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More on Neutrosophic Norms and Conorms

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Abstract**.** In 1995, Smarandache talked for the first time about neutrosophy and he defined one of the most important new mathematical tool which is a neutrosophic set theory as a new mathematical tool for handling problems involving imprecise, indeterminacy, and inconsistent data. He also defined the neutrosophic norm and conorms namely N-norm and N-conorm respectively. In this paper we give generating theorems for N-norm and N-conorm. Given an N-norm we can generate a class of N-norms and N-cnorms, and given an N-conorm we can generate a class of N-conorms and N-norms. We also give bijective generating theorems for N-norms and N-conorms.

Keywords: N-norm; N-conorm; generating theorem; bijective generating theorem.

1 Introduction

Since Zadeh [10] defined fuzzy set with *min* and *max* as the respective intersection and union operators, various alternatives operators have been proposed Dubois & Prade [2]; Yager [9]. The proposed operators are examples of the triangular norm and conorm (*t*-norm and *t*-conorm or *s*-norm) and hence fuzzy sets with these t-norms and s-norms as generalization of intersection and union are discussed in Klement [5], Waber [7], Wang [8] and Lowen [6]. In 2007 Alkhazaleh and Salleh [1] gave two generating theorems for *s*-norms and *t*-norms, namely given an *s*-norm we can generate a class of *s*-norms and *t*-norms, and given a *t*-norm we can generate a class of *t*-norms and *s*-norms. We also give two bijective generating theorems for *s*-norms and *t*-norms, that is given a bijective function under certain condition, we can generate new *s*-norm and *t*-norm from a given *s*-norm and also from a given *t*-norm. In 1995, Smarandache talked for the first time about neutrosophy and he in 1999 and 2005 [4, 3] defined one of the most important new mathematical tools which is a neutrosophic set theory as a new mathematical tool for handling problems involving imprecise, indeterminacy, and inconsistent data. He also presented the N-norms/N-conorms in neutrosophic logic and set as extensions of T-norms/T-conorms in fuzzy logic and set. In this paper we give two generating theorems for *N*-norms and *N*-conorms, namely given an *N*-norm we can generate a class of *N*-norms and *N*-conorms, and given a *N*-conorm we can generate a class of *N*-conorms and *N*-norms. We also give two bijective generating theorems for *N*-norms and *N*-conorms, that is given a bijective function under certain condition, we can generate new N-norm and N-conorm from a given N-norm and also from a given N-conorm.

2 Preliminaries

In this section, we recall some basic notions in fuzzy set and neutrosophic set theory.
For a fuzzy set we have the following s-norm and t-norm:

Definition 2.1 *The function* $s:[0,1]\times[0,1]\rightarrow[0,1]$ *is called an s-norm if it satisfies the following four requirements:*
Axiom s1. $s(x,y)=s(y,x)$ *(commutative condition).*
Axiom s2. $s(s(x, y), z) = s(x, s(y, z))$ *(associative condition).*
Axiom s3. If $x_1 \geq x_2$ *and* $y_1 \geq y_2$, *then* $s(x_1,y_1) \geq s(x_2,y_2)$ *(nondecreasing condition).*
Axiom s4. $s(1, 1) = 1$, $s(x, 0) = s(0, x) = x$ *(boundary condition).*

Definition 2.2 *The function* $t:[0,1]\times[0,1]\rightarrow[0,1]$ *is called a* ***t-norm*** *if it satisfies the following four requirements:*
Axiom t1. $t(x, y) = t(y, x)$ *(commutative condition).*
Axiom t2. $t(t(x, y), z) = t(x, t(y, z))$ *(associative condition).*
Axiom t3. If $x_1 \geq x_2$ *and* $y_1 \geq y_2$, *then* $t(x_1,y_1) \geq t(x_2,y_2)$ *(nondecreasing condition).*
Axiom t4 $t(x, 1) = x$ *(boundary condition).*

	***s*-norm**	***t*-norm**
Basic	max(x,y)	min(x,y)
Bounded	$\min(x+y,1)$	$\max(x+y-1,0)$
Algebraic	$x+y-xy$	xy

Definition 2.3 [6] *A neutrosophic set* A *on the universe of discourse* X *is defined as*

$A = \{< x, T_A(x), I_A(x), F_A(x) >, x \in X\}$ *where* T , I , $F : X \to]^-0,1^+[$ *and* $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

As a generalization of T-norm and T-conorm from the Fuzzy Logic and Set, Smarandache introduced the **N-norms and N-conorms** for the Neutrosophic Logic and Set. He defined a *partial relation order* on the neutrosophic set/logic in the following way:

$x(T_1, I_1, F_1) \leq y(T_2, I_2, F_2)$ iff $T_1 \leq T_2$, $I_1 \geq I_2$, $F_1 \geq F_2$ for crisp components. And, in general, for subunitary set components:

$x(T_1, I_1, F_1) \leq y(T_2, I_2, F_2)$ *iff*

$\inf T_1 \leq \inf T_2, \ \sup T_1 \leq \sup T_2,$

$\inf I_1 \geq \inf I_2, \ \sup I_1 \geq \sup I_2,$

$\inf F_1 \geq \inf F_2, \ \sup F_1 \geq \sup F_2.$

Definition 2.4 [4] N-norms

$N_n : ( ]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[ )^2 \to ( ]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[ )$

$N_n(x(T_1,I_1,F_1), y(T_2,I_2,F_2)) = (N_nT(x,y), N_nI(x,y), N_nF(x,y)),$

where $N_nT(.,.), N_nI(.,.), N_nF(.,.)$ *are the truth /membership, indeterminacy, and respectively falsehood /nonmembership components.* N_n *have to satisfy, for any* x, y, z *in the neutrosophic logic/set* M *of the universe of discourse* U *, the following axioms:*

a) Boundary Conditions: $N_n(x,0) = 0, \ N_n(x,1) = x.$

b) Commutativity: $N_n(x, y) = N_n(y, x).$

c) Monotonicity: If $x \leq y$, *then* $N_n(x, z) \leq N_n(y, z).$

d) Associativity: $N_n(N_n(x, y), z) = N_n(x, N_n(y, z)).$

N_n *represent the and operator in neutrosophic logic, and respectively the intersection operator in neutrosophic set theory.*

Definition 2.5 [4] N-conorms

$N_c : ( ]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[ )^2 \to ( ]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[ )$

$N_c(x(T_1,I_1,F_1), y(T_2,I_2,F_2)) = (N_cT(x,y), N_cI(x,y), N_cF(x,y)),$

where $N_cT(.,.), N_cI(.,.), N_cF(.,.)$ are the truth /membership, indeterminacy, and respectively falsehood /nonmembership components. N_c have to satisfy, for any x, y, z in the neutrosophic logic/set M of the universe of discourse U , the following axioms:

a) Boundary Conditions: $N_c(x,0) = x, \ N_c(x,1) = 1.$

b) Commutativity: $N_c(x, y) = N_c(y, x).$

c) Monotonicity: If $x \leq y$, *then* $N_c(x, z) \leq N_c(y, z).$

d) Associativity: $N_c(N_c(x, y), z) = N_c(x, N_c(y, z)).$

A general example of N-norm would be this.

