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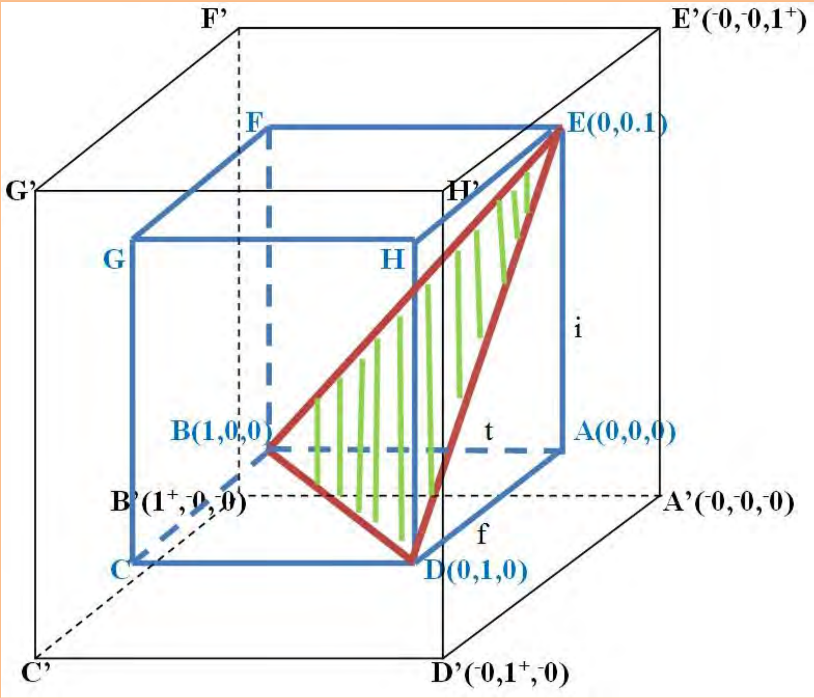

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*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Bèzier Curve Modeling for Neutrosophic Data Problem

Ferhat Taş[1], Selçuk Topal[2]

[1]Department of Mathematics, Istanbul University, Istanbul, Turkey. E-mail: tasf@istanbul.edu.tr

[2]Department of Mathematics, Bitlis Eren University, Bitlis, Turkey. E-mail: s.topal@beu.edu.tr

Abstract: Neutrosophic set concept is defined with membership, non-membership and indeterminacy degrees. This concept is the solution and representation of the problems with various fields. In this paper, a geometric model is introduced for Neutrosophic data problem for the first time. This model is based on neutrosophic sets and neutrosophic relations. Neutrosophic control points are defined according to these points, resulting in neutrosophic Bèzier curves.

Keywords: Neutrosophic Sets, Neutrosophic Logic, Bèzier Curve

1 Introduction

While today's technologies are rapidly developing, the contribution of mathematics is fundamental and leading the science. In particular, the developments in geometry are not only modeling the mathematics of the objects but also being geometrically modeled in most abstract concepts. What is the use of these abstract concepts in modeling? In the future of science, there will be artificial intelligence. For the development of this technology, many branches of science work together and especially the topics such as logic, data mining, quantum physics, machine learning come to the forefront. Of course, the place where these areas can cooperate is the computer environment. Data can be transferred in various ways. One of them is to transfer the data as a geometric model. The first method that comes to mind in terms of a geometric model is the Bzier technique. Although this method is generally used for curve and surface designs, it is used in many disciplines ranging from the solution of differential equations to robot motion planning.

The embodied state of the adventure of obtaining meaning and mathematical results from uncertainty states (fuzzy) was begun by Zadeh [1]. Fuzzy sets proposed by Zadeh provided a new dimension to the concept of classical sets. Atanassov introduced intuitionistic fuzzy sets dealing with membership and non-membership degrees [2]. Neutrosophy was proposed by Smarandache as a mathematical application of the concept neutrality [3]. Neutrosophic set concept is defined with membership, non-membership and indeterminacy degrees. Neutrosophic set concept is separated from intuitionistic fuzzy set by the difference as follow: intuitionistic fuzzy sets are defined by degree of membership and non-membership degree and, uncertainty degrees by the 1- (membership degree plus non-membership degree), while degree of uncertainty are considered independently of the degree of membership and non-membership in neutrosophic sets. Here, membership, non-membership and uncertainty (indeterminacy) degrees can be judged according to the interpretation in the spaces to be used, such as truth and falsity degrees. It depends entirely on subject space (discourse universe). In this sense, the concept of neutrosophic set is the solution and representation of the problems with various fields.

Recently, geometric interpretations of data that uncertain truth were presented by Wahab and friends [4, 5, 6, 7]. They studied geometric models of fuzzy and intuitionistic fuzzy data and gave fuzzy interpolation and Bèzier curve modeling. In this paper, we consider a geometric modeling of neutrosophic data.

2 Preliminaries

In this section, we will first give some fundamental definitions dealing with Bzier curve and Neutrosophic sets (elements). We will then introduce the new definitions needed to form a *Neutrosophic Bèzier curve*.

Definition 1 *Let* $\boldsymbol{P}_i, (i = 0, 1, 2, ..., n), \boldsymbol{P}_i \in E^3$ *be the set of points. A Bézier curve with degree n is defined by*

$$\boldsymbol{B}(t) = B_i^n(t)\boldsymbol{P}_i, t \in [0, 1] \tag{1}$$

where $B_i^n(t) = \sum_{i=0}^n \binom{n}{i}(1-t)^{n-i}t^i$ *and* $\boldsymbol{P}_i$ *are the Bernstein polynomial function and the control points, respectively. Notice that there are (n + 1)-control points for a Bèzier curve with degree n. Because* $n-$*interpolation is done with* $(n+1)$*-control points [8, 9, 10, 11].*

Definition 2 *Let* E *be a universe and* $A \subseteq E$. $N = \{(x, T(x), I(x), F(x)) : x \in A\}$ *is a neutrosophic element where* $T_p = N \to [0, 1]$ *(membership function),* $I_p = N \to [0, 1]$ *(indeterminacy function) and* $F_p = N \to [0, 1]$ *(non-membership function).*

Definition 3 *Let* $A^* = \{(x, T(x), I(x), F(x)) : x \in A\}$ *and* $B^* = \{(y, T(y), I(y), F(y)) : y \in B\}$ *be neutrosophic elements.* $NR = \{((x, y), T(x, y), I(x, y), F(x, y)) : (x, y) \in A \times B\}$ *is a neutrosophic relation on* A^* *and* B^*.

3 Neutrosophic Bèzier Model

Definition 4 *NS of P^* in space N is NCP and $P^* = \{P_i^*\}$ where $i = 0, ..., n$ is a set of $NCPs$ where there exists $T_p = N \to [0,1]$ as membership function, $I_p = N \to [0,1]$ as indeterminacy function and $F_p = N \to [0,1]$ as non-membership function with*

$$T_p(P^*) = \begin{cases} 0 & \text{if } P_i \notin N \\ a \in (0,1) & \text{if } P_i \tilde{\in} N \\ 1 & \text{if } P_i \in N \end{cases}$$

$$F_p(P^*) = \begin{cases} 0 & \text{if } P_i \notin N \\ c \in (0,1) & \text{if } P_i \tilde{\in} N \\ 1 & \text{if } P_i \in N \end{cases}$$

$$I_p(P^*) = \begin{cases} 0 & \text{if } P_i \notin N \\ e \in (0,1) & \text{if } P_i \tilde{\in} N \\ 1 & \text{if } P_i^- \in N. \end{cases}$$

Bèzier Neutrosophic curves are generated based on the control points from one of $TC = \{(x, y, T(x, y))\}$, $IC = \{(x, y, I(x, y))\}$ and $FC = \{(x, y, F(x, y))\}$ sets. Thus, there will be three different neutrosophic Bèzier curve models for a neutrosophic relation and variables x and y. A neutrosophic control point relation can be defined as a set of n+1 points that shows a position and coordinate of a location and is used to described three curve which are denoted by
$NR_{p_i} = \{NR_{p_0}, NR_{p_1}, ..., NR_{p_n}\}$
and can be written as
$\{((x_0, y_0), T(x_0, y_0), I(x_0, y_0), F(x_0, y_0))$
$, ..., ((x_n, y_n), T(x_n, y_n), I(x_n, y_n), F(x_n, y_n))\}$
in order to control the shape of a curve from a neutrosophic data.

Definition 5 *A neutrosophic Bézier curve with degree n is defined by*

$$\boldsymbol{NB}(t) = B_i^n(t) NR_{pi}, t \in [0,1] \quad (2)$$

Every set of $TC = \{(x, y, T(x, y))\}$, $IC = \{(x, y, I(x, y))\}$ and $FC = \{(x, y, F(x, y))\}$ determines a Bézier curve. Thus we get three Bézier curves. A Neutrosophic Bézier curve is defined by these three curves. So it is a set of curves just like in its definition.

As an illustrative example, we can consider a neutrosophic data in Table 1. One can see there are three qubic Bézier curves.

4 Conclusion and Future Work

Visualization or geometric modeling of data plays an important role in data mining, databases, stock market, economy, and stochastic processes. In this article, we used the Bézier technique for visualizing neutrosophic data. This model is suitable for statisticians, data scientists, economists and engineers. Furthermore, the differential geometric properties of this model can be investigated as in [8] for classification of neutrosophic data. On the other hand, transforming the images of objects into neutrosophic data is an important problem [12]. In our model, the curve and the data can be transformed into each other by the blossoming method, which can be used in neutrosophic image processing. This and similar applications can be studied in the future.

Point	Truth degree	Indeterminacy degree	Falsity degree
(2,3)	0.6	0.4	0.7
(1,3)	0.5	0.6	0.2
(4,6)	0.7	0.5	0.3
(3,5)	0.3	0.2	0.7

Table 1: A neutrosophic data example

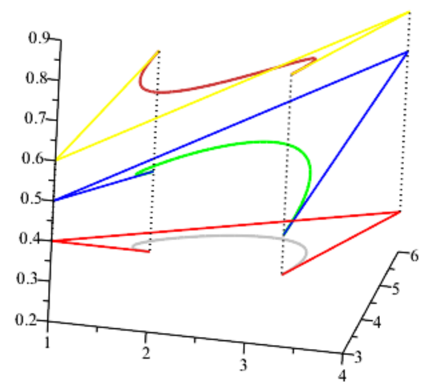

Figure 1: Neutrosophic Bézier curves for data in Table 1.

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NSS University of New Mexico

A Study on Neutrosophic Frontier and Neutrosophic Semi-frontier in Neutrosophic Topological Spaces

P. Iswarya[1] and Dr. K. Bageerathi[2]

[1] Department of Mathematics, Govindammal Aditanar College for Women, Tiruchendur, India
E mail ID : iswaryap3@gmail.com

[2]Department of Mathematics, Aditanar College of Arts and Science, Tiruchendur, India
E mail ID : sivarathi_2006@yahoo.in

ABSTRACT. In this paper neutrosophic frontier and neutrosophic semi-frontier in neutrosophic topology are introduced and several of their properties, characterizations and examples are established.

MATHEMATICS SUBJECT CLASSIFICATION (2010) : 03E72

KEYWORDS : Neutrosophic frontier and Neutrosophic semi-frontier.

I. INTRODUCTION

Theory of Fuzzy sets [21], Theory of Intuitionistic fuzzy sets [2], Theory of Neutrosophic sets [10] and the theory of Interval Neutrosophic sets [13] can be considered as tools for dealing with uncertainties. However, all of these theories have their own difficulties which are pointed out in [10]. In 1965, Zadeh [21] introduced fuzzy set theory as a mathematical tool for dealing with uncertainties where each element had a degree of membership. The Intuitionistic fuzzy set was introduced by Atanassov [2] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. The neutrosophic set was introduced by Smarandache [10] and explained, neutrosophic set is a generalization of Intuitionistic fuzzy set. In 2012, Salama, Alblowi [18], introduced the concept of Neutrosophic topological spaces. They introduced neutrosophic topological space as a generalization of Intuitionistic fuzzy topological space and a Neutrosophic set besides the degree of membership, the degree of indeterminacy and the degree of non-membership of each element.

The concepts of neutrosophic semi-open sets, neutrosophic semi-closed sets, neutrosophic semi-interior and neutrosophic semi-closure in neutrosophic topological spaces were introduced by P. Iswarya and Dr. K. Bageerathi [12] in 2016. Frontier and semifrontier in intuitionistic fuzzy topological spaces were introduced by Athar Kharal [4] in 2014. In this paper, we are extending the above concepts to neutrosophic topological spaces. We study some of the basic properties of neutrosophic frontier and neutrosophic semi-frontier in neutrosophic topological spaces with examples. Properties of neutrosophic semi-interior, neutrosophic semi-closure, neutrosophic frontier and neutrosophic semi-frontier have been obtained in neutrosophic product related spaces.

II. NEUTROSOPHIC FRONTIER

In this section, the concepts of the neutrosophic frontier in neutrosophic topological space are introduced and also discussed their characterizations with some related examples.

Definition 2.1 Let $\alpha, \beta, \lambda \in [0, 1]$ and $\alpha + \beta + \lambda \leq 1$. A neutrosophic point [*NP* for short] $x_{(\alpha,\beta,\lambda)}$ of X is a *NS* of X which is defined by

$$x_{(\alpha,\beta,\lambda)} = \begin{cases} (\alpha,\beta,\lambda), & y = x \,, \\ (0,0,1), & y \neq x \,. \end{cases}$$

In this case, x is called the support of $x_{(\alpha,\beta,\lambda)}$ and α, β and λ are called the value, intermediate value and the non-value of $x_{(\alpha,\beta,\lambda)}$, respectively. A *NP* $x_{(\alpha,\beta,\lambda)}$ is said to belong to a *NS* $A = \langle \mu_A, \sigma_A, \gamma_A \rangle$ in X, denoted by $x_{(\alpha,\beta,\lambda)} \in A$ if $\alpha \leq \mu_A(x)$, $\beta \leq \sigma_A(x)$ and $\lambda \geq \gamma_A(x)$. Clearly a neutrosophic point can be represented by an ordered triple of neutrosophic points as follows : $x_{(\alpha,\beta,\lambda)} = (x_\alpha , x_\beta , C (x_{C(\lambda)}))$. A class of all *NPs* in X is denoted as *NP* (X).

Definition 2.2 Let X be a *NTS* and let $A \in NS$ (X). Then $x_{(\alpha,\beta,\lambda)} \in NP$ (X) is called a neutrosophic frontier point [*NFP* for short] of A if $x_{(\alpha,\beta,\lambda)} \in NCl(A) \cap NCl(C(A))$. The intersection of all the *NFPs* of A is called a neutrosophic frontier of A and is denoted by $NFr(A)$. That is,
$NFr(A) = NCl(A) \cap NCl(C(A))$.

Proposition 2.3 For each $A \in NS(X)$, $A \cup NFr(A) \subseteq NCl(A)$.

Proof : Let A be the NS in the neutrosophic topological space X. Then by Definition 2.2,

$A \cup NFr(A) = A \cup (NCl(A) \cap NCl(C(A)))$
$= (A \cup NCl(A)) \cap (A \cup NCl(C(A)))$
$\subseteq NCl(A) \cap NCl(C(A))$
$\subseteq NCl(A)$

Hence $A \cup NFr(A) \subseteq NCl(A)$.

From the above proposition, the inclusion cannot be replaced by an equality as shown by the following example.

Example 2.4 Let X = { a, b } and τ = { 0_N, A, B, C, D, 1_N }. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are C (τ) = { 1_N, E, F, G, H, 0_N } where
A = ⟨ (0.5, 1, 0.1), (0.9, 0.2, 0.5) ⟩,
B = ⟨ (0.2, 0.5, 0.9), (0, 0.5, 1) ⟩,
C = ⟨ (0.5, 1, 0.1), (0.9, 0.5, 0.5) ⟩,
D = ⟨ (0.2, 0.5, 0.9), (0, 0.2, 1) ⟩,
E = ⟨ (0.1, 0, 0.5), (0.5, 0.8, 0.9) ⟩,
F = ⟨ (0.9, 0.5, 0.2), (1, 0.5, 0) ⟩,
G = ⟨ (0.1, 0, 0.5), (0.5, 0.5, 0.9) ⟩ and
H = ⟨ (0.9, 0.5, 0.2), (1, 0.8, 0) ⟩.
Here NCl (A) = 1_N and NCl (C (A)) = NCl (E) = E. Then by Definition 2.2, NFr (A) = E.
Also $A \cup NFr(A)$ = ⟨ (0.5, 1, 0.1), (0.9, 0.8, 0.5) ⟩ $\subseteq$ 1_N. Therefore $NCl(A) = 1_N \nsubseteq$ ⟨ (0.5, 1, 0.1), (0.9, 0.8, 0.5) ⟩.

Theorem 2.5 For a NS A in the NTS X, $NFr(A) = NFr(C(A))$.

Proof : Let A be the NS in the neutrosophic topological space X. Then by Definition 2.2,

$NFr(A) = NCl(A) \cap NCl(C(A))$
$= NCl(C(A)) \cap NCl(A)$
$= NCl(C(A)) \cap NCl(C(C(A)))$

Again by Definition 2.2,
$= NFr(C(A))$

Hence $NFr(A) = NFr(C(A))$.

Theorem 2.6 If a NS A is a NCS, then $NFr(A) \subseteq A$.

Proof : Let A be the NS in the neutrosophic topological space X. Then by Definition 2.2,

$NFr(A) = NCl(A) \cap NCl(C(A))$
$\subseteq NCl(A)$

By Definition 4.4 (a) [18] ,
$= A$

Hence $NFr(A) \subseteq A$, if A is NCS in X.

The converse of the above theorem needs not be true as shown by the following example.

Example 2.7 From Example 2.4, NFr (C) = G $\subseteq$ C. But C $\notin$ C (τ).

Theorem 2.8 If a NS A is NOS, then $NFr(A) \subseteq C(A)$.

Proof : Let A be the NS in the neutrosophic topological space X. Then by Definition 4.3 [18] , A is NOS implies C (A) is NCS in X. By Theorem 2.6, $NFr(C(A)) \subseteq C(A)$ and by Theorem 2.5, we get $NFr(A) \subseteq C(A)$.

The converse of the above theorem is not true as shown by the following example.

Example 2.9 From Example 2.4, NFr (G) = G $\subseteq$ C (G) = C. But G $\notin$ τ.

Theorem 2.10 For a NS A in the NTS X, $C(NFr(A)) = NInt(A) \cup NInt(C(A))$.

Proof : Let A be the NS in the neutrosophic topological space X. Then by Definition 2.2,

$C(NFr(A)) = C(NCl(A) \cap NCl(C(A)))$

By Proposition 3.2 (1) [18] ,
$= C(NCl(A)) \cup C(NCl(C(A)))$

By Proposition 4.2 (b) [18] ,
$= NInt(C(A)) \cup NInt(A)$

Hence $C(NFr(A)) = NInt(A) \cup NInt(C(A))$.

Theorem 2.11 Let $A \subseteq B$ and $B \in NC(X)$ (resp., $B \in NO(X)$). Then $NFr(A) \subseteq B$ (resp., $NFr(A) \subseteq C(B)$), where NC (X) (resp., NO (X)) denotes the class of neutrosophic closed (resp., neutrosophic open) sets in X.

Proof : By Proposition 1.18 (d) [12] , $A \subseteq B$, $NCl(A) \subseteq NCl(B)$ -------------------- (1).

By Definition 2.2,
$NFr(A) = NCl(A) \cap NCl(C(A))$
$\subseteq NCl(B) \cap NCl(C(A))$ by (1)
$\subseteq NCl(B)$

By Definition 4.4 (b) [18] ,
$= B$

Hence $NFr(A) \subseteq B$.

Theorem 2.12 Let A be the NS in the NTS X. Then $NFr(A) = NCl(A) - NInt(A)$.

Proof : Let A be the NS in the neutrosophic topological space X. By Proposition 4.2 (b) [18] , $C(NCl(C(A))) = NInt(A)$ and by Definition 2.2,

$NFr(A) = NCl(A) \cap NCl(C(A))$
$= NCl(A) - C(NCl(C(A)))$
by using $A - B = A \cap C(B)$

By Proposition 4.2 (b) [18] ,
$= NCl(A) - NInt(A)$

Hence $NFr(A) = NCl(A) - NInt(A)$.

Theorem 2.13 For a *NS* A in the *NTS* X, $NFr\ (NInt\ (A)) \subseteq NFr\ (A)$.
Proof : Let A be the *NS* in the neutrosophic topological space X. Then by Definition 2.2,
$NFr\ (NInt\ (A)) = NCl\ (NInt\ (A)) \cap NCl\ (C\ (NInt\ (A)))$
By Proposition 4.2 (a) [18] ,
$= NCl\ (NInt\ (A)) \cap NCl\ (NCl\ (C\ (A)))$
By Definition 4.4 (b) [18] ,
$= NCl\ (NInt\ (A)) \cap NCl\ (C\ (A))$
By Definition 4.4 (a) [18] ,
$\subseteq NCl\ (A) \cap NCl\ (C\ (A))$
Again by Definition 2.2,
$= NFr\ (A)$
Hence $NFr\ (NInt\ (A)) \subseteq NFr\ (A)$.

The converse of the above theorem is not true as shown by the following example.

Example 2.14 Let X = { a, b } and τ = { 0_N, A, B, C, D, 1_N }. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are C (τ) = { 1_N, E, F, G, H, 0_N } where
A = ⟨ (0.5, 0.6, 0.7), (0.1, 0.9, 0.4) ⟩,
B = ⟨ (0.3, 0.9, 0.2), (0.4, 0.1, 0.6) ⟩,
C = ⟨ (0.5, 0.9, 0.2), (0.4, 0.9, 0.4) ⟩,
D = ⟨ (0.3, 0.6, 0.7), (0.1, 0.1, 0.6) ⟩,
E = ⟨ (0.7, 0.4, 0.5), (0.4, 0.1, 0.1) ⟩,
F = ⟨ (0.2, 0.1, 0.3), (0.6, 0.9, 0.4) ⟩,
G = ⟨ (0.2, 0.1, 0.5), (0.4, 0.1, 0.4) ⟩ and
H = ⟨ (0.7, 0.4, 0.3), (0.6, 0.9, 0.1) ⟩.
Define A_1 = ⟨ (0.4, 0.2, 0.8), (0.4, 0.5, 0.1) ⟩. Then C (A_1) = ⟨ (0.8, 0.8, 0.4), (0.1, 0.5, 0.4) ⟩.
Therefore by Definition 2.2, $NFr\ (A_1) = H \not\subseteq 0_N = NFr\ (NInt\ (A_1))$.

Theorem 2.15 For a *NS* A in the *NTS* X, $NFr\ (NCl\ (A)) \subseteq NFr\ (A)$.
Proof : Let A be the *NS* in the neutrosophic topological space X. Then by Definition 2.2,
$NFr\ (NCl\ (A)) = NCl\ (NCl\ (A)) \cap NCl\ (C\ (NCl\ (A)))$
By Proposition 1.18 (f) [12] and 4.2 (b) [18] ,
$= NCl\ (A) \cap NCl\ (NInt\ (C\ (A)))$
By Proposition 1.18 (a) [12] ,
$\subseteq NCl\ (A) \cap NCl\ (C\ (A))$
Again by Definition 2.2,
$= NFr\ (A)$
Hence $NFr\ (NCl\ (A)) \subseteq NFr\ (A)$.

The converse of the above theorem is not true as shown by the following example.

Example 2.16 From Example 2.14, let A_2 = ⟨ (0.7, 0.9, 0.2), (0.5, 0.9, 0.3) ⟩.
Then C (A_2) = ⟨ (0.2, 0.1, 0.7), (0.3, 0.1, 0.5) ⟩. Then by Definition 2.2, $NFr\ (A_2) = G$.
Therefore $NFr\ (A_2) = G \not\subseteq 0_N = NFr\ (NCl\ (A_2))$.

Theorem 2.17 Let A be the *NS* in the *NTS* X. Then $NInt\ (A) \subseteq A - NFr\ (A)$.
Proof : Let A be the *NS* in the neutrosophic topological space X. Now by Definition 2.2,
$A - NFr\ (A) = A - (NCl\ (A) \cap NCl\ (C\ (A)))$
$= (A - NCl\ (A)) \cup (A - NCl\ (C\ (A)))$
$= A - NCl\ (C\ (A))$
$\supseteq NInt\ (A)$.
Hence $NInt\ (A) \subseteq A - NFr\ (A)$.

Example 2.18 From Example 2.14, $A_1 - NFr\ (A_1)$ = ⟨ (0.3, 0.2, 0.8), (0.1, 0.1, 0.6) ⟩.
Therefore $A_1 - NFr\ (A_1)$ = ⟨ (0.3, 0.2, 0.8), (0.1, 0.1, 0.6) ⟩ $\not\subseteq 0_N = NInt\ (A_1)$.

Remark 2.19 In general topology, the following conditions are hold :
$NFr\ (A) \cap NInt\ (A) = 0_N$,
$NInt\ (A) \cup NFr\ (A) = NCl\ (A)$,
$NInt\ (A) \cup NInt\ (C\ (A)) \cup NFr\ (A) = 1_N$.
But the neutrosophic topology, we give counter-examples to show that the conditions of the above remark may not be hold in general.

Example 2.20 From Example 2.14,
$NFr\ (A_2) \cap NInt\ (A_2) = G \cap C = G \neq 0_N$.

$NInt\ (A_2) \cup NFr\ (A_2) = C \cup G = C \neq 1_N = NCl\ (A_2)$.

$NInt\ (A_2) \cup NInt\ (C\ (A_2)) \cup NFr\ (A_2) = C \cup 0_N \cup G = C \neq 1_N$.

Theorem 2.21 Let A and B be the two *NSs* in the *NTS* X. Then $NFr\ (A \cup B) \subseteq NFr\ (A) \cup NFr\ (B)$.
Proof : Let A and B be the two *NSs* in the *NTS* X. Then by Definition 2.2,
$NFr\ (A \cup B) = NCl\ (A \cup B) \cap NCl\ (C\ (A \cup B))$
By Proposition 3.2 (2) [18] ,
$= NCl\ (A \cup B) \cap NCl\ (\ C\ (A) \cap C\ (B)\ )$
by Proposition 1.18 (h) and (o) [12] ,
$\subseteq (NCl\ (A) \cup NCl\ (B)) \cap (NCl\ (C\ (A)) \cap NCl(C\ (B)))$
$= [(NCl\ (A) \cup NCl\ (B)\ ) \cap NCl\ (C\ (A))\ ]$
$\cap [\ (\ NCl\ (A) \cup NCl\ (B)\ ) \cap NCl\ (C\ (B))\ ]$
$= [(NCl\ (A) \cap NCl\ (C\ (A))) \cup (NCl\ (B) \cap NCl(C\ (A)))]$
$\cap [(NCl\ (A) \cap NCl\ (C\ (B))) \cup (NCl(B) \cap NCl(C\ (B)))]$
Again by Definition 2.2,
$= [NFr\ (A) \cup (\ NCl\ (B) \cap NCl\ (C\ (A)))\ ]$
$\cap\ [\ (\ NCl\ (A)\ \cap\ NCl\ (C\ (B))\ )\ \cup\ NFr(B)\ ]$
$= (\ NFr\ (A) \cup NFr\ (B)) \cap [\ (\ NCl\ (B) \cap NCl\ (C\ (A))\ )$
$\cup (\ NCl\ (A) \cap NCl\ (C\ (B))\ )\ ]$

$\subseteq NFr(A) \cup NFr(B)$.
Hence $NFr(A \cup B) \subseteq NFr(A) \cup NFr(B)$.

The converse of the above theorem needs not be true as shown by the following example.

Example 2.22 By Example 2.14, we define
$A_1 = \langle (0.2, 0, 0.5), (0.4, 0.1, 0.1) \rangle$,
$A_2 = \langle (0.7, 0.9, 0.2), (0.5, 0.9, 0.3) \rangle$,
$A_1 \cup A_2 = A_3 = \langle (0.7, 0.9, 0.2), (0.5, 0.9, 0.1) \rangle$ and
$A_1 \cap A_2 = A_4 = \langle (0.2, 0, 0.5), (0.4, 0.1, 0.3) \rangle$. Then
$C(A_1) = \langle (0.5, 1, 0.2), (0.1, 0.9, 0.4) \rangle$,
$C(A_2) = \langle (0.2, 0.1, 0.7), (0.3, 0.1, 0.5) \rangle$,
$C(A_3) = \langle (0.2, 0.1, 0.7), (0.1, 0.1, 0.5) \rangle$ and
$C(A_4) = \langle (0.5, 1, 0.2), (0.3, 0.9, 0.4) \rangle$.
Therefore $NFr(A_1) \cup NFr(A_2) = E \cup G = E \nsubseteq G = NFr(A_3) = NFr(A_1 \cup A_2)$.

Note 2.23 The following example shows that $NFr(A \cap B) \nsubseteq NFr(A) \cap NFr(B)$ and
$NFr(A) \cap NFr(B) \nsubseteq NFr(A \cap B)$.

Example 2.24 From Example 2.22, $NFr(A_1 \cap A_2) = NFr(A_4) = E \nsubseteq G = NFr(A_1) \cap NFr(A_2)$.
From Example 2.14, We define $B_1 = \langle (0.4, 0.5, 0.1), (0.2, 0.9, 0.5) \rangle$,
$B_2 = \langle (0.5, 0.2, 0.9), (0.8, 0.4, 0.7) \rangle$,
$B_1 \cup B_2 = B_3 = \langle (0.5, 0.5, 0.1), (0.8, 0.9, 0.5) \rangle$ and
$B_1 \cap B_2 = B_4 = \langle (0.4, 0.2, 0.9), (0.2, 0.4, 0.7) \rangle$. Then
$C(B_1) = \langle (0.1, 0.5, 0.4), (0.5, 0.1, 0.2) \rangle$,
$C(B_2) = \langle (0.9, 0.8, 0.5), (0.7, 0.6, 0.8) \rangle$,
$C(B_3) = \langle (0.1, 0.5, 0.5), (0.5, 0.1, 0.8) \rangle$ and
$C(B_4) = \langle (0.9, 0.8, 0.4), (0.7, 0.6, 0.2) \rangle$.
Therefore $NFr(B_1) \cap NFr(B_2) = 1_N \cap 1_N = 1_N \nsubseteq H = NFr(B_4) = NFr(B_1 \cap B_2)$.

Theorem 2.25 For any *NSs* *A* and *B* in the *NTS* X, $NFr(A \cap B) \subseteq (NFr(A) \cap NCl(B)) \cup (NFr(B) \cap NCl(A))$.
Proof : Let *A* and *B* be the two *NSs* in the *NTS* X. Then by Definition 2.2,
$NFr(A \cap B) = NCl(A \cap B) \cap NCl(C(A \cap B))$
By Proposition 3.2 (1) [18] ,
$= NCl(A \cap B) \cap NCl(C(A) \cup C(B))$
By Proposition 1.18 (n) and (h) [12] ,
$\subseteq (NCl(A) \cap NCl(B)) \cap (NCl(C(A)) \cup NCl(C(B)))$
$= [(NCl(A) \cap NCl(B)) \cap NCl(C(A))]$
$\cup [(NCl(A) \cap NCl(B)) \cap NCl(C(B))]$
Again by Definition 2.2,
$= (NFr(A) \cap NCl(B)) \cup (NFr(B) \cap NCl(A))$
Hence $NFr(A \cap B) \subseteq (NFr(A) \cap NCl(B)) \cup (NFr(B) \cap NCl(A))$.

The converse of the above theorem needs not be true as shown by the following example.

Example 2.26 From Example 2.24, $(NFr(B_1) \cap NCl(B_2)) \cup (NFr(B_2) \cap NCl(B_1)) = (1_N \cap 1_N) \cup (1_N \cap 1_N) = 1_N \cup 1_N = 1_N \nsubseteq H = NFr(B_1 \cap B_2)$.

Corollary 2.27 For any *NSs* *A* and *B* in the *NTS* X, $NFr(A \cap B) \subseteq NFr(A) \cup NFr(B)$.
Proof : Let *A* and *B* be the two *NSs* in the *NTS* X. Then by Definition 2.2,
$NFr(A \cap B) = NCl(A \cap B) \cap NCl(C(A \cap B))$
By Proposition 3.2 (1) [18] ,
$= NCl(A \cap B) \cap NCl(C(A) \cup C(B))$
By Proposition 1.18 (n) and (h) [12] ,
$\subseteq (NCl(A) \cap NCl(B)) \cap (NCl(C(A)) \cup NCl(C(B)))$
$= (NCl(A) \cap NCl(B) \cap NCl(C(A)))$
$\cup (NCl(A) \cap NCl(B) \cap NCl(C(B)))$
Again by Definition 2.2,
$= (NFr(A) \cap NCl(B)) \cup (NCl(A) \cap NFr(B))$
$\subseteq NFr(A) \cup NFr(B)$
Hence $NFr(A \cap B) \subseteq NFr(A) \cup NFr(B)$.

The equality in the above corollary may not hold as seen in the following example.

Example 2.28 From Example 2.24, $NFr(B_1) \cup NFr(B_2) = 1_N \cup 1_N = 1_N \nsubseteq H = NFr(B_4) = NFr(B_1 \cap B_2)$.

Theorem 2.29 For any *NS* *A* in the *NTS* X,
(1) $NFr(NFr(A)) \subseteq NFr(A)$,
(2) $NFr(NFr(NFr(A))) \subseteq NFr(NFr(A))$.
Proof : (1) Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 2.2,
$NFr(NFr(A)) = NCl(NFr(A)) \cap NCl(C(NFr(A)))$
Again by Definition 2.2,
$= NCl(NCl(A) \cap NCl(C(A))) \cap$
$NCl(C(NCl(A) \cap NCl(C(A))))$
By Proposition 1.18 (f) [12] and by 4.2 (b) [18] ,
$\subseteq (NCl(NCl(A)) \cap NCl(NCl(C(A))))$
$\cap NCl(NInt(C(A)) \cup NInt(A))$
By Proposition 1.18 (f) [12] ,
$= (NCl(A) \cap NCl(C(A))) \cap (NCl(NInt(C(A)))$
$\cup NCl(NInt(A))$
$\subseteq NCl(A) \cap NCl(C(A))$
By Definition 2.2,
$= NFr(A)$
Therefore $NFr(NFr(A)) \subseteq NFr(A)$.

(2) By Definition 2.2,
$NFr(NFr(NFr(A))) = NCl(NFr(NFr(A))) \cap$
$NCl(C(NFr(NFr(A))))$

By Proposition 1.18 (f) [12] ,
$\subseteq$ (*NFr* (*NFr* (*A*))) $\cap$ *NCl* (C (*NFr* (*NFr* (*A*))))
$\subseteq$ *NFr* (*NFr* (*A*)).
Hence *NFr* (*NFr* (*NFr* (*A*))) $\subseteq$ *NFr* (*NFr* (*A*)).

Remark 2.30 From the above theorem, the converse of (1) needs not be true as shown by the following example and no counter-example could be found to establish the irreversibility of inequality in (2).

Example 2.31 Let X = {*a*, *b*} and τ = {0_N, A, B, 1_N }. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are C (τ) = { 1_N, C, D, 0_N } where
A = ⟨ (0.8, 0.4, 0.5), (0.4, 0.6, 0.7) ⟩,
B = ⟨ (0.4, 0.2, 0.9), (0.1, 0.4, 0.9) ⟩,
C = ⟨ (0.5, 0.6, 0.8), (0.7, 0.4, 0.4) ⟩ and
D = ⟨ (0.9, 0.8, 0.4), (0.9, 0.6, 0.1) ⟩. Define
A_1 = ⟨ (0.6, 0.7, 0.8), (0.5, 0.4, 0.5) ⟩. Then
C (A_1) = ⟨ (0.8, 0.3, 0.6), (0.5, 0.6, 0.5) ⟩.
Therefore by Definition 2.2, *NFr* (A_1) = D $\nsubseteq$ C = *NFr* (*NFr* (A_1)).

Theorem 2.32 Let *A*, *B*, *C* and *D* be the *NSs* in the *NTS* X. Then (*A* ∩ *B*) × (*C* ∩ *D*) = (*A* × *D*) ∩ (*B* × *C*).
Proof : Let *A*, *B*, *C* and *D* be the *NSs* in the *NTS* X. Then by Definition 2.2 [12] ,
$\mu_{(A\cap B)\times(C\cap D)}(x, y)$
$= \min \{\mu_{(A\cap B)}(x), \mu_{(C\cap D)}(y)\}$
$= \min \{ \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_C(y), \mu_D(y) \} \}$
$= \min \{ \min \{ \mu_A(x), \mu_D(y) \}, \min \{ \mu_B(x), \mu_C(y) \} \}$
$= \min \{ \mu_{(A\times D)}(x, y), \mu_{(B\times C)}(x, y) \}$.
Thus $\mu_{(A\cap B)\times(C\cap D)}(x, y) = \mu_{(A\times D)\cap(B\times C)}(x, y)$.
Similarly
$\sigma_{(A\cap B)\times(C\cap D)}(x, y)$
$= \min \{\sigma_{(A\cap B)}(x), \sigma_{(C\cap D)}(y)\}$
$= \min \{ \min \{ \sigma_A(x), \sigma_B(x) \}, \min \{ \sigma_C(y), \sigma_D(y) \} \}$
$= \min \{ \min \{ \sigma_A(x), \sigma_D(y) \}, \min \{ \sigma_B(x), \sigma_C(y) \} \}$
$= \min \{ \sigma_{(A\times D)}(x, y), \sigma_{(B\times C)}(x, y) \}$.
Thus $\sigma_{(A\cap B)\times(C\cap D)}(x, y) = \sigma_{(A\times D)\cap(B\times C)}(x, y)$.
And also
$\gamma_{(A\cap B)\times(C\cap D)}(x, y)$
$= \max \{ \gamma_{(A\cap B)}(x), \gamma_{(C\cap D)}(y)\}$
$= \max \{ \max \{ \gamma_A(x), \gamma_B(x) \}, \max \{ \gamma_C(y), \gamma_D(y) \} \}$
$= \max \{ \max \{ \gamma_A(x), \gamma_D(y) \}, \max \{ \gamma_B(x), \gamma_C(y) \} \}$
$= \max \{ \gamma_{(A\times D)}(x, y), \gamma_{(B\times C)}(x, y) \}$.
Thus $\gamma_{(A\cap B)\times(C\cap D)}(x, y) = \gamma_{(A\times D)\cap(B\times C)}(x, y)$.
Hence (*A* ∩ *B*) × (*C* ∩ *D*) = (*A* × *D*) ∩ (*B* × *C*).

Theorem 2.33 Let X_i, $i = 1, 2, \ldots, n$ be a family of neutrosophic product related *NTSs*. If each A_i is a *NS* in X_i. Then *NFr* ($\prod_{i=1}^{n} A_i$) = [*NFr* (A_1) × *NCl* (A_2) × · · · × *NCl* (A_n)] ∪ [*NCl* (A_1) × *NFr* (A_2) × *NCl* (A_3) × · · · × *NCl* (A_n)] ∪ · · · ∪ [*NCl* (A_1) × *NCl* (A_2) × · · · × *NFr* (A_n)].
Proof : It suffices to prove this for $n = 2$. Let A_i be the *NS* in the neutrosophic topological space X_i. Then by Definition 2.2,
NFr ($A_1 \times A_2$) = *NCl* ($A_1 \times A_2$) ∩ *NCl* (C ($A_1 \times A_2$))
By Proposition 4.2 (a) [18] ,
= *NCl* ($A_1 \times A_2$) ∩ C (*NInt* ($A_1 \times A_2$))
By Theorem 2.17 (1) and (2) [12] ,
= (*NCl* (A_1) × *NCl* (A_2)) ∩ C (*NInt* (A_1) × *NInt* (A_2))
= (*NCl* (A_1) × *NCl* (A_2)) ∩
C[(*NInt* (A_1) ∩ *NSCl* (A_1)) ×(*NInt* (A_2) ∩ *NCl* (A_2))]
By Lemma 2.3 (iii) [12] ,
= (*NCl* (A_1) × *NCl* (A_2)) ∩ [C (*NInt* (A_1) ∩ *NCl* (A_1)) × 1_N ∪ 1_N × C (*NInt* (A_2) ∩ *NCl* (A_2))]
= (*NCl* (A_1) × *NCl* (A_2)) ∩ [(*NCl* (C (A_1)) ∪ *NInt* (C (A_1))) × 1_N ∪ 1_N × (*NCl* (C (A_2)) ∪ *NInt* (C (A_2)))]
= (*NCl* (A_1) × *NCl* (A_2)] ∩ [(*NCl* (C (A_1)) × 1_N) ∪ (1_N × *NCl* (C (A_2)))]
= [(*NCl* (A_1) × *NCl* (A_2)) ∩ (*NCl* (C (A_1)) × 1_N)] ∪ [(*NCl* (A_1) × *NCl* (A_2)) ∩ (1_N × *NCl* (C (A_2)))]
By Theorem 2.32,
= [(*NCl* (A_1) ∩ *NCl* (C (A_1))) × (1_N ∩ *NCl* (A_2))] ∪ [(*NCl* (A_1) ∩ 1_N) × (*NCl* (A_2) ∩ *NCl* (C (A_2)))]
= (*NFr* (A_1) × *NCl* (A_2)) ∪ (*NCl* (A_1) × *NFr* (A_2)).
Hence *NFr* ($A_1 \times A_2$) = (*NFr* (A_1) × *NCl* (A_2)) ∪ (*NCl* (A_1) × *NFr* (A_2)).

III. NEUTROSOPHIC SEMI-FRONTIER

In this section, we introduce the neutrosophic semi-frontier and their properties in neutrosophic topological spaces.

Definition 3.1 Let *A* be a *NS* in the *NTS* X. Then the neutrosophic semi-frontier of *A* is defined as *NSFr* (*A*) = *NSCl* (*A*) ∩ *NSCl* (C (*A*)). Obviously *NSFr* (*A*) is a *NSC* set in X.

Theorem 3.2 Let *A* be a *NS* in the *NTS* X. Then the following conditions are holds :
(i) *NSCl* (*A*) = *A* ∪ *NInt* (*NCl* (*A*)),
(ii) *NSInt* (*A*) = *A* ∩ *NCl* (*NInt* (*A*)).
Proof : (i) Let *A* be a *NS* in X. Consider
NInt (*NCl* (*A* ∪ *NInt* (*NCl* (*A*))))
= *NInt* (*NCl* (*A*) ∪ *NCl* (*NInt* (*NCl* (*A*))))
= *NInt* (*NCl* (*A*))
$\subseteq$ *A* ∪ *NInt* (*NCl* (*A*))
It follows that *A* ∪ *NInt* (*NCl* (*A*)) is a *NSC* set in X.
Hence *NSCl* (*A*) $\subseteq$ *A* ∪ *NInt* (*NCl* (*A*)) ------------ (1)

By Proposition 6.3 (ii) [12], *NSCl* (*A*) is *NSC* set in X. We have *NInt* (*NCl* (*A*)) $\subseteq$ *NInt* (*NCl* (*NSCl* (*A*))) $\subseteq$ *NSCl* (*A*).
Thus *A* $\cup$ *NInt* (*NCl* (*A*)) $\subseteq$ *NSCl* (*A*) -------------- (2).
From (1) and (2), *NSCl* (*A*) = *A* $\cup$ *NInt* (*NCl* (*A*)).

(ii) This can be proved in a similar manner as (i).

Theorem 3.3 For a *NS A* in the *NTS* X, *NSFr* (*A*) = *NSFr* (C (*A*)).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
NSFr (*A*) = *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
= *NSCl* (C (*A*)) $\cap$ *NSCl* (*A*)
= *NSCl* (C (*A*)) $\cap$ *NSCl* (C (C (*A*)))
Again by Definition 3.1,
= *NSFr* (C (*A*))
Hence *NSFr* (*A*) = *NSFr* (C (*A*)).

Theorem 3.4 If *A* is *NSC* set in X, then *NSFr* (*A*) $\subseteq$ *A*.
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
NSFr (*A*) = *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
$\subseteq$ *NSCl* (*A*)
By Proposition 6.3 (ii) [12] ,
= *A*
Hence *NSFr* (*A*) $\subseteq$ *A*, if *A* is *NSC* in X.

The converse of the above theorem is not true as shown by the following example.

Example 3.5 Let X = { *a*, *b*, *c* } and τ = { 0_N, A, B, C, D, 1_N }. Then (X, τ) is a neutrosophic topological space. The neutrosophic closed sets are C (τ) = { 1_N, F, G, H, I, 0_N } where
A = ⟨ (0.5, 0.6, 0.7), (0.1, 0.8, 0.4) , (0.7, 0.2, 0.3) ⟩,
B = ⟨ (0.8, 0.8, 0.5), (0.5, 0.4, 0.2) , (0.9, 0.6, 0.7) ⟩,
C = ⟨ (0.8, 0.8, 0.5), (0.5, 0.8, 0.2) , (0.9, 0.6, 0.3) ⟩,
D = ⟨ (0.5, 0.6, 0.7), (0.1, 0.4, 0.4) , (0.7, 0.2, 0.7) ⟩,
E = ⟨ (0.8, 0.8, 0.4), (0.5, 0.8, 0.1) , (0.9, 0.7, 0.2) ⟩,
F = ⟨ (0.7, 0.4, 0.5), (0.4, 0.2, 0.1) , (0.3, 0.8, 0.7) ⟩,
G = ⟨ (0.5, 0.2, 0.8), (0.2, 0.6, 0.5) , (0.7, 0.4, 0.9) ⟩,
H = ⟨ (0.5, 0.2, 0.8), (0.2, 0.2, 0.5) , (0.3, 0.4, 0.9) ⟩,
I = ⟨ (0.7, 0.4, 0.5), (0.4, 0.6, 0.1) , (0.7, 0.8, 0.7) ⟩ and
J = ⟨ (0.4, 0.2, 0.8), (0.1, 0.2, 0.5) , (0.2, 0.3, 0.9) ⟩.
Here E and J are neutrosophic semi-open and neutrosophic semi-closed set respectively. Therefore the neutrosophic semi-open and neutrosophic semi-closed set topologies are τ_{NSO} = 0_N, A, B, C, D, E, 1_N and C $(\tau)_{NSC}$ = 1_N, F, G, H, I, J, 0_N . Therefore *NSFr* (C) = H $\subseteq$ C. But C $\notin$ C $(\tau)_{NSC}$.

Theorem 3.6 If *A* is *NSO* set in X, then *NSFr* (*A*) $\subseteq$ C (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Proposition 4.3 [12] , *A* is *NSO* set implies C (*A*) is *NSC* set in X. By Theorem 3.4, *NSFr* (C (*A*)) $\subseteq$ C (*A*) and by Theorem 3.3, we get *NSFr* (*A*) $\subseteq$ C (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.7 From Example 3.5, *NSFr* (J) = J $\subseteq$ C (J) = E. But J $\notin$ τ_{NSO}.

Theorem 3.8 Let *A* $\subseteq$ *B* and *B* $\in$ *NSC* (X) (resp., *B* $\in$ *NSO* (X)). Then *NSFr* (*A*) $\subseteq$ *B* (resp., *NSFr* (*A*) $\subseteq$ C(*B*)), where *NSC* (X) (resp., *NSO* (X)) denotes the class of neutrosophic semi-closed (resp., neutrosophic semi-open) sets in X.
Proof : By Proposition 6.3 (iv) [12] , *A* $\subseteq$ *B* , *NSCl* (*A*) $\subseteq$ *NSCl* (*B*) -------------------- (1).
By Definition 3.1,
NSFr (*A*) = *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
$\subseteq$ *NSCl* (*B*) $\cap$ *NSCl* (C (*A*)) by (1)
$\subseteq$ *NSCl* (*B*)
By Proposition 6.3 (ii) [12] ,
= *B*
Hence *NSFr* (*A*) $\subseteq$ *B*.

Theorem 3.9 Let *A* be the *NS* in the *NTS* X. Then C (*NSFr* (*A*)) = *NSInt* (*A*) $\cup$ *NSInt* (C (*A*)).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
C (*NSFr* (*A*)) = C (*NSCl* (*A*) $\cap$ *NSCl* (C (*A*)))
By Proposition 3.2 (1) [18] ,
= C (*NSCl* (*A*)) $\cup$ C (*NSCl* (C (*A*)))
By Proposition 6.2 (ii) [12] ,
= *NSInt* (C (*A*)) $\cup$ *NSInt* (*A*)
Hence C (*NSFr* (*A*)) = *NSInt* (*A*) $\cup$ *NSInt* (C (*A*)).

Theorem 3.10 For a *NS A* in the *NTS* X, then *NSFr* (*A*) $\subseteq$ *NFr* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Proposition 6.4 [12] , *NSCl* (*A*) $\subseteq$ *NCl* (*A*) and *NSCl* (C (*A*)) $\subseteq$ *NCl* (C (*A*)).
Now by Definition 3.1,
NSFr (*A*) = *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
$\subseteq$ *NCl* (*A*) $\cap$ *NCl* (C (*A*))
By Definition 2.2,
= *NFr* (*A*)
Hence *NSFr* (*A*) $\subseteq$ *NFr* (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.11 From Example 3.5, let $A_1 = \langle$ (0.4, 0.1, 0.9), (0.1, 0.2, 0.6) , (0.1, 0.3, 0.9) $\rangle$, then
C (A_1) = $\langle$ (0.9, 0.9, 0.4), (0.6, 0.8, 0.1) , (0.9, 0.7, 0.1) $\rangle$. Therefore *NFr* (A_1) = H $\not\subseteq$ J = *NSFr* (A_1).

Theorem 3.12 For a *NS A* in the *NTS* X, then *NSCl* (*NSFr* (*A*)) $\subseteq$ *NFr* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
NSCl (*NSFr* (*A*)) = *NSCl* (*NSCl* (*A*) $\cap$ *NSCl* (C (*A*)))
$\subseteq$ *NSCl* (*NSCl* (*A*)) $\cap$ *NSCl* (*NSCl* (C (*A*)))
By Proposition 6.3 (iii) [12] ,
= *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
By Definition 3.1,
= *NSFr* (*A*)
By Theorem 3.10,
$\subseteq$ *NFr* (*A*)
Hence *NSCl* (*NSFr* (*A*)) $\subseteq$ *NFr* (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.13 From Example 3.5, *NFr* (A_1) = H $\not\subseteq$ J = *NSCl* (*NSFr* (A_1)).

Theorem 3.14 Let *A* be a *NS* in the *NTS* X. Then *NSFr* (*A*) = *NSCl* (*A*) – *NSInt* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. By Proposition 6.2 (ii) [12] ,
C (*NSCl* (C (*A*))) = *NSInt* (*A*) and by Definition 3.1,
NSFr (*A*) = *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
= *NSCl* (*A*) – C (*NSCl* (C (*A*)))
by using $A - B = A \cap$ C (B)
By Proposition 6.2 (ii) [12] ,
= *NSCl* (*A*) – *NSInt* (*A*)
Hence *NSFr* (*A*) = *NSCl* (*A*) – *NSInt* (*A*).

