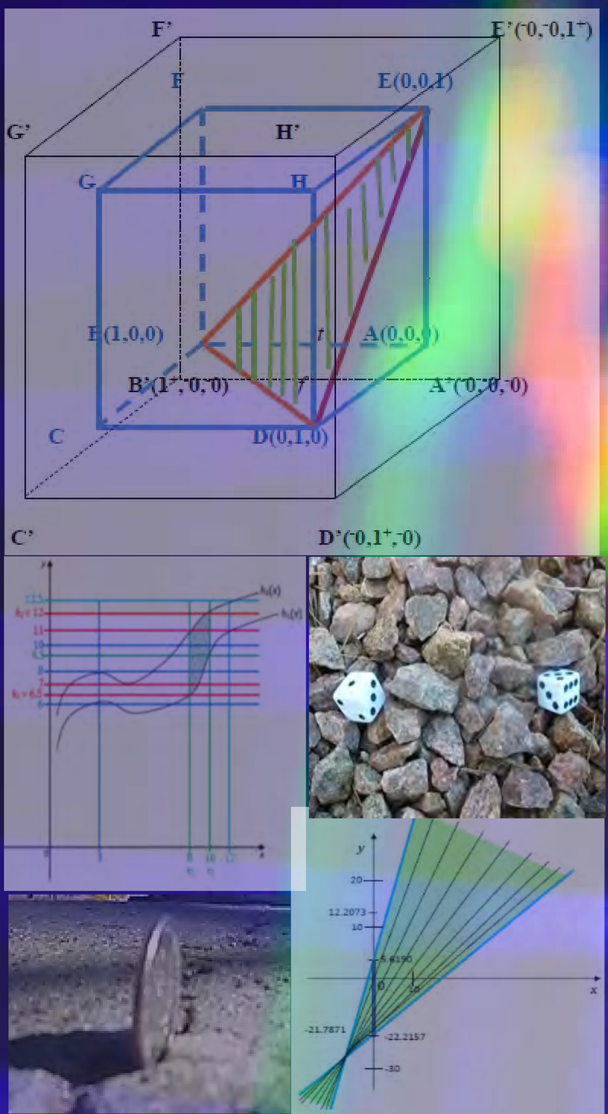
Volume 48, 2022

# Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering



<A> <neutA> <antiA>

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Editors-in-Chief

ISSN 2331-6055 (Print)
ISSN 2331-608X (Online)

Neutrosophic Science
International Association (NSIA)

ISSN 2331-6055 (print)

ISSN 2331-608X (online)

# Neutrosophic
# Sets
# and
# Systems

**An International Journal in Information Science and Engineering**

University of New Mexico

ISSN 2331-6055 (print) ISSN 2331-608X (online)

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# Neutrosophic Sets and Systems

**An International Journal in Information Science and Engineering**

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*Neutrosophy* is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

*Neutrosophic Set* and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth ($T$), a degree of indeterminacy ($I$), and a degree of falsity ($F$), where $T, I, F$ are standard or non-standard subsets of $]^-0, 1^+[$.

*Neutrosophic Probability* is a generalization of the classical probability and imprecise probability.

*Neutrosophic Statistics* is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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ISSN 2331-6055 (print) ISSN 2331-608X (online)

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**An International Journal in Information Science and Engineering**

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**Google Dictionaries** have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

*ISSN 2331-6055 (print)* **NSS** *ISSN 2331-608X (online)*
***University of New Mexico***

# Contents

# MCGDM Approach Using the Weighted Hyperbolic Sine Similarity Measure of Neutrosophic (Indeterminate Fuzzy) Multivalued Sets for the Teaching Quality Assessment of Teachers

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**Abstract:** A neutrosophic (indeterminate fuzzy) multivalued set (NMS) can be effectively described by neutrosophic number sequences with identical or different neutrosophic numbers $z_i = \mu_i + v_iI \subseteq [0, 1]$ ($i$ = 1, 2, …, $q$) for $\mu, v \in R$ and $I \in [I^-, I^+]$. Therefore, NMS is a stronger and more valuable tool for describing indeterminate fuzzy multivalued information. In this article, we propose the weighted hyperbolic sine similarity measure of NMSs to deal with the multi-criteria group decision-making (MCGDM) issue of teaching quality assessment with different indeterminate ranges of decision makers. To do so, first according to the hyperbolic sine function, we propose a hyperbolic sine similarity measure of NMSs and a weighted hyperbolic sine similarity measure of NMSs and investigate their desirable properties. Second, we develop a MCGDM approach with some indeterminate ranges in terms of the proposed weighted hyperbolic sine similarity measure of NMSs. Lastly, an illustrative example on the teaching quality assessment of teachers is presented to illustrate the applicability of the developed approach, then the developed approach is compared with the existing related approach to reveal the effectiveness of the developed approach for the teaching quality assessment of teachers in the environment of NMSs.

**Keywords:** neutrosophic (indeterminate fuzzy) multivalued set; neutrosophic number; hyperbolic sine similarity measure; group decision making; teaching quality assessment

## 1. Introduction

Fuzzy set (FS) [1] is represented by the degree of membership, which occurs only once for each element. Since FS can describe problems related to imprecise and ambiguous judgments, it has been used in various applications [2-7]. To express that an element occurs more than once with identical or different membership values, a fuzzy multiset (FM) [8-10] was proposed as the generalization of FS. Then, FMs were used for some applications, such as decision making and data analysis, clustering analysis, and medical diagnosis [11-16].

To describe the vagueness and indeterminacy of human judgments in real life environment, the neutrosophic number $z = \mu + vI$ for $\mu, v \in R$ and $I \in [I^-, I^+]$ introduced by Smarandache [17-19] can flexibly indicate indeterminate information according to an indeterminate range of $I$. Therefore, it is also regarded as a variable neutrosophic number, depending on indeterminate ranges of $I$. In a multi-criteria group decision-making (MCGDM) problem, to express multiple evaluation values of a criterion to an alternative given by multiple decision makers, Du and Ye [20] proposed a neutrosophic (indeterminate fuzzy) multivalued set (NMS), which is described by neutrosophic number sequences (NNS) with different and/or identical neutrosophic numbers ($z_i = [\mu_i + v_iI^-, \mu_i + v_iI^+] \subseteq [0, 1]$ ($i = 1, 2, \ldots, q$) for $\mu$, $v \in R$ and $I \in [I^-, I^+]$), as a particularly challenging generalization of FM, and then they developed the parameterized correlation coefficients (PCCs) of NMSs to perform MCGDM problems with some indeterminate ranges of decision makers. In indeterminate MCGDM problems, NMS implies its highlighting advantage in expressing indeterminate fuzzy multivalued problems with indeterminate ranges of $I \in [I^-, I^+]$. However, it is worth noting that the similarity measure is a key mathematical tool in decision-making problems, but unfortunately there is no similarity measure between NMSs in the current research. Therefore, this paper proposes a hyperbolic sine similarity measure (HSSM) between two NMSs and a weighted HSSM between two NMSs in view of the hyperbolic sine function, and then develops a MCGDM approach using the weighted HSSM to solve the assessment problem of teachers' teaching quality with some indeterminate ranges of decision makers in the setting of NMSs.

The rest of this article consists of the following sections. Section 2 reviews some notions of NMSs. In Section 3, the HSSM and weighted HSSM of NMSs are proposed according to the hyperbolic sine function. Section 4 develops a MCGDM approach using the weighted HSSM of NMSs along with specific indeterminate ranges of decision makers in the environment of NMSs. In Section 5, the developed MCGDM approach is applied in an illustrative example on the teaching quality assessment of teachers with some indeterminate ranges of decision makers. In Section 6, a comparative analysis with the related method is given to reveal the efficiency of the developed MCGDM approach in the environment of NMSs. Conclusions and future research are addressed in Section 7.

## 2. Some Notions of NMSs

**Definition 1** [20]. Let $Z = \{z_1, z_2, \ldots, z_m\}$ be a fixed set. A neutrosophic (indeterminate fuzzy) multivalued set (NMS) $E$ on $Z$ is denoted by $E = \{\langle z_k, e_E(z_k, I)\rangle \mid z_k \in Z, I \in [I^-, I^+]\}$, where $e_E(z_k, I)$ is the increasing neutrosophic number sequence $e_E(z_k, I) = \left(e^1(z_k, I), e^2(z_k, I), \ldots, e^{p_k}(z_k, I)\right)$ with identical and/or different neutrosophic numbers $e_E^i(z_k, I) = \mu_k^i + v_k^i I \subseteq e_E^{i+1}(z_k, I) = \mu_k^{i+1} + v_k^{i+1} I \subseteq [0,1]$ ($i = 1, 2, \ldots, p_k$; $k = 1, 2, \ldots, m$) of an element $z_k$ to the set $E$ for $I \in [I^-, I^+]$, $\mu_k^i$, $v_k^i \in R$ and $z_k \in Z$.

For convenient expression, each element $e_E(z_k, I)$ ($k = 1, 2, \ldots, m$) in $Z$ is simply represented as the NNS $e_{Ek}(I) = (e_k^1(I), e_k^2(I), \ldots, e_k^{p_k}(I))$ for $I \in [I^-, I^+]$. If $e_k^i(I) = [\mu_k^i + v_k^i I^-, \mu_k^i + v_k^i I^+] \subseteq [0,1]$ or $e_k^i(I) = \mu_k^i + v_k^i I \in [0,1]$ ($i = 1, 2, \ldots, p_k$; $k = 1, 2, \ldots, m$) in $e_{Ek}(I)$, the NNS $e_{Ek}(I)$ can contain an interval-valued fuzzy sequence or a single-valued fuzzy sequence depending on a range/value of $I$. It is obvious that NMS contains the fuzzy multivalued set and interval-valued fuzzy multivalued set.

**Definition 2** [20]. Let two NNSs be $e_{1k}(I) = (e_{1k}^1(I), e_{1k}^2(I), \ldots, e_{1k}^{p_k}(I))$ and $e_{2k}(I) = (e_{2k}^1(I), e_{2k}^2(I), \ldots, e_{2k}^{p_k}(I))$ with neutrosophic numbers $e_{1k}^i(I) = \mu_{1k}^i + v_{1k}^i I \subseteq [0,1]$ and $e_{2k}^i(I) = \mu_{2k}^i + v_{2k}^i I \subseteq [0,1]$ for $I \in [I^-, I^+]$ and $\mu_{1k}^i, v_{1k}^i, \mu_{2k}^i, v_{2k}^i \in R$ ($i = 1, 2, \ldots, p_k$; $k = 1, 2, \ldots, m$). Then, their relations are indicated below:

(1) $e_{1k}(I) \supseteq e_{2k}(I) \Leftrightarrow e_{1k}^i(I) = \mu_{1k}^i + v_{1k}^i I \supseteq e_{2k}^i(I) = \mu_{2k}^i + v_{2k}^i I$;

(2) $e_{1k}(I) = e_{2k}(I) \Leftrightarrow e_{1k}(I) \supseteq e_{2k}(I)$ and $e_{2k}(I) \supseteq e_{1k}(I)$;

(3)
$$e_{1k}(I) \cup e_{2k}(I) = \left(e_{1k}^1(I) \cup e_{2k}^1(I), e_{1k}^2(I) \cup e_{2k}^2(I), \ldots, e_{1k}^{p_k}(I) \cup e_{2k}^{p_k}(I)\right)$$
$$= \begin{pmatrix} [\mu_{1k}^1 + v_{1k}^1 I^- \vee \mu_{2k}^1 + v_{2k}^1 I^-, \mu_{1k}^1 + v_{1k}^1 I^+ \vee \mu_{2k}^1 + v_{2k}^1 I^+], \\ [\mu_{1k}^2 + v_{1k}^2 I^- \vee \mu_{2k}^2 + v_{2k}^2 I^-, \mu_{1k}^2 + v_{1k}^2 I^+ \vee \mu_{2k}^2 + v_{2k}^2 I^+], \ldots, \\ [\mu_{1k}^{p_k} + v_{1k}^{p_k} I^- \vee \mu_{2k}^{p_k} + v_{2k}^{p_k} I^-, \mu_{1k}^{p_k} + v_{1k}^{p_k} I^+ \vee \mu_{2k}^{p_k} + v_{2k}^{p_k} I^+] \end{pmatrix};$$

$$e_{1j}(I)\cap e_{2j}(I)=\left(e_{1k}^{1}(I)\cap e_{2k}^{1}(I), e_{1k}^{2}(I)\cap e_{2k}^{2}(I),...,e_{1k}^{p_k}(I)\cap e_{2k}^{p_k}(I)\right)$$

(4)
$$=\begin{pmatrix}[\mu_{1k}^{1}+v_{1k}^{1}I^{-}\wedge\mu_{2k}^{1}+v_{2k}^{1}I^{-},\mu_{1k}^{1}+v_{1k}^{1}I^{+}\wedge\mu_{2k}^{1}+v_{2k}^{1}I^{+}],\\ [\mu_{1k}^{2}+v_{1k}^{2}I^{-}\wedge\mu_{2k}^{2}+v_{2k}^{2}I^{-},\mu_{1k}^{2}+v_{1k}^{2}I^{+}\wedge\mu_{2k}^{2}+v_{2k}^{2}I^{+}],...,\\ [\mu_{1k}^{p_k}+v_{1k}^{p_k}I^{-}\wedge\mu_{2k}^{p_k}+v_{2k}^{p_k}I^{-},\mu_{1k}^{p_k}+v_{1k}^{p_k}I^{+}\wedge\mu_{2k}^{p_k}+v_{2k}^{p_k}I^{+}]\end{pmatrix};$$

(5)
$$e_{1k}^{c}(I)=\begin{pmatrix}[1-(\mu_{1k}^{p_k}+v_{1k}^{p_k}I^{+}),1-(\mu_{1k}^{p_k}+v_{1k}^{p_k}I^{-})],[1-(\mu_{1k}^{p_k-1}+v_{1k}^{p_k-1}I^{+}),1-(\mu_{1k}^{p_k-1}+v_{1k}^{p_k-1}I^{-})],\\ ...,[1-(\mu_{1k}^{2}+v_{1k}^{2}I^{+}),1-(\mu_{1k}^{2}+v_{1k}^{2}I^{-})],[1-(\mu_{1k}^{1}+v_{1k}^{1}I^{+}),1-(\mu_{1k}^{1}+v_{1k}^{1}I^{-})]\end{pmatrix}$$

(Complement of $e_{1k}(I)$).

Suppose that there are two NMSs $E_1 = \{e_{11}(I), e_{12}(I), …, e_{1m}(I)\}$ and $E_2 = \{e_{21}(I), e_{22}(I), …, e_{2m}(I)\}$, where $e_{1k}(I)=\left(e_{1k}^{1}(I),e_{1k}^{2}(I),...,e_{1k}^{p_k}(I)\right)$ and $e_{2k}(I)=\left(e_{2k}^{1}(I),e_{2k}^{2}(I),...,e_{2k}^{p_k}(I)\right)$ ($k = 1, 2, …, m$) are two collections of NNSs with neutrosophic numbers $e_{1k}^{i}(I)=\mu_{1k}^{i}+v_{1k}^{i}I\subseteq[0,1]$ and $e_{2k}^{i}(I)=\mu_{2k}^{i}+v_{2k}^{i}I\subseteq[0,1]$ for $I \in [I^-, I^+]$ and $\mu_{1k}^{i},v_{1k}^{i},\mu_{2k}^{i},v_{2k}^{i}\in R$ ($i = 1, 2, …, p_k$; $k = 1, 2, …, m$). Then, the importance of the NNS $e_{jk}(I)$ ($j = 1, 2$; $k = 1, 2, …, m$) in $E_1$ and $E_2$ is specified by its weight $\varphi_k \in [0, 1]$ with $\sum_{k=1}^{m}\varphi_k=1$. Thus, Du and Ye [20] proposed the weighted PCCs of NMSs $E_1$ and $E_2$ with an indeterminate parameter $\rho \in [0, 1]$ below:

$$R_{w1}^{\rho}(E_1,E_2)=\frac{\sum_{k=1}^{m}\varphi_k\left\{\begin{array}{l}[\mu_{1k}^{1}+v_{1k}^{1}I^{-}+\rho v_{1k}^{1}(I^{+}-I^{-})][\mu_{2k}^{1}+v_{2k}^{1}I^{-}+\rho v_{2k}^{1}(I^{+}-I^{-})]\\ +[\mu_{1j}^{2}+v_{1j}^{2}I^{-}+\rho v_{1j}^{2}(I^{+}-I^{-})][\mu_{2j}^{2}+v_{2j}^{2}I^{-}+\rho v_{2j}^{2}(I^{+}-I^{-})]+\\ ...+[\mu_{1k}^{p_k}+v_{1k}^{p_k}I^{-}+\rho v_{1k}^{p_k}(I^{+}-I^{-})][\mu_{2k}^{p_k}+v_{2k}^{p_k}I^{-}+\rho v_{2k}^{p_k}(I^{+}-I^{-})]\end{array}\right\}}{\sqrt{\sum_{k=1}^{m}\varphi_k\left\{\begin{array}{l}[\mu_{1j}^{1}+v_{1j}^{1}I^{-}+\rho v_{1j}^{1}(I^{+}-I^{-})]^2\\ +[\mu_{1j}^{2}+v_{1j}^{2}I^{-}+\rho v_{1j}^{2}(I^{+}-I^{-})]^2+\\ ...+[\mu_{1k}^{p_k}+v_{1k}^{p_k}I^{-}+\rho v_{1k}^{p_k}(I^{+}-I^{-})]^2\end{array}\right\}}\times\sqrt{\sum_{k=1}^{m}\varphi_k\left\{\begin{array}{l}[\mu_{2j}^{1}+v_{2j}^{1}I^{-}+\rho v_{2j}^{1}(I^{+}-I^{-})]^2\\ +[\mu_{2j}^{2}+v_{2j}^{2}I^{-}+\rho v_{2j}^{2}(I^{+}-I^{-})]^2+\\ ...+[\mu_{2k}^{p_k}+v_{2k}^{p_k}I^{-}+\rho v_{2k}^{p_k}(I^{+}-I^{-})]^2\end{array}\right\}}}, \quad (1)$$

$$R_{w2}^{\rho}(E_1,E_2)=\frac{\sum_{k=1}^{m}\varphi_k\left\{\begin{array}{l}[\mu_{1k}^{1}+v_{1k}^{1}I^{-}+\rho v_{1k}^{1}(I^{+}-I^{-})][\mu_{2k}^{1}+v_{2k}^{1}I^{-}+\rho v_{2k}^{1}(I^{+}-I^{-})]\\ +[\mu_{1j}^{2}+v_{1j}^{2}I^{-}+\rho v_{1j}^{2}(I^{+}-I^{-})][\mu_{2j}^{2}+v_{2j}^{2}I^{-}+\rho v_{2j}^{2}(I^{+}-I^{-})]+\\ ...+[\mu_{1k}^{p_k}+v_{1k}^{p_k}I^{-}+\rho v_{1k}^{p_k}(I^{+}-I^{-})][\mu_{2k}^{p_k}+v_{2k}^{p_k}I^{-}+\rho v_{2k}^{p_k}(I^{+}-I^{-})]\end{array}\right\}}{\max\left\{\sum_{k=1}^{m}\varphi_k\left\{\begin{array}{l}[\mu_{1j}^{1}+v_{1j}^{1}I^{-}+\rho v_{1j}^{1}(I^{+}-I^{-})]^2\\ +[\mu_{1j}^{2}+v_{1j}^{2}I^{-}+\rho v_{1j}^{2}(I^{+}-I^{-})]^2+\\ ...+[\mu_{1k}^{p_k}+v_{1k}^{p_k}I^{-}+\rho v_{1k}^{p_k}(I^{+}-I^{-})]^2\end{array}\right\},\sum_{k=1}^{m}\varphi_k\left\{\begin{array}{l}[\mu_{2j}^{1}+v_{2j}^{1}I^{-}+\rho v_{2j}^{1}(I^{+}-I^{-})]^2\\ +[\mu_{2j}^{2}+v_{2j}^{2}I^{-}+\rho v_{2j}^{2}(I^{+}-I^{-})]^2+\\ ...+[\mu_{2k}^{p_k}+v_{2k}^{p_k}I^{-}+\rho v_{2k}^{p_k}(I^{+}-I^{-})]^2\end{array}\right\}\right\}}.$$

(2)

## 3. Hyperbolic Sine Similarity Measures of NMSs

According to the hyperbolic sine function, this section proposes the HSSM and weighted HSSM between two NMSs.

**Definition 3.** Set two NMSs as $E_1 = \{e_{11}(I), e_{12}(I), …, e_{1m}(I)\}$ and $E_2 = \{e_{21}(I), e_{22}(I), …, e_{2m}(I)\}$, where $e_{1k}(I)=\left(e_{1k}^{1}(I),e_{1k}^{2}(I),...,e_{1k}^{p_k}(I)\right)$ and $e_{2k}(I)=\left(e_{2k}^{1}(I),e_{2k}^{2}(I),...,e_{2k}^{p_k}(I)\right)$ ($k = 1, 2, …, m$) are two collections of NNSs with neutrosophic numbers $e_{1k}^{i}(I)=\mu_{1k}^{i}+v_{1k}^{i}I\subseteq[0,1]$ and $e_{2k}^{i}(I)=\mu_{2k}^{i}+v_{2k}^{i}I\subseteq[0,1]$ for $I \in [I^-, I^+]$ and $\mu_{1k}^{i},v_{1k}^{i},\mu_{2k}^{i},v_{2k}^{i}\in R$ ($i = 1, 2, …, p_k$; $k = 1, 2, …, m$). Thus, HSSM between two NMSs $E_1$ and $E_2$ is expressed below:

$$Sh(E_1,E_2)=1-\frac{1}{m}\sum_{k=1}^{m}\sinh\left[\frac{\ln(1+\sqrt{2})}{2p_k}\sum_{i=1}^{p_k}\begin{pmatrix}\left|(\mu_{1k}^i+v_{1k}^iI^-)-(\mu_{2k}^i+v_{2k}^iI^-)\right|\\+\left|(\mu_{1k}^i+v_{1k}^iI^+)-(\mu_{2k}^i+v_{2k}^iI^+)\right|\end{pmatrix}\right]. \quad (3)$$

**Proposition 1.** The HSSM $Sh(E_1, E_2)$ reveals the following properties:

(C1) $Sh(E_1, E_2) = Sh(E_2, E_1)$;

(C2) $0 \le Sh(E_1, E_2) \le 1$;

(C3) $Sh(E_1, E_2) = 1$ if only if $E_1 = E_2$;

(C4) If $E_1 \subseteq E_2 \subseteq E_3$ for any NMSs $E_1$, $E_2$, $E_3$, then $Sh(E_1, E_2) \ge Sh(E_1, E_3)$ and $Sh(E_2, E_3) \ge Sh(E_1, E_3)$.

**Proof:**

(C1) It is straightforward.

(C2) Since the values of $\left|(\mu_{1k}^i+v_{1k}^iI^-)-(\mu_{2k}^i+v_{2k}^iI^-)\right|$ and $\left|(\mu_{1k}^i+v_{1k}^iI^+)-(\mu_{2k}^i+v_{2k}^iI^+)\right|$ ($i$ = 1, 2, …, $p_k$; $k$ = 1, 2, …, $m$) are between 0 and 1, the value of the hyperbolic sine function in Eq. (3) falls in the interval [0, 1], and then the value of Eq. (3) also falls in the interval [0, 1]. Therefore, there is $0 \le Sh(E_1, E_2) \le 1$.

(C3) If $E_1 = E_2$, this reveals $e_{1k}(I) = e_{2k}(I)$, and then there is $e_{1k}^i(I)=\mu_{1k}^i+v_{1k}^iI$ = $e_{2k}^i(I)=\mu_{2k}^i+v_{2k}^iI$ ($i$ = 1, 2, …, $p_k$; $k$ = 1, 2, …, $m$) for $I \in [I^-, I^+]$. Thus, there are $\left|(\mu_{1k}^i+v_{1k}^iI^-)-(\mu_{2k}^i+v_{2k}^iI^-)\right|=0$ and $\left|(\mu_{1k}^i+v_{1k}^iI^+)-(\mu_{2k}^i+v_{2k}^iI^+)\right|=0$. Hence, $Sh(E_1, E_2) = 1$ exists.

If $Sh(E_1, E_2) = 1$, this reveals $\sinh(x) = 0$ in Eq. (3), then there are $\left|(\mu_{1k}^i+v_{1k}^iI^-)-(\mu_{2k}^i+v_{2k}^iI^-)\right|=0$ and $\left|(\mu_{1k}^i+v_{1k}^iI^+)-(\mu_{2k}^i+v_{2k}^iI^+)\right|=0$ ($i$ = 1, 2, …, $p_k$; $k$ = 1, 2, …, $m$). Thus, there is $e_{1k}^i(I)=\mu_{1k}^i+v_{1k}^iI$ = $e_{2k}^i(I)=\mu_{2k}^i+v_{2k}^iI$. Therefore, there exists $e_{1k}(I) = e_{2k}(I)$. It is obvious that $E_1 = E_2$ exists.

(C4) Since $E_1 \subseteq E_2 \subseteq E_3$, there are $e_{1k}(I) \subseteq e_{2k}(I) \subseteq e_{3k}(I)$, then there is also $e_{1k}^i(I)=\mu_{1k}^i+v_{1k}^iI$ $\subseteq$ $e_{2k}^i(I)=\mu_{2k}^i+v_{2k}^iI$ $\subseteq$ $e_{3k}^i(I)=\mu_{3k}^i+v_{3k}^iI$. Therefore, there are $\left|(\mu_{1k}^i+v_{1k}^iI^-)-(\mu_{2k}^i+v_{2k}^iI^-)\right|$ $\le$ $\left|(\mu_{1k}^i+v_{1k}^iI^-)-(\mu_{3k}^i+v_{3k}^iI^-)\right|$, $\left|(\mu_{2k}^i+v_{2k}^iI^-)-(\mu_{3k}^i+v_{3k}^iI^-)\right|$ $\le$ $\left|(\mu_{1k}^i+v_{1k}^iI^-)-(\mu_{3k}^i+v_{3k}^iI^-)\right|$, $\left|(\mu_{1k}^i+v_{1k}^iI^+)-(\mu_{2k}^i+v_{2k}^iI^+)\right|$ $\le$ $\left|(\mu_{1k}^i+v_{1k}^iI^+)-(\mu_{3k}^i+v_{3k}^iI^+)\right|$, and $\left|(\mu_{2k}^i+v_{2k}^iI^+)-(\mu_{3k}^i+v_{3k}^iI^+)\right|$ $\le$ $\left|(\mu_{1k}^i+v_{1k}^iI^+)-(\mu_{3k}^i+v_{3k}^iI^+)\right|$. Since the hyperbolic sine function $\sinh(x)$ for $x \ge 0$ is an increasing function, there are $Sh(E_1, E_2) \ge Sh(E_1, E_3)$ and $Sh(E_2, E_3) \ge Sh(E_1, E_3)$ corresponding to Eq. (3).

When the weight of $e_{ik}(I)$ ($i$ = 1,2; $k$ = 1, 2, …, $m$) is specified by $\varphi_k$ with $\varphi_k \in [0, 1]$ and $\sum_{k=1}^{m}\varphi_k=1$, we give the weighted HSSM of NMSs:

$$Sh_W(E_1,E_2)=1-\sum_{k=1}^{m}\varphi_k\sinh\left[\frac{\ln(1+\sqrt{2})}{2p_k}\sum_{i=1}^{p_k}\begin{pmatrix}\left|(\mu_{1k}^i+v_{1k}^iI^-)-(\mu_{2k}^i+v_{2k}^iI^-)\right|\\+\left|(\mu_{1k}^i+v_{1k}^iI^+)-(\mu_{2k}^i+v_{2k}^iI^+)\right|\end{pmatrix}\right]. \quad (4)$$

It is obvious that the weighted HSSM $Sh_W(E_1, E_2)$ also reveals the following properties:

(C1) $Sh_W(E_1, E_2) = Sh_W(E_2, E_1)$;

(C2) $0 \le Sh_W(E_1, E_2) \le 1$;

(C3) $Sh_W(E_1, E_2) = 1$ if only if $E_1 = E_2$;

(C4) If $E_1 \subseteq E_2 \subseteq E_3$ for NMSs $E_1$, $E_2$, $E_3$, then $Sh_W(E_1, E_2) \ge Sh_W(E_1, E_3)$ and $Sh_W(E_2, E_3) \ge Sh_W(E_1, E_3)$.

## 4. MCGDM Approach Using the Weighted HSSM of NMSs

The section develops a MCGDM approach using the weighted HSSM of NMSs with some indeterminate ranges of decision makers in the environment of NMSs.

When performing a MCGDM issue, a set of alternatives $F = \{F_1, F_2, \ldots, F_q\}$ is preliminarily provided and assessed by a set of criteria $Z = \{z_1, z_2, \ldots, z_m\}$. The weight vector of $Z$ is given by $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m)$. Thus, we can carry out the MCGDM issue in terms of the following steps.

**Step 1:** Every alternative $F_j$ ($j$ = 1, 2, …, $q$) is assessed over the criteria $z_k$ ($k$ = 1, 2, …, $m$) by a group of $p$ decision makers/experts, and then their evaluation values are represented by the NNSs $e_{jk}(I) = \left(e_{jk}^1(I), e_{jk}^2(I), \ldots, e_{jk}^p(I)\right)$ for $e_{jk}^i(I) = \mu_{jk}^i + v_{jk}^i I \subseteq [0,1]$ ($i$ = 1, 2, …, $p$; $j$ = 1, 2, ..., $q$; $k$ = 1, 2, …, $m$) and $I \in \{[I_1^-, I_1^+], [I_2^-, I_2^+], \ldots, [I_s^-, I_s^+]\}$. Thus, all the NNSs $E_j = \{e_{j1}(I), e_{j2}(I), \ldots, e_{jm}(I)\}$ ($j$ = 1, 2, …, $q$) is constructed as the NMS decision matrix $E = (e_{jk}(I))_{q\times m}$.

**Step 2:** The ideal solution is given by the ideal NMS:

$$E^* = \left\{ \underbrace{\left( \overbrace{[1,1],[1,1],\ldots,[1,1]}^{p} \right), \left( \overbrace{[1,1],[1,1],\ldots,[1,1]}^{p} \right), \ldots, \left( \overbrace{[1,1],[1,1],\ldots,[1,1]}^{p} \right)}_{m} \right\}.$$

Thus, the weighted HSSM of the NMSs $E_j$ and $E^*$ for $F_j$ ($j$ = 1, 2, ..., $q$) is given by the following equation:

$$Sh_W(E_j, E^*) = 1 - \sum_{k=1}^{m} \varphi_k \sinh\left[ \frac{\ln(1+\sqrt{2})}{2p} \sum_{i=1}^{p} \left(2 - (\mu_{jk}^i + v_{jk}^i I^-) - (\mu_{1k}^i + v_{1k}^i I^+)\right) \right]. \quad (5)$$

**Step 3:** In terms of the weighted HSSM values, the alternatives are sorted in descending order, and the best one is chosen.

**Step 4:** End.

## 5. Illustrative Example on the Teaching Quality Evaluation of Teachers

In the process of university education, the teaching quality of teachers is a key issue, because it will affect students' career choices, employment, and professional status. Establishing a teaching quality evaluation system in colleges and universities is an effective operating mechanism and management strategy to improve teaching quality. Since the teaching quality evaluation of teachers is a MCGDM issue with some indeterminacy, this section applies the developed MCGDM approach to an illustrative example on the teaching quality assessment of teachers to reveal the applicability and efficiency of the developed MCGDM approach in the environment of NMSs.

A university hopes to select one teacher with the best teaching quality from the School of Mechanical and Electrical Engineering. The school preliminarily provides four potential teachers, which are indicated by a set of alternatives $F = \{F_1, F_2, F_3, F_4\}$. To assess their teaching quality, they must satisfy the requirements of four criteria: teaching ability ($z_1$), teaching method ($z_2$), teaching attitude ($z_3$), and student satisfaction ($z_4$). Then, the weight vector of the four criteria is specified as $\varphi$ = (0.3, 0.25, 0.2, 0.25). The decision steps are described below.

First, the evaluation values of each alternative with respect to the four criteria are given by three experts/decision makers and expressed as the NNSs $e_{jk}(I) = \left(e_{jk}^1(I), e_{jk}^2(I), e_{jk}^3(I)\right)$ for $e_{jk}^i(I) = \mu_{jk}^i + v_{jk}^i I \subseteq [0,1]$ ($i$ = 1, 2, 3; $j, k$ = 1, 2, 3, 4) and $I \in$ {[0, 0.1], [0, 0.3], [0, 0.6]}. Then, the NMS decision matrix $E = (e_{jk}(I))_{4\times 4}$ is tabulated in Table 1.

Next, using Eq. (5) for $I \in$ {[0, 0.1], [0, 0.3], [0, 0.6]}, the weighted HSSM values of the NMSs $E_j$ and $E^*$ for $F_j$ ($j$ = 1, 2, 3, 4) and the decision results are given in Table 2.

In view of the decision results in Table 2, all sorting orders are the same and reveal their robustness corresponding to some indeterminate ranges of *I*. Then, the best teacher is $F_4$.

**Table 1.** The decision matrix of NMSs

| | $z_1$ | $z_2$ | $z_3$ | $z_4$ |
|---|---|---|---|---|
| $F_1$ | (0.5+0.1*I*, 0.7+0.2*I*, 0.8+0.1*I*) | (0.6+0.3*I*, 0.7+0.2*I*, 0.8+0.1*I*) | (0.7+0.2*I*, 0.8+0.2*I*, 0.9+0.1*I*) | (0.6+0.1*I*, 0.7+0.2*I*, 0.7+0.1*I*) |
| $F_2$ | (0.7+0.3*I*, 0.8+0.1*I*, 0.8+0.1*I*) | (0.7+0.2*I*, 0.7+0.1*I*, 0.8+0.2*I*) | (0.6+0.1*I*, 0.7+0.1*I*, 0.8+0.2*I*) | (0.7+0.2*I*, 0.8+0.2*I*, 0.8+0.2*I*) |
| $F_3$ | (0.6+0.4*I*, 0.7+0.2*I*, 0.8+0.1*I*) | (0.5+0.2*I*, 0.6+0.1*I*, 0.6+0.1*I*) | (0.7+0.1*I*, 0.8+0.2*I*, 0.8+0.1*I*) | (0.7+0.1*I*, 0.8+0.2*I*, 0.8+0.1*I*) |
| $F_4$ | (0.7+0.3*I*, 0.8+0.1*I*, 0.8+0.2*I*) | (0.7+0.2*I*, 0.8+0.1*I*, 0.8+0.1*I*) | (0.8+0.3*I*, 0.8+0.2*I*, 0.8+0.1*I*) | (0.7+0.2*I*, 0.8+0.3*I*, 0.8+0.2*I*) |

**Table 2.** The decision results for $I \in \{[0, 0.1], [0, 0.3], [0, 0.6]\}$

| *I* | $Shw(E_1, E^*)$, $Shw(E_2, E^*)$, $Shw(E_3, E^*)$, $Shw(E_4, E^*)$ | Sorting order | The best teacher |
|---|---|---|---|
| *I* = [0, 0.1] | 0.7409, 0.7809, 0.7362, 0.8075 | $F_4 > F_2 > F_1 > F_3$ | $F_4$ |
| *I* = [0, 0.3] | 0.7551, 0.7960, 0.7511, 0.8247 | $F_4 > F_2 > F_1 > F_3$ | $F_4$ |
| *I* = [0, 0.6] | 0.7764, 0.8187, 0.7733, 0.8503 | $F_4 > F_2 > F_1 > F_3$ | $F_4$ |

## 6. Comparison with the Related MCGDM Approach

This section compares the developed MCGDM approach with the related MCGDM approach [20] to reveal the efficiency of the developed MCGDM approach by the illustrative example on the teaching quality evaluation of teachers in the setting of NMSs.

Using Eqs. (1) and (2), the values of the weighted PCCs $R_{w1}^{\rho}(E_j, E^*)$ and $R_{w2}^{\rho}(E_j, E^*)$ for *I* = [0, 1] and $\rho$ = 0.1, 0.3, 0.6 and their decision results are shown in Tables 3 and 4.

**Table 3.** The decision results corresponding to $R_{w1}^{\rho}(E_j, E^*)$ for *I* = [0, 1] and $\rho$ = 0.1, 0.3, 0.6

| $\rho$ | $R_{w1}^{\rho}(E_1, E^*)$, $R_{w1}^{\rho}(E_2, E^*)$, $R_{w1}^{\rho}(E_3, E^*)$, $R_{w1}^{\rho}(E_4, E^*)$ | Sorting order | The best teacher |
|---|---|---|---|
| $\rho$ = 0.1 | 0.9898, 0.9967, 0.9908, 0.9986 | $F_4 > F_2 > F_3 > F_1$ | $F_4$ |
| $\rho$ = 0.3 | 0.9907, 0.9967, 0.9920, 0.9987 | $F_4 > F_2 > F_3 > F_1$ | $F_4$ |
| $\rho$ = 0.6 | 0.9914, 0.9962, 0.9926, 0.9984 | $F_4 > F_2 > F_3 > F_1$ | $F_4$ |

**Table 4.** The decision results corresponding to $R_{w2}^{\rho}(E_j, E^*)$ for *I* = [0, 1] and $\rho$ = 0.1, 0.3, 0.6

| $\rho$ | $R_{w2}^{\rho}(E_1, E^*)$, $R_{w2}^{\rho}(E_2, E^*)$, $R_{w2}^{\rho}(E_3, E^*)$, $R_{w2}^{\rho}(E_4, E^*)$ | Sorting order | The best teacher |
|---|---|---|---|
| $\rho$ = 0.1 | 0.7173, 0.7618, 0.7130, 0.7925 | $F_4 > F_2 > F_1 > F_3$ | $F_4$ |
| $\rho$ = 0.3 | 0.7487, 0.7955, 0.7457, 0.8308 | $F_4 > F_2 > F_1 > F_3$ | $F_4$ |
| $\rho$ = 0.6 | 0.7957, 0.8460, 0.7947, 0.8883 | $F_4 > F_2 > F_1 > F_3$ | $F_4$ |

In view of the sorting results in Tables 2-4, the sorting orders in Tables 2 and 4 are the same, but slightly different from the sorting orders in Table 3. However, the best teacher is always $F_4$ among all decision results. It is obvious that the developed MCGDM approach is effective in the MCGDM example with some indeterminate ranges of decision makers.

## 7. Conclusions

According to the hyperbolic sine function, this article proposed the HSSM and weighted HSSM between NMSs. Then, a MCGDM approach with some indeterminate ranges was developed in terms of the weighted HSSM of NMSs. Next, the developed MCGDM approach was applied to an illustrative example on the teaching quality evaluation of teachers in the setting of NMSs. Through the comparison of the developed MCGDM approach with the related MCGDM approach, the results revealed the efficiency of the developed MCGDM approach for the teaching quality evaluation of teachers in the setting of NMSs. However, the proposed HSSMs and MCGDM approach will also be used for pattern recognition, clustering analysis, and medical diagnosis in the environment of NMSs, which are considered as the future research targets.

**Conflicts of Interest:** The authors declare no conflict of interest.

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Received: Nov 1, 2021. Accepted: Feb 5, 2022

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# Positive implicative ideals of $BCK$-algebras based on neutrosophic sets and falling shadows

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**Abstract:** Neutrosophy is introduced by F. Smarandache in 1980 which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy considers a proposition, theory, event, concept, or entity, "A" in relation to its opposite, "Anti-A" and that which is not A, "Non-A", and that which is neither "A" nor "Anti-A", denoted by "Neut-A". Neutrosophy is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, and neutrosophic statistics. In this article, we apply the notion of neutrosophic set theory to (positive implicative) ideals in $BCK$-algebras by using the concept of falling shadows. The notions of a positive implicative $(\in, \in)$-neutrosophic ideal and a positive implicative falling neutrosophic ideal are introduced, and several properties are investigated. Characterizations of a positive implicative $(\in, \in)$-neutrosophic ideal are considered, and relations between a positive implicative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal are discussed. Conditions for an $(\in, \in)$-neutrosophic ideal to be a positive implicative $(\in, \in)$-neutrosophic ideal are provided, and relations between a positive implicative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and a positive implicative falling neutrosophic ideal are studied. Conditions for a falling neutrosophic ideal to be positive implicative are provided.

**Keywords:** neutrosophic random set, neutrosophic falling shadow, (positive implicative) $(\in, \in)$-neutrosophic ideal, (positive implicative) falling neutrosophic ideal.

# 1 Introduction

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [5] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [26] introduced the theory of falling shadows which directly relates probability concepts to the membership function of fuzzy sets. Falling shadow representation theory shows us a method of selection relaid on the joint degree distributions. It is a reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [27]. Tan et al. [24, 25] established a theoretical approach for defining a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Neutrosophic set (NS) developed by Smarandache [20, 21, 22] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site http://fs.gallup.unm.edu/neutrosophy.htm. Jun, Bordbar, Borumand Saeid and Öztürk studied neutrosophic subalgebras/ideals in $BCK/BCI$-algebras based on neutrosophic points (see [3], [4], [9], [11], [15], [17], [19] and [23]). It is a reasonable and convenient approach for the theoretical development and the practical applications of neutrosophic sets and neutrosophic logics. Jun et al. [12] introduced the notion of neutrosophic random set and neutrosophic falling shadow. Using these notions, they introduced the concept of falling neutrosophic subalgebra and falling neutrosophic ideal in $BCK/BCI$-algebras, and investigated related properties. They discussed relations between falling neutrosophic subalgebra and falling neutrosophic ideal, and established a characterization of falling neutrosophic ideal [13]. Jun et al. [14] introduced the concepts of a commutative $(\in, \in)$-neutrosophic ideal and a commutative falling neutrosophic ideal, and investigate several properties. They obtained characterizations of a commutative $(\in, \in)$-neutrosophic ideal, and discussed relations between a commutative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal. They provided conditions for an $(\in, \in)$-neutrosophic ideal to be a commutative $(\in, \in)$-neutrosophic ideal, and considered relations between a commutative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and a commutative falling neutrosophic ideal. They also gave conditions for a falling neutrosophic ideal to be commutative [18].

In this paper, we introduce the concepts of a positive implicative $(\in, \in)$-neutrosophic ideal and a positive implicative falling neutrosophic ideal, and investigate several properties. We obtain characterizations of a positive implicative $(\in, \in)$-neutrosophic ideal, and discuss relations between a positive implicative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal. We provide conditions for an $(\in, \in)$-neutrosophic ideal to be a positive implicative $(\in, \in)$-neutrosophic ideal, and consider relations between a positive implicative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and a positive implicative falling neutrosophic ideal. We give conditions for a falling neutrosophic ideal to be positive implicative.

# 2 Preliminaries

A $BCK/BCI$-algebra is an important class of logical algebras introduced by K. Iséki (see [6] and [7]) and was extensively investigated by several researchers.

By a *$BCI$-algebra*, we mean a set $X$ with a special element $0$ and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a $BCI$-algebra $X$ satisfies the following identity:

(V) $(\forall x \in X)\,(0 * x = 0)$,

then $X$ is called a *$BCK$-algebra*. Any $BCK/BCI$-algebra $X$ satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$
$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\; z * y \leq z * x)\,, \tag{2.2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$. A $BCK$-algebra $X$ is said to be *positive implicative* if the following assertion is valid.

$$(\forall x, y, z \in X)\,((x * z) * (y * z) = (x * y) * z)\,. \tag{2.5}$$

A nonempty subset $S$ of a $BCK/BCI$-algebra $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for all $x, y \in S$. A subset $I$ of a $BCK/BCI$-algebra $X$ is called an *ideal* of $X$ if it satisfies:

$$0 \in I, \tag{2.6}$$
$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{2.7}$$

A subset $I$ of a $BCK$-algebra $X$ is called a *positive implicative ideal* (see [16]) of $X$ if it satisfies (2.6) and

$$(\forall x, y, z \in X)(((x * y) * z \in I,\; y * z \in I \;\Rightarrow\; x * z \in I)\,. \tag{2.8}$$

Observe that every positive implicative ideal is an ideal, but the converse is not true (see [16]).

We refer the reader to the books [8, 16] for further information regarding $BCK/BCI$-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let $X$ be a non-empty set. A *neutrosophic set* (NS) in $X$ (see [21]) is a structure of the form:

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A_\sim = (A_T, A_I, A_F)$ for the neutrosophic set

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a set $X$, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$T_{\in}(A_{\sim};\alpha) := \{x \in X \mid A_T(x) \geq \alpha\},$$
$$I_{\in}(A_{\sim};\beta) := \{x \in X \mid A_I(x) \geq \beta\},$$
$$F_{\in}(A_{\sim};\gamma) := \{x \in X \mid A_F(x) \leq \gamma\}.$$

We say $T_{\in}(A_{\sim};\alpha)$, $I_{\in}(A_{\sim};\beta)$ and $F_{\in}(A_{\sim};\gamma)$ are *neutrosophic $\in$-subsets*.

A neutrosophic set $A_{\sim} = (A_T, A_I, A_F)$ in a $BCK/BCI$-algebra $X$ is called an $(\in, \in)$-*neutrosophic subalgebra* of $X$ (see [9]) if the following assertions are valid.

$$(\forall x, y \in X)\left(\begin{array}{l} x \in T_{\in}(A_{\sim};\alpha_x),\ y \in T_{\in}(A_{\sim};\alpha_y) \ \Rightarrow\ x * y \in T_{\in}(A_{\sim};\alpha_x \wedge \alpha_y), \\ x \in I_{\in}(A_{\sim};\beta_x),\ y \in I_{\in}(A_{\sim};\beta_y) \ \Rightarrow\ x * y \in I_{\in}(A_{\sim};\beta_x \wedge \beta_y), \\ x \in F_{\in}(A_{\sim};\gamma_x),\ y \in F_{\in}(A_{\sim};\gamma_y) \ \Rightarrow\ x * y \in F_{\in}(A_{\sim};\gamma_x \vee \gamma_y) \end{array}\right) \tag{2.9}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

A neutrosophic set $A_{\sim} = (A_T, A_I, A_F)$ in a $BCK/BCI$-algebra $X$ is called an $(\in, \in)$-*neutrosophic ideal* of $X$ (see [19]) if the following assertions are valid.

$$(\forall x \in X)\left(\begin{array}{l} x \in T_{\in}(A_{\sim};\alpha_x) \ \Rightarrow\ 0 \in T_{\in}(A_{\sim};\alpha_x) \\ x \in I_{\in}(A_{\sim};\beta_x) \ \Rightarrow\ 0 \in I_{\in}(A_{\sim};\beta_x) \\ x \in F_{\in}(A_{\sim};\gamma_x) \ \Rightarrow\ 0 \in F_{\in}(A_{\sim};\gamma_x) \end{array}\right) \tag{2.10}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} x * y \in T_{\in}(A_{\sim};\alpha_x),\ y \in T_{\in}(A_{\sim};\alpha_y) \ \Rightarrow\ x \in T_{\in}(A_{\sim};\alpha_x \wedge \alpha_y) \\ x * y \in I_{\in}(A_{\sim};\beta_x),\ y \in I_{\in}(A_{\sim};\beta_y) \ \Rightarrow\ x \in I_{\in}(A_{\sim};\beta_x \wedge \beta_y) \\ x * y \in F_{\in}(A_{\sim};\gamma_x),\ y \in F_{\in}(A_{\sim};\gamma_y) \ \Rightarrow\ x \in F_{\in}(A_{\sim};\gamma_x \vee \gamma_y) \end{array}\right) \tag{2.11}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

In what follows, let $X$ and $\mathcal{P}(X)$ denote a $BCK/BCI$-algebra and the power set of $X$, respectively, unless otherwise specified.

For each $x \in X$ and $D \in \mathcal{P}(X)$, let

$$\bar{x} := \{C \in \mathcal{P}(X) \mid x \in C\}, \tag{2.12}$$

and

$$\bar{D} := \{\bar{x} \mid x \in D\}. \tag{2.13}$$

An ordered pair $(\mathcal{P}(X), \mathcal{B})$ is said to be a *hyper-measurable structure* on $X$ if $\mathcal{B}$ is a $\sigma$-field in $\mathcal{P}(X)$ and $\bar{X} \subseteq \mathcal{B}$.

Given a probability space $(\Omega, \mathcal{A}, P)$ and a hyper-measurable structure $(\mathcal{P}(X), \mathcal{B})$ on $X$, a *neutrosophic random set* on $X$ (see [12]) is defined to be a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which $\xi_T$, $\xi_I$ and $\xi_F$ are mappings from

$\Omega$ to $\mathcal{P}(X)$ which are $\mathcal{A}$-$\mathcal{B}$ measurables, that is,

$$(\forall C \in \mathcal{B}) \begin{pmatrix} \xi_T^{-1}(C) = \{\omega_T \in \Omega \mid \xi_T(\omega_T) \in C\} \in \mathcal{A} \\ \xi_I^{-1}(C) = \{\omega_I \in \Omega \mid \xi_I(\omega_I) \in C\} \in \mathcal{A} \\ \xi_F^{-1}(C) = \{\omega_F \in \Omega \mid \xi_F(\omega_F) \in C\} \in \mathcal{A} \end{pmatrix}. \tag{2.14}$$

Given a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on $X$, consider functions:

$$\begin{aligned} &\tilde{H}_T : X \to [0,1],\ x_T \mapsto P(\omega_T \mid x_T \in \xi_T(\omega_T)), \\ &\tilde{H}_I : X \to [0,1],\ x_I \mapsto P(\omega_I \mid x_I \in \xi_I(\omega_I)), \\ &\tilde{H}_F : X \to [0,1],\ x_F \mapsto 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)). \end{aligned}$$

Then $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a neutrosophic set on $X$, and we call it a *neutrosophic falling shadow* (see [12]) of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$, and $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *neutrosophic cloud* (see [12]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

For example, consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ where $\mathcal{A}$ is a Borel field on $[0,1]$ and $m$ is the usual Lebesgue measure. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic set in $X$. Then a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\begin{aligned} &\xi_T : [0,1] \to \mathcal{P}(X), \alpha \mapsto T_\in(\tilde{H}; \alpha), \\ &\xi_I : [0,1] \to \mathcal{P}(X), \beta \mapsto I_\in(\tilde{H}; \beta), \\ &\xi_F : [0,1] \to \mathcal{P}(X), \gamma \mapsto F_\in(\tilde{H}; \gamma) \end{aligned}$$

is a neutrosophic random set and $\xi := (\xi_T, \xi_I, \xi_F)$ is a neutrosophic cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$. We will call $\xi := (\xi_T, \xi_I, \xi_F)$ defined above as the *neutrosophic cut-cloud* (see [12]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on $X$. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are subalgebras (resp., ideals) of $X$ for all $\omega_T, \omega_I, \omega_F \in \Omega$, then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *falling neutrosophic subalgebra* (resp., *falling neutrosophic ideal*) of $X$ (see [12]).

# 3 Positive implicative $(\in, \in)$-neutrosophic ideals

**Definition 3.1.** A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a $BCK$-algebra $X$ is called a *positive implicative* $(\in, \in)$*-neutrosophic ideal* of $X$ if it satisfies the condition (2.10) and

$$\begin{array}{l} (x*y)*z \in T_\in(A_\sim; \alpha_x),\ y*z \in T_\in(A_\sim; \alpha_y) \ \Rightarrow\ x*z \in T_\in(A_\sim; \alpha_x \wedge \alpha_y) \\ (x*y)*z \in I_\in(A_\sim; \beta_x),\ y*z \in I_\in(A_\sim; \beta_y) \ \Rightarrow\ x*z \in I_\in(A_\sim; \beta_x \wedge \beta_y) \\ (x*y)*z \in F_\in(A_\sim; \gamma_x),\ y*z \in F_\in(A_\sim; \gamma_y) \ \Rightarrow\ x*z \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array} \tag{3.1}$$

for all $x, y, z \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

**Example 3.2.** Consider a set $X = \{0,1,2,3,4\}$ with the binary operation $*$ which is given in Table 1 Then $(X; *, 0)$ is a $BCK$-algebra (see [16]). Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in $X$ defined by Table 2

Table 1: Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 2 | 2 | 2 | 0 | 0 | 2 |
| 3 | 3 | 3 | 3 | 0 | 3 |
| 4 | 4 | 4 | 4 | 4 | 0 |

Table 2: Tabular representation of $A_\sim = (A_T, A_I, A_F)$

| $X$ | $A_T(x)$ | $A_I(x)$ | $A_F(x)$ |
|---|---|---|---|
| 0 | 0.8 | 0.6 | 0.1 |
| 1 | 0.7 | 0.6 | 0.4 |
| 2 | 0.6 | 0.5 | 0.4 |
| 3 | 0.4 | 0.2 | 0.6 |
| 4 | 0.2 | 0.3 | 0.9 |

Routine calculations show that $A_\sim = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of $X$.

**Theorem 3.3.** *Every positive implicative $(\in, \in)$-neutrosophic ideal of a $BCK$-algebra $X$ is an $(\in, \in)$-neutrosophic ideal of $X$.*

*Proof.* It is clear by taking $z = 0$ in (3.1) and using (2.1). □

**Theorem 3.4.** *For a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a $BCK$-algebra $X$, the following are equivalent.*

(1) *The non-empty $\in$-subsets $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are positive implicative ideals of $X$ for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.*

(2) *$A_\sim = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X)\left(\ A_T(0) \geq A_T(x),\, A_I(0) \geq A_I(x),\, A_F(0) \leq A_F(x)\ \right) \tag{3.2}$$

*and*

$$(\forall x, y, z \in X)\left(\begin{array}{l} A_T(x * z) \geq A_T((x * y) * z) \wedge A_T(y * z) \\ A_I(x * z) \geq A_I((x * y) * z) \wedge A_I(y * z) \\ A_F(x * z) \leq A_F((x * y) * z) \vee A_F(y * z) \end{array}\right) \tag{3.3}$$

*Proof.* Assume that the non-empty $\in$-subsets $T_\in(A_\sim; \alpha)$, $I_\in(A_\sim; \beta)$ and $F_\in(A_\sim; \gamma)$ are positive implicative ideals of $X$ for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. If $A_T(0) < A_T(a)$ for some $a \in X$, then $a \in T_\in(A_\sim; A_T(a))$ and $0 \notin T_\in(A_\sim; A_T(a))$. This is a contradiction, and so $A_T(0) \geq A_T(x)$ for all $x \in X$. Similarly, $A_I(0) \geq A_I(x)$ for all $x \in X$. Suppose that $A_F(0) > A_F(a)$ for some $a \in X$. Then $a \in F_\in(A_\sim; A_F(a))$ and

$0 \notin F_{\in}(A_{\sim}; A_F(a))$. This is a contradiction, and thus $A_F(0) \leq A_F(x)$ for all $x \in X$. Therefore (3.2) is valid. Assume that there exist $a, b, c \in X$ such that

$$A_T(a * c) < A_T((a * b) * c) \wedge A_T(b * c).$$

Taking $\alpha := A_T((a * b) * c) \wedge A_T(b * c)$ implies that $(a * b) * c \in T_{\in}(A_{\sim}; \alpha)$ and $b * c \in T_{\in}(A_{\sim}; \alpha)$ but $a * c \notin T_{\in}(A_{\sim}; \alpha)$, which is a contradiction. Hence

$$A_T(x * z) \geq A_T((x * y) * z) \wedge A_T(y * z)$$

for all $x, y, z \in X$. By the similar way, we can verify that

$$A_I(x * z) \geq A_I((x * y) * z) \wedge A_I(y * z)$$

for all $x, y, z \in X$. Now suppose there are $x, y, z \in X$ such that

$$A_F(x * z) > A_F((x * y) * z) \vee A_F(y * z) := \gamma.$$

Then $(x * y) * z \in F_{\in}(A_{\sim}; \gamma)$ and $y * z \in F_{\in}(A_{\sim}; \gamma)$ but $x * z \notin F_{\in}(A_{\sim}; \gamma)$, a contradiction. Thus

$$A_F(x * z) \leq A_F((x * y) * z) \vee A_F(y * z)$$

for all $x, y, z \in X$.

Conversely, let $A_{\sim} = (A_T, A_I, A_F)$ be a neutrosophic set in $X$ satisfying two conditions (3.2) and (3.3). Assume that $T_{\in}(A_{\sim}; \alpha)$, $I_{\in}(A_{\sim}; \beta)$ and $F_{\in}(A_{\sim}; \gamma)$ are nonempty for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Let $x \in T_{\in}(A_{\sim}; \alpha)$, $a \in I_{\in}(A_{\sim}; \beta)$ and $u \in F_{\in}(A_{\sim}; \gamma)$ for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(a) \geq \beta$, and $A_F(0) \leq A_F(u) \leq \gamma$ by (3.2). It follows that $0 \in T_{\in}(A_{\sim}; \alpha)$, $0 \in I_{\in}(A_{\sim}; \beta)$ and $0 \in F_{\in}(A_{\sim}; \gamma)$. Let $a, b, c \in X$ be such that $(a * b) * c \in T_{\in}(A_{\sim}; \alpha)$ and $b * c \in T_{\in}(A_{\sim}; \alpha)$ for $\alpha \in (0, 1]$. Then

$$A_T(a * c) \geq A_T((a * b) * c) \wedge A_T(b * c) \geq \alpha$$

by (3.3), and so $a * c \in T_{\in}(A_{\sim}; \alpha)$. If $(x * y) * z \in I_{\in}(A_{\sim}; \beta)$ and $y * z \in I_{\in}(A_{\sim}; \beta)$ for all $x, y, z \in X$ and $\beta \in (0, 1]$, then $A_I((x * y) * z) \geq \beta$ and $A_I(y * z) \geq \beta$. Hence the condition (3.3) implies that

$$A_I(x * z) \geq A_I((x * y) * z) \wedge A_I(y * z) \geq \beta,$$

that is, $x * z \in I_{\in}(A_{\sim}; \beta)$. Finally, suppose that $(x * y) * z \in F_{\in}(A_{\sim}; \gamma)$ and $y * z \in F_{\in}(A_{\sim}; \gamma)$ for all $x, y, z \in X$ and $\gamma \in (0, 1]$. Then $A_F((x * y) * z) \leq \gamma$ and $A_F(y * z) \leq \gamma$, which imply from the condition (3.3) that

$$A_F(x * z) \leq A_F((x * y) * z) \vee A_F(y * z) \leq \gamma.$$

Hence $x * z \in F_{\in}(A_{\sim}; \gamma)$. Therefore the non-empty $\in$-subsets $T_{\in}(A_{\sim}; \alpha)$, $I_{\in}(A_{\sim}; \beta)$ and $F_{\in}(A_{\sim}; \gamma)$ are positive implicative ideals of $X$ for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. □

**Theorem 3.5.** *Let $A_{\sim} = (A_T, A_I, A_F)$ be a neutrosophic set in a $BCK$-algebra $X$. Then $A_{\sim} = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of $X$ if and only if the non-empty neutrosophic $\in$-subsets $T_{\in}(A_{\sim}; \alpha)$, $I_{\in}(A_{\sim}; \beta)$ and $F_{\in}(A_{\sim}; \gamma)$ are positive implicative ideals of $X$ for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.*

*Proof.* Let $A_\sim = (A_T, A_I, A_F)$ be a positive implicative $(\in, \in)$-neutrosophic ideal of $X$ and assume that $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are nonempty for $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Then there exist $x, y, z \in X$ such that $x \in T_\in(A_\sim;\alpha)$, $y \in I_\in(A_\sim;\beta)$ and $z \in F_\in(A_\sim;\gamma)$. It follows from (2.10) that $0 \in T_\in(A_\sim;\alpha)$, $0 \in I_\in(A_\sim;\beta)$ and $0 \in F_\in(A_\sim;\gamma)$. Let $x, y, z, a, b, c, u, v, w \in X$ be such that $(x * y) * z \in T_\in(A_\sim;\alpha)$, $y * z \in T_\in(A_\sim;\alpha)$, $(a * b) * c \in I_\in(A_\sim;\beta)$, $b * c \in I_\in(A_\sim;\beta)$, $(u * v) * w \in F_\in(A_\sim;\gamma)$ and $v * w \in F_\in(A_\sim;\gamma)$. Then $x * z \in T_\in(A_\sim;\alpha \wedge \alpha) = T_\in(A_\sim;\alpha)$, $a * c \in I_\in(A_\sim;\beta \wedge \beta) = I_\in(A_\sim;\beta)$, and $u * w \in F_\in(A_\sim;\gamma \vee \gamma) = F_\in(A_\sim;\gamma)$ by (3.1). Hence the non-empty neutrosophic $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are positive implicative ideals of $X$ for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$.

Conversely, let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in $X$ for which $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are nonempty and are positive implicative ideals of $X$ for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Obviously, (2.10) is valid. Let $x, y, z \in X$ and $\alpha_x, \alpha_y \in (0,1]$ be such that $(x*y)*z \in T_\in(A_\sim;\alpha_x)$ and $y*z \in T_\in(A_\sim;\alpha_y)$. Then $(x * y) * z \in T_\in(A_\sim;\alpha)$ and $y * z \in T_\in(A_\sim;\alpha)$ where $\alpha = \alpha_x \wedge \alpha_y$. Since $T_\in(A_\sim;\alpha)$ is a positive implicative ideal of $X$, it follows that $x * z \in T_\in(A_\sim;\alpha) = T_\in(A_\sim;\alpha_x \wedge \alpha_y)$. Similarly, if $(x * y) * z \in I_\in(A_\sim;\beta_x)$ and $y * z \in I_\in(A_\sim;\beta_y)$ for all $x, y, z \in X$ and $\beta_x, \beta_y \in (0,1]$, then $x * z \in I_\in(A_\sim;\beta_x \wedge \beta_y)$. Now, suppose that $(x * y) * z \in F_\in(A_\sim;\gamma_x)$ and $y * z \in F_\in(A_\sim;\gamma_y)$ for all $x, y, z \in X$ and $\gamma_x, \gamma_y \in [0,1)$. Then $(x*y)*z \in F_\in(A_\sim;\gamma)$ and $y*z \in F_\in(A_\sim;\gamma)$ where $\gamma = \gamma_x \vee \gamma_y$. Hence $x*z \in F_\in(A_\sim;\gamma) = F_\in(A_\sim;\gamma_x \vee \gamma_y)$ since $F_\in(A_\sim;\gamma)$ is a positive implicative ideal of $X$. Therefore $A_\sim = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of $X$. □

**Corollary 3.6.** *Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in a $BCK$-algebra $X$. Then $A_\sim = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of $X$ if and only if it satisfies two conditions (3.2) and (3.3).*

**Lemma 3.7** ([18])**.** *Every $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of a $BCK/BCI$-algebra $X$ satisfies the following assertion.*

$$(\forall x, y \in X) \left( x \le y \;\Rightarrow \left\{ \begin{array}{l} A_T(x) \ge A_T(y) \\ A_I(x) \ge A_I(y) \\ A_F(x) \le A_F(y) \end{array} \right. \right). \tag{3.4}$$

**Lemma 3.8** ([18])**.** *Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a $BCK/BCI$-algebra $X$, the following assertions are equivalent.*

(1) $A_\sim = (A_T, A_I, A_F)$ *is an $(\in, \in)$-neutrosophic ideal of $X$.*

(2) $A_\sim = (A_T, A_I, A_F)$ *satisfies the following assertions.*

$$(\forall x \in X)\left( \; A_T(0) \ge A_T(x),\, A_I(0) \ge A_I(x),\, A_F(0) \le A_F(x) \; \right) \tag{3.5}$$

*and*

$$(\forall x, y \in X) \left( \begin{array}{l} A_T(x) \ge A_T(x * y) \wedge A_T(y) \\ A_I(x) \ge A_I(x * y) \wedge A_I(y) \\ A_F(x) \le A_F(x * y) \vee A_F(y) \end{array} \right) \tag{3.6}$$

**Proposition 3.9.** *Every positive implicative $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of a $BCK$-algebra*

*X satisfies the following assertions.*

$$(\forall x, y \in X) \left( \begin{array}{l} A_T(x * y) \geq A_T((x * y) * y) \\ A_I(x * y) \geq A_I((x * y) * y) \\ A_F(x * y) \leq A_F((x * y) * y) \end{array} \right), \tag{3.7}$$

$$(\forall x, y \in X) \left( \begin{array}{l} A_T((x * z) * (y * z)) \geq A_T((x * y) * z) \\ A_I((x * z) * (y * z)) \geq A_I((x * y) * z) \\ A_F((x * z) * (y * z)) \leq A_F((x * y) * z) \end{array} \right), \tag{3.8}$$

*and*

$$(\forall x, y \in X) \left( \begin{array}{l} A_T(x * y) \geq A_T(((x * y) * y) * z) \wedge A_T(z) \\ A_I(x * y) \geq A_I(((x * y) * y) * z) \wedge A_I(z) \\ A_F(x * y) \leq A_F(((x * y) * y) * z) \vee A_F(z) \end{array} \right). \tag{3.9}$$

*Proof.* Let $A_\sim = (A_T, A_I, A_F)$ be a positive implicative $(\in, \in)$-neutrosophic ideal of a $BCK$-algebra $X$. Then $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of a $BCK$-algebra $X$ (see Theorem 3.3). Since $x * x = 0$ for all $x \in X$, putting $z = y$ in (3.3) and using (3.2) induce (3.7). Since

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

for all $x, y, z \in X$, we have

$$\begin{aligned} A_T((x * z) * (y * z)) &= A_T((x * (y * z)) * z) \\ &\geq A_T(((x * (y * z)) * z) * z) \\ &\geq A_T((x * y) * z), \end{aligned}$$

$$\begin{aligned} A_I((x * z) * (y * z)) &= A_I((x * (y * z)) * z) \\ &\geq A_I(((x * (y * z)) * z) * z) \\ &\geq A_I((x * y) * z) \end{aligned}$$

and

$$\begin{aligned} A_F((x * z) * (y * z)) &= A_F((x * (y * z)) * z) \\ &\leq A_F(((x * (y * z)) * z) * z) \\ &\leq A_F((x * y) * z) \end{aligned}$$

by (2.3), (3.7) and Lemma 3.7. Thus (3.8) is valid. Note that

$$(x * y) * z = ((x * z) * y) * (y * y)$$

for all $x, y \in X$. It follows from Lemma 3.8, (3.8) and (2.3) that

$$\begin{aligned} A_T(x * y) &\geq A_T((x * y) * z) \wedge A_T(z) \\ &= A_T(((x * z) * y) * (y * y)) \wedge A_T(z) \\ &\geq A_T(((x * z) * y) * y) \wedge A_T(z) \\ &= A_T(((x * y) * y) * z) \wedge A_T(z), \end{aligned}$$

$$\begin{aligned} A_I(x * y) &\geq A_I((x * y) * z) \wedge A_I(z) \\ &= A_I(((x * z) * y) * (y * y)) \wedge A_I(z) \\ &\geq A_I(((x * z) * y) * y) \wedge A_I(z) \\ &= A_I(((x * y) * y) * z) \wedge A_I(z), \end{aligned}$$

and

$$\begin{aligned} A_F(x * y) &\leq A_F((x * y) * z) \vee A_F(z) \\ &= A_F(((x * z) * y) * (y * y)) \vee A_F(z) \\ &\leq A_F(((x * z) * y) * y) \vee A_F(z) \\ &= A_F(((x * y) * y) * z) \vee A_F(z) \end{aligned}$$

for all $x, y, z \in X$. Therefore (3.9) is valid. □

The converse of Theorem 3.3 is not true as seen in the following example.

**Example 3.10.** Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 3

Table 3: Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 1 |
| 2 | 2 | 2 | 0 | 2 | 0 |
| 3 | 3 | 1 | 3 | 0 | 3 |
| 4 | 4 | 4 | 4 | 4 | 0 |

Then $(X; *, 0)$ is a $BCK$-algebra (see [16]). Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in $X$ defined by Table 4

Routine calculations show that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X$. Neutrosophic $\in$-

Table 4: Tabular representation of $A_{\sim} = (A_T, A_I, A_F)$

| $X$ | $A_T(x)$ | $A_I(x)$ | $A_F(x)$ |
|---|---|---|---|
| 0 | 0.7 | 0.9 | 0.3 |
| 1 | 0.4 | 0.7 | 0.5 |
| 2 | 0.5 | 0.6 | 0.4 |
| 3 | 0.4 | 0.7 | 0.5 |
| 4 | 0.1 | 0.4 | 0.6 |

subsets are given as follows.

$$T_{\in}(A_{\sim};\alpha) = \begin{cases} \emptyset & \text{if } \alpha \in (0.7, 1], \\ \{0\} & \text{if } \alpha \in (0.5, 0.7], \\ \{0, 2\} & \text{if } \alpha \in (0.4, 0.5], \\ \{0, 1, 2, 3\} & \text{if } \alpha \in (0.1, 0.4], \\ X & \text{if } \alpha \in (0, 0.1], \end{cases}$$

$$I_{\in}(A_{\sim};\beta) = \begin{cases} \emptyset & \text{if } \beta \in (0.9, 1], \\ \{0\} & \text{if } \beta \in (0.7, 0.9], \\ \{0, 1, 3\} & \text{if } \beta \in (0.6, 0.7], \\ \{0, 1, 2, 3\} & \text{if } \beta \in (0.4, 0.6], \\ X & \text{if } \beta \in (0, 0.4], \end{cases}$$

and

$$F_{\in}(A_{\sim};\gamma) = \begin{cases} X & \text{if } \gamma \in [0.6, 1), \\ \{0, 1, 2, 3\} & \text{if } \gamma \in [0.5, 0.6), \\ \{0, 2\} & \text{if } \gamma \in [0.4, 0.5), \\ \{0\} & \text{if } \gamma \in [0.3, 0.4), \\ \emptyset & \text{if } \gamma \in [0, 0.3). \end{cases}$$

If $\alpha \in (0.4, 0.5]$ and $\gamma \in [0.4, 0.5)$, then $T_{\in}(A_{\sim};\alpha)$ and $F_{\in}(A_{\sim};\gamma)$ are not positive implicative ideals of $X$. Thus $A_{\sim} = (A_T, A_I, A_F)$ is not a positive implicative $(\in, \in)$-neutrosophic ideal of $X$ by Theorems 3.4 and 3.5.

We provide conditions for an $(\in, \in)$-neutrosophic ideal to be a positive implicative $(\in, \in)$-neutrosophic ideal.

**Theorem 3.11.** *Given a neutrosophic set $A_{\sim} = (A_T, A_I, A_F)$ in a $BCK$-algebra $X$, the following assertions are equivalent.*

(1) $A_{\sim} = (A_T, A_I, A_F)$ *is a positive implicative $(\in, \in)$-neutrosophic ideal of $X$.*

(2) $A_{\sim} = (A_T, A_I, A_F)$ *is an $(\in, \in)$-neutrosophic ideal of $X$ that satisfies the condition (3.7).*

(3) $A_\sim = (A_T, A_I, A_F)$ *is an* $(\in, \in)$*-neutrosophic ideal of* $X$ *that satisfies the condition* (3.8).

(4) $A_\sim = (A_T, A_I, A_F)$ *satisfies two conditions* (3.2) *and* (3.9).

*Proof.* Assume that $A_\sim = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of $X$. Then $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X$ by Theorem 3.3. If we take $z = y$ in (3.3) and use (3.2), then we get the condition (3.7). Suppose that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X$ satisfying the condition (3.7). Note that

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

for all $x, y, z \in X$. It follows from (2.3), (3.7) and Lemma 3.7 that

$$\begin{aligned} A_T((x * z) * (y * z)) &= A_T((x * (y * z)) * z) \\ &\geq A_T(((x * (y * z)) * z) * z) \\ &\geq A_T((x * y) * z), \end{aligned}$$

$$\begin{aligned} A_I((x * z) * (y * z)) &= A_I((x * (y * z)) * z) \\ &\geq A_I(((x * (y * z)) * z) * z) \\ &\geq A_I((x * y) * z), \end{aligned}$$

and

$$\begin{aligned} A_F((x * z) * (y * z)) &= A_F((x * (y * z)) * z) \\ &\leq A_F(((x * (y * z)) * z) * z) \\ &\leq A_F((x * y) * z). \end{aligned}$$

Hence (3.8) is valid. Assume that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X$ satisfying the condition (3.8). It is clear that $A_\sim = (A_T, A_I, A_F)$ satisfies the condition (3.2). Using (3.6), (III), (2.3) and (3.8), we have

$$\begin{aligned} T_\in(x * y) &\geq T_\in((x * y) * z) \wedge T_\in(z) \\ &= T_\in(((x * z) * y) * (y * y)) \wedge T_\in(z) \\ &\geq T_\in(((x * z) * y) * y) \wedge T_\in(z) \\ &= T_\in(((x * y) * y) * z) \wedge T_\in(z), \end{aligned}$$

$$\begin{aligned} I_\in(x * y) &\geq I_\in((x * y) * z) \wedge I_\in(z) \\ &= I_\in(((x * z) * y) * (y * y)) \wedge I_\in(z) \\ &\geq I_\in(((x * z) * y) * y) \wedge I_\in(z) \\ &= I_\in(((x * y) * y) * z) \wedge I_\in(z), \end{aligned}$$

and

$$\begin{aligned} F_{\in}(x*y) &\leq F_{\in}((x*y)*z) \vee F_{\in}(z) \\ &= F_{\in}(((x*z)*y)*(y*y)) \vee F_{\in}(z) \\ &\leq F_{\in}(((x*z)*y)*y) \vee F_{\in}(z) \\ &= F_{\in}(((x*y)*y)*z) \vee F_{\in}(z) \end{aligned}$$

for all $x, y, z \in X$. Thus (3.9) is valid. Finally suppose that $A_{\sim} = (A_T, A_I, A_F)$ satisfies two conditions (3.2) and (3.9). Using (2.1) and (3.9), we get

$$\begin{aligned} A_T(x) &= A_T(x*0) \\ &\geq A_T(((x*0)*0)*y) \wedge A_T(y) \\ &= A_T(x*y) \wedge A_T(y), \end{aligned}$$

$$\begin{aligned} A_I(x) &= A_I(x*0) \\ &\geq A_I(((x*0)*0)*y) \wedge A_I(y) \\ &= A_I(x*y) \wedge A_I(y), \end{aligned}$$

and

$$\begin{aligned} A_F(x) &= A_F(x*0) \\ &\leq A_F(((x*0)*0)*y) \vee A_F(y) \\ &= A_F(x*y) \vee A_F(y) \end{aligned}$$

for all $x, y \in X$. Hence $A_{\sim} = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of $X$. Since

$$((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$$

for all $x, y, z \in X$, it follows from (3.9) and (3.4) that

$$\begin{aligned} A_T(x*z) &\geq A_T(((x*z)*z)*(y*z)) \wedge A_T(y*z) \\ &\geq A_T((x*y)*z) \wedge A_T(y*z), \end{aligned}$$

$$\begin{aligned} A_I(x*z) &\geq A_I(((x*z)*z)*(y*z)) \wedge A_I(y*z) \\ &\geq A_I((x*y)*z) \wedge A_I(y*z), \end{aligned}$$

and

$$\begin{aligned} A_F(x*z) &\leq A_F(((x*z)*z)*(y*z)) \vee A_F(y*z) \\ &\leq A_F((x*y)*z) \vee A_F(y*z) \end{aligned}$$

for all $x, y, z \in X$. Therefore $A_{\sim} = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of $X$. □

# 4 Positive implicative falling neutrosophic ideals

**Definition 4.1.** Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on a $BCK$-algebra $X$. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are positive implicative ideals of $X$ for all $\omega_T, \omega_I, \omega_F \in \Omega$, then the neutrosophic shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on $X$, that is,

$$\begin{aligned} &\tilde{H}_T(x_T) = P(\omega_T \mid x_T \in \xi_T(\omega_T)), \\ &\tilde{H}_I(x_I) = P(\omega_I \mid x_I \in \xi_I(\omega_I)), \\ &\tilde{H}_F(x_F) = 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)) \end{aligned} \tag{4.1}$$

is called a *positive implicative falling neutrosophic ideal* of $X$.

**Example 4.2.** Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 5

Table 5: Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 |
| 2 | 2 | 2 | 0 | 2 | 0 |
| 3 | 3 | 3 | 3 | 0 | 3 |
| 4 | 4 | 4 | 4 | 4 | 0 |

Then $(X; *, 0)$ is a $BCK$-algebra (see [16]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on $X$ which is given as follows:

$$\xi_T : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0, 3\} & \text{if } t \in [0, 0.25), \\ \{0, 1\} & \text{if } t \in [0.25, 0.55), \\ \{0, 1, 2\} & \text{if } t \in [0.55, 0.85), \\ \{0, 1, 3\} & \text{if } t \in [0.85, 0.95), \\ X & \text{if } t \in [0.95, 1], \end{cases}$$

$$\xi_I : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0, 1, 2\} & \text{if } t \in [0, 0.45), \\ \{0, 1, 2, 3\} & \text{if } t \in [0.45, 0.75), \\ \{0, 1, 2, 4\} & \text{if } t \in [0.75, 1], \end{cases}$$

and

$$\xi_F : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.9, 1], \\ \{0, 3\} & \text{if } t \in (0.7, 0.9], \\ \{0, 1, 2\} & \text{if } t \in (0.5, 0.7], \\ \{0, 1, 2, 3\} & \text{if } t \in (0.3, 0.5], \\ \{0, 1, 2, 4\} & \text{if } t \in [0, 0.3]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are positive implicative ideals of $X$ for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a positive implicative falling neutrosophic ideal of $X$, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.75 & \text{if } x = 1, \\ 0.35 & \text{if } x = 2, \\ 0.4 & \text{if } x = 3, \\ 0.05 & \text{if } x = 4, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x \in \{0,1,2\}, \\ 0.3 & \text{if } x = 3, \\ 0.25 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.7 & \text{if } x \in \{1,2\}, \\ 0.4 & \text{if } x = 3, \\ 0.3 & \text{if } x = 4. \end{cases}$$

Given a probability space $(\Omega, \mathcal{A}, P)$, let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$. For $x \in X$, let

$$\begin{aligned} \Omega(x;\xi_T) &:= \{\omega_T \in \Omega \mid x \in \xi_T(\omega_T)\}, \\ \Omega(x;\xi_I) &:= \{\omega_I \in \Omega \mid x \in \xi_I(\omega_I)\}, \\ \Omega(x;\xi_F) &:= \{\omega_F \in \Omega \mid x \in \xi_F(\omega_F)\}. \end{aligned}$$

Then $\Omega(x;\xi_T), \Omega(x;\xi_I), \Omega(x;\xi_F) \in \mathcal{A}$ (see [12]).

**Proposition 4.3.** *Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a $BCK$-algebra $X$. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a positive implicative falling neutrosophic ideal of $X$, then*

$$(\forall x, y, z \in X) \begin{pmatrix} \Omega((x*y)*z;\xi_T) \cap \Omega(y*z;\xi_T) \subseteq \Omega(x*z;\xi_T) \\ \Omega((x*y)*z;\xi_I) \cap \Omega(y*z;\xi_I) \subseteq \Omega(x*z;\xi_I) \\ \Omega((x*y)*z;\xi_F) \cap \Omega(y*z;\xi_F) \subseteq \Omega(x*z;\xi_F) \end{pmatrix}, \tag{4.2}$$

$$(\forall x, y, z \in X) \begin{pmatrix} \Omega(x*z;\xi_T) \subseteq \Omega((x*y)*z;\xi_T) \\ \Omega(x*z;\xi_I) \subseteq \Omega((x*y)*z;\xi_I) \\ \Omega(x*z;\xi_F) \subseteq \Omega((x*y)*z;\xi_F) \end{pmatrix}. \tag{4.3}$$

*Proof.* Let $\omega_T \in \Omega((x*y)*z;\xi_T) \cap \Omega(y*z;\xi_T)$, $\omega_I \in \Omega((x*y)*z;\xi_I) \cap \Omega(y*z;\xi_I)$ and $\omega_F \in \Omega((x*y)*z;\xi_F) \cap \Omega(y*z;\xi_F)$ for all $x, y, z \in X$. Then

$(x*y)*z \in \xi_T(\omega_T)$ and $y*z \in \xi_T(\omega_T)$,
$(x*y)*z \in \xi_I(\omega_I)$ and $y*z \in \xi_I(\omega_I)$,

$(x * y) * z \in \xi_F(\omega_F)$ and $y * z \in \xi_F(\omega_F)$.
Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are positive implicative ideals of $X$, it follows from (2.8) that $x * z \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$ and so that $\omega_T \in \Omega(x * z; \xi_T)$, $\omega_I \in \Omega(x * z; \xi_I)$ and $\omega_F \in \Omega(x * z; \xi_F)$. Hence (4.2) is valid. Now let $x, y, z \in X$ be such that $\omega_T \in \Omega(x * z; \xi_T)$, $\omega_I \in \Omega(x * z; \xi_I)$, and $\omega_F \in \Omega(x * z; \xi_F)$. Then $x * z \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$. Note that

$$\begin{aligned}&((x * y) * z) * (x * z) = ((x * y) * (x * z)) * z\\ &\leq (z * y) * z = (z * z) * y\\ &= 0 * y = 0,\end{aligned}$$

which yields

$$((x * y) * z) * (x * z) = 0 \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are positive implicative ideals and hence ideals of $X$, it follows that $(x * y) * z \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$. Hence $\omega_T \in \Omega((x*y)*z; \xi_T)$, $\omega_I \in \Omega((x*y)*z; \xi_I)$, and $\omega_F \in \Omega((x*y)*z; \xi_F)$. Therefore (4.3) is valid. □

Given a probability space $(\Omega, \mathcal{A}, P)$, let

$$\mathcal{F}(X) := \{f \mid f : \Omega \to X \text{ is a mapping}\}. \tag{4.4}$$

Define a binary operation $\circledast$ on $\mathcal{F}(X)$ as follows:

$$(\forall \omega \in \Omega)\,((f \circledast g)(\omega) = f(\omega) * g(\omega)) \tag{4.5}$$

for all $f, g \in \mathcal{F}(X)$. Then $(\mathcal{F}(X); \circledast, \theta)$ is a $BCK/BCI$-algebra (see [10]) where $\theta$ is given as follows:

$$\theta : \Omega \to X,\ \omega \mapsto 0.$$

For any subset $A$ of $X$ and $g_T, g_I, g_F \in \mathcal{F}(X)$, consider the followings:

$$\begin{aligned}A_T^g &:= \{\omega_T \in \Omega \mid g_T(\omega_T) \in A\},\\ A_I^g &:= \{\omega_I \in \Omega \mid g_I(\omega_I) \in A\},\\ A_F^g &:= \{\omega_F \in \Omega \mid g_F(\omega_F) \in A\}\end{aligned}$$

and

$$\begin{aligned}&\xi_T : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_T \mapsto \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in A\},\\ &\xi_I : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_I \mapsto \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in A\},\\ &\xi_F : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_F \mapsto \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in A\}.\end{aligned}$$

Then $A_T^g, A_I^g, A_F^g \in \mathcal{A}$ (see [12]).

**Theorem 4.4.** *If $K$ is a positive implicative ideal of a $BCK$-algebra $X$, then*

$$\xi_T(\omega_T) = \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in K\},$$
$$\xi_I(\omega_I) = \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in K\},$$
$$\xi_F(\omega_F) = \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in K\}$$

*are positive implicative ideals of $\mathcal{F}(X)$.*

*Proof.* Assume that $K$ is a positive implicative ideal of a $BCK$-algebra $X$. Since $\theta(\omega_T) = 0 \in K$, $\theta(\omega_I) = 0 \in K$ and $\theta(\omega_F) = 0 \in K$ for all $\omega_T, \omega_I, \omega_F \in \Omega$, we have

$$\theta \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Let $f_T, g_T, h_T \in \mathcal{F}(X)$ be such that $(f_T \circledast g_T) \circledast h_T \in \xi_T(\omega_T)$ and $g_T \circledast h_T \in \xi_T(\omega_T)$. Then

$$(f_T(\omega_T) * g_T(\omega_T)) * h_T(\omega_T) = ((f_T \circledast g_T) \circledast h_T)(\omega_T) \in K$$

and $g_T(\omega_T) * h_T(\omega_T) \in K$. Since $K$ is a positive implicative ideal of $X$, it follows from (2.8) that

$$(f_T \circledast h_T)(\omega_T) = f_T(\omega_T) * h_T(\omega_T) \in K,$$

that is, $f_T \circledast h_T \in \xi_T(\omega_T)$. Hence $\xi_T(\omega_T)$ is a positive implicative ideal of $\mathcal{F}(X)$. Similarly, we can verify that $\xi_I(\omega_I)$ is a positive implicative ideal of $\mathcal{F}(X)$. Now, let $f_F, g_F, h_F \in \mathcal{F}(X)$ be such that $(f_F \circledast g_F) \circledast h_F \in \xi_F(\omega_F)$ and $g_F \circledast h_F \in \xi_F(\omega_F)$. Then

$$(f_F(\omega_F) * g_F(\omega_F)) * h_F(\omega_F) = ((f_F \circledast g_F) \circledast h_F)(\omega_F) \in K$$

and $g_F(\omega_F) * h_F(\omega_F) \in K$. Then

$$(f_F \circledast h_F)(\omega_F) = f_F(\omega_F) * h_F(\omega_F) \in K,$$

and so $f_F \circledast h_F \in \xi_F(\omega_F)$. Hence $\xi_F(\omega_F)$ is a positive implicative ideal of $\mathcal{F}(X)$. This completes the proof. □

**Theorem 4.5.** *If we consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, then every positive implicative $(\in, \in)$-neutrosophic ideal of a $BCK$-algebra is a positive implicative falling neutrosophic ideal.*

*Proof.* Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a positive implicative $(\in, \in)$-neutrosophic ideal of a $BCK$-algebra $X$. Then $T_\in(\tilde{H}; \alpha)$, $I_\in(\tilde{H}; \beta)$ and $F_\in(\tilde{H}; \gamma)$ are positive implicative ideals of $X$ for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ by Theorem 3.5. Hence a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\xi_T : [0,1] \to \mathcal{P}(X),\ \alpha \mapsto T_\in(\tilde{H}; \alpha),$$
$$\xi_I : [0,1] \to \mathcal{P}(X),\ \beta \mapsto I_\in(\tilde{H}; \beta),$$
$$\xi_F : [0,1] \to \mathcal{P}(X),\ \gamma \mapsto F_\in(\tilde{H}; \gamma)$$

is a neutrosophic cut-cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$, and so $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a positive implicative falling neutrosophic ideal of $X$. □

The converse of Theorem 4.5 is not true as seen in the following example.

Table 6: Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 1 |
| 2 | 2 | 2 | 0 | 0 | 2 |
| 3 | 3 | 2 | 1 | 0 | 3 |
| 4 | 4 | 4 | 4 | 4 | 0 |

**Example 4.6.** Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 6

Then $(X; *, 0)$ is a $BCK$-algebra (see [16]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on $X$ which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,1\} & \text{if } t \in [0, 0.2), \\ \{0,2\} & \text{if } t \in [0.2, 0.55), \\ \{0,2,4\} & \text{if } t \in [0.55, 0.75), \\ \{0,1,2,3\} & \text{if } t \in [0.75, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,2\} & \text{if } t \in [0, 0.26), \\ \{0,4\} & \text{if } t \in [0.26, 0.68), \\ \{0,1,2,3\} & \text{if } t \in [0.68, 1] \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.77, 1], \\ \{0,1\} & \text{if } t \in (0.66, 0.77], \\ \{0,2\} & \text{if } t \in (0.48, 0.66], \\ \{0,2,4\} & \text{if } t \in (0.23, 0.48], \\ \{0,1,2,3\} & \text{if } t \in [0, 0.23]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are positive implicative ideals of $X$ for all $t \in [0, 1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a positive implicative falling neutrosophic ideal of $X$, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.45 & \text{if } x = 1, \\ 0.8 & \text{if } x = 2, \\ 0.25 & \text{if } x = 3, \\ 0.2 & \text{if } x = 4, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.32 & \text{if } x \in \{1, 3\}, \\ 0.58 & \text{if } x = 2, \\ 0.42 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.66 & \text{if } x = 1, \\ 0.34 & \text{if } x = 2, \\ 0.77 & \text{if } x = 3, \\ 0.75 & \text{if } x = 4. \end{cases}$$

But $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not a positive implicative $(\in, \in)$-neutrosophic ideal of $X$ since

$$\tilde{H}_T(3 * 4) = \tilde{H}_T(3) = 0.25 < 0.8 = \tilde{H}_T((3 * 2) * 4) \wedge \tilde{H}_T(2 * 4)$$

and/or

$$\tilde{H}_T(3 * 4) = \tilde{H}_T(3) = 0.77 > 0.66 = \tilde{H}_T((3 * 1) * 4) \vee \tilde{H}_T(1 * 4).$$

We provide relations between a falling neutrosophic ideal and a positive implicative falling neutrosophic ideal .

**Theorem 4.7.** *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a $BCK$-algebra $X$. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a positive implicative falling neutrosophic ideal of $X$, then it is a falling neutrosophic ideal of $X$.*

*Proof.* Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a positive implicative falling neutrosophic ideal of a $BCK$-algebra $X$. Then $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are positive implicative ideals of $X$, and so $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are ideals of $X$ for all $\omega_T, \omega_I, \omega_F \in \Omega$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of $X$. □

The following example shows that the converse of Theorem 4.7 is not true in general.

**Example 4.8.** Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 7

Table 7: Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 1 |
| 2 | 2 | 2 | 0 | 0 | 2 |
| 3 | 3 | 3 | 2 | 0 | 3 |
| 4 | 4 | 4 | 4 | 4 | 0 |

Then $(X;*,0)$ is a $BCK$-algebra (see [16]). Consider $(\Omega,\mathcal{A},P)=([0,1],\mathcal{A},m)$ and let $\xi:=(\xi_T,\xi_I,\xi_F)$ be a neutrosophic random set on $X$ which is given as follows:

$$\xi_T:[0,1]\to\mathcal{P}(X),\quad x\mapsto\begin{cases}\{0,4\} & \text{if } t\in[0,0.37),\\ \{0,1,2,3\} & \text{if } t\in[0.37,0.67),\\ \{0,1,4\} & \text{if } t\in[0.67,1],\end{cases}$$

$$\xi_I:[0,1]\to\mathcal{P}(X),\quad x\mapsto\begin{cases}\{0,4\} & \text{if } t\in[0,0.45),\\ \{0,1,2,3\} & \text{if } t\in[0.45,1],\end{cases}$$

and

$$\xi_F:[0,1]\to\mathcal{P}(X),\quad x\mapsto\begin{cases}\{0\} & \text{if } t\in(0.74,1],\\ \{0,1\} & \text{if } t\in(0.66,0.74],\\ \{0,4\} & \text{if } t\in(0.48,0.66],\\ \{0,1,2,3\} & \text{if } t\in[0,0.48].\end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are ideals of $X$ for all $t\in[0,1]$. Hence the neutrosophic falling shadow $\tilde{H}:=(\tilde{H}_T,\tilde{H}_I,\tilde{H}_F)$ of $\xi:=(\xi_T,\xi_I,\xi_F)$ is a falling neutrosophic ideal of $X$. But it is not a positive implicative falling neutrosophic ideal of $X$ because if $\alpha\in[0.67,1]$, $\beta\in[0,0.45)$ and $\gamma\in(0.66,0.74]$, then $\xi_T(\alpha)=\{0,1,4\}$, $\xi_I(\beta)=\{0,4\}$ and $\xi_F(\gamma)=\{0,1\}$ are not positive implicative ideals of $X$ respectively.

Since every ideal is positive implicative in a positive implicative $BCK$-algebra, we have the following theorem.

**Theorem 4.9.** *Let $(\Omega,\mathcal{A},P)$ be a probability space and let $\tilde{H}:=(\tilde{H}_T,\tilde{H}_I,\tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi:=(\xi_T,\xi_I,\xi_F)$ on a positive implicative $BCK$-algebra. If $\tilde{H}:=(\tilde{H}_T,\tilde{H}_I,\tilde{H}_F)$ is a falling neutrosophic ideal of $X$, then it is a positive implicative falling neutrosophic ideal of $X$.*

**Corollary 4.10.** *Let $(\Omega,\mathcal{A},P)$ be a probability space. For any $BCK$-algebra $X$ which satisfies one of the following assertions*

$$\begin{aligned}&(\forall x,y\in X)(x*y=(x*y)*y),\\ &(\forall x,y\in X)((x*(x*y))*(y*x)=x*(x*(y*(y*x)))),\\ &(\forall x,y\in X)(x*y=(x*y)*(x*(x*y))),\\ &(\forall x,y\in X)(x*(x*y)=(x*(x*y))*(x*y)),\\ &(\forall x,y\in X)((x*(x*y))*(y*x)=(y*(y*x))*(x*y)),\end{aligned}$$

*let $\tilde{H}:=(\tilde{H}_T,\tilde{H}_I,\tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi:=(\xi_T,\xi_I,\xi_F)$ on $X$. If $\tilde{H}:=(\tilde{H}_T,\tilde{H}_I,\tilde{H}_F)$ is a falling neutrosophic ideal of $X$, then it is a positive implicative falling neutrosophic ideal of $X$.*

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Received: July 17, 2021 Accepted: : Feb 1, 2022.

# Neutrosophic Fuzzy Strong Bi-ideals of Near-Subtraction Semigroups

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**Abstract:** The theory of Neutrosophy fuzzy set is the extension of the fuzzy set that deals with imprecise and indeterminate data Neutrosophy is a new branch of Philosophy. We already conceptualized the Neutrosophic fuzzy bi-ideals of Near –subtraction Semigroups(NFBI).In this article, We extend our study to strong bi-ideals. We examine some of its fundamentals and algebraic structures. Our aim of this manuscript are given as follows:

(i)To explore the new ideas in Neutrosophic fuzzy Near-subtraction semigroups of said bi-ideals and strong bi-ideals.

(ii)To examine the some basic properties and fundamentals.

(iii)Also expand the direct product and regularity of Neutrosophic fuzzy strong bi-ideals(NFSBI) of a Near- Subtraction Semigroups.

**Keywords:** Neutrosophic Fuzzy sub algebra, Neutrosophic fuzzy X-sub algebra, Neutrosophic fuzzy bi-ideal, Neutrosophic fuzzy strong bi-ideal.

## 1.Introduction

The fuzzy set was first introduced by L.A. Zadeh [18] .It was conceptualized the grade of truth values belonging to a unit interval.The fuzzy sub nearrings and fuzzy ideals of near-rings was introduced by Abou zaid[1]. V.Chinnadurai and S.Kadalarasi[4] examined the direct product of fuzzy subnearring, fuzzy ideal and fuzzy R-subgroups. Atanassov[3] expanded the intuitionstic fuzzy set to deal with complicated version.It explained the truth and false membership functions.It may applicable in various fields such as medicine, decision making techniques.

Later, Florentin Smarandache[13]introduced the concept of Neutrosophy. Neutrosophy is an extension of fuzzy logic in which indeterminancy also included. In Neutrosophic logic, we may have truth membership functions, false membership function and indeterminate functions. This idea of neutrosophic set have a remarkable achievement in various fields like medical diagnosis, image processing, decision making problem,robotics and so on. I.Arockiarani[8] consider the neutrosophic set with value from the subset of [0,1] and extended the research in fuzzy neutrosophic set. We gained inspiration from the advantages of Neutrosophy fuzzy set.J.Sivaranjini, V. Mahalakshmi[10]examined the concept of fuzzy bi-ideals in Near-Subtraction Semigroups.The results obtained are entirely more beneficial to the researchers.

## 2. Preliminaries

The aim of this section is to recall some basic definitions.

***2.1 Definition[7]***

A non-empty set X together with the binary operation '-' and'•' is said to be a right(left) ***near-subtraction semigroup*** if it satisfies the following.

(i)( X ,-)is a subtraction algebra (ii)(X ,•)is a semigroup (iii)(p-q)r=pr-qr for all p,q,r in X. It is clear that 0p=0 for all p in X. Similarly, we can define for left near-subtraction semigroup.

***2.2 Definition[12]***

A ***Neutrosophic Fuzzy*** Set S on the universe of discourse X Characterised by a truth membership function $T_S(p)$, a indeterminacy function $I_S(p)$ and a non- membership function $F_S(p)$ is defined as S={<p, $T_S(p)$, $I_S(p)$, $F_S(p)$>/p$\epsilon$X} where $T_S$, $I_S$, $F_S$:X→[0,1] and $0\leq T_S(p)+ I_S(p)+ F_S(p)\leq 3$.

***2.3 Definition[12]***

If V is said to be ***Neutrosophic fuzzy sub algebra*** of a near Subtraction Semigroup X , then it satisfies the following conditions:

(i)$T_V(p-q) \geq \min\{T_V(p), T_V(q)\}$ (ii)$I_V(p-q) \leq \max\{I_V(p), I_V(q)\}$

(iii)$F_V(p-q) \leq \max\{ F_V(p), F_V(q)\}$ for all p,q, in V.

***2.4 Definition[14]***

A near- subtraction Semigroup X is said to be ***left permutable*** if pqr=qpr for all p,q,r in X.

***2.5 Definition[12]***

Let S and V be any two Neutrosophic Fuzzy Sets of X and p$\epsilon$ X. Then

(1)S∪V={<p, $T_{S\cup V}$ (p), $I_{S\cup V}$ (p), $F_{S\cup V}$ (p)>/p$\epsilon$X}

(i) $T_{S\cup V}$ (p)=max{$T_S(p)$, $T_V(p)$} (ii)$I_{S\cup V}$(p)=min{ $I_S(p)$, $I_V(p)$} (iii)$F_{S\cup V}$ (p)=min{ $F_S(p)$, $F_V(p)$}

(2)$S \cap V$={<p, $T_{S\cap V}$ (p), $I_{S\cap V}$ (p), $F_{S\cap V}$ (p)>/p$\epsilon$X}where,

(i) $T_{S\cap V}$ (p)=min{ $T_S(p)$, $T_V(p)$} (ii)$I_{S\cap V}$ (p)=max{ $I_S(p)$, $I_V(p)$} (iii)$F_{S\cap V}$ (p)=max{ $F_S(p)$, $F_V(p)$}

**2.6Definition[10]**

An fuzzy sub algebra is deal to be ***fuzzy bi-ideal*** of X if $\mu$ (pqr) ≥ min {$\mu$ (p) , $\mu$(r)} where p,q, r in X.

***2.7 Definition[10]***

A Neutrosophic Fuzzy Sub algebra S in a near Subtraction Semigroup X is said to be ***Neutrosophic Fuzzy Bi-ideal*** of X if it satisfies the following conditions:

(i) $T_S(pqr)\geq \min\{T_S(p), T_S(r)\}$
(ii) $I_S(pqr)\leq \max\{I_S(p), I_S(r)\}$

(iii) $F_S(pqr) \leq \max\{F_S(p), F_S(r)\}$ for all p,q,r ϵX

***2.7 Definition[10]***

A Neutrosophic fuzzy set S of X is said to be ***Neutrosophic fuzzy right(left)X-sub algebra*** of X if

(i)$T_S(p-q) \geq \min\{T_S(p), T_S(q)\}$ ;$T_S(pq) \geq T_S(p)$[ $T_S(pq) \geq T_S(q)$]

(ii)$I_S(p-q) \leq \max\{I_S(p), I_S(q)\}$ ; $I_S(pq) \leq I_S(p)$ [$I_S(pq) \leq I_S(q)$]

(iii)$F_S(p-q) \leq \max\{F_S(p), F_S(q)\}$;$F_S(pq) \leq F_S(p)$, [$F_S(pq) \leq F_S(q)$] for all p,q, in X.

***2.8 Definition[14]***

Let S and V be any two Neutrosophic Fuzzy subsets of Near Subtraction Semigroups X and Y respectively. Then the ***direct product is*** defined by

$S\times V=\{<(p,q), T_{S\times V}(p,q), I_{S\times V}(p,q), F_{S\times V}(p,q)>/p \in X, q \in Y\}$where,

$T_{S\times V}(p,q)=\min\{T_S(p),T_V(q)\}$;$I_{S\times V}(p,q)=\max\{I_S(p),I_V(q)\}$;$F_{S\times V}(p,q)=\max\{F_S(p),F_V(q)\}$

**3. Neutrosophic Fuzzy Strong Bi-ideals of Near-Subtraction Semigroups**

The aim of this section is to explore the idea of this concept.