Let $x(T_1, I_1, F_1)$ and $y(T_1, I_1, F_1)$ be in the neutrosophic set/logic M. Then: $N_n(x, y) = (T_1 \wedge T_2, I_1 \vee I_2, F_1 \vee F_2).$

A general example of N-conorm would be this. Let $x(T_1, I_1, F_1)$ and $y(T_1, I_1, F_1)$ be in the neutrosophic set/logic M. Then: $N_c(x, y) = (T_1 \vee T_2, I_1 \wedge I_2, F_1 \wedge F_2)$ where the "/\" operator, acting on two (standard or non-standard) subunitary sets, is a N-norm (verifying the above N-norms axioms); while the "\/" operator, also acting on two (standard or non-standard) subunitary sets, is a N-conorm (verifying the above N-conorms axioms). For example, /\ can be the Algebraic Product T-norm/N-norm, so T1/\T2 = T1·T2; and \/ can be the Algebraic Product T-conorm/N-conorm, so T1\/T2 = T1+T2-T1·T2. Or /\ can be any T-norm/N-norm, and \/ any T-conorm/N-conorm from the above and below; for example the easiest way would be to consider the min for crisp components (or inf for subset components) and respectively max for crisp components (or sup for subset components).

In the folowing we recall some theorems and corollaries given by Shawkat and Salleh 2007 in [1]

Theorem 2.6 *For any s-norm* $s(x, y)$ *and for all* $\alpha \geq 1$, *we get the following s-norms and t-norms*:

1. $S_\alpha^s(x, y) = \sqrt[\alpha]{s(x^\alpha, y^\alpha)}$,
2. $T_\alpha^s(x, y) = 1 - \sqrt[\alpha]{s((1-x)^\alpha, (1-y)^\alpha)}$.

Theorem 2.7 *For any t-norm* $t(x, y)$ *and for all* $\alpha > 1$, *we get the following t-norms and s-norms*:

1. $T_\alpha^t(x, y) = \sqrt[\alpha]{t(x^\alpha, y^\alpha)}$,
2. $S_\alpha^t(x, y) = 1 - \sqrt[\alpha]{t((1-x)^\alpha, (1-y)^\alpha)}$

Theorem 2.8 *Let* $f, g : [0,1] \to [0,1]$ *be bijective functions such that* $f(0) = 0$, $f(1) = 1$, $g(0) = 1$ *and* $g(1) = 0$. *For any s-norm* $s(x, y)$ *we get the following s-norm and t-norm*:

1. $S_f^s(x, y) = f^{-1}\left[s\left(f(x), f(y)\right)\right]$,
2. $T_g^s(x, y) = g^{-1}\left[s\left(g(x), g(y)\right)\right]$.

Corollary 2.9 *Let* $f(x) = \sin\frac{\pi}{2}x$ *and* $g(x) = \cos\frac{\pi}{2}x$ *then*

1. $S_{\sin}^s(x, y) = \frac{2}{\pi}\sin^{-1} s\left(\sin\frac{\pi}{2}x, \sin\frac{\pi}{2}y\right)$ is an *s*-norm
2. $T_{\cos}^s(x, y) = \frac{2}{\pi}\cos^{-1} s\left(\cos\frac{\pi}{2}x, \cos\frac{\pi}{2}y\right)$ is a *t*-norm

Theorem 2.10 *Let* $f, g : [0,1] \to [0,1]$ *be bijective functions such that* $f(0) = 0$, $f(1) = 1$, $g(0) = 1$ *and* $g(1) = 0$.

For any t-norm $t(x,y)$ we get the following t-norm and s-norm:

1. $T_f^t(x,y)=f^{-1}\left[t\left(f(x),f(y)\right)\right]$,
2. $S_g^t(x,y)=g^{-1}\left[t\left(g(x),g(y)\right)\right]$

Corollary 2.10 *Let f(x)=* $\sin\frac{\pi}{2}x$ *and g(x)=* $\cos\frac{\pi}{2}x$ *then*

1. $T_{\sin}^t(x,y)=\frac{2}{\pi}\sin^{-1}\left(t\left(\sin\frac{\pi}{2}x,\sin\frac{\pi}{2}y\right)\right)$ is a *t*-norm
2. $S_{\cos}^t(x,y)=\frac{2}{\pi}\cos^{-1}\left(t\left(\cos\frac{\pi}{2}x,\cos\frac{\pi}{2}y\right)\right)$ is an *s*-norm

3 Generating Theorems

In this section we give two generating theorems to generate N-norms and N-conorms by using any N-norms and N-conorms. Without loss of generality, we will rewrite the Smarandache's N-norm and N-conorm as it follows:

Definition 3.1

$T_n:\ (\ ]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\ )^2\rightarrow(\ ]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\ )$

$T_n\left(x(T,I,F),\ y(T,I,F)\right)=\left(t(x_T,y_T),\ s(x_I,y_I),\ s(x_F,y_F)\right),$

where $t(x_T,y_T),\ s(x_I,y_I),\ s(x_F,y_F)$ are the truth /membership, indeterminacy, and respectively falsehood /nonmembership components and *s* and *t* are the fuzzy s-norm and fuzzy t-norm respectively. T_n have to satisfy, for any $x,\ y,\ z$ in the neutrosophic logic/set M of the universe of discourse U, the following axioms:

a) Boundary Conditions: $T_n(x,0)=0,\ T_n(x,1)=x.$

b) Commutativity: $T_n(x,\ y)=T_n(y,\ x).$

c) Monotonicity: If $x\le y$, *then* $T_n(x,\ z)\le T_n(y,\ z).$

d) Associativity: $T_n\left(T_n\ (x,\ y),\ z\right)=T_n\left(x,\ T_n(y,\ z)\right).$

Definition 3.2 N-conorms

$S_n:\ (\ ]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\ )^2\rightarrow(\ ]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[\ )$

$S_n\left(x(T,I,F),\ y(T,I,F)\right)=\left(s(x_T,y_T),\ t(x_I,y_I),\ t(x_F,y_F)\right),$

where $s(x_T,y_T),\ t(x_I,y_I),\ t(x_F,y_F)$ are the truth /membership, indeterminacy, and respectively falsehood /nonmembership components and *s* and *t* are the fuzzy s-norm and fuzzy t-norm respectively. S_n have to satisfy, for any $x,\ y,\ z$ in the neutrosophic logic/set M of the universe of discourse U, the following axioms:

a) Boundary Conditions: $S_n(x,0)=x,\ S_n(x,1)=1.$

b) Commutativity: $S_n(x,\ y)=S_n(y,\ x).$

c) Monotonicity: If $x\le y$, *then* $S_n(x,\ z)\le S_n(y,\ z).$

d) Associativity: $S_n\left(S_n\ (x,\ y),\ z\right)=S_n\left(x,\ S_n(y,\ z)\right).$

From now we use the following notation for N-norm and N-conorm respectively $T_n-(x,y)$ and $S_n-(x,y)$.