Theorem 3.15 For a *NS A* in the *NTS* X, then *NSFr* (*NSInt* (*A*)) $\subseteq$ *NSFr* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
NSFr (*NInt* (*A*))=*NSCl*(*NInt*(*A*))$\cap$*NSCl*(C(*NSInt* (*A*)))
By Proposition 6.2 (i) [12] ,
=*NSCl*(*NSInt*(*A*))$\cap$*NSCl*(*NSCl*(C(*A*)))
By Proposition 6.3 (iii) [12] ,
= *NSCl* (*NSInt* (*A*)) $\cap$ *NSCl* (C (*A*))
By Proposition 5.2 (ii) [12] ,
$\subseteq$ *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
By Definition 3.1,
= *NSFr* (*A*)
Hence *NSFr* (*NSInt* (*A*)) $\subseteq$ *NSFr* (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.16 Let X = { *a*, *b*, *c* } and τ_{NSO} = 0_N, A, B, C, D, E, 1_N and C $(\tau)_{NSC}$ = 1_N, F, G, H, I, J, 0_N where
A = $\langle$ (0.3, 0.4, 0.2), (0.5, 0.6, 0.7), (0.9, 0.5, 0.2) $\rangle$,
B = $\langle$ (0.3, 0.5, 0.1), (0.4, 0.3, 0.2), (0.8, 0.4, 0.6) $\rangle$,
C = $\langle$ (0.3, 0.5, 0.1), (0.5, 0.6, 0.2), (0.9, 0.5, 0.2) $\rangle$,
D = $\langle$ (0.3, 0.4, 0.2), (0.4, 0.3, 0.7), (0.8, 0.4, 0.6) $\rangle$,
E = $\langle$ (0.5, 0.6, 0.1), (0.6, 0.7, 0.1), (0.9, 0.5, 0.2) $\rangle$,
F = $\langle$ (0.2, 0.6, 0.3), (0.7, 0.4, 0.5), (0.2, 0.5, 0.9) $\rangle$,
G = $\langle$ (0.1, 0.5, 0.3), (0.2, 0.7, 0.4), (0.6, 0.6, 0.8) $\rangle$,
H = $\langle$ (0.1, 0.5, 0.3), (0.2, 0.4, 0.5), (0.2, 0.5, 0.9) $\rangle$,
I = $\langle$ (0.2, 0.6, 0.3), (0.7, 0.7, 0.4), (0.6, 0.6, 0.8) $\rangle$ and
J = $\langle$ (0.1, 0.4, 0.5), (0.1, 0.3, 0.6), (0.2, 0.5, 0.9) $\rangle$.
Define A_1 = $\langle$ (0.2, 0.3, 0.4), (0.4, 0.5, 0.6) , (0.3, 0.4, 0.8) $\rangle$.
Then C (A_1) = $\langle$ (0.4, 0.7, 0.2), (0.6, 0.5, 0.4) , (0.8, 0.6, 0.3) $\rangle$. Therefore *NSFr* (A_1) = I $\not\subseteq$ 0_N = *NSFr* (*NSInt* (A_1)).

Theorem 3.17 For a *NS A* in the *NTS* X, then *NSFr* (*NSCl* (*A*)) $\subseteq$ *NSFr* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,
NSFr(*NSCl*(*A*))=*NSCl*(*NSCl*(*A*))$\cap$*NSCl*(C(*NSCl* (*A*)))
By Proposition 6.3 (iii) and Proposition 6.2 (ii) [12] ,
= *NSCl* (*A*) $\cap$ *NSCl* (*NSInt* (C (*A*)))
By Proposition 5.2 (i) [12] ,
$\subseteq$ *NSCl* (*A*) $\cap$ *NSCl* (C (*A*))
By Definition 3.1,
= *NSFr* (*A*)
Hence *NSFr* (*NSCl* (*A*)) $\subseteq$ *NSFr* (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.18 From Example 3.16, let A_2 = $\langle$ (0.2, 0.6, 0.2), (0.3, 0.4, 0.6) , (0.3, 0.4, 0.8) $\rangle$. Then C (A_2) = $\langle$ (0.2, 0.4, 0.2), (0.6, 0.6, 0.3) , (0.8, 0.6, 0.3) $\rangle$. Therefore *NSFr* (A_2) = 1_N $\not\subseteq$ 0_N = *NSFr* (*NSCl* (A_2)).

Theorem 3.19 Let *A* be the *NS* in the *NTS* X. Then *NSInt* (*A*) $\subseteq$ *A* – *NSFr* (*A*).
Proof : Let *A* be the *NS* in the neutrosophic topological space X. Now by Definition 3.1,
A – *NSFr* (*A*) = *A* − (*NSCl* (*A*) $\cap$ *NSCl* (C (*A*)))
= (*A* – *NSCl* (*A*)) $\cup$ (*A* – *NSCl* (C (*A*)))
= *A* – *NSCl* (C (*A*))
$\supseteq$ *NSInt* (*A*).
Hence *NSInt* (*A*) $\subseteq$ *A* – *NSFr* (*A*).

The converse of the above theorem is not true as shown by the following example.

Example 3.20 From Example 3.16, A_1 – *NSFr* (A_1) = ⟨ (0.2, 0.3, 0.4), (0.4, 0.3, 0.7) , (0.3, 0.4, 0.8) ⟩ $\nsubseteq 0_N$ = *NSInt* (A_1).

Remark 3.21 In general topology, the following conditions are hold :
NSFr $(A) \cap$ *NSInt* $(A) = 0_N$,
NSInt $(A) \cup$ *NSFr* (A) = *NSCl* (A),
NSInt $(A) \cup$ *NSInt* $(C(A)) \cup$ *NSFr* $(A) = 1_N$.
But the neutrosophic topology, we give counter-examples to show that the conditions of the above remark may not be hold in general.

Example 3.22 From Example 3.16, define A_1 = ⟨ (0.4, 0.6, 0.1), (0.5, 0.8, 0.3) , (0.9, 0.6, 0.2) ⟩. Then C (A_1) = ⟨ (0.1, 0.4, 0.4), (0.3, 0.2, 0.5) , (0.2, 0.4, 0.9) ⟩. Therefore *NSFr* $(A_1) \cap$ *NSInt* (A_1) = F $\cap$ D = ⟨ (0.2, 0.4, 0.3), (0.4, 0.3, 0.7), (0.2, 0.4, 0.9) ⟩ $\neq 0_N$.

NSInt $(A_1) \cup$ *NSFr* (A_1) = D $\cup$ F = ⟨ (0.3, 0.6, 0.2), (0.7, 0.4, 0.5), (0.8, 0.5, 0.6) ⟩ $\neq 1_N$ = *NSCl* (A_1).

NSInt $(A_1) \cup$ *NSInt* $(C(A_1)) \cup$ *NSFr* (A_1) = D $\cup 0_N$ $\cup$ F = ⟨ (0.3, 0.6, 0.2), (0.7, 0.4, 0.5), (0.8, 0.5, 0.6) ⟩ $\neq 1_N$.

Theorem 3.23 Let *A* and *B* be *NSs* in the *NTS* X. Then *NSFr* $(A \cup B) \subseteq$ *NSFr* $(A) \cup$ *NSFr* (B).
Proof : Let *A* and *B* be *NSs* in the *NTS* X. Then by Definition 3.1,
NSFr $(A \cup B)$ = *NSCl* $(A \cup B) \cap$ *NSCl* $(C(A \cup B))$
By Proposition 3.2 (2) [18] ,
= *NSCl* $(A \cup B) \cap$ *NSCl* $(C(A) \cap C(B))$
By Proposition 6.5 (i) and (ii) [12] ,
$\subseteq (NSCl(A) \cup NSCl(B)) \cap (NSCl(C(A)) \cap NSCl(C(B)))$
$= [(NSCl(A) \cup NSCl(B)) \cap NSCl(C(A))] \cap$
$[(NSCl(A) \cup NSCl(B)) \cap NSCl(C(B))]$
$= [(NSCl(A) \cap NSCl(C(A))) \cup (NSCl(B) \cap NSCl(C(A)))]$
$\cap [(NSCl(A) \cap NSCl(C(B))) \cup (NSCl(B) \cap NSCl(C(B)))]$
By Definition 3.1,
$= [NSFr(A) \cup (NSCl(B) \cap NSCl(C(A)))] \cap$
$[(NSCl(A) \cap NSCl(C(B))) \cup NSFr(B)]$
$= (NSFr(A) \cup NSFr(B)) \cap [(NSCl(B) \cap$
$NSCl(C(A))) \cup (NSCl(A) \cap NSCl(C(B)))]$
$\subseteq NSFr(A) \cup NSFr(B)$.
Hence *NSFr* $(A \cup B) \subseteq$ *NSFr* $(A) \cup$ *NSFr* (B).

The converse of the above theorem needs not be true as shown by the following example.

Example 3.24 Let X = { *a* } with τ_{NSO} = 0_N, A, B, C, D, 1_N and C $(\tau)_{NSC}$ = 1_N, E, F, G, H, 0_N where
A = ⟨ (0.6, 0.8, 0.4) ⟩,
B = ⟨ (0.4, 0.9, 0.7) ⟩,
C = ⟨ (0.6, 0.9, 0.4) ⟩,
D = ⟨ (0.4, 0.8, 0.7) ⟩,
E = ⟨ (0.4, 0.2, 0.6) ⟩,
F = ⟨ (0.7, 0.1, 0.4) ⟩,
G = ⟨ (0.4, 0.1, 0.6) ⟩ and
H = ⟨ (0.7, 0.2, 0.4) ⟩. Now we define
B_1 = ⟨ (0.7, 0.6, 0.5) ⟩,
B_2 = ⟨ (0.6, 0.8, 0.2) ⟩,
$B_1 \cup B_2 = B_3$ = ⟨ (0.7, 0.8, 0.2) ⟩ and
$B_1 \cap B_2 = B_4$ = ⟨ (0.6, 0.6, 0.5) ⟩. Then
C (B_1) = ⟨ (0.5, 0.4, 0.7) ⟩,
C (B_2) = ⟨ (0.2, 0.2, 0.6) ⟩,
C (B_3) = ⟨ (0.2, 0.2, 0.7) ⟩ and
C (B_4) = ⟨ (0.5, 0.4, 0.6) ⟩.
Therefore *NSFr* $(B_1) \cup$ *NSFr* $(B_2) = 1_N \cup E = 1_N \nsubseteq E$ = *NSFr* (B_3) = *NSFr* $(B_1 \cup B_2)$.

Note 3.25 The following example shows that *NSFr* $(A \cap B) \nsubseteq$ *NSFr* $(A) \cap$ *NSFr* (B) and
NSFr $(A) \cap$ *NSFr* $(B) \nsubseteq$ *NSFr* $(A \cap B)$.

Example 3.26 From Example 3.24, we define
A_1 = ⟨ (0.5, 0.1, 0.9) ⟩,
A_2 = ⟨ (0.3, 0.5, 0.6) ⟩,
$A_1 \cup A_2 = A_3$ = ⟨ (0.5, 0.5, 0.6) ⟩, and
$A_1 \cap A_2 = A_4$ = ⟨ (0.3, 0.1, 0.9) ⟩. Then
C (A_1) = ⟨ (0.9, 0.9, 0.5) ⟩,
C (A_2) = ⟨ (0.6, 0.5, 0.3) ⟩,
C (A_3) = ⟨ (0.6, 0.5, 0.5) ⟩ and
C (A_4) = ⟨ (0.9, 0.9, 0.3) ⟩.
Therefore *NSFr* $(A_1) \cap$ *NSFr* $(A_2) = F \cap 1_N = F \nsubseteq G$ = *NSFr* (A_4) = *NSFr* $(A_1 \cap A_2)$.

Also *NSFr* $(B_1 \cap B_2)$ = *NSFr* $(B_4) = 1_N \nsubseteq E = 1_N \cap E$ = *NSFr* $(B_1) \cap$ *NSFr* (B_2).

Theorem 3.27 For any *NSs A* and *B* in the *NTS* X,
NSFr $(A \cap B) \subseteq$ (*NSFr* $(A) \cap$ *NSCl* (B)) $\cup$ (*NSFr* $(B) \cap$ *NSCl* (A)).
Proof : Let *A* and *B* be *NSs* in the *NTS* X. Then by Definition 3.1,
NSFr $(A \cap B)$ = *NSCl* $(A \cap B) \cap$ *NSCl* $(C(A \cap B))$
By Proposition 3.2 (1) [18] ,
= *NSCl* $(A \cap B) \cap$ *NSCl* $(C(A) \cup C(B))$
By Proposition 6.5 (ii) and (i) [12] ,
$\subseteq (NSCl(A) \cap NSCl(B)) \cap (NSCl(C(A)) \cup NSCl(C(B)))$
$= [(NSCl(A) \cap NSCl(B)) \cap NSCl(C(A))] \cup$
$[(NSCl(A) \cap NSCl(B)) \cap NSCl(C(B))]$
By Definition 3.1,
$= (NSFr(A) \cap NSCl(B)) \cup (NSFr(B) \cap NSCl(A))$
Hence *NSFr* $(A \cap B) \subseteq$ (*NSFr* $(A) \cap$ *NSCl* (B)) $\cup$ (*NSFr* $(B) \cap$ *NSCl* (A)).

The converse of the above theorem is not true as shown by the following example.

Example 3.28 From Example 3.24, ($NSFr$ $(A_1) \cap NSCl$ (A_2)) $\cup$ ($NSFr$ $(A_2) \cap NSCl$ (A_1)) = $(F \cap 1_N) \cup (1_N \cap F) = F \cup F = F \not\subseteq G = NSFr\ (A_1 \cap A_2)$.

Corollary 3.29 For any *NSs* A and B in the *NTS* X, $NSFr\ (A \cap B) \subseteq NSFr\ (A) \cup NSFr\ (B)$.

Proof : Let A and B be *NSs* in the *NTS* X. Then by Definition 3.1,

$NSFr\ (A \cap B) = NSCl\ (A \cap B) \cap NSCl\ (C\ (A \cap B))$

By Proposition 3.2 (1) [18] ,

$= NSCl\ (A \cap B) \cap NSCl\ (\ C\ (A) \cup C\ (B)\ )$

By Proposition 6.5 (ii) and (i) [12] ,

$\subseteq (NSCl(A) \cap NSCl(B)) \cap (NSCl(C(A)) \cup NSCl(C(B)))$

$= (\ NSCl\ (A) \cap NSCl\ (B) \cap NSCl\ (C\ (A))\ ) \cup (\ NSCl\ (A) \cap NSCl\ (B) \cap NSCl\ (C\ (B))\ )$

By Definition 3.1,

$= (\ NSFr\ (A) \cap NSCl\ (B)\ ) \cup (\ NSCl\ (A) \cap NSFr\ (B))$

$\subseteq NSFr\ (A) \cup NSFr\ (B)$.

Hence $NSFr\ (A \cap B) \subseteq NSFr\ (A) \cup NSFr\ (B)$.

The equality in the above theorem may not hold as seen in the following example.

Example 3.30 From Example 3.24, $NSFr\ (A_1) \cup NSFr\ (A_2) = F \cup 1_N = 1_N \not\subseteq G = NSFr\ (A_4) = NSFr\ (A_1 \cap A_2)$.

Theorem 3.31 For any *NS* A in the *NTS* X,

(1) $NSFr\ (NSFr\ (A)) \subseteq NSFr\ (A)$,

(2) $NSFr\ (NSFr\ (NSFr\ (A))) \subseteq NSFr\ (NSFr\ (A))$.

Proof : (1) Let A be the *NS* in the neutrosophic topological space X. Then by Definition 3.1,

$NSFr\ (NSFr\ (A))$

$= NSCl\ (NSFr\ (A)) \cap NSCl\ (C\ (NSFr\ (A)))$

By Definition 3.1,

$= NSCl\ (\ NSCl\ (A) \cap NSCl\ (C\ (A))\ ) \cap NSCl\ (\ C\ (\ NSCl\ (A) \cap NSCl\ (C\ (A))\ )\ )$

By Proposition 6.3 (iii) and 6.2 (ii) [12] ,

$\subseteq (\ NSCl\ (NSCl\ (A)) \cap NSCl\ (NSCl\ (C\ (A)))\ ) \cap NSCl\ (\ NSInt\ (C\ (A)) \cup NSInt\ (A)\ )$

By Proposition 6.3 (iii) [12] ,

$= (NSCl\ (A) \cap NSCl\ (C\ (A))) \cap (NSCl\ (NSInt(C\ (A))) \cup NSCl\ (NSInt\ (A))$

$\subseteq NSCl\ (A) \cap NSCl\ (C\ (A))$

By Definition 3.1,

$= NSFr\ (A)$

Therefore $NSFr\ (NSFr\ (A)) \subseteq NSFr\ (A)$.

(2) By Definition 3.1,

$NSFr\ (NSFr\ (NSFr\ (A))) = NSCl\ (NSFr\ (NSFr\ (A))) \cap NSCl\ (C\ (NSFr\ (NSFr\ (A))))$

By Proposition 6.3 (iii) [12] ,

$\subseteq (NSFr\ (NSFr\ (A))) \cap NSCl\ (C\ (NSFr\ (NSFr\ (A))))$

$\subseteq NSFr\ (NSFr\ (A))$.

Hence $NSFr\ (NSFr\ (NSFr\ (A))) \subseteq NSFr\ (NSFr\ (A))$.

Remark 3.32 From the above theorem, the converse of (1) needs not be true as shown by the following example and no counter-example could be found to establish the irreversibility of inequality in (2).

Example 3.33 From Example 3.16, $NSFr\ (A_2) = 1_N \not\subseteq 0_N = NSFr\ (\ NSFr\ (A_2))$.

Theorem 3.34 Let X_i, $i = 1, 2, \ldots, n$ be a family of neutrosophic product related *NTSs*. If each A_i is a *NS* in X_i, then $NSFr\ (\ \prod_{i=1}^{n} A_i\ ) = [\ NSFr\ (A_1) \times NSCl\ (A_2) \times \cdots \times NSCl\ (A_n)\ ] \cup [\ NSCl\ (A_1) \times NSFr\ (A_2) \times NSCl\ (A_3) \times \cdots \times NSCl\ (A_n)\ ] \cup \cdots \cup [\ NSCl\ (A_1) \times NSCl\ (A_2) \times \cdots \times NSFr\ (A_n)\ ]$.

Proof : It suffices to prove this for $n = 2$. Let A_i be the *NS* in the neutrosophic topological space X_i.

Then by Definition 3.1,

$NSFr\ (A_1 \times A_2) = NSCl\ (A_1 \times A_2) \cap NSCl\ (C\ (A_1 \times A_2))$

By Proposition 6.2 (i) [12] ,

$= NSCl\ (A_1 \times A_2) \cap C\ (\ NSInt\ (A_1 \times A_2)\ )$

By Theorem 6.9 (i) and (ii) [12] ,

$= (NSCl(A_1) \times NSCl(A_2)) \cap C(NSInt(A_1) \times NSInt\ (A_2))$

$= (\ NSCl\ (A_1) \times NSCl\ (A_2)\ ) \cap C\ [\ (\ NSInt\ (A_1) \cap NSCl\ (A_1)\ ) \times (\ NSInt\ (A_2) \cap NSCl\ (A_2)\ )\ ]$

By Lemma 2.3 (iii) [12] ,

$= (\ NSCl\ (A_1) \times NSCl\ (A_2)\ ) \cap [\ C\ (\ NSInt\ (A_1) \cap NSCl\ (A_1)\ ) \times 1_N \cup 1_N \times C(\ NSInt\ (A_2) \cap NSCl\ (A_2)\ )\ ]$

$=(NSCl(A_1) \times NSCl(A_2)) \cap [(NSCl(C(A_1)) \cup NSInt(C(A_1)\ )) \times 1_N \cup 1_N \times (NSCl\ (C\ (A_2)) \cup NSInt\ (C\ (A_2))\ )\ ]$

$= (\ NSCl\ (A_1) \times NSCl\ (A_2)\ ] \cap [\ (\ NSCl\ (C\ (A_1)) \times 1_N\ ) \cup (\ 1_N \times NSCl\ (C\ (A_2))\ )\ ]$

$= [(\ NSCl\ (A_1) \times NSCl\ (A_2)\ ) \cap (NSCl\ (C\ (A_1)) \times 1_N\ )] \cup [(NSCl\ (A_1) \times NSCl\ (A_2)\ ) \cap (1_N \times NSCl\ (C\ (A_2))\ )\ ]$

By Theorem 2.32,

$= [\ (NSCl\ (A_1) \cap NSCl\ (C\ (A_1))\ ) \times (\ 1_N \cap NSCl\ (A_2))] \cup [(NSCl\ (A_1) \cap 1_N\ ) \times (NSCl\ (A_2) \cap NSCl\ (C\ (A_2)))]$

$= (NSFr\ (A_1) \times NSCl\ (A_2)) \cup (NSCl\ (A_1) \times NSFr\ (A_2))$

Hence $NSFr\ (A_1 \times A_2) = (\ NSFr\ (A_1) \times NSCl\ (A_2)\ ) \cup (\ NSCl\ (A_1) \times NSFr\ (A_2)\ )$.

CONCLUSION

In this paper, we studied the concepts of frontier and semi-frontier in neutrosophic topological spaces. In future, we plan to extend this neutrosophic topology concepts by neutrosophic continuous, neutrosophic semi-continuous, neutrosophic almost continuous and neutrosophic weakly continuous in neutrosophic topological spaces, and also to expand this neutrosophic concepts by nets, filters and borders.

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University of New Mexico

On Some New Notions and Functions in Neutrosophic Topological Spaces

[1] I. Arokiarani, [2] R. Dhavaseelan, [3]S. Jafari, [4]M. Parimala

[1]Department of Mathematics, Nirmala College for women,Coimbatore, Tamil Nadu, India. E-mail :stell11960@yahoo.co.in

[2] Department of Mathematics, Sona College of Technology, Salem-636005,Tamil Nadu,India. E-mail: dhavaseelan.r@gmail.com

[3] Department of Mathematics, College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark. E-mail: jafaripersia@gmail.com

[4] Department of Mathematics, Bannari Amman Institute of Technology,Sathyamangalam-638401 Tamil Nadu, India. E-mail: rishwanthpari@gmail.com

Abstract: In this paper, we define the notion of neutrosophic semiopen (resp. preopen and α-open) functions and investigate relation among them. We give a characterization of neutrosophic α-open set, and provide conditions for a neutrosophic set to be a neutrosophic α-open set. We discuss characterizations of neutrosophic pre-continuous (resp. α-continuous) functions. We give a condition for a function of neutrosophic topological spaces to be a neutrosophic α-continuous function.

Keywords:neutrosophic α-open set;neutrosophic semiopen; neutrosophic preopen; neutrosophic pre-continuous ; neutrosophic α-continuous.

1 Introduction and Preliminaries

After the advent of the notion of fuzzy set by Zadeh[11], C. L. Chang [4] introduced the notion of fuzzy topological space and many researchers converted, among others, general topological notions in the context of fuzzy topology. The notion of intuitionistic fuzzy set introduced by Atanassov [1, 2, 3] is one of the generalizations of the notion of fuzzy set. Later, Coker [5] by using the notion of the intuitionistic fuzzy set, offered the useful notion of intuitionistic fuzzy topological space. Joung Kon Jeon et al.[7] introduced and studied the notions of intuitionistic fuzzy α-continuity and pre-continuity which we will investigate in the context of neutrosophic topology. After the introduction of the concepts of neutrosophy and neutrosophic set by F. Smarandache [[9], [10]], the concepts of neutrosophic crisp set and neutrosophic crisp topological spaces were introduced by A. A. Salama and S. A. Alblowi[8].

In this paper, we define the notion of neutrosophic semiopen (resp. preopen and α-open) functions and investigate relation among them. We give a characterization of neutrosophic α-open set, and provide conditions for which a neutrosophic set is neutrosophic α-open. We discuss characterizations of neutrosophic precontinuous (resp. α-continuous) functions.

Definition 1.1. [6] A neutrosophic topology (NT) on a nonempty set X is a family T of neutrosophic sets in X satisfying the following axioms:

(i) $0_N, 1_N \in T$,

(ii) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$,

(iii) $\cup G_i \in T$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq T$.

In this case the ordered pair (X, T) or simply X is called a neutrosophic topological space (briefly NTS) and each neutrosophic set in T is called a neutrosophic open set (briefly NOS). The complement $\overline{A}$ of a NOS A in X is called a neutrosophic closed set (briefly NCS) in X. Each neutrosophic supra set (briefly NS) which belongs to (X, T) is called a neutrosophic supra open set (briefly NSOS) in X. The complement $\overline{A}$ of a NSOS A in X is called a neutrosophic supra closed set (briefly IFSCS) in X.

Definition 1.2. [6] Let A be a neutrosophic set in a neutrosophic topological space X. Then

$Nint(A) = \bigcup\{G \mid G$ is a neutrosophic open set in X and $G \subseteq A\}$ is called the neutrosophic interior of A;

$Ncl(A) = \bigcap\{G \mid G$ is a neutrosophic closed set in X and $G \supseteq A\}$ is called the neutrosophic closure of A.

Definition 1.3. [6] Let X be a nonempty set. If r, t, s be real standard or non standard subsets of $]0^-, 1^+[$ then the neutrosophic set $x_{r,t,s}$ is called a neutrosophic point(in short NP)in X given by

$$x_{r,t,s}(x_p) = \begin{cases} (r, t, s), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p \end{cases}$$

for $x_p \in X$ is called the support of $x_{r,t,s}$.where r denotes the degree of membership value ,t denotes the degree of indeterminacy and s is the degree of non-membership value of $x_{r,t,s}$.

2 Definitions

Definition 2.1. A neutrosophic set A in a neutrosophic topological space (X, T) is called

1) a neutrosophic semiopen set (briefly NSOS) if $A \subseteq Ncl(Nint(A))$.

2) a neutrosophic α-open set (briefly $N\alpha OS$) if $A \subseteq Nint(Ncl(Nint(A)))$.

3) a neutrosophic preopen set (briefly NPOS) if $A \subseteq Nint(Ncl(A))$.

4) a neutrosophic regular open set (briefly NROS) if $A = Nint(Ncl(A))$.

5) a neutrosophic semipreopen or β-open set (briefly $N\beta OS$) if $A \subseteq Ncl(Nint(Ncl(A)))$.

A neutrosophic set A is called neutrosophic semiclosed (resp. neutrosophic α-closed, neutrosophic preclosed, neutrosophic regular closed and neutrosophic β-closed) (briefly NSCS, NαCS, NPCS, NRCS and NβCS) if the complement of A is a neutrosophic semiopen (resp. neutrosophic α-open, neutrosophic preopen, neutrosophic regular open and neutrosophic β-open).

Example 2.1. Let $X = \{a, b, c\}$. Define the neutrosophic sets A, B, C, D and E in X as follows:
$A = \langle x, (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.5}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.5}), (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.5})\rangle$,
$B = \langle x, (\frac{a}{0.5}, \frac{b}{0.55}, \frac{c}{0.5}), (\frac{a}{0.5}, \frac{b}{0.55}, \frac{c}{0.5}), (\frac{a}{0.5}, \frac{b}{0.55}, \frac{c}{0.5})\rangle$,
$C = \langle x, (\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.5}), (\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.5}), (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.5})\rangle$. Then $T = \{0_N, 1_N, A, B, C\}$ is neutrosophic topology on X. Thus, (X, T) is neutrosophic topological space. Observe that $D = \langle x, (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.5}), (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.5}), (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.5})\rangle$ is both semiopen and α-open in (X, T) and $E = \langle x, (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.4}), (\frac{a}{0.5}, \frac{b}{0.4}, \frac{c}{0.4}), (\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{0.6})\rangle$ is both preopen and β-open in (X, T).

Proposition 2.1. Let (X, T) be a neutrosophic topological space. If A is a neutrosophic α-open set then it is a neutrosophic semiopen set.

Proposition 2.2. Let (X, T) be a neutrosophic topological space. If A is a neutrosophic α-open set then it is a neutrosophic preopen set.

Proposition 2.3. Let A be a neutrosophic set in a neutrosophic topological spaces (X, T). If B is a neutrosophic semiopen set such that $B \subseteq A \subseteq Nint(Ncl(B))$, then A is a neutrosophic α-open set.

Proof. Since B is a neutrosophic semiopen set, we have $B \subseteq Ncl(Nint(B))$. Thus, $A \subseteq Nint(Ncl(B)) \subseteq Nint(Ncl(Ncl(Nint(B)))) = Nint(Ncl(Nint(B))) \subseteq Nint(Ncl(Nint(A)))$, and so A is a neutrosophic α-open set. □

Lemma 2.1. Any union of NS α-open sets (resp. neutrosophic preopen sets) is a NS α-open sets (resp., neutrosophic preopen sets).

The Proof is straightforward.

Proposition 2.4. A neutrosophic set A in a neutrosophic topological space X is neutrosophic α-open (resp. neutrosophic preopen) iff for every neutrosophic point $x_{r,t,s} \in A$, there exists a neutrosophic α-open set (resp. neutrosophic preopen set) $B_{x_{r,t,s}}$ such that $x_{r,t,s} \in B_{x_{r,t,s}} \subseteq A$.

Proof. If A is a neutrosophic α-open set (resp. neutrosophic preopen set), then we may take $B_{x_{r,t,s}} = A$ for every $x_{r,t,s} \in A$. Conversely assume that for every neutrosophic point $x_{r,t,s} \in A$, there exists a neutrosophic α-open set (resp., neutrosophic preopen set), $B_{x_{r,t,s}}$ such that $x_{r,t,s} \in B_{x_{r,t,s}} \subseteq A$. Then, $A = \cup\{x_{r,t,s} | x_{r,t,s} \in A\} \subseteq \cup\{B_{x_{r,t,s}} | x_{r,t,s} \in A\} \subseteq A$, and so $A = \cup\{B_{x_{r,t,s}} | x_{r,t,s} \in A\}$, which is a neutrosophic α-open set (resp. neutrosophic preopen set) by Lemma 2.1. □

Definition 2.2. Let f be a function from a neutrosophic topological spaces (X, T) and (Y, S). Then f is called

(i) a neutrosophic open function if $f(A)$ is a neutrosophic open set in Y for every neutrosophic open set A in X.

(ii) a neutrosophic α-open function if $f(A)$ is a neutrosophic α-open set in Y for every neutrosophic open set A in X.

(iii) a neutrosophic preopen function if $f(A)$ is a neutrosophic preopen set in Y for every neutrosophic open set A in X.

(iv) a neutrosophic semiopen function if $f(A)$ is a neutrosophic semiopen set in Y for every neutrosophic open set A in X.

Proposition 2.5. Let $(X, T), (Y, S)$ and (Z, R) be three neutrosophic topological spaces, let $f : (X, T) \to (Y, S)$ and $g : (Y, S) \to (Z, R)$ be functions. If f is neutrosophic open and g is neutrosophic α-open(resp., neutrosophic preopen), then $g \circ f$ is neutrosophic α-open(resp. neutrosophic preopen).

Proof. The Proof is straightforward. □

Proposition 2.6. Let (X, T) and (Y, S) are neutrosophic topological spaces. If $f : (X, T) \to (Y, S)$ is neutrosophic α-open then it is neutrosophic semiopen.

Proof. Assume that f is neutrosophic α-open and let A be a neutrosophic open set in X. Then, $f(A)$ is a neutrosophic α-open set in Y. It follows from Proposition 2.1 that $f(A)$ is a neutrosophic semiopen set so that f is a neutrosophic semiopen function. □

Proposition 2.7. Let (X, T) and (Y, S) are neutrosophic topological spaces. If $f : (X, T) \to (Y, S)$ is neutrosophic α-open then it is neutrosophic preopen.

3 Neutrosophic Continuity

Definition 3.1. Let f be a function from a neutrosophic topological space (X, T) to a neutrosophic topological space (Y, S). Then f is called a neutrosophic pre-continuous function if $f^{-1}(B)$ is a neutrosophic preopen set in X for every neutrosophic open set B in Y.

Proposition 3.1. For a function f from a neutrosophic topological spaces (X, T) to an (Y, S), the following are equivalent.

(i) f is neutrosophic pre-continuous.

(ii) $f^{-1}(B)$ is a neutrosophic preclosed set in X for every neutrosophic closed set B in Y.

(iii) $Ncl(Nint(f^{-1}(A))) \subseteq f^{-1}(Ncl(A))$ for every neutrosophic set A in Y.

Proof. $(i) \Rightarrow (ii)$ The Proof is straightforward.
$(ii) \Rightarrow (iii)$ Let A be a neutrosophic set in Y. Then $Ncl(A)$ is neutrosophic closed. It follows from (ii) that $f^{-1}(Ncl(A))$ is a neutrosophic preclosed set in X so that $Ncl(Nint(f^{-1}(A))) \subseteq Ncl(Nint(f^{-1}(Ncl(A)))) \subseteq f^{-1}(Ncl(A))$.
$(iii) \Rightarrow (i)$ Let A be a neutrosophic open set in Y. Then $\overline{A}$ is a neutrosophic closed set in Y, and so $Ncl(Nint(f^{-1}(\overline{A}))) \subseteq f^{-1}(Ncl(\overline{A})) = f^{-1}(\overline{A})$. This implies that $\overline{Nint(Ncl(f^{-1}(A)))} = Ncl(\overline{Ncl(f^{-1}(A))}) = Ncl(Nint(\overline{f^{-1}(A)})) = Ncl(Nint(f^{-1}(\overline{A}))) \subseteq f^{-1}(\overline{A}) = \overline{f^{-1}(A)}$, and thus $f^{-1}(A) \subseteq Nint(Ncl(f^{-1}(A)))$. Hence $f^{-1}(A)$ is a neutrosophic preopen set in X, and f is neutrosophic precontinuous. □

Definition 3.2. Let $x_{r,t,s}$ be a neutrosophic point of a neutrosophic topological space (X, T). A neutrosophic set A of X is called neutrosophic neighbourhood of $x_{r,t,s}$ if there exists a neutrosophic open set B in X such that $x_{r,t,s} \in B \subseteq A$.

Proposition 3.2. Let f be a function from a neutrosophic topological space (X, T) to a neutrosophic topological space (Y, S). Then the following assertions are equivalent.

(i) f is a neutrosophic pre-continuous function.

(ii) For each neutrosophic point $x_{r,t,s} \in X$ and every neutrosophic neighbourhood A of $f(x_{r,t,s})$, there exists a neutrosophic preopen set B in X such that $x_{r,t,s} \in B \subseteq f^{-1}(A)$.

(iii) For each neutrosophic point $x_{r,t,s} \in X$ and every neutrosophic neighbourhood A of $f(x_{r,t,s})$, there exists a neutrosophic preopen set B in X such that $x_{r,t,s} \in B$ and $f(B) \subseteq A$

Proof. $(i) \Rightarrow (ii)$ Let $x_{r,t,s}$ be a neutrosophic point in X and let A be a neutrosophic neighbourhood of $f(x_{r,t,s})$. Then there exists a neutrosophic open set B in Y such that $f(x_{r,t,s}) \in B \subseteq A$. Since f is a neutrosophic pre-continuous function, we know that $f^{-1}(B)$ is a neutrosophic preopen set in X and $x_{r,t,s} \in f^{-1}(f(x_{r,t,s})) \subseteq f^{-1}(B) \subseteq f^{-1}(A)$. Consequently (ii) is valid.
$(ii) \Rightarrow (iii)$ Let $x_{r,t,s}$ be a neutrosophic point in X and let A be a neutrosophic neighbourhood of $f(x_{r,t,s})$. The condition (ii) implies that there exists a neutrosophic preopen set B in X such that $x_{r,t,s} \in B \subseteq f^{-1}(A)$ so that $x_{r,t,s} \in B$ and $f(B) \subseteq f(f^{-1}(A)) \subseteq A$. Hence (iii) is true.
$(iii) \Rightarrow (i)$ Let B be a neutrosophic open set in Y and let $x_{r,t,s} \in f^{-1}(B)$. Then $f(x_{r,t,s}) \in B$, and so B is a neutrosophic neighbourhood of $f(x_{r,t,s})$ since B is neutrosophic open set. It follows from (iii) that there exists a neutrosophic preopen set A in X such that $x_{r,t,s} \in A$ and $f(A) \subseteq B$ so that $x_{r,t,s} \in A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(B)$. Applying Propostion 2.4 induces that $f^{-1}(B)$ is a neutrosophic preopen set in X. Therefore, f is a neutrosophic pre-continuous function. □

Definition 3.3. Let f be a function from a neutrosophic topological space (X, T) to a neutrosophic topological space (Y, S). Then f is called a neutrosophic α-continuous function if $f^{-1}(B)$ is a neutrosophic α-open set in X for every neutrosophic open set B in Y.

Proposition 3.3. Let f be a function from a neutrosophic topological space (X, T) to a neutrosophic topological space (Y, S) that satisfies $Ncl(Nint(Ncl(f^{-1}(B)))) \subseteq f^{-1}(Ncl(B))$ for every neutrosophic set B in Y. Then f is a neutrosophic α-continuous function.

Proof. Let B be an neutrosophic open set in Y. Then $\overline{B}$ is a neutrosophic closed set in Y, which implies that from hypothesis that $Ncl(Nint(Ncl(f^{-1}(\overline{B})))) \subseteq f^{-1}(Ncl(\overline{B})) = f^{-1}(\overline{B})$. It follows that

$$\begin{aligned}\overline{Nint(Ncl(Nint(f^{-1}(B))))} &= Ncl(\overline{Ncl(Nint(f^{-1}(B)))}) \\ &= Ncl(Nint(\overline{Nint(f^{-1}(B))})) \\ &= Ncl(Nint(Ncl(\overline{f^{-1}(B)}))) \\ &= Ncl(Nint(Ncl(f^{-1}(\overline{B})))) \subseteq f^{-1}(\overline{B}) \\ &= \overline{f^{-1}(B)}\end{aligned}$$

so that $f^{-1}(B) \subseteq Nint(Ncl(Nint(f^{-1}(B))))$. This shows that $f^{-1}(B)$ is a neutrosophic α-open set in X. Hence, f is a neutrosophic α-continuous function. □

Proposition 3.4. Let f be a function from a neutrosophic topological space (X, T) to a neutrosophic topological space (Y, S). Then the following assertions are equivalent.

(i) f is neutrosophic α-continuous.

(ii) For each neutrosophic point $x_{r,t,s} \in X$ and every neutrosophic neighbourhood A of $f(x_{r,t,s})$, there exists a neutrosophic α-open set B in X such that $x_{r,t,s} \in B \subseteq f^{-1}(A)$.

(iii) For each neutrosophic point $x_{r,t,s} \in X$ and every neutrosophic neighbourhood A of $f(x_{r,t,s})$, there exists a neutrosophic α-open set B in X such that $x_{r,t,s} \in B$ and $f(B) \subseteq A$

Proof. $(i) \Rightarrow (ii)$ Let $x_{r,t,s}$ be a neutrosophic point in X and let A be a neutrosophic neighbourhood of $f(x_{r,t,s})$. Then there exists a neutrosophic open set B in Y such that $f(x_{r,t,s}) \in B \subseteq A$. Since f is neutrosophic α-continuous, we know that $f^{-1}(B)$ is a neutrosophic α-open set in X and $x_{r,t,s} \in f^{-1}(f(x_{r,t,s})) \subseteq f^{-1}(B) \subseteq f^{-1}(A)$. Consequently (ii) is valid.
$(ii) \Rightarrow (iii)$ Let $x_{r,t,s}$ be a neutrosophic point in X and let A be a neutrosophic neighbourhood of $f(x_{r,t,s})$. The condition (ii) implies that there exists a neutrosophic α-open set B

in X such that $x_{r,t,s} \in B \subseteq f^{-1}(A)$ so that $x_{r,t,s} \in B$ and $f(B) \subseteq f(f^{-1}(A)) \subseteq A$. Hence (iii) is true.
$(iii) \Rightarrow (i)$ Let B be a neutrosophic open set in Y and let $x_{r,t,s} \in f^{-1}(B)$. Then $f(x_{r,t,s}) \in B$, and so B is a neutrosophic neighbourhood of $f(x_{r,t,s})$ since B is neutrosophic open set. It follows from (iii) that there exists a neutrosophic α-open set A in X such that $x_{r,t,s} \in A$ and $f(A) \subseteq B$ so that $x_{r,t,s} \in A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(B)$. Applying Proposition 2.4 induces that $f^{-1}(B)$ is a neutrosophic α-open set in X. Therefore, f is a neutrosophic α-continuous function. □

Proposition 3.5. Let f be a function from a neutrosophic topological space (X, T) to a neutrosophic topological space (Y, S). If f is neutrosophic α-continuous, then it is neutrosophic semi-continuous.

Proof. Let B be a neutrosophic open set in Y. Since f is neutrosophic α-continuous, $f^{-1}(B)$ is a neutrosophic semiopen set in X. It follows from Proposition 2.1 that $f^{-1}(B)$ is a neutrosophic semiopen set in X so that f is a neutrosophic semi-continuous function. □

Proposition 3.6. Let f be a function from a neutrosophic topological space (X, T) to a neutrosophic topological space (Y, S). If f is neutrosophic α-continuous, then it is neutrosophic pre-continuous.

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Neutrosophic Baire Spaces

[1] R. Dhavaseelan, [2]S. Jafari ,[3]R. Narmada Devi, [4]Md. Hanif Page

[1] Department of Mathematics, Sona College of Technology, Salem-636005, Tamil Nadu, India. E-mail: dhavaseelan.r@gmail.com

[2] Department of Mathematics, College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark. E-mail: jafaripersia@gmail.com

[3]Department of Mathematics, Lady Doak College, Madurai, Tamil Nadu, India. E-mail :narmadadevi23@gmail.com

[4] Department of Mathematics, KLE Technological University, Hubli-31, Karnataka, India. E-mail: hanif01@yahoo.com

Abstract: In this paper, we introduce the concept of neutrosophic Baire space and present some of its characterizations.

Keywords:neutrosophic first category ; neutrosophic second category; neutrosophic residual set; neutrosophic Baire space.

1 Introduction and Preliminaries

The fuzzy idea has invaded all branches of science as far back as the presentation of fuzzy sets by L. A. Zadeh [17]. The important concept of fuzzy topological space was offered by C. L. Chang [6] and from that point forward different ideas in topology have been reached out to fuzzy topological space. The concept of "intuitionistic fuzzy set" was first presented by Atanassov [1]. He and his associates studied this useful concept [2, 3, 4]. Afterward, this idea was generalized to "intuitionistic L - fuzzy sets" by Atanassov and Stoeva [5]. The idea of somewhat fuzzy continuous functions and somewhat fuzzy open hereditarily irresolvable were introduced and investigated by by G. Thangaraj and G. Balasubramanian in [15]. The idea of intuitionistic fuzzy nowhere dense set in intuitionistic fuzzy topological space presented and studied by by Dhavaseelan and et al. in [16]. The concepts of neutrosophy and neutrosophic set were introduced by F. Smarandache [[13], [14]]. Afterwards, the works of Smarandache inspired A. A. Salama and S. A. Alblowi[12] to introduce and study the concepts of neutrosophic crisp set and neutrosophic crisp topological spaces. The Basic definitions and Proposition related to neutrosophic topological spaces was introduced and discussed by Dhavaseelan et al. [9]. In this paper the concepts of neutrosophic Baire spaces are introduced and characterizations of neutrosophic baire spaces are studied.

Definition 1.1. [13, 14] Let T,I,F be real standard or non standard subsets of $]0^-, 1^+[$, with $sup_T = t_{sup}, inf_T = t_{inf}$
$sup_I = i_{sup}, inf_I = i_{inf}$
$sup_F = f_{sup}, inf_F = f_{inf}$
$n - sup = t_{sup} + i_{sup} + f_{sup}$
$n - inf = t_{inf} + i_{inf} + f_{inf}$. T,I,F are neutrosophic components.

Definition 1.2. [13, 14] Let X be a nonempty fixed set. A neutrosophic set [briefly NS] A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represents the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set A.

Remark 1.1. [13, 14]

(1) A neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ can be identified to an ordered triple $\langle \mu_A, \sigma_A, \gamma_A\rangle$ in $]0^-, 1^+[$ on X.

(2) For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \sigma_A, \gamma_A\rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.3. [13, 14] Let X be a nonempty set and the neutrosophic sets A and B in the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;

(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(c) $\bar{A} = \{\langle x, \gamma_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$; [Complement of A]

(d) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$;

(e) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$;

(f) $[\,]A = \{\langle x, \mu_A(x), \sigma_A(x), 1 - \mu_A(x)\rangle : x \in X\}$;

(g) $\langle\rangle A = \{\langle x, 1 - \gamma_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.4. [13, 14] Let $\{A_i : i \in J\}$ be an arbitrary family of neutrosophic sets in X. Then

(a) $\bigcap A_i = \{\langle x, \wedge\mu_{A_i}(x), \wedge\sigma_{A_i}(x), \vee\gamma_{A_i}(x)\rangle : x \in X\}$;

(b) $\bigcup A_i = \{\langle x, \vee\mu_{A_i}(x), \vee\sigma_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle : x \in X\}$.

Since our main purpose is to construct the tools for developing neutrosophic topological spaces, we introduce the neutrosophic sets 0_N and 1_N in X as follows:

Definition 1.5. [13, 14] $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$.

Definition 1.6. [9] A neutrosophic topology (NT) on a nonempty set X is a family T of neutrosophic sets in X satisfying the following axioms:

(i) $0_N, 1_N \in T$,

(ii) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$,

(iii) $\cup G_i \in T$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq T$.

In this case the ordered pair (X, T) or simply X is called a neutrosophic topological space (briefly NTS) and each neutrosophic set in T is called a neutrosophic open set (briefly NOS). The complement $\overline{A}$ of a NOS A in X is called a neutrosophic closed set (briefly NCS) in X.

Definition 1.7. [9] Let A be a neutrosophic set in a neutrosophic topological space X. Then

$Nint(A) = \bigcup\{G \mid G$ is a neutrosophic open set in X and $G \subseteq A\}$ is called the neutrosophic interior of A;

$Ncl(A) = \bigcap\{G \mid G$ is a neutrosophic closed set in X and $G \supseteq A\}$ is called the neutrosophic closure of A.

Definition 1.8. [9] Let X be a nonempty set. If r, t, s be real standard or non standard subsets of $]0^-, 1^+[$ then the neutrosophic set $x_{r,t,s}$ is called a neutrosophic point(in short NP)in X given by

$$x_{r,t,s}(x_p) = \begin{cases} (r, t, s), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p \end{cases}$$

for $x_p \in X$ is called the support of $x_{r,t,s}$.where r denotes the degree of membership value, t denotes the degree of indeterminacy and s is the degree of non-membership value of $x_{r,t,s}$.

Definition 1.9. [11] A neutrosophic set A in neutrosophic topological space (X, T) is called neutrosophic dense if there exists no neutrosophic closed set B in (X, T) such that $A \subset B \subset 1_N$

Proposition 1.1. [11] If A is a neutrosophic nowhere dense set in (X, T), then $\overline{A}$ is a neutrosophic dense set in (X, T) .

Proposition 1.2. [11] Let A be a neutrosophic set. If A is a neutrosophic closed set in (X, T) with $Nint(A) = 0_N$, then A is a neutrosophic nowhere dense set in (X, T).

2 Neutrosophic Baire Spaces

Definition 2.1. Let (X, T) be a neutrosophic topological space. A neutrosophic set A in (X, T) is called neutrosophic first category if $A = \bigcup_{i=1}^{\infty} B_i$, where B_i's are neutrosophic nowhere dense sets in (X, T). Any other neutrosophic set in (X, T) is said to be of neutrosophic second category.

Definition 2.2. A neutrosophic topological space (X, T) is called neutrosophic first category space if the neutrosophic set 1_N is a neutrosophic first category set in (X, T).That is, $1_N = \bigcup_{i=1}^{\infty} A_i$ where A_i's are neutrosophic nowhere dense sets in (X, T). Otherwise (X, T) will be called a neutrosophic second category space.

Proposition 2.1. If A be a neutrosophic first category set in (X, T),then $\overline{A} = \bigcap_{i=1}^{\infty} B_i$ where $Ncl(B_i) = 1_N$.

Proof. Let A be a neutrosophic first category set in (X, T). Then $A = \bigcup_{i=1}^{\infty} A_i$, where A_i's are neutrosophic nowhere dense sets in (X, T). Now $\overline{A} = \overline{\bigcup_{i=1}^{\infty} A_i} = \bigcap_{i=1}^{\infty}(\overline{A_i})$. Now A_i is a neutrosophic nowhere dense set in (X, T). Then, by Proposition 1.1, we have $\overline{A_i}$ is a neutrosophic dense set in (X, T). Let us put $B_i = \overline{A_i}$. Then $\overline{A} = \bigcap_{i=1}^{\infty} B_i$ where $Ncl(B_i) = 1_N$. □

Definition 2.3. Let A be a neutrosophic first category set in (X, T). Then $\overline{A}$ is called a neutrosophic residual set in (X, T).

Definition 2.4. Let (X, T) be a neutrosophic topological space. Then (X, T) is said to neutrosophic Baire space if $Nint(\bigcup_{i=1}^{\infty} A_i) = 0_N$, where A_i's are neutrosophic nowhere dense sets in (X, T).

Example 2.1. Let $X = \{a, b, c\}$. Define the neutrosophic sets A, B, C and D as follows :
$A = \langle x, (\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.5}), (\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.5}), (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.5})\rangle$,
$B = \langle x, (\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.6}), (\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.6}), (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.3})\rangle$,
$C = \langle x, (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.4}), (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.4}), (\frac{a}{0.7}, \frac{b}{0.7}, \frac{c}{0.4})\rangle$ and
$D = \langle x, (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.3}), (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.3}), (\frac{a}{0.7}, \frac{b}{0.7}, \frac{c}{0.7})\rangle$. Then the family $T = \{0_N, 1_N, A\}$ is a neutrosophic topologies on X. Thus, (X, T) is a neutrosophic topological spaces. Now $\overline{A}, \overline{B}, C$ and D are neutrosophic nowhere dense sets in (X, T). Also $Nint(\overline{A} \cup \overline{B} \cup C \cup D) = 0_N$. Hence (X, T) is a neutrosophic Baire space.

Proposition 2.2. If $Nint(\bigcup_{i=1}^{\infty} A_i) = 0_N$ where $Nint(A_i) = 0_N$ and $A_i \in T$, then (X, T) is a neutrosophic Baire space.

Proof. Now $A_i \in T$ implies that A_i is a neutrosophic open set in (X, T). Since $Nint(A_i) = 0_N$. By Proposition 1.2, A_i is a neutrosophic nowhere dense set in (X, T). Therefore $Nint(\bigcup_{i=1}^{\infty} A_i) = 0_N$. where A_i's are neutrosophic nowhere dense set in (X, T). Hence (X, T) is a neutrosophic Baire space. □

Proposition 2.3. If $Ncl(\bigcap_{i=1}^{\infty} A_i) = 1_N$ where A_i's are neutrosophic dense and neutrosophic open sets in (X, T), then (X, T) is a neutrosophic Baire Space.

Proof. Now $Ncl(\bigcap_{i=1}^{\infty} A_i) = 1_N$ implies that $\overline{Ncl(\bigcap_{i=1}^{\infty} A_i)} = 0_N$. Then we have $Nint(\overline{\bigcap_{i=1}^{\infty} A_i}) = 0_N$. Which implies that $Nint(\bigcup_{i=1}^{\infty} \overline{A_i}) = 0_N$. Let $B_i = \overline{A_i}$. Then $Nint(\bigcup_{i=1}^{\infty} B_i) =$

0_N. Now $A_i \in T$ implies that $\overline{A_i}$ is a neutrosophic closed set in (X,T) and hence B_i is a neutrosophic closed and $Nint(B_i) = Nint(\overline{A_i}) = \overline{Ncl(A_i)} = 0_N$. Hence By Proposition 1.2, B_i is a neutrosophic nowhere dense set in (X,T). Hence $Nint(\bigcup_{i=1}^{\infty} B_i) = 0_N$ where B_i's are neutrosophic nowhere dense sets, implies that (X,T) is a neutrosophic Baire space. □

Proposition 2.4. Let (X,T) be a neutrosophic topological space. Then the following are equivalent

(i) (X,T) is a neutrosophic Baire space.

(ii) $Nint(A) = 0_N$, for every neutrosophic first category set A in (X,T).