***3.1.Definition***

A Neutrosophic Fuzzy Bi-Ideal S of X is said to be ***Neutrosophic Fuzzy Strong Bi- Ideal*** (NFSBI) of X if it satisfies the following conditions:

(i)$T_S(pqr) \geq \min\{T_S(q), T_S(r)\}$ (ii)$I_S(pqr) \leq \max\{I_S(q), I_S(r)\}$ (iii)$F_S(pqr) \leq \max\{F_S(q), F_S(r)\}$ for all p,q,r ϵX.

***3.2 Example***

Assume that X={0,p,q,r} in which '-' and '•' defined by

| - | 0 | p | q | R |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| p | P | 0 | p | 0 |
| q | Q | q | 0 | 0 |
| r | R | q | p | 0 |

| • | 0 | P | q | r |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| P | 0 | Q | 0 | q |
| Q | 0 | 0 | 0 | 0 |
| R | 0 | Q | 0 | q |

Consider the Fuzzy set S:X→[0,1] be a fuzzy subset of X defined by

$T_S(0)=.7$ $T_S(p)=.5$ $T_S(q)=.3$ $T_S(r)=.2$ ;$I_S(0)=.3$ $I_S(p)=.4$ $I_S(q)=.6$ $I_S(r)=.8$;$F_S(0)=.2$ $F_S(p)=.3$ $F_S(q)=.7$ $F_S(r)=.9$.

***3.3Theorem***

Consider S=( $T_S$, $I_S$, $F_S$) to be a NFSBI of X iff XTT⊆T(XII⊇I,XFF⊇F)

**Proof**: Assume that S is a NFSBI of X. Let p,q,l,m,a$\epsilon$X.

Consider a=pq and p=lm. We already prove that T is a NFBI X[10]. Therefore

XTT(a)=$sup_{a=pq}$\{min\{(XT)(p), T(q)\}\}

=$sup_{a=pq}$\{min\{$sup_{p=lm}$\{min\{X(l),T(m)\},T(q)\}=$sup_{a=pq}$\{min\{$sup_{p=lm}$\{T(m)\},T(q)\}

Since T is a NFBIof X.

=$sup_{a=pq}$min\{T(m),T(q)\}≤$sup_{p=lmq}$\{T(lmq)\}=T(lmq)=T(a)

We have, XTT⊆T. Conversely, Assume that XTT⊆T

If a cannot expressed as a=pq then, XTT(a)=0≤T(a) .In both cases XTT⊆T. Choose p,q,r,a,b,c $\epsilon$X such that a=pqr. Then

T(pqr)=T(a)≥XTT(a)

= $sup_{a=bc}$min\{(XT)(b),T(c)\}≥min\{X(p),T(q),T(r)\}=min\{T(q),T(r)\}

XII(a)=$inf_{a=pq}$\{max\{(XI)(p), I(q)\}\}

=$inf_{a=pq}$\{max\{$inf_{p=lm}$\{max\{X(l),I(m)\},I(q)\}

=$inf_{a=pq}$\{max\{$inf_{p=lm}$\{I(m)\},I(q)\}

Since I is a NFSBI of X.

=$inf_{a=pq}$max\{I(m),I(q)\}≥$inf_{p=lmq}$\{I(lmq)\}=I(lmq)=I(a)

We have, IXI⊇I. If a cannot expressed as a=pq then XII(a)=0≥I(a).In both cases, XII⊇I

Conversely, Assume that XII⊇I.Choose p,q,r,a,b,c $\epsilon$X such that a=pqr. Then

I(pqr)=I(a)≤XII(a)

= $inf_{a=bc}$max\{(XI)(b),I(c)\} ≤max\{X(p),I(q),I(r)\}=max\{I(q),I(r)\}

FXF(a)=$inf_{a=pq}$\{max\{(XF)(p), F(q)\}\}

=$inf_{a=pq}$\{max\{$inf_{p=lm}$\{max\{X(l),F(m)\},F(q)\}

$=inf_{a=pq}\{\max\{inf_{p=lm}\{F(m)\},F(q)\}$

Since F is a Neutrosophic Fuzzy strong bi-ideal of X.

$=inf_{a=pq}\max\{F(m),F(q)\}\geq inf_{p=lmq}\{F(lmq)\}=F(lmq)=F(a)$

Hence $FXF\supseteq F$ If a cannot expressed as a=pq then $XFF(a)=0\geq F(a)$.In both cases, $XFF\supseteq F$

Conversely, Assume $FXF\supseteq F$.Choose p,q,r,a,b,c $\epsilon X$ such that a=pqr.Then

$F(pqr)=F(a)\leq XFF(a)$

$= inf_{a=bc}\max\{(XF)(b),F(c)\}\leq \max\{X(p),F(q),F(r)\}=\max\{F(q),F(r)\}$

***3.4 Theorem***

The Direct Product of any two NFSBI of a Near-Subtraction Semigroups is again a NFSBI of X×Y.

**Proof:**

Consider S and V be any two NFSBI of X and Y respectively.We already prove that S×V is a NFBI of X×Y[10].

Now $p=(p_1,p_2)$ $q=(q_1,q_2)$ $r=(r_1,r_2)\epsilon X\times Y$ respectively.

(i) $T_{S\times V}((p_1,p_2),(q_1,q_2),(r_1,r_2))= T_{S\times V}(p_1q_1r_1,p_2q_2r_2)$

$=\min\{T_S(p_1q_1r_1), T_V(p_2q_2r_2)\}$

$\geq\min\{\min\{T_S(q_1), T_S(r_1)\},\min\{T_V(q_2), T_V(r_2)\}\}$

$=\min\{ T_{S\times V}(q_1,q_2), T_{S\times V}(r_1,r_2)\}$

(ii) $I_{S\times V}((p_1,p_2),(q_1,q_2),(r_1,r_2))= I_{S\times V}(p_1q_1r_1,p_2q_2r_2)$

$=\max\{I_S(p_1q_1r_1), I_V(p_2q_2r_2)\}$

$\leq\max\{\max\{I_S(q_1), I_S(r_1)\},\min\{I_V(q_2), I_V(r_2)\}\}$

$=\max\{ I_{S\times V}(q_1,q_2), I_{S\times V}(r_1,r_2)\}$

(iii) $F_{S\times V}((p_1,p_2),(q_1,q_2),(r_1,r_2))= F_{S\times V}(p_1q_1r_1,p_2q_2r_2)$

$=\max\{F_S(p_1q_1r_1), F_V(p_2q_2r_2)\}$

$\leq\max\{\max\{F_S(q_1), F_S(r_1)\},\min\{F_V(q_2), F_V(r_2)\}\}$

$=\max\{ F_{S\times V}(q_1,q_2), F_{S\times V}(r_1,r_2)\}$

Hence, S×V is a NFSBI of X×Y.

***3.5Theorem***

If $S\times V=(T_{S\times V}, I_{S\times V}, F_{S\times V})$ be a NFSBI of X×Y .Then $S\times V=(T_{S\times V}, I_{S\times V}, F^{C}_{S\times V})$ is a NFSBI of X×Y.

**Proof:**

Consider $S\times V=(T_{S\times V}, I_{S\times V}, F_{S\times V})$ be a NFSBI of X×Y.

Now $p=(p_1,p_2)$ $q=(q_1,q_2)$ $r=(r_1,r_2)\epsilon X\times Y$

By [Theorem 3.4] $T_{S\times V}$, $I_{S\times V}$ and $F_{S\times V}$ are NFSBI of X ×Y.

Now it is enough to prove $T_{S\times V}^{C}(p_1,p_2)(q_1,q_2)(r_1,r_2)\le\max\{T_{S\times V}(q_1,q_2), T_{S\times V}(r_1,r_2)\}$

Now, $T^{C}_{S\times V}(p_1,p_2)(q_1,q_2)(r_1,r_2)$ $=1- T_{S\times V}(p_1,p_2)(q_1,q_2)(r_1,r_2)$

$\le 1-\min\{T_{S\times V}(q_1,q_2), T_{S\times V}(r_1,r_2)\}$

$=\max\{1- T_{S\times V}(q_1,q_2), 1-T_{S\times V}(r_1,r_2)\}$

$=\max\{T^{C}_{S\times V}(q_1,q_2), T^{C}_{S\times V}(r_1,r_2)\}$

Thus, $S\times V=(T_{S\times V}, I_{S\times V}, F^{C}_{S\times V})$ is a NFSBI of X ×Y.

***3.6Corollary***

If $S\times V=(T_{S\times V}, I_{S\times V}, F_{S\times V})$ be a NFSBI of X×Y.Then $S\times V=(F^{C}_{S\times V}, I_{S\times V}, T^{C}_{S\times V})$ is NFSBI of X×Y.

***3.7Corollary***

Consider $S\times V=(T_{S\times V}, I_{S\times V}, F_{S\times V})$ be a NFSBI of X×Y.Then $S\times V=(F_{S\times V}^{C}, I_{S\times V}, I_{S\times V})$ is a NFSBI of X×Y.

***3.8 Theorem***

Let X be a Strong regular Near –Subtraction Semigroup. Let $S=(T_S, I_S, F_S)$ be a NFSBI of X,then XTT=T, XII=I and XFF=F

**Proof:**

Consider $S=(T_S, I_S, F_S)$ be a NFSBI of X. Choose pϵX. Since X is a strong regular near subtraction semigroup there exists a ϵX such that $p=ap^2$.

Now, $XTT(p)=XTT(ap^2)$.

(i)$XTT(p)=sup_{p=app}\{\min\{(XT)(ap), T(p)\}\}\ge\min\{XT(ap),T(p)\}$

$=\min\{sup_{ap=lm}\{\min\{X(l),T(m)\},T(p)\}\}$

$\ge\min\{\min\{X(a),T(p)\},T(p)\}=\min\{T(p),T(p)\}=T(p)$

Also we know that $XTT\subseteq T$.From that, XTT=T

(ii) XII(p)=$inf_{p=app}$ {max{(XI)(ap), I(p)}}

≤max{XI(ap),I(p)}

=max{$inf_{ap=lm}$ {min{X(l),I(m)},I(p)}}

≤max{max{X(a),I(p)},I(p)}=max{I(p),I(p)}=I(p)

Also we know that XII⊇I.From that, XII=I

(iii) XFF(p)=$inf_{p=app}$ {max{(XF)(ap), F(p)}}

≤max{XF(ap),F(p)}

=max{$inf_{ap=lm}$ {min{X(l),F(m)},F(p)}}

≤max{max{X(a),F(p)},F(p)}=max{F(p),F(p)}=F(p)

Also we know that XFF⊇F.From that, XFF=F

***3.9 Theorem***

Every left permutable fuzzy right X-sub algebra of X is a NFSBI of X.

**Proof:**

Consider S=( $T_S$, $I_S$, $F_S$) be a Neutrosophic fuzzy right X-sub algebra of X.First we prove S is a NFBI of X.Choose a,p,q,l,mϵX.Also a=pq,p=lm

TXT(a) =$sup_{a=pq}$ {min{(TX)(p), T(q)}}=$sup_{a=pq}$ {min{$sup_{p=lm}$ {min{T(l),X(m)},T(q)}

=$sup_{a=pq}$ {min{$sup_{p=lm}$ {T(l)},T(q)}=$sup_{a=pq}$ min{T(l),T(q)}

Since T is a Neutrosophic fuzzy right X-sub algebra T(pq)=T((lm)q)≥T(l)

≤$sup_{a=pq}$ min{T(pq),X(q)}sinceX(q)=1=T(pq)=T(a)

Therefore, TXT⊆T

IXI(a)=$inf_{a=pq}${max{(IX)(p), I(q)}}

=$inf_{a=pq}${max{$inf_{p=lm}$ {max{I(l),X(m)},I(q)}

=$inf_{a=pq}${max{$inf_{p=lm}$ {I(l)},I(q)}=$inf_{a=pq}$max{I(l),I(q)}

Since I is a Neutrosophic fuzzy right X-sub algebra I(pq)=I((lm)q)≤I(l)

$\geq inf_{a=pq}$max{I(pq),X(q)}sinceX(q)=0=I(pq)=I(a)

Therefore, IXI⊒I

FXF(a) $=inf_{a=pq}${max{(FX)(p), F(q)}}

$=inf_{a=pq}${max{$inf_{p=lm}${max{F(l),X(m)},F(q)}

$=inf_{a=pq}${max{$inf_{p=lm}${F(l)},F(q)} $=inf_{a=pq}$max{F(l),F(q)}

Since I is a Neutrosophic fuzzy right X-sub algebra F(pq)=F((lm)q)≤F(l)

$\geq inf_{a=pq}$max{F(pq),X(q)}sinceX(q)=0=F(pq) =F(a)

Therefore, FXF⊒F

XTT(a) $=sup_{a=pq}${min{(XT)(p), T(q)}}$=sup_{a=pq}${min{$sup_{p=lm}${min{X(l),T(m)},T(q)}

$=sup_{a=pq}${min{$sup_{p=lm}${T(m)},T(q)}

Since T is a left permutable Neutrosophic Fuzzy right X-Sub algebra of X.T(pq)=T((lm)q)=T(mlq)≥T(m)≤$sup_{p=lmq}${min{T(pq),X(q)}.Since X(q)=1=T(pq)=T(a)

XII(a)$=inf_{a=pq}${max{(XI)(p), I(q)}}

$=inf_{a=pq}${max{$inf_{p=lm}${max{X(l),I(m)},I(q)}

$=inf_{a=pq}${max{$inf_{p=lm}${I(m)},I(q)}

Since I is a left permutable Neutrosophic Fuzzy right X-sub algebra of X.

I(pq)=I((lm)q)=I(mlq)≤I(m)

$\geq inf_{a=pq}$max{I(pq),X(q)}. Since X(q)=0=I(pq)=I(a)

We have, XII⊒I

XFF(a) $=inf_{a=pq}${max{(XF)(p), F(q)}}

$=inf_{a=pq}${max{$inf_{p=lm}${max{X(l),F(m)},F(q)}

$=inf_{a=pq}${max{$inf_{p=lm}${F(m)},F(q)}

Since F is a left permutable Neutrosophic Fuzzy right X-sub algebra of X. F(pq)=I((lm)q)=F(mlq)≤F(m)

$\geq inf_{a=pq}$max{F(pq),X(q)}. Since X(q)=0=F(pq)=F(a)

We have, FXX⊒I

***3.10Theorem***

Every left permutable fuzzy left X-sub algebra of X is a NFSBI of X.

**Proof:** Consider S=( Ts, Is, Fs) be a Neutrosophic fuzzy left X-sub algebra of X.First we prove S is a NFBI of X.Choose a,p,q,l,mϵX. Also a=pq,p=lm

TXT(a) $=sup_{a=pq}$\{min{(T)(p), XT(q)}}

$=sup_{a=pq}$\{min{T(p), {$sup_{q=lm}$min{X(l)},T(m)}}

$=sup_{a=pq}$\{min{T(p), $sup_{q=lm}$T(m)}=$sup_{a=pq}$min{T(p),T(m)}

Since T is a Neutrosophic fuzzy left X-sub algebra T(pq)=T((pl)m)≥T(m)

$\leq sup_{a=pq}$min{X(p),T(pq)}sinceX(q)=1=T(pq)=T(a)

Therefore, TXT⊆T

IXI(a) $=inf_{a=pq}$\{max{I(p), XI(q)}}=$inf_{a=pq}$\{max {I(p), $inf_{q=lm}$max{X(l),I(m)}

$=inf_{a=pq}$\{max{I(p), {$inf_{p=lm}$I(m)}

$=inf_{a=pq}$max{I(p),I(m)}

Since I is a Neutrosophic fuzzy left X-sub algebra I(pq)=I((pl)m)≤I(m)

$\geq inf_{a=pq}$max{X(p),I(pq)}sinceX(q)=0=I(pq)=I(a)

Therefore, IXI⊒I

FXF(a) $=inf_{a=pq}$\{max{F(p), XF(q)}}

$=inf_{a=pq}$\{max {F(p), $inf_{q=lm}$max{X(l),F(m)}

$=inf_{a=pq}$\{max{F(p), {$inf_{p=lm}$F(m)}=$inf_{a=pq}$max{F(p),F(m)}

Since I is a Neutrosophic fuzzy left X-sub algebra F(pq)=F((pl)m)≤F(m)

$\geq inf_{a=pq}$max{X(p),F(pq)}sinceX(q)=0=F(pq)=F(a)

Therefore, FXF⊇F

XTT(a) $=sup_{a=pq}${min{(X)(p), TT(q)}}=$sup_{a=pq}${min{X(p), $sup_{q=lm}$min{T(l),T(m)}}

Since T is a left permutable Neutrosophic Fuzzy left X-Sub algebra of X.T(pq)=T(plm)=T((lp)m)≥T(m)

$\leq sup_{a=pq}${min{X(l),T(pq)}.Since X(l)=1=T(pq) =T(a)

XII(a)=$inf_{a=pq}${max{(X)(p), II(q)}}=$inf_{a=pq}${max{X(p), $inf_{q=lm}$max{I(l),I(m)}}

Since I is a left permutable Neutrosophic Fuzzy left X-Sub algebra of X.I(pq)=I(plm)=I((lp)m)≤I(m)

$\geq inf_{a=pq}${max{X(l),I(pq)}.Since X(l)=0 =I(pq)=I(a)

XFF(a) $=inf_{a=pq}${max{(X)(p), FF(q)}}=$inf_{a=pq}${max{X(p), $inf_{q=lm}$max{F(l),F(m)}}

Since F is a left permutable Neutrosophic Fuzzy left X-Sub algebra of X.F(pq)=F(plm)=F((lp)m)≤F(m)

$\geq inf_{a=pq}${max{X(l),F(pq)}.Since X(l)=0=F(pq)=F(a)

We have, FXX⊇F

***3.11Theorem***

Every Neutrosophic fuzzy two-sided (left and right) X- sub algebra of X is a NFSBI of X.

**Proof:** Straight forward

**Conclusion**

The theory of Neutrosophy fuzzy set is basically the extension of the Intuitionistic fuzzy set. In the present manuscript, we have defined the Union, direct product, Intersection, Homomorphism of Neutrosophic fuzzy Strong Biideal in Near subtraction Semi group In future, we will investigate the Neutrosophy fuzzy Ideals and their fundamentals.

**Funding:** This research received no external funding

**Acknowledgments:** I convey my grateful acknowledgement and thanks to my mother for motivating me.I also thank to my reviewer and Editor.

**Conflicts of Interest:** The authors declare no conflict of interest.

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Received: Nov 18, 2021. Accepted: Feb 4, 2022

Neutrosophic Sets and Systems, Vol. 48, 2022
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# Neutrosophic Nano RW-Closed Sets in Neutrosophic Nano Topological Spaces

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***Abstract:*** The main objective of this study is to introduce a new class of closed sets namely Neutrosophic Nano RW-closed sets and Neutrosophic Nano RW-continuous functions in Neutrosophic Nano topological spaces. Some of its properties and interrelationship with some existing Neutrosophic nano closed sets have been discussed.

***Keywords:*** $N_N$RW-closed set, $N_N$RW-open set, $N_N$RW$T_{1/2}$ space, $N_N$RW-connected space, $N_N$RW-continuous, $N_N$RW-irresolute, $N_N$RW-open and $N_N$- closed maps.

## 1. Introduction

The theory of neutrosophic sets with three components namely, membership T (Truth), Indeterminacy I, and non-membership F (Falsehood), one of the interesting generalizations of theory of fuzzy sets and Intuitionistic fuzzy sets introduced by F.Smarandache [8]. In 2012, A.A. Salama and S.A. Alblowi [13] introduced and studied the theory of neutrosophic topological spaces. Since then several mathematicians contributed many papers to this area. Various results in ordinary topological spaces have been put in the neutrosophic setting, and also various departures have been observed. Neutrosophic set is a powerful tool to deal with indeterminate and inconsistent data. The concept of nano topology explored by M. Lellis Thivagar et. al[11] can be described as a collection of nano approximations for which equivalence classes are building blocks. In 2018, M. Lellis Thivagar et. al. [12] introduced a new concept called as Neutrosophic Nano topology and discussed neutrosophic nano interior and neutrosophic nano closure.

In 2007, S.S. Benchalli and R.S. Wali [4] introduced RW-closed sets in topological spaces. The authors D. Savithiri and C. Janaki [15] introduced the concept of Neutrosophic RW-closed sets in Neutrosophic topological spaces. In this article we introduce Neutrosophic Nano RW-closed sets and discuss some of its properties.

## 2 PRELIMINARIES

The following recalls requisite ideas and preliminaries necessary in the sequel of our work.

**Definition 2.1:**[9] Let X be a non-empty fixed set a Neutrosophic set **(NS for short)** A is an object having the form A = < x, $\mu_A(x)$,$\sigma_A(x)$,$\gamma_A(x)$> , x∈X where $\mu_A(x)$, $\sigma_A(x)$, $\gamma_A(x)$ which represents the degree of membership function, the degree of indeterminacy and the degree of non-membership function respectively of each element x∈X to the set A.

**Definition 2.2:[11]** Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U,R) is said to be the approximation space. Let X ⊆ U.

(i) The lower approximation of X with respect to R is the set of all objects, which can be classified as X with respect to R and it is denoted by $L_R(X)$. That is $L_R(X) = \bigcup_{x\in U}\{(R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.

(ii) The upper approximation of X with respect to R is the set of all objects which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is $U_R(X) = \bigcup_{x\in U}\{R(x) : R(x) \cap X \neq \phi\}$.

(iii) The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not X with respect to R and it is denoted by $B_R(X)$. That is $B_R(X) = U_R(X) – L_R(X)$.

**Remark 2.3:[11]**

(i) $L_R(X) \subseteq X \subseteq U_R(X)$.

(ii) $L_R(\phi) = U_R(\phi) = \phi$ and $L_R(U) = U_R(U) = U$.

(iii) $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$.

(iv) $L_R(X \cap Y) = L_R(X) \cap L_R(Y)$.

(v) $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$.

(vi) $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$.

(vii) $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$, whenever $X \subseteq Y$.

(viii) $U_R(X^C) = [L_R(X)]^C$ and $L_R(X^C) = [U_R(X)]^C$.

(ix) $U_RU_R(X) = L_RU_R(X) = U_R(X)$.

(x) $L_RL_R(X) = L_RU_R(X) = L_R(X)$.

**Definition 2.4:[11]** Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where X ⊆ U. $\tau_R(X)$ satisfies the following axioms:

(i) U and $\phi \in \tau_R(X)$.

(ii) The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

(iii) The intersection of the elements of any finite sub collection $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ forms a topology on U called the nano topology on U with respect to X. We call (U, $\tau_R(X)$) as the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets.

**Definition 2.5:[12]** Let U be a non-empty set and R be an equivalence relation on U. Let S be a neutrosophic set in U with the membership function $\mu_S$, the indeterminacy function $\sigma_S$, and the non-membership function $\gamma_S$. The neutrosophic nano lower, neutrosophic nano upper approximation and neutrosophic nano boundary of S in the approximation (U,R) denoted by $\underline{N}(S), \overline{N}(S) and\ B(S)$ are respectively defined as follows:

(i) $\underline{N}(S) = \{\langle x, \mu_{\underline{R}(A)}(x),\ \sigma_{\underline{R}(A)}(x), \gamma_{\underline{R}(A)}(x)\rangle / y \in [x]_R, x \in U\}$.

(ii) $\overline{N}(S) = \{\langle x, \mu_{\overline{R}(A)}(x), \sigma_{\overline{R}(A)}(x), \gamma_{\overline{R}(A)}(x)\rangle /\ y\ \in [x]_R, x \in U\}$.

(iii) B(S) = $\overline{N}(S) - \underline{N}(S)$.

where $\mu_{\underline{R}(A)}(x) = \ \wedge_{y\in[x]_R} \mu_A(y),\ \sigma_{\overline{R}(A)}(x) = \ \wedge_{y\in[x]_R} \sigma_A(y), \gamma_{\underline{R}(A)}(x) = \ \vee_{y\in[x]_R} \gamma_A(y)$,

$\mu_{\bar{R}(A)}(x) = \vee_{y\in[x]_R} \mu_A(y), \sigma_{\bar{R}(A)}(x) = \vee_{y\in[x]_R} \sigma_A(y), \gamma_{\bar{R}(A)}(x) = \wedge_{y\in[x]_R} \gamma_A(y)$.

**Definition 2.6:[12]** Let U be an universe, R be an equivalence relation on U and S be a neutrosophic set in U and if the collection $\tau_N(S)$ = {$0_N$, $1_N$, $\underline{N}(S), \overline{N}(S), B(S)$} forms a topology then it is said to be a neutrosophic nano topology. We call (U, $\tau_N(S)$) as the neutrosophic nano topological space **(Briefly NNTS)**. The elements of $\tau_N(S)$ are called as neutrosophic nano open **(In Short $N_N$O)** sets.

**Remark 2.7:[12]**[$\tau_N(S)$]$^C$ is called as dual neutrosophic nano topology of $\tau_N(S)$. The elements of [$\tau_N(S)$]$^C$ are called neutrosophic nano closed **(In Short $N_N$C)** sets.

**Remark 2.8:[12]** In neutrosophic nano topological space, the neutrosophic nano boundary cannot be empty. Since the difference between neutrosophic nano upper and neutrosophic nano lower approximations is defined as the maximum and minimum of the values in the neutrosophic sets.

**Proposition 2.9:[12]** Let U be a non-empty finite universe and S be a neutrosophic set on U. Then the following statements hold:

(i) The collection $\tau_N(S)$ = {$0_N$, $1_N$}, is the indiscrete neutrosophic nano topology on U.

(ii) If $\underline{N}(S) = \overline{N}(S) = B(S)$, then the neutrosophic nano topology, $\tau_N(S)$ = {$0_N$, $1_N$, $\underline{N}(S), B(S)$}.

(iii) If $\underline{N}(S) = B(S)$, then $\tau_N(S)$ = {$0_N$, $1_N$, $\underline{N}(S), \overline{N}(S)$} is a neutrosophic nano topology.

(iv) If $\overline{N}(S) = B(S)$, then $\tau_N(S)$ = {$0_N$, $1_N$, $\underline{N}(S), B(S)$}.

(v) The collection $\tau_N(S) = \{0_N, 1_N, \underline{N}(S), \overline{N}(S), B(S)\}$ is the discrete neutrosophic nano topology on U

**Definition 2.10:[12]** Let (U, $\tau_N$(S)) be NNTS and A = < x, $\mu_A(x),\sigma_A(x),\gamma_A(x)$, x $\in$ U> be a NNS in X. Then the neutrosophic nano closure and neutrosophic nano interior of A are defined by

$N_N$Cl (A) = $\cap$ { K : K is a $N_N$CS in X and A $\subseteq$ K }

$N_N$Int (A) = $\cup$ { G : G is a $N_N$OS in X and G $\subseteq$ A }.

**Definition 2.11:[12]** A subset A of a neutrosophic nano topological space Let (U, $\tau_N$(S)) is said to be

(i) a neutrosophic nano pre closed **($N_N$pre-closed)** set if $N_N$Cl($N_N$Int(A)) $\subseteq$ A.

(ii) a neutrosophic nano semi-closed **($N_N$semi-closed)** set if $N_N$Int($N_N$Cl(A)) $\subseteq$ A.

(iii) a neutrosophic nano regular open **(In short $N_N$RO)** set if A =$N_N$Int($N_N$Cl(A)) and regular closed **(In short $N_N$RC)** set if A = $N_N$Cl($N_N$Int(A)) .

(iv) a neutrosophic regular semi open **(In short NRSO)** if there exists a NRO set U such that U $\subseteq$ A $\subseteq$ NCl(A)

(v) a neutrosophic nano $\alpha$-closed **($N_N\alpha$-closed)** set if $N_N$Cl($N_N$Int($N_N$Cl(A))) $\subseteq$ A.

(vi) a neutrosophic nano g-closed **($N_N$ g-closed)** set if $N_N$Cl(A) $\subseteq$ F whenever A $\subseteq$ F and F is NNO in U.

**Definition 2.11:[6]** The difference between two neutrosophic nano sets A and B is defined as

A\B (S) = {x, min [($\mu_A(x)$, $\gamma_B(x)$)], min [($\sigma_A(x)$,1- $\sigma_B(x)$)], max [$\gamma_A(x)$, $\mu_B(x)$]].

## 3. NEUTROSOPHIC NANO RW-CLOSED SETS

**Definition 3.1:** A subset A of a neutrosophic nano topological space (U, $\tau_N$(S)) is called as neutrosophic nano regular weakly closed **(In short $N_N$RW-closed)** set, if $N_N$Cl(A) $\subseteq$ V whenever A $\subseteq$ V and V is a neutrosophic nano regular open in U.

**Definition 3.2:** The neutrosophic nano RW-closure and neutrosophic nano RW-interior of A are defined by

$N_N$RWCl (A) = $\cap$ { K : K is a $N_N$RWCS in X and A $\subseteq$ K }

$N_N$RWInt (A) = $\cup$ { G : G is a $N_N$RWOS in X and G $\subseteq$ A }.

**Definition 3.3:** (i) neutrosophic nano RG- Closed set **(shortly $N_N$RG – closed set)** of X if there exists a neutrosophic nano regular open set U such that $N_N$Cl(A) $\subseteq$ U whenever A $\subseteq$ U.

(ii) neutrosophic nano RWG- closed set **(shortly $N_N$RWG – closed set)** of X if there exists a neutrosophic nano regular open set U such that $N_N$Cl($N_N$Int(A)) $\subseteq$ U whenever A $\subseteq$ U.

(iii) neutrosophic nano W-closed set **(shortly NNW – closed set)** of X if there exists a neutrosophic nano semi-open set U such that NNCl(A) ⊆ U whenever A ⊆ U.

(iv) neutrosophic nano g-closed set **(shortly NNG – closed set)** of X if there exists a neutrosophic open set U such that NNCl(A) ⊆ U whenever A ⊆ U.

**Proposition 3.3:** (i) Every NN-closed set is NNRW-closed.

(ii) Every NN- regular closed set is NNRW-closed.

(iii) Every NN- πclosed set is NNRW-closed.

(iv) Every NNW-closed set is NNRW-closed.

**Proof:** Follows from [4].

The following example makes clear that the converse of the Proposition 3.3 need not be true.

**Example 3.4**: Let U = $\{p_1, p_2, p_3\}$ be the universe set and the equivalence relation U\R = $\{\{p_1, p\}, \{p_2\}\}$. Let $S = \left\{\langle\frac{p_1}{(0.1,0.2,0.3)}\rangle, \langle\frac{p_2}{(0.2,0.3,0.4)}\rangle, \langle\frac{p_3}{(0.1,0.6,0.4)}\rangle\right\}$ be a neutrosophic nano subset of U. Then $\overline{N}(S) = \left\{\langle\frac{p_1,p_3}{(0.1,0.6,0.3)}\rangle, \langle\frac{p_2}{(0.2,0.3,0.4)}\rangle\right\}$, $\underline{N}(S) = \left\{\langle\frac{p_1,p_3}{(0.1,0.2,0.4)}\rangle, \langle\frac{p_2}{(0.2,0.3,0.4)}\rangle\right\}$ and B $(S) = \left\{\langle\frac{p_1,p_3}{(0.1,0.6,0.3)}\rangle, \langle\frac{p_2}{(0.2,0.3,0.4)}\rangle\right\}$. So the neutrosophic nano topology τN = $\{0_N, 1_N, \underline{N}, B\}$ where the neutrosophic closed sets are τN$^C$ = $\{0_N, 1_N, \underline{N}^C, B^C\}$. Let $Q_1 = \left\{\langle\frac{p_1}{(0.2,0.1,0.3)}\rangle, \langle\frac{p_2}{(0.3,0.1,0.2)}\rangle, \langle\frac{p_3}{(0.1,0.2,0.3)}\rangle\right\}$, then $Q_1$ is NNRW-closed but it is not an NN-closed set in U. $Q_2 = \left\{\langle\frac{p_1}{(0.2,0.3,0.5)}\rangle, \langle\frac{p_2}{(0.3,0.6,0.5)}\rangle, \langle\frac{p_3}{(0.2,0.3,0.3)}\rangle\right\}$, $Q_2$ is NNRW-closed but it is neither NN Regular-closed nor NNπ-closed set and $Q_3 = \left\{\langle\frac{p_1}{(0.1,0.3,0.6)}\rangle, \langle\frac{p_2}{(0.2,0.6,0.6)}\rangle, \langle\frac{p_3}{(0.1,0.2,0.6)}\rangle\right\}$, then $Q_3$ is NNRW-closed but not NNW-closed set.

**Proposition 3.5:** (i) Every NNRW-closed set is NNRG-closed.

(ii) Every NNRW-closed set is NNGPR-closed.

(iii) Every NNRW-closed set is NNRWG-closed.

**Proof:** Follows from [4].

The converse of the Proposition 3.4 need not be true.

**Example 3.6**: * Let U = $\{p_1, p_2, p_3, p_4, p_5\}$ be the universe set and the equivalence relation U\R = $\{\{p_1, p_3\}, \{p_2\}, \{p_4, p_5\}\}$ . Let S $= \left\{\langle\frac{p_1}{(0.4,0.3,0.4)}\rangle, \langle\frac{p_2}{(0.5,0.3,0.5)}\rangle, \langle\frac{p_3}{(0.5,0.3,0.5)}\rangle, \langle\frac{p_4}{(0.6,0.3,0.1)}\rangle, \langle\frac{p_5}{(0.5,0.3,0.1)}\rangle\right\}$ be a neutrosophic nano subset of U $\overline{N}(S) = \left\{\langle\frac{p_1,p_3}{(0.5,0.3,0.2)}\rangle, \langle\frac{p_2}{(0.5,0.3,0.5)}\rangle, \langle\frac{p_4,p_5}{(0.6,0.3,0.1)}\rangle\right\}$ , $\underline{N}(S) =$

$\left\{\langle\frac{p_1,p_3}{(0.4,0.3,0.4)}\rangle, \langle\frac{p_2}{(0.5,0.3,0.5)}\rangle, \langle\frac{p_4,p_5}{(0.5,0.3,0.1)}\rangle\right\}$ and B $(S) = \left\{\langle\frac{p_1,p_3}{(0.4,0.3,0.4)}\rangle, \langle\frac{p_2}{(0.5,0.3,0.5)}\rangle, \langle\frac{p_4,p_5}{(0.1,0.3,0.6)}\rangle\right\}$. The neutrosophic nano topology τN = $\{0_N, 1_N, \underline{N}, \overline{N}, B\}$. Let $R_1 = \left\{\langle\frac{p_1}{(0.3,0.3,0.7)}\rangle, \langle\frac{p_2}{(0.2,0.3,0.6)}\rangle, \langle\frac{p_3}{(0.2,0.3,0.5)}\rangle, \langle\frac{p_4}{(0.1,0.2,0.7)}\rangle, \langle\frac{p_5}{(0.1,0.3,0.8)}\rangle\right\}$. Then $R_1$ is both NNGPR-closed and NNRWG – closed but it is not an NNRW-closed.

* In example 3.4, let $R_2 = \left\{\langle\frac{p_1}{(0.3,0.7,0.5)}\rangle, \langle\frac{p_2}{(0.3,0.4,0.6)}\rangle, \langle\frac{p_3}{(0.2,0.5,0.5)}\rangle, \langle\frac{p_4}{(0.1,0.5,0.6)}\rangle, \langle\frac{p_5}{(0.1,0.6,0.7)}\rangle\right\}$, then $R_2$ is NNRG-closed but not an NNRW-closed.

**Proposition 3.7:** The finite union of NNRW – closed subsets of U is also an NNRW – closed subset of U.

**Proof:** Assume that P and Q are NNRW –closed sets in U. Let R be an NNRSO set in X such that P ∪ Q ⊆ R. Then P ⊆ R and Q ⊆ R. Since P and Q are NNRW – closed sets, NNCl(P) ⊆ R and NNCl (Q) ⊆ R. Then NNCl(P∪Q) = NNCl(P) ∪ NNCl(Q) ⊆ R. Hence P ∪ Q is an NNRW – closed set in U.

**Remark 3.8:** The intersection of two NNRW-closed sets in (U, τN(S)) need not be an NNRW-closed set in U.

**Example 3.9:** Let U $=\{p_1, p_2, p_3, p_4, p_5\}$ be the universe set and the equivalence relation U\R = $\{\{p_1, p_3\}, \{p_2\}, \{p_4, p_5\}\}$. Let S= $\left\{\langle\frac{p_1}{(0.4,0.3,0.4)}\rangle, \langle\frac{p_2}{(0.5,0.3,0.5)}\rangle, \langle\frac{p_3}{(0.5,0.3,0.5)}\rangle, \langle\frac{p_4}{(0.6,0.3,0.1)}\rangle, \langle\frac{p_5}{(0.5,0.3,0.1)}\rangle\right\}$ be a neutrosophic nano subset of U $\overline{N}(S) = \left\{\langle\frac{p_1,p_3}{(0.5,0.3,0.2)}\rangle, \langle\frac{p_2}{(0.5,0.3,0.5)}\rangle, \langle\frac{p_4,p_5}{(0.6,0.3,0.1)}\rangle\right\}$, $\underline{N}(S) = \left\{\langle\frac{p_1,p_3}{(0.4,0.3,0.4)}\rangle, \langle\frac{p_2}{(0.5,0.3,0.5)}\rangle, \langle\frac{p_4,p_5}{(0.5,0.3,0.1)}\rangle\right\}$ and B$(S) = \left\{\langle\frac{p_1,p_3}{(0.4,0.3,0.4)}\rangle, \langle\frac{p_2}{(0.5,0.3,0.5)}\rangle, \langle\frac{p_4,p_5}{(0.1,0.3,0.6)}\rangle\right\}$. The neutrosophic nano topology τN = $\{0_N, 1_N, \underline{N}, \overline{N}, B\}$. $R_1 = \left\{\langle\frac{p_1}{(0.6,0.3,0.3)}\rangle, \langle\frac{p_2}{(0.5,0.3,0.3)}\rangle, \langle\frac{p_3}{(0.5,0.2,0.3)}\rangle, \langle\frac{p_4}{(0.3,0.3,0.1)}\rangle, \langle\frac{p_5}{(0.4,0.4,0.1)}\rangle\right\}$, $R_2 = \left\{\langle\frac{p_1}{(0.2,0.3,0.5)}\rangle, \langle\frac{p_2}{(0.3,0.5,0.7)}\rangle, \langle\frac{p_3}{(0.2,0.3,0.5)}\rangle, \langle\frac{p_4}{(0.1,0.5,0.6)}\rangle, \langle\frac{p_5}{(0.1,0.7,0.6)}\rangle\right\}$. Then $R_1$ and $R_2$ are NNRW-closed sets but $R_1 \cap R_2$ is not an NNRW-closed set.

**Proposition 3.10:** If a subset A of U is NNRW – closed set in U, then NNCl(A)\A does not contain any non-empty neutrosophic nano regular semi-open set in U.

**Proof:** Suppose that A is an NNRW –closed set in U. We shall prove by contradiction. Let R be an NNRSO set such that NNCl(A)\A ⊃ R which implies R ⊆ U\A i.e., A ⊆ U\R. Since R is NNRSO, U\R is also NNRSO set in U. Since A is an NNRW – closed set, NNCl(A) ⊆ U\R. So R ⊆ U\NNCl(A) also R ⊆ NNCl(A) implies R = ϕ. Hence NNCl (A)\A does not contain any non-empty NNRSO set in U.

The converse of the Proposition 3.10 need not be true as shown in the following example.

**Example 3.11:** In example 3.9, in the neutrosophic nano topological space (U, τN(S)), let $A = \left\{\langle\frac{p_1}{(0.3,0.2,0.5)}\rangle, \langle\frac{p_2}{(0.3,0.2,0.6)}\rangle, \langle\frac{p_3}{(0.2,0.3,0.5)}\rangle, \langle\frac{p_4}{(0.1,0.2,0.7)}\rangle, \langle\frac{p_5}{(0.1,0.3,0.8)}\rangle\right\}$, then NNCl(A)\A does not contain any non-empty NNRSO set, but A is not an NNRW-closed set in U.

**Corollary 3.12:** If a subset A of U is $N_N$RW – closed set in U, then $N_N$Cl(A)\A does not contain any non-empty neutrosophic nano regular- open set in U.

**Proof:** Follows from the Proposition 3.10 and the fact that every $N_N$RO set is $N_N$RSO in U.

**Proposition 3.13:** If A is $N_N$RO and $N_N$RW-closed, then A is $N_N$RC set and hence $N_N$-clopen.

**Proof:** Suppose A is $N_N$RO and $N_N$RW – closed. As every $N_N$RO set is $N_N$RSO and A ⊂ A, we have $N_N$Cl(A) ⊂ A. Also A ⊂ $N_N$Cl(A), thus $N_N$Cl(A) = A. Hence A is a $N_N$C set. Since A is $N_N$RO it is $N_N$O set. Now $N_N$Cl($N_N$Int(A) = $N_N$Cl(A) = A. Therefore A is $N_N$RC and Neutrosophic nano clopen.

**Proposition 3.14:** If A is an $N_N$RW – closed subset of U such that A ⊆ B ⊆ NCl(A), then B is an $N_N$RW – closed set in U.

**Proof:** Let A be an $N_N$RW – closed set of U such that A ⊆ B ⊆ $N_N$Cl(A). Let R be $N_N$RSO set of U such that B ⊆ R. Then A ⊆ R. Since A is $N_N$RW –closed set, we have $N_N$Cl(A) ⊆ R and $N_N$Cl(B) ⊆ $N_N$Cl($N_N$Cl(A)) ⊆ R. Therefore B is also an $N_N$RW – closed set in U.

The following example shows that the converse of the Proposition 3.13 need not be true.

**Example 3.15:** Let U = $\{n_1, n_2, n_3\}$ be the universe set and the equivalence relation U\R = $\{\{n_1, n_3\}, \{n_2\}\}$. Let $S = \left\{\langle\frac{x_1}{(0.1,0.2,0.3)}\rangle, \langle\frac{x_2}{(0.2,0.3,0.4)}\rangle, \langle\frac{x_3}{(0.1,0.6,0.4)}\rangle\right\}$ be a neutrosophic nano subset of U. Then $\overline{N}(S) = \left\{\langle\frac{x_1,x_3}{(0.1,0.6,0.3)}\rangle, \langle\frac{x_2}{(0.2,0.3,0.4)}\rangle\right\}$, $\underline{N}(S) = \left\{\langle\frac{x_1,x_3}{(0.1,0.2,0.4)}\rangle, \langle\frac{x_2}{(0.2,0.3,0.4)}\rangle\right\}$ and $B(S) = \left\{\langle\frac{x_1,x_3}{(0.1,0.6,0.3)}\rangle, \langle\frac{x_2}{(0.2,0.3,0.4)}\rangle\right\}$. So the neutrosophic nano topology $\tau_N = \{0_N, 1_N, \underline{N}, B\}$ and the neutrosophic closed sets are $\tau_N^C = \{0_N, 1_N, \underline{N}^C, B^C\}$. Let A = $\left\{\langle\frac{x_1}{(0.1,0.3,0.6)}\rangle, \langle\frac{x_2}{(0.2,0.6,0.6)}\rangle, \langle\frac{x_3}{(0.1,0.2,0.6)}\rangle\right\}$ and B = $\left\{\langle\frac{x_1}{(0.2,0.3,0.5)}\rangle, \langle\frac{x_2}{(0.3,0.6,0.5)}\rangle, \langle\frac{x_3}{(0.2,0.3,0.3)}\rangle\right\}$. Then A and B are $N_N$RW-closed sets in (U, $\tau_N$(S)), but A ⊂ B is not a subset of $N_N$Cl(A).

**Proposition 3.16:** Let A be an $N_N$RW-closed in (U, $\tau_N$(S)). Then A is $N_N$-closed if and only if $N_N$Cl(A)\A is $N_N$RSO.

**Proof:** Let A be an $N_N$-closed in (U, $\tau_N$(S)). Then $N_N$Cl(A)\A = ϕ which is $N_N$RSO.

Conversely, suppose $N_N$Cl(A)\A is $N_N$RSO in U. By hypothesis, A is $N_N$RW-closed implies $N_N$Cl(A)\A does not contain any non-empty $N_N$RSO in U. Then $N_N$Cl(A)\A = ϕ which implies that A is $N_N$-closed in U.

**Proposition 3.17:** If A is $N_N$RO and $N_N$RG closed, then A is $N_N$RW-closed in U.

**Proof:** Let A be an $N_N$RO and $N_N$RG-closed. Let Q be any $N_N$RSO set in U such that A ⊆ R. since A is $N_N$RO and $N_N$RG we have $N_N$Cl(A) ⊆ A ⊆ R. Therefore A is $N_N$RW-closed.

**Proposition 3.18:** If a subset A of a neutrosophic nano topological space U is both $N_N$RSO and $N_N$RW-closed, then it is $N_N$-closed.

**Proof:** Suppose A be a subset of a neutrosophic nano topological space U is both $N_N$RSO and $N_N$RW-closed. Then $A \subset A$ and $N_N Cl(A) \subseteq A$ which implies A is $N_N$-closed.

**Remark 3.19:** The concept of $N_N$RW-closed set is independent with the concepts of (i) $N_N$semi –closed (ii) $N_N$RW-preclosed (iii) $N_N\alpha$-closed (iv) $N_N$WG - closed sets which is shown by the following example.

**Example 3.20:** Let U = $\{n_1, n_2, n_3\}$ be the universe set. U\R = $\{\{n_1\}, \{n_2, n_3\}\}$ be an equivalence relation. Let $S = \left\{\langle\frac{n_1}{(0.1,0.2,0.3)}\rangle, \langle\frac{n_2}{(0.3,0.4,0.5)}\rangle, \langle\frac{n_3}{(0.6,0.4,0.1)}\rangle\right\}$ be a neutrosophic nano subset of U. Then $\overline{N}(S) = \left\{\langle\frac{n_1,n_3}{(0.1,0.2,0.3)}\rangle, \langle\frac{n_2}{(0.6,0.4,0.1)}\rangle\right\}$, $\underline{N}(S) = \left\{\langle\frac{n_1,n_3}{(0.1,0.2,0.3)}\rangle, \langle\frac{n_2}{(0.3,0.4,0.5)}\rangle\right\}$ and $B(S) = \left\{\langle\frac{n_1,n_3}{(0.1,0.2,0.3)}\rangle, \langle\frac{n_2}{(0.1,0.4,0.6)}\rangle\right\}$. So the neutrosophic nano topology $\tau_N = \{0_N, 1_N, \underline{N}, \overline{N}, B\}$. In the neutrosophic nano topology (U, $\tau_N$(S)),

- Let $A = \left\{\langle\frac{n_1}{(0.2,0.5,0.3)}\rangle, \langle\frac{n_2}{(0.1,0.5,0.6)}\rangle, \langle\frac{n_3}{(0.1,0.4,0.7)}\rangle\right\}$ and $B = \left\{\langle\frac{n_1}{(0.2,0.7,0.4)}\rangle, \langle\frac{n_2}{(0.5,0.6,0.4)}\rangle, \langle\frac{n_3}{(0.4,0.5,0.4)}\rangle\right\}$, then A is $N_N$semi-closed but not an $N_N$RW-closed and B is $N_N$RW-closed but it is not an $N_N$semi-closed.
- Let $C = \left\{\langle\frac{n_1}{(0.1,0.2,0.4)}\rangle, \langle\frac{n_2}{(0.3,0.3,0.6)}\rangle, \langle\frac{n_3}{(0.1,0.3,0.5)}\rangle\right\}$ and $D = \left\{\langle\frac{n_1}{(0.1,0.4,0.7)}\rangle, \langle\frac{n_2}{(0.1,0.6,0.7)}\rangle, \langle\frac{n_3}{(0.3,0.4,0.4)}\rangle\right\}$, then C is both $N_N$pre-closed set and $N_N$WG-closed but not an $N_N$RW-closed and D is $N_N$RW-closed but it is neither $N_N$pre-closed nor an $N_N$WG-closed sets.
- In example 3.8, in the topological space (U,$\tau_N$(S)), $E = \left\{\langle\frac{n_1}{(0.6,0.3,0.3)}\rangle, \langle\frac{n_2}{(0.5,0.3,0.3)}\rangle, \langle\frac{n_3}{(0.5,0.2,0.3)}\rangle, \langle\frac{n_4}{(0.3,0.3,0.1)}\rangle, \langle\frac{n_5}{(0.4,0.4,0.1)}\rangle\right\}$ and $F = \left\{\langle\frac{n_1}{(0.3,0.3,0.7)}\rangle, \langle\frac{n_2}{(0.2,0.3,0.6)}\rangle, \langle\frac{n_3}{(0.2,0.3,0.5)}\rangle, \langle\frac{n_4}{(0.1,0.2,0.7)}\rangle, \langle\frac{n_5}{(0.1,0.3,0.8)}\rangle\right\}$, E is $N_N$RW-closed set but not an $N_N\alpha$-closed set and F is $N_N\alpha$-closed but it is not an $N_N$RW-closed set.

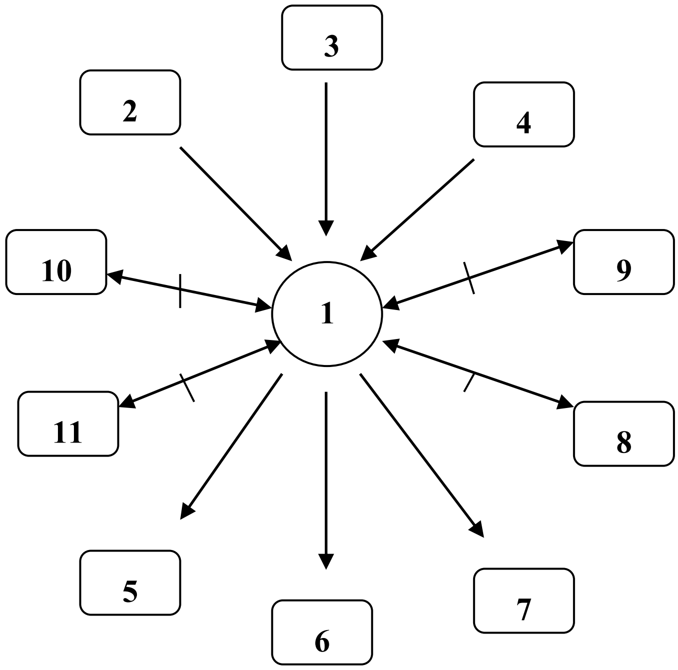
**Proposition 3.21:** If an $N_N$ subset A is both $N_N$-open and $N_N$G-closed in (U, $\tau_N$(S)), then it is $N_N$RW-closed in U.

**Proof:** Let A be $N_N$-open and $N_N$G-closed in U. Let $A \subset U$ and U be an $N_N$RSO in U. Now, $A \subset A$. By hypothesis, $N_N Cl(A) \subset U$. Thus A is $N_N$RW-closed.

**Remark 3.22:** If A is both NN-open and NNRW-closed in U, then A need not be NNG-closed in general which is shown in the following example.

**Example 3.23:** In example 3.8, the $N_N$-open set B is $N_N$RW-closed but it is not an $N_N$G-closed set.

**The above discussions are implicated in the following diagram.**




1. $N_N$RW-closed 2. $N_N$-closed 3. $N_N$R-closed 4. $N_N\pi$-closed 5. $N_N$RG-closed
6. $N_N$RWG-closed 7. $N_N$GPR-closed 8. $N_N$semi-closed 9. $N_N$pre-closed
10. $N_N\alpha$-closed 11. $N_N$WG-closed.

**Proposition 3.24:** If a subset A of a neutrosophic nano topological space U is both $N_N$-open and $N_N$WG-closed, then it is $N_N$RW-closed.

**Proof:** Suppose a subset A of U is both NN-open and $N_N$WG-closed. Let $A \subset U$ and U is $N_N$RSO. Then $N_N$Cl ($N_N$Int(A)) = $A \subset A$, since A is NN-open. Hence $N_N$Cl(A) $\subset U$ implies that A is an $N_N$RW-closed in U.

**Definition 3.25:** A neutrosophic nano subset A of a neutrosophic nano topological space $(U,\tau_N(S))$ is called an $N_N$RW-open if and only if its complement $A^C$ is $N_N$RW-closed.

**Proposition 3.26:** An $N_N$ set A of a topological space $(U,\tau_N(S))$ is $N_N$RW-open if $F \subseteq$ NNInt(A) whenever F is $N_N$RSO and $F \subset A$.

**Proof:** Follows from the definition 3.1.

**Proposition 3.27:** Let A be an $N_N$RW-open set of neutrosophic nano topological space $(U,\tau_N(S))$ and $N_N$Int(A) $\subseteq B \subseteq A$. Then B is $N_N$RW-open.

**Proof:** Suppose that A is an $N_N$RW-open in U and $N_N$Int(A) $\subseteq B \subseteq A$ implies $A^c \subseteq B^c \subseteq N_N$Cl($A^c$). Since $A^c$ is $N_N$RW-closed, by Proposition 3.14, $B^c$ is $N_N$RW-closed. Hence B is $N_N$RW-open.

**Proposition 3.28:** Let $(U,\tau_N(S))$ be a neutrosophic nano topological space and $N_N$RSO(X) and $N_N$C(X) be the family of all $N_N$RSO sets and $N_N$C sets respectively. Then $N_N$RSO(X) $\subseteq N_N$C(X) if and only if every neutrosophic nano set of U is $N_N$RW-closed.

**Proof: Necessity:** Suppose that $N_N$RSO(X) ⊆ $N_N$C(X) and let A be an $N_N$ – set of U such that A ⊆ R ∈ $N_N$RSO(X). Then $N_N$Cl(A) ⊆ $N_N$Cl(R) = R, by hypothesis. Hence $N_N$Cl(A) ⊆ R when A ⊆ R and R is $N_N$RSO which implies that A is $N_N$RW-closed.

**Sufficiency:** Assume that every neutrosophic nano set of U is $N_N$RW-closed. Let R ∈ $N_N$RSO(X). Then since R ⊆ R and R is $N_N$RW-closed, $N_N$Cl(R) ⊆ R then R ∈ $N_N$Cl(X). Therefore $N_N$RSO(X) ⊆ $N_N$Cl(X).

**Definition 3.29:** A neutrosophic nano topological space (U,$\tau_N$(S)) is called as $N_N$RW-connected if there is no proper $N_N$-subset of U which is both $N_N$RW-open $N_N$RW-closed.

**Proposition 3.30:** Every $N_N$RW-connected space is $N_N$-connected.

**Proof:** Let (U,$\tau_N$(S)) be an $N_N$RW-connected and suppose that (U,$\tau_N$(S)) is not $N_N$-connected. Then there exists a proper $N_N$-set A (A ≠ $0_N$, A ≠ $1_N$) such that A is both $N_N$-open and $N_N$-closed set. Since every $N_N$-open and $N_N$-closed set is $N_N$RW-open and $N_N$RW-closed, (U,$\tau_N$(S)) is not an $N_N$RW-connected which is a contradiction. This shows that U is $N_N$-connected.

**Proposition 3.31:** A $N_N$T space is $N_N$RW- connected if and only if there exists no non-zero $N_N$RW- open sets A and B in X such that A = $B^C$.

**Proof: Necessity:** Suppose that A and B are $N_N$RW-open sets such that A ≠ $0_N$ ≠ B. and A = $B^C$. Since B = $A^C$, A is $N_N$RW-closed set and B ≠ $0_N$ implies $B^C$ ≠ $1_N$, i.e., A ≠ $1_N$. Hence there exists a proper $N_N$ –set A which is both $N_N$RW-open and $N_N$RW-closed which is a contradiction to the fact that U is $N_N$RW-connected.

**Sufficiency:** Let (U,$\tau_N$(S)) be an $N_N$TS and A is both $N_N$RW-open and $N_N$RW-closed set in U such that $0_N$ ≠ A ≠ $1_N$. Take B = $A^C$ implies that B is $N_N$RW-open and A ≠ $1_N$ ⇒B = $A^C$ ≠ $0_N$ which is a contradiction. Hence there is no proper $N_N$-subset of U which is both $N_N$RW-open and $N_N$RW-closed. Therefore $N_N$TS (U,$\tau_N$(S)) is $N_N$RW-connected.

**Definition 3.32:** A neutrosophic nano topological space(U,$\tau_N$(S)) is said to be an $N_N$RW$T_{1/2}$-space if every $N_N$RW-closed set in U is $N_N$-closed in U.

**Proposition 3.33:** A neutrosophic nano topological space (U,$\tau_N$(S)) is $N_N$RW$T_{1/2}$ space, then the following statements are equivalent:

(i) U is $N_N$RW-connected (ii) U is $N_N$-connected.

**Proof: (i) ⇒ (ii):** Follows from the Proposition 3.29.

**(ii) ⇒ (i):** Assume that U is $N_N$RW$T_{1/2}$-space, and $N_N$-connected. Suppose that U is not an $N_N$RW-connected, then there exists a proper $N_N$-set A which is both $N_N$RW-open and $N_N$RW-closed. Since (U,$\tau_N$(S)) is $N_N$RW$T_{1/2}$, A is both $N_N$-open and $N_N$-closed which is a contradiction to the fact that U is $N_N$-connected. This shows that U is $N_N$RW-connected.

## 4. $N_N$RW-CONTINUOUS FUNCTIONS

**Definition 4.1: (i)** A function f: (U,$\tau_N$(S)) → (V,$\tau_N$(T)) is said to be a neutrosophic nano RW-continuous **(In short $N_N$RW-continuous)** if the inverse image of $N_N$-closed set of V is $N_N$RW-closed in (U,$\tau_N$(S)).

**(ii)** A function f: (U,$\tau_N$(S)) → (V,$\tau_N$(T)) is said to be a neutrosophic nano RW-irresolute **(In short $N_N$RW-irresolute)** if the inverse image of $N_N$RW-closed set of V is $N_N$RW-closed in (U,$\tau_N$(S)).

**Proposition 4.2:** A mapping f: (U,$\tau_N$(S)) → (V,$\tau_N$(T)) is $N_N$RW-continuous if and only if the inverse image of every $N_N$-open set of V is $N_N$RW-open in U.

**Proof:** It is obvious because $f^{-1}(A^c) = [f^{-1}(A)]^c$ for every $N_N$-set A of V.

**Proposition 4.3:** If f: (U,$\tau_N$(S)) → (V,$\tau_N$(T)) is $N_N$RW-continuous, then f($N_N$RWCl(A)) ⊆ $N_N$Cl(f(A)) for every $N_N$-set A of U.

**Proof:** Let A be an $N_N$-set of U. Then $N_N$Cl(f(A)) is an $N_N$-closed set of V. Since f is an $N_N$RW-continuous function, $f^{-1}$($N_N$Cl(f(A)) is $N_N$RW-closed in U. Clearly A ⊆ $f^{-1}$($N_N$Cl(f(A)). Therefore $N_N$RWCl(A) ⊆ $N_N$RWCl ($f^{-1}$($N_N$Cl f(A))) = $f^{-1}$($N_N$Cl(f(A)). Hence f($N_N$RWCl(A)) ⊆ $N_N$Cl(f(A)) for every $N_N$-set A of U.

**Proposition 4.4**: (i) Every $N_N$-continuous map is $N_N$RW-continuous.

(ii) Every $N_N$- regular continuous map is $N_N$RW-continuous.

(iii) Every $N_N$- π-continuous set is $N_N$RW-continuous.

(iv) Every $N_N$W-continuous map is $N_N$RW-continuous.

(v) Every $N_N$RW-irresolute map is $N_N$RW-continuous.

**Proof:** Obvious.

**Remark 4.4:** The following example makes clear that the converse of the Proposition 4.4 may not be true.

**Example 4.5:** Let U = $\{n_1, n_2, n_3\}$ = V be the universe sets. U\R1 = $\{\{n_1\},\{n_2,n_3\}\}$ and U\R2 = $\{\{n_1,n_3\},\{n_2\}\}$ be equivalence relations. Let $S_1 = \left\{\langle\frac{n_1}{(0.3,0.4,0.3)}\rangle,\langle\frac{n_2}{(0.6,0.3,0.1)}\rangle,\langle\frac{n_3}{(0.2,0.6,0.2)}\rangle\right\}$ , $S_2 = \left\{\langle\frac{n_1}{(0.1,0.2,0.3)}\rangle,\langle\frac{n_2}{(0.2,0.3,0.4)}\rangle,\langle\frac{n_3}{(0.1,0.6,0.4)}\rangle\right\}$ be a neutrosophic nano subsets of U. Then $\tau_N(S_1)$ = $\{0_N, \overline{N}(S_1), \underline{N}(S_1), B(S_1), 1_N\}$, $\tau_N(S_2)$ = $\{0_N, \overline{N}(S_2), B(S_2), 1_N\}$ be the neutrosophic nano topologies on U and V respectively. Define an identity map f: (U,$\tau_N(S_1)$) → (V.$\tau_N(S_2)$). Then f is $N_N$RW-continuous but is neither $N_N$-continuous nor $N_N$W-continuous. Similarly it's not an $N_N$R-continuous, $N_N$π -continuous and $N_N$RW-irresolute.

**Proposition 4.6:** (i) Every $N_N$RW-continuous map is $N_N$RG-continuous.

(ii) Every $N_N$RW- continuous map is $N_N$GPR- continuous.

(iii) Every $N_N$RW- continuous map is $N_N$RWG- continuous.

**Proposition 4.7:** If f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is $N_N$RW-continuous and g: $(V,\tau_N(T)) \rightarrow (W,\tau_N(R))$ is $N_N$-continuous. Then g°f: f: $(U,\tau_N(S)) \rightarrow (W,\tau_N(R))$ is $N_N$RW-continuous.

**Proof:** Let A be an $N_N$-closed in W. Then $g^{-1}(A)$ is $N_N$-closed in V, because g is $N_N$-continuous. Therefore $(g°f)^{-1}(A) = f^{-1}(g^{-1}(A))$ is $N_N$RW-closed in U. Hence g°f is $N_N$RW-continuous.

**Proposition 4.8:** If f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is $N_N$RW-continuous and g: $(V,\tau_N(T)) \rightarrow (W,\tau_N(R))$ is $N_N$G-continuous and $(V,\tau_N(T))$ is $N_NT_{1/2}$ then g°f: $(U,\tau_N(S)) \rightarrow (W,\tau_N(R))$ is $N_N$RW-continuous.

**Proof:** Let A be an $N_N$-closed set in W, then $g^{-1}(A)$ is $N_N$G-closed in V. Since V is $N_NT_{1/2}$ then $g^{-1}(A)$ is $N_N$-closed in V. Hence, $(g°f)^{-1}(A) = f^{-1}(g^{-1}(A))$ is $N_N$RW-closed in U. Hence g°f is $N_N$RW-continuous.

**Proposition 4.9:** If f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is $N_N$RG - irresolute and g: $(V,\tau_N(V)) \rightarrow (W,\tau_N(R))$ is $N_N$RW-continuous, then g°f: $(U,\tau_N(S)) \rightarrow (W,\tau_N(R))$ is $N_N$RG-continuous.

**Proof:** Let A be an $N_N$-closed set in W, then $g^{-1}(A)$ is $N_N$RW-closed in V, since g is $N_N$RW-continuous. Every $N_N$RW-closed set is $N_N$RG-closed, $g^{-1}(V)$ is $N_N$RG-closed set in V. Then $(g°f)^{-1}(A) = f^{-1}(g^{-1}(A))$ is NNRG-closed in U, by hypothesis. Hence g°f: $(U,\tau_N(S)) \rightarrow (W,\tau_N(R))$ is $N_N$RG-continuous.

**Proposition 4.10:** If f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is $N_N$RW-continuous surjection and U is $N_N$RW-connected then V is $N_N$-connected.

**Proof:** Assume that V is not an $N_N$-connected space. Then there exists a proper $N_N$-subset F of V which is both $N_N$-open and $N_N$-closed. Therefore, by hypothesis, $f^{-1}(F)$ is a proper $N_N$-set of U which is both $N_N$RW-open and $N_N$RW-closed in U implies that U is not an $N_N$RW-connected which is a contradiction. This shows that V is $N_N$-connected.

**Definition 4.11:** (i) A mapping f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is said to be $N_N$RW-open map if the image of every $N_N$-open set of U is $N_N$RW-open set in V.

(ii) A mapping f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is said to be $N_N$RW-closed map if the image of every $N_N$-closed set of U is $N_N$RW-closed set in V.

**Proposition 4.12:** A mapping f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is $N_N$RW-open if and only if for every $N_N$-set A of U, $f(N_N Int(A)) \subseteq N_N RW\ Int(f(A))$.

**Proof: Necessity:** Let f be an $N_N$RW-open map and A is an $N_N$-open set in U, $N_N Int(A) \subseteq A$ which implies that $f(N_N Int(A)) \subseteq f(A)$. Since f is an $N_N$RW-open mapping, $f(N_N Int(A))$ is $N_N$RW-open set in V such that $f(N_N Int(A)) \subseteq f(A)$. Therefore $f(N_N Int(A)) \subseteq N_N RWInt\ f(A)$.

**Sufficiency:** Suppose that A is an $N_N$-open set of U. Then $f(A) = f(N_N Int(A) \subseteq N_N RWInt\ f(A)$. But $N_N RWInt\ (f(A)) \subseteq f(A)$. Consequently $f(A) = N_N RWInt(A)$ which implies that f(A) is an $N_N$RW-open set of V and hence f is an $N_N$RW-open map.

**Proposition 4.13:** A mapping f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is $N_N$RW-open if and only if for every neutrosophic nano set A of V and for each $N_N$-closed set B of U containing $f^{-1}(A)$ there is a $N_N$RW-closed set F of V such that $A \subseteq F$ and $f^{-1}(F) \subseteq B$.

**Proof: Necessity:** Suppose that f is $N_N$RW-open map. Let A be a $N_N$-closed set of V and B be a $N_N$C set of U such that $f^{-1}(A) \subseteq B$. Then $F = f^{-1}(B^c)^c$ is a $N_N$RW- closed set of V such that $f^{-1}(F) \subseteq B$.

**Sufficiency:** Let F be a $N_N$O set of U. Then $f^{-1}(f(F))^c \subseteq F^c$ and $F^c$ is a $N_N$C set in X. By hypothesis there is an $N_N$RW- closed set G of V such that $(f(F))^c \subseteq G$ and $f^{-1}(G) \subseteq F^c$. Therefore $F \subseteq (f^{-1}(G))^c$. Hence $G^c \subseteq f(F) \subseteq f((f^{-1}(G))^c) \subseteq G^c$ i.e., $f(F) = G^c$ which is $N_N$RW-open in V and thus f is $N_N$RW-open map.

**Proposition 4.14:** If a mapping f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is $N_N$RW-open, then $N_N Int(f^{-1}(G)) \subseteq f^{-1}(N_N RWInt(G))$ for every neutrosophic nano set G of Y.

**Proof:** Let G be neutrosophic nano set of V. Then $N_N Intf^{-1}(G)$ is a $N_N$O set in U. Since f is $N_N$RW – open $f(N_N Intf^{-1}(G)) \subseteq N_N RWInt(f(f^{-1}(G)) \subseteq N_N RWInt(G))$. Thus $N_N Int(f^{-1}(G)) \subseteq f^{-1}(N_N RWInt(G))$.

**Proposition 4.15:** A mapping f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is $N_N$RW-closed if and only if for every neutrosophic nano set A of V and for each $N_N$O set B of U containing $f^{-1}(A)$ there is a $N_N$RW -open set F of V such that $A \subseteq F$ and $f^{-1}(F) \subseteq B$.

**Proof: Necessity:** Suppose that f is $N_N$RW -closed map. Let A be a $N_N$C set of V and B be a $N_N$O set of U such that $f^{-1}(A) \subseteq B$. Then $F = V \setminus f^{-1}(B^c)$ is a $N_N$RW –open set of V such that $f^{-1}(F) \subseteq B$.

**Sufficiency:** Let F be a $N_N$C set of X. Then $f^{-1}(f(F))^c \subseteq F^c$ and $F^c$ is a $N_N$O set in U. By hypothesis there is an $N_N$RW - open set R of V such that $(f(F))^c \subseteq R$ and $f^{-1}(R) \subseteq F^c$. Therefore $F \subseteq (f^{-1}(R))^c$. Hence $R^c \subseteq f(F) \subseteq f((f^{-1}(R))^c) \subseteq R^c$ i.e., $f(F) = R^c$ which is $N_N$RW -closed in V. Thus f is $N_N$RW -closed map.

**Proposition 4.16:** If f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is NN-almost irresolute and $N_N$RW-closed map. If A is $N_N$RW-closed set of U, then f(A) is $N_N$RW-closed in V.

**Proof:** Let $f(A) \subseteq R$ where R is an $N_N$RSO set of V. since f is an $N_N$-almost irresolute, $f^{-1}(R)$ is an $N_N$SO set of U such that $A \subseteq f^{-1}(R)$. Since A is $N_N$W-closed set of U which implies that $N_N Cl(A) \subseteq f^{-1}(R) \Rightarrow f(N_N Cl(A)) \subseteq R$, i.e., $N_N Cl(f(N_N Cl(A)) \subseteq R$. Therefore $N_N Cl(f(A)) \subseteq R$ whenever $f(A) \subseteq R$ where R is an $N_N$RSO set of V. Hence f(A) is an $N_N$RW-closed set of V.

**Proposition 4.17:** If f: $(U,\tau_N(S)) \rightarrow (V,\tau_N(T))$ is $N_N$-closed and g: $(V,\tau_N(T)) \rightarrow (W,\tau_N(R))$ is $N_N$RW-closed then $g^\circ f$: $(U,\tau_N(S)) \rightarrow (W,\tau_N(R))$ is $N_N$RW-closed.

**Proof:** Let F be an $N_N$-closed set of neutrosophic nano topological space $(U, _N(S))$. Then f(F) is an $N_N$-closed set of $(V,\tau_N(T))$. By hypothesis, $g^\circ f(F) = g(f(F))$ is an $N_N$RW-closed set in $N_N$-topological space W. Thus $g^\circ f$: $(U,\tau_N(S)) \rightarrow (W,\tau_N(R))$ is $N_N$RW-closed.

**Conclusions:** In this article, the authors have introduced and studied the concepts such as, Neutrosophic nano RW- closed set, $N_N$RW-open set, $N_N RWT_{1/2}$ space, $N_N$RW-connected space, $N_N$RW-continuous, $N_N$RW-irresolute, $N_N$RW-open and $N_N$- closed maps. In future it can be extended to some new forms of continuous functions and homeomorphisms.

**Funding:** This research received no external funding.

**Acknowledgements:** The authors are highly grateful to the Referees for their valuable suggestions.

**Conflicts of Interest**: The authors declare no conflict of interest.

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**Received: Nov 1, 2021. Accepted: Feb 5, 2022**

Neutrosophic Sets and Systems, Vol. 48, 2022
University of New Mexico

# A new servicizing business model of transportation: Comparing the new and existing alternatives via neutrosophic Analytic Hierarchy Process

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**Abstract:** The new normal of the world has been shaped by the COVID-19 outbreak by avoiding public transportation in order to prevent the spread of the disease. Due to the high financial burden of purchasing a car, new business models have been developed in order to make possible of utilizing vehicles to meet the transportation needs in pay-per-use base concept called "servicizing" or "servicization" which is based on presenting a product as a service, and selling the functionality of that product instead of the product itself. In order to meet the increasing demand for individual vehicle use, the existing car rental service providers have provided a new mobile application controlled business model which makes the rental process easier. The aim of this study is to evaluate the customers' preferences of purchasing, renting through an agency, or mobile application supported new pay-as-you-go business model use, in order to determine which criterion is prominent in the decision-making process, and to identify the weights of these criteria. Due to the uncertain and indeterminate attitudes of the customers in decision making, the data were collected as neutrosophic data sets and analyzed with a novel neutrosophic Analytic Hierarchy Process (nAHP) approach. The study provides implications both theoretically and practically in terms of revealing new servicization possibilities and analyzing real user judgments.

**Keywords:** **s**ervicization; servicizing business model; car sharing program; neutrosophic sets; neutrosophic Analytic Hierarchy Process.

## 1. Introduction

Circular Economy which based upon the reuse, remanufacture and recycling of the products is a well-known and well-accepted movement of sustainable operations management research [1]. The servicizing business models, i.e. servicization or product-as-a-service concept, grounds on selling the functionality of a product / item / device instead of selling the product itself to the customers. This is a phenomenon converting the products into services [2], or transforming the consumers into users [3] by bringing the functionalization into the forefront. In this case, companies don't transfer the product ownership to the customers, instead, they charge the them in pay-per-use base.

Servicizing business models have been drawn attention with its sustainable and environmental side owing to the durability and reliability requirement of these repeatedly in use products, and they have been defined as an "opportunity to research" [1] in the literature. Besides, the companies have made serious investments for this business model recently [4]. However, the COVID-19 pandemic has caused a serious decrease in individual purchasing power, and the companies have developed a new servicization versions in order to minimize the face-to-face communication and contracting process with an easier way of payment via mobile applications.

This change in the way of business has motivated this research to analyze the customer perception and attitude towards different individual transportation options. Hence, this study aims to develop a decision model for evaluating the customers' decisions on purchasing, renting through an agency (walk-in or using the website of provider or a website comparing all providers), or new mobile application controlled way of renting alternatives of driving in order to determine which criterion is more important in the decision-making process, and to identify the weights of these criteria.

Since the decision criteria have often vague, uncertain, indeterminate or inconsistent information, the data were collected as neutrosophic data sets from the real customers having experiences in both purchasing, renting through an agency and renting through the mobile application alternatives were analyzed with a neutrosophic AHP approach. The fuzzy AHP provides a wide range of application areas and remarkable results for many sectors [5-9]. The study provides theoretical and practical implications by revealing new servicization alternatives and analyzing real customer attitudes.

The literature points out that there is an obvious research gap in the field of study [42-46]. The researchers investigating and doing research on this topic especially for the sake of sustainability. The topic is important owing to the significance of achieving sustainable supplier selection, green supply chain management practices, and sustainability evaluation of transportation technologies.

This study introduces a new way of servicizing business model as a contribution to the literature with real customer preferences shaping the decision making process. The analysis results addressed the weights of criteria and alternative ranking by real user preferences.

The following sections include literature review, objective of the study, methodology, analysis and conclusion parts.

## 2. Materials and Methods

### *2.1. Literature Review*

Current servicization literature focuses on the intensions of the organizations towards servicizing [10-12], product-as-a-service [13], device-as-a-service [3, 14], the potential of Industry 4.0 adoption in servicizing [15-16].

There are successful examples in servicization such as Xerox printing services, Runway car rental, Michelin fleet solutions, Philips' lighting solutions, Rolls-Royce's total care solutions [17], and Bundles' household appliance services [1].

Servicization studies implementing AHP discuss construction servicization [18], design requirements for plumbing services [19], prioritization of product-service business model elements at aerospace industry [20], and cloud manufacturing [21]. Moreover, there are Neutrosophic AHP papers addressing system selection [22-23], AHP-SWOT analysis for strategic planning and decision-making [24], AHP and TOPSIS framework [25], AHP and DEA methodology [26], and performance analysis [27], comparative analysis of AHP, FAHP and Neutrosophic-AHP [41],

However, the new mobile application driven pay-as-you-go model of servicization research is missing in the literature. Besides, there are limited number of AHP studies applied neutrosophic sets. Therefore, the priorities of the customers having experiences in both purchasing and renting cars will be examined in this study with neutrosophic sets in order to serve as a good example of neutrosophic AHP for servicizing.

### *2.2. Methodology*

The evaluation criteria that the real customers consider in transportation through driving alternatives have been specified via an in-depth interview with a car rental service provider X representative. The model is based on the literature review and information provided by the company X representative. The goal, criteria and alternatives are presented in Figure 1.

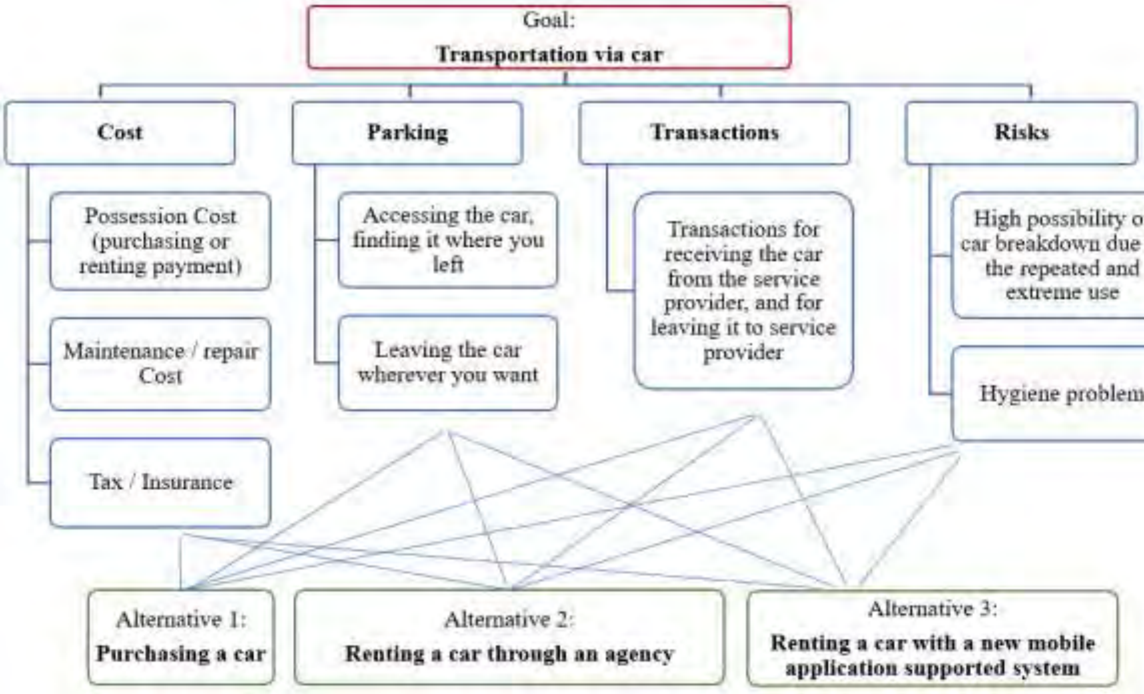


**Figure 1.** Developed AHP model.

The cost criterion includes the sub criteria of possessing cost by purchasing or renting payment [28], maintenance / repair cost [29], tax / insurance cost [30]. The parking criterion consists of two sub criteria such as accessing the car and finding it where you left, and leaving the car wherever you want [31]. The transaction criterion refers to receiving the car from the service provider, and leaving it to again the service provider [32]. Moreover, the risk criterion forms from hygiene sub criterion due to the COVID-19 pandemics, and the high possibility of car breakdown due to the repeated and extreme use [33].

In order to obtain the customer judgements, a user survey has been used, and neutrosophic sets have been used to gather the preferences. The experts were selected from the car rental service provider X's real users who had comments about the mobile application in the website of the company. 36 users were identified as candidate experts, and just 3 of them accepted to state their opinions.

### 2.2.1. Preliminaries

Neutrosophic sets (NSs) are proposed by Smarandache [34] as a general form of fuzzy sets and intuitionistic fuzzy set. This is a powerful technique to handle incomplete, indeterminate and inconsistent information that is valid in the real world applications. Besides, there are many neutrosophic sets: single valued, interval-valued, multi-valued, bipolar, hesitant, refined, simplified, rough and hyper-complex neutrosophic sets [35]. Basic definitions and operations of neutrosophic sets:

Definition 1. A neutrosophic set A in E (let E be a universe) is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$ where $x \in E$.

A can be defined as $A=\{\langle x, T_A(x), I_A(x), F_A(x), \mid x \in E \rangle\}$

where $T_A(x), I_A(x), F_A(x) \in ]0^-,1^+[$ such that $0^- \leq T_A(x), I_A(x), F_A(x) \leq 3^+$.

Definition 2. A single-valued neutrosophic set A is a subclass of NS and is stated as

$A=\{\langle x, T_A(x), I_A(x), F_A(x) \mid x \in E \rangle\}$ where $T_A, I_A, F_A : X \rightarrow [0,1]$

such that $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

In particular, if E has only 1 element, A is called a simplified neutrosophic number (SNN), which is represented as $A=\langle T_A, I_A, F_A \rangle$ [36].

Definition 3. Let A and B be two SNN, and p(A) be the complement of A, the following operations are valid [22, 36].

$A \oplus B = \langle T_A + T_B - T_A * T_B, I_A * I_B, F_A * F_B \rangle$

$A \otimes B = \langle T_A * T_B, I_A + I_B - I_A * I_B, F_A + F_B - F_A * F_B \rangle$

$A / B = \langle T_A / T_B, I_B - I_A / 1 - I_A, F_B - F_A / 1 - F_A \rangle$

$\alpha A = \langle 1-(1-T_A)^{\alpha}, I_A^{\alpha}, (F_A^{\alpha}) \rangle, \alpha>0$

$A/\alpha = \langle 1-(1-T_A)^{1/\alpha}, I_A^{1/\alpha}, (F_A^{1/\alpha}) \rangle, \alpha>0$

$p(A) = \langle F_A, 1 - I_A, T_A \rangle$

Definition 4. The score function is defined as $s(A) = (2 + T_A - I_A - F_A) / 3$ for a SNN to deneutrosophicate or rank [35].

Definition 5. Geometric means are defined as [26]:

$T_1 = [1 \times T_{12} \times \ldots \times T_{1n}]^{1/n}, \ldots, T_n = [T_{1n} \times \ldots \times 1]^{1/n}$

$I_{1m} = [1 \times I_{12m} \times \ldots \times I_{1nm}]^{1/n}, \ldots, I_{im} = [I_{n1m} \times \ldots \times 1]^{1/n}$

$F_{1m} = [1 \times F_{12m} \times \ldots \times F_{1nm}]^{1/n}, \ldots, F_{im} = [F_{n1m} \times \ldots \times 1]^{1/n}$

Definition 6. Aggregation formula is [35]: $F_w(A_1, A_2, \ldots, A_n) =$

$\langle 1 - \prod_{j=1}^{n}\left(1 - T_{A_j}(x)\right)^{w_j}, 1 - \prod_{j=1}^{n}\left(1 - I_{A_j}(x)\right)^{w_j}, 1 - \prod_{j=1}^{n}(1 - F_{A_j}(x))^{w_j} \rangle$ where $W = (w_1, w_2, \ldots, w_n)$ is the weight vector of $A_j (j = 1, 2, \ldots, n)$, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

The truth-membership $T_A$ stands for "the possibility in which the statement is true", the indeterminacy-membership $I_A$ is "the degree in which he/she is not sure", and the falsity-membership $F_A$ means that "the statement is false" [37].

All of the above definitions will be applied to the proposed nAHP methodology in the following sections.

### 2.2.2. Procedure in Gathering and Aggregating the Individual Evaluations

There are different proposed scales for the neutrosophic linguistic variable such as [22] and [26]. However, there is also a fair criticism for these scales due to the defined structure of them. For example, the aforementioned Radwan et al. [22] scale defines "extremely highly preferred" as <.9 .1 .1>. The truth-membership can be thought as the reverse of falsity-membership; this is acceptable by definition. However, since the indeterminacy means "the degree in which one is not sure", we cannot define this indeterminacy proportional to the truth-membership value with a scale.

Participants should express "the degree in which he/she is not sure". Therefore, this study gathers the truth and indeterminacy values separately from the participants instead of using these defined tables in order to deal with this criticism.

In order to aggregate the individual neutrosophic evaluations into group evaluations, the captured expert opinions have been processed with the proposed formula of [26] (the definition 6). There are nAHP papers use the neutrosophic weighted arithmetic average aggregation operator of [37], such as [38]. However, since the average operator is problematic in terms of finding reciprocals, this study prefers to adopt a geometric mean based formulation in aggregating the expert opinions.

2.2.3. Steps of the Methodology

The steps of the nAHP used in this study:

Step 1. Defining the problem, criteria and alternatives with a structured hierarchy.

Step 2. Gathering the expert evaluations by taking truth- and indeterminacy-membership values separately via a survey in order to obtain pairwise comparisons of criteria and alternatives.

Step 3. Checking the consistency of pairwise matrices by Eigenvector solution.

Step 4. Aggregating the individual evaluations into group decision.

Step 5. Obtaining the weights of each criteria. Repeating these steps for the alternatives' pairwise comparisons.

Step 6. Ranking the alternatives with respect to the calculated weights.

## 3. Application

The defined problem with criteria and alternatives in a structured hierarchy is provided in Figure 1 previously by fulfilling the Step 1.

Step 2. The user survey provided real users' judgements on the goal "transportation via car" and the alternative ways of transportation. Table 1 presents the individual judgements of the experts.

**Table 1.** Pairwise comparison matrix with respect to goal by experts.

| | Expert # | Cost | Parking | Transactions | Risks |
|---|---|---|---|---|---|
| Cost | 1 | < .5 .5 .5 > | < .7 .2 .3 > | < .7 .2 .3 > | < .4 .7 .6 > |
| | 2 | < .5 .5 .5 > | < .9 .1 .1 > | < .9 .1 .1 > | < .9 .1 .1 > |
| | 3 | < .5 .5 .5 > | < .9 .1 .1 > | < .9 .1 .1 > | < .7 .2 .3 > |
| Parking | 1 | < .3 .8 .7 > | < .5 .5 .5 > | < .7 .2 .3 > | < .3 .8 .7 > |
| | 2 | < .1 .9 .9 > | < .5 .5 .5 > | < .9 .1 .1 > | < .6 .2 .4 > |
| | 3 | < .1 .9 .9 > | < .5 .5 .5 > | < .8 .1 .2 > | < .5 .1 .5 > |
| Transactions | 1 | < .3 .8 .7 > | < .3 .8 .7 > | < .5 .5 .5 > | < .2 .8 .8 > |
| | 2 | < .1 .9 .9 > | < .1 .9 .9 > | < .5 .5 .5 > | < .9 .1 .1 > |
| | 3 | < .1 .9 .9 > | < .2 .9 .8 > | < .5 .5 .5 > | < .7 .1 .3 > |
| Risks | 1 | < .6 .3 .4 > | < .7 .2 .3 > | < .8 .2 .2 > | < .5 .5 .5 > |
| | 2 | < .1 .9 .9 > | < .4 .8 .6 > | < .1 .9 .9 > | < .5 .5 .5 > |
| | 3 | < .3 .8 .7 > | < .5 .9 .5 > | < .3 .9 .7 > | < .5 .5 .5 > |

Step 3. The consistency was checked with the score function value definition for each participant evaluations via Eigenvector solution procedure [39].

The score function was applied to deneutrosophicate the evaluations into crisp values. The sum of each column was taken, next, each element of the matrix was divided into the sum of its columns in order to have normalized relative weights. Then, the normalized principal Eigenvector (also called priority vector) is obtained by averaging across the rows. This calculation provides the experts' priorities with respect to goal. For example, while the risk criterion is the priority of the expert 1, cost criterion is the most important criteria for expert 2 and 3. Besides of the relative weight calculation, this procedure paves the way for checking the consistency of participants' answers. Here, one needs Principal Eigen value ($\lambda_{max}$) obtaining from summation of products between each element of Eigen vector and sum of columns of the reciprocal matrix. Table 2 states the score function values, normalization, weights and Principal Eigen value.

The largest Eigen value equals to the size of comparison matrix, or $\lambda_{max}$ = n [40], which gives a measure of consistency named Consistency Index (CI = ($\lambda_{max}$ – n)/(n-1)). The CI values should be compared with Random Consistency Index as a previously defined index of sample size 500, and RI is 0.89 for n=4 (4×4 matrix). The Consistency Ratio CR was calculated (CR = CI / RI), and if the CR is ≤ 10% in comparison with the CI, the inconsistency is acceptable. Accordingly, while the evaluations of expert 1 and 3 are within the acceptable inconsistency limits, the evaluations of expert 2 cannot be taken into consideration due to the CR = 23%.

**Table 2.** Score function values, normalization, weights and principal Eigen value.

| wrt. Goal | | **Score function values** | | | | **x / sum values** | | | | **w** | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | C | P | T | R | C | P | T | R | Row average | $\lambda_{max}$ |
| E1 | C | 0,500 | 0,733 | 0,733 | 0,367 | 0,300 | 0,328 | 0,265 | 0,275 | 0,292 | 3,681 |
| | P | 0,267 | 0,500 | 0,733 | 0,267 | 0,160 | 0,224 | 0,265 | 0,200 | 0,212 | |
| | T | 0,267 | 0,267 | 0,500 | 0,200 | 0,160 | 0,119 | 0,181 | 0,150 | 0,153 | |
| | R | 0,633 | 0,733 | 0,800 | 0,500 | 0,380 | 0,328 | 0,289 | 0,375 | 0,343 | |
| | Sum | 1,667 | 2,233 | 2,767 | 1,333 | 1 | 1 | 1 | 1 | | |
| E2 | C | 0,500 | 0,900 | 0,900 | 0,900 | 0,313 | 0,429 | 0,338 | 0,303 | 0,345 | 4,409 |
| | P | 0,367 | 0,500 | 0,900 | 0,667 | 0,229 | 0,238 | 0,338 | 0,225 | 0,257 | |
| | T | 0,367 | 0,367 | 0,500 | 0,900 | 0,229 | 0,175 | 0,188 | 0,303 | 0,224 | |
| | R | 0,367 | 0,333 | 0,367 | 0,500 | 0,229 | 0,159 | 0,138 | 0,169 | 0,173 | |
| | Sum | 1,600 | 2,100 | 2,667 | 2,967 | 1 | 1 | 1 | 1 | 1,000 | |
| E3 | C | 0,500 | 0,900 | 0,900 | 0,733 | 0,333 | 0,466 | 0,365 | 0,278 | 0,361 | 4,002 |
| | P | 0,367 | 0,500 | 0,833 | 0,633 | 0,244 | 0,259 | 0,338 | 0,241 | 0,270 | |
| | T | 0,367 | 0,167 | 0,500 | 0,767 | 0,244 | 0,086 | 0,203 | 0,291 | 0,206 | |
| | R | 0,267 | 0,367 | 0,233 | 0,500 | 0,178 | 0,190 | 0,095 | 0,190 | 0,163 | |
| | Sum | 1,500 | 1,933 | 2,467 | 2,633 | 1 | 1 | 1 | 1 | 1,000 | |

Step 4. In order to aggregate the individual evaluations into group decision, the aggregation definition 6 was used (see Table 3).

Step 5. The weights of each criterion were obtained, ant the step was repeated for the alternatives' and sub-criteria's pairwise comparisons.

Step 6. The alternatives were ranked with respect to the calculated weights.

According to the analysis results, renting through an agency was the most preferred alternative in terms of the cost criterion. Secondly the new system, and then the purchasing option was preferred by the weight values. When the parking criterion was considered, the ranking was purchasing, renting through an agency and new system, respectively. Similarly, in case we had a focus on the transactions, the same ranking was valid. However, participants addressed the new system as the most risky alternative, next renting through an agency and then the purchasing option, respectively.

**Table 3.** Aggregating the individual evaluations into group decision.

| | Cost | | | Parking | | | Transactions | | | Risks | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| wrt. Goal | T | I | F | T | I | F | T | I | F | T | I | F |
| Cost | 0,4 | 0,4 | 0,4 | 0,7 | 0,1 | 0,1 | 0,7 | 0,1 | 0,1 | 0,4 | 0,3 | 0,3 |
| Parking | 0,1 | 0,3 | 0,6 | 0,3 | 0,3 | 0,3 | 0,5 | 0,1 | 0,1 | 0,2 | 0,3 | 0,4 |
| Transactions | 0,1 | 0,2 | 0,5 | 0,1 | 0,5 | 0,4 | 0,2 | 0,2 | 0,2 | 0,2 | 0,2 | 0,3 |
| Risks | 0,3 | 0,3 | 0,3 | 0,4 | 0,4 | 0,2 | 0,5 | 0,4 | 0,2 | 0,3 | 0,3 | 0,3 |

The sub criteria analysis revealed that there was a tax/insurance, maintenance / repair cost, and possession cost sequence with respect to cost criterion. Moreover, "hygiene problem" sub criterion had a greater importance than the "high possibility of car breakdown due to the repeated and extreme use" in terms of risks criterion. Besides, the "accessing the car, finding it where you left" sub criterion and the "leaving the car wherever you want" sub criterion had close weights as 0,51 and 0,49.

When the criteria weights and alternatives were combined, this analysis resulted that the effect of alternatives on the goal was identified with the weights as renting through an agency (0.358), purchasing option (0.326), and the new system (0.316).

## 4. Conclusions

This study introduces a new way of servicizing business model as a contribution to the literature with real customer preferences shaping the decision making process. The analysis results addressed the weights of criteria and alternative ranking by real user preferences.

The cost, parking, transactions and risks parameters have been investigated via a user survey provided real users' judgements on the goal "transportation via car" and the alternative ways of transportation. The results point out that;

- Renting through an agency was the most preferred alternative in terms of the cost criterion.
- Secondly the new system, and then the purchasing option was preferred by the weight values.
- When the parking criterion was considered, the ranking was purchasing, renting through an agency and new system, respectively.
- Similarly, in case we had a focus on the transactions, the same ranking was valid.
- However, participants addressed the new system as the most risky alternative, next renting through an agency and then the purchasing option, respectively.

As a theoretical implication, this study tries to handle the criticism of previously defined linguistic variable tables by a different way of data gathering. In addition, the study adopts the score functions to deneutrosophicate the fuzzy sets in analysis procedure as a new approach.

The practical implications of the paper provide a real world customer preference point of view for the industry representatives. Since the new normal of the world requires new way of business models, this analysis addresses new initiatives to overcome the burden of this hard time. One can infer from these results that the companies can introduce new way servicization by taking the defined significant criteria into consideration.

The number of company representatives, number of participants, and the possibility of biased attitudes of the both these representatives and the participants are the main limitations of this study. Hence, this study tries to select the real participants who have experienced these services previously in order to reflect the real world case. In addition, the participants were asked whether they are willing to participate the survey, or they are feeling obliged at the beginning of the survey questions.

Furthermore, this paper serves both theoretical implications by using the neutrosophic sets to AHP and practical implications by presenting the real user priorities. One can infer from the study to understand which criteria is prominent in contrast with the others, and the theoretical background can be applied to different decision making problems.

Further researches may have a large number of participants and representatives, or different mathematical assumptions can be utilized in the calculations. This study differs from the existing ones by gathering the indeterminacy values of neutrosophic sets by the participants instead of using the defined linguistic variable tables.

**Conflicts of Interest:** The authors declare no conflict of interest.

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Received: Nov 7, 2021. Accepted: Feb 3, 2022

# Anti Neutrosophic multi fuzzy ideals of $\gamma$ near ring

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**Abstract:** The theory of anti neutrosophic multi fuzzy ideals of $\gamma$ near ring is dispensed in this work and various algebraic properties such as intersection, union of anti neutrosophic multi fuzzy ideals of $\gamma$ near ring are examined. Further we examined the direct anti product of anti neutrsophic multi fuzzy ideals of $\gamma$ near ring and also we proved the homorphic images and pre images of anti neutrosophic multi fuzy ideals of $\gamma$ near ring.

**Keywords:** Anti neutrosophic fuzzy set, $\gamma$ near ring, anti neutrosophic multi fuzzy set, anti neutrosophic multi fuzzy ideal of $\gamma$ near ring, anti product of anti neutrosophic multi fuzzy ideals of near ring. $\gamma$

## 1. Introduction

In 1965, Zadeh[25] proposed the notion of fuzzy set. Later A.Rosenfeld[16] developed fuzzy groups. The numerous authors like Bh.Satyanarayana[3,4,5] proposed the concept of fuzzy $\gamma$ near ring. The authors S. Ragamai, Y. Bhargavi, T. Eswarlal[19] developed theory of fuzzy and L fuzzy ideals of $\gamma$ near ring. Later the properties of $\gamma$ near ring in multi fuzzy sets were extended by K. Hemabala and Srinivasa kumar[13]. After the theory of fuzzy sets, Florentin Smarandache[7,8] established as a new field of philosophy which is a neutrosophic theory, in 1995.The main base of neutrosophic logic is neutrosophy that includes indeterminacy. It is an agumentation of fuzzy set and intuitionstic fuzzy set. In neutrosophic logic each proposition is estimated by three components T,I,F. The neutrosophic set theory have seen great triumph in several fields such as image processing ,medical diagnosis, robotic, decision making problem and so on. I. Arockiarani[3] extended the theory of neutrosophic fuzzy set. A.Solairaju and S.Thiruveni[2] verified the algebraic properties of fuzzy neutrosophic set in near rings. In fuzzy neutrosophic set, the three components T,I,F can take single values between 0 and 1. There is some ambiguity irrespective of the distance to the element is. The neutrosophic fuzzy set theory on its own is not sufficient to study real world problems. F. Smarandache[9] developed notion of neutrosophic multi sets, an extension of neutrosophic set, in 2016. Authors like Vakkas Ulucay and Memet sahin[23] verified the concepts of neutrosophic multi fuzzy set in groups and verified the group properties. We carry the neutrosophic multi fuzzy notion in $\gamma$ near ring and hence some properties of algebra are verified.

## 2. Preliminaries:

Basic definitions of fuzzy set, multi fuzzy set, neutrosophic set and neutrosophic multi set, $\gamma$ near ring are presenting in this section. Fuzzy set can take a single value between [0,1]

### 2.1 Definition:

Let $E$ be a non empty set and $\tilde{A}$ be a fuzzy set over $E$ is defined by[25]

$\tilde{A} = \{\tilde{A}(a) / a \in E\}$ where $\tilde{A}: E \to [0,1]$.

**2.2 Definition:**

Let E be a non empty set and $M_f$ be a multi fuzzy set over E is defined as[20,21]

$M_f=\{<a, M_f^1(a), M_f^2(a), M_f^3(a), \ldots M_f^i(a)>: a \in E\}$ where $M_f^n: E \to [0,1]$ for all $n \in \{1,2,\ldots i\}$ and $a \in E$

2.3 **Definition:**

Let E be a non empty set then neutrosophic fuzzy set $\tilde{N}$[7] in E is defined as

$\tilde{N}=\{a, \ddot{t}_{\tilde{N}}(a), \ddot{i}_{\tilde{N}}(a), \ddot{f}_{\tilde{N}}(a)>: a \in E$ and $\ddot{t}_{\tilde{N}}(a), \ddot{i}_{\tilde{N}}(a), \ddot{f}_{\tilde{N}}(a) \in [0,1]\}$

Where $\ddot{t}_{\tilde{N}}(a)$ is the truth membership function, $\ddot{i}_{\tilde{N}}(a)$ is the indeterminancy membership function and $\ddot{f}_{\tilde{N}}$ falsity membership function and $0 \leq \ddot{t}_{\tilde{N}}(x) + \ddot{i}_{\tilde{N}}(x) + \ddot{f}_{\tilde{N}} \leq 1$.

**2.4 Definition**:

Let E be a non empty set. A neutrosophic multi fuzzy set $\tilde{N}$ on E can be defined as follows

$\tilde{N}=\{< a, (\ddot{t}_{\tilde{N}}^1(a), \ddot{t}_{\tilde{N}}^2(a), \ldots \ddot{t}_{\tilde{N}}^i(a)), (\ddot{i}_{\tilde{N}}^1(a), \ddot{i}_{\tilde{N}}^2(a) \ldots \ddot{i}_{\tilde{N}}^i(a)), (\ddot{f}_{\tilde{N}}^1(a), \ddot{f}_{\tilde{N}}^2(a), \ldots \ddot{f}_{\tilde{N}}^i(a))>: a \in E\}$

Where $\ddot{t}_{\tilde{N}}^1(a), \ddot{t}_{\tilde{N}}^2(a), \ldots \ddot{t}_{\tilde{N}}^i(a): E \to [0,1]$

$\ddot{i}_{\tilde{N}}^1(a), \ddot{i}_{\tilde{N}}^2(a) \ldots \ddot{i}_{\tilde{N}}^i(a): E \to [0,1]$

$\ddot{f}_{\tilde{N}}^1(a), \ddot{f}_{\tilde{N}}^2(a), \ldots \ddot{f}_{\tilde{N}}^i(a): E \to [0,1]$

$0 \leq \sup \ddot{t}_{\tilde{N}}^n(a) + \sup \ddot{i}_{\tilde{N}}^n(a) + \sup \ddot{f}_{\tilde{N}}^n(a) \leq 1$ for n=1 to i

$(\ddot{t}_{\tilde{N}}^1(a), \ddot{t}_{\tilde{N}}^2(a) \ldots \ddot{t}_{\tilde{N}}^i(a))$, $(\ddot{i}_{\tilde{N}}^1(a), \ddot{i}_{\tilde{N}}^2(a) \ldots \ddot{i}_{\tilde{N}}^i(a))$, $(\ddot{f}_{\tilde{N}}^1(a), \ddot{f}_{\tilde{N}}^2(a), \ldots \ddot{f}_{\tilde{N}}^i(a))$ are the sequences of truth membership values, indeterminacy membership values and falsity membership values. In addition i is called the dimension of neutrosophic multi fuzzy set $\tilde{N}$ denoted by $d(\tilde{N})$. The sequence of truth membership values are arranged in decreasing order, but the corresponding indeterminacy membership and falsity membership values may not be in any order.