Remark: We will use the following border: $0(0,1,1)$ and $1(1,0,0)$.

Theorem 3.3. *For any* $S_n-(x,y)$ *and for all* $\alpha\ge1$, by *using any fuzzy union s-norm we get the following* $S_n-(x,y)$ *and* $T_n-(x,y)$:

1. $$\underset{s}{S_n^\alpha}(x,y)=\left\langle\begin{array}{l}\sqrt[\alpha]{s(x_T{}^\alpha,y_T{}^\alpha)},\\ 1-\sqrt[\alpha]{s((1-x_I)^\alpha,(1-y_I)^\alpha)},\\ 1-\sqrt[\alpha]{s((1-x_F)^\alpha,(1-y_F)^\alpha)}\end{array}\right\rangle$$

2. $$\underset{s}{T_n^\alpha}(x,y)=\left\langle\begin{array}{l}1-\sqrt[\alpha]{s\left(\left(1-x_T{}^\alpha\right),\left(1-y_T{}^\alpha\right)\right)},\\ \sqrt[\alpha]{s\left(x_I{}^\alpha,y_I{}^\alpha\right)},\\ \sqrt[\alpha]{s\left(x_F{}^\alpha,y_F{}^\alpha\right)}\end{array}\right\rangle.$$

Where *s* any s-norm (fuzzy union).

Proof. 1.

Axiom 1.

$$\underset{s}{S_n^\alpha}(0,x)=\left\langle\begin{array}{l}\sqrt[\alpha]{s(0^\alpha,x_T{}^\alpha)},\\ 1-\sqrt[\alpha]{s((1-1)^\alpha,(1-x_I)^\alpha)},\\ 1-\sqrt[\alpha]{s((1-1)^\alpha,(1-x_F)^\alpha)}\end{array}\right\rangle=x(x_T,x_I,x_F).$$

$$\underset{s}{S_n^\alpha}(1,x)=\left\langle\begin{array}{l}\sqrt[\alpha]{s(1^\alpha,x_T{}^\alpha)},\\ 1-\sqrt[\alpha]{s((1-0)^\alpha,(1-x_I)^\alpha)},\\ 1-\sqrt[\alpha]{s((1-0)^\alpha,(1-x_F)^\alpha)}\end{array}\right\rangle=1(1,0,0)$$

Axiom 2.

$$\underset{s}{S_n^\alpha}(x,y)=\left\langle\begin{array}{l}\sqrt[\alpha]{s(x_T{}^\alpha,y_T{}^\alpha)},\\ 1-\sqrt[\alpha]{s((1-x_I)^\alpha,(1-y_I)^\alpha)},\\ 1-\sqrt[\alpha]{s((1-x_F)^\alpha,(1-y_F)^\alpha)}\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\sqrt[\alpha]{s(y_T{}^\alpha,x_T{}^\alpha)},\\ 1-\sqrt[\alpha]{s((1-y_I)^\alpha,(1-x_I)^\alpha)},\\ 1-\sqrt[\alpha]{s((1-y_F)^\alpha,(1-x_F)^\alpha)}\end{array}\right\rangle.$$

$$=\underset{s}{S_n^\alpha}(y,x)$$

Axiom 3. Let $x(x_1, x_2, x_3) \le y(y_1, y_2, y_3)$ then $x_1 \le y_1, x_2 \ge y_2, x_3 \ge y_3$ and $s(x_1^\alpha, z_1^\alpha) \ge s(y_1^\alpha, z_1^\alpha)$ which implies $\sqrt[\alpha]{s(x_1^\alpha, z_1^\alpha)} \ge \sqrt[\alpha]{s(y_2^\alpha, z_2^\alpha)}$. (1)

Also we have $(1-x_2)^\alpha \le (1-y_2)^\alpha$ then $s\left((1-x_2)^\alpha, (1-z_2)^\alpha\right) \le s\left((1-y_2)^\alpha, (1-z_2)^\alpha\right)$, which implies that

$$1-\sqrt[\alpha]{s\left((1-x_2)^\alpha, (1-z_2)^\alpha\right)} \ge 1-\sqrt[\alpha]{s\left((1-y_2)^\alpha, (1-z_2)^\alpha\right)} \quad (2)$$

And we have $(1-x_3)^\alpha \le (1-y_3)^\alpha$ then $s\left((1-x_3)^\alpha, (1-z_3)^\alpha\right) \le s\left((1-y_3)^\alpha, (1-z_3)^\alpha\right)$, which implies that

$$1-\sqrt[\alpha]{s\left((1-x_3)^\alpha, (1-z_3)^\alpha\right)} \ge 1-\sqrt[\alpha]{s\left((1-y_3)^\alpha, (1-z_3)^\alpha\right)} \quad (3)$$

From (1), (2) and (3) we have $\underset{s}{S_n^\alpha}(x,z) \ge \underset{s}{S_n^\alpha}(y,z)$.