(iii) $Ncl(B) = 1_N$, for every neutrosophic residual set B in (X,T).

Proof. $(i) \Rightarrow (ii)$ Let A be a neutrosophic first category set in (X,T). Then $A = (\bigcup_{i=1}^{\infty} A_i)$ where A_i's are neutrosophic nowhere dense sets in (X,T). Now $Nint(A) = Nint(\bigcup_{i=1}^{\infty} A_i) = 0_N$. Since (X,T) is a neutrosophic Baire space. Therefore $Nint(A) = 0_N$.

$(ii) \Rightarrow (iii)$ Let B be a neutrosophic residual set in (X,T). Then $\overline{B}$ is a neutrosophic first category set in (X,T). By hypothesis $Nint(\overline{B}) = 0_N$ which implies that $\overline{Ncl(A)} = 0_N$. Hence $Ncl(A) = 1_N$.

$(iii) \Rightarrow (i)$ Let A be a neutrosophic first category set in (X,T). Then $A = (\bigcup_{i=1}^{\infty} A_i)$ where A_i's are neutrosophic nowhere dense sets in (X,T). Now A is a neutrosophic first category set implies that $\overline{A}$ is a neutrosophic residual set in (X,T). By hypothesis, we have $Ncl(\overline{A}) = 1_N$, which implies that $\overline{Nint(A)} = 1_N$. Hence $Nint(A) = 0_N$. That is, $Nint(\bigcup_{i=1}^{\infty} A_i) = 0_N$, where A_i's are neutrosophic nowhere dense sets in (X,T). Hence (X,T) is a neutrosophic Baire space. □

Proposition 2.5. A neutrosophic topological space (X,T) is a neutrosophic Baire space if and only if $(\bigcup_{i=1}^{\infty} A_i) = 1_N$, where A_i's is a neutrosophic closed set in (X,T) with $Nint(A_i) = 0_N$, implies that $Nint(\bigcup_{i=1}^{\infty} A_i) = 0_N$.

Proof. Let (X,T) be a neutrosophic Baire space. Now A_i is a neutrosophic closed in (X,T) and $Nint(A_i) = 0_N$, implies that A_i is a neutrosophic nowhere dense set in (X,T). Now $\bigcup_{i=1}^{\infty} A_i = 1_N$ implies that 1_N is a neutrosophic first category set in (X,T). Since (X,T) is a neutrosophic Baire space space, by Proposition 2.4, $Nint(1_N) = 0_N$. That is, $Nint(\bigcup_{i=1}^{\infty} A_i) = 0_N$.

Conversely suppose that $Nint(\bigcup_{i=1}^{\infty} A_i) = 0_N$ where A_i. By Proposition 1.2, A_i is a neutrosophic nowhere dense set in (X,T). Hence $Nint(\bigcup_{i=1}^{\infty} A_i) = 0_N$ implies that (X,T) is a neutrosophic Baire space. □

Definition 2.5. Let (X,T) and (Y,S) be any two neutrosophic topological spaces. A map $f : (X,T) \to (Y,S)$ is said to be a neutrosophic open if the image of every neutrosophic open set A in (X,T) is neutrosophic open $f(A)$ in (Y,S).

Definition 2.6. [10] Let (X,T) and (Y,S) be any two neutrosophic topological spaces. A map $f : (X,T) \to (Y,S)$ is called neutrosophic contra continuous if the inverse image of every neutrosophic open set in (Y,S) is neutrosophic closed in (X,T).

Proposition 2.6. Let (X,T) and (Y,S) be any two neutrosophic topological spaces. If $f : (X,T) \to (Y,S)$ is an onto neutrosophic contra continuous and neutrosophic open then (Y,S) is a neutrosophic Baire space.

Proof. Let A be a neutrosophic first category set in (Y,S). Then $A = (\bigcup_{i=1}^{\infty} A_i)$ where A_i are neutrosophic nowhere dense sets in (Y,S). Suppose $Nint(A) \neq 0_N$. Then there exists a neutrosophic open set $B \neq 0_N$ in (Y,S), such that $B \subseteq A$. Then $f^{-1}(B) \subseteq f^{-1}(A) = f^{-1}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$. Hence

$$f^{-1}(B) \subseteq \bigcup_{i=1}^{\infty} f^{-1}(Ncl(A_i)). \tag{2.1}$$

Since f is neutrosophic contra continuous and $Ncl(A_i)$ is a neutrosophic closed set in (Y,S), $f^{-1}(Ncl(A_i))$ is a neutrosophic open in (X,T). From (2.1) we have

$$f^{-1}(B) \subseteq \bigcup_{i=1}^{\infty} f^{-1}(Ncl(A_i)) = \bigcup_{i=1}^{\infty} Nint(f^{-1}(Ncl(A_i))). \tag{2.2}$$

Since f is intuitionsitic fuzzy open and onto, $Nint(f^{-1}(A_i)) \subseteq f^{-1}(Nint(A_i))$. From 2.2, we have $f^{-1}(B) \subseteq \bigcup_{i=1}^{\infty} f^{-1}(NintNcl(A_i)) \subseteq \bigcup_{i=1}^{\infty} f^{-1}(0_N) = 0_N$. Since A_i is a neutrosophic nowhere dense. That is, $f^{-1}(B) \subseteq 0_N$ and hence $f^{-1}(B) = 0_N$ which implies that $B = 0_N$, which is a contradiction to $B \neq 0_N$. Hence $Nint(A) = 0_N$ where A is a neutrosophic first category set in (Y,S). Hence by Proposition 2.4, (Y,S) is a neutrosophic Baire space.

□

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University of New Mexico

A Knowledge-based Recommendation Framework using SVN Numbers

Roddy Cabezas Padilla[1], José González Ruiz[2], Milton Villegas Alava[3], Maikel Leyva Vázquez[4]

[1]Universidad de Guayaquil, Facultad de Ciencias Administrativas, Guayaquil Ecuador. E-mail: roddy.cabezasp@ug.edu.ec
[2]Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. E-mail: jose.gonzalezr@ug.edu.ec
[3]Universidad de Guayaquil, Facultad de Ciencias Administrativas, Guayaquil Ecuador. E-mail: milton.villegasa@ug.edu.ec
[4]Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. E-mail: mleyvaz@gmail.com

Abstract:

Current knowledge based recommender systems, despite proven useful and having a high impact, persist with some shortcomings. Among its limitations are the lack of more flexible models and the inclusion of indeterminacy of the factors involved for computing a global similarity. In this paper, a new knowledge based recommendation models based SVN number is presented. It includes database construction, client profiling, products filtering and generation of recommendation. Its implementation makes possible to improve reliability and include indeterminacy in product and user profile. An illustrative example is shown to demonstrate the model applicability.

Keywords: recommendation systems, neutrosophy, SVN numbers.

1 Introduction

Recommendation systems are useful in decision making process providing the user with a group of options that meet expectations [1]. Based on the information and the algorithms used to generate the recommendations,, various techniques can be distinguish [2, 3]:

Knowledge Based Recommender Systems use the knowledge about users' necessities to infer recommendations not requiring a great amount of data like another approaches [4]. They use cased based reasoning techniques frequently. In this paper, a new framework for including neutrosophic in knowledge based recommender system is presented.

This paper is structured as follows: Section 2 reviews some important preliminary concepts about Single valued neutrosophic numbers (SVN number). In Section 3, is presented a knowledge based recommendation model framework based on SVN numbers. Section 4 shows a case study of the proposed model. The paper ends with conclusions and further work recommendations.

2.2 SVN-numbers

Neutrosophy [5] is a mathematical theory developed for dealing with indeterminacy. Neutrosophy has been the base for developing new methods to handle indeterminate and inconsistent information like neutrosophic sets an neutrosophic logic [6, 7] .

The truth value in neutrosophic set is as follows [8]:

Definition 1. Let N be a set defined as: $N = \{(T, I, F) : T, I, F \subseteq [0, 1]\}$, a neutrosophic valuation n is a mapping from the set of propositional formulas to N , that is for each sentence p we have v (p) $= (T, I, F)$.

Single valued neutrosophic set (SVNS) [9] were developed with the goal of facilitate the real world applications of neutrosophic set and set-theoretic operators.

A single valued neutrosophic set (SVNS) has been defined as follows [9]:

Definition 2. Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form:

$$A = \{\langle x, uA(x), rA(x), vA(x)\rangle : x \in X\} \qquad (1)$$

where $u_A(x): X \to [0,1]$, $r_A(x) : X \to [0,1]$ and $v_A(x): X \to [0,1]$ with $0 \le u_A(x) + r_A(x) + v_A(x) :\le 3$ for all $x \in X$. The intervals $u_A(x)$, $r_A(x)$ y $v_A(x)$ denote the truth- membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.

Single valued neutrosophic numbers (SVN number) is denoted by $A= (a, \text{b}, c)$, where $a, b, c \in [0,1]$ and $a+b+c \le 3$.

Euclidean distance in SVN is defined as follows[12, 13]:

Definition 3. Let$A^* = (A_1^*, A_2^*, \ldots, A_n^*)$ be a vector of n SVN numbers such that $A_j^* = (a_j^*, b_j^*, c_j^*)$ j=(1,2, … , n) and $B_i = (B_{i1}, B_{i2}, \ldots, B_{im})$ (i = 1,2, … , m) be m vectors of n SVN numbers such that $B_{ij} = (a_{ij}, b_{ij}, c_{ij})$ (i = 1,2,

… , m), (j = 1,2, … , n). Then the separation measure between $B_i's$ y A^* is defined as follows:

$$s_I=\left(\frac{1}{3}\sum_{j=1}^{n}\left\{\left(\left|a_{ij}-a_j^*\right|\right)^2+\left(\left|b_{ij}-b_j^*\right|\right)^2+\left(\left|c_{ij}-c_j^*\right|\right)^2\right\}\right)^{\frac{1}{2}} \quad (2)$$

(i = 1,2, … , m)

In this paper linguistic variables[14] are represented using single valued neutrosophic numbers [13] for developing knowledge based recommender system.

3 Proposed framework

The proposed framework is presented in Figure 1. It is based mainly on the proposal made by Cordon [15] for recommendation systems based on content/knowledge adapted to SVN numbers.

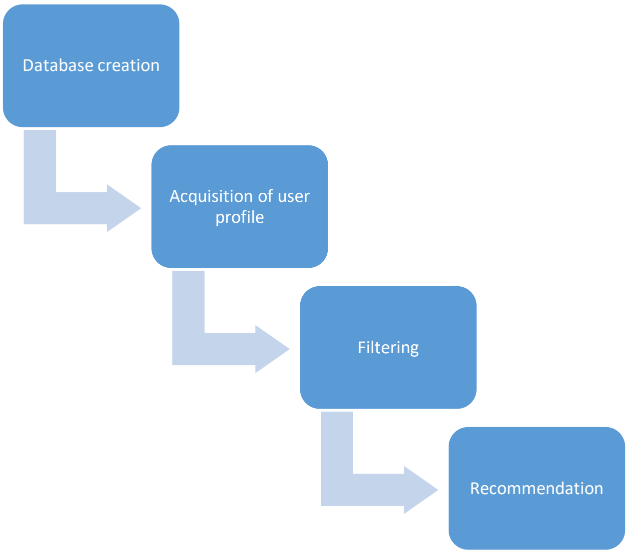

Figure. 1 Proposed framework

3.1 Database creation

A key for a recommendation model is the creation of the database. Each product a_i will be described by a set of characteristics that make up the profile:

$$C = \{c_1, \dots, c_k, \dots, c_l\} \quad (3)$$

Each product will be described by a vector of features:

$$F_{a_j} = \{v_1^j, \dots, v_k^j, \dots v_l^j\}, j = 1, \dots n \quad (4)$$

There are techniques for generating these profiles automatically or semi-automatically for recommendation systems [15]. In this case, an expert or group of experts is suggested.

Profiles of product a_j, is expressed using the linguistic scale expressed S, $v_k^j \in S$ where $S = \{s_1, \dots, s_g\}$ is the linguistic term set for evaluating the characteristic c_k using SVN.

Having described the products:

$$A = \{a_1, \dots, a_j, \dots, a_n\} \quad (5)$$

Then, are stored in a database.

3.2 Acquisition of the user profile

The proposed framework presents a fundamental difference with previous proposals, it is focused in the fact that most of this information is collected using SVN numbers this information is stored in the database.

$$P_e = \{p_1^e, \dots, p_k^e, \dots, p_l^e\} \quad (6)$$

This profile will be composed of a set of attributes:

$$C^e = \{c_1^e, \dots, c_k^e, \dots, c_l^e\} \quad (7)$$

3.3 Filtering

In this activity, products according to the similarity with the user profile are filtered to find out which are the most appropriate for the student.

The similarity between user profile, P_e, product a_j is calculated. For the calculation of the overall similarity

The similarity measure can be obtained from a distance measurement, if $d(x, y)\epsilon[0, max]$ then[16] :

$$sim\left(p_k^e, v_k^j\right) = 1 - \frac{d(dp_k^e, v_k^j)}{max} \quad (8)$$

In this case similarity is calculated as follows:

$$S_i = 1 - \left(\left(\frac{1}{3}\sum_{j=1}^{n}\left\{\left(\left|a_{ij}-a_j^*\right|\right)^2+\left(\left|b_{ij}-b_j^*\right|\right)^2+\left(\left|c_{ij}-c_j^*\right|\right)^2\right\}\right)^{\frac{1}{2}}\right) \quad (9)$$

Where function S calculate similarity among user profile and products profiles [17].

3.4 Recommending

In this activity, a set of products that match with the user profiles is suggested. After calculating the similarity products are ordered and represented with the following similarity vector:

$$S = (s_1, \dots, s_n) \quad (10)$$

The best is that best meet the needs of the user profile (greater similarity).

4 Case study

To show the applicability of the model, a case study is developed.

Initially a database of products is created:

$$A = \{a_1, a_2, a_3, a_4, a_5\}$$

described with the following attributes:

$$C = \{c_1, c_2, c_3, c_{4,}, c_5\}$$

Attributes are evaluated in the linguistic scale show in Table 1 and stored in the database.

Linguistic terms	SVNSs
Extremely good (EG)	(1,0,0)
Very very good (VVG)	(0.9, 0.1, 0.1)
Very good (VG)	(0.8,0,15,0.20)
Good (G)	(0.70,0.25,0.30)
Medium good (MG)	(0.60,0.35,0.40)
Medium (M)	(0.50,0.50,0.50)
Medium bad (MB)	(0.40,0.65,0.60)
Bad (B)	(0.30,0.75,0.70)
Very bad (VB)	(0.20,0.85,0.80)
Very very bad (VVB)	(0.10,0.90,0.90)
Extremely bad (EB)	(0,1,1)

Table 1. Linguistic terms used to provide the assessments [13].

Database used in this example is shown in Table 2.

	c_1	c_2	c_3	c_4
a_1	MDB	M	MMB	B
a_2	B	MD	MB	M
a_3	MMB	M	M	B
a_4	M	B	MMB	B

Table 2: Products database.

If user u_e, wish to receive recommendation expressing his/her preferences in this case:

$$P_e = \{\text{MDB}, \text{MB}, \text{MMB}, \text{MB}\}$$

The next step in this case is the calculation of similarity between user profile and products profiles stored in database.

a_1	a_2	a_3	a_4
0.44	0.76	0.42	0.84

Table 3: Similarity calculation

A ranking of products based on similarity calculation is:

$$\{a_4, a_2, a_1, a_3\}$$

In case that the recommendation of two products was needed it is as follows:

$$a_4, a_2$$

This example shows the applicability of the proposal

5 Conclusions

In this paper, a product recommendation model was presented following the knowledge-based approach. It is based on the use of SVN numbers to express linguistic terms.

Future work will be related to the creation of the database from multiple experts, as well as obtaining the weights of the characteristics using group evaluations. In addition, we will work on the integration of more complex aggregation models, as well as hybridization with other models of recommendation.

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NSS
University of New Mexico

An Improved Framework for Diagnosing Confusable Diseases Using Neutrosophic Based Neural Network

Okpako Abugor Ejaita
University of Port Harcourt,
Choba Rivers State, Nigeria.
okpako.ejaita@gmail.com

Asagba P.O.
University of Port Harcourt,
Choba Rivers State, Nigeria.
lopito2013@yahoo.com

ABSTRACT

The two major motivations in medical science are to prevent and diagnose diseases. Diagnosis of disease must be done with care since it is the first stage of therapeutic actions towards eventual management of the disease; a mistake at this stage is disastruous, and such, adequate care must be ensured. Diagnosis becomes difficult in medical domain due to influence of medical uncertainties that arises from confus-ability in disease symptomatic presentation between two diseases. This confusability of these diseases stems from the overlaps in the disease symptomatic presentation and has led to misdiagnosis with various degrees of associated costs and in worst cases led to death. In this research, we present the analysis of the existing systems and finally present a framework for the diagnosis of confusable disease using neutrosophic-based neural network.

General Terms

Clinical Decision Support System, Medical Diagnosis, Machine Learning, Soft Computing.

Keywords

Decision Support System, Medical Uncertainties, Neutrosophic Logic, Confusable Diseases

1. INTRODUCTION

Decision making in medical science is unique and is quite different from other science disciplines since it is a known fact that scientists tend to look for typical, normal phenomena while medical sciences look out for the atypical, abnormal, morbid phenomena. Medical decision making is a collaborative process between Physicians, Patients, and lab technologists typically through exchange of information that would ultimately guide the physician to make appropriate and proper therapeutic recommendations. There is an exponential amount of data generated daily in the medical domain thereby opening doors for all forms of uncertainties such as incompleteness of information, inconsistent description of disease symptoms, overlapping diseases symptoms, just to mention a few and has led to difficulties in properly diagnosing diseases in such situations. Medical uncertainty is an inherent phenomenon in medical science; it is what fuels medical research, prompts patients to seek medical attention and stimulate medical intervention notwithstanding, it poses challenges in diagnostic decision making. In recent times, the negative effect of medical uncertainties has attracted attention due to the emerging realities of this period in medical sciences where evidence based, shared decision making and patient-centered care has brought to fore the limitation of scientific knowledge. The effect of uncertainties in the medical domain has been acknowledged by researchers since the 1950's when the sociologist Renee Fox conducted a seminal studies documenting how physician struggle with uncertainty during their trainings. Brause (2001) highlighted that almost all the physicians are confronted during their formative years by the task of learning to diagnose. Central to good diagnosis, is the ability of an experienced physician to know what symptoms or vitals to throw away and what to keep in the diagnostic process.

The ability of the physician(s) to thoroughly scan through the series of laboratory tests and symptoms of a patient which are time varying as the case may be and pick out meaningful and useful information that 'stand-out', for proper identification of a disease (amongst several diseases which would sometimes share common symptom) makes a good physician. It is not overly out of place to say that perception plays a central role amidst skills and experiences garnered by an expert physician during his or her education pursuit, in order to perform a near accurate or accurate diagnosis of a disease. Sisson et al (2007), opined that medical diagnosis is both science and arts where the art is what separate between two well-trained medical personnel thus is very necessary to talk of it if we are aiming at developing an application that would sieve through data and provide semantically relevant information amidst the wide range of uncertainties in a manner that simulate a human expert physician.

A pertinent question would be "how computers have helped in medical diagnosis?" and "how can we improve on the existing systems". Computers have been employed widely in the medical sector in recent time, from local and global patient and medicine databases to emergency networks, or as digital archives. Meanwhile, in the case of medical diagnosis, due to the complexity of the task, it has not been realistic to expect a fully automatic, computer-based, medical diagnosis system. However, recent advances in the field of intelligent systems are materializing into a wider usage of computers, armed with Artificial Intelligence (AI) techniques. It is therefore imperative to have a decision support to assist in the diagnostic decision making. A decision support system in this context is a computer based information system that supports medical staff in diagnostic decision making. A properly designed medical decision support systems is interactive software whose intent is to help medical practitioners to semantically sieve through a deluge of raw data in order to identify and solve medical problems.

In the purview of computing, decision making in medical diagnosis is all about problem solving strategies which is done by taking potential candidate solutions from the possibilities of various solutions. But often times one is faced with the problem of how to choose from the abundant alternatives that have confusing or conflicting symptoms. If physician's premises are wrong, then the final decision is also wrong which ultimately leads to cases of misdiagnosis whose cost is obvious. It is pertinent to note that we can successfully select the numbers of features that would optimally help in the diagnostic process but as to what values this features can have, which areas needs further probing cannot be empirically ascertained. Medical uncertainties come in different flavors and shapes, but its impact which comes along the lines of class overlap or confusable symptoms is of interest to us. It has continually affected the diagnostic decision of diseases which have ultimately led to performance degradation amidst the supposedly high percentage of accuracy of some re-known classifiers mostly when considered in relation to practical implementation in medical domain. The complexity of the management of low prevalent diseases in the midst of high prevalent ones is to a larger extent attributed to the fact that other diseases have signs and symptoms that are similar to those presented by patients of low prevalent ones. For example Typhoid which is highly prevalent in the Niger Delta region of Nigeria and Hepatitis disease which is low prevalent have some common symptoms and sometimes could be very confusing to novice practitioners and patients in rural areas to diagnose correctly and as such as in most cases would overly conclude it for Typhoid. It should also be noted that in medical decision making, different types of misclassifications or misdiagnosis have different costs. For example, in Hepatitis diagnosis, a false positive decision translates into an unnecessary biomarkers test or liver biopsy which is associated with both emotional, financial cost and other inherent complications. False negative decision on the other hand, however, means a missed Hepatitis-positive which in turn can be deadly.

Medical diagnosis must therefore take into consideration issues of uncertainty and class imbalance which comes either in form of confusability or overlaps ,incomplete information, vagueness, inconsistency or indeterminacy, disease prevalence in order to make a reliable decision towards the prediction and eventual treatment of a disease. Neutrosophic logic is a new logic which is an extended and general framework to measure the truth, indeterminacy, and falsehoodness of the information and as such suitable for handling issues of uncertainties thus giving fair estimate about the reliability of information. This research work proposes a framework that uses the tripartite membership power of Neutrosophic logic and combining it with the conventional Neural Networks in order to estimate a confusability measurement for two confusable diseases resulting from class overlap in lieu of providing an innovative approach that might be useful to support decisions about medical diagnoses for confusable diseases.

2.0 RELATED LITERATURE

Evans and Gadd [3], describe four different levels into which clinical knowledge is organized in a medical problem solving context. They stated that **Observations** are units of information that are recognized as potentially relevant in a problem solving context, however they do not constitute clinically useful facts. **Findings** are observations that have potential clinical significance (e.g. symptoms). **Facet**s are clusters of findings that are suggestive of pre-diagnostic interpretations while **clinical diagnosis** is the level of classification that encompasses and explains all levels beneath it. The model is hierarchical with facets and diagnoses serving to establish a context in which observations and findings are interpreted, and to provide a basis for anticipating and searching for confirming or discriminating findings.

Oguntimelehin et al [17] opined that medical diagnosis is simply the task of categorization which allows physician to make predictions using clinical situations and to determine appropriate cause of action. They said it is a complex decision process that involves a lot of vagueness and uncertainty management especially when the disease has multiple symptoms. Diagnosis has been seen generally as the identification of the nature and cause of a certain phenomenon. Several disciplines make use of it but we are only considering it in the parlance of medical science and to put it in more simplistic form, it is the answer to the question of whether a system(in this case human body) is malfunctioning or not, and to the process of computing the answer. Expert diagnosis would not be trivialized in this regard, which is majorly based on experience with the system. Using this experience, a mapping is built that efficiently associates the observations to the corresponding diagnoses.

2.1: Medical Uncertainties

Mishel[13] defined uncertainty in illness as the inability to determine the meaning of illness-related events. McCormick [11] opines that uncertainty is a component of all illness experiences and it is believed to affect psychosocial adaptation and outcomes of disease and as such high levels of uncertainty are related to high emotional distress, anxiety and depression. Peter Szolovits [19] opines that "Uncertainty is the central, critical fact about medical reasoning. Patients cannot describe exactly what has happened to them or how they feel, doctors and nurses cannot tell exactly what they observe, laboratories report results only with some degree of error, physiologists do not understand precisely how the human body works, medical researchers cannot precisely characterize how diseases alter the normal functioning of the body, pharmacologists do not fully understand the mechanisms accounting for the effectiveness of drugs, and no one can precisely determine one's prognosis". Paul et al(2011) opine that irrespective of the visible negative effect of uncertainty in various domain and most importantly to the medical domain, there is limited comprehensible way of addressing the problems it poses in relation to layperson, physicians and patients and health policy makers. According to Smithson [26] this knowledge gaps reflect limitations in empirical evidence; however, a more fundamental problem is the absence of a shared concept of uncertainty, and a lack of integration of insights from different disciplines. Uncertainty is not a monolithic phenomenon and such in considering it, the varied meanings and synonyms should also be considered. Bammer et al [1] opined that there are multiple varieties of uncertainty, which may have distinct psychological effects and thus warrant different courses of action, thus there is, need to have an organized conceptual framework that categorizes these multiple varieties of uncertainty in a coherent, useful way.

2.2: Confusable Diseases

This research work pointed out the serious effect of uncertainty, yet how it affect medical diagnosis needs to be elucidated. When two or more diseases have some overlapping symptoms which make it naturally difficult for a physician to establish the right diagnosis, it is referred to as confusable diseases in medical parlance. Fries et al.[5] opined that in order to diagnose confusable diseases properly, a diagnostic criterion for a particular disease is needed so as not to confuse it with other diseases because of shared symptoms. Joop [8] opined that for a diagnosis to be effective in this regard, the target disease has to be recognized in a pool of confusable diseases and suggested two ways to handle this: by recognition of the combination of symptoms of the target disease or by exclusion of confusable disease as the cause of the symptoms .

Confusable disease is poised with the following problems outline herewith.

a. Confusable disease manifests the same symptoms thereby leading to imprecise or incomplete diagnosis by the physician.
b. A disease at one stage can manifest similar symptoms with a different disease at another stage.
c. Failure to correctly diagnose a confusable disease would lead to a physician giving the wrong treatment to the patient.
d. Patients may be suffering from more than one confusable disease.

2.3 Clinical Decision and Support Systems

In literature, many researchers have given their definitions of Clinical Decision Support Systems (CDDS). Musen [15] defined a CDSS as any piece of software that takes information about a clinical situation as inputs and that produces inferences as outputs that can assist practitioners in their decision making and that would be judged as "intelligent" by the program's users. Miller and Geissbuhler [12] defined a CDSS as a computer-based algorithm that assists a clinician with one or more component steps of the diagnostic process. Sim et al [22] defined CDSS as a software that is designed to be a direct aid to clinical decision-making, in which the characteristics of an individual patient are matched to a computerized clinical knowledge base and patient specific assessments or recommendations are then presented to the clinician or the patient for a decision. In more recent studies, researchers have been trying to classify CDSSs in the literature so as to provide a holistic picture of CDSSs. For example, Berlin et al [2] did research on CDSS taxonomy to describe the technical, workflow, and contextual characteristics of CDSSs, and the research results are very useful for researchers to have a comprehensive understanding of various designs and functions of CDSSs.

A general model of all clinical and decision support system is shown in Fig 2.1. the interaction is simple: A patient clinical signs and symptoms or lab tests is fed into the system having the inference mechanism component which in turn in consultation with the knowledge base proffer a diagnostic and therapeutic recommendation to the doctor who in turn advise the patient accordingly.

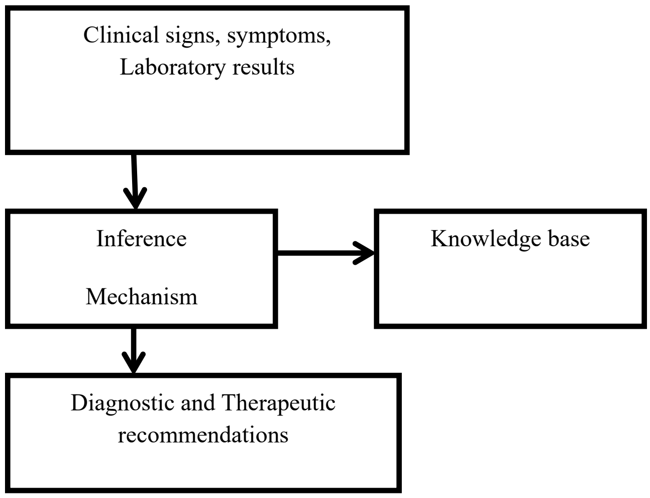

Fig. 2.1: A general model of CDSSs (Source: Lincoln 1999, Reggia 1983)

2.4: Neutrosophic Logic

Neutrosophic Logic represents an alternative to the existing logics as a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction. It is a non-classical logic. It is a logic in which each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F, where T, I, F are defined above, is called Neutrosophic Logic.

A neutrosophic set *A* in *X* is characterized by a truth membership function *TA*, a indeterminacy- membership function *IA* and a falsity-membership function *FA*. *TA(x), IA(x)* and *FA(x)* are real standard or non-standard subsets of]-0, 1+ [.

That is

TA: X ε]-0, 1+ [

IA: X ε]-0, 1+ [

FA: X ε]-0, 1+ [

There is no restriction on the sum of *TA(x), IA(x)* and *FA(x),* so

-0≤supTA(x) + sup IA(x) + sup FA(x) ≤ 3+……………………………………………..2.1

2.5: Conditional Probabilities

In medical diagnosis, there are many variables that contribute to the diagnostic process of arriving at a particular disease with varied values of the variables which ultimately in most cases leads to some forgivable errors. As good as this may sound, there is a level of tolerable errors that would be associated with every instance of diagnosis of such disease but it is very unrealistic to quantify the errors for all instances of the disease owing to the fact that we would have just a handful of sample data (due to low prevalence) and as such there is going to be many evaluation of the decision variables. In order to accomplish this feat in less time and space, conditional probabilities become handy.

Conditional distributions are one of the key tools in probability theory for reasoning about uncertainty. They specify the distribution of a random variable when the value of another random variable is known (or more generally, when some event is known to be true).

Formally, conditional probability of *X = e given Y = d* is defined as

P(X = e|Y = d) =P(X = e, Y = d) /P(Y = b) …………………………………………………. .2.2

Note that this is not defined when the probability of *Y* = d is 0. The idea of conditional probability extends naturally to the case when the distribution of a random variable is conditioned on several variables.

As for notations, we write *P*(*X*|*Y* = *d*) to denote the distribution of random variable *X* when *Y* = *d*. We may also write *P*(*X*|*Y*) to denote a set of distributions of *X*, one for each of the different values that *Y* can take.

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3.0: Analysis of Existing Systems

Proper diagnoses and prevention is the major concerns in medical science, it is there imperative to have systems that assist in medical diagnosis with such an accuracy comparable to human physicians. Many existing system have employed different approaches in ameliorating the effect of uncertainties yet there is still room for improvement so as to handle the diagnosis of confusable diseases.

A detailed review and analysis of existing system was carried out in order to bring to fore areas to improve on, in order to tackle the embarrassing effect of confusable disease diagnosis. We reviewed the following:-

i. The approaches and methods used in the existing system in knowledge construction
ii. The inference mechanism in handling uncertainties
iii. Support for diagnostic criteria for reliability of prediction of disease in a two class of diseases diagnosis with confusable symptoms

3.1: Architecture of the Existing systems Using Neural Network

A typical architecture for diagnosis of disease used in existing system using an Artificial Neural Network is shown in Fig 3.1.

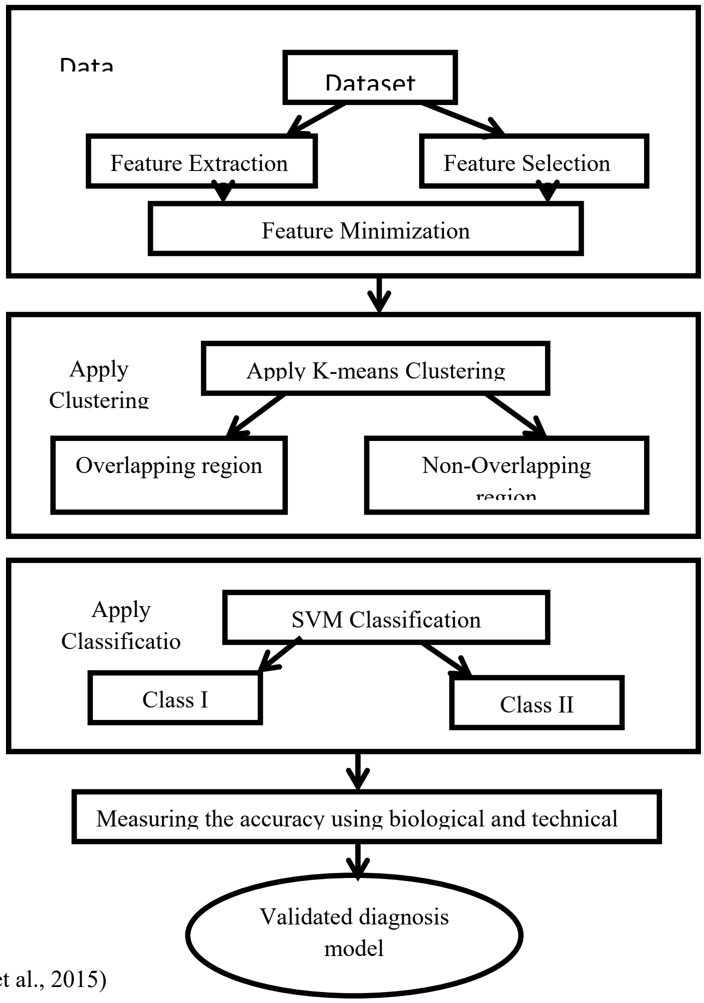

Fig. 3.1: Architecture of the Existing System (Source: Mohammed et al., 2015)

3.2 Limitations of the Existing System

The existing system has some limitations which prevent it from having a practically good performance as needed. The salient findings include

1. Though some of the existing system ensures multiple belongingness of a particular element to multiple classes with varied degree but capturing the neutralities due to class overlap or confusability which could degrade the prediction performance is missing.

2. The existing system is mute or unable to classify instances that falls under overlapping region and as such refers them for further medical probe. This clearly defeats timeliness and quality of service delivery we are seeking for in clinical diagnosis and as such not suitable to handle confusable diseases whose features are overlapped.

3. In diagnosing confusability in disease classes, some of the existing system used only unsupervised statistical approach such as k-means to separate the overlapping region from the non-overlapping region. K-means is very poor when it comes to data with serious overlapping; is unable to handle noisy data and outliers as well as not suitable for non-linear data sets. Supervised machine learning using neural network is more suitable for complex nature of biological systems and non-linear data sets.

4. There is no reliability or justification metric for the decision of the classification which serves as a diagnostic criterion that allows a disease to be definitely diagnosed or definitely excluded in cases of non-linear decision boundary cases.

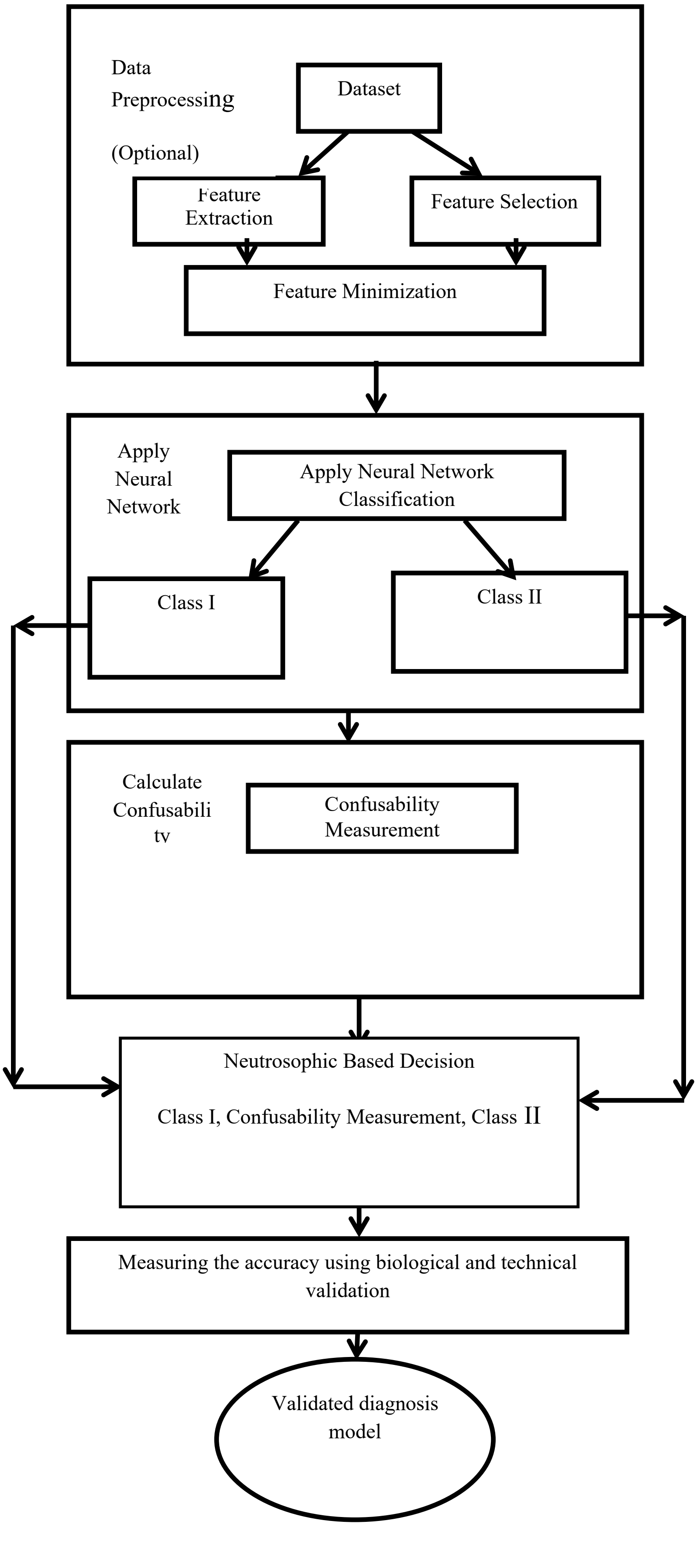

Fig. 3.2: Proposed System Architecture

4.1: Brief Description of the Components of the Proposed System

This section talks about the brief description of the various components in the proposed system architecture.

1. Patient Symptoms and Signs Subcomponent

Disease symptoms are the biological indicators which are associated with the clinical presentation of disease as learnt from medical literature and expert physicians. George et al (2000) opine that a symptom is a visible or even a measurable condition indicating the presence of a disease and thus can be regard as an aid towards diagnosis. It is based on this clinical presentation that a doctor or physician makes a tentative judgment about the state of the patient and consequently a test for confirmation.

2. Feature Selection Sub Component

It is important to note that the essence of feature selection in this research is to help reduce the dimension of a dataset of features potentially relevant with respect to the diagnosis of the diseases, finding the best minimum subset without transforming the data into a new set. The feature selection process points out all the input features relevant for the diagnosis of the diseases, and it is an indispensable data preprocessing step. The difficulty of extracting the most relevant variables is due mainly to the large dimension of the original feature set, the correlations between inputs which cause redundancy and finally the presence of variables which do not affect the diseases. In this research, we will employ feature selection using genetic algorithm for the feature searching techniques. The genetic algorithm was originally used to select binary string but it has been used been used in recent times to explore the inter-dependencies between the bits in the string, hence the choice of its usage. Singh et al (2016) have successfully used it for feature selection and its performance was superlative.

3. Confusability Measurement

There are two components that make up this component- Vagueness and multidimensional interpolation of the errors. The confusability measurement provides information on amount of uncertainty associated with such a classification that would have degraded the performance and is on this basis that final diagnosis is made. Confusability Measurement **is 1-|Tm(class I) –Tm(Class I|)|**, where Tm means the truth membership.

4. Inference Engine/Decision support Component

The decision made by the inference system through the neural work is optimized by this component by taking the result of the inference sub-component as input and with the aid of the result confusability measurement, a decision is ultimately made. The supporting components for the confusability measurement are multidimensional interpolation and vagueness calculated from the two networks which objectively influences the result of the proposed system thereby optimizing the practical implementation of the system in regards to sensitivity and specificity in an environment poised with class overlaps.

5.0: Conclusion

To make proper, reasonable and appropriate medical decision in the diagnosis of confusable diseases, the knowledge base and the inference mechanism play an indispensable role as they are the heart of clinical decision support systems. Once such clinical decision and support systems are built, we are faced in most times with a large feature set of symptoms which needs to be pruned to improve the performance of the system with regards to accuracy of classification. The key quality in this study is to achieve a better and proper diagnosis of confusable diseases. A genetic algorithm is applied in the feature selection phase. In quantifying the confusability, a multidimensional interpolation of error is plotted in the multidimensional feature space while the vagueness is calculated from the two class Neural Network as |1-(class A-class B)|, both vagueness and the errors form the confusability measurement. The inference mechanism is also improved by employing the concept of neutrosophic logic thereby having a tripartite membership (Degree of class A, Confusability Measurement, Degree of class B) rather than just two in order to make therapeutic recommendations. With these consideration, it is hope that there is going to be an obvious improvement in the system performance in terms of handling confusability in disease symptomatic presentations and eventually renders a proper diagnosis. Therefore, in this study, the architecture for diagnosing confusable disease was developed using the concept of neutrosophic logic in combination with neural network. This will be able to capture and quantify the confusability in this situation and ultimately being used in the decision making process.

6.0: Future Work

In the paper, analysis of the existing systems was carried out and some limitations were highlighted for consideration. The proposed architecture provides an interface where a patient's symptom is captured by the system, the confusability measure is calculated and in consultation with the knowledge base, the inference mechanism makes its therapeutic recommendation to the doctors who in turn advise the patient accordingly. Future work will delve into the implementation procedure of the framework for the diagnosis of confusable diseases using two confusable diseases and the result from the implementation and evaluation will be provided. The interface for the system based on patients' symptoms will also be presented.

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University of New Mexico

Compact Open Topology and Evaluation Map via Neutrosophic Sets

R. Dhavaseelan[1], S. Jafari[2], F. Smarandache[3]

[1] Department of Mathematics, Sona College of Technology, Salem-636005,Tamil Nadu,India. E-mail dhavaseelan.r@gmail.com
[2] Department of Mathematics, College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark. E-mail: jafaripersia@gmail.com
[3] Mathematics & Science Department, University of New Maxico, 705 Gurley Ave, Gallup, NM 87301, USA. E-mail: fsmarandache@gmail.com

Abstract: The concept of neutrosophic locally compact and neutrosophic compact open topology are introduced and some interesting propositions are discussed.

Keywords: neutrosophic locally Compact Hausdorff space; neutrosophic product topology; neutrosophic compact open topology; neutrosophic homeomorphism; neutrosophic evaluation map; Exponential map.

1 Introduction and Preliminaries

In 1965, Zadeh [19] introduced the useful notion of a fuzzy set and Chang [6] three years later offered the notion of fuzzy topological space. Since then, several authors have generalized numerous concepts of general topology to the fuzzy setting. The concept of intuitionistic fuzzy set was introduced and studied by Atanassov [1] and subsequently some important research papers published by him and his colleagues [2,3,4]. The concept of fuzzy compact open topology was introduced by S.Dang and A . Behera[9]. The concepts of intuitionistic evaluation maps by R.Dhavaseelan et al[9]. After the introduction of the concepts of neutrosophy and neutrosophic set by F. Smarandache [[11], [12]], the concepts of neutrosophic crisp set and neutrosophic crisp topological spaces were introduced by A. A. Salama and S. A. Alblowi[10].

In this paper the notion of neutrosophic compact open topology is introduced. Some interesting properties are discussed. Moreover, neutrosophic local compactness and neutrosophic product topology are developed. We have also utilized the notion of fuzzy locally compactness due to Wong[17], Christoph [8] and fuzzy product topology due to Wong [18].

Throughout this paper neutrosophic topological spaces (X, T),(Y, S) and (Z, R) will be replaced by X,Y and Z respectively.

Definition 1.1. Let T,I,F be real standard or non standard subsets of $]0^-, 1^+[$, with $sup_T = t_{sup}, inf_T = t_{inf}$
$sup_I = i_{sup}, inf_I = i_{inf}$
$sup_F = f_{sup}, inf_F = f_{inf}$
$n - sup = t_{sup} + i_{sup} + f_{sup}$
$n-inf = t_{inf}+i_{inf}+f_{inf}$. T,I,F are neutrosophic components.

Definition 1.2. Let X be a nonempty fixed set. A neutrosophic set [briefly NS] A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set A.

Remark 1.1. (1) A neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ can be identified to an ordered triple $\langle \mu_A, \sigma_A, \gamma_A\rangle$ in $]0^-, 1^+[$ on X.

(2) For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \sigma_A, \gamma_A\rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

We introduce the neutrosophic sets 0_N and 1_N in X as follows:

Definition 1.3. $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$.

Definition 1.4. [8] A neutrosophic topology (NT) on a nonempty set X consists of a family T of neutrosophic sets in X which satisfies the following:

(i) $0_N, 1_N \in T$,

(ii) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$,

(iii) $\cup G_i \in T$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq T$.

In this case the ordered pair (X, T) or simply X is called a neutrosophic topological space (NTS) and each neutrosophic set in T is called a neutrosophic open set (NOS). The complement $\overline{A}$ of a NOS A in X is called a neutrosophic closed set (NCS) in X.

Definition 1.5. [8] Let A be a neutrosophic subset of a neutrosophic topological space X. The neutrosophic interior and neutrosophic closure of A are denoted and defined by
$Nint(A) = \bigcup\{G \mid G$ is a neutrosophic open set in X and

$G \subseteq A\}$;

$Ncl(A) = \bigcap\{G \mid G$ is a neutrosophic closed set in X and $G \supseteq A\}$.

2 Neutrosophic Locally Compact and Neutrosophic Compact Open Topology

Definition 2 .1. Let X be a nonempty set and $x \in X$ a fixed element in X. If $r, t \in I_0 = (0, 1]$ and $s \in I_1 = [0, 1)$ are fixed real numbers such that $0 < r + t + s < 3$, then $x_{r,t,s} = \langle x, r, t, s\rangle$ is called a neutrosophic point (in short NP) in X, where r denotes the degree of membership of $x_{r,t,s}$, t denotes the degree of indeterminacy and s denotes the degree of nonmembership of $x_{r,t,s}$ and $x \in X$ the support of $x_{r,t,s}$.

The neutrosophic point $x_{r,t,s}$ is contained in the neutrosophic $A(x_{r,t,s} \in A)$ if and only if $r < \mu_A(x), t < \sigma_A(x), s > \gamma_A(x)$.

Definition 2.2. A neutrosophic set $A = \langle x, \mu_A, \sigma_A, \gamma_A\rangle$ in a neutrosophic topological space (X, T) is said to be a neutrosophic neighbourhood of a neotrosophic point $x_{r,t,s}, x \in X$, if there exists a neutrosophic open set $B = \langle x, \mu_B, \sigma_B, \gamma_B\rangle$ with $x_{r,t,s} \subseteq B \subseteq A$.

Definition 2.3. *Let X and Y be neutrosophic topological spaces.A mapping $f : X \to Y$ is said to be a neutrosophic homeomorphism if f is bijective, neutrosophic continuous and neutrosophic open.*

Definition 2.4. *An neutrosophic topological space (X, T) is called a neutrosophic Hausdorff space or T_2-space if for any pair of distinct neutrosophic points(i.e., neutrosophic points with distinct supports) $x_{r,t,s}$ and $y_{u,v,w}$,there exist neutrosophic open sets U and V such that $x_{r,t,s} \in U$,$y_{u,v,w} \in V$ and $U \wedge V = 0_N$*

Definition 2.5. *An neutrosophic topological space (X, T) is said to be neutrosophic locally compact if and only if for every neutrosophic point $x_{r,t,s}$ in X, there exists a neutrosophic open set $U \in T$ such that $x_{r,t,s} \in U$ and U is neutrosophic compact,i.e., each neutrosophic open cover of U has a finite subcover.*

Definition 2.6. *Let $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle$ and $B = \langle y, \mu_B(y), \sigma_B(y), \gamma_B(y)\rangle$ be neutrosophic sets of X and Y respectively.The product of two neutrosophic sets A and B in a neutrosophic topological space X is defined as*
$(A \times B)(x, y) = \langle (x, y), min(\mu_A(x), \mu_B(y)), min(\sigma_A(x), \sigma_B(y)), max(\gamma_A(x), \gamma_B(y))\rangle$ *for all* $(x, y) \in X \times Y$.

Definition 2.7. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$. The product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is defined by: $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))\ \forall (x_1, x_2) \in X_1 \times X_2$.*

Lemma 2.1. *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be functions and U, V are neutrosophic sets of Y_1, Y_2, respectively, then $(f_1 \times f_2)^{-1}(U \times V) = f_1^{-1}(U) \times f_2^{-1}(V)\ \forall\, U \times V \in Y_1 \times Y_2$*

Definition 2.8. *A mapping $f : X \to Y$ is neutrosophic continuous iff for each neutrosophic point $x_{r,t,s}$ in X and each neutrosophic neighbourhood B of $f(x_{r,t,s})$ in Y,there is a neutrosophic neighbourhood A of $x_{r,t,s}$ in X such that $f(A) \subseteq B$.*

Definition 2.9. *A mapping $f : X \to Y$ is said to be neutrosophic homeomorphism if f is bijective ,neutrosophic continuous and neutrosophic open.*

Definition 2.10. *A neutrosophic topological space X is called a neutrosophic Hausdorff space or T_2 space if for any distinct neutrosophic points $x_{r,t,s}$ and $y_{u,v,w}$,there exists neutrosophic open sets G_1 and G_2, such that $x_{r,t,s} \in G_1$,$y_{u,v,w} \in G_2$ and $G_1 \cap G_2 = 0_\sim$*

Definition 2.11. *A neutrosophic topological space X is said to be a neutrosophic locally compact iff for any neutrosophic point $x_{r,t,s}$ in X, there exists a neutrosophic open set $U \in T$ such that $x_{r,t,s} \in U$ and U is neutrosophic compact that is, each neutrosophic open cover of U has a finite subcover.*

Proposition 2.1. In a neutrosophic Hausdorff topological space X, the following conditions are equivalent.

(a) X is a neutrosophic locally compact

(b) for each neutrosophic point $x_{r,t,s}$ in X, there exists a neutrosophic open set G in X such that $x_{r,t,s} \in G$ and $Ncl(G)$ is neutrosophic compact

Proof. $(a) \Rightarrow (b)$ By hypothesis for each neutrosophic point $x_{r,t,s}$ in X, there exists a neutrosophic open set G which is neutrosophic compact.Since X is neutrosophic Hausdorff (neutrosophic compact subspace of neutrosophic Hausdorff space is neutrosophic closed), G is neutrosophic closed, thus $G = Ncl(G)$. Hence $x_{r,t,s} \in G$ and $Ncl(G)$ is neutrosophic compact.
$(b) \Rightarrow (a)$ Proof is simple. □

Proposition 2.2. Let X be a neutrosophic Hausdorff topological space.Then X is neutrosophic locally compact at a neutrosophic point $x_{r,t,s}$ in X iff for every neutrosophic open set G containing $x_{r,t,s}$ there exists a neutrosophic open set V such that $x_{r,t,s} \in V$, $Ncl(V)$ is neutrosophic compact and $Ncl(V) \subseteq G$.