**2.5 Definition:**

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy sets

where $\mathcal{L}$={($\ddot{t}^1_{\mathcal{L}}(a), \ddot{t}^2_{\mathcal{L}}(a),..\ddot{t}^i_{\mathcal{L}}(a)$),($\ddot{i}^1_{\mathcal{L}}(a), \ddot{i}^2_{\mathcal{L}}(a), \ldots, \ddot{i}^i_{\mathcal{L}}(a)$),($\ddot{f}^1_{\mathcal{L}}(a), \ddot{f}^2_{\mathcal{L}}(a),\ldots\ddot{f}^i_{\mathcal{L}}(a)$)} and

$\mathcal{R}$={($\ddot{t}^1_{\mathcal{R}}(a), \ddot{t}^2_{\mathcal{R}}(a),..\ddot{t}^i_{\mathcal{R}}(a)$),($\ddot{i}^1_{\mathcal{R}}(a), \ddot{i}^2_{\mathcal{R}}(a), \ldots, \ddot{i}^i_{\mathcal{R}}(a)$),($\ddot{f}^1_{\mathcal{R}}(a), \ddot{f}^2_{\mathcal{R}}(a),\ldots\ddot{f}^i_{\mathcal{R}}(a)$)} then we have the following relations and operations

1) $\mathcal{L} \subseteq \mathcal{R}$iff $\ddot{t}^n_{\mathcal{L}}(a) \leq \ddot{t}^n_{\mathcal{R}}(a)$ , $\ddot{i}^n_{\mathcal{L}}(a) \geq \ddot{i}^n_{\mathcal{R}}(a)$, $\ddot{f}^n_{\mathcal{L}}(a) \geq \ddot{f}^n_{\mathcal{R}}(a)$ ,$a \in$ E and n= 1 toi.

2) $\mathcal{L} = \mathcal{R}$iff$\ddot{t}^n_{\mathcal{L}}(a) = \ddot{t}^n_{\mathcal{R}}(a)$ , $\ddot{i}^n_{\mathcal{L}}(a) = \ddot{i}^n_{\mathcal{R}}(a)$, $\ddot{f}^n_{\mathcal{L}}(a) = \ddot{f}^n_{\mathcal{R}}(a)$ ,$a \in$ E and n= 1 to i.

3) $\mathcal{L} \cup \mathcal{R}$= {$a$, max ($\ddot{t}^n_{\mathcal{L}}(a), \ddot{t}^n_{\mathcal{R}}(a)$), min($\ddot{i}^n_{\mathcal{L}}(a), \ddot{i}^n_{\mathcal{R}}(a)$),min($\ddot{f}^n_{\mathcal{L}}(a)\ddot{f}^n_{\mathcal{R}}(a)$)}, $a \in$ E and n= 1 to i.

4) $\mathcal{L} \cap \mathcal{R}${$a$, min ($\ddot{t}^n_{\mathcal{L}}(a), \ddot{t}^n_{\mathcal{R}}(a)$), max ($\ddot{i}^n_{\mathcal{L}}(a), \ddot{i}^n_{\mathcal{R}}(a)$),max ($\ddot{f}^n_{\mathcal{L}}(a)\ddot{f}^n_{\mathcal{R}}(a)$)}, $a \in$ E and n= 1 to i.

**2.6 Definition:**

A non empty set E with the binary operations '+'(addition) and '.'(multiplication) is called a near ring[3] if the following conditions hold:

1) (E, +) is a group

2) (E, .) is a semigroup

3) $(e_1 + e_2).e_3 = e_1.e_3 + e_2.e_3$ for all $e_1, e_2, e_3 \in$ E

To be precise, it is called right near ring .Since it satisfies the right distributive law. But the word near ring is intended to mean right near ring. We use gh instead of g. h

A $\gamma$ near ring $M$ is a triple ($M$,+, $\gamma$) where

1) ($M$,+) is a group

2) $\gamma$is a non empty set of binary operations on $M$ such that $\tau \in \gamma$, ($M$,+,$\tau$) is a near ring.

3) $e_1\tau(e_2\sigma e_3) = (e_1\tau e_2)\sigma e_3$ for all $e_1, e_2, e_3 \in M$and $\tau,\sigma \in \gamma$.

## 3. Anti Neutrosophic multi fuzzy set of $\gamma$ near ring

In this section, we introduce the definition of anti neutrosophic multi fuzzy sets of $\gamma$ near ring. We proved that union of two anti neutrosophic multi fuzzy ideals of $\mathcal{L}$and$\mathcal{R}$is an anti neutrosophic multi fuzzy ideal. We also prove that the intersection of two anti neutrosophic multi fuzzy ideals of $\mathcal{L}$and$\mathcal{R}$is also an anti neutrosophic multi fuzzy ideal.

### 3.1 Definition:

A neutrosophic multi fuzzy set

$\mathcal{L}$= ={($(\ddot{t}_{\mathcal{L}}^{1}(a),\ddot{t}_{\mathcal{L}}^{2}(a),..\ddot{t}_{\mathcal{L}}^{i}(a)),(\ddot{i}_{\mathcal{L}}^{1}(a),\ddot{i}_{\mathcal{L}}^{2}(a),\ldots,\ddot{i}_{\mathcal{L}}^{i}(a)),(\ddot{f}_{\mathcal{L}}^{1}(a),\ddot{f}_{\mathcal{L}}^{2}(a),\ldots..\ddot{f}_{\mathcal{L}}^{i}(a))$} in a $\gamma$ near ring $M$ is called anti neutrosophic multi fuzzy sub $\gamma$ near ring of $M$ if

1) $\ddot{t}_{\mathcal{L}}^{n}(a-\mathrm{b}) \leq \max(\ddot{t}_{\mathcal{L}}^{n}(a), \ddot{t}_{\mathcal{L}}^{n}(\mathrm{b}))$,

   $\ddot{i}_{\mathcal{L}}^{n}(a-\mathrm{b}) \geq \min(\ddot{i}_{\mathcal{L}}^{n}(a), \ddot{i}_{\mathcal{L}}^{n}(\mathrm{b}))$,

   $\ddot{f}_{\mathcal{L}}^{n}(a-\mathrm{b}) \geq \min(\ddot{f}_{\mathcal{L}}^{n}(a), \ddot{f}_{\mathcal{L}}^{n}(\mathrm{b}))$, n= 1 to i.

2) $\ddot{t}_{\mathcal{L}}^{n}(a\tau\mathrm{b}) \leq \max(\ddot{t}_{\mathcal{L}}^{n}(a), \ddot{t}_{\mathcal{L}}^{n}(\mathrm{b}))$,

   $\ddot{i}_{\mathcal{L}}^{n}(a\tau\mathrm{b}) \geq \min(\ddot{i}_{\mathcal{L}}^{n}(a), \ddot{i}_{\mathcal{L}}^{n}(\mathrm{b}))$,

   $\ddot{f}_{\mathcal{L}}^{n}(a\tau\mathrm{b}) \geq \min(\ddot{f}_{\mathcal{L}}^{n}(a), \ddot{f}_{\mathcal{L}}^{n}(\mathrm{b}))$, n= 1 to i.

### 3.2 Definition:

Let $M$ be a $\gamma$ near ring. An anti neutrosophic multi fuzzy set $\mathcal{L}$in a $\gamma$ near ring $M$ is called anti neutrosophic multi fuzzy left(resp. right) ideal of $M$ if for all $a$, b,$\rho_1, \rho_2 \in M$, $\tau \in \gamma$, n=1,2,…i

1) $\ddot{t}_{\mathcal{L}}^{n}(a-\mathrm{b}) \leq \max(\ddot{t}_{\mathcal{L}}^{n}(a), \ddot{t}_{\mathcal{L}}^{n}(\mathrm{b}))$,

   $\ddot{i}_{\mathcal{L}}^{n}(a-\mathrm{b}) \geq \min(\ddot{i}_{\mathcal{L}}^{n}(a), \ddot{i}_{\mathcal{L}}^{n}(\mathrm{b}))$,

$$\ddot{f}^n_{\mathcal{L}}(a-\mathrm{b}) \geq \min(\ddot{f}^n_{\mathcal{L}}(a), \ddot{f}^n_{\mathcal{L}}(\mathrm{b}))$$

2) $$\ddot{t}^n_{\mathcal{L}}(\mathrm{b}+a-\mathrm{b}) \leq \ddot{t}^n_{\mathcal{L}}(a),$$

$$\ddot{\iota}^n_{\mathcal{L}}(\mathrm{b}+a-\mathrm{b}) \geq \ddot{\iota}^n_{\mathcal{L}}(a),$$

$$\ddot{f}^n_{\mathcal{L}}(\mathrm{b}+a-\mathrm{b}) \geq \ddot{f}^n_{\mathcal{L}}(a),$$

3) $$\ddot{t}^n_{\mathcal{L}}(\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2) \leq \ddot{t}^n_{\mathcal{L}}(a),$$

$$\ddot{\iota}^n_{\mathcal{L}}(\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2) \geq \ddot{\iota}^n_{\mathcal{L}}(a),$$

$$\ddot{f}^n_{\mathcal{L}}(\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2) \geq \ddot{f}^n_{\mathcal{L}}(a),$$

[resp. right

$$\ddot{t}^n_{\mathcal{L}}(a\tau\rho_1) \leq \ddot{t}^n_{\mathcal{L}}(a)$$

$$\ddot{\iota}^n_{\mathcal{L}}(a\tau\rho_1) \geq \ddot{\iota}^n_{\mathcal{L}}(a),$$

$$\ddot{f}^n_{\mathcal{L}}(a\tau\rho_1) \geq \ddot{f}^n_{\mathcal{L}}(a)]$$

$\mathcal{L}$ is called a anti neutrosophic multi fuzzy ideal of $M$ if $\mathcal{L}$ both left and right anti neutrosophic multi fuzzy ideal of $M$.

**3.1 Theorem**:

Let $\mathcal{L}$ and $\mathcal{R}$ anti neutrosophic multi fuzzy left ideal of $M$. Then $\mathcal{L} \cup \mathcal{R}$ is a anti neutrosophic multi fuzzy left ideal of $M$.

**Proof:**

Let $\mathcal{L}$ and $\mathcal{R}$ anti neutrosophic multi fuzzy left ideal of $M$.

Let $a, \mathrm{b}, \rho_1, \rho_2 \in M$, $\tau \in \gamma$

1) $\ddot{t}^n_{\mathcal{L}\cup\mathcal{R}}(a-\mathrm{b}) = \max\{\ddot{t}^n_{\mathcal{L}}(a-\mathrm{b}), \ddot{t}^n_{\mathcal{R}}(a-\mathrm{b})\}$

$$\leq \max\{\{\max(\ddot{t}^{n}_{\mathcal{L}}(a),\ddot{t}^{n}_{\mathcal{L}}(\mathrm{b})),\max(\ddot{t}^{n}_{\mathcal{R}}(a),\ddot{t}^{n}_{\mathcal{R}}(\mathrm{b}))\}\}$$

$$\leq \max\{\{\max(\ddot{t}^{n}_{\mathcal{L}}(a),\ddot{t}^{n}_{\mathcal{R}}(a)),\max(\ddot{t}^{n}_{\mathcal{L}}(\mathrm{b}),\ddot{t}^{n}_{\mathcal{R}}(\mathrm{b}))\}\}$$

$$\leq \max(\ddot{t}^{n}_{\mathcal{L}\cup\mathcal{R}}(a),\ \ddot{t}^{n}_{\mathcal{L}\cup\mathcal{R}}(\mathrm{b}))$$

$$\ddot{i}^{n}_{\mathcal{L}\cup\mathcal{R}}(a-\mathrm{b}) = \min\{\ddot{i}^{n}_{\mathcal{L}}(a-\mathrm{b}),\ \ddot{i}^{n}_{\mathcal{R}}(a-\mathrm{b})\}$$

$$\geq \min\{\{\min(\ddot{i}^{n}_{\mathcal{L}}(a),\ddot{i}^{n}_{\mathcal{L}}(\mathrm{b})),\min(\ddot{i}^{n}_{\mathcal{R}}(a),\ddot{i}^{n}_{\mathcal{R}}(\mathrm{b}))\}\}$$

$$\geq \min\{\{\min(\ddot{i}^{n}_{\mathcal{L}}(a),\ddot{i}^{n}_{\mathcal{R}}(a)),\min(\ddot{i}^{n}_{\mathcal{L}}(\mathrm{b}),\ddot{i}^{n}_{\mathcal{R}}(\mathrm{b}))\}\}$$

$$\geq \min(\ddot{i}^{n}_{\mathcal{L}\cup\mathcal{R}}(a),\ \ddot{i}^{n}_{\mathcal{L}\cup\mathcal{R}}(\mathrm{b}))$$

$$\ddot{f}^{n}_{\mathcal{L}\cup\mathcal{R}}(a-\mathrm{b}) = \min\{\ddot{f}^{n}_{\mathcal{L}}(a-\mathrm{b}),\ \ddot{f}^{n}_{\mathcal{R}}(a-\mathrm{b})\}$$

$$\geq \min\{\{\min(\ddot{f}^{n}_{\mathcal{L}}(a),\ddot{f}^{n}_{\mathcal{L}}(\mathrm{b})),\min(\ddot{f}^{n}_{\mathcal{R}}(a),\ddot{f}^{n}_{\mathcal{R}}(\mathrm{b}))\}\}$$

$$\geq \min\{\{\min(\ddot{f}^{n}_{\mathcal{L}}(a),\ddot{f}^{n}_{\mathcal{R}}(a)),\min(\ddot{f}^{n}_{\mathcal{L}}(\mathrm{b}),\ddot{f}^{n}_{\mathcal{R}}(\mathrm{b}))\}\}$$

$$\geq \min(\ddot{f}^{n}_{\mathcal{L}\cup\mathcal{R}}(a),\ \ddot{f}^{n}_{\mathcal{L}\cup\mathcal{R}}(\mathrm{b}))$$

2) $$\ddot{t}^{n}_{\mathcal{L}\cup\mathcal{R}}(\mathrm{b}+a-\mathrm{b}) = \max\{\ddot{t}^{n}_{\mathcal{L}}(\mathrm{b}+a-\mathrm{b}),\ \ddot{t}^{n}_{\mathcal{R}}(\mathrm{b}+a-\mathrm{b})\}$$

$$\leq \max\{\ddot{t}^{n}_{\mathcal{L}}(a),\ddot{t}^{n}_{\mathcal{R}}(a)\}$$

$$\leq \ddot{t}^{n}_{\mathcal{L}\cup\mathcal{R}}(a)$$

$$\ddot{i}^{n}_{\mathcal{L}\cup\mathcal{R}}(\mathrm{b}+a-\mathrm{b}) = \min\{\ddot{i}^{n}_{\mathcal{L}}(\mathrm{b}+a-\mathrm{b}),\ \ddot{i}^{n}_{\mathcal{R}}(\mathrm{b}+a-\mathrm{b})\}$$

$$\geq \min\{\ddot{i}^{n}_{\mathcal{L}}(a),\ddot{i}^{n}_{\mathcal{R}}(a)\}$$

$$\geq \ddot{i}^{n}_{\mathcal{L}\cup\mathcal{R}}(a)$$

$$\ddot{f}^{n}_{\mathcal{L}\cup\mathcal{R}}(\mathrm{b}+a-\mathrm{b}) = \min\{\ddot{f}^{n}_{\mathcal{L}}(\mathrm{b}+a-\mathrm{b}),\ \ddot{f}^{n}_{\mathcal{R}}(\mathrm{b}+a-\mathrm{b})\}$$

$\geq \min\{\ddot{f}^n_{\mathcal{L}}(a), \ddot{f}^n_{\mathcal{R}}(a)\}$

$\geq \ddot{f}^n_{\mathcal{L}\cup\mathcal{R}}(a)$

3. $\ddot{t}^n_{\mathcal{L}\cup\mathcal{R}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big)$

$= \max\{\ddot{t}^n_{\mathcal{L}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big), \ddot{t}^n_{\mathcal{R}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big)\}$

$\leq \max\{\ddot{t}^n_{\mathcal{L}}(a), \ddot{t}^n_{\mathcal{R}}(a)\}$

$\leq \ddot{t}^n_{\mathcal{L}\cup\mathcal{R}}(a)$

$\ddot{\imath}^n_{\mathcal{L}\cup\mathcal{R}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big)$

$= \min\{\ddot{\imath}^n_{\mathcal{L}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big), \ddot{\imath}^n_{\mathcal{R}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big)\}$

$\geq \min\{\ddot{\imath}^n_{\mathcal{L}}(a), \ddot{\imath}^n_{\mathcal{R}}(a)\}$

$\geq \ddot{\imath}^n_{\mathcal{L}\cup\mathcal{R}}(a)$

$\ddot{f}^n_{\mathcal{L}\cup\mathcal{R}}\big((\theta_1\tau(\text{ӿ}+\theta_2)-\theta_1\tau\theta_2\big)$

$= \min\{\ddot{f}^n_{\mathcal{L}}\big((\theta_1\tau(\text{ӿ}+\theta_2)-\theta_1\tau\theta_2\big), \ddot{f}^n_{\mathcal{R}}\big((\theta_1\tau(\text{ӿ}+\theta_2)-\theta_1\tau\theta_2\big)\}$

$\geq \min\{\ddot{f}^n_{\mathcal{L}}(a), \ddot{f}^n_{\mathcal{R}}(a)\}$

$\geq \ddot{f}^n_{\mathcal{L}\cup\mathcal{R}}(a)$

$\therefore$ $\mathcal{L}\cup\mathcal{R}$ is a anti neutrosophic multi fuzzy left ideal of $M$.

**3.2 Theorem**:

Let $\mathcal{L}$and$\mathcal{R}$ anti neutrosophic multi fuzzy right ideal of $M$then $\mathcal{L}\cup\mathcal{R}$ is a anti neutrosophic multi fuzzy right ideal of $M$.

**Proof:**

Let $\mathcal{L}$and$\mathcal{R}$ neutrosophic multi fuzzy right ideal of $M$.

Let $a$ , b, $\rho_1, \rho_2 \in M$, $\tau \in \gamma$

1. $\ddot{t}^n_{\mathcal{L}\cup\mathcal{R}}(a-\text{b})= \max\{\ddot{t}^n_{\mathcal{L}}(a-\text{b}) , \ddot{t}^n_{\mathcal{R}}(a-\text{b})\}$

$\leq \max\{\{\max(\ddot{t}^n_{\mathcal{L}}(a),\ddot{t}^n_{\mathcal{L}}(\text{b})) ,\max( \ddot{t}^n_{\mathcal{R}}(a),\ddot{t}^n_{\mathcal{R}}(\text{b}))\}$

$\leq \max\{\{ \max (\ddot{t}^n_{\mathcal{L}}(a),\ddot{t}^n_{\mathcal{R}}(a)) , \max ( \ddot{t}^n_{\mathcal{L}}(\text{b}),\ddot{t}^n_{\mathcal{R}}(\text{b}))\}$

$\leq \max(\ddot{t}^n_{\mathcal{L}\cup\mathcal{R}}(a) , \ddot{t}^n_{\mathcal{L}\cup\mathcal{R}}(\text{b}))$

$\ddot{i}^n_{\mathcal{L}\cup\mathcal{R}}(a-\text{b}) = \min\{\ddot{i}^n_{\mathcal{L}}(a-\text{b}) , \ddot{i}^n_{\mathcal{R}}(a-\text{b})\}$

$\geq \min\{\{ \min (\ddot{i}^n_{\mathcal{L}}(a),\ddot{i}^n_{\mathcal{L}}(\text{b})) , \min ( \ddot{i}^n_{\mathcal{R}}(a),\ddot{i}^n_{\mathcal{R}}(\text{b}))\}$

$\geq \min \{\{ \min (\ddot{i}^n_{\mathcal{L}}(a),\ddot{i}^n_{\mathcal{R}}(a)) , \min( \ddot{i}^n_{\mathcal{L}}(\text{b}),\ddot{i}^n_{\mathcal{R}}(\text{b}))\}$

$\geq \min (\ddot{i}^n_{\mathcal{L}\cup\mathcal{R}}(a) , \ddot{i}^n_{\mathcal{L}\cup\mathcal{R}}(\text{b}))$

$\ddot{f}^n_{\mathcal{L}\cup\mathcal{R}}(a-\text{b})= \min\{\ddot{f}^n_{\mathcal{L}}(a-\text{b}) , \ddot{f}^n_{\mathcal{R}}(a-\text{b})\}$

$\geq \min\{\{ \min (\ddot{f}^n_{\mathcal{L}}(a),\ddot{f}^n_{\mathcal{L}}(\text{b})) , \min ( \ddot{f}^n_{\mathcal{R}}(a),\ddot{f}^n_{\mathcal{R}}(\text{b}))\}$

$\geq \min \{\{ \min (\ddot{f}^n_{\mathcal{L}}(a),\ddot{f}^n_{\mathcal{R}}(a)) , \min ( \ddot{f}^n_{\mathcal{L}}(\text{b}),\ddot{f}^n_{\mathcal{R}}(\text{b}))\}$

$\geq \min (\ddot{f}^n_{\mathcal{L}\cup\mathcal{R}}(a) , \ddot{f}^n_{\mathcal{L}\cup\mathcal{R}}(\text{b}))$

2. $\ddot{t}^n_{\mathcal{L}\cup\mathcal{R}}(\text{b}+a-\text{b}) = \max\{\ddot{t}^n_{\mathcal{L}}(\text{b}+a-\text{b}) , \ddot{t}^n_{\mathcal{R}}(\text{b}+a-\text{b})\}$

$\leq \max\{\ddot{t}^n_{\mathcal{L}}(a),\ddot{t}^n_{\mathcal{R}}(a)\}$

$\leq \ddot{t}^n_{\mathcal{L}\cup\mathcal{R}}(a)$

$\ddot{i}^n_{\mathcal{L}\cup\mathcal{R}}(\text{b}+a-\text{b}) = \min\{\ddot{i}^n_{\mathcal{L}}(\text{b}+a-\text{b}) , \ddot{i}^n_{\mathcal{R}}(\text{b}+a-\text{b})\}$

$\geq \min\{\ddot{i}^n_{\mathcal{L}}(a),\ddot{i}^n_{\mathcal{R}}(a)\}$

$\geq \ddot{i}^n_{\mathcal{L}\cup\mathcal{R}}(a)$

$$\ddot{f}^{n}_{\mathcal{L}\cup\mathcal{R}}(\mathrm{b}+a-\mathrm{b}) = \min\{\ddot{f}^{n}_{\mathcal{L}}(\mathrm{b}+a-\mathrm{b}),\ \ddot{f}^{n}_{\mathcal{R}}(\mathrm{b}+a-\mathrm{b})\}$$

$$\geq \min\{\ddot{f}^{n}_{\mathcal{L}}(a), \ddot{f}^{n}_{\mathcal{R}}(a)\}$$

$$\geq \ddot{f}^{n}_{\mathcal{L}\cup\mathcal{R}}(a)$$

3. $\ddot{t}^{n}_{\mathcal{L}\cup\mathcal{R}}(a\ \tau\rho_1) = \max\{\ddot{t}^{n}_{\mathcal{L}}(a\ \tau\rho_1),\ \ddot{t}^{n}_{\mathcal{R}}(a\ \tau\rho_1)\}$

$$\leq \max\{\ddot{t}^{n}_{\mathcal{L}}(a), \ddot{t}^{n}_{\mathcal{R}}(a)\}$$

$$\leq \ddot{t}^{n}_{\mathcal{L}\cup\mathcal{R}}(a)$$

$$\ddot{i}^{n}_{\mathcal{L}\cup\mathcal{R}}(a\ \tau\rho_1) = \min\{\ddot{i}^{n}_{\mathcal{L}}(\ddot{x}\ \tau\rho_1),\ \ddot{i}^{n}_{\mathcal{R}}(\ddot{x}\ \tau\rho_1)\}$$

$$\geq \min\{\ddot{i}^{n}_{\mathcal{L}}(a), \ddot{i}^{n}_{\mathcal{R}}(a)\}$$

$$\geq \ddot{i}^{n}_{\mathcal{L}\cup\mathcal{R}}(a)$$

$$\ddot{f}^{n}_{\mathcal{L}\cup\mathcal{R}}(a\ \tau\rho_1) = \min\{\ddot{f}^{n}_{\mathcal{L}}(a\ \tau\rho_1),\ \ddot{f}^{n}_{\mathcal{R}}(a\ \tau\rho_1)\}$$

$$\geq \min\{\ddot{f}^{n}_{\mathcal{L}}(a), \ddot{f}^{n}_{\mathcal{R}}(a)\}$$

$$\geq \ddot{f}^{n}_{\mathcal{L}\cup\mathcal{R}}(a)$$

$\therefore$ $\mathcal{L}\cup\mathcal{R}$ is a anti neutrosophic multi fuzzy right ideal of $M$.

**3.3 Theorem:**

Let $\mathcal{L}$ and $\mathcal{R}$ anti neutrosophic multi fuzzy ideal of $M$ then $\mathcal{L}\cup\mathcal{R}$ is a anti neutrosophic multi fuzzy ideal of $M$.

Proof: It is clear.

**3.4 Theorem**:

Let $\mathcal{L}$ and $\mathcal{R}$ anti neutrosophic multi fuzzy left ideal of $M$ and then $\mathcal{L}\cap\mathcal{R}$ is a anti neutrosophic multi fuzzy left ideal of $M$.

**Proof:**

Let $\mathcal{L}$and$\mathcal{R}$ neutrosophic multi fuzzy left ideal of $\xi$.

Let$a$ , b,$\rho_1, \rho_2 \in M$, $\tau \in \gamma$

1. $\ddot{t}^{n}_{\mathcal{L}\cap\mathcal{R}}(a-\mathrm{b})$ = min{$\ddot{t}^{n}_{\mathcal{L}}(a-\mathrm{b})$ , $\ddot{t}^{n}_{\mathcal{R}}(a-\mathrm{b})$}

$\leq$ min{{max($\ddot{t}^{n}_{\mathcal{L}}(a)$,$\ddot{t}^{n}_{\mathcal{L}}(\mathrm{b})$) ,max( $\ddot{t}^{n}_{\mathcal{R}}(a)$,$\ddot{t}^{n}_{\mathcal{R}}(\mathrm{b})$)}}

$\leq$max{{ min ($\ddot{t}^{n}_{\mathcal{L}}(a)$,$\ddot{t}^{n}_{\mathcal{R}}(a)$) , min ( $\ddot{t}^{n}_{\mathcal{L}}(\mathrm{b})$,$\ddot{t}^{n}_{\mathcal{R}}(\mathrm{b})$)}}

$\leq$max($\ddot{t}^{n}_{\mathcal{L}\cap\mathcal{R}}(a)$ , $\ddot{t}^{n}_{\mathcal{L}\cap\mathcal{R}}(\mathrm{b})$)

$\ddot{i}^{n}_{\mathcal{L}\cap\mathcal{R}}(a-\mathrm{b}) =$ max{$\ddot{i}^{n}_{\mathcal{L}}(a-\mathrm{b})$ , $\ddot{i}^{n}_{\mathcal{R}}(a-\mathrm{b})$}

$\geq$ max{{ min ($\ddot{i}^{n}_{\mathcal{L}}(a)$,$\ddot{i}^{n}_{\mathcal{L}}(\mathrm{b})$) , min ( $\ddot{i}^{n}_{\mathcal{R}}(a)$,$\ddot{i}^{n}_{\mathcal{R}}(\mathrm{b})$)}}

$\geq$ min {{ max ($\ddot{i}^{n}_{\mathcal{L}}(a)$,$\ddot{i}^{n}_{\mathcal{R}}(a)$) , max( $\ddot{i}^{n}_{\mathcal{L}}(\mathrm{b})$,$\ddot{i}^{n}_{\mathcal{R}}(\mathrm{b})$)}}

$\geq$ min ($\ddot{i}^{n}_{\mathcal{L}\cap\mathcal{R}}(a)$ , $\ddot{i}^{n}_{\mathcal{L}\cap\mathcal{R}}(\mathrm{b})$)

$\ddot{f}^{n}_{\mathcal{L}\cap\mathcal{R}}(a-\mathrm{b})$ = max{$\ddot{f}^{n}_{\mathcal{L}}(a-\mathrm{b})$ , $\ddot{f}^{n}_{\mathcal{R}}(a-\mathrm{b})$}

$\geq$ max{{ min ($\ddot{f}^{n}_{\mathcal{L}}(a)$,$\ddot{f}^{n}_{\mathcal{L}}(\mathrm{b})$) , min ( $\ddot{f}^{n}_{\mathcal{R}}(a)$,$\ddot{f}^{n}_{\mathcal{R}}(\mathrm{b})$)}}

$\geq$ min {{ max ($\ddot{f}^{n}_{\mathcal{L}}(a)$,$\ddot{f}^{n}_{\mathcal{R}}(a)$) , max ( $\ddot{f}^{n}_{\mathcal{L}}(\mathrm{b})$,$\ddot{f}^{n}_{\mathcal{R}}(\mathrm{b})$)}}

$\geq$ min ($\ddot{f}^{n}_{\mathcal{L}\cap\mathcal{R}}(a)$ , $\ddot{f}^{n}_{\mathcal{L}\cap\mathcal{R}}(\mathrm{b})$)

2. $\ddot{t}^{n}_{\mathcal{L}\cap\mathcal{R}}(\mathrm{b}+a-\mathrm{b})$ = min{$\ddot{t}^{n}_{\mathcal{L}}(\mathrm{b}+a-\mathrm{b})$ , $\ddot{t}^{n}_{\mathcal{R}}(\mathrm{b}+a-\mathrm{b})$}

$\leq$ min{$\ddot{t}^{n}_{\mathcal{L}}(a)$,$\ddot{t}^{n}_{\mathcal{R}}(a)$}

$\leq \ddot{t}^{n}_{\mathcal{L}\cap\mathcal{R}}(a)$

$\ddot{i}^{n}_{\mathcal{L}\cap\mathcal{R}}(\mathrm{b}+a-\mathrm{b})$ = max{$\ddot{i}^{n}_{\mathcal{L}}(\mathrm{b}+a-\mathrm{b})$ , $\ddot{i}^{n}_{\mathcal{R}}(\mathrm{b}+a-\mathrm{b})$}

$\geq$ max{$\ddot{i}^{n}_{\mathcal{L}}(a)$,$\ddot{i}^{n}_{\mathcal{R}}(a)$}

$$\geq \ddot{i}^{n}_{\mathcal{L}\cap\mathcal{R}}(a)$$

$$\ddot{f}^{n}_{\mathcal{L}\cap\mathcal{R}}(\mathrm{b}+a-\mathrm{b}) = \max\{\ddot{f}^{n}_{\mathcal{L}}(\mathrm{b}+a-\mathrm{b}) , \ddot{f}^{n}_{\mathcal{R}}(\mathrm{b}+a-\mathrm{b})\}$$

$$\geq \max\{\ddot{f}^{n}_{\mathcal{L}}(a),\ddot{f}^{n}_{\mathcal{R}}(a)\}$$

$$\geq \ddot{f}^{n}_{\mathcal{L}\cap\mathcal{R}}(a)$$

3. $\ddot{t}^{n}_{\mathcal{L}\cap\mathcal{R}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big)$

$$= \min\{\ddot{t}^{n}_{\mathcal{L}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big), \ddot{t}^{n}_{\mathcal{R}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big)\}$$

$$\leq \min\{\ddot{t}^{n}_{\mathcal{L}}(a),\ddot{t}^{n}_{\mathcal{R}}(a)\}$$

$$\leq \ddot{t}^{n}_{\mathcal{L}\cap\mathcal{R}}(a)$$

$$\ddot{i}^{n}_{\mathcal{L}\cap\mathcal{R}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big)$$

$$= \max\{\ddot{i}^{n}_{\mathcal{L}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big) , \ddot{i}^{n}_{\mathcal{R}}\big((\rho_1\tau(a+\rho_2)-\rho_1\tau\rho_2\big)\}$$

$$\geq \max\{\ddot{i}^{n}_{\mathcal{L}}(a),\ddot{i}^{n}_{\mathcal{R}}(a)\}$$

$$\geq \ddot{i}^{n}_{\mathcal{L}\cap\mathcal{R}}(a)$$

$$\ddot{f}^{n}_{\mathcal{L}\cap\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)$$

$$= \max\{\ddot{f}^{n}_{\mathcal{L}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big) , \ddot{f}^{n}_{\mathcal{R}}\big((\theta_1\tau(\mathfrak{x}+\theta_2)-\theta_1\tau\theta_2\big)\}$$

$$\geq \max\{\ddot{f}^{n}_{\mathcal{L}}(a),\ddot{f}^{n}_{\mathcal{R}}(a)\}$$

$$\geq \ddot{f}^{n}_{\mathcal{L}\cap\mathcal{R}}(a)$$

$\therefore$ $\mathcal{L}\cap\mathcal{R}$is a anti neutrosophic multi fuzzy left ideal of $M$.

**3.5 Theorem:**

Let $\mathcal{L}$ and $\mathcal{R}$ anti neutrosophic multi fuzzy right ideal of $M$then $\mathcal{L}\cap\mathcal{R}$is a anti neutrosophic multi fuzzy right ideal of $M$.

**Proof:**

Let $\mathcal{L}$ and $\mathcal{R}$ neutrosophic multi fuzzy right ideal of $M$.

Let Let$a$ , b,$\rho_1, \rho_2 \in M$, $\tau \in \gamma$

1. $\ddot{t}^n_{\mathcal{L}\cap\mathcal{R}}(a-\text{b})$= min{$\ddot{t}^n_{\mathcal{L}}(a-\text{b})$ , $\ddot{t}^n_{\mathcal{R}}(a-\text{b})$}

$\leq$ min{{max($\ddot{t}^n_{\mathcal{L}}(a)$,$\ddot{t}^n_{\mathcal{L}}(\text{b})$) ,max( $\ddot{t}^n_{\mathcal{R}}(a)$,$\ddot{t}^n_{\mathcal{R}}(\text{b})$)}}

$\leq$ max{{ min ($\ddot{t}^n_{\mathcal{L}}(a)$,$\ddot{t}^n_{\mathcal{R}}(a)$) , min ( $\ddot{t}^n_{\mathcal{L}}(\text{b})$,$\ddot{t}^n_{\mathcal{R}}(\text{b})$)}}

$\leq$ max($\ddot{t}^n_{\mathcal{L}\cap\mathcal{R}}(a)$ , $\ddot{t}^n_{\mathcal{L}\cap\mathcal{R}}(\text{b})$)

$\ddot{i}^n_{\mathcal{L}\cap\mathcal{R}}(a-\text{b})$ = max{$\ddot{i}^n_{\mathcal{L}}(a-\text{b})$ , $\ddot{i}^n_{\mathcal{R}}(a-\text{b})$}

$\geq$ max{{ min ($\ddot{i}^n_{\mathcal{L}}(a)$,$\ddot{i}^n_{\mathcal{L}}(\text{b})$) , min ( $\ddot{i}^n_{\mathcal{R}}(a)$,$\ddot{i}^n_{\mathcal{R}}(\text{b})$)}}

$\geq$ min {{ max ($\ddot{i}^n_{\mathcal{L}}(a)$,$\ddot{i}^n_{\mathcal{R}}(a)$) , max( $\ddot{i}^n_{\mathcal{L}}(\text{b})$,$\ddot{i}^n_{\mathcal{R}}(\text{b})$)}}

$\geq$ min ($\ddot{i}^n_{\mathcal{L}\cap\mathcal{R}}(a)$ , $\ddot{i}^n_{\mathcal{L}\cap\mathcal{R}}(\text{b})$)

$\ddot{f}^n_{\mathcal{L}\cap\mathcal{R}}(a-\text{b})$ = max{$\ddot{f}^n_{\mathcal{L}}(a-\text{b})$ , $\ddot{f}^n_{\mathcal{R}}(a-\text{b})$}

$\geq$ max{{ min ($\ddot{f}^n_{\mathcal{L}}(a)$,$\ddot{f}^n_{\mathcal{L}}(\text{b})$) , min ( $\ddot{f}^n_{\mathcal{R}}(a)$,$\ddot{f}^n_{\mathcal{R}}(\text{b})$)}}

$\geq$ min {{ max ($\ddot{f}^n_{\mathcal{L}}(a)$,$\ddot{f}^n_{\mathcal{R}}(a)$) , max ( $\ddot{f}^n_{\mathcal{L}}(\text{b})$,$\ddot{f}^n_{\mathcal{R}}(\text{b})$)}}

$\geq$ min ($\ddot{f}^n_{\mathcal{L}\cap\mathcal{R}}(a)$ , $\ddot{f}^n_{\mathcal{L}\cap\mathcal{R}}(\text{b})$)

2. $\ddot{t}^n_{\mathcal{L}\cap\mathcal{R}}(\text{b}+a-\text{b})$ = min{$\ddot{t}^n_{\mathcal{L}}(\text{b}+a-\text{b})$ , $\ddot{t}^n_{\mathcal{R}}(\text{b}+a-\text{b})$}

$\leq$ min{$\ddot{t}^n_{\mathcal{L}}(a)$,$\ddot{t}^n_{\mathcal{R}}(a)$}

$\leq \ddot{t}^n_{\mathcal{L}\cap\mathcal{R}}(a)$

$$\ddot{i}^n_{\mathcal{L}\cap\mathcal{R}}(b+a-b) = \max\{\ddot{i}^n_{\mathcal{L}}(b+a-b)\ ,\ \ddot{i}^n_{\mathcal{R}}(b+a-b)\}$$

$$\geq \max\{\ddot{i}^n_{\mathcal{L}}(a), \ddot{i}^n_{\mathcal{R}}(a)\}$$

$$\geq \ddot{i}^n_{\mathcal{L}\cap\mathcal{R}}(a)$$

$$\ddot{f}^n_{\mathcal{L}\cap\mathcal{R}}(b+a-b) = \max\{\ddot{f}^n_{\mathcal{L}}(b+a-b)\ ,\ \ddot{f}^n_{\mathcal{R}}(b+a-b)\}$$

$$\geq \max\{\ddot{f}^n_{\mathcal{L}}(a), \ddot{f}^n_{\mathcal{R}}(a)\}$$

$$\geq \ddot{f}^n_{\mathcal{L}\cap\mathcal{R}}(a)$$

3. $$\ddot{t}^n_{\mathcal{L}\cap\mathcal{R}}(a\,\tau\rho_1) = \min\{\ddot{t}^n_{\mathcal{L}}(a\,\tau\rho_1)\ ,\ \ddot{t}^n_{\mathcal{R}}(a\,\tau\rho_1)\}$$

$$\leq \min\{\ddot{t}^n_{\mathcal{L}}(a), \ddot{t}^n_{\mathcal{R}}(a)\}$$

$$\leq \ddot{t}^n_{\mathcal{L}\cap\mathcal{R}}(a)$$

$$\ddot{i}^n_{\mathcal{L}\cap\mathcal{R}}(a\,\tau\rho_1) = \max\{\ddot{i}^n_{\mathcal{L}}(a\tau\rho_1)\ ,\ \ddot{i}^n_{\mathcal{R}}(a\,\tau\rho_1)\}$$

$$\geq \max\{\ddot{i}^n_{\mathcal{L}}(a), \ddot{i}^n_{\mathcal{R}}(a)\}$$

$$\geq \ddot{i}^n_{\mathcal{L}\cap\mathcal{R}}(a)$$

$$\ddot{f}^n_{\mathcal{L}\cap\mathcal{R}}(a\,\tau\rho_1) = \max\{\ddot{f}^n_{\mathcal{L}}(a\,\tau\rho_1)\ ,\ \ddot{f}^n_{\mathcal{R}}(a\,\tau\rho_1)\}$$

$$\geq \max\{\ddot{f}^n_{\mathcal{L}}(a), \ddot{f}^n_{\mathcal{R}}(a)\}$$

$$\geq \ddot{f}^n_{\mathcal{L}\cap\mathcal{R}}(a)$$

$\therefore$ $\mathcal{L}\cap\mathcal{R}$ is a anti neutrosophic multi fuzzy right ideal of $M$.

**3.6 Theorem:**

Let $\mathcal{L}$ and $\mathcal{R}$ anti neutrosophic multi fuzzy ideal of $M$ then $\mathcal{L}\cap\mathcal{R}$ is also a anti neutrosophic multi fuzzy ideal of $M$.

**Proof**: It is clear.

## 4. Anti Product of anti neutrosophic multi fuzzy ideals

In this section we define anti product of anti neutrosophic multi fuzzy $\gamma$ near ring M. We proved that anti product of anti neutrosophic multi fuzzy ideals of M is a anti neutrosophic multi fuzzy ideal of M.

**4.1 Definition:**

Let $\mathcal{L}$ and $\mathcal{R}$ are two anti neutrosophic multi fuzzy ideals of $\gamma$ near rings $M_1$ and $M_2$ resp. Then the anti product of anti neutrosophic multi fuzzy subset of $\gamma$ near ring is defined by

$\mathcal{L}x\mathcal{R}: M_1 x M_2 \rightarrow [0,1]$ such that

$\mathcal{L}x\mathcal{R} = \{< (a,b), \ddot{t}^n_{\mathcal{L}x\mathcal{R}}(a,b), \ddot{i}^n_{\mathcal{L}x\mathcal{R}}(a,b), \ddot{f}^n_{\mathcal{L}x\mathcal{R}}(a,b) >: a \in M_1, b \in M_2\}$

$Where\ \ddot{t}^n_{\mathcal{L}x\mathcal{R}}(a,b) = \max\{\ddot{t}^n_{\mathcal{L}}(a), \ddot{t}^n_{\mathcal{R}}(b)\}$

$$\ddot{i}^n_{\mathcal{L}x\mathcal{R}}(a,b) = \min\{\ddot{i}^n_{\mathcal{L}}(a), \ddot{i}^n_{\mathcal{R}}(b)\}$$

$$\ddot{f}^n_{\mathcal{L}x\mathcal{R}}(a,b) = \min\{\ddot{f}^n_{\mathcal{L}}(a), \ddot{f}^n_{\mathcal{R}}(b)\}$$

**4.2 Theorem:**

Let $\mathcal{L}$ and $\mathcal{R}$ anti neutrosophic multi fuzzy left ideal of $\gamma$ near rings $M_1$ and $M_2$ then $\mathcal{L}x\mathcal{R}$ is also a anti neutrosophic multi fuzzy left ideal of $M_1 x M_2$.

Proof:

Let $\mathcal{L}$ and $\mathcal{R}$ be anti neutrosophic fuzzy left ideals of $M_1 x M_2$ respectively

Let $(a_1, a_2), (b_1, b_2), (\rho_1, \rho_2) \in M_1 x M_2$

1. $\ddot{t}^n_{\mathcal{L}x\mathcal{R}}((a_1, a_2) - (b_1, b_2)) = \ddot{t}^n_{\mathcal{L}x\mathcal{R}}(a_1 - b_1, a_2 - b_2)$

$$= \max(\ddot{t}^n_{\mathcal{L}}(a_1 - b_1), \ddot{t}^n_{\mathcal{R}}(a_2 - b_2))$$

$\leq \max \{ \max (\ddot{t}_{\mathcal{L}}^{n} (a_1), \ddot{t}_{\mathcal{L}}^{n} (b_1))], \max [\ddot{t}_{\mathcal{R}}^{n} (a_2), \ddot{t}_{\mathcal{R}}^{n} (b_2))\}$

$\leq \max \{ \max [\ddot{t}_{\mathcal{L}}^{n} (a_1), \ddot{t}_{\mathcal{R}}^{n} (a_2)], \max [\ddot{t}_{\mathcal{L}}^{n} (b_1), \ddot{t}_{\mathcal{R}}^{n} (b_2))\}$

$\leq \max (\ddot{t}_{\mathcal{L}x\mathcal{R}}^{n} (a_1, a_2), \ddot{t}_{\mathcal{L}x\mathcal{R}}^{n} (b_1, b_2))$

$\ddot{i}_{\mathcal{L}x\mathcal{R}}^{n} ((a_1, a_2) - (b_1, b_2)) = \ddot{i}_{\mathcal{L}x\mathcal{R}}^{n} (a_1 - b_1, a_2 - b_2)$

$= \min (\ddot{i}_{\mathcal{L}}^{n} (a_1 - b_1), \ddot{i}_{\mathcal{R}}^{n} (a_2 - b_2))$

$\geq \min \{ \min (\ddot{i}_{\mathcal{L}}^{n} (a_1), \ddot{i}_{\mathcal{L}}^{n} (b_1))], \min [\ddot{i}_{\mathcal{R}}^{n} (a_2), \ddot{i}_{\mathcal{R}}^{n} (b_2))\}$

$\geq \min \{ \min [\ddot{i}_{\mathcal{L}}^{n} (a_1), \ddot{i}_{\mathcal{R}}^{n} (a_2)], \min [\ddot{i}_{\mathcal{L}}^{n} (b_1), \ddot{i}_{\mathcal{R}}^{n} (b_2))\}$

$\geq \min (\ddot{i}_{\mathcal{L}x\mathcal{R}}^{n} (a_1, a_2), \ddot{i}_{\mathcal{L}x\mathcal{R}}^{n} (b_1, b_2))$

$\ddot{f}_{\mathcal{L}x\mathcal{R}}^{n} ((a_1, a_2) - (b_1, b_2)) = \ddot{f}_{\mathcal{L}x\mathcal{R}}^{n} (a_1 - b_1, a_2 - b_2)$

$= \min (\ddot{f}_{\mathcal{L}}^{n} (a_1 - b_1), \ddot{f}_{\mathcal{R}}^{n} (a_2 - b_2))$

$\geq \min \{ \min (\ddot{f}_{\mathcal{L}}^{n} (a_1), \ddot{f}_{\mathcal{L}}^{n} (b_1))], \min [\ddot{f}_{\mathcal{R}}^{n} (a_2), \ddot{f}_{\mathcal{R}}^{n} (b_2))\}$

$\geq \min \{ \min [\ddot{f}_{\mathcal{L}}^{n} (a_1), \ddot{f}_{\mathcal{R}}^{n} (a_2)], \min [\ddot{f}_{\mathcal{L}}^{n} (b_1), \ddot{f}_{\mathcal{R}}^{n} (b_2))\}$

$\geq \min (\ddot{f}_{\mathcal{L}x\mathcal{R}}^{n} (a_1, a_2), \ddot{f}_{\mathcal{L}x\mathcal{R}}^{n} (b_1, b_2))$

2. $\ddot{t}_{\mathcal{L}x\mathcal{R}}^{n}((b_1,b_2) + (a_1,a_2) - (b_1,b_2)) = \ddot{t}_{\mathcal{L}x\mathcal{R}}^{n}(b_1 + a_1 - b_1, b_2 + a_2 - b_2)$

$= \max(\ddot{t}_{\mathcal{L}}^{n}(b_1 + a_1 - b_1), \ddot{t}_{\mathcal{R}}^{n}(b_2 + a_2 - b_2))$

$\leq \max \{\ddot{t}_{\mathcal{L}}^{n}(a_1), \ddot{t}_{\mathcal{R}}^{n}(a_2)\}$

$\leq \ddot{t}_{\mathcal{L}x\mathcal{R}}^{n}(a_1, a_2)$

$\ddot{i}_{\mathcal{L}x\mathcal{R}}^{n}((b_1,b_2) + (a_1,a_2) - (b_1,b_2)) = \ddot{i}_{\mathcal{L}x\mathcal{R}}^{n}(b_1 + a_1 - b_1, b_2 + a_2 - b_2)$

$\geq \min\{\ddot{i}_{\mathcal{L}}^{n}(a_1), \ddot{i}_{\mathcal{R}}^{n}(a_2)\}$

$\geq \ddot{i}_{\mathcal{L}x\mathcal{R}}^{n}(a_1, a_2)$

$$\ddot{f}^n_{\mathcal{L}x\mathcal{R}}((b_1,b_2)+(a_1,a_2)-(b_1,b_2)) = \ddot{f}^n_{\mathcal{L}x\mathcal{R}}(b_1+a_1-b_1, b_2+a_2-b_2)$$

$$\geq \min\{\ddot{f}^n_{\mathcal{L}}(a_1), \ddot{f}^n_{\mathcal{R}}(a_2)\}$$

$$\geq \ddot{f}^n_{\mathcal{L}x\mathcal{R}}(a_1,a_2)$$

3. $\ddot{t}^n_{\mathcal{L}x\mathcal{R}}\big((\rho_1\tau((a_1,a_2)+\rho_2)-\rho_1\tau\rho_2\big)$

$$=\ddot{t}^n_{\mathcal{L}x\mathcal{R}}(\rho_1\tau(a_1+\rho_2)-\rho_1\tau\rho_2, \rho_1\tau(a_2+\rho_2)-\rho_1\tau\rho_2)$$

$$= \max\{\ddot{t}^n_{\mathcal{L}}\big((\rho_1\tau(a_1+\rho_2)-\rho_1\tau\rho_2\big), \ddot{t}^n_{\mathcal{R}}\big((\rho_1\tau(a_2+\rho_2)-\rho_1\tau\rho_2\big)\}$$

$$\leq \max\{\ddot{t}^n_{\mathcal{L}}(a_1), \ddot{t}^n_{\mathcal{R}}(a_2)\}$$

$$\leq \ddot{t}^n_{\mathcal{L}\,x\mathcal{R}}(a_1,a_2)$$

$\ddot{i}^n_{\mathcal{L}\,x\mathcal{R}}\big((\rho_1\tau(a_1,a_2+\rho_2)-\rho_1\tau\rho_2\big)$

$$=\ddot{i}^n_{\mathcal{L}x\mathcal{R}}(\rho_1\tau(a_1+\rho_2)-\rho_1\tau\rho_2, \rho_1\tau(a_2+\rho_2)-\rho_1\tau\rho_2)$$

$$= \min\{\ddot{i}^n_{\mathcal{L}}\big((\rho_1\tau(a_1+\rho_2)-\rho_1\tau\rho_2\big), \ddot{i}^n_{\mathcal{R}}\big((\rho_1\tau(a_2+\rho_2)-\rho_1\tau\rho_2\big)\}$$

$$\geq \min\{\ddot{i}^n_{\mathcal{L}}(a_1), \ddot{i}^n_{\mathcal{R}}(a_2)\}$$

$$\geq \ddot{i}^n_{\mathcal{L}x\mathcal{R}}(a_1,a_2)$$

$\ddot{f}^n_{\mathcal{L}\,x\mathcal{R}}\big((\rho_1\tau(a_1,a_2+\rho_2)-\rho_1\tau\rho_2\big)$

$$=\ddot{f}^n_{\mathcal{L}x\mathcal{R}}(\rho_1\tau(a_1+\rho_2)-\rho_1\tau\rho_2, \rho_1\tau(a_2+\rho_2)-\rho_1\tau\rho_2)$$

$$= \min\{\ddot{f}^n_{\mathcal{L}}\big((\rho_1\tau(a_1+\rho_2)-\rho_1\tau\rho_2\big), \ddot{f}^n_{\mathcal{R}}\big((\rho_1\tau(a_2+\rho_2)-\rho_1\tau\rho_2\big)\}$$

$$\geq \min\{\ddot{f}^n_{\mathcal{L}}(a_1), \ddot{f}^n_{\mathcal{R}}(a_2)\}$$

$$\geq \ddot{f}^n_{\mathcal{L}x\mathcal{R}}(a_1,a_2)$$

$\therefore\ \mathcal{L}x\mathcal{R}$ is also a anti neutrosophic multi fuzzy left ideal of $M_1xM_2$.

**4.3 Theorem:**

Let $\mathcal{L}$ and $\mathcal{R}$ anti neutrosophic multi fuzzy right ideal of $\gamma$ near rings $M_1$ and $M_2$then $\mathcal{L}x\mathcal{R}$is also a anti neutrosophic multi fuzzy right ideal of $M_1xM_2$.

Proof:

Let $\mathcal{L}$ and $\mathcal{R}$ be anti neutrosophic fuzzy right ideals of $M_1xM_2$ respectively

Let $(a_1, a_2),(b_1, b_2),(\rho_1, \rho_2) \in M_1xM_2$

1. $\ddot{t}^n_{\mathcal{L}x\mathcal{R}}((a_1, a_2)-(b_1, b_2)) = \ddot{t}^n_{\mathcal{L}x\mathcal{R}}(a_1 - b_1, a_2 - b_2)$

$= \max(\ddot{t}^n_{\mathcal{L}}(a_1 - b_1), \ddot{t}^n_{\mathcal{R}}(a_2 - b_2))$

$\leq \max\{\max(\ddot{t}^n_{\mathcal{L}}(a_1), \ddot{t}^n_{\mathcal{L}}(b_1))], \max[\ddot{t}^n_{\mathcal{R}}(a_2), \ddot{t}^n_{\mathcal{R}}(b_2))\}$

$\leq \max\{\max[\ddot{t}^n_{\mathcal{L}}(a_1), \ddot{t}^n_{\mathcal{R}}(a_2)], \max[\ddot{t}^n_{\mathcal{L}}(b_1), \ddot{t}^n_{\mathcal{R}}(b_2))\}$

$\leq \max(\ddot{t}^n_{\mathcal{L}x\mathcal{R}}(a_1, a_2), \ddot{t}^n_{\mathcal{L}x\mathcal{R}}(b_1, b_2))$

$\ddot{i}^n_{\mathcal{L}x\mathcal{R}}((a_1, a_2)-(b_1, b_2)) = \ddot{i}^n_{\mathcal{L}x\mathcal{R}}(a_1 - b_1, a_2 - b_2)$

$= \min(\ddot{i}^n_{\mathcal{L}}(a_1 - b_1), \ddot{i}^n_{\mathcal{R}}(a_2 - b_2))$

$\geq \min\{\min(\ddot{i}^n_{\mathcal{L}}(a_1), \ddot{i}^n_{\mathcal{L}}(b_1))], \min[\ddot{i}^n_{\mathcal{R}}(a_2), \ddot{i}^n_{\mathcal{R}}(b_2))\}$

$\geq \min\{\min[\ddot{i}^n_{\mathcal{L}}(a_1), \ddot{i}^n_{\mathcal{R}}(a_2)], \min[\ddot{i}^n_{\mathcal{L}}(b_1), \ddot{i}^n_{\mathcal{R}}(b_2))\}$

$\geq \min(\ddot{i}^n_{\mathcal{L}x\mathcal{R}}(a_1, a_2), \ddot{i}^n_{\mathcal{L}x\mathcal{R}}(b_1, b_2))$

$\ddot{f}^n_{\mathcal{L}x\mathcal{R}}((a_1, a_2)-(b_1, b_2)) = \ddot{f}^n_{\mathcal{L}x\mathcal{R}}(a_1 - b_1, a_2 - b_2)$

$= \min(\ddot{f}^n_{\mathcal{L}}(a_1 - b_1), \ddot{f}^n_{\mathcal{R}}(a_2 - b_2))$

$\geq \min\{\min(\ddot{f}^n_{\mathcal{L}}(a_1), \ddot{f}^n_{\mathcal{L}}(b_1))], \min[\ddot{f}^n_{\mathcal{R}}(a_2), \ddot{f}^n_{\mathcal{R}}(b_2))\}$

$\geq \min\{\min[\ddot{f}^n_{\mathcal{L}}(a_1), \ddot{f}^n_{\mathcal{R}}(a_2)], \min[\ddot{f}^n_{\mathcal{L}}(b_1), \ddot{f}^n_{\mathcal{R}}(b_2))\}$

$\geq \min(\ddot{f}^n_{\mathcal{L}x\mathcal{R}}(a_1, a_2), \ddot{f}^n_{\mathcal{L}x\mathcal{R}}(b_1, b_2))$

2. $\ddot{t}^{n}_{\mathcal{L}x\mathcal{R}}((b_1,b_2)+(a_1,a_2)-(b_1,b_2)) = \ddot{t}^{n}_{\mathcal{L}x\mathcal{R}}(b_1+a_1-b_1,b_2+a_2-b_2)$

$= \max(\ddot{t}^{n}_{\mathcal{L}}(b_1+a_1-b_1),\ddot{t}^{n}_{\mathcal{R}}(b_2+a_2-b_2))$

$\leq \max\{\ddot{t}^{n}_{\mathcal{L}}(a_1),\ddot{t}^{n}_{\mathcal{R}}(a_2)\}$

$\leq \ddot{t}^{n}_{\mathcal{L}x\mathcal{R}}(a_1,a_2)$

$\ddot{i}^{n}_{\mathcal{L}x\mathcal{R}}((b_1,b_2)+(a_1,a_2)-(b_1,b_2)) = \ddot{i}^{n}_{\mathcal{L}x\mathcal{R}}(b_1+a_1-b_1,b_2+a_2-b_2)$

$\geq \min\{\ddot{i}^{n}_{\mathcal{L}}(a_1),\ddot{i}^{n}_{\mathcal{R}}(a_2)\}$

$\geq \ddot{i}^{n}_{\mathcal{L}x\mathcal{R}}(a_1,a_2)$

$\ddot{f}^{n}_{\mathcal{L}x\mathcal{R}}((b_1,b_2)+(a_1,a_2)-(b_1,b_2)) = \ddot{f}^{n}_{\mathcal{L}x\mathcal{R}}(b_1+a_1-b_1,b_2+a_2-b_2)$

$\geq \min\{\ddot{f}^{n}_{\mathcal{L}}(a_1),\ddot{f}^{n}_{\mathcal{R}}(a_2)\}$

$\geq \ddot{f}^{n}_{\mathcal{L}x\mathcal{R}}(a_1,a_2)$

3. $\ddot{t}^{n}_{\mathcal{L}x\mathcal{R}}((a_1,a_2)\tau(\rho_1,\rho_2)) = \ddot{t}^{n}_{\mathcal{L}x\mathcal{R}}\{(a_1\tau\rho_1),(a_2\tau\rho_2)\}$

$\leq \max\{\ddot{t}^{n}_{\mathcal{L}}(a_1),\ddot{t}^{n}_{\mathcal{R}}(a_2)\}$

$\leq \ddot{t}^{n}_{\mathcal{L}x\mathcal{R}}(a_1,a_2)$

$\ddot{i}^{n}_{\mathcal{L}x\mathcal{R}}((a_1,a_2)\tau(\rho_1,\rho_2)) = \ddot{i}^{n}_{\mathcal{L}x\mathcal{R}}\{(a_1\tau\rho_1),(a_2\tau\rho_2)\}$

$\geq \min\{\ddot{i}^{n}_{\mathcal{L}}(a_1),\ddot{i}^{n}_{\mathcal{R}}(a_2)\}$

$\geq \ddot{i}^{n}_{\mathcal{L}x\mathcal{R}}(a_1,a_2)$

$\ddot{f}^{n}_{\mathcal{L}x\mathcal{R}}((a_1,a_2)\tau(\rho_1,\rho_2)) = \ddot{f}^{n}_{\mathcal{L}x\mathcal{R}}\{(a_1\tau\rho_1),(a_2\tau\rho_2)\}$

$\geq \min\{\ddot{f}^{n}_{\mathcal{L}}(a_1),\ddot{i}^{n}_{\mathcal{R}}(a_2)\}$

$\geq \ddot{f}^{n}_{\mathcal{L}x\mathcal{R}}(a_1,a_2)$

∴ $\mathcal{L}x\mathcal{R}$ is also a anti neutrosophic multi fuzzy right ideal of $M_1xM_2$.

**4.4 Theorem:**

Let $\mathcal{L}$ and $\mathcal{R}$ anti neutrosophic multi fuzzy ideal of $\gamma$ near rings $M_1$ and $M_2$then $\mathcal{L}x\mathcal{R}$is also a anti neutrosophic multi fuzzy ideal of $M_1xM_2$.

**Proof:** It is clear

**5. Conclusion**

To conclude, the notion of neutrosophic multi fuzzy gamma near-ring, neutrosophic multi fuzzy ideals of gamma near-rings have been discussed. The proof for the theorem that states "Union and Intersectionof two neutrosophic multi fuzzy ideals of gamma near-ring is also a Neutrosophic multi fuzzy ideal of gamma near-ring" has been provided.

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Received: Nov 10, 2021. Accepted: Feb 5, 2022

Neutrosophic Sets and Systems, Vol. 48, 2022
University of New Mexico

# Two-warehouse system for trapezoidal bipolar neutrosophic disparate expeditious worsen items with power demand pattern

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**Abstract**: This paper is to introduce a two-warehouse system for trapezoidal bipolar neutrosophic disparate expeditious worsen items with power demand pattern is dealt. Inventory worsens in the both storehouses disparate fixed amount. Demand is considered three different models (a) increasing demand (b) decreasing demand (c) linear demand. The model effectiveness in identifying the optimal order time that minimized overall costs is improved by the trapezoidal bipolar Neutrosophic number representation of the parameters. The Worsen in self-warehouse at earliest but in other scenario rented warehouse mostly we have more provision and potentiality provides for better growth in inventory. Finally, the model is executed using the trapezoidal bipolar neutrosophic number with numerical examples. Affectability analysis of the minimize solution of a total cost is effective to various types of the model is provide the output are furnished in detail.

**Keywords:** Inventory, Tropezoidal Bipolar Neutrosophic number, Two Ware-House, Power demand, Shortages, worsen, shortage, logarithmic demand.

## 1. INTRODUCTION

In inventory models of EOQ is consequence considered request of product is linear regrettably. This may not be attainable in a few circumstances. It'll be best to considered that the demand changes with time. Knowledge on stock demonstrate with power demand pattern is vital since it permit to germane uncover with the behaviours and evolution of the stock.

Consumers want just-finished food items, so demand for baked or ready-made goods such as cakes, cookies, candy and streamed food is increased level at the start of the scheduling period ($m > 1$). Fresh meat, fish, fruits, vegetables, yoghurts, and other foods may all experience this form of demand. Since deals are decreased when the arrange of rot approaches. Request for unused things with a solid specialized parameter is expanded at the begin of the cycle than at the end. Mobile phones, smart phones, and computers, for example, are in higher demand as they first come out on the market because of the creativity and new implementation they provide.

Other items, on other hand, have higher demand during end of the inventory period ($m< 1$). Condition arises when a commodity becomes unavailable, such as gasoline or diesel oil. Flour, coffee, gasoline, milk, water, and sugar are examples of essential household products that fall into this category. Increases in demand happen when the amount of inventory on sale starts to deplete due to daily needs. Other sources request for theatres tickets, cinemas, musical events, sporting events, and other events, it's often positive by closing accounting year, or when it's time to revel it.

Adaraniwon et al. [1] discussed the EOQ model was researched for deferred corruption and missed deals. Request rate, reliable pace of rot, and fractional overabundance pace of request amount, just as absolute stock expense per unit time, are completely expressed. A. K. Bhunia et al.[2] manages a solitary declining thing stock model with two separate stockrooms with various conservation offices. D. Nagarajan et al.[13] developed Dombi Interval Valued Neutrosophic Graph and its Role in Traffic Control Management. J. Kavikumar & D. Nagarajan [16] deal with Neutrosophic General Finite Automata. S. Broumi et al. [21] Analyzing Age Group and Time of the Day Using Interval Valued Neutrosophic Sets. Abdel-Basset et al. [18] manages a Bipolar Neutrosophic Multi Criteria Decision Making Framework for Professional Selection.

S. Agrawal et al. talked about slope type interest, with the capacity to go about as a solitary stockroom stock framework or two distribution center stock frameworks relying upon the model boundaries. Chakraborty & Sankar [7] by allowing The Pentagonal Fuzzy Number: Its Different Representations, Properties, Ranking, Defuzzification and Application in Game Problems. Chakraborty & Broumi [8] introduced Some properties of Pentagonal Neutrosophic Numbers and its Applications in Transportation Problem Environment. Ganesan et al.[15] developed by An integrated new threshold FCMs Markov chain based forecasting model for analyzing the power of stock trading trend. J. Sicilia, et al. [17] examined Stock is deterministic, differs as per time in each solicitation period, and follows a force request design. Chakraborty & Mondal by including Different linear and non-linear form of Trapezoidal Neutrosophic Numbers, De-Neutrosophication Techniques and its Application in time cost optimization technique, sequencing problem. Chakraborty Mondal, S. Broumi deal with De-Neutrosophication technique of pentagonal neutrosophic number and application in minimal spanning tree. S. Broumi & D. Nagarajan introduced by Implementation of Neutrosophic Function Memberships Using MATLAB Program.D.Nagarajan et al[31] explained the nutrosophic multiple regression. S.Broumi et al [32] presented A new distance measure for trapezoidal fuzzy neutrosophic numbers based on the centroid.

R. B. Krishnaraj et al. [20] introduced the two-boundary Weibull circulation decay deterministic stock model for power request designs without deficiencies. S. Te Jung, et al. [26] determined an EOQ model for things Weibull dispersed corruption, deficiencies and force request design.

S. Pradhan, , et al.[25] presented the impact of swelling on the force request design was explored, with two boundaries of the Weibull appropriation for crumbling being thought of, just as a confined pay approach with dynamic trade credit. S.Gomathy et al.[27] by introducing Plithogenic sets and their application in decision making. Muhammad Saqlain & Smarandache [28] show that with Octagonal Neutrosophic Number: Its Different Representations, Properties, Graphs and De-Neutrosophication. Saqlain et al. consider Linear and Non-Linear Octagonal Neutrosophic Numbers: Its Representation, $\alpha$-Cut and Applications. N. Rajeswari, et al. solved the overabundance pace of neglected interest is thought to be a diminishing outstanding capacity of holding up time in the investigation. Evaluated mean portrayal, marked distance, and centroid strategies are utilized to defuzzify the all-out cost.

## 2. ASSUMPTIONS AND NOTATIONS

(i) $D(t) = \frac{\varepsilon t^{\frac{1}{m}-1}}{mT^{\frac{1}{m}}}$ where ε is positive constant, 0 <m< 1 , T is the planning horizon.

(ii) Shortages are permitted.
(iii) The lead time is negligible.
(iv) Worsen rate of RW ($\theta$) disparate than the decay rate of OW ($\emptyset$)
(v) The inventory system deals with single item only.
(vi) The cost of holding at RW is less than the holding cost at OW ($IC_{H_0} > IC_{H_r}$).
(vii) The OW is restricted number W units, while the RW is infinite number. For business purposes, RW products are consumed first, followed by OW items.

**The accompanying documentations are utilized all through the paper:**

$\mathfrak{I}_r(t)$ : Stock volume in RW at time t , t≥0.
$\mathfrak{I}_0(t)$ : Stock volume in OW at time t, t≥0.
S(t) : Scarcity Stock volume at time t, t≥0
A : Ordering cost per order per year.
T : Cycle of length.
$t_1$ : Time of the Stock level it vanish in RW
$t_2$ : Time of the Stock level it vanish in OW
$IC_{H_0}$ : Holding cost per unit for OW $IC_{H_0} > IC_{H_r}$
$IC_{H_r}$ : Holding cost per unit for RW
$\mathcal{H}_r$ : Inventory holding cost per unit of RW
$\mathcal{H}_0$ : Inventory holding cost per unit of OW
S : Inventory Shortage cost per cycle.
$IC_S$ :Shortagecost per unit .
$IC_d$ : Worsen cost per unit.
ε : Demand index during the constant cycle time T
m : Demand index
Z : Stock level Higher at the beginning of the cycle.
W : Warehouse capacity of OW.
Q : Total order Quantity per cycle.
TC ($t_1$, T) : Optimum total cost per unit.

## 3. MATHEMATICAL FORMULATION

**Definition :3.1**

**Fuzzy set:** A set s $\hat{S}$, defined as $\hat{S} = \{(A, \emptyset_{\hat{S}}) : A \in S \quad \emptyset_{\hat{S}}(A) \in [0,1]\}$ and usually denoted by the pair as $(A, \emptyset_{\hat{S}}(A))$, $A \in S$ and $\emptyset_{\hat{S}}(A) \in [0,1]$ then $\hat{S}$ is said to be a fuzzy set.

**Definition 3.2 Neutrosophic set: [ 5 ]** A set $\hat{T}$ is identified as a neutrosophic set if $\hat{T} = \{\langle p; \alpha_{\hat{T}}(p), \beta_{\hat{T}}(p), \gamma_{\hat{T}}(p) \,]\rangle$ : x ∈ P, P = universal set} , where $\alpha_{\hat{T}}(p)$ : P → [0, 1] signifes the scale of confidence $\beta_{\widehat{neuBiT}}(p)$ (p) : P → [0, 1] signifies the scale of hesitation and $\gamma_{\hat{T}}(p)$ : P → [0, 1] signifes the scale of falseness. Where, $\alpha_{\hat{T}}(p), \beta_{\hat{T}}(p), and\ \gamma_{\hat{T}}(p)$ satisfies the relation:

$$0 \leq \alpha_{\hat{T}}(p) + \beta_{\hat{T}}(p), + \gamma_{\hat{T}}(p) \leq 3$$

**Definition :3.3 Single-Valued Neutrosophic..Set: $Chakraborty$ [ 4]**

A set of Neutrosphic is $\widetilde{Ns}$ in the?definition 3.1. is claimed to be a single-Valued neutrosophic set ($\widetilde{SVTrNs}$ ) if $x$ may be single-valued independent variable. $\widetilde{SVTrNs}$ = {⟨$x$; [$\rho_{\widetilde{SVTrNs}}(x)$, $\sigma_{\widetilde{SVTrNs}}(x)$, $\tau_{\widetilde{SVTrNs}}(x)$]⟩: $x$∈$X$}, where $\rho_{\widetilde{SVTrNs}}(x)$, $\sigma_{\widetilde{SVTrNs}}(x)$, $\tau_{\widetilde{SVTrNs}}(x)$ provided the method of accuracy, dubiety and falsehood-memberships function0respectively.

**Definition :3.4 (.Tropezoidal&Single Valued Neutrosophic Number&)**

Neutrosophic number with tropezoidal Single Valued ($\tilde{\Omega}$) is defined a $\tilde{\Omega}$=< $(r_1, r_2, r_3, r_4: \Upsilon), (u_1, u_2, u_3, u_4: \lambda), (q_1, q_2, q_3, q_4: \eta)$ > , where$\mu$, $\vartheta$, $\zeta \in$ [0,1]. The real/membership function $\rho_{\tilde{\Omega}}$: R → [0, Υ], the dubiety/membership function $\sigma_{\tilde{\Omega}}$: R → [$\lambda$, 1] and the falsehood/membership function $\tau_{\tilde{\Omega}}$: : R+→//[$\eta$, 1] are characterized as follows:

$$\pi_{\tilde{\Omega}} = \begin{cases} \vartheta_{\tilde{\Omega}l}(x), & r_1 \le x < r_2 \\ \Upsilon, & r_2 \le x < r_3 \\ \vartheta_{\tilde{\Omega}r}(x), & r_3 < x \le r_4 \\ 0, & otherwise > . \end{cases}$$

$$\theta_{\tilde{\Omega}} = \begin{cases} \varepsilon_{\tilde{\Omega}l}(x), & u_1 \le x < u_2 \\ \lambda, & u_2 \le x < u_3 \\ \varepsilon_{\tilde{\Omega}r}(x), & u_3 < x \le u_4 \\ 1, & otherwise > . \end{cases}$$

$$\eta_{\tilde{\Omega}} = \begin{cases} \ell_{\tilde{\Omega}l}(x), & q_1 \le x < q_2 \\ \eta, & q_2 \le x < q_3 \\ \ell_{\tilde{\Omega}r}(x), & q_3 < x \le q_4 \\ 1, & otherwise > . \end{cases}$$

**Definition :3.5 Bipolar neutrosophic set:** A set $\hat{T}$ is identified as a neutrosophic set if $\widehat{T_{neuB\iota}} = \{\langle p; \alpha_{\widehat{T_{neuB\iota}}}(p), \beta_{\widehat{T_{neuB\iota}}}(p), \gamma_{\widehat{T_{neuB\iota}}}(p) ]\rangle$ : x ∈ P, P = universal set} , where $\alpha_{\widehat{T_{neuB\iota}}}{}^{+}(p)$ : P → [0, 1], $\alpha_{\widehat{T_{neuB\iota}}}{}^{-}(p)$ : P → [-1, 0], signifies the scale of confidence $\beta_{\widehat{T_{neuB\iota}}}{}^{+}(p)$ : P → [0, 1], $\beta_{\widehat{T_{neuB\iota}}}{}^{-}(p)$ : P → [-1, 0] signifies the scale of hesitation $\gamma_{\widehat{T_{neuB\iota}}}{}^{+}(p)$ : P → [0, 1], $\gamma_{\widehat{T_{neuB\iota}}}{}^{-}(p)$ : P → [-1, 0],signifies the scale of falseness. Where, $\alpha_{\widehat{T_{neuB\iota}}}(p), \beta_{\widehat{T_{neuB\iota}}}(p), and\ \gamma_{\widehat{T_{neuB\iota}}}(p)$ satisfies the relation:
$-0 \le \alpha_{\widehat{T_{neuB\iota}}}(p), \beta_{\widehat{T_{neuB\iota}}}(p), and\ \gamma_{\widehat{T_{neuB\iota}}}(p) \le 3+$

**Definition :3.6 De- Bipolar neutrosophication of Tropezoidal. Neutrosophic0 number:**

This..system, the expulsion region procedure executed to assess the de-neutrosophication worth of tropezoidal single esteemed neutrosophic number is
$\widehat{T_{neuB\iota}}$ =< $(r_1, r_2, r_3, r_4: \Upsilon), (u_1, u_2, u_3, u_4: \lambda), (q_1, q_2, q_3, q_4: \eta)$ > , de-neutrosophic form $\tilde{S}$ is provided as
$\widehat{T_{DneuB\iota}} = \left(\frac{r_1 + r_2 + r_3 + r_4 + u_1 + u_2 + u_3 + u_4 + q_1 + q_2 + q_3 + q_4}{6}\right)$

The inventory model is created in the following manner: At the start of each period, Z units of goods arrived in the stock system. The W units are kept in OW, while the rest are kept in RW. The items in OW are devoured solely after the products in RW have been burned-through. The stock level is diminishing in the RW during the time span $[o, t_1]$, because of the request rate and disintegration.

The stock model is advanced as follows: Z units of object arrived inventory model at the start of each period. W units are kept in OW and the rest is put absent in RW. The things of OW are eaten up exclusively after burning-through the items kept in RW. In the RW, during the time span $[o, t_1]$, stock level is diminishing because of the interest rate and disintegration and the stock level is lessening to zero at $t_1$. The stock W diminishes during $[o, t_1]$, because of decay just, while during $[t_1, t_2]$, the stock is exhausted because of both interest and crumbling. The Stock level is dropping to zero at $t_2$. Worsen rate of RW ($\theta$) is disparate than the worsen rate of OW ($\emptyset$).The holding cost at RW is less than the holding cost at OW ($IC_{H_0} > IC_{H_r}$). Finally, a shortfall happens due to demand during the time span $[t_1, T]$.

**Case (i) System with increasing demand $0 < m < 1$**

The total request during the span is ε units. When $0 < m < 1$, a larger portion of request at end of inventory cycle.

Here the Stock level at RW reduce due to increasing demand rate and constant worsen rate in the interval $(0, t_1)$ and reaches zero at $t_1$.

Hence, the Stock level in RW for $t \epsilon (0, t_1)$ fulfil the differential equations

$$\frac{d\Im_r(t)}{dt} = -\frac{\varepsilon t^{\frac{1}{m}-1}}{mT^{\frac{1}{m}}} - \theta\, \Im_r(t) \qquad 0 \le t \le t_1 \qquad (1)$$

Stock level at OW diminishes, because of weakening over the span $(0, t_1)$, and because of expanding request rate and consistent decay rate over the stretch $(t_1, t_2)$ and arrives at zero at $t_2$. In this manner it fulfils the differential conditions

$$\frac{d\Im_0(t)}{dt} = -\emptyset \Im_0(t) \qquad 0 \le t \le t_1 \qquad (2)$$

$$\frac{d\Im_0(t)}{dt} = -\frac{\varepsilon t^{\frac{1}{m}-1}}{mT^{\frac{1}{m}}} - \emptyset \Im_0(t) \qquad t_1 \le t \le t_2 \qquad (3)$$

The range of pending shortages over $( t_2, T)$ satisfies the derivative

$$\frac{dS(t)}{dt} = -\frac{\varepsilon\, t^{\frac{1}{m}-1}}{mT^{\frac{1}{m}}} \qquad t_2 \le t \le T \qquad (4)$$

The actions of the stock system during the entire span [0, T] is shown in Figures 1

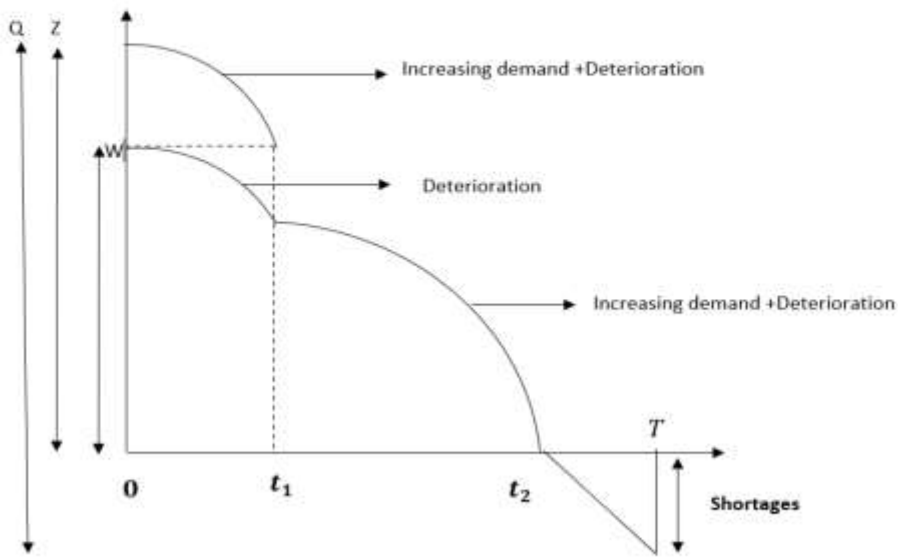


Figure 1. Graphical representation of a two-warehouse inventory system with Increasing Demand

Solving the above differential equations with the boundary conditions

$\Im_r(t_1) = 0, \qquad 0 \le t \le t_1$

$\Im_o(0) = W, \qquad 0 \le t \le t_1$

$\Im_0(t_2) = 0\,, \qquad t_1 \le t \le t_2$

$S(t_2) = 0, \qquad t_2 \le t \le T$

The solutions to Equations (1)-(4) are

$$\Im_r(t) = \frac{\varepsilon}{T^{\frac{1}{m}}} \left\{ \left\{ t_1^{\frac{1}{m}} - t^{\frac{1}{m}} \right\} - \frac{m\,\theta}{m+1} \left\{ t_1^{\frac{1}{m}+1} - t^{\frac{1}{m}+1} \right\} \right\} \qquad 0 \le t \le t_1 \qquad (5)$$

$$\Im_0(t) = W - \emptyset t \qquad 0 \le t \le t_1 \qquad (6)$$

$$\Im_0(t) = \frac{\varepsilon}{T^{\frac{1}{m}}} \left\{ \left\{ t_2^{\frac{1}{m}} - t^{\frac{1}{m}} \right\} - \frac{m\emptyset}{m+1} \left\{ t_2^{\frac{1}{m}+1} - t^{\frac{1}{m}+1} \right\} \right\} \qquad t_1 \le t \le t_2 \qquad (7)$$

$$S(t) = \frac{\varepsilon}{T^{\frac{1}{m}}} \left\{ t_2^{\frac{1}{m}} - t^{\frac{1}{m}} \right\} \qquad t_2 \le t \le T \qquad (8)$$

Applying the boundary condition $\Im_r(0) = Z - W$ the value of Z is

$$Z = W + \frac{\varepsilon}{T^{\frac{1}{m}}} \left\{ 1 - \frac{m\,\theta\, t_1}{m+1} \right\} \left\{ t_1^{\frac{1}{m}} \right\} \qquad (9)$$

The maximum shortage inventory, S(T) is obtained from equation (8)

Q= Z - S(T)

Finally, from equation (9) we have

$$Q = W + \frac{\varepsilon}{T^{\frac{1}{m}}}\left\{1 - \frac{m\,\emptyset\, t_1}{m+1}\right\}\left\{t_1^{\frac{1}{m}}\right\} - \frac{\varepsilon}{T^{\frac{1}{m}}}\left\{t_2^{\frac{1}{m}} - T^{\frac{1}{m}}\right\} \quad (10)$$

Total similar inventory cost per span consists of the following cost parameters:

1. **The invoice cost is A**
2. **The holding cost of inventory in RW is resulting**

$$\mathcal{H}_r = IC_{H_r}\{\int_0^{t_1} \Im_r(t)\, dt\}$$

$$\mathcal{H}_r = \frac{IC_{H_r}\,\varepsilon}{T^{\frac{1}{m}}}\left\{ \left\{\left(\frac{1}{m+1}\right) t_1^{\frac{1}{m}+1}\right\} - \left\{\left(\frac{\theta m}{(2m+1)}\right) t_1^{\frac{1}{m}+2}\right\}\right\} \quad (11)$$

3. **The holding cost of inventory in OW is resulting**

$$\mathcal{H}_0 = IC_{H_0}\{\int_0^{t_2} \Im_0(t)\, dt\}$$

$$\mathcal{H}_0 = IC_{H_0}\left\{W\, t_1 - \emptyset \frac{t_1^2}{2}\right\} + \frac{IC_{H_0}\varepsilon}{T^{\frac{1}{m}}}\left\{ \left\{\left(\frac{1}{m+1}\right) t_2^{\frac{1}{m}+1}\right\} - \left\{\left(\frac{\emptyset m}{(2m+1)}\right) t_2^{\frac{1}{m}+2}\right\}\right\} - \frac{IC_{H_0}\,\varepsilon}{T^{\frac{1}{m}}}\left\{ \left\{t_1\; t_2^{\frac{1}{m}} - \left\{\left(\frac{m}{m+1}\right) t_1^{\frac{1}{m}+1}\right\}\right\} - \left(\frac{\emptyset m}{(m+1)}\right)\left\{t_1\; t_2^{\frac{1}{m}+1} - \left\{\left(\frac{m}{(2m+1)}\right) t_1^{\frac{1}{m}+2}\right\}\right\}\right\} \quad (12)$$

4. **The shortage cost per cycle is**

$$S = IC_s\left\{\int_{t_2}^{T} \frac{\varepsilon}{T^{\frac{1}{m}}}\left\{t_2^{\frac{1}{m}} - t^{\frac{1}{m}}\right\}\; dt\right\}$$

$$S = IC_s\frac{\mathrm{IC}}{T^{\frac{1}{m}}}\left\{t_2^{\frac{1}{m}}T - \left(\frac{m}{m+1}\right) T^{\frac{1}{m}+1} - \left(\frac{1}{m+1}\right) t_2^{\frac{1}{m}+1}\right\} \quad (13)$$

5. **The cost of worsen products in RW and OW during $(\mathbf{0}, \boldsymbol{t_2})$ are**

$$D = IC_d\{\theta \int_0^{t_1} \Im_r(t)\, dt + \emptyset \int_0^{t_2} \Im_0(t)\, dt\}$$

$$D = \frac{IC_d\;\varepsilon\,\theta}{T^{\frac{1}{m}}}\left\{ \left\{\left(\frac{1}{m+1}\right) t_1^{\frac{1}{m}+1}\right\} - \left\{\left(\frac{\theta m}{(2m+1)}\right) t_1^{\frac{1}{m}+2}\right\}\right\} + IC_d\,\emptyset\left\{W\, t_1 - \emptyset\frac{t_1^2}{2}\right\} + \frac{IC_d\;\varepsilon\,\emptyset}{T^{\frac{1}{m}}}\left\{ \left\{\left(\frac{1}{m+1}\right) t_2^{\frac{1}{m}+1}\right\} - \left\{\left(\frac{\emptyset m}{(2m+1)}\right) t_2^{\frac{1}{m}+2}\right\}\right\} - \frac{IC_d\,\varepsilon}{T^{\frac{1}{m}}}\left\{ \left\{t_1\; t_2^{\frac{1}{m}} - \left\{\left(\frac{m}{m+1}\right) t_1^{\frac{1}{m}+1}\right\}\right\} - \left(\frac{\emptyset m}{(m+1)}\right)\left\{t_1\; t_2^{\frac{1}{m}+1} - \left\{\left(\frac{m}{(2m+1)}\right) t_1^{\frac{1}{m}+2}\right\}\right\}\right\} \quad (14)$$

Finally, the Total inventory cost per unit time is resulting

TC ($t_2$)= $\frac{1}{T}$ (Invoice cost + Holding cost +Shortage cost +Worsen cost )

$$\begin{aligned}\mathrm{TC}\,(t_2) = \frac{1}{T}\Bigg\{ &(A) + \frac{(IC_{H_r} + IC_d\,\theta)\,\varepsilon}{T^{\frac{1}{m}}}\left\{ \left\{\left(\frac{1}{m+1}\right) t_1^{\frac{1}{m}+1}\right\} - \left\{\left(\frac{\theta m}{(2m+1)}\right) t_1^{\frac{1}{m}+2}\right\}\right\} \\ &+ \left(IC_{H_0} + IC_d\;\emptyset\right)\left\{W\, t_1 - \emptyset\frac{t_1^2}{2}\right\} \\ &+ \frac{(IC_{H_0} + IC_d\;\emptyset)\varepsilon}{T^{\frac{1}{m}}}\left\{ \left\{\left(\frac{1}{m+1}\right) t_2^{\frac{1}{m}+1}\right\} - \left\{\left(\frac{\emptyset m}{(2m+1)}\right) t_2^{\frac{1}{m}+2}\right\}\right\} \\ &- \frac{(IC_{H_0} + IC_d\;\emptyset)\varepsilon}{T^{\frac{1}{m}}}\left\{ \left\{t_1\; t_2^{\frac{1}{m}} - \left\{\left(\frac{m}{m+1}\right) t_1^{\frac{1}{m}+1}\right\}\right\} \right. \\ &\left. - \left(\frac{\emptyset m}{(m+1)}\right)\left\{t_1\; t_2^{\frac{1}{m}+1} - \left\{\left(\frac{m}{(2m+1)}\right) t_1^{\frac{1}{m}+2}\right\}\right\}\right\} \\ &+ \frac{IC_s}{T^{\frac{1}{m}}}\left\{t_2^{\frac{1}{m}}T - \left(\frac{m}{m+1}\right) T^{\frac{1}{m}+1} - \left(\frac{1}{m+1}\right) t_2^{\frac{1}{m}+1}\right\}\Bigg\} \end{aligned} \quad (15)$$

**Numerical examples**

We represent the proposed model for certain mathematical models as given below.

**Example 1 [Case(i) increasing demand 0<m<1]**

W=100, A=50, $\varepsilon$=100, $\theta = 0.02$ ,$\emptyset = 0.03$ , $IC_{H_r} = \$\ 3/unit/\ year$, $IC_{H_0} = 5$, $IC_s = 12$, $IC_d =$ 10, m=0.5 , $t_1$= 0.8, T=1 in appropriate units. The result is obtained as follows $t_2 = 1.15337$, $\mathrm{TC}(t_2{}^{*}) = 618.872$ $Q = 130.462\ units.$

**Example 2 [Case(ii) decreasing demand m>1]**

As like the same Example 1 with m= 1,

$t_2 = 1.1487$, $\mathrm{TC}(t_2{}^{*}) = 593.214$ $Q = 164.147\ units$

**Example 3 [Case(iii) Linear demand m=1]**

As like the same Example 1 with m= 2,

The result obtained is as follows like $t_1 = 0.8\ months,\ T = 1\ years,$

$t_2 = 1.14419$ , $\mathrm{TC}(t_2{}^{*}) = 572.59$ $Q = 175.254\ unit$

**Solution procedure**

We came to know that the nonlinear equations. Here, we use MATHEMATICA 9.0 tool find the optimum solution of $t_1{}^{*}$ and $T^{*}$ using equation (15)
$D(t_1{}^{*}, T^{*}) = \frac{\partial^2 TC\,(t_1,T)}{\partial t_1{}^2}\frac{\partial^2 TC\,(t_1,T)}{\partial T^2} - \left[\frac{\partial^2 TC\,(t_1,T)}{\partial t_1\,\partial T}\right]^2 > 0$ we recommended "D-test" for optimizing functions of two variables $t_1$ and T such that

$$\frac{\partial TC\,(t_1,T)}{\partial T} = \Bigg\{\left(-\frac{A}{T^2}\right) - \left\{1+\frac{1}{m}\right\} \frac{(IC_{H_r}+\theta IC_d)\ IC}{T^{\frac{1}{m}+2}}\Bigg\{ \left\{\left(\frac{1}{m+1}\right) t_1^{\frac{1}{m}+1}\right\} + \left\{\left(\frac{2\theta m}{(m+1)(m+2)}\right) t_1^{\frac{1}{m}+2}\right\}\Bigg\} + (IC_{H_0} + \emptyset\ IC_d)W\left\{t_1 - \emptyset\frac{t_1{}^2}{2}\right\}\left(-\frac{1}{T^2}\right) - \left\{1+\frac{1}{m}\right\} \frac{(IC_{H_0}+\emptyset\ IC_d)\ IC}{T^{\frac{1}{m}+2}}\Bigg\{ \left\{\left(\frac{1}{m+1}\right) t_2^{\frac{1}{m}+1}\right\} + \left\{\left(\frac{2\theta m}{(m+1)(m+2)}\right) t_2^{\frac{1}{m}+2}\right\}\Bigg\} + IC_s\frac{IC}{T^{\frac{1}{m}+1}}\left\{-\frac{1}{m}t_2^{\frac{1}{m}}\right\} - \left\{\frac{1}{m}\ \frac{IC_s\ IC}{T^{\frac{1}{m}+2}}\ t_2^{\frac{1}{m}+1}\right\}\Bigg\} \quad (16)$$

$$\frac{\partial TC\,(t_1,T)}{\partial t_1} == \frac{(IC_{H_r}+\theta IC_d)\ IC}{T^{\frac{1}{m}+1}}\Bigg\{ \left\{\left(\frac{1}{m}\right) t_1^{\frac{1}{m}}\right\} + \left\{\left(\frac{2\theta(2m+1)}{(m+1)(m+2)}\right) t_1^{\frac{1}{m}+1}\right\}\Bigg\} + \left(\frac{1}{T}\right)(IC_{H_0} + \emptyset\ IC_d)W\{1 - \emptyset\ t_1\} \quad (17)$$

$$\frac{\partial TC\,(t_1,T)}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial TC\,(t_1,T)}{\partial T} = 0$$

If $\frac{\partial^2 TC\,(t_1{}^{*},T^{*})}{\partial t_1{}^2} > 0$ then $TC\,(t_1{}^{*}, T^{*})$ is minimum Value

**Case (ii)** Model **with decreasing demand** ( $m > 1$ )
The total requires during the time period is IC units. At the point when m > 1, a larger part of request happens at the start of the time span.

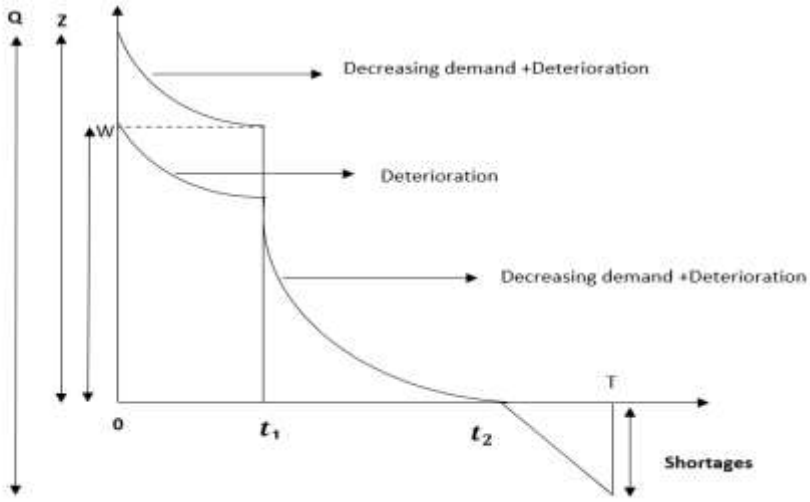


Figure 2 . Graphical representation of a two-warehouse inventory system with Decreasing Demand

**Case (iii) Model with Linear demand $m = 1$**

The total requires during the time span is IC units. When *m* = 1, the request follows a uniform system.

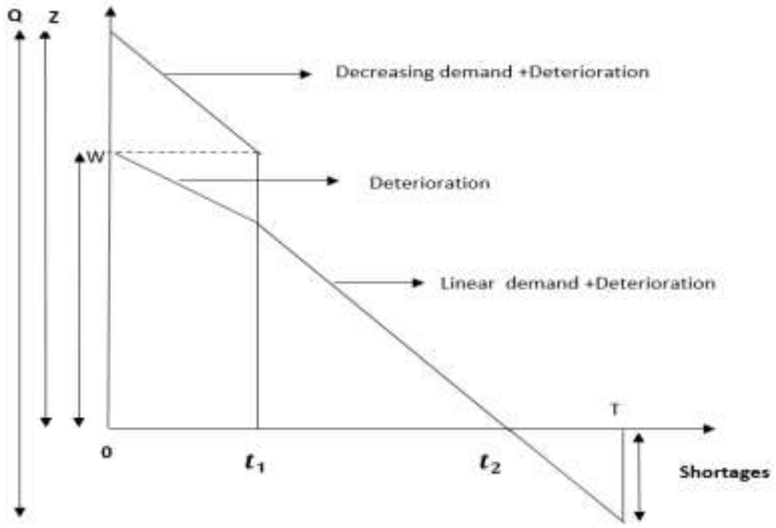


Figure 3 . Graphical representation of a two-warehouse inventory system with Linear Demand

$$\widehat{TC}_{T_{\widehat{neuB\iota}}}(t_2) = \frac{1}{T}\left\{ (A) + \frac{(\widehat{IC}_{H_r}+\widehat{IC}_d\,\theta)\,\varepsilon}{T^{\frac{1}{m}}}\left\{ \left\{\left(\frac{1}{m+1}\right) t_1^{\frac{1}{m}+1}\right\} - \left\{\left(\frac{\theta m}{(2m+1)}\right) t_1^{\frac{1}{m}+2}\right\}\right\} + (\widehat{IC}_{H_0}+\widehat{IC}_d\ \emptyset)\left\{W\,t_1 - \emptyset\frac{t_1^{2}}{2}\right\} + \frac{(\widehat{IC}_{H_0}+\widehat{IC}_d\ \emptyset)\varepsilon}{T^{\frac{1}{m}}}\left\{ \left\{\left(\frac{1}{m+1}\right) t_2^{\frac{1}{m}+1}\right\} - \left\{\left(\frac{\emptyset m}{(2m+1)}\right) t_2^{\frac{1}{m}+2}\right\}\right\} - \frac{(\widehat{IC}_{H_0}+\widehat{IC}_d\ \emptyset)\varepsilon}{T^{\frac{1}{m}}}\left\{ \left\{t_1\ t_2^{\frac{1}{m}} - \left\{\left(\frac{m}{m+1}\right) t_1^{\frac{1}{m}+1}\right\}\right\} - \left(\frac{\emptyset m}{(m+1)}\right)\left\{t_1\ t_2^{\frac{1}{m}+1} - \left\{\left(\frac{m}{(2m+1)}\right) t_1^{\frac{1}{m}+2}\right\}\right\}\right\} + \frac{IC_s}{T^{\frac{1}{m}}}\left\{t_2^{\frac{1}{m}}T - \left(\frac{m}{m+1}\right) T^{\frac{1}{m}+1} - \left(\frac{1}{m+1}\right) t_2^{\frac{1}{m}+1}\right\}\right\} \quad (15)$$

**The effects of trapezoidal bipolar neutrosophic numbers:**

Here, deteriotion cost $\widehat{IC_d}$, holding cost in owned warehouse $\widehat{IC_{H_o}}$, holding cost in rented warehouse $\widehat{IC_{H_r}}$ have been considerede.as trapezoidal bipolar neutrosophic fuzzy set. Thus, the parameters of bipolar neutrosophic numbers are:

$$\widehat{IC_d} < (2.25, 2.72, 3.26, 4.82); (1.76, 2.88, 3.14, 4.37), (1.44, 2.76, 3.38, 4.02) >$$
$$\widehat{IC_{H_o}} = < (2.29, 3.26, 5.02, 6.82); (2.06, 3.39, 4.92, 5.65), (3.85, 4.48, 6.34, 8.10\ ) >$$
$$\widehat{IC_{H_r}} = < (3.11, 4.15, 5.81, 6.80); (1.57, 2.73, 3.42, 4.04), (0.04, 1.26, 2.35, 3.22) >$$

We can generate outcomes into neutrosophic numbers based on the result of trapezoidal bipolar neutrosophic number and its membership functions as the De-neutrosophication technology develops.

$$\widehat{T_{DneuBi}} = \left(\frac{r_1 + r_2 + r_3 + r_4 + u_1 + u_2 + u_3 + u_4 + q_1 + q_2 + q_3 + q_4}{6}\right)$$

**Numerical examples**

We represent the proposed model for certain mathematical models as given below.

**Example 1 [Case(i) increasing demand 0<m<1]**

W=100, A=50, $\varepsilon$=100, $\theta = 0.02$ ,$\emptyset = 0.03$ , $\widehat{IC_{H_r}} = \$\ 6.75/unit/\ year$, $\widehat{IC_{H_0}} = 9.36$ , $\widehat{IC_d} = 6.13$, $IC_s = 12$ , m=0.5 , $t_1$= 0.7, T=1 year in right units. The results extracted as follows $t_2 = 1.078$, $\mathrm{TC}(\mathrm{t_2}^*) = 510.572$ $Q = 163.90\ units$.

