definition 4.1.** Let $g : X \times Q \to Y \times Q$ and $h : A \to B$ be two functions where $A$ and $B$ are parameter sets for the sets $X \times Q$ and $Y \times Q$, respectively. Then, the pair $(g, h)$ is called a Q-NS fn from $X \times Q$ to $Y \times Q$.

**Definition 4.2.** Let $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ be two Q-NSSs defined over $X \times Q$ and $Y \times Q$, respectively, and $(g, h)$ be a Q-NS fn from $X \times Q$ to $Y \times Q$. Then,

1. The image of $(\Gamma_Q, A)$ under $(g, h)$, denoted by $(g, h)(\Gamma_Q, A)$, is a Q-NSS over $Y \times Q$ and is defined by:

$$(g,h)(\Gamma_Q, A) = \Big(g(\Gamma_Q), h(A)\Big) = \Big\{\Big\langle b, g(\Gamma_Q)(b) : b \in h(A)\Big\rangle\Big\},$$

where for all $b \in h(A), t \in Y, p \in Q$,

$$T_{g(\Gamma_Q)(b)}(t,p) = \begin{cases} \max_{g(s,p)=(t,p)} \max_{h(a)=b}[T_{\Gamma_Q(a)}(s,p)] & \text{if } (s,p) \in g^{-1}(t,p), \\ 0 & otherwise, \end{cases}$$

$$I_{g(\Gamma_Q)(b)}(t,p) = \begin{cases} \min_{g(s,p)=(t,p)} \min_{h(a)=b}[I_{\Gamma_Q(a)}(s,p)] & \text{if } (s,p) \in g^{-1}(t,p), \\ 1 & otherwise, \end{cases}$$

$$F_{g(\Gamma_Q)(b)}(t,p) = \begin{cases} \min_{g(s,p)=(t,p)} \min_{h(a)=b}[F_{\Gamma_Q(a)}(s,p)] & \text{if } (s,p) \in g^{-1}(t,p), \\ 1 & otherwise, \end{cases}$$

2. The preimage of $(\Psi_Q, B)$ under $(g, h)$, denoted by $(g, h)^{-1}(\Psi_Q, B)$, is a Q-NSS over $X \times Q$ and is defined by:

$$(g,h)^{-1}(\Psi_Q, B) = \Big(g^{-1}(\Psi_Q), h^{-1}(B)\Big) = \Big\{\Big\langle a, g^{-1}(\Psi_Q)(a) : a \in h^{-1}(B)\Big\rangle\Big\},$$

where, for all $a \in h^{-1}(B), s \in X, p \in Q$,

$$\begin{aligned} T_{g^{-1}(\Psi_Q)(a)}(s,p) &= T_{\Psi_Q[h(a)]}(g(s,p)),\\ I_{g^{-1}(\Psi_Q)(a)}(s,p) &= I_{\Psi_Q[h(a)]}(g(s,p)),\\ F_{g^{-1}(\Psi_Q)(a)}(s,p) &= F_{\Psi_Q[h(a)]}(g(s,p)). \end{aligned}$$

If $g$ and $h$ are injective (surjective), then $(g, h)$ is injective (surjective).

**Definition 4.3.** Let $(g, h)$ be a Q-NS fn from $X \times Q$ to $Y \times Q$. If $g$ is a homomorphism from $X \times Q$ to $Y \times Q$, then $(g, h)$ is said to be a Q-NS hom. If $g$ is an isomorphism from $X \times Q$ to $Y \times Q$ and $h$ is a one-to-one mapping from $A$ to $B$, then $(g, h)$ is said to be a Q-neutrosophic soft isomorphism.

**Theorem 4.4.** *Let $(\Gamma_Q, A)$ be a Q-NSG over a group $G_1$ and $(g, h)$ be a Q-NS hom from $G_1 \times Q$ to $G_2 \times Q$. Then, $(g, h)(\Gamma_Q, A)$ is a Q-NSG over $G_2$.*

*Proof.* Let $b \in h(E), t_1, t_2 \in G_2$ and $p \in Q$. For $g^{-1}(t_1, p) = \phi$ or $g^{-1}(t_2, p) = \phi$, the proof is clear.