Proof. Suppose that X is neutrosophic locally compact at a neutrosophic point $x_{r,t,s}$. By Definition 2.11, there exists a neutrosophic open set G such that $x_{r,t,s} \in G$ and G is neutrosophic compact. Since X is a neutrosophic Hausdorff space,(neutrosophic compact subspace of neutrosophic Hausdorff space is neutrosophic closed), G is neutrosophic closed. Thus $G = Ncl(G)$. Consider a neutrosophic point $x_{r,t,s} \in \overline{G}$. Since X is neutrosophic Hausdorff space, by Definition 2.10, there exist neutrosophic open sets C and D such that $x_{r,t,s} \in C$, $y_{u,v,w} \in D$ and $C \cap D = 0_\sim$. Let $V = C \cap G$. Hence $V \subseteq G$ implies $Ncl(V) \subseteq Ncl(G) = G$. Since $Ncl(V)$ is neutrosophic closed and G is neutrosophic compact, (every neutrosophic closed subset of a neutrosophic compact space is neutrosophic compact) it follows that $Ncl(V)$ is neutrosophic compact. Thus $x_{r,t,s} \in Ncl(V) \subseteq G$ and $Ncl(G)$ is neutrosophic compact.

The converse follows from Proposition 2.1(b). □

Definition 2.12. *Let X and Y be two neutrosophic topological spaces.The function $T : X \times Y \to Y \times X$ defined by $T(x, y) = (y, x)$ for each $(x, y) \in X \times Y$ is called a switching map.*

Proposition 2.3. The switching map $T : X \times Y \to Y \times X$ defined as above is neutrosophic continuous.

We now introduce the concept of a neutrosophic compact open topology in the set of all neutrosophic continuous functions from a neutrosophic topological space X to a neutrosophic topological space Y.

Definition 2.13. *Let X and Y be two neutrosophic topological spaces and let $Y^X = \{f : X \to Y$ such that f is neutrosophic continuous$\}$. We give this class Y^X a topology called the neutrosophic compact open topology as follows: Let $\mathcal{K} = \{K \in I^X :$ K is neutrosophic compact on $X\}$ and $\mathcal{V} = \{V \in I^Y : V$ is neutrosophic open in $Y\}$.For any $K \in \mathcal{K}$ and $V \in \mathcal{V}$,let $S_{K,V} = \{f \in Y^X : f(K) \subseteq V\}$.*

The collection of all such $\{S_{K,V} : K \in \mathcal{K}, V \in \mathcal{V}\}$ is a neutrosophic subbase to generate a neutrosophic topology on the class Y^X. The class Y^X with this topology is called a neutrosophic compact open topological space.

3 Neutrosophic Evaluation Map and Exponential Map

We now consider the neutrosophic product topological space $Y^X \times X$ and define a neutrosophic continuous map from $Y^X \times X$ into Y.

Definition 3.1. *The mapping $e : Y^X \times X \to Y$ defined by $e(f, x_{r,t,s}) = f(x_{r,t,s})$ for each neutrosophic point $x_{r,t,s} \in X$ and $f \in Y^X$ is called the neutrosophic evaluation map.*

Definition 3.2. *Let X,Y,Z be neutrosophic topological spaces and $f : Z \times X \to Y$ be any function. Then the induced map $\widehat{f} : X \to Y^Z$ is defined by $(\widehat{f}(x_{r,t,s}))(z_{t,u,v}) = f(z_{t,u,v}, x_{r,t,s})$ for neutrosophic point $x_{r,t,s} \in X$ and $z_{t,u,v} \in Z$.*
Conversely, given a function $\widehat{f} : X \to Y^Z$, a corresponding function f can also be defined by the same rule.

Proposition 3.1. Let X be a neutrosophic locally compact Hausdorff space. Then the neutrosophic evaluation map $e : Y^X \times X \to Y$ is neutrosophic continuous.

Proof. Consider $(f, x_{r,t,s}) \in Y^X \times X$,where $f \in Y^X$ and $x_{r,t,s} \in X$.Let V be a neutrosophic open set containing $f(x_{r,t,s}) = e(f, x_{r,t,s})$ in Y. Since X is neutrosophic locally compact and f is neutrosophic continuous, by Proposition 2.2, there exists a neutrosophic open set U in X such that $x_{r,t,s} \in Ncl(U)$ is neutrosophic compact and $f(Ncl(U)) \subseteq V$.

Consider the neutrosophic open set $S_{Ncl(U),V} \times U$ in $Y^X \times X$. Clearly $(f, x_{r,t,s}) \in S_{Ncl(U),V} \times U$.Let $(g, x_{t,u}) \in S_{Ncl(U),V} \times U$ be arbitrary. Thus $g(Ncl(U)) \subseteq V$. Since $x_{t,u} \in U$,we have $g(x_{t,u}) \in V$ and $e(g, x_{t,u}) = g(x_{t,u}) \in V$.Thus $e(S_{Ncl(U),V} \times U) \subseteq V$.Hence e is neutrosophic continuous. □

Proposition 3.2. Let X and Y be two neutrosophic topological spaces with Y being neutrosophic compact. Let $x_{r,t,s}$ be any neutrosophic point in X and N be a neutrosophic open set in the neutrosophic product space $X \times Y$ containing $\{x_{r,t,s}\} \times Y$. Then there exists some neutrosophic neighbourhood W of $x_{r,t,s}$ in X such that $\{x_{r,t,s}\} \times Y \subseteq W \times Y \subseteq N$.

Proposition 3.3. Let Z be a neutrosophic locally compact Hausdorff space and X, Y be arbitrary neutrosophic topological spaces. Then a map $f : Z \times X \to Y$ is neutrosophic continuous iff $\widehat{f} : X \to Y^Z$ is neutrosophic continuous,where $\widehat{f}$ is defined by the rule $(\widehat{f}(x_{r,t,s}))(z_{t,u,v}) = f(z_{t,u,v}, x_{r,t,s})$.

Proposition 3.4. Let X and Z be a neutrosophic locally compact Hausdorff spaces. Then for any neutrosophic topological space Y,the function $E : Y^{Z\times X} \to (Y^Z)^X$ defined by $E(f) = \widehat{f}$(that is $E(f)(x_{r,t,s})(z_{t,u,v}) = f(z_{t,u,v}, x_{r,t,s}) = (\widehat{f}(x_{r,t,s})(z_{t,u,v})))$ for all $f : Z \times X \to Y$ is a neutrosophic homeomorphism.

Proof. (a) Clearly E is onto.

(b) For E to be injective, let $E(f) = E(g)$ for $f, g : Z \times X \to Y$. Thus $\widehat{f} = \widehat{g}$, where $\widehat{f}$ and $\widehat{g}$ are the induced map of f and g, respectively. Now for any neutrosophic point $x_{r,t,s}$ in X and any neutrosophic point $z_{t,u,v}$ in Z, $f(z_{t,u,v}, x_{r,t,s}) = (\widehat{f}(x_{r,t,s})(z_{t,u,v})) = (\widehat{g}(x_{r,t,s})(z_{t,u,v})) = g(z_{t,u,v}, x_{r,t,s})$. Thus $f = g$.

(c) For proving the neutrosophic continuity of E, consider any neutrosophic subbasis neighbourhood V of $\widehat{f}$ in $(Y^Z)^X$, i.e V is of the form $S_{K,W}$ where K is a neutrosophic compact subset of X and W is neutrosophic open in Y^Z. Without loss of generality, we may assume that $W = S_{L,U}$, where L is a neutrosophic compact subset of Z and U is a neutrosophic open set in Y. Then $\widehat{f}(K) \subseteq S_{L,U} = W$ and this implies that $\widehat{f}(K)(L) \subseteq U$. Thus for any neutrosophic point $x_{r,t,s}$ in K and for every neutrosophic point $z_{t,u,v}$ in L, we have $(\widehat{f}(x_{r,t,s}))(z_{t,u,v}) \in U$, that is $f(z_{t,u,v}, x_{r,t,s}) \in U$ and therefore $f(L \times K) \subseteq U$. Now since L is a neutrosophic compact in Z and K is a neutrosophic compact in X, $L \times K$ is also a neutrosophic compact in $Z \times X$[7] and since U is a neutrosophic open set in Y, we conclude that $f \in S_{L\times K,U} \subseteq Y^{Z\times X}$. We assert that $E(S_{L\times K,U}) \subseteq S_{K,W}$. Let $g \in S_{L\times K,U}$ be arbitrary. Thus $g(L \times K) \subseteq U$, i.e $g(z_{t,u,v}, x_{r,t,s}) = (\widehat{g}(x_{r,t,s}))(z_{t,u,v}) \in U$ for all neutrosophic points $z_{t,u,v} \in L \subseteq Z$ and for every neutrosophic point $x_{r,t,s} \in L \subseteq X$. So $(\widehat{g}(x_{r,t,s}))(L) \subseteq U$ for every neutrosophic point $x_{r,t,s} \in K \subseteq X$, that is $(\widehat{g}(x_{r,t,s})) \in S_{L,U} = W$ for every neutrosophic points $x_{r,t,s} \in K \subseteq X$, that is $\widehat{g}(x_{r,t,s}) \in S_{L,U} = W$ for every neutrosophic point $x_{r,t,s} \in K \subseteq U$. Hence we have $\widehat{g}(K) \subseteq W$, that is $\widehat{g} = E(g) \in S_{K,W}$ for any $g \in S_{L\times K,U}$.

Thus $E(S_{L\times K,U}) \subseteq S_{K,W}$. This proves that E is a neutrosophic continuous.

(d) For proving the neutrosophic continuity of E^{-1},we consider the following neutrosophic evaluation maps: $e_1 : (Y^Z)^X \times X \to Y^Z$ defined by $e_1(\widehat{f}, x_{r,t,s}) = \widehat{f}(x_{r,t,s})$ where $\widehat{f} \in (Y^Z)^X$ and $x_{r,t,s}$ is any neutrosophic point in X and $e_2 : Y^Z \times Z \to Y$ defined by $e_2(g, z_{t,u,v}) = g(z_{t,u,v})$, where $g \in Y^Z$ and $z_{t,u,v}$ is a neutrosophic point in Z. Let ψ denote the composition of the following neutrosophic continuous functions $\psi : (Z \times X) \times (Y^Z)^X \xrightarrow{T} (Y^Z)^X \times (Z \times X) \xrightarrow{i\times t} (Y^Z)^X \times (X \times Z) \xrightarrow{=} ((Y^Z)^X \times X) \times Z \xrightarrow{e_1 \times i_Z} (Y^Z) \times Z \xrightarrow{e_2} Y$, where i, i_Z denote the neutrosophic identity maps on $(Y^Z)^X$ and Z, respectively and T, t denote the switching maps. Thus $\psi : (Z \times X) \times (Y^Z)^X \to Y$, that is $\psi \in Y^{(Z\times X)\times (Y^Z)^X}$. We consider the map $\widetilde{E} : Y^{(Z\times X)\times (Y^Z)^X} \to (Y^{(Z\times X)})^{(Y^Z)^X}$ (as defined in the statement of the Proposition 3.4 in fact it is E). So $\widetilde{E}(\psi) : (Y^Z)^X \to Y^{(Z\times X)}$. Now for any neutrosophic points $z_{t,u,v} \in Z, x_{r,t,s} \in X$ and $f \in Y^{(Z\times X)}$, again we have that $(\widetilde{E}(\psi) \circ E)(f)(z_{t,u,v}, x_{r,t,s}) = f(z_{t,u,v}, x_{r,t,s})$;hence $\widetilde{E}(\psi) \circ E$=identity. Similarly for any $\widehat{g} \in (Y^Z)^X$ and neutrosophic points $x_{r,t,s} \in X, z_{t,u,v} \in Z$, so we have that $(E \circ \widetilde{E}(\psi))(\widehat{g})(x_{r,t,s}, z_{t,u,v}) = (\widehat{g}(x_{r,t,s}))(z_{t,u,v})$;hence $E \circ \widetilde{E}(\psi)$=identity. Thus E is a neutrosophic homeomorphism.

□

Definition 3.3. *The map E in Proposition 3.4 is called the exponential map.*

As easy consequence of Proposition 3.4 is as follows.

Proposition 3.5. Let X, Y, Z be neutrosophic locally compact Hausdorff spaces. Then the map $N : Y^X \times Z^Y \to Z^X$ defined by $N(f, g) = g \circ f$ is neutrosophic continuous.

Proof. Consider the following compositions: $X \times Y^X \times Z^Y \xrightarrow{T} Y^X \times Z^Y \times X \xrightarrow{t\times i_X} Z^Y \times Y^X \times X \xrightarrow{=} Z^Y \times (Y^X \times X) \xrightarrow{i\times e_2} Z^Y \times Y \xrightarrow{e_2} Z$, where T, t denote the switching maps, i_X, i denote the neutrosophic identity functions on X and Z^Y, respectively and e_2 denotes the neutrosophic evaluation maps. Let $\varphi = e_2 \circ (i \times e_2) \circ (t \times i_X) \circ T$. By proposition 3.4, we have an exponential map $E : Z^{X\times Y^X \times Z^Y} \to (Z^X)^{Y^X \times Z^Y}$. Since $\varphi \in Z^{X\times Y^X \times Z^Y}$, $E(\varphi) \in (Z^X)^{Y^X \times Z^Y}$. Let $N = E(\varphi)$ that is, $N : Y^X \times Z^Y \to Z^X$ is neutrosophic continuous. For $f \in Y^X, g \in Z^Y$ and for any neutrosophic point $x_{r,t,s} \in X$,it easy to see that $N(f, g)(x_{r,t,s}) = g(f(x_{r,t,s}))$. □

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University of New Mexico

On Neutrosophic Semi-Supra Open Set and Neutrosophic Semi-Supra Continuous Functions

R. Dhavaseelan[1], M. Parimala[2], S. Jafari[3], F. Smarandache[4]

[1] Department of Mathematics, Sona College of Technology, Salem-636005,Tamil Nadu,India. E-mail dhavaseelan.r@gmail.com

[2] Department of Mathematics, Bannari Amman Institute of Technology Sathyamangalam-638401,Tamil Nadu, India . E-mail: rishwanthpari@gmail.com

[3] Department of Mathematics, College of Vestsjaelland South, Herrestraede 11, 4200 Slagelse, Denmark. E-mail: jafaripersia@gmail.com

[4] Mathematics & Science Department, University of New Maxico, 705 Gurley Ave, Gallup, NM 87301, USA. E-mail: fsmarandache@gmail.com

Abstract: In this paper, we introduce and investigate a new class of sets and functions between topological space called neutrosophic semi-supra open set and neutrosophic semi-supra open continuous functions respectively.

Keywords: Supra topological spaces; neutrosophic supra-topological spaces; neutrosophic semi-supra open set.

1 Introduction and Preliminaries

Intuitionistic fuzzy set is defined by Atanassov [2] as a generalization of the concept of fuzzy set given by Zadesh [14]. Using the notation of intuitionistic fuzzy sets, Coker [3] introduced the notion of an intuitionistic fuzzy topological space. The supra topological spaces and studied s-continuous functions and s^*-continuous functions were introduced by A. S. Mashhour [6] in 1993. In 1987, M. E. Abd El-Monsef et al. [1] introduced the fuzzy supra topological spaces and studied fuzzy supra continuous functions and obtained some properties and characterizations. In 1996, Keun Min [13] introduced fuzzy s-continuous, fuzzy s-open and fuzzy s-closed maps and established a number of characterizations. In 2008, R. Devi et al. [4] introduced the concept of supra α-open set, and in 1983, A. S. Mashhour et al. introduced the notion of supra-semi open set, supra semi-continuous functions and studied some of the basic properties for this class of functions. In 1999, Necla Turan [11] introduced the concept of intuitionistic fuzzy supra topological space. The concept of intuitionistic fuzzy semi-supra open set was introduced by Parimala and Indirani [7]. After the introduction of the concepts of neutrosophy and a neutrosophic se by F. Smarandache [[9], [10]], A. A. Salama and S. A. Alblowi[8] introduced the concepts of neutrosophic crisp set and neutrosophic topological spaces.

The purpose of this paper is to introduce and investigate a new class of sets and functions between topological space called neutrosophic semi-supra open set and neutrosophic semi-supra open continuous functions, respectively.

Definition 1.1. Let T, I, F be real standard or non standard subsets of $]0^-, 1^+[$, with $sup_T = t_{sup}, inf_T = t_{inf}$
$sup_I = i_{sup}, inf_I = i_{inf}$
$sup_F = f_{sup}, inf_F = f_{inf}$
$n - sup = t_{sup} + i_{sup} + f_{sup}$
$n - inf = t_{inf} + i_{inf} + f_{inf}$. T, I, F are neutrosophic components.

Definition 1.2. Let X be a nonempty fixed set. A neutrosophic set [briefly NS] A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set A.

Remark 1.1. (1) A neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ can be identified to an ordered triple $\langle \mu_A, \sigma_A, \gamma_A\rangle$ in $]0^-, 1^+[$ on X.

(2) For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \sigma_A, \gamma_A\rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.3. Let X be a nonempty set and the neutrosophic sets A and B in the form
$A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;

(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(c) $\bar{A} = \{\langle x, \gamma_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$; [Complement of A]

(d) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$;

(e) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$;

(f) $[\,]A = \{\langle x, \mu_A(x), \sigma_A(x), 1 - \mu_A(x)\rangle : x \in X\}$;

(g) $\langle\rangle A = \{\langle x, 1 - \gamma_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 1.4. Let $\{A_i : i \in J\}$ be an arbitrary family of neutrosophic sets in X. Then

(a) $\bigcap A_i = \{\langle x, \wedge\mu_{A_i}(x), \wedge\sigma_{A_i}(x), \vee\gamma_{A_i}(x)\rangle : x \in X\}$;

(b) $\bigcup A_i = \{\langle x, \vee\mu_{A_i}(x), \vee\sigma_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle : x \in X\}$.

Since our main purpose is to construct the tools for developing neutrosophic topological spaces, we must introduce the neutrosophic sets 0_N and 1_N in X as follows:

Definition 1.5. $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$.

Definition 1.6. [5] A neutrosophic topology (NT) on a nonempty set X is a family T of neutrosophic sets in X satisfying the following axioms:

(i) $0_N, 1_N \in T$,

(ii) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$,

(iii) $\cup G_i \in T$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq T$.

In this case the ordered pair (X, T) or simply X is called a neutrosophic topological space (NTS) and each neutrosophic set in T is called a neutrosophic open set (NOS). The complement $\overline{A}$ of a NOS A in X is called a neutrosophic closed set (NCS) in X.

Definition 1.7. [5] Let A be a neutrosophic set in a neutrosophic topological space X. Then

$Nint(A) = \bigcup\{G \mid G$ is a neutrosophic open set in X and $G \subseteq A\}$ is called the neutrosophic interior of A;

$Ncl(A) = \bigcap\{G \mid G$ is a neutrosophic closed set in X and $G \supseteq A\}$ is called the neutrosophic closure of A.

Definition 1.8. Let X be a nonempty set. If r, t, s be real standard or non standard subsets of $]0^-, 1^+[$, then the neutrosophic set $x_{r,t,s}$ is called a neutrosophic point(in short NP)in X given by

$$x_{r,t,s}(x_p) = \begin{cases} (r, t, s), & \text{if } x = x_p \\ (0, 0, 1), & \text{if } x \neq x_p \end{cases}$$

for $x_p \in X$ is called the support of $x_{r,t,s}$, where r denotes the degree of membership value ,t denotes the degree of indeterminacy and s is the degree of non-membership value of $x_{r,t,s}$.

Now we shall define the image and preimage of neutrosophic sets. Let X and Y be two nonempty sets and $f : X \to Y$ be a function.

Definition 1.9. [5]

(a) If $B = \{\langle y, \mu_B(y), \sigma_B(y), \gamma_B(y)\rangle : y \in Y\}$ is a neutrosophic set in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is the neutrosophic set in X defined by
$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\sigma_B)(x), f^{-1}(\gamma_B)(x)\rangle : x \in X\}$.

(b) If $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ is a neutrosophic set in X, then the image of A under f, denoted by $f(A)$, is the neutrosophic set in Y defined by
$f(A) = \{\langle y, f(\mu_A)(y), f(\sigma_A)(y), (1 - f(1 - \gamma_A))(y)\rangle : y \in Y\}$. where

$$f(\mu_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$f(\sigma_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \sigma_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$(1 - f(1 - \gamma_A))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \gamma_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 1, & \text{otherwise,} \end{cases}$$

For the sake of simplicity, let us use the symbol $f_-(\gamma_A)$ for $1 - f(1 - \gamma_A)$.

Corollary 1.1. [5] Let A , $A_i (i \in J)$ be neutrosophic sets in X, B, $B_i (i \in K)$ be neutrosophic sets in Y and $f : X \to Y$ a function. Then

(a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$,

(b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,

(c) $A \subseteq f^{-1}(f(A))$ { If f is injective,then $A = f^{-1}(f(A))$ } ,

(d) $f(f^{-1}(B)) \subseteq B$ { If f is surjective,then $f(f^{-1}(B)) = B$ },

(e) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$,

(f) $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$,

(g) $f(\bigcup A_i) = \bigcup f(A_i)$,

(h) $f(\bigcap A_i) \subseteq \bigcap f(A_i)$ { If f is injective,then $f(\bigcap A_i) = \bigcap f(A_i)$},

(i) $f^{-1}(1_N) = 1_N$,

(j) $f^{-1}(0_N) = 0_N$,

(k) $f(1_N) = 1_N$, if f is surjective

(l) $f(0_N) = 0_N$,

(m) $\overline{f(A)} \subseteq f(\overline{A})$, if f is surjective,

(n) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

2 Main Results

Definition 2.1. A neutrosophic set A in a neutrosophic topological space (X,T) is called

1) a neutrosophic semiopen set (NSOS) if $A \subseteq Ncl(Nint(A))$.
2) a neutrosophic α open set ($N\alpha OS$) if $A \subseteq Nint(Ncl(Nint(A)))$.
3) a neutrosophic preopen set (NPOS) if $A \subseteq Nint(Ncl(A))$.
4) a neutrosophic regular open set (NROS) if $A = Nint(Ncl(A))$.
5) a neutrosophic semipre open or β open set ($N\beta OS$) if $A \subseteq Ncl(Nint(Ncl(A)))$.

A neutrosophic set A is called a neutrosophic semiclosed set, neutrosophic α closed set, neutrosophic preclosed set, neutrosophic regular closed set and neutrosophic β closed set, respectively (NSCS, NαCS, NPCS, NRCS and NβCS, resp), if the complement of A is a neutrosophic semiopen set, neutrosophic α-open set, neutrosophic preopen set, neutrosophic regular open set, and neutrosophic β-open set, respectively.

Definition 2.2. Let (X,T) ba a neutrosophic topological space. A neutrosophic set A is called a neutrosophic semi-supra open set (briefly NSSOS) if $A \subseteq s\text{-}Ncl(s\text{-}Nint(A))$. The complement of a neutrosophic semi-supra open set is called a neutrosophic semi-supra closed set.

Proposition 2.1. Every neutrosophic supra open set is neutrosophic semi-supra open set.

Proof. Let A be a neutrosophic supra open set in (X,T). Since $A \subseteq s\text{-}Ncl(A)$, we get $A \subseteq s\text{-}Ncl(s\text{-}Nint(A))$. Then $s\text{-}Nint(A) \subseteq s\text{-}Ncl(s\text{-}Nint(A))$. Hence $A \subseteq s\text{-}Ncl(s\text{-}Nint(A))$. □

The converse of Proposition 2.1., need not be true as shown in Example 2.1.

Example 2.1. Let $X = \{a,b\}$. Define the neutrosophic sets A, B and C in X as follows:
$A = \langle x, (\frac{a}{0.2}, \frac{b}{0.4}), (\frac{a}{0.2}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.6})\rangle$, $B = \langle x, (\frac{a}{0.6}, \frac{b}{0.2}), (\frac{a}{0.6}, \frac{b}{0.2}), (\frac{a}{0.3}, \frac{b}{0.4})\rangle$
and $C = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.4}, \frac{b}{0.4})\rangle$. Then the families $T = \{0_N, 1_N, A, B, A \cup B\}$ is neutrosophic topology on X. Thus, (X,T) is a neutrosophic topological space. Then C is called neutrosophic semi-supra open but not neutrosophic supra open set.

Proposition 2.2. Every neutrosophic α-supra open is neutrosophic semi-supra open

Proof. Let A be a neutrosophic α-supra open in (X,T), then $A \subseteq s\text{-}Nint(s\text{-}Ncl(s\text{-}Nint(A)))$. It is obvious that $s\text{-}Nint(s\text{-}Ncl(s\text{-}Nint(A))) \subseteq s\text{-}Ncl(s\text{-}Nint(A))$. Hence $A \subseteq s\text{-}Ncl(s\text{-}Nint(A))$.

The converse of Proposition 2.2., need not be true as shown in Example 2.2. □

Example 2.2. Let $X = \{a,b\}$. Define the neutrosophic sets A, B and C in X as follows:
$A = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.5}, \frac{b}{0.3})\rangle$, $B = \langle x, (\frac{a}{0.1}, \frac{b}{0.2}), (\frac{a}{0.1}, \frac{b}{0.2}), (\frac{a}{0.6}, \frac{b}{0.5})\rangle$
and $C = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3})\rangle$. Then the families $T = \{0_N, 1_N, A, B, A \cup B\}$ is neutrosophic topology on X.Thus, (X,T) is a neutrosophic topological space. Then C is called neutrosophic semi-supra open but not neutrosophic α-supra open set.

Proposition 2.3. Every neutrosophic regular supra open set is neutrosophic semi-supra open set

Proof. Let A be a neutrosophic regular supra open set in (X,T). Then $A \subseteq (s\text{-}Ncl(A))$. Hence $A \subseteq s\text{-}Ncl(s\text{-}Nint(A))$.

The converse of Proposition 2.3., need not be true as shown in Example 2.3. □

Example 2.3. Let $X = \{a,b\}$. Define the neutrosophic sets A, B and C in X as follows:
$A = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.5}, \frac{b}{0.3})\rangle$, $B = \langle x, (\frac{a}{0.1}, \frac{b}{0.2}), (\frac{a}{0.1}, \frac{b}{0.2}), (\frac{a}{0.6}, \frac{b}{0.5})\rangle$
and $C = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3})\rangle$. Then the families $T = \{0_N, 1_N, A, B, A \cup B\}$ is neutrosophic topology on X. Thus, (X,T) is a neutrosophic topological space. Then C is neutrosophic semi-supra open but not neutrosophic regular-supra open set.

Definition 2.3. The neutrosophic semi-supra closure of a set A is denoted by $semi\text{-}s\text{-}Ncl(A) = \bigcup\{$ G :G is aneutrosophic semi-supra open set in X and $G \subseteq A\}$ and the neutrosophic semi-supra interior of a set A is denoted by $semi\text{-}s\text{-}Nint(A) = \bigcap\{$G :G is a neutrosophic semi-supra closed set in X and $G \supseteq A\}$.

Remark 2.1. It is clear that $semi\text{-}s\text{-}Nint(A)$ is a neutrosophic semi-supra open set and $semi\text{-}s\text{-}Ncl(A)$ is a neutrosophic semi-supra closed set.

Proposition 2.4. i) $\overline{semi - s - Nint(A)}$ = semi s-Ncl $(\overline{A})$

ii) $\overline{semi - s - Ncl(A)}$ = semi s-int $(\overline{A})$

iii) if $A \subseteq B$ then $semi\text{-}s\text{-}Ncl(A) \subseteq semi\text{-}s\text{-}Ncl(B)$ and $semi\text{-}s\text{-}Nint(A) \subseteq semi\text{-}s\text{-}Nint(B)$

Proof. It is obvious. □

Proposition 2.5. (i) The intersection of a neutrosophic supra open set and a neutrosophic semi-supra open set is a neutrosophic semi- supra open set.

(ii) The intersection of a neutrosophic semi-supra open set and aneutrosophic pre-supra open set is a neutrosophic pre-supra open set.

Proof. It is obvious. □

Definition 2.4. Let (X, T) and (Y, S) be two neutrosophic semi-supra open sets and R be a associated supra topology with T. A map $f : (X, T) \to (Y, S)$ is called neutrosophic semi- supra continuous map if the inverse image of each neutrosophic open set in Y is a neutrosophic semi- supra open in X.

Proposition 2.6. Every neutrosophic supra continuous map is neutrosophic semi-supra continuous map.

Proof. Let $f : (X, T) \to (Y, S)$ be a neutrosophic supra continuous map and A is a neutrosophic open set in Y. Then $f^{-1}(A)$ is a neutrosophic open set in X. Since R is associated with T. Then $T \subseteq R$. Therefore $f^{-1}(A)$ is a neutrosophic supra open set in X which is a neutrosophic supra open set in X. Hence f is aneutrosophic semi-supra continuous map. □

Remark 2.2. Every neutrosophic semi-supra continuous map need not be neutrosophic supra continuous map.

Proposition 2.7. Let (X, T) and (Y, S) be two neutrosophic topological spaces and R be a associated neutrosophic supra topology with T. Let f be a map from X into Y. Then the following are equivalent.

i) f is a neutrosophic semi-supra continuous map.

ii) The inverse image of a neutrosophic closed sets in Y is a neutrosophic semi closed set in X.

iii) Semi-s-Ncl($f^{-1}(A)$) $\subseteq f^{-1}(Ncl(A))$ for every neutrosophic set A in Y.

iv) $f(semi\text{-}s\text{-}Ncl(A)) \subseteq Ncl(f(A))$ for every neutrosophic set A in X.

v) $f^{-1}(Nint(B)) \subseteq semi\text{-}s\text{-}Nint(f^{-1}(B))$ for every neutrosophic set B in Y.

Proof. $(i) \Rightarrow (ii)$: Let A be a neutrosophic closed set in Y. Then $\overline{A}$ is neutrosophic open in Y, Thus $f^{-1}(\overline{A}) = \overline{f^{-1}(A)}$ is neutrosophic semi-open in X. It follows that $f^{-1}(A)$ is a neutrosophic semi-s closed set of X.
$(ii) \Rightarrow (iii)$: Let A be any subset of X. Since $Ncl(A)$ is neutrosophic closed in Y then it follows that $f^{-1}(Ncl(A))$ is neutrosophic semi-s closed in X. Therefore, $f^{-1}(Ncl(A)) = semi\text{-}s\text{-}Ncl(f^{-1}(Ncl(A)) \supseteq semi\text{-}s\text{-}Ncl(f^{-1}(A))$
$(iii) \Rightarrow (iv)$: Let A be any subset of X. By (iii) we obtain $f^{-1}(Ncl(f((A))) \supseteq semi\text{-}s\text{-}Ncl(f^{-1}(f(A))) \supseteq semi\text{-}s\text{-}Ncl(A)$ and hence $f(semi\text{-}s\text{-}Ncl(A)) \subseteq Ncl(f(A))$.
$(iv) \Rightarrow (v)$: Let $f(semi\text{-}s\text{-}Ncl(A)) \subseteq f(Ncl(A)$ for every neutrosophic set A in X. Then $semi\text{-}s\text{-}Ncl(A)) \subseteq f^{-1}(Ncl(f(A))$. $\overline{semi-s-Ncl(A)} \supseteq \overline{f^{-1}(Ncl(f(A)))}$ and $semi\text{-}s\text{-}Nint(\overline{A}) \supseteq f^{-1}(Nint(\overline{f(A)}))$. Then $semi\text{-}s\text{-}Nint(f^{-1}(B)) \supseteq f^{-1}(Nint(B))$. Therefore $f^{-1}(Nint(B)) \subseteq s\text{-}Nint(f^{-1}(B))$ for every B in Y.
$(v) \Rightarrow (i)$: Let A be a neutrosophic open set in Y. Therefore $f^{-1}(Nint(A)) \subseteq semi\text{-}s\text{-}Nint(f^{-1}(A))$, hence $f^{-1}(A) \subseteq semi\text{-}s\text{-}Nint(f^{-1}(A))$. But we know that $semi\text{-}s\text{-}Nint(f^{-1}(A)) \subseteq f^{-1}(A)$, then $f^{-1}(A) = semi\text{-}s\text{-}Nint(f^{-1}(A))$. Therefore $f^{-1}(A)$ is a neutrosophic semi-s-open set. □

Proposition 2.8. If a map $f : (X, T) \to (Y, S)$ is a neutrosophic semi-s-continuous and $g : (Y, S) \to (Z, R)$ is neutrosophic continuous, Then $g \circ f$ is neutrosophic semi-s-continuous.

Proof. Obvious. □

Proposition 2.9. Let a map $f : (X, T) \to (Y, S)$ be a neutrosophic semi-supra continuous map, then one of the following holds

i) $f^{-1}(semi\text{-}s\text{-}Nint(A)) \subseteq Nint(f^{-1}(A))$ for every neutrosophic set A in Y.

ii) $Ncl(f^{-1}(A)) \subseteq f^{-1}(semi\text{-}s\text{-}Ncl(A))$ for every neutrosophic set A in Y.

iii) $f(Ncl(B)) \subseteq semi\text{-}s\text{-}Ncl(f(B))$ for every neutrosophic set B in X.

Proof. Let A be any neutrosophic open set of Y, then condition (i) is satisfied, then $f^{-1}(semi\text{-}s\text{-}Nint(A)) \subseteq Nint(f^{-1}(A))$. We get, $f^{-1}(A) \subseteq Nint(f^{-1}(A))$. Therefore $f^{-1}(A)$ is a neutrosophic supra open set. Every neutrosophic supra open set is a neutrosophic semi supra open set. Hence f is a neutrosophic semi-s-continuous function. If condition (ii) is satisfied, then we can easily prove that f is a neutrosophic semi -s continuous function if condition (iii) is satisfied, and A is any neutrosophic open set of Y, then $f^{-1}(A)$ is a set in X and $f(Ncl(f^{-1}(A)) \subseteq semi\text{-}s\text{-}Ncl(f(f^{-1}(A)))$. This implies $f(Ncl(f^{-1}(A))) \subseteq semi\text{-}s\text{-}Ncl(A)$. This is nothing but condition (ii). Hence f is a neutrosophic semi-s-continuous function. □

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University of New Mexico

Neutrosophic Cubic MCGDM Method Based on Similarity Measure

Surapati Pramanik[1], Shyamal Dalapati[2], Shariful Alam[3], Tapan Kumar Roy[4], Florentin Smarandache[5]

[1] Department of Mathematics, Nandalal Ghosh B.T. College, Panpur, P.O.-Narayanpur, District –North 24 Parganas, Pin code-743126, West Bengal, India. E-mail: sura_pati@yahoo.co.in

[2] Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, P.O.-Botanic Garden, Howrah-711103, West Bengal, India. E-mail: dalapatishyamal30@gmail.com

[3] Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, P.O.-Botanic Garden, Howrah-711103, West Bengal, India. E-mail: salam50@yahoo.co.in

[4] Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, P.O.-Botanic Garden, Howrah-711103, West Bengal, India. E-mail: roy_t_k@yahoo.co.in

[5] University of New Mexico. Mathematics & Science Department, 705 Gurley Ave., Gallup, NM 87301, USA. Email:fsmarandache@gmail.com

Abstract. The notion of neutrosophic cubic set is originated from the hybridization of the concept of neutrosophic set and interval valued neutrosophic set. We define similarity measure for neutrosophic cubic sets and prove some of its basic properties. We present a new multi criteria group decision making method with linguistic variables in neutrosophic cubic set environment. Finally, we present a numerical example to demonstrate the usefulness and applicability of the proposed method.

Keywords: Cubic set, Neutrosophic cubic set, similarity measure, multi criteria group decision making.

1. Introduction

In practical life we frequently face decision making problems with uncertainty that cannot be dealt with the classical methods. Therefore sophisticated techniques are required for modification of classical methods to deal decision making problems with uncertainty. L. A. Zadeh [1] first proposed the concept of fuzzy set to deal non-statistical uncertainty called fuzziness. K. T. Atanassov [2, 3] introduced the concept of intuitionistic fuzzy set (IFS) to deal with uncertainty by introducing the non-membership function as an independent component. F. Smarandache [4, 5, 6, 7, 8] introduced the notion of neutrosophic set by introducing indeterminacy as independent component. The theory of neutrosophic sets is a powerful tool to deal with incomplete, indeterminate and inconsistent information involed in real world decision making problem. Wang et al. [9] defined single valued neutrosophic set (SVNS) which is an instance of neutrosophic set. SVNS can independently express a truth-membership degree, an indeterminacy-membership degree and non-membership (falsity-membership) degree. SVNS is capable of representing human thinking due to the imperfection of knowledge received from real world problems. SVNS is obviously suitable for representing incomplete, inconsistent and indeterminate information.

Neutrosophic sets and SVNSs have become hot research topics in different areas of research such as conflict resolution [10], clustering analysis [11, 12], decision making [13-41], educational problem [42, 43], image processing [44, 45, 46], medical diagnosis [47], optimization [48-53], social problem [54, 55].

By combining neutrosophic sets and SVNS with other sets, several neutrosophic hybrid sets have been proposed in the literature such as neutrosophic soft sets [56, 57, 58, 59, 60, 61], neutrosophic soft expert set [62, 63], single valued neutrosophic hesitant fuzzy sets [64, 65, 66, 67, 68], interval neutrosophic hesitant sets [69], interval neutrosophic linguistic sets [70], single valued neutrosophic linguistic sets [71], rough neutrosophic set [72, 73, 74, 75, 76, 77, 78, 79], interval rough neutrosophic set [80, 81, 82], bipolar neutrosophic set [83, 84], bipolar rough neutrosophic set [85] Tri-complex rough neutrosophic set[86], hyper complex rough neutrosophic set [87]. Neutrosophic refined set [88, 89, 90, 91, 92, 93], Bipolar neutrosophic refined sets [94], rough complex set neutrosophic cubic set [95].

Jun et al. [96] put forward the concept of cubic set in fuzzy environment and defined external and internal cubic set. Ali et al. [95] proposed neutrosophic cubic set and defined external and internal neutrosophic cubic sets and their basic properties.

Similarity measure is a vital topic in fuzzy set theory, Chen and Hsiao [97] presented comparisons of similarity measures of fuzzy sets. Pramanik and Mondal [98] studied weighted fuzzy similarity measure based on tangent function and presented its application to medical diagnosis. Hwang and Yang [99] constructed a new similarity measure between intuitionistic fuzzy sets based on lower, upper and middle fuzzy sets. Pramanik and Mondal [100] developed tangent similarity measures in intuitionistic fuzzy environment and applied to medical diagnosis. Ren and Wang [101] proposed similarity measures in interval-valued intuitionistic fuzzy environment and applied it to multi attribute decision making problems. Baccour et al. [102] presented survey of similarity measures for intuitionistic fuzzy sets. Baroumi and Smarandache [103] disicussed several similarity measures of neutrosophic sets. Mondal and Pramanik [104] extended the concept of intuitionistic tangent similarity measure to neutrosophic environment. Biswas et al. [105] studied cosine similarity measure with trapezoidal fuzzy neutrosophic number and its applied to multi attribute decision making problems. Pramanik and Mondal [106] proposed cosine similarity measure of rough neutrosophic set and applied it to medical diagnosis problems. Pramanik and Mondal [107]developed ccotangent similarity measure of rough neutrosophic sets and its application to medical diagnosis J. Ye [108] proposed a similarity measures under interval neutrosophic domain using hamming distance and Euclidean distance. P. Majumdar and S. K. Samanta [109] introduced some measures of similarity and entropy of single valued neutrosophic sets. Ali aydogdu [110] proposed similarity and entropy measure of single valued neutrosophic sets. Ali aydogdu [111] also defined entropy and similarity measures of interval neutrosophic sets. Mukherjee and Sarkar [112] proposed similarity measures, weighted similarity measure and developed an algorithm in interval valued neutrosophic soft set setting for supervised pattern recognition problem. In neutrosophic cubic set environment, similarity measure is yet to appear.

In this paper we define similarity measures in neutrosophic cubic set environment and develop a multi criteria group decision making (MCGDM) method in neutrosophic cubic set setting. The decision makers' weights and criteria (attributes) weights are described by neutrosophic cubic numbers using linguistic variables. The ranking of alternatives is presented in descending order. Finally, illustrate numerical example MCGDM problem in neutrosophic cubic set environment is dolved to show the effectiveness of the proposed method.

Rest of the paper is presented as follows. Section 2 presents some basic definition of fuzzy sets, interval-valued fuzzy sets, neutrosophic sets, interval valued neutrosophic sets, cubic set, neutrosophic cubic sets and their basic operations. Section 3 is devoted to prove the basic properties of similarity measure for neutrosophic cubic sets. Section 4 presents a MCGDM method based on similarity measure in neutrosophic cubic set environment. Section 5 presents a numerical example for a MCGDM problem. Finally, section 6 presents conclusion and future scope of research.

2 Preliminaries

In this section, we recall some basic definitions which are relevant to develop the paper.

Definition 2.1 [1] Fuzzy set

Let U be a universal set. Then a fuzzy set Z over U is defined by Z = {(u, μ_Z (u)): u $\in$ U}

Where μ_Z : U $\rightarrow$ [0, 1] is called membership function of Z and μ_Z (u) specifies the grade or degree to which any element u in Z, μ_Z (u) $\in$ [0, 1]. Larger values of μ_Z (u) indicate higher degrees of membership.

Definition 2.2 [113] Interval valued fuzzy set

Let U be a universal set, then an interval valued fuzzy set $\tilde{Z}$ over U is defined by $\tilde{Z}$ = {[Z^-(u), Z^+ (u)] /u: u $\in$ U }, where Z^-(u), Z^+ (u) represent respectively the lower and upper degrees of membership values for u $\in$ U and $0 \leq Z^-(u) + Z^+(u) \leq 1$.

Definition 2.3 [96] Cubic set

Let G be a non-empty set. A cubic set C (G) in G is defined by

C (G) = {g, $\tilde{Z}$ (g), Z (g)/g $\in$ G}

Where $\tilde{Z}$ (g) and Z (g) be the interval valued fuzzy set and fuzzy set in G.

Definition 2.4 [4] Neutrosophic set (NS)

Let U be a space of points (objects) with a generic element in U denoted by u i.e. u $\in$ U. A neutrosophic set R in U is characterized by truth-membership function t_R , a indeterminacy membership function i_R and falsity-membership function f_R . Where t_R , i_R , f_R are the functions from U to $]^-0, 1^+[$ i.e. $t_R, i_R, f_R: U \rightarrow\ ]^-0, 1^+[$ that means $t_R(u), i_R(u), f_R(u)$ are the real standard or non-standard subset of $]^-0, 1^+[$. Neutrosophic set can be expressed as R = {<u, (t_R (u), i_R (u), f_R (u))>: u $\in$ U}. Since $t_R(u), i_R(u), f_R(u)$ are the subset of $]^-0, 1^+[$ then the sum ($t_R(u) + i_R(u) + f_R(u)$) lies between $^-0$ and 3^+, where $^-0 = 0 - \varepsilon$ and $3^+ = 3 + \varepsilon$, $\varepsilon > 0$ and $\varepsilon \rightarrow 0$.

Definition 2.5 [9] Single valued neutrosophic set

Let U be a space of points (objects) with a generic element in U denoted by u. A single valued neutrosophic set H in U is expressed by H = {<u, (t_H (u), i_H (u), f_H (u))>, u $\in$ U},where $t_H(u), i_H(u), f_H(u): U \rightarrow [0, 1]$
Therefore for each u $\in$ U, $t_H(u), i_H(u), f_H(u) \in [0, 1]$ and $0 \leq t_H(u) + i_H(u) + f_H(u) \leq 3$.

Definition 2.6 [4] Complement of neutrosophic set

The complement of neutrosophic set R denoted by R' and defined as R' = {< u, $t_{R'}(u), i_{R'}(u), f_{R'}(u)$>: u $\in$ U}, where $t_{R'}(u) = f_R(u)$, $i_{R'}(u) = \{1^+\} - i_R(u)$, $f_{R'}(u) = t_R(u)$.

Definition 2.7 [8]Containment

A neutrosophic set R_1 is contained in another neutrosophic set R_2 i.e. $R_1 \subseteq R_2$ iff $t_{R_1}(u) \leq t_{R_2}(u)$, $i_{R_1}(u) \leq i_{R_2}(u)$ and $f_{R_1}(u) \geq f_{R_2}(u)$, $\forall$ u $\in$ U.

Definition 2.8 [4] Equality

Two single valued neutrosophic set R_1 and R_2 are equal iff $R_1 \subseteq R_2$ and $R_2 \subseteq R_1$.

Definition 2.9 [4] Union

The union of two single valued neutrosophic set R_1 and R_2 is a neutrosophic set R_3 (say) written as $R_3 = R_1 \cup R_2$.
$t_{R_3}(u) = \max\{t_{R_1}(u), t_{R_2}(u)\}$, $i_{R_3}(u) = \max\{i_{R_1}(u), i_{R_2}(u)\}$, $f_{R_1}(u) = \min\{f_{R_1}(u), f_{R_2}(u)\}$, $\forall$ u $\in$ U.

Definition 2.10 [4] Intersection

The intersection of two single valued neutrosophic set R_1 and R_2 denoted by R_4 and written as $R_4 = R_1 \cap R_2$defined by $t_{R_4}(u) = \min\{t_{R_1}(u), t_{R_2}(u)\}$, $i_{R_4}(u) = \min\{i_{R_1}(u), i_{R_2}(u)\}$
$f_{R_4}(u) = \max\{f_{R_1}(u), f_{R_2}(u)\}$, $\forall$ u $\in$ U.

Definition 2.11 [114] Interval neutrosophic set (INS)

Let G be a non-empty set. An interval neutrosophic set $\tilde{G}$ in G is characterized by truth-membership function $t_{\tilde{G}}$, the indeterminacy function $i_{\tilde{G}}$ and falsity membership function $f_{\tilde{G}}$. For each g $\in$ G, $t_{\tilde{G}}(g), i_{\tilde{G}}(g), f_{\tilde{G}}(g) \subseteq [0, 1]$ and $\tilde{G}$ defined as
$\tilde{G}$ = {<g; [$t^-_{\tilde{G}}(g), t^+_{\tilde{G}}(g)$], [$i^-_{\tilde{G}}(g), i^+_{\tilde{G}}(g)$], [$f^-_{\tilde{G}}(g), f^+_{\tilde{G}}(g)$]: g $\in$ G}.

Definition 2.12 [114] Containment

Let G_1 and G_2 be two interval neutrosophic set defined by $\tilde{G}_1$ = {<g, [$t^-_{\tilde{G}_1}(g), t^+_{\tilde{G}_1}(g)$], [$i^-_{\tilde{G}_1}(g), i^+_{\tilde{G}_1}(g)$], [$f^-_{\tilde{G}_1}(g), f^+_{\tilde{G}_1}(g)$]>: g $\in$ G}
and $\tilde{G}_2$ = {<g, [$t^-_{\tilde{G}_2}(g), t^+_{\tilde{G}_2}(g)$], [$i^-_{\tilde{G}_2}(g), i^+_{\tilde{G}_2}(g)$], [$f^-_{\tilde{G}_2}(g), f^+_{\tilde{G}_2}(g)$]>: g $\in$ G}
then, (i) $\tilde{G}_1 \subseteq \tilde{G}_2$ defined as
$t^-_{\tilde{G}_1}(g) \leq t^-_{\tilde{G}_2}(g)$, $t^+_{\tilde{G}_1}(g) \leq t^+_{\tilde{G}_2}(g)$
$i^-_{\tilde{G}_1}(g) \leq i^-_{\tilde{G}_2}(g)$, $i^+_{\tilde{G}_1}(g) \leq i^+_{\tilde{G}_2}(g)$
$f^-_{\tilde{G}_1}(g) \geq f^-_{\tilde{G}_2}(g)$, $f^+_{\tilde{G}_1}(g) \geq f^+_{\tilde{G}_2}(g)$ for all g $\in$ G.

Definition 2.13 [114] Equality

$\tilde{G}_1 = \tilde{G}_2$ iff $\tilde{G}_1 \subseteq \tilde{G}_2$ and $\tilde{G}_2 \subseteq \tilde{G}_1$ that means $t^-_{\tilde{G}_1}(g) = t^-_{\tilde{G}_2}(g)$, $t^+_{\tilde{G}_1}(g) = t^+_{\tilde{G}_2}(g)$, $i^-_{\tilde{G}_1}(g) = i^-_{\tilde{G}_2}(g)$, $i^+_{\tilde{G}_1}(g) = i^+_{\tilde{G}_2}(g)$, $f^-_{\tilde{G}_1}(g) = f^-_{\tilde{G}_2}(g)$, $f^+_{\tilde{G}_1}(g) = f^+_{\tilde{G}_2}(g)$ for all g $\in$ G.

Definition 2.14 [114] Compliment

Compliment of an interval neutrosophic set $\tilde{G}_1$ denoted by $\tilde{G}_1'$ and defined by
$\tilde{G}_1'$ = {<g, [$t^-_{\tilde{G}_1'}(g), t^+_{\tilde{G}_1'}(g)$], [$i^-_{\tilde{G}_1'}(g), i^+_{\tilde{G}_1'}(g)$], [$f^-_{\tilde{G}_1'}(g), f^+_{\tilde{G}_1'}(g)$]>: g $\in$ G},Where, $t^-_{\tilde{G}_1'}(g) = f^-_{\tilde{G}_1}(g)$, $t^+_{\tilde{G}_1'}(g) = f^+_{\tilde{G}_1}(g)$, $i^-_{\tilde{G}_1'}(g) = \{1\} - i^-_{\tilde{G}_1}(g)$, $i^+_{\tilde{G}_1'}(g) = \{1\} - i^+_{\tilde{G}_1}(g)$,
$f^-_{\tilde{G}_1'}(g) = t^-_{\tilde{G}_1}(g)$, $f^+_{\tilde{G}_1'}(g) = f^+_{\tilde{G}_1}(g)$.

Definition 2.15 [114] Union

The union of two interval neutrosophic sets $\tilde{G}_1$, and $\tilde{G}_2$ is denoted by $\tilde{G}_3 = \tilde{G}_1 \cup \tilde{G}_2$ and defined as

$\tilde{G}_3$ = {<g, [max { $t^{-}_{\tilde{G}_1}$ (g), $t^{-}_{\tilde{G}_2}$ (g)},max { $t^{+}_{\tilde{G}_1}$ (g), $t^{+}_{\tilde{G}_2}$ (g)}], [max { $i^{-}_{\tilde{G}_1}$ (g), $i^{-}_{\tilde{G}_2}$ (g)}, max { $i^{+}_{\tilde{G}_1}$ (g), $i^{+}_{\tilde{G}_2}$ (g)}], [min { $f^{-}_{\tilde{G}_1}$ (g), $f^{-}_{\tilde{G}_2}$ (g)}, min { $f^{+}_{\tilde{G}_1}$ (g), $f^{+}_{\tilde{G}_2}$ (g)}]>: g $\in$ G}.

Definition 2.16 [114] Intersection

The intersection of two interval neutrosophic set $\tilde{G}_1$, $\tilde{G}_2$ is denoted by $\tilde{G}_4 = \tilde{G}_1 \cap \tilde{G}_2$ and defined as

$\tilde{G}_4$ = {<g, [min { $t^{-}_{\tilde{G}_1}$ (g), $t^{-}_{\tilde{G}_2}$ (g)},min { $t^{+}_{\tilde{G}_1}$ (g), $t^{+}_{\tilde{G}_2}$ (g)}], [min { $i^{-}_{\tilde{G}_1}$ (g), $i^{-}_{\tilde{G}_2}$ (g)}, min { $i^{+}_{\tilde{G}_1}$ (g), $i^{+}_{\tilde{G}_2}$ (g)}], [max { $f^{-}_{\tilde{G}_1}$ (g), $f^{-}_{\tilde{G}_2}$ (g)}, max { $f^{+}_{\tilde{G}_1}$ (g), $f^{+}_{\tilde{G}_2}$ (g)}]>: g $\in$ G}.