**Example 2 [Case(ii) decreasing demand m>1]**

As like the same Example 1 with m= 2,

The results were obtained as follows $t_1 = 0.8\ months$, $T = 1\ years$, $t_2 = 1.624$, $\mathrm{TC}(\mathrm{t_2}^*) = 564.854$ $Q = 148.152\ units$

**Example 3 [Case(iii) Linear demand m=1]**

As like the same Example 1 with m= 1,

The results were obtained as follows $t_1 = 0.8\ years$, $T = 1\ years$, $t_2 = 1.14419$, $\mathrm{TC}(\mathrm{t_2}^*) = 546.21$ $Q = 150.724\ units$

**Table 1**
**Impacts of changes within the different sort of the show.**

| $\alpha^*$ | $\beta^*$ | $\gamma^*$ | $t_1$ | T | $\widehat{TC}_{\widehat{T_{neuBt}}}(t_2)$ | Z | Q |
|---|---|---|---|---|---|---|---|
| W | 120 | +20 | 0.3476 | 0.5034 | 474.505 | 167.569 | 219.89 |
| | 110 | +10 | 0.3758 | 0.515 | 454.438 | 163.114 | 209.867 |
| | 100 | 0 | 0.3815 | 0.5326 | 417.787 | 151.178 | 199.87 |
| | 90 | -10 | 0.3951 | 0.5464 | 388.933 | 142.149 | 189.862 |
| | 80 | -20 | 0.3987 | 0.5558 | 322.461 | 126.996 | 179.872 |
| $\varepsilon$ | 120 | +20 | 0.3976 | 0.4798 | 532.097 | 183.68 | 219.776 |
| | 110 | +10 | 0.394 | 0.4743 | 515.487 | 175.707 | 209.801 |
| | 100 | 0 | 0.3926 | 0.4696 | 501.177 | 169.712 | 199.817 |
| | 90 | -10 | 0.3728 | 0.4687 | 408.982 | 156.797 | 189.858 |
| | 80 | -20 | 0.3416 | 0.4642 | 398.818 | 148.627 | 189.789 |
| A | 60 | +20 | 0.3939 | 0.6308 | 271.028 | 138.891 | 199.898 |
| | 55 | +10 | 0.3726 | 0.6203 | 267.199 | 135.992 | 199.91 |
| | 50 | 0 | 0.3349 | 0.6146 | 255.416 | 129.626 | 199.934 |
| | 45 | -10 | 0.3108 | 0.6041 | 251.385 | 126.415 | 199.945 |
| | 40 | -20 | 0.3024 | 0.5968 | 247.617 | 125.623 | 199.948 |
| M | 0.6 | +20 | 0.394 | 0.652 | 251.134 | 143.065 | 199.872 |
| | 0.55 | +10 | 0.3905 | 0.6442 | 250.102 | 140.135 | 199.888 |
| | 0.5 | 0 | 0.3666 | 0.6311 | 248.016 | 133.661 | 199.918 |
| | 0.45 | -10 | 0.3448 | 0.6254 | 239.399 | 126.572 | 199.943 |
| | 0.4 | -20 | 0.3321 | 0.5901 | 238.567 | 121.624 | 199.959 |
| $t_2$ | 0.48 | +20 | 0.3684 | 0.5295 | 456.028 | 148.288 | 199.881 |
| | 0.44 | +10 | 0.3726 | 0.5353 | 397.648 | 148.329 | 199.88 |
| | 0.4 | 0 | 0.3805 | 0.5396 | 341.843 | 149.568 | 199.874 |
| | 0.36 | -10 | 0.3847 | 0.5451 | 283.619 | 149.679 | 199.872 |
| | 0.32 | -20 | 0.3872 | 0.5516 | 228.066 | 149.686 | 199.871 |
| $\widehat{IC}_{H_r}$ | 2.4 | +20 | 0.348 | 0.5651 | 310.475 | 137.835 | 199.912 |
| | 2.2 | +10 | 0.3549 | 0.5754 | 299.415 | 137.953 | 199.91 |
| | 2 | 0 | 0.3682 | 0.5911 | 285.082 | 138.706 | 199.905 |
| | 1.8 | -10 | 0.3839 | 0.6142 | 264.511 | 138.968 | 199.9 |
| | 1.6 | -20 | 0.3917 | 0.622 | 256.041 | 139.554 | 199.896 |
| $\widehat{IC}_{H_0}$ | 4.8 | +20 | 0.3965 | 0.5866 | 340.075 | 145.567 | 199.871 |
| | 4.4 | +10 | 0.3924 | 0.5696 | 335.495 | 147.335 | 199.872 |
| | 4 | 0 | 0.384 | 0.5442 | 333.16 | 149.663 | 199.873 |
| | 3.6 | -10 | 0.3757 | 0.5313 | 327.115 | 149.873 | 199.875 |
| | 3.2 | -20 | 0.3628 | 0.5126 | 319.867 | 149.972 | 199.879 |
| $IC_s$ | 14.4 | +20 | 0.3784 | 0.5359 | 338.189 | 149.732 | 199.874 |
| | 13.2 | +10 | 0.3734 | 0.52 | 357.968 | 151.435 | 199.872 |
| | 12 | 0 | 0.3687 | 0.5053 | 375.68 | 153.11 | 199.869 |
| | 10.8 | -10 | 0.3624 | 0.4913 | 391.145 | 154.279 | 199.868 |
| | 9.6 | -20 | 0.3568 | 0.47 | 415.109 | 157.494 | 199.863 |
| $\widehat{IC}_d$ | 12 | +20 | 0.2886 | 0.4986 | 348.947 | 133.439 | 199.936 |
| | 11 | +10 | 0.3024 | 0.5108 | 346.59 | 134.977 | 199.929 |
| | 10 | 0 | 0.3176 | 0.5281 | 331.785 | 136.092 | 199.923 |
| | 9 | -10 | 0.3262 | 0.5308 | 329.747 | 137.684 | 199.918 |
| | 8 | -20 | 0.3428 | 0.5381 | 325.654 | 140.491 | 199.907 |
| $\theta$ | 0.024 | +20 | 0.3985 | 0.6005 | 289.962 | 143.898 | 199.86 |
| | 0.022 | +10 | 0.3956 | 0.5984 | 290.647 | 143.578 | 199.873 |
| | 0.02 | 0 | 0.3867 | 0.5953 | 299.51 | 142.494 | 199.903 |
| | 0.018 | -10 | 0.3756 | 0.58 | 301.548 | 141.842 | 199.905 |
| | 0.016 | -20 | 0.3734 | 0.5684 | 302.033 | 141.594 | 199.917 |
| $\emptyset$ | 0.036 | +20 | 0.3213 | 0.5659 | 291.871 | 130.792 | 199.934 |
| | 0.033 | +10 | 0.327 | 0.5765 | 293.236 | 132.103 | 199.93 |
| | 0.03 | 0 | 0.3327 | 0.5796 | 296.559 | 133.685 | 199.925 |
| | 0.027 | -10 | 0.3789 | 0.5826 | 298.656 | 142.19 | 199.893 |
| | 0.024 | -20 | 0.3998 | 0.5859 | 300.648 | 146.439 | 199.876 |

**Note: $\alpha^*$= Parameters, $\beta^*$=Values, $\gamma^*$=%Changes**

To understand the impact of different parameters, on the optimal cost given by the considered strategy. Affectability result is executed by changing (increasing and decreasing) 10 % in every parameter. The effect of the parameters is detailed below.

As the result of the above table ,

(i) Increases in the value of the parameter W then $t_1$, $T$ *is decreased and* $\widehat{TC}_{\widehat{T_{neuB\imath}}}(t_2)$, $Z$ , $Q$ is increased.

(ii) Increases in the value of the parameter $\delta$ then $t_1$, $T$ , $\widehat{TC}_{\widehat{T_{neuB\imath}}}(t_2)$ , $Z$ , $Q$ is increased.

(iii) Increases in the values of either of the parameters A, m then $t_1$, $T$ , $\widehat{TC}_{\widehat{T_{neuB\imath}}}(t_2)$ , $Z$ *is increased and* $Q$ is decreased.

(iv) Increases in the values of either of the parameters $IC_{H_r}$, $\widehat{IC}_d$, $t_2$ then $t_1$, $T$ , $Z$ is decreased and $Q$ , $\widehat{TC}_{\widehat{T_{neuB\imath}}}(t_2)$ is increased.

(v) Increases in the values of the parameter $\widehat{IC}_{H_0}$ then $t_1$, $T$ , $\widehat{TC}_{\widehat{T_{neuB\imath}}}(t_2)$ is increased and $Q$ , $Z$ is decreased.

(vi) Increases in the values of the parameter $IC_s$ then $t_1$, $T$ , $Q$ is decreased and $\widehat{TC}_{\widehat{T_{neuB\imath}}}(t_2)$ , $Z$ is increased.

(vii) Increases in the values of the parameter $\theta$ then $t_1$, $T$ , $Z$ is increased and $\widehat{TC}_{\widehat{T_{neuB\imath}}}(t_2)$ , $Q$ is decreased

(viii) Increases in the values of the parameter $\emptyset$ then $t_1$, $T$ , $\widehat{TC}_{\widehat{T_{neuB\imath}}}(t_2)$ , $Z$ is decreased *and* $Q$ is increased.

**Changing the parameter values and different total cost of power demand pattern**

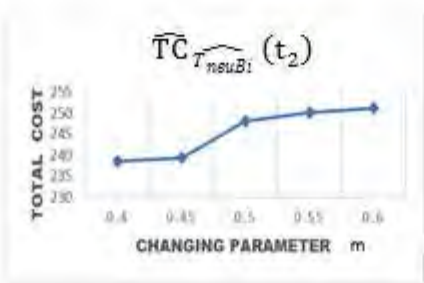


**Figure 4**

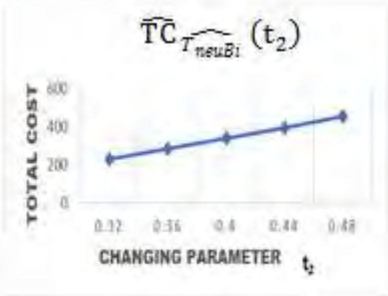


**Figure 5**

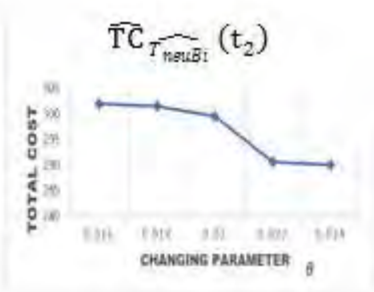


**Figure 6**

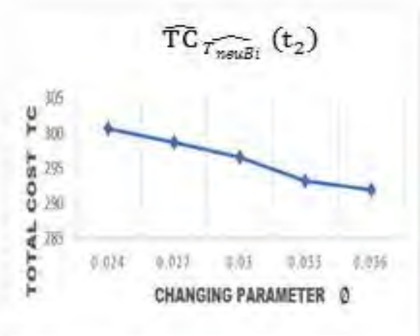


**Figure 7**

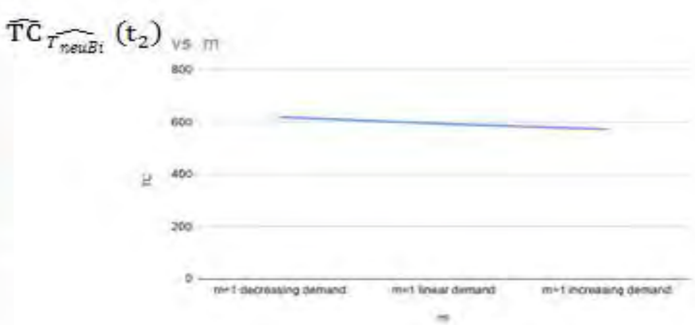


**Figure 8**

**CONCLUSON**

In this article, two-warehouse system for trapezoidal bipolar neutrosophic disparate expeditious worsen items with power demand pattern. Because of the various preservation conditions, OW and RW have trapezoidal bipolar neutrosophic parameters holding costs and exacerbate costs. The impact of demand pattern index optimal policy is dependent on whether the request sample index is lower than, equal to, or more than 1.0, according to our findings. Furthermore, when the request sample index is $0<m<1$, a similar structure emerges. However, when m = 1, a different structure emerges. However, when $m > 1$, The proposed demonstrate joins some practical highlights that are probably going to be associated for certain sorts of stock. Likewise, this model can be embraced in the stock control of retail business like food ventures, convenient garments household goods, car accessories, electronic items etc. increases quantity and decreases the cost of total amount. In future this paper can be extended in the Nero fuzzy environment and can be elongate the EPQ model is considering variable worsen with index of power demand and shortages not allow.

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Received: Nov 20, 2021. Accepted: Feb 6, 2022

*Neutrosophic Sets and Systems, Vol. 48, 2022*
*University of New Mexico*

# Telafer University's Experiment in E-learning as Vague Neutrosophic Experiment

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**Abstract.** This paper is dedicated to studying the stages that Telafer University has gone through at using E-learning. The challenges of converting traditional learning (i.e. classical learning with realistic student attendance in their teaching programs) to E-learning over google classrooms. The efforts to maintain the quality teaching and their outcomes were a very ambiguous experiment that led the authors after more than two years for spreading the Covid-19 Pandemic to evaluate the University's performance using the most modern mathematical tools in uncertainty systems that called the Neutrosophic theory and logic. Finally, the flexibility of the neutrosophic mathematical methods has been applied to analyze the recorded issued data of E-learning in the University.

**Keywords:** E-Learning in Telafer University, Neutrosophic Theory, Neutrosophic Logic, Neutrosophic Soft Sets, Comparison Matrix.

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## 1. Introduction

Telafer University has been established since 2014, exactly after four months of establishing Telafer university, civilians suffered from forced immigration because of the ISIS occupation. this led to the infrastructures of the university have vanished. However, now we do all administrative issues and teaching tasks in alternative buildings despite challenges and bad situations.

At the beginning of quarantine in Iraq (i.e. Feb. 28, 2020) Telafer University represented by all academic staff, and their employees were eager to provide a private electronic platform for all facilities of the university, since the spread of the COVID-19 pandemic, the providing of private virtual platform was an urgent requirement to reconvert the traditional teaching to remote e-learning for enabling the teachers and students to communicate smoothly. However, the console of Telafer University's platform has been administrated by the team of engineering that they enable to provide almost all supporting programs for both teachers and students to avoid any lacking in the teaching procedures, as well as the private domain for the university has the extension (@uotelafer.edu.iq) to use the Google Workspace. The committee of e-learning in Telafer University has equipped e-mails accounts within the domain of the university to all students and the university's members, as well as, they uploaded all lecturers (either synchronous or non-synchronous lectures) for the academic staff to the google classrooms, the university e-learning council guided the examinations committees in the scientific departments by follow up the google classrooms to present help for the teachers and students at holding the examinations and any other logistic help for them.

The neutrosophic theory, neutrosophic probability, neutrosophic sets, neutrosophic mathematical programming and neutrosophic logic have firstly originated by the polymath Florentin Smarandache, the mathematical professor in New Mexico University at 1995 by his first publications [1- 4], the main notion that neutrosophic theory stands on is that every problem can be formulated by three functions, truth function, indeterminacy function and its falsity function, this broad insight gives the neutrosophic theory the flexibility and wide ability to analyses the data giving problems solving in new modern mathematics, the following example regarded as a good demo for the readers to understand how the neutrosophic logic and theory can view and solve the problems.

The example that firstly stated in [5], Let's consider the population of a country $C1$. Most people in this country have only the citizenship of the country, therefore they belong 100% to $C1$. But there are people that have double citizenships, of countries $C1\ and\ C2$. Those people belong 50% to $C1$, and 50% to $C2$. While citizens with triple citizenships of countries $C1, C2$, and $C3$ belong only 33.33% to each country. Of course, considering various criteria these percentages may differ. Also, there are countries with autonomous zones, whose citizens in these zones may not entirely consider themselves as belonging to those countries. But there is another category of people that have been stripped from their $C1$ citizenship for political reasons and they have other citizenship, while still living (temporarily) in $C1$. They are called paria, and they do not belong to $C1$ (not having citizenship), but still belong to $C1$ (because they still living in $C1$). They form the indeterminate part of neutrosophic population of country $C1$.

This paper has been arranged to recognized the performance of the scientific departments in the colleges of Telafer University in Iraq versus to its efforts in e-learning and how these departments implementation the new methods of remote teaching, where the section 2 has been dedicated to the basic mathematical notions which represents the basic tools for the next section, while section 3 represents the core of the article containing a case study using collected data for three studying courses during the spread of COVID-19 pandemic

## 2. Mathematical Preliminaries

In this section, the authors will focus on recalling the essential mathematical tools that should be used in the upcoming section to make a fairly estimation to evaluate the performance of the e-learning at Telafer University, it is worthy to know the notions of the neutrosophic soft sets where the soft set theory was firstly introduced by D. Molodtsov at 1999 [7], while the neutrosophic soft set was set up by P. K. Maji at 2013 [8], also there are some other definitions as follows:

**2.1 Definition [6]**

A neutrosophic set $A$ on the universe of discourse $X$ is defined as $A = \{< x, T_A(x), I_A(x),\ F_A(x) >, x \in X\}$, where $T, I, F: X \to ]\,^{-}0, 1^{+}[$ and $^{-}0 \le T_A(x), +I_A(x) + F_A(x) \le 3^{+}$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]\,^{-}0, 1^{+}[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real non-standard subset of $]\,^{-}0, 1^{+}[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0,1]$.

**2.2 Definition [7]** Let $U$ be an initial universe set and $E$ be a set of parameters. Let $P(U)$ denotes the power set of $U$. Consider a nonempty set $A, A \subset E$. A pair $(F, A)$ is called a soft set over $U$, where $F$ is a mapping given by $F: A \to P(U)$.

**2.3 Definition [8]**

Let $U$ be an initial universe set and $E$ be a set of parameters. Consider $A \subset E$. Let $P(U)$ denotes the set of neutrosophic sets of $U$. The collection $(F, A)$ is termed to be the soft neutrosophic set over $U$, where $F$ is a mapping given by $F: A \rightarrow P(U)$.

The following definitions have been adapted to consider the upcoming case study section in which the best scientific department in Telafer University who can apply the strategical of e-learning in their classes to get the best performance in the studying pedagogy. In this paper we have six numbers of parameters out of five numbers of studying fields (i.e. Mathematics, Arabic Language, General Nursing, Field Crops, Animal Production). We still have the assumptions that the parameters are $d_1, d_2, d_3, d_4, d_5, and\ d_6$, while the scientific departments are $u_1, u_2, u_3, u_4, and\ u_5$.

**2.4 Definition [8]**

A comparison matrix is a matrix whose rows labelled by the object names $d_1, d_2, d_3, d_4, d_5, d_6$ and the columns are labelled by the parameters $u_1, u_2, u_3, u_4, u_5$. The entries $c_{ij}$ are calculated by $c_{ij} = a + b - c$, where $'a'$ is the integer calculated as ' how many times $T_{d_i}(u_j)$ exceeds or equal to $T_{d_k}(u_j)'$, for $d_i \neq d_k, \forall\, d_k \in U$, 'b' is the integer calculated as ' how many times $I_{d_i}(u_j)$ exceeds or equal to $I_{d_k}(u_j)'$ for $d_i \neq d_k, \forall\, d_k \in U$ and 'c' is the integer calculated as ' how many times $F_{d_i}(u_j)$ exceeds or equal to $F_{d_k}(u_j)'$, for $d_i \neq d_k, \forall\, d_k \in U$.

**2.5 Definition [8]**

The score of an object $d_i$ is $S_i$ and is calculated as $S_i = \sum_j c_{ij}$, then the most appropriated or best selection of an object $u_j$ which own to the maximum value of $S_i$.

## 3. Algorithm

The following algorithm will be the basic road map for the upcoming case study in section 4,

Step-1- Consider the logical parameters $D = \{d_1, d_2, d_3, d_4, d_5, d_6\}$.

Step-2- Consider the department names $u = \{u_1, u_2, u_3, u_4, u_5\}$.

Step-3- Compute all $T_{d_1}, \dots, T_{d_6}, I_{d_1}, \dots, I_{d_1}, F_{d_1}, \dots, F_{d_6}$ for all departments $u_1, u_2, u_3, u_4, u_5$, the result of this computation step are 30 elements of the kind $(T_{d_j}(u_i), I_{d_j}(u_i), F_{d_j}(u_i))$.

Step -4- For each column j=1,2,3,4,5,6 compute how many times that the $\boldsymbol{T_{d_k}} \geq \boldsymbol{T_{dj}}\ \forall\, \boldsymbol{k} \neq \boldsymbol{j}\ \ (\mathbf{1})$ , consider $''\ a\ ''$ is the times for the satisfaction of condition (1).

Step -5- For each column j=1,2,3,4,5,6 compute how many times that the $\boldsymbol{I_{d_k}} \geq \boldsymbol{I_{dj}}\ \ \forall\, \boldsymbol{k} \neq \boldsymbol{j}\ \ \ (\mathbf{2})$, consider $''\ b\ ''$ is the times for the satisfaction of condition (2).

Step -6- For each column j=1,2,3,4,5,6 compute how many times that the $\boldsymbol{F_{d_k}} \geq \boldsymbol{F_{dj}}\ \forall\, \boldsymbol{k} \neq \boldsymbol{j}\ \ (\mathbf{3})$, consider $''\ c\ ''$ is the times for the satisfaction of condition (3).

Step-7- Compute $''\ a + b - c\ ''$ that have been considered for all departments of the table 2.

Step- 8- The score of the performance for each department is the summation of the corresponding row in table (2). The results of these summations labeled in table (3).

Step -9- Reorder the best performance to the poor performance depending upon the scores of these departments from maximum score to the minimum score.

Step -10- End

## 4. Case Study to Evaluate the Performance of Telafer University in E-learning

This section has been originated to summarize the performance of the e-learning in five scientific departments (math dept., Arabic language dept., general nursing dept., animal production dept., and fields crops dept.) belonging to three colleges (College of Basic Education, College of Nursing, and College of Agriculture) in Telafer University during the spread of COVID-19 pandemic and its several mutations through the time period from Feb. 28, 2020, to present, where the coronavirus actually entered to Iraq since Feb. 2020, and the quarantine processes were applied which led the Iraqi universities to adopt the e-learning.

### 4.1 Example

Let $U$ be the set of five Scientific Departments in three colleges of Telafer University as follow:

$u = \{u_1, u_2, u_3, u_4, u_5\}$ where

$u_1$ represents the department of general nursing.

$u_2$ represents the department of Arabic language.

$u_3$ represents the mathematical dept.

$u_4$ represents the animal's production dept.

$u_5$ represents the field crops dept.

Let $D = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ be the set of parameters, where each parameter is a neutrosophic sentence involving neutrosophic words defining as follow:

$d_1$= The percentage of the internet speed for the districts of the students' resident and it was ranged between (zero to 2.69 Mbps) depending upon the information that available in the website (https://www.cable.co.uk/broadband/speed/worldwide-speed-league/ )

$d_2$ = The percentage of the students' attendance in the whole e-lectures through the studying course.

$d_3$ = Designing the e-lectures and harmonising them with principles of pedagogy and set up interactive courses.

$d_4$= The percentage of the syllabus coverage through the whole course by the lecturers.

$d_5$= The procedure that taken to reduce cheating during the performance of students' electronic examinations.

$d_6$ = Percentage of student satisfaction by launching questionnaires to measure the students' understanding for the e-lectures.

Table 1: This table demonstrates the performance of all scientific departments in e-learning using neutrosophic soft sets.

| D \ U | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |
|---|---|---|---|---|---|---|
| $u_1$ | (0.8, 0.6,0.1) | (0.7, 0.5,0.2) | (0.6, 0.4,0.3) | (0.85, 0.53,0.12) | (0.4, 0.9,0.5) | (0.7, 0.51,0.4) |
| $u_2$ | (0.75, 0.55,0.16) | (0.63, 0.5,0.25) | (0.3, 0.6,0.4) | (0.81, 0.37,0.29) | (0.42, 0.87,0.56) | (0.9, 0.4,0.14) |
| $u_3$ | (0.9, 0.44,0.05) | (0.99, 0.16,0.12) | (0.89, 0.4,0.33) | (0.75, 0.44,0.17) | (0.6, 0.49,0.33) | (0.64, 0.7,0.85) |
| $u_4$ | (0.72, 0.57,0.3) | (0.66, 0.3,0.99) | (0.32, 0.53,0.99) | (0.77, 0.49,0.18) | (0.2, 0.4,0.9) | (0.67, 0.57,0.41) |
| $u_5$ | (0.73, 0.61,0.32) | (0.76, 0.45,0.82) | (0.35, 0.44,0.83) | (0.64, 0.77,0.32) | (0.34, 0.52,0.69) | (0.71, 0.61,0.39) |

Table 2: This table demonstrates the comparison matrix in the neutrosophic soft sets $(u, D)$

| $u \backslash D$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |
|---|---|---|---|---|---|---|
| $u_1$ | 5 | 5 | 4 | 7 | 5 | 1 |
| $u_2$ | 1 | 2 | 2 | 0 | 4 | 3 |
| $u_3$ | 4 | 4 | 4 | 1 | 5 | 0 |
| $u_4$ | -1 | -2 | 0 | 2 | -4 | 0 |
| $u_5$ | 1 | 2 | -1 | 0 | 0 | 4 |

Table 3: The score of performance in each department $u_i$ , where the values are the summation of each row for the above comparison matrix:

| $U$ | sum |
|---|---|
| $u_1$ | 27 |
| $u_2$ | 12 |
| $u_3$ | 18 |
| $u_4$ | -5 |
| $u_5$ | 6 |

These values illustrate that the best performance in the e-learning issue was for the general nursing department which has 27 degrees, while in the second level was for the mathematical department, in the third grid was the department of the Arabic language. It is worthy to note that the department of animals' production which is one of agriculture college departments should review the strategy of teaching and trying to improve it.

# 4 Conclusion

This paper comes as an urgent need due to the ongoing global quarantine situation in the COVID-19 pandemic. where Telafer University's procedures in converting traditional learning to e-learning faced many challenges in different trends as qualifying the teaching staff, students and providing an electronic learning platform for the university, also edification in spreading the ethics of the e-learning, all these reasons led the authors to use the most modern mathematical logic that named Neutrosophic Theory to analyses the data which has been collected during the period Feb. 2020 to present, this article gave analysis, good feedback and deep insight to evaluate the experiment of e-learning in Telafer University.

**Acknowledgement:** This research is supported by the Neutrosophic Science International Association (NSIA) in both of its headquarter in New Mexico University and its Iraqi branch at Telafer University, for more details about (NSIA) see the URL http://neutrosophicassociation.org/ .

**Funding Statement:** This research received no external funding.

**Conflicts of Interest:** The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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Received: Aug. 20, 2021. Accepted: Feb. 1, 2021

Neutrosophic Sets and Systems, Vol. 48, 2022
University of New Mexico

# Vulnerability Parameters in Neutrosophic Graphs

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**Abstract:**. Let $G = (\mathrm{U}, \mathrm{V})$ be a Single valued Neutrosophic graph. A subset $S \in U(G)$ is a said to be score equitable set if the score value of any two nodes in S differ by at most one. That is, $|s(u) - s(v)| \leq 1, u, v \in S$. If e is an edge with end vertices u and v and score of u is greater than or equal to score of v then we say u strongly dominates v. If every vertex of V − S is strongly influenced by some vertex of S then S is called strong score set of G. The minimum cardinality of a strong dominating set is called the strong score number of G. The equitable integrity of Single valued Neutrosophic graph G which is defined as $\mathrm{E}I(G) = min\{|S| + m(G - S): S$ is a score equitable set in $G\}$, where $m(G - S)$ denotes the order of the largest component in $G - S$. The strong integrity of Single valued Neutrosophic graph G which is defined as $\mathrm{S}I(G) = min\{|S| + m(G - S):$ $S$ is a strong score set in $G\}$. In this paper, we study the concepts of equitable integrity and strong equitable integrity in different classes of regular Neutrosophic graphs and discussed the upper and lower bounds.

**Keywords:** Score equitable sets, Strong Score Equitable Sets, Equitable integrity, Strong Equitable integrity

## 1. Introduction

Real-life problems in any communication network, social network, supply chain network and brain network analysis can be modelled as a graph. The objects and the relations between objects are represented by the vertices and edges of the graph. In many real life problems, loss of information, a lack of evidence, imperfect statistical data and insufficient information can be converted by using classical set theory, which was presented by Cantor. Any vertex or edge in the classical graphs is having two possibilities, is either in the graph or it is not in the graph. Therefore, uncertain optimization problems cannot be modelled as a classical graph. An extended version of the classical sets is the fuzzy sets, where the objects have varying membership degrees. It gives different membership degrees between zero and one to its objects. The membership describes membership in vaguely-defined sets but not the same as probability. Zadeh [1] introduced the degree of membership/truth (T) in 1965 and defined the fuzzy set. The concept of fuzziness in graph theory was described by Kaufmann [2] using the fuzzy relation. Rosenfeld [3] introduced some concepts such as bridges, cycles, paths, trees, and the connectedness of the fuzzy graph and described some of the properties of the fuzzy graph. Samanta and Pal [4] and Rashmanlou and Pal [5] presented the concept of the irregular and regular fuzzy graph. They also described some applications of those graphs.

Intuitionistic fuzzy sets( IFS) considers not only the membership grade (degree), but also independent membership grade and non-membership grade for any entity, and the only requirement is that the sum of non-membership and membership degree values be no greater than one. The idea of the intuitionistic fuzzy set (IFS) as a modified version of the classical fuzzy set was introduced by Atanassov [6–8]. The idea of the IFS relation and the intuitionistic fuzzy graphs (IFG) and discussed many theorems, proofs, and proprieties were presented by Shannon and Atanassov [9]. Parvathi et al. [10–12] presented many different operations such as the join, union, and product of two IFGs. Some products such as strong, direct, and lexicographic products for two IFGs were presented by Rashmanlou et. al. [13]. In real-word problems, uncertainties due to inconsistent and indeterminate information about a problem cannot be represented properly by the fuzzy graph or IFG. To overcome this situation, a new concept introduced which is called the neutrosophic sets.

Smarandache [15] introduced the degree of indeterminacy/neutrality (I) as independent component in 1995 and defined the neutrosophic set on three components *(T, I, F)*=(Truth, Indeterminacy, Falsity). Neutrosophic sets are identified by three functions called truth-membership (T), indeterminacy-membership (I) and falsity-membership (F) whose values are real standard or non-standard subset of unit interval ]−0, 1+[. Single-valued neutrosophic set (SVNS) which takes the value from the subset of [0, 1] and is an instance of neutrosophic set and can be used expediently to deal with real-world problems, especially in decision support. The neutrosophic set can work with uncertain, indeterminate, vague, and inconsistent information of any uncertain real-life problem. The neutrosophic graph can efficiently model the inconsistent information about any real-life problem. Recently, many researchers have more actively worked on neutrosophic graph theory; for instance, Ye [15], Yang et al. [16], Naz et al. [17], Broumi [18-19], and Akram [20–23].

Section 2 briefly introduces the concepts and operations of NSs, SVNSs, and INSs. In Section 3, define a new set of vulnerability parameters based score functions and discussed some basic bounds. Then in Section 4, two examples are presented to illustrate the proposed parameters and its applications. Finally, Section 6 concludes the paper.

## 2. Preliminaries

In this section, we provide the basic concepts and definitions in neutrosophic sets and graphs and different types of neutrosophic sets and graphs. In 1999, Smarandache, F. introduced the following definition for Neutrosophic sets [NS]

.

*2.1. Definition [14]*

A Neutrosophic set A in X is defined by its "truth membership function" ($T_A$), an "indeterminacy-membership function" ($I_A(x)$), and a "falsity membership function" (FA(x)) where all are the subset of ]$^-$0, 1$^+$[ such that $^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$ for all $x \in X$.

2.2. *Definition [40]*

A NS $A$ in X is defined as $A = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\}$, and is called as SNS where $T_A(x), I_A(x), F_A(x) \in [0, 1]$. SNS is also denoted by
$A = \{< T_A(x), I_A(x), F_A(x) >\} or\ A = < a, b, c >$.

2.3. *Definition* [41]

An INS A in X is defined as
$A = \{< [inf\ T_A\ (x), sup\ T_A(x)], [inf\ I_A(x), sup\ I_A(x)], [inf\ F_A(x), sup\ F_A(x)] > | x \in X\}$,
Where $T_A(x), I_A(x), F_A(x) \in [0, 1]$ and $0 \leq sup\ T_A(x) + sup\ I_A(x) + sup\ F_A(x) \leq 3, x \in X$. An INS is also denoted by $A = < [a^L, a^U], [b^L, b^U], [c^L, c^U] >$.

## 3. Rank & Score Functions

Ranking of uncertainty numbers is an important issue in fuzzy set theory. Then numerical values are represented in uncertain nature termed as fuzzy numbers, a comparison of these numerical values is not easy. There are various methods have been introduced in literature to rank fuzzy numbers. An intuitionistic fuzzy number (IFN) is a generalization of fuzzy numbers. Many ranking methods for ordering of IFNs have been introduced in the literature. IFNs are treated as two families of metrics and developed a ranking method for IFNs by Grzegorzewski [42,43]. A ranking method to order triangular intuitionistic fuzzy numbers (TIFNs) proposed Mitchell [46] by accepting a statistical viewpoint and interpreting each IFN as ensemble of ordinary fuzzy numbers. Ranking of TIFN on the basis of value index to ambiguity index is proposed by Li [45] and solved a multi attribute decision-making problem.

A ranking function based on score function was proposed and the same used to solve intuitionistic fuzzy linear programming (IFLP), in which the data parameters are TIFNs. In the past, Nayagam et al. [47] introduced TIFNs and proposed a method to rank them. He has also [48] defined new intuitionistic fuzzy scoring method for the intuitionistic fuzzy number. Wang et al. in [49] proposed Intuitionistic trapezoidal fuzzy weighted arithmetic averaging operator and weighted geometric averaging operator.

The expected values, score function, and the accuracy function of intuitionistic trapezoidal fuzzy numbers are also defined. By comparing the score function and the accuracy function values of integrated fuzzy numbers, a ranking of the whole alternative set was attained. A ranking technique for TIFN using a,b-cut, score function and accuracy function was introduced by Nagoorgani et al. [51], is validated by applying the concept to solve the intuitionistic fuzzy variable linear programming problem. K. Arun Prakash et al. [52] introduced the method of ranking trapezoidal intuitionistic fuzzy numbers with centroid index uses the geometric center of a trapezoidal intuitionistic fuzzy number.

Decision making problems are one of the most widely used tools in any real time problems. In this process, several steps involve reaching the final destination and some of them may be vague in nature. The decision makers are facing several difficulties to make a decision within a reasonable time by using uncertain, imprecise, and vague information.

Researchers give more attention to the fuzzy set (FS) theory and corresponding extensions such as intuitionistic fuzzy set (IFS) theory, interval-valued IFS (IVIFS), Neutrosophic set (NS), etc.

for handling these situations. IFSs and IVIFSs have been widely applied by the various researchers in different decision-making problems. An aggregation operator for handling the different preferences of the decision makers towards the alternatives under IFS environment proposed by some of the authors proposed. Garg [26-30] presented a generalized score function for ranking the IVIFSs. Garg presented some series of geometric aggregation operator under an intuitionistic multiplicative set environment. He also presented [33] an accuracy function for interval-valued Pythagorean fuzzy sets. Garg studied a novel correlation coefficient between the Pythagorean fuzzy sets.

3.1. *Definition* [36]

Consider SNS $A = < a, b, c >$ then in order to rank the NS, score functions [35] have been defined as

$$K(A) = \frac{1 + a - 2b - c}{2} \; ; \; K(A) \in [0, 1]$$

$$I(A) = a - 2b - c \; ; \; I(A) \in [-3, 1].$$

These score functions I(A) and K(A) are unable to give the best alternative under some special cases. So, a new score function for ranking NS and INS by overcoming the shortcoming of the above functions has been proposed by Nancy & Harish Garg [36].

3.2. *Definition[36]*

Let $A = < a, b, c >$ be a SNS, a score function N(·), based on the "truth-membership degree" (a), "indeterminacy-membership degree" (b), and "falsity membership degree" (c) which is defined as

$$N(A) = \frac{1 + (a - 2b - c)(2 - a - c)}{2}$$

Clearly, if in some cases SNS has $a + c = 1$ then $N(A)$ reduces to K(A). Based on it, a prioritized comparison method for any two SNSs $A_1$ and $A_2$ is defined as

(i) if $K(A_1) < K(A_2)$ then $A_1 \prec A_2$,

(ii) if $K(A_1) = K(A_2)$ then

- if $N(A_1) < N(A_2)$ then $A_1 \prec A_2$
- if $N(A_1) > N(A_2)$ then $A_1 \succ A_2$
- if $N(A_1) = N(A_2)$ then $A_1 \sim A_2$

*3.3. Definition[36]*

Let *G* = (*U,V*) be a SVNG, where *U* is a single-valued neutrosophic vertex set of *G* and *V* is called single-valued neutrosophic edge set of *G*, such that $U = \{T_U(x), I_U(x), F_U(x) \; : x \in X\}$ is a *SVN*. The *score function* of *SVNG* is computed using the value of truth membership $T_U(x)$, indeterminacy membership $I_U(x)$ and falsity membership $F_U(x)$ and is defined by

$$S(u) = \frac{1 + pq}{2} \qquad \dots \dots \dots (1)$$

Where $p = T_U(x) - 2I_U(x) - F_U(x)$ and $q = 2 - T_U(x) - F_U(x)$

*3.4. Observations*

Case 1: if $B = (1,0,0)$ then $S(B) = 1$

Case 2: if $B = (0,0,1)$ then $S(B) = 0$

Case 3: if $B = (0,1,0)$ then $S(B) = -1.5$

Case 4: if $B = (1,1,0)$ then $S(B) = 0$

Case 5: if $B = (0,1,1)$ then $S(B) = -1$

Case 6: if $B = (1,0,1)$ then $S(B) = 0.5$

Case 7: if $B = (0,0,0)$ then $S(B) = 0.5$

Case 8: if $B = (1,1,1)$ then $S(B) = 0.5$

Therefore the bounds are sharp $-1.5 \leq S(B) \leq 1$

*3.5. Definition [19,40]*

A single-valued neutrosophic (SVNG) graph on a nonempty set $X$ is a pair $G = (U,V)$, where $U$ is single-valued neutrosophic set in $X$ and $V$ is single-valued Neutrosophic relation on $X$ such that

$$T_V(x,y) \leq \min\{T_U(x), T_U(y)\},$$
$$I_V(x,y) \leq \min\{I_U(x), I_U(y)\},$$
$$F_V(x,y) \leq \max\{F_U(x), F_U(y)\},$$

For all $x, y \in X$ . $U$ is said to be single-valued neutrosophic vertex set of $G$ and $V$ is called single-valued neutrosophic edge set of $G$, respectively.

*3.6. Definition*

The *order* and the *size* of a SVNG G are denoted by $O(G)$ and $S(G)$, respectively and are defined by

$$O(G) = \left(\sum_{x \in X} T_U(x), \sum_{x \in X} I_U(x), \sum_{x \in X} F_U(x)\right),$$

$$S(G) = \left(\sum_{xy \in V} T_V(x,y), \sum_{xy \in V} I_V(x,y), \sum_{xy \in V} F_V(x,y)\right),$$

*3.7. Definition*

The *degree* and the *total degree* of a vertex $x$ of a SVNG G are defined by

$$d_G(x) = \left(\sum_{x \neq y} T_V(x,y), \sum_{x \neq y} I_V(x,y), \sum_{x \neq y} F_V(x,y)\right),$$

and

$$Td_G(x) = \left(\sum_{x \neq y} T_V(x,y) + T_U(x), \sum_{x \neq y} I_V(x,y) + I_U(x), \sum_{x \neq y} F_V(x,y) + F_U(x)\right),$$

For $xy \in V$ and $x \in X$, is denoted by $d_G(x) = \big(d_T(x), d_I(x), d_F(x)\big)$ and $Td_G(x) = \big(Td_T(x), Td_I(x), Td_F(x)\big)$, respectively.

*3.8. Definition*

The *maximum degree* of a SVNG $G$ is defined as $\Delta(G) = (\Delta_T(G), \Delta_I(G), \Delta_F(G))$, where

$$\Delta_T(G) = \max\{d_T(x): x \in X\}$$
$$\Delta_I(G) = \max\{d_I(x): x \in X\}$$
$$\Delta_F(G) = \max\{d_F(x): x \in X\}$$

*3.9. Definition*

The *minimum degree* of a SVNG G is defined as $\delta(G) = (\delta_T(G), \delta_I(G), \delta_F(G))$, where

$$\delta_T(G) = \min\{d_T(x): x \in X\}$$
$$\delta_I(G) = \min\{d_I(x): x \in X\}$$
$$\delta_F(G) = \min\{d_F(x): x \in X\}$$

*3.10. Definition*

A SVNG G is called a regular if each vertex has same degree, (i.e.)

$$d_G(x) = (m_1, m_2, m_3), \qquad for\ all\ x \in X$$

Example:

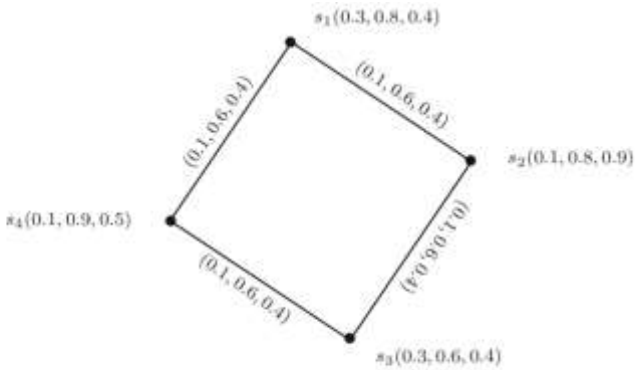


Fig. 1. Regular SVNG

| Vertices | T | I | F | p | q | S(u) |
|---|---|---|---|---|---|---|
| **S1** | 0.3 | 0.8 | 0.4 | -1.7 | 1.3 | **-0.605** |
| **S2** | 0.1 | 0.8 | 0.9 | -2.4 | 1 | **-0.7** |
| **S3** | 0.3 | 0.6 | 0.4 | -1.3 | 1.3 | **-0.345** |
| **S4** | 0.1 | 0.9 | 0.5 | -2.2 | 1.4 | **-1.04** |

Table: 1 Score value of regular SVNG

*3.11. Definition*

Let $G = (U, V)$ be an SVNG. G is said to be a strong SVNG if:

$$T_V(x, y) = \min(T_U(x), T_U(y))$$
$$I_V(x, y) = \min(I_U(x), I_U(y))$$
$$F_V(x, y) = \max(F_U(x), F_U(y)), \forall (x, y) \in E$$

Example:

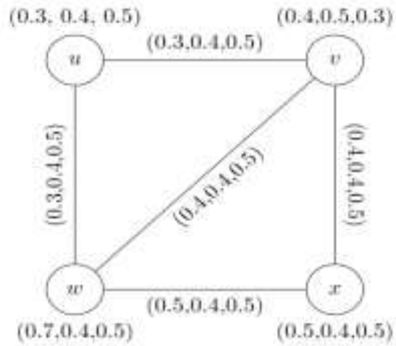


Fig. 2. Strong SVNG

| Vertices | T | I | F | p | q | S(u) |
|---|---|---|---|---|---|---|
| **u** | 0.3 | 0.4 | 0.5 | -1 | 1.2 | **-0.1** |
| **v** | 0.4 | 0.5 | 0.3 | -0.9 | 1.3 | **-0.085** |
| **x** | 0.5 | 0.4 | 0.5 | -0.8 | 1 | **0.1** |
| **w** | 0.7 | 0.4 | 0.5 | -0.6 | 0.8 | **0.26** |

Table: 2 Score value of Strong SVNG

*3.12. Definition*

A SVNG $G = (U, V)$ is called *complete* if the following conditions are satisfied:

$$T_V(x,y) = \min\{T_U(x), T_U(y)\}$$
$$I_V(x,y) = \min\{I_U(x), I_U(y)\}$$
$$F_V(x,y) = \max\{F_U(x), F_U(y)\}, \forall (x,y) \in E$$

Example:

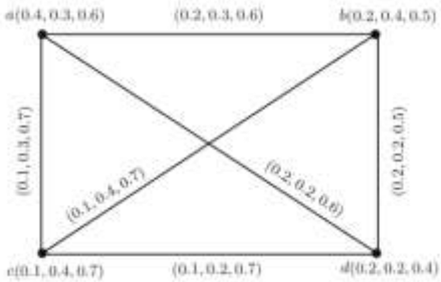

Fig. 3. Complete SVNG

| Vertices | T | I | F | p | q | S(u) |
|---|---|---|---|---|---|---|
| **a** | 0.4 | 0.3 | 0.6 | -0.8 | 1 | **0.1** |
| **b** | 0.2 | 0.4 | 0.5 | -1.1 | 1.3 | **-0.215** |
| **c** | 0.1 | 0.4 | 0.7 | -1.4 | 1.2 | **-0.34** |
| **d** | 0.2 | 0.2 | 0.4 | -0.6 | 1.4 | **0.08** |

Table: 3 Score value of Complete SVNG

*3.13. Definition*

A SVNG $G$ = ($U$, $V$) is called *complete bipartite neutrosophic graph* if the vertex set V can be divided into two nonempty sets, such that for every $v_1$, $v_2 \in V_1$ $or$ $V_2$ and for every $u \in V_1$ $and$ $v \in V_2$

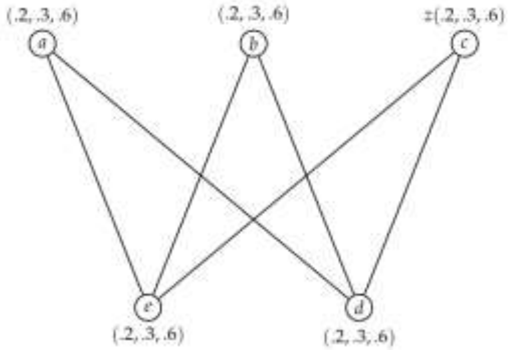


Fig. 4. Complete bipartite SVNG

| Vertices | T | I | F | p | q | S(u) |
|---|---|---|---|---|---|---|
| **a** | 0.2 | 0.3 | 0.6 | -1 | 1.2 | **-0.1** |
| **b** | 0.2 | 0.3 | 0.6 | -1 | 1.2 | **-0.1** |
| **c** | 0.2 | 0.3 | 0.6 | -1 | 1.2 | **-0.1** |
| **d** | 0.2 | 0.3 | 0.6 | -1 | 1.2 | **-0.1** |
| **e** | 0.2 | 0.3 | 0.6 | -1 | 1.2 | **-0.1** |

Table: 4 Score value of complete bipartite SVNG

## 4. Score Equitable Integrity and Strong Score Equitable Integrity of SVNG

*4.1 Definition*

Let $G = (\mathrm{U}, \mathrm{V})$ be a Single valued Neutrosophic graph. A subset $S \in U(G)$ is a said to be score equitable set if the score value of any two nodes in S differ by at most one. (i.e.) $|s(u) - s(v)| \leq 1, u, v \in S$. If e is an edge with end vertices u and v and score of u is greater than or equal to score of v then we say u strongly dominates v. If every vertex of V − S is strongly influenced by some vertex of S then S is called strong score set of G. The minimum cardinality of a strong dominating set is called the strong score number of G.

*4.2 Definition*

The equitable integrity of Single valued Neutrosophic graph G which is defined as $\mathrm{E}I(G) = min\{|S| + m(G - S): S$ is a score equitable set in $G\}$, where $m(G - S)$ denotes the order of the largest component in $G - S$.

*4.3 Definition*

The strong integrity of Single valued Neutrosophic graph G which is defined as $SI(G) = min\{|S| + m(G - S): S$ is a strong score set in $G\}$, where $m(G - S)$ denotes the order of the largest component in $G - S$.

*4.4 Example*

Consider the *SVNG* in Figure 5.

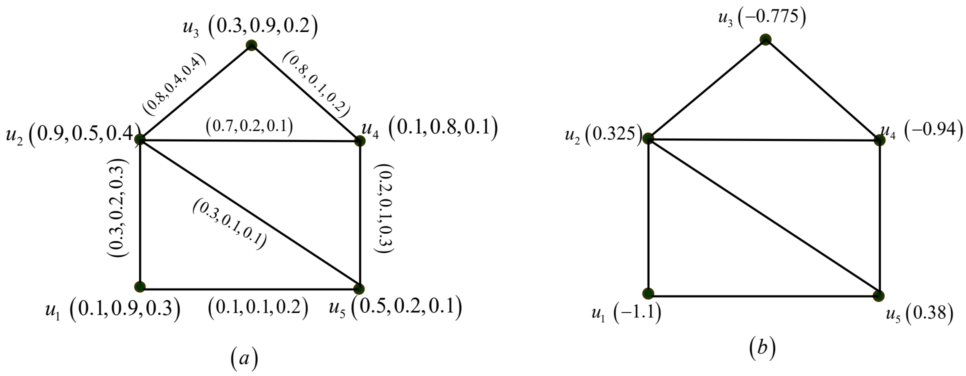


Fig. 5. Example of SVNG and its Score value

Using Eq. 1 we can compute score value of all the nodes. Figure 1.(b) shows the score value of each node. The score equitable sets are $S_1 = \{u_1, u_3\}, S_2 = \{u_1, u_4\}, S_3 = \{u_2, u_5\}, S_4 = \{u_3, u_4\}, S_5 = \{u_1, u_3, u_4\}$ and score equitable integrity is calculated by $EI(G) = min\{[2 + 3 = 5], [2 + 3 = 5], [2 + 2 = 4], [2 + 3 = 5], [3 + 2 = 5]\} = 4$. From this the score equitable integrity value is 4 and corresponding set is $S_3 = \{u_2, u_5\}$. The strong score equitable set is $S_3 = \{u_2, u_5\}$ and also strong equitable integrity is 4.

*4.5 Theorem:*

Let G be SVNG then

(i) $EI(G) = \text{n}\ if\ and\ only\ if\ G \cong K_n$

(ii) $SI(G) = \text{n}\ if\ and\ only\ if\ G \cong K_n$

Proposition: Every score equitable integrity and strong score equitable integrity of complete SVNG is equal to score equitable integrity and strong score equitable integrity of regular SVNG.

## 5. Case Study

### 5.1 Detection of a Safe Root for an Airline Journey

We consider a neutrosophic set of five countries: Germany, China, USA, Brazil and Mexico. Suppose we want to travel between these countries through an airline journey. The airline companies aim to facilitate their passengers with high quality of services. Air traffic controllers have to make sure that company planes must arrive and depart at right time. This task is possible by planning efficient routes for the planes. A neutrosophic graph of airline network among these five countries is shown in Fig.6 in which vertices and edges represent the countries and flights, respectively.

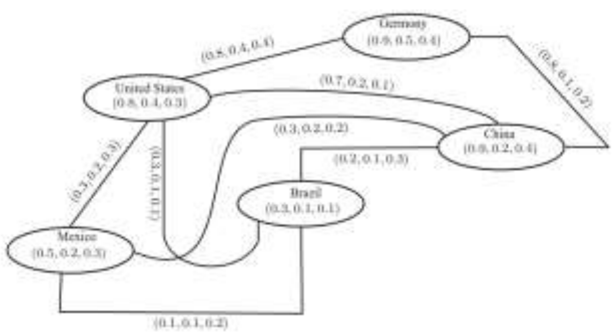

Fig.6. Neutrosophic Graph of Airline Network among these five Countries

| Country | T | I | F | p | q | S(u) |
|---|---|---|---|---|---|---|
| United States | 0.8 | 0.4 | 0.3 | -0.3 | 0.9 | 0.365 |
| Germany | 0.9 | 0.5 | 0.4 | -0.5 | 0.7 | 0.325 |
| China | 0.9 | 0.2 | 0.4 | 0.1 | 0.7 | 0.535 |
| Brazil | 0.3 | 0.1 | 0.1 | 0 | 1.6 | 0.5 |
| Mexico | 0.5 | 0.2 | 0.3 | -0.2 | 1.2 | 0.38 |

Table: 5 Score value of Airline Network

The truth-membership degree of each vertex indicates the strength of that country's airline system. The indeterminacy-membership degree of each vertex demonstrates how much the system is uncertain. The falsity-membership degree of each vertex tells the flaws of that system. The truth-membership degree of each edge interprets that how much the flight is safe. The indeterminacy-membership degree of each edge shows the uncertain situations during a flight such as weather conditions, mechanical error and sabotage. The falsity-membership degree of each edge indicates the flaws of that flight. For example, the edge between Germany and China indicates that the flight chosen for this travel is 80% safe, 10% depending on uncertain systems and 20% unsafe.

The truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of each edge are calculated by using the following relations.

$$T_V(x,y) \leq \min\{T_U(x), T_U(y)\},$$
$$I_V(x,y) \leq \min\{I_U(x), I_U(y)\},$$
$$F_V(x,y) \leq \max\{F_U(x), F_U(y)\},$$

Sometimes due to weather conditions, technical issues a passenger missed his direct flight between two particular countries. So, if he has to go somewhere urgently, then he has to choose indirect route as there are indirect routes between these countries.

Using Eq. 1 we can compute score value of all the nodes. Table 5. shows the score value of each node. We observe that all the sets are score and strong score equitable sets, and by computation the equitable integrity is, $\mathrm{EI}(G) = \min\{|S| + m(G - S)\} = 4$, where S = {China, USA} The strong score equitable set is S ={China, USA} and strong score equitable integrity is $\mathrm{SI}(G) = \min\{|S| + m(G - S)\} = 4$.

## 6. Conclusion

In this paper, Score Equitable Integrity and Strong Score Equitable Integrity of SVNG is introduced as a new vulnerability parameter in Neutrosophic graphs and some fundamental results in some standard graphs are established. Also the application on airline systems related to EI and SI parameters are dealt with real time scenario pertaining to the safety measures of flights connecting any two countries. We will focus on the study of EI and SI regular strong SVNG, $d_m$ regular SVNG $td_m$ regular SVNG, soft graphs and so on.

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Received: Nov 17, 2021. Accepted: Feb 6, 2022

# On Neurosophic Delta Generated Per-Continuous Functions in Neutrosophic Topological Spaces

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**Abstract:** In this work, we investigate new type of neutrosophic continuity, it is called neutrosophic almost $\delta gp$ −continuity functions, which is stronger than the conception of neutrosophic almost gpr-continuous functions. Also, new notions like neutrosophic $\delta gp$-compact, neutrosophic $\delta gp$-compact relative to neutrosophic space and neutrosophic strongly $\delta gp$ −closed for graph of neutrosophic functions are shown. Furthermore, some of its interest properties are shown and studied.

**Keywords:** neutrosophic sets, neutrosophic topological space, neutrosophic $\delta gp$ −continuity functions, neutrosophic almost gpr-continuous functions.

## 1. Introduction

As an expansion of Fuzzy sets given in 1965 by Zadeh [1] and Intuitionistic Fuzzy sets given in 1983 by Atanassav [2], the Neutrosophic sets (NSs) have been shown and explained by Smarandache. A (NS) is depicted by a truth value (membershis), an indeterminacy value and a falsity value (non-membershis). Salama and Alblowi [3] introduced the new concept of neutrosophic topological space (NTS) in 2012, which had been investigated recently. In 2018, Parimala M et al. explain the concept of Neutrosophic homeomorphism and Neutrosophic αψ homeomorphism in (NTS) [4]. In 2020, the notions of Ngpr homeomorphism and Nigpr homeomorphism in (NTS) are introduced and studied [5]. There are some sets in topological spaces their expansion in non-classical are studied, like soft sets [6-13], fuzzy sets [14-19], permutation sets [20-26], neutrosophic sets [27-30] nano sets [31,32] and others [33,34]. Here, we will use the conception of neutrosophic to study our expansion in non-classical. The neutrosophic closure and neutrosophic interior of any (NS) $A$ in (NTS) $(\Psi,\tau)$ are defined as $Ncl(A) = \cap \{A \subseteq B;\ B^c \in \tau\}$ and $Nint(A) = \cup \{B \subseteq A;\ \ B \in \tau\}$, respectively. The neutrosophic class of neutrosophic $\delta$gp-open (resp. neutrosophic δgp-closed, neutrosophic open, closed, neutrosophic regular closed, neutrosophic regular open, neutrosophic δ-preopen, neutrosophic δ-semiopen, neutrosophic preopen, neutrosophic semiopen, neutrosophic $e^*$-open and neutrosophic $\beta$-open) sets of $(\Psi,\tau)$ containing a point s∈ $\Psi$ is denoted by NδGPO($\Psi$,s) (resp. NδGPC($\Psi$,s), NO($\Psi$,s), NC($\Psi$,s), NRC($\Psi$,s), NRO($\Psi$,s), NδPO($\Psi$,s), NδSO($\Psi$,s), NPO($\Psi$,s), NSO($\Psi$,s), N$e^*$O($\Psi$,s) and N$\beta$ O($\Psi$,s)). That means if A is neutrosophic q-open (q-closed) set in neutrosophic topological space $(\Psi,\tau)$, where q is any property for the neutrosophic set A and s∈A for some s∈ $\Psi$, then it is denoted by NqO($\Psi$,s) (NqC($\Psi$,s)). In this paper, We're looking into a new kind of neutrosophic continuity, it is known as neutrosophic almost $\delta gp-$continuity functions, which is stronger than the conception of neutrosophic almost gpr-continuous functions. Also, some characteristics of neutrosophic almost $\delta gp$ $-$continuity functions are explained and discussed.

## 2. Preliminaries

Basic definitions and notations can be found here, which are used in this section are referred from the references [3,35-37].

**Definition 2.1:**

Assume $\Psi \neq \varphi$ . A neutrosophic set (NS) $\theta$ is defined as $\theta = \{\langle \alpha, \partial_\theta(\alpha),\ \omega_\theta(\alpha),\ \ell_\theta(\alpha)\rangle : \alpha \in \Psi\}$ where $\partial_\theta(\alpha)$ is the degree of membership, $\omega_\theta(\alpha)$ is the degree of indeterminacy and $\ell_\theta(\alpha)$ is the degree of non-membership, $\forall \alpha \in \Psi$ to $\theta$. Let $D = \{\langle \alpha, \partial_D(\alpha), \omega_D(\alpha), \ell_D(\alpha)\rangle : \alpha \in \Psi\}$ be the second (NS), then $\theta \cap D = \{\langle \alpha,\ \min\{\partial_\theta(\alpha), \partial_D(\alpha)\},\ \max\{\omega_\theta(\alpha), \omega_D(\alpha)\}, \max\{\ell_\theta(\alpha), \ell_D(\alpha)\}\rangle : \alpha \in \Psi\}$ and $\theta \cup D = \{\langle \alpha, \max\{\partial_\theta(\alpha), \partial_D(\alpha)\}, \min\{\omega_\theta(\alpha), \omega_D(\alpha)\}, \min\{\ell_\theta(\alpha), \ell_D(\alpha)$

$\}\rangle: \alpha \in \Psi\}$ . Also, $\theta \subseteq D$ if and only if $\partial_\theta(\alpha) \leq \partial_D(\alpha)$ , $\omega_\theta(\alpha) \geq \omega_D(\alpha)$ and $\ell_\theta(\alpha) \geq \ell_D(\alpha)$. The complement of $\theta$ is $\theta^c = \{\langle \alpha, \ell_\theta(\alpha), 1-\omega_\theta(\alpha), \partial_\theta(\alpha)\rangle: \alpha \in \Psi\}$

**Definition 2.2**: We say $(\Psi, \tau)$ is a neutrosophic topological space (NTS) if and only if $\tau$ is a collection of (NSs) in $\Psi$ and it such that:

(1) $1_N, 0_N \in \tau$, where $0_N = \{\langle \alpha, (0,1,1)\rangle: \alpha \in \Psi\}$ and $1_N = \{\langle \alpha, (1,0,0)\rangle: \alpha \in \Psi\}$.

(2) $\theta \cap \beta \in \tau$ for any $\theta, \beta \in \tau$,

(3) $\bigcup_{i\in I} \theta_i \in \tau$ for any arbitrary family $\{\theta_i | i \in I\} \subseteq \tau$. Also, any $\theta \in \tau$ is called neutrosophic open set (NOS) and we say neutrosophic closed set (NCS) for its complement.

**Definition 2.3**. Let $\Gamma \subseteq X$be (NS) in (NTS) $X$. We say $\Gamma$ is neutrosophic pre-closed (NP-C) (resp. neutrosophic regular-closed (NR-C), neutrosophic semi-closed (NS-C), neutrosophic $\beta$ -closed (N$\beta$ -C)) if $Ncl(int(\Gamma)) \subseteq \Gamma$ (resp. $\Gamma = Ncl(N\,int(\Gamma))$ , $Ncl(Nint(\Gamma)) \subseteq \Gamma$ and $Nint(Ncl(N\,int(\Gamma)) \subseteq \Gamma$).

**Definition 2.4**. Let $\Gamma \subseteq X$be (NS) in (NTS) $X$. We say $\Gamma$ is neutrosophic $\delta$-closed ($N\delta$-C), if $\Gamma = Ncl_\delta(\Gamma)$ where $Ncl_\delta(\Gamma) = \{p \in X: Nint(Ncl(D) \cap \Gamma \neq \varphi,\ D \in \tau$ and $p \in D\}$.

**Definition 2.5.** Let $\Gamma \subseteq X$be (NS) in (NTS) $X$. We say $\Gamma$ is neutrosophic $\delta$-preclosed (N$\delta P$-C) (resp. neutrosophic $e^*$-closed (N$e^* - C$), neutrosophic $\delta$ -semiclosed (N $\delta$S-C) and neutrosophic $\alpha$ -closed (N $\alpha$ -C)) if $Ncl(Nint_\delta(\Gamma)) \subseteq \Gamma$ (resp. $Nint(Ncl(Nint_\delta(\Gamma)) \subseteq \Gamma$ , $Nint(cl_\delta(\Gamma)) \subseteq \Gamma$ and $Ncl(Nint(Ncl_\delta(\Gamma))) \subseteq \Gamma$).

**Definition 2.6.** Let $\Gamma \subseteq X$be (NS) in (NTS) $X$. We say $\Gamma$ is;

(i) neutrosophic $\delta gp$ -closed ($N\delta gp$-C) (resp. neutrosophic $gpr$ -closed ($Ngpr$-C) and neutrosophic $gp$ -closed (N$gp$ -C)) if N$pcl(\Gamma) \subseteq L$ whenever $\Gamma \subseteq L$ and $L$ is neutrosophic $\delta$

-open (N$\delta$ -O) (resp. neutrosophic regular open (NR-O) and neutrosophic open (NO)) in $X$, where $\text{N}pcl(\Gamma) = \cap \{\Gamma \subseteq B;\ B\ \ is\ (\text{NP} - \text{C})\}$

(ii) neutrosophic $g\delta s$ -closed (N $g\delta s$ -C) if $Nscl(\Gamma) \subseteq L$ whenever $\Gamma \subseteq L$ and $L$ is (N$\delta$ -O) in $X$, where $\text{N}scl(\Gamma) = \cap \{\Gamma \subseteq B;\ B\ \ is\ (\text{NS} - \text{C})\}$.

The neutrosophic open sets are the complements of the previously described neutrosophic closed sets.

**Definition 2.7.** Assume $W$and $V$ are (NTSs) and $h: W \to V$ is a neutrosophic map (NM). We say $h$ is;

(i) Neutrosophic $R$ -map (NR-M) (resp. neutrosophic $\delta$ -continuous (N$\delta$ -CO), neutrosophic almost continuous (NA-CO), neutrosophic almost $pre$ -continuous (NAP-CO), neutrosophic almost $gp$ -continuous (NA$gp$ -CO), neutrosophic almost $G$ -continuous (NA$G$ -CO) and neutrosophic almost $g\delta s$ -continuous (NA$g\delta s$ -CO) if $h^{-1}(L)$ of any (NR-O) set $L$ of $V$ is (NR-O) set (resp. (N$\delta$ -O), (NO), (NP-O), (N$gp$ -O), (N$G$ -O) and (N $g\delta s$ -O)) set in $W$,

(ii) Neutrosophic $\delta gp$ -continuous (N $\delta gp$ -CO) if $h^{-1}(L)$ of any (NO) set $L$ of $V$ is neutrosophic $\delta gp$ -open (N$\delta gp$ -O) in $W$,

(iii) Neutrosophic almost contra continuous (NAC-CO) (resp. neutrosophic almost contra $sup\,e\,r$ -continuous (NACsup-CO) and neutrosophic contr $R$ -map (NCR-M)) if $h^{-1}(L)$ of any (NR-C) set $L$ of $V$ is (NO) (resp. ($N\delta$ -O) and (NR-O)) in $W$,

(iv) Neutrosophic almost perfectly-continuous (NAperf-CO) if the inverse image of any (NR-C) set $L$ of $V$ is neutrosophic clopen in $W$,

(v) Neutrosophic almost contra $\delta gp$ -continuous (NAC$\delta gp$ -CO) (resp. neutrosophic contra $\delta gp$ -continuous (NC$\delta$GP-CO) and neutrosophic $\delta gp$ -irresolute (N$\delta gp$ -IR), if $h^{-1}(L)$ of any (NR-O) (resp. (NO) and (N$\delta gp$ -C)) set $L$ of $V$ is (N$\delta gp$ -C) in $W$.

**Definition 2.8**. Let $\Omega$ be a (NTS), $NGPRO(\Omega) = \{A \subseteq \Omega \mid A\ is\ (NGPR - O) in\ \Omega\}$, $N\delta GPO(\Omega) = \{A \subseteq \Omega \mid A\ is\ (N\delta gp - O) in\ \Omega\}$ and $NPO(\Omega) = \{A \subseteq \Omega \mid A\ is\ (NP - O)\ in\ \Omega\}$. We say $\Omega$ is;

(i) Neutrosophic preregular $T_{\frac{1}{2}}$-space (Npr-reg-$T_{\frac{1}{2}}$-S) if $NGPRO(\Omega) = NPO(\Omega)$,

(ii) Neutrosophic $T_{\delta gp}$ -space (N$T_{\delta gp}$-S) if $N\delta GPO(\Omega) = NPO(\Omega)$

(iii) Neutrosophic $\delta gpT_{\frac{1}{2}}$-space (N$\delta gpT_{\frac{1}{2}}$-S) if N$\delta GPO(\Omega) = NPO(\Omega)$, ,

(iv) Neutrosophic extremely disconnected (NED) if the closure of any (NO) subset of $\Omega$ is (NO),

(v) Neutrosophic submaximal space (N-submax-S) if any (NP-O) set is (NO),

(vi) Neutrosophic strongly irresolvable (N-si) if any neutrosophic open subspace of $\Omega$ is irresolvable,

(vii) Neutrosophic nearly compact space (N-NCom-S) if any (NR-O) cover of $\Omega$ has a finite subcover,

(viii) Neutrosophic $r - T_1$-space (N-$r - T_1$-S) if for each $\sigma_1 \neq \sigma_2$ two points in $\Omega$, there exist (NR-O) sets $\lambda_1$ and $\lambda_2$ such that $\sigma_1 \in \lambda_1$, $\sigma_2 \notin \lambda_1$ and $\sigma_1 \notin \lambda_2$, $\sigma_2 \in \lambda_2$,

(ix) Neutrosophic $r - T_2$-space (N-$r - T_2$-S) if for each $\sigma_1 \neq \sigma_2$ in $\Omega$, there exist (NR-O) sets $\lambda_1$ and $\lambda_2$ such that $x \in U$, $y \in V$and $U \cap V = \varphi$,

(x) Neutrosophic $\delta gp - T_1$ -space (N $\delta gp - T_1$ -S) if for each $p \neq q$ in $\Omega$, there exist $\Psi_1, \Psi_2 \in N\delta GPO(\Omega)$ such that $P \in \Psi_1$, $q \notin \Psi_1$and $q \in \Psi_2$, $p \notin \Psi_2$,

(xi) Neutrosophic Hausdorff space (NH-S) (resp., Neutrosophic $\delta gp$-Hausdorff, space (N$\delta gp$-H-S)) if for each $\sigma_1 \neq \sigma_2$ in $\Omega$, there exist $\Psi_1$,$\Psi_2 \in NO(\Omega)$ (resp., $\Psi_1$,$\Psi_2 \in \delta GPO(\Omega)$ ) such that $x \in G$, $y \in H$and $G \cap H = \varphi$

(xii) Neutrosophic $\delta gp$ -additive space (N-$\delta gp$ -add-S) if $\delta GPC(\Omega)$ is closed under arbitrary intersections.

**Definition 2.9**. Let $\Omega$ be a (NTS) and $\lambda \subseteq \Omega$ . We say $\Omega$ is Neutrosophic $N$ -closed relative (NN-Cl-R) to $\lambda$ if any cover of $\lambda$ by (NR-O) sets of $\Omega$ has a finite subcover.

**Theorem 2.10**. (i)If $\lambda_1$ and $\lambda_2$ are (N$\delta gp$-O) subsets of a (N-submax-S)$\lambda$, then $\lambda_1 \cap \lambda_2$ is (N$\delta gp$-O) in $\Omega$.

(ii) Let $\Omega$ be a (N-$\delta gp$ -add-S). Then $\lambda_1 \subseteq \Omega$is (N$\delta gp$-C) if and only if N$\delta gp - cl(\lambda_1) = \lambda_1$, where N$\delta gp - cl(\lambda_1) = \cap \{\lambda_1 \subseteq B;\ B \ \ is\ (\text{N}\delta gp - \text{C})\}$.

**Definition 2.11**. Assume $\Omega$ is a (NTS). We say $\Omega$ is a neutrosophic locally indiscrete space (N-li-S) if $NO(\Omega) = NRO(\Omega)$, where $NO(\Omega) = \{A \subseteq \Omega \mid A\ is\ (NO) in\ \Omega\}$ and $NRO(\Omega) = \{A \subseteq \Omega \mid A\ is\ (NR - O) in\ \Omega\}$.

**Lemma 2.12**. Let $\Omega$ be a (NTS) and $\lambda \subseteq \Omega$ . Then these terms are true:

(i) $\lambda \ \ \in NPO(\Omega)$ if and only if $Nscl(\lambda\ \ ) = Nint(Ncl(\lambda\ \ ))$.

(ii) $p \in N\delta gpcl(\lambda)$ if and only if $B \cap \lambda \neq \varphi$ for any (N$\delta gp$ -O) set $B$ containing $r$.

**Remark: 2.13:** For any (NS) $\lambda \subseteq \Omega$ in (NTS) $\Omega$, we consider that:

(1) $Ncl(N\,int_\delta(\lambda)) = N\,cl_\delta(\,(N\,int_\delta(\,\lambda)),$

(2) $Nint(N\,cl_\delta(\,\lambda)) = N\,int_\delta(\,(N\,cl_\delta(\,\lambda)).$

(3) $N\,int_\delta(\,\Omega\backslash\lambda) = \Omega(N\,cl_\delta(\,\lambda)) \in NRO(\Omega), if\ \lambda\ is\ (\text{N}e^* - O).$

## 3. Neutrosophic Almost $\delta gp$ -Continuous Functions.

**Definition 3.1.** Let $h: \Omega \rightarrow \mu$ be a (NM). We say $h$ is neutrosophic almost $\delta gp$ -continuous (NA$\delta gp$ -CO) if $h^{-1}(\lambda) \in N\delta GPC(\Omega)$ for each (NR-C) set $\boldsymbol{\lambda}$of $\boldsymbol{\mu}$.

**Example 3.2.** Define the neutrosophic sets $D_1, D_2, D_3, D_4$ and $H_1, H_2, H_3, H_4, H_5$ as follows:

$D_1$={⟨$a$, (0,1,0.3)⟩, ⟨$b$, (0.3,0.5,1)⟩, ⟨$c$, (0,0.6,1)⟩, ⟨$d$, (0.5,1,0.8)⟩}

$D_2$={⟨$a$, (0.2,0.4,1)⟩, ⟨$b$, (0,1,0.3)⟩, ⟨$c$, (0.7,0.1,0.6)⟩, ⟨$d$, (0,0.5,1)⟩}

$D_3$={⟨$a$, (0.2,0.4,0.3)⟩, ⟨$b$, (0.3,0.5,0.3)⟩, ⟨$c$, (0.7,0.6,0.6)⟩, ⟨$d$, (0.5,0.5,0.8)⟩}

$D_4$={⟨$a$, (0.3,0.3,0.2)⟩, ⟨$b$, (0.4,0.4,0.3)⟩, ⟨$c$, (0.8,0.5,0.5)⟩, ⟨$d$, (0.6,0.4,0.7)⟩}

And

$H_1$={⟨$a$, (0.2,0.4,1)⟩, ⟨$b$, (0,1,0.3)⟩, ⟨$c$, (0.7,0.1,0.6)⟩, ⟨$d$, (0,0.5,1)⟩}

$H_2$={⟨$a$, (0,1,0.3)⟩, ⟨$b$, (0.3,0.5,1)⟩, ⟨$c$, (0,0.6,1)⟩, ⟨$d$, (0.5,1,0.8)⟩}

$H_3$={⟨$a$, (0.3,0.3,0.2)⟩, ⟨$b$, (0.4,0.4,0.3)⟩, ⟨$c$, (0.8,0.5,0.5)⟩, ⟨$d$, (0.6,0.4,0.7)⟩}

$H_4$={⟨$a$, (0.2,0.4,0.3)⟩, ⟨$b$, (0.3,0.5,0.3)⟩, ⟨$c$, (0.7,0.6,0.6)⟩, ⟨$d$, (0.5,0.5,0.8)⟩}

Now, let t={$1_N, 0_N, D_1, D_2, D_3, D_4$} and h={$1_N, 0_N, H_1, H_2, H_3, H_4$} then (X,t) and (Y,h) are (NTSs), where X = {a,b,c,d} = Y . Define $f: X \to Y$ by $f(a) = f(c) = b, f(b) = a, f(d) = c$. We consider that $f$ is neutrosophic almost $\delta gp$ -continuous.

**Theorem 3.3**. Let $h: X \to Y$ be (NM). Then $h$ is (NA$\delta gp$ -CO) if and only if $h^{-1}(\mu)$ of any (NR-O) set $\mu$ of *Y* is (N$\delta gp$ -O) in *X*.

**Proof:** since the complement for any (NO) is (NC) and by Definition (3.1). Then the theorem is held.

**Example 3.4.** Define the neutrosophic sets $D_1, D_2, D_3, D_4$ and $H_1, H_2, H_3, H_4, H_5$ as follows:

$D_1$={⟨$a$, (0.1,1,0.4)⟩, ⟨$b$, (0.4,0.6,1)⟩, ⟨$c$, (0.1,0.7,1)⟩, ⟨$d$, (0.6,1,0.9)⟩}

$D_2$={⟨$a$, (0.3,0.5,1)⟩, ⟨$b$, (0.1,1,0.4)⟩, ⟨$c$, (0.8,0.2,0.7)⟩, ⟨$d$, (0.1,0.6,1)⟩}

$D_3$={⟨$a$, (0.3,0.5,0.4)⟩, ⟨$b$, (0.4,0.6,0.4)⟩, ⟨$c$, (0.8,0.7,0.7)⟩, ⟨$d$, (0.6,0.6,0.9)⟩}

$D_4$={⟨$a$, (0.4,0.4,0.3)⟩, ⟨$b$, (0.5,0.5,0.4)⟩, ⟨$c$, (0.9,0.6,0.6)⟩, ⟨$d$, (0.7,0.5,0.8)⟩}

And

$H_1$={⟨$a$, (0.3,0.5,1)⟩, ⟨$b$, (0.1,1,0.4)⟩, ⟨$c$, (0.8,0.2,0.7)⟩, ⟨$d$, (0.1,0.6,1)⟩}

$H_2$={⟨$a$, (0.1,1,0.4)⟩, ⟨$b$, (0.4,0.6,1)⟩, ⟨$c$, (0.1,0.7,1)⟩, ⟨$d$, (0.6,1,0.9)⟩}

$H_3$={⟨$a$, (0.4,0.4,0.3)⟩, ⟨$b$, (0.5,0.5,0.4)⟩, ⟨$c$, (0.9,0.6,0.6)⟩, ⟨$d$, (0.7,0.5,0.8)⟩}

$H_4$={⟨$a$, (0.3,0.5,0.4)⟩, ⟨$b$, (0.4,0.6,0.4)⟩, ⟨$c$, (0.8,0.7,0.7)⟩, ⟨$d$, (0.6,0.6,0.9)⟩}

Now, let t=$\{1_N, 0_N, D_1, D_2, D_3, D_4\}$ and h=$\{1_N, 0_N, H_1, H_2, H_3, H_4\}$ then (X,t) and (Y,h) are (NTSs), where $X$= {$a$,b,c,d} = $Y$. Define $h: X \to Y$ by $h(a) = h(c) = b, h(b) = a, h(d) = c$. Then we consider that $h$ is (NA$\delta gp$ -CO). Also, $h^{-1}(\mu)$ is (N$\delta gp$ -O) in $X$ for any (NR-O) set $\mu$ of $Y$ .

**Remark 3.4**. Let $h: \Omega \to \mu$ be a (NM). Then by Definitions (2.7) and (3.1), we consider diagram (1) as follows:

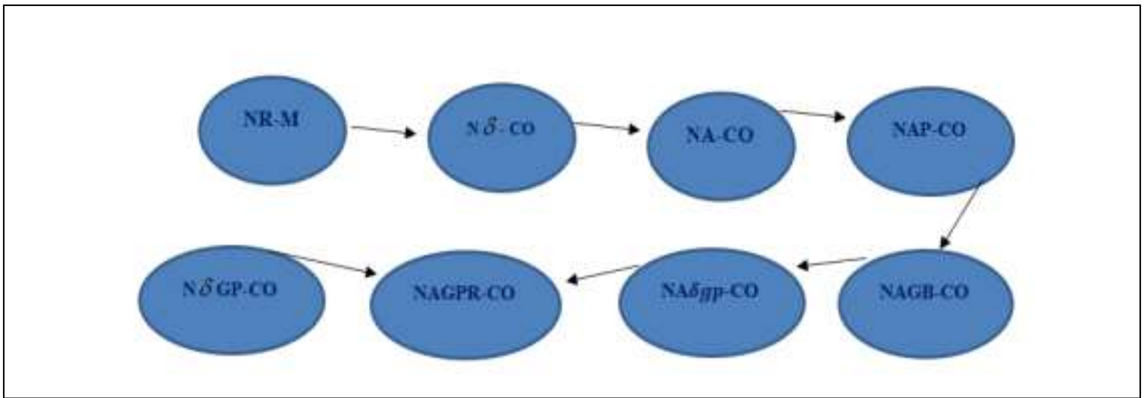


**Diagram (1):** The relationships among some classes of neutrosophic continuous

**Theorem 3.5**. If $\boldsymbol{f}: \boldsymbol{\mu} \to \boldsymbol{\eta}$ is (NA$\boldsymbol{\delta gp}$ -CO) and $\boldsymbol{\eta}$ is (N-li-S), then $\boldsymbol{f}$ is (N$\boldsymbol{\delta gp}$ -CO).

**Proof.** Let $\boldsymbol{\lambda}$ be (NO) set in $\boldsymbol{\eta}$, then $\boldsymbol{\lambda}$ is (NR-O) in $\boldsymbol{\eta}$. Since $\boldsymbol{f}$ is (NA$\boldsymbol{\delta gp}$ -CO), then $\boldsymbol{f^{-1}(\lambda)}$ is (N$\boldsymbol{\delta gp}$ -O) in $\boldsymbol{\mu}$. Hence $\boldsymbol{f}$ is (N$\boldsymbol{\delta gp}$ -CO)

**Theorem 3.6**. Let $\boldsymbol{\Omega}$ be a (N-li-S), then these terms are equivalent:

(i) $f: \Omega \to \mu$is (N$gpr$ -CO),

(ii) $f: \Omega \to \mu$ is (NA$\delta gp$ -CO),

(iii) $f: \Omega \to \mu$ is (NA$gp$ -CO).

**Proof.** Follows from the Definitions (2.11), (2.7) and (3.1).

**Remark 3.7.** It is clear from the definitions in section 2, we consider that all of the theorems [(3.8)-(3.13)] are held.

**Theorem 3.8**. (i) If $f: \Omega \to \mu$is (NA$g\delta s$ -CO) with $\Omega$ as (NED), then it is (NA$\delta gP$ -CO).

(ii) If $f: \Omega \to \mu$is (NA$\delta gP$ -CO) with $\Omega$ as (N-si). Then it is (NA$g\delta s$ -CO).

**Theorem 3.9**. All of these terms are equivalent:

(i) $f: \Omega \to \eta$ is (NAperf-CO),

(ii) $f: \Omega \to \eta$ is (NAC-CO) and (NAP-CO),

(iii) $f: \Omega \to \eta$ is (NAC-CO) and (NA$gp$ -CO),

(iv) $f: \Omega \to \eta$ is (NACsup-CO) and (NA$\delta gp$ -CO),

(v) $f: \Omega \to \eta$ is (NCR-M) and (NA$gpr$ -CO),

(vi) $f: \Omega \to \eta$ is (NCR-M) and (NAP-CO),

(vii) $f: \Omega \to \eta$ is (NACsup-CO) and (NAP-CO).

**Theorem 3.10.** Let $\Omega$ be a(N$\delta gpT_{\frac{1}{2}}$-S). Then all of these terms are equivalent:

(i) $f: \Omega \to \eta$ is (NAP-CO),

(ii) $f: \Omega \to \eta$ is (NA$gp$ -CO),

(iii) $f: \Omega \to \eta$ is (NA$\delta gp$ -CO).

**Theorem 3.11**. Let $\Omega$ be a (Npr-reg-$T_{\frac{1}{2}}$-S). Then All of these terms are equivalent:

(i) $f: \Omega \to \eta$ is (NAP-CO),

(ii) $f: \Omega \to \eta$ is (NA$gp$ -CO),

(iii) $\boldsymbol{f}: \boldsymbol{\Omega} \rightarrow \boldsymbol{\eta}$ is (NA$\boldsymbol{\delta gp}$ -CO),

(iv) $\boldsymbol{f}: \boldsymbol{\Omega} \rightarrow \boldsymbol{\eta}$ is (NA$\boldsymbol{gpr}$ -CO).

**Theorem 3.12.** Let $\boldsymbol{\Omega}$ be a $\boldsymbol{T_{\delta gp}}$-space. Then these terms are equivalent:

(i) $\boldsymbol{f}: \boldsymbol{\Omega} \rightarrow \boldsymbol{\mu}$ is (NA-CO);

(ii) $\boldsymbol{f}: \boldsymbol{\Omega} \rightarrow \boldsymbol{\mu}$ is (NA$\boldsymbol{pre}$ -CO),

(iii) $\boldsymbol{f}: \boldsymbol{\Omega} \rightarrow \boldsymbol{\mu}$ is (NA$\boldsymbol{gp}$ -CO),

(iv) $\boldsymbol{f}: \boldsymbol{\Omega} \rightarrow \boldsymbol{\mu}$ is (NA$\boldsymbol{\delta gp}$ - CO),

(v) $\boldsymbol{f}: \boldsymbol{\Omega} \rightarrow \boldsymbol{\mu}$ is (NA$\boldsymbol{gpr}$ - CO).

**Theorem 3.13.** The following are equivalent:

(i) $\boldsymbol{f}: \boldsymbol{\Omega} \rightarrow \boldsymbol{\mu}$ is (NA$\boldsymbol{\delta gp}$ -CO) and $\boldsymbol{\Omega}$ is (N$\boldsymbol{\delta gp}$ -add-S),

(ii) for each $\boldsymbol{\sigma} \in \boldsymbol{\Omega}$ and each open set $\lambda_1$ containing $f(p)$, there exists (N$\delta gp$ -O) set $\lambda_2$ containing $\sigma$ such that $f(\lambda_2) \subset \text{N}int(Ncl(\lambda_1))$.

**Theorem 3.14**. All of these terms are equivalent:

(i) $\boldsymbol{f}: \boldsymbol{\Omega} \rightarrow \boldsymbol{\mu}$ is (NA$\boldsymbol{\delta gp}$ -CO) and $\boldsymbol{\Omega}$ is (N$\boldsymbol{\delta gp}$ -add-S),

(ii) For each $\sigma \in \Omega$ and each $\lambda_1 \in NO(\mu, f(\sigma))$, there exists $\lambda_2 \in \text{N}\delta GPO(\Omega, \sigma)$ such that $f(\lambda_2) \subset Nscl(\lambda_1)$;

(iii) For each $\sigma \in \Omega$ and each $\lambda_3 \in NO(\mu, f(\sigma))$,there exists $\gamma_1 \in \text{N}\delta GPO(\Omega, \sigma)$ such that $f(\gamma_1) \subset \lambda_3$;

(iv) For each $\sigma \in \Omega$ and each $\gamma_2 \in N\delta O(\mu, f(\sigma))$, there exists $\Sigma \in \text{N}\delta GPO(\Omega, \sigma)$ such that $f(\Sigma) \subset \gamma_2$;

(v) For each $\sigma \in \Omega$ and each $\gamma_2 \in N\delta C(\mu, f(\sigma))$, there exists $\Sigma \in N\delta GPC(\Omega, \sigma)$ such that $f(\Sigma) \subset \gamma_2$;

**Proof.** (i) → (ii): Let $\sigma \in \Omega$and $N$ be (NO) set of $\mu$ containing $f(\sigma)$. By (i) and Theorem 3.13, there exists $\lambda_2 \in$ N$\delta GPO(\Omega,\sigma)$ such that $f(\lambda_2) \subset Nint(Ncl(\lambda_1))$. Since $\lambda_2$ is preopen, then by Lemma 2.12(i), $f(\lambda_2) \subset Nscl(\lambda_1)$.

(ii) → (iii): Let $\sigma \in \Omega$ and $\lambda_1 \in NRO(\mu, f(\sigma))$. Then $\lambda_1 \in NO(\mu, f(\sigma))$. By (ii), there exists $\lambda_2 \in N\delta GPO(\Omega,\sigma)$ such that $f(\lambda_2) \subset Nscl(\lambda_1)$. Since $\lambda_3$ is (NP-O), then by Lemma 2.12 (i), $f(\lambda_2) \subset N\,int(Ncl(\lambda_1)) = \lambda_1$.

(iii) → (iv): Let $\sigma \in \Omega$ and $\lambda_1 \in N\delta O(\mu, f(\sigma))$, then there exists $\lambda_2 \in NO(\Omega, f(\sigma))$ such that $M \subset N\,int(Ncl(M)) \subset N$. Since $N\,int(Ncl(M)) \in NRO(Y, f(p))$, by (iii), there exists $\Sigma \in N\delta GPO(\Omega,\sigma)$ such that $f(\Sigma) \subset N\,int(N\,cl(\lambda_2)) \subset \lambda_1$.

(iv)→ (i): Let $\sigma \in \Omega$ and $\lambda_1 \in NO(\mu, f(\sigma))$. Then $N\,int(Ncl(\lambda_1)) \in N\delta O(\mu, f(\sigma))$. By (iv), there exists $\lambda_2 \in N\delta GPO(\Omega,\sigma)$ such that $f(\lambda_2) \subset N\,int(Ncl(\lambda_1))$. Hence $f$ is (NA$\delta gp$ -CO).

(iv) ↔ (v): Obvious.

**Remark 3.15.** If $\Omega$ is a (N$\delta gp$ -add-S), then $\lambda \subseteq \Omega$ is (N$\delta gp$ -C) (resp. (N$\delta gp$ -O)) if and only if N $\delta gp - cl(\lambda) = \lambda$ (resp. N $\delta gp - int(\lambda) = \lambda$ )., where N$\delta gp - cl(\lambda) = \cap \{\lambda \subseteq B;\ B\ \ is\ (\text{N}\delta gp - \text{C})\}$ and N$\delta gp - int(\lambda) = \cap \{B \subseteq \lambda;\ B\ \ is\ (\text{N}\delta gp - \text{O})\}$

**Theorem 3.16.** All of these terms are equivalent:

(i) $f: \Omega \to \mu$ is (NA$\delta gp$ -CO) and $\Omega$is (N$\delta gp$ -add-S),

(ii) $f(N\delta gp - cl(\lambda_2)) \subset Ncl_\delta(f(\lambda_1))$ for each $\lambda_1 \subseteq \Omega$;

(iii) N$\delta gp - cl(f^{-1}(\lambda_2)) \subset f^{-1}(Ncl_\delta(\lambda_2))$ for each $\lambda_2 \subseteq \mu$;

(iv) $f^{-1}(\beta_1) \in N\delta GPC(\Omega)$ for each $\beta_2 \in N\delta C(\mu)$;

(v) $f^{-1}(\gamma_1) \in N\delta GPO(\Omega)$ for each $\gamma_1 \in N\delta O(\mu)$;

**Proof.** (i) → (ii) Suppose that $\lambda_2 \in N\delta C(\mu)$; such that $f(\lambda_1) \subset \lambda_2$ . Observe that $\lambda_1 = Ncl_\delta(\lambda_1) = \cap \{\gamma_2 : \lambda_2 \subset \gamma_2$ and $\gamma_2 \in NRC(\mu)\}$ and so $f^{-1}(\lambda_2) = \cap \{f^{-1}(\gamma_2) : \lambda_2 \subset \gamma_2\}$ . By (i) and Definition 2.8 (xii), we have $f^{-1}(\lambda_2) \in N\delta GPC(\Omega)$ and $\lambda_1 \subset f^{-1}(\lambda\ )$ . Hence N $\delta gp - cl(\lambda_1)$ $\subset f^{-1}(\lambda_2)$ , and it follows that $f(N\delta gp - cl(\lambda_1)) \subset \lambda_2$. Since this is true for any (N$\delta$ -C) set $\lambda_2$ containing $f(\lambda_1)$, we have $f(N\delta gp - cl(\lambda_1)) \subset Ncl_\delta(f(\lambda_1))$..

(ii) → (iii) Let $\beta_1 \subset \mu$, then $f^{-1}(\beta_1) \subset \Omega$. By (ii),

$f\left(N\delta gp - cl(f^{-1}(\beta_1))\right) \subset$ N $cl_\delta\left(f(f^{-1}(\beta_1))\right) \subset$ N $\delta gp - cl(\beta_1)$ . So that N$\delta gp - cl(f^{-1}(\beta_1)) \subset f^{-1}(Ncl_\delta(\beta_1))$

(iii) → (iv) Let $\beta_2 \in N\delta C(\mu)$. Then by (iii), N $\delta gp - cl(f^{-1}(\beta_2))$ $\subset f^{-1}(Ncl_\delta(\beta_2))$ $= f^{-1}(\beta_2)$.In consequence, N$\delta gp - cl(f^{-1}(\beta_2))$ $= f^{-1}(\beta_2)$ and hence by remark ( 3.15), $f^{-1}(\beta_2) \in$ N$\delta GPC(\Omega)$.

(iv) → (v): Clear.

(v) → (i): Let $\lambda_2 \in NRO(\mu)$ Then $\lambda_2 \in N\delta O(\mu)$. By (v), $f^{-1}(\lambda_2) \in$ N$\delta GPO(\Omega)$. Hence by Theorem 3.3, $f$ is (NA$\delta gp$ -CO).

**Theorem 3.17**. All of these terms are equivalent:

(i) $f : \Omega \rightarrow \eta$ is almost $\delta gp$ -continuous and $\Omega$is (N$\delta gp$ -add-S),

(ii) For any $\lambda \in NO(\eta)$,$f^{-1}(N\,int\,(Ncl(\lambda) \in N\delta GPO(\Omega)$;

(iii) For any $\gamma \in NC(\eta)$,$f^{-1}(Ncl(N\,int\,(\gamma) \in N\delta GPC(\Omega)$;

(iv) For any $\lambda \in N\beta O(\eta)$, N$\delta gpcl(f^{-1}(\lambda)) \subset f^{-1}(Ncl(\lambda))$;

(v) For any $\gamma \in N\beta C(\eta)$,$f^{-1}(N\,int\,(\gamma) \subset N\delta gp\,int\,(^{-1}\,(\gamma))$;

(vi) For any $\gamma \in NSC(\eta), f^{-1}(N\,int(\gamma)) \subset N\delta gp\,int(f^{-1}(\gamma))$;

(vii) For any $\lambda \in NSO(\eta)$, N$\delta gpcl\left(f^{-1}(\lambda)\right) \subset f^{-1}\left(Ncl(\lambda)\right)$;

(viii) For any $\gamma \in NPO(\eta))$, $f^{-1}(\gamma) \subset N\delta gp\,int(f^{-1}\left(N\,int(Ncl(\gamma)\right)$

Proof. (i) $\leftrightarrow$ (ii): Let $\lambda \in NO(\eta)$ .Since N $int(N\,cl(\lambda)) \in NRO(\eta)$ Then by (i), $f^{-1}\left(N\,int(Ncl(\lambda)\right) \in N\delta GPO(\Omega)$. The converse is similar.

(i) $\leftrightarrow$ (iii) It is similar to (i) $\leftrightarrow$ (ii).

(i) $\rightarrow$ (iv): Let $\lambda \in N\beta O(\eta)$, then N$cl(\lambda) \in NRC(\eta)$ so by (i), $f^{-1}\left(Ncl(\lambda)\right) \in N\delta GPC(\Omega)$.

Since $f^{-1}(\lambda) \subset f^{-1}\left(Ncl(\lambda)\right)$ which implies $N\delta gpcl\left(f^{-1}(\lambda)\right) \subset f^{-1}\left(Ncl(\lambda)\right)$.

(iv) $\rightarrow$ (v) and (vi) $\rightarrow$ (vii):Obvious

(v) $\rightarrow$ (vi): It follows from the fact that N$SC(\eta) \subset N\beta C(\eta)$.

(vii) $\rightarrow$ (i): It follows from the fact that N$RC(\eta) \subset NSO(\eta)$.

(i) $\leftrightarrow$ (viii): Let $\lambda \in NPO(\eta)$. Since N$int(\,Ncl(\lambda)) \in NRO(\eta)$, then by (i),

$f^{-1}\left(Nint(\,Ncl(N)\right) \quad \in N\delta GPO(X)$ and hence $f^{-1}(\lambda) \subset f^{-1}(N\,int(\,Ncl(\lambda)))$ $= N\delta gp\,int(\,f^{-1}(N\,int(\,Ncl(\lambda))))$. Conversely, let $\lambda \in NRO(\eta)$. Since $\lambda \in NPO(\eta)$, $f^{-1}(\eta) \subset \quad N\delta gp\,int(\,f^{-1}(N\,int(\,cl(\eta)))) \quad = N\delta gp\,int(\,f^{-1}(\lambda))$, in consequence, N$\delta gp\,int(\,f^{-1}(\lambda)) = f^{-1}(\lambda)$ and by remark (3.15), $f^{-1}(\lambda) \in N\delta GPO(\Omega)$.

**Theorem 3.18**. The following are equivalent:

(i) $f: \mu \rightarrow \eta$ is (NA$\delta gp$ -CO) and $\mu$ is (N$\delta gp$ -add-S),

(ii) For any (N$e^*$-O) set $\alpha$ of $\eta$, $f^{-1}\left(Ncl_\delta(\alpha)\right)$ is (N$\delta gp$ -C) in $\mu$,

(iii) For any (N$\delta$S-O) subset $\alpha$ of $\eta$, $f^{-1}\left(Ncl_\delta(\alpha)\right)$ is (N$\delta gp$ -C) in $\mu$;

(iv) For any (N$\delta$P-O) subset $\alpha$ of $\eta$, $f^{-1}(N\,int(N\,cl_\delta(\alpha)))$ is (N$\delta gp$ -O)in $\mu$;

(v) For any (NO) subset $\alpha$ of $\alpha$, $f^{-1}(N\,int(N\,cl_\delta(\alpha)))$ is (N$\delta gp$ -O) in $\mu$;

(vi) For any (NC) subset $\alpha$ of $Y$, $f^{-1}(Ncl(N\,int_{\delta}(\alpha)))$ is (N$\delta gp$ -C) in $\mu$.

**Proof.** (i) → (ii): Let $\alpha \in Ne^{*}O(\eta)$ . Then by remark (2.13), N$cl_{\delta}(\alpha) \in NRC(\eta)$. By (i), $f^{-1}(Ncl_{\delta}(\eta)) \in N\delta GPC(\mu)$.

(ii) → (iii): Obvious since N$\delta SO(\eta) \subset Ne^{*}O(\eta)$.

(iii) → (iv): Let $\alpha \in N\delta PO(\eta)$,then N$int_{\delta}(\eta\backslash\alpha) \in N\delta SO(\eta)$). By (iii), $f^{-1}(Ncl_{\delta}(N\,int_{\delta}(\eta\backslash\alpha)) \in N\delta GPC(\mu)$ which implies $f^{-1}(N\,int(Ncl_{\delta}(\alpha)) \in N\delta GPO(\mu)$.

(iv) → (v): Obvious since N$O(\eta) \subset N\delta PO(\eta)$.

(v) → (vi):Clear

(vi) → (i): Let N$\alpha \in NRO(\eta)$. Then $\alpha = Nint(Ncl_{\delta}(\alpha))$ and hence $(\eta\backslash\alpha) \in NC(\eta)$. By (vi), $f^{-1}(\eta\backslash\alpha) = \mu\backslash f^{-1}(N\,int(N\,cl_{\delta}(\alpha)) = f^{-1}(Ncl(N\,int_{\delta}(\eta\backslash\alpha))) \in N\delta GPC(\mu)$. Thus $f^{-1}(\alpha) \in N\delta GPO(\mu)$.

**Theorem 3.19**. If $\boldsymbol{f}:\boldsymbol{\Omega} \to \boldsymbol{\mu}$ is (NA$\boldsymbol{\delta gp}$ -CO) injective function and $\boldsymbol{\mu}$ is (N-$r - T_1$-S), then $\boldsymbol{X}$ is (N$\delta gp - T_1$-S).

**Proof.** Let $(\mu,\sigma)$ be (N-$r - T_1$-S) and $p, q \in \Omega$, with $p \neq q$. Then there exist (NR-O) subsets $\lambda$, $\gamma$ in $Y$ such that $f(p) \in \lambda$,$f(q) \notin \lambda$, $f(p) \notin \gamma$ and $f(q) \in \gamma$. Since $f$ is (NA$\delta gp$ -CO), $f^{-1}(\lambda)$ and $f^{-1}(\gamma) \in$ N$\delta GPO(\Omega)$ satisfy $p \in f^{-1}(\lambda)$, $q \notin f^{-1}(\lambda)$, $p \notin f^{-1}(\gamma)$ and $q \in f^{-1}(\gamma)$. Hence $\Omega$ is (N$\delta gp - T_1$-S).

**Theorem 3.20**. If $\boldsymbol{f}:\boldsymbol{\Omega} \to \boldsymbol{\eta}$ is (NA$\boldsymbol{\delta gp}$ -CO) injective function and $\boldsymbol{\eta}$ is(N-$\boldsymbol{r} - \boldsymbol{T_2} - S$), then $\Omega$ is $(N\delta gp - T_2 - S)$.

**Proof.** The proof is the same way of Theorem (3.20).

**Theorem 3.21**. If $\boldsymbol{f}, \boldsymbol{g}:\boldsymbol{\Omega} \to \boldsymbol{\eta}$ are (NA$\boldsymbol{\delta gp}$ -CO) with $\boldsymbol{\Omega}$ as (N-submax-S) and (N$\boldsymbol{\delta gp}$ -add-S) and $\boldsymbol{\eta}$ is (NH-S), then the set $\{\boldsymbol{x} \in \Omega : \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})\}$ is $\boldsymbol{\delta gp}$ -closed in $\boldsymbol{\Omega}$.

**Proof.** Let $E = \{x \in \Omega : f(x) = g(x)\}$ and $x \notin (\Omega \backslash \lambda)$. Then $f(x) \neq g(x)$. Since $\eta$ (NH-S), there exist (NO) sets $\lambda_1$ and $\lambda_2$ of $\eta$ satisfy $f(x) \in \lambda_1$ , $g(x) \in \lambda_2$ and $\lambda_1 \cap \lambda_2 = \varphi$, hence N $int(N\,cl(\lambda_1)) \cap Nint(\,Ncl(\lambda_2)) = \varphi$ . Since $f$ and $g$ are (NA $\delta gp$ -CO), there exist $\gamma_1, \gamma_2 \in N\delta GPO(\Omega, x))$ satisfy $f(\gamma_1) \subseteq N\,int(N\,cl(\lambda_1))$ and $g(\gamma_2) \subseteq Nint(\,Ncl(\lambda_2))$ . Now, put $\Sigma = \gamma_1 \cap \gamma_2$ , then $\Sigma \in N\delta GPO(\Omega, x)$ and $f(\Sigma) \cap g(\Sigma) \subseteq N\,int(\,Ncl(\lambda_1)) \cap Nint(N\,cl(\lambda_2)) = \varphi$. Thus, we get $\Sigma \cap \lambda = \varphi$ and hence $x \notin N\delta gp - cl(E)$ then $\lambda = N\delta gp - cl(\lambda)$. Since $\Omega$ is (N$\delta gp$ -add-S), $\lambda$is (N$\delta gp$ -C) in $\Omega$.

**Definition 3.22.** A space $\boldsymbol{\mu}$ is called neutrosophic $\boldsymbol{\delta gp}$ -compact (N$\boldsymbol{\delta gp}$ -Com) if any cover of $\boldsymbol{\mu}$ by $\boldsymbol{\delta gp}$ -open sets has a finite subcover.

**Definition 3.23**. Let $\boldsymbol{\lambda}$ be (NS) in (NTS) $\boldsymbol{\Omega}$**.** We say $\boldsymbol{\lambda}$ is neutrosophic $\boldsymbol{\delta gp}$ -compact relative (N$\boldsymbol{\delta gp}$ -Com-R) to $\boldsymbol{\Omega}$ if any cover of $\boldsymbol{\lambda}$ by (N$\boldsymbol{\delta gp}$ -O) sets of $\boldsymbol{\Omega}$ has a finite subcover.

**Theorem 3.24**. If $\boldsymbol{f}: \boldsymbol{\mu} \rightarrow \boldsymbol{\eta}$ is (NA$\boldsymbol{\delta gp}$ -CO) and $\boldsymbol{\lambda}$ is (N$\boldsymbol{\delta gp}$ -Com-R) to $\boldsymbol{\mu}$, then $\boldsymbol{f(\lambda)}$ is (NN-Cl-R) to $\boldsymbol{\eta}$.