Axiom 4.

$$\underset{s}{S_n^\alpha}\left(\underset{s}{S_n^\alpha}(x,y), z\right) = S_n^\alpha\left(\left\langle \begin{array}{l} \sqrt[\alpha]{s(x_T{}^\alpha, y_T{}^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-x_I)^\alpha, (1-y_I)^\alpha)}, \\ 1-\sqrt[\alpha]{s((1-x_F)^\alpha, (1-y_F)^\alpha)} \end{array} \right\rangle, z\right)$$

$$= \left\langle \begin{array}{l} \sqrt[\alpha]{s\left(\left(\sqrt[\alpha]{s\left(x_T{}^\alpha, y_T{}^\alpha\right)}\right)^\alpha, z_T{}^\alpha\right)}, \\ 1-\sqrt[\alpha]{s\left(\left(1-\left(1-\sqrt[\alpha]{s\left((1-x_I)^\alpha, (1-y_I)^\alpha\right)}\right)\right)^\alpha, (1-z_I)^\alpha\right)}, \\ 1-\sqrt[\alpha]{s\left(\left(1-\left(1-\sqrt[\alpha]{s\left((1-x_F)^\alpha, (1-y_F)^\alpha\right)}\right)\right)^\alpha, (1-z_F)^\alpha\right)} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \sqrt[\alpha]{s\left(s\left(x_T{}^\alpha, y_T{}^\alpha\right), z_T{}^\alpha\right)}, \\ 1-\sqrt[\alpha]{s\left(s\left(\left(1-x_I{}^\alpha\right), \left(1-y_I{}^\alpha\right)\right), \left(1-z_I{}^\alpha\right)\right)}, \\ 1-\sqrt[\alpha]{s\left(s\left(\left(1-x_F{}^\alpha\right), \left(1-y_F{}^\alpha\right)\right), \left(1-z_F{}^\alpha\right)\right)} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \sqrt[\alpha]{s\left(x_T{}^\alpha, s\left(y_T{}^\alpha, z_T{}^\alpha\right)\right)}, \\ 1-\sqrt[\alpha]{s\left(\left(1-x_I{}^\alpha\right), s\left(\left(1-y_I{}^\alpha\right), \left(1-z_I{}^\alpha\right)\right)\right)}, \\ 1-\sqrt[\alpha]{s\left(\left(1-x_F{}^\alpha\right), s\left(\left(1-y_F{}^\alpha\right), \left(1-z_F{}^\alpha\right)\right)\right)} \end{array} \right\rangle$$

$$= \underset{s}{S_n^\alpha}\left(x, S_n^\alpha(y,z)\right)$$

Therefore $\underset{s}{S_n^\alpha}(x,y)$ is an N-conorm.

2. The proof is similar to Proof 1. □

Theorem 3.4. *For any $T_n-(x,y)$ and for all $\alpha \ge 1$, by using any fuzzy intersection t-norm we get the following $S_n-(x,y)$ and $T_n-(x,y)$:*

1. $$\underset{t}{S_n^\alpha}(x,y) = \left\langle \begin{array}{l} 1-\sqrt[\alpha]{t\left(\left(1-x_T{}^\alpha\right), \left(1-y_T{}^\alpha\right)\right)}, \\ \sqrt[\alpha]{t\left(x_I{}^\alpha, y_I{}^\alpha\right)}, \\ \sqrt[\alpha]{t\left(x_F{}^\alpha, y_F{}^\alpha\right)} \end{array} \right\rangle$$

2. $$\underset{t}{T_n^\alpha}(x,y) = \left\langle \begin{array}{l} \sqrt[\alpha]{t\left(x^\alpha \; y_T{}^\alpha\right)}, \\ 1-\sqrt[\alpha]{t\left(\left(1-x_I{}^\alpha\right), \left(1-y_I{}^\alpha\right)\right)}, \\ 1-\sqrt[\alpha]{t\left(\left(1-x_F{}^\alpha\right), \left(1-y_F{}^\alpha\right)\right)} \end{array} \right\rangle.$$

Where t any t-norm (fuzzy intersection).

Proof. The proof is similar to Proof of theorem 3.3. □

4. Bijective Generating Theorems

In this section we give two generating theorems to generate *N-norms* and *N-conorms* from any *N-norms* and *N-conorms*. By these theorems we can generate infinitely many *N-norms* and *N-conorms* by using two bijective functions with certain conditions.

Theorem 4.1. *Let $f, g:[0,1] \to [0,1]$ be bijective functions such that $f(0)=0$, $f(1)=1$, $g(0)=1$ and $g(1)=0$. For any $S_n-(x,y)$ and by using any fuzzy union s-norm we get the following $S_n-(x,y)$ and $T_n-(x,y)$:*

1. $$\underset{f,g}{S_n^s}(x,y) = \left\langle \begin{array}{l} f^{-1}\left[s\left(f(x_T), f(y_T)\right)\right], \\ g^{-1}\left[s\left(g(x_I), g(y_I)\right)\right], \\ g^{-1}\left[s\left(g(x_F), g(y_F)\right)\right] \end{array} \right\rangle,$$

2. $$\underset{f,g}{T_n^s}(x,y) = \left\langle \begin{array}{l} g^{-1}\left[s\left(g(x_T), g(y_T)\right)\right], \\ f^{-1}\left[s\left(f(x_I), f(y_I)\right)\right], \\ f^{-1}\left[s\left(f(x_F), f(y_F)\right)\right] \end{array} \right\rangle.$$

Proof. 1.

Axiom 1.

$$\underset{f,g}{S_n^s}(x,0)=\left\langle\begin{array}{l} f^{-1}\left[s\left(f\left(x_T\right),f\left(0\right)\right)\right],\\ g^{-1}\left[s\left(g\left(x_I\right),g\left(1\right)\right)\right],\\ g^{-1}\left[s\left(g\left(x_F\right),g\left(1\right)\right)\right]\end{array}\right\rangle=x.$$

$$\underset{f,g}{S_n^s}(x,1)=\left\langle\begin{array}{l} f^{-1}\left[s\left(f\left(x_T\right),f\left(1\right)\right)\right],\\ g^{-1}\left[s\left(g\left(x_I\right),g\left(0\right)\right)\right],\\ g^{-1}\left[s\left(g\left(x_F\right),g\left(0\right)\right)\right]\end{array}\right\rangle=1\left(1,0,0\right).$$

Axiom 2.

$$\underset{f,g}{S_n^s}(x,y)=\left\langle\begin{array}{l} f^{-1}\left[s\left(f\left(x_T\right),f\left(y_T\right)\right)\right],\\ g^{-1}\left[s\left(g\left(x_I\right),g\left(y_I\right)\right)\right],\\ g^{-1}\left[s\left(g\left(x_F\right),g\left(y_F\right)\right)\right]\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l} f^{-1}\left[s\left(f\left(y_T\right),f\left(x_T\right)\right)\right],\\ g^{-1}\left[s\left(g\left(y_I\right),g\left(x_I\right)\right)\right],\\ g^{-1}\left[s\left(g\left(y_F\right),g\left(x_F\right)\right)\right]\end{array}\right\rangle=\underset{f,g}{S_n^s}(y,x).$$

Axiom 3. Let $x\le y$. Since f is bijective on the interval $[0,1]$ and by Axiom s3 we have

$s\left(f\left(x_T\right),f\left(z_T\right)\right)\le s\left(f\left(y_T\right),f\left(z_T\right)\right)$ then