Definition 2.17 [95] Neutrosophic cubic set (NCS)

A neutrosophic cubic set Q (N) in a universal set G is defined as

Q (N) = {<g, $\tilde{G}$ (g), R (g)>: g $\in$ G}, where $\tilde{G}$ is an interval neutrosophic set and R is a neutrosophic set in G. In this paper,we represent neutrosophic cubic set in the following form:

Q (N) = < $\tilde{G}$, R > as order pair, set of all neutrosophic cubic sets in G, we denote it by NCS (G).

Definition 2.18 Another definition of neutrosophic cubic set

Let G be a universal set, then the neutrosophic cubic set Q (N) in G is expressed as the pair

< $\tilde{G}$, R > , where $\tilde{G}$ and R be the mappings represented by $\tilde{G}$: G $\rightarrow$ INS (G), R: $\rightarrow$ NS (G)

Combining the two mappings, NCS can be expressed as Q (N) = $\tilde{G}^R$: G $\rightarrow$ [INS (G), NS (G)] and defined as Q (N) = $\tilde{G}^R$ = {< g/ < $\tilde{G}$ (g), R (g)>>: g $\in$ G}.

Definition 2.19 [95] Containment

Let Q_1 (N) = ($\tilde{G}_1^{R_1}$) and Q_2 (N) = ($\tilde{G}_2^{R_2}$) be any two NCSs in G, then Q_1 (N) contained in Q_2 (N) i.e. Q_1 (N) $\subseteq Q_2$ (N) iff $\tilde{G}_1 \subseteq \tilde{G}_2$ and $R_1 \subseteq R_2$.

Definition 2.20 [95] Equality

Assume that Q_1 (N) = ($\tilde{G}_1^{R_1}$) and Q_2 (N) = ($\tilde{G}_2^{R_2}$) be the two NCSs in G. They are said to be equal iff Q_1 (N) $\subseteq$ Q_2 (N) and Q_2 (N) $\subseteq$ Q_1 (N) that means $\tilde{G}_1 = \tilde{G}_2$ and $R_1 = R_2$.

Definition 2.21 [95] Union

The union of two NCSs Q_1 (N) = ($\tilde{G}_1^{R_1}$) and Q_2 (N) = ($\tilde{G}_2^{R_2}$) in G is denoted by

Q_1 (N) $\cup$ Q_2 (N) = Q_3 (N) (say) and defined as

Q_3 (N) = {<g, ($\tilde{G}_1 \cup \tilde{G}_2$) (g), ($R_1 \cup R_2$) (g)>: g $\in$ G}.

Definition 2.22 [95] Intersection

The intersection of two NCS Q_1 (N) = ($\tilde{G}_1^{R_1}$) and Q_2 (N) = ($\tilde{G}_2^{R_2}$) in G is denoted by Q_1 (N) $\cup$ Q_2 (N) = Q_4 (N) (say) and defined as Q_4 (N) = {<g, ($\tilde{G}_1 \cap \tilde{G}_2$) (g), ($R_1 \cap R_2$) (g)>: g $\in$ G}.

Definition 2.23 [95]Complement

Let Q_1 (N) be a NCS. Then complement of Q_1 (N) is denoted by Q'_1 (N) = {<g, $\tilde{G}_1'$ (g), $\tilde{R}_1'$ (g)>: g $\in$ G}.

3 Similarity measure of NCS

We define similarity measure for neutrosophic cubic set.

Definition3.1

Let Q_1 and Q_2 be two NCSs in G. Similarity measure for $Q_1$and Q_2 is defined as a mapping

SM: NCS (G) × NCS (G) $\rightarrow$ [0, 1] that satisfies the following conditions:

(1) $0 \leq$ SM (Q_1, Q_2) ≤ 1
(2) SM (Q_1, Q_2) = 1 iff $Q_1 = Q_2$
(3) SM (Q_1, Q_2) = SM (Q_2, Q_1)
(4) If $Q_1 \subseteq Q_2 \subseteq Q_3$ then SM (Q_1, Q_3) $\leq$ SM (Q_1, Q_2) and SM (Q_1, Q_3) $\leq$ SM (Q_2, Q_3) for all Q_1, Q_2, Q_3 $\in$ NCS (G).

Similarity measure for two NCSs Q_1 and Q_2 expressed as

$$SM(Q_1, Q_2) = \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{D_i}{9}\right),$$

where D_i = ($|t^{-}_{\tilde{G}_1}(g_i) - t^{-}_{\tilde{G}_2}(g_i)| + |t^{+}_{\tilde{G}_1}(g_i) - t^{+}_{\tilde{G}_2}(g_i)| + |i^{-}_{\tilde{G}_1}(g_i) - i^{-}_{\tilde{G}_2}(g_i)| + |i^{+}_{\tilde{G}_1}(g_i) - i^{+}_{\tilde{G}_2}(g_i)| + |f^{-}_{\tilde{G}_1}(g_i) - f^{-}_{\tilde{G}_2}(g_i)| + |f^{+}_{\tilde{G}_1}(g_i) - f^{+}_{\tilde{G}_2}(g_i)| + |t_{R_1}(g_i) - t_{R_2}(g_i)| + |i_{R_1}(g_i) - i_{R_2}(g_i)| + |f_{R_1}(g_i) - f_{R_2}(g_i)|$).

We now prove that the similarity measure satisfies the four stated conditions:

(1) $0 \leq$ SM (Q_1, Q_2) ≤ 1

Proof: If D_i has extreme value i.e. $D_i = 0$ or 9, then SM $(Q_1, Q_2) = 1$ or 0 (1)

If D_i lies between 0 and 9 i.e $0<D_i<9$, then $0<\frac{D_i}{9}<1$

$\Rightarrow 0> -\frac{D_i}{9} > -1$

Adding 1 each part of the above inequality, we obtain

$0< 1-\frac{D_i}{9}<1$

$$\frac{1}{n}\sum_{i=1}^{n}0<\frac{1}{n}\sum_{i=1}^{n}(1-\frac{D_i}{9})<\frac{1}{n}\sum_{i=1}^{n}1=1$$

$$\Rightarrow 0<\frac{1}{n}\sum_{i=1}^{n}(1-\frac{D_i}{9})<1$$

$\Rightarrow 0<SM(Q_1, Q_2)<1$ (2)

Combining (1) and (2), we get $0\leq SM(Q_1, Q_2)\leq 1$

(2) $SM(Q_1, Q_2) = 1$ iff $Q_1 = Q_2$

Proof:

If $Q_1 = Q_2$, then $D_i = 0$ by the definition of equality.

$$SM(Q_1, Q_2) = \frac{1}{n}\sum_{i=1}^{n}(1-\frac{D_i}{9}) = 1.$$

(3) $SM(Q_1, Q_2) = SM(Q_2, Q_1)$

Proof: $SM(Q_1, Q_2) = \frac{1}{n}\sum_{i=1}^{n}(1-\frac{D_i}{9})$,

where $D_i(Q_1, Q_2) = (|t^-_{\tilde{G}_1}(g_i) - t^-_{\tilde{G}_2}(g_i)| + |t^+_{\tilde{G}1}(g_i) - t^+_{\tilde{G}_2}(g_i)| + |i^-_{\tilde{G}_1}(g_i) - i^-_{\tilde{G}_2}(g_i)| + |i^+_{\tilde{G}_1}(g_i) - i^+_{\tilde{G}_2}(g_i)| + |f^-_{\tilde{G}_1}(g_i) - f^-_{\tilde{G}_2}(g_i)| + |f^+_{\tilde{G}_1}(g_i) - f^+_{\tilde{G}_2}(g_i)| + |t_{R_1}(g_i) - t_{R_2}(g_i)| + |i_{R_1}(g_i) - i_{R_2}(g_i)| + |f_{R_1}(g_i) - f_{R_2}(g_i)|)$

since, $|t^-_{\tilde{G}_1}(g_i) - t^-_{\tilde{G}_2}(g_i)| = |t^-_{\tilde{G}_2}(g_i) - t^-_{\tilde{G}_1}(g_i)|$, $|t^+_{\tilde{G}1}(g_i) - t^+_{\tilde{G}_2}(g_i)| = |t^+_{\tilde{G}2}(g_i) - t^+_{\tilde{G}_1}(g_i)|$, $|i^-_{\tilde{G}_1}(g_i) - i^-_{\tilde{G}_2}(g_i)| = |i^-_{\tilde{G}_2}(g_i) - i^-_{\tilde{G}_1}(g_i)|$, $|i^+_{\tilde{G}_1}(g_i) - i^+_{\tilde{G}_2}(g_i)| = |i^+_{\tilde{G}_2}(g_i) - i^+_{\tilde{G}_1}(g_i)|$, $|f^-_{\tilde{G}_1}(g_i) - f^-_{\tilde{G}_2}(g_i)| = |f^-_{\tilde{G}_2}(g_i) - f^-_{\tilde{G}_1}(g_i)|$, $|f^+_{\tilde{G}_1}(g_i) - f^+_{\tilde{G}_2}(g_i)| = |f^+_{\tilde{G}_2}(g_i) - f^+_{\tilde{G}_1}(g_i)|$, $|t_{R_1}(g_i) - t_{R_2}(g_i)| = |t_{R_2}(g_i) - t_{R_1}(g_i)|$, $|i_{R_1}(g_i) - i_{R_2}(g_i)| = |i_{R_2}(g_i) - i_{R_1}(g_i)|$, $|f_{R_1}(g_i) - f_{R_2}(g_i)| = |f_{R_2}(g_i) - f_{R_1}(g_i)|$.

$\Rightarrow D_i(Q_1, Q_2) = D_i(Q_2, Q_1)$

Therefore, $SM(Q_1, Q_2) = SM(Q_2, Q_1)$.

(4) If $Q_1 \subseteq Q_2 \subseteq Q_3$, then $SM(Q_1, Q_3) \leq SM(Q_1, Q_2)$ and $SM(Q_1, Q_3) \leq SM(Q_2, Q_3)$ for all $Q_1, Q_2, Q_3 \in$ NCS (G).

Proof:

Let $Q_1 \subseteq Q_2 \subseteq Q_3$ then ,

$t^-_{\tilde{G}_1}(g_i) \leq t^-_{\tilde{G}_2}(g_i) \leq t^-_{\tilde{G}_3}(g_i)$, $t^+_{\tilde{G}1}(g_i) \leq t^+_{\tilde{G}_2}(g_i) \leq t^+_{\tilde{G}_3}(g_i)$, $i^-_{\tilde{G}_1}(g_i) \leq i^-_{\tilde{G}_2}(g_i) \leq i^-_{\tilde{G}_3}(g_i)$

$i^+_{\tilde{G}_1}(g_i) \leq i^+_{\tilde{G}_2}(g_i), \leq i^+_{\tilde{G}_3}(g_i)$, $f^-_{\tilde{G}_1}(g_i) \geq f^-_{\tilde{G}_2}(g_i) \geq f^-_{\tilde{G}_3}(g_i)$, $f^+_{\tilde{G}_1}(g_i) \geq f^+_{\tilde{G}_2}(g_i) \geq f^+_{\tilde{G}_3}(g_i)$ $t_{R_1}(g_i)$ $\leq t_{R_2}(g_i) \leq t_{R_3}(g_i)$, $i_{R_1}(g_i) \leq i_{R_2}(g_i) \leq i_{R_3}(g_i)$, $f_{R_1}(g_i) \geq f_{R_2}(g_i) \geq f_{R_3}(g_i)$ (3)

Now $D_i(Q_1, Q_2) = (|t^-_{\tilde{G}_1}(g_i) - t^-_{\tilde{G}_2}(g_i)| + |t^+_{\tilde{G}1}(g_i) - t^+_{\tilde{G}_2}(g_i)| + |i^-_{\tilde{G}_1}(g_i) - i^-_{\tilde{G}_2}(g_i)| + |i^+_{\tilde{G}_1}(g_i) - i^+_{\tilde{G}_2}(g_i)| + |f^-_{\tilde{G}_1}(g_i) - f^-_{\tilde{G}_2}(g_i)| + |f^+_{\tilde{G}_1}(g_i) - f^+_{\tilde{G}_2}(g_i)| + |t_{R_1}(g_i) - t_{R_2}(g_i)| + |i_{R_1}(g_i) - i_{R_2}(g_i)| + |f_{R_1}(g_i) - f_{R_2}(g_i)|)$

And $D_i(Q_1, Q_3) = (|t^-_{\tilde{G}_1}(g_i) - t^-_{\tilde{G}_3}(g_i)| + |t^+_{\tilde{G}1}(g_i) - t^+_{\tilde{G}_3}(g_i)| + |i^-_{\tilde{G}_1}(g_i) - i^-_{\tilde{G}_3}(g_i)| + |i^+_{\tilde{G}_1}(g_i) - i^+_{\tilde{G}_3}(g_i)| + |f^-_{\tilde{G}_1}(g_i) - f^-_{\tilde{G}_3}(g_i)| + |f^+_{\tilde{G}_1}(g_i) - f^+_{\tilde{G}_3}(g_i)| + |t_{R_1}(g_i) - t_{R_2}(g_i)| + |i_{R_1}(g_i) - i_{R_3}(g_i)| + |f_{R_1}(g_i) - f_{R_3}(g_i)|)$

From (3), we conclude that

$D_i(Q_1, Q_3) \geq D_i(Q_1, Q_2)$

$$\Rightarrow \frac{D_i(Q_1,Q_3)}{9} \geq \frac{D_i(Q_1,Q_2)}{9}$$

$$\Rightarrow -\frac{D_i(Q_1,Q_3)}{9} \leq -\frac{D_i(Q_1,Q_2)}{9}$$

$$\Rightarrow [1-\frac{D_i(Q_1,Q_3)}{9}] \leq [1-\frac{D_i(Q_1,Q_2)}{9}]$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}[1-\frac{D_i(Q_1,Q_3)}{9}] \leq \frac{1}{n}\sum_{i=1}^{n}[1-\frac{D_i(Q_1,Q_3)}{9}]$$

$\Rightarrow SM(Q_1, Q_3) \leq SM(Q_1, Q_2)$

Similarly we can shows that $SM(Q_1, Q_3) \leq SM(Q_2, Q_3)$, hence the proof.

4 MCGDM methods based on similarity measure in NCS environment

In this section we propose a new MCGDM method based on similarity measure in NCS environment. Assume that

$\alpha = \{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\}$ be a set of n alternatives with criteria $\beta = \{\beta_1, \beta_2, \beta_3, ..., \beta_m\}$ and $\gamma = \{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_r\}$ be the r decision makers. Let $\Psi = \{\Psi_1, \Psi_2, \Psi_3, ..., \Psi_r\}$ be the weight vestor of decision makers, where $\Psi_k > 0$ and $\sum_{k=1}^{r} \Psi_k = 1$. Proposed MCGDM method is presented using the following steps.

Step1. Formation of ideal NCS decision matrix

Ideal NCS decision matrix is an important matrix for similarity measure of MCGDM. Here we construct an ideal NCS matrix in the form

$$M = \begin{pmatrix} & \beta_1 & \beta_2 & \dots\dots & \beta_m \\ \alpha_1 & Q_{11} & Q_{12} & \dots & Q_{1m} \\ \alpha_2 & Q_{21} & Q_{22} & & Q_{2m} \\ . & . & & \dots & . \\ \alpha_n & Q_{n1} & Q_{n2} & \dots & .Q_{nm} \end{pmatrix} \quad (4)$$

Where $Q_{ij} = \langle G_{ij}, R_{ij} \rangle$, i = 1, 2, 3,…, n. j = 1, 2, 3, …, m.

Step 2. Construction of NCS decision matrix

Since r decision makers are involved in the decision making process, the k-th (k = 1, 2, 3,…, r) decision maker provides the evaluation information of the alternative α_i (i= 1, 2, 3,…, n) with respect to criteria β_j (j= 1, 2, 3,…, m) in terms of the NCS. The k-th decision matrix denoted by M^k (See eq. (5)) is constreucted as follows:

$$M^k = \langle Q_{ij}^k \rangle = \begin{pmatrix} & \beta_1 & \beta_2 & \dots\dots & \beta_m \\ \alpha_1 & Q_{11}^k & Q_{12}^k & \dots & Q_{1m}^k \\ \alpha_2 & Q_{21}^k & Q_{22}^k & & Q_{2m}^k \\ . & . & & \dots & . \\ \alpha_n & Q_{n1}^k & Q_{n2}^k & \dots & .Q_{nm}^k \end{pmatrix} \quad (5)$$

Where k = 1, 2, 3,…, r. i = 1, 2, 3,…, n. j = 1, 2, 3,…, m.

Step 3. Determination of attribute weight

All attribute are not equally important in decision making situation. Every decision maker provides their own opinion regarding to the attribute weight in terms of linguistic variables that can be converted into NCS. Let $w_k(\beta_j)$ be the attribute weight for the attribute β_j given by the k-th decision maker in term of NCS. We convert $w_k(\beta_j)$ into fuzzy number as follows:

$$w_k^F(\beta_j) = \begin{cases} (1-\sqrt{\dfrac{V_{kj}}{9}}), \text{if } \beta_j \in \beta \\ 0 \quad \text{otherwise} \end{cases} \quad (6)$$

where V_{kj} =

$$\sqrt{\begin{Bmatrix} (1-t_k^-(\beta_j))^2 + (1-t_k^+(\beta_j))^2 + (i_k^-(\beta_j))^2 + (i_k^+(\beta_j))^2 \\ +(f_k^-(\beta_j))^2 + (f_k^+(\beta_j))^2 + (1-t_k(\beta_j))^2 \\ +(i_k(\beta_j))^2 + (f_k(\beta_j))^2 \end{Bmatrix}}.$$

Then aggregate weight for the criteria β_j can be determined as:

$$W_j = \frac{(1-\prod_{k=1}^{r}(1-w_k^F(\beta_j))}{\sum_{k=1}^{r}(1-\prod_{k=1}^{r}(1-w_k^F(\beta_j))} \quad (7)$$

Here $\sum_{k=1}^{r} W_j = 1$.

Step 4. Calculation of weighted similarity measure

We now calculate weighted similarity measure between idel matrix M and M^k as follows:

$$S^w(M, M^k) = \langle \lambda_i^k \rangle$$

$$= (\lambda_1^k, \lambda_2^k, ..., \lambda_n^k)^T = \left(\frac{1}{m} \sum_{j=1}^{m} (1 - \frac{D_{ij}^k}{9}) W_j \right)_{i=1}^{n} \quad (8)$$

Here, k =1, 2, 3,…, r.

Step 5. Ranking of alternatives

In order to rank alternatives, we propose the formula (see eq.9):

$$\rho_i = \sum_{k=1}^{r} \psi_k \lambda_i^k \quad (9)$$

We arrange alternatives according to the descending order values of ρ_i. The highest value of ρ_i (i= 1, 2, 3,…, n) reflects the best alternative.

5 Numerical example

We solve a MCGDM problem adapted from [108] to demonstrate the applicability and effectiveness of the proposed method. Assume that an investment company wants to invest a sum of money in the best option. The investment company forms a decision making committee comprising of three members (k_1, k_2, k_3) to make a panel of four alternatives to invest money. The alternatives are Car company (α_1), Food company (α_2), Computer company

(α_3) and Arm company (α_4). Decision makers take decision based on the criteria namely, risk analysis (β_1), growth analysis (β_2), environment impact (β_3) and criterion weights are provided by the decision makers in terms of linguistic variables that can be converted into NCS.(See Table 1).

Table 1: Linguistic term for rating of attribute/ criterion

Linguistic terms	NCS
Very important (VI)	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>
Important (I)	<[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)>
Medium (M)	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>
Unimportant (UI)	<[.3, .4], [.5, .6], [.5, .7], (.4, .6, .7)>
Very unimportant (VUI)	<[.1, .2], [.6, .8], [.7, .9], (.2, .8, .9)>

Step1. Formation of ideal NCS decision matrix

We construct ideal NCS decision matrix (see eq.(10).

$$M=\begin{pmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \\ \alpha_2 & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \\ \alpha_3 & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \\ \alpha_4 & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> & <[1,1],[0,0],[0,0],(1,0,0)> \end{pmatrix} \quad (10)$$

Step 2. Construction of NCS decision matrix

The NCS decision matrices are constructed for four alternatives with respect to the three criteria.

Decision matrix for k_1 in NCS form

M^1 =

$$M^1=\begin{pmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> \\ \alpha_2 & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \\ \alpha_3 & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> \\ \alpha_4 & <[.3,.4],[.5,.6],[.5,.7],(.4,.6,.7)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \end{pmatrix}$$

Decision matrix for k_2 in NCS form

M^2 =

$$\begin{pmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & <[.3,.4],[.5,.6],[.5,.7],(.4,.6,.7)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \\ \alpha_2 & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \\ \alpha_3 & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> \\ \alpha_4 & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \end{pmatrix}$$

Decision matrix for k_3 in NCS form

M^3 =

$$\begin{pmatrix} & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> \\ \alpha_2 & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> \\ \alpha_3 & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> & <[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)> \\ \alpha_4 & <[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)> & <[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)> & <[.3,.4],[.5,.6],[.5,.7],(.4,.6,.7)> \end{pmatrix}$$

Step 3. Determination of attribute weight

The linguistic terms shown in Table 1 are used to evaluate each attribute. The importance of each attribute for every decision maker is rated with linguistic terms shown in Table 2. Linguistic terms are converted into NCS (See Table 3.) .

Table 2. Attribute rating in linguistic variables

	β_1	β_2	β_3
K_1	VI	M	I
K_2	VI	VI	M
K_3	M	VI	M

Table 3. Attribute rating in NCS

	β_1	β_2	β_3
K_1	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>	<[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)>
K_2	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>
K_3	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>

Using eq. (6) and eq. (7), we obtain the attribute weights as follows: $w_1 = .36, w_2 = .37, w_3 = .27$. (11)

Step 4. Calculation of weighted similarity measures

We now calculate weighted similarity measures using the formula (8).

$$S^w(M, M^1) = \begin{pmatrix} .25 \\ .22 \\ .19 \\ .24 \end{pmatrix}, S^w(M, M^2) = \begin{pmatrix} .18 \\ .20 \\ .25 \\ .22 \end{pmatrix}, S^w(M, M^3) = \begin{pmatrix} .20 \\ .21 \\ .25 \\ .20 \end{pmatrix} \quad (12)$$

Step 5. Ranking of alternatives

We rank the alternatives according to the descending value of ρ_i (i = 1, 2, 3, 4) using eq.(10), eq.(11), and eq. (12).

We obtain $\rho_1 = .202, \rho_2 = .206, \rho_3 = .232, \rho_4 = .216$, Therefore the ranking order is

$\rho_3 > \rho_4 > \rho_2 > \rho_1 \Rightarrow \alpha_3 > \alpha_4 > \alpha_2 > \alpha_1$.

Hence Computer company (α_3) is the best alternative for money investment.

6 Conclusion

In this paper we have defined similarity measure between neutrosophic cubic sets and proved its basic properties . We have developed a new multi criteria group decision making method basd on the proposed similarity measure. We also provide an illustrative example for multi criteria group decision making method to show its applicability and effectiveness. We have employed linguistic variables to present criteria weights and presented conversion of linguistic variables into neutrosophic cubic numbers. We have also proposed a conversion formula for neutrosophic cubic number into fuzzy number. The poposed method can be applied to other MCGDM making problems in neutrosophic cubic set environment such as banking system, engineering problems, school choice problems, teacher selection problem, etc. We also hope that the proposed method will open up a new direction of research work in neutrosophic cubic set environment.

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NSS University of New Mexico

Neutrosophic Crisp Mathematical Morphology

Eman.M.El-Nakeeb,[1,a] Hewayda ElGhawalby,[1,b] A.A.Salama,[2,c] S.A.El-Hafeez[2,d]

[1,2]Port Said University, Faculty of Engineering, Physics and Engineering Mathematics Department, Egypt

a) emanmarzouk1991@gmail.com

b) *hewayda2011@eng.psu.edu.eg*

[3,4]Port Said University , Faculty of Science, Department of Mathematics and Computer Science, Egypt

c) *drsalama44@gmail.com*

d) samyabdelhafeez@yahoo.com

Abstract In this paper, we aim to apply the concepts of the neutrosophic crisp sets and its operations to the classical mathematical morphological operations, introducing what we call "Neutrosophic Crisp Mathematical Morphology". Several operators are to be developed, including the neutrosophic crisp dilation, the neutrosophic crisp erosion, the neutrosophic crisp opening and the neutrosophic crisp closing.
Moreover, we extend the definition of some morphological filters using the neutrosophic crisp sets concept. For instance, we introduce the neutrosophic crisp boundary extraction, the neutrosophic crisp Top-hat and the neutrosophic crisp Bottom-hat filters.
The idea behind the new introduced operators and filters is to act on the image in the neutrosophic crisp domain instead of the spatial domain.

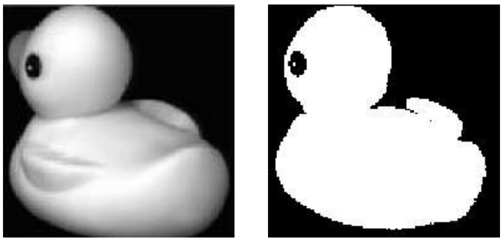
Keywords: Neutrosophic Crisp Set, Neutrosophic Sets, Mathematical Morphology, Filter Mathematical Morphology.

1 Introduction

In late 1960's, a relatively separate part of image analysis was developed; eventually known as "The Mathematical Morphology". Mostly, it deals with the mathematical theory of describing shapes using sets in order to extract meaningful information's from images, the concept of neutrosophy was first presented by Smarandache [14]; as the study of original, nature and scape of neutralities, as well as their interactions with different ideational spectra. The mathematical treatment for the neutrosophic phenomena, which already exists in our real world, was introduced in several studies; such as in [2].
The authors in [15], introduced the concept of the neutrosophic set to deduce. Neutrosophic mathematical morphological operations as an extension for the fuzzy mathematical morphology.
In [9] Salama introduced the concept of neutrosophic crisp sets, to represent any event by a triple crisp structure. In this paper, we aim to use the idea of the neutrosophic crisp sets to develop an alternative extension of the binary morphological operations. The new proposed neutrosophic crisp morphological operations is to be used for image analysis and processing in the neutrosophic domain. To commence, we review the classical operations and some basic filters of mathematical morphology in both §2 and § 3.
A revision of the concepts of neutrosophic crisp sets and its basic operations, is presented in §4 . the remaining sections, (§5, §6 and §7), are devoted for presenting our new concepts for "Neutrosophic crisp mathematical morphology" and its basic operations, as well as some basic neutrosophic crisp morphological filters.

2 Mathematical Morphological Operations:

In this section, we review the definitions of the classical binary morphological operators as given by Heijmans [6]; which are consistent with the original definitions of the Minkowski addition and subtraction [4].
For the purpose of visualizing the effect of these operators, we will use the binary image show in Fig.1(b); which is deduced form the original gray scale image shown in Fig.1(a).



a) b)

Fig.1: a) *the Original grayscale image* b) *the Binary image*

2.1 Binary Dilation: (Minkowski addition)

Based on the concept of Minkowski addition, the dilation is considered to be one of the basic operations in mathematical morphology, the dilations is originally developed for binary images [5]. To commence, we consider any Euclidean space E and a binary image A in E, the Dilation of A by some structuring element B is defined by: $A \oplus B = \bigcup_{b \in B} A_b$, where A_b is the translate of the set A along the vector *b*, i.e., $A_b = \{a + b \in E / a \in A, b \in B\}$ A_b

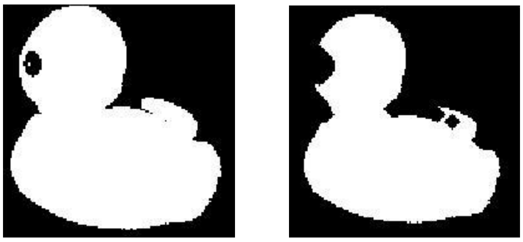
The Dilation is commutative, and may also be given by:
$A \oplus B = B \oplus A = \cup_{a \in A} B_a$
$A \oplus B = B \oplus B = \bigcup_{a \in A} B_a$

An interpretation of the Dilation of *A* by *B* can be understood as, if we put a copy of B at each pixel in A and union all of the copies, then we get $A \oplus B$.
The Dilation can also be obtained by: $A \oplus B = \{b \in E \mid (-B) \cap A \neq \emptyset\}$, where (–*B*) denotes the reflection of B, that is,
$-B = \{x \in E / -x \in B\}$
Where the reflection satisfies the following property: $-(A \oplus B) = (-A) \oplus (-B)$
$-(A \oplus B) = (-A) \oplus (-B)$
$-(A \oplus B) = (-A) \oplus (-B)$.

a) b)

Fig.2: *Applying the dilation operator: a) the Original binary image b) the dilated image.*

2. 2 Binary Erosion: (Minkowski subtraction)
Strongly related to the Minkowski subtraction, the erosion of the binary image A by the structuring element B is defined by: $A \ominus B = \bigcap_{b \in B} A_{-b}$ $A \Theta B = \bigcup_{b \in B} A_{-b}$ Unlike dilation, erosion is not commutative, much like how addition is commutative while subtraction is not [5]. hence $A \Theta B$ is all pixels in *A* that these copies were translated to. The erosion of *A* by *B* is also may be given by the expression:
$A \ominus B = \{p \in E \mid B_p \subseteq A\}$ where B_p is the translation of B by the vector p, i.e., $B_p = \{b + p \in E / b \in B\}, \forall p \in E$
$B_p = \{b + p \in E \mid b \in B\}, \forall p \in E$.



a) b)

Fig.3: *Applying the erosion operator: a) the Original binary image b) the eroted image.*

2. 3 Binary Opening [5]:
The Opening of A by B is obtained by the erosion of A by B, followed by dilation of the resulting image by B: $A \circ B = (A \Theta B) \oplus B$. $A \circ B = (A \ominus B) \oplus B$ The opening is also given by $A \circ B = \bigcup_{B_x \subseteq A} B_x$ $A \circ B = \cup_{B_x \subseteq A} B_x$, which means that, an opening can be consider to be the union of all translated copies of the structuring element that can fit inside the object. Generally, openings can be used to remove small objects and connections between objects.

a) b)

Fig.4: *Applying the opening operator: a) the Original binary image b) the image opening.*

2.4 Binary Closing [5]:

The closing of A by B is obtained by the dilation of A by B, followed by erosion of the resulting structure by B: $A \bullet B = (A \oplus B)\Theta B$ $A \bullet B = (A \oplus B) \ominus B$.

The closing can also be obtained by $A \bullet B = co(coA \circ co(-B))$ $A \bullet B = (A^c \circ (-B))^c$, where coA denotes the complement of A relative to E (that is, $coA = \{a \in E / a \notin A\}$ $A^c = \{a \in E \mid a \notin A\}$).

Whereas opening removes all pixels where the structuring element won't fit inside the image foreground, closing fills in all places where the structuring element will not fit in the image background, that is opening removes small objects, while closing removes small holes.

a) b)

Fig.5: *Applying the closing operator: a) the Original binary image b) the image closing.*

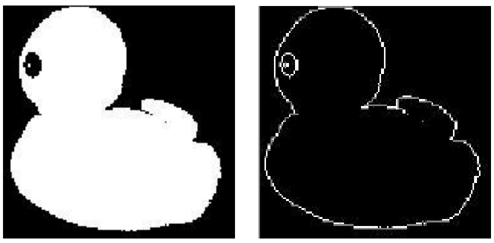
3. Mathematical Morphological Filters [13]:

In image processing and analysis, it is important to extract features of objects, describe shapes, and recognize patterns. Such tasks often refer to geometric concepts, such as size, shape, and orientation. Mathematical Morphology takes these concept from set theory, geometry, and topology to analyse the geometric structures in an image. Most essential image-processing algorithms can be represented in the form of Morphological operations.

In this section we review some basic Morphological filters, such as: the boundary extraction, and the Top-hat and the Bottom-hat filters.

3.1 The Boundary External [13]:

Boundary extraction of a set A requires first the dilating of A by a structuring element B and then taking the set difference between dilation and A. That is, the boundary of a set A is obtained by: $.\partial A = A - (A\Theta B)$



a) b)

Fig.6: *Applying the External Boundary: a) the Original binary image b) the External Boundary.*

3.2 The Hat Filters [13]:

In Mathematical Morphology and digital image processing, top-hat transform is an operation that extracts small elements and details from given images. There exist two types of hat filters: The Top-hat filter is defined as the difference between the input image and its opening by some structuring element; The Bottom-hat filter is defined dually as the difference between the closing and the input image. Top-hat filter are used for various image processing tasks, such as feature extraction, background equalization and image enhancement.

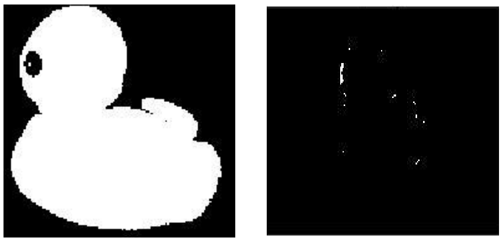
If an opening removes small structures, then the difference of the original image and the opened image should bring them out. This is exactly what the white Top-hat filter does, which is defined as the residue of the original and opening:

Top-hat filter: $T_{hat} = A - (A \circ B\ )$

The counter part of the Top-hat filter is the Bottom-hat filter which is defined as the residue of closing and the original:

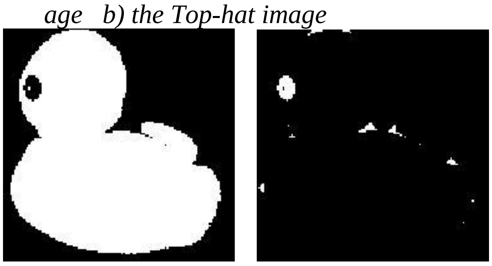
Bottom-hat filter: $B_{hat} = (A \bullet B\ ) - A$

These filters preserve the information removed by the Opening and Closing operations, respectively. They are often cited as white top-hat and black top-hat, respectively.



a) b)

Fig.7: *Applying the Top-hat: a) the Original binary image b) the Top-hat image*



a) b)

Fig.8: *Applying the Bottom-hat filter:*
a) the Original binary image b) Bottom-hat filter image

4. Neutrosophic Crisp Sets Theory [9]:

In this section we review some basic concepts of neutrosophic crisp sets and its operations.

4.1 Neutrosophic Crisp Sets:

4.1.1 Definition [9]

Let X X be a non-empty fixed set, a neutrosophic crisp set *A* (NCS for short), can be defined as a triple of the form $\langle A^1, A^2, A^3 \rangle$ $\langle A^1,\ A^2,\ A^3 \rangle$, where $A^1, A^2 and\ A^3$ and A^3 are crisp subsets of X. The three components represent a classification of the elements of the space X according to some event A; the subset A^1 contains all the elements of X that are supportive to A, A^3 contains all the elements of X that are against A, and A^2 contains all the elements of X that stand in a distance from being with or against A. A. Consequently, every crisp event A in X can be considered as a NCS having the form: $A = \langle A^1, A^2, A^3 \rangle$. The set of all neutrosophic crisp sets of X will be denoted $\mathcal{NC}(X)$.

Fig.9: *Neutrosophic Crisp Image:tivelycrespe* $\langle A^1, A^2, A^3 \rangle$

4.1.2 Definition [7, 9]:

The null (empty) neutrosophic set φ_N, the absolute (universe) neutrosophic set X_N and
the complement of a neutrosophic crisp set are defined as follows:

1) φ_N may be defined as one of the following two ϕ_N types:

Type 1:,$\varphi_N = \langle \varphi,\ \varphi,\ X \rangle$ $\phi_N = \langle \phi, \phi, X \rangle$

Type 2: $\phi_N = \langle \phi, X, X \rangle$ $\varphi_N = \langle \varphi,\ X,\ X \rangle$

2) 2) X_N may be defined as one of the following two types:

Type 1: $X_N = \langle X, X, \varphi \rangle$,

Type 2: $X_N = \langle X, \varphi, \varphi \rangle$.

3) The complement of a NCS (coA *co* A for short) may be defined as one of the following two types:

Type 1: $coA = \langle coA^1,\ coA^2,\ coA^3 \rangle$,

Type 2: $coA = \langle A^3,\ coA^2, A^1 \rangle$.

4.2. Neutrosophic Crisp Sets Operations:

In [6, 14], the authors extended the definitions of the crisp sets operations to be defined over Neutrosophic Crisp Sets (in short NCSs). In the following definitions we consider a non-empty set X, and any two Neutrosophic Crisp Sets of X, A and B, where $A = \langle A^1, A^2, A^3 \rangle$ and $B = \langle B^1,\ B^2,\ B^3 \rangle$.

4.2.1 Definition [8, 9]:

For any two sets A, B $\in \mathcal{NC}(X)$, A is said to be a neutrosophic crisp subset of the NCS B, i.e., ($A \subseteq B$), and may be defined as one of the following two types:

$$Type1: A \subseteq B \Leftrightarrow \langle A^1 \subseteq B^1, A^2 \subseteq B^2 and A^3 \supseteq B^3 \rangle$$

$$A \subseteq B \Leftrightarrow A^1 \subseteq B^1, A^2 \subseteq B^2 \text{ and } A^3 \supseteq B^3$$

$$A \subseteq B \Leftrightarrow A^1 \subseteq B^1, A^2 \supseteq B^2 \text{ and } A^3 \supseteq B^3$$

4.2.2 Proposition [7, 9]:

For any neutrosophic crisp set A, the following properties are hold:

a) $\phi_N \subseteq A$ and $\phi_N \subseteq \phi_N$

b) $A \subseteq X_N$ and $X_N \subseteq X_N$

4.2.3 Definition [7, 9]:

The neutrosophic intersection and neutrosophic union of any two neutrosophic crisp sets A, B $\in \mathcal{NC}(X)$, may be defined as follows:

1. The neutrosophic intersection $A \cap B$, may be defined as one of the following two types:

Type1: $A \cap B = \langle A^1 \cap B^1, A^2 \cap B^2, A^3 \cup B^3 \rangle$.

Type2: $A \cap B = \langle A^1 \cap B^1, A^2 \cup B^2, A^3 \cup B^3 \rangle$.

2. The neutrosophic union $A \cup B$, may be defined as one of the following two types:

Type 1: $A \cup B = \langle A^1 \cup B^1, A^2 \cup B^2, A^3 \cap B^3 \rangle$.

Type2: $A \cup B = \langle A^1 \cup B^1, A^2 \cap B^2, A^3 \cap B^3 \rangle$.

4.2.4 Proposition [7, 9]:

For any two neutrosophic crisp sets A, $B \in \mathcal{NC}(X)$, then:

$co(A \cap B) = coA \cup coB$

$co(A \cap B) = coA \cup coB$ and

$co(A \cup B) = coA \cap coB$

$co(A \cup B) = coA \cap coB$.

Proof: We can easily prove that the two statements are true for both the complement operators. Defined in definition 4.1.2.

4.2.5 Proposition [9]:

For any arbitrary family $\{A_i : i \in I\}$, of neutrosophic crisp subsets of X, a generalization for the neutrosophic intersection and for the neutrosophic union given in Definition 4.2.3, can be defined as follows:

1) $\bigcap_{i\in I} A_i$ may be defined as one of the following two types:

$$Type1: \bigcap_{i\in I} A_i = \left\langle \bigcap_{i\in I} A_i^1, \bigcap_{i\in I} A_i^2, \bigcup_{i\in I} A_i^3 \right\rangle$$

$$Type2: \bigcap_{i\in I} A_i = \left\langle \bigcap_{i\in I} A_i^1, \bigcup_{i\in I} A_i^2, \bigcup_{i\in I} A_i^3 \right\rangle$$

2) may be defined as one of the $\bigcup_{i\in I} A_i$ $\bigcup_{i\in I} A_i$ following two types:

$$Type1: \bigcup_{i\in I} A_i = \left\langle \bigcup_{i\in I} A_i^1, \bigcup_{i\in I} A_i^2, \bigcap_{i\in I} A_i^3 \right\rangle$$

$$Type2: \bigcup_{i\in I} A_i = \left\langle \bigcup_{i\in I} A_i^1, \bigcap_{i\in I} A_i^2, \bigcap_{i\in I} A_i^3 \right\rangle$$

5. Neutrosophic Crisp Mathematical Morphology:

As a generalization of the classical mathematical morphology, we present in this section the basic operations for the neutrosophic crisp mathematical morphology. To commence, we need to define the translation of a neutrosophic set.

5.1.1 Definition:

Consider the Space X=R^n or Z^n $X = R^n$ or Z^n With origin 0 = (0,...,0) given The reflection of the structuring element B mirrored in its Origin is defined as: $-B = \langle -B^1, -B^2, -B^3 \rangle$.

5.1 Definition:

For every the $p \in A$, translation by p is the map $p: X \to X, a \to a + p$ $p: X \to X,\ a \mapsto a + p$ it transforms any Subset A of X into its translate by $p \in Z^2$, $-B = \langle -B^1, -B^2, -B^3 \rangle$, $A_n = \langle A^1{}_n, A^2{}_n, A^3{}_n \rangle A_n = \langle A^1{}_n, A^2{}_n, A^3{}_n \rangle$

Where

$A_p^1 = \{u + p : u \in A^1, p \in B^1\}$

$= \{u + p : u \in A^1,\ p \in B^1\}$

$A^1{}_n(u) = \{u + p : u \in A^1,\ p \in B^1\}$

$A_p^2 = \{u + p : u \in A^2, p \in B^3\}$

$= \{u + p : u \in A^2,\ p \in B^2\}$

$A_p^3 = \{u + p : u \in A^3, p \in B^3\}$

$\{u + p : u \in A^3,\ p \in B^3\}$

5.2 Neutrosophic Crisp Mathematical Morphological Operations:

5.2.1 Neutrosophic Crisp Dilation Operator:

let A, B $\in \mathcal{NC}(X)$, then we define two types of the neutrosophic crisp dilation as follows:

Type1:

$A \tilde{\oplus} B = \langle A^1 \oplus B^1, A^2 \oplus B^2, A^3 \Theta B^3 \rangle$

$(A \oplus B) = \langle A^1 \oplus B^1, A^2 \oplus B^2, A^3 \ominus B^3 \rangle$, where for each u and $v \in Z^2 \in Z^2$.

$A^1 \oplus B^1 = \bigcup_{b\in B^1} A_b^1$ $\quad A^2 \Theta B^2 = \bigcup_{b\in B^2} A_{-b}^2$

$A^3 \Theta B^3 = \bigcap_{b\in B^3} A_{-b}^3$

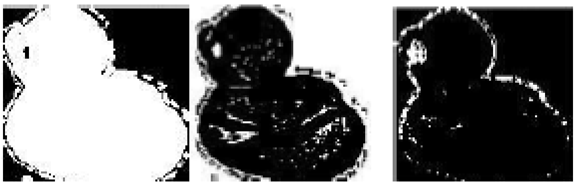


Fig.10(i): *Neutrosophic Crisp Dilation components in type 1* $\langle A^1, A^2, A^3 \rangle$ *respectively*

Type2:

$(A \oplus B) = \langle A^1 \oplus B^1, A^2 \ominus B^2, A^3 \ominus B^3 \rangle$,

where for each u and $v \in Z^2 \in Z^2$.

$A^1 \oplus B^1 = \bigcup_{b\in B^1} A_b^1$, $\quad A^2 \oplus B^2 = \bigcap_{b\in B^3} A_{-b}^2$

$A^3 \Theta B^3 = \bigcap_{b\in B^3} A_{-b}^3$

Fig.10(ii): *Neutrosophic Crisp Dilation components in type 2* $\langle A^1, A^2, A^3 \rangle$ *respectively*

5.2.2 Neutrosophic Crisp Erosion Operation:

let A and B $\in \mathcal{NC}(X)$; then the neutrosophic dilation is given as two type:

Type 1:

$(A \ominus B) = \langle A^1 \ominus B^1, A^2 \ominus B^2, A^3 \oplus B^3 \rangle$,

where for each u and $v \in Z^2 \cdot \in Z^2$ $A^1 \Theta B^1 = \bigcap_{b\in B^3} A_{-b}^1$

$A^2 \Theta B^2 = \bigcap_{b\in B^3} A_{-b}^2$ and $A^3 \oplus B^3 = \bigcup_{b\in B^1} A_b^3$

Fig.11(i): *Neutrosophic Crisp Erosion components in type 1* $\langle A^1, A^2, A^3 \rangle$ *respectively*

$A^1 \ominus B^1 = \bigcap_{b \in B^1} A^1_{-b}$, $A^2 \ominus B^2 = \bigcap_{b \in B^2} A^2_{-b}$, $A^3 \oplus B^3 = \bigcup_{b \in B^3} A^3_{b}$

Type2:

$(A \ominus B) = \langle A^1 \ominus B^1, A^2 \oplus B^2, A^3 \oplus B^3 \rangle$,

where for each u and $v \in Z^2 \in Z^2$. $A^1 \Theta B^1 = \bigcap_{b \in B^3} A^1_{-b}$

$A^2 \oplus B^2 = \bigcup_{b \in B^1} A^2_b$ and $A^3 \oplus B^3 = \bigcup_{b \in B^1} A^3_b$

Fig.11(ii): *Neutrosophic Crisp Erosion components in type2* $\langle A^1, A^2, A^3 \rangle$ *respectively*

5.2.3 Neutrosophic Crisp Opening Operation:

let A, B $\in \mathcal{NC}(X)$; then we define two types of the neutrosophic crisp dilation operator as follows:

Type 1: $A \tilde{\circ} B = \langle A^1 \circ B^1, A^2 \circ B^2, A^3 \bullet B^3 \rangle$

$A^1 \circ B^1 = (A^1 \Theta B^1) \oplus B^1$

$A^2 \circ B^2 = (A^2 \Theta B^2) \oplus B^2$

$A^3 \circ B^3 = (A^3 \oplus B^3) \Theta B^3$

$A^1 \circ B^1 = (A^1 \ominus B^1) \oplus B^1$

$A^2 \circ B^2 = (A^2 \ominus B^2) \oplus B^2$

Fig.12(i): *Neutrosophic Crisp opening components in type1* $\langle A^1, A^2, A^3 \rangle$ *respectively*

$A^3 \bullet B^3 = (A^3 \oplus B^3) \ominus B^3$

Type 2:

$A \tilde{\circ} B = \langle A^1 \circ B^1, A^2 \bullet B^2, A^3 \bullet B^3 \rangle$

$A^1 \circ B^1 = (A^1 \ominus B^1) \oplus B^1$

$A^1 \circ B^1 = (A^1 \Theta B^1) \oplus B^1$

$A^2 \bullet B^2 = (A^2 \oplus B^2) \Theta B^2$

$A^3 \bullet B^3 = (A^3 \oplus B^3) \Theta B^3$

Fig.12(ii): *Neutrosophic Crisp opening components in type2* $\langle A^1, A^2, A^3 \rangle$ *respectively*

5.2.4 Neutrosophic Crisp Closing Operation:

let A and B $\in \mathcal{NC}(X)$; then the neutrosophic dilation is given as two types:

Type 1: $A \tilde{\bullet} B = \langle A^1 \bullet B^1, A^2 \bullet B^2, A^3 \circ B^3 \rangle$

$A^1 \bullet B^1 = (A^1 \ominus B^1) \oplus B^1$

$A^3 \circ B^3 = (A^3 \oplus B^3) \ominus B^3$

$A^3 \circ B^3 = (A^3 \Theta B^3) \oplus B^3$

Fig.13(i): *Neutrosophic Crisp closing components in type1* $\langle A^1, A^2, A^3 \rangle$ *respectively*

Type 2: $A \tilde{\bullet} B = \langle A^1 \bullet B^1, A^2 \circ B^2, A^3 \circ B^3 \rangle$

$$A^1 \bullet B^1 = (A^1 \ominus B^1) \oplus B^1$$

$$A^3 \circ B^3 = (A^3 \oplus B^3) \ominus B^3$$

$$A^3 \circ B^3 = (A^3 \Theta B^3) \oplus B^3$$

Fig.13(ii): *Neutrosophic Crisp closing components in type2* $\langle A^1, A^2, A^3 \rangle$ *respectively*

6. Algebraic Properties in Neutrosophic Crisp:

In this section, we investigate some of the algebraic properties of the neutrosophic crisp erosion and dilation, as well as the neutrosophic crisp opening and closing operator [15].

6.1 Properties of the Neutrosophic Crisp Erosion Operator:

6.1.1 Proposition:

The Neutrosophic erosion satisfies the monotonicity for all A, B ∈ $\mathcal{NC}(Z^2)$.

$Type1: a) A \subseteq B \Rightarrow \langle A^1\Theta C^1, A^2\Theta C^2, A^3\Theta C^3 \rangle$

$\subseteq \langle B^1\Theta C^1, B^2\Theta C^2, B^3\Theta C^3 \rangle$

$A^1\Theta C^1 \subseteq B^1\Theta C^1, A^2\Theta C^2 \subseteq B^2\Theta C^2$

$A^3\Theta C^3 \supseteq A^3\Theta C^3$

$b) A \subseteq B \Rightarrow \langle C^1\Theta A^1, C^2\Theta A^2, C^3\Theta A^3 \rangle$

$\subseteq \langle C^1\Theta B^1, C^2\Theta B^2, C^3\Theta B^3 \rangle$

$C^1\Theta A^1 \subseteq C^1\Theta B^1, C^2\Theta A^2 \subseteq C^2\Theta B^2$

$C^3\Theta A^3 \supseteq C^3\Theta A^3$

$Type2: a) A \subseteq B \Rightarrow \langle A^1\Theta C^1, A^2\Theta C^2, A^3\Theta C^3 \rangle$

$\subseteq \langle B^1\Theta C^1, B^2\Theta C^2, B^3\Theta C^3 \rangle$

$A^1\Theta C^1 \subseteq B^1\Theta C^1, A^2\Theta C^2 \supseteq B^2\Theta C^2$

$A^3\Theta C^3 \supseteq A^3\Theta C^3$

$b) A \subseteq B \Rightarrow \langle C^1\Theta A^1, C^2\Theta A^2, C^3\Theta A^3 \rangle$

$\subseteq \langle C^1\Theta B^1, C^2\Theta B^2, C^3\Theta B^3 \rangle$

$C^1\Theta A^1 \subseteq C^1\Theta B^1, C^2\Theta A^2 \supseteq C^2\Theta B^2$

$C^3\Theta A^3 \supseteq C^3\Theta A^3$

Note that: Dislike the Neutrosophic crisp dilation operator, the Neutrosophic crisp erosion does not satisfy commutativity and the associativity properties.

6.1.2 Proposition: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ $(A_i | i \in I)$ *in* $\mathcal{N}(Z^2)$ and B $\mathcal{NC}(Z^2)$.