**Proof.** Let $\{\boldsymbol{A_\alpha}: \boldsymbol{\alpha} \in \boldsymbol{\Omega}\}$ be any cover of $\boldsymbol{f(\lambda)}$ by (NR-O) sets of $\boldsymbol{\eta}$ . Then $\{\boldsymbol{f^{-1}(A_\alpha)}: \boldsymbol{\alpha} \in \boldsymbol{\Omega}\}$ is a cover of $\boldsymbol{\lambda}$ by (N$\boldsymbol{\delta gp}$ -O) sets of $\boldsymbol{\mu}$. Hence there exists a finite subset $\boldsymbol{\Omega_0}$ of $\boldsymbol{\Omega}$ such that $\boldsymbol{\lambda} \subset \cup \{\boldsymbol{f^{-1}(A\alpha)}: \boldsymbol{\alpha} \in \boldsymbol{\Omega_0}\}$. Therefore, we obtain $\boldsymbol{f(\lambda)} \subset \{\boldsymbol{A_\alpha}: \boldsymbol{\alpha} \in \boldsymbol{\Omega_0}\}$. This shows that $\boldsymbol{f(\lambda)}$ is (NN-Cl-R) to $\boldsymbol{\eta}$ .

**Corollary 3.25**. If $\boldsymbol{f}: \boldsymbol{\Omega} \rightarrow \boldsymbol{\mu}$ is (NA$\boldsymbol{\delta gp}$ -CO) surjection and $\boldsymbol{\Omega}$ is (N$\boldsymbol{\delta gp}$ -Com) and (N$\boldsymbol{\delta gp}$ -add-S), then $\boldsymbol{\mu}$ is (N-NCom-S).

**Lemma 3.26**. Let $\boldsymbol{\mu}$ be (N$\boldsymbol{\delta gp}$ -Com). If $\boldsymbol{\lambda} \subset \boldsymbol{\mu}$ is (N$\boldsymbol{\delta gp}$ -C), then $\boldsymbol{\lambda}$ is (N$\boldsymbol{\delta gp}$ -Com-R) to $\boldsymbol{\mu}$.

**Proof.** Let $\{\beta_\alpha : \alpha \in \Omega\}$ be a cover of $N$ by (N$\delta gp$ -O) sets of $\mu$. Note that $(\mu - N)$ is (N$\delta gp$ -O) and that the (NS) $(\mu - N) \cup \{\beta_\alpha : \alpha \in \Omega\}$ is a cover of $\mu$ by (N$\delta gp$ -O) sets. Since $\mu$ is (N$\delta gp$ -Com), there exists a finite $\Omega_0$ subset of $\Omega$ such that the (NS) $(\mu - N) \cup \{\beta_\alpha : \alpha \in \Omega_0\}$ is a cover of $\mu$ by (N$\delta gp$ -O) sets in $\mu$. Hence $\{\beta_\alpha : \alpha \in \Omega_0\}$ is a finite cover of $N$ by (N$\delta gp$ -O) sets in $\mu$.

**Theorem 3.27** If the graph function $g : \Omega \to \Omega \times \mu$ of $f : \Omega \to \mu$, defined by $g(\sigma) = (\sigma, f(\sigma))$ for each $\sigma \in \Omega$ is (NA$\delta gp$ -CO) Then $f$ is (NA$\delta gp$ -CO)

**Proof.** Let $\lambda \in NRO(\mu)$, then $\Omega \times \mu \in NRO(\Omega \times \mu)$. As $g$ is (NA $\delta gp$ -CO), $f^{-1}(\lambda) = g^{-1}(\Omega \times \lambda) \in N\delta GPO(\Omega)$.

**Theorem 3.28**. Let $\Omega, \eta$ be (NTSs) and $g : \Omega \to \Omega \times \eta$ be graph neutrosophic function of $f : \Omega \to \eta$, defined by $g(\sigma) = (\sigma, f(\sigma))$ for each $\sigma \in \Omega$. If $\Omega$ is a (N-submax-S) and (N$\delta gp$ -add-S), then g is (NA$\delta gp$ -CO) if and only if $f$ is (NA$\delta gp$ -CO).

**Proof.** We only prove the sufficiency. Let $\sigma \in \Omega$ and $W \in RO(\Omega \times \eta)$. Then there exist (NR-O) sets $\lambda_1$ and $V$ in $\Omega$ and $\eta$, respectively such that $\lambda_1 \times V \subset W$. If $f$ is (NA$\delta gp$ -CO), so there exists a (N$\delta gp$ -O) set $\lambda_2$in $\Omega$ satisfies $\sigma \in \lambda_2$ and $f(\lambda_2) \subset V$. Put $\lambda = (\lambda_1 \cap \lambda_2)$.Then $\lambda$ is (N$\delta gp$ -O) and $g(\lambda) \subset \lambda_1 \times V \subset W$. Thus $g$ is (NA$\delta gp$ -CO).

**Definition 3.29.** A graph $G_f = \{(\Omega, F(\sigma) : \sigma \in \Omega\} \subset \Omega \times \eta$of a neutrosophic function $f : \Omega \to \mu$ is said to be neutrosophic strongly $\delta gp$ -closed (N-Str-$\delta gp$ **-C)** if for each $(\rho, \theta) \notin G_f$, there exist $\lambda \in N\delta GPO(\Omega, \rho)$ and $V \in NRO(\mu, \theta)$ satisfy $(\lambda \times V) \cap G_f = \varphi$.

**Lemma 3.30**. For a graph $G_f$ of a neutrosophic function $f : \Omega \to \mu$, the following properties are equivalent:

(i) $G_f$ is (N-Str-$\delta gp$ -C) in $\Omega \times \mu$;

(ii)For each $(\rho,\theta) \notin G_f$, there exist $\lambda \in N\delta GPO(\Omega,\rho)$ and $V \in$ N $RO(\mu,\theta)$ such that $f(\lambda) \cap V = \varphi$.

**Theorem 3.31.** Let $\boldsymbol{f}: \boldsymbol{\Omega} \to \boldsymbol{\mu}$ have a (N-Str-$\delta gp$ -C) graph $G_f$ . If $f$ is injective, then $\Omega$ is (N$\delta gp - T_1$-S).

**Proof.** Let $\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2} \in \boldsymbol{\Omega}$ with $\boldsymbol{\sigma_1} \neq \boldsymbol{\sigma_2}$ .Then $\boldsymbol{f(\sigma_1)} \neq \boldsymbol{f(\sigma_2)}$ as $\boldsymbol{f}$ is injective so that $(\boldsymbol{\sigma_1}, \boldsymbol{f(\sigma_2)}) \notin \boldsymbol{G_f}$. Thus there exist $\lambda_1 \in N\delta GPO(\Omega,\sigma_1)$ and $\lambda_2 \in NRO(\mu, f(\sigma_2))$ such that $f(\lambda_1) \cap \lambda_2 = \varphi$ . Then $f(\sigma_2) \notin f(\lambda_1)$ implies $\sigma_2 \notin \lambda_1$ and it follows that $\Omega$ is (N$\delta gp - T_1$-S).

**Theorem 3.32.**

(i) If $\boldsymbol{f}: \boldsymbol{\Omega} \to \boldsymbol{\mu}$ is (NA$\boldsymbol{\delta gp}$ -CO) and $\boldsymbol{g}: \boldsymbol{\mu} \to \boldsymbol{\eta}$is (N$\boldsymbol{R}$ -M), then $\boldsymbol{g} \circ \boldsymbol{f}: \boldsymbol{\Omega} \to \boldsymbol{\eta}$is (NA$\boldsymbol{\delta gp}$ -CO).

(ii) If $\boldsymbol{f}: \boldsymbol{\Omega} \to \boldsymbol{\mu}$ is (N$\boldsymbol{\delta gp}$ -CO) and $\boldsymbol{g}: \boldsymbol{\mu} \to \boldsymbol{\eta}$ is (NA-CO), then $\boldsymbol{g} \circ \boldsymbol{f}: \boldsymbol{\Omega} \to \boldsymbol{\eta}$ is (NA$\boldsymbol{\delta gp}$ -CO).

(iii) If $\boldsymbol{f}: \boldsymbol{\Omega} \to \boldsymbol{\mu}$ is (N$\delta gp$ -IR) and $\boldsymbol{g}: \boldsymbol{\mu} \to \boldsymbol{\eta}$ is (NA$\boldsymbol{\delta gp}$ -CO), then $\boldsymbol{g} \circ \boldsymbol{f}: \boldsymbol{\Omega} \to \boldsymbol{\eta}$ is (NA$\boldsymbol{\delta gp}$ -CO).

**Proof.** (i) Let $\boldsymbol{\lambda} \in NRO(\eta)$. Then $g^{-1}(\lambda) \in NRO(\mu)$ since g is (N$R$ -M). The (NA$\delta gp$ -CO) of *f* implies $f^{-1}[g^{-1}(\lambda))] = (g \circ f)^{-1}((\lambda)) \in N\delta GPO(\Omega)$. Hence $g \circ f$ is (NA$\delta gp$ -CO).

The proofs of (ii) and (iii) are similar to (i).

**Theorem 3.33.** If $\boldsymbol{f}: \boldsymbol{\Omega} \to \boldsymbol{\mu}$ is a pre $\boldsymbol{\delta gp}$ -open surjection and $\boldsymbol{g}: \boldsymbol{\mu} \to \boldsymbol{\eta}$ is a function such that $\boldsymbol{g} \circ \boldsymbol{f}: \boldsymbol{\mu} \to \boldsymbol{\eta}$ is (NA$\boldsymbol{\delta gp}$ -CO), then g is (NA$\boldsymbol{\delta gp}$ -CO).

**Proof.** Let $\boldsymbol{\theta} \in \boldsymbol{\eta}$ and $\boldsymbol{\sigma} \in \boldsymbol{\Omega}$ such that $\boldsymbol{f(\sigma)} = \boldsymbol{\theta}$. Let $\boldsymbol{G} \in \boldsymbol{RO}(\boldsymbol{\eta}, (\boldsymbol{g} \circ \boldsymbol{f})(\boldsymbol{\sigma}))$. Then there exists $\boldsymbol{U} \in \boldsymbol{\delta GPO}(\boldsymbol{\Omega}, \boldsymbol{\sigma})$ such that such that $\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{U})) \subset \boldsymbol{G}$. Since $\boldsymbol{f}$ is pre $\boldsymbol{\delta gp}$ -open in$\boldsymbol{\mu}$, we have that $\boldsymbol{g}$ is (NA$\boldsymbol{\delta gp}$ -CO) at $\boldsymbol{\mu}$.

## Conclusion

In this paper, some new notions of neurosophic delta generated pre-continuous functions in neutrosophic topological spaces are given and discussed, which is a very interesting topic in nature. It will open up many avenues for the researchers work neutrosophic topological spaces, we can in future work extend and study these our notions for this paper in soft setting form.

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Received: Nov 18, 2021. Accepted: Feb 5, 2022

Neutrosophic Sets and Systems, Vol. 48, 2022
University of New Mexico

# Neutrosophic in Multi-Criteria Decision Making for Location Selection

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**Abstract:** Neutrosophic set (NS) theory has a diverse nature in dealing with impreciseness of real life events and has a wider range of applications in logic, algebra, topology, operation research, pattern recognition, artificial intelligence, neural networks and several other fields.In this paper, we combine single valued neutrosophic with we use the score and accuracy function and hybrid score accuracy function of single- valued neutrosophic number and ranking method for single- valued neutrosophic numbers to model logistics center location problem . The combined values of each alternative have been ordered with the help of score function to find the best attributes. Finally, an illustrative example has been provided to validate the proposed approach for multi attribute decision making problem.

Keywords: Neutrosophic Logic; decision making ; Interval Valued Neutrosophic Set.

## 1. Introduction

The concept of fuzzy sets (FS) was introduced by L.Zaheh(1965), where each element had degree of membership. Since the fuzzy sets and fuzzy logic have been applied in many real life problems in uncertain and ambiguous environment. The traditional fuzzy sets is characterized by the membership value and the grade of membership value. the concept of interval valued fuzzy sets was proposed by Turksen(1986) to capture the uncertain of grade of membership value. Neither the fuzzy sets nor the interval valued fuzzy sets is appropriate for such a situation. A tool which represents the partnership or relationship function is called a Fuzzy Set (FS) and handles the real world problems in which generally some type of uncertainty exists. This concept was generalized by Atanassov to intuitionistic fuzzy set (IFS) which is determined in terms of membership (MS) and non-membership (NMS) functions, the characteristic functions of the set. Intuitionistic fuzzy sets (IFS) introduced by Atanassov(1986) as a generalization of FS, where besides the degree of membership $\mu_A(x)\epsilon[0,1]$ for each element $x\epsilon X$ to a set A there was considered a degree of non-membership$\vartheta_A(x)\epsilon[0,1]$, such that$\forall x \in X, \mu_A(x) + \vartheta_A(x) \leq 1$ the neutrosophic set (NS) was introduced by F.Smarandache who introduced the degree of indeterminacy(I) as independent component in 1998. In this paper, we combine single valued neutrosophic with we use the score and accuracy function and hybrid score accuracy function of single- valued neutrosophic number and ranking method for single- valued neutrosophic numbers to model logistics center location problem The combined values of each alternative have been ordered with the help of score function to find the best attributes. Finally, an an illustrative example has been provided to validate the proposed approach for multi attribute decision making problem.

Multi-criteria decision-making (MCDM) is a common offshoot of decision-making science. There are a huge number of MCDM techniques which assist individuals in constructing and solving decision problems that concern multiple criteria. Each technique has its own physiognomies and no single one is the best. The proper MCDM technique should be designated consistent with the problem structure. It is recognized that without integrating preference information, no unique optimal solution to an MCDM problem can be acquired. Regardless of the chosen MCDM technique for the problem we are dealing with, the significant phase is to achieve the decision factor weights. Either the subjective or objective method can regulate the criteria weights in MCDM techniques.

## 2. Review of Literature

The author in, [1] analyzed Spatially explicit seasonal forecasting using fuzzy spatiotemporal clustering of long-term daily rainfall and temperature data. And the authors of, [2] analyzed Ambient Atmospheric Temperature Prediction Using Fuzzy Knowledge –Rule Base for Inland Cities in India. [3] Analysis a new Approach and Applications, International Journal of Research in Computer and Communication Technology. [4] examined Project Schedule Uncertainty Analysis Using Fuzzy Logic, Project Management Institute. [5] Analyzed the Power Flow Analysis Using Fuzzy Logic [6] proposed Types of Neutrosophic Graphs and neutrosophic Algebraic Structures together with their Applications in Technology. [7] proposed a new approach for Single Valued Neutrosophic Graphs. [8] Proposed a method for On Bipolar Single Valued Neutrosophic

[1] Graph. [9] Proposed various types of Interval Valued Neutrosophic Graphs.[10] proposed Isolated Single Valued Neutrosophic Graphs [11] examined bipolar single valued neutrosophic graphs. [12] Proposed Single-Valued Neutrosophic Minimum Spanning Tree and Its Clustering Method. [13] proposed Fuzzy based approach for weather advisory system. [14] provided Weather Forecasting using Fuzzy Neural Network (FNN) and Hierarchy Particle Swarm Optimization

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## 3. Preliminaries

### 3.1 Neutrosophic set

Neutrosophy is a branch of philosophy identified by Florentin Smarandache in 1980.

Definition 1:

Assume that X be an universe o discourse.then a neutrosophic sets N can be dehined as follows;

$$N = \{< x: T_N(x), I_N(x), F_N(x) >/ \ x\in X\} \qquad (1)$$

Here the functions T, I, F define respectively the membership degree, indeterminacy degree and the non-membership degree of the element $x\in X$ to the set N. the three functions T, I and F satisfy the following the conditions:

$T, I, F{:}X \to ]0^-, 1^+[$

$$0^- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+ \qquad (2)$$

For two neutrosophic sets

$M = \{< x: T_M(x), I_M(x), F_M(x) >/ \ x\in X\}$ and $N = \{< x: T_N(x), I_N(x), F_N(x) >/ \ x \in X\}$ the two relations are defined as follows:

$M \subseteq N$ if and only if $T_M(x) \le T_N(x), I_M(x) \ge I_N(x), F_M(x) \ge F_N(x)$

$M = N$ if and only if $T_M(x) = T_N(x), I_M(x) = I_N(x), F_M(x) = F_N(x)$

Definition 2:

Complement of neutrosophic sets:

For any set $M = \{< x: T_M(x), I_M(x), F_M(x) >/ \ x\in X\}$ ,then

$M' = \{< x: F_M(x), 1 - I_M(x), T_M(x) > / \ x \in X\}$ (3)

Definition 3:

Single valued neutrosophic number (SVNN)

Let X be a universe of discourse with generic element in X denoted by x. A SVNS M in X is characterized by a truth-membership function $T_M(x)$, a indeterminacy-membership function $I_M(x)$ and a falsity-membership function $F_M(x)$. Then, a SVNS M can be written as follows:

$M = \{< x: T_M(x), I_M(x), F_M(x) > / \ x \in X\}$ where $T_M(x), I_M(x), F_M(x) \in [0,1]$ for each point x in X. Since no restriction is imposed in the sum of t M (p), i M (p) and f M (p),

it satisfies $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$ . For a SVNS M in the triple $< T_M(x), I_M(x), F_M(x) >$ is called single valued neutrosophic number (SVNN).

4. MCGDM method based on hybrid – score accuracy functions under single valued neutrosophic environment

Assume that B = {$B_1$, …, $B_n$ }($n \geq 2$) be the set of logistics centers, K = {$K_1$, $K_2$, ..., Kq } ($q \geq 2$) be the set of criteria and E = {$E_1$, $E_2$, ..., $E_m$} ($m \geq 2$) be the set of decision makers or experts.

The weights of the decision makers and criteria are not previously assigned, where the information about the weights of the decision- makers is completely unknown and information about the weights of the criteria is incompletely known in the group decision making problem. In such a case, we develop a method based on the hybrid score – accuracy function for MCDM problem with unknown weights under single-valued neutrosophic environment using linguistic variables.

The steps for solving MCGDM by proposed approach have been presented below is discussed

Algorithm

Step 1:

Formation of the decision matrix

Step 2:

Calculate hybrid score accuracy matrix

Step 3:

Calculate the average matrix

Step 4:

Determination of decision maker's weights

Step 5:

Calculate collective hybrid score accuracy matrix

Step 6:

Weight model for criteria

Step 7:

Ranking of alternatives

Step 8:

End

5. Methodology:

The following illustration is suppose that a state government wants to construct tan eco-tourism park for the development of state tourism and especially for the mental refreshment of children. After initial screening three potential spots namely spot-1 ($P_1$), spot-2 ($P_2$), spot-3 ($P_3$). A team of three decision makers namely $D_1$,$D_2$,$D_3$ has been constructed a neutrosophic sets for selecting the most suitable spot with respect to the following attributes:

Ecology ($C_1$)
Cost ($C_2$)
Technical facility ($C_3$)
Transport ($C_4$)
Risk factors ($C_5$)

Assume that a new modern logistic center is required in a town. There are three spot $P_1$ , $P_2$ ,$P_3$ . A committee of four decision makers or experts namely, $D_1$ , $D_2$ , $D_3$ has been formed to select the most appropriate location on the basis of five criteria adopted from the study [6] namely, $C_1$,$C_2$,$C_3$,$C_4$,$C_5$.Thus the three decision makers use linguistic variables to rate the alternatives with respect to the criterion and construct the decision matrices as follows:
Step 1:

Formation of the decision matrix

| $D_1$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $P_1$ | (0.7,0.4,0.4) | (0.7,0.4,0.3) | (0.8,0.1,0.1) | (0.7,0.2,0.1) | (0.6,0.5,0.5) |
| $P_2$ | (0.4,0.3,0.6) | (0.5,0.2,0.5) | (0.6,0.2,0.2) | (0.7,0.3,0.3) | (0.4,0.3,0.4) |
| $P_3$ | (0.4,0.2,0.3) | (0.8,0.1,0.3) | (0.5,0.4,0.4) | (0.5,0.2,0.2) | (0.7,0.3,0.2) |

| $D_2$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $P_1$ | (0.5,0.2,0.3) | (0.7,,0.4,0.4) | (0.8,0.2,0.2) | (0.5,0.2,0.2) | (0.5,0.5,0.4) |
| $P_2$ | (0.5,0.4,0.4) | (0.5,0.2,0.4) | (0.5,0.3,0.3) | (0.8,0.3,0.3) | (0.4,0.1,0.4) |
| $P_3$ | (0.4,0.2,0.5) | (0.8,0.2,0.2) | (0.5,0.3,0.3) | (0.7,0.2,0.2) | (0.7,0.4,0.2) |

| $D_3$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $P_1$ | (0.7,0.4,0.3) | (0.8,0.2,0.1) | (0.6,0.3,0.3) | (0.7,0.2,0.5) | (0.5,0.6,0.5) |
| $P_2$ | (0.6,0.2,0.3) | (0.5,0.1,0.3) | (0.7,0.4,0.4) | (0.5,0.3,0.4) | (0.3,0.4,0.4) |

| | | | | | |
|---|---|---|---|---|---|
| $P_3$ | (0.6,0.2,0.3) | (0.6,0.4,0.2) | (0.5,0.3,0.3) | (0.7,0.4,0.2) | (0.5,0.6,0.4) |

Step 2:

Calculate hybrid score accuracy matrix

Now we use the above method for single valued neutrophic group decision making to choice appropriate location. We take $\alpha$= 0.5 for demonstrating the computing procedure

Calculate hybrid score – accuracy matrix

Hybrid score- accuracy matrix can be obtained from above decision matrix using equation

$$h_{ij}^{s} = \frac{1}{2}\alpha(1 + t_{ij}^{s} - f_{ij}^{s}) + \frac{1}{3}(1 - \alpha)(2 + t_{ij}^{s} - i_{ij}^{s} - f_{ij}^{s})$$

are given below respectively.

HYBRID SCORE MATRIX FOR $D_1$

$H_1$

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $P_1$ | 0.6417 | 0.6833 | 0.8583 | 0.8000 | 0.5417 |
| $P_2$ | 0.4500 | 0.5500 | 0.7167 | 0.7000 | 0.5333 |
| $P_3$ | 0.5917 | 0.7750 | 0.5583 | 0.6750 | 0.7417 |

HYBRID SCORE MATRIX FOR $D_2$

$H_2$

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $P_1$ | 0.6333 | 0.6417 | 0.8000 | 0.6750 | 0.5417 |
| $P_2$ | 0.5583 | 0.5917 | 0.6167 | 0.7417 | 0.5667 |
| $P_3$ | 0.5083 | 0.8000 | 0.6167 | 0.7583 | 0.7250 |

HYBRID SCORE MATRIX FOR $D_3$

$H_3$

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $P_1$ | 0.6833 | 0.8417 | 0.6583 | 0.6333 | 0.4833 |
| $P_2$ | 0.6750 | 0.6500 | 0.6417 | 0.5750 | 0.4750 |
| $P_3$ | 0.6750 | 0.6833 | 0.6167 | 0.7250 | 0.5250 |

Step 3:

Calculate the average matrix . Form the above hybrid score-accuracy matrix by using equation $h_{ij}^{*} = \frac{1}{m}\sum_{s=1}^{m}(h_{ij}^{s})$ .

We form the average matrix H*

The average matrix

| | | | | | |
|---|---|---|---|---|---|
| H* | 0.6528 | 0.7222 | 0.7722 | 0.7028 | 0.5222 |
| | 0.5611 | 0.5972 | 0.6583 | 0.6722 | 0.5250 |
| | 0.5917 | 0.7528 | 0.5972 | 0.7194 | 0.6639 |

The collective correlation co-efficient between $H^s$ and H* express follows by equation

$$\Omega_s = \sum_{i=1}^{n} \frac{\sum_{j=i}^{\rho} h_{ij}^{s} h_{ij}^{*}}{\sqrt{\sum_{j=1}^{\rho}\left(h_{ij}^{s}\right)^2}} \frac{1}{\sqrt{\sum_{j=1}^{\rho}\left(h_{ij}^{*}\right)^2}}$$

Here s={1,2,3}

To find $\Omega_1$ ,

| $H_1$ x H* | | | | | $\sum_{j=i}^{\rho} h_{ij}^{1} h_{ij}^{*}$ | $\sqrt{\sum_{j=1}^{\rho}\left(h_{ij}^{1}\right)^2}$ | $\sqrt{\sum_{j=1}^{\rho}\left(h_{ij}^{*}\right)^2}$ |
|---|---|---|---|---|---|---|---|
| 0.4189 | 0.4935 | 0.6628 | 0.5622 | 0.2829 | 2.4203 | 1.5965 | 1.5201 |
| 0.2525 | 0.3285 | 0.4718 | 0.4706 | 0.2800 | 1.8033 | 1.3391 | 1.3537 |
| 0.3501 | 0.5834 | 0.3334 | 0.4856 | 0.4924 | 2.2449 | 1.5060 | 1.4939 |

$$\Omega_1 = \frac{2.4203}{1.5965 \times 1.5201} + \frac{1.8033}{1.3391 \times 1.3537} + \frac{2.2449}{1.3391 \times 1.4939} = 0.4278$$

$\Omega_2 = 2.9937$

$\Omega_3 = 2.9781$

Step – 4

Determination decision maker's weights

$$\gamma_s = \frac{\Omega_s}{\sum_{s=1}^{m} \Omega_s}, 0 \leq \gamma_s \leq 1 \; for \; s = 1,2,3,\ldots,m$$

From the equation we determine the weight of the four decision makers as follows :-

$$\Omega_1 + \Omega_2 + \Omega_3 = 6.3996$$

$\gamma_1 = \frac{\Omega_1}{\Omega_1+\Omega_2+\Omega_3} = \frac{0.4278}{6.3996} = 0.0669$

$\gamma_2 = 0.4678$

$\gamma_3 = 0.4654$

Step 5:

Calculate collective hybrid score – accuracy matrix

Hence the hybrid score-accuracy values of the different decision makers choice are aggregated by equation $\mathcal{H} = (h_{ij})n \times p = \sum_{s=1}^{m} \gamma_s h_{ij}^{s}$ and the collective hybrid score-accuracy matrix can be formulated as follows:

$\gamma_1 \times H_1$

| | | | | |
|---|---|---|---|---|
| 0.0327 | 0.0349 | 0.0438 | 0.0408 | 0.0276 |
| 0.0230 | 0.0281 | 0.0366 | 0.0357 | 0.0272 |
| 0.0302 | 0.0395 | 0.0285 | 0.0344 | 0.0378 |

$\gamma_2 \times H_2$

| | | | | |
|---|---|---|---|---|
| 0.3008 | 0.3048 | 0.3800 | 0.3206 | 0.2573 |
| 0.2652 | 0.2810 | 0.2929 | 0.3523 | 0.2692 |
| 0.2415 | 0.3800 | 0.2929 | 0.3602 | 0.3444 |

$\gamma_3 \times H_3$

| | | | | |
|---|---|---|---|---|
| 0.3239 | 0.3990 | 0.312 | 0.3002 | 0.2291 |
| 0.3200 | 0.3081 | 0.3042 | 0.2726 | 0.2252 |
| 0.3200 | 0.3239 | 0.2923 | 0.3437 | 0.2489 |

By adding all the above

$\mathcal{H} =$

| | | | | |
|---|---|---|---|---|
| 0.657 | 0.739 | 0.736 | 0.662 | 0.514 |
| 0.608 | 0.617 | 0.634 | 0.661 | 0.522 |
| 0.592 | 0.743 | 0.614 | 0.738 | 0.631 |

Step 6

Weight model for criteria

Assume that the information about criteria weights is incompletely known given as follows: weight vectors, Using the linear programming model $\text{Max}\,\omega = \frac{1}{n}\sum_{j=1}^{\rho}\omega_j h_{ij}$, we obtain the weight vector of the criteria as $\omega$ =[0.1, 0.1, 0.25, 0.2, 0.15].

Step 7

Ranking of alternatives

Using equation $\psi(P_i) = \sum_{j=1}^{\rho}\omega^i h^{ij}$ we calculate the over all hybrid score-accuracy values $\psi(P_i)$ (i = 1, 2, 3):

$\psi(P_1)$=0.533

$\psi(P_2)$=0.491

$\psi(P_3)$=0.529

Based on the above values of $\psi(P_i)$ (i = 1, 2, 3, ) the ranking order of the locations are as follows:

$P_1 > P_3 > P_2$

Therefore the location $P_1$ is the best location.

## 6. Conclusions

In this paper, the concept of single valued Neutrosophic set used with location problem tested with the help of score function. A possible application has been tackled through the usage of SVNS which will not only prove useful by itself but will help out keen researchers to solve other problems of uncertainties through similar procedures. The following paper demonstrated a new solution procedure to solve neutrosophic fuzzy sets with the contraction value based on real life decision making problems. This procedure proves quite feasible in many real life scenarios where else of decision making is the goal in mind.

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Received: Nov 23, 2021. Accepted: Feb 4, 2022

Neutrosophic Sets and Systems, Vol. 48, 2022
University of New Mexico

# Single-Valued Pentapartitioned Neutrosophic Dice Similarity Measure and Its Application in the Selection of Suitable Metal Oxide Nano-Additive for Biodiesel Blend on Environmental Aspect

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**Abstract:** In this paper, an attempt has been made to introduce a new similarity measure namely single-valued pentapartitioned neutrosophic dice similarity measure (SVPNDSM) under the single-valued pentapartitioned neutrosophic set (SVPNS) environment, and to formulate several interesting results on SVPNDSM and SVPNWDSM. In this present work, the SVPNDSM under the SVPNS framework is combined with a multi-attribute decision making (MADM) strategy. This proposed method is used to select suitable metal oxide nano-additive for biodiesel blends on the basis of environmental aspects. The effects of nano-additives on engine emissions have been reported here from six different literatures. The SVPNDSM applied under the SVPNS environment enables the selection of the best nano-additive among relevant literatures. Alternative $L_4$ comes out as the best from the proposed method. The proposed MADM method is shown to be well suited to this problem after it has been compared with two existing methods.

**Keywords:** *Neutrosophic Set*; *Indeterminacy*; *SVPNS*; *Dice Similarity*.

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**1. Introduction:** The notion of fuzzy set (FS) theory was grounded by Zadeh [49] to deal with the events having uncertainty, where every element has membership value between 0 and 1. Later on, Atanassov [2] introduced the concept of intuitionistic fuzzy set (IFS) by generalizing the notion of FS in the year 1986, where every element has membership and non-membership values. Till now many researchers around the globe have applied the concept of FS and IFS in the area of theoretical and practical research.

We come across many situations involving indeterminacy, incompleteness which cannot be easily determined by the degrees of membership and non-membership. Keeping in mind, Smarandache [40] introduced the notion of neutrosophic set (NS) on generalizing the idea of FS and IFS to deal with the events having indeterminacy. In an NS, every element has three independent memberships values namely truth, indeterminacy and false membership values respectively, those lie between 0 and 1 each. In the recent past, many researcher around the globe used the concept of NS and their extensions for theoretical research [4-6, 8-11, 14, 24, 42, etc.]. Degree of indeterminacy membership of a mathematical expression plays an important role in every MADM problem of this real world. Afterwards, Wang et al. [45] introduced the notion of single-valued neutrosophic set (SVNS) in 2010, which is a subclass of NS. The notion of SVNS is more useful to deal with the situation involving incomplete and indeterminate information. Till now, many researcher have applied SVNS and their extensions in different branches of real-world such as fault diagnosis [46, 47], medical diagnosis [35, 36], decision-making problems [3, 7, 12-13, 15, 17-23, 25-28, 32-34, 43, 48], etc.

Recently, Mallick and Pramanik [24] investigated the notion of single-valued pentapartitioned neutrosophic set (SVPNS) by splitting the degree of indeterminacy membership into three independent components namely contradiction membership, ignorance membership and unknown membership. Das et al. [6] presented the notion of pentapartitioned neutrosophic Q-ideals of Q-algebra in 2021. Das et al. [13] proposed a MADM strategy based on the tangent similarity measure of SVPNS. Later on, Das et al. [12] established a MADM strategy based on grey relational analysis under the SVPNS environment. A MADM strategy based on cosine similarity measure of SVPNSs was established by Majumder et al. [23] to identify the most significant risk factor of COVID-19 in economy.

In this article, a new similarity measure called SVPNDSM is proposed used to select suitable metal oxide nano-additive for biodiesel blends on the basis of environmental aspects under the SVPNS environment and generate several interesting results. In addition, a MADM technique is established based on SVPNDSM within the SVPNS environment.

**Research Gap:** In the literature review, no study is found relating to SVPNDSM based MADM strategy in SVPNS.

**Motivation:** To explore the unexplored MADM strategy in SVPNS environment, a new MADM strategy under SVPNS environment based on SVPNDSM between SVPNSs is presented in this present work.

The rest of this paper has been split into the following sections:

Section-2 presents several basic definitions and operations on SVPNSs those are very useful for developing the main results of this paper. In section-3, Single-Valued Pentapartitioned Neutrosophic Dice Similarity Measure and Single-Valued Pentapartitioned Neutrosophic Weighted Dice Similarity Measure under the SVPNS environment is proposed. Further, we formulate some interesting results on SVPNDSM and SVPNWDSM. A MADM strategy using SVPNWDSM under the SVPNS environment is discussed in section-4. In section-5 the proposed MADM strategy is applied to a real world problem. Finally, in section 6, a comparative study has been conducted to validate the results obtained from the proposed method. In section-7, wrap up the work presented in this article.

List of abbreviations are shown in below:

| **List of abbreviations** | |
|---|---|
| **Full Form** | **Short Form** |
| Fuzzy Set | FS |
| Intuitionistic Fuzzy Set | IFS |
| Neutrosophic Set | NS |
| Single Valued Neutrosophic Set | SVNS |
| Single Valued Pentapartitioned Neutrosophic Set | SVPNS |
| Multi-Attribute Decision Making | MADM |
| Single Valued Pentapartitioned Neutrosophic Dice Similarity Measure | SVPNDSM |
| Single Valued Pentapartitioned Neutrosophic Weighted Dice Similarity Measure | SVPNWDSM |
| Positive Ideal Solution | PIS |

## 2. Basic Preliminaries:

In this section some basic definitions and results are described.

**Definition 2.1.**[24] Suppose that X be a fixed set. Then R, an SVPNS over X is defined as follows:

$$R = \{(\delta, \Delta_R(\delta), \Gamma_R(\delta), \Pi_R(\delta), \Omega_R(\delta), \Phi_R(\delta)) : \delta \in X\},$$

where $\Delta_R$, $\Gamma_R$, $\Pi_R$, $\Omega_R$, $\Phi_R$: $X \to [0, 1]$ represents the truth, contradiction, ignorance, unknown and falsity membership functions respectively such that $0 \leq \Delta_R(\delta) + \Gamma_R(\delta) + \Pi_R(\delta) + \Omega_R(\delta) + \Phi_R(\delta) \leq 5$, for all $\delta \in X$.

**Remark 2.1.**[24] Suppose that $R = \{(\delta, \Delta_R(\delta), \Gamma_R(\delta), \Pi_R(\delta), \Omega_R(\delta), \Phi_R(\delta)) : \delta \in X\}$ be an SVPNS over X. Then, for any $\delta \in X$, $(\Delta_R(\delta), \Gamma_R(\delta), \Pi_R(\delta), \Omega_R(\delta), \Phi_R(\delta))$ is called an single-valued pentapartitioned neutrosophic number (SVPNN) over X.

**Definition 2.2.**[24] Suppose that W={(δ, $\Delta_W(\delta)$, $\Gamma_W(\delta)$, $\Pi_W(\delta)$, $\Omega_W(\delta)$, $\Phi_W(\delta)$) : δ∈X} and M={(δ, $\Delta_M(\delta)$, $\Gamma_M(\delta)$, $\Pi_M(\delta)$, $\Omega_M(\delta)$, $\Phi_M(\delta)$) : δ∈X} be two SVPNSs over X. Then,

i. $W\subseteq M \Leftrightarrow \Delta_W(\delta)\leq\Delta_M(\delta)$, $\Gamma_W(\delta)\leq\Gamma_M(\delta)$, $\Pi_W(\delta)\geq\Pi_M(\delta)$, $\Omega_W(\delta)\geq\Omega_M(\delta)$, $\Phi_W(\delta)\geq\Phi_M(\delta)$, for all δ∈X;

ii. $W^c$={(δ, $\Phi_W(\delta)$, $\Omega_W(\delta)$, 1-$\Pi_W(\delta)$, $\Gamma_W(\delta)$, $\Delta_W(\delta)$) : δ∈X} and $M^c$={(δ, $\Phi_M(\delta)$, $\Omega_M(\delta)$, 1-$\Pi_M(\delta)$, $\Gamma_M(\delta)$, $\Delta_M(\delta)$) : δ∈X};

iii. W∪M={(δ, max {$\Delta_W(\delta)$, $\Delta_M(\delta)$}, max {$\Gamma_W(\delta)$, $\Gamma_M(\delta)$}, min {$\Pi_W(\delta)$, $\Pi_M(\delta)$}, min {$\Omega_W(\delta)$, $\Omega_M(\delta)$}, min {$\Phi_W(\delta)$, $\Phi_M(\delta)$}) : δ∈X};

iv. W∩M={(δ, min {$\Delta_W(\delta)$, $\Delta_M(\delta)$}, min {$\Gamma_W(\delta)$, $\Gamma_M(\delta)$}, max {$\Pi_W(\delta)$, $\Pi_M(\delta)$}, max {$\Omega_W(\delta)$, $\Omega_M(\delta)$}, max {$\Phi_W(\delta)$, $\Phi_M(\delta)$}) : δ∈X}.

**Definition 2.3.**[24] The absolute SVPNS ($1_X$) and null SVPNS ($0_X$) over a fixed set X are defined by:

i. $1_X$= {(δ, 1, 1, 0, 0, 0) : δ∈X};

ii. $0_X$= {(δ, 0, 0, 1, 1, 1) : δ∈X}.

Clearly, $0_X\subseteq R\subseteq 1_X$, for any SVPNS R over X.

## 3. Single-Valued Pentapartitioned Neutrosophic Dice Similarity Measure:

This section introduces the notion of Single-Valued Pentapartitioned Neutrosophic Dice Similarity Measure and Single-Valued Pentapartitioned Neutrosophic Weighted Dice Similarity Measure, and formulates several interesting results on them under the SVPNS environment.

**Definition 3.1.** Assume that W = {(θ, $\Delta_W(\theta)$, $\Gamma_W(\theta)$, $\Pi_W(\theta)$, $\Omega_W(\theta)$, $\Phi_W(\theta)$) : θ∈U} and M = {(θ, $\Delta_M(\theta)$, $\Gamma_M(\theta)$, $\Pi_M(\theta)$, $\Omega_M(\theta)$, $\Phi_M(\theta)$) : θ∈U} be two SVPNSs over a fixed set U. Then, the SVPNDSM between W and M is defined by:

$D_{SVPNDSM}$(W, M)

$$=\frac{1}{n}\sum_{\theta\in U}\frac{2[\Delta_W(\theta).\Delta_M(\theta)+\Gamma_W(\theta).\Gamma_M(\theta)+\Pi_W(\theta).\Pi_M(\theta)+\Omega_W(\theta).\Omega_M(\theta)+\Phi_W(\theta).\Phi_M(\theta)]}{[(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2+(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2]} \quad (1)$$

**Example 3.1.** Let W={(a,0.6,0.3,0.1,0.2,0.1), (b,0.9,0.1,0.0,0.1,0.2)} and M={(a,1.0,0.0,0.1,0.1, 0.2), (b,0.8,0.0,0.0,0.1,0.0)}be two SVPNSs over U={a, b}. Then, SVPNDSM (W, M) = 0.8942758967.

**Theorem 3.1.** Suppose that $D_{SVPNDSM}$ (W, M) be the SVPNDSM between the SVPNSs W and M. Then,

(i) 0 ≤ $D_{SVPNDSM}$ (W, M) ≤ 1;

(ii) $D_{SVPNDSM}$ (W, M) = $D_{SVPNDSM}$ (M, W);

(iii) W = M ⇒$D_{SVPNDSM}$ (W, M) = 1.

**Proof.** (i) Let $D_{SVPNDSM}$ (W, M) be the SVPNDSM between W and M.

Therefore, $D_{SVPNDSM}$ (W, M)

$$= \frac{1}{n}\sum_{\theta\in U}\frac{2[\Delta_W(\theta).\Delta_M(\theta)+\Gamma_W(\theta).\Gamma_M(\theta)+\Pi_W(\theta).\Pi_M(\theta)+\Omega_W(\theta).\Omega_M(\theta)+\Phi_W(\theta).\Phi_M(\theta)]}{[(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2+(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2]}.$$

It is known that, $0 \leq\Delta_W(\theta) \leq 1$, $0 \leq\Delta_M(\theta) \leq 1$, $0 \leq\Gamma_W(\theta) \leq 1$, $0 \leq\Gamma_M(\theta) \leq 1$, $0 \leq\Pi_W(\theta) \leq 1$, $0 \leq\Pi_M(\theta) \leq 1$, $0 \leq\Omega_W(\theta) \leq 1$, $0 \leq\Omega_M(\theta) \leq 1$, $0 \leq\Phi_W(\theta) \leq 1$ and $0 \leq\Phi_M(\theta) \leq 1$, for each θ∈U.

$\Rightarrow$ $0 \le \Delta_W(\theta).\Delta_M(\theta) \le 1$, $0 \le \Gamma_W(\theta).\Gamma_M(\theta) \le 1$, $0 \le \Pi_W(\theta).\Pi_M(\theta) \le 1$, $0 \le \Omega_W(\theta).\Omega_M(\theta) \le 1$, $0 \le \Phi_W(\theta).\Phi_M(\theta) \le 1$, $0 \le (\Delta_W(\theta))^2 \le 1$, $0 \le (\Gamma_W(\theta))^2 \le 1$, $0 \le (\Pi_W(\theta))^2 \le 1$, $0 \le (\Omega_W(\theta))^2 \le 1$, $0 \le (\Phi_W(\theta))^2 \le 1$, for each $\theta \in U$.

$\Rightarrow$ $0 \le (\Delta_W(\theta))^2 + (\Gamma_W(\theta))^2 + (\Pi_W(\theta))^2 + (\Omega_W(\theta))^2 + (\Phi_W(\theta))^2 + (\Delta_M(\theta))^2 + (\Gamma_M(\theta))^2 + (\Pi_M(\theta))^2 + (\Omega_M(\theta))^2 + (\Phi_M(\theta))^2 \le 10$, $0 \le \Delta_W(\theta).\Delta_M(\theta) + \Gamma_W(\theta).\Gamma_M(\theta) + \Pi_W(\theta).\Pi_M(\theta) + \Omega_W(\theta).\Omega_M(\theta) + \Phi_W(\theta).\Phi_M(\theta) \le 5$, for each $\theta \in U$.

$$\Rightarrow 0 \le \frac{2[\Delta_W(\theta).\Delta_M(\theta)+\Gamma_W(\theta).\Gamma_M(\theta)+\Pi_W(\theta).\Pi_M(\theta)+\Omega_W(\theta).\Omega_M(\theta)+\Phi_W(\theta).\Phi_M(\theta)]}{[(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2+(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2]} \le 1,\ \forall \theta \in U.$$

$$\Rightarrow 0 \le \sum_{\theta \in U} \frac{2[\Delta_W(\theta).\Delta_M(\theta)+\Gamma_W(\theta).\Gamma_M(\theta)+\Pi_W(\theta).\Pi_M(\theta)+\Omega_W(\theta).\Omega_M(\theta)+\Phi_W(\theta).\Phi_M(\theta)]}{[(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2+(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2]} \le n.$$

$$\Rightarrow 0 \le \frac{1}{n} \sum_{\theta \in U} \frac{2[\Delta_W(\theta).\Delta_M(\theta)+\Gamma_W(\theta).\Gamma_M(\theta)+\Pi_W(\theta).\Pi_M(\theta)+\Omega_W(\theta).\Omega_M(\theta)+\Phi_W(\theta).\Phi_M(\theta)]}{[(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2+(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2]} \le 1$$

$\Rightarrow 0 \le D_{SVPNDSM}(W, M) \le 1$.

Therefore, $0 \le D_{SVPNDSM}(W, M) \le 1$.

(ii) We have, $D_{SVPNDSM}(W, M)$

$$= \frac{1}{n} \sum_{\theta \in U} \frac{2[\Delta_W(\theta).\Delta_M(\theta)+\Gamma_W(\theta).\Gamma_M(\theta)+\Pi_W(\theta).\Pi_M(\theta)+\Omega_W(\theta).\Omega_M(\theta)+\Phi_W(\theta).\Phi_M(\theta)]}{[(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2+(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2]}$$

$$= \frac{1}{n} \sum_{\theta \in U} \frac{2[\Delta_M(\theta).\Delta_W(\theta)+\Gamma_M(\theta).\Gamma_W(\theta)+\Pi_M(\theta).\Pi_W(\theta)+\Omega_M(\theta).\Omega_W(\theta)+\Phi_M(\theta).\Phi_W(\theta)]}{[(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2+(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2]}$$

$= D_{SVPNDSM}(M, W)$

Therefore, $D_{SVPNDSM}(W, M) = D_{SVPNDSM}(M, W)$.

(iii) Suppose that W and M be two SVPNSs over a fixed set U such that W = M. Let $D_{SVPNDSM}(W, M)$ be the SVPNDSM between the SVPNSs W and M.

Now, W = M

$\Rightarrow \Delta_W(\theta)=\Delta_M(\theta)$, $\Gamma_W(\theta)=\Gamma_M(\theta)$, $\Pi_W(\theta)=\Pi_M(\theta)$, $\Omega_W(\theta)=\Omega_M(\theta)$ and $\Phi_W(\theta)=\Phi_M(\theta)$, for each $\theta \in U$.

$\Rightarrow$ $\Delta_W(\theta).\Delta_M(\theta) = (\Delta_W(\theta))^2$, $\Gamma_W(\theta).\Gamma_M(\theta) = (\Gamma_W(\theta))^2$, $\Pi_W(\theta).\Pi_M(\theta) = (\Pi_W(\theta))^2$, $\Omega_W(\theta).\Omega_M(\theta)=(\Omega_W(\theta))^2$ and $\Phi_W(\theta).\Phi_M(\theta)=(\Phi_W(\theta))^2$, for each $\theta \in U$.

$\Rightarrow$ $2[\Delta_W(\theta).\Delta_M(\theta) + \Gamma_W(\theta).\Gamma_M(\theta) + \Pi_W(\theta).\Pi_M(\theta) + \Omega_W(\theta).\Omega_M(\theta) + \Phi_W(\theta).\Phi_M(\theta)] = [(\Delta_W(\theta))^2 + (\Gamma_W(\theta))^2 + (\Pi_W(\theta))^2 + (\Omega_W(\theta))^2 + (\Phi_W(\theta))^2 + (\Delta_M(\theta))^2 + (\Gamma_M(\theta))^2 + (\Pi_M(\theta))^2 + (\Omega_M(\theta))^2 + (\Phi_M(\theta))^2]$, for each $\theta \in U$.

$$\Rightarrow \frac{2[\Delta_M(\theta).\Delta_W(\theta)+\Gamma_M(\theta).\Gamma_W(\theta)+\Pi_M(\theta).\Pi_W(\theta)+\Omega_M(\theta).\Omega_W(\theta)+\Phi_M(\theta).\Phi_W(\theta)]}{[(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2+(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2]} = 1,\ \forall \theta \in U.$$

$$\Rightarrow \sum_{\theta \in U} \frac{2[\Delta_M(\theta).\Delta_W(\theta)+\Gamma_M(\theta).\Gamma_W(\theta)+\Pi_M(\theta).\Pi_W(\theta)+\Omega_M(\theta).\Omega_W(\theta)+\Phi_M(\theta).\Phi_W(\theta)]}{[(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2+(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2]} = n.$$

$$\Rightarrow \frac{1}{n} \sum_{\theta \in U} \frac{2[\Delta_M(\theta).\Delta_W(\theta)+\Gamma_M(\theta).\Gamma_W(\theta)+\Pi_M(\theta).\Pi_W(\theta)+\Omega_M(\theta).\Omega_W(\theta)+\Phi_M(\theta).\Phi_W(\theta)]}{[(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2+(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2]} = 1.$$

$\Rightarrow D_{SVPNDSM}(M, W) = 1$.

**Definition 3.2.** Assume that W = {(θ, $\Delta_W(\theta)$, $\Gamma_W(\theta)$, $\Pi_W(\theta)$, $\Omega_W(\theta)$, $\Phi_W(\theta)$) : θ∈U} and M = {(θ, $\Delta_M(\theta)$, $\Gamma_M(\theta)$, $\Pi_M(\theta)$, $\Omega_M(\theta)$, $\Phi_M(\theta)$) : θ∈U} be two SVPNSs over a fixed set U. Then, the SVPNWDSM between two SVPNSs W and M is defined by:

$D_{SVPNWDSM}$ (W, M)

$$= \sum_{\theta\in U} w_\theta \cdot \frac{2[\Delta_W(\theta).\Delta_M(\theta)+\Gamma_W(\theta).\Gamma_M(\theta)+\Pi_W(\theta).\Pi_M(\theta)+\Omega_W(\theta).\Omega_M(\theta)+\Phi_W(\theta).\Phi_M(\theta)]}{[(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2+(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2]} \quad (2)$$

**Example 3.2.** Consider the SVPNSs W and M on U given in Example 3.1. Assume that $w_1$=0.6, $w_2$= 0.4 be the corresponding weights of the SVPNSs W and M respectively. Then, SVPNWDSM (W, M) = 0.8810258129.

**Theorem 3.2.** Suppose that $D_{SVPNWDSM}$ (W, M) be the SVPNWDSM between the SVPNSs W and M. Then, the following holds:

(i) 0 ≤ $D_{SVPNWDSM}$ (W, M) ≤ 1;

(ii) $D_{SVPNWDSM}$ (W, M) = $D_{SVPNWDSM}$ (M, W);

(iii) W = M ⇒ $D_{SVPNWDSM}$ (W, M) = 1.

**Proof.** (i) Suppose that $D_{SVPNWDSM}$ (W, M) be the SVPNWDSM between the SVPNSs W and M, where $D_{SVPNWDSM}$ (W, M)

$$= \sum_{\theta\in U} w_\theta \cdot \frac{2[\Delta_W(\theta).\Delta_M(\theta)+\Gamma_W(\theta).\Gamma_M(\theta)+\Pi_W(\theta).\Pi_M(\theta)+\Omega_W(\theta).\Omega_M(\theta)+\Phi_W(\theta).\Phi_M(\theta)]}{[(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2+(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2]}.$$

It is known that,

$0\le \Delta_W(\theta)\le 1$, $0\le\Delta_M(\theta)\le 1$, $0\le \Gamma_W(\theta)\le 1$, $0\le \Gamma_M(\theta)\le 1$, $0\le \Pi_W(\theta)\le 1$, $0\le \Pi_M(\theta)\le 1$, $0\le \Omega_W(\theta)\le 1$, $0\le \Omega_M(\theta)\le 1$, $0\le \Phi_W(\theta)\le 1$ and $0\le \Phi_M(\theta)\le 1$, for each θ∈U.

$\Rightarrow 0 \le \Delta_W(\theta).\Delta_M(\theta) \le 1$, $0 \le \Gamma_W(\theta).\Gamma_M(\theta) \le 1$, $0 \le \Pi_W(\theta).\Pi_M(\theta) \le 1$, $0 \le \Omega_W(\theta).\Omega_M(\theta) \le 1$, $0 \le \Phi_W(\theta).\Phi_M(\theta) \le 1$, $0 \le (\Delta_W(\theta))^2 \le 1$, $0 \le (\Gamma_W(\theta))^2 \le 1$, $0 \le (\Pi_W(\theta))^2 \le 1$, $0 \le (\Omega_W(\theta))^2 \le 1$, $0 \le (\Phi_W(\theta))^2 \le 1$, for each θ∈U.

$\Rightarrow 0\le (\Delta_W(\theta))^2 + (\Gamma_W(\theta))^2 + (\Pi_W(\theta))^2 + (\Omega_W(\theta))^2 + (\Phi_W(\theta))^2 + (\Delta_M(\theta))^2 + (\Gamma_M(\theta))^2 + (\Pi_M(\theta))^2 + (\Omega_M(\theta))^2 + (\Phi_M(\theta))^2 \le 10$, and $0\le \Delta_W(\theta).\Delta_M(\theta) + \Gamma_W(\theta).\Gamma_M(\theta) + \Pi_W(\theta).\Pi_M(\theta) + \Omega_W(\theta).\Omega_M(\theta) + \Phi_W(\theta).\Phi_M(\theta)\le 5$, for each θ∈U.

$$\Rightarrow 0 \le \frac{2[\Delta_W(\theta).\Delta_M(\theta)+\Gamma_W(\theta).\Gamma_M(\theta)+\Pi_W(\theta).\Pi_M(\theta)+\Omega_W(\theta).\Omega_M(\theta)+\Phi_W(\theta).\Phi_M(\theta)]}{[(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2+(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2]} \le 1, \ \forall\theta\in U$$

$$\Rightarrow 0\le\sum_{\theta\in U} w_\theta \cdot \frac{2[\Delta_W(\theta).\Delta_M(\theta)+\Gamma_W(\theta).\Gamma_M(\theta)+\Pi_W(\theta).\Pi_M(\theta)+\Omega_W(\theta).\Omega_M(\theta)+\Phi_W(\theta).\Phi_M(\theta)]}{[(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2+(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2]}\le 1$$

⇒ 0 ≤ $D_{SVPNWDSM}$ (W, M) ≤ 1

Therefore, we have 0 ≤ $D_{SVPNWDSM}$ (W, M) ≤ 1.

(ii) We have, $D_{SVPNWDSM}$ (W, M)

$$= \sum_{\theta\in U} w_\theta \cdot \frac{2[\Delta_W(\theta).\Delta_M(\theta)+\Gamma_W(\theta).\Gamma_M(\theta)+\Pi_W(\theta).\Pi_M(\theta)+\Omega_W(\theta).\Omega_M(\theta)+\Phi_W(\theta).\Phi_M(\theta)]}{[(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2+(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2]}$$

$= \sum_{\theta\in U} w_\theta . \frac{2[\Delta_M(\theta).\Delta_W(\theta)+\Gamma_M(\theta).\Gamma_W(\theta)+\Pi_M(\theta).\Pi_W(\theta)+\Omega_M(\theta).\Omega_W(\theta)+\Phi_M(\theta).\Phi_W(\theta)]}{[(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2+(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2]}$

$= D_{SVPNWDSM}(M, W)$

Therefore, $D_{SVPNWDSM}(W, M) = D_{SVPNWDSM}(M, W)$.

(iii) Suppose that W and M be two SVPNSs over a fixed set U such that W = M. Assume that $D_{SVPNWDSM}(W, M)$ be the SVPNWDSM between W and M.

Now, W = M

$\Rightarrow$ $\Delta_W(\theta)=\Delta_M(\theta)$, $\Gamma_W(\theta)=\Gamma_M(\theta)$, $\Pi_W(\theta)=\Pi_M(\theta)$, $\Omega_W(\theta)=\Omega_M(\theta)$ and $\Phi_W(\theta)=\Phi_M(\theta)$, for each $\theta\in U$.

$\Rightarrow$ $\Delta_W(\theta).\Delta_M(\theta) = (\Delta_W(\theta))^2$, $\Gamma_W(\theta).\Gamma_M(\theta)=(\Gamma_W(\theta))^2$, $\Pi_W(\theta).\Pi_M(\theta)=(\Pi_W(\theta))^2$, $\Omega_W(\theta).\Omega_M(\theta) = (\Omega_W(\theta))^2$, $\Phi_W(\theta).\Phi_M(\theta) = (\Phi_W(\theta))^2$, for each $\theta\in U$.

$\Rightarrow$ $2[\Delta_W(\theta).\Delta_M(\theta) + \Gamma_W(\theta).\Gamma_M(\theta) + \Pi_W(\theta).\Pi_M(\theta) + \Omega_W(\theta).\Omega_M(\theta) + \Phi_W(\theta).\Phi_M(\theta)] = [(\Delta_W(\theta))^2 + (\Gamma_W(\theta))^2 + (\Pi_W(\theta))^2 + (\Omega_W(\theta))^2 + (\Phi_W(\theta))^2 + (\Delta_M(\theta))^2 + (\Gamma_M(\theta))^2 + (\Pi_M(\theta))^2 + (\Omega_M(\theta))^2 + (\Phi_M(\theta))^2]$, for each $\theta\in U$.

$\Rightarrow \frac{2[\Delta_M(\theta).\Delta_W(\theta)+\Gamma_M(\theta).\Gamma_W(\theta)+\Pi_M(\theta).\Pi_W(\theta)+\Omega_M(\theta).\Omega_W(\theta)+\Phi_M(\theta).\Phi_W(\theta)]}{[(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2+(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2]} = 1, \forall\theta\in U$

$\Rightarrow \sum_{\theta\in U} w_\theta . \frac{2[\Delta_M(\theta).\Delta_W(\theta)+\Gamma_M(\theta).\Gamma_W(\theta)+\Pi_M(\theta).\Pi_W(\theta)+\Omega_M(\theta).\Omega_W(\theta)+\Phi_M(\theta).\Phi_W(\theta)]}{[(\Phi_W(\theta))^2+(\Delta_M(\theta))^2+(\Gamma_M(\theta))^2+(\Pi_M(\theta))^2+(\Omega_M(\theta))^2+(\Phi_M(\theta))^2+(\Delta_W(\theta))^2+(\Gamma_W(\theta))^2+(\Pi_W(\theta))^2+(\Omega_W(\theta))^2]} = 1$

$\Rightarrow D_{SVPNWDSM}(M, W) = 1$

Therefore, W = M $\Rightarrow D_{SVPNWDSM}(W, M) = 1$.

## 4. MADM-Strategy Based on SVPNWDSM under SVPNS Environment:

The main focus of this section is to propose a MADM-strategy using the SVPNWDSM between two SVPNSs under the SVPNS environment. Figure-1 represents the proposed MADM-strategy.

Let us consider a MADM-problem, where L = {$L_1$, $L_2$, …….,$L_p$} is a set of possible alternatives and A = {$A_1$, $A_2$, ..., $A_q$} is the family of attributes. Then, the decision maker can give their evaluation information for each alternative $L_i$(i = 1, 2,..., p)against the attribute $A_j$ (j = 1, 2,..., q) by using SVPNS.

Then, the proposed MADM-strategy is designed in the following steps:

**Step-1: Decision Matrix Formation using SVPNS.**

Suppose, the decision maker gives their evaluation information by using the SVPNS $E_{L_i}$= {($A_j$, $\Delta_{ij}(L_i, A_j)$, $\Gamma_{ij}(L_i, A_j)$, $\Pi_{ij}(L_i, A_j)$, $\Omega_{ij}(L_i, A_j)$, $\Phi_{ij}(L_i, A_j)$): $A_j \in A$} for each alternative $L_i$ against the corresponding attributes $A_j$(j = 1, 2, ..., q), where ($\Delta_{ij}(L_i,A_j)$, $\Gamma_{ij}(L_i,A_j)$, $\Pi_{ij}(L_i,A_j)$, $\Omega_{ij}(L_i,A_j)$, $\Phi_{ij}(L_i,A_j)$) = $(L_i,A_j)$(i = 1, 2, ..., p and j = 1, 2, ..., q) is an SVPNN. By using all these evaluation information, a decision matrix ($D^M$) is billed as follows.

The decision matrix can be expressed as follows:

| $D^M$ | $A_1$ | $A_2$ | ……. | …….. | $A_q$ |
|---|---|---|---|---|---|

| | | | | | |
|---|---|---|---|---|---|
| $L_1$ | ($L_1$, $A_1$) | ($L_1$, $A_1$) | …….. | ……. | ($L_1$, $A_q$) |
| $L_2$ | ($L_2$, $A_1$) | ($L_2$, $A_2$) | …… | …….. | ($L_2$, $A_q$) |
| …… | …….. | …….. | ……. | ……. | ……… |
| $L_p$ | ($L_p$, $A_1$) | ($L_p$, $A_2$) | ……….. | ……… | ($L_p$, $A_q$) |

**Step-2: Selection of the Positive Ideal Solution for the Decision Matrix.**

The Positive Ideal Solution (PIS) for the decision matrix is defined as follows:

$L^+ = [(\Delta_1^+, \Gamma_1^+, \Pi_1^+, \Omega_1^+, \Phi_1^+), (\Delta_2^+, \Gamma_2^+, \Pi_2^+, \Omega_2^+, \Phi_2^+), \ldots\ldots\ldots, (\Delta_m^+, \Gamma_m^+, \Pi_m^+, \Omega_m^+, \Phi_m^+)]$, (3)

where $\Delta_j^+ = \max\{\Delta_{ij}(L_i, A_j): i=1, 2, 3, \ldots, n\}$, $\Gamma_j^+ = \max\{\Gamma_{ij}(L_i, A_j): i=1, 2, 3, \ldots, n\}$, $\Pi_j^+ = \min\{\Pi_{ij}(L_i, A_j): i=1, 2, 3, \ldots, n\}$, $\Omega_j^+ = \min\{\Omega_{ij}(L_i, A_j): i=1, 2, 3, \ldots, n\}$ and $\Phi_j^+ = \min\{\Phi_{ij}(L_i, A_j): i=1, 2, 3, \ldots, n\}$.

**Step-3: Calculation of Attribute's Weight.**

In any MADM problem, the decision maker can use the compromise function as tools for the calculation of the weight of each attribute those are completely unknown.

The compromise function is defined as follows:

$\Psi_j = \sum_{i=1}^{p} (3 + \Delta_{ij}(L_i, A_j) + \Gamma_{ij}(L_i, A_j) - \Pi_{ij}(L_i, A_j) - \Omega_{ij}(L_i, A_j) - \Phi_{ij}(L_i, A_j)) / 5.$ (4)

Then, the weight of the j-th attribute is defined by $w_j = \frac{\Psi_j}{\sum_{j=1}^{q} \Psi_j}$ (5)

Here, $\sum_{j=1}^{q} w_j = 1$.

**Step-4: Determination of the SVPNWDSM between PIS and $E_{L_i}$ (i = 1, 2,..., p).**

In this step, the SVPNWDSM between the decision elements from the decision matrix and the PIS is calculated by using eq. (2).

**Step-5: Ranking Order of the Alternatives.**

Finally, the ranking order of alternatives is determined based on the ascending order of SVPNWDSM between the PIS and the decision elements from the decision matrix. The alternative associated with the highest SVPNWDSM value is the most suitable alternatives.

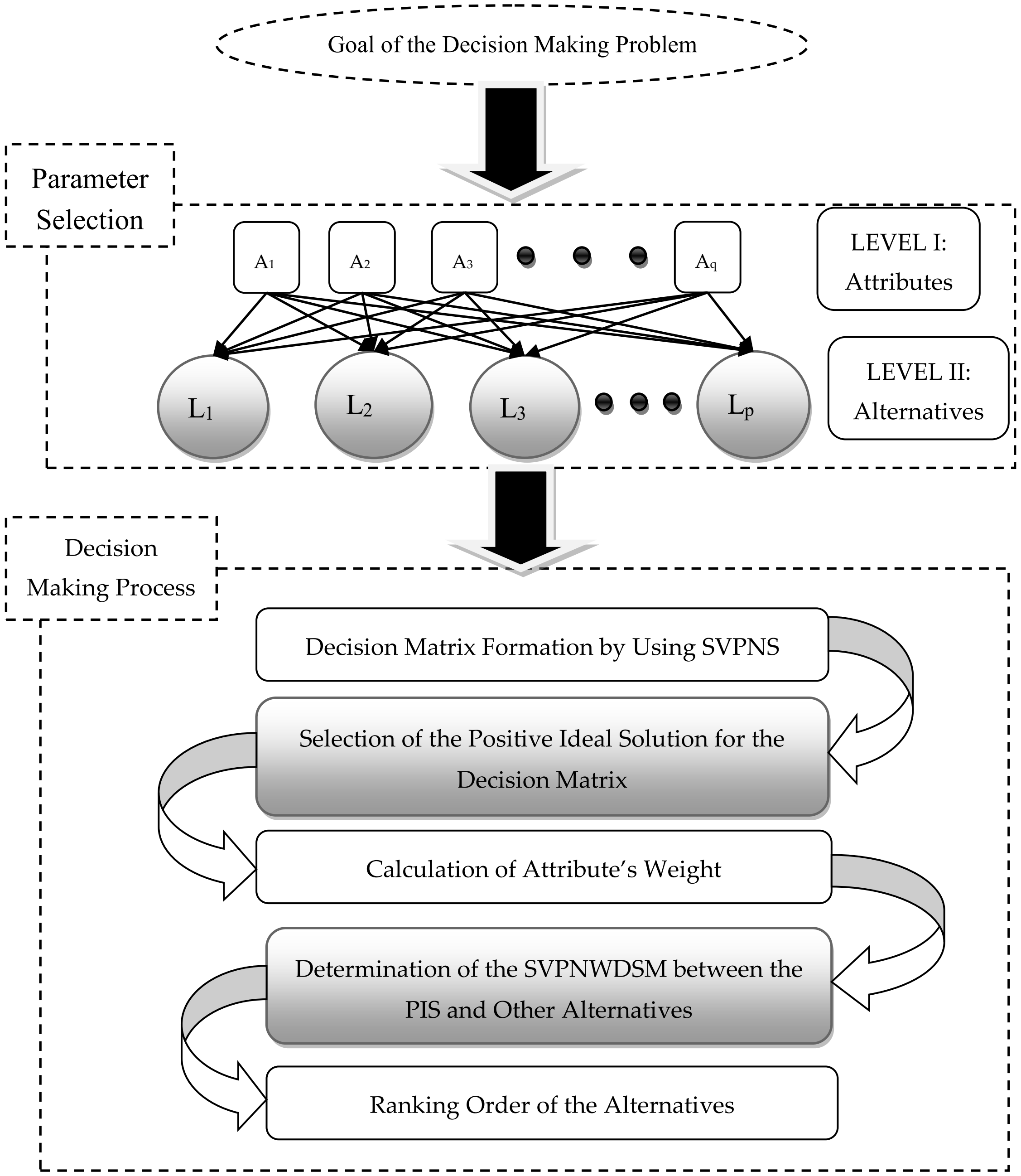


**Figure-1:** Proposed MADM-Strategy

## 5. Application of the Proposed MADM Strategy in the Selection of Suitable Metal Oxide Nano-Additive for Biodiesel Blend on Environmental Aspect under the SVPNS Environment:

The search for a potential alternative fuel has flourished due to the global demand for fossil fuels and environmental problems. Among all alternative fuels, biodiesel has become more popular in many nations. But main problem with biodiesel have their low calorific value and low heat value, which result low engine performance. In different research studies, nano-additives have been proposed for improving the performance and emission characteristics of biodiesel. Metal oxide nano-additives are basically used frequently for the improvement of combustion quality.

The proposed research work focused on the selection of suitable metal oxide nano-additive for biodiesel blend on environmental aspect under the SVPNS environment.

For the current research work five attributes namely (*i*) CO emission, (*ii*) HC emission (*iii*) $SO_2$ emission, (*iv*) $NO_x$ emission and (*v*) Smoke emission, and six alternatives namely (*i*) GO, (*ii*) $SiO_2$.,(*iii*) CuO, (*iv*) $Al_2O_3$, (*v*) $Fe_2O_3$ and (*vi*) $TiO_2$ are chosen from different literature [1, 16, 31, 39, 41, 44]. Best alternative among them was chosen with the proposed MADM strategy under the SVPNS environment.

In addition of $TiO_2$ nano-particles the values of CO, HC and smoke opacity emission reduced, while emission of $CO_2$ and NO are increased. This happen due to the intensified combustion process as compared to Bio diesel blends without $TiO_2$ [31]. CuO nano-particles shows good impact in reduction of CO, HC and smoke emission though in addition of CuO nano-additive $CO_2$ emission increase while $NO_x$ emission increase slightly [39]. A comparative study was done by Tomar and Kumar [41] between $Al_2O_3$ and $Fe_2O_3$. Reduction of all kind emission was observed with both nano-additives though $Fe_2O_3$ is slightly more effective in reduction of CO emission but in the case of $SO_2$ and NOx emission reduction, $Al_2O_3$ is more effective. Effect of GO was studied by Hoseini et al. [16]. In addition of GO, the emission of HC and CO decrease with a penalty of increased $NO_x$ emission. The effect of $Al_2O_3$ nano-particles was studied separately and it was observed that all the emission i.e., HC, CO, smoke and $NO_x$ emission reduced significantly at different loading condition. Ağbuluta et al. [1] have done a comparative study among three nano-particles metallic oxide namely $Al_2O_3$, $TiO_2$ and $SiO_2$. In [1], the authors reported, emission of CO, HC and $NO_x$ were reduced in the presence of three nano-additives with blend though the highest reduction of CO emission observed with $Al_2O_3$ nano-particles and highest $NO_x$ emission with $TiO_2$ nano-particles. Table-1 represents the list of nano-Particles added to biodiesel and their corresponding engine emissions.

**Table-1: List of nano-particles added to biodiesel and their corresponding engine emissions**

| Nano-particles & Dosage | Operating Condition | CO | HC | $NO_x$ | $CO_2$ | Smoke |
|---|---|---|---|---|---|---|

| | | | | | | |
|---|---|---|---|---|---|---|
| $TiO_2$ 0.01% by mass [31] | 1000 rpm, 1500 rpm, 2000rpm, 2500 rpm, 3000 rpm | B20+$TiO_2$ reduce the 25.56% CO emission compared to Diesel | B20+$TiO_2$ reduce HC emission around 34.12% at 3000 rpm | $TiO_2$ dramatically increasing the in cylinder pressure and temperature since the rising of NO emission | -- | Average reduction 25.07% |
| CuO, 25, 50, and 75 ppm[39] | Different load condition | Reduced | Reduced | Increase slightly | Increase | Reduced |
| $Fe_2O_3$,$Al_2O_3$, 30;60;90 ppm[41] | 1800 rpm and at 50% load condition | Reduced | | Reduced(up to 24%Reduction found with $Al_2O_3$) | | 10-15% lower at 300 ppm |
| GO, 30;60;90 ppm [16] | 2100 rpm and different load condition | Reduced | Reduced | Increase | Increase | |
| $Al_2O_3$, 25;50 ppm [44] | 1500 rpm and different load condition | Reduced (Maximum Reduction found with 50 ppm $Al_2O_3$) | Reduced (Maximum Reduction found with 50 ppm $Al_2O_3$) | Reduced (Maximum Reduction found with 50 ppm $Al_2O_3$) | Reduced (Maximu m Reductio n found with 50 ppm $Al_2O_3$) | Reduced (Maximu m Reductio n found with 50 ppm $Al_2O_3$) |
| $Al_2O_3$;$TiO_2$,Si $O_2$ppm [1] | 2000 rpm and different load condition | Reduced (Maximum Reduction found with $Al_2O_3$) | Reduced (Maximum Reduction found with $Al_2O_3$) | Increase (Maximum increment found with $TiO_2$) | | |

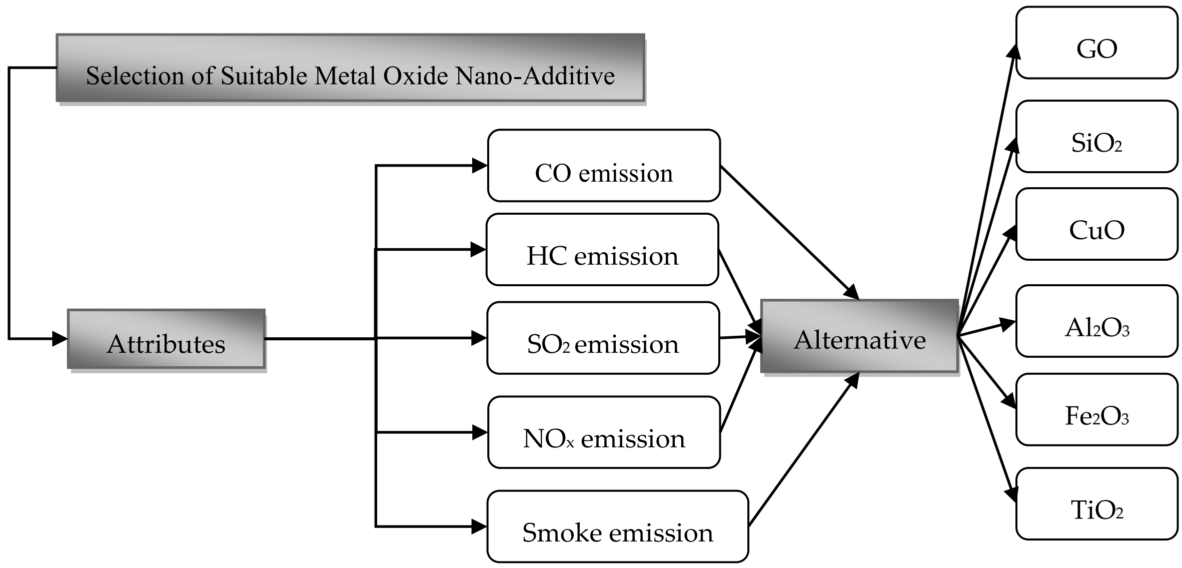


**Figure-2:** Decision Hierarchy of the Current MADM Problem

Figure-2 represents decision hierarchy of the current MADM problem and steps involve in the current MADM problem is presented as follows:

By using the evaluation information for all alternatives given by the decision makers, prepare the decision matrix in Table-2.

**Table-2: Decision Matrix**

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |
|---|---|---|---|---|---|
| $L_1$ | (1.0,0.2,0.2,0.0,0.0) | (0.8,0.1,0.1,0.1,0.1) | (0.9,0.0,0.2,0.1,0.1) | (1.0,0.0,0.2,0.1,0.1) | (0.9,0.0,0.0,0.0,0.1) |
| $L_2$ | (1.0,0.1,0.2,0.1,0.2) | (0.9,0.1,0.2,0.2,0.1) | (0.8,0.1,0.0,0.0,0.1) | (0.9,0.0,0.0,0.1,0.1) | (0.9,0.1,0.0,0.2,0.0) |
| $L_3$ | (1.0,0.1,0.0,0.0,0.1) | (0.9,0.0,0.1,0.1,0.2) | (0.9,0.1,0.1,0.1,0.1) | (0.8,0.1,0.2,0.1,0.1) | (1.0,0.2,0.0,0.0,0.1) |
| $L_4$ | (0.9,0.2,0.1,0.1,0.0) | (1.0,0.1,0.0,0.0,0.1) | (0.9,0.1,0.1,0.1,0.1) | (1.0,0.1,0.0,0.0,0.1) | (0.8,0.1,0.1,0.2,0.1) |
| $L_5$ | (1.0,0.2,0.1,0.1,0.1) | (0.8,0.1,0.0,0.0,0.1) | (0.7,0.2,0.0,0.1,0.1) | (0.8,0.1,0.0,0.0,0.1) | (1.0,0.1,0.0,0.0,0.1) |
| $L_6$ | (0.8,0.1,0.1,0.2,0.1) | (1.0,0.0,0.1,0.2,0.1) | (0.9,0.0,0.2,0.1,0.1) | (0.8,0.1,0.1,0.2,0.1) | (1.0,0.1,0.0,0.0,0.1) |

Now, by using the eq. (3), the PIS ($L^{+}$) is formed for the decision matrix in Table-3.

**Table-3: Positive Ideal Solution**

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |
|---|---|---|---|---|---|
| $L^+$ | (1.0,0.2,0.0,0.0,0.0) | (1.0,0.1,0.0,0.0,0.1) | (0.9,0.2,0.0,0.0,0.1) | (1.0,0.1,0.0,0.0,0.1) | (1.0,0.2,0.0,0.0,0.0) |

Weights of the attributes are determined by using the eq. (4) & eq. (5). The weights of the attribute are $w_1$=0.2042819, $w_2$=0.1962533, $w_3$=0.1953613, $w_4$=0.1971454, $w_5$=0.2069581.

By using the eq. (2), obtained SVPNWDSM of similarities between the PIS and the decision elements from the decision matrix as follows:

$D_{SVPNWDSM}(L_1, L^+)$ =0.966693;

$D_{SVPNWDSM}(L_2, L^+)$ = 0.968999;

$D_{SVPNWDSM}(L_3, L^+)$ = 0.978151;

$D_{SVPNWDSM}(L_4, L^+)$ = 0.980355;

$D_{SVPNWDSM}(L_5, L^+)$ = 0.978792;

$D_{SVPNWDSM}(L_6, L^+)$ = 0.95907.

The ascending order of the SVPNWDSM between the PIS and the decision elements from the decision matrix is as follows:

$D_{SVPNWDSM}(L_6, L^+) < D_{SVPNWDSM}(L_1, L^+) < D_{SVPNWDSM}(L_2, L^+) < D_{SVPNWDSM}(L_3, L^+) < D_{SVPNWDSM}(L_5, L^+) < D_{SVPNWDSM}(L_4, L^+)$.

Hence, the alternative $L_4$ i.e., $Al_2O_3$ is the most suitable metal oxide nano-additive for the biodiesel blend on environmental aspect under the SVPNS environment.

## 6. Comparative Study:

To verify the proposed result based on the SVPNWDSM, an investigation has been conducted for the purpose of comparison with the existing MADM techniques [13, 23].

From the comparative table (see Table-4) it is observed that the existing methods support the same performance as per the proposed method for best attribute. According to the Table-4 it is clear that the weighted values of all attribute are much closed for two existing methods. In case of proposed technique the weighted values of all attribute is not closed compare to existing tool, it helps to take better decision for considering attributes. So the proposed method is more effective compare to considering MADM methods.

**Table-4: Comparative Study**

| Methods | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | $L_6$ | Ranking Order |
|---|---|---|---|---|---|---|---|
| MADM Strategy Based on Tangent Similarity Measure under SVPNS Environment [13] | 0.976292 | 0.978664 | 0.981409 | 0.985021 | 0.982482 | 0.975305 | $L_6 < L_1 < L_2 < L_3 < L_5 < L_4$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| MADM Strategy Based on Cosine Similarity Measure under SVPNS Environment [23] | 0.834713 | 0.834963 | 0.834974 | 0.836300 | 0.835798 | 0.834743 | $L_1 < L_6 < L_2 < L_3 < L_5 < L_4$ |
| Proposed MADM Strategy | 0.966693 | 0.968999 | 0.978151 | 0.980355 | 0.978792 | 0.959070 | $L_6 < L_1 < L_2 < L_3 < L_5 < L_4$ |

From the above comparison Table-4, it is clear that $L_4$ is the most appropriate alternative in all the MADM strategies.

**7. Conclusions:**

In this article, a novel MADM is proposed for selecting suitable nano-additives for biodiesel to enhance performance and emissions characteristics of internal combustion engines. Using this method, a ranking among the alternatives is generated. The ranking order $L_6 < L_1 < L_2 < L_3 < L_5 < L_4$ is derived by the proposed method. It is obvious from the ranking order generated by the new method that alternative $L_4$ is the best among all alternatives. A comparison of the results obtained by the new MADM method is performed using different existing methods. Based on all methods, alternative $L_4$ is the best, and therefore, it is concluded that the proposed method is well suited for solving such a problem.

In a future study, the nano-additive selected from this present work will be applied to biodiesel in different concentrations and its performance and emission characteristics will be examined experimentally. Further, it is hoped that, the proposed MADM-strategy can also be used to deal with the other real life problems such as Data Mining [30], Medical Diagnosis [35-36], Fault Diagnosis [46-47], and decision-making problems such as Tender Selection [7], Electronic Goods Selection [12], Plot Selection [13], Weaver Selection [15], Brick Selection [25, 29], Logistic Center Location Selection [32-33], Teacher Selection [37], etc.

**Conflict of Interest:** The authors declare that they have no conflict of interest.

**Authors Contribution:** The authors declare that all the authors have equal contribution for the preparation of this article.

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Received: Nov 22, 2021. Accepted: Feb 2, 2022

Neutrosophic Sets and Systems, Vol. 48, 2022
University of New Mexico

# An Intelligent Model to Rank Risks of Cloud Computing based on Firm's Ambidexterity Performance under Neutrosophic Environment

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**Abstract**: In recent years, cloud computing has emerged as a revolutionary technology that offers several benefits to businesses; nevertheless, like any other technology, it comes with significant risks. Firms can gain a competitive advantage by investing in cloud computing while simultaneously exploring new opportunities and leveraging their existing knowledge and capabilities. Cloud computing dangers, on the other hand, may limit these capabilities. We have shown that prospective cloud computing risks have a considerable impact on organizations' performance in two key areas of explorative and exploitative innovation using the ambidexterity theoretical lens. To achieve these goals, the Neutrosophic VlseKriterijumska Optimizcija I Kaompromisno Resenje in Serbian (VIKOR) and multi-attributive border approximation area comparison (MABAC) techniques were used, in which the Neutrosophic approach aids experts in expressing their opinions using linguistic variables, and the VIKOR and MABAC techniques rank cloud computing risks based on ambidexterity criteria. There are eight criteria and ten alternatives are used in this study.

**Keywords**: Cloud Computing; Risks; Neutrosophic; Uncertainty; VIKOR; MABAC

## 1. Introduction

Firms have placed a greater emphasis on public computing infrastructure in recent years [1]. Based on cloud computing, it is estimated that organizations have experienced a $3.3 trillion shift in their computer performance [2]. Cloud computing is a computing model that involves the deployment of enormous data centers with efficient processor equipment [3]. By implementing cloud computing technologies, businesses can reap numerous benefits, including reduced investment costs [4]. Cloud computing has also improved the firm's agility by providing flexibility and on-demand services [5-6]. Cloud computing has been cited as a good example of how to improve your business [7]. Though cloud computing is gaining a lot of traction in many industries, it, like any other technology, comes with significant hazards [8]. The most significant hazards of cloud computing implementation, according to past research, are "authentication," "data security, and privacy." [9-11] “confidentiality,” “integrity,” “availability” [12], “accountability,” and “accessibility” [13]. Because risks can have direct and indirect negative effects on service quality, it's critical to have a thorough awareness of them, especially for a newly created technology [10]. Cloud computing plays an important role in strong company innovation since it provides a huge number of innovation opportunities, such as novel computing capabilities and solutions [14]. Though, cloud computing systems' innovative performance is affected by unpredictability and risk issues.

Exploration and exploitation are two methods for obtaining innovative results. The former relates to gathering information and benefiting from new opportunities by investigating new possibilities; the latter, on the other hand, focuses on producing value by taking into account current prospects [15]. Businesses that use both exploration and exploitation at the same time might profit from ambidexterity performance in this way [16]. To put it another way, while exploitation focuses on increasing business productivity and efficiency by deploying current knowledge, exploration focuses on getting innovative and recent technologies and resources by producing and acquiring new knowledge [17-18]. Exploration and exploitation innovations rely heavily on information technology, which may lead to the development of new goods and services for new consumers as well as the extension of existing products and services for existing customers. As a consequence, businesses may achieve long-term success in a changing environment [19].

Exploitation competency may be gained by conserving and leveraging current innovative skills, processes, and knowledge, whereas exploration competency can be gained through recreating knowledge and abilities [20]. The capacity of a company to explore and exploit new opportunities while reacting quickly to market changes results in ambidextrous success [21]. Ambidexterity characteristics help cloud computing corporations to be flexible in an unpredictable market, suggesting that businesses can gain a competitive edge by leveraging dynamic skills. In moderately dynamic markets, exploration capabilities such as deploying routines and codified knowledge are expected; however, in high-velocity markets, exploration capabilities should be strengthened [22]. Because risks influence how businesses spend their dualities (exploration and exploitation) [23]. To ensure that cloud computing systems work well, researchers focused on limiting the influence of risk factors [24]. For example, the cloud computing environment's cyber security risk results in poor service level performance [25]. Another research found that IT infrastructure improves ambidexterity performance and helps firms function more efficiently [26], [27]. Firms' flexibility, agility, cost-effectiveness, and scalability may all benefit from cloud computing, according to it is cited. Furthermore, it can be beneficial in facilitating the rapid introduction of startups to the market, although cloud computing risks might have a detrimental impact on a firm's performance [28]. According to another research, organizations place a high value on data kept in cloud computing infrastructures, which are vulnerable to a variety of dangers. As a result, if such risks materialize, corporations will encounter major problems in carrying out their exploratory and exploitative performance [29]. Furthermore, successful cloud computing adoption may have a favorable impact on a company's performance since it merges internal IT skills, human, and physical resources to operate and improve operations [2]. When security concerns are taken into account, cloud computing encourages inventive performance, particularly when it comes to bringing new goods and services to market [30]. Indeed, cloud computing may lead to inter-organizational innovation that makes use of external knowledge, skills, and production facilities while also maximizing internal knowledge and production capabilities. Various cloud computing concerns, including as economic risks, service availability risks, and data security risks, might be overlooked. It will be steered toward a low adoption rate [31]. It is reasonable to assume that if cloud computing infrastructure is exposed to hazards, this will have a negative influence on business performance.

As a result, the primary purpose of this study is to identify cloud computing risks, followed by a gap analysis of the influence of risks on company performance using organizational ambidexterity theory. As a result, the given theory is used to answer the following research question: what are the top cloud computing risks? To answer the study's main issue, we first assemble previously researched cloud computing risk indicators, then rank them using neutrosophic VIKOR and MABAC approaches based on ambidexterity measurements (Exploration and Exploitation). In various fields, neutrosophic VIKOR and MABAC have been effectively employed to solve neutrosophic multi-criteria decision-making problems [32-39]. However, it has never been used to mitigate the hazards associated with cloud computing. This research makes several contributions. For starters, cloud computing risk concerns have been discovered from a much broader perspective. Second, selected risk variables are prioritized using ambidexterity measures using the neutrosophic VIKOR and MABAC approaches, which is the first research of cloud computing risk factors.

Section 2 provides the related works of cloud computing risks. Section 3 shows the methodology of this paper. Section 4 shows the case study and application of methodology. Section 5 refers to the conclusion of this paper.

## 2. Related Works

Table 1. show the previous research on the risks of cloud computing.

**Table 1.** Prior study on cloud computing risk ranking

| **Reference** | **Prioritizing cloud computing risks** |
|---|---|
| Dutta et al .[40] | Cloud computing dangers were found in this study, and the ten most significant ones were chosen by creating a risk score based on three factors: chance, effect, and frequency. |
| Elzamly et al. [41] | Based on Delphi research, the study identified and prioritized important security concerns in cloud computing for financial firms. |
| Boutkhoum et al. [42] | In this study, a fuzzy AHP-PROMETHEE is employed to determine the best appropriate cloud computing for large data. |
| Boutkhoum et al. [43] | The authors employed the AHP-TOPSIS approach to assessing cloud computing services to better manage big data in this study. |
| Amini et al. [44] | To rank cloud computing hazards, fuzzy logic was used in this study. The severity and likelihood criteria were used for assessment. |
| Henriques de Gusma˜o et al [45] | The authors of this work examine cyber security threats using fault tree analysis and fuzzy decision theory. |
| Patel and Alabisi [46] | Cloud computing threats were classified in this study into many categories. Customers, service providers, and the government are all taken into account when identifying hazards. |
| Krishnaveni and Prabakaran [47] | Researchers used machine learning classifier methods to classify cloud computing network intrusion and assaults in this study. SVM, Naive Bayes, and Logistic regression algorithms were used, and the approaches were assessed based on accuracy and reaction time. |

| | |
|---|---|
| Swathy Akshaya and Padmavathi [48] | A taxonomy of cloud computing dangers has been presented in this work. The service delivery paradigms "software as a service," "platform as a service," and "infrastructure as a service" were used to create the categorization. |
| Jouini et al. [49] | Security threats associated with cloud computing infrastructures were categorized in this study, and new information security metrics were provided based on quantitative analysis. |
| Sheehan et al. [50] | The cyber security risk of cloud computing has been categorized in this study. In addition, proactive and reactive obstacles to minimizing such hazards have been identified. To assess cyber security risk, likelihood and severity/impact criteria have been implemented, which aid in quantifying those risks. |
| Mohammad Taghi Taghavifard & Setareh Majidian [51] | This study used the Fuzzy VIKOR Technique to identify cloud computing risks based on a firm's ambidexterity performance. |
| This study | In this study, we used the neutrosophic sets hybrid with the MCDM methods like neutrosophic VIKOR and MABAC to compute the weights of criteria and rank of risks (alternatives). |

## 3. Methodology

In this section, we provide some definitions in neutrosophic sets and we introduce the neutrosophic VIKOR and MABAC methods. we use $P_a = \{1,2, \ldots, a\}$ and $P_b = \{1,2, \ldots, b\}$ as an index set for $a \in \mathbb{N}$ and $b \in \mathbb{N}$, respectively.

### 3.1 Definitions

**Definition 1:** [52] Make X become a universe. The definition of a neutrosophic set Y over X is:

$$Y = \{< V, (T_Y(V), I_Y(V), F_Y(V)) >: V \in X\}.$$

where $T_Y(V)$, $I_Y(V)$,, and $F_Y(V)$ are the truth-membership, indeterminacy-membership, and falsity membership functions, respectively. They are described as follows:

$$T_Y: X \rightarrow ]\ 0^-, 1^+[, I_Y: X \rightarrow ]\ 0^-, 1^+[, F_Y: X \rightarrow ]\ 0^-, 1^+[$$

Such that $0^- \leq T_Y(V) + I_Y(V) + F_Y(V) \leq 3^+$.

**Definition 2:** [52] Assume X be a universe. A single-valued neutrosophic set (SVN-set) over X is a neutrosophic set over X, but the truth-membership function, indeterminacy membership function, and falsity-membership function are respectively described as:

$$T_Y: X \longrightarrow [0,1],\ \ I_Y: X \longrightarrow [0,1], F_Y: X \longrightarrow [0,1]$$

Such that $0 \leq T_Y(V) + I_Y(V) + F_Y(V) \leq 3$

**Definition 3**: [52] Assume $h_{\overline{k}}, g_{\overline{k}}, j_{\overline{k}} \in [0,1]$ be any real numbers, $n_w, m_w, o_w, q_w \in \mathbb{R}$ and , $n_w \leq m_w \leq o_w \leq q_w$ $(w = 1,2,3)$ Then a single valued neutrosophic number (SVNN)

$$\hat{y} = \langle\left((n_1, m_1, o_1, q_1), h_{\overline{k}}\right), \left((n_2, m_2, o_2, q_2), g_{\overline{k}}\right), \left((n_3, m_3, o_3, q_3), j_{\overline{k}}\right)\rangle$$

is a special neutrosophic set on the set of real numbers $\mathbb{R}$, whose truth-membership function $b_{\overline{k}}$, indeterminacy membership function $c_{\overline{k}}$ and falsity-membership function $d_{\overline{k}}$ are respectively described as:

$$b_{\overline{k}}: \mathbb{R} \longrightarrow \left[0, h_{\overline{k}}\right], b_{\overline{k}}(V) = \begin{cases} f_b^1(V), & n_1 \leq V < m_1 \\ h_{\overline{k}}, & m_1 \leq V < o_1 \\ f_b^e(V), & o_1 \leq V < q_1 \\ 0, & otherwise \end{cases}$$

$$c_{\overline{k}}: \mathbb{R} \longrightarrow \left[g_{\overline{k}}, 1\right], c_{\overline{k}}(V) = \begin{cases} f_c^1(V), & n_2 \leq V < m_2 \\ g_{\overline{k}}, & m_2 \leq V < o_2 \\ f_c^e(V), & o_2 \leq V < q_2 \\ 1, & otherwise \end{cases}$$

$$d_{\overline{k}}: \mathbb{R} \longrightarrow \left[j_{\overline{k}}, 1\right], d_{\overline{k}}(V) = \begin{cases} f_d^1(V), & n_3 \leq V < m_3 \\ j_{\overline{k}}, & m_3 \leq V < o_3 \\ f_d^e(V), & o_3 \leq V < q_3 \\ 1, & otherwise \end{cases}$$

Where the functions $f_b^1: [n_1, m_1] \longrightarrow \left[0, h_{\overline{k}}\right]$ , $f_c^e: [o_2, q_2] \longrightarrow \left[g_{\overline{k}}, 1\right]$, $f_d^e: [o_3, q_3] \longrightarrow \left[j_{\overline{k}}, 1\right]$ are continuous and non-decreasing , and satisfy the conditions: $f_b^1(n_1) = 0, f_b^1(m_1) = h_{\overline{k}},\ f_c^e(o_2) = g_{\overline{k}},\ f_c^e(q_2) = 1, f_d^e(o_3) = j_{\overline{k}}, f_d^e(q_3) = 1$ functions $f_b^e: [o_1, q_1] \longrightarrow \left[0, h_{\overline{k}}\right]$ , $f_c^1: [n_2, m_2] \longrightarrow \left[g_{\overline{k}}, 1\right]$, $f_d^1: [n_3, m_3] \longrightarrow \left[j_{\overline{k}}, 1\right]$ are continuous and nodecreasing , and satisfy the conditions: $f_b^e(o_1) = h_{\overline{k}}, f_b^r(q_1) = 0,\ f_c^1(n_2) = 1,\ f_c^1(m_2) = g_{\overline{k}}, f_d^1(n_3) = 1, f_d^1(m_3) = j_{\overline{k}}$. $[m_1, o_1], n_1$ *and* $q_1$ For the truth-membership function, the mean interval and the lower and higher limits of the general neutrosophic number $\overline{k}$, respectively. $[m_2, o_2], n_2$ *and* $q_2$ For the indeterminacy-membership function, the mean interval, and the lower and higher limits of the general neutrosophic number $\overline{k}$, respectively. $[m_3, o_3], n_3$ *and* $q_3$ For the falsity-membership function, the mean interval, and the lower and higher limits of the general neutrosophic number $\overline{k}$, respectively. The maximum truth-membership degree, minimum indeterminacy-membership degree, and minimum falsity-membership degree are $h_{\overline{k}}$, $g_{\overline{k}}$, and $j_{\overline{k}}$, respectively.

**Definition 4:** [53] Assume $\overline{k} = \langle (n_1, m_1, o_1, q_1); h_{\overline{k}}, g_{\overline{k}}, j_{\overline{k}} \rangle$, $\overline{kk} = \langle (n_2, m_2, o_2, q_2); h_{\overline{kk}}, g_{\overline{kk}}, j_{\overline{kk}} \rangle$ be two SVNNs and a constant $s \neq 0$ be any real number then:

$$\overline{k} + \overline{kk} = \langle (n_1 + n_2, m_1 + m_2, o_1 + o_2, q_1 + q_2); h_{\overline{k}} \wedge h_{\overline{kk}}, g_{\overline{k}} \vee g_{\overline{kk}}, j_{\overline{k}} \vee j_{\overline{kk}} \rangle$$

$$\overline{k}\,\overline{kk} = \begin{cases} \langle (n_1 n_2, m_1 m_2, o_1 o_2, q_1 q_2); h_{\overline{k}} \wedge h_{\overline{kk}}, g_{\overline{k}} \vee g_{\overline{kk}}, j_{\overline{k}} \vee j_{\overline{kk}} \rangle & (q_1 > 0, q_2 > 0) \\ \langle (n_1 q_2, m_1 o_2, o_1 m_2, q_1 n_2); h_{\overline{k}} \wedge h_{\overline{kk}}, g_{\overline{k}} \vee g_{\overline{kk}}, j_{\overline{k}} \vee j_{\overline{kk}} \rangle & (q_1 < 0, q_2 > 0) \\ \langle (q_1 q_2, o_1 o_2, m_1 m_2, n_1 n_2); h_{\overline{k}} \wedge h_{\overline{kk}}, g_{\overline{k}} \vee g_{\overline{kk}}, j_{\overline{k}} \vee j_{\overline{kk}} \rangle & (q_1 < 0, q_2 > 0) \end{cases}$$

$$s\overline{k} = \begin{cases} \langle (sn_1, sm_1, so_1, sq_1); h_{\overline{k}}, g_{\overline{k}}, j_{\overline{k}} \rangle & (s > 0) \\ \langle (sq_1, so_1, sm_1, sn_1); h_{\overline{k}}, g_{\overline{k}}, j_{\overline{k}} \rangle & (s < 0) \end{cases}$$

**Definition 5:** Assume $\overline{k} = \langle (n_1, m_1, o_1, ); h_{\overline{k}}, g_{\overline{k}}, j_{\overline{k}} \rangle$, $\overline{kk} = \langle (n_2, m_2, o_2, ); h_{\overline{kk}}, g_{\overline{kk}}, j_{\overline{kk}} \rangle$ be two SVNNs and a constant $s \neq 0$ be any real number then:

$$\overline{k} + \overline{kk} = \langle (n_1 + n_2, m_1 + m_2, o_1 + o_2); h_{\overline{k}} \wedge h_{\overline{kk}}, g_{\overline{k}} \vee g_{\overline{kk}}, j_{\overline{k}} \vee j_{\overline{kk}} \rangle$$

$$\overline{k}\,\overline{kk} = \begin{cases} \langle (n_1 n_2, m_1 m_2, o_1 o_2); h_{\overline{k}} \wedge h_{\overline{kk}}, g_{\overline{k}} \vee g_{\overline{kk}}, j_{\overline{k}} \vee j_{\overline{kk}} \rangle & (o_1 > 0, o_2 > 0) \\ \langle (n_1 o_2, m_1 m_2, o_1 n_2); h_{\overline{k}} \wedge h_{\overline{kk}}, g_{\overline{k}} \vee g_{\overline{kk}}, j_{\overline{k}} \vee j_{\overline{kk}} \rangle & (o_1 < 0, o_2 > 0) \\ \langle (o_1 o_2, m_1 m_2, n_1 n_2); h_{\overline{k}} \wedge h_{\overline{kk}}, g_{\overline{k}} \vee g_{\overline{kk}}, j_{\overline{k}} \vee j_{\overline{kk}} \rangle & (o_1 < 0, o_2 > 0) \end{cases}$$

$$s\overline{k} = \begin{cases} \langle (sn_1, sm_1, so_1); h_{\overline{k}}, g_{\overline{k}}, j_{\overline{k}} \rangle & (s > 0) \\ \langle (sq_1, so_1, sm_1); h_{\overline{k}}, g_{\overline{k}}, j_{\overline{k}} \rangle & (s < 0) \end{cases}$$

**Definition 6:** A single valued trapezoidal neutrosophic number $\overline{k} = \langle (n, m, o, q); h_{\overline{k}}, g_{\overline{k}}, j_{\overline{k}} \rangle$

is a special neutrosophic set on the set of real numbers $\mathbb{R}$, whose truth-membership function, indeterminacy membership function and falsity-membership function are respectively described as:

$$b_{\overline{k}}(V) = \begin{cases} (v - n) h_{\overline{k}} / (m - n), & n \leq V < m \\ h_{\overline{k}}, & m \leq V < o \\ (q - v) h_{\overline{k}} / (q - o), & o \leq V < q \\ 0, & otherwise \end{cases}$$

$$c_{\overline{k}}(V) = \begin{cases} \big(m - v + c_{\overline{k}}(v - n)\big) / (m - n), & n \leq V < m \\ c_{\overline{k}}, & m \leq V < o \\ \big(v - o + c_{\overline{k}}(q - v)\big) / (q - o), & o \leq V < q \\ 0, & otherwise \end{cases}$$

$$d_{\overline{k}}(V) = \begin{cases} (m - v + d_{\overline{k}}(V)(v-n))/(m-n), & n \le V < m \\ c_{\overline{k}}, & m \le V < o \\ (v - o + d_{\overline{k}}(V)(q-v))/(q-o), & o \le V < q \\ 0, & otherwise \end{cases}$$

respectively.

**Definition 7:** A single valued trapezoidal neutrosophic number $\overline{k} = \langle (n, m, o); h_{\overline{k}}, g_{\overline{k}}, j_{\overline{k}} \rangle$

is a special neutrosophic set on the set of real numbers $\mathbb{R}$, whose truth-membership function, indeterminacy membership function and falsity-membership function are respectively described as:

$$b_{\overline{k}}(V) = \begin{cases} (v-n)h_{\overline{k}}/(m-n), & n \le V < m \\ (o-v)h_{\overline{k}}/(o-m), & m \le V \le o \\ 0, & otherwise \end{cases}$$

$$c_{\overline{k}}(V) = \begin{cases} (m - v + c_{\overline{k}}(v-n))/(m-n), & n \le V < m \\ ((m-o)c_{\overline{k}}(o-m))/(o-m), & m \le V \le o \\ 0, & otherwise \end{cases}$$

$$d_{\overline{k}}(V) = \begin{cases} (m - v + d_{\overline{k}}(V)(v-n))/(m-n), & n \le V < m \\ ((v-m)d_{\overline{k}}(V)(o-v))/(o-m), & m \le V \le o \\ 0, & otherwise \end{cases}$$

respectively.