So, suppose there exist $s_1, s_2 \in G_1$ and $p \in Q$ such that $g(s_1, p) = (t_1, p)$ and $g(s_2, p) = (t_2, p)$. Then,

$$\begin{aligned} T_{g(\Gamma_Q)(b)}(t_1t_2,p) &= \max_{g(s,p)=(t_1t_2,p)} \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(s,p)\Big]\\ &\geq \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(s_1s_2,p)\Big]\\ &\geq \max_{h(a)=b} \Big[\min\Big\{T_{\Gamma_Q(a)}(s_1,p), T_{\Gamma_Q(a)}(s_2,p)\Big\}\Big]\\ &= \min\Big\{\max_{h(a)=b}\Big[T_{\Gamma_Q(a)}(s_1,p)\Big], \max_{h(a)=b}\Big[T_{\Gamma_Q(a)}(s_2,p)\Big]\Big\} \end{aligned}$$

$$\begin{aligned} T_{g(\Gamma_Q)(b)}(t_1^{-1},p) &\geq \max_{g(s,p)=(t_1^{-1},p)} \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(s,p)\Big]\\ &\geq \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(s_1^{-1},p)\Big]\\ &\geq \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(s_1,p)\Big]. \end{aligned}$$

Since, the inequality is hold for each $s_1, s_2 \in G_1$ and $p \in Q$, which satisfy $g(s_1, p) = (t_1, p)$ and $g(s_2, p) = (t_2, p)$. Then,

$$\begin{aligned} T_{g(\Gamma_Q)(b)}(t_1t_2,p) &\geq \min\Big\{\max_{g(s_1,p)=(t_1,p)} \max_{h(a)=b}\Big[T_{\Gamma_Q(a)}(s_1,p)\Big], \max_{g(s_2,p)=(t_1,p)} \max_{h(a)=b}\Big[T_{\Gamma_Q(a)}(s_2,p)\Big]\Big\}\\ &= \min\Big\{T_{g(\Gamma_Q)(b)}(t_1,p), T_{g(\Gamma_Q)(b)}(t_2,p)\Big\}. \end{aligned}$$

Also,

$$T_{g(\Gamma_Q)(b)}(t_1^{-1},p) \geq \max_{g(s_1,p)=(t_1,p)} \max_{h(a)=b} \left[T_{\Gamma_Q(a)}(s_1,p)\right] = T_{g(\Gamma_Q)(b)}(t_1,p).$$

Similarly, we can obtain

$$I_{g(\Gamma_Q)(b)}(t_1t_2,p) \leq \max\left\{I_{g(\Gamma_Q)(b)}(t_1,p), I_{g(\Gamma_Q)(b)}(t_2,p)\right\}, I_{g(\Gamma_Q)(b)}(t_1^{-1},p) \leq I_{g(\Gamma_Q)(b)}(t_1,p),$$
$$F_{g(\Gamma_Q)(b)}(t_1t_2,p) \leq \max\left\{F_{g(\Gamma_Q)(b)}(t_1,p), F_{g(\Gamma_Q)(b)}(t_2,p)\right\}, F_{g(\Gamma_Q)(b)}(t_1^{-1},p) \leq F_{g(\Gamma_Q)(b)}(t_1,p).$$

This completes the proof. □

**Theorem 4.5.** *Let $(\Psi_Q, B)$ be a Q-NSG over a group $G_2$ and $(g,h)$ be a Q-NS hom from $G_1 \times Q$ to $G_2 \times Q$. Then, $(g,h)^{-1}(\Psi_Q, B)$ is a Q-NSG over $G_1$.*

*Proof.* For $a \in h^{-1}(B)$, $s_1, s_2 \in G_1$ and $p \in Q$, we have

$$\begin{aligned} T_{g^{-1}(\Psi_Q)(a)}(s_1s_2,p) &= T_{\Psi_Q[h(a)]}(g(s_1s_2,p)) \\ &= T_{\Psi_Q[h(a)]}(g(s_1,p)g(s_2,p)) \\ &\geq \min\left\{T_{\Psi_Q[h(a)]}(g(s_1,p)), T_{\Psi_Q[h(a)]}(g(s_2,p))\right\} \\ &= \min\left\{T_{g^{-1}(\Psi_Q)(a)}(s_1,p), T_{g^{-1}(\Psi_Q)(a)}(s_2,p)\right\} \end{aligned}$$