Type1: a) $\cap_{i\in I} A_i \tilde{\ominus} B)$ $\cap_{i\in I} A_i \tilde{\ominus} B = \cap_{i\in I}(A_i \tilde{\ominus} B)$

$$= \left\langle \bigcap_{i\in I} A_i^1\Theta B^1, \bigcap_{i\in I} A_i^2\Theta B^2, \bigcap_{i\in I} A_i^3 \oplus B^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(A_i^1\Theta B^1), \bigcap_{i\in I}(A_i^2\Theta B^2), \bigcap_{i\in I}(A_i^3 \oplus B^3) \right\rangle$$

b) $B\tilde{\Theta}\bigcap_{i\in I} A_i = \bigcap_{i\in I}(B\Theta A_i)$ $B \tilde{\ominus} \cap_{i\in I} A_i$

$$= \left\langle B^1\Theta\bigcap_{i\in I} A_i^1, B^2\Theta\bigcap_{i\in I} A_i^2, B^3 \oplus \bigcap_{i\in I} A_i^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(B^1\Theta A_i^1), \bigcap_{i\in I}(B^2\Theta A_i^2), \bigcap_{i\in I}(B^3 \oplus A_i^3) \right\rangle$$

Type2: a) $\cap_{i\in I} A_i \tilde{\ominus} B)$ $\cap_{i\in I} A_i \tilde{\ominus} B = \cap_{i\in I}(A_i \tilde{\ominus} B)$

$$= \left\langle \bigcap_{i\in I} A_i^1\Theta B^1, \bigcap_{i\in I} A_i^2 \oplus B^2, \bigcap_{i\in I} A_i^3 \oplus B^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(A_i^1\Theta B^1), \bigcap_{i\in I}(A_i^2 \oplus B^2), \bigcap_{i\in I}(A_i^3 \oplus B^3) \right\rangle$$

b) $B\tilde{\Theta}\bigcap_{i\in I} A_i = \bigcap_{i\in I}(B\Theta A_i)$ $B \tilde{\ominus} \cap_{i\in I} A_i$

$$= \left\langle B^1\Theta\bigcap_{i\in I} A_i^1, B^2 \oplus \bigcap_{i\in I} A_i^2, B^3 \oplus \bigcap_{i\in I} A_i^3 \right\rangle$$

$$= \left\langle \bigcap_{i\in I}(B^1\Theta A_i^1), \bigcap_{i\in I}(B^2 \oplus A_i^2), \bigcap_{i\in I}(B^3 \oplus A_i^3) \right\rangle$$

Proof: a) in two type:

Type 1: $\cap_{i\in I} A_i \oplus B \; \bigcap_{i\in I} A_i \Theta B$

$$= \left\langle \bigcap_{b\in B} (\bigcap_{i\in I} A^1_{ib}), \bigcap_{b\in B} (\bigcap_{i\in I} A^2_{ib}), \bigcup_{b\in B} (\bigcap_{i\in I} A^3_{i(-b)}) \right\rangle$$

$$\cap_{i\in I} A_i \oplus B = \langle \cup (\cap A^1_{ib}), \cup (\cap A^2_{ib}), \cap (\cap A^3_{i(-b)}) \rangle$$

$$= \left\langle \bigcap_{i\in I} (\bigcap_{b\in B} A^1_{i(-b)}), \bigcap_{i\in I} (\bigcup_{b\in B} A^2_{i(-b)}), \bigcap_{i\in I} (\bigcup_{b\in B} A^3_{ib}) \right\rangle$$

$$\langle \cap_{i\in I} (\cup_{b\in B} A^1_{ib}), \cap_{i\in I} (\cup_{b\in B} A^2_{ib}), \cap_{i\in I} (\cap_{b\in B} A^3_{i(-b)}) \rangle = \cap_{i\in I} (A_i \oplus B).$$

$$= \bigcap_{i\in I} (A_i \Theta B)$$

Type 2: similarity, we can show that it is true in type 2,

b) The proof is similar to point a).

6.1.3 Proposition: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$

Type 1: $a) \bigcup_{i\in I} A_i \tilde{\Theta} B = \bigcup_{i\in I} (A_i \tilde{\Theta} B)$

$$= \left\langle \bigcup_{i\in I} A^1_i \Theta B^1, \bigcup_{i\in I} A^2_i \Theta B^2, \bigcup_{i\in I} A^3_i \oplus B^3 \right\rangle$$

$$= \left\langle \bigcup_{i\in I} (A^1_i \Theta B^1), \bigcup_{i\in I} (A^2_i \Theta B^2), \bigcup_{i\in I} (A^3_i \oplus B^3) \right\rangle$$

$$b) B \tilde{\Theta} \bigcup_{i\in I} A_i = \bigcup_{i\in I} (B \tilde{\Theta} A_i)$$

$$= \left\langle B^1 \Theta \bigcup_{i\in I} A^1_i, B^2 \Theta \bigcup_{i\in I} A^2_i, B^3 \oplus \bigcup_{i\in I} A^3_i \right\rangle$$

$$= \left\langle \bigcup_{i\in I} (B^1 \Theta A^1_i), \bigcup_{i\in I} (B^2 \Theta A^2_i), \bigcup_{i\in I} (B^3 \oplus A^3_i) \right\rangle$$

Type 2: $a) \bigcup_{i\in I} A_i \tilde{\Theta} B = \bigcup_{i\in I} (A_i \tilde{\Theta} B)$

$$A \subseteq B \Rightarrow \langle A^1 \oplus C^1, A^2 \oplus C^2, A^3 \oplus C^3 \rangle \subseteq \langle B^1 \oplus C^1, B^2 \oplus C^2, B^3 \oplus C^3 \rangle$$

$$= \left\langle \bigcup_{i\in I} A^1_i \Theta B^1, \bigcup_{i\in I} A^2_i \oplus B^2, \bigcup_{i\in I} A^3_i \oplus B^3 \right\rangle$$

$$= \left\langle \bigcup_{i\in I} (A^1_i \Theta B^1), \bigcup_{i\in I} (A^2_i \oplus B^2), \bigcup_{i\in I} (A^3_i \oplus B^3) \right\rangle$$

$$b) B \tilde{\Theta} \bigcup_{i\in I} A_i = \bigcup_{i\in I} (B \tilde{\Theta} A_i)$$

$$= \left\langle B^1 \Theta \bigcup_{i\in I} A^1_i, B^2 \oplus \bigcup_{i\in I} A^2_i, B^3 \oplus \bigcup_{i\in I} A^3_i \right\rangle$$

$$= \left\langle \bigcup_{i\in I} (B^1 \Theta A^1_i), \bigcup_{i\in I} (B^2 \oplus A^2_i), \bigcup_{i\in I} (B^3 \oplus A^3_i) \right\rangle$$

Proof: a)

$$Type 1: \bigcup_{i\in I} A_i \tilde{\Theta} B =$$

$$\left\langle \bigcap_{b\in B} (\bigcup_{i\in I} A^1_{i(-b)}), \bigcap_{b\in B} (\bigcup_{i\in I} A^1_{i(-b)}), \bigcup_{b\in B} (\bigcup_{i\in I} A^1_i) \right\rangle$$

$$\left\langle \bigcup_{i\in I} (\bigcap_{b\in B} A^1_{i(-b)}), \bigcup_{i\in I} (\bigcap_{b\in B} A^2_{i(-b)}), \bigcup_{i\in I} (\bigcup_{b\in B} A^3_{ib}) \right\rangle = \bigcup_{i\in I} (A_i \tilde{\Theta} B)$$

Type 2: can be verified in a similar way as in type 1.

b) The proof is similar to point a)

6.2 Proposition: (Properties of the Neutrosophic Crisp Dilation Operator):

6.2.1 Proposition:

The neutrosophic Dilation satisfies the following properties: $\forall A, B \in \mathcal{NC}(Z^2)$ $\forall A, B \in \mathcal{N}(Z^2)$

i) Commutativity: $A \tilde{\oplus} B = B \tilde{\oplus} A$ $A \oplus B = B \oplus A$

ii) Associativity: $(A \tilde{\oplus} B) \tilde{\oplus} B = A \tilde{\oplus} (B \tilde{\oplus} B)$ $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

iii) Monotonicity: (increasing in both arguments):

Type1:

a) $A \subseteq B \Rightarrow \langle A^1 \oplus C^1, A^2 \oplus C^2, A^3 \oplus C^3 \rangle$

$$\subseteq \langle B^1 \oplus C^1, B^2 \oplus C^2, B^3 \oplus C^3 \rangle$$

$A^1 \oplus C^1 \subseteq B^1 \oplus C^1, A^2 \oplus C^2 \subseteq B^2 \oplus C^2$ and

$A^3 \oplus C^3 \supseteq B^3 \oplus C^3$

$C^1 \oplus A^1 \subseteq C^1 \oplus B^1$, $C^2 \oplus A^2 \subseteq C^2 \oplus B^2$ *and* $C^3 \oplus A^3 \supseteq C^3 \oplus B^3$

$$b) A \subseteq B \Rightarrow \langle C^1 \oplus A^1, C^2 \oplus A^2, C^3 \oplus A^3 \rangle$$

$$\subseteq \langle C^1 \oplus B^1, C^2 \oplus B^2, C^3 \oplus B^3 \rangle$$

$C^1 \oplus A^1 \subseteq C^1 \oplus B^1, C^2 \oplus A^2 \subseteq C^2 \oplus B^2$ and

$C^3 \oplus A^3 \supseteq C^3 \oplus B^3$

$A \subseteq B \Rightarrow$
$\langle C^1 \oplus A^1, C^2 \oplus A^2, C^3 \oplus A^3 \rangle \subseteq$
$\langle C^1 \oplus B^1, C^2 \oplus B^2, C^3 \oplus B^3 \rangle$

Type2:

$$a) A \subseteq B \Rightarrow \langle A^1 \oplus C^1, A^2 \oplus C^2, A^3 \oplus C^3 \rangle$$

$\subseteq \langle B^1 \oplus C^1, B^2 \oplus C^2, B^3 \oplus C^3 \rangle$

$A^1 \oplus C^1 \subseteq B^1 \oplus C^1, A^2 \oplus C^2 \supseteq B^2 \oplus C^2$ and

$A^3 \oplus C^3 \supseteq B^3 \oplus C^3$

$C^1 \oplus A^1 \subseteq C^1 \oplus B^1,\ C^2 \oplus A^2 \subseteq C^2 \oplus B^2$ *and* $C^3 \oplus A^3 \supseteq C^3 \oplus B^3$

$b) A \subseteq B \Rightarrow \langle C^1 \oplus A^1, C^2 \oplus A^2, C^3 \oplus A^3 \rangle$

$\subseteq \langle C^1 \oplus B^1, C^2 \oplus B^2, C^3 \oplus B^3 \rangle$

and $C^1 \oplus A^1 \subseteq C^1 \oplus B^1, C^2 \oplus A^2 \supseteq C^2 \oplus B^2$

$C^3 \oplus A^3 \supseteq C^3 \oplus B^3$

6.2.2 Proposition: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and B $\in \mathcal{NC}(Z^2)$ *and* $B \in \mathcal{N}(Z^2)$

Type 1: a) $\cap_{i\in I} A_i \tilde{\oplus} B = \cap_{i\in I}(A_i \tilde{\oplus} B)$

$= \langle \bigcap_{i\in I} A_i^1 \oplus B^1, \bigcap_{i\in I} A_i^2 \oplus B^2, \bigcap_{i\in I} A_i^3 \Theta B^3 \rangle$

$= \langle \bigcap_{i\in I} (A_i^1 \oplus B^1), \bigcap_{i\in I} (A_i^2 \oplus B^2), \bigcap_{i\in I} (A_i^3 \Theta B^3) \rangle$

b) $B \tilde{\oplus} \cap_{i\in I} A_i = \cap_{i\in I}(B \tilde{\oplus} A_i)$

$= \langle B^1 \oplus \bigcap_{i\in I} A_i^1, B^2 \oplus \bigcap_{i\in I} A_i^2, B^3 \Theta \bigcap_{i\in I} A_i^3 \rangle$

$= \langle \bigcap_{i\in I} (B^1 \oplus A_i^1), \bigcap_{i\in I} (B^2 \oplus A_i^2), \bigcap_{i\in I} (B^3 \Theta A_i^3) \rangle$

$\langle B^1 \oplus \cap_{i\in I} A^1{}_i, B^2 \oplus \cap_{i\in I} A^2{}_i, B^3 \ominus \cap_{i\in I} A^3{}_i \rangle = \langle \cap_{i\in I} (B^1 \oplus A^1{}_i), \cap_{i\in I} (B^2 \oplus A^2{}_i), \cap_{i\in I} (B^3 \ominus A^3{}_i) \rangle$

Type 2: a) $\cap_{i\in I} A_i \tilde{\oplus} B = \cap_{i\in I}(A_i \tilde{\oplus} B)$

$= \langle \bigcap_{i\in I} A_i^1 \oplus B^1, \bigcap_{i\in I} A_i^2 \Theta B^2, \bigcap_{i\in I} A_i^3 \Theta B^3 \rangle$

$= \langle \bigcap_{i\in I} (A_i^1 \oplus B^1), \bigcap_{i\in I} (A_i^2 \Theta B^2), \bigcap_{i\in I} (A_i^3 \Theta B^3) \rangle$

b) $B \tilde{\oplus} \cap_{i\in I} A_i = \cap_{i\in I}(B \tilde{\oplus} A_i)$

$= \langle B^1 \oplus \bigcap_{i\in I} A_i^1, B^2 \Theta \bigcap_{i\in I} A_i^2, B^3 \Theta \bigcap_{i\in I} A_i^3 \rangle$

$= \langle \bigcap_{i\in I} (B^1 \oplus A_i^1), \bigcap_{i\in I} (B^2 \Theta A_i^2), \bigcap_{i\in I} (B^3 \Theta A_i^3) \rangle$

Proof: we will prove this property for the two types of the neutrosophic crisp intersection operator:

Type 1: $\cap_{i\in I} A_i \oplus B \bigcap_{i\in I} A_i \oplus B$

$= \langle \bigcup_{b\in B} (\bigcap_{i\in I} A_{ib}^1), \bigcup_{b\in B} (\bigcap_{i\in I} A_{ib}^2), \bigcap_{b\in B} (\bigcap_{i\in I} A_{i(-b)}^3) \rangle$

$\cap_{i\in I} A_i \oplus B = \langle \cup (\cap A^1{}_{ib}), \cup (\cap A^2{}_{ib}), \cap (\cap A^3{}_{i(-b)}) \rangle$

$= \langle \bigcap_{i\in I} (\bigcup_{b\in B} A_{ib}^1), \bigcap_{i\in I} (\bigcup_{b\in B} A_{ib}^2), \bigcap_{i\in I} (\bigcap_{b\in B} A_{i(-b)}^3) \rangle$

$\langle \cap_{i\in I} (\cup_{b\in B} A^1{}_{ib}), \cap_{i\in I} (\cup_{b\in B} A^2{}_{ib}), \cap_{i\in I} (\cap_{b\in B} A^3{}_{i(-b)}) \rangle = \cap_{i\in I}(A_i \oplus B).$

$= \bigcap_{i\in I} (A_i \oplus B)$

Type 2: $\cap_{i\in I} A_i \oplus B$

$\langle$ MOGDM base Si $\rangle$

$= \langle \bigcap_{i\in I} (\bigcup_{b\in B} A_{ib}^1), \bigcap_{i\in I} (\bigcap_{b\in B} A_{ib}^2), \bigcap_{i\in I} (\bigcap_{b\in B} A_{i(-b)}^3) \rangle$

$\cap_{i\in I} A_i \oplus B = \langle \cup (\cap A^1{}_{ib}), \cap (\cap A^2{}_{i(-b)}), \cap (\cap A^3{}_{i(-b)}) \rangle$

$\bigcap_{i\in I} (A_i \oplus B)$ =

b) The proof is similar to a)

6.2.3 Proposition: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and B $\in \mathcal{NC}(Z^2)$

Type 1: $a) \bigcup_{i\in I} A_i \tilde{\oplus} B = \bigcup_{i\in I} (A_i \tilde{\oplus} B)$

$\langle \bigcup_{i\in I} A_i^1 \oplus B^1, \bigcup_{i\in I} A_i^2 \oplus B^2, \bigcup_{i\in I} A_i^3 \Theta B^3 \rangle$

$= \langle \bigcup_{i\in I} (A_i^1 \oplus B^1), \bigcup_{i\in I} (A_i^2 \oplus B^2), \bigcup_{i\in I} (A_i^3 \Theta B^3) \rangle$

$b) B \tilde{\oplus} \bigcup_{i\in I} A_i = \bigcup_{i\in I} (B \tilde{\oplus} A_i)$

$\langle B^1 \oplus \bigcup_{i\in I} A_i^1, B^2 \oplus \bigcup_{i\in I} A_i^2, B^3 \Theta \bigcup_{i\in I} A_i^3 \rangle$

$= \langle \bigcup_{i\in I} (B^1 \oplus A_i^1), \bigcup_{i\in I} (B^2 \oplus A_i^2), \bigcup_{i\in I} (B^3 \Theta A_i^3) \rangle$

Type 2: $a) \bigcup_{i\in I} A_i \tilde{\oplus} B = \bigcup_{i\in I} (A_i \tilde{\oplus} B)$

$\langle \bigcup_{i\in I} A_i^1 \oplus B^1, \bigcup_{i\in I} A_i^2 \Theta B^2, \bigcup_{i\in I} A_i^3 \Theta B^3 \rangle$

$= \langle \bigcup_{i\in I} (A_i^1 \oplus B^1), \bigcup_{i\in I} (A_i^2 \Theta B^2), \bigcup_{i\in I} (A_i^3 \Theta B^3) \rangle$

$b) B \tilde{\oplus} \bigcup_{i\in I} A_i = \bigcup_{i\in I} (B \tilde{\oplus} A_i)$

$= \langle B^1 \oplus \bigcup_{i\in I} A_i^1, B^2 \Theta \bigcup_{i\in I} A_i^2, B^3 \Theta \bigcup_{i\in I} A_i^3 \rangle$

$$=\left\langle \bigcup_{i\in I}(B^1\oplus A_i^1),\bigcup_{i\in I}(B^2\Theta A_i^2),\bigcup_{i\in I}(B^3\Theta A_i^3)\right\rangle$$

Proof: a) we will prove this property for the two types of the neutrosophic crisp union operator:

$$Type1:\bigcup_{i\in I}A_i\,\tilde{\oplus}\,B=\left\langle \bigcup_{b\in B}(\bigcup_{i\in I}A_{ib}^1),\bigcup_{b\in B}(\bigcup_{i\in I}A_{ib}^2),\bigcap_{b\in B}(\bigcup_{i\in I}A_{i(-b)}^3)\right\rangle$$

$$=\bigcup_{i\in I}(A_i\,\tilde{\oplus}\,B)\;=\left\langle \bigcup_{i\in I}(\bigcup_{b\in B}A_i^1),\bigcup_{i\in I}(\bigcup_{b\in B}A_{ib}^2),\bigcup_{i\in I}(\bigcap_{b\in B}A_{i(-b)}^3)\right\rangle$$

$$Type\ 2:\ \bigcup_{i\in I}A_i\,\tilde{\oplus}\,B=\left\langle \bigcup_{b\in B}(\bigcup_{i\in I}A_i^1),\bigcap_{b\in B}(\bigcup_{i\in I}A_{i(-b)}^2),\bigcap_{b\in B}(\bigcup_{i\in I}A_{i(-b)}^3)\right\rangle$$

$$=\bigcup_{i\in I}(A_i\,\tilde{\oplus}\,B)=\left\langle \bigcup_{i\in I}(\bigcup_{b\in B}A_i^1),\bigcup_{i\in I}(\bigcap_{b\in B}A_{i(-b)}^2),\bigcup_{i\in I}(\bigcap_{b\in B}A_{i(-b)}^3)\right\rangle$$

b) **The proof** is similar to (a)

6.2.4 Proposition (*Duality Theorem of Neutrosophic Crisp Dilation*):

let A, B $\in\mathcal{NC}(Z^2)$. Neutrosophic crisp Erosion and Dilation are dual operations i.e.

Type1:

$co(coA\oplus B)=\langle co(coA^1\oplus B^1),co(coA^2\oplus B^2),co(coA^3\ominus B^3)\rangle$

$$co(coA\tilde{\oplus}B)=\left\langle co(coA^1\oplus B^1),co(coA^2\oplus B^2),co(coA^3\Theta B^3)\right\rangle$$

$$=\left\langle A^1\Theta B^1,A^2\Theta B^2,A^3\oplus B^3\right\rangle$$

$\langle A^1\ominus B^1,A^2\ominus B^2,A^3\oplus B^3\rangle\ A\tilde{\Theta}B=$

Type2:

$co(coA\oplus B)=\langle co(coA^1\oplus B^1),co(coA^2\ominus B^2),co(coA^3\ominus B^3)\rangle$

=

$$\left\langle A^1\Theta B^1,A^2\oplus B^2,A^3\oplus B^3\right\rangle$$

$\langle A^1\ominus B^1,A^2\oplus B^2,A^3\oplus B^3\rangle$

$\langle A^1\ominus B^1,A^2\ominus B^2,A^3\oplus B^3\rangle\ A\tilde{\Theta}B=$

6.3 Properties of the Neutrosophic Crisp Opening Operator:

6.3.1 Proposition:

The neutrosophic opening satisfies the monotonicity $\forall A,B\in\mathcal{NC}(Z^2)$

$$Type1:\ A\subseteq B\Rightarrow\left\langle A^1\circ C^1,A^2\circ C^2,A^3\circ C^3\right\rangle$$

$$\subseteq\left\langle B^1\circ C^1,B^2\circ C^2,B^3\circ C^3\right\rangle$$

$$A^1\circ C^1\subseteq B^1\circ C^1,A^2\circ C^2\subseteq B^2\circ C^2,$$

$$A^3\circ C^3\supseteq B^3\circ C^3$$

$$Type2:\ A\subseteq B\Rightarrow\left\langle A^1\circ C^1,A^2\circ C^2,A^3\circ C^3\right\rangle$$

$$\subseteq\left\langle B^1\circ C^1,B^2\circ C^2,B^3\circ C^3\right\rangle$$

$$A^1\circ C^1\subseteq B^1\circ C^1,A^2\circ C^2\supseteq B^2\circ C^2$$

$$A^3\circ C^3\supseteq B^3\circ C^3$$

6.3.2 Proposition: for any family in $(A_i|i\in I)$ $(A_i|i\in I)$ $\mathcal{NC}(Z^2)$ and B $\in\mathcal{NC}(Z^2)$

$$Type1:\ \bigcap_{i\in I}A_i\,\tilde{\circ}\,B=\bigcap_{i\in I}(A_i\,\tilde{\circ}\,B)\qquad)\ \cap_{i\in I}A_i\,\tilde{\ominus}\,B$$

$\cap_{i\in I}(A_i\,\tilde{\ominus}\,B)$

$$\left\langle \bigcap_{i\in I}A_i^1\circ B^1,\bigcap_{i\in I}A_i^2\circ B^2,\bigcap_{i\in I}A_i^3\bullet B^3\right\rangle$$

$$=\left\langle \bigcap_{i\in I}(A_i^1\circ B^1),\bigcap_{i\in I}(A_i^2\circ B^2),\bigcap_{i\in I}(A_i^3\bullet B^3)\right\rangle$$

$$Type2:\ \bigcap_{i\in I}A_i\,\tilde{\circ}\,B=\bigcap_{i\in I}(A_i\,\tilde{\circ}\,B)\qquad)\ \cap_{i\in I}A_i\,\tilde{\ominus}\,B$$

$\cap_{i\in I}(A_i\,\tilde{\ominus}\,B)$

$$\left\langle \bigcap_{i\in I}A_i^1\circ B^1,\bigcap_{i\in I}A_i^2\bullet B^2,\bigcap_{i\in I}A_i^3\bullet B^3\right\rangle$$

$$=\left\langle \bigcap_{i\in I}(A_i^1\circ B^1),\bigcap_{i\in I}(A_i^2\bullet B^2),\bigcap_{i\in I}(A_i^3\bullet B^3)\right\rangle$$

6.3.3 Proposition: for any family $(A_i|i\in I)$ in $\mathcal{NC}(Z^2)$ and B $\in$ $\mathcal{NC}(Z^2)$

$$Type1:\ \bigcup_{i\in I}A_i\,\tilde{\circ}\,B=\bigcup_{i\in I}(A_i\,\tilde{\circ}\,B)\qquad)\ \cap_{i\in I}A_i\,\tilde{\ominus}\,B$$

$\cap_{i\in I}(A_i\,\tilde{\ominus}\,B)$

$$=\left\langle \bigcup_{i\in I}A_i^1\circ B^1,\bigcup_{i\in I}A_i^2\circ B^2,\bigcup_{i\in I}A_i^3\bullet B^3\right\rangle$$

$$=\left\langle \bigcup_{i\in I}(A_i^1\circ B^1),\bigcup_{i\in I}(A_i^2\circ B^2),\bigcup_{i\in I}(A_i^3\bullet B^3)\right\rangle$$

$$Type2:\bigcup_{i\in I}A_i\,\tilde{\circ}\,B=\bigcup_{i\in I}(A_i\,\tilde{\circ}\,B)$$

$$=\left\langle \bigcup_{i\in I}A_i^1\circ B^1,\bigcup_{i\in I}A_i^2\bullet B^2,\bigcup_{i\in I}A_i^3\bullet B^3\right\rangle$$

$$=\left\langle \bigcup_{i\in I}(A_i^1\circ B^1),\bigcup_{i\in I}(A_i^2\bullet B^2),\bigcup_{i\in I}(A_i^3\bullet B^3)\right\rangle$$

Proof: Is similar to the procedure used to prove the propositions given in § 6.1.3 and § 6.2.3.

6.4 Properties of the Neutrosophic Crisp Closing

6.4.1 Proposition:
The neutrosophic closing satisfies the monotonicity
$\forall A, B \in \mathcal{NC}(Z^2)$

Type1:

a) $A \subseteq B \Rightarrow \langle A^1 \bullet C^1, A^2 \bullet C^2, A^3 \bullet C^3 \rangle$
$\subseteq \langle B^1 \bullet C^1, B^2 \bullet C^2, B^3 \bullet C^3 \rangle$
$A^1 \bullet C^1 \subseteq B^1 \bullet C^1, A^2 \bullet C^2 \subseteq B^2 \bullet C^2,$
$A^3 \bullet C^3 \supseteq B^3 \bullet C^3$

Type2:

a) $A \subseteq B \Rightarrow \langle A^1 \bullet C^1, A^2 \bullet C^2, A^3 \bullet C^3 \rangle$
$\subseteq \langle B^1 \bullet C^1, B^2 \bullet C^2, B^3 \bullet C^3 \rangle$
$A^1 \bullet C^1 \subseteq B^1 \bullet C^1, A^2 \bullet C^2 \supseteq B^2 \bullet C^2,$
$A^3 \bullet C^3 \supseteq B^3 \bullet C^3$

6.4.2 Proposition: for any family $(A_i | i \in I)$ $(A_i | i \in I)$
In $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$

Type1: $\bigcap_{i\in I} A_i \,\tilde{\bullet}\, B = \bigcap_{i\in I} (A_i \,\tilde{\bullet}\, B)$) $\cap_{i\in I} A_i \tilde{\ominus} B$
$\cap_{i\in I}(A_i \tilde{\ominus} B)$
$\left\langle \bigcap_{i\in I} A_i^1 \bullet B^1, \bigcap_{i\in I} A_i^2 \bullet B^2, \bigcap_{i\in I} A_i^3 \circ B^3 \right\rangle$
$= \left\langle \bigcap_{i\in I} (A_i^1 \bullet B^1), \bigcap_{i\in I} (A_i^2 \bullet B^2), \bigcap_{i\in I} (A_i^3 \circ B^3) \right\rangle$

Type2: $\bigcap_{i\in I} A_i \,\tilde{\bullet}\, B = \bigcap_{i\in I} (A_i \,\tilde{\bullet}\, B)$) $\cap_{i\in I} A_i \tilde{\ominus} B$
$\cap_{i\in I}(A_i \tilde{\ominus} B)$
$\left\langle \bigcap_{i\in I} A_i^1 \bullet B^1, \bigcap_{i\in I} A_i^2 \circ B^2, \bigcap_{i\in I} A_i^3 \circ B^3 \right\rangle$
$= \left\langle \bigcap_{i\in I} (A_i^1 \bullet B^1), \bigcap_{i\in I} (A_i^2 \circ B^2), \bigcap_{i\in I} (A_i^3 \circ B^3) \right\rangle$

6.4.3 Proposition: for any family $(A_i | i \in I)$ $(A_i | i \in I)$
in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$

Type1: $\bigcup_{i\in I} A_i \,\tilde{\bullet}\, B = \bigcup_{i\in I} (A_i \,\tilde{\bullet}\, B)$) $\cap_{i\in I} A_i \tilde{\ominus} B$
$\cap_{i\in I}(A_i \tilde{\ominus} B)$
$= \left\langle \bigcup_{i\in I} A_i^1 \bullet B^1, \bigcup_{i\in I} A_i^2 \bullet B^2, \bigcup_{i\in I} A_i^3 \circ B^3 \right\rangle$
$= \left\langle \bigcup_{i\in I} (A_i^1 \bullet B^1), \bigcup_{i\in I} (A_i^2 \bullet B^2), \bigcup_{i\in I} (A_i^3 \circ B^3) \right\rangle$

Type2: $\bigcup_{i\in I} A_i \,\tilde{\bullet}\, B = \bigcup_{i\in I} (A_i \,\tilde{\bullet}\, B)$
$= \left\langle \bigcup_{i\in I} A_i^1 \bullet B^1, \bigcup_{i\in I} A_i^2 \circ B^2, \bigcup_{i\in I} A_i^3 \circ B^3 \right\rangle$
$= \left\langle \bigcup_{i\in I} (A_i^1 \bullet B^1), \bigcup_{i\in I} (A_i^2 \circ B^2), \bigcup_{i\in I} (A_i^3 \circ B^3) \right\rangle$

Proof: Is similar to the procedure used to prove the propositions given in § 6.1.3.

6.4.4 Proposition (*Duality theorem of Closing*):
let A, $B \in \mathcal{NC}(Z^2)$; Neutrosophic erosion and dilation are dual operations i.e.

Type1:
$co(coA \,\tilde{\bullet}\, B) = \langle co(coA^1 \bullet B^1), co(coA^2 \bullet B^2), co(coA^3 \circ B^3) \rangle$
$co(coA \,\tilde{\bullet}\, B) = \langle co(coA^1 \bullet B^1), co(coA^2 \bullet B^2), co(coA^3 \circ B^3) \rangle$
$= \langle A^1 \circ B^1, A^2 \circ B^2, A^3 \bullet B^3 \rangle = A \,\tilde{\circ}\, B$

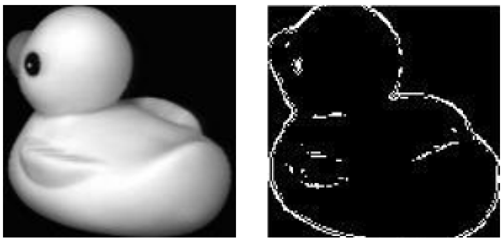
Type2:
$co(coA \,\tilde{\bullet}\, B) = \langle co(coA^1 \bullet B^1), co(coA^2 \circ B^2), co(coA^3 \circ B^3) \rangle$
$= \langle A^1 \circ B^1, A^2 \bullet B^2, A^3 \bullet B^3 \rangle = A \,\tilde{\circ}\, B$

7. Neutrosophic Crisp Mathematical Morphological Filters:

7.1 Neutrosophic Crisp External Boundary:

Where A^1 A^1 is the set of all pixels that belong to the foreground of the picture, A^3 A^3 contains the pixels that belong to the background whilecontains those A^2 A^2 pixel which do not belong to neither. A^3 nor A^1 A^1 A^3
Let A, $B \in \mathcal{NC}(Z^2)$, such that $A = \langle A^1, A^2, A^3 \rangle$ and B is some structure element of the form $B = \langle B^1, B^2, B^3 \rangle$; then the NC boundary extraction filter is defined to be:

$\partial_1 A^1 = A^1 - (A^1 \Theta B^1)$
$\partial_3 A^3 = (A^3 \Theta B^3) - A^3,$
$\partial(A) = A^2 - (\partial_1 A^1 \cup \partial_3 A^3)$
$\partial^*(A) = A^2 - [(A^3 \oplus B^3) - (A^1 \Theta B^1)]$
$b(A) = \partial^*(A) \cap \partial(A)$



a) b)

Fig. 14: *Applying the neutrosophic crisp External boundary: a) the Original image b) Neutrosophic crisp boundary.*

7.2 Neutrosophic Crisp Top-hat Filter:

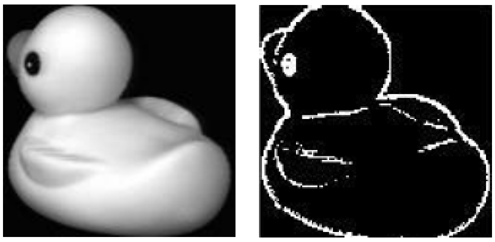
$$B_1(A^1) = A^1 - (A^1 \circ B^1)$$

$$B_3(A^3) = (A^3 \bullet B^3) - A^3$$

$$B(A) \ = A^2 - (B_1(A^1) \cup B_3(A^3))$$

$$B^*(A) \ = A^2 - \left[(A^1 \circ B^1) - (A^3 \bullet B^3)\right]$$

$$\tilde{Top}_{hat}(A) = B(A) \cap B^*(A)$$



a) b)

Fig. 15: *Applying the Neutrosophic crisp top-hat filter: a) Original image b) Neutrosophic Crisp components $\langle A^1, A^2, A^3 \rangle$ respectively*

7.3 Bottom-hat filter:

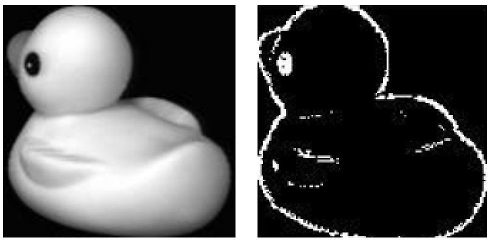
$$B_1(A^1) = (A^1 \bullet B^1) - A^1$$

$$B_3(A^3) = A^3 - (A^3 \circ B^3)$$

$$B(A) \ = A^2 - (B_1(A^1) \cup B_3(A^3))$$

$$B^*(A) \ = A^2 - \left[(A^1 \bullet B^1) - (A^3 \circ B^3)\right]$$

$$Bott\tilde{o}m_{hat}(A) = B(A) \cap B^*(A)$$



a) b)

Fig. 16: *Applying the Neutrosophic crisp Bottom-hat filter: Neutrosophic Crisp components $\langle A^1, A^2, A^3 \rangle$ respectively*

8 Conclusion:

In this paper we established a foundation for what we called "Neutrosophic Crisp Mathematical Morphology". Our aim was to generalize the concepts of the classical mathematical morphology.
For this purpose, we developed serval neutrosophic crisp morphological operators; namley, the neutrosophic crisp dilation, the neutrosophic crisp erosion, the neutrosophic crisp opening and the neutrosophic crisp closing operators. These operators were presented in two different types, each type is determined according to the behaviour of the seconed component of the triple strucure of the operator.
Furthermore, we developed three neutrosophic crisp morphological filters; namely, the neutrosophic crisp boundary extraction, the neutrosophic crisp Top-hat and the neutrosophic crisp Bottom-hat filters.
Some promising expermintal results were presented to visualise the effect of the new introduced operators and filters on the image in the neutrosophic domain instead of the spatial domain.

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NSS University of New Mexico

Neutrosophic Rough Soft Set – A Decision Making Approach to Appendicitis Problem

Kanika Bhutani
Department of Computer Engineering
NIT Kurukshetra
Kurukshetra, India
kanikabhutani91@gmail.com

Swati Aggarwal
COE
NSIT, Dwarka
Delhi, India
swati1178@gmail.com

***Abstract*—Classification based on fuzzy logic techniques can handle uncertainty to a certain extent as it provides only the fuzzy membership of an element in a set. This paper implements the extension of fuzzy logic: Neutrosophic logic to handle indeterminacy, uncertainty effectively. Classification is done on various techniques based on Neutrosophic logic i.e. Neutrosophic soft set, rough Neutrosophic set, Neutrosophic ontology to provide better results in comparison to fuzzy logic based techniques. It is proved that rough neutrosophic soft set will handle indeterminacy effectively that exists in the medical domain as it provides the minimum and maximum degree of truth, indeterminacy, falsity for every element.**

Keywords—Fuzzy set; Neutrosophic soft set; Rough Neutrosophic set, Rough Neutrosophic soft set.

I. Introduction

Classification can be described as a procedure in which different items are identified, differentiated and inferenced [1]. Classification is followed by collecting the instances of appendicitis disease of different patients so that we would be able to do a comparative study on the various symptoms of the disease. There exist many techniques which are used for classification and give a practical answer to feasible inputs [2]. Fuzzy logic is of great interest because of its ability to deal with non-statistical ambiguity. In decision making, ambiguous data is treated probabilistically in numerical format. Indeterminacy is present everywhere in real life. If a die is tossed on a irregular surface then there is no clear face to see. Indeterminacy occurs due to defects in creation of physical space or defective making of physical items involved in the events. Indeterminacy occurs when we are not sure of any event. Neutrosophic logic will help us to consider this indeterminacy.

This paper is written to concentrate on the classification of ambiguous, uncertain and incomplete data. Authors here propose a new technique of classification based on Neutrosophic rough soft set to handle indeterminacy. Neutrosophic rough soft set helps us to calculate the lower and upper approximation for every class.

II. Preliminaries & Basic Definitions

This section provides the definition of various techniques based on fuzzy logic and Neutrosophic logic. In further sections, these techniques are used for classification of data. Fuzzy logic was described by L.A.Zadeh in 1965[3].Fuzzy logic is a multivalued logic in which the membership of truth lies in 0-1[3].

Definition 1. Fuzzy set

A fuzzy set *X* over *U*which is considered as Universe is a function defined as[4]:

$$X = \{\mu_x(u)/u : u \in U\} \quad (1)$$

where $\mu_x : U \to [0,1]$ μ_x is known as the membership function of *X*, the value $\mu_x(u)$ is known as the degree of membership of $u \in U$ .Membership value can lie between *0* and *1*.

Definition 2. Neutrosophic set[5]

A Neutrosophic set *A* in *U*which is considered as a space of items, is described by a truth-membership function T_A, a indeterminacy-membership function I_A and a falsity-membership function F_A[5].An element belonging to *U*is represented by*u*.

$$A = \{< x, (T_A(x), I_A(x), F_A(x)) >: x \in U, T_A(u), I_A(u), F_A(u) \subseteq [0,1]\} \quad (2)$$

There is no restriction on the sum of $T_A(u)$, $I_A(u)$ and $F_A(u)$, so $^-0 \le \sup T_A(u) + \sup I_A(u) + \sup F_A(u) \le 3$. The sum of the three degrees has no restriction as it can lie from 0-3.

Definition 3. Soft set[6]

A soft set F_A over *U*which is considered as Universe, is a set defined by a set valued function f_A representing a mapping

$$f_A : E \to P(U) \text{ such that } f_A(x) = \varnothing \text{ if } x \in E - A \quad (3)$$

where f_A is called approximate function of soft set F_A.

$$F_A = \{(x, f_A(x)) : x \in E, f_A(x) = \varnothing \text{ if } x \in E - A \quad (4)$$

*E*is the set of parameters that describe the elements of *U* and $A \in E$. The subscript *A* in f_A indicates that f_A is approximate function of F_A and is called as called *x-element* of soft set for every $x \in E$.

Definition 4. Neutrosophic soft set (NSS)[7]

Let U be a universe, $N(U)$ is the set of all neutrosophic sets on U, E is the set of parameters that describe the elements of U and . A Neutrosophic soft set N over U is a set described by a set valued function f_N representing mapping

$$f_N : A \to N(U) \text{ such that } f_N(x) = \emptyset \text{ if } x \in E - A \quad (5)$$

where f_N is called approximate function of Neutrosophic soft set N.

$$N = \{(x, f_N(x)) : x \in E, f\ (x) = \emptyset \text{ if } x \in E - A\} \quad (6)$$

Definition 5. Rough Neutrosophic set (RNS)[8]

Let U be a Universe of non-null values and R is any equivalence relation on U. Consider F is any Neutrosophic set in U with its belonginess, ambiguity and non-belonginess function. The lower and higher approximations of F in the approximation(U,R) which is represented by and $\overline{N}(F)$ are defined as

$$\underline{N}(F) = \{< x, \mu_{\underline{N}(F)}(x), \nu_{\underline{N}(F)}(x), \omega_{\underline{N}(F)}(x) > \mid y \in [x]_R, x \in U\} \quad (7)$$

$$\overline{N}(F) = \{< x, \ _{\overline{N}(F)}(x), \nu_{\overline{N}(F)}(x), \omega_{\overline{N}(F)}(x) > \mid y \in [x]_R, x \in U\} \quad (8)$$

where

$$\mu_{\underline{N}(F)}(x) = \wedge_{y \in [x]_R} \mu_F(y), \nu_{\underline{N}(F)}(x) = \vee_{y \in [x]_R} \nu_F(y), \omega_{\underline{N}(F)}(x) = \vee_{y \in [x]_R} \omega_F(y) \quad (9)$$

$$\mu_{\overline{N}(F)}(x) = \vee_{y \in [x]_R} \mu_F(y), \nu_{\overline{N}(F)}(x) = \wedge_{y \in [x]_R} \nu_F(y), \omega_{\overline{N}(F)}(x) = \wedge_{y \in [x]_R} \omega_F(y) \quad (10)$$

where and mean min and max operators. The pair $(\underline{N}(F), \overline{N}(F))$ is called rough Neutrosophic set in (U,R). R is an equivalence relation over U.

Definition 6. Rough Neutrosophic soft set (RNSS)

Authors here propose a new technique of Neutrosophic rough soft set by combining the concept of Neutrosophic soft set and rough Neutrosophic set. RNSS will provide the lower and upper approximations for every class available.

Let U be a Universe of non-null values and R is any equivalence relation on U. Consider F is a set of every neutrosophic set in U with its belonginess, ambiguity and non-belonginess function. Thelower and upper approximations of all the Neutrosophic sets can be calculated with min and max operator using eq. 9,10.

Indeterminacy is present everywhere in real life. If weather experts will say that there is a chance of rain tomorrow is 60% then it does not specify that the chance of not raining is 40% as there are many factors like weather fronts etc which are not considered in weather reports. Various doctors may have different opinions on the same disease diagnosis so, indeterminacy can be seen in real life.

Neutrosophic logic was proposed by Florentine Smarandache to present mathematical model of uncertainty and indeterminacy.In Neutrosophic logic, each idea is estimated to have the percentage of truth, indeterminacy and falsity. [5]

Consider U be any set of buildings and E is the set of parameters. Every parameter is a Neutrosophic word. Consider E= {wooden, expensive, beautiful, cheap}. To define a Neutrosophic soft set, there is a need to point out wooden buildings, expensive buildings and so on. Let us assume that there are three buildings in the universe U given by U={b1,b2,b3} and set of parameters A={e1,e2,e3,e4} where e1 represents wooden, e2 represents expensive and so on.

F(wooden)={<b1,0.6,0.3,0.4>,<b2,0.4,0.6,0.6>,<b3,0.6,0.4,0.2>},

F(expensive)={<b1,0.7,0.4,0.5>,<b2,0.6,0.2,0.4>,<b3,0.7,0.4,0.3>},

F(beautiful)={<b1,0.8,0.2,0.1>,<b2,0.6,0.7,0.6>,<b3,0.8,0.4,0.3>},

F(cheap)={<b1,0.8,0.2,0.7>,<b2,0.4,0.6,0.4>,<b3,0.7,0.3,0.2>}.

F(e1) means buildings(wooden) whose value of function is the Neutrosophic set {<b1,0.6,0.3,0.4>, <b2,0.4,0.5,0.6>,<b3,0.6,0.4,0.2>}.

Each approximation has two parts: predicate p and an approximate value-set v. For the approximation 'wooden buildings={<b1,0.6,0.3,0.4>,<b2,0.4,0.6,0.6>,<b3,0.6,0.4,0.2>}, predicate is wooden buildings and approximate value set is {<b1,0.6,0.3,0.4>, <b2,0.4,0.6,0.6>,<b3,0.6,0.4,0.2>}.

The concept rough neutrosophic concept is introduced by combining both rough set and Neutrosophic set. These are the generalizations of rough fuzzy sets and rough intuitionistic fuzzy sets[8].

Let U={p1,p2,p3,p4} be a universe and R be an equivalence relation its partition of U is given as

U/R = {{p1,p2},p4}

Let N(F)={(p1.(0.3,0.2,0.5)),(p2,(0.3,0.2,0.5)),(p3,(0.4,0.5,0.2))}.

$\underline{N}(F)$ ={(p1,(0.3,0.2,0.5)),(p2,(0.3,0.2,0.5)),(p3,(0.4,0.5,0.2))}

$\overline{N}(F)$ ={(p1,(0.3,0.2,0.5)),(p2,(0.3,0.2,0.5)),(p3,(0.4,0.5,0.2))}

RNSS will calculate the lower and upper approximations for all the elements of universe U. All elements must exist in one of those partition elements.

III. How Rough Neutrosophic Rough Set is Better Than Fuzzy Set

Rough Neutrosophic soft set is combination of Neutrosophic soft set and rough Neutrosophic set. RNSS is based on Neutrosophic logic and fuzzy set is based on fuzzy logic instituted by L.A. Zadeh. In this logic, every proposition is estimated to have the degree of truth, indeterminacy and falsity (T,I,F). Neutrosophic soft set will provide predicate and approximate value set for every instance of classification data. Fuzzy set is a subset of Neutrosophic set and it provides the degree of membership and non-membership of any instance.

Rough Neutrosophic soft set provides the lower and upper approximations i.e. minimum and maximum degree of truth, indeterminacy and falsity.

For example, In case of fuzzy logic if a person is suffering from dengue havingdegree of membership as 0.6 i.e. Person is said to be having 60% chance of dengue and 40 % chance of not suffering from dengue. So, fuzzydegree of membership to a class is represented by fuzzy set.

In case of Neutrosophic logic if a person is suffering from dengue havinga membership value of 0.6 i.e. Person is said to be having 60% chances of dengue but not necessarily having 40% chances of not suffering from dengue, no inference can be made about the 40%. In reality Neutrosophic logic is effective in providing the degree of truth, indeterminacy, falsity that a person has in favour of dengue as there are many indeterminate factors which are not considered by doctors. Authors here propose torepresent Neutrosophic logic by experimenting with Rough Neutrosophic soft set, that suitably captures the indeterminacy, which is not captured by fuzzy set.

IV. Details of Appendicitis Dataset

Appendicitis dataset is chosen here for research from knowledge extraction based on evolutionary learning (KEEL)[9]. This dataset has 7 attributes which are defined in 2 classes and are of real- value type. It has 106 instances as shown in Fig. 1. The seven different attributes are standardised in the range of 0-100 by multiplying each attribute by 100.

The various attributes to be tested are WBC1, MNEP, MNEA, MBAP, MBAA, HNEP, HNEA.

Classes to be classify:-

0 means the patient suffers from appendicitis.

1 means the patient does not sufferfrom appendicitis.

In this research, we have collected the appendicitis dataset samples from knowledge extraction based on evolutionary learning. Using some training we have designed a fuzzy inference system that is able to classify an unknown appendicitis sample and on the behalf of the learning tuples it is able to predict the class to which that particular unknown sample belongs to whether the patient has appendicitis or not. Pursuing this research further will contribute us in designing a Neutrosophic inference system or Neutrosophic classifier. It has been suggested on the lines of fuzzy logic but instead of giving one defuzzified value, output value in neutrosophic classifier takes the neutrosophic format of the type: output (truthness, indeterminacy, falsity) . Then we will be able to predict more accurately in the overlapping sections of the attributes. Here, 96 instances are used for training and 10 instances which are randomly selected are used for testing i.e. 9:1.

V. Fuzzy set Based Classification

Fuzzy set is a component of standard information theory. It shows vague probabilities with ties to concepts of random sets. It shares the frequent attribute of all uncertain probability models, the indeterminacy of an object is described in terms of probability or with bounds on probability. Fuzzy logic is a many-valued logic that deals with reasoning which is approximate not exact. Comparing with traditional binary sets, fuzzy logic variables may have a truth value that ranges between 0 and 1. Fuzzy classification is the process of collecting elements into a fuzzy set whose membership function is described by the truth value of a fuzzy propositional function. In fuzzy classification, a sample can have membership in various classes to varying degrees.Typically, the membership values are restricted so that all of the membership values for a specific sample sum to Linguistic rules related to the control system composing two parts; an antecedent part (between the IF and THEN) and a consequent part (after THEN). A variable is fuzzy if its ambiguity arises as a consequence of imprecision and vagueness and is describes by a membership function. There can be unlimited number of membership functions that can be used to represent a fuzzy set. For fuzzy sets, membership function increases the flexibility by sacrificing distinctiveness as we can regulate a membershipfunction so as to expand the service for a specific purpose. We use membership function as a curve or shape to describethe degree of membership each point in the input zone or universe of discourse. The mandatory condition for a membership function to satisfy is that it must be in the range of [0,1]. The membership functions constitute of different types of mathematical expressions and geometric shapes like triangular, trapezoidal, bell etc. We can choose a membership function from a wide selection range provided by MATLAB Fuzzy Logic ToolBox. There are 11 in-built membership functions included in Fuzzy Logic ToolBox, Triangular and Trapezoidal membership functions

A. Determination of fuzzy membership and non-membership values

Fuzzy logic determines the basis of classification for fuzzy set. For all the attributes and output classes of appendicitis dataset, suitable rules are designed to account for the overlapping expected in fuzzy logic. As per observation, in the inference system, three types of outputs are produced after defuzzification as shown by Fig. 1. Defuzzified value or crisp value is obtained by applying various defuzzification techniques [10] to fuzzified value given by the inference module.

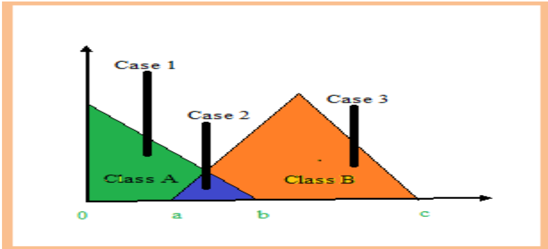

Fig.1. Criteria for assigning fuzzy values

Case 1. It provides the grade of membership and non-membership to class A. So, an output which belongs in the range of 0-a will support greater membership value for class A and smaller membership value for class B.

Case 2. There is some degree of indeterminacy for the output value lying in the overlapping range of a-b. Higher membership to class A is shown by range a-a+b/2, greaterdegree of belongingness to class B is shown by range a+b/2-b. Equal degree of membership to both classes is shown at point a+b/2, that cannot be classifiedinto any class. Neutrosophic logic is applied in the overlapping region where we are not sure about the existence of instance to class A or class B. In neutrosophic logic, every proposition is estimated to have some grade of truth, indeterminacy and falsity (T,I,F)[5]. Thus, to find the solution in overlapping areas, Neutrosophic logic comes to the rescue.

Case 3. It provides the grade of membership and non-membership to class B.So, an output lying in the range of b-c, will support greaterdegree of membership for class B and smaller degree of membership for class A.

VI. Rough Neutrosophic Soft Setbased Classification

Rough Neutrosophic soft set is a description of each instance that belongs to the overlapping area. Each instance of rough Neutrosophic soft sethelps us to examine the probability of existence to a class with grade of truth, indeterminacy, falsity in that range.In the medical domain, there is a lot of ambiguity, indeterminacy and uncertainty as different doctors have different opinions on the same diagnosis. So, Neutrosophic logic would prove effective by considering the existing indeterminacy in medical domain and by providing the grade of indeterminacy for each instance. Hence by classifying the appendicitis data into three classes, the Neutrosophic logic will provide better results.

A. Determination of Neutrosophic membership values

Rough Neutrosophic soft set works on the same dimension like fuzzy set, however it differs in the representation of output value. Output value after defuzzification, is described in the triplet format i.e. truthness, indeterminacy, falsity [5]. After obtaining the value in triplet form, it calculates the lower and upper approximations for every class existing in the universe. Neutrosophic logic will be applied in the overlapping regions to check whether the instance exists in class appendicitis or not. The design of Neutrosophic components is described in Fig. 2.