### 3.2 Phases of the proposed model for Cloud Computing

In this subsection, we provide two phases

**Phase I: The Neutrosophic VIKOR Procedure**

**Stage 1:** Form a committee of experts to decide on the aim, alternatives, and criteria.

**Stage 2:** Draw and create the language scales that will be used to characterize experts, as well as the alternatives.

**Stage 3:** Collect the opinions of the experts on each component.

**Stage 4**: Covert opinions of experts to the SVNNs

**Stage 5:** Compute the score function, by converting the three values of SVNNs into a one value by

$$S(V) = \frac{2 + T(V) - I(V) - F(V)}{3}$$

**Stage 6:** Compute the weights of criteria by the average method as:

$$W_a = \frac{S_1(V) + S_2(V) + \cdots . S_a(V))}{a}$$

Where a refers to number of criteria.

**Stage 7:** Construct an evaluation matrix by opinions of experts then average these opinions to obtain one decision matrix

$$V = \begin{pmatrix} V_{11} & \cdots & V_{1a} \\ \vdots & \ddots & \vdots \\ V_{b1} & \cdots & V_{ba} \end{pmatrix}$$

**Stage 8:** Compute the best and worst solution

${L_a}^+ = \max V_{ba}$ for positive criteria

${L_a}^- = \min V_{ba}$ for negative criteria

**Stage 9:** Compute the $Z_a, U_a$ values:

$$Z_a = \sum_{b=1}^{a} W_a * \frac{{L_a}^+ - V_{ba}}{{L_a}^+ - {L_a}^-}$$

$$U_a = \max_b (W_a * \frac{{L_a}^+ - V_{ba}}{{L_a}^+ - {L_a}^-})$$

**Stage 10:** Compute the value of $R_a$ as:

$$R_a = d\left(\frac{Z_a - \min_b Z_a}{\max_b Z_a - \min_b Z_a}\right) + (1-d)\left(\frac{U_a - \min_b U}{\max_b U_a - \min_b U_a}\right)$$

Where d =0.5

**Stage 11:** Rank alternatives according to ascending order of the previous step

**Phase II: The Neutrosophic MABAC Procedure**

**Stage A:** Use the previous steps to obtain the opinions of experts then convert them into a single value by a score function, then aggregate these opinions into one matrix.

**Stage B:** Normalize the decision matrix as:

$\boldsymbol{N_{ba}} = \frac{V_{ba}-L_a^-}{L_a^+-L_a^-}$ for positive criteria

$N_{ba} = \frac{V_{ba}-L_a^+}{L_a^--L_a^+}$ for cost criteria

**Stage C:** Compute the weighted normalized decision matrix as:

$$WN_{ba} = W_a + W_a * N_{ba}$$

**Stage D:** Compute the border approximation area as:

$$Bor_{ba} = (\prod_{a=1}^{b} WN_{ba})^{1/b}$$

**Stage E:** Compute the distance from the $Bor_{ba}$

$$DIS_{ba} = WN_{ba} - Bor_{ba}$$

**Stage F:** The alternatives are ranked based on the descending value of the previous step.


**Fig 1**. The eight criteria used in this study

## 4. Case Study: Results and Analysis

Based on the literature, cloud computing risks have been highlighted in this study. The case study is made in a firm in Egypt, which is a new cloud computing company. Experts are a group of three people. Experts will evaluate eight criteria and ten alternatives. The criteria and alternatives in Fig 1 and Fig 2. Then replace their opinions with the scale of SVNNs as in [54]. Then apply the steps of neutrosophic VIKOR and MABAC methods to obtain the weights of criteria and rank of alternatives.

**Phase I:** Obtaining the weights of criteria by applying the score function to obtain one value then applying the average method. The weights of the criteria are presented in Table 2.

**Table 2.** The weights of criteria.

| Criteria | $COM_1$ | $COM_2$ | $COM_3$ | $COM_4$ | $COM_5$ | $COM_6$ | $COM_7$ | $COM_8$ |
|---|---|---|---|---|---|---|---|---|
| Weights | 0.1744 | 0.0817 | 0.0817 | 0.0604 | 0.1744 | 0.1744 | 0.1921 | 0.0604 |

**Phase II**: Rank alternatives by the VIKOR and MABAC. Let experts evaluate the decision matrix, then apply the score function to obtain one value, then aggregate three decision matrix into one matrix, Table 3 show the aggregated decision matrix. All criteria are positive. Then apply steps of the neutrosophic VIKOR method to obtain the values of $Z_a, U_a, R_a$, then rank alternatives. Data security and privacy is the highest rank and Business continuity is the lowest rank by the VIKOR method. Table 4 show the values of $Z_a, U_a, R_a$ and rank of alternatives. Fig. 3 shows the rank of alternatives.

**Table 3.** The aggregated decision matrix.

| Criteria/Alternatives | $COM_1$ | $COM_2$ | $COM_3$ | $COM_4$ | $COM_5$ | $COM_6$ | $COM_7$ | $COM_8$ |
|---|---|---|---|---|---|---|---|---|
| $RCOM_1$ | 0.6999 | 0.8445 | 0.8722 | 0.6666 | 0.8722 | 0.8612 | 0.8445 | 0.8722 |
| $RCOM_2$ | 0.2830 | 0.8167 | 0.9000 | 0.9000 | 0.6999 | 0.6388 | 0.3830 | 0.4609 |
| $RCOM_3$ | 0.8722 | 0.6943 | 0.8445 | 0.5276 | 0.5220 | 0.8445 | 0.6666 | 0.5220 |
| $RCOM_4$ | 0.4609 | 0.6666 | 0.6721 | 0.6388 | 0.3163 | 0.5553 | 0.9000 | 0.8445 |
| $RCOM_5$ | 0.4942 | 0.8167 | 0.3497 | 0.6721 | 0.5943 | 0.2830 | 0.8167 | 0.8167 |
| $RCOM_6$ | 0.2830 | 0.4887 | 0.8167 | 0.8722 | 0.5220 | 0.4277 | 0.9000 | 0.4887 |
| $RCOM_7$ | 0.8167 | 0.5220 | 0.8722 | 0.6999 | 0.4887 | 0.5220 | 0.5610 | 0.8722 |
| $RCOM_8$ | 0.6943 | 0.4609 | 0.7277 | 0.8167 | 0.5666 | 0.5220 | 0.5220 | 0.3887 |
| $RCOM_9$ | 0.5000 | 0.6943 | 0.8445 | 0.8722 | 0.6943 | 0.9000 | 0.4766 | 0.9000 |
| $RCOM_{10}$ | 0.4887 | 0.6333 | 0.5276 | 0.7277 | 0.6666 | 0.7277 | 0.8445 | 0.6943 |

**Table 4.** The values of $Z_a, U_a, R_a$ and rank of alternatives.

| Criteria/Alternatives | $Z_a$ | $U_a$ | $R_a$ | $Rank$ |
|---|---|---|---|---|
| $RCOM_1$ | 0.127915 | 0.051025 | 0 | 1 |
| $RCOM_2$ | 0.552401 | 0.192184 | 0.972253 | 10 |
| $RCOM_3$ | 0.357762 | 0.109915 | 0.464308 | 2 |

| | | | | |
|---|---|---|---|---|
| $RCOM_4$ | 0.514447 | 0.174461 | 0.86725 | 7 |
| $RCOM_5$ | 0.539084 | 0.174461 | 0.894659 | 8 |
| $RCOM_6$ | 0.559294 | 0.174461 | 0.917143 | 9 |
| $RCOM_7$ | 0.478351 | 0.126017 | 0.655498 | 5 |
| $RCOM_8$ | 0.577341 | 0.140514 | 0.816979 | 6 |
| $RCOM_9$ | 0.368216 | 0.157403 | 0.644143 | 4 |
| $RCOM_{10}$ | 0.400119 | 0.113567 | 0.524365 | 3 |

Then apply the neutrosophic MABAC method. Start with the Table 3. Then normalize the decision matrix and obtain the weighted normalized decision matrix, then obtain the border approximation area to attain the distance from the border approximation area in Table5, then obtain the total distance and rank alternatives according to the descending value of total distance in Table 6. According to Table 6 Data security and privacy is the highest rank and Provider lock-in is the lowest rank alternative. Fig. 4 shows the rank of alternatives.

**Table 5.** The distance from the border approximation area.

| Criteria/Alternatives | $COM_1$ | $COM_2$ | $COM_3$ | $COM_4$ | $COM_5$ | $COM_6$ | $COM_7$ | $COM_8$ |
|---|---|---|---|---|---|---|---|---|
| $RCOM_1$ | -0.801 | -0.859 | -0.875 | -0.912 | -0.752 | -0.767 | -0.755 | -0.878 |
| $RCOM_2$ | -0.924 | -0.865 | -0.871 | -0.874 | -0.806 | -0.830 | -0.926 | -0.927 |
| $RCOM_3$ | -0.750 | -0.891 | -0.879 | -0.934 | -0.862 | -0.772 | -0.821 | -0.919 |
| $RCOM_4$ | -0.872 | -0.897 | -0.905 | -0.916 | -0.927 | -0.854 | -0.734 | -0.881 |
| $RCOM_5$ | -0.862 | -0.865 | -0.953 | -0.911 | -0.839 | -0.931 | -0.765 | -0.884 |
| $RCOM_6$ | -0.924 | -0.935 | -0.883 | -0.878 | -0.862 | -0.890 | -0.734 | -0.923 |
| $RCOM_7$ | -0.766 | -0.928 | -0.875 | -0.906 | -0.872 | -0.863 | -0.860 | -0.878 |
| $RCOM_8$ | -0.802 | -0.941 | -0.896 | -0.887 | -0.848 | -0.863 | -0.875 | -0.935 |
| $RCOM_9$ | -0.860 | -0.891 | -0.879 | -0.878 | -0.808 | -0.756 | -0.892 | -0.875 |
| $RCOM_{10}$ | -0.863 | -0.904 | -0.926 | -0.902 | -0.817 | -0.805 | -0.755 | -0.899 |

**Table 6.** The values of $Z_a, U_a, R_a$ and rank of alternatives.

| Criteria/Alternatives | $Total\ Distance$ | $Rank$ |
|---|---|---|
| $RCOM_1$ | 0.127915 | 1 |
| $RCOM_2$ | 0.552401 | 8 |
| $RCOM_3$ | 0.357762 | 2 |
| $RCOM_4$ | 0.514447 | 6 |
| $RCOM_5$ | 0.539084 | 7 |
| $RCOM_6$ | 0.559294 | 9 |
| $RCOM_7$ | 0.478351 | 5 |
| $RCOM_8$ | 0.577341 | 10 |

| | | |
|---|---|---|
| $RCOM_9$ | 0.368216 | 3 |
| $RCOM_{10}$ | 0.400119 | 4 |


**Fig 2.** The ten alternatives are used in this study.

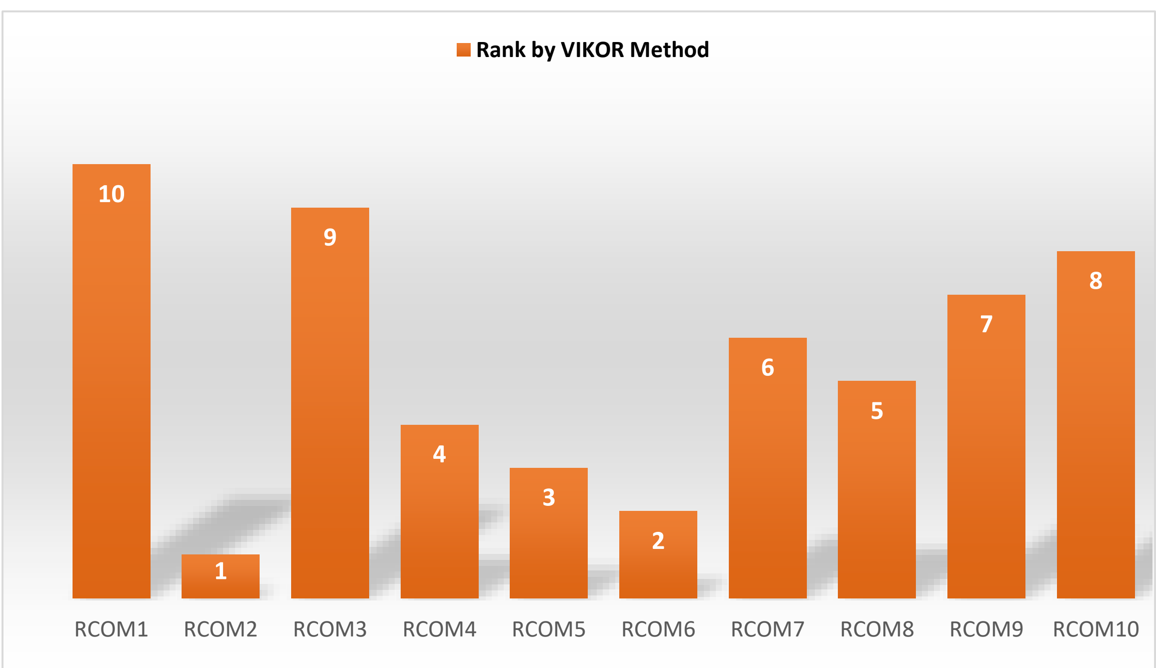


**Fig 3**. The rank of alternatives by the VIKOR method

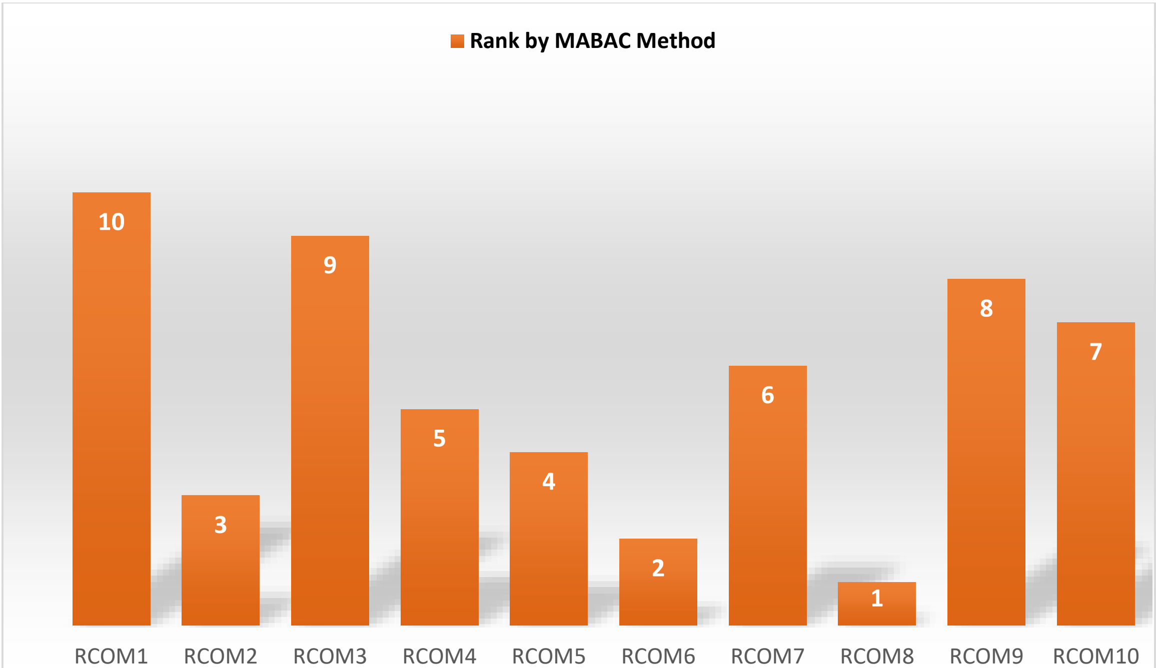


**Fig 4.** The rank of alternatives by the VIKOR method

## 5. Sensitivity Analysis

In this section, we would change the weights of criteria to show the robust of the model. When we change the weights of criteria, the rank of alternatives will change. In this section, we used five cases changes of weights of criteria. We applied these cases in the neutrosophic VIKOR and MABAC model and show the rank of alternatives. Table 7. Show the five cases. In the neutrosophic VIKOR method, case 1,2,4,5 is agreed in highest rank ($RCOM_1$), but in case 3 the height rank is $RCOM_3$. In the neutrosophic MABAC, all cases agreed ($RCOM_1$) is the highest rank. Table 8. Show the rank of alternatives after changing in weights of criteria.

**Table 7.** Five case changes of weights

| | $COM_1$ | $COM_2$ | $COM_3$ | $COM_4$ | $COM_5$ | $COM_6$ | $COM_7$ | $COM_8$ |
|---|---|---|---|---|---|---|---|---|
| **Case 1** | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 |
| **Case 2** | 0.0714 | 0.0714 | 0.0714 | 0.0714 | 0.0714 | 0.0714 | 0.5 | 0.0714 |
| **Case 3** | 0.5 | 0.0714 | 0.0714 | 0.0714 | 0.0714 | 0.0714 | 0.0714 | 0.0714 |
| **Case 4** | 0.0714 | 0.0714 | 0.0714 | 0.0714 | 0.5 | 0.0714 | 0.0714 | 0.0714 |
| **Case 5** | 0.0714 | 0.0714 | 0.0714 | 0.0714 | 0.0714 | 0.5 | 0.0714 | 0.0714 |

**Table 8. Rank of alternatives based on five cases**

| VIKOR Case1 | VIKOR Case 2 | VIKOR Case 3 | VIKOR Case 4 | VIKOR Case 5 | MABAC Case1 | MABAC Case 2 | MABAC Case 3 | MABAC Case 4 | MABAC Case 5 |
|---|---|---|---|---|---|---|---|---|---|
| $RCOM_1$ | $RCOM_1$ | $RCOM_3$ | $RCOM_1$ | $RCOM_1$ | $RCOM_1$ | $RCOM_1$ | $RCOM_1$ | $RCOM_1$ | $RCOM_1$ |
| $RCOM_{10}$ | $RCOM_{10}$ | $RCOM_7$ | $RCOM_9$ | $RCOM_9$ | $RCOM_9$ | $RCOM_4$ | $RCOM_3$ | $RCOM_9$ | $RCOM_9$ |
| $RCOM_9$ | $RCOM_4$ | $RCOM_1$ | $RCOM_2$ | $RCOM_3$ | $RCOM_3$ | $RCOM_{10}$ | $RCOM_7$ | $RCOM_2$ | $RCOM_3$ |
| $RCOM_7$ | $RCOM_6$ | $RCOM_8$ | $RCOM_{10}$ | $RCOM_{10}$ | $RCOM_{10}$ | $RCOM_6$ | $RCOM_9$ | $RCOM_{10}$ | $RCOM_{10}$ |
| $RCOM_3$ | $RCOM_5$ | $RCOM_9$ | $RCOM_5$ | $RCOM_2$ | $RCOM_7$ | $RCOM_5$ | $RCOM_8$ | $RCOM_5$ | $RCOM_2$ |
| $RCOM_2$ | $RCOM_3$ | $RCOM_{10}$ | $RCOM_3$ | $RCOM_4$ | $RCOM_2$ | $RCOM_3$ | $RCOM_{10}$ | $RCOM_3$ | $RCOM_7$ |
| $RCOM_4$ | $RCOM_7$ | $RCOM_5$ | $RCOM_8$ | $RCOM_7$ | $RCOM_4$ | $RCOM_9$ | $RCOM_5$ | $RCOM_7$ | $RCOM_4$ |
| $RCOM_5$ | $RCOM_9$ | $RCOM_4$ | $RCOM_6$ | $RCOM_8$ | $RCOM_5$ | $RCOM_7$ | $RCOM_4$ | $RCOM_8$ | $RCOM_8$ |
| $RCOM_6$ | $RCOM_8$ | $RCOM_2$ | $RCOM_7$ | $RCOM_6$ | $RCOM_6$ | $RCOM_8$ | $RCOM_2$ | $RCOM_6$ | $RCOM_6$ |
| $RCOM_8$ | $RCOM_2$ | $RCOM_6$ | $RCOM_4$ | $RCOM_5$ | $RCOM_8$ | $RCOM_2$ | $RCOM_6$ | $RCOM_4$ | $RCOM_5$ |

## 6. Comparative Analysis

In this section, we made a comparison with the neutrosophic TOPSIS method to show the robust of this model. We use this data to apply with the TOPSIS method. After applying this comparison, we found that the heights rank is constant in two method. Table 9. Show the comparison between VIKOR, MABAC and TOPSIS methods.

**Table 9. Rank of alternatives based on comparative analysis.**

| MABAC | TOPSIS | VIKOR |
|---|---|---|
| $RCOM_1$ | $RCOM_1$ | $RCOM_1$ |
| $RCOM_3$ | $RCOM_3$ | $RCOM_3$ |
| $RCOM_9$ | $RCOM_{10}$ | $RCOM_{10}$ |
| $RCOM_{10}$ | $RCOM_9$ | $RCOM_9$ |
| $RCOM_7$ | $RCOM_7$ | $RCOM_7$ |
| $RCOM_4$ | $RCOM_8$ | $RCOM_8$ |
| $RCOM_5$ | $RCOM_4$ | $RCOM_4$ |
| $RCOM_2$ | $RCOM_5$ | $RCOM_5$ |
| $RCOM_6$ | $RCOM_2$ | $RCOM_6$ |
| $RCOM_8$ | $RCOM_6$ | $RCOM_2$ |

## 7. Managerial Implications

Cloud computing surround many risks. That effect on market, companies, good and other. So, these risks should be identified and ranked. The hybrid model introduced by this study to identify and rank alternatives. The hybrid model contains the VIKOR and MABAC methods. This study provides the rank of risks of cloud computing.

## 8. Conclusions and Future work

Risks associated with cloud computing might limit a company's exploration efforts, such as entering a new market, generating new goods and services, locating new clients, and absorbing new information. Furthermore, cloud computing hazards might obstruct a company's exploitation operations, which include competing in the present market with current customers, current goods, and current expertise. In this study, we used eight criteria and ten alternatives. The SVN is used to obtain the rank of alternatives. The neutrosophic set is hybrid with the VIKOR and MABAC methods to obtain the weights of criteria and rank of risks. In future work, we suggest this model be used with other problems like energy selection and others and can use other MCDM methods such as TOPSIS, AHP, and others.

**Conflict of interest:** Authors declare that there is no conflict of interest about the research.

**Funding:** This research has no funding source.

**Ethical approval:** None of the authors' experimented with human subjects or animals during this research.

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Received: Nov 20, 2021. Accepted: Feb 2, 2022

NSS

Neutrosophic Sets and Systems, Vol. 48, 2022

University of New Mexico

# Neutrosophic Entropy Based Heavy Metal Contamination Indices for Impact Assessment of Sarsa River Water Quality Within County of District Baddi, India

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**Abstract:** This study contributes a novel fuzzy and neutrosophic entropy-based procedure to identify the most contaminated sampling spot and to assess the impact of heavy metals concentration, before and after the amalgamation of pharmaceutical effluents treated in common effluent treatment plant. in river water samples. It is observed that the concentration of heavy metals, which were within permissible limits before amalgamation, dwindled gradually after amalgamation, owing to the decrease in fuzzy and neutrosophic entropy values. To identify the most contaminated sampling spot, responsible for heavy metal contamination, the proposed trigonometric fuzzy and single valued neutrosophic entropy measures are fascinated for assigning weights to each monitored heavy metal concentration reading with respect to four sampling spots and thereafter coupled with the relative sub-indices of each heavy metal to construct fuzzy and neutrosophic entropy weighted heavy metal contamination indices (FHCI and NHCI). The maximum (or minimum) FHCI and NHCI score among each sampling spot is designated to the "most contaminated" or "least contaminated" sampling spot accordingly. The proposed entropy-based contamination indices are superior in providing a better insight in classifying the desired contaminated sampling spot in comparison with the existing Deluca-Termini fuzzy entropy-based contamination index which may indicate uncertainty in the quality analysis of heavy metal contamination in river water samples.

**Keywords:** Neutrosophic Entropy, Deluca-Termini Entropy, Pharmaceutical Effluents, Heavy Metals Contamination.

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## 1. Introduction

Heavy metals contamination in river water is a serious problem, not only in India, but also all over the world. The possible reasons behind this could be the increasing human population and excessive use of fertilizers in agriculture that causes pollution of fresh water resources with diverse and detrimental contaminants leading to the spread of water borne diseases. Contamination of water resources, available for domestic and drinking purposes with heavy metals and harmful bacteria, leads to health problems which, of course, may be life threatening. Waste water could be full of significant amounts of toxic heavy metals, that might not only pollute the soil, but also carries deleterious effects on food quality after entering in to the food chain. All of the above issues reinforce the necessity of establishing an efficient methodology which can assess the impact of heavy metal concentration and enhance the quality analysis of heavy metal contamination in river water. Our main endeavor is to establish a novel entropy based heavy metal contamination evaluation methodology, the aftermaths of which can be utilized to control the spread of water borne deceases, reduce the risk of water and soil pollution, increase the ecological and aesthetical qualities of lakes and rivers, etc. The Baddi-Barotiwala region, better known as the industrial and commercial district of Himachal Pradesh, India, forms a part of the Shivalik's and lies in the lap of outer Himalayas. It has become an industrial hub due to huge investments in Baddi, Barotiwala, Nalagarh and Parwanoo districts. The Sarsa River-a tributary of the Sutlej River of district Baddi (Himachal Pradesh, India)- has turned out to be a grave for aquatic life due to uncontrolled industrial growth of the area and alleged discharge of toxic effluents from pharmaceutical industries. Likewise, flora, fauna and the surrounding environment are adversely affected by the harmful chemicals released from pharmaceutical industries and from the improperly treated effluents from waste water treatment plants (WWTPs). Besides industrial pollution, drug pollution is a major contributor in killing of fish, amphibians and amphibian path morphology. The ongoing research has shown that continuing exposure of compound pharmaceutical effluents to rivulet biota may result in severe and persistent problems, behavioral issues, buildup in tissues, effect on reproductive system, impact on cell propagation & multiplication. Severely elevated chloride concentrations in the water bodies are unfavorable for aquatic life and can amplify metal toxicity and another bioactive composites. High levels of nitrogen and phosphorus appearing from sewage, animal wastes, fertilizer and agriculture may affect rivulet biota. Excess of phosphates cause stepped up eutrophication as they overfertilize the aquatic plants; choke the waterways due to excessive algal growth and other wild plants. In warm weather, fast growth of algae and floating aquatic weeds is stimulated by the nutrients resulting into deterioration of water quality. All these suffocated activities decline the ecological and aesthetic qualities of lakes, rivers etc. A systematic analysis of the river Nile in Egypt was computed to assess the water suitability for aquatic life and domestic purposes. Studies have shown that applications of electrochemical technologies like Electro-coagulation & electro-floatation, in wastewater treatment are very effective in recuperating toxic heavy metals from wastewater. These techniques perform better than the conventional techniques being used in removing colloidal particles as well as other organic pollutants. Electro-oxidation is also being used in treatment of wastewater by combining it with other contamination techniques.

Recently, Alam et al. **[1]** deployed pollution indices and geographical accumulation index approach for evaluating heavy metal contamination in water resources of an open landfill area. The heavy metal accumulation in the samples was observed to be slightly higher than the standard one, indicating a danger to the humanity and environment. Vardhan et al. **[2]** discussed an environment friendly adsorption analysis approach for removing the toxicity among heavy metals available in the aquatic system for the purpose of avoiding illness of human beings. Khangembam and Kshetrimayum **[3]** utilized some statistical measures including scatter diagrams, trilinear and Gibbs plots for evaluating the water quality index of ground water samples and indicated the unsuitability of ground water for drinking purposes because of the excessive availability of highly toxic heavy metals. Yang et al. **[4]** employed Information entropy and cloud model theory for assessing the complexity of heavy metal pollution in agriculture soils of mining zones. Huu et al. **[5]** investigated geo chemical and distribution factors to assess the contamination status of heavy metals in estuarine system, which indicated an increase of arsenic metal in the samples. Hussain et al. **[6]** experimented the heavy metal contamination through the analytical hierarchy process and validated that, among all heavy metals available in the river water samples of Godavari; the concentration of Zn was highest and of Cd was lowest. Sabbir et al. **[7]** calculated the concentration of arsenic, chromium, lead, mercury and cadmium available in freshwater fish muscles and sediments of the Rupsha river and evaluated the suitability of River water. Alidadi et al. **[8]** calculated the carcinogenic and non-carcinogenic risks of arsenic for adults and kids based on the LCR (life time cancer risk) factor, hazard index and hazard quotient by chemically analyzing and testing of toxicity of arsenic in water resources of north-east regions of Iran.

Recently, Singh et al. **[9]** determined some entropy weighted heavy metal contamination indices (EHCI) of various sampling spots of the Brahmaputra River by quantifying Shannon's probabilistic entropy and evaluated the impact of heavy metal contamination. Basset et al. **[10]** developed an aggregation operator based on neutrosophic numbers of type 2 and selected the best banking facilities. The authors also modified the existing TOPSIS method under neutrosophic environment and selected the best corporation importing supplier. In another work, Basset et al. **[11]** integrated the enduring ANP and VIKOR method by taking into consideration the triangular neutrosophic numbers and demonstrated a case study of selecting the best supplier for importing. Furthermore, Basset et al. **[12]** suggested a novel robust ranking procedure by integrating trapezoidal neutrosophic numbers with GSCM approach, intended to predict the environmental and economical practices to be implemented in industry. Also, Basset et al. **[13]** presented a hybrid model that could combine EDAS, DEMATEC and neutrosophic numbers for the purpose of classifying the most sustainable bioenergy technique under ambiguous and inconsistent situations. In addition, Basset et al. **[14]** utilized a hybrid approach by combining TOPSIS and VIKOR methods for classifying the most sustainable RESs under neutrosophic treatments.

### 1.1 Motivation

To represent the macroscopic state of heavy metal concentration in river water and to construct entropy weighted heavy metal contamination indices (EHCI) by deploying Shannon's entropy, there may occur a problematic situation because this fascinating entropy is facing a major drawback because of its assumption $0\log0=0$. Due to this fancy assumption, Shannon's entropy is facing intrinsic conflicts and hence indicating lack of macroscopic view in the quality analysis of concentration and contamination of heavy metals in river water samples. Zadeh's **[15]** fuzzy set theory has become an indispensable tool for reflecting the complexity of heavy metal contamination. A fuzzy entropy measure can represent the macroscopic state of heavy metal concentration in a broader way. Dubois and Prade **[16]** established many variants of fuzzy sets, one of which is related to the grade or membership degree of the underlying fuzzy set. Thereafter, many equivalents of fuzzy sets have been developed and utilized for dealing with assessment problems of heavy metal contamination in river water samples. A neutrosophic set (NS), which is hinged on three variants-truth, indeterminacy and falsity membership degrees of a fuzzy set, can represent the macroscopic state of heavy metal contamination in an efficient way. Smarandache's neutrosophic set theory **[17]** can play a vital role in classifying the most contaminated sampling spots with respect to each heavy metal concentration in river water samples. Motivated by Zadeh's fuzzy set theory and Smarandache's neutrosophic set theory, an effort has been accomplished in this path way by construct trigonometric fuzzy and single valued neutrosophic entropy weighted heavy metal contamination indices. The desired goal is achieved by establishing a superior contamination evaluation methodology and its applicability which can provide a better insight in classifying the most contaminated sampling spot with respect to each heavy metal concentration in river water samples.

**1.2 Novelties**

The identification of the most contaminated sampling spot through the proposed methodology can help in reducing the risk of water and soil pollution. The following points have been addresses by the proposed research work.

- To construct a novel trigonometric fuzzy entropy measure.
- To construct a novel symmetric trigonometric fuzzy cross entropy (TFE) measure.
- To establish a novel trigonometric single valued neutrosophic entropy (TNE) measure.
- To assess the concentration of heavy metals in river water samples though the proposed TFE and TNE measures.
- To construct fuzzy and single valued neutrosophic entropy weighted heavy metal contamination indices (FHCI and NHCI).
- To identify the most contaminated sampling spot through the proposed FHCI and NHCI.

The major contributions delivered in this study can be summarized as follows.

- Because of the fancy assumption 0log0=0 deliberated to Shannon's probabilistic entropy as it may represent macroscopic view of contamination in a narrow way, our TFE measure has been found efficient in assessing the accurate impact of heavy metal concentration and thus representing the macroscopic view of heavy metal concentration in a broader way.

- Because of the limitation deliberated to Deluca and Termini's entropy measure as it may return meaningless results in certain mathematical treatments, especially, when the grade or degree of membership conceived by its membership function is zero or unity, our TFE measure can perform well under fuzzy environment and provide consistent and specified results in certain mathematical treatments.
- To expand the applicability of Smarandache's neutrosophic set theory **[17]** and to enhance the quality analysis of heavy metal contamination under neutrosophic environment, our TNE measure has been found capable in reckoning the most contaminated sampling spot with respect to each heavy metal concentration in river water samples.
- The findings of the proposed study can be utilized for controlling the spread of water borne deceases, reducing the risk of water and soil pollution, increasing the ecological and aesthetical qualities of lakes and rivers, etc. The rest of the proposed research work is organized as follows:

**Section 2** introduces in brief the basic concepts of Information theory required for understanding the proposed heavy metal contamination evaluation methodology. **Sections 3-4** are dedicated for the establishment of novel trigonometric fuzzy entropy and single valued neutrosophic entropy measures consecutively. **Section 5** is devoted to assess the impact of heavy metal concentration through the experimental investigations and proposed trigonometric fuzzy entropy measure (TFE) and single valued neutrosophic entropy (TNE) measure consecutively**. Section 6** introduces a novel entropy-based heavy metal contamination evaluation methodology by means of fuzzy and single valued neutrosophic entropy weighted heavy metal contamination indices (FHCI and NHCI). **Section 7** validates the effectiveness of the proposed methodology by identifying the most contaminated sampling spot responsible for heavy metal contamination with respect to each heavy metal concentration in river water samples. **Section 8** finally summarizes the concrete conclusions of this study.

**2. Preliminaries:**

This section deals with the introduction of basic prerequisites required for understanding the propounded study.

**Def. 2.1Fuzzy Set (FS) [18]** A fuzzy set $A_{FS}^{w} \subseteq U$ in a finite discourse of universe $U = \left(x_1, x_2, ..., x_n\right)$ is an object of the form: $A_{FS}^{w} = \left(< x_i, \tilde{\mu}_{A^w}(x_i) > | \, x_i \in U\right)$, where $\tilde{\mu}_{A^w}(x_i) : U \to [0,1]$ represents true membership function and satisfy $0 \le \tilde{\mu}_{A^w}(x_i) \le 1$ Further, the complement $C\left(A_{FS}^{w}\right)$ of $A_{FS}^{w} \subseteq U$ is an object of the form defined by $C\left(A_{FS}^{w}\right) = \left(< x_i, 1 - \tilde{\mu}_{A^w}(x_i) > | \, x_i \in U\right)$.

**Def. 2.2 Fuzzy Entropy Measure [18]** Suppose $S(U)$ represents the collection of all fuzzy sets in $U=(x_1,x_2,...,x_n)$ and $A_{FS}^{w}=(<x_i,\tilde{\mu}_{A^w}(x_i)>|\,x_i\in U)$ be any fuzzy set quantified by its truth membership functions $\tilde{\mu}_{A^w}(x_i):U\to[0,1]$ satisfying $0\le\tilde{\mu}_{A^w}(x_i)\le 1$. Then a function $T_E:S(U)\to R^+(\text{set of positive reals})$ is called as fuzzy entropy measure if $(i)\,T_E(A_{FS}^{w})\ge 0\,\forall A_{FS}^{w}\subseteq U$ with equality if $\tilde{\mu}_{A^w}(x_i)=0$ or $1$ *(ii)* $T_E(A_{FS}^{w})$ does not change whenever $\tilde{\mu}_{A^w}(x_i)$ is replaced by $1-\tilde{\mu}_{A^w}(x_i)$ *(iii)* $T_E(A_{FS}^{w})$ is a concave function of $\tilde{\mu}_{A^w}(x_i)$ *(iv)* $T_E(A_{FS}^{w})$ possesses its maximum value which arises when $\tilde{\mu}_{A^w}(x_i)=\frac{1}{2}$.

**Def.2.3Symmetric Fuzzy Cross Entropy Measure [19]** Let $A_{FS}^{w}=(<x_i,\tilde{\mu}_{A^w}(x_i)>|\,x_i\in U)$ and $B_{FS}^{w}=(<x_i,\tilde{\mu}_{B^w}(x_i)>|\,x_i\in U)$ are any two fuzzy sets in $U=(x_1,x_2,...,x_n)$ quantified by their truth membership functions $\tilde{\mu}_{A^w}(x_i),\tilde{\mu}_{A^w}(x_i):U\to[0,1]$ satisfying $0\le\tilde{\mu}_{A^w}(x_i),\tilde{\mu}_{B^w}(x_i)\le 1$. Then a function $T_{CE}:S(U)\times S(U)\to R^+(\text{set of positive reals})$ is called as symmetric fuzzy cross entropy or discrimination information measure between two fuzzy sets $A_{FS}^{w}$ and $B_{FS}^{w}$ if $(i)\;T_{CE}(A_{FS}^{w},B_{FS}^{w})\ge 0\,\forall A_{FS}^{w},B_{FS}^{w}\in S(U)$ with equality if $A_{FS}^{w}=B_{FS}^{w}$. $(ii)\;T_{CE}(A_{FS}^{w},B_{FS}^{w})=T_{CE}(B_{FS}^{w},A_{FS}^{w})$ and $(iii)T_{CE}(A_{FS}^{w},B_{FS}^{w})$ does not change whenever $\tilde{\mu}_{A^w}(x_i),\tilde{\mu}_{B^w}(x_i)$ are replaced by their counterparts $1-\tilde{\mu}_{A^w}(x_i),1-\tilde{\mu}_{B^w}(x_i)$.

**Def. 2.4 Single Valued Neutrosophic Set (SVNS) [17]**. A SVNS $A_{FS}^{w}\subseteq U$ is defined as $A_{SV}^{w}=(<x_i,\tilde{\mu}_{A^w}(x_i),\tilde{i}_{A^w}(x_i),\tilde{f}_{A^w}(x_i)>|\,x_i\in U)$ where $\tilde{\mu}_{A^w}(x_i),\tilde{i}_{A^w}(x_i),\tilde{f}_{A^w}(x_i):U\to[0,1]$ satisfy $0\le\tilde{\mu}_{A^w}(x_i),\tilde{i}_{A^w}(x_i),\tilde{f}_{A^w}(x_i)\le 3$ and respectively called as truth, indeterminacy and falsity membership functions. Further, the complement $C(A_{SV}^{w})$ of $A_{SV}^{w}\subseteq U$ is defined as $C(A_{SV}^{w})=(<x_i,\tilde{f}_{A^w}(x_i),1-\tilde{i}_{A^w}(x_i),\tilde{\mu}_{A^w}(x_i)>|\,x_i\in U)$.

**Def. 2.5 Single Valued Neutrosophic Entropy Measure [17]** Let $R(U)$ be a well-defined collection of all single valued neutrosophic sets $A_{FS}^{w} \subseteq U$ ,then a function $R_N : R(U) \to R^{+}$ is called as single valued neutrosophic entropy measure if

$(i)\, R_N\left(A_{SV}^{w}\right) \geq 0 \forall A_{SV}^{w} \in R(U)$ with equality if either $\tilde{\mu}_{A^w}(x_i) = 1, \tilde{i}_{A^w}(x_i) = 0, \tilde{f}_{A^w}(x_i) = 0$ or $\tilde{\mu}_{A^w}(x_i) = 0, \tilde{i}_{A^w}(x_i) = 0, \tilde{f}_{A^w}(x_i) = 1$ $(ii)\, R_N\left(C\left(A_{SV}^{w}\right)\right) = R_N\left(A_{SV}^{w}\right)$ $(iii)\, R_N\left(A_{SV}^{w}\right)$ exhibits its concavity property for each $\tilde{\mu}_{A^w}(x_i), \tilde{i}_{A^w}(x_i), \tilde{f}_{A^w}(x_i)$ $(iv)\, R_N\left(A_{SV}^{w}\right)$ possesses its maximum value which arises when each $\tilde{\mu}_{A^w}(x_i) = \tilde{i}_{A^w}(x_i) = \tilde{f}_{A^w}(x_i) = \frac{1}{2}$.

**Def. 2.6 Symmetric Single Valued Neutrosophic Cross Entropy Measure [17]**

A function $R_{CE} : R(U) \times R(U) \to R$ is called as symmetric single valued neutrosophic cross entropy measure between two SVNSs $A_{SV}^{w}$ and $B_{SV}^{w}$ if

$(i) R_{CE}\left(A_{SV}^{w}, B_{SV}^{w}\right) \geq 0 \forall A_{SV}^{w}, B_{SV}^{w} \in R(U)$ with equality if and only if $A_{SV}^{w} = B_{SV}^{w}$.

$(ii)\, R_{CE}\left(A_{SV}^{w}, B_{SV}^{w}\right) = R_{CE}\left(B_{SV}^{w}, A_{SV}^{w}\right)$ $(iii)\, R_{CE}\left(C\left(A_{SV}^{w}\right), C\left(B_{SV}^{w}\right)\right) = R_{CE}\left(A_{SV}^{w}, B_{SV}^{w}\right) \forall A_{SV}^{w}, B_{SV}^{w} \in R(U)$.

## 3. A Novel Trigonometric Fuzzy Entropy Measure

We shall, here, develop a novel trigonometric fuzzy entropy (TFE) measure (**Theorems. 3.1**) followed by trigonometric symmetric fuzzy cross entropy (FCE) measure hinged on two fuzzy sets (**Theorems. 3.2),** the outcomes of which will be utilized to establish the proposed single valued neutrosophic entropy (TNE) measure.

**Theorem.3.1** Let $A_{FS}^{w} = \left(< x_i, \tilde{\mu}_{A^w}(x_i) > | x_i \in U\right)$ be any fuzzy set in $U$ with cardinality $n$.

Then, $H_F(A_{FS}^{w})$ is an authentic trigonometric fuzzy entropy measure **[Def. 2.2]** defined as

$$H_F(A_{FS}^{w}) = -\sum_{i=1}^{n} \left[ \tan\left( \frac{2\sqrt{2} + 2\sqrt{\tilde{\mu}_{A^w}^{2}(x_i) + \left(1 - \tilde{\mu}_{A^w}(x_i)\right)^2} - \sqrt{2\tilde{\mu}_{A^w}(x_i)\left(1 - \tilde{\mu}_{A^w}(x_i)\right)}}{5} \right) - \tan\left(\frac{2\sqrt{2}+2}{5}\right) \right] \quad \ldots (1)$$

with minimum zero and maximum as $\left(\tan\left(\frac{2\sqrt{2}+2}{5}\right) - \tan\left(\frac{1}{\sqrt{2}}\right)\right) n.$

Here, the generic entity $'x_i'$ represents the $i^{th}$ macroscopic level of heavy metal contamination and $H_F(A_{FS}^{w})$ indicates the fuzzy entropy of heavy metal contamination indicated by the fuzzy set $A_{FS}^{w}$.

**Proof** In view of **[Def. 2.2],**

(i) $H_F^w\left(A_{FS}^w\right) \geq 0 \forall \tilde{\mu}_{A^w}(x_i) \in [0,1]$ with equality if $\tilde{\mu}_{A^w}(x_i)=0$ or $1$ for each $i=1,2,\ldots,n$.

(ii) $H_F(A_{FS}^w)$ remains unchanged whenever $\tilde{\mu}_{A^w}(x_i)$ is replaced by $1-\tilde{\mu}_{A^w}(x_i)$.

(iii) **Concavity:** The fact that $H_F\left(A_{FS}^w\right)$ is concave in nature can be seen from its three-dimensional rotational plot (**Fig. 1).** Also, the finite series of positive terms in **(1)** can be partially differentiated with respect to each $\tilde{\mu}_{A^w}(x_i)$ because of its uniform and absolute convergence. Mathematica (software from Wolfram) yields $\dfrac{\partial^2 H_F\left(A_{FS}^w\right)}{\partial \tilde{\mu}_{A^w}^2(x_i)} < 0 \forall \tilde{\mu}_{A^w}(x_i) \in [0,1]$, which also justifies the concavity of $H_F\left(A_{FS}^w\right)$ with respect to each $\tilde{\mu}_{A^w}(x_i)$.

(iv)With the aid of concavity property of $H_F\left(A_{FS}^w\right)$ with respect to $\tilde{\mu}_{A^w}(x_i)$., there exists its maximum value which arises when

$$\frac{\partial H_F\left(A_{FS}^w\right)}{\partial \tilde{\mu}_{A^w}(x_i)} = \frac{1}{5}\left(\begin{array}{c}\dfrac{1-2\tilde{\mu}_{A^w}(x_i)}{\sqrt{2\tilde{\mu}_{A^w}(x_i)\left(1-\tilde{\mu}_{A^w}(x_i)\right)}} \\ +\dfrac{2-4\tilde{\mu}_{A^w}(x_i)}{\sqrt{\tilde{\mu}_{A^w}^2(x_i)+\left(1-\tilde{\mu}_{A^w}(x_i)\right)^2}}\end{array}\right)\sec^2\left(\frac{2\sqrt{2}+2\sqrt{\tilde{\mu}_{A^w}^2(x_i)+\left(1-\tilde{\mu}_{A^w}(x_i)\right)^2}-\sqrt{2\tilde{\mu}_{A^w}(x_i)\left(1-\tilde{\mu}_{A^w}(x_i)\right)}}{5}\right)=0$$

which yields $\tilde{\mu}_{A^w}(x_i)=\dfrac{1}{2}$. In view of **(1),**

$$\text{Max.}H_F\left(A_{FS}^w\right) = H_F\left(A_{FS}^w\right)\Big|_{\tilde{\mu}_{A^w}(x_i)=\frac{1}{2}} = \left(\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\tan\left(\frac{1}{\sqrt{2}}\right)\right)n \qquad \ldots (2)$$

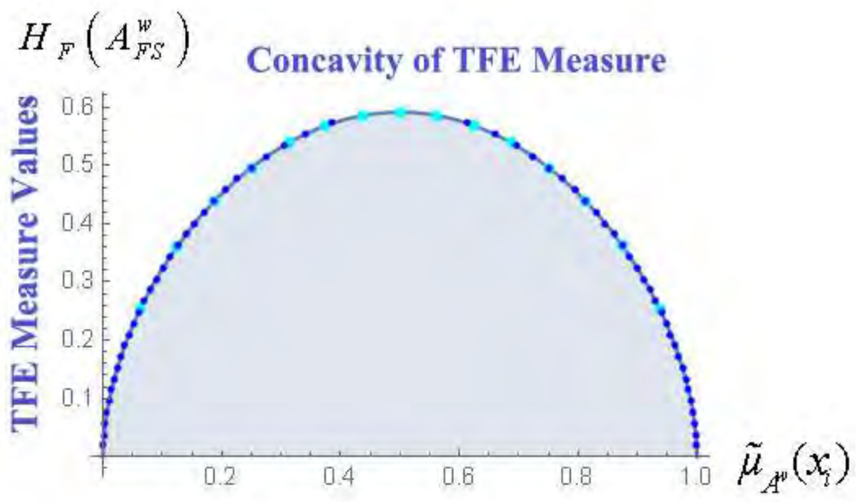


**Fig.1** Concavity property exhibited by TFE measure $H_F\left(A_{FS}^w\right)$ with respect to $\tilde{\mu}_{A^w}(x_i)$

**Theorem.3.2** Let $A_{FS}^w=\left(<x_i,\tilde{\mu}_{A^w}(x_i)>| x_i\in U\right)$ and $B_{FS}^w=\left(<x_i,\tilde{\mu}_{B^w}(x_i)>| x_i\in U\right)$ be any two fuzzy sets with same cardinality $n$. Show that $H_{CE}^{\mu}\left(A_{FS}^w,B_{FS}^w\right)$ is a valid trigonometric symmetric fuzzy cross entropy measure (**Def. 2.3)** between two fuzzy sets $A_{FS}^w$ and $B_{FS}^w$ defined by

$$H_{CE}^{\mu}\left(A_{FS}^w,B_{FS}^w\right)=\sum_{i=1}^{n}\left[\begin{array}{l}-10\tan\frac{1}{\sqrt{2}}+\left(4+\tilde{\mu}_{A^w}(x_i)+\tilde{\mu}_{B^w}(x_i)\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{\mu}_{A^w}^2(x_i)+\tilde{\mu}_{B^w}^2(x_i)}-\sqrt{2\tilde{\mu}_{A^w}(x_i)\tilde{\mu}_{B^w}(x_i)}}{4+\tilde{\mu}_{A^w}(x_i)+\tilde{\mu}_{B^w}(x_i)}\right)\\ +\left(6-\tilde{\mu}_{A^w}(x_i)-\tilde{\mu}_{B^w}(x_i)\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{+\left(1-\tilde{\mu}_{A^w}(x_i)\right)^2+\left(1-\tilde{\mu}_{B^w}(x_i)\right)^2}-\sqrt{\left(1-\tilde{\mu}_{A^w}(x_i)\right)\left(1-\tilde{\mu}_{B^w}(x_i)\right)}}{6-\tilde{\mu}_{A^w}(x_i)-\tilde{\mu}_{B^w}(x_i)}\right)\end{array}\right] \quad \ldots (3)$$

Here, $H_{CE}^{\mu}\left(A_{FS}^w,B_{FS}^w\right)$ indicates the amount of true membership degree of symmetric discrimination of the fuzzy set $A_{FS}^w$ against $B_{FS}^w$.

**Proof**. It is easy to verify that $(i)\,H_{CE}^{\mu}\left(C\left(A_{FS}^w\right),C\left(B_{FS}^w\right)\right)=H_{CE}^{\mu}\left(A_{FS}^w,B_{FS}^w\right)$ and $H_{CE}^{\mu}\left(A_{FS}^w,B_{FS}^w\right)=H_{CE}^{\mu}\left(B_{FS}^w,A_{FS}^w\right)\ \forall A_{FS}^w,B_{FS}^w\in S(U)$. To establish the non-negativity of $H_{CE}^{\mu}\left(A_{FS}^w,B_{FS}^w\right)$, we first divert to develop the following **Lemma 3.1.**

**Lemma 3.1** Define

$$A_1\left(\tilde{\mu}_{A^w}(x_i),\tilde{\mu}_{B^w}(x_i)\right)=\frac{\tilde{\mu}_{A^w}(x_i)+\tilde{\mu}_{B^w}(x_i)}{2},\ N_1\left(\tilde{\mu}_{A^w}(x_i),\tilde{\mu}_{B^w}(x_i)\right)=\left(\frac{\sqrt{\tilde{\mu}_{A^w}(x_i)}+\sqrt{\tilde{\mu}_{B^w}(x_i)}}{2}\right)^2,$$

$S_1\left(\tilde{\mu}_{A^w}(x_i),\tilde{\mu}_{B^w}(x_i)\right)=\sqrt{\frac{\tilde{\mu}_{A^w}^2(x_i)+\tilde{\mu}_{B^w}^2(x_i)}{2}}$. Then there exist the inequalities: $S_1\geq A_1\geq N_1$ with equality if and only if $\tilde{\mu}_{A^w}(x_i)=\tilde{\mu}_{B^w}(x_i)\forall\tilde{\mu}_{A^w}(x_i),\tilde{\mu}_{B^w}(x_i)\in[0,1](i=1,2,...,n)$

**Proof.** In view of our notations,

$$(i)\ S_1^2-A_1^2=\frac{\tilde{\mu}_{A^w}^2(x_i)+\tilde{\mu}_{B^w}^2(x_i)}{2}-\left(\frac{\tilde{\mu}_{A^w}(x_i)+\tilde{\mu}_{B^w}(x_i)}{2}\right)^2=\left(\frac{\tilde{\mu}_{A^w}(x_i)-\tilde{\mu}_{B^w}(x_i)}{2}\right)^2\geq 0\Rightarrow S_1^2-A_1^2\Rightarrow S_1\geq A_1 \quad \ldots (4)$$

$$(ii)\, A_1^2 - N_1^2 = \left(\frac{\tilde{\mu}_{A^w}(x_i)+\tilde{\mu}_{B^w}(x_i)}{2}\right)^2 - \left(\frac{\sqrt{\tilde{\mu}_{A^w}(x_i)}+\sqrt{\tilde{\mu}_{B^w}(x_i)}}{2}\right)^2 = \left(\frac{\sqrt{\tilde{\mu}_{A^w}(x_i)}-\sqrt{\tilde{\mu}_{B^w}(x_i)}}{2}\right)^2 \ge 0 \Rightarrow A_1^2 - N_1^2 \Rightarrow A_1 \ge N_1$$

… (5)

Combining the resulting inequalities **(4)** and **(5)** to obtain $S_1 \ge A_1 \ge N_1$ with equality if and only if $\tilde{\mu}_{A^w}(x_i) = \tilde{\mu}_{B^w}(x_i) \forall \tilde{\mu}_{A^w}(x_i), \tilde{\mu}_{B^w}(x_i) \in [0,1]$.

Thus, in view of Lemma **3.1,** the resulting inequality $S_1 \ge A_1 \ge N_1$ can be re-scheduled to give

$$S_1\left(\tilde{\mu}_{A^w}(x_i), \tilde{\mu}_{B^w}(x_i)\right) \ge N_1\left(\tilde{\mu}_{A^w}(x_i), \tilde{\mu}_{B^w}(x_i)\right)$$

$$\Rightarrow \sqrt{\frac{\tilde{\mu}^2_{A^w}(x_i)+\tilde{\mu}^2_{B^w}(x_i)}{2}} \ge \left(\frac{\sqrt{\tilde{\mu}_{A^w}(x_i)}+\sqrt{\tilde{\mu}_{B^w}(x_i)}}{2}\right)^2 = \frac{\tilde{\mu}_{A^w}(x_i)+\tilde{\mu}_{B^w}(x_i)}{4} + \frac{\sqrt{\tilde{\mu}_{A^w}(x_i)\tilde{\mu}_{B^w}(x_i)}}{2}$$

$$\Rightarrow \sqrt{\frac{\tilde{\mu}^2_{A^w}(x_i)+\tilde{\mu}^2_{B^w}(x_i)}{2}} - \frac{\sqrt{\tilde{\mu}_{A^w}(x_i)\tilde{\mu}_{B^w}(x_i)}}{2} + 1 \ge \frac{\tilde{\mu}_{A^w}(x_i)+\tilde{\mu}_{B^w}(x_i)}{4} + 1$$

$$\Rightarrow \frac{2\sqrt{2}+2\sqrt{\tilde{\mu}^2_{A^w}(x_i)+\tilde{\mu}^2_{B^w}(x_i)}-\sqrt{2\tilde{\mu}_{A^w}(x_i)\tilde{\mu}_{B^w}(x_i)}}{4+\tilde{\mu}_{A^w}(x_i)+\tilde{\mu}_{B^w}(x_i)} \ge \frac{1}{\sqrt{2}} \quad \dots (6)$$

Employing the monotonicity property of tangent function over $[0,1]$, the inequality **(6)** yields

$$\left(\left(4+\tilde{\mu}_{A^w}(x_i)+\tilde{\mu}_{B^w}(x_i)\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{\mu}^2_{A^w}(x_i)+\tilde{\mu}^2_{B^w}(x_i)}-\sqrt{2\tilde{\mu}_{A^w}(x_i)\tilde{\mu}_{B^w}(x_i)}}{4+\tilde{\mu}_{A^w}(x_i)+\tilde{\mu}_{B^w}(x_i)}\right)\right) \ge \left(4+\tilde{\mu}_{A^w}(x_i)+\tilde{\mu}_{B^w}(x_i)\right)\tan\left(\frac{1}{\sqrt{2}}\right)$$

… (7)

Replacement of $\tilde{\mu}_{A^w}(x_i), \tilde{\mu}_{B^w}(x_i)$ with $\left(1-\tilde{\mu}_{A^w}(x_i)\right), \left(1-\tilde{\mu}_{B^w}(x_i)\right)$ into **(7)** yields

$$\left(\begin{array}{l}\left(6-\tilde{\mu}_{A^w}(x_i)-\tilde{\mu}_{B^w}(x_i)\right)\times \\ \tan\left(\dfrac{2\sqrt{2}+2\sqrt{\left(1-\tilde{\mu}_{A^w}(x_i)\right)^2+\left(1-\tilde{\mu}_{B^w}(x_i)\right)^2}-\sqrt{2\left(1-\tilde{\mu}_{A^w}(x_i)\right)\left(1-\tilde{\mu}_{B^w}(x_i)\right)}}{6-\tilde{\mu}_{A^w}(x_i)-\tilde{\mu}_{B^w}(x_i)}\right)\end{array}\right) \ge \left(6-\tilde{\mu}_{A^w}(x_i)-\tilde{\mu}_{B^w}(x_i)\right)\tan\left(\frac{1}{\sqrt{2}}\right)$$

… (8)

Simply adding the inequalities **(7) & (8)** and taking the sum over $i=1$ to $i=n$ to obtain

$H^{\mu}_{CE}\left(A^w_{FS}, B^w_{FS}\right) \ge 0 \forall \tilde{\mu}_{A^w}(x_i), \tilde{\mu}_{B^w}(x_i) \in [0,1]$ with equality if $\tilde{\mu}_{A^w}(x_i) = \tilde{\mu}_{B^w}(x_i) \forall \tilde{\mu}_{A^w}(x_i), \tilde{\mu}_{B^w}(x_i) \in [0,1]$.

We next divert to discuss the situation under which out TFE measure $H^{\mu}_{CE}\left(A^w_{FS}, B^w_{FS}\right)$ admits its extreme values as shown in the following **Theorem. 3.3.**

**Theorem3.3** Let $A^w_{FS}$ and $B^w_{FS}$ be any two fuzzy sets with same cardinality *n.* then there exists the inequality: $0 \le H^{\mu}_{CE}\left(A^w_{FS}, B^w_{FS}\right) \le 10\left(\tan\left(\frac{2\sqrt{2}+2}{5}\right) - \tan\left(\frac{1}{\sqrt{2}}\right)\right)n.$

**Proof.** In view of Def**. 2.1,** the resulting **Theorem 3.2** yields

$$H_{CE}^{\mu}\left(A_{FS}^{w},C\left(A_{FS}^{w}\right)\right)=\sum_{i=1}^{n}\left[-10\tan\left(\frac{1}{\sqrt{2}}\right)+10\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{\mu}_{A^w}^{2}(x_i)+\left(1-\tilde{\mu}_{A^w}(x_i)\right)^2}-\sqrt{2\tilde{\mu}_{A^w}(x_i)\left(1-\tilde{\mu}_{A^w}(x_i)\right)}}{5}\right)\right]$$

$$=\sum_{i=1}^{n}\left[10\tan\left(\frac{2\sqrt{2}+2}{5}\right)-10Tan\left(\frac{1}{\sqrt{2}}\right)-10\left\{\begin{array}{l}\tan\left(\frac{2\sqrt{2}+2}{5}\right)\\-\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{\mu}_{A^w}^{2}(x_i)+\left(1-\tilde{\mu}_{A^w}(x_i)\right)^2}-\sqrt{2\tilde{\mu}_{A^w}(x_i)\left(1-\tilde{\mu}_{A^w}(x_i)\right)}}{5}\right)\end{array}\right\}\right]$$

$$=10\,\text{Max.}H_F(A_{FS}^{w})-10\,H_F(A_{FS}^{w}) \qquad \dots (9)$$

Since $H_F\left(A_{FS}^{w}\right)\geq 0$ (**Theorem. 3.1)**, therefore, the resulting expression **(9)** yields

$$H_F\left(A_{FS}^{w}\right)=\text{Max.}H_F\left(A_{FS}^{w}\right)-\frac{1}{10}H_{CE}^{\mu}\left(A_{FS}^{w},C\left(A_{FS}^{w}\right)\right)\geq 0 \qquad \dots. (10)$$

$$\Rightarrow 0\leq H_{CE}^{\mu}\left(A_{FS}^{w},C\left(A_{FS}^{w}\right)\right)\leq 10\left(\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\tan\left(\frac{1}{\sqrt{2}}\right)\right)n \qquad \dots (11)$$

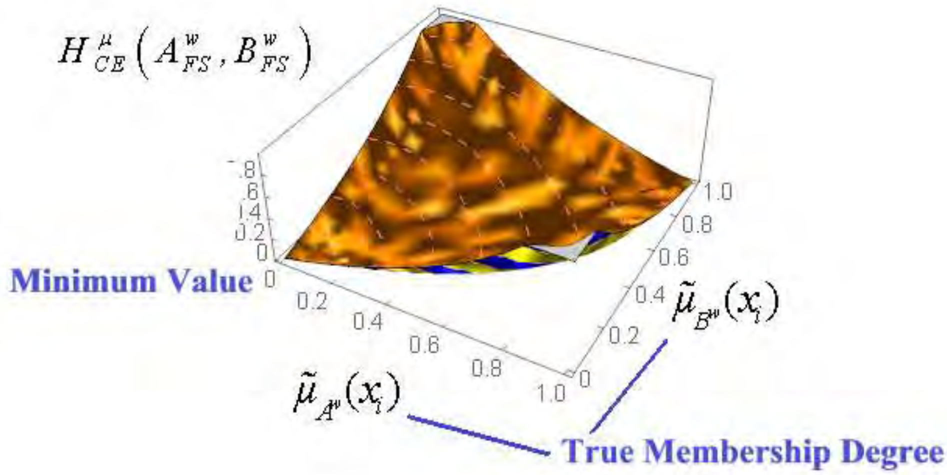
Inequality **(11)** suggests that $H_{CE}^{\mu}\left(A_{FS}^{w},C\left(A_{FS}^{w}\right)\right)$ is finite. Hence, it is easy to establish that $H_{CE}^{\mu}\left(A_{FS}^{w},B_{FS}^{w}\right)$ is also finite and satisfy $0\leq H_{CE}^{\mu}\left(A_{FS}^{w},B_{FS}^{w}\right)\leq 10\left(\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\tan\left(\frac{1}{\sqrt{2}}\right)\right)n$ for a fixed *n*. This implies that $\text{Max.}H_{CE}^{\mu}\left(A_{FS}^{w},B_{FS}^{w}\right)=10\left(\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\tan\left(\frac{1}{\sqrt{2}}\right)\right)n$ and this value completely depends only on the cardinality of $U$. The fact that, $H_{CE}^{\mu}\left(A_{FS}^{w},B_{FS}^{w}\right)$ affirms its minimum value zero can be seen from its three-dimensional plot as shown in **Fig.2 (a).** Furthermore, the three-dimensional plots represented in **Fig. 2(b)** depicts that $H_{CE}^{\mu}\left(A_{FS}^{w},B_{FS}^{w}\right)$ increases whenever $\left|A_{FS}^{w}-B_{FS}^{w}\right|$ increases, attains its maximum value as $10\left(\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\tan\left(\frac{1}{\sqrt{2}}\right)\right)n$ at the points (1, 0) and (0, 1) and minimum value zero whenever $A_{FS}^{w}=B_{FS}^{w}$.

The findings of resulting **Theorems 3.1 & 3.2** will be utilized to establish one more important **Theorem 4.1,** the outcomes of which will play an eminent role in understanding the macroscopic state of heavy metal pollution as follows.

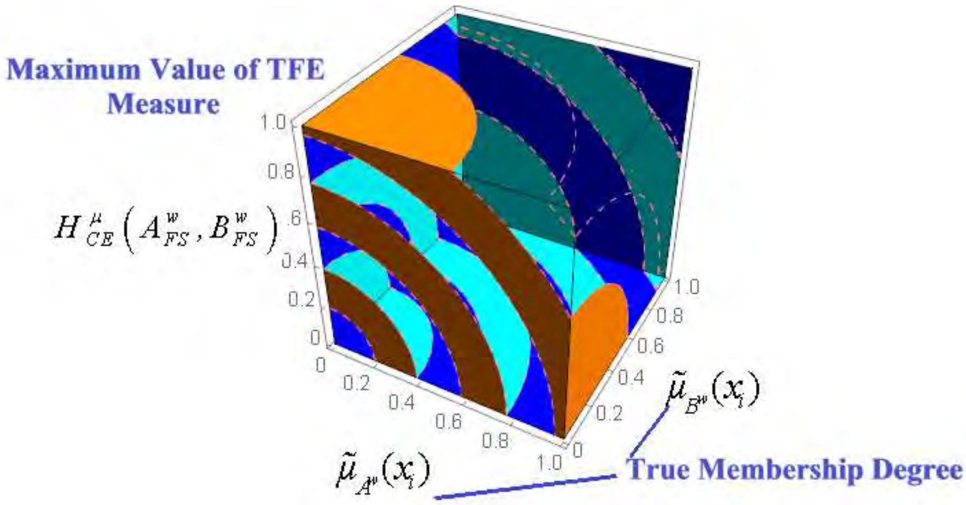
**4. A Trigonometric Single Valued Neutrosophic Cross Entropy Measure**

We shall now, equally will, extend the newly discovered trigonometric symmetric fuzzy cross entropy measure **(Theorem 3.2**) hinged on two fuzzy sets to this measure hinged on two single-valued neutrosophic sets.

**Def.4.1** Let $A_{SV}^{w} \subseteq U$ and $B_{SV}^{w} \subseteq U$ be any two single valued neutrosophic sets given by $A_{SV}^{w} = \left( < x_i, \tilde{\mu}_{A^w}(x_i), \tilde{i}_{A^w}(x_i), \tilde{f}_{A^w}(x_i) > | \, x_i \in U \right); B_{SV}^{w} = \left( < x_i, \tilde{\mu}_{B^w}(x_i), \tilde{i}_{B^w}(x_i), \tilde{f}_{B^w}(x_i) > | \, x_i \in U \right)$. The amount of true membership degree between two fuzzy sets $A_{FS}^{w}$ and $B_{FS}^{w}$, represented by $H_{CE}^{\mu}\left(A_{FS}^{w}, B_{FS}^{w}\right)$, is established in **Theorem 3.2.** Similarly, the amount of indeterminacy degree between two fuzzy sets $A_{FS}^{w}$ and $B_{FS}^{w}$ can be represented by $H_{CE}^{i}\left(A_{FS}^{w}, B_{FS}^{w}\right)$ and is defined as




**(a)**




**(b)**

**Fig.2** Minimum value of symmetric fuzzy cross entropy measure $H^{\mu}_{CE}\left(A^{w}_{FS}, B^{w}_{FS}\right)$

$$H^{i}_{CE}\left(A^{w}_{FS}, B^{w}_{FS}\right) = \sum_{i=1}^{n}\left[\begin{array}{l} -10\tan\frac{1}{\sqrt{2}} + \left(4+\tilde{i}_{A^w}(x_i)+\tilde{i}_{B^w}(x_i)\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{i}^2_{A^w}(x_i)+\tilde{i}^2_{B^w}(x_i)}-\sqrt{2\tilde{i}_{A^w}(x_i)\tilde{i}_{B^w}(x_i)}}{4+\tilde{i}_{A^w}(x_i)+\tilde{i}_{B^w}(x_i)}\right) \\ +\left(6-\tilde{i}_{A^w}(x_i)-\tilde{i}_{B^w}(x_i)\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{+\left(1-\tilde{i}_{A^w}(x_i)\right)^2+\left(1-\tilde{i}_{B^w}(x_i)\right)^2}-\sqrt{\left(1-\tilde{i}_{A^w}(x_i)\right)\left(1-\tilde{i}_{B^w}(x_i)\right)}}{6-\tilde{i}_{A^w}(x_i)-\tilde{i}_{B^w}(x_i)}\right)\end{array}\right]$$

… (12)

Furthermore, the amount of falsity membership degree between two fuzzy sets $A^{w}_{FS}$ and $B^{w}_{FS}$ can be represented by $H^{f}_{CE}\left(A^{w}_{FS}, B^{w}_{FS}\right)$ and is defined as

$$H^{f}_{CE}\left(A^{w}_{FS}, B^{w}_{FS}\right) = \sum_{i=1}^{n}\left[\begin{array}{l} -10\tan\frac{1}{\sqrt{2}} + \left(4+\tilde{f}_{A^w}(x_i)+\tilde{f}_{B^w}(x_i)\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{f}^2_{A^w}(x_i)+\tilde{f}^2_{B^w}(x_i)}-\sqrt{2\tilde{f}_{A^w}(x_i)\tilde{f}_{B^w}(x_i)}}{4+\tilde{f}_{A^w}(x_i)+\tilde{f}_{B^w}(x_i)}\right) \\ +\left(6-\tilde{f}_{A^w}(x_i)-\tilde{f}_{B^w}(x_i)\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{+\left(1-\tilde{f}_{A^w}(x_i)\right)^2+\left(1-\tilde{f}_{B^w}(x_i)\right)^2}-\sqrt{\left(1-\tilde{f}_{A^w}(x_i)\right)\left(1-\tilde{f}_{B^w}(x_i)\right)}}{6-\tilde{f}_{A^w}(x_i)-\tilde{f}_{B^w}(x_i)}\right)\end{array}\right]$$

… (13)

Hence, the proclaimed single valued neutrosophic cross entropy measure hinged on two single-valued neutrosophic sets (SVNSs) $A^{w}_{SV}$ and $B^{w}_{SV}$ can be obtained by simply adding the resulting equations **(3), (12)** and **(13).** Thus,

$$R_{SV}\left(A^{w}_{SV}, B^{w}_{SV}\right) = H^{\mu}_{CE}\left(A^{w}_{FS}, B^{w}_{FS}\right)\textbf{(Eq.3)} + H^{i}_{CE}\left(A^{w}_{FS}, B^{w}_{FS}\right)\textbf{(Eq.12)} + H^{i}_{CE}\left(A^{w}_{FS}, B^{w}_{FS}\right)\ \textbf{(Eq.13)} \quad \text{… (14)}$$

Here, $R_{SV}\left(A_{\mathrm{SV}}^{w}, B_{\mathrm{SV}}^{w}\right)$ satisfies all the conditions (i), (ii) and (iii) of **Def. 2.6** and hence a valid single valued neutrosophic entropy measure hinged on two single-valued neutrosophic sets $A_{\mathrm{SV}}^{w}$ and $B_{\mathrm{SV}}^{w}$.

**Theorem 4.1** Let

$A_{SV}^{w}=\left(<x_i,\tilde{\mu}_{A^w}(x_i),\tilde{i}_{A^w}(x_i),\tilde{f}_{A^w}(x_i)>|\, x_i\in U\right); B_{SV}^{w}=\left(<x_i,\tilde{\mu}_{B^w}(x_i),\tilde{i}_{B^w}(x_i),\tilde{f}_{B^w}(x_i)>|\, x_i\in U\right)$ be any two single-valued neutrosophic sets with same cardinality $n$. There exist the inequality

$$0\le R_{\mathrm{SV}}\left(A_{\mathrm{SV}}^{w}, B_{\mathrm{SV}}^{w}\right)\le 30\left(\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\tan\left(\frac{1}{\sqrt{2}}\right)\right)n.$$

**Proof.** In view of equations **(3), (12)** and **(13)** and replacement of $B_{\mathrm{SV}}^{w}$ with $C\left(A_{\mathrm{SV}}^{w}\right)$ into **(14)** yields

$$R_{SV}\left(A_{\mathrm{SV}}^{w}, C\left(A_{\mathrm{SV}}^{w}\right)\right)$$

$$=\sum_{i=1}^{n}\left[\begin{array}{l}30\tan\left(\frac{2\sqrt{2}+2}{5}\right)-30\tan\left(\frac{1}{\sqrt{2}}\right)\\ -10\left(\begin{array}{l}3\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\left(\frac{4+\tilde{\mu}_{A^w}(x_i)+\tilde{f}_{A^w}(x_i)}{5}\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{\mu}_{A^w}^2(x_i)+\tilde{f}_{A^w}^2(x_i)}-\sqrt{2\tilde{\mu}_{A^w}(x_i)\tilde{f}_{A^w}(x_i)}}{4+\tilde{\mu}_{A^w}(x_i)+\tilde{f}_{A^w}(x_i)}\right)\\ -\left(\frac{6-\tilde{\mu}_{A^w}(x_i)-\tilde{f}_{A^w}(x_i)}{5}\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{\left(1-\tilde{\mu}_{A^w}(x_i)\right)^2+\left(1-\tilde{f}_{A^w}(x_i)\right)^2}-\sqrt{2\left(1-\tilde{\mu}_{A^w}(x_i)\right)\left(1-\tilde{f}_{A^w}(x_i)\right)}}{6-\tilde{\mu}_{A^n}(x_i)-\tilde{f}_{A^n}(x_i)}\right)\\ -\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{i}_{A^w}^2(x_i)+\left(1-\tilde{i}_{A^w}(x_i)\right)^2}-\sqrt{2\tilde{i}_{A^w}(x_i)\left(1-\tilde{i}_{A^w}(x_i)\right)}}{5}\right)\end{array}\right)\end{array}\right]$$

$$=10\,\mathrm{Max}.R_N\left(A_{SV}^{w}\right)-10\,R_N\left(A_{SV}^{w}\right);\text{where} \quad \ldots (15)$$

$$R_N\left(A_{SV}^{w}\right)$$

$$=\sum_{i=1}^{n}\left[\begin{array}{l}3\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\left(\frac{4+\tilde{\mu}_{A^w}(x_i)+\tilde{f}_{A^w}(x_i)}{5}\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{\mu}_{A^w}^2(x_i)+\tilde{f}_{A^w}^2(x_i)}-\sqrt{2\tilde{\mu}_{A^w}(x_i)\tilde{f}_{A^w}(x_i)}}{4+\tilde{\mu}_{A^w}(x_i)+\tilde{f}_{A^w}(x_i)}\right)\\ -\left(\frac{6-\tilde{\mu}_{A^w}(x_i)-\tilde{f}_{A^w}(x_i)}{5}\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{\left(1-\tilde{\mu}_{A^w}(x_i)\right)^2+\left(1-\tilde{f}_{A^w}(x_i)\right)^2}-\sqrt{2\left(1-\tilde{\mu}_{A^w}(x_i)\right)\left(1-\tilde{f}_{A^w}(x_i)\right)}}{6-\tilde{\mu}_{A^n}(x_i)-\tilde{f}_{A^n}(x_i)}\right)\\ -\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{i}_{A^w}^2(x_i)+\left(1-\tilde{i}_{A^w}(x_i)\right)^2}-\sqrt{2\tilde{i}_{A^w}(x_i)\left(1-\tilde{i}_{A^w}(x_i)\right)}}{5}\right)\end{array}\right].$$

… (16)

The resulting mathematical expression **(16)** is the desired single valued neutrosophic entropy measure since it satisfies all the necessary conditions (i), (ii) and (iii) as laid down in **Def. 2.6.**

Since $\mathrm{R}_{\mathrm{N}}\left(A_{\mathrm{SV}}^{w}\right)\geq 0$ (in view of **Fig.3)** for each $A_{\mathrm{SV}}^{w}\subseteq R(U)$, the resulting inequality **(15)** yields

$$\mathrm{R}_{\mathrm{N}}\left(A_{\mathrm{SV}}^{w}\right)=\mathrm{Max.R}_{\mathrm{N}}\left(A_{\mathrm{SV}}^{w}\right)-\frac{1}{10}\mathrm{R}_{\mathrm{SV}}\left(A_{\mathrm{SV}}^{w},C\left(A_{\mathrm{SV}}^{w}\right)\right)\geq 0 \quad \ldots (17)$$

$$\Rightarrow 0\leq R_{\mathrm{SV}}\left(A_{\mathrm{SV}}^{w},C\left(A_{\mathrm{SV}}^{w}\right)\right)\leq 30\left(\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\tan\left(\frac{1}{\sqrt{2}}\right)\right)n \quad \ldots (18)$$

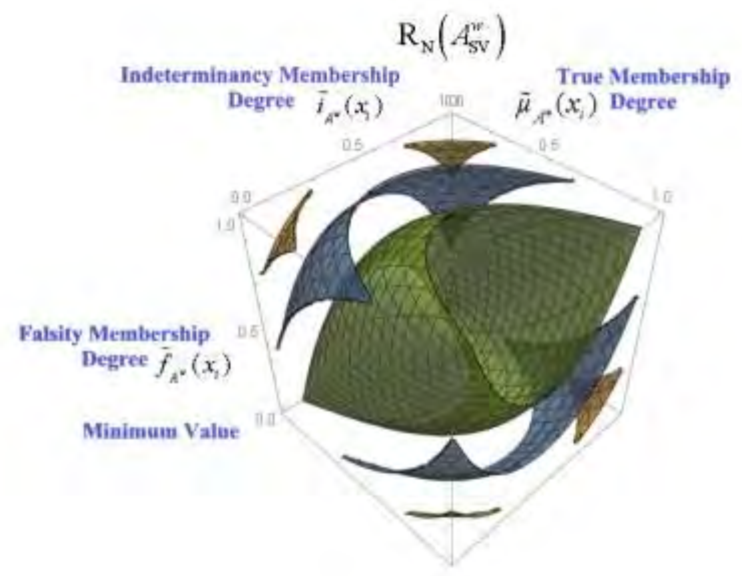



**Fig.3** Three -dimensional contour plot for non-negativity of $\mathrm{R}_{\mathrm{N}}\left(A_{\mathrm{SV}}^{w}\right)$

**Discussion** The resulting inequality **(18)** justifies that $R_{\mathrm{SV}}\left(A_{\mathrm{SV}}^{w},C\left(A_{\mathrm{SV}}^{w}\right)\right)$ is a finite quantity. Following the similar pattern as deploying to obtain **(18),** it is reasonable to establish that $0\leq R_{\mathrm{SV}}\left(A_{\mathrm{SV}}^{w},B_{\mathrm{SV}}^{w}\right)\leq 30\left(\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\tan\left(\frac{1}{\sqrt{2}}\right)\right)n$, where $n$ is a fixed natural number. Thus, $\mathrm{Max.}R_{\mathrm{SV}}\left(A_{\mathrm{SV}}^{w},B_{\mathrm{SV}}^{w}\right)=30\left(\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\tan\left(\frac{1}{\sqrt{2}}\right)\right)n$ and this value completely depends on the cardinality of $U$.

The overhead discussion has put us in a conclusive position to deploy the newly discovered trigonometric fuzzy entropy (TFE) and single valued neutrosophic entropy (TNE) measures, represented by **(1)** and **(14),** to assess the impact of heavy metal concentration in river water as follows.

**5. Impact Assessment of Heavy Metal**

The underlying research work is initiated by collecting Sarsa River water samples, before and after the amalgamation of pharmaceutical effluents (PE) treated in common effluent treatment plant (CETP). To reckon the quality of river water for drinking purposes, we have done a lot of experimenatation investigations, data comparsion and expressed the concentration of each heavy metal in terms of $\mu g / mL$ **(Fig. 4(a-b))**. In this study, the impact of concentration of heavy metals like cadmium (Cd), manganese (Mn), cobalt (Co), lead (Pb), copper (Cu), zinc (Zn) and iron (Fe), have been done through experimental investigations as well as the proposed TFE and TNE measures.

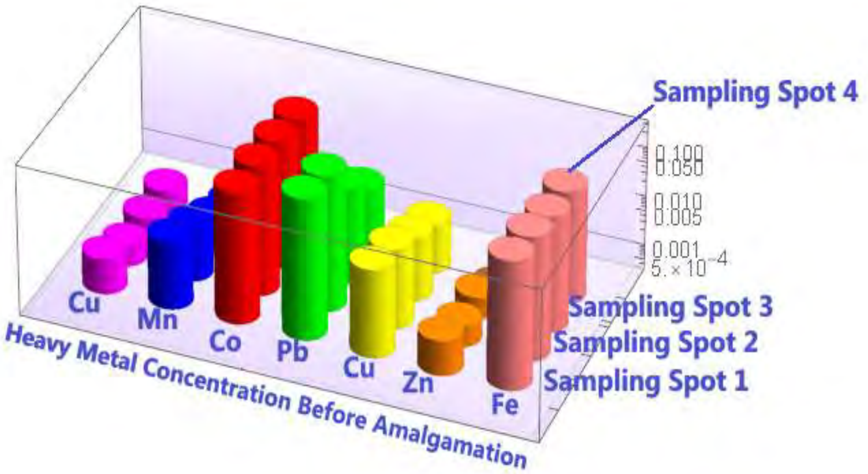
**5.1 Assessment of Heavy Metal Concentration Based on Experimental Observations:** The following observations were made before and after amalgamating pharmaceutical effluents (PE) treated in common effluent treatment plant (CETP) into river water samples.

**(a) Cadmium.** Before amalgamation, the concentration of cadmium, depcited in **Fig. 4(a),** was $0.002\ (S_1)$, followed by $0.001\ (S_2, S_3, S_4)$ with an average concentration of $0.00125\ \mu g / mL.$.After amalgamation, no cadmium was detected in the recorded observations as can be seen in **Fig. 4 (b).**

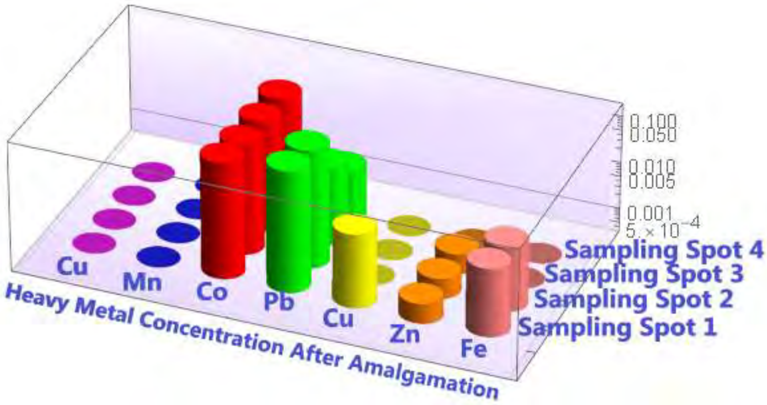
**(b) Manganese.**Spatial variations were observed in the concentration of manganese, before and after amalgamation, as depicited in **Fig. 4(a, b).** Before amalgamation, the concentration of Mn was $0.01\,(S_1)$, $0.005\ (S_2)$, $0.002\,(S_3)$, and $0.001\,(S_4)$ with an average concentration of $0.0045\ \mu g / mL.$ After introducing pharmaceutical effluents into river water samples, Mn was found to be absent.

**(c) Cobalt.** The cobalt's concentration varied spatially in Sarsa river water samples, taken before and after amalgamation, as shown in **Fig. 4(a, b)**. The concentration of cobalt decreased from $0.225$ to $0.143\ (S_1)$, $0.214$ to $0.0107\,(S_2)$, $0.18$ to $0.1\,(S_3)$ and $0.147$ to $0.1\,(S_4)$ respectively and it was within permissible limits**.**

**(d) Lead.** The water of Sarsa river recorded lead concentration of 0.36 before amalgamation and it decreased to $0.19\ (S_1)$ after amalgamation, as indicated by **Fig.4(a, b**). Similarly, the treatment was effective in reducing lead concentration from $0.28$ to $0.158\ (S_2)$, $0.04$ to $0.02\ (S_3)$ and $0.01$ to $0.005\ (S_4)$ respectively.




**(a)**




**(b)**

**Fig 4**. Heavy metal concentration of (a) Cd, (b) Mn, (c) Co, (d) Pb, (e) Cu, (f) Zn and (g) Fe in Sarsa river water samples

**(e) Copper.** The results depcited by **Fig. 4 (a, b**) indicate that before amalgamation of pharmaceutical effluents into river water samples, the concentration of Cu was $0.027\ (S_1)$ which reduced to $0.016\ (S_1)$ after amalgamation. The concentration further decreased from $0.018$ to nil $(S_2)$, $0.008$ to nil ($(S_3)$) and $0.004$ to nil $(S_4)$ respectively. Dilution of the effluents with river water could be the factor responsible for complete copper removal in the river water.

**(f) Zinc.** The results of **Fig. 4(a, b**) clearly indicate that before and after amalgamation, the concentration of zinc decreased from $0.002$ to $0.001\ (S_1)$, from $0.001$ to nil $(S_2, S_3)$ and no zinc was detected in $S_4$ before and after amalgamation.

**(g) Iron.** While comparing the concentration of iron (Fe) represented by **Fig. 4(a, b**), it was found that, before and after amalgamation, the concentration of Fe decreased from $0.195$ to $0.012$ $(S_1)$, $0.155$ to $0.01$ $S_2$, $0.104$ to nil $(S_3)$ and $0.09$ to nil $(S_4)$ consecutively.

The overhead discussion concludes that the concentration of heavy metals in Sarsa river water samples, which were within the permissible limits before amalgamation, dwindled gradually after the amalgamation of pharmaceutical effluents into river water samples **(Fig.4a, 4b**). The effectiveness of the proposed TFE and TNE measures can be confirmed only if these entropy measures can be proven capable in justifying the similar ressutls as obtained through experimental investigations.

**5.2Assessment of Heavy Metal Concentration Based on TFE and TNE Measures**

To evaluate the impact of heavy metals concentration in river water samples through our proposed trigonometric fuzzy and single valued neutrosophic entropy measures, we have represented each heavy metal by the set $B=(B_1,B_2,B_3,B_4,B_5,B_6,B_7)$ where $B_1$ = Cadmium (Cd), $B_2$ = Manganese (Mn), $B_3$ = Cobalt (Co), $B_4$ =Lead (Pb), $B_5$ = Copper (Cu), $B_6$ = Zinc (Zn) and $B_7$ = Iron (Fe) consecutively. After doing a lot of data comparison and experimental investigations, we have, equally well, extracted the lower (minimum) and upper (maximum) bounds for each monitored heavy metal concentration reading. Let $\tilde{\mu}_{B_K^w}(x_1)$ and $\tilde{U}_{B_K^w}(x_1)$ respectively be the lower and upper bounds extracted from $K^{th}$ heavy metal concentration. In this study, we have constructed the concentration intervals $\lfloor \tilde{\mu}_{B_K^w}(x), \tilde{U}_{B_K^w}(x) \rfloor$, before and after amalgamation, for each heavy metal concentration represented by $B_K (K=1,2,3,4,5,6,7)$ and the results are displayed in **Table. 1(a).**

Let $\tilde{f}_{B_K^w}(x)=1-\tilde{U}_{B_K^w}(x)$ , $\tilde{i}_{B_K^w}(x)=1-\tilde{f}_{B_K^w}(x)-\tilde{U}_{B_K^w}(x)$ denote the amount of fuzziness based on the falsity and indeterminacy membership degree of $K^{th}$ heavy metal concentration. If we restrict the value(s) of $\tilde{i}_{B_K^w}(x)$ to $0\cdot 0001$ if it is less than or equal to zero, then the set $B_K$ can be extended into the forms of single valued neutrosophic set (SVNS) represented by $\lfloor \tilde{\mu}_{B_K^w}(x), \tilde{i}_{B_K^w}(x), \tilde{f}_{B_K^w}(x) \rfloor$ and the results are displayed in **Table. 1(b).** Let $H_F(B_K)$ and $R_N(B_K)$ denote the trigonometric fuzzy entropy and single valued neutrosophic entropy measures value of $K^{th}$ heavy metal concentration.

Taking $i=1$ and replacing $\tilde{\mu}_{A^w}(x_i)$ with $\tilde{\mu}_{B_K^w}(x)$ into the **(1),** Then, $H_F(B_K)(K=1,2,3,4,5,6,7)$, after modification, takes the form as shown in **(17).** The results are displayed in **Table 1(a).** Thus,

$$H_F(B_K)=\tan\left(\frac{1}{4}\right)\tan\left[\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{\mu}^2_{B_K^w}(x)+\left(1-\tilde{\mu}_{B_K^w}(x)\right)^2}-\sqrt{2\tilde{\mu}_{B_K^w}(x)\left(1-\tilde{\mu}_{B_K^w}(x)\right)}}{5}\right)\right]$$

…(19)

Similarly, $R_N(B_K)$ ; $(K=1,2,...,7)$ can also be modified by taking $i=1$ and replacing $\tilde{\mu}_{A^w}(x_1),\tilde{i}_{A^w}(x_1),\tilde{f}_{A^w}(x_1)$ with $\tilde{\mu}_{B_K^w}(x),\tilde{\mu}_{B_K^w}(x),\tilde{f}_{B_K^w}(x)$ into the resulting equation **(16). T**he results are displayed in **Table 1(b).** Thus,

$$R_N(B_K)=3\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\left(\frac{4+\tilde{\mu}_{B_K^w}(x)+\tilde{f}_{B_K^w}(x)}{5}\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{\mu}^2_{B_K^w}(x)+\tilde{f}^2_{B_K^w}(x)}-\sqrt{2\tilde{\mu}_{B_K^w}(x)\tilde{f}_{B_K^w}(x)}}{4+\tilde{\mu}_{B_K^w}(x)+\tilde{f}_{B_K^w}(x)}\right)$$
$$-\left(\frac{6-\tilde{\mu}_{B_K^w}(x)-\tilde{f}_{B_K^w}(x)}{5}\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{\left(1-\tilde{\mu}_{B_K^w}(x)\right)^2+\left(1-\tilde{f}_{B_K^w}(x)\right)^2}-\sqrt{2\left(1-\tilde{\mu}_{B_K^w}(x)\right)\left(1-\tilde{f}_{B_K^w}(x)\right)}}{6-\tilde{\mu}_{B_K^w}(x)-\tilde{f}_{B_K^w}(x)}\right) \quad (20)$$
$$-\tan\left(\frac{2\sqrt{2}+2\sqrt{\tilde{i}^2_{B_K^w}(x)+\left(1-\tilde{i}_{B_K^w}(x)\right)^2}-\sqrt{2\tilde{i}_{B_K^w}(x)\left(1-\tilde{i}_{B_K^w}(x)\right)}}{5}\right)$$

The trigonometric fuzzy entropy (TFE) measure values $H_F(B_K)$ as well as single valued neutrosophic entropy (TNE) measure values $R_N(B_K)$ for each heavy metal $B_K$ can be evaluated employing equations **(17)** and **(18)**. A comparative analysis of the results depicted in **Table. 1)a)** and **Table. 1(b)** reveal that, before amalgamation, the heavy metal concentration was found to be more macroscopic (owing to high TFE and TNE values as shown in **Fig. 5**) which became less macroscopic after amalgamation (owing to low TFE and TNE values as shown in **Fig. 5**). In other words, the concentration of each heavy metal, which was within the permissible limits before amalgamation, dwindled gradually (owing to negative change in TFE and TNE values) after amalgamating pharmaceutical effluents into River water samples.

**Table 1(a).** Concentration intervals and TFE values of each $B_K$ before and after amalgamation of pharmaceutical effluents

| Heavy Metal | Concentration Interval Before Amalgamation | Concentration Interval After Amalgamation | TFE Value Before Amalgamation | TFE Value After Amalgamation | Change in TFE Values |
|---|---|---|---|---|---|
| Cadmium | [0.001,0.002] | [0.000,0.000] | 0.0073 | 0.0000 | -0.0073 |
| Manganese | [0.001,0.010] | [0.000,0.000] | 0.0073 | 0.0000 | -0.0073 |
| Cobalt | [0.147,0.225] | [0.100,0.143] | 0.1000 | 0.0825 | -0.0174 |
| Lead | [0.010,0.360] | [0.005,0.190] | 0.0073 | 0.0000 | -0.0073 |
| Copper | [0.004,0.027] | [0.000,0.016] | 0.0073 | 0.0168 | 0.0095 |
| Zinc | [0.000,0.002] | [0.000,0.001] | 0.0000 | 0.0000 | 0.0000 |
| Iron | [0.090,0.195] | [0.000,0.012] | 0.0073 | 0.0000 | -0.0073 |

**Table 1(b).** Conversion of concentration intervals into the forms of SVNSs TNE values of each $B_K$ before and after amalgamation of pharmaceutical effluents

| Heavy Metal | SVNSs Before Amalgamation | SVNSs After Amalgamation | TNE Values Before Amalgamation | TNE Values After Amalgamation | Change in TNE Values |
|---|---|---|---|---|---|
| Cadmium | [0.0010,0.0010,0.9980] | [0.0010,0.0001,0.9999] | 0.0249 | 0.0045 | -0.0204 |
| Manganese | [0.0010,0.0090,0.9900] | [0.0010,0.0001,0.9999] | 0.0546 | 0.0045 | -0.0500 |
| Cobalt | [0.1470,0.0780,0.7750] | [0.1000,0.0430,0.8570] | 0.2953 | 0.2347 | -0.0606 |
| Lead | [0.0010,0.3590,0.6400] | [0.0050,0.1850,0.8100] | 0.3334 | 0.2541 | -0.0793 |
| Copper | [0.0010,0.0017,0.9973] | [0.0000,0.0160,0.9840] | 0.0290 | 0.0628 | 0.0338 |
| Zinc | [0.0000,0.0020,0.9980] | [0.0000,0.0010,0.9990] | 0.0208 | 0.0146 | 0.0063 |
| Iron | [0.0010,0.1940,0.8050] | [0.0000,0.0120,0.9880] | 0.2508 | 0.0537 | -0.1971 |

**Discussion** After amalgamation, the concentration of cadmium and manganese was found to be negligible **(Fig.4)**. The possible reasons for absence of these heavy metals in river water samples could be the use of physico-chemical processes-adsorption, membrane filtration, electro dialysis etc., which further diluted the river water after amalgamation and made heavy metal presence almost negligible in the river.

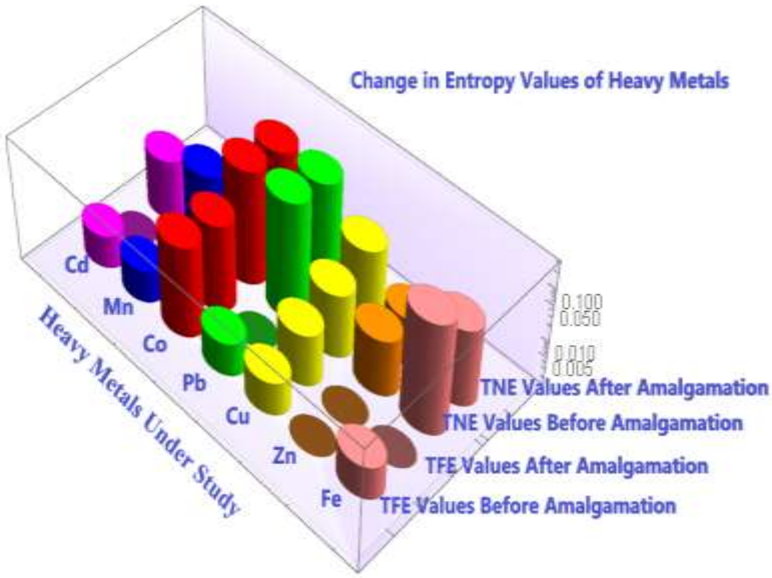



**Fig.5** TFE and TNE values of each heavy metal concentration before and after amalgamating pharmaceutical effluents into river water samples

The similar observations were experienced from the experimental investigations. Hence, the effectiveness and validity of our proposed TFE and TNE measures have been justified.

We next switch to establish the proclaimed heavy metal contamination evaluation methodology, intended to identify the most contaminated sampling spot responsible for heavy metal contamination in Sarsa river water.

**6. Heavy Metal Contamination Evaluation Methodology**

To reckon the most contaminated sampling spot by the proposed methodology, we proceed as follows.

**Step: -1 Collection of River Water Samples**

The water samples were collected from Sarsa river by covering a stretch of 20 km from four sampling spots $S_1, S_2, S_3$ & $S_4$, before and after the amalgamation of CETP treated pharmaceutical effluents (PE) into river water samples. The samples were stored in high-grade polythene bottles of one-liter capacity. Representative water sample from selected sites was collected and transported to the laboratory for experimentation investigations by keeping in mind that the comparative concentrations of all related components were same in all the samples. All precautions were taken to avoid any significant alteration in sample composition before experiments were performed. Analytical studies were carried out by the methods of American Public Health Association **[20]**.

**Step: -2 Normalization of Monitored Heavy Metal Concentration**

Suppose the number of parameters (heavy metals) to be studied is denoted by "$n$". Let the number of sampling spots under study is denoted by "$m$". Let $l_{ji}$ denotes the monitored concentration reading of $j^{th}$ heavy metal at $i^{th}$ sampling spot. To ensure the quality of various quantity grades, it becomes essential, before fuzzification, to normalize each monitored heavy metal concentration reading. Let $p_{ji}$ denotes the normalization concentration function for the concentration of $j^{th}$ heavy metal at $i^{th}$ sampling spot. Then

$$p_{ji} = \frac{l_{ji} - \text{Min}.l_{ji}}{\text{Max}.l_{ji} - \text{Min}.l_{ji}}; j = 1,2,3,4...,n; i = 1,2,3,4...,m. \quad \dots (21)$$

**Step: 3 Determination of Fuzzy and Neutrosophic Entropy Weights**

After the introduction of fuzzy sets theory by Zadeh **[15]**, Information theory started receiving recognition from different quarters. In the existing literature, many fuzzy entropy measures have been investigated and characterized by researchers, but with some demerits and limitations. De Luca and Termini **[19]** suggested the first non-additive measure of fuzzy entropy:

$$H(A) = -\frac{1}{\log m}\sum_{j=1}^{n}\left[\mu_A(x_j)\log\mu_A(x_j) + (1-\mu_A(x_j))\log(1-\mu_A(x_j))\right] \quad \dots (22)$$

where $A = (<x_i, \mu_A(x_i)> | x_i \in U)$ is the corresponding fuzzy set satisfying $\mu_A(x_j): X \to [0,1]$ and $"m"$ is any fixed natural number. The fuzzy entropy measure **(22)** has been found capable for analyzing the macroscopic view of heavy metal pollution in river water samples**.** Unfortunately, the entropy measure **(22)** is facing a major drawback as it is unknowingly based on the fancy assumption $0 \times log 0 = 0$ and hence indicates less macroscopic view of heavy metal contamination. To represent macroscopic view of heavy metal contamination in a broader way and to meet the exigency, we have successfully deployed our proposed TFE and TNE measures to construct fuzzy and single valued neutrosophic entropy weights for various sampling spot with respect to each heavy metal concentration as follows.

Let $T_{ji}$ denotes the amount of fuzziness based on true membership degree of $j^{th}$ heavy metal concentration at $i^{th}$ sampling spot. Then,

$$T_{ji} = \frac{p_{ji}}{\sum_{j=1}^{n} p_{ji}}; j = 1,2,...,n, i = 1,2,..,m \quad \dots (23)$$

**(a)** The weights $w_{ji}^{(0)}$ for $j^{th}$ heavy metal concentration at $i^{th}$ sampling spot employing Deluca and Termini **(22)** can be evaluated as follows. Let $"m"$ be the number of sampling spots, then

$$w_{ji}^{(0)} = \frac{1 - E_{ji}^{(0)}}{\sum_{j=1}^{n} E_{ji}^{(0)}}, \text{ where} \quad \dots (24)$$

$$E_{ji}^{(0)} = -\frac{1}{\log m}\sum_{j=1}^{n}\left[T_{ji}\log T_{ji} + (1-T_{ji})\log(1-T_{ji})\right] \quad \dots (25)$$

**(b)** The weights $w_{ji}^{(1)}$ for $j^{th}$ heavy metal concentration at $i^{th}$ sampling spot employing the proposed trigonometric fuzzy entropy (TFE) measure (**1**) can be evaluated as follows: Let "$m$" be the number of sampling spots, then

$$w_{ji}^{(1)} = \frac{1 - E_{ji}^{(1)}}{\sum_{j=1}^{n} E_{ji}^{(1)}}, \text{ where} \quad \dots (26)$$

$$E_{ji}^{(1)} = -\tan\left(m^{-1}\right)\sum_{j=1}^{n}\left[\tan\left(\frac{2\sqrt{2}+2\sqrt{T_{ji}^{2}+\left(1-T_{ji}\right)^{2}}-\sqrt{2T_{ji}\left(1-T_{ji}\right)}}{5}\right)-\tan\left(\frac{2\sqrt{2}+2}{5}\right)\right] \quad \dots (27)$$

**(c)**The weights $w_{ji}^{(2)}$ for $j^{th}$ heavy metal concentration at $i^{th}$ sampling spot employing the proposed single valued neutrosophic entropy (TNE) measure (**16**) can be evaluated as follows.