$$\begin{aligned} T_{g^{-1}(\Psi_Q)(a)}(s_1^{-1},p) &= T_{\Psi_Q[h(a)]}(g(s_1^{-1},p)) \\ &= T_{\Psi_Q[h(a)]}(g(s_1,p)^{-1}) \\ &\geq T_{\Psi_Q[h(a)]}(g(s_1,p))\} \\ &= T_{g^{-1}(\Psi_Q)(a)}(s_1,p). \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} I_{g^{-1}(\Psi_Q)(a)}(s_1s_2,p) &\leq \min\left\{I_{g^{-1}(\Psi_Q)(a)}(s_1,p), I_{g^{-1}(\Psi_Q)(a)}(s_2,p)\right\}, \\ I_{g^{-1}(\Psi_Q)(a)}(s_1^{-1},p) &= I_{g^{-1}(\Psi_Q)(a)}(s_1,p), \\ F_{g^{-1}(\Psi_Q)(a)}(s_1s_2,p) &=\leq \min\left\{F_{g^{-1}(\Psi_Q)(a)}(s_1,p), F_{g^{-1}(\Psi_Q)(a)}(s_2,p)\right\}, \\ F_{g^{-1}(\Psi_Q)(a)}(s_1^{-1},p) &= F_{g^{-1}(\Psi_Q)(a)}(s_1,p). \end{aligned}$$

Thus, the theorem is proved. □

# 5 Cartesian product of Q-neutrosophic soft groups

In this section, we introduce the cartesian product of Q-NSGs and discuss some of its properties.

**Definition 5.1.** Let $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ be two Q-NSGs over the groups $G_1$ and $G_2$, respectively. Then their cartesian product is $(\Gamma_Q, A) \times (\Psi_Q, B) = (\Omega_Q, A \times B)$ where $\Omega_Q(a, b) = \Gamma_Q(a) \times \Psi_Q(b)$ for $(a, b) \in A \times B$. Analytically,

$$\Omega_Q(a,b) = \Big\{\Big\langle((s,t),p), T_{\Omega_Q(a,b)}\Big((s,t),p\Big), I_{\Omega_Q(a,b)}\Big((s,t),p\Big), F_{\Omega_Q(a,b)}\Big((s,t),p\Big)\Big\rangle : s \in G_1, t \in G_2, p \in Q\Big\}$$

where,

$$\begin{aligned}
T_{\Omega_Q(a,b)}\Big((s,t),p\Big) &= \min\Big\{T_{\Gamma_Q(a)}(s,p), T_{\Psi_Q(b)}(t,p)\Big\},\\
I_{\Omega_Q(a,b)}\Big((s,t),p\Big) &= \max\Big\{I_{\Gamma_Q(a)}(s,p), I_{\Psi_Q(b)}(t,p)\Big\},\\
F_{\Omega_Q(a,b)}\Big((s,t),p\Big) &= \max\Big\{F_{\Gamma_Q(a)}(s,p), F_{\Psi_Q(b)}(t,p)\Big\}.
\end{aligned}$$

**Theorem 5.2.** *Let $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ be two Q-NSGs over the groups $G_1$ and $G_2$. Then their cartesian product $(\Gamma_Q, A) \times (\Psi_Q, B) = (\Omega_Q, A \times B)$ is also a Q-NSG over $G_1 \times G_2$.*

*Proof.* Let $(\Gamma_Q, A) \times (\Psi_Q, B) = (\Omega_Q, A \times B)$ where $\Omega_Q(a, b) = \Gamma_Q(a) \times \Psi_Q(b)$ for $(a, b) \in A \times B$. Then for $\Big((s_1, t_1), p\Big), \Big((s_2, t_2), p\Big) \in (G_1 \times G_2) \times Q$

$$\begin{aligned}
&T_{\Omega_Q(a,b)}\Big((s_1,t_1)(s_2,t_2),p\Big)\\
&\quad= T_{\Omega_Q(a,b)}\Big((s_1s_2,t_1t_2),p\Big)\\
&\quad= \min\Big\{T_{\Gamma_Q(a)}(s_1s_2,p), T_{\Psi_Q(b)}(t_1t_2,p)\Big\}\\
&\quad\geq \min\Big\{\min\big\{T_{\Gamma_Q(a)}(s_1,p), T_{\Gamma_Q(a)}(s_2,p)\big\}, \min\big\{T_{\Psi_Q(b)}(t_1,p), T_{\Psi_Q(b)}(t_2,p)\big\}\Big\}\\
&\quad= \min\Big\{\min\big\{T_{\Gamma_Q(a)}(s_1,p), T_{\Psi_Q(b)}(t_1,p)\big\}, \min\big\{T_{\Gamma_Q(a)}(s_2,p), T_{\Psi_Q(b)}(t_2,p)\big\}\Big\}\\
&\quad= \min\Big\{T_{\Omega_Q(a,b)}\Big((s_1,t_1),p\Big), T_{\Omega_Q(a,b)}\Big((s_2,t_2),p\Big)\Big\},
\end{aligned}$$