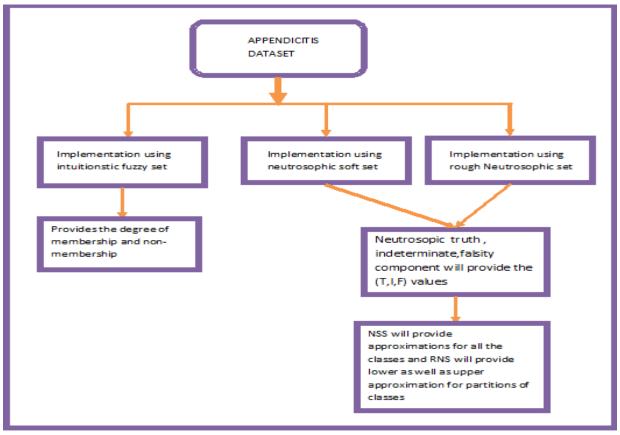

Fig. 2.Block diagram of neutrosophic components

Data using Rough Neutrosophic soft set is classified using the following steps:

1. The training sets and the testing sets are createdfor each class. Out of the 106 instances, 96 instances i.e. 90% of the total are used for training and 10 instances i.e. 10% of the total are used for testing.

2. Three components are used to express Neutrosophic logic: Neutrosphic truth, neutrosophic indeterminacy and neutrosophic falsity component[11]

3. Truth component of Neutrosophic logic is descibed as follows:

a) For all the variables (input and output), membership functions are designedso that there is no overlap between the two defined membership functions.

b) Using rule editor, appropriate rules are produced.

4. Indeterminacy component of Neutrosophic logic is designed as follows:

a) For all the variables (input and output), membership functions are designed in such a way as to overcome the overlapping regions. The other two components i.e. indeterminacy component and falsity components are designed for overlapping regions

b) Using rule editor, appropriate rules are produced.

5. Falsity component of Neutrosophic logic using training set is designed similarto indeterminacy component. In falsity component, the maximum value of every membership function i.e. height is considered as 0.5.

6. After training is done, all the three components i.e. truth, indeterminate and falsity are verified using the 10 testing instances.

7. All these values will help us to determine the NSS i.e. predicate and approximate value-set for all testing instances.

8. After creation of approximation value set, lower and upper approximations are calculated for RNSS.

VII. MATLAB Implementation of Fuzzy and Rough Neutrosophic Soft Set On Datase

There are various techniques available for classification of data[12]. Here, fuzzy and Neutrosophic logic are used for the classification of data. Fuzzy and neutrosophic components are designed for appendicitis dataset as described below:

1) Trapezoidal membership functions are designed for input variable 1 which is ranging between 0 to 100 as shown below in Fig. 3.

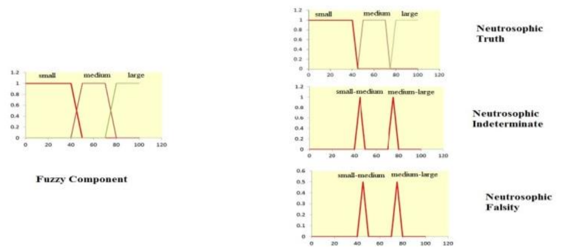

Fig. 3. Trapezoidal Membership function for input 1

2) Input membership function is defined for all other attributes.

3) Output membership function is designed for two classes i.e. 1 and 0 represented by A and B as shown below in Fig. 4.

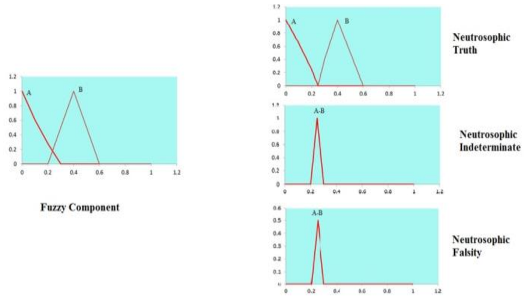

Fig. 4. Triangular Membership function for output class.

4) The fuzzy component contains 35 If-Then Rules. The rule base of neutrosophic truth, neutrosophic indeterminate and neutrosophic falsity component contain 40, 11 and 11 rules.

5) Fuzzy component will provide the degree of membership of belonginess. Its non-membership can be calculated as Non-membership= 1 – membership.

6) Neutrosophic components will provide the (T,I,F) values. Then the Neutrosophic result is calculated for all classes.

7) Lower and upper approximations are calculated with the approximations available.

VIII. Experiments and Results

The Table I shows the details of the testing instances for fuzzy component on appendicitis dataset.

TABLE I. DETAILS OF TESTING INSTANCES USING FUZZY SET

S.no.	Instance/class	Degree of membership	Degree of non-membership	Analysis
1.	[21.3 55.4 20.7 0 0 74.9 22]/A	0.08	0.92	Here, all instances lies in their classes correctly except two instances. Instance 5 and 6 lies in overlapping range. In the overlapping region we cannot surely say about the belongingness of an instance so, fuzzy logic cannot handle indeterminate data.
2.	[5.8 58.9 8.7 58.3 19.6 57.6 6]/A	0.08	0.92	
3.	[14.2 58.9 15.7 70.8 32.5 93.8 18.6]/B	0.41	0.59	
4.	[53.8 73.2 54.9 5.6 5.8 88.2 55.8]/B	0.41	0.59	
5.	[32.9 66.1 33.4 15.3 11.2 67.4 30.4]/B	0.29	0.71	
6.	[75.1 82.1 79.7 29.2 39.2 74.7 70]/B	0.29	0.71	
7.	[57.3 75 59 36.1 39.2 95.6 61.9]/B	0.41	0.59	
8.	[51.6 76.8 54.4 13.9 13.9 66.7 46.2]/B	0.42	0.58	
9.	[47.1 83.9 53.1 11.1 10.4 84.5 48.1]/B	0.42	0.58	
10.	[62.2 75 63.5 26.4 30.6 78.7 60.1]/B	0.42	0.58	

The Table II shows the details of testing instances using Neutrosophic soft set. As we can see here that all instances exist in their classes accurately but 2 instances are having their membership values in overlapping areas. So, neutrosophic logic will be applied on those instances to get better results.

TABLE II. DETAILS OF TESTING INSTANCES USING NEUTROSOPHIC SOFT SET

Instance	(T,I,F) values generated after defuzzification	Neutrosophic result of appendicitis class (class A)	Neutrosophic result of non-appendicitis class (class B)
[21.3 55.4 20.7 0 0 74.9 22]	(0.08,0.5,0.5)	(1,0,0)	(0,1,1)
[5.8 58.9 8.7 58.3 19.6 57.6 6]	(0.08,0.5,0.5)	(1,0,0)	(0,1,1)
[14.2 58.9 15.7 70.8 32.5 93.8 18.6]	(0.41,0.5,0.5)	(0,1,1)	(1,0,0)
[53.8 73.2 54.9 5.6 5.8 88.2 55.8]	(0.20,0.4,0.4)	(0,1,0.1)	(0.1,0,0)
[32.9 66.1 33.4 15.3 11.2 67.4 30.4]	(0.2901,0.25,0.25)	(0.5,0,0.1)	(0.1,1,0.5)
[75.1 82.1 79.7 29.2 39.2 74.7 70]	(0.29,0.25,0.25)	(0.5,0,0.1)	(0.1,1,0.5)
[57.3 75 59 36.1 39.2 95.6 61.9]	(0.4204,0.5,0.5)	(0,1,1)	(1,0,0)
[51.6 76.8 54.4 13.9 13.9 66.7 46.2]	(0.2865,0.5,0.5)	(0,1,0.2)	(0.2,0,0)
[47.1 83.9 53.1 11.1 10.4 84.5 48.1]	(0.2841,0.5,0.5)	(0,1,0.1)	(0.1,0,0)
[62.2 75 63.5 26.4 30.6 78.7 60.1]	(0.2877,0.5,0.5)	(0,1,0.1)	(0.1,0,0)

As it can be seen in Table II, NSS will provide the predicate and approximate value-set for every instance of every parameter. Predicate is class A instances and approximation value-set is <1,0,0>, <1,0,0>. Predicate is class B instances and approximation value-set for third instance is <0,1,1> and so on. Instance 5 and 6 lies in the overlapping region, Neutrosophic result is (0.1,1,0.5). So, it provides maximum value of indeterminacy in class B.

The neutrosophic components will provide the Neutrosophic results for instances of class A and B. The Neutrosophic result of instances of class A can be calculated for class B using the complement. The complement can be calculated as:

$$T\ (x) = F_B(x)$$

$$I\ (x) = 1\quad I_B(x)$$

$$F_A(x) = T\ (x)$$

The Table III shows the details of testing instances using Rough Neutrosophic soft set. Here, U be a universe of 10 instances and R be an equivalence relation its partition of U is given as

U/R = {{p1,p2}}, p1 and p2 are the classes A and B.

TABLE III. DETAILS OF TESTING INSTANCES USING ROUGH NEUTROSOPHIC SOFT SET

Instance	Neutrosophic result of appendicitis class(p1)	Neutrosophic result of non-appendicitis class(p2)	Lower approximation	Higher approximation
[21.3 55.4 20.7 0 0 74.9 22]	(1,0,0)	(0,1,1)	(0,1,1)	(1,0,0)
[5.8 58.9 8.7 58.3 19.6 57.6 6]	(1,0,0)	(0,1,1)	(0,1,1)	(1,0,0)
[14.2 58.9 15.7 70.8 32.5 93.8 18.6]	(0,1,1)	(1,0,0)	(0,1,1)	(1,0,0)
[53.8 73.2 54.9 5.6 5.8 88.2 55.8]	(0,1,0.1)	(0.1,0,0)	(0,1,0.1)	(0.1,0,0)
[32.9 66.1 33.4 15.3 11.2 67.4 30.4]	(0.5,0,0.1)	(0.1,1,0.5)	(0.1,1,0.5)	(0.5,0,0.1)
[75.1 82.1 79.7 29.2 39.2 74.7 70]	(0.5,0,0.1)	(0.1,1,0.5)	(0.1,1,0.5)	(0.5,0,0.1)
[57.3 75 59 36.1 39.2 95.6 61.9]	(0,1,1)	(1,0,0)	(0,1,1)	(1,0,0)
[51.6 76.8 54.4 13.9 13.9 66.7 46.2]	(0,1,0.2)	(0.2,0,0)	(0,1,0.2)	(0.2,0,0)
[47.1 83.9 53.1 11.1 10.4 84.5 48.1]	(0,1,0.1)	(0.1,0,0)	(0,1,0.1)	(0.1,0,0)
[62.2 75 63.5 26.4 30.6 78.7 60.1]	(0.1,0,0)	(0,1,0.1)	(0,1,0.1)	(0.1,0,0)

Lower and higher approximation provide the minimum and maximum value of truth, indeterminacy and falsity component for every instance. Lower and higher approximations can be calculated using eq. 9,10.

IX. Discussion of results

Classification using RNS i.e. rough neutrosophic sets presents more realistic results as it classifies the dataset into three classes. If it belongs to overlapping regions, we cannot be sure about its existence in either class. It is discussed in section 8 that various instances are having results in overlapping areas which can be handled with neutrosophic logic easily. Rough neutrosophic soft set has prons over fuzzy set which are discussed as:

1. Neutrosophic logic can handle indeterminacy of overlapping areas which is used byRough Neutrosophic soft set.

2. Membership value and non-membership value for every instance is considered by fuzzy logic whereasRough Neutrosophic soft set considers the membership value in truth class, indeterminate class and falsity class.

3. Lower as well as upper approximations are provided by Rough Neutrosophic soft set.

X. Conclusion

The proposed rough Neutrosophic soft set divides the classification domain into overlapping and non-overlapping sections. RNSS will provide better results as it allows us to consider the indeterminacy present in the medical domain. There are many cases in which the doctors may vary in their decisions and cannot surely say whether the person suffers from that disease or not, so indeterminacy exists in medical field. Neutrosophic logic based techniques provide the grade of truth, indeterminacy, falsity for every instance but fuzzy logic based techniques provides the degree of membership and non-membership. Also, the results generated by RNSS provide the minimum and maximum degree of truth, indeterminacy and falsity. Here, authors have confined the application of RNSS to a small dataset.

As the results are encouraging, it can be applied on other complex datasets or which are having more ambiguous results which can be provided solution with Neutrosophic logic. Hybridization of other soft computing techniques with techniques based on neutrosophic logic can be done to analyze the indeterminacy present in the data.

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NSS University of New Mexico

PCR5 and Neutrosophic Probability in Target Identification (revisited)

Florentin Smarandache
Mathematics & Science Department
University of New Mexico
Gallup, NM, USA
smarand@unm.edu

Nassim Abbas, Youcef Chibani, Bilal Hadjadji, Zayen Azzouz Omar
University of Science and Technology
Algiers, Algeria
nabbas@usthb.dz
ychibani@usthb.dz
bhadjadji@usthb.dz

Abstract. In this paper, we use PCR5 in order to fusion the information of two sources providing subjective probabilities of an event A to occur in the following form: chance that A occurs, indeterminate chance of occurrence of A, chance that A does not occur.

Keywords. Target Identification, PCR5, neutrosophic measure, neutrosophic probability, normalized neutrosophic probability.

I. Introduction

Neutrosophic Probability [1] was defined in 1995 and published in 1998, together with neutrosophic set, neutrosophic logic, and neutrosophic probability.

The words “neutrosophy” and “neutrosophic” were introduced by F. Smarandache in his 1998 book. Etymologically, “neutrosophy” (noun) [French neutre < Latin neuter, neutral, and Greek sophia, skill/wisdom] means knowledge of neutral thought. While “neutrosophic” (adjective), means having the nature of, or having the characteristic of Neutrosophy.

Neutrosophy is a new branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

Zadeh introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set.

Atanassov introduced the degree of nonmembership/ falsehood (f) in 1986 and defined the intuitionistic fuzzy set.

Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set. He has coined the words “neutrosophy” and “neutrosophic”. In 2013 he refined/split the neutrosophic set to n components: t_1, t_2, …t_j; i_1, i_2, …, i_k; f_1, f_2, …, f_l, with $j+k+l = n > 3$. And, as particular cases of refined neutrosophic set, he split the fuzzy set truth into t_1, t_2, …; and the intuitionistic fuzzy set into t_1, t_2, … and f_1, f_2, … .

See: http://fs.gallup.unm.edu/neutrosophy.htm.

For single valued neutrosophic logic, the sum of the components is:

$0 \le t+i+f \le 3$ when all three components are independent;

$0 \le t+i+f \le 2$ when two components are dependent, while the third one is independent from them;

$0 \le t+i+f \le 1$ when all three components are dependent.

When three or two of the components T, I, F are independent, one leaves room for incomplete information (sum < 1), paraconsistent and contradictory information (sum > 1), or complete information (sum = 1).

If all three components T, I, F are dependent, then similarly one leaves room for incomplete information (sum < 1), or complete information (sum = 1).

II. Definition of Neutrosophic Measure

A neutrosophic space is a set which has some indeterminacy with respect to its elements.

Let X be a neutrosophic space, and Σ a σ-neutrosophic algebra over X. A *neutrosophic measure* ν is defined by for neutrosophic set $A \in \Sigma$ by

$$\nu : X \to R^3,$$
$$\nu(A) = (m(A), m(neutA), m(antiA)), \qquad (1)$$

with *antiA* = the opposite of A, and *neutA* = the neutral (indeterminacy), neither A nor anti A (as defined above); for any $A \subseteq X$ and $A \in \Sigma$,

m(A) means *measure of the determinate part of A;*
m(neutA) means measure of indeterminate part of A;

and *m(antiA)* means *measure of the determinate part of antiA*; where ν is a function that satisfies the following two properties:

a) Null empty set: $\nu(\Phi) = (0,0,0)$.
b) Countable additivity (or σ -additivity): For all countable collections $\{A_n\}_{n\in L}$ of disjoint neutrosophic sets in Σ, one has:

$$\nu\left(\bigcup_{n\in L} A_n\right) = \left(\sum_{n\in L} m(A_n), \sum_{n\in L} m(neutA_n), \sum_{n\in L} m(antiA_n) - (n-1)m(X)\right) \quad (2)$$

where X is the whole neutrosophic space, and

$$\sum_{n\in L} m(antiA_n) - (n-1)m(X) = m(X) - \sum_{n\in L} m(A_n) = m(\bigcap_{n\in L} antiA_n). \quad (3)$$

A neutrosophic measure space is a triplet (X, Σ, ν).

III. NORMALIZED NEUTROSOPHIC MEASURE

A neutrosophic measure is called normalized if

$$\nu(X) = (m(X), m(neutX), m(antiX)) = (x_1, x_2, x_3), \quad (4)$$

with $x_1 + x_2 + x_3 = 1$, and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$, where, of course, X is the whole neutrosophic measure space.

As a particular case of neutrosophic measure ν is the neutrosophic probability measure, i.e. a neutrosophic measure that measures probable/possible propositions

$$0 \leq \nu(X) \leq 3,$$

where X is the whole neutrosophic probability sample space.

For single valued neutrosophic logic, the sum of the components is:

$0 \leq x_1+x_2+x_3 \leq 3$ when all three components are independent;

$0 \leq x_1+x_2+x_3 \leq 2$ when two components are dependent, while the third one is independent from them;

$0 \leq x_1+x_2+x_3 \leq 1$ when all three components are dependent.

When three or two of the components x_1, x_2, x_3 are independent, one leaves room for incomplete information (sum < 1), paraconsistent and contradictory information (sum > 1), or complete information (sum = 1).

If all three components x_1, x_2, x_3 are dependent, then similarly one leaves room for incomplete information (sum < 1), or complete information (sum = 1).

IV. NORMALIZED PROBABILITY

We consider the case when the sum of the components m(A) + m(neutA) + m(antiA) =1.

We may denote the normalized neutrosophic probability of an event A as $NP(\mathcal{A}) = (t, i, f)$, where t is the chance that $\mathcal{A}$ occurs, i is indeterminate chance of occurrence of $\mathcal{A}$, and f is the chance that $\mathcal{A}$ does not occur.

V. THE PCR5 FORMULA

Let the frame of discernment $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}, n \geq 2$. Let $G = (\Theta, \cup, \cap, C)$ be the super-power set, which is Θ closed under union, intersection, and respectively complement.

Let's consider two masses provided by 2 sources:

$$m_1, m_2 : G \rightarrow [0, 1].$$

The conjunctive rule is defined as

$$m_{12}(X) = \sum_{X_1, X_2 \in G} m_1(X_1) m_2(X_2). \quad (5)$$

Then the Proportional Conflict Redistribution Rule (PCR) #5 formula for 2 sources of information is defined as follows:

$$\forall X \in G \setminus \{\phi\},$$

$$m_{PCR5}(X) = m_{12}(X) + \sum_{Y \in G \setminus \{X\}} \left[\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_1(X) + m_2(Y)}\right] \quad (6)$$

where all denominators are different from zero.

If a denominator is zero, that fraction is discarded.

VI. APPLICATION IN INFORMATION FUSION

Suppose an airplane A is detected by the radar. What is the chance that A is friendly, neutrally, or enemy?

Let's have two sources that provide the following information:

$$NP_1^{(A)}(t_1, i_1, f_1), \text{ and } NP_2^{(A)}(t_2, i_2, f_2).$$

Then:

$$[NP_1 \oplus NP_2](t) = t_1 t_2 + \left(\frac{t_1^2 i_2}{t_1+i_2} + \frac{t_2^2 i_1}{t_2+i_1}\right) + \left(\frac{t_1^2 f_2}{t_1+f_2} + \frac{t_2^2 f_1}{t_2+f_1}\right) \quad (7)$$

Because: $t_1 i_2$ is redistributed back to the truth (t) and indeterminacy proportionally with respect to t_1 and respectively i_2:

$$\frac{x_1}{t_1} = \frac{y_1}{i_2} = \frac{t_1 i_2}{t_1+i_2}, \quad (8)$$

whence $x_1 = \frac{t_1^2 i_2}{t_1+i_2}$, $y_1 = \frac{t_1 i_2^2}{t_1+i_2}$. (9)

Similarly, $t_2 i_1$ is redistributed back to t and i proportionally with respect to t_2 and respectively i_1:

$$\frac{x_2}{t_2} = \frac{y_2}{i_1} = \frac{t_2 i_1}{t_2+i_1}, \quad (10)$$

whence $x_2 = \frac{t_2^2 i_1}{t_2+i_1}$, $y_2 = \frac{t_2 i_1^2}{t_2+i_1}$. (11)

Similarly, $t_1 f_2$ is redistributed back to t and f (falsehood) proportionally with respect to t_1 and respectively f_2:

$$\frac{x_3}{t_1} = \frac{z_1}{f_2} = \frac{t_1 f_2}{t_1+f_2}, \quad (12)$$

whence $x_3 = \frac{t_1^2 f_2}{t_1+f_2}$, $z_1 = \frac{t_1 f_2^2}{t_1+f_2}$. (13)

Again, similarly $t_2 f_1$ is redistributed back to t and f proportionally with respect to t_2 and respectively f_1:

$$\frac{x_4}{t_2} = \frac{z_2}{f_1} = \frac{t_2 f_1}{t_2+f_1}, \quad (14)$$

whence $x_4 = \frac{t_2^2 f_1}{t_2+f_1}$, $z_2 = \frac{t_2 f_1^2}{t_2+f_1}$. (15)

In the same way, $i_1 f_2$ is redistributed back to i and f proportionally with respect to i_1 and respectively f_2:

$$\frac{y_3}{i_1} = \frac{z_3}{f_2} = \frac{i_1 f_2}{i_1+f_2}, \quad (16)$$

whence $y_3 = \frac{i_1^2 f_2}{i_1+f_2}$, $z_3 = \frac{i_1 f_2^2}{i_1+f_2}$. (17)

While $i_2 f_1$ is redistributed back to i and t proportionally with respect to i_2 and respectively f_1:

$$\frac{y_4}{i_2} = \frac{z_4}{f_1} = \frac{i_2 f_1}{i_2+f_1}, \quad (18)$$

whence $y_4 = \frac{i_2^2 f_1}{i_2+f_1}$, $z_4 = \frac{i_2 f_1^2}{i_2+f_1}$. (19)

Then

$$[NP_1 \oplus NP_2](i) = i_1 i_2 + \left(\frac{i_1^2 t_2}{i_{11}+t_2} + \frac{i_2^2 t_1}{i_2+t_1}\right) + \left(\frac{i_1^2 f_2}{i_1+f_2} + \frac{i_2^2 f_1}{i_2+f_1}\right), \quad (20)$$

and

$$[NP_1 \oplus NP_2](f) = f_1 f_2 + \left(\frac{f_1^2 t_2}{f_1+t_2} + \frac{f_2^2 t_1}{f_2+t_1}\right) + \left(\frac{f_1^2 i_2}{f_1+i_2} + \frac{f_2^2 i_1}{f_2+i_1}\right). \quad (21)$$

VII. EXAMPLE

Let's compute: $(0.6, 0.1, 0.3) \wedge_N (0.2, 0.3, 0.5)$.

$t_1 = 0.6, i_1 = 0.1, f_1 = 0.3$, and

$t_2 = 0.2, i_2 = 0.3, f_2 = 0.5$,

are replaced into the three previous neutrosophic logic formulas:

- *(using PCR5 rule)*

$$[NP_1 \oplus nm_2](t) = 0.6(0.2) + \left(\frac{0.6^2(0.3)}{0.6+0.3} + \frac{0.2^2(0.1)}{0.2+0.1}\right) + \left(\frac{0.6^2(0.5)}{0.6+0.5} + \frac{0.2^2(0.3)}{0.2+0.3}\right) \simeq 0.44097$$

.

$$[NP_1 \oplus NP_2](i) = 0.1(0.3) + \left(\frac{0.1^2(0.2)}{0.1+0.2} + \frac{0.3^2(0.6)}{0.3+0.6}\right) + \left(\frac{0.1^2(0.5)}{0.1+0.5} + \frac{0.3^2(0.3)}{0.3+0.3}\right) \simeq 0.15000$$

.

$$[NP_1 \oplus NP_2](f) = 0.3(0.5) + \left(\frac{0.3^2(0.2)}{0.3+0.2} + \frac{0.5^2(0.6)}{0.5+0.6}\right) + \left(\frac{0.3^2(0.3)}{0.3+0.3} + \frac{0.5^2(0.1)}{0.5+0.1}\right) \simeq 0.40903$$

.

- *(using Dempster's rule)*

Conj. rule:

0.12	0.03	0.15

Dempster's rule:

0.40	0.10	0.50

This is actually a PCR5 formula for a frame of discernment $\Omega = \{\theta_1, \theta_2, \theta_3\}$ whose all intersections are empty.

We can design a PCR6 formula too for the same frame.

Another method will be to use the neutrosophic $N-norm$, which is a generalization of fuzzy $T-norm$.

If we have two neutrosophic probabilities

	Friend	Neutral	Enemy
NP_1	t_1	i_1	f_1
NP_2	t_2	i_2	f_2

then

$$NP_1 \oplus NP_2 = (t_1 + i_1 + f_1) \cdot (t_2 + i_2 + f_2) =$$

$$t_1 t_2 + t_1 i_2 + t_2 i_1 + i_1 i_2 + t_1 f_1 + t_1 f_2 + t_2 f_1 + i_1 f_2 + i_2 f_1 + f_1 f_2$$

Of course, the quantity of $t_1 t_2$ will go to Friend, quantity of $i_1 i_2$ will go to Neutral, and quantity of $f_1 f_2$ will go to Enemy.

The other quantities will go depending on the pessimistic or optimistic way:

a) In the pessimistic way (lower bound) $t_1 i_2 + t_2 i_1$ will go to Neutral, and $t_1 f_2 + t_2 f_1 + i_1 f_2 + i_2 f_1$ to Enemy.

b) In the optimistic way (upper bound) $t_1 i_2 + t_2 i_1$ will go to Friend, and $t_1 f_2 + t_2 f_1 + i_1 f_2 + i_2 f_1$ to Neutral. About $t_1 f_2 + t_2 f_1$, we can split it half-half to Friend and respectively Enemy.

We afterwards put together the pessimistic and optimistic ways as an interval neutrosophic probability.

c) Of course, the reader or expert can use different transfers of intermediate mixed quantities $t_1 i_2 + t_2 i_1$, and respectively $t_1 f_2 + t_2 f_1 + i_1 f_2 + i_2 f_1$ to Friend, Neutral, and Enemy.

CONCLUSION

We have introduced the application of neutrosophic probability into information fusion, using the combination of information provided by two sources using the PCR5.

Other approaches can be done, for example the combination of the information using the N-norm and N-conorm, which are generalizations of the T-norm and T-conorm from the fuzzy theory to the neutrosophic theory.

More research is needed in this direction.

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NSS University of New Mexico

Rough Neutrosophic Multisets

Suriana Alias[1], Daud Mohamad[2], Adibah Shuib[3]

[1] Faculty of Computer and Mathematical Sciences, Universiti Teknologi Mara (UiTM) Kelantan, Campus Machang, Kelantan, 18500, Malaysia. E-mail: suria588@kelantan.uitm.edu.my

[2] Faculty of Computer and Mathematical Sciences, Universiti Teknologi Mara (UiTM) Shah Alam, Shah Alam, 40450, Malaysia. E-mail: daud@tmsk.uitm.edu.my

[3] Faculty of Computer and Mathematical Sciences, Universiti Teknologi Mara (UiTM) Shah Alam, Shah Alam, 40450, Malaysia. E-mail: adibah@tmsk.uitm.edu.my

Abstract. Many past studies largely described the concept of neutrosophic sets, neutrosophic multisets, rough sets, and rough neutrosophic sets in many areas. However, no paper has discussed about rough neutrosophic multisets. In this paper, we present some definition of rough neutrosophic multisets such as complement, union and intersection. We also have examined some desired properties of rough neutrosophic multisets based on these definitions. We use the hybrid structure of rough set and neutrosophic multisets since these theories are powerful tool for managing uncertainty, indeterminate, incomplete and imprecise information.

Keywords: Neutrosophic set, neutrosophic multiset, rough set, rough neutrosophic set, rough neutrosophic multisets

1 Introduction

In our real-life problems, there are situations with uncertain data that may be not be successfully modelled by the classical mathematics. For example, the opinion about “beauty”, which is can be describe by more beauty, beauty, beauty than, or less beauty. Therefore, there are some mathematical tools for dealing with uncertainties such as fuzzy set theory introduced by Zadeh [1], intuitionistic fuzzy set theory introduced by Attanasov [2], rough set theory introduced by Pawlak [3], and soft set theory initiated by Molodtsov [4]. Rough set theory introduced by Pawlak in 1981/1982, deals with the approximation of sets that are difficult to describe with the available information. It is expressed by a boundary region of set and also approach to vagueness. After Pawlak’s work several researcher were studied on rough set theory with applications [5], [6].

However, these concepts cannot deal with indeterminacy and inconsistent information. In 1995, Smarandache [7] developed a new concept called neutrosophic set (NS) which generalizes probability set, fuzzy set and intuitionistic fuzzy set. There are three degrees of membership described by NS which is membership degree, indeterminacy degree and non-membership degree. This theory and their hybrid structures has proven useful in many different field [8], [9], [10], [11],[12], [13],[14].

Broumi et al. [15] proposed a hybrid structure called neutrosophic rough set which is combination of neutrosophic set [7] and rough set [3] and studied their properties. Later, Broumi et al. [16] introduced interval neutrosophic rough set that combines interval- valued neutrosophic sets and rough sets. It studies roughness in interval- valued neutrosophic sets and some of its properties. After the introduction of rough neutrosophic set theory, many interesting application have been studied such as in medical organisation [17], [18], [19].

But until now, there have been no study on rough neutrosophic multisets (RNM). Therefore, the objective of this paper is to study the concept of RNM which is combination of rough set [3] and neutrosophic multisets [20] as a generalization of rough neutrosophic sets [15].

This paper is arranged in following manner. In section 2, some mathematical preliminary concepts were recall for more understanding about RNM. In section 3, the concepts of RNM and some of their properties are presented with examples. Finally, we conclude the paper.

2 Mathematical Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets [7], [21], [22], neutrosophic multisets [23], [24], [20], [25], rough set [3], and rough neutrosophic set [15], [17], that relevant to the present work and for further details and background.

Definition 2.1 (Neutrosophic Set) [7] Let X be an universe of discourse, with a generic element in X denoted by x, the neutrosophic (NS) set is an object having the form

$$A = \{\langle x, (T_A(x), I_A(x), F_A(x))\rangle \mid x \in X\}$$

where the functions $T, I, F : X \to]^-0, 1^+[$ define respectively the degree of membership (or Truth), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $x \in X$ to the set A with the condition

$$^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$$

From a philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. So, instead of $]^-0, 1^+[$ we need to take the interval [0, 1] for technical applications, because $]^-0, 1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems. Therefore, we have

$$A = \{\langle x, (T_A(x), I_A(x), F_A(x))\rangle \mid x \in X, T_A(x), I_A(x), F_A(x) \in [0, 1]\}.$$

There is no restriction on the sum of $T_A(x)$; $I_A(x)$ and $F_A(x)$, so

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$$

For two NS,

$$A = \{\langle x, (T_A(x), I_A(x), F_A(x))\rangle \mid x \in X\} \text{ and}$$

$$B = \{\langle x, (T_B(x), I_B(x), F_B(x))\rangle \mid x \in X\}$$

the relations are defined as follows:

(i) $A \subseteq B$ if and only if $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, $F_A(x) \ge F_B(x)$,

(ii) $A = B$ if and only if $T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$,

(iii) $A \cap B = \{\langle x, \min(T_A(x),T_B(x)), \max(I_A(x),I_B(x)), \max(F_A(x),F_B(x))\rangle \mid x \in X\}$,

(iv) $A \cup B = \{\langle x, \max(T_A(x),T_B(x)), \min(I_A(x),I_B(x)), \min(F_A(x),F_B(x))\rangle \mid x \in X\}$,

(v) $A^c = \{\langle x, F_A(x), 1- I_A(x), T_A(x)\rangle \mid x \in X\}$

(vi) $0_n = (0, 1, 1)$ and $1_n = (1, 0, 0)$.

As an illustration, let us consider the following example.

Example 2.2. Assume that the universe of discourse $U = \{x_1, x_2, x_3\}$, where x_1 characterizes the capability, x_2 characterizes the trustworthiness and x_3 indicates the prices of the objects. It may be further assumed that the values of x_1, x_2, and x_3 are in [0, 1] and they are obtained from some questionnaires of some experts. The experts may impose their opinion in three components which is the degree of goodness, the degree of indeterminacy and that of poorness to explain the characteristics of the objects. Suppose A is a neutrosophic set (NS) of U, such that,

$$A = \{\langle x_1, (0.3, 0.4, 0.5)\rangle, \langle x_2, (0.5, 0.1, 0.4)\rangle, \langle x_3, (0.4, 0.3, 0.5)\rangle\},$$

where the degree of goodness of prices is 0.4, degree of indeterminacy of prices is 0.3 and degree of poorness of prices is 0.5 etc.

The following definitions are refer to [25].

Definition 2.3 (Neutrosophic Multisets) Let E be a universe. A neutrosophic multiset (NMS) A on E can be defined as follows:

$$A = \{\langle x, (T_A^1(x), T_A^2(x), \ldots, T_A^p(x)), (I_A^1(x), I_A^2(x), \ldots, I_A^p(x)), (F_A^1(x), F_A^2(x), \ldots, F_A^p(x))\rangle : x \in E\}$$

where, the truth membership sequence $(T_A^1(x), T_A^2(x), \ldots, T_A^p(x))$, the indeterminacy membership sequence $(I_A^1(x), I_A^2(x), \ldots, I_A^p(x))$ and the falsity membership sequence $(F_A^1(x), F_A^2(x), \ldots, F_A^p(x))$ may be in decreasing or increasing order, and the sum of $T_A^i(x), I_A^i(x), F_A^i(x) \in [0,1]$ satisfies the condition $0 \le T_A^i(x) + I_A^i(x) + F_A^i(x) \le 3$ for any $x \in E$ and $i = 1, 2, \ldots, p$. Also, p is called the dimension (cardinality) of NMS A.

For convenience, a NMS A can be denoted by the simplified form:

$$A = \{\langle x, (T_A^i(x), I_A^i(x), F_A^i(x)\rangle \mid x \in E, i = 1,2,\ldots,p\}$$

Definition 2.4 Let $A,B \in$ NMS(E). Then,

(i) A is said to be NM subset of B is denoted by $A \,\tilde{\subseteq}\, B$ if $T_A^i(x) \le T_B^i(x)$, $I_A^i(x) \ge I_B^i(x)$, $F_A^i(x) \ge F_B^i(x)$, $\forall x \in E$.

(ii) A is said to be neutrosophic equal of B is denoted by $A = B$ if

$$T_A^i(x) = T_B^i(x),\ I_A^i(x) = I_B^i(x),\ F_A^i(x) = F_B^i(x),\ \forall x \in E.$$

(iii) The complement of A denoted by $A^{\tilde{C}}$ is defined by

$$A^{\tilde{C}} = \{\langle x, (F_A^1(x), F_A^2(x), \ldots, F_A^p(x)), (1 - I_A^1(x), 1 - I_A^2(x), \ldots, 1 - I_A^p(x)), (T_A^1(x), T_A^2(x), \ldots, T_A^p(x))\rangle : x \in E\}$$

(iv) If $T_A^i(x)=0$ and $I_A^i(x)=F_A^i(x)=1$ for all $x\in E$ and $i=1,2,\ldots,p$, then A is called null ns-set and denoted by $\widetilde{\Phi}$.

(iv) If $T_A^i(x)=1$ and $I_A^i(x)=F_A^i(x)=0$ for all $x\in E$ and $i=1,2,\ldots,p$, then A is called universal ns-set and denoted by $\widetilde{E}$.

Definition 2.5 Let $A,B\in$ NMS(E). Then,

(i) The union of A and B is denoted by $A\widetilde{\bigcup}B=C$ is defined by

$$C=\{\langle x,(T_C^1(x),T_C^2(x),\ldots,T_C^p(x)),\ (I_C^1(x),I_C^2(x),\ldots,I_C^p(x)),\ (F_C^1(x),F_C^2(x),\ldots,F_C^p(x))\rangle : x\in E\}$$

where

$$T_C^i(x)=T_A^i(x)\vee T_B^i(x),\quad I_C^i(x)=I_A^i(x)\wedge I_B^i(x),\quad F_C^i(x)=F_A^i(x)\wedge F_B^i(x),$$

for $\forall x\in E$ and $i=1,2,\ldots,p$.

(ii) The intersection of A and B is denoted by $A\widetilde{\bigcap}B=D$ and is defined by

$$D=\{\langle x,(T_D^1(x),T_D^2(x),\ldots,T_D^p(x)),\ (I_D^1(x),I_D^2(x),\ldots,I_D^p(x)),\ (F_D^1(x),F_D^2(x),\ldots,F_D^p(x))\rangle \mid x\in E\}$$

where

$$T_D^i(x)=T_A^i(x)\wedge T_B^i(x),\quad I_D^i(x)=I_A^i(x)\vee I_B^i(x),\quad F_D^i(x)=F_A^i(x)\vee F_B^i(x),$$

for $\forall x\in E$ and $i=1,2,\ldots,p$.

(iii) The addition of A and B is denoted by $A\widetilde{+}B=G$ and is defined by

$$G=\{\langle x,(T_G^1(x),T_G^2(x),\ldots,T_G^p(x)),\ (I_G^1(x),I_G^2(x),\ldots,I_G^p(x)),\ (F_G^1(x),F_G^2(x),\ldots,F_G^p(x))\rangle \mid x\in E\}$$

where

$$T_G^i(x)=T_A^i(x)+T_B^i(x)-T_A^i(x)\cdot T_B^i(x),$$
$$I_G^i(x)=I_A^i(x)\cdot I_B^i(x),$$
$$F_G^i(x)=F_A^i(x)\cdot F_B^i(x),$$

for $\forall x\in E$ and $i=1,2,\ldots,p$.

(iv) The multiplication of A and B is denoted by $A\widetilde{\times}B=H$ and is defined by

$$H=\{\langle x,(T_H^1(x),T_H^2(x),\ldots,T_H^p(x)),\ (I_H^1(x),I_H^2(x),\ldots,I_H^p(x)),\ (F_H^1(x),F_H^2(x),\ldots,F_H^p(x))\rangle \mid x\in E\}$$

where

$$T_H^i(x)=T_A^i(x)\cdot T_B^i(x),$$
$$I_H^i(x)=I_A^i(x)+I_B^i(x)-I_A^i(x)\cdot I_B^i(x),$$
$$F_H^i(x)=F_A^i(x)+F_B^i(x)-F_A^i(x)\cdot F_B^i(x),$$

for $\forall x\in E$ and $i=1,2,\ldots,p$.

Here $\wedge,\vee,+,\cdot,-$ denotes minimum, maximum, addition, multiplication, subtraction of real numbers respectively.

Definition 2.6 (Rough Set) [3] Let R be an equivalence relation on the universal set U. Then, the pair (U, R) is called a Pawlak's approximation space. An equivalence class of R containing x will be denotes by $[x]_R$. Now, for $X\subseteq U$, the upper and lower approximation of X with the respect to (U, R) are denoted by, respectively $A_1(x)$ and $A_2(x)$ and defined by

$$A_1(x)=\{x:[x]_R\subseteq X\} \text{ and } A_2(x)=\{x:[x]_R\cap X\neq\emptyset\}$$

Now, if $A_1(x)=A_2(x)$, then X is called definable; otherwise, the pair $A(X)=(A_1(x), A_2(x))$ is called the rough set of X in U.

Example 2.7 [5] Let $A=(U, R)$ be an approximate space where $U=\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the relation R on U be definable aRb iff $a\equiv b \pmod 5$ for all $a, b\in U$. Let us consider a subset $X=\{1, 2, 6, 7, 8, 9\}$ of U. Then, the rough set of X is $A(x)=(\underline{A}(x), \overline{A}(x))$ where $\underline{A}(x)=\{1, 2, 6, 7\}$ and $\overline{A}(x)=\{1, 2, 3, 4, 6, 7, 8, 9\}$. Here, the equivalence classes are

$[0]_R=[5]_R=[10]_R=\{0, 5, 10\}$

$[1]_R=[6]_R=\{1, 6\}$

$[2]_R=[7]_R=\{2, 7\}$

$[3]_R=[8]_R=\{3, 8\}$

$[4]_R=[9]_R=\{4, 9\}$

Thus, $\underline{A}(x)=\{x\in U: [x]_R\subseteq X\}=\{1, 2, 6, 7\}$ and

$\overline{A}(x)=\{x:[x]_R\cap X\neq\emptyset\}=\{1, 2, 3, 4, 6, 7, 8, 9\}$.

The following definitions are refer to [15].

Definition 2.8 Let $A = (A_1,A_2)$ and $B = (B_1,B_2)$ be two rough sets in the approximation space $S = (U, R)$. Then,

(i) $A \cup B = (A_1 \cup B_1, A_2 \cup B_2)$,

(ii) $A \cap B = (A_1 \cap B_1, A_2 \cap B_2)$,

(iii) $A \subseteq B$ if $A \cap B = A$,

(iv) $\sim A = \{U - A_2, U - A_1\}$.

Definition 2.9 (Rough Neutrosophic Set) Let U be a non-null set and R be an equivalence relation on U. Let A be neutrosophic set in U with the membership function T_A, indeterminacy function I_A and non-membership function F_A. The lower and the upper approximations of A in the approximation (U, R) denoted by $\underline{N}(A)$ and $\overline{N}(A)$ are respectively defined as follows:

$$\underline{N}(A) = \{\langle x, (T_A(x), I_A(x), F_A(x))\rangle \mid y \in [x]_R, x \in U\},$$

$$\overline{N}(A) = \{\langle x, (T_A(x), I_A(x), F_A(x))\rangle \mid y \in [x]_R, x \in U\}$$

where

$$T_{\underline{N}(A)}(x) = \wedge_{y\in[x]_R} T_A(y),\ \ I_{\underline{N}(A)}(x) = \vee_{y\in[x]_R} I_A(y),$$
$$F_{\underline{N}(A)}(x) = \vee_{y\in[x]_R} F_A(y)$$

$$T_{\overline{N}(A)}(x) = \vee_{y\in[x]_R} T_A(y),\ \ I_{\overline{N}(A)}(x) = \wedge_{y\in[x]_R} I_A(y),$$
$$F_{\overline{N}(A)}(x) = \wedge_{y\in[x]_R} F_A(y)$$

So,

$$0 \le T_{\underline{N}(A)}(x) + I_{\underline{N}(A)}(x) + F_{\underline{N}(A)}(x) \le 3, \quad \text{and}$$

$$0 \le T_{\overline{N}(A)}(x) + I_{\overline{N}(A)}(x) + F_{\overline{N}(A)}(x) \le 3$$

Here $\wedge$ and $\vee$ denote "min" and "max'' operators respectively. $T_A(y)$, $I_A(y)$ and $F_A(y)$ are the membership, indeterminacy and non-membership degrees of y with respect to A. $\underline{N}(A)$ and $\overline{N}(A)$ are two neutrosophic sets in U.

Thus, NS mappings $\underline{N}, \overline{N} : N(U) \rightarrow N(U)$ are, respectively, referred to as the upper and lower rough NS approximation operators, and the pair is $(\underline{N}(A), \overline{N}(A))$ called the rough neutrosophic set in (U, R).

Based on the above definition, it is observed that $\underline{N}(A)$ and $\overline{N}(A)$ have a constant membership on the equivalence classes of U, if $\underline{N}(A) = \overline{N}(A)$; i.e.,

$$T_{\underline{N}(A)}(x) = T_{\overline{N}(A)}(x),\ \ I_{\underline{N}(A)}(x) = I_{\overline{N}(A)}(x),$$
$$F_{\underline{N}(A)}(x) = I_{\overline{N}(A)}(x)$$

For any $x \in U$, A is called a definable neutrosophic set in the approximation (U, R). Obviously, zero neutrosophic set (0_N) and unit neutrosophic sets (1_N) are definable neutrosophic sets. Let consider the example in the following.

Example 2.10 Let $U = \{ p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$ be the universe of discourse. Let R be an equivalence relation its partition of U is given by

$$U/R = \{ \{p_1, p_4\}, \{p_2, p_3, p_6\}, \{ p_5\}, \{ p_7, p_8\} \}.$$

Let $N(A) = \{ (p_1, (0.3, 0.4, 0.5), (p_4, (0.4, 0.6, 0.5)), (p_5, (0.5, 0.7, 0.3)), (p_7, (0.2, 0.4, 0.6)) \}$ be a neutrosophic set of U. By definition 2.6 and 2.9, we obtain:

$\underline{N}(A) = \{ (p_1, (0.3, 0.6, 0.5)), (p_4, (0.3, 0.6, 0.5)), (p_5, (0.5, 0.7, 0.3)) \}$ and

$\overline{N}(A) = \{ (p_1, (0.3, 0.4, 0.5)), (p_4, (0.3, 0.4, 0.5)), (p_5, (0.5, 0.7, 0.3)), (p_7, (0.2, 0.4, 0.6)), (p_8, (0.2, 0.4, 0.6)) \}$.

For another neutrosophic sets,

$N(B) = \{ (p_1, (0.3, 0.4, 0.5), (p_4, (0.3, 0.4, 0.5)), (p_5, (0.5, 0.7, 0.3)) \}$.

The lower approximation and upper approximation of $N(B)$ are calculated as

$\underline{N}(B) = \{ (p_1, (0.3, 0.4, 0.5)), (p_4, (0.3, 0.4, 0.5)), (p_5, (0.5, 0.7, 0.3)) \}$ and

$\overline{N}(B) = \{ (p_1, (0.3, 0.4, 0.5)), (p_4, (0.3, 0.4, 0.5)), (p_5, (0.5, 0.7, 0.3)) \}$.

Obviously, $\underline{N}(B) = \overline{N}(B)$ is a definable neutrosophic set in the approximation space (U, R).

Definition 2.11 If $N(A) = (\underline{N}(A), \overline{N}(A))$ is a rough neutrosophic set in (U, R), the rough complement of N(A) is the rough neutrosophic set denoted by $\sim N(A) = (\underline{N}(A)^c, \overline{N}(A)^c)$ where $\underline{N}(A)^c$, $\overline{N}(A)^c$ are the complements of neutrosophic sets $\underline{N}(A)$ and $\overline{N}(A)$ respectively,

$$\underline{N}(A)^c = \{\langle x, F_{\underline{N}(A)}(x), 1 - I_{\underline{N}(A)}(x), T_{\underline{N}(A)}(x)\rangle \mid x \in U\},$$

$$\overline{N}(A)^c = \{\langle x, F_{\overline{N}(A)}(x), 1 - I_{\overline{N}(A)}(x), T_{\overline{N}(A)}(x)\rangle \mid x \in U\}$$

Definition 2.12 If $N(F_1)$ and $N(F_2)$ are two rough neutrosophic set of the neutrosophic sets F_1 and F_2 respectively in U, then we define the following:

(i) $N(F_1) = N(F_2)$ iff $\underline{N}(F_1) = \underline{N}(F_2)$ and $\overline{N}(F_1) = \overline{N}(F_2)$

(ii) $N(F_1) \subseteq N(F_2)$ iff $\underline{N}(F_1) \subseteq \underline{N}(F_2)$ and $\overline{N}(F_1) \subseteq \overline{N}(F_2)$

(iii) $N(F_1) \bigcup N(F_2) = \left\langle \underline{N}(F_1) \bigcup \underline{N}(F_2), \overline{N}(F_1) \bigcup \overline{N}(F_2) \right\rangle$

(iv) $N(F_1) \bigcap N(F_2) = \left\langle \underline{N}(F_1) \bigcap \underline{N}(F_2), \overline{N}(F_1) \bigcap \overline{N}(F_2) \right\rangle$

(v) $N(F_1) + N(F_2) = \left\langle \underline{N}(F_1) + \underline{N}(F_2), \overline{N}(F_1) + \overline{N}(F_2) \right\rangle$

(vi) $N(F_1) \cdot N(F_2) = \left\langle \underline{N}(F_1) \cdot \underline{N}(F_2), \overline{N}(F_1) \cdot \overline{N}(F_2) \right\rangle$

If *N*, *M*, *L* are rough neutrosophic set in (*U*, *R*), then the results in the following proposition are straightforward from definitions.

Proposition 2.13.

(i) $\sim N(\sim N) = N$

(ii) $N \bigcup M = M \bigcup N$, $N \bigcap M = M \bigcap N$

(iii) $(N \bigcup M) \bigcup L = N \bigcup (M \bigcup L)$, and $(N \bigcap M) \bigcap L = N \bigcap (M \bigcap L)$

(iv) $(N \bigcup M) \bigcap L = (N \bigcup M) \bigcap (N \bigcup L)$, and $(N \bigcap M) \bigcup L = (N \bigcap M) \bigcup (N \bigcap L)$

De Morgan's Laws are satisfied for rough neutrosophic sets:

Proposition 2.14.

(i) $\sim (N(F_1) \cup N(F_2)) = (\sim N(F_1)) \cap (\sim N(F_2))$

(ii) $\sim (N(F_1) \cap N(F_2)) = (\sim N(F_1)) \cup (\sim N(F_2))$

Proposition 2.15. If F_1 and F_2 are two neutrosophic sets in *U* such that $F_1 \subseteq F_2$, then $N(F_1) \subseteq N(F_2)$

(i) $N(F_1 \bigcup F_2) \supseteq N(F_1) \bigcup N(F_2)$

(ii) $N(F_1 \bigcap F_2) \subseteq N(F_1) \bigcap N(F_2)$

Proposition 2.16.

(i) $N(F) = \sim \overline{N}(\sim F)$

(ii) $\overline{N}(\sim F) = \sim \underline{N}(\sim F)$

(iii) $\underline{N}(F) \subseteq \overline{N}(F)$

3 Rough Neutrosophic Multisets

Based on the equivalence relation on the universe of discourse, we introduce the lower and upper approximations of neutrosophic multisets [20] in a Pawlak's approximation space [3] and obtained a new notion called rough neutrosophic multisets (RNM). Its basic operations such as complement, union and intersection also discuss over them with the examples. Some of it is quoted from [15], [25],[20], [26].

Definition 3.1 Let *U* be a non-null set and *R* be an equivalence relation on *U*. Let *A* be neutrosophic multisets in *U* with the truth membership sequence T_A^i, indeterminacy membership sequences I_A^i and falsity membership sequences F_A^i. The lower and the upper approximations of *A* in the approximation (*U*, *R*) denoted by $\underline{Nm}(A)$ and $\overline{Nm}(A)$ are respectively defined as follows:

$$\underline{Nm}(A) = \{\langle x, (T^i_{\underline{Nm}(A)}(x), I^i_{\underline{Nm}(A)}(x), F^i_{\underline{Nm}(A)}(x))\rangle \mid y \in [x]_R, x \in U\},$$

$$\overline{Nm}(A) = \{\langle x, (T^i_{\overline{Nm}(A)}(x), I^i_{\overline{Nm}(A)}(x), F^i_{\overline{Nm}(A)}(x))\rangle \mid y \in [x]_R, x \in U\},$$

where

$i = 1, 2, .., p,$

$$T^i_{\underline{Nm}(A)}(x) = \bigwedge_{y \in [x]_R} T^i_A(y),$$
$$I^i_{\underline{Nm}(A)}(x) = \bigvee_{y \in [x]_R} I^i_A(y),$$
$$F^i_{\underline{Nm}(A)}(x) = \bigvee_{y \in [x]_R} F^i_A(y),$$

$$T^i_{\overline{Nm}(A)}(x) = \bigvee_{y \in [x]_R} T^i_A(y),$$
$$I^i_{\overline{Nm}(A)}(x) = \bigwedge_{y \in [x]_R} I^i_A(y),$$
$$F^i_{\overline{Nm}(A)}(x) = \bigwedge_{y \in [x]_R} F^i_A(y)$$

such that,

$$T^i_{\underline{Nm}(A)}(x), I^i_{\underline{Nm}(A)}(x), F^i_{\underline{Nm}(A)}(x) \in [0,1],$$

$$T^i_{\overline{Nm}(A)}(x), I^i_{\overline{Nm}(A)}(x), F^i_{\overline{Nm}(A)}(x) \in [0,1],$$

$$0 \le T^i_{\underline{Nm}(A)}(x) + I^i_{\underline{Nm}(A)}(x) + F^i_{\underline{Nm}(A)}(x) \le 3, \text{ and}$$

$$0 \le T^i_{\overline{Nm}(A)}(x) + I^i_{\overline{Nm}(A)}(x) + F^i_{\overline{Nm}(A)}(x) \le 3$$

Here $\wedge$ and $\vee$ denote "min" and "max'' operators respectively. $T_A^i(y)$, $I_A^i(y)$ and $F_A^i(y)$ are the membership sequences, indeterminacy sequences and non-membership sequences of y with respect to A and i =1, 2, …, p.