$f^{-1}\left[s\left(f\left(x_T\right),f\left(z_T\right)\right)\right]\le f^{-1}\left[s\left(f\left(y_T\right),f\left(z_T\right)\right)\right]$ (1)

Also since g is bijective on the interval $[0,1]$ and by Axiom t3 we have

$s\left(g\left(x_I\right),g\left(z_I\right)\right)\ge s\left(g\left(y_I\right),g\left(z_I\right)\right)$ then

$g^{-1}\left[s\left(g\left(x_I\right),g\left(z_I\right)\right)\right]\ge g^{-1}\left[s\left(g\left(y_I\right),g\left(z_I\right)\right)\right]$ (2)

And

$s\left(g\left(x_F\right),g\left(z_F\right)\right)\ge s\left(g\left(y_F\right),g\left(z_F\right)\right)$ then

$g^{-1}\left[s\left(g\left(x_F\right),g\left(z_F\right)\right)\right]\ge g^{-1}\left[s\left(g\left(y_F\right),g\left(z_F\right)\right)\right]$ (3)

From (1), (2) and (3) we have $\underset{f,g}{S_n^s}(x,z)\le\underset{f,g}{S_n^s}(y,z)$

Axiom 4.

$$\underset{f,g}{S_n^s}\left(\underset{f,g}{S_n^s}(x,y),z\right)=\underset{f,g}{S_n^s}\left(\left\langle\begin{array}{l} f^{-1}\left[s\left(f\left(x_T\right),f\left(y_T\right)\right)\right],\\ g^{-1}\left[s\left(g\left(x_I\right),g\left(y_I\right)\right)\right],\\ g^{-1}\left[s\left(g\left(x_F\right),g\left(y_F\right)\right)\right]\end{array}\right\rangle,z\right)$$

$$=\left\langle\begin{array}{l} f^{-1}\left(f\left(f^{-1}\left[s\left(f\left(x_T\right),f\left(y_T\right)\right)\right]\right),f\left(z_T\right)\right),\\ g^{-1}\left(g\left(g^{-1}\left[s\left(g\left(x_I\right),g\left(y_I\right)\right)\right]\right),g\left(z_I\right)\right),\\ g^{-1}\left(g\left(g^{-1}\left[s\left(g\left(x_F\right),g\left(y_F\right)\right)\right]\right),g\left(z_F\right)\right)\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l} f^{-1}\left(s\left(f\left(x_T\right),f\left(y_T\right)\right),f\left(z_T\right)\right),\\ g^{-1}\left(s\left(g\left(x_I\right),g\left(y_I\right)\right),g\left(z_I\right)\right),\\ g^{-1}\left(s\left(g\left(x_F\right),g\left(y_F\right)\right),g\left(z_F\right)\right)\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l} f^{-1}\left(f\left(x_T\right),s\left(f\left(y_T\right),f\left(z_T\right)\right)\right),\\ g^{-1}\left(g\left(x_I\right),s\left(g\left(y_I\right),g\left(z_I\right)\right)\right),\\ g^{-1}\left(g\left(x_F\right),s\left(g\left(y_F\right),g\left(z_F\right)\right)\right)\end{array}\right\rangle$$

$$=S_n^s\left(x,\underset{f,g}{S_n^s}(y,z)\right).$$

Therefore $\underset{f,g}{S_n^s}(x,y)$ is an $S_n-(x,y)$ □

2. The Proof is similar to Proof 1.

Corollary 4.2. *Let* $f(x)=\sin\frac{\pi}{2}x$ *and* $g(x)=\cos\frac{\pi}{2}x$ *then*

1. $$\underset{\sin,\cos}{S_n^s}(x,y)=\left\langle\begin{array}{l} \frac{2}{\pi}\sin^{-1}s\left(\sin\frac{\pi}{2}x_T,\sin\frac{\pi}{2}y_T\right),\\ \frac{2}{\pi}\cos^{-1}s\left(\cos\frac{\pi}{2}x_I,\cos\frac{\pi}{2}y_I\right),\\ \frac{2}{\pi}\cos^{-1}s\left(\cos\frac{\pi}{2}x_F,\cos\frac{\pi}{2}y_F\right)\end{array}\right\rangle$$

is an $S_n-(x,y)$

2. $$T^s_{\substack{n\\ \sin,\cos}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\cos^{-1}s\left(\cos\frac{\pi}{2}x_T,\cos\frac{\pi}{2}y_T\right),\\ \frac{2}{\pi}\sin^{-1}s\left(\sin\frac{\pi}{2}x_I,\sin\frac{\pi}{2}y_I\right),\\ \frac{2}{\pi}\sin^{-1}s\left(\sin\frac{\pi}{2}x_F,\sin\frac{\pi}{2}y_F\right)\end{array}\right\rangle$$

is a $T_n-(x,y)$

Theorem 4.3. *Let* $f,\ g:[0,1]\to[0,1]$ *be bijective functions such that* $f(0)=0$, $f(1)=1$, $g(0)=1$ *and* $g(1)=0$. *For any* $T_n-(x,y)$ *and by using any fuzzy intersection t-norm we get the following* $S_n-(x,y)$ *and* $T_n-(x,y)$:

1. $$S^t_{\substack{n\\ f,g}}(x,y)=\left\langle\begin{array}{l}g^{-1}\left[t\left(g(x_T),g(y_T)\right)\right],\\ f^{-1}\left[t\left(f(x_I),f(y_I)\right)\right],\\ f^{-1}\left[t\left(f(x_F),f(y_F)\right)\right]\end{array}\right\rangle,$$

2. $$T^t_{\substack{n\\ f,g}}(x,y)=\left\langle\begin{array}{l}f^{-1}\left[t\left(f(x_T),f(y_T)\right)\right],\\ g^{-1}\left[t\left(g(x_I),g(y_I)\right)\right],\\ g^{-1}\left[t\left(g(x_F),g(y_F)\right)\right]\end{array}\right\rangle.$$

Proof. The proof is similar to Proof of theorem 4.1. □

Corollary 4.4. *Let* $f(x)=\sin\frac{\pi}{2}x$ *and* $g(x)=\cos\frac{\pi}{2}x$ *then*

1. $$S^t_{\substack{n\\ \sin,\cos}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\cos^{-1}t\left(\cos\frac{\pi}{2}x_T,\cos\frac{\pi}{2}y_T\right),\\ \frac{2}{\pi}\sin^{-1}t\left(\sin\frac{\pi}{2}x_I,\sin\frac{\pi}{2}y_I\right),\\ \frac{2}{\pi}\sin^{-1}t\left(\sin\frac{\pi}{2}x_F,\sin\frac{\pi}{2}y_F\right)\end{array}\right\rangle$$

is an $S_n-(x,y)$

2. $$T^t_{\substack{n\\ \sin,\cos}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\sin^{-1}t\left(\sin\frac{\pi}{2}x_T,\sin\frac{\pi}{2}y_T\right),\\ \frac{2}{\pi}\cos^{-1}t\left(\cos\frac{\pi}{2}x_I,\cos\frac{\pi}{2}y_I\right),\\ \frac{2}{\pi}\cos^{-1}t\left(\cos\frac{\pi}{2}x_F,\cos\frac{\pi}{2}y_F\right)\end{array}\right\rangle$$

is a $T_n-(x,y)$

5. Examples

In this section we generate some new $S_n-(x,y)$ and $T_n-(x,y)$ from existing $S_n-(x,y)$ and $T_n-(x,y)$ using the Generating Theorems and the Bijective Generating Theorems.