Let $F_{ji} = 1 - T_{ji}$ and $I_{ji} = 1 - T_{ji} - F_{ji}$ denote the amount of fuzziness based on the indeterminacy and falsity membership degree of $j^{th}$ heavy metal concentration at $i^{th}$ sampling spot. Here, the values of $I_{ji}$ are restricted to $0 \cdot 001$ if it is less than or equal to zero and "$m$" is the number of sampling spots. Then,

$$w_{ji}^{(2)} = \frac{1 - E_{ji}^{(2)}}{\sum_{j=1}^{n} E_{ji}^{(2)}}, \text{where} \quad \dots (28)$$

$$E_{ji}^{(2)} = \tan\left(m^{-1}\right)\sum_{j=1}^{n}\left|\begin{array}{l} 3\tan\left(\frac{2\sqrt{2}+2}{5}\right)-\left(\frac{4+T_{ji}+F_{ji}}{5}\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{T_{ji}^{2}+F_{ji}^{2}}-\sqrt{2T_{ji}F_{ji}}}{4+T_{ij}+F_{ij}}\right) \\ -\left(\frac{6-T_{ji}-F_{ji}}{5}\right)\tan\left(\frac{2\sqrt{2}+2\sqrt{\left(1-T_{ji}\right)^{2}+\left(1-F_{ji}\right)^{2}}-\sqrt{2\left(1-T_{ji}\right)\left(1-F_{ji}\right)}}{6-T_{ij}-F_{ij}}\right) \\ -\tan\left(\frac{2\sqrt{2}+2\sqrt{I_{ji}^{2}+\left(1-I_{ji}\right)^{2}}-\sqrt{2I_{ji}\left(1-I_{ji}\right)}}{5}\right) \end{array}\right| \quad \dots (29)$$

**Step: -4 Calculations of Relative Sub-Indices**

The quality of river water parameters (heavy metals) can be well described by means of two types of sub-indices-absolute and relative-which are being used by the eminent researchers. Since absolute (or relative) sub-indexing approaches are fully independent (or dependent) on water quality standards, the relative sub-indexing approach has been empowered in this study. Let $Q_{ji}$ = Relative sub-index, $S_{ji}$ = Maximum permissible concentration limit and $l_{ji}$ = Monitored concentration reading of the $j^{th}$ heavy metal at $i^{th}$ sampling spot respectively. Then, the relative sub-indices of each heavy metal with respect to various sampling spots are assigned as

$$Q_{ji} = \frac{l_{ji}}{S_{ji}} \times 100; j = 1,2,3,4...n, i = 1,2,3,4..,m. \quad \ldots (30)$$

**Step: -5 Constructions of FHCI and NHCI**

The enduring Deluca and Termini entropy (**[19]**, proposed trigonometric fuzzy entropy weighted and single valued neutrosophic entropy weighted heavy metal contamination indices (EHCI, FHCI and NHCI), before and after the amalgamation of pharmaceutical effluents into river water samples, can be constructed as follows:

$$\text{EHCI at } i^{th} \text{ Sampling Spot} = \sum_{j=1}^{n} w_{ji}^{(0)} Q_{ji} \quad \ldots (31)$$

$$\text{FHCI at } i^{th} \text{ Sampling Spot} = \sum_{j=1}^{n} w_{ji}^{(1)} Q_{ji} \quad \ldots (32)$$

$$\text{NHCI at } i^{th} \text{ Sampling Spot} = \sum_{j=1}^{n} w_{ji}^{(2)} Q_{ji} \quad \ldots (33)$$

**Step: -6 Identifying the Most Contaminated Sampling Spot**

The maximum(or minimum) EHCI, FHCI or NHCI score among each sampling spot is designated to the"most (or least) contaminated sampling spot" accordingly.

We finally deploy the proclaimed fuzzy entropy weighted heavy metal contamination index (EHCI) and single valued neutrosophic entropy weighted heavy metal contamination index (NHCI) to identify the most contaminated sampling spot responsible for heavy metal contamination in Sarsa river water.

## 7. Application of TFE and TNE Based Heavy Metal Contamination Evaluation Methodology

To reckon the most contaminated sampling spot, responsible for heavy metal contamination in Sarsa river water, we have computed the enduring Deluca and Termini entropy **[19]**, proposed trigonometric fuzzy entropy and single valued neutrosophic entropy weighted heavy metal contamination indices (FHCI, EHCI and NHCI) through the proposed methodology as explained in **Section. 6** and the results are displayed in **Tables.2-4(a, b).**

### 7.1 Identification of Most Contaminated Sampling Spot Through EHCI

The Deluca and Termini fuzzy entropy weighted heavy metal contamination index (EHCI) at each sampling spot $S_1, S_2, S_3, S_4$, before and after the amalgamation of pharmaceutical effluents into river water samples, is computed by deploying **(31)** and the results are depicted in **Table.2(a, b).** The monitored concentration reading of each heavy metal is expressed in terms of $mg/L$. In this study, the number of parameters (heavy metals) is seven $(n = 7)$ and the number of sampling spots is four $(m = 4)$. The normalization concentration function $p_{ji}\,(j = 1,2,...,7; i = 1,2,3,4)$ for each heavy metal at various sampling spots and the results are depicted in **Table.2(a, b).** As per W.H.O. **[20]**, the maximum permissible limits $S_{ji}\left(j=1,2,..7, i=1,2,3,4\right)$ of each heavy metal at various sampling spots are considered as $0.005\text{(Cd)}, 0.1\text{(Mn)}, 2\text{(Co)}, 0.05\text{(Pb)}, 1\text{(Cu)}, 5\text{(Zn)}, 0.1\text{(Fe)}(mg/L)$.

**Observations.** The results depicted in **Table 2(a, b)** and **Fig. 5 i**ndicate that the trend of EHCI scores reduced gradually, from 0.3623 to0.1673 $\left(S_1\right)$, from 0.2816 to0.1559 ($S_2$), from 0.1067 to 0.0440 $\left(S_3\right)$ and from 0.0766 to 0.0468 $\left(S_4\right)$ consecutively.

**7.2 Identification of Most Contaminated Sampling Spot Through FHCI**

The trigonometric fuzzy entropy weighted heavy metal contamination index (EHCI) at each sampling spot $S_1, S_2, S_3, S_4$, before and after the amalgamation of pharmaceutical effluents into river water samples, is computed by deploying **(32)** and the results are depicted in **Table.3(a, b).**

**Observations.** The results depicted in **Table 3(a, b)** and **Fig. 5** reveal that the proposed EHCI exhibited the similar trend as returned by Deluca and Termini entropy weighted heavy metal contamination index (FHCI). The FHCI scores at each sampling spot $S_1, S_2, S_3, S_4$ reduced gradually, from 1.6865 to0.8343 $\left(S_1\right)$, from 1.3276 to0.8237$\left(S_2\right)$, from 0.4186 to 0.1895 $\left(S_3\right)$ and from 0.3064 to 0.1266 $\left(S_4\right)$ consecutively.

**7.3 Identification of Most Contaminated Sampling Spot Through NHCI**

The single valued neutrosophic entropy weighted heavy metal contamination index (NHCI) at each sampling spot $S_1, S_2, S_3, S_4$, before and after the amalgamation of pharmaceutical effluents into river water samples, is computed by deploying **(33)** and the results are depicted in **Table.4(a, b).**

**Observations** The results depicted in **Table 4(a, b)** and **Fig. 5** reveal that NHCI scores at each sampling spot $S_1, S_2, S_3, S_4$ reduced gradually, from 0.7093 to 0.3216 $\left(S_1\right)$, from 0.5575 to 0.3031$\left(S_2\right)$, from 0.1841 to 0.0673 $\left(S_3\right)$ and from 0.1342 to 0.0481$\left(S_4\right)$consecutively.

**Results and Discussions** The accumulated trend of EHCI, FHCI and NHCI scores depcited by **Table. 2-4(a, b)** and **Fig 6** leads to wind-up the conclusion that, before amalgamation, the EHCI, FHCI and NHCI scores, which were on higher side, dwindled gradually after amalgamation. In other words, before amalgamation, the quality of river water was imperfect" or "unfavourable" but after amalgamation, resulted into " impeccable" and " favourable" (still not suitable for drinking purposes and could be owed to dilution of the river water). In view of and **Fig 6**, the sampling spot $S_1$ was found to be most contaminated owing to its maximum EHCI (before amalgamation: 0.3623, after amalgamation: 0.1673) scores, FHCI (before amalgamation: 1.6865, after amalgamation: 0.8343) scores and NHCI (before amalgamation: 0.7093, after amalgamation: 0.3216) scores.

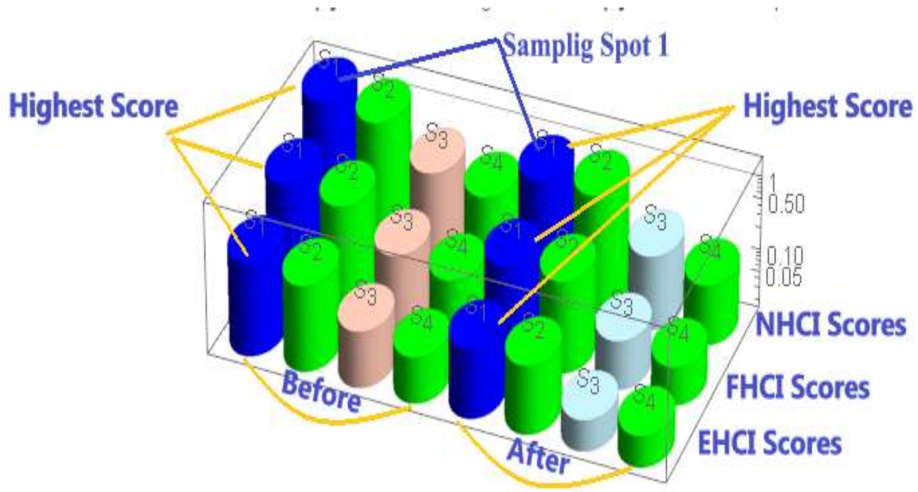



**Fig.6** Trend of EHCI, FHCI and NHCI at four sampling spots before andafter amalgamating pharmaceutical effluents

**Table 2(a)**: Calculation of EHCI scores employing Deluca and Termini entropy **[19]** before amalgamation

| Heavy Metals | Construction Function | Fuzziness Values | Entropy Values | Assigned Weights | Relative Sub-Indices | EHCI Score |
|---|---|---|---|---|---|---|
| | | | **Sampling Spot 1** | | | |
| Cd | 0.0050 | 0.0024 | 0.0123 | 0.6629 | 0.0400 | |
| Mn | 0.0250 | 0.0122 | 0.0475 | 0.6393 | 0.0100 | |
| Co | 0.5625 | 0.2741 | 0.4236 | 0.3868 | 0.0113 | |
| Pb | 0.9000 | 0.4385 | 0.4945 | 0.3392 | 0.7200 | **0.3623** |
| Cu | 0.0675 | 0.0329 | 0.1043 | 0.6011 | 0.0027 | |
| Zn | 0.0050 | 0.0024 | 0.0123 | 0.6629 | 0.0000 | |
| Fe | 0.4875 | 0.2375 | 0.3954 | 0.4057 | 0.1950 | |
| | | | **Sampling Spot 2** | | | |
| Cd | 0.0003 | 0.0015 | 0.0080 | 0.6767 | 0.0200 | |
| Mn | 0.0017 | 0.0074 | 0.0316 | 0.6606 | 0.0050 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| Co | 0.0713 | 0.3175 | 0.4508 | 0.3746 | 0.0107 | |
| Pb | 0.0933 | 0.4154 | 0.4896 | 0.3482 | 0.5600 | 0.2816 |
| Cu | 0.0060 | 0.0267 | 0.0888 | 0.6216 | 0.0018 | |
| Zn | 0.0003 | 0.0015 | 0.0080 | 0.6767 | 0.0000 | |
| Fe | 0.0517 | 0.2300 | 0.3890 | 0.4168 | 0.1550 | |
| **Sampling Spot 3** | | | | | | |
| Cd | 0.0005 | 0.0030 | 0.0146 | 0.7329 | 0.0200 | |
| Mn | 0.0010 | 0.0060 | 0.0263 | 0.7242 | 0.0020 | |
| Co | 0.0900 | 0.5357 | 0.4982 | 0.3733 | 0.0090 | |
| Pb | 0.0200 | 0.1190 | 0.2633 | 0.5479 | 0.0800 | 0.1067 |
| Cu | 0.0040 | 0.0238 | 0.0812 | 0.6834 | 0.0008 | |
| Zn | 0.0005 | 0.0030 | 0.0146 | 0.7329 | 0.0000 | |
| Fe | 0.0520 | 0.3095 | 0.4463 | 0.4118 | 0.1040 | |
| **Sample Spot 4** | | | | | | |
| Cd | 0.0100 | 0.0040 | 0.0186 | 0.8346 | 0.0200 | |
| Mn | 0.0100 | 0.0040 | 0.0186 | 0.8346 | 0.0010 | |
| Co | 1.4700 | 0.5810 | 0.4905 | 0.4333 | 0.0074 | |
| Pb | 0.1000 | 0.0395 | 0.1201 | 0.7483 | 0.0200 | 0.0766 |
| Cu | 0.0400 | 0.0158 | 0.0586 | 0.8006 | 0.0004 | |
| Zn | 0.0000 | 0.0000 | 0.0000* | 0.8504* | 0.0000 | |
| Fe | 0.9000 | 0.3557 | 0.4695 | 0.4511 | 0.0900 | |

*At $S_4$, the entropy value 0.8504 of Zinc is based on the assumption: $0 \times \log 0 = 0$.

**Table 2(b)**: Calculation of EHCI scores employing Deluca and Termini [19] entropy after amalgamation

| Heavy Metals | Monitored Values | Fuzziness Values | Entropy Values | Assigned Weights | Relative Sub-Indices | EHCI Score |
|---|---|---|---|---|---|---|
| **Sampling Spot 1** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0000* | 0.8114* | 0.0000 | |
| Mn | 0.0000 | 0.0000 | 0.0000* | 0.8114* | 0.0000 | |
| Co | 0.7150 | 0.3950 | 0.4840 | 0.4187 | 0.0072 | |
| Pb | 0.9500 | 0.5249 | 0.4991 | 0.4064 | 0.3800 | **0.1673** |
| Cu | 0.0800 | 0.0442 | 0.1306 | 0.7054 | 0.0016 | |
| Zn | 0.0050 | 0.0028 | 0.0137 | 0.8003 | 0.0000 | |
| Fe | 0.0600 | 0.0331 | 0.1050 | 0.7262 | 0.0120 | |
| **Sampling Spot 2** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0000* | 0.9060* | 0.0000 | |
| Mn | 0.0000 | 0.0000 | 0.0000* | 0.9060* | 0.0000 | |
| Co | 0.5350 | 0.3877 | 0.4816 | 0.4697 | 0.0054 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| Pb | 0.7900 | 0.5725 | 0.4924 | 0.4599 | 0.3160 | 0.1559 |
| Cu | 0.0000 | 0.0000 | 0.0000 | 0.9060* | 0.0000 | |
| Zn | 0.0000 | 0.0036 | 0.0173 | 0.8904 | 0.0000 | |
| Fe | 0.0500 | 0.0362 | 0.1124 | 0.8042 | 0.0100 | |
| **Sampling Spot 3** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0000* | 1.4476* | 0.0000 | |
| Mn | 0.0000 | 0.0000 | 0.0000* | 1.4476* | 0.0000 | |
| Co | 1.0000 | 0.8264 | 0.3329 | 0.9657 | 0.0050 | |
| Pb | 0.2000 | 0.1653 | 0.3234 | 0.9794 | 0.0400 | 0.0440 |
| Cu | 0.0000 | 0.0000 | 0.0000* | 1.4476* | 0.0000 | |
| Zn | 0.0000 | 0.0083 | 0.0345 | 1.3976 | 0.0000 | |
| Fe | 0.0000 | 0.0000 | 0.0000* | 1.4476* | 0.0000 | |
| **Sampling Spot 4** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0000* | 3.6206* | 0.0000 | |
| Mn | 0.0000 | 0.0000 | 0.0000* | 3.6206* | 0.0000 | |
| Co | 1.0000 | 0.9524 | 0.1381 | 3.1206 | 0.0050 | |
| Pb | 0.0500 | 0.0476 | 0.1381 | 3.1206 | 0.0100 | 0.0468 |
| Cu | 0.0000 | 0.0000 | 0.0000* | 3.6206* | 0.0000 | |
| Zn | 0.0000 | 0.0000 | 0.0000* | 3.6206* | 0.0000 | |
| Fe | 0.0000 | 0.0000 | 0.0000* | 3.6206* | 0.0000 | |

* Values are based on the assumption: $0 \times \log 0 = 0$ during calculation of $E_j^{(0)}$.

| Heavy Metals | Construction Function | Fuzziness Values | Entropy Values | Assigned Weights | Relative Sub-Indices | FHCI Score |
|---|---|---|---|---|---|---|
| **Sampling Spot 1** | | | | | | |
| Cd | 0.0050 | 0.0024 | 0.0115 | 1.9734 | 0.0400 | |
| Mn | 0.0250 | 0.0122 | 0.0269 | 1.9427 | 0.0100 | |
| Co | 0.5625 | 0.2741 | 0.1313 | 1.7342 | 0.0113 | |
| Pb | 0.9000 | 0.4385 | 0.1496 | 1.6978 | 0.7200 | **1.6865** |
| Cu | 0.0675 | 0.0329 | 0.0459 | 1.9048 | 0.0027 | |
| Zn | 0.0050 | 0.0024 | 0.0115 | 1.9734 | 0.0000 | |
| Fe | 0.4875 | 0.2375 | 0.1242 | 1.7485 | 0.1950 | |
| **Sampling Spot 2** | | | | | | |
| Cd | 0.0003 | 0.0015 | 0.0089 | 2.0284 | 0.0200 | |
| Mn | 0.0017 | 0.0074 | 0.0207 | 2.0043 | 0.0050 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| Co | 0.0713 | 0.3175 | 0.1383 | 1.7636 | 0.0107 | |
| Pb | 0.0933 | 0.4154 | 0.1483 | 1.7432 | 0.5600 | 1.3276 |
| Cu | 0.0060 | 0.0267 | 0.0410 | 1.9627 | 0.0018 | |
| Zn | 0.0003 | 0.0015 | 0.0089 | 2.0284 | 0.0000 | |
| Fe | 0.0517 | 0.2300 | 0.1225 | 1.7959 | 0.1550 | |
| **Sampling Spot 3** | | | | | | |
| Cd | 0.0005 | 0.0030 | 0.0128 | 2.1447 | 0.0200 | |
| Mn | 0.0010 | 0.0060 | 0.0184 | 2.1325 | 0.0020 | |
| Co | 0.0900 | 0.5357 | 0.1505 | 1.8455 | 0.0090 | |
| Pb | 0.0200 | 0.1190 | 0.0902 | 1.9766 | 0.0800 | 0.4186 |
| Cu | 0.0040 | 0.0238 | 0.0386 | 2.0887 | 0.0008 | |
| Zn | 0.0005 | 0.0030 | 0.0128 | 2.1447 | 0.0000 | |
| Fe | 0.0520 | 0.3095 | 0.1371 | 1.8746 | 0.1040 | |
| **Sampling Spot 4** | | | | | | |
| Cd | 0.0100 | 0.0040 | 0.0148 | 2.4452 | 0.0200 | |
| Mn | 0.0100 | 0.0040 | 0.0148 | 2.4452 | 0.0010 | |
| Co | 1.4700 | 0.5810 | 0.1485 | 2.1134 | 0.0074 | |
| Pb | 0.1000 | 0.0395 | 0.0506 | 2.3564 | 0.0200 | 0.3064 |
| Cu | 0.0400 | 0.0158 | 0.0309 | 2.4052 | 0.0004 | |
| Zn | 0.0000 | 0.0000 | 0.0000 | 2.4820 | 0.0000 | |
| Fe | 0.9000 | 0.3557 | 0.1431 | 2.1268 | 0.0900 | |

**Table 3(a):** Calculation of FHCI scores employing the proposed trigonometric fuzzy entropy (TFE) measure before amalgamation

**Table 3(b):** Calculation of FHCI score employing the proposed trigonometric fuzzy entropy (TFE) measure after amalgamation

| Heavy Metals | Construction Function | Fuzziness Values | Entropy Values | Assigned Weights | Relative Sub-Indices | FHCI Score |
|---|---|---|---|---|---|---|
| **Sampling Spot 1** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0000 | 2.4408 | 0.0400 | |
| Mn | 0.0000 | 0.0000 | 0.0000 | 2.4408 | 0.0100 | |
| Co | 0.7150 | 0.3950 | 0.1468 | 2.0824 | 0.0113 | |
| Pb | 0.9500 | 0.5249 | 0.1507 | 2.0729 | 0.7200 | **0.8343** |
| Cu | 0.0800 | 0.0442 | 0.0537 | 2.3096 | 0.0027 | |
| Zn | 0.0050 | 0.0028 | 0.0123 | 2.4108 | 0.0000 | |
| Fe | 0.0600 | 0.0331 | 0.0461 | 2.3284 | 0.1950 | |
| **Sampling Spot 2** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0000 | 2.7956 | 0.0200 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| Mn | 0.0000 | 0.0000 | 0.0000 | 2.7956 | 0.0050 | |
| Co | 0.5350 | 0.3877 | 0.1462 | 2.3869 | 0.0107 | |
| Pb | 0.7900 | 0.5725 | 0.1490 | 2.3791 | 0.5600 | 0.8237 |
| Cu | 0.0000 | 0.0000 | 0.0000 | 2.7956 | 0.0018 | |
| Zn | 0.0000 | 0.0036 | 0.0142 | 2.7560 | 0.0000 | |
| Fe | 0.0500 | 0.0362 | 0.0483 | 2.6606 | 0.1550 | |
| **Sampling Spot 3** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0000 | 4.2409 | 0.0200 | |
| Mn | 0.0000 | 0.0000 | 0.0000 | 4.2409 | 0.0020 | |
| Co | 1.0000 | 0.8264 | 0.1082 | 3.7821 | 0.0090 | |
| Pb | 0.2000 | 0.1653 | 0.1057 | 3.7925 | 0.0800 | 0.1895 |
| Cu | 0.0000 | 0.0000 | 0.0000 | 4.2409 | 0.0008 | |
| Zn | 0.0000 | 0.0083 | 0.0219 | 4.1480 | 0.0000 | |
| Fe | 0.0000 | 0.0000 | 0.0000 | 4.2409 | 0.1040 | |
| **Sampling Spot 4** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0000 | 8.9366 | 0.0200 | |
| Mn | 0.0000 | 0.0000 | 0.0000 | 8.9366 | 0.0010 | |
| Co | 1.0000 | 0.9524 | 0.0559 | 8.4368 | 0.0074 | |
| Pb | 0.0500 | 0.0476 | 0.0559 | 8.4368 | 0.0200 | 0.1266 |
| Cu | 0.0000 | 0.0000 | 0.0000 | 8.9366 | 0.0004 | |
| Zn | 0.0000 | 0.0000 | 0.0000 | 8.9366 | 0.0000 | |
| Fe | 0.0000 | 0.0000 | 0.0000 | 8.9366 | 0.0900 | |

**Table 4(a)** calculation of NHCI score employing the proposed single valued neutrosophic entropy (TNE) measure before amalgamation

| Heavy Metals | Construction Function | Fuzziness Values | Entropy Values | Assigned Weights | Relative Sub-Indices | NHCI Score |
|---|---|---|---|---|---|---|
| **Sampling Spot 1** | | | | | | |
| Cd | 0.0050 | 0.0024 | 0.0230 | 0.9753 | 0.0400 | |
| Mn | 0.0250 | 0.0122 | 0.0538 | 0.9446 | 0.0100 | |
| Co | 0.5625 | 0.2741 | 0.2627 | 0.7361 | 0.0113 | |
| Pb | 0.9000 | 0.4385 | 0.2991 | 0.6997 | 0.7200 | **0.7093** |
| Cu | 0.0675 | 0.0329 | 0.0917 | 0.9067 | 0.0027 | |
| Zn | 0.0050 | 0.0024 | 0.0230 | 0.9753 | 0.0000 | |
| Fe | 0.4875 | 0.2375 | 0.2483 | 0.7504 | 0.1950 | |
| **Sampling Spot 2** | | | | | | |
| Cd | 0.0003 | 0.0015 | 0.0178 | 1.0051 | 0.0200 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| Mn | 0.0017 | 0.0074 | 0.0414 | 0.9810 | 0.0050 | |
| Co | 0.0713 | 0.3175 | 0.2766 | 0.7403 | 0.0107 | |
| Pb | 0.0933 | 0.4154 | 0.2966 | 0.7198 | 0.5600 | 0.5575 |
| Cu | 0.0060 | 0.0267 | 0.0820 | 0.9394 | 0.0018 | |
| Zn | 0.0003 | 0.0015 | 0.0178 | 1.0051 | 0.0000 | |
| Fe | 0.0517 | 0.2300 | 0.2450 | 0.7726 | 0.1550 | |
| **Sampling Spot 3** | | | | | | |
| Cd | 0.0005 | 0.0030 | 0.0256 | 1.0583 | 0.0200 | |
| Mn | 0.0010 | 0.0060 | 0.0368 | 1.0462 | 0.0020 | |
| Co | 0.0900 | 0.5357 | **0.3010** | 0.7592 | 0.0090 | 0.1841 |
| Pb | 0.0200 | 0.1190 | 0.1803 | 0.8903 | 0.0800 | |
| Cu | 0.0040 | 0.0238 | 0.0771 | 1.0024 | 0.0008 | |
| Zn | 0.0005 | 0.0030 | 0.0256 | 1.0583 | 0.0000 | |
| Fe | 0.0520 | 0.3095 | 0.2743 | 0.7882 | 0.1040 | |
| **Sampling Spot 4** | | | | | | |
| Cd | 0.0100 | 0.0040 | 0.0297 | 1.2043 | 0.0200 | |
| Mn | 0.0100 | 0.0040 | 0.0297 | 1.2043 | 0.0010 | |
| Co | 1.4700 | 0.5810 | 0.2970 | 0.8725 | 0.0074 | |
| Pb | 0.1000 | 0.0395 | 0.1012 | 1.1155 | 0.0200 | 0.1342 |
| Cu | 0.0400 | 0.0158 | 0.0619 | 1.1643 | 0.0004 | |
| Zn | 0.0000 | 0.0000 | 0.0000 | 1.2412 | 0.0000 | |
| Fe | 0.9000 | 0.3557 | 0.2862 | 0.8859 | 0.0900 | |

**Table4(b):** Calculation of NHCI score employing the proposed single valued neutrosophic

| Heavy Metals | Construction Function | Fuzziness Values | Entropy Values | Assigned Weights | Relative Sub-Indices | NHCI Score |
|---|---|---|---|---|---|---|
| **Sampling Spot 1** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0073 | 1.1407 | 0.0400 | |
| Mn | 0.0000 | 0.0000 | 0.0073 | 1.1407 | 0.0100 | |
| Co | 0.7150 | 0.3950 | 0.3009 | 0.8033 | 0.0113 | |
| Pb | 0.9500 | 0.5249 | 0.3088 | 0.7943 | 0.7200 | **0.3216** |
| Cu | 0.0800 | 0.0442 | 0.1148 | 1.0172 | 0.0027 | |
| Zn | 0.0050 | 0.0028 | 0.0319 | 1.1124 | 0.0000 | |
| Fe | 0.0600 | 0.0331 | 0.0994 | 1.0348 | 0.1950 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| **Sampling Spot 2** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0073 | 1.2955 | 0.0200 | |
| Mn | 0.0000 | 0.0000 | 0.0073 | 1.2955 | 0.0050 | |
| Co | 0.5350 | 0.3877 | 0.2997 | 0.9138 | 0.0107 | |
| Pb | 0.7900 | 0.5725 | 0.3053 | 0.9066 | 0.5600 | 0.3031 |
| Cu | 0.0000 | 0.0000 | 0.0073 | 1.2955 | 0.0018 | |
| Zn | 0.0000 | 0.0036 | 0.0356 | 1.2585 | 0.0000 | |
| Fe | 0.0500 | 0.0362 | 0.1039 | 1.1694 | 0.1550 | |
| **Sampling Spot 3** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0073 | 1.9000 | 0.0200 | |
| Mn | 0.0000 | 0.0000 | 0.0073 | 1.9000 | 0.0020 | |
| Co | 1.0000 | 0.8264 | 0.2236 | 1.4859 | 0.0090 | |
| Pb | 0.2000 | 0.1653 | 0.2187 | 1.4952 | 0.0800 | 0.0673 |
| Cu | 0.0000 | 0.0000 | 0.0073 | 1.9000 | 0.0008 | |
| Zn | 0.0000 | 0.0083 | 0.0511 | 1.8162 | 0.0000 | |
| Fe | 0.0000 | 0.0000 | 0.0073 | 1.9000 | 0.1040 | |
| **Sampling Spot 4** | | | | | | |
| Cd | 0.0000 | 0.0000 | 0.0073 | 3.6152 | 0.0200 | |
| Mn | 0.0000 | 0.0000 | 0.0073 | 3.6152 | 0.0010 | |
| Co | 1.0000 | 0.9524 | 0.1191 | 3.2078 | 0.0074 | |
| Pb | 0.0500 | 0.0476 | 0.1191 | 3.2078 | 0.0200 | 0.0481 |
| Cu | 0.0000 | 0.0000 | 0.0073 | 3.6152 | 0.0004 | |
| Zn | 0.0000 | 0.0000 | 0.0073 | 3.6152 | 0.0000 | |
| Fe | 0.0000 | 0.0000 | 0.0073 | 3.6152 | 0.0900 | |

entropy (TNE) measure after amalgamation

It is informative to note that while calculating EHCI score, as depcited in **Table.2 (a, b)**, the values $E_1^{(0)}, E_2^{(0)}$ at sampling spots $S_1, S_2, S_3, S_4$; $E_5^{(0)}$ at $S_2, S_3, S_4$; $E_5^{(0)}$ at $S_3, S_4$ and $E_7^{(0)}$ at $S_4$ are based on the fancy assumption $0 \times \log 0 = 0$. This indicates major conflicts and lack of macroscopic view in the quality analysis of heavy metal contamination in river water samples. However, our trigonometric fuzzy entropy (TFE) and single valued neutrosophic entropy (TNE) measures have been proven capable for providing a superior contamination evaluation methodology.

**Acknowledgment:** We extend our sincere thanks to Guru Jambeshwar University, Hisar, Haryana, for providing excellent laboratory facilities for heavy metal estimation.

**Funding:** This project was granted no funding from anywhere.

**Conflict of Interest:** Authors declare no conflict of interest

**CONCLUSIONS**

To assess the impact of concentration of heavy metals (cadmium, manganese, cobalt, lead, copper, zinc and iron) and to identify the most contaminated sampling spot responsible for heavy metal contamination in river water samples, a novel trigonometric fuzzy entropy as well as single valued neutrosophic entropy measures are established and thereafter deployed to construct fuzzy and neutrosophic entropy weighted heavy metal contamination indices (FHCI and NHCI). The novelty of our contaminated sampling spot identification methodology lies in the fact that our heavy metal contamination indices are superior and capable in classifying the most contaminated sampling spot, whereas the existing Deluca-Termini fuzzy entropy weighted heavy metal contamination index (EHCI) exhibits assumption-based results which can affect the identification accuracy of the selected contaminated sampling spot with respect to each heavy metal concentration in river water samples. It is concluded that

- The concentration of each heavy metal, which was within the permissible limits before amalgamation, dwindled gradually (owing to negative change in TFE and TNE values), after amalgamating pharmaceutical effluents into the river water samples.
- The concentration of cadmium and manganese is found to be negligible. The possible reasons could be the use of physico-chemical processes which further diluted the river water after amalgamation.
- The sampling spot $S_1$ was found to be the most contaminated, owing to its maximum EHCI (before amalgamation: 0.3623, after amalgamation: 0.1673) score, FHCI (before amalgamation: 1.6865, after amalgamation: 0.8343) score and NHCI (before amalgamation: 0.7093, after amalgamation: 0.3216) scores.

Moreover, the findings of the underlying study can be utilized for controlling the spread of water borne deceases, reducing the risk of water and soil pollution, increasing the ecological and aesthetical qualities of lakes and rivers, etc.

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Received: Nov 29, 2021. Accepted: Feb 2, 2022

Neutrosophic Sets and Systems, Vol. 48, 2022
University of New Mexico

# Interval-Valued Intuitionistic Hypersoft Sets and Their Algorithmic Approach in Multi-criteria Decision Making

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**Abstract:** Hypersoft set(HSS) is one of the recent topics developed by Smarandache in 2018 to be presented by replacing the single attribute function, used in soft set(SS), with a multi attribute function i.e a function can be further bifurcated using HSS. So, HSS provides more options to the decision-makers than SS to make precise and valid decisions. Also, the interval-valued intuitionistic fuzzy set (IVIFS) is developed to counter a kind of uncertain complex decision-making problem where the membership and non-membership values of a certain element are not precise. Basically, the IVIFS is the generalization of a fuzzy set(FS), interval-valued fuzzy set(IVFS), and intuitionistic fuzzy set(IFS). Therefore, the mixture of HSS and IVIFS will surely give a new field of study for the decision-makers to enhance their critical thinking ability to make a conclusive decisions. The main aim of the paper is to present the notion of interval-valued intuitionistic fuzzy hypersoft sets (IVIFHSSs) and study some fundamental operations on them which are worthy in critical decision making. The IVIFHSSs can be viewed as a hybrid structure that can be formed by combining interval-valued intuitionistic fuzzy sets (IVIFSs) and hypersoft sets (HSSs). On the idea of IVIFHSSs and their kinds, different operators such as complement, union, intersection, OR, AND etc have been introduced, and by using these operators we can encounter real-life-based problems that contain incomplete and parameterized information or data. A new algorithm based on IVIFHSSs has been initiated. Finally, a numerical example is employed to check the reliability and validity of the algorithm. In the future, we use the proposed concept practically in medical diagnosis, personality selection, weather forecasting, data clustering, parameter reduction, decision making, etc.
**Keywords:** Interval-valued intuitionistic fuzzy set; Hypersoft set; Interval-valued intuitionistic hypersoft set; Decision making.

## 1. Introduction

In most real-life problems, there is an existence of a considerable amount of ambiguity and it is due to the uncertainty involved in the information. That is uncertainty arises when the information is not precise and accurate. The classical mathematical tools can't measure such kinds of data. So, there is a serious need to introduce a powerful tool that is capable to measure uncertainty without any fail. Finally, the invention of the fuzzy set(FS) by Zadeh[1] in 1965 helped us to deal with uncertainty in a structured manner. In FS, each element of the universe has a membership degree $\mu_A(x)\in[0,1]$. After the introduction of FS, it has been developed rapidly and it has an extensive application in various fields of knowledge. Some of the recent works and applications associated with FS are discussed in the literature given in [2-5]. The FS theory captures the attention of the researchers over the decades and they are motivated a lot which gives rise to other mathematical tools namely rough set[6], fuzzy logic[7], vague set[8], etc. We know that hesitancy is an integral part of human thinking and FS theory does not measure hesitancy due to its inbuilt difficulties. Such difficulties were removed with the creation of an intuitionistic fuzzy set( IFS) by Atanassov[9]. The IFS is formed by adding a non-membership degree to the FS in such a manner that the sum of the membership and the non-membership degree can't exceed one. Decision-making is a scientific approach to select the best alternative among the set of attributes and the approach of the decision-making process by the decision-makers depends upon the nature of the fuzzy environment. This leads to the introduction of the interval-valued fuzzy set(IVFS)[10], interval-valued intuitionistic fuzzy set(IVIFS)[11], hesitant fuzzy set(HFS)[12], picture fuzzy set(PFS)[13], Pythagorean fuzzy set(PFS)[14], etc.

To work under fuzzy environment, there is always a challenge to construct membership function and there exist some issues in real-world that can't be solved with an aid of membership function because recently we are encountered the kind of data that are parametric and there is an inadequacy in FSs and their variants to parameterize data. To overcome such difficulties, in 1999, Molodtsov[15] introduced the soft set(SS) theory. There is a lot of instances in a real-life situation where SS theory proved to be more functional than FS theory to describe uncertain parametric information without any effort. The SS theory removes the difficulty of constructing membership function in each event. So, we claim that SS is a more functional general framework than FS to model uncertainty without assigning membership function. Later on, Maji et al.[16] presented several assertions on SS, Cagman et al.[17] used SS in decision-making, Ali et al.[18] introduced some new operations on SS etc. An amalgamation of two or more concepts provides more information to the decision-makers to make their decisions more vulnerable. Because of this, some new hybrid structures such as fuzzy soft sets (FSSs), intuitionistic fuzzy soft sets(IFSSs),interval-valued fuzzy soft sets(IVFSSs), interval-valued intuitionistic fuzzy soft sets(IVIFSSs),etc. are introduced. Some of the works related to these are the following: Agarwal et al.[19] introduced generalized IFSSs and their applications in decision-making, FSS theory and its application given in[20], Cagman et al.[21] applied IFSS in decision-making, Chetia et al.[22] presented an application based on IVFSS, Jiang et al.[23] discussed IVIFSSs and their related properties, entropy on IFSSs and IVFSSs are proposed in [24], Ma et al.[25] introduced the parameter reduction of IVFSSs and its related algorithms, Majumder et al.[26] presented generalized FSSs, Maji et al.[27] initiated more on IFSSs, algorithms for IVFSSs in emergency decision-making shown in [28], a complete model for evaluation system based on IVFSS given in [29], Roy et al.[30] introduced an FSS theoretic approach to decision-making problems, Tripathy et al.[31] given a new approach to IVFSSs and its application in decision-making, Yang et al.[32] studied combination of IVFS and SS, a novel approach to IVIFSS is initiated by Zhang et al. in [33].

In 2018, Smarandache[34] has extended SS to the hypersoft set(HSS) and pilthogenic hypersoft set(PHSS). The HSS is introduced by transforming the single attribute function $F$ to a multi attribute function $F_1 \times F_2 \times ....... \times F_n$ , where each attribute has some preference values such that $F_i \cap F_j = \varnothing, for\ i \neq j$ . However, HSS provides more options to the decision-makers than SS to make their decisions more constructive and meaningful. Some recent works based on HSSs are given in [35-40]. In HSS, the belongingness of an element is denoted by 1 and non- belongingness is denoted by 0 i.e. values are crisp. To deal with uncertainty under the hypersoft environment, a fuzzy hypersoft set (FHSS)[41,42] is introduced, and to handle hesitancy under the hypersoft environment, an intuitionistic fuzzy hypersoft set(IFHSS)[43] is introduced. Some more recent works based on HSSs are given in [44-51].

In 2010, Jiang et al.[23] introduced IVIFSSs and their properties and in 2021, Yolcu et al.[43] introduced IFHSS. In IVIFSS, there is only one attribute function, but there is some urgency to solve certain types of problems where there is more than one attribute or an attribute is further bifurcated. To address such issues there is a demand to introduce IVIFHSSs. On the other hand, in IFHSS, the membership and non-membership values are precise, but in real-life decision-making problems, we find the existence of the environment where the membership and non-membership degrees are uncertain i.e they are subjective. This situation also IVIFHSSs solve the purpose which cannot be handled by IFHSS. Therefore, there are two aspects of introducing IVIFHSS in the proposed study. Moreover, the following diagrammatic illustration will give an insight into the proposed study:

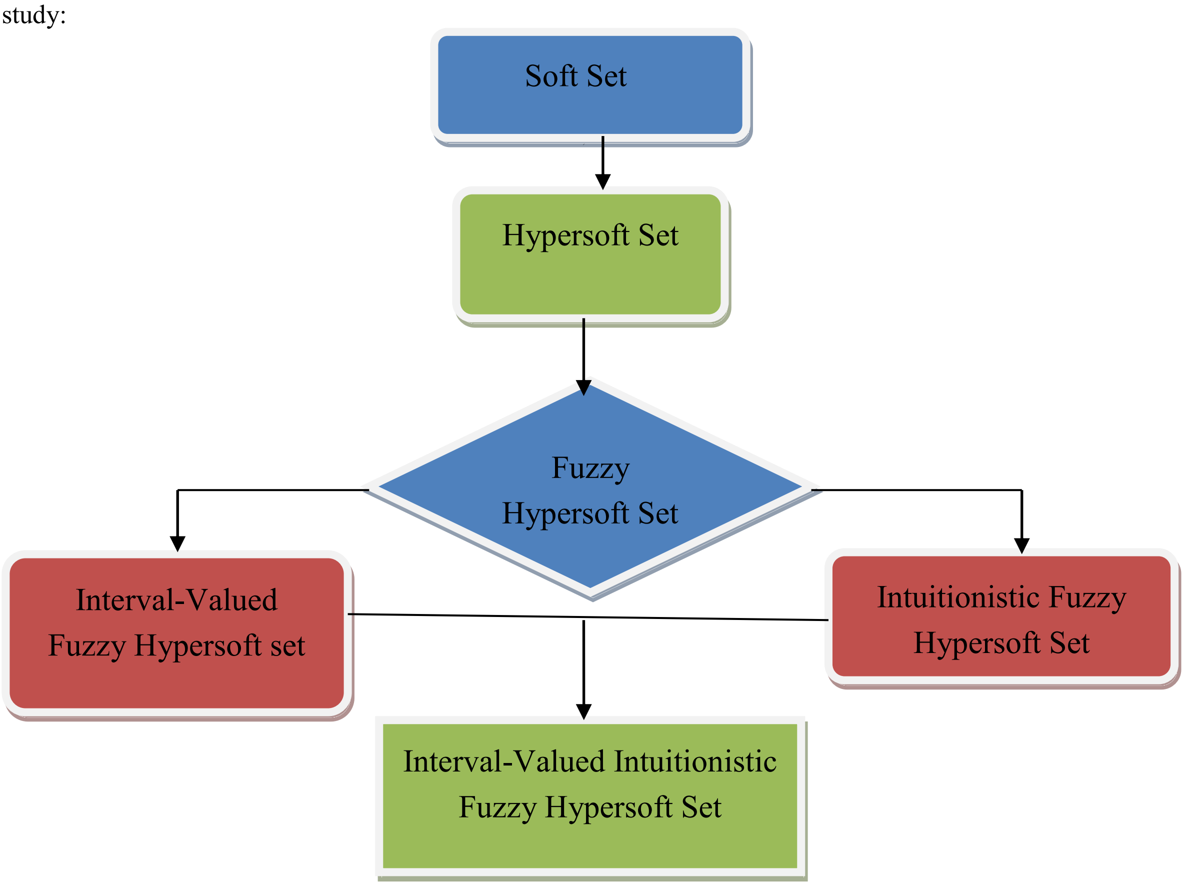


**Fig 1** Diagrammatic representation of the soft set and its generalization for the proposed study

There is no research work yet to be done on IVIFHSS. This gives us the motivation to present the paper.
The rest of the paper is organized as follows:

Section 2 provides an overview of the earlier research works that are useful for the present study. In section 3, we establish the IVIFHSSs and obtain some properties and important results on them. In section 4, an algorithm is being constructed for multi-criteria decision-making problems using the notion of membership score function, non-membership score function, and the total score function under the IVIFHSS environment. Conclusion and future work are added in section 5.

**2. Literature Review**

In this section, we give some basic definitions and results that are useful for the rest of the paper.

**Definition 2.1 [11]** An interval-valued intuitionistic fuzzy set(IVIFS) $\mathrm{H}$ over the universe of discourse $\mathrm{X}$ is an object of the form $\{\langle x, \mu_H(x), \gamma_H(x)\rangle : x \in X\}$ where, $\mu_H$ , $\gamma_H : X \to Int([0,1])$ , where $Int([0,1])$ stands for the set of all close subintervals of $[0,1]$ satisfying the following condition: $\forall x \in X, \sup(\mu_H(x)) + \sup(\gamma_H(x)) \le 1$ . Further, $\mu_H(x)$ and $\gamma_H(x)$ can be written as $\mu_H(x) = [\mu^l{}_H(x), \mu^u{}_H(x)]$ and $\gamma_H(x) = [\gamma^l{}_H(x), \gamma^u{}_H(x)]$ . The class of all IVIFS is denoted by $IVIFS(X)$.

**Definition 2.2 [15]** Let $U$ be an initial universe and $E$ be a set of parameters. Also, $P(U)$ denotes the power set of $U$ and $A \subseteq E$ . Then the pair $(F, A)$ where $F : A \to P(U)$ is called the soft set over $U$ . A SS is a parameterized family of subsets over the universe $U$ .

**Definition 2.3 [23]** Let $U$ be the universe of discourse and $E$ be the set of parameters and $IVIFS(U)$ denote the set of all IVIFSs over $U$ . Also, let $A \subseteq E$ . Then the pair $(F, A)$ is called an IVIFSS over $U$ where $F : A \to IVIFS(U)$ .

**Example 2.3.1** Let $U = \{c_1, c_2, c_3, c_4, c_5\}$ be a set of cars under consideration and $E = \{e_1 = size, e_2 = color, e_3 = fuel\ efficiency, e_4 = \exp ensive, e_5 = style, e_6 = comfortable\}$ be a set of parameters and $A = \{e_1, e_2, e_3, e_6\} \subseteq E$. Under the advice of a decision-maker, Mr. X wants to purchase a car. The IVIFSS is denoted by $(F, A)$ which describes the "attractiveness of the cars" to the decision-maker. Then, the tabular representation of $(F, A)$ is given in Table 1.

| $(F,A)$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
|---|---|---|---|---|---|
| $e_1$ | $\langle[0.4,0.6],[0.2,0.3]\rangle$ | $\langle[0.5,0.6],[0.1,0.2]\rangle$ | $\langle[0.3,0.5],[0.2,0.5]\rangle$ | $\langle[0.3,0.4],[0.4,0.5]\rangle$ | $\langle[0.2,0.4],[0.3,0.5]\rangle$ |
| $e_2$ | $\langle[0.5,0.6],[0.2,0.3]\rangle$ | $\langle[0.2,0.3],[0.3,0.5]\rangle$ | $\langle[0.4,0.5],[0.3,0.4]\rangle$ | $\langle[0.6,0.7],[0.1,0.2]\rangle$ | $\langle[0.1,0.3],[0.6,0.7]\rangle$ |
| $e_3$ | $\langle[0.2,0.3],[0.4,0.5]\rangle$ | $\langle[0.2,0.3],[0.4,0.5]\rangle$ | $\langle[0.1,0.3],[0.5,0.6]\rangle$ | $\langle[0.3,0.6],[0.3,0.4]\rangle$ | $\langle[0.5,0.6],[0.3,0.4]\rangle$ |
| $e_6$ | $\langle[0.4,0.5],[0.3,0.4]\rangle$ | $\langle[0.2,0.4],[0.3,0.5]\rangle$ | $\langle[0.7,0.8],[0.1,0.2]\rangle$ | $\langle[0.5,0.7],[0.1,0.2]\rangle$ | $\langle[0.4,0.5],[0.2,0.3]\rangle$ |

**Table1.** Tabular representation of IVIFSS $(F,A)$

The above representation is very useful for storage such big data in a computer as it consumes less memory and it is handy for numerical calculation which solves the purpose of the decision maker to make a precise decision.

**Definition 2.4 [34, 39]** Let $S$ denotes the set of the universe and $P(S)$ is the power set of $S$. Let $e_1, e_2, ...., e_n$ be n distinct attributes, where $n \geq 1$, whose corresponding attribute values are respectively the sets $E_1, E_2, ...., E_n$ with $E_i \cap E_j = \varnothing, i \neq j$ and $i, j \in N$. Then the hypersoft set(HSS) is denoted by $(\Omega, E_1 \times E_2 \times .... \times E_n)$ where $\Omega : E_1 \times E_2 \times .... \times E_n \to P(S)$. For simplicity, we represent the HSS by $(\Omega, E)$ where $E = E_1 \times E_2 \times .... \times E_n$. Thus, in HSS, the attribute function can be further split until it is not suitable for the decision-maker in a certain environment. Therefore, HSS qualitatively enhanced the decision-making process.

**Example 2.4.1** Let $S = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ be a set of journals and the set of attributes are $E_1$ =citation style, $E_2$ =indexing and abstracting, $E_3$ =article processing charge(APC), $E_4$ = impact factor(IF) and their respective attribute values are given by:

$E_1$ =citation style= $\{APA(e_1), MLA(e_2), Har\operatorname{var} d(e_3), Bibtex(e_4)\}$ , $E_2$ =indexing and abstracting= $\{SCI(e_5), Scopus(e_6), Web\ of\ Science(e_7), Taylor\ and\ Francis(e_8)\}$ , $E_3$ =article processing charge= $\{Nil(e_9)\}$ , $E_4$ = impact factor= $\{high(e_{10}), low(e_{11}), medium(e_{12})\}$.

Let $A_i \subseteq E_i, where\ i = 1,2,3,4$ and suppose $A_1 = \{e_1, e_4\}$ , $A_2 = \{e_5, e_6, e_7\}$ , $A_3 = \{e_9\}$ and $A_3 = \{e_{10}, e_{12}\}$. Then the HSS $(\Omega, A_1 \times A_2 \times A_3 \times A_4)$ defined as

$$(\Omega, A_1 \times A_2 \times A_3 \times A_4) =$$

$$\left\{\begin{array}{l}\langle(e_1,e_5,e_9,e_{10}),\{\varsigma_1,\varsigma_3\}\rangle,\langle(e_1,e_5,e_9,e_{12}),\{\varsigma_1,\varsigma_2\}\rangle,\langle(e_1,e_6,e_9,e_{10}),\{\varsigma_1,\varsigma_3,\varsigma_4\}\rangle,\\ \langle(e_1,e_6,e_9,e_{12}),\{\varsigma_2,\varsigma_4\}\rangle,\langle(e_1,e_7,e_9,e_{10}),\{\varsigma_1,\varsigma_2\}\rangle,\langle(e_1,e_7,e_9,e_{12}),\{\varsigma_3,\varsigma_4\}\rangle,\\ \langle(e_4,e_5,e_9,e_{10}),\{\varsigma_1,\varsigma_3,\varsigma_4\}\rangle,\langle(e_4,e_5,e_9,e_{12}),\{\varsigma_2,\varsigma_4\}\rangle,\langle(e_4,e_6,e_9,e_{10}),\{\varsigma_1,\varsigma_2,\varsigma_3\}\rangle,\\ \langle(e_4,e_6,e_9,e_{12}),\{\varsigma_3,\varsigma_4\}\rangle,\langle(e_4,e_7,e_9,e_{10}),\{\varsigma_2,\varsigma_3\}\rangle,\langle(e_4,e_7,e_9,e_{12}),\{\varsigma_3,\varsigma_4\}\rangle\end{array}\right\}$$

Therefore, the HSS $(\Omega, A_1 \times A_2 \times A_3 \times A_4)$ is not a normal set, it is a multiple parameterized family of sets over $S$. It is a new scientific approach of representing bifurcated parametric representation and it's a very useful model that provides sufficient information to the decision-maker to make their decisions elegantly. It is a more powerful and sophisticated tool than SS to deal with a wide range of problems related to various fields.

**Definition 2.5 [41, 42]** Let $F^S$ be the set of all fuzzy subsets of the universe set $S$ and let $E_1 \times E_2 \times .... \times E_n$ be the set of parameters where $E_i \cap E_j = \emptyset, i \neq j$ ,and $i, j \in N$ . For every $\varepsilon \in E_1 \times E_2 \times .... \times E_n$ , the pair $(\Omega, E_1 \times E_2 \times ...... \times E_n)$ is called the fuzzy hypersoft set(FHSS) over $S$ , where $\Omega : E_1 \times E_2 \times .... \times E_n \to F^S$ and the FHSS defined as

$$(\Omega, E_1 \times E_2 \times ...... \times E_n) = \left\{ \left\langle \varepsilon, \left( \frac{x}{(\mu_{\Omega(\varepsilon)}(x))} \right) \right\rangle : \varepsilon \in E_1 \times E_2 \times ...... \times E_n\ and\ x \in S \right\},$$ where

$\mu_{\Omega(\varepsilon)}(x)$ denotes the membership value such that $\mu_{\Omega(\varepsilon)}(x) \in [0,1]$.

**Example 2.5.1** If we take the same sets of attributes proposed in **example 2.4.1**, then the representation of FHSS $(\Omega, A_1 \times A_2 \times A_3 \times A_4)$ in the following form:

$\left(\Omega, A_1 \times A_2 \times A_3 \times A_4\right)$ =

$$\left\{\begin{array}{l}\left\langle\left(e_1, e_5, e_9, e_{10}\right),\left\{\frac{\varsigma_1}{0.4}, \frac{\varsigma_3}{0.6}\right\}\right\rangle,\left\langle\left(e_1, e_5, e_9, e_{12}\right),\left\{\frac{\varsigma_1}{0.5}, \frac{\varsigma_2}{0.8}\right\}\right\rangle,\left\langle\left(e_1, e_6, e_9, e_{10}\right),\left\{\frac{\varsigma_1}{0.6}, \frac{\varsigma_3}{0.7}, \frac{\varsigma_4}{0.8}\right\}\right\rangle, \\ \left\langle\left(e_1, e_6, e_9, e_{12}\right),\left\{\frac{\varsigma_2}{0.2}, \frac{\varsigma_4}{0.4}\right\}\right\rangle,\left\langle\left(e_1, e_7, e_9, e_{10}\right),\left\{\frac{\varsigma_1}{0.8}, \frac{\varsigma_2}{0.3}\right\}\right\rangle,\left\langle\left(e_1, e_7, e_9, e_{12}\right),\left\{\frac{\varsigma_3}{0.3}, \frac{\varsigma_4}{0.4}\right\}\right\rangle, \\ \left\langle\left(e_4, e_5, e_9, e_{10}\right),\left\{\frac{\varsigma_1}{0.4}, \frac{\varsigma_3}{0.3}, \frac{\varsigma_4}{0.6}\right\}\right\rangle,\left\langle\left(e_4, e_5, e_9, e_{12}\right),\left\{\frac{\varsigma_2}{0.8}, \frac{\varsigma_4}{0.9}\right\}\right\rangle,\left\langle\left(e_4, e_6, e_9, e_{10}\right),\left\{\frac{\varsigma_1}{0.3}, \frac{\varsigma_2}{0.5}, \frac{\varsigma_3}{0.6}\right\}\right\rangle, \\ \left\langle\left(e_4, e_6, e_9, e_{12}\right),\left\{\frac{\varsigma_3}{0.5}, \frac{\varsigma_4}{0.6}\right\}\right\rangle,\left\langle\left(e_4, e_7, e_9, e_{10}\right),\left\{\frac{\varsigma_2}{0.4}, \frac{\varsigma_3}{0.5}\right\}\right\rangle,\left\langle\left(e_4, e_7, e_9, e_{12}\right),\left\{\frac{\varsigma_3}{0.4}, \frac{\varsigma_4}{0.8}\right\}\right\rangle\end{array}\right\}$$

By FHSS we represent the multi-parameterized family of uncertain data.

**Definition 2.6 [43]** Let $IF^S$ be the set of all intuitionistic fuzzy subsets of the universe set $S$ and let $E_1 \times E_2 \times .... \times E_n$ be the set of parameters where $E_i \cap E_j = \emptyset, i \neq j$ ,and $i, j \in N$ . For every $\varepsilon \in E_1 \times E_2 \times .... \times E_n$, the pair $\left(\Omega, E_1 \times E_2 \times ...... \times E_n\right)$ is called the intuitionistic fuzzy hypersoft set(IFHSS) over $S$, where $\Omega: E_1 \times E_2 \times .... \times E_n \rightarrow IF^S$ and the FHSS defined as,

$$\left(\Omega, E_1 \times E_2 \times ...... \times E_n\right)=\left\{\left\langle\varepsilon,\left(\frac{x}{\left\langle\mu_{\Omega(\varepsilon)}(x), \gamma_{\Omega(\varepsilon)}(x)\right\rangle}\right)\right\rangle: \varepsilon \in E_1 \times E_2 \times ...... \times E_n \text{ and } x \in S\right\}$$ ,where $\mu_{\Omega(\varepsilon)}(x)$ and $\gamma_{\Omega(\varepsilon)}(x)$ respectively denotes the membership and the non-membership values where $\mu_{\Omega(\varepsilon)}(x), \gamma_{\Omega(\varepsilon)}(x) \in [0,1]$ such that $0 \leq \mu_{\Omega(\varepsilon)}(x) + \gamma_{\Omega(\varepsilon)}(x) \leq 1$ . The hesitancy is determined by $\Pi_{\Omega(\varepsilon)}(x) = 1 - \mu_{\Omega(\varepsilon)}(x) - \gamma_{\Omega(\varepsilon)}(x)$.

**Example 2.6.1** Revisiting **example 2.4.1**, we address the IFHSS $\left(\Omega, A_1 \times A_2 \times A_3 \times A_4\right)$ in the following manner:

$$\left(\Omega, A_1 \times A_2 \times A_3 \times A_4\right) =$$

$$\left\{\begin{array}{l}
\left\langle\left(e_1, e_5, e_9, e_{10}\right),\left\{\frac{\varsigma_1}{\langle 0.3,0.6\rangle}, \frac{\varsigma_3}{\langle 0.2,0.5\rangle}\right\}\right\rangle,\left\langle\left(e_1, e_5, e_9, e_{12}\right),\left\{\frac{\varsigma_1}{\langle 0.4,0.6\rangle}, \frac{\varsigma_2}{\langle 0.4,0.5\rangle}\right\}\right\rangle, \\
\left\langle\left(e_1, e_6, e_9, e_{10}\right),\left\{\frac{\varsigma_1}{\langle 0.2,0.3\rangle}, \frac{\varsigma_3}{\langle 0.4,0.5\rangle}, \frac{\varsigma_4}{\langle 0.6,0.2\rangle}\right\}\right\rangle,\left\langle\left(e_1, e_6, e_9, e_{12}\right),\left\{\frac{\varsigma_2}{\langle 0.2,0.4\rangle}, \frac{\varsigma_4}{\langle 0.4,0.7\rangle}\right\}\right\rangle, \\
\left\langle\left(e_1, e_7, e_9, e_{10}\right),\left\{\frac{\varsigma_1}{\langle 0.7,0.2\rangle}, \frac{\varsigma_2}{\langle 0.1,0.4\rangle}\right\}\right\rangle,\left\langle\left(e_1, e_7, e_9, e_{12}\right),\left\{\frac{\varsigma_3}{\langle 0.3,0.7\rangle}, \frac{\varsigma_4}{\langle 0.4,0.5\rangle}\right\}\right\rangle, \\
\left\langle\left(e_4, e_5, e_9, e_{10}\right),\left\{\frac{\varsigma_1}{\langle 0.4,0.5\rangle}, \frac{\varsigma_3}{\langle 0.8,0.1\rangle}, \frac{\varsigma_4}{\langle 0.6,0.4\rangle}\right\}\right\rangle,\left\langle\left(e_4, e_5, e_9, e_{12}\right),\left\{\frac{\varsigma_2}{\langle 0.2,0.3\rangle}, \frac{\varsigma_4}{\langle 0.5,0.4\rangle}\right\}\right\rangle, \\
\left\langle\left(e_4, e_6, e_9, e_{10}\right),\left\{\frac{\varsigma_1}{\langle 0.4,0.5\rangle}, \frac{\varsigma_2}{\langle 0.4,0.8\rangle}, \frac{\varsigma_3}{\langle 0.2,0.8\rangle}\right\}\right\rangle,\left\langle\left(e_4, e_6, e_9, e_{12}\right),\left\{\frac{\varsigma_3}{\langle 0.1,0.5\rangle}, \frac{\varsigma_4}{\langle 0.6,0.3\rangle}\right\}\right\rangle, \\
\left\langle\left(e_4, e_7, e_9, e_{10}\right),\left\{\frac{\varsigma_2}{\langle 0.2,0.7\rangle}, \frac{\varsigma_3}{\langle 0.5,0.4\rangle}\right\}\right\rangle,\left\langle\left(e_4, e_7, e_9, e_{12}\right),\left\{\frac{\varsigma_3}{\langle 0.5,0.3\rangle}, \frac{\varsigma_4}{\langle 0.6,0.3\rangle}\right\}\right\rangle
\end{array}\right\}$$

By IFHSS, we represent the multi-parameterized family of hesitant data.

## 3. Interval-Valued Intuitionistic Fuzzy Hypersoft Sets (IVIFHSSs)

In this section, first, we give the basic definition of IVIFHSS and its associated sets with real-life-based examples. Then, we defined different operators such as union, intersection, complement, OR, AND,etc on IVIFHSSs and investigated their properties.

**Definition 3.1** Let $X$ be the universal set and $IVIF^X$ denote the collection of all interval-valued intuitionistic fuzzy (IVIF) subsets of X. Again, let $C_1, C_2, ...., C_n$ for $n \geq 1$ be $n$ well-defined attributes, whose corresponding preferences are denoted by the sets $P_1, P_2, ...., P_n$ with $P_i \cap P_j = \varnothing, i \neq j$ and $i, j \in N$ . Let $P_i$ be non-empty subsets of $C_i$ for each $i \in N$ . Then the IVIFHSS is denoted as the pair $\left(Y, P_1 \times P_2 \times .... \times P_n\right)$ where $Y : P_1 \times P_2 \times .... \times P_n \rightarrow IVIF^X$ and defined as

$$Y\left(P_1 \times P_2 \times .... \times P_n\right) = \left\{\left\langle \eta, \frac{x}{\left\langle \varphi_{Y(\eta)}(x), \psi_{Y(\eta)}(x)\right\rangle}\right\rangle : x \in X, \eta \in P_1 \times P_2 \times .... \times P_n \subseteq C_1 \times C_2 \times .... \times C_n\right\}$$

, where

$\varphi_{Y(\eta)}(x)=\left[\varphi^{l}{}_{Y(\eta)}(x),\varphi^{u}{}_{Y(\eta)}(x)\right]$, and $\psi_{Y(\eta)}(x)=\left[\psi^{l}{}_{Y(\eta)}(x),\psi^{u}{}_{Y(\eta)}(x)\right]$ are the membership and non-membership intervals such that $0\le\varphi^{u}{}_{Y(\eta)}(x)+\psi^{u}{}_{Y(\eta)}(x)\le 1$ and $\varphi_{Y(\eta)},\psi_{Y(\eta)}:X\to D([0,1])$. Here $D([0,1])$ denotes the set of all closed subintervals of $[0,1]$.

For simplification, we write the symbol $\Sigma$ for $C_1\times C_2\times....\times C_n$, $\Gamma$ for $P_1\times P_2\times....\times P_n$ and $\eta$ for any element of the set $\Gamma$.

Thus, $(Y,P_1\times P_2\times....\times P_n)$ represent a multi-parameterized family whose universe of discourse is $IVIF^X$.

**Example 3.1.1** Let $X=\{x_1,x_2,x_3\}$ be the set of cars under consideration and the sets of attribute are

$$C_1=Quality=\{good(c_1),\,very\ good(c_2),\,excellent(c_3)\},$$

$$C_2=\mathrm{Re}\,liability=\{satisfactory(c_4)\},$$

$$C_3=Color=\{\mathrm{red}(c_5),\mathrm{black}(c_6),blue(c_7),yellow(c_8)\},C_4=Fuel\ Efficiency=\{economical(c_9),\,high(c_{10})\}$$

Suppose, $P_i$ and $Q_i$ are subsets of $C_i\ (i=1,2,3,4)$ such that

$$P_1=\{c_2,c_3\},P_2=\{c_4\},P_3=\{c_5,c_7\},P_4=\{c_9\},\text{ and }Q_1=\{c_3\},Q_2=\{c_4\},Q_3=\{c_5,c_6\},Q_4=\{c_9\}$$

Then the IVIFHSSs with respect to $P_i$ and $Q_i$ denoted as $(Y,\Gamma_1)$, and $(Z,\Gamma_2)$ respectively and defined in the following:

$$(Y,\Gamma_1)=\left\{\begin{array}{l}\left\langle(c_2,c_4,c_5,c_9),\left\{\frac{x_1}{\langle[0.3,0.4],[0.5,0.6]\rangle},\frac{x_3}{\langle[0.5,0.7],[0.1,0.2]\rangle}\right\}\right\rangle,\\ \left\langle(c_2,c_4,c_7,c_9),\left\{\frac{x_1}{\langle[0.4,0.6],[0.3,0.4]\rangle},\frac{x_2}{\langle[0.7,0.8],[0.1,0.2]\rangle}\right\}\right\rangle\\ \left\langle(c_3,c_4,c_5,c_9),\left\{\frac{x_2}{\langle[0.45,0.55],[0.23,0.35]\rangle},\frac{x_3}{\langle[0.35,0.55],[0.25,0.4]\rangle}\right\}\right\rangle,\\ \left\langle(c_2,c_4,c_7,c_9),\left\{\frac{x_1}{\langle[0.6,0.8],[0.1,0.2]\rangle},\frac{x_2}{\langle[0.3,0.5],[0.25,0.45]\rangle}\right\}\right\rangle\end{array}\right\}$$

$$\left(Z,\Gamma_2\right)=\left\{\begin{array}{l}\left\langle\left(c_3,c_4,c_5,c_9\right),\left\{\frac{x_2}{\langle[0.6,0.7],[0.1,0.3]\rangle},\frac{x_3}{\langle[0.35,0.45],[0.45,0.5]\rangle}\right\}\right\rangle,\\ \left\langle\left(c_2,c_4,c_6,c_9\right),\left\{\frac{x_1}{\langle[0.2,0.3],[0.3,0.4]\rangle},\frac{x_2}{\langle[0.6,0.7],[0.1,0.2]\rangle}\right\}\right\rangle\end{array}\right\}$$

**Note:** By adding HSS with IVIFS, a new structure called IVIFHSS is introduced and it is viable to increase the features of selecting an object which a decision-maker could not imagine with an aid of multi-attribute function. Every IVIFHSS is IVIFSS and in IVIFHSS, when the lower and upper membership and lower and upper non-membership coincide then IVIFHSS is reduced to IFHSS. However, the IVIFHSS is preferable for the environment where there is no precise membership and non-membership value i.e vagueness involved in assigning the membership and non-membership values. For example, suppose if we want to describe the attractiveness of a house by IVIFHSS, then by membership interval, we can measure its minimum attractiveness and maximum attractiveness, on the other hand, the non-membership interval tells the minimum non-attractiveness and maximum non-attractiveness. So, IVIFHSS is more functional than IFHSS to represent parameterized hesitant information.

**Definition 3.2** An IVIFHSS $\left(Y,\Sigma\right)$ over $X$ is said to be null IVIFHSS if for all $x\in X$ and $\eta\in\Sigma$, $\varphi_{Y(\eta)}\left(x\right)=\left[0,0\right]$ and $\psi_{Y(\eta)}\left(x\right)=\left[1,1\right]$ and it is denoted by $\Phi_{(Y,\Sigma)}$.

On the other hand, $\left(Y,\Sigma\right)$ over $X$ is called a universal IVIFHSS if for all $x\in X$ and $\eta\in\Sigma$ $\varphi_{Y(\eta)}\left(x\right)=\left[1,1\right]$ and $\psi_{Y(\eta)}\left(x\right)=\left[0,0\right]$ and it is denoted by $\Lambda_{(Y,\Sigma)}$.

**Definition 3.3** Let $X$ be the set of universe and $\left(Y,\Gamma_1\right)$ and $\left(Z,\Gamma_2\right)$ be two IVIFHSSs over $X$. Then, we say that $\left(Y,\Gamma_1\right)$ is an IVIFHS subset of $\left(Z,\Gamma_2\right)$ if

(i) $\Gamma_1\subseteq\Gamma_2$

(ii) For any $\eta\in\Gamma_1$, $Y\left(\eta\right)\subseteq Z\left(\eta\right)$ and it is denoted by and denoted by $\left(Y,\Gamma_1\right)\tilde{\subseteq}\left(Z,\Gamma_2\right)$.

That is, for all $x\in X$ and $\eta\in\Gamma_1$,

$\varphi^{l}_{Y(\eta)}\left(x\right)\leq\varphi^{l}_{Z(\eta)}\left(x\right),\varphi^{u}_{Y(\eta)}\left(x\right)\leq\varphi^{u}_{Z(\eta)}\left(x\right)$,and $\psi^{l}_{Y(\eta)}\left(x\right)\geq\psi^{l}_{Z(\eta)}\left(x\right),\psi^{u}_{Y(\eta)}\left(x\right)\geq\psi^{u}_{Z(\eta)}\left(x\right)$.

**Definition 3.4** Let $X$ be the set of universe and $(Y,\Gamma_1)$ and $(Z,\Gamma_2)$ be two IVIFHSSs over $X$. Then, we say that $(Y,\Gamma_1)$ is said to be equal to $(Z,\Gamma_2)$ if $(Y,\Gamma_1)$ is an IVIFHS subset of $(Z,\Gamma_2)$ and conversely and it is denoted by $(Y,\Gamma_1)\tilde{=}(Z,\Gamma_2)$.

Otherwise, $(Y,\Gamma_1)$ and $(Z,\Gamma_2)$ are said to be equal if for all $x\in X$ ,and $\eta\in\Sigma$,

$\varphi_{Y(\eta)}(x)=\varphi_{Z(\eta)}(x)\Rightarrow\varphi^{l}{}_{Y(\eta)}(x)=\varphi^{l}{}_{Z(\eta)}(x)$ ,and $\varphi^{u}{}_{Y(\eta)}(x)=\varphi^{u}{}_{Z(\eta)}(x)$

And $\psi_{Y(\eta)}(x)=\psi_{Z(\eta)}(x)\Rightarrow\psi^{l}{}_{Y(\eta)}(x)=\psi^{l}{}_{Z(\eta)}(x)$ and $\psi^{u}{}_{Y(\eta)}(x)=\psi^{u}{}_{Z(\eta)}(x)$

**Theorem 3.5** Let $X$ be an initial universe and $(Y,\Gamma_1)$ , $(W,\Gamma_2)$ and $(Z,\Gamma_3)$ be three IVIHSSs over $X$ and $\Gamma_1,\Gamma_2,\Gamma_3\subseteq\Sigma$. Then

(i) $(Y,\Gamma_1)\tilde{\subseteq}\Lambda_{(Y,\Sigma)}$

(ii) $\Phi_{(Y,\Sigma)}\tilde{\subseteq}(Y,\Gamma_1)$

(iii) $(Y,\Gamma_1)\tilde{\subseteq}(W,\Gamma_2)$,and $(W,\Gamma_2)\tilde{\subseteq}(Z,\Gamma_3)\Rightarrow(Y,\Gamma_1)\tilde{\subseteq}(Z,\Gamma_3)$.

**Proof:** All proofs are straightforward.

**Definition 3.6** The Complement of IVIFHSS $(Y,\Sigma)$ over $X$ is denoted by $(Y,\Sigma)^c$ and defined as $(Y,\Sigma)^c=(Y^c,\neg\Sigma)$, where $Y^c:\neg\Sigma\rightarrow IVIF^X$ and the set-theoretic presentation is given by

$$(Y,\Sigma)^c=\left\{\left\langle\eta,\left(\frac{x}{\left\langle\left[\psi^{l}_{Y(\eta)}(x),\psi^{u}_{Y(\eta)}(x)\right],\left[\varphi^{l}_{Y(\eta)}(x),\varphi^{u}_{Y(\eta)}(x)\right]\right\rangle}\right)\right\rangle:x\in X,\eta\in\Sigma\right\}$$

**Theorem 3.7** Let $(Y,\Sigma)$ be any IVIFHSS over the initial universe $X$. Then

(i) $\left((Y,\Sigma)^c\right)^c=(Y,\Sigma)$

(ii) $\Phi^{c}_{(Y,\Sigma)}=\Lambda_{(Y,\Sigma)}$

(iii) $\Lambda^{c}_{(Y,\Sigma)} = \Phi_{(Y,\Sigma)}$

**Proof:** Proofs are obvious.

**Definition 3.8** If $(Y,\Gamma_1)$ and $(Z,\Gamma_2)$ be two IVIFHSSs over a common universe $X$, then "$(Y,\Gamma_1)$ **AND** $(Z,\Gamma_2)$" is denoted by $(Y,\Gamma_1)\tilde{\wedge}(Z,\Gamma_2)$ and is defined by

$(Y,\Gamma_1)\tilde{\wedge}(Z,\Gamma_2)=(W,\Gamma_1\times\Gamma_2)$, where $W(\alpha,\beta)=Y(\alpha)\cap Z(\beta),\forall(\alpha,\beta)\in\Gamma_1\times\Gamma_2$ such a way that,

$$W(\alpha,\beta)(x)=\left(\left\langle\begin{matrix}\left[\inf\left(\varphi^{l}_{Y(\alpha)}(x),\varphi^{l}_{Z(\beta)}(x)\right),\inf\left(\varphi^{u}_{Y(\alpha)}(x),\varphi^{u}_{Z(\beta)}(x)\right)\right],\\ \left[\sup\left(\psi^{l}_{Y(\alpha)}(x),\psi^{l}_{Z(\beta)}(x)\right),\sup\left(\psi^{u}_{Y(\alpha)}(x),\psi^{u}_{Z(\beta)}(x)\right)\right]\end{matrix}\right\rangle:\forall(\alpha,\beta)\in\Gamma_1\times\Gamma_2,x\in X\right)$$

**Definition 3.9** If $(Y,\Gamma_1)$ and $(Z,\Gamma_2)$ be two IVIFHSSs over a common universe $X$, then "$(Y,\Gamma_1)$ **OR** $(Z,\Gamma_2)$" is denoted by $(Y,\Gamma_1)\tilde{\vee}(Z,\Gamma_2)$ and is defined by

$(Y,\Gamma_1)\tilde{\vee}(Z,\Gamma_2)=(W,\Gamma_1\times\Gamma_2)$, where $W(\alpha,\beta)=Y(\alpha)\cup Z(\beta),\forall(\alpha,\beta)\in\Gamma_1\times\Gamma_2$ such a way that,

$$W(\alpha,\beta)(x)=\left(\left\langle\begin{matrix}\left[\sup\left(\varphi^{l}_{Y(\alpha)}(x),\varphi^{l}_{Z(\beta)}(x)\right),\sup\left(\varphi^{u}_{Y(\alpha)}(x),\varphi^{u}_{Z(\beta)}(x)\right)\right],\\ \left[\inf\left(\psi^{l}_{Y(\alpha)}(x),\psi^{l}_{Z(\beta)}(x)\right),\inf\left(\psi^{u}_{Y(\alpha)}(x),\psi^{u}_{Z(\beta)}(x)\right)\right]\end{matrix}\right\rangle:\forall(\alpha,\beta)\in\Gamma_1\times\Gamma_2,x\in X\right)$$

**Theorem 3.10** Let $(Y,\Gamma_1),(Z,\Gamma_2)$ and $(W,\Gamma_3)$ be three IVIFHSSs over $X$, then we have the following:

(i) $\left((Y,\Gamma_1)\tilde{\wedge}(Z,\Gamma_2)\right)^{c}=(Y,\Gamma_1)^{c}\tilde{\vee}(Z,\Gamma_2)^{c}$

(ii) $\left((Y,\Gamma_1)\tilde{\vee}(Z,\Gamma_2)\right)^{c}=(Y,\Gamma_1)^{c}\tilde{\wedge}(Z,\Gamma_2)^{c}$

(iii) $(Y,\Gamma_1)\tilde{\wedge}\left((Z,\Gamma_2)\tilde{\wedge}(W,\Gamma_3)\right)=\left((Y,\Gamma_1)\tilde{\wedge}(Z,\Gamma_2)\right)\tilde{\wedge}(W,\Gamma_3)$

(iv) $(Y,\Gamma_1)\tilde{\vee}\left((Z,\Gamma_2)\tilde{\vee}(W,\Gamma_3)\right)=\left((Y,\Gamma_1)\tilde{\vee}(Z,\Gamma_2)\right)\tilde{\vee}(W,\Gamma_3)$

**Proof:** (i) $(Y,\Gamma_1)\tilde{\wedge}(Z,\Gamma_2)=(W,\Gamma_1\times\Gamma_2)$

Then, $\left((Y,\Gamma_1)\tilde{\wedge}(Z,\Gamma_2)\right)^c=(W,\Gamma_1\times\Gamma_2)^c=\left(W^c,\neg(\Gamma_1\times\Gamma_2)\right)$

Similarly, $(Y,\Gamma_1)^c=(Y^c,\neg\Gamma_1)$ and $(Z,\Gamma_2)^c=(Z^c,\neg\Gamma_2)$

Then, $(Y,\Gamma_1)^c\tilde{\vee}(Z,\Gamma_2)^c=(Y^c,\neg\Gamma_1)\tilde{\vee}(Z^c,\neg\Gamma_2)=(H,\neg\Gamma_1\times\neg\Gamma_2)=(H,\neg(\Gamma_1\times\Gamma_2))$ where $\forall(\neg\alpha,\neg\beta)\in\neg\Gamma_1\times\neg\Gamma_2,\ x\in X$. Then we have,

$$\varphi_{H(\neg\alpha,\neg\beta)}(x)=\left\lfloor\sup\left(\varphi^l_{Y^c(\neg\alpha)}(x),\varphi^l_{Z^c(\neg\beta)}(x)\right),\sup\left(\varphi^u_{Y^c(\neg\alpha)}(x),\varphi^u_{Z^c(\neg\beta)}(x)\right)\right\rfloor$$

$$\psi_{H(\neg\alpha,\neg\beta)}(x)=\left\lfloor\inf\left(\psi^l_{Y^c(\neg\alpha)}(x),\psi^l_{Z^c(\neg\beta)}(x)\right),\inf\left(\psi^u_{Y^c(\neg\alpha)}(x),\psi^u_{Z^c(\neg\beta)}(x)\right)\right\rfloor$$

Now, $Y^c(\neg\alpha)=\left\langle x,\psi_{Y(\alpha)}(x),\varphi_{Y(\alpha)}(x)\right\rangle$, $Z^c(\neg\beta)=\left\langle x,\psi_{Z(\beta)}(x),\varphi_{Z(\beta)}(x)\right\rangle$.

Then,

$$\varphi^l_{Y^c(\neg\alpha)}(x)=\psi^l_{Y(\alpha)}(x),\varphi^l_{Z^c(\neg\beta)}(x)=\psi^l_{Z(\beta)}(x),\varphi^u_{Y^c(\neg\alpha)}(x)=\psi^u_{Y(\alpha)}(x),\varphi^u_{Z^c(\neg\beta)}(x)=\psi^u_{Z(\beta)}(x)$$

Therefore, we have the following,

$$\varphi_{H(\neg\alpha,\neg\beta)}(x)=\left\lfloor\sup\left(\psi^l_{Y(\alpha)}(x),\psi^l_{Z(\beta)}(x)\right),\sup\left(\psi^u_{Y(\alpha)}(x),\psi^u_{Z(\beta)}(x)\right)\right\rfloor$$

$$\psi_{H(\neg\alpha,\neg\beta)}(x)=\left\lfloor\inf\left(\varphi^l_{Y(\alpha)}(x),\varphi^l_{Z(\beta)}(x)\right),\sup\left(\varphi^u_{Y(\alpha)}(x),\varphi^u_{Z(\beta)}(x)\right)\right\rfloor$$

We have, $(\neg\alpha,\neg\beta)\in\neg(\Gamma_1\times\Gamma_2)$. Since, $(W,\Gamma_1\times\Gamma_2)^c=\left(W^c,\neg(\Gamma_1\times\Gamma_2)\right)$, then we can write

$W^c(\neg\alpha,\neg\beta)=\left\langle x,\psi_{W(\alpha,\beta)}(x),\varphi_{W(\alpha,\beta)}(x)\right\rangle$.

Thus, $\varphi_{W^c(\neg\alpha,\neg\beta)}(x)=\psi_{W(\alpha,\beta)}(x)$ and $\psi_{W^c(\neg\alpha,\neg\beta)}(x)=\varphi_{W(\alpha,\beta)}(x)$.

Since, $(Y,\Gamma_1)\tilde{\wedge}(Z,\Gamma_2)=(W,\Gamma_1\times\Gamma_2)$, then

$$W_{(\alpha,\beta)}(x)=\left\langle \begin{array}{l}\left[Inf\left(\varphi^{l}_{Y(\alpha)}(x),\varphi^{l}_{Z(\beta)}(x)\right),\inf\left(\varphi^{u}_{Y(\alpha)}(x),\varphi^{u}_{Z(\beta)}(x)\right)\right],\\ \left[\sup\left(\psi^{l}_{Y(\alpha)}(x),\psi^{l}_{Z(\beta)}(x)\right),\sup\left(\psi^{u}_{Y(\alpha)}(x),\psi^{u}_{Z(\beta)}(x)\right)\right]\end{array}\right\rangle$$

Thus, $\varphi_{H(\alpha,\beta)}(x)=\left\lfloor Inf\left(\varphi^{l}_{Y(\alpha)}(x),\varphi^{l}_{Z(\beta)}(x)\right),\inf\left(\varphi^{u}_{Y(\alpha)}(x),\varphi^{u}_{Z(\beta)}(x)\right)\right\rfloor$ and

$\psi_{H(\alpha,\beta)}(x)=\left\lfloor \sup\left(\psi^{l}_{Y(\alpha)}(x),\psi^{l}_{Z(\beta)}(x)\right),\sup\left(\psi^{u}_{Y(\alpha)}(x),\psi^{u}_{Z(\beta)}(x)\right)\right\rfloor$

So, we can say that operators $W^{c}$ and $H$ are same.

Therefore, $\left((Y,\Gamma_1)\tilde{\wedge}(Z,\Gamma_2)\right)^{c}=(Y,\Gamma_1)^{c}\tilde{\vee}(Z,\Gamma_2)^{c}$

Proofs of (ii) to (iv) are left as an exercise for the readers.

**Definition 3.11** Let $X$ be the universe of discourse and $\Gamma_1,\Gamma_2\subseteq\Sigma$. Let $(Y,\Gamma_1)$ and $(Z,\Gamma_2)$ be two IVIFHSSs over $X$. Then the union(relative) of $(Y,\Gamma_1)$ and $(Z,\Gamma_2)$ is denoted by $(Y,\Gamma_1)\tilde{\cup}(Z,\Gamma_2)=(W,\Gamma_3)$ where $\Gamma_3=\Gamma_1\bigcup\Gamma_2$ and defined as follows:

$$\varphi_{W(\eta)}(x)=\begin{cases}\varphi_{Y(\eta)}(x),\ if\ \eta\in\Gamma_1-\Gamma_2\\ \varphi_{Z(\eta)}(x),\ if\ \eta\in\Gamma_2-\Gamma_1\\ \left[\sup\left(\varphi^{l}_{Y(\eta)}(x),\varphi^{l}_{Z(\eta)}(x)\right),\sup\left(\varphi^{u}_{Y(\eta)}(x),\varphi^{u}_{Z(\eta)}(x)\right)\right],if\ \eta\in\Gamma_1\bigcap\Gamma_2,x\in X\end{cases}$$

$$\psi_{W(\eta)}(x)=\begin{cases}\psi_{Y(\eta)}(x),\ if\ \eta\in\Gamma_1-\Gamma_2\\ \psi_{Z(\eta)}(x),\ if\ \eta\in\Gamma_2-\Gamma_1\\ \left[\inf\left(\psi^{l}_{Y(\eta)}(x),\psi^{l}_{Z(\eta)}(x)\right),\inf\left(\psi^{u}_{Y(\eta)}(x),\psi^{u}_{Z(\eta)}(x)\right)\right],if\ \eta\in\Gamma_1\bigcap\Gamma_2,x\in X\end{cases}$$

**Definition 3.12** Let $X$ be the universe of discourse and $\Gamma_1,\Gamma_2\subseteq\Sigma$. Let $(Y,\Gamma_1)$ and $(Z,\Gamma_2)$ be two IVIFHSSs over $X$. Then the intersection of $(Y,\Gamma_1)$ and $(Z,\Gamma_2)$ is denoted by $(Y,\Gamma_1)\tilde{\cap}(Z,\Gamma_2)=(W,\Gamma_3)$ where $\Gamma_3=\Gamma_1\bigcup\Gamma_2$ and defined as follows:

$$\varphi_{W(\eta)}(x)=\begin{cases}\varphi_{Y(\eta)}(x), if\, \eta\in\Gamma_1-\Gamma_2\\ \varphi_{Z(\eta)}(x), if\, \eta\in\Gamma_2-\Gamma_1\\ \left[\inf\left(\varphi^l_{Y(\eta)}(x),\varphi^l_{Z(\eta)}(x)\right),\inf\left(\varphi^u_{Y(\eta)}(x),\varphi^u_{Z(\eta)}(x)\right)\right], if\ \eta\in\Gamma_1\cap\Gamma_2, x\in X\end{cases}$$

$$\psi_{W(\eta)}(x)=\begin{cases}\psi_{Y(\eta)}(x), if\, \eta\in\Gamma_1-\Gamma_2\\ \psi_{Z(\eta)}(x), if\, \eta\in\Gamma_2-\Gamma_1\\ \left[\sup\left(\psi^l_{Y(\eta)}(x),\psi^l_{Z(\eta)}(x)\right),\sup\left(\psi^u_{Y(\eta)}(x),\psi^u_{Z(\eta)}(x)\right)\right], if\ \eta\in\Gamma_1\cap\Gamma_2, x\in X\end{cases}$$

**Theorem 3.13** Let $X$ be the set of the universe and $\Gamma_1,\Gamma_2,\Gamma_3\subseteq\Sigma$. Let $(Y,\Gamma_1)$, $(Z,\Gamma_2)$ and $(W,\Gamma_3)$ be three IVIFHSSs over $X$, then we have the following properties

(i) $(Y,\Gamma_1)\tilde{\cup}(Y,\Gamma_1)=(Y,\Gamma_1)$

(ii) $\Phi_{(Y,\Sigma)}\tilde{\cup}(Y,\Gamma_1)=(Y,\Gamma_1)$

(iii) $(Y,\Gamma_1)\tilde{\cup}\Lambda_{(Y,\Sigma)}=\Lambda_{(Y,\Sigma)}$

(iv) $(Y,\Gamma_1)\tilde{\cup}(Z,\Gamma_2)=(Z,\Gamma_2)\tilde{\cup}(Y,\Gamma_1)$

(v) $\left((Y,\Gamma_1)\tilde{\cup}(Z,\Gamma_2)\right)\tilde{\cup}(W,\Gamma_3)=(Y,\Gamma_1)\tilde{\cup}\left((Z,\Gamma_2)\tilde{\cup}(W,\Gamma_3)\right)$

**Proof:** Proofs are obvious.

**Theorem 3.14** Let $X$ be the set of the universe and $\Gamma_1,\Gamma_2,\Gamma_3\subseteq\Sigma$. Let $(Y,\Gamma_1)$, $(Z,\Gamma_2)$ and $(W,\Gamma_3)$ be three IVIFHSSs over $X$, then we have the following properties

(i) $(Y,\Gamma_1)\tilde{\cap}(Y,\Gamma_1)=(Y,\Gamma_1)$

(ii) $\Phi_{(Y,\Sigma)}\tilde{\cap}(Y,\Gamma_1)=\Phi_{(Y,\Sigma)}$

(iii) $(Y,\Gamma_1)\tilde{\cap}\Lambda_{(Y,\Sigma)}=(Y,\Gamma_1)$

(iv) $(Y,\Gamma_1)\tilde{\cap}(Z,\Gamma_2)=(Z,\Gamma_2)\tilde{\cap}(Y,\Gamma_1)$

(v) $\left((Y,\Gamma_1)\tilde{\cap}(Z,\Gamma_2)\right)\tilde{\cap}(W,\Gamma_3)=(Y,\Gamma_1)\tilde{\cap}\left((Z,\Gamma_2)\tilde{\cap}(W,\Gamma_3)\right)$

(vi) $(Y,\Gamma_1)\tilde{\cap}\left((Z,\Gamma_2)\tilde{\cup}(W,\Gamma_3)\right)=\left((Y,\Gamma_1)\tilde{\cap}(Z,\Gamma_2)\right)\tilde{\cup}\left((Y,\Gamma_1)\tilde{\cap}(W,\Gamma_3)\right)$

(vii) $(Y,\Gamma_1)\tilde{\cup}\left((Z,\Gamma_2)\tilde{\cap}(W,\Gamma_3)\right)=\left((Y,\Gamma_1)\tilde{\cup}(Z,\Gamma_2)\right)\tilde{\cap}\left((Y,\Gamma_1)\tilde{\cup}(W,\Gamma_3)\right)$

**Proof:** All are straightforward.

**Definition 3.15** Let $X$ be the universal set, $\Gamma_1,\Gamma_2 \subseteq \Sigma$ and $(Y,\Gamma_1)$, $(Z,\Gamma_2)$ be two IVIFHSSs over $X$. Then, the difference between $(Y,\Gamma_1)$ and $(Z,\Gamma_2)$ is denoted by $(Y,\Gamma_1)\tilde{\setminus}(Z,\Gamma_2)=(W,\Gamma_3)$ where $(Y,\Gamma_1)\tilde{\cap}(Z,\Gamma_2)^c=(Z,\Gamma_2)=(W,\Gamma_3)$.

**Theorem 3.16** Let $X$ be the universal set, $\Gamma_1,\Gamma_2 \subseteq \Sigma$ and $(Y,\Gamma_1)$ and $(Z,\Gamma_2)$ be two IVIFHSSs over $X$. Then we have the following properties:

(i) $\left((Y,\Gamma_1)\tilde{\cup}(Z,\Gamma_2)\right)^c=(Y,\Gamma_1)^c\tilde{\cap}(Z,\Gamma_2)^c$

(ii) $\left((Y,\Gamma_1)\tilde{\cap}(Z,\Gamma_2)\right)^c=(Y,\Gamma_1)^c\tilde{\cup}(Z,\Gamma_2)^c$

**Proof:** (1) Let $(Y,\Gamma_1)\tilde{\cup}(Z,\Gamma_2)=(W,\Gamma_3)$, where $\Gamma_3=\Gamma_1\bigcup\Gamma_2$ and for all $\eta\in\Gamma_3$, we have the following

$$\varphi_{W(\eta)}(x)=\begin{cases}\varphi_{Y(\eta)}(x), if\,\eta\in\Gamma_1-\Gamma_2\\ \varphi_{Z(\eta)}(x), if\,\eta\in\Gamma_2-\Gamma_1\\ \left[\sup\left(\varphi^l_{Y(\eta)}(x),\varphi^l_{Z(\eta)}(x)\right),\sup\left(\varphi^u_{Y(\eta)}(x),\varphi^u_{Z(\eta)}(x)\right)\right], if\ \eta\in\Gamma_1\bigcap\Gamma_2, x\in X\end{cases}$$

$$\psi_{W(\eta)}(x)=\begin{cases}\psi_{Y(\eta)}(x),if\,\eta\in\Gamma_1-\Gamma_2\\ \psi_{Z(\eta)}(x),if\,\eta\in\Gamma_2-\Gamma_1\\ \left[\inf\left(\psi^l_{Y(\eta)}(x),\psi^l_{Z(\eta)}(x)\right),\inf\left(\psi^u_{Y(\eta)}(x),\psi^u_{Z(\eta)}(x)\right)\right],if\ \eta\in\Gamma_1\bigcap\Gamma_2,x\in X\end{cases}$$

Now, $\left((Y,\Gamma_1)\tilde{\cup}(Z,\Gamma_2)\right)^c = (W,\Gamma_3)^c=(W^c,\neg\Gamma_3)$, where $W^c(\neg\eta)=\left\langle x,\psi_{W(\eta)}(x),\varphi_{W(\eta)}(x)\right\rangle$ for all $x\in X$ and $\neg\eta\in\neg\Gamma_3=\neg(\Gamma_1\bigcup\Gamma_2)=\neg\Gamma_1\bigcup\neg\Gamma_2$.

Then we have,

$$\varphi^c{}_{W(\neg\eta)}(x)=\begin{cases}\psi_{Y(\eta)}(x),if\,\eta\in\Gamma_1-\Gamma_2\\ \psi_{Z(\eta)}(x),if\,\eta\in\Gamma_2-\Gamma_1\\ \left[\inf\left(\psi^l_{Y(\eta)}(x),\psi^l_{Z(\eta)}(x)\right),\inf\left(\psi^u_{Y(\eta)}(x),\psi^u_{Z(\eta)}(x)\right)\right],if\ \eta\in\Gamma_1\bigcap\Gamma_2,x\in X\end{cases}$$

$$\psi^c{}_{W(\neg\eta)}(x)=\begin{cases}\varphi_{Y(\eta)}(x),if\,\eta\in\Gamma_1-\Gamma_2\\ \varphi_{Z(\eta)}(x),if\,\eta\in\Gamma_2-\Gamma_1\\ \left[\sup\left(\varphi^l_{Y(\eta)}(x),\varphi^l_{Z(\eta)}(x)\right),\inf\left(\varphi^u_{Y(\eta)}(x),\varphi^u_{Z(\eta)}(x)\right)\right],if\ \eta\in\Gamma_1\bigcap\Gamma_2,x\in X\end{cases}$$

Since, $(Y,\Gamma_1)^c=(Y^c,\neg\Gamma_1)$ and $(Z,\Gamma_2)^c=(Z^c,\neg\Gamma_2)$ then

$(Y,\Gamma_1)^c\tilde{\cap}(Z,\Gamma_2)^c=(Y^c,\neg\Gamma_1)\tilde{\cap}(Z^c,\neg\Gamma_2)=(H,\Gamma_4)$ (say), where $\Gamma_4=\neg\Gamma_3=\neg\Gamma_1\cup\neg\Gamma_2$ and for all $\neg\eta\in\Gamma_4$,

$$\varphi_{H(\neg\eta)}(x)=\begin{cases}\varphi_{Y^c(\neg\eta)}(x), if \neg\eta\in\neg\Gamma_1-\neg\Gamma_2\\ \varphi_{Z^c(\neg\eta)}(x), if \neg\eta\in\neg\Gamma_2-\neg\Gamma_1\\ \left[\inf\left(\varphi^l_{Y^c(\neg\eta)}(x),\varphi^l_{Z^c(\neg\eta)}(x)\right),\inf\left(\varphi^u_{Y^c(\neg\eta)}(x),\varphi^u_{Z^c(\neg\eta)}(x)\right)\right], if \neg\eta\in\neg\Gamma_1\cap\neg\Gamma_2, x\in X\end{cases}$$

$$=\begin{cases}\psi_{Y(\eta)}(x), if \eta\in\Gamma_1-\Gamma_2\\ \psi_{Z(\eta)}(x), if \eta\in\Gamma_2-\Gamma_1\\ \left[\inf\left(\psi^l_{Y(\eta)}(x),\psi^l_{Z(\eta)}(x)\right),\inf\left(\psi^u_{Y(\eta)}(x),\psi^u_{Z(\eta)}(x)\right)\right], if\ \eta\in\Gamma_1\cap\Gamma_2, x\in X\end{cases}$$

$$\psi_{H(\neg\eta)}(x)=\begin{cases}\psi_{Y^c(\neg\eta)}(x), if \neg\eta\in\neg\Gamma_1-\neg\Gamma_2\\ \psi_{Z^c(\neg\eta)}(x), if \neg\eta\in\neg\Gamma_2-\neg\Gamma_1\\ \left[\sup\left(\psi^l_{Y^c(\neg\eta)}(x),\varphi^l_{Z^c(\neg\eta)}(x)\right),\inf\left(\psi^u_{Y^c(\neg\eta)}(x),\psi^u_{Z^c(\neg\eta)}(x)\right)\right], if \neg\eta\in\neg\Gamma_1\cap\neg\Gamma_2, x\in X\end{cases}$$

$$=\begin{cases}\varphi_{Y(\eta)}(x), if \eta\in\Gamma_1-\Gamma_2\\ \varphi_{Z(\eta)}(x), if \eta\in\Gamma_2-\Gamma_1\\ \left[\sup\left(\varphi^l_{Y(\eta)}(x),\varphi^l_{Z(\eta)}(x)\right),\sup\left(\varphi^u_{Y(\eta)}(x),\varphi^u_{Z(\eta)}(x)\right)\right], if\ \eta\in\Gamma_1\cap\Gamma_2, x\in X\end{cases}$$

Therefore, $W^c$ and $H$ are the same operators. Thus, $\left((Y,\Gamma_1)\tilde{\cup}(Z,\Gamma_2)\right)^c=(Y,\Gamma_1)^c\tilde{\cap}(Z,\Gamma_2)^c$.

(ii) Similar to that of (i)

## 4. An Algorithmic Approach for Multi-criteria Decision Making Based on IVIFHSSs

A variety of real-based decision-making problems in different fields such as engineering, social science, economics, weather forecasting, risk management, medical science, etc. contains imprecise fuzzy data and it is due to diverse types of uncertainties present in the system. Day to day the problem becomes more and more complicated. There is a requirement to introduce another new tool that can handle a large amount of imprecision involved in a system. The introduction of IVIFHSSs is capable enough to encounter such problems. So, we present an algorithm to handle fuzzy decision-making problems based on IVIFHSSs, which is very much helpful for the decision-makers to obtain the optimal choice. Firstly, we give some definitions that are related to the proposed algorithm in the following:

**Definition 4.1** Let $(Y,\Sigma)$ be an IVIFHSS over the set of the universe $X=\{x_1,x_2,....,x_n\}$ where $\Sigma=E_1\times E_2\times......\times E_n$. For any $\eta\in\Sigma$, $\varphi_{Y(\eta)}(x_i)=\left\langle\left[\varphi^l{}_{Y(\eta)}(x_i),\varphi^u{}_{Y(\eta)}(x_i)\right]\right\rangle$ denotes the degree of membership of an element $x_i$ via $Y(\eta)$.

Then the score of membership degree of $x_i$ for each $\eta$ is denoted and defined as

$$S^M_{Y(\eta)}(x_i)=\sum_{k=1}^{n}\left[\left(\varphi^l_{Y(\eta)}(x_i)+\varphi^u_{Y(\eta)}(x_i)\right)-\left(\varphi^l_{Y(\eta)}(x_k)+\varphi^u_{Y(\eta)}(x_k)\right)\right]$$

**Definition 4.2** Let $(Y,\Sigma)$ be an IVIFHSS over the set of the universe $X=\{x_1,x_2,....,x_n\}$ where $\Sigma=E_1\times E_2\times......\times E_m$. For any $\eta_j\in\Sigma$, $\psi_{Y(\eta_j)}(x_i)=\left\langle\left[\varphi^l{}_{Y(\eta_j)}(x_i),\varphi^u{}_{Y(\eta_j)}(x_i)\right]\right\rangle$ denotes the degree of non-membership of an element $x_i$ via $Y\left(\eta_j\right)$. Then the score of non-membership degree of $x_i$ for each $\eta_j$ is denoted and defined as

$$S^N_{Y(\eta_j)}(x_i)=-\sum_{k=1}^{n}\left[\left(\psi^l_{Y(\eta_j)}(x_i)+\psi^u_{Y(\eta_j)}(x_i)\right)-\left(\psi^l_{Y(\eta_j)}(x_k)+\psi^u_{Y(\eta_j)}(x_k)\right)\right]$$

**Definition 4.3** Let $(Y,\Sigma)$ be an IVIFHSS over the set of the universe $X=\{x_1,x_2,....,x_n\}$ where $\Sigma=E_1\times E_2\times......\times E_n$. For any $\eta_j\in\Sigma$, the score of the membership and non-membership degree of each $x_i$ denoted by $S^M_{Y(\eta_j)}(x_i)$ and $S^N_{Y(\eta_j)}(x_i)$ respectively. Then the total score of $x_i$ is denoted by $T_{Y(\eta_j)}(x_i)$ and is defined as

$$T_{Y(\eta_j)}(x_i)=S^M_{Y(\eta_j)}(x_i)+S^N_{Y(\eta_j)}(x_i)$$

The steps of the algorithms, based on these definitions are discussed below:

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**Algorithm:**

---

**Step1:** Input an IVIFHSS $(Y, \Sigma)$ over $X$

**Step2:** Compute the score of membership degrees $S^{M}_{Y(\eta j)}(x_i)$ and the score of non-membership degrees $S^{N}_{Y(\eta j)}(x_i)$ for every $\eta \in \Sigma$.