Since $\underline{Nm}(A)$ and $\overline{Nm}(A)$ are two neutrosophic multisets in U, thus neutrosophic multisets mappings $\underline{Nm}, \overline{Nm} : Nm(U) \rightarrow Nm(U)$ are respectively referred to as the upper and lower rough neutrosophic multisets approximation operators, and the pair is $(\underline{Nm}(A), \overline{Nm}(A))$ called the rough neutrosophic multisets in (U, R).

From the above definition, we can see that $\underline{Nm}(A)$ and $\overline{Nm}(A)$ have constant membership on the equivalence classes of U, if $\underline{Nm}(A) = \overline{Nm}(A)$; i.e.,

$$T^i_{\underline{Nm}(A)}(x) = T^i_{\overline{Nm}(A)}(x),$$

$$I^i_{\underline{Nm}(A)}(x) = I^i_{\overline{Nm}(A)}(x),$$

$$F^i_{\underline{Nm}(A)}(x) = F^i_{\overline{Nm}(A)}(x).$$

Let consider the following example.

Example 3.2 Let $U = \{ p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8 \}$ be the universe of discourse. Let R be an equivalence relation its partition of U is given by

$U/R = \{\{p_1, p_4\}, \{p_2, p_3, p_6\}, \{ p_5\}, \{ p_7, p_8\}\}$.

Let $Nm(A)$ = {(<p_1, (0.8, 0.6, 0.5), (0.3, 0.2, 0.1), (0.4, 0.2, 0.1)>, <p_4, (0.5, 0.4, 0.3), (0.4, 0.4, 0.3), (0.6, 0.3, 0.3)>), <p_5, (0.2, 0.1, 0.0), (0.3, 0.2, 0.2), (0.8, 0.7, 0.7)>, <p_7, (0.7, 0.6, 0.5), (0.3, 0.2, 0.1), (0.4, 0.3, 0.2)> } be a neutrosophic multisets of U. By definition 3.1 we obtain:

$\underline{Nm}(A) = \{p_1, p_4, p_5\}$

={<p_1, (0.5, 0.6, 0.5), (0.3, 0.4, 0.3), (0.4, 0.3, 0.3)>, <p_4, (0.5, 0.6, 0.5), (0.3, 0.4, 0.3), (0.4, 0.3, 0.3)>, <p_5, (0.2, 0.1, 0.0), (0.3, 0.2, 0.2), (0.8, 0.7, 0.7)>} and

$\overline{Nm}(A) = \{p_1, p_4, p_5, p_7, p_8\}$

{<p_1, (0.8, 0.4, 0.3), (0.4, 0.2, 0.1), (0.6, 0.2, 0.1)>, <p_4, (0.8, 0.4, 0.3), (0.4, 0.2, 0.1), (0.6, 0.2, 0.1)>, <p_5, (0.2, 0.1, 0.0), (0.3, 0.2, 0.2), (0.8, 0.7, 0.7)>, <p_7, (0.7, 0.6, 0.5), (0.3, 0.2, 0.1), (0.4, 0.3, 0.2)>, <p_8, (0.7, 0.6, 0.5), (0.3, 0.2, 0.1), (0.4, 0.3, 0.2)>}.

For another neutrosophic multisets

$Nm(B)$ = {<p_1, (0.8, 0.6, 0.5), (0.3, 0.2, 0.1), (0.4, 0.2, 0.1)>, <p_4, (0.5, 0.4, 0.3), (0.4, 0.4, 0.3), (0.6, 0.3, 0.3)>, <p_5, (0.2, 0.1, 0.0), (0.3, 0.2, 0.2), (0.8, 0.7, 0.7)>}.

The lower approximation and upper approximation of $Nm(B)$ are calculated as

$\underline{Nm}(B) = \{p_1, p_4, p_5\}$

={<p_1, (0.5, 0.6, 0.5), (0.3, 0.4, 0.3), (0.4, 0.3, 0.3)>, <p_4, (0.5, 0.6, 0.5), (0.3, 0.4, 0.3), (0.4, 0.3, 0.3)>, <p_5, (0.2, 0.1, 0.0), (0.3, 0.2, 0.2), (0.8, 0.7, 0.7)>} and

$\overline{Nm}(B) = \{p_1, p_4, p_5\}$

={<p_1, (0.5, 0.6, 0.5), (0.3, 0.4, 0.3), (0.4, 0.3, 0.3)>, <p_4, (0.5, 0.6, 0.5), (0.3, 0.4, 0.3), (0.4, 0.3, 0.3)>, <p_5, (0.2, 0.1, 0.0), (0.3, 0.2, 0.2), (0.8, 0.7, 0.7)>}

Obviously, $\underline{Nm}(B) = \overline{Nm}(B)$ is a definable neutrosophic multisets in the approximation space (U, R).

Definition 3.3 Let $Nm(A) = (\underline{Nm}(A), \overline{Nm}(A))$ be a rough neutrosophic multisets in (U, R). The rough complement of $Nm(A)$ is denoted by $\sim Nm(A) = (\underline{Nm}(A)^c, \overline{Nm}(A)^c)$ where $\underline{Nm}(A)^c$ and $\overline{Nm}(A)^c$ are the complements of neutrosophic multisets of $\underline{Nm}(A)$ and $\overline{Nm}(A)$ respectively,

$$\underline{Nm}(A)^c = \{\langle x, (F^i_{\underline{Nm}(A)}(x), 1 - I^i_{\underline{Nm}(A)}(x), T^i_{\underline{Nm}(A)}(x))\rangle \mid x \in U\},$$

$$\overline{Nm}(A)^c = \{\langle x, (F^i_{\overline{Nm}(A)}(x), 1 - I^i_{\overline{Nm}(A)}(x), T^i_{\overline{Nm}(A)}(x))\rangle \mid x \in U\}$$

where i = 1, 2, …, p.

Example 3.4 Consider RNM, $Nm(A)$ in the set $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $y \in [x]_R$ is equivalence relation and i = 1, 2, 3.

Let $Nm(A)$ = {⟨x_1, [(0.6, 0.4, 0.4), (0.7, 0.3, 0.4)], [(0.8, 0.4, 0.5), (0.7, 0.6, 0.5)], [(0.4, 0.3, 0.5), (0.3, 0.2, 0.7)] ⟩, ⟨ x_2, [(0.4, 0.3, 0.3), (0.5, 0.3, 0.4)], [(0.2, 0.4, 0.4), (0.3, 0.3, 0.5)], [(0.7, 0.8, 0.4), (0.7, 0.1, 0.5)]⟩, ⟨ x_4, [(0.2, 0.5, 0.7), (0.7, 0.8, 0.0)], [(1.0, 1.0, 0.0), (0.9, 0.2, 0.5)], [(0.1, 0.5, 0.3), (0.2, 0.8, 0.5)] ⟩}

Then the complement of $Nm(A)$ is defined as $\sim Nm(A) = (\underline{Nm}(A)^c, \overline{Nm}(A)^c)$ =

{⟨x_1, [(0.4, 0.6, 0.6), (0.4, 0.7, 0.7)], [(0.5, 0.6, 0.8), (0.5, 0.4, 0.7)], [(0.5, 0.7, 0.4), (0.7, 0.8, 0.3)] ⟩, ⟨x_2, [(0.3, 0.7, 0.4), (0.4, 0.7, 0.5)], [(0.4, 0.6, 0.2), (0.5, 0.7, 0.3)], [(0.4, 0.2, 0.7), (0.5, 0.9, 0.7)]⟩, ⟨ x_4, [(0.7, 0.5, 0.2), (0.0, 0.2, 0.7)], [(0.0, 0.0, 1.0), (0.5, 0.8, 0.9)], [(0.3, 0.5, 0.1), (0.5, 0.2, 0.2)] ⟩}.

Definition 3.5 Let $Nm(A)$ and $Nm(B)$ are RNM respectively in U, then the following definitions hold:

(i) $Nm(A) = Nm(B)$ iff $\underline{Nm}(A) = \underline{Nm}(B)$ and $\overline{Nm}(A) = \overline{Nm}(B)$

(ii) $Nm(A) \subseteq Nm(B)$ iff $\underline{Nm}(A) \subseteq \underline{Nm}(B)$ and $\overline{Nm}(A) \subseteq \overline{Nm}(B)$

(iii) $Nm(A) \bigcup Nm(B) =$

$$\left\langle \underline{Nm}(A) \bigcup \underline{Nm}(B), \overline{Nm}(A) \bigcup \overline{Nm}(B) \right\rangle$$

(iv) $Nm(A) \bigcap Nm(B) =$

$$\left\langle \underline{Nm}(A) \bigcap \underline{Nm}(B), \overline{Nm}(A) \bigcap \overline{Nm}(B) \right\rangle$$

(v) $Nm(A) + Nm(B) =$

$$\left\langle \underline{Nm}(A) + \underline{Nm}(B), \overline{Nm}(A) + \overline{Nm}(B) \right\rangle$$

(vi) $Nm(A) \cdot Nm(B) =$

$$\left\langle \underline{Nm}(A) \cdot \underline{Nm}(B), \overline{Nm}(A) \cdot \overline{Nm}(B) \right\rangle$$

Example 3.6 Consider $Nm(A)$ in Example 3.4 and $Nm(B)$ are two RNM.

$Nm(B)$ = {⟨x_1, [(0.6, 0.1, 0.2), (0.3, 0.3, 0.3)], [(0.7, 0.2, 0.5), (0.8, 0.6, 0.5)], [(0.7, 0.3, 0.5), (1.0, 0.2, 0.7)] ⟩, ⟨x_2, [(0.4, 0.4, 0.7), (0.6, 0.5, 0.6)], [(0.3, 0.4, 0.4), (0.6, 0.2, 0.5)], [(0.7, 0.8, 0.4), (0.6, 0.1, 0.5)]⟩, ⟨x_3, [(0.3, 0.4, 0.5), (0.6, 0.4, 0.0)], [(1.0, 1.0, 0.0), (0.7, 0.2, 0.5)], [(0.1, 0.5, 0.3), (0.2, 0.8, 0.5)] ⟩, ⟨x_4, [(0.4, 0.5, 0.6), (0.7, 0.8, 0.2)], [(1.0, 1.0, 0.0), (0.9, 0.2, 0.1)], [(0.6, 0.5, 0.3), (0.2, 0.2, 0.7)] ⟩}

Then, we have

(i) $Nm(A) \subseteq Nm(B)$

(ii) $Nm(A) \bigcup Nm(B)$

= {⟨x_1, [(0.6, 0.1, 0.2), (0.7, 0.3, 0.3)], [(0.8, 0.2, 0.5), (0.7, 0.6, 0.5)], [(0.7, 0.3, 0.5), (1.0, 0.2, 0.7)] ⟩, ⟨x_2, [(0.4, 0.3, 0.3), (0.6, 0.3, 0.4)], [(0.3, 0.4, 0.4), (0.6, 0.2, 0.5)], [(0.7, 0.8, 0.4), (0.7, 0.1, 0.5)]⟩, ⟨x_3, [(0.3, 0.4, 0.5), (0.6, 0.4, 0.0)], [(1.0, 1.0, 0.0), (0.7, 0.2, 0.5)], [(0.1, 0.5, 0.3), (0.2, 0.8, 0.5)]⟩, ⟨x_4, [(0.4, 0.5, 0.6), (0.7, 0.8, 0.0)], [(1.0, 1.0, 0.0), (0.9, 0.2, 0.1)], [(0.6, 0.5, 0.3), (0.2, 0.2, 0.5)] ⟩}.

(iii) $Nm(A) \bigcap Nm(B)$

= {⟨x_1, [(0.6, 0.4, 0.4), (0.3, 0.3, 0.4)], [(0.7, 0.4, 0.5), (0.7, 0.6, 0.5)], [(0.4, 0.3, 0.5), (0.3, 0.2, 0.7)]⟩, ⟨x_2, [(0.4, 0.4, 0.7), (0.5, 0.5, 0.6)], [(0.2, 0.4, 0.4), (0.3, 0.3, 0.5)], [(0.7, 0.8, 0.4), (0.6, 0.1, 0.5)]⟩, ⟨x_3, [(0.3, 0.4, 0.5), (0.6, 0.4, 0.0)], [(1.0, 1.0, 0.0), (0.7, 0.2, 0.5)], [(0.1, 0.5, 0.3), (0.2, 0.8, 0.5)]⟩, ⟨x_4, [(0.2, 0.5, 0.7), (0.7, 0.8, 0.2)], [(1.0, 1.0, 0.0), (0.9, 0.2, 0.5)], [(0.1, 0.5, 0.3), (0.2, 0.8, 0.7)] ⟩}.

Proposition 3.7 If Nm, Mm, Lm are the RNM in (U, R), then the following propositions are stated from definitions.

(i) $\sim (\sim Nm) = Nm$

(ii) $Nm \bigcup Mm = Mm \bigcup Nm$, $Nm \bigcap Mm = Mm \bigcap Nm$

(iii) $(Nm \bigcup Mm) \bigcup Lm = Nm \bigcup (Mm \bigcup Lm)$, and $(Nm \bigcap Mm) \bigcap Lm = Nm \bigcap (Mm \bigcap Lm)$

(iv) $(Nm \bigcup Mm) \bigcap Lm = (Nm \bigcup Mm) \bigcap (Nm \bigcup Lm)$, and

$(Nm \bigcap Mm) \bigcup Lm = (Nm \bigcap Mm) \bigcup (Nm \bigcap Lm)$

Proof (i):

$\sim (\sim Nm(A)) = \sim (\sim (\underline{Nm}(A), \overline{Nm}(A)))$

$= \sim (\underline{Nm}(A)^c, \overline{Nm}(A)^c)$

$= (\underline{Nm}(A), \overline{Nm}(A))$

$= Nm(A)$

Proof (ii – iv) : The proofs is straightforward from definition.

Proposition 3.8 De Morgan's Law are satisfied for rough neutrosophic multisets:

(i) $\sim (Nm(A) \bigcup Nm(B)) = (\sim Nm(A)) \bigcap (\sim Nm(B))$

(ii) $\sim (Nm(A) \bigcap Nm(B)) = (\sim Nm(A)) \bigcup (\sim Nm(B))$

Proof (i):

$(Nm(A) \bigcup Nm(B))$

$= \sim (\{ \underline{Nm}(A) \bigcup \underline{Nm}(B) \}, \{ \overline{Nm}(A) \bigcup \overline{Nm}(B) \})$

$= (\sim\{ \underline{Nm}(A) \bigcup \underline{Nm}(B) \}, \sim\{ \overline{Nm}(A) \bigcup \overline{Nm}(B) \})$

$= (\{ \underline{Nm}(A) \bigcup \underline{Nm}(B) \}^c, \{ \overline{Nm}(A) \bigcup \overline{Nm}(B) \}^c)$

$= (\sim\{ \underline{Nm}(A) \bigcap \underline{Nm}(B) \}, \sim\{ \overline{Nm}(A) \bigcap \overline{Nm}(B) \})$

$= (\sim Nm(A)) \bigcap (\sim Nm(B))$.

Proof (ii): Similar to the proof of (i).

Proposition 3.9. If A and B are two neutrosophic multisets in U such that $A \subseteq B$, then $Nm(A) \subseteq Nm(B)$

(i) $Nm(A \bigcup B) \supseteq Nm(A) \bigcup Nm(B)$

(ii) $Nm(A \bigcap B) \subseteq Nm(A) \bigcap Nm(B)$

Proof (i):

$$T^i_{\underline{Nm}(A\cup B)}(x) = \inf\{T^i_{Nm(A\cup B)}(x) \mid x \in X\}$$

$$= \inf(\max\{T^i_{Nm(A)}(x), T^i_{Nm(B)}(x) \mid x \in X\})$$

$$= \max\{\inf\{T^i_{Nm(A)}(x) \mid x \in X\}, \inf\{T^i_{Nm(B)}(x) \mid x \in X\}\}$$

$$= \max\{(T^i_{\underline{Nm}(A)}(x), T^i_{\underline{Nm}(B)}(x)) \mid x \in X\}$$

$$= (T^i_{\underline{Nm}(A)} \cup T^i_{\underline{Nm}(B)})(x)$$

Similarly,

$$I^i_{\underline{Nm}(A\cup B)}(x) \leq (I^i_{\underline{Nm}(A)} \cup I^i_{\underline{Nm}(B)})(x),$$

$$F^i_{\underline{Nm}(A\cup B)}(x) \leq (F^i_{\underline{Nm}(A)} \cup F^i_{\underline{Nm}(B)})(x)$$

Thus, $\underline{Nm}(A \cup B) \supseteq \underline{Nm}(A) \cup \underline{Nm}(B)$

Hence,

$$Nm(A \cup B) \supseteq Nm(A) \cup Nm(B)$$

Proof (ii): Similar to the proof of (i).

Proposition 3.10.

(i) $\underline{Nm}(A) = \sim \overline{Nm}(\sim A)$

(ii) $\overline{Nm}(A) = \sim \underline{Nm}(\sim A)$

(iii) $\underline{Nm}(A) \subseteq \overline{Nm}(A)$

Proof (i): According to Definition 3.1, we can obtain

$$A = \{\langle x, (T^i_A(x), I^i_A(x), F^i_A(x))\rangle \mid x \in X\}$$

$$\sim A = \{\langle x, (F^i_A(x), 1 - I^i_A(x), T^i_A(x))\rangle \mid x \in X\}$$

$$\overline{Nm}(\sim A) = \{\langle x, (F^i_{\overline{Nm}(\sim A)}(x), 1 - I^i_{\overline{Nm}(\sim A)}(x), T^i_{\overline{Nm}(\sim A)}(x))\rangle \mid y \in [x]_R, x \in U\}$$

$$\sim \overline{Nm}(\sim A) = \{\langle x, (T^i_{\overline{Nm}(\sim A)}(x), 1 - (1 - I^i_{\overline{Nm}(\sim A)}(x)), F^i_{\overline{Nm}(\sim A)}(x))\rangle \mid y \in [x]_R, x \in U\}$$

$$= \{\langle x, (T^i_{\overline{Nm}(\sim A)}(x), I^i_{\overline{Nm}(\sim A)}(x), F^i_{\overline{Nm}(\sim A)}(x))\rangle \mid y \in [x]_R, x \in U\}$$

where

$$T^i_{\overline{Nm}(\sim A)}(x) = \bigwedge_{y\in[x]_R} T^i_A(y),$$

$$I^i_{\overline{Nm}(\sim A)}(x) = \bigvee_{y\in[x]_R} I^i_A(y),$$

$$F^i_{\overline{Nm}(\sim A)}(x) = \bigvee_{y\in[x]_R} F^i_A(y),$$

Hence $\underline{Nm}(A) = \sim \overline{Nm}(\sim A)$.

Proof (ii): Similar to the proof of (i).

Proof (iii): For any $y \in \underline{Nm}(A)$, we can have

$$T^i_{\underline{Nm}(A)}(x) = \bigwedge_{y\in[x]_R} T^i_A(y) \leq \bigvee_{y\in[x]_R} T^i_A(y),$$

$$I^i_{\underline{Nm}(A)}(x) = \bigvee_{y\in[x]_R} I^i_A(y) \geq \bigwedge_{y\in[x]_R} I^i_A(y), \text{ and}$$

$$F^i_{\underline{Nm}(A)}(x) = \bigvee_{y\in[x]_R} F^i_A(y) \geq \bigwedge_{y\in[x]_R} F^i_A(y)$$

Hence $\underline{Nm}(A) \subseteq \overline{Nm}(A)$.

Conclusion

This paper firstly defined the rough neutrosophic multisets (RNM) theory and their properties and operations were studied. The RNM are the extension of rough neutrosophic sets [15]. The future work will cover the others operation in rough set, neutrosophic multisets and rough neutrosophic set that is suitable for RNM theory such as the notion of inverse, symmetry, and relation.

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University of New Mexico

Competencies Interdepencies Analysis based on Neutrosophic Cognitive Mapping

Evelyn Jazmín Henríquez Antepara[1] , Jenny Elizabeth Arízaga Gamboa[2] , Mélida Rocio Campoverde Méndez[3] , Miriam Elizabeth Peña González[4]

[1] Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. Email: evelyn.henriqueza@ug.edu.ec
[2] Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. Email: ing_jennyarizaga@hotmail.com
[3] Universidad Laica Vicente Rocafuerte de Guayaquil, Facultad de Educación, Guayaquil Ecuador. Email: rociocampoverde@gmail.com
[4] Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. Email: mepg60@gmail.com

Abstract. Recently, there has been increasing interest in competency-based education. Additionally, neutrosophic cognitive maps and its application in decision making have become a topic of significant importance for researchers and practitioners. In this paper, a framework 33based on static analysis of neutrosophic cognitive maps applied to competencies modelling and prioritization in presented. A case study based on modelling and prioritization of transversal competencies in system engineering is developed. The paper ends with conclusion and future research directions.
.

Keywords: : information systems, competencies, neutrosophic cognitive mapping, prioritization.

1 Introduction

Recently, there has been increasing interest in competency-based education [1]. Competency-based education is known to improve employability in students [2]There are many interdependencies among competencies, determining the interrelationship of competencies is very import for in evaluation [3].

Neutrosophic sets and logic is a generalization of fuzzy set and logic based on neutrosophy [4]. Neutrosophy can handle indeterminate and inconsistent information, while fuzzy sets and intuitionistic fuzzy sets cannot describe them appropriately [5]. In this paper a new model for competencies analysis based on neutrosophic cognitive maps(NCM) [6] is presented giving methodological support and the possibility of dealing with interdependence, feedback and indeterminacy.

This paper is structured as follows: Section 2 reviews some important preliminaries concepts about Neutrosophic cognitive maps. In Section 3, a framework for competencies interrelation analysis based on NCM static analysis is presented. Section 4 shows a case study of the proposed model. The paper ends with conclusions and further work recommendations.

2 Neutrosophic cognitive maps

Neutrosophic Logic (NL) was introduced in 1995 as a generalization of the fuzzy logic, especially of the intuitionistic fuzzy logic [7]. A logical proposition P is characterized by three neutrosophic components:

$$NL(P) = (T, I, F) \qquad (1)$$

where T is the degree of truth, F the degree of falsehood, and I the degree of indeterminacy.
A neutrosophic matrix is a matrix where the elements $a = (a_{ij})$ have been replaced by elements in $\langle R \cup I \rangle$, where $\langle R \cup I \rangle$ is the neutrosophic integer ring [8].A neutrosophic graph is a graph in which at least one edge is a neutrosophic edge [9]. If indeterminacy is introduced in cognitive mapping it is called Neutrosophic Cognitive Map (NCM) [10].
NCM are based on neutrosophic logic to represent uncertainty and indeterminacy in cognitive maps [4]. A NCM is a directed graph in which at least one edge is an indeterminacy denoted by dotted lines [11] (Figure 2.).

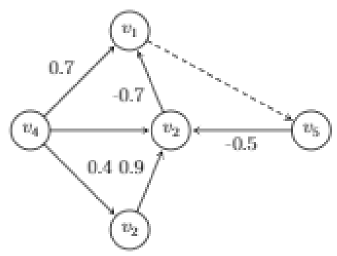

Fig. 1. NCM example

In [12] and in [13] a static analysis of mental model framework in the form of NCM is presented. The result of the static analysis result is in the form neutrosophic numbers (a+bI, where I = indeterminacy) [14]. Finally a the de-neutrosophication process as proposes by Salmeron and Smarandache [15] is applied to given the final ranking value. In this paper, this model is extended and detailed to deal with factors prioritization.

3 Proposed Framework

Our aim is to develop a framework for competencies interdependencies analysis based on NCM. The model consists of the following phases

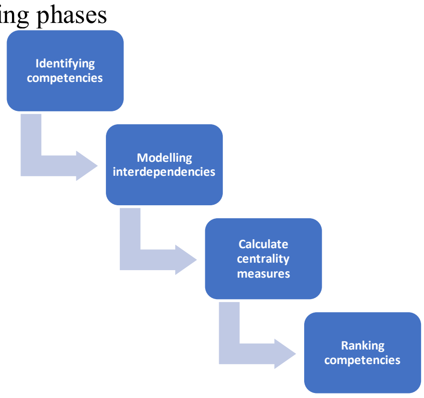

Fig. 2. Proposed framework for PEST analysis.

1.1 Identifying competencies

In this step, the relevant competencies are identified. Different techniques can be used, for example the Delphi technique[16].

1.2 Modelling interdependencies

Causal interdependencies among competencies are modelled. This step consists in the formation of NCM, according to the views of the evaluator.

1.3 Calculate centrality measures

The following measures are calculated[17] with absolute values of the NCM adjacency matrix [18]:

Outdegree $od(v_i)$ is the row sum of absolute values of a variable in the neutrosophic adjacency matrix. It shows the cumulative strengths of connections (c_{ij}) exiting the variable.

$$od(v_i) = \sum_{i=1}^{N} c_{ij} \qquad (2)$$

Indegree $id(v_i)$ is the column sum of absolute values of a variable. It shows the cumulative strength of variables entering the variable.

$$id(v_i) = \sum_{i=1}^{N} c_{ji} \qquad (3)$$

The centrality (total degree $td(v_i)$), of a variable is the summation of its indegree (in-arrows) and outdegree (out-arrows)

$$td(v_i) = od(v_i) + id(v_i) \qquad (4)$$

1.4 Ranking competencies

A de-neutrosophication process gives an interval number for centrality. This one is based on max-min values of I . A neutrosophic value is transformed in an interval with two values, the maximum and the minimum value $I \in [0,1]$.

The contribution of a variable in a cognitive map can be understood by calculating its degree centrality, which shows how connected the variable is to other variables and what the cumulative strength of these connections are. The median of the extreme values [19] is used to give a centrality value :

$$\lambda([a_1, a_2]) = \frac{a_1 + a_2}{2} \qquad (5)$$

Then

$$A > B \Leftrightarrow \frac{a_1 + a_2}{2} > \frac{b_1 + b_2}{2} \qquad (6)$$

Finally, a ranking of variables is given. The numerical value it used for factor prioritization and/or reduction [20].

4 Case study

In this case, the relationship between competencies are represented by a subset of so-called transversal competencies in system engineering

Competencies	Description
c_1	Ability to solve mathematical problems
c_2	Understanding and mastering the basic concepts of information technology
c_3	Basic knowledge about the use and programming of computers
c_4	Ability to solve problems within your area of study

c_5	Be motivated by professional achievement and to face new challenges.
c_6	Use of the English language at written and oral level.

Table 1. Competencies analyzed

The NCM is developed by capturing expert's causal knowledge. The generated neutrosophic adjacency matrix is shown in Table 2.

0	0.7	0.4	I	0	0
0	0	0.9	0.7	0	0
0	0	0	0.9	0	0
0	0.5	0	0	0.9	0
0	I	0	0.7	0	0
0	0.9	0.6	0.7	I	0

Table 2: Adjacency. matrix

The centrality measures calculated are shown below.

c_1	1.1+I
c_2	1.6+I
c_3	0.9
c_4	1.4
c_5	0.7
c_6	2.2+I

Table 3: Outdegree

c_1	A	0
c_2	B	2.1+I
c_3	C	1.9
c_4	D	3+I
c_5	E	0.9+I
c_6	F	0

Table 4: Indegree

c_1	A	1.1+I
c_2	B	3.7+2I
c_3	C	2.18
c_4	D	3.4+I
c_5	E	1.6+I
c_6	F	2.2+I

Table 5: Total degree

A static analysis in NCM [10] which gives as result initially neutrophic number of the form (a + bI, where I = indeterminacy) . Finally, a de-neutrosification process as proposed by Salmerón and Smarandache [12] is developed. I ∈ [0,1] is replaced by its maximum and minimum values.

c_1	[1.1, 2.1]
c_2	[3.7, 5.7]
c_3	2.18
c_4	[3.4, 4.4]
c_5	[1.6, 2.6]
c_6	[2.2, 3.2]

Table 6: de-neutrosification

Finally, we work with the mean of the extreme values to obtain a single value [19] .

c_1	1.6
c_2	4,7
c_3	2.18
c_4	3,9
c_5	2,1
c_6	2.7

Table 7. Median of the extreme values

From these numerical values, the following ranking is obtained:

$$c_2 \succ c_4 \succ c_6 \succ c_3 \succ c_5 \succ c_1$$

In this case the most important competence is: "Understanding and mastering the basic concepts of information technology".

5 Conclusion

In the work, a model was presented to analyze the interrelationships between competencies and giving a priority is using the static analysis of neutrosophic cognitive maps. In the case study developed was determined as the most important : Understanding and mastering the basic concepts on the laws of information technology.

A future work is to analyze new competencies in the proposed framework. Incorporating scenario analysis and developing a software tool is another area of research.

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NSS University of New Mexico

Support-Neutrosophic Set: A New Concept in Soft Computing

Nguyen Xuan Thao,[1] Florentin Smarandache,[2] Nguyen Van Dinh[1]

[1]Faculty of Information Technology, Vietnam National University of Agriculture (VNUA)
Email: nxthao2000@gmail.com, nvdinh2000@gmail.com
[2]Department of Mathematics University of New Mexico Gallup, NM, USA.
E-mail: smarand@unm.edu

Abstract. Today, soft computing is a field that is used a lot in solving real-world problems, such as problems in economics, finance, banking... With the aim to serve for solving the real problem, many new theories and/or tools which were proposed, improved to help soft computing used more efficiently. We can mention some theories as fuzzy sets theory (L. Zadeh, 1965), intuitionistic fuzzy set (K Atanasov, 1986), neutrosophic set (F. Smarandache 1999). In this paper, we introduce a new notion of support-neutrosophic set (SNS), which is the combination a neutrosophic set with a fuzzy set. So, SNS set is a direct extension of fuzzy set and neutrosophic sets (F. Smarandache). Then, we define some operators on the support-neutrosophic sets, and investigate some properties of these operators.

Keywords: support-neutrosophic sets, support-neutrosophic fuzzy relations, support- neutrosophic similarity relations

1 Introduction

In 1998, Prof. Smarandache gave the concept of the *neutrosophic set* (NS) [3] which generalized fuzzy set [10] and intuitionistic fuzzy set [1]. It is characterized by a degree of truth (T), a degree of indeterminacy (I) and a degree of falsity (F). Over time, the sub-class of the neutrosophic set were proposed to capture more advantageous in practical applications. Wang et al. [5] proposed the interval neutrosophic set and its operators. Wang et al. [6] proposed a single-valued neutrosophic set as an instance of the neutrosophic set accompanied with various set theoretic operators and properties. Ye [8] defined the concept of simpli-fied neutrosophic set whose elements of the universe have a degree of truth, indeterminacy and falsity respectively that lie between [0, 1]. Some operational laws for the simplified neutrosophic set and two aggregation operators, including a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator were presented.

In 2015, Nguyen et al. [2] introduced a Support-intuitionistic fuzzy set, it combines a intuitionistic fuzzy set with a fuzzy set (the support of an intuitionistic). Apter, Young et al [9] applied support – intuitionistic in decision making.

Practically, lets' consider the following case: a customer is interested in two products A and B. The customer has one rating of good (i), indeterminacy (ii) or not good (iii) for each of the products. These ratings (i),(ii) and (iii) (known as neutrosophic ratings) will affect the customer's decision of which product to buy. However, the customer's financial capacity will also affect her decision. This factor is called the support factor, with the value is between 0 and 1. Thus, the decision of which product to buy are determined by truth factors (i), indeterminacy factors (ii), falsity factors (iii) and support factor (iv). If a product is considered good and affordable, it is the best situation for a buying decision. The most unfavorable situation is when a product is considered bad and not affordable (support factor is bad),in this case, it would be easy to refuse to buy the product.

Another example, the business and purchase of cars in the Vietnam market. For customers, they will care about the quality of the car (good, bad and indeterminacy, they are neutrosophic) and prize, which are considered as supporting factors for car buyers. For car dealers, they are also interested in the quality of the car, the price and the government's policy on importing cars such as import duties on cars. Price and government policies can be viewed as supporting components of the car business.

In this paper, we combine a neutrosophic set with a fuzzy set. This raise a new concept called support-neutrosophic set (SNS). In which, there are four membership functions of an element in a given set. The remaining of this paper was structured as follows: In section 2, we introduce the concept of support-neutrosophic set and study some properties of SNS. In section 3, we give some distances between two SNS sets. Finally, we construct the distance of two support-neutrosophic sets.

2 Support-Neutrosophic set

Throughout this paper, U will be a nonempty set called the universe of discourse. First, we recall some the concept about fuzzy set and neutrosophic set. Here, we use mathematical operations on real numbers. Let S_1 and S_2 be two real standard or non-standard subsets, then

$$S_1 + S_2 = \{x | x = s_1 + s_2, s_1 \in S_1, s_2 \in S_2\}$$

$$S_1 - S_2 = \{x | x = s_1 - s_2, s_1 \in S_1, s_2 \in S_2\}$$

$$\bar{S}_2 = \{1^+\} - S_2 = \{x | x = 1^+ - s_2, s_2 \in S_2\}$$

$$S_1 \times S_2 = \{x | x = s_1 \times s_2, s_1 \in S_1, s_2 \in S_2\}$$

$$S_1 \vee S_2 = [\max\{infS_1, infS_2\}, \max\{supS_1, supS_2\}]$$

$$S_1 \wedge S_2 = [\min\{infS_1, infS_2\}, \min\{supS_1, supS_2\}]$$

$$d(S_1, S_2) = \inf_{s_1 \in S_1, s_2 \in S_2} d(s_1, s_2)$$

Remark: $\overline{S_1 \wedge S_2} = \overline{S_1} \vee \overline{S_2}$ and $\overline{S_1 \vee S_2} = \overline{S_1} \wedge \overline{S_2}$. Indeed, we consider two cases:

+ if $infS_1 \le infS_2$ and $\sup S_1 \le \sup S_2$ then $1 - infS_2 \le 1 - infS_1$, $1 - supS_2 \le 1 - supS_1$ and $S_1 \wedge S_2 = S_1$, $S_1 \vee S_2 = S_2$. So that $\overline{S_1 \wedge S_2} = \overline{S_1} = \overline{S_1} \vee \overline{S_2}$ and $\overline{S_1 \vee S_2} = \overline{S_2} = \overline{S_1} \wedge \overline{S_2}$.

+ if $infS_1 \le infS_2 \le supS_2 \le \sup S_1$. Then $S_1 \wedge S_2 = [infS_1, supS_2]$ and $\overline{S_1} \vee \overline{S_2} = [1 - supS_2, 1 - infS_1]$. Hence $\overline{S_1 \wedge S_2} = \overline{S_1} \vee \overline{S_2}$. Similarly, we have $\overline{S_1 \wedge S_2} = \overline{S_1} \vee \overline{S_2}$.

Definition 1. A fuzzy set A on the universe U is an object of the form

$$A = \{(x, \mu_A(x)) | x \in U\}$$

where $\mu_A(x) (\in [0,1])$ is called the degree of membership of x in A.

Definition 2. A neutrosophic set A on the universe U is an object of the form

$$A = \{(x, T_A(x), I_A(x), F_A(x)) | x \in U\}$$

where T_A is a truth –membership function, I_A is an indeterminacy-membership function, and F_A is falsity – membership function of A. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$, that is

$$T_A: U \to]0^-, 1^+[$$
$$I_A: U \to]0^-, 1^+[$$
$$F_A: U \to]0^-, 1^+[$$

In real applications, we usually use

$$T_A: U \to [0,1]$$
$$I_A: U \to [0,1]$$
$$F_A: U \to [0,1]$$

Now, we combine a neutrosophic set with a fuzzy set. That leads to a new concept called support-neutrosophic set (SNS). In which, there are four membership functions of each element in a given set. This new concept is stated as follows:

Definition 3. A support – neutrosophic set (SNS) A on the universe U is characterized by a truth –membership function T_A, an indeterminacy-membership function I_A, a falsity – membership function F_A and support-membership function s_A. For each $x \in U$ we have $T_A(x), I_A(x)$, $F_A(x)$ and $s_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$, that is

$$T_A: U \to]0^-, 1^+[$$
$$I_A: U \to]0^-, 1^+[$$
$$F_A: U \to]0^-, 1^+[$$
$$s_A: U \to]0^-, 1^+[$$

We denote support – neutrosophic set (SNS)

$$A = \{(x, T_A(x), I_A(x), F_A(x), s_A(x)) | x \in U\}.$$

There is no restriction on the sum of $T_A(x), I_A(x)$, $F_A(x)$, so $0^- \le sup T_A(x) + sup I_A(x) + sup\ F_A(x) \le 3^+$, and $0^- \le s_A(x) \le 1^+$.

When U is continuous, a SNS can be written as

$$A = \int_U {< T_A(x), I_A(x), F_A(x), s_A(x) >}/{x}$$

When $U = \{x_1, x_2, .., x_n\}$ is discrete, a SNS can be written as

$$A = \sum_{i=1}^{n} \frac{< T_A(x_i), I_A(x_i), F_A(x_i), s_A(x_i) >}{x_i}$$

We denote $SNS(U)$ is the family of SNS sets on U.

Remarks:

+ The element $x_* \in U$ is called “worst element” in A if $T_A(x_*) = 0, I_A(x_*) = 0, F_A(x_*) = 1, s_A(x_*) = 0$. The element $x^* \in U$ is called “best element” in A if

$$T_A(x^*) = 1, I_A(x^*) = 1, F_A(x_*) = 0, s_A(x_*) = 1$$

(if there is restriction $sup\mathrm{T_A}(\mathrm{x}) + sup\mathrm{I_A}(\mathrm{x}) + sup\ \mathrm{F_A}(\mathrm{x}) \leq 1$ then the element $x^* \in U$ is called "best element" in A if

$T_A(x^*) = 1, I_A(x^*) = 0, F_A(x_*) = 0, s_A(x_*) = 1)$.

+ the support – neutrosophic set A reduce an neutrosophic set if $s_A(x) = c \in [0,1], \forall x \in U$.

+ the support – neutrosophic set A is called a support-standard neutrosophic set if

$\mathrm{T_A}(\mathrm{x}), \mathrm{I_A}(\mathrm{x}), \mathrm{F_A}(\mathrm{x}) \in [0,1]$ and

$\mathrm{T_A}(\mathrm{x}) + \mathrm{I_A}(x) + \mathrm{F_A}(\mathrm{x}) \leq 1$

for all $x \in U$.

+ the support – neutrosophic set A is a support-intuitionistic fuzzy set if $\mathrm{T_A}(\mathrm{x}), \mathrm{F_A}(\mathrm{x}) \in [0,1], \mathrm{I_A}(x) = 0$ and $\mathrm{T_A}(\mathrm{x}) + \mathrm{F_A}(\mathrm{x}) \leq 1$ for all $x \in U$.

+ A constant SNS set

$(\alpha, \widehat{\beta, \theta}, \gamma) = \{(x, \alpha, \beta, \theta, \gamma) | x \in U$

where $0 \leq \alpha, \beta, \theta, \gamma \leq 1\}$.

+ the SNS universe set is

$U = 1_U = (1, \widehat{1,0}, 1) = \{(x, 1,1,0,1) | x \in U\}$

+ the SNS empty set is

$U = 0_U = (0, \widehat{0,1}, 0) = \{(x, 0,0,1,0) | x \in U\}$

Definition 4. The complement of a SNS A is denoted by $c(A)$ and is defined by

$$T_{C(A)}(x) = F_A(x)$$
$$I_{C(A}(x) \quad \{1^+\} - I_A(x)$$
$$F_{C(A)}(x) = T_A(x)$$
$$s_{C(A)}(x) \quad \{1^+\} - s_A(x)$$

for all $x \in U$.

Definition 5. A SNS A is contained in the other SNS B, denote $A \subseteq B$, if and only if

$$inf T_A(x) \leq inf T_B(x), \qquad sup T_A(x) \leq \sup T_B(x)$$
$$inf\ F_A(x) \geq inf\ F_B(x), \qquad sup F_A(x) \geq sub\ F_B(x)$$
$$inf s_A(x) \leq inf s_B(x), \qquad sup s_A(x) \leq \sup s_B(x)$$

for all $x \in U$.

Definition 6. The union of two SNS A and B is a SNS $C = A \cup B$, that is defined by

$$T_C = T_A \vee T_B$$
$$I_C = I_A \vee I_B$$
$$F_C = F_B \wedge F_B$$
$$s_C = s_A \vee s_B$$

Definition 7. The intersection of two SNS A and B is a SNS $D = A \cap B$, that is defined by

$$T_D = T_A \wedge T_B$$
$$I_D = I_A \wedge I_B$$
$$F_D = F_B \vee F_B$$
$$s_D = s_A \wedge s_B$$

Example 1. Let $U = \{x_1, x_2, x_3, x_4\}$ be the universe. Suppose that

$$A = \frac{\langle [0.5, 0.8], [0.4, 0.6], [0.2, 0.7], [0.7, 0.9] \rangle}{x_1} + \frac{\langle [0.4, 0.5], [0.45, 0.6], [0.3, 0.6], [0.5, 0.8] \rangle}{x_2} + \frac{\langle [0.5, 0.9], [0.4, 0.5], [0.6, 0.7], [0.2, 0.6] \rangle}{x_3} + \frac{\langle [0.5, 0.9], [0.3, 0.6], [0.4, 0.8], [0.1, 0.6] \rangle}{x_4}$$

and

$$B = \frac{\langle [0.2, 0.6], [0.3, 0.5], [0.3, 0.6], [0.6, 0.9] \rangle}{x_1} + \frac{\langle [0.45, 0.7], [0.4, 0.8], [0.9, 1], [0.4, 0.9] \rangle}{x_2} + \frac{\langle [0.1, 0.7], [0.4, 0.8], [0.6, 0.9], [0.2, 0.7] \rangle}{x_3} + \frac{\langle [0.5, 1], [0.2, 0.9], [0.3, 0.7], [0.1, 0.5] \rangle}{x_4}$$

are two support –neutrosophic set on U.

We have

+ complement of A, denote $c(A)$ or $\sim A$, defined by

$$c(A) = \frac{\langle[0.2,0.7],[0.4,0.6],[0.5,0.8],[0.1,0.3]\rangle}{x_1} + \frac{\langle[0.3,0.6],[0.4,0.55],[0.4,0.5],[0.2,0.5]\rangle}{x_2} + \frac{\langle[0.6,0.7],[0.5,0.6],[0.5,0.9],[0.4,0.8]\rangle}{x_3} + \frac{\langle[0.4,0.8],[0.4,0.7],[0.5,0.9],[0.4,0.9]\rangle}{x_4}$$

+ Union $C = A \cup B$:

$$C = \frac{\langle[0.5,0.8],[0.4,0.6],[0.2,0.6],[0.7,0.9]\rangle}{x_1} + \frac{\langle[0.45,0.7],[0.45,0.8],[0.3,0.6],[0.4,0.9]\rangle}{x_2} + \frac{\langle[0.5,0.9],[0.4,0.8],[0.6,0.7],[0.2,0.7]\rangle}{x_3} + \frac{\langle[0.5,1],[0.3,0.9],[0.3,0.7],[0.1,0.6]\rangle}{x_4}$$

+ the intersection $D = A \cap B$:

$$D = \frac{\langle[0.2,0.6],[0.3,0.5],[0.3,0.7],[0.6,0.9]\rangle}{x_1} + \frac{\langle[0.4,0.5],[0.4,0.6],[0.9,1],[0.4,0.8]\rangle}{x_2} + \frac{\langle[0.1,0.7],[0.4,0.5],[0.6,0.9],[0.2,0.6]\rangle}{x_3} + \frac{\langle[0.5,0.9],[0.2,0.6],[0.4,0.8],[0.1,0.5]\rangle}{x_4}$$

Proposition 1. For all $A, B, C \in SNS(U)$, we have

(a) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$,
(b) $c(c(A)) = A$,
(c) Operators $\cap$ and $\cup$ are commutative, associative, and distributive,
(d) Operators $\cap, \sim$ and $\cup$ satisfy the law of De Morgan. It means that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Proof.

It is easy to verify that (a), (b), (c) is truth.

We show that (d) is correct. Indeed, for each

$$T_{\sim(A\cap B)} = F_{A\cap B} = F_A \vee F_B = T_{\sim A} \vee T_{\sim B}$$
$$I_{\sim(A\cap B)} = \{1^+\} - I(A\cap B) = \overline{I(A) \wedge I(B)}$$
$$= \overline{I(A)} \vee \overline{I(B)} = I_{\sim A} \vee I_{\sim B}$$
$$F_{\sim(A\cap B)} = T_{A\cap B} = T_A \wedge T_B = F_{\sim A} \wedge F_{\sim B}$$
$$s_{\sim(A\cap B)} = \{1^+\} - s(A\cap B) = \overline{s(A) \wedge s(B)}$$
$$= \overline{s(A)} \vee \overline{s(B)} = s_{\sim A} \vee s_{\sim B}$$

So that $\overline{A \cap B} = \overline{A} \cup \overline{B}$. By same way, we have $\overline{A \cup B} = \overline{A} \cap \overline{B}$. □

3 The Cartesian product of two SNS

Let U, V be two universe sets.

Definition 8. Let A, B two SNS on U, V, respectively. We define the Cartesian product of these two SNS sets:

a)

$$A \times B = \left\{ \left(\begin{array}{l} (x,y), T_{A\ B}(x,y), I_{A\times B}(x,y), \\ F_{A\ B}(x,y), s_{A\times B}(x,y) \end{array} \right) \mid x \in U, y \in V \right\}$$

where

$$T_{A\times B}(x,y) = T_A(x)T_B(y),$$
$$I_{A\times B}(x,y) = I_A(x)I_B(y),$$
$$F_{A\times B}(x,y) = F_A(x)F_B(y)$$

and

$$s_{A\times B}(x,y) = s_A(x)s_B(y), \forall x \in U, y \in V.$$

$$A \otimes B = \left\{ \left(\begin{array}{l} (x,y), T_{A\ B}(x,y), I_{A\otimes B}(x,y), \\ F_{A\ B}(x,y), s_{A\otimes B}(x,y) \end{array} \right) \mid x \in U, y \in V \right\}$$

Where

$$T_{A\otimes B}(x,y) = T_A(x)\text{ě}\ T_B(y),$$

$$I_{A\otimes B}(x,y) = I_A(x)\text{ě}\ I_B(y),$$

$$F_{A\otimes B}(x,y) = F_A(x)\text{ę}\ F_B(y)$$

and

$$s_{A\otimes B}(x,y) = s_A(x)\text{ě}\ s_B(y), \forall x \in U, y \in V\ .$$

Example 2. Let $U = \{x_1, x_2\}$ be the universe set. Suppose that

$$A = \frac{\langle [0.5, 0.8], [0.4, 0.6], [0.2, 0.7], [0.7, 0.9] \rangle}{x_1}$$
$$+ \frac{\langle [0.4, 0.5], [0.45, 0.6], [0.3, 0.6], [0.5, 0.8] \rangle}{x_2}$$

and

$$B = \frac{\langle [0.2, 0.6], [0.3, 0.5], [0.3, 0.6], [0.6, 0.9] \rangle}{x_1}$$
$$+ \frac{\langle [0.45, 0.7], [0.4, 0.8], [0.9, 1], [0.4, 0.9] \rangle}{x_2}$$

are two SNS on U. Then we have

$$A \times B \quad \frac{\langle [0.25, 0.72], [0.16, 0.3], [0.12, .49], [0.14, 0.54] \rangle}{(x_1, x\ )}$$
$$+ \frac{\langle [0.225, 0.56], [0.16, 0.48], [0.18, 0.7], [0.28, 1] \rangle}{(x_1, x\ )}$$
$$+ \frac{\langle [0.2, 0.45], [0.18, 0.3], [0.18, 0.42], [0.1, 0.48] \rangle}{(x_2, x\ )}$$
$$+ \frac{\langle [0.2, 0.45], [0.135, 0.36], [0.12, 0.48], [0.05, 0.48] \rangle}{(x_2, x\ )}$$

and

$$A \otimes B \quad \frac{\langle [0.5, 0.8], [0.4, 0.5], [0.6, 0.7], [0.2, 0.6] \rangle}{(x_1, x\ )}$$
$$+ \frac{\langle [05, 0.8], [0.3, 0.6], [0.4, 0.8], [0.1, 0.6] \rangle}{(x_1, x\ )}$$
$$+ \frac{\langle [0.4, 0.5], [0.4, 0.5], [0.6, 0.7], [0.2, 0.6] \rangle}{(x_2, x\ )}$$
$$+ \frac{\langle [0.4, 0.5], [0.3, 0.6], [0.4, 0.8], [0.1, 0.6] \rangle}{(x_2, x\ )}$$

Proposition 2. For every three universes U, V, W and three universe sets A on U, B on V, C on W. We have

a) $A \times B = B \times A$ and $A \otimes B = B \otimes A$
b) $(A \times B) \times C = A \times (B \times C)$
   and $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

Proof. It is obvious.

4 Distance between support-neutrosophic sets

In this section, we define the distance between two support-neutrosophic sets in the sene of Szmidt and Kacprzyk are presented:

Definition 9. Let $U = \{x_1, x_2, \dots, x_n\}$ be the universe set. Given $A, B \in SNS(U)$, we define

a) The Hamming distance
$$d_{SNS}(A, B) = \frac{1}{n}\sum_{i=1}^{n}[d(T_A(x_i), T_B(x_i)) + d(I_A(x_i), I_B(x_i)) + d(F_A(x_i), F_B(x_i)) + d(s_A(x_i), s_B(x_i))]$$

b) The Euclidean distance
$$e_{SNS}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\left[d^2(T_A(x_i), T_B(x_i)) + d^2(I_A(x_i), I_B(x_i)) + d^2(F_A(x_i), F_B(x_i)) + d^2(s_A(x_i), s_B(x_i))\right]^{\frac{1}{2}}$$

Example 3. Let $U = \{x_1, x_2\}$ be the universe set. Two SNS $A, B \in SNS(U)$ as in example 2 we have $d_{SNS}(A, B) = 0.15$; $e_{SNS}(A, B) = 0.15$.

If

$$C = \frac{\langle [0.5, 0.7], [0.4, 0.6], [0.2, 0.7], [0.7, 0.9] \rangle}{x_1}$$
$$+ \frac{\langle [0.4, 0.5], [0.45, 0.6], [0.3, 0.6], [0.5, 0.8] \rangle}{x_2}$$

and

$$D = \frac{\langle [0.2, 0.4], [0.3, 0.5], [0.3, 0.6], [0.6, 0.9] \rangle}{x_1}$$
$$+ \frac{\langle [0.6, 0.7], [0.4, 0.8], [0.9, 1], [0.4, 0.9] \rangle}{x_2}$$

then $d_{SNS}(C, D) = 0{,}25$ and $e_{SNS}(C, D) = 0{,}2081$.

Conclusion

In this paper, we introduce a new concept: support-neutrosophic set. We also study operators on the support-neutrosophic set and their initial properties. We have given the distance and the Cartesian product of two support – neutrosophic sets. In the future, we will study more results on the support-neutrosophic set and their applications.

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