also

$$\begin{aligned}
&I_{\Omega_Q(a,b)}\Big((s_1,t_1)(s_2,t_2),p\Big)\\
&\quad= I_{\Omega_Q(a,b)}\Big((s_1s_2,t_1t_2),p\Big)\\
&\quad= \max\Big\{I_{\Gamma_Q(a)}(s_1s_2,p), I_{\Psi_Q(b)}(t_1t_2,p)\Big\}\\
&\quad\leq \max\Big\{\max\big\{I_{\Gamma_Q(a)}(s_1,p), I_{\Gamma_Q(a)}(s_2,p)\big\}, \max\big\{I_{\Psi_Q(b)}(t_1,p), I_{\Psi_Q(b)}(t_2,p)\big\}\Big\}\\
&\quad= \max\Big\{\max\big\{I_{\Gamma_Q(a)}(s_1,p), I_{\Psi_Q(b)}(t_1,p)\big\}, \max\big\{I_{\Gamma_Q(a)}(s_2,p), I_{\Psi_Q(b)}(t_2,p)\big\}\Big\}\\
&\quad= \max\Big\{I_{\Omega_Q(a,b)}\Big((s_1,t_1),p\Big), I_{\Omega_Q(a,b)}\Big((s_2,t_2),p\Big)\Big\},
\end{aligned}$$

similarly, $F_{\Omega_Q(a,b)}\Big((s_1,t_1)(s_2,t_2),p\Big) \leq \max\Big\{F_{\Omega_Q(a,b)}\Big((s_1,t_1),p\Big), F_{\Omega_Q(a,b)}\Big((s_2,t_2),p\Big)\Big\}$.

Next,

$$\begin{aligned} T_{\Omega_Q(a,b)}\Big((s_1,t_1)^{-1},p\Big) &= T_{\Omega_Q(a,b)}\Big((s_1^{-1},t_1^{-1}),p\Big) \\ &\geq \min\Big\{T_{\Gamma_Q(a)}(s_1^{-1},p), T_{\Psi_Q(b)}(t_1^{-1},p)\Big\} \\ &\geq \min\Big\{T_{\Gamma_Q(a)}(s_1,p), T_{\Psi_Q(b)}(t_1,p)\Big\} \\ &= T_{\Omega_Q(a,b)}\Big((s_1,t_1),p\Big), \end{aligned}$$

also

$$\begin{aligned} I_{\Omega_Q(a,b)}\Big((s_1,t_1)^{-1},p\Big) &= I_{\Omega_Q(a,b)}\Big((s_1^{-1},t_1^{-1}),p\Big) \\ &\leq \max\Big\{I_{\Gamma_Q(a)}(s_1^{-1},p), I_{\Psi_Q(b)}(t_1^{-1},p)\Big\} \\ &\leq \max\Big\{I_{\Gamma_Q(a)}(s_1,p), I_{\Psi_Q(b)}(t_1,p)\Big\} \\ &= I_{\Omega_Q(a,b)}\Big((s_1,t_1),p\Big), \end{aligned}$$

similarly, $F_{\Omega_Q(a,b)}\Big((s_1,t_1)^{-1},p\Big) \leq F_{\Omega_Q(a,b)}\Big((s_1,t_1),p\Big)$. Hence, this proves that $(\Gamma_Q, A) \times (\Psi_Q, B)$ is a Q-NSG over $G_1 \times G_2$. □

# 6 Conclusions

A Q-NSS is a NSS over two-dimensional universal set. Thus, a Q-NSS is a set with three components that can handle two-dimensional and indeterminate data simultaneously. The main goal of the current work is to utilize Q-NSSs to group theory. This study conceptualizes the notion of Q-NSGs as a new algebraic structure that deals with two-dimensional universal set. Some relevant properties and basic characteristics are explored. We define the Q-level soft set of a Q-NSS, which acts as a bridge between Q-neutrosophic soft groups and soft groups. Also, the concepts of image and preimage of a Q-NSG are investigated. Moreover, the cartesian product of Q-NSGs is discussed. The defined notion serves as the base for applying Q-NSSs to different algebraic structures such as semigroups, rings, hemirings, fields, lie subalgebras, BCK/BCI-algebras and in hyperstructure theory such as hypergroups and hyperrings following the discussion in [50, 51, 52, 53]. Moreover, these topics may be discussed using $t$-norm and $s$-norm.

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