**Step3:** Compute the total score $T_{Y(\eta j)}(x_i)$.

**Step4:** Obtain $\lambda$, for which $T_{\lambda} = \max_{x_i \in X}\left(T_{Y(\eta j)}(x_i)\right)$. Thus, $x_{\lambda} \in X$ is the optimal choice for the decision-maker.

---

**Example 4.4** Considering example **3.1.1,** we have

**Step1:**

$$(Y, \Gamma_1) = \left\{ \begin{array}{l} \left\langle (c_2, c_4, c_5, c_9), \left\{ \frac{x_1}{\langle [0.3, 0.4], [0.5, 0.6] \rangle}, \frac{x_3}{\langle [0.5, 0.7], [0.1, 0.2] \rangle} \right\} \right\rangle, \\ \left\langle (c_2, c_4, c_7, c_9), \left\{ \frac{x_1}{\langle [0.4, 0.6], [0.3, 0.4] \rangle}, \frac{x_2}{\langle [0.7, 0.8], [0.1, 0.2] \rangle} \right\} \right\rangle \\ \left\langle (c_3, c_4, c_5, c_9), \left\{ \frac{x_2}{\langle [0.45, 0.55], [0.23, 0.35] \rangle}, \frac{x_3}{\langle [0.35, 0.55], [0.25, 0.4] \rangle} \right\} \right\rangle, \\ \left\langle (c_2, c_4, c_7, c_9), \left\{ \frac{x_1}{\langle [0.6, 0.8], [0.1, 0.2] \rangle}, \frac{x_2}{\langle [0.3, 0.5], [0.25, 0.45] \rangle} \right\} \right\rangle \end{array} \right\}$$

**Step2:**

$$S^{M}_{Y(\eta_j)}(x_1) = \left[(0.3+0.4)-(0.5+0.7)\right]+\left[(0.4+0.6)-(0.7+0.8)\right]+\left[(0.6+0.8)-(0.3+0.5)\right]$$
$$= (-0.5-0.5+0.6) = -0.4$$

$$S^{M}_{Y(\eta_j)}(x_2) = \left[(0.7+0.8)-(0.4+0.6)\right]+\left[(0.45+0.55)-(0.35+0.55)\right]+\left[(0.3+0.5)-(0.6+0.8)\right]$$
$$= (0.5+0.1-0.6) = 0.0$$

$$S^{M}_{Y(\eta_j)}(x_3) = \left[(0.5+0.7)-(0.3+0.4)\right]+\left[(0.35+0.55)-(0.45+0.55)\right]$$
$$= (0.5-0.1) = 0.4$$

$$S^{N}_{Y(\eta_j)}(x_1) = -\{[(0.5+0.6)-(0.1+0.2)]+[(0.3+0.4)-(0.1+0.2)]+[(0.1+0.2)-(0.25+0.45)]\}$$
$$= -(0.8+0.4-0.4) = -0.8$$
$$S^{N}_{Y(\eta_j)}(x_2) = -\{[(0.1+0.2)-(0.3+0.4)]+[(0.23+0.35)-(0.25+0.4)]+[(0.25+0.45)-(0.1+0.2)]\}$$
$$= -(-0.4-0.07+0.4) = 0.07$$
$$S^{N}_{Y(\eta_j)}(x_3) = -\{[(0.1+0.2)-(0.5+0.6)]+[(0.25+0.4)-(0.23+0.35)]\}$$
$$= -(-0.8+0.07) = 0.73$$

**Step3:**

$$T_{Y(\eta_j)}(x_1) = -1.2, T_{Y(\eta_j)}(x_2) = 0.07, T_{Y(\eta_j)}(x_3) = 1.13$$

**Step4:**

$$T_{\lambda} = \text{Max } \{-1.2, 0.07, 1.13\} = 1.13$$

Thus, $x_3$ is the optimal choice for the decision-maker. If there is a tie, then we reassess all the attributes and repeat all the steps.

## 5. Conclusion and Future Scope

In this work, a new mathematical model called IVIFHSSs has been introduced. The IVIFHSSs are the extensions of IVIFSSs, HSSs, FHSSs, IFHSSs etc. We also studied some basic operations such as union, intersection, complement, difference, AND, OR on them. Further, some properties of IVIFHSSs are investigated. We present an algorithm based on IVIFHSSs to solve real-world problems. In the end, to check the feasibility of the proposed algorithm a numerical example is employed.

For future direction, there is a scope to introduce parameterized reduction method, TOPSIS method, Similarity measures, weight operators, entropy method, cluster analysis method,etc. on IVIFHSSs to solve vivid types of decision making problems.

**Conflicts of Interest:** Author declares no conflict of interest.

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Received: Nov 25, 2021. Accepted: Feb 7, 2022

Neutrosophic Sets and Systems, Vol. 48, 2022

University of New Mexico

# Pythagorean $m$-polar Fuzzy Neutrosophic Topology with Applications

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**Abstract**. The overarching structures like intuitionistic fuzzy sets, Pythagorean fuzzy sets, $m$-polar fuzzy sets, and neutrosophic fuzzy sets etc. have their own inadequacies and impediments. These models are unable to do work because of their impediments in many real life situations. To overcome these deficiencies, in this paper, we introduce a set entitled Pythagorean $m$-polar fuzzy neutrosophic set (P$m$FNS), as a hybrid model of Pythagorean fuzzy set, $m$-polar fuzzy set and single-valued neutrosophic set. We define some notions related to P$m$FNS with the help of illustrations. We also present some concept of Pythagorean $m$-polar fuzzy neutrosophic topology alongside its leading characteristics. We render two applications of P$m$FNS of scarcity of water and uplifting economy ruined due to COVID-19 using TOPSIS.

**Keywords:** Pythagorean $m$-polar fuzzy neutrosophic set; Pythagorean $m$-polar fuzzy neutrosophic topology; TOPSIS; COVID-19

## 1. Introduction

The methods usually working in classical mathematics are not generally advantageous for the reason that uncertainties and unclearness being there, to tackle real world difficulties. There are numerous methods to handle such circumstances. Unfortunately, all these models have their own restrictions and drawbacks. In 1965, the thought of fuzzy sets as an augmentation of the conventional crisp set was inaugurated by Zadeh [18], to overcome these deficiencies, by associating the membership function $\mu_A : X \to [0,1]$. Hence, in this new outline, we face the problems relating to topology, the study on them form the subjects of fuzzy topology. In 1968, Chang explained fuzzy topology, as a branch merging ordered structure with topological structure, on fuzzy set. Pao-Ming and Ying-Ming [10] defined the formation of neighborhood of fuzzy-point. In 1983, Atanassov [2, 3] provided the idea of intuitionistic fuzzy sets (IFSs). Later, intuitionistic fuzzy topological spaces via intuitionistic fuzzy sets were obtained by Çoker *et al.* [5]. Lee and Lee [7] gave the outlook of intuitionistic fuzzy points accompanied by the notion of intuitionistic fuzzy neighborhoods. They discovered the characteristics of continuous, open and closed maps in the intuitionistic fuzzy topological spaces. In 2013, Yager [15]- [17] presented Pythagorean fuzzy sets as an expansion of intuitionistic fuzzy sets with a wider scope of applications and presented Pythagorean membership grades with their practical implementations to the multi-criteria decision making (MCDM). Olgun *et al.* [9] introduced the idea of Pythagorean fuzzy topological space.

In 2005, the model of neutrosophic sets, which is the broad view of intuitionistic fuzzy sets, for handling with difficulties involving exaggeration, indeterminacy and irregularity was explored by Smarandache [13]. The notion of fuzzy neutrosophic sets was presented by Arockiarani *et al.* [1]. Recently, Jansi and Mohana [6] coined the notion of pairwise Pythagorean neutrosophic bitopological spaces treating truth and falsity membership functions as dependent neutrosophic components. Neutrosophic set was protracted to Plithogenic set [14] by Smarandache, which is a collection whose each element is regarded as by many attribute values and every attribute value has either a fuzzy, intuitionistic fuzzy or neutrosophic degree of appurtenance to the set. Chen *et al.* [4] expanded the view of bipolar fuzzy sets to $m$-polar fuzzy sets and provided some of its practical implementations in day-to-day situations. In 2019, Naeem *et al.* [8] explored the notions of Pythagorean $m$-polar fuzzy sets (P$m$FSs) along with some of their foremost features. They also gave an application of P$m$FSs in decision making difficulty of selection of most suitable manner of the advertisement using the conventional tool TOPSIS (Technique based on Order Preference by Similarity to Ideal Solution). Later, Riaz *et al.* [11] extended the notion to corresponding soft sets.

The main aspiration behind this article is to study some features of Pythagorean $m$-polar fuzzy neutrosophic sets and construct topology on it. There appear several circumstances where data contains multi-polar facts and figures. Pythagorean $m$-polar fuzzy neutrosophic sets (P$m$FNSs) is one of the utmost suitable tools for managing such conditions. It can be used to illustrate the ambiguous facts further satisfactorily and exactly. It has been used in many areas for example in aggregation operators, information measures, and decision making. Because of such an evolution, we present an outline on Pythagorean $m$-polar fuzzy neutrosophic sets with goal of offering a clear outlook on the different tools, concepts and trends related to their extensions. The rest of the paper is systemized as: Elementary notions are dealt with in Section 2. Section 3 presents some notions of Pythagorean $m$-polar fuzzy neutrosophic sets. The topological structure on our proposed model along with its prime attributes is presented in Section 4. Two applications of decision making are rendered in Section 5.

## 2. Preliminaries

**Definition 2.1.** [18] A collection of orderly pairs $(\hbar, T_{\mathcal{F}}(\hbar))$, $\hbar$ being an element of the underlying universe $X$ and $T_{\mathcal{F}}$ (the affiliation, association or membership function) is a well-defined map, that drives members of $X$ to $[0,1]$, is entitled as a *fuzzy set* (FS) $\mathcal{F}$ over $X$. In other words

$$T(\hbar) = \begin{cases} 1, & \text{if } \hbar \in \mathcal{F} \\ 0, & \text{if } \hbar \notin \mathcal{F} \\ ]0,1[, & \text{if } \hbar \text{ is partially in } \mathcal{F} \end{cases}$$

**Definition 2.2.** [2,3] An *intuitionistic fuzzy set* (IFS) $G$ in $X$ is an object having the form

$$G = \{\langle \hbar, T(\hbar), F(\hbar)\rangle : \hbar \in X\}$$

where the membership function $T(\hbar) : X \to [0,1]$ and the non-membership function $F(\hbar) : X \to [0,1]$ for every $x \in X$ obey the constraint

$$0 \leq T(\hbar) + F(\hbar) \leq 1.$$

**Definition 2.3.** [15,16] A *Pythagorean fuzzy set*, shortened as PFS, is a collection defined by

$$P = \Big\{ < \hbar, T_P(\hbar), F_P(\hbar) >: \hbar \in X \Big\}$$

where $T_P$ and $F_P$ are mappings from a set $X$ to [0, 1] obeying the restriction $0 \leq T_P^2(\hbar) + F_P^2(\hbar) \leq 1$, representing correspondingly the affiliation and dissociation grades of $\hbar \in X$ to $P$. The ordered pair $p = (T_p, F_p)$ is accredited as Pythagorean fuzzy number (PFN). The quantity $\exists(\hbar) = \sqrt{1 - \{T^2(\hbar) + F^2(\hbar)\}}$ is famous as the hesitation margin.

**Definition 2.4.** [12,13] A *neutrosophic set* $\mathbb{N}$ on the underlying set $X$ is defined as

$$\mathbb{N} = \{< \hbar, T_{\mathbb{N}}(\hbar), I_{\mathbb{N}}(\hbar), F_{\mathbb{N}}(\hbar) >: \hbar \in X\}$$

where $T, I, F : X \mapsto ]^{-}0, 1^{+}[$ accompanied by the constraint $^{-}0 \leq T_{\mathbb{N}}(\hbar) + I_{\mathbb{N}}(\hbar) + F_{\mathbb{N}}(\hbar) \leq 3^{+}$. Here $T_{\mathbb{N}}(\hbar)$, $I_{\mathbb{N}}(\hbar)$ and $F_{\mathbb{N}}(\hbar)$ are the degrees of membership, indeterminacy and falsity (non-membership) of members of the given set, respectively. $T$, $I$ and $F$ are acknowledged as the neutrosophic components.

**Definition 2.5.** [1] A *fuzzy neutrosophic set* ($\mathfrak{fn}$-set) over $X$ is delineated as

$$A = \{< \hbar, T_A(\hbar), I_A(\hbar), F_A(\hbar) >: \hbar \in X\}$$

where $T, I, F : X \mapsto [0,1]$ in such a way that $0 \leq T_A(\hbar) + I_A(\hbar) + F_A(\hbar) \leq 3$.

**Definition 2.6.** [8] Suppose that $m \in \mathbb{N}$. A *Pythagorean m-polar fuzzy set* (P$m$FS) $\mathcal{P}$ over $X$ is regarded as by the mappings $T^{(i)}_{\mathcal{P}} : X \mapsto [0,1]$ (the membership functions) and $F^{(i)}_{\mathcal{P}} : X \mapsto [0,1]$ (the non-membership functions) with the limitation that

$$0 \leq \left(T^{(i)}_{\mathcal{P}}(\hbar)\right)^2 + \left(F^{(i)}_{\mathcal{P}}(\hbar)\right)^2 \leq 1$$

for integral values of $i$ ranging from 1 to $m$.
A P$m$FS may be articulated as

$$\mathcal{P} = \left\{\left\langle \hbar, \left(\left(T^{(1)}_{\mathcal{P}}(\hbar), F^{(1)}_{\mathcal{P}}(\hbar)\right), \cdots, \left(T^{(m)}_{\mathcal{P}}(\hbar), F^{(m)}_{\mathcal{P}}(\hbar)\right)\right)\right\rangle : \hbar \in X\right\}$$

or more conveniently as

$$\begin{aligned} \mathcal{P} &= \left\{\frac{\hbar}{\left(\left(T^{(1)}_{\mathcal{P}}(\hbar), F^{(1)}_{\mathcal{P}}(\hbar)\right), \cdots, \left(T^{(m)}_{\mathcal{P}}(\hbar), F^{(m)}_{\mathcal{P}}(\hbar)\right)\right)} : \hbar \in X\right\} \\ &= \left\{\frac{\hbar}{\left(\left(T^{(i)}_{\mathcal{P}}(\hbar), F^{(i)}_{\mathcal{P}}(\hbar)\right)\right)} : \hbar \in X; i = 1, 2, \cdots, m\right\} \end{aligned}$$

The tabular materialization of $\mathcal{P}$ is

| $\mathcal{P}$ | | | |
|---|---|---|---|
| $\hbar_1$ | $(T^{(1)}_{\mathcal{P}}(\hbar_1), F^{(1)}_{\mathcal{P}}(\hbar_1))$ | $\cdots$ | $(T^{(m)}_{\mathcal{P}}(\hbar_1), F^{(m)}_{\mathcal{P}}(\hbar_1))$ |
| $\hbar_2$ | $(T^{(1)}_{\mathcal{P}}(\hbar_2), F^{(1)}_{\mathcal{P}}(\hbar_2))$ | $\cdots$ | $(T^{(m)}_{\mathcal{P}}(\hbar_2), F^{(m)}_{\mathcal{P}}(\hbar_2))$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $\hbar_k$ | $(T^{(1)}_{\mathcal{P}}(\hbar_k), F^{(1)}_{\mathcal{P}}(\hbar_k))$ | $\cdots$ | $(T^{(m)}_{\mathcal{P}}(\hbar_k), F^{(m)}_{\mathcal{P}}(\hbar_k))$ |

and in matrix format as

$$\mathcal{P} = \begin{bmatrix} (T^{(1)}_{\mathcal{P}}(\hbar_1), F^{(1)}_{\mathcal{P}}(\hbar_1)) & \cdots & (T^{(m)}_{\mathcal{P}}(\hbar_1), F^{(m)}_{\mathcal{P}}(\hbar_1)) \\ (T^{(1)}_{\mathcal{P}}(\hbar_2), F^{(1)}_{\mathcal{P}}(\hbar_2)) & \cdots & (T^{(m)}_{\mathcal{P}}(\hbar_2), F^{(m)}_{\mathcal{P}}(\hbar_2)) \\ \vdots & \ddots & \vdots \\ (T^{(1)}_{\mathcal{P}}(\hbar_k), F^{(1)}_{\mathcal{P}}(\hbar_k)) & \cdots & (T^{(m)}_{\mathcal{P}}(\hbar_k), F^{(m)}_{\mathcal{P}}(\hbar_k)) \end{bmatrix}$$

This matrix of order $k \times m$ is reckoned as *PmF-matrix*.

**Definition 2.7.** Let $X \neq \phi$ be a crisp set. A family $\tau$ of subsets of $X$ is called a *topology* on $X$ if

(i) $\phi$ and $X$ itself belong to $\tau$.
(ii) The union of any number of members of $\tau$ is again in $\tau$.
(iii) The intersection of any finite number of members of $\tau$ belong to $\tau$.

If $\tau$ is a topology on $X$, then $(X, \tau)$ is known as a *topological space*.

**Example 2.8.** Let $X = \{s, f\}$, then $\tau_1 = \{\phi, X\}$, $\tau_2 = \{\phi, \{s\}, X\}$, $\tau_3 = \{\phi, \{f\}, X\}$ and $\tau_4 = \{\phi, \{s\}, \{f\}, X\}$ are topologies on $X$.
Likewise, if $\tau$ is the union of all open intervals in the set $\mathbb{R}$ of reals, then $\tau$ is a topology (called *real topology*) on $\mathbb{R}$. $\mathbb{R}$ with this topology is called the *real line*.

## 3. Pythagorean $m$-Polar Fuzzy Neutrosophic Sets

In this section, we introduce the notion of Pythagorean $m$-polar fuzzy neutrosophic set along with its prime characteristics and illustrations.

**Definition 3.1.** A *Pythagorean m-polar fuzzy neutrosophic set* (P$m$FNS) $\Im$ over a basic set $X$ is marked by three mappings $T^{(i)}_{\Im} : X \to [0,1]^m$, $I^{(i)}_{\Im} : X \to [0,1]^m$ and $F^{(i)}_{\Im} : X \to [0,1]^m$, where $m$ is a natural number, $\forall i = 1, 2, \cdots, m$, with the limitation that

$$0 \leq \left(T^{(i)}_{\Im}(\hbar)\right)^2 + \left(I^{(i)}_{\Im}(\hbar)\right)^2 + \left(F^{(i)}_{\Im}(\hbar)\right)^2 \leq 2$$

for all $\hbar \in X$.
A P$m$FNS may be expressed as

$$\begin{aligned} \Im &= \left\{(\hbar, \left((T^{(1)}_{\Im}(\hbar), I^{(1)}_{\Im}(\hbar), F^{(1)}_{\Im}(\hbar)), \cdots, (T^{(m)}_{\Im}(\hbar), I^{(m)}_{\Im}(\hbar), F^{(m)}_{\Im}(\hbar))\right) : \hbar \in X\right\} \\ &= \left\{\frac{\hbar}{(T^{(1)}_{\Im}(\hbar), I^{(1)}_{\Im}(\hbar), F^{(1)}_{\Im}(\hbar)), \cdots, (T^{(m)}_{\Im}(\hbar), I^{(m)}_{\Im}(\hbar), F^{(m)}_{\Im}(\hbar))} : \hbar \in X\right\} \\ &= \left\{\frac{\hbar}{(T^{(i)}_{\Im}(\hbar), I^{(i)}_{\Im}(\hbar), F^{(i)}_{\Im}(\hbar))} : \hbar \in X, i = 1, 2, \cdots, m\right\} \end{aligned}$$

If cardinality of $X$ is $l$, then tabular structure of $\Im$ is as in Table 1:

TABLE 1. Tabular representation of P$m$FNS $\Im$

| $\Im$ | | | | |
|---|---|---|---|---|
| $\hbar_1$ | $(T^{(1)}_{\Im}(\hbar_1), I^{(1)}_{\Im}(\hbar_1), F^{(1)}_{\Im}(\hbar_1))$ | $(T^{(2)}_{\Im}(\hbar_1), I^{(2)}_{\Im}(\hbar_1), F^{(2)}_{\Im}(\hbar_1))$ | $\cdots$ | $(T^{(m)}_{\Im}(\hbar_1), I^{(m)}_{\Im}(\hbar_1), F^{(m)}_{\Im}(\hbar_1))$ |
| $\hbar_2$ | $(T^{(1)}_{\Im}(\hbar_2), I^{(1)}_{\Im}(\hbar_2), F^{(1)}_{\Im}(\hbar_2))$ | $(T^{(2)}_{\Im}(\hbar_2), I^{(2)}_{\Im}(\hbar_2), F^{(2)}_{\Im}(\hbar_2))$ | $\cdots$ | $(T^{(m)}_{\Im}(\hbar_2), I^{(m)}_{\Im}(\hbar_2), F^{(m)}_{\Im}(\hbar_2))$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $\hbar_l$ | $(T^{(1)}_{\Im}(\hbar_l), I^{(1)}_{\Im}(\hbar_l), F^{(1)}_{\Im}(\hbar_l))$ | $(T^{(2)}_{\Im}(\hbar_l), I^{(2)}_{\Im}(\hbar_l), F^{(2)}_{\Im}(\hbar_l))$ | $\cdots$ | $(T^{(m)}_{\Im}(\hbar_l), I^{(m)}_{\Im}(\hbar_l), F^{(m)}_{\Im}(\hbar_l))$ |

The corresponding matrix format is

$$\Im = \begin{pmatrix} (T^{(1)}_{\Im}(\hbar_1), I^{(1)}_{\Im}(\hbar_1), F^{(1)}_{\Im}(\hbar_1)) & (T^{(2)}_{\Im}(\hbar_1), I^{(2)}_{\Im}(\hbar_1), F^{(2)}_{\Im}(\hbar_1)) & \cdots & (T^{(m)}_{\Im}(\hbar_1), I^{(m)}_{\Im}(\hbar_1), F^{(m)}_{\Im}(\hbar_1)) \\ (T^{(1)}_{\Im}(\hbar_2), I^{(1)}_{\Im}(\hbar_2), F^{(1)}_{\Im}(\hbar_2)) & (T^{(2)}_{\Im}(\hbar_2), I^{(2)}_{\Im}(\hbar_2), F^{(2)}_{\Im}(\hbar_2)) & \cdots & (T^{(m)}_{\Im}(\hbar_2), I^{(m)}_{\Im}(\hbar_2), F^{(m)}_{\Im}(\hbar_2)) \\ \vdots & \vdots & \ddots & \vdots \\ (T^{(1)}_{\Im}(\hbar_l), I^{(1)}_{\Im}(\hbar_l), F^{(1)}_{\Im}(\hbar_l)) & (T^{(2)}_{\Im}(\hbar_l), I^{(2)}_{\Im}(\hbar_l), F^{(2)}_{\Im}(\hbar_l)) & \cdots & (T^{(m)}_{\Im}(\hbar_l), I^{(m)}_{\Im}(\hbar_l), F^{(m)}_{\Im}(\hbar_l)) \end{pmatrix}$$

This $l \times m$ matrix is known as *PmFN matrix*. The assortment of each P$m$FNS characterized over universe would be designated by P$m$FNS(X).

**Example 3.2.** If X=$\{e, f\}$ be a crisp set, then

$$\Im = \left\{\frac{e}{(0.57, 0.52, 0.91), (0.09, 0.37, 0.47), (0.00, 0.49, 0.81)}, \frac{f}{(0.79, 0.33, 0.67), (1.00, 0.00, 0.07), (0.77, 0.99, 1.00)}\right\}$$

is a P3FNS defined over $X$. The tabular form of this set is as in Table 2:

TABLE 2. Tabular representation of P3FNS $\Im$

| $\Im$ | | | |
|---|---|---|---|
| $e$ | (0.57, 0.52, 0.91) | (0.09, 0.37, 0.47) | (0.00, 0.49, 0.81) |
| $f$ | (0.79, 0.33, 0.67) | (1.00, 0.00, 0.07) | (0.77, 0.39, 1.00) |

The matrix form of this set is

$$\Im = \begin{pmatrix} (0.57, 0.52, 0.91) & (0.09, 0.37, 0.47) & (0.00, 0.49, 0.81) \\ (0.79, 0.33, 0.67) & (1.00, 0.00, 0.07) & (0.77, 0.39, 1.00) \end{pmatrix}$$

**Definition 3.3.** Let $\Im_1$ and $\Im_2$ be P$m$FNSs over $X$. $\Im_1$ is acknowledged as a *subset* of $\Im_2$, written as $\Im_1 \subseteq \Im_2$, $\forall \Im \in X$ and each values of $i$ ranging from 1 to $m$, if

1) $T^{(i)}_{\Im_1}(\hbar) \leq T^{(i)}_{\Im_2}(\hbar)$,
2) $I^{(i)}_{\Im_1}(\hbar) \geq I^{(i)}_{\Im_2}(\hbar)$,
3) $F^{(i)}_{\Im_1}(\hbar) \geq F^{(i)}_{\Im_2}(\hbar)$.

$\Im_1$ and $\Im_2$ are said to be *equal* if $\Im_1 \subseteq \Im_2 \subseteq \Im_1$ and is written as $\Im_1 = \Im_2$.

**Example 3.4.** Let

$$\Im_1 = \begin{pmatrix} (0.41, 0.29, 1.00) & (0.71, 0.09, 0.88) & (0.49, 0.23, 0.00) \\ (0.39, 0.76, 0.97) & (0.00, 1.00, 0.66) & (0.01, 0.59, 0.77) \\ (0.5, 0.02, 0.03) & (0.04, 0.43, 0.61) & (0.82, 0.03, 0.2) \end{pmatrix}$$

and

$$\Im_2 = \begin{pmatrix} (0.58, 0.06, 0.00) & (0.89, 0.04, 0.19) & (1.00, 0.21, 0.00) \\ (0.92, 0.04, 0.11) & (0.17, 0.00, 0.29) & (1.00, 0.33, 0.23) \\ (0.73, 0.02, 0.01) & (0.64, 0.22, 0.03) & (0.91, 0.01, 0.06) \end{pmatrix}$$

be P$m$FNSs over some set X, then $\Im_1 \subseteq \Im_2$.

**Definition 3.5.** A P$m$FNS $\Im$ over $X$ is known as *null PmFNS* if $T^{(i)}_{\Im}(\hbar) = 0$ , $I^{(i)}_{\Im}(\hbar) = 1$ and $F^{(i)}_{\Im}(\hbar) = 1$, $\forall \hbar \in X$ and all acceptable values of $i$. It is designated by $\Phi$.

Thus,

$$\Phi = \begin{pmatrix} (0,1,1) & (0,1,1) & \cdots & (0,1,1) \\ (0,1,1) & (0,1,1) & \cdots & (0,1,1) \\ \vdots & \vdots & \ddots & \vdots \\ (0,1,1) & (0,1,1) & \cdots & (0,1,1) \end{pmatrix}.$$

**Definition 3.6.** A P$m$FNS $\Im$ over $X$ is called an *absolute PmFNS* if $T^{(i)}_{\Im}(\hbar) = 1$, $I^{(i)}_{\Im}(\hbar) = 0$, and $F^{(i)}_{\Im}(\hbar) = 0$, $\forall \hbar \in X$. It is denoted by $\breve{\chi}$.
Thus,

$$\breve{\chi} = \begin{pmatrix} (1,0,0) & (1,0,0) & \cdots & (1,0,0) \\ (1,0,0) & (1,0,0) & \cdots & (1,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (1,0,0) & (1,0,0) & \cdots & (1,0,0) \end{pmatrix}.$$

**Definition 3.7.** The *complement* of a P$m$FNS

$$\Im = \left\{ \frac{\hbar}{(T^{(i)}_{\Im}(\hbar), I^{(i)}_{\Im}(\hbar), F^{(i)}_{\Im}(\hbar))} : \hbar \in X, i = 1, \cdots, m \right\}$$

over X is defined as

$$\Im^c = \left\{ \frac{\hbar}{(F^{(i)}_{\Im}(\hbar), 1 - I^{(i)}_{\Im}(\hbar), T^{(i)}_{\Im}(\hbar))} : \hbar \in X, i = 1, \cdots, m \right\}.$$

**Example 3.8.** The complement of the P$m$FNS $\Im$ given in example 3.2 is

$$\Im^c = \begin{pmatrix} (0.91, 0.48, 0.57) & (0.47, 0.63, 0.09) & (0.81, 0.51, 0.00) \\ (0.67, 0.67, 0.79) & (0.07, 1.00, 1.00) & (1.00, 0.01, 0.77) \end{pmatrix}.$$

**Remark 3.9.** It may be observed from the entry at $(2,2)$ position of the matrix given in Example 3.8 that $0.07^2 + 1.00^2 + 1.00^2 \nleq 2$. Thus, we may infer that the complement of a P$m$FNS is not always a P$m$FNS. Further, the complement of a P$m$FNS will be a P$m$FNS iff the sum of squares of the three neutrosophic components does not exceed $2I^{(i)} + 1$ i.e. $(T^{(i)})^2 + (I^{(i)})^2 + (F^{(i)})^2 \leq 2I^{(i)} + 1$.

**Definition 3.10.** The *union* of any P$m$FNSs $\Im_1$ and $\Im_2$ expressed over the same universe $X$ is represented as

$$\Im_1 \cup_{\mathfrak{M}} \Im_2 = \left\{ \frac{\hbar}{(\max(T^{(i)}_{\Im 1}(\hbar), T^{(i)}_{\Im 2}(\hbar)), \min(I^{(i)}_{\Im 1}(\hbar), I^{(i)}_{\Im 2}(\hbar)), \min(F^{(i)}_{\Im 1}(\hbar), F^{(i)}_{\Im 2}(\hbar))} : \hbar \in X, i = 1, \cdots, m \right\}$$

**Definition 3.11.** The *intersection* of any P$m$FNSs $\Im_1$ and $\Im_2$ expressed over the same universe $X$ is represented as

$$\Im_1 \cap_{\mathfrak{M}} \Im_2 = \left\{ \frac{\hbar}{(\min(T^{(i)}_{\Im 1}(\hbar), T^{(i)}_{\Im 2}(\hbar)), \max(I^{(i)}_{\Im 1}(\hbar), I^{(i)}_{\Im 2}(\hbar), \max(F^{(i)}_{\Im 1}(\hbar), F^{(i)}_{\Im 2}(\hbar))} : \hbar \in X, i = 1, \cdots, m \right\}$$

**Example 3.12.** If

$$\Im_1 = \begin{pmatrix} (0.57, 0.61, 0.19) & (0.74, 0.61, 0.00) & (0.00, 0.55, 0.22) \\ (0.11, 0.88, 1.00) & (0.49, 0.99, 0.10) & (0.92, 0.67, 0.80) \\ (0.00, 0.36, 0.29) & (0.70, 0.20, 1.00) & (1.00, 0.00, 0.46) \end{pmatrix}$$

and

$$\Im_2 = \begin{pmatrix} (1.00, 0.59, 0.32) & (0.50, 0.72, 1.00) & (0.33, 1.00, 0.70) \\ (0.78, 0.09, 0.50) & (0.00, 0.66, 0.11) & (0.54, 0.61, 0.00) \\ (0.60, 0.00, 0.85) & (0.28, 0.43, 0.90) & (0.83, 0.40, 0.14) \end{pmatrix}$$

are two P$m$FNSs defined over the same universe of discourse $X$, then

$$\Im_1 \cup_{\mathfrak{M}} \Im_2 = \begin{pmatrix} (1.00, 0.59, 0.19) & (0.74, 0.61, 0.00) & (0.33, 0.55, 0.22) \\ (0.78, 0.09, 0.50) & (0.49, 0.66, 0.10) & (0.92, 0.61, 0.00) \\ (0.60, 0.00, 0.29) & (0.70, 0.20, 0.90) & (1.00, 0.00, 0.14) \end{pmatrix}$$

and

$$\Im_1 \cap_{\mathfrak{M}} \Im_2 = \begin{pmatrix} (0.57, 0.61, 0.32) & (0.50, 0.72, 1.00) & (0.00, 1.00, 0.70) \\ (0.11, 0.88, 1.00) & (0.00, 0.99, 0.11) & (0.54, 0.67, 0.80) \\ (0.00, 0.36, 0.85) & (0.28, 0.43, 1.00) & (0.83, 0.40, 0.46) \end{pmatrix}$$

**Proposition 3.13.** *If $\Im, \Im_1, \Im_2, \Im_3$ are P$m$FNSs over X, then*

(1) $\Phi \cup_{\mathfrak{M}} \Im = \Im$
(2) $\Phi \cap_{\mathfrak{M}} \Im = \Phi$
(3) $\breve{\chi} \cup_{\mathfrak{M}} \Im = \breve{\chi}$
(4) $\breve{\chi} \cap_{\mathfrak{M}} \Im = \Im$
(5) $\Im \cup_{\mathfrak{M}} \Im = \Im$
(6) $\Im \cap_{\mathfrak{M}} \Im = \Im$
(7) $\Im_1 \cup_{\mathfrak{M}} \Im_2 = \Im_2 \cup_{\mathfrak{M}} \Im_1$
(8) $\Im_1 \cap_{\mathfrak{M}} \Im_2 = \Im_2 \cap_{\mathfrak{M}} \Im_1$
(9) $\Im_1 \cup_{\mathfrak{M}} (\Im_2 \cup_{\mathfrak{M}} \Im_3) = (\Im_1 \cup_{\mathfrak{M}} \Im_2) \cup_{\mathfrak{M}} \Im_3$
(10) $\Im_1 \cap_{\mathfrak{M}} (\Im_2 \cap_{\mathfrak{M}} \Im_3) = (\Im_1 \cap_{\mathfrak{M}} \Im_2) \cap_{\mathfrak{M}} \Im_3$
(11) $\Im_1 \cup_{\mathfrak{M}} (\Im_2 \cap_{\mathfrak{M}} \Im_3) = (\Im_1 \cup_{\mathfrak{M}} \Im_2) \cap_{\mathfrak{M}} (\Im_1 \cup_{\mathfrak{M}} \Im_3)$
(12) $\Im_1 \cap_{\mathfrak{M}} (\Im_2 \cup_{\mathfrak{M}} \Im_3) = (\Im_1 \cap_{\mathfrak{M}} \Im_2) \cup_{\mathfrak{M}} (\Im_1 \cap_{\mathfrak{M}} \Im_3)$

*Proof.* Here, we prove only (11). We may assume, without losing the generality, that $\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar)) = T^{(i)}_{\Im_1}(\hbar)$, $\max(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)) = I^{(i)}_{\Im_1}(\hbar)$ and $\max(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar)) =$

$F^{(i)}_{\Im_1}(\hbar)$. Then, $\forall \hbar \in X$ and $i = 1, 2, \cdots, m$.

$$\begin{aligned}
\Im_2 \cap_{\mathfrak{M}} \Im_3 &= \left\{ \frac{\hbar}{\big(\min(T^{(i)}_{\Im_2}(\hbar), T^{(i)}_{\Im_3}(\hbar)), \max(I^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_3}(\hbar)), \max(F^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_3}(\hbar))\big)} \right\} \\
&= \left\{ \frac{\hbar}{\big(T^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_2}(\hbar)\big)} \right\} \\
\therefore \Im_1 \cup_{\mathfrak{M}} (\Im_2 \cap_{\mathfrak{M}} \Im_3) &= \left\{ \frac{\hbar}{\big(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar)\big)} \right\} \cup_{\mathfrak{M}} \left\{ \frac{\hbar}{\big(T^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_2}(\hbar)\big)} \right\} \\
&= \left\{ \frac{\hbar}{\big(\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar)), \min(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)), \min(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar))\big)} \right\} \\
&= \left\{ \frac{\hbar}{\big(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar)\big)} \right\}
\end{aligned}$$

and

$$\begin{aligned}
\Im_1 \cup_{\mathfrak{M}} \Im_2 &= \left\{ \frac{\hbar}{\big(\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar)), \min(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)), \min(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar)\big)} \right\} \\
&= \left\{ \frac{\hbar}{\big(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar)\big)} \right\} \\
\Im_1 \cup_{\mathfrak{M}} \Im_3 &= \left\{ \frac{\hbar}{\big(\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_3}(\hbar)), \min(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_3}(\hbar)), \min(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_3}(\hbar))\big)} \right\} \\
&= \left\{ \frac{\hbar}{\big(T^{(i)}_{\Im_3}(\hbar), I^{(i)}_{\Im_3}(\hbar), F^{(i)}_{\Im_3}(\hbar)\big)} \right\} \\
\therefore (\Im_1 \cup_{\mathfrak{M}} \Im_2) \cap_{\mathfrak{M}} (\Im_1 \cup_{\mathfrak{M}} \Im_3) &= \left\{ \frac{\hbar}{\big(\min(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_3}(\hbar)), \max(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_3}(\hbar)), \max(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_3}(\hbar))\big)} \right\} \\
&= \left\{ \frac{\hbar}{\big(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar)\big)} \right\}
\end{aligned}$$

0.1cm□

**Corollary 3.14.** (1) $\Phi \cup_{\mathfrak{M}} \breve{\chi} = \breve{\chi}$

(2) $\Phi \cap_{\mathfrak{M}} \breve{\chi} = \Phi$

**Proposition 3.15.** *If $\Im_1$ and $\Im_2$ are PmFNSs over $X$, then*

(1) $\Im_1 \cap_{\mathfrak{M}} \Im_2 \subseteq \Im_1 \subseteq \Im_1 \cup_{\mathfrak{M}} \Im_2$

(2) $\Im_1 \cap_{\mathfrak{M}} \Im_2 \subseteq \Im_2 \subseteq \Im_1 \cup_{\mathfrak{M}} \Im_2$

*Proof.* The results are easy consequences of properties of max and min. 0.1cm□

**Proposition 3.16.** *Let $\Im_1$, $\Im_2$ be PmFNSs over universe set X, then De Morgan laws hold i.e.*

(1) $(\Im_1 \cup_{\mathfrak{M}} \Im_2)^c = \Im_1^c \cap_{\mathfrak{M}} \Im_2^c$.

(2) $(\Im_1 \cap_{\mathfrak{M}} \Im_2)^c = \Im_1^c \cup_{\mathfrak{M}} \Im_2^c$.

*Proof.* : Here, we demonstrate only (1). The verification of (2) perhaps provided in the same way. We may assume, without losing the generality, that $\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar)) = T^{(i)}_{\Im_1}(\hbar)$, $\max(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)) = I^{(i)}_{\Im_1}(\hbar)$ and $\max(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar)) = F^{(i)}_{\Im_1}(\hbar)$. Then, $\forall\ \hbar \in X$ and $i = 1, 2, \cdots, m$.

$$\begin{aligned}
(\Im_1 \cup_{\mathfrak{M}} \Im_2)^c &= \left\{\frac{\hbar}{(\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar)), \min(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)), \min(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar)))}\right\}^c \\
&= \left\{\frac{\hbar}{(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar))}\right\}^c \\
&= \left\{\frac{\hbar}{(F^{(i)}_{\Im_1}(\hbar), 1 - I^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_1}(\hbar))}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\Im_1^c \cap_{\mathfrak{M}} \Im_2^c &= \left\{\frac{\hbar}{(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar))}\right\}^c \cap_{\mathfrak{M}} \left\{\frac{\hbar}{(T^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_2}(\hbar))}\right\}^c \\
&= \left\{\frac{\hbar}{(F^{(i)}_{\Im_1}(\hbar), 1 - I^{(i)}_{\Im_2}(\hbar), T^{(i)}_{\Im_1}(\hbar))}\right\} \cap_{\mathfrak{M}} \left\{\frac{\hbar}{(F^{(i)}_{\Im_2}(\hbar), 1 - I^{(i)}_{\Im_2}(\hbar), T^{(i)}_{\Im_2}(\hbar))}\right\} \\
&= \left\{\frac{\hbar}{(\min(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar)), 1 - \min(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)), \max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar))}\right\} \\
&= \left\{\frac{\hbar}{(F^{(i)}_{\Im_1}(\hbar), 1 - I^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_1}(\hbar))}\right\}
\end{aligned}$$

0.1cm□

**Remark 3.17.** Let $\Im$ is a P$m$FNS over universe set X. Then

*(1)* $\Im \cup_{\mathfrak{M}} \Im^c \neq \breve{\chi}$

*(2)* $\Im \cap_{\mathfrak{M}} \Im^c \neq \Phi$

**Proposition 3.18.** (1) $\Phi^c = \breve{\chi}$

(2) $\breve{\chi}^c = \Phi$

(3) $(\Im^c)^c = \Im$

*Proof.* Straight forward. 0.1cm□

**Definition 3.19.** The *difference* of two P$m$FNS $\Im_1$ and $\Im_2$ expressed over the same universe X is represented as

$$\Im_1 \setminus \Im_2 = \left\{\frac{\hbar}{(\min(T^{(i)}_{\Im 1}(\hbar), F^{(i)}_{\Im 2}(\hbar)), \min(I^{(i)}_{\Im 1}(\hbar), I^{(i)}_{\Im 2}(\hbar)), \max(F^{(i)}_{\Im 1}(\hbar), T^{(i)}_{\Im 2}(\hbar))} : \hbar \in X, i = 1, 2, \cdots, m\right\}$$

**Example 3.20.** For $\Im_1$ and $\Im_2$ given in Example 3.12, we have

$$\Im_1 \setminus \Im_2 = \begin{pmatrix} (0.32, 0.59, 1.00) & (0.74, 0.61, 0.50) & (0.00, 0.55, 0.33) \\ (0.11, 0.09, 1.00) & (0.11, 0.66, 0.10) & (0.00, 0.61, 0.80) \\ (0.00, 0.00, 0.60) & (0.70, 0.20, 1.00) & (0.14, 0.00, 0.83) \end{pmatrix}$$

**Definition 3.21.** The *symmetric difference* of two P$m$FNSs $\Im_1$ and $\Im_2$ is set of elements which are either in $\Im_1$ or in $\Im_2$ but not in both i.e.

$$\Im_1 \triangle \Im_2 = (\Im_1 \setminus \Im_2) \cup_{\mathfrak{M}} (\Im_2 \setminus \Im_1)$$

**Example 3.22.** Let

$$\Im_1 = \begin{pmatrix} (0.57, 0.61, 0.19) & (0.74, 0.61, 0.00) & (0.00, 0.55, 0.22) \\ (0.11, 0.88, 1.00) & (0.49, 0.99, 0.10) & (0.92, 0.67, 0.80) \\ (0.00, 0.36, 0.29) & (0.70, 0.20, 1.00) & (1.00, 0.00, 0.46) \end{pmatrix}$$

and

$$\Im_2 = \begin{pmatrix} (1.00, 0.59, 0.32) & (0.50, 0.72, 1.00) & (0.33, 1.00, 0.70) \\ (0.78, 0.09, 0.50) & (0.00, 0.66, 0.11) & (0.54, 0.61, 0.00) \\ (0.60, 0.00, 0.85) & (0.28, 0.43, 0.90) & (0.83, 0.40, 0.14) \end{pmatrix}$$

so that

$$\Im_1 \setminus \Im_2 = \begin{pmatrix} (0.32, 0.59, 1.00) & (0.74, 0.61, 0.50) & (0.00, 0.55, 0.33) \\ (0.11, 0.09, 1.00) & (0.11, 0.66, 0.10) & (0.00, 0.61, 0.80) \\ (0.00, 0.00, 0.60) & (0.70, 0.20, 1.00) & (0.14, 0.00, 0.83) \end{pmatrix}$$

and

$$\Im_2 \setminus \Im_1 = \begin{pmatrix} (0.19, 0.59, 0.57) & (0.00, 0.61, 1.00) & (0.22, 0.55, 0.70) \\ (0.78, 0.09, 0.50) & (0.00, 0.66, 0.49) & (0.54, 0.61, 0.92) \\ (0.29, 0.00, 0.85) & (0.28, 0.20, 0.90) & (0.46, 0.00, 1.00) \end{pmatrix}$$

$$\begin{aligned} \therefore (\Im_1 \setminus \Im_2) \cup_{\mathfrak{M}} (\Im_2 \setminus \Im_1) &= \begin{pmatrix} (0.32, 0.59, 0.57) & (0.74, 0.61, 0.50) & (0.22, 0.55, 0.33) \\ (0.78, 0.09, 0.91) & (0.11, 0.66, 0.10) & (0.54, 0.61, 0.80) \\ (0.29, 0.00, 0.60) & (0.70, 0.20, 0.90) & (0.46, 0.00, 0.83) \end{pmatrix} \\ &= \Im_1 \triangle \Im_2 \end{aligned}$$

**Definition 3.23.** The *sum* of two P$m$FNSs $\Im_1$ and $\Im_2$ chosen from same universe $X$ is represented as

$$\Im_1 \oplus \Im_2 = \left\{ \frac{\hbar}{\left(\sqrt{(T^{(i)}_{\Im_1}(\hbar))^2 + (T^{(i)}_{\Im_2}(\hbar))^2 - (T^{(i)}_{\Im_1}(\hbar) T^{(i)}_{\Im_2}(\hbar))^2}, I^{(i)}_{\Im_1}(\hbar) I^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_1}(\hbar) F^{(i)}_{\Im_2}(\hbar)\right)} \right\}$$

where $\hbar \in X$ and $i$ runs from 1 to $m$.

**Example 3.24.** For $\Im_1$ and $\Im_2$ given in Example 3.12, we have

$$\Im_1 \oplus \Im_2 = \begin{pmatrix} (1.00, 0.36, 0.06) & (0.81, 0.44, 0.00) & (0.33, 0.55, 0.15) \\ (0.78, 0.08, 0.50) & (0.49, 0.65, 0.01) & (0.94, 0.41, 0.00) \\ (0.60, 0.00, 0.25) & (0.73, 0.09, 0.90) & (1.00, 0.00, 0.06) \end{pmatrix}$$

**Definition 3.25.** The *product* of two P$m$FNSs $\Im_1$ & $\Im_2$ take off the same universe X is explained as

$$\Im_1 \otimes \Im_2 = \left\{ \frac{\hbar}{(T^{(i)}_{\Im_1}(\hbar) T^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_1}(\hbar) I^{(i)}_{\Im_2}(\hbar), \sqrt{(F^{(i)}_{\Im_1}(\hbar))^2 + (F^{(i)}_{\Im_2}(\hbar))^2 - (F^{(i)}_{\Im_1}(\hbar) F_{\Im_2}(\hbar))^2})} \right\}$$

for $\hbar \in X$ and $i$ runs from 1 to $m$.

**Example 3.26.** For $\Im_1$ and $\Im_2$ given in Example 3.12, we have

$$\Im_1 \otimes \Im_2 = \begin{pmatrix} (0.57, 0.36, 0.37) & (0.37, 0.44, 1.00) & (0.00, 0.55, 0.72) \\ (0.09, 0.08, 1.00) & (0.00, 0.65, 0.15) & (0.49, 0.41, 0.8) \\ (0.00, 0.00, 0.86) & (0.19, 0.09, 1.00) & (0.83, 0.00, 0.48) \end{pmatrix}$$

**Definition 3.27.** If $\Im_1 = \Im_2$ in Definition 3.25, then we express $\Im_1 \otimes \Im_1$ by $\Im_1^2$. Thus,

$$\begin{aligned} \Im^2 &= \left\{ \frac{\hbar}{((T^{(i)}_{\Im}(\hbar))^2, (I^{(i)}_{\Im}(\hbar))^2, \sqrt{2(F^{(i)}_{\Im}(\hbar))^2 - (F^{(i)}_{\Im}(\hbar))^4})} : \hbar \in X; i = 1, 2, \cdots, m \right\} \\ &= \left\{ \frac{\hbar}{((T^{(i)}_{\Im}(\hbar))^2, (I^{(i)}_{\Im}(\hbar))^2, \sqrt{1 - (1 - ((F^{(i)}_{\Im}(\hbar))^2)^2)})} : \hbar \in X; i = 1, 2, \cdots, m \right\} \end{aligned}$$

The set $\Im^2$ is called as *concentration* of $\Im$, written as $con(\Im)$. If $k \in [0, \infty)$, in general, then

$$\Im^k = \left\{ \frac{\hbar}{((T^{(i)}_{\Im}(\hbar))^k, (I^{(i)}_{\Im}(\hbar))^k, \sqrt{1 - (1 - ((F^{(i)}_{\Im}(\hbar))^2)^k)})} : \hbar \in X; i = 1, 2, \cdots, m \right\}$$

The set

$$\Im^{1/2} = \left\{ \frac{\hbar}{(\sqrt{T^{(i)}_{\Im}(\hbar)}, \sqrt{I^{(i)}_{\Im}(\hbar)}, \sqrt{1 - \sqrt{(1 - (F^{(i)}_{\Im}(\hbar))^2)}})} : \hbar \in X; i = 1, 2, \cdots, m \right\}$$

is called as *dilation* of $\Im$, denoted as $dil(\Im)$.

**Example 3.28.** For P$m$FNS $\Im_1$ given in Example 3.12, we have

$$con(\Im) = \begin{pmatrix} (0.32, 0.37, 0.27) & (0.55, 0.37, 0.00) & (0.00, 0.30, 0.31) \\ (0.01, 0.77, 1.00) & (0.24, 0.98, 0.14) & (0.85, 0.45, 0.93) \\ (0.00, 0.13, 0.40) & (0.49, 0.04, 1.00) & (1.00, 0.00, 0.62) \end{pmatrix}$$

and

$$dil(\Im) = \begin{pmatrix} (0.75, 0.78, 0.13) & (0.86, 0.78, 0.00) & (0.00, 0.74, 0.16) \\ (0.33, 0.94, 1.00) & (0.70, 0.99, 0.07) & (0.96, 0.67, 0.63) \\ (0.00, 0.60, 0.21) & (0.84, 0.45, 1.00) & (1.00, 0.00, 0.33) \end{pmatrix}$$

**Definition 3.29.** The *Cartesian product* of two P$m$FNSs $\Im_1$ and $\Im_2$ over $X$ is characterized as

$$\Im_1 \times \Im_2 = \left\{ \frac{(\hbar_1, \hbar_2)}{\left(T^{(i)}_{\Im_1}(\hbar)T^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_1}(\hbar)I^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_1}(\hbar)F^{(i)}_{\Im_2}(\hbar)\right)} : \hbar_1, \hbar_2 \in X; i = 1, 2, \cdots, m \right\}$$

**Example 3.30.** For $\Im_1$ and $\Im_2$ given in Example 3.12, we have

$$\Im_1 \times \Im_2 = \begin{pmatrix} (0.57, 0.36, 0.06) & (0.37, 0.44, 0.00) & (0.00, 0.55, 0.15) \\ (0.44, 0.05, 0.10) & (0.00, 0.40, 0.00) & (0.00, 0.34, 0.00) \\ (0.34, 0.00, 0.16) & (0.21, 0.26, 0.00) & (0.00, 0.22, 0.03) \\ (0.11, 0.52, 0.32) & (0.25, 0.71, 0.10) & (0.30, 0.67, 0.56) \\ (0.09, 0.08, 0.50) & (0.00, 0.65, 0.01) & (0.50, 0.41, 0.00) \\ (0.07, 0.00, 0.85) & (0.14, 0.43, 0.09) & (0.76, 0.27, 0.11) \\ (0.00, 0.21, 0.09) & (0.35, 0.14, 1.00) & (0.33, 0.00, 0.32) \\ (0.00, 0.03, 0.15) & (0.00, 0.13, 0.11) & (0.54, 0.00, 0.00) \\ (0.00, 0.00, 0.25) & (0.20, 0.09, 0.90) & (0.83, 0.00, 0.06) \end{pmatrix}$$

### 3.1. *Superiority of the proposed work*

The superiority of our suggested work is exhibited in Table 3, which is self explanatory. The same applies for the corresponding topology.

TABLE 3. Concise comparison of P$m$FNS set with some prevailing structures

| Set | Membership function | Indeterminacy | Non-membership function | Multiple membership function |
|---|---|---|---|---|
| Fuzzy set [18] | ✓ | × | × | × |
| Intuitionistic fuzzy set [2] | ✓ | × | ✓ | × |
| Pythagorean fuzzy set [15, 16] | ✓ | × | ✓ | × |
| $m$-polar fuzzy set [4] | ✓ | × | × | ✓ |
| Pythagorean $m$-polar fuzzy set [8] | ✓ | × | ✓ | ✓ |
| P$m$FNS (proposed) | ✓ | ✓ | ✓ | ✓ |

## 4. **Pythagorean $m$-polar fuzzy neutrosophic topology**

In this section, we present Pythagorean $m$-polar fuzzy neutrosophic topology on Pythagorean $m$-polar fuzzy neutrosophic set and elongate numerous characteristics of crisp topology towards Pythagorean $m$-polar fuzzy neutrosophic topology. Separation axioms in P$m$FNSs are also discussed.

**Definition 4.1.** Let P$m$FNS($\underset{\sim}{X}$) be the collection of all P$m$FN-subsets of the absolute P$m$FNS $\underset{\sim}{X}_A$. For $\underset{\sim}{S}$, $\underset{\sim}{T} \subseteq \underset{\sim}{A}$, a subcollection $\underset{\sim}{\beth}_{pn}$ of P$m$FNS($\underset{\sim}{X}$) is known as *Pythagorean m-polar fuzzy neutrosophic topology* (P$m$FNT) on $\underset{\sim}{X}$ if the following needs are satisfied:

(i) $\underset{\sim}{\emptyset}$, $\underset{\sim}{X}_A \Subset \underset{\sim}{\beth}_{pn}$,

(ii) $\jmath_S, \jmath_T \Subset \daleth_{pn}$ then $\jmath_S \Cap \jmath_T \Subset \daleth_{pn}$,

(iii) $\jmath_i \Subset \daleth_{pn}, \forall i \Subset \underline{\mathrm{I}}$, then $\Cup_{i \in \underline{\mathrm{I}}} \jmath_i \Subset \daleth_{pn}$.

The doublet $(\underline{\mathrm{X}}, \daleth_{pn})$ or simply $\daleth_{pn}$, where $\underline{\mathrm{X}}$ is a non-empty P$m$FNS and $\daleth_{pn}$ is a Pythagorean $m$-polar fuzzy neutrosophic topology on $\underline{\mathrm{X}}$, is known as *Pythagorean m-polar fuzzy neutrosophic topological space* (P$m$FNTS).

**Example 4.2.** Let $\underline{\mathrm{X}} = \{\hbar_1, \hbar_2\}$ be a universal P3FNS with $\underline{\mathrm{S}}$ and $\underline{\mathrm{T}}$ be as shown in table 4 and table 5 below:

Table 4. P3FNS $\underline{\mathrm{S}}$

| $\underline{\mathrm{S}}$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.401, 0.210, 0.216) | (0.221, 0.100, 0.363) | (0.632, 0.029, 0.216) |
| $\hbar_2$ | (0.626, 0.111, 0.162) | (0.432, 0.000, 0.163) | (0.221, 0.012, 0.108) |

Table 5. P3FNS $\underline{\mathrm{T}}$

| $\underline{\mathrm{T}}$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.126, 0.621, 0.623) | (0.063, 0.920, 0.706) | (0.276, 0.636, 0.591) |
| $\hbar_2$ | (0.168, 0.702, 0.668) | (0.165, 0.761, 0.726) | (0.149, 0.712, 0.561) |

Then $\daleth_{pn5} = \{\underline{\emptyset}, \underline{\mathrm{S}}, \underline{\mathrm{T}}, \underline{\mathrm{X}}_A\}$ is a P3FNT on $\underline{\mathrm{X}}$.

**Definition 4.3.** The members of $\daleth_{pn}$ are called *Pythagorean m-polar fuzzy neutrosophic open sets* (P$m$FN-open sets). The complements of Pythagorean $m$-polar fuzzy neutrosophic open sets are called *Pythagorean m-polar fuzzy neutrosophic closed sets* (P$m$FN-closed sets) and P$m$FN-open set as well as P$m$FN-closed set is called *Pythagorean m-polar fuzzy neutrosophic clopen sets* (P$m$FN-clopen sets).

**Example 4.4.** For the P3FNTS $\daleth_{pn5}$ given in Example 4.2, we have $\underline{\emptyset}, \underline{\mathrm{S}}, \underline{\mathrm{T}}, \underline{\mathrm{X}}_A$ are P3FN-open sets because they are members of $\daleth_{pn5}, (\underline{\mathrm{X}}_A)^c = \underline{\emptyset} \Subset \daleth_{pn}$ is a P3FN-closed set and $\underline{\emptyset}$,$\underline{\mathrm{X}}_A$ are P3FN$-$clopen sets as $\underline{\emptyset}^c = \underline{\mathrm{X}}_A - \underline{\emptyset} = \underline{\mathrm{X}}_A$ and $\underline{\mathrm{X}}_A^c = \underline{\mathrm{X}}_A - \underline{\mathrm{X}}_A = \underline{\emptyset}$

**Example 4.5.** Consider the P3FNSs $\underline{\mathrm{X}}, \underline{\mathrm{S}}$ and $\underline{\mathrm{T}}$ given in Example 4.2 and

Table 6. P3FNS $\underline{\mathrm{U}}$

| $\underline{\mathrm{U}}$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.221, 0.561, 0.524) | (0.172, 0.603, 0.367) | (0.307, 0.633, 0.336) |
| $\hbar_2$ | (0.267, 0.623, 0.201) | (0.380, 0.529, 0.419) | (0.162, 0.560, 0.333) |

We have,

$$\begin{aligned}
\daleth_{pn1} &= \{\emptyset, X_A\}\\
\daleth_{pn2} &= \{\emptyset, S, X_A\}\\
\daleth_{pn3} &= \{\emptyset, T, X_A\}\\
\daleth_{pn4} &= \{\emptyset, U, X_A\}\\
\daleth_{pn5} &= \{\emptyset, S, T, X_A\}\\
\daleth_{pn6} &= \{\emptyset, T, U, X_A\}\\
\daleth_{pn7} &= \{\emptyset, S, U, X_A\}\\
\daleth_{pn8} &= \{\emptyset, S, T, U, X_A\}
\end{aligned}$$

are Pythagorean 3-polar fuzzy neutrosophic topologies over $X$. Here, both $\emptyset$ & $X_A$ are P3FN-open set as well as P3FN-closed set so it is a P3FN-clopen set.

**Definition 4.6.** Let $(X, \daleth_{pn1})$ and $(X, \daleth_{pn2})$ be two P$m$FNTSs on $X$. $\daleth_{pn2}$ is contained in $\daleth_{pn1}$ i.e $\daleth_{pn2} \subseteq \daleth_{pn1}$ if $\kappa \in \daleth_{pn1}$ for every $\kappa \in \daleth_{pn2}$. In such case, $\daleth_{pn2}$ is known as *Pythagorean m−polar fuzzy neutrosophic coarser or weaker* (P$m$FN-coaser/weaker) than $\daleth_{pn1}$ and $\daleth_{pn1}$ is called *Pythagorean m−polar fuzzy neutrosophic finer or stronger* P$m$FN-finer/stronger than $\daleth_{pn2}$. $\daleth_{pn1}$ and $\daleth_{pn2}$ in such a case are known as *comparable*. In Example 4.5, $\daleth_{pn2}$ is P$m$FN-coarser than $\daleth_{pn5}$ and $\daleth_{pn5}$ is P$m$FN-stronger than $\daleth_{pn2}$. Hence $\daleth_{pn2}$ and $\daleth_{pn5}$ are comparable.

**Definition 4.7.** The P$m$FNT $\daleth_{pn(indiscrete)} = \{\emptyset, X_A\}$ is known as *indiscrete Pythagorean m-polar fuzzy neutrosophic topology* (indiscrete-P$m$FNT) & $\daleth_{pn(discrete)} = P(X_A)$ (power set of $X_A$) is known as *discrete Pythagorean m-polar fuzzy neutrosophic topology* (discrete-P$m$FNT) over $X$.

**Remark 4.8.** On $X$, the smallest P$m$FNT is $\daleth_{pn(indiscrete)}$ whereas the largest P$m$FNT is $\daleth_{pn(discrete)}$.

**Definition 4.9.** Suppose that $(X, \daleth_{pnX})$ be a P$m$FNTS. A few $Y \subseteq X$ and P$m$FN-open sets are $S^{\star}_n = S_n \cap Y_A$ of P$m$FNT $\daleth_{pnY}$ on $Y$ where $S_n$ are P$m$FN-open sets of $\daleth_{pnX}$ & $Y_A$ is absolute P$m$FNS on $Y$ then $\daleth_{pnY}$ is reserved as the *Pythagorean m-polar fuzzy neutrosophic subspace* (P$m$FN-subspace) of $\daleth_{pnX}$. It can be written as:

$$\daleth_{pnY} = \{S^{\star}_n : S^{\star}_n = S_n \cap Y_A, S_n \in \daleth_{pnX}\}$$

**Example 4.10.** Let $\daleth_{pnX} = \{\emptyset, S, T, X_A\}$, then $\daleth_{pnX}$ is a P3FNT on $X$. P3FNS on $Y = \{S\} \subseteq X$ is

Table 7. P3FNS $\underline{Y}_A$

| $\underline{Y}_A$ | | | |
|---|---|---|---|
| $\hbar_1$ | (1.000, 0.000, 0.000) | (1.000, 0.000, 0.000) | (1.000, 0.000, 0.000) |

Since

$$\underline{Y}_A \cap \underline{\emptyset} = \underline{\emptyset}$$
$$\underline{Y}_A \cap \underline{S} = \underline{S}$$
$$\underline{Y}_A \cap \underline{T} = \underline{T}$$
$$\underline{Y}_A \cap \underline{X}_A = \underline{Y}_A$$

So, $\underline{\beth}_{pnY} = \{\underline{\emptyset}, \underline{S}, \underline{T}, \underline{Y}_A\}$ is a *Pythagorean 3-polar fuzzy neutrosophic subtopology* (P3FN-subtopology) of $\underline{\beth}_{pnX}$ (i.e $\underline{\beth}_{pnY} \subseteq \underline{\beth}_{pnX}$).

**Remark 4.11.** (1) A P$m$FN-subtopology i.e. $\underline{\beth}_{pnZ}$ of a P$m$FN-subtopology $\underline{\beth}_{pnY}$ of a P$m$FNTS $\underline{\beth}_{pnX}$ is also a P$m$FN-subtopology of $\underline{\beth}_{pnX}$.
(2) Every P$m$FN-subspace of a discrete-P$m$FNTS is always discrete-P$m$FNTS. Similarly, every P$m$FN-subspace of indiscrete-P$m$FNTS ia also an indiscrete-P$m$FNTS.

**Definition 4.12.** Let $(\underline{X}, \underline{\beth}_{pn})$ be a P$m$FNTS and $\underline{V} \subseteq PmFNS(\underline{X})$. The *Pythagorean m-polar fuzzy neutrosophic interior* (P$m$FN-interior) $\underset{\cdot}{\underline{V}}$ of $\underline{V}$ is P$m$FNS which is the union of all P$m$FNS-open subsets (i.e that are contained in $\underline{V}$) of $\underline{X}$.

**Example 4.13.** If

Table 8. P3FNS $\underline{V}$

| $\underline{V}$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.233, 0.449, 0.496) | (0.276, 0.507, 0.365) | (0.332, 0.501, 0.312) |
| $\hbar_2$ | (0.314, 0.416, 0.308) | (0.389, 0.501, 0.402) | (0.267, 0.517, 0.223) |

and $\underline{\beth}_{pn8} = \{\underline{\emptyset}, \underline{S}, \underline{T}, \underline{U}, \underline{X}_A\}$, then $\underset{\cdot}{\underline{V}} = \underline{T} \cup \underline{U} = \underline{U}$
or

Table 9. P3FN-interior $\underset{\cdot}{\underline{V}}$

| $\underset{\cdot}{\underline{V}}$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.221, 0.561, 0.524) | (0.172, 0.603, 0.367) | (0.307, 0.633, 0.336) |
| $\hbar_2$ | (0.267, 0.623, 0.401) | (0.380, 0.529, 0.419) | (0.162, 0.560, 0.333) |

**Definition 4.14.** Let $(\underline{X}, \underline{\beth}_{pn})$ be a P$m$FNTS and $\underline{V} \subseteq$ P$m$FN$(\underline{X})$. Then the *Pythagorean m-polar fuzzy neutrosophic closure* (P$m$FN-closure) $\dot{\underline{V}}$ of $\underline{V}$ is the P$m$FNS which is intersection of all P$m$FN-closed supersets (i.e that contain $\underline{V}$) of $\underline{V}$.

**Example 4.15.** Let $\underline{\beth}_{pn8} = \{\underline{\emptyset}, \underline{S}, \underline{T}, \underline{U}, \underline{X}_A\}$, then first of all we've to find $\underline{\emptyset}^c, \underline{S}^c, \underline{T}^c, \underline{U}^c$, $\underline{X}_A^c$.
$\underline{\emptyset}^c = \underline{X}_A, \underline{S}^c = \underline{S}_1, \underline{T}^c = \underline{T}_1, \underline{U}^c = \underline{U}_1, \underline{X}_A^c = \underline{\emptyset}$ where

Table 10. P3FNS $\underline{S}^c/\underline{S}_1$

| $\underline{S}^c$=$\underline{S}_1$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.216, 0.790, 0.401) | (0.363, 0.900, 0.221) | (0.216, 0.971, 0.632) |
| $\hbar_2$ | (0.162, 0.889, 0.626) | (0.163, 1.000, 0.432) | (0.108, 0.988, 0.221) |

Table 11. P3FNS $\underline{T}^c/\underline{T}_1$

| $\underline{T}^c$=$\underline{T}_1$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.623, 0.379, 0.126) | (0.706, 0.080, 0.063) | (0.591, 0.364, 0.276) |
| $\hbar_2$ | (0.368, 0.298, 0.368) | (0.726, 0.239, 0.165) | (0.561, 0.288, 0.149) |

and

Table 12. P3FNS $\underline{U}^c/\underline{U}_1$

| $\underline{U}^c$=$\underline{U}_1$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.524, 0.439, 0.221) | (0.367, 0.397, 0.172) | (0.336, 0.367, 0.307) |
| $\hbar_2$ | (0.401, 0.377, 0.267) | (0.419, 0.471, 0.380) | (0.333, 0.440, 0.162) |

As $\underline{X}_A$ is the only P3FN-closed supersets of $\underline{V}$ i.e $\underline{V}$ is contained only in $\underline{X}_A$. Thus, $\dot{\underline{V}} = \underline{X}_A$

**Remark 4.16.** Largest P$m$FN-open subset of $\underline{V}$ is $\underset{\cdot}{\underline{V}}$ whereas the smallest P$m$FN-closed superset of $\underline{V}$ is $\dot{\underline{V}}$.

**Definition 4.17.** Let $(\underline{X}, \underline{\beth}_{pn})$ be a P$m$FNTS and $\underline{V} \subseteq$ P$m$FN$(\underline{X})$. Then the *Pythagorean m-polar fuzzy neutrosophic frontier or boundary* (P$m$FN-frontier/boundary) $F^\diamond(\underline{V})$ of $\underline{V}$ is defined as:

$$F^\diamond(\underline{V}) = \dot{\underline{V}} \cap \dot{\underline{V}}^c$$

**Example 4.18.** For the P3FNS $\underline{V}$ given in Example 4.13, we have

TABLE 13. P3FNS $\underline{V}^c$

| $\underline{V}^c$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.496, 0.551, 0.233) | (0.365, 0.493, 0.276) | (0.312, 0.499, 0.332) |
| $\hbar_2$ | (0.308, 0.584, 0.314) | (0.402, 0.499, 0.389) | (0.223, 0.483, 0.267) |

$$\dot{\underline{V}} = \underline{\emptyset}^c \underline{\cap} \underline{T}^c \underline{\cap} \underline{U}^c = \underline{U}^c$$
$$\dot{\underline{V}} = \underline{X} \underline{\cap} \underline{T}_1 \underline{\cap} \underline{U}_1 = \underline{U}_1$$
$$\Rightarrow F^{\diamond}(\underline{V}) = \underline{X} \underline{\cap} \underline{U}_1 = \underline{U}_1$$

**Definition 4.19.** Let $(\underline{X}, \underline{\gimel}_{pn})$ be a P$m$FNTS and $\underline{V} \underline{\subseteq}$ P$m$FN($\underline{X}$). Then the *Pythagorean m-polar fuzzy neutrosophic exterior* (P$m$FN-exterior) $E^{\diamond}(\underline{V})$ of $\underline{V}$ is defined as:

$$E^{\diamond}(\underline{V}) = \underset{\cdot}{\underline{V}}^c$$

From Example 4.5 and 4.15, we get $\underset{\cdot}{\underline{V}}^c = \underline{S}^c \underline{\cup} \underline{\emptyset} = \underline{S}^c = \underline{S}_1$

**Example 4.20.** For the P3FNSs $\underline{S}, \underline{T}, \underline{U}, \underline{V}$ given in Examples 4.5,4.13, and

TABLE 14. P3FNS $\underline{W}$

| $\underline{W}$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.721, 0.110, 0.116) | (0.662, 0.100, 0.265) | (0.621, 0.010, 0.116) |
| $\hbar_2$ | (0.765, 0.011, 0.062) | (0.571, 0.000, 0.006) | (0.795, 0.002, 0.008) |

(i) $\underset{\cdot}{\underline{V}} \underline{\subseteq} \underline{V} \underline{\subseteq} \dot{\underline{V}}$ (See Table 8 and Table 9) and as we know

TABLE 15. P3FNS $\underline{X}$

| $\underline{X}$ | | | |
|---|---|---|---|
| $\hbar_1$ | (1.000, 0.000, 0.000) | (1.000, 0.000, 0.000) | (1.000, 0.000, 0.000) |
| $\hbar_2$ | (1.000, 0.000, 0.000) | (1.000, 0.000, 0.000) | (1.000, 0.000, 0.000) |

(ii) $\underset{:}{\underline{V}} = \underset{\cdot}{\underline{V}}$

$\underset{:}{\underline{V}} = \underline{T} \underline{\cup} \underline{U} = \underline{U}$

$\underset{\cdot}{\underline{V}} = \underline{U}$

From above equations we get, $\underset{:}{\underline{V}} = \underset{\cdot}{\underline{V}}$

(iii) $\overset{:}{\underline{V}} = \dot{\underline{V}}$

$\overset{:}{\underline{V}} = \underline{X}$ and $\dot{\underline{V}} = \underline{X}$

(iv) $\underset{\cdot}{\underline{\mathrm{X}}} = \underline{\mathrm{X}}$

$\dot{\underline{\mathrm{X}}} = \underline{\mathrm{S}}\underline{\cup}\underline{\mathrm{T}}\underline{\cup}\underline{\mathrm{U}}\underline{\cup}\underline{\mathrm{X}} = \underline{\mathrm{X}}$

(v) $\dot{\underline{\emptyset}} = \underline{\emptyset}$

As $\underline{\emptyset}$ is superset of itself only.

(vi) $\underline{\mathrm{V}} \subseteq \underline{\mathrm{W}} \Rightarrow \underset{\cdot}{\underline{\mathrm{V}}} \subseteq \underset{\cdot}{\underline{\mathrm{W}}}$ and $\dot{\underline{\mathrm{V}}} \subseteq \dot{\underline{\mathrm{W}}}$

We know that, $\underset{\cdot}{\underline{\mathrm{V}}} = \underline{\mathrm{U}}$ and $\underset{\cdot}{\underline{\mathrm{W}}} = \underline{\mathrm{S}}\underline{\cup}\underline{\mathrm{T}}\underline{\cup}\underline{\mathrm{U}} = \underline{\mathrm{S}} \Rightarrow \underset{\cdot}{\underline{\mathrm{V}}} \subseteq \underset{\cdot}{\underline{\mathrm{W}}} (\because \underline{\mathrm{U}} \subseteq \underline{\mathrm{S}})$

Now, $\dot{\underline{\mathrm{V}}} = \underline{\mathrm{X}}$ also $\dot{\underline{\mathrm{W}}} = \underline{\mathrm{X}} \Rightarrow \dot{\underline{\mathrm{V}}} \subseteq \dot{\underline{\mathrm{W}}} (\because \underline{\mathrm{X}} \subseteq \underline{\mathrm{X}})$

(vii) $(\mathrm{V}\underset{\cdot}{\underline{\cap}}\mathrm{W}) = \underset{\cdot}{\underline{\mathrm{V}}}\underline{\cap}\underset{\cdot}{\underline{\mathrm{W}}}$

TABLE 16. P3FNS $\underline{\mathrm{V}}\underline{\cap}\underline{\mathrm{W}}$

| $\underline{\mathrm{V}}\underline{\cap}\underline{\mathrm{W}}$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.233, 0.449, 0.496) | (0.276, 0.507, 0.365) | (0.332, 0.501, 0.312) |
| $\hbar_2$ | (0.314, 0.416, 0.308) | (0.389, 0.501, 0.402) | (0.267, 0.517, 0.223) |

$(\mathrm{V}\underset{\cdot}{\underline{\cap}}\mathrm{W}) = \underline{\mathrm{T}}\underline{\cup}\underline{\mathrm{U}} = \underline{\mathrm{U}}$ and $\underset{\cdot}{\underline{\mathrm{V}}}\underline{\cap}\underset{\cdot}{\underline{\mathrm{W}}} = \underline{\mathrm{S}}\underline{\cap}\underline{\mathrm{U}} = \underline{\mathrm{U}}$

From above equations, we get the result, $(\mathrm{V}\underset{\cdot}{\underline{\cap}}\mathrm{W}) = \underset{\cdot}{\underline{\mathrm{V}}}\underline{\cap}\underset{\cdot}{\underline{\mathrm{W}}}$

**Proposition 4.21.** *Let $(\underline{X}, \underline{\beth}_{pn})$ be a PmFNTS and $\underline{Q} \subseteq \underline{X}$, then*

(i) $(\underset{\cdot}{\underline{Q}})^c = \dot{\underline{Q}}^c$

(ii) $(\dot{\underline{Q}})^c = \underset{\cdot}{\underline{Q}}^c$

*Proof.* (i) $\underline{\mathrm{Q}} = \left\{ \frac{\hbar_1}{(T^{(i)}(\hbar_1), I^{(i)}(\hbar_1), F^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{\mathrm{X}}, i = 1, 2, \cdots, m \right\}$

Let P$m$FN-open sets contained in $\underline{\mathrm{Q}}$ be indexed by the collection

$$\left\{ \frac{\hbar_1}{(T_j^{(i)}(\hbar_1), I_j^{(i)}(\hbar_1), F_j^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{\mathrm{X}}, i = 1, 2, \cdots, m; j \underline{\in} \underline{\mathrm{J}} \right\}.$$

By definition,

$$\underset{\cdot}{\underline{\mathrm{Q}}} = \left\{ \frac{\hbar_1}{(\max\ T_j^{(i)}(\hbar_1), \min\ I_j^{(i)}(\hbar_1), \min\ F_j^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{\mathrm{X}}, i = 1, 2, \cdots, m; j \underline{\in} \underline{\mathrm{J}} \right\}$$

and

$$(\underset{\cdot}{\underline{\mathrm{Q}}})^c = \left\{ \frac{\hbar_1}{(\min\ F_j^{(i)}(\hbar_1), 1 - \min\ I_j^{(i)}(\hbar_1), \max\ T_j^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{\mathrm{X}}, i = 1, 2, \cdots, m; j \underline{\in} \underline{\mathrm{J}} \right\}$$

$$\because \underline{\mathrm{Q}}^c = \left\{ \frac{\hbar_1}{(F^{(i)}(\hbar_1), 1 - I^{(i)}(\hbar_1), T^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{\mathrm{X}}, i = 1, 2, \cdots, m \right\}$$

Also, $T_j^{(i)}(\hbar_1) \leq T^{(i)}(\hbar_1), 1 - (I_j^{(i)}(\hbar_1)) \geq 1 - (I^{(i)}(\hbar_1)), F_j^{(i)}(\hbar_1) \geq F^{(i)}(\hbar_1), \forall$ values of $i$ & $j \underline{\in} \underline{\mathrm{J}}$, so it develops that

$$\left\{ \frac{\hbar_1}{(F_j^{(i)}(\hbar_1), 1 - I_j^{(i)}(\hbar_1), T_j^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{\mathrm{X}}, i = 1, 2, \cdots, m; j \underline{\in} \underline{\mathrm{J}} \right\}$$

is the entire of P$m$FN-closed sets contained contained $\underset{\sim}{Q}^c$ i.e.

$$\dot{\underset{\sim}{Q}}^c = \left\{ \frac{\hbar_1}{(\min\ F_j^{(i)}(\hbar_1), 1-(\min\ I_j^{(i)}(\hbar_1)), \max\ T_j^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i=1,2,\cdots,m; j \underset{\sim}{\in} \underset{\sim}{J} \right\}$$

which completes the proof.

(ii) $\underset{\sim}{Q} = \left\{ \frac{\hbar_1}{(T^{(i)}(\hbar_1), I^{(i)}(\hbar_1), F^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i=1,2,\cdots,m \right\}$

Let P$m$FN-closed supersets of $\underset{\sim}{Q}$ be indexed by the collection

$$\left\{ \frac{\hbar_1}{(T_j^{(i)}(\hbar_1), I_j^{(i)}(\hbar_1), F_j^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i=1,2,\cdots,m; j \underset{\sim}{\in} \underset{\sim}{J} \right\}$$

By definition,

$$\dot{\underset{\sim}{Q}} = \left\{ \frac{\hbar_1}{(min\ T_j^{(i)}(\hbar_1), max\ I_j^{(i)}(\hbar_1), max\ F_j^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i=1,2,\cdots,m; j \underset{\sim}{\in} \underset{\sim}{J} \right\}$$

and

$$(\dot{\underset{\sim}{Q}})^c = \left\{ \frac{\hbar_1}{(max\ F_j^{(i)}(\hbar_1), 1-(max\ I_j^{(i)}(\hbar_1)), min\ T_j^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i=1,2,\cdots,m; j \underset{\sim}{\in} \underset{\sim}{J} \right\}$$

Now,

$$\because \underset{\sim}{Q}^c = \left\{ \frac{\hbar_1}{(F^{(i)}(\hbar_1), 1-I^{(i)}(\hbar_1), T^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i=1,2,\cdots,m \right\}$$

and $T_j^{(i)}(\hbar_1) \leq T^{(i)}(\hbar_1), 1-(I_j^{(i)}(\hbar_1)) \geq 1-(I^{(i)}(\hbar_1)), F_j^{(i)}(\hbar_1) \geq F^{(i)}(\hbar_1), \forall$ values of $i$ and $j \underset{\sim}{\in} \underset{\sim}{J}$ so it follows that

$$\underset{\dot{\sim}}{Q}^c = \left\{ \frac{\hbar_1}{(max\ F_j^{(i)}(\hbar_1), 1-(max\ I_j^{(i)}(\hbar_1)), min\ T_j^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i=1,2,\cdots,m; j \underset{\sim}{\in} \underset{\sim}{J} \right\}$$

which completes the proof.

0.1cm□

**Proposition 4.22.**

*(i)* $\underset{\dot{\sim}}{Q} \neq \underset{\sim}{Q} - \dot{\underset{\sim}{Q}}^c$

*(ii)* $E^\diamond(\underset{\sim}{Q})^c = \underset{\dot{\sim}}{Q}$

*(iii)* $E^\diamond(\underset{\sim}{Q}) = \underset{\dot{\sim}}{Q}^c$

*(iv)* $E^\diamond(\underset{\sim}{Q}) \underset{\sim}{\cup} F^\diamond(\underset{\sim}{Q}) \underset{\sim}{\cup} \underset{\dot{\sim}}{Q} \neq \underset{\sim}{X}_A$

*(v)* $F^\diamond(\underset{\sim}{Q}) = F^\diamond(\underset{\sim}{Q}^c)$

*(vi)* $\underset{\dot{\sim}}{Q} \underset{\sim}{\cap} F^\diamond(\underset{\sim}{Q}) \neq \underset{\sim}{\emptyset}$

*(vii)* $\dot{\underset{\sim}{Q}} \neq \underset{\sim}{Q} \underset{\sim}{\cup} F^\diamond(\underset{\sim}{Q})$

*(viii)* $\dot{\underset{\sim}{Q}} \neq \underset{\dot{\sim}}{Q} \underset{\sim}{\cup} F^\diamond(\underset{\sim}{Q})$

*Proof.* Follows directly from definition. 0.1cm□

**Proposition 4.23.** *Let $(\underset{\sim}{X}, \daleth_{pn})$ be a PmFNTS and $\underset{\sim}{Q} \subseteq \underset{\sim}{X}$, then $F^{\diamond}(\underset{\sim}{Q})$=$F^{\diamond}(\underset{\sim}{Q}^c)$*

*Proof.* By definition; $F^{\diamond}(\underset{\sim}{Q}) = \dot{\underset{\sim}{Q}} \cap \dot{\underset{\sim}{Q}}^c = \dot{\underset{\sim}{Q}}^c \cap \dot{\underset{\sim}{Q}} = \dot{\underset{\sim}{Q}}^c \cap (\dot{\underset{\sim}{Q}}^c)^c = F^{\diamond}(\underset{\sim}{Q}^c)$ 0.1cm□

**Remark 4.24.** The intersection of two or more P$m$FNTSs is always a P$m$FNTS but it is not necessary that their union is also a P$m$FNTS.

**Example 4.25.** Let $\underset{\sim}{X} = \{\hbar_1, \hbar_2\}$ be a universal non-empty P3FNS and let

TABLE 17. P3FNS $\underset{\sim}{Q}_1$

| $\underset{\sim}{Q}_1$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.211, 0.301, 0.451) | (0.251, 0.321, 0.420) | (0.021, 0.567, 0.481) |
| $\hbar_2$ | (0.100, 0.500, 0.256) | (0.257, 0.421, 0.000) | (0.424, 0.567, 0.291) |

TABLE 18. P3FNS $\underset{\sim}{Q}_2$

| $\underset{\sim}{Q}_2$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.312, 0.217, 0.111) | (0.171, 0.367, 0.582) | (0.361, 0.272, 0.391) |
| $\hbar_2$ | (0.111, 0.421, 0.156) | (0.167, 0.568, 0.721) | (0.321, 0.666, 0.382) |

be P3FNSs over $\underset{\sim}{X}$, then $\daleth_{pno1} = \{\underset{\sim}{\emptyset}, \underset{\sim}{Q}_1, \underset{\sim}{X}_A\}$ and $\daleth_{pno2} = \{\underset{\sim}{\emptyset}, \underset{\sim}{Q}_2, \underset{\sim}{X}_A\}$ are two P3FNTs over $\underset{\sim}{X}$. However, $\daleth_{pno1} \cup \daleth_{pno2} = \{\underset{\sim}{\emptyset}, \underset{\sim}{Q}_1, \underset{\sim}{Q}_2, \underset{\sim}{X}_A\}$ fails to be P3FNT on $\underset{\sim}{X}$ and intersection of P3FNT over $\underset{\sim}{X}$, $\daleth_{pno1} \cap \daleth_{pno2} = \{\underset{\sim}{\emptyset}, \underset{\sim}{X}_A\}$ is also a P3FNT.

**Theorem 4.26.** *Let $(\underset{\sim}{X}, \daleth_{pn})$ be a PmFNTS then the following conditions are satisfied:*

(1) *$\underset{\sim}{\emptyset}$, $\underset{\sim}{X}_A$ are PmFN−open sets.*
(2) *Union of any number of PmFN−open sets is PmFN−open set.*
(3) *Intersection of any number of PmFN−closed sets is PmFN−closed set.*
(4) *The intersection of any two PmFN−open sets (and hence of any finite number of PmFN−open sets) is PmFN−open set.*
(5) *The union of any two PmFN−closed sets (and hence of any finite number of PmFN−closed sets) is PmFN−closed set.*
(6) *$\underset{\sim}{\emptyset}$, $\underset{\sim}{X}_A$ are PmFN−closed set.*

*Proof.* (1) The proof is obvious.

(2) Let $\{< \hbar, (\daleth_1^{(i)}(\hbar), \ddot{\underset{\sim}{I}}_1^{(i)}(\hbar), \underset{\sim}{F}_1^{(i)}(\hbar)) >: \hbar \in \underset{\sim}{X}\}$ be a collection of P$m$FN-open sets.
Also, $\underset{\sim}{Y} = \bigcup_{\hbar \in \underset{\sim}{X}} \{< \hbar, (\daleth_1^{(i)}(\hbar), \ddot{\underset{\sim}{I}}_1^{(i)}(\hbar), \underset{\sim}{F}_1^{(i)}(\hbar)) >\}$

Let $\hbar^{\dagger}\Subset \c{Y}$ implies that $\hbar^{\dagger}\Subset\{<\hbar,(\daleth_1^{(i)}(\hbar),\ddot{I}_1^{(i)}(\hbar),\digamma_1^{(i)}(\hbar))>\}$ for some $\hbar\Subset\underline{X}$ and $\c{B}(\underline{y},\underline{r})\subseteq\{<\hbar,(\daleth_1^{(i)}(\hbar),\ddot{I}_1^{(i)}(\hbar),\digamma_1^{(i)}(\hbar))>\}\subseteq\bigcup_{\hbar\Subset\underline{X}}\{<\hbar,(\daleth_1^{(i)}(\hbar),\ddot{I}_1^{(i)}(\hbar),\digamma_1^{(i)}(\hbar))>\}=\c{Y}$

$\Rightarrow \c{Y}$ is P$m$FN-open set.

(3) Let $\{<\hbar,(\daleth_1^{(i)}(\hbar),\ddot{I}_1^{(i)}(\hbar),\digamma_1^{(i)}(\hbar))>:\hbar\Subset\underline{X}\}$ be any number of P$m$FN-closed sets.

We shall show that $\bigcap_{\hbar\Subset\underline{X}}\{<\hbar,(\daleth_1^{(i)}(\hbar),\ddot{I}_1^{(i)}(\hbar),\digamma_1^{(i)}(\hbar))>\}$, is P$m$FN-closed set, by proving that its complement is P$m$FN-open set.

By De Morgan's law,

$$[\bigcap_{\hbar\Subset\underline{X}}\{<\hbar,(\daleth_1^{(i)}(\hbar),\ddot{I}_1^{(i)}(\hbar),\digamma_1^{(i)}(\hbar))>\}]^c=\bigcup_{\hbar\Subset\underline{X}}\{<\hbar,(\daleth_1^{(i)}(\hbar),\ddot{I}_1^{(i)}(\hbar),\digamma_1^{(i)}(\hbar))>\}^c$$

Since each $<\hbar,(\daleth_1^{(i)}(\hbar),\ddot{I}_1^{(i)}(\hbar),\digamma_1^{(i)}(\hbar))>$ is P$m$FN-closed set, each $\{<\hbar,(\daleth_1^{(i)}(\hbar),$ $\ddot{I}_1^{(i)}(\hbar),\digamma_1^{(i)}(\hbar))>\}^c$ is a P$m$FN-open set ( by definition of P$m$FN-closed set).

So, $\bigcup_{\hbar\Subset\underline{X}}\{<\hbar,(\daleth_1^{(i)}(\hbar),\ddot{I}_1^{(i)}(\hbar),\digamma_1^{(i)}(\hbar))>\}^c$ is P$m$FN-open set.

Hence $\bigcap_{\hbar\Subset\underline{X}}\{<\hbar,(\daleth_1^{(i)}(\hbar),\ddot{I}_1^{(i)}(\hbar),\digamma_1^{(i)}(\hbar))>\}$ is P$m$FN-closed set.

(4) and (5) may be established in the similar way.

(6) The complement of $\underline{X}_A$ is the P$m$FN-open set $\underline{\emptyset}$ and the complement of $\underline{\emptyset}$ is the P$m$FN-open set $\underline{X}_A$. So, $\underline{X}_A$ and $\underline{\emptyset}$ are P$m$FN-closed sets.

0.1cm□

**Definition 4.27.** Let $(\underline{X},\underline{\beth}_{pn})$ be a P$m$FNTS and let $\hbar$ be a P$m$FN-point of $\underline{X}$. $\aleph^{\dagger}\subseteq\underline{X}$ is called a *neighborhood* of $\hbar$ iff there exists a P$m$FN-open set $\text{Ł}^{\dagger}$ s.t. $\hbar\Subset\text{Ł}^{\dagger}$ and $\text{Ł}^{\dagger}\Subset\aleph^{\dagger}$ (or, for short, $\hbar\Subset\text{Ł}^{\dagger}\subseteq\aleph^{\dagger}$). In other words, $\aleph^{\dagger}$ is a neighborhood of $\hbar$, iff it contains some P$m$FN-open set to which $\hbar$ belongs.

**Example 4.28.** Let $\mathbb{X}=\{\underline{e},\underline{f},\underline{g}\}$ be a universal P3FNS and $\underline{\beth}_{pn}=\{\underline{\emptyset},\mathbb{D}_1,\mathbb{D}_2,\mathbb{X}_A\}$ where,

TABLE 19. P3FNS $\mathbb{D}_1$

| $\mathbb{D}_1$ | | | |
|---|---|---|---|
| $\underline{e}$ | (0.672, 0.421, 0.221) | (0.567, 0.420, 0.111) | (0.242, 0.121, 0.199) |
| $\underline{f}$ | (0.211, 0.467, 0.520) | (0.562, 0.721, 0.221) | (0.444, 0.333, 0.111) |
| $\underline{g}$ | (0.167, 0.437, 0.561) | (0.466, 0.167, 0.321) | (0.252, 0.467, 0.490) |

and,

Table 20. P3FNS $\mathbb{D}_2$

| $\mathbb{D}_2$ | | | |
|---|---|---|---|
| $\underline{f}$ | (0.115, 0.226, 0.421) | (0.462, 0.621, 0.221) | (0.555, 0.222, 0.001) |
| $\underline{g}$ | (0.267, 0.337, 0.461) | (0.366, 0.017, 0.421) | (0.452, 0.376, 0.241) |

$\mathbb{X}_A$ is the only P3FN−open set of

Table 21. P3FNS $\hbar_1^*$

| $\hbar_1^*$ | | | |
|---|---|---|---|
| $\underline{e}$ | (1.000, 0.000, 0.000) | (1.000, 0.000, 0.000) | (1.000, 0.000, 0.000) |

So, $\mathbb{X}_A$ is the only neighborhood of $\hbar_1^*$.
The P3FN−point

Table 22. P3FNS $\hbar_2^*$

| $\hbar_2^*$ | | | |
|---|---|---|---|
| $\underline{f}$ | (0.111, 0.562, 0.621) | (0.461, 0.921, 0.178) | (0.321, 0.642, 0.316) |

has three neighborhoods, namely, $\mathbb{D}_1, \mathbb{D}_2$ and $\mathbb{X}_A$.
Similarly, the P3FN−point

Table 23. P3FNS $\hbar_3^*$

| $\hbar_3^*$ | | | |
|---|---|---|---|
| $\underline{e}$ | (0.462, 0.562, 0.398) | (0.367, 0.572, 0.192) | (0.120, 0.499, 0.400) |

has two neighborhoods $\mathbb{D}_1$ and $\mathbb{X}_A$.

**Remark 4.29.** In an indiscrete−PmFNTS, each P$m$FN-point has a single neighborhood which is the ground P$m$FNS itself.

The following example illustrate the P$m$FN-point that a neighborhood of a P$m$FN-point may not be P$m$FN-open set.

**Example 4.30.** Let $\mathbb{X} = \{\underline{e}, \underline{f}, \underline{g}\}$ be an universal non-empty P3FNS and $\daleth_{pn} = \{\underline{\emptyset}, \mathbb{D}_4, \mathbb{X}_A\}$ where,

Table 24. P3FNS $\mathbb{D}_4$

| $\mathbb{D}_4$ | | | |
|---|---|---|---|
| $\underline{f}$ | (0.315, 0.226, 0.421) | (0.162, 0.621, 0.221) | (0.555, 0.222, 0.001) |
| $\underline{g}$ | (0.267, 0.337, 0.461) | (0.366, 0.017, 0.421) | (0.452, 0.376, 0.241) |

clearly the P3FNS

Table 25. P3FNS $\mathbb{D}_3$

| $\mathbb{D}_3$ | | | |
|---|---|---|---|
| $\underline{e}$ | (0.672, 0.421, 0.221) | (0.567, 0.420, 0.111) | (0.242, 0.121, 0.199) |
| $\underline{f}$ | (1.000, 0.000, 0.000) | (0.715, 0.421, 0.226) | (1.000, 0.000, 0.000) |
| $\underline{g}$ | (0.452, 0.421, 0.324) | (1.000, 0.210, 0.000) | (0.667, 0.210, 0.140) |

is a neighborhood of $\mathbb{D}_4$, but it is not P3FN-open set because it is not an element of $\underline{\beth}_{pn}$.

The following theorem enables us to recognize P$m$FN-open sets by knowing all the neighborhoods of a point and conversely. Thus, knowledge about P$m$FN-open sets enables us to determine the neighborhood of a point and conversely.

**Theorem 4.31.** *If $(\underline{X}, \underline{\beth}_{pn})$ is a PmFNTS, then a PmFN-subset $\mathbb{A}$ of $\underline{X}$ is PmFN-open set, iff $\mathbb{A}$ is a neighborhood of each of its PmFN-points.*

*Proof.* Assume that $\mathbb{A}$ is P$m$FN-open set. We shall show that $\mathbb{A}$ is a neighborhood of each of its P$m$FN-points. Let $\underline{\varkappa}$ be any P$m$FN-point of $\mathbb{A}$, then $\mathbb{A}$ itself can play the role of the P$m$FN-open set, whose existence qualifies $\mathbb{A}$ to be a neighborhood of $\underline{\varkappa}$. Symbolically, $\underline{\varkappa} \underline{\in} \mathbb{A} \subseteq \mathbb{A}$ where $\mathbb{A}$ is P$m$FN-open set. It follows that $\mathbb{A}$ is neighborhood of each of its P$m$FN-points.
Conversely, if $\mathbb{A}$ is a neighborhood of every P$m$FN-point belonging to it, then for each $\underline{\varkappa} \underline{\in} \mathbb{A}$ there exists a P$m$FN-open set $\chi$ such that $\underline{\varkappa} \underline{\in} \chi \subseteq \mathbb{A}$. Then

$$\mathbb{A} = \underline{\cup}\{< \hbar, (\daleth^{(i)}_{A1}(\hbar), \ddot{\underline{I}}^{(i)}_{A1}(\hbar), \underline{F}^{(i)}_{A1}(\hbar)) >: \hbar \underline{\in} \mathbb{A}\}$$
$$\subseteq \underline{\cup}\{< \hbar, (\daleth^{(i)}_{A2}(\hbar), \ddot{\underline{I}}^{(i)}_{A2}(\hbar), \underline{F}^{(i)}_{A2}(\hbar)) >: \hbar \underline{\in} \mathbb{A}\} \subseteq \mathbb{A}$$

The simultaneous validity of

$$\mathbb{A} \subseteq \underline{\cup}\{< \hbar, (\daleth^{(i)}_{A2}(\hbar), \ddot{\underline{I}}^{(i)}_{A2}(\hbar), \underline{F}^{(i)}_{A2}(\hbar)) >: \hbar \underline{\in} \mathbb{A}\}$$

and

$$\underline{\cup}\{< \hbar, (\daleth^{(i)}_{A2}(\hbar), \ddot{\underline{I}}^{(i)}_{A2}(\hbar), \underline{F}^{(i)}_{A2}(\hbar)) >: \hbar \underline{\in} \mathbb{A}\} \subseteq \mathbb{A}$$
$$\Rightarrow \mathbb{A} = \underline{\cup}\{< \hbar, (\daleth^{(i)}_{A2}(\hbar), \ddot{\underline{I}}^{(i)}_{A2}(\hbar), \underline{F}^{(i)}_{A2}(\hbar)) >: \hbar \underline{\in} \mathbb{A}\}$$

Since the union of P$m$FN-open sets is also P$m$FN-open set, it follows that $\mathbb{A}$ is P$m$FN-open set. 0.1cm□

The most important properties of neighborhoods in a P$m$FNTS are established in the following:

**Definition 4.32.** Let $\underline{\varkappa}$ be a P$m$FN-point in a P$m$FNTS $(\underline{\mathrm{X}}, \daleth_{pn})$. Then the set of all neighborhoods of $\underline{\varkappa}$ is called the *neighborhood system* of the P$m$FN-point $\underline{\varkappa}$ and is denoted by $\aleph^{\dagger}(\underline{\varkappa})$.

**Definition 4.33.** Let $(\underline{\mathrm{X}}, \daleth_{pn})$ be a P$m$FNTS and $\mathbb{A}$ is a P$m$FN−subset of $\underline{\mathrm{X}}$. A point $\underline{\varkappa} \Subset \underline{\mathrm{X}}$ is known as *Pythagorean m-polar fuzzy neutrosophic limit point* (P$m$FN-limit point) or *Pythagorean m-polar fuzzy neutrosophic cluster point* or *Pythagorean m−polar fuzzy neutrosophic accumulation point* $\mathbb{A}$ if every P$m$FN-open set,containing $\underline{\varkappa}$ contains a P$m$FN-point of $\mathbb{A}$ different from $\underline{\varkappa}$.

**Example 4.34.** Let $(\underline{\mathrm{X}}, \daleth_{pn})$ is a P3FNTS, $\mathbb{X} = \{\underline{\mathrm{e}}, \underline{\mathrm{f}}, \underline{\mathrm{g}}\}$ be an universal non-empty P3FNS and

Table 26. P3FNS $\mathbb{C}$

| $\mathbb{C}$ | | | |
|---|---|---|---|
| $\underline{\mathrm{e}}$ | (0.000, 1.000, 1.000) | (0.000, 1.000, 1.000) | (0.000, 1.000, 1.000) |
| $\underline{\mathrm{f}}$ | (0.511, 0.062, 0.211) | (0.312, 0.270, 0.137) | (0.921, 0.266, 0.152) |
| $\underline{\mathrm{g}}$ | (0.232, 0.101, 0.431) | (0.466, 0.352, 0.121) | (0.368, 0.572, 0.400) |

Table 27. P3FNS $\hbar_4^*$

| $\hbar_4^*$ | | | |
|---|---|---|---|
| $\underline{\mathrm{e}}$ | (0.417, 0.312, 0.356) | (0.312, 0.270, 0.137) | (0.012, 0.374, 0.436) |
| $\underline{\mathrm{f}}$ | (0.412, 0.117, 0.362) | (0.333, 0.672, 0.491) | (0.068, 0.772, 0.221) |

and,

Table 28. P3FNS $\hbar_5^*$

| $\hbar_5^*$ | | | |
|---|---|---|---|
| $\underline{\mathrm{e}}$ | (0.324, 0.467, 0.576) | (0.247, 0.657, 0.421) | (0.001, 0.476, 0.891) |

then,

TABLE 29. P3FNS $\hbar_4^* - \hbar_5^*$

| $\hbar_4^* - \hbar_5^*$ | | | |
|---|---|---|---|
| $\underline{e}$ | (0.417, 0.312, 0.356) | (0.312, 0.270, 0.247) | (0.012, 0.374, 0.436) |
| $\underline{f}$ | (0.000, 0.117, 1.000) | (0.000, 0.672, 1.000) | (0.000, 0.772, 1.000) |

TABLE 30. P3FNS $(\hbar_4^* - \hbar_5^*) \Cap \mathbb{C}$

| $(\hbar_4^* - \hbar_5^*) \Cap \mathbb{C}$ | | | |
|---|---|---|---|
| $\underline{e}$ | (0.000, 1.000, 1.000) | (0.000, 1.000, 1.000) | (0.000, 1.000, 1.000) |
| $\underline{f}$ | (0.000, 0.117, 1.000) | (0.000, 0.672, 1.000) | (0.000, 0.772, 1.000) |

As $(\hbar_4^* - \hbar_5^*) \Cap \mathbb{C} \neq \underline{\emptyset}$. So, $\hbar_5^*$ is the P3FN-limit point of $\mathbb{C}$.

**Definition 4.35.** Let $(\underline{X}, \daleth_{pn})$ be a P$m$FNTS then *Pythagorean m-polar fuzzy neutrosophic basis* (P$m$FN-basis) $\underline{\mathbb{B}}^{\circledcirc} \subseteq \daleth_{pn}$ for $\daleth_{pn}$ if for each $\underline{¥} \Subset \