5. 1. BOUNDED SUM GENERATING CLASSES

New $S_n-(x,y)$ and $T_n-(x,y)$ from the bounded sum *s*-norm.

$$S^\alpha_{\substack{n\\ s}}(x,y)=\left\langle\begin{array}{l}\sqrt[\alpha]{\min(x_T{}^\alpha+y_T{}^\alpha,1)},\\ 1-\sqrt[\alpha]{\min((1-x_I)^\alpha+(1-y_I)^\alpha,1)},\\ 1-\sqrt[\alpha]{\min((1-x_F)^\alpha+(1-y_F)^\alpha,1)}\end{array}\right\rangle$$

$$T^\alpha_{\substack{n\\ bs}}(x,y)=\left\langle\begin{array}{l}1-\sqrt[\alpha]{\min\left((1-x_T)^\alpha+(1-\ _T)^\alpha,1\right)},\\ \sqrt[\alpha]{\min\left(x_I{}^\alpha+y_I{}^\alpha,1\right)},\\ \sqrt[\alpha]{\min\left(x_F{}^\alpha+y_F{}^\alpha,1\right)}\end{array}\right\rangle.$$

$$S^{bs}_{\substack{n\\ \sin,\cos}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\sin^{-1}\left(\min\left(\left(\sin\frac{\pi}{2}x_T\right)+\left(\sin\frac{\pi}{2}y_T\right),1\right)\right),\\ \frac{2}{\pi}\cos^{-1}\left(\min\left(\left(\cos\frac{\pi}{2}x_I\right)+\left(\cos\frac{\pi}{2}y_I\right),1\right)\right),\\ \frac{2}{\pi}\cos^{-1}\left(\min\left(\left(\cos\frac{\pi}{2}x_F\right)+\left(\cos\frac{\pi}{2}y_F\right),1\right)\right)\end{array}\right\rangle$$

$$T^{bs}_{\underset{\sin,\cos}{n}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\cos^{-1}\left(\min\left(\left(\cos\frac{\pi}{2}x_T\right)+\left(\cos\frac{\pi}{2}y_T\right),1\right)\right),\\ \frac{2}{\pi}\sin^{-1}\left(\min\left(\left(\sin\frac{\pi}{2}x_I\right)+\left(\sin\frac{\pi}{2}y_I\right),1\right)\right),\\ \frac{2}{\pi}\sin^{-1}\left(\min\left(\left(\sin\frac{\pi}{2}x_F\right)+\left(\sin\frac{\pi}{2}y_F\right),1\right)\right)\end{array}\right.$$

5.2. ALGEBRAIC SUM GENERATING CLASSES

New $S_n-(x,y)$ and $T_n-(x,y)$ from the algebraic sum *s*-norm.

$$S^{\alpha}_{\underset{as}{n}}(x,y)=\left\langle\begin{array}{l}\sqrt[\alpha]{x_T{}^{\alpha}+y_T{}^{\alpha}-x_T{}^{\alpha}.y_T{}^{\alpha}},\\ 1-\sqrt[\alpha]{(1-x_I)^{\alpha}+(1-y_I)^{\alpha}-(1-x_I)^{\alpha}.(1-y_I)^{\alpha}},\\ 1-\sqrt[\alpha]{(1-x_F)^{\alpha}+(1-y_F)^{\alpha}-(1-x_F)^{\alpha}.(1-y_F)^{\alpha}}\end{array}\right\rangle$$

$$T^{\alpha}_{\underset{as}{n}}(x,y)=\left\langle\begin{array}{l}1-\sqrt[\alpha]{(1-x_T)^{\alpha}+(1-y_T)^{\alpha}-(1-x_T)^{\alpha}(1-y_T)^{\alpha}},\\ \sqrt[\alpha]{x_I{}^{\alpha}+y_I{}^{\alpha}-x_I{}^{\alpha}.y_I{}^{\alpha}},\\ \sqrt[\alpha]{x_F{}^{\alpha}+y_F{}^{\alpha}-x_F{}^{\alpha}.y_F{}^{\alpha}}\end{array}\right\rangle.$$

$$S^{as}_{\underset{\sin,\cos}{n}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\sin^{-1}\left(\left(\sin\frac{\pi}{2}x_T\right)+\left(\sin\frac{\pi}{2}y_T\right)-\left(\sin\frac{\pi}{2}x_T\right)\left(\sin\frac{\pi}{2}y_T\right)\right),\\ \frac{2}{\pi}\cos^{-1}\left(\left(\cos\frac{\pi}{2}x_I\right)+\left(\cos\frac{\pi}{2}y_I\right)-\left(\cos\frac{\pi}{2}x_I\right)\left(\cos\frac{\pi}{2}y_I\right)\right),\\ \frac{2}{\pi}\cos^{-1}\left(\left(\cos\frac{\pi}{2}x_F\right)+\left(\cos\frac{\pi}{2}y_F\right)-\left(\cos\frac{\pi}{2}x_F\right)\left(\cos\frac{\pi}{2}y_F\right)\right)\end{array}\right\rangle$$

$$T^{as}_{\underset{\sin,\cos}{n}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\cos^{-1}\left(\left(\cos\frac{\pi}{2}x_T\right)+\left(\cos\frac{\pi}{2}y_T\right)-\left(\cos\frac{\pi}{2}x_T\right)\left(\cos\frac{\pi}{2}y_T\right)\right),\\ \frac{2}{\pi}\sin^{-1}\left(\left(\sin\frac{\pi}{2}x_I\right)+\left(\sin\frac{\pi}{2}y_I\right)-\left(\sin\frac{\pi}{2}x_I\right)\left(\sin\frac{\pi}{2}y_I\right)\right),\\ \frac{2}{\pi}\sin^{-1}\left(\left(\sin\frac{\pi}{2}x_F\right)+\left(\sin\frac{\pi}{2}y_F\right)-\left(\sin\frac{\pi}{2}x_F\right)\left(\sin\frac{\pi}{2}y_F\right)\right)\end{array}\right\rangle.$$

5.3. EINSTEIN SUM GENERATING CLASSES

New $S_n-(x,y)$ and $T_n-(x,y)$ from the Einstein sum *s*-norm.

$$S^{\alpha}_{\underset{es}{n}}(x,y)=\left\langle\begin{array}{l}\sqrt[\alpha]{\frac{x_T{}^{\alpha}+y_T{}^{\alpha}}{1+x_T{}^{\alpha}y_T{}^{\alpha}}},\\ 1-\sqrt[\alpha]{\frac{(1-x_I)^{\alpha}+(1-y_I)^{\alpha}}{1+(1-x_I)^{\alpha}(1-y_I)^{\alpha}}},\\ 1-\sqrt[\alpha]{\frac{(1-x_F)^{\alpha}+(1-y_F)^{\alpha}}{1+(1-x_F)^{\alpha}(1-y_F)^{\alpha}}}\end{array}\right\rangle$$

$$T^{\alpha}_{\underset{es}{n}}(x,y)=\left\langle\begin{array}{l}1-\sqrt[\alpha]{\frac{(1-x_T)^{\alpha}+(1-y_T)^{\alpha}}{1+(1-x_T)^{\alpha}(1-y_T)^{\alpha}}},\\ \sqrt[\alpha]{\frac{x_I{}^{\alpha}+y_I{}^{\alpha}}{1+x_I{}^{\alpha}y_I{}^{\alpha}}},\\ \sqrt[\alpha]{\frac{x_F{}^{\alpha}+y_F{}^{\alpha}}{1+x_F{}^{\alpha}y_F{}^{\alpha}}}\end{array}\right\rangle.$$

$$S^{bs}_{\underset{\sin,\cos}{n}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_T\right)+\left(\sin\frac{\pi}{2}y_T\right)}{1+\left(\sin\frac{\pi}{2}x_T\right)\left(\sin\frac{\pi}{2}y_T\right)}\right),\\ \frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_I\right)+\left(\cos\frac{\pi}{2}y_I\right)}{1+\left(\cos\frac{\pi}{2}x_I\right)\left(\cos\frac{\pi}{2}y_I\right)}\right),\\ \frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_F\right)+\left(\cos\frac{\pi}{2}y_F\right)}{1+\left(\cos\frac{\pi}{2}x_F\right)\left(\cos\frac{\pi}{2}y_F\right)}\right)\end{array}\right\rangle$$

$$T^{bs}_{\underset{\sin,\cos}{n}}(x,y)=\left\langle\begin{array}{l}\frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_T\right)+\left(\cos\frac{\pi}{2}y_T\right)}{1+\left(\cos\frac{\pi}{2}x_T\right)\left(\cos\frac{\pi}{2}y_T\right)}\right),\\ \frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_I\right)+\left(\sin\frac{\pi}{2}y_I\right)}{1+\left(\sin\frac{\pi}{2}x_I\right)\left(\sin\frac{\pi}{2}y_I\right)}\right),\\ \frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_F\right)+\left(\sin\frac{\pi}{2}y_F\right)}{1+\left(\sin\frac{\pi}{2}x_F\right)\left(\sin\frac{\pi}{2}y_F\right)}\right)\end{array}\right\rangle.$$

5. 4. BOUNDED PRODUCT GENERATING CLASSES

New $S_n-(x,y)$ and $T_n-(x,y)$ from the bounded product *t*-norm.

$$T_{\underset{bp}{n}}^{\alpha}\left(x,y\right)=\left\langle \begin{array}{l} \sqrt[\alpha]{\max\left(x_T{}^{\alpha}\quad {}_T{}^{\alpha}-1,0\right)}, \\ 1-\sqrt[\alpha]{\max\left(\left(1-x_I\right)^{\alpha}+\left(1-y_I{}^{\alpha}\right)-1,0\right)}, \\ 1-\sqrt[\alpha]{\max\left(\left(1-x_F\right)^{\alpha}+\left(1-y_F\right)^{\alpha}-1,0\right)} \end{array}\right\rangle .$$

$$S_{\underset{bp}{n}}^{\alpha}\left(x,y\right)=\left\langle \begin{array}{l} 1-\sqrt[\alpha]{\max\left(\left(1-x_T\right)^{\alpha}+\left(1-\quad {}_T\right)^{\alpha}-1,0\right)}, \\ \sqrt[\alpha]{\max\left(x_I{}^{\alpha}+y_I{}^{\alpha}-1,0\right)}, \\ \sqrt[\alpha]{\max\left(x_F{}^{\alpha}+y_F{}^{\alpha}-1,0\right)} \end{array}\right\rangle$$

$$T_{\underset{\sin,\cos}{n}}^{bp}\left(x,y\right)=\left\langle \begin{array}{l} \frac{2}{\pi}\sin^{-1}\left(\max\left(\left(\sin\frac{\pi}{2}x_T\right)+\left(\sin\frac{\pi}{2}y_T\right)-1,0\right)\right), \\ \frac{2}{\pi}\cos^{-1}\left(\max\left(\left(\cos\frac{\pi}{2}x_I\right)+\left(\cos\frac{\pi}{2}y_I\right)-1,0\right)\right), \\ \frac{2}{\pi}\cos^{-1}\left(\max\left(\left(\cos\frac{\pi}{2}x_F\right)+\left(\cos\frac{\pi}{2}y_F\right)-1,0\right)\right) \end{array}\right\rangle .$$

$$S_{\underset{\sin,\cos}{n}}^{t}\left(x,y\right)=\left\langle \begin{array}{l} \frac{2}{\pi}\cos^{-1}\left(\max\left(\left(\cos\frac{\pi}{2}x_T\right)+\left(\cos\frac{\pi}{2}y_T\right)-1,0\right)\right), \\ \frac{2}{\pi}\sin^{-1}\left(\max\left(\left(\sin\frac{\pi}{2}x_I+\sin\frac{\pi}{2}y_I\right)-1,0\right)\right), \\ \frac{2}{\pi}\sin^{-1}\left(\max\left(\left(\sin\frac{\pi}{2}x_F+\sin\frac{\pi}{2}y_F\right)-1,0\right)\right) \end{array}\right\rangle .$$

.

5.5. EINSTEIN PRODUCT GENERATING CLASSES

New $S_n-\left(x,y\right)$ and $T_n-\left(x,y\right)$ from the Einstein product *t*-norm.

$$T_{\underset{ep}{n}}^{\alpha}\left(x,y\right)=\left\langle \begin{array}{l} \sqrt[\alpha]{\frac{x_T{}^{\alpha}y_T{}^{\alpha}}{2-\left(x_T{}^{\alpha}+y_T{}^{\alpha}-x_T{}^{\alpha}y_T{}^{\alpha}\right)}}, \\ 1\quad\sqrt[\alpha]{\frac{\left(1-x\ \right)^{\alpha}\left(1-y_I\right)^{\alpha}}{2-\left(\left(1-x_I\right)^{\alpha}+\left(1-y_I\right)^{\alpha}-\left(1-x_I\right)^{\alpha}\left(1-y_I\right)^{\alpha}\right)}}, \\ 1-\sqrt[\alpha]{\frac{\left(1-x_F\right)^{\alpha}\left(1-y_F\right)^{\alpha}}{2-\left(\left(1-x_F\right)^{\alpha}+\left(1-y_F\right)^{\alpha}-\left(1-x_F\right)^{\alpha}\left(1-y_F\right)^{\alpha}\right)}} \end{array}\right\rangle .$$

$$S_{\underset{ep}{n}}^{\alpha}\left(x,y\right)=\left\langle \begin{array}{l} 1-\sqrt[\alpha]{\frac{\left(1-x_T\right)^{\alpha}\left(1-y_T\right)^{\alpha}}{2-\left(\left(1-x_T\right)^{\alpha}+\left(1-y_T\right)^{\alpha}-\left(1-x_T\right)^{\alpha}\left(1-y_T\right)^{\alpha}\right)}}, \\ \sqrt[\alpha]{\frac{x_I{}^{\alpha}y^{\alpha}}{2-\left(x_I{}^{\alpha}+y_I{}^{\alpha}-x_I{}^{\alpha}y_I{}^{\alpha}\right)}}, \\ \sqrt[\alpha]{\frac{x_F{}^{\alpha}y_F{}^{\alpha}}{2-\left(x_F{}^{\alpha}+y_F{}^{\alpha}-x_F{}^{\alpha}y_F{}^{\alpha}\right)}} \end{array}\right\rangle$$

$$T_{\underset{\sin,\cos}{n}}^{ep}\left(x,y\right)=\left\langle \begin{array}{l} \frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_T\right)\left(\sin\frac{\pi}{2}y_T\right)}{2-\left(\left(\sin\frac{\pi}{2}x_T\right)+\left(\sin\frac{\pi}{2}y_T\right)-\left(\sin\frac{\pi}{2}x_T\right)\left(\sin\frac{\pi}{2}y_T\right)\right)}\right), \\ \frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_I\right)\left(\cos\frac{\pi}{2}y_I\right)}{2-\left(\left(\cos\frac{\pi}{2}x_I\right)+\left(\cos\frac{\pi}{2}y_I\right)-\left(\cos\frac{\pi}{2}x_I\right)\left(\cos\frac{\pi}{2}y_I\right)\right)}\right), \\ \frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_F\right)\left(\cos\frac{\pi}{2}y_F\right)}{2-\left(\left(\cos\frac{\pi}{2}x_F\right)+\left(\cos\frac{\pi}{2}y_F\right)-\left(\cos\frac{\pi}{2}x_F\right)\left(\cos\frac{\pi}{2}y_F\right)\right)}\right) \end{array}\right\rangle$$

$$s_{\underset{\sin,\cos}{n}}^{ep}\left(x,y\right)=\left\langle \begin{array}{l} \frac{2}{\pi}\cos^{-1}\left(\frac{\left(\cos\frac{\pi}{2}x_T\right)\left(\cos\frac{\pi}{2}y_T\right)}{2-\left(\left(\cos\frac{\pi}{2}x_T\right)+\left(\cos\frac{\pi}{2}y_T\right)-\left(\cos\frac{\pi}{2}x_T\right)\left(\cos\frac{\pi}{2}y_T\right)\right)}\right), \\ \frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_I\right)\left(\sin\frac{\pi}{2}y_I\right)}{2-\left(\left(\sin\frac{\pi}{2}x_I\right)+\left(\sin\frac{\pi}{2}y_I\right)-\left(\sin\frac{\pi}{2}x_I\right)\left(\sin\frac{\pi}{2}y_I\right)\right)}\right), \\ \frac{2}{\pi}\sin^{-1}\left(\frac{\left(\sin\frac{\pi}{2}x_F\right)\left(\sin\frac{\pi}{2}y_F\right)}{2-\left(\left(\sin\frac{\pi}{2}x_F\right)+\left(\sin\frac{\pi}{2}y_F\right)-\left(\sin\frac{\pi}{2}x_F\right)\left(\sin\frac{\pi}{2}y_F\right)\right)}\right) \end{array}\right\rangle$$

Remark. Note that for the *s*-norms max and drastic sum and *t*-norms min, algebraic product and drastic product we get the same norms

6 Conclusion

In this paper, we gave generating theorems for N-norm and N-conorm. By given any N-norm we generated a class of N-norms and N-cnorms, and by given any N-conorm we generated a class of N-conorms and N-norms. We also gave bijective generating theorems for N-norms and N-conorms.

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References

[1] Shawkat Alkhazaleh & Abdul Razak Salleh, Generating theorems for s-norms and t-norms, Journal of Quality Measurement and Analysis, 1 (1), (2007), 1-12.

[2] Dubois, D. & Prade, H. 1979. Outline of fuzzy set theory. In Gupta et al. (Ed.). Advances in fuzzy set theory and applications. North-Holland.

[3] F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Inter. J. Pure Appl. Math., 24 (2005), 287-297.

[4] F. Smarandache, A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth, fourth edition, 2005.

[5] Klement, E.P. 1982. Operation on fuzzy sets-an axiomatic approach.Information Sciences 27, 221-232(1982).

[6] Lowen, R. 1996. Fuzzy set theory. Netherland: Kluwer Academic Publishers.

[7] Waber, S. 1983. A general concept of fuzzy connectives, negations and implications based on t-norms and t-conorms. Fuzzy Sets and Systems 11: 115-134.

[8] Wang, Li-Xin. 1997. A course in fuzzy systems and control. New Jersey: Prentice Hall.

[9] Yager, R.R. 1980. On a general class of fuzzy connectives. Fuzzy Sets and Systems 4: 235-342.

[10] Zadeh, L.A. 1965. Fuzzy sets. Inform. and Control 8: 338-353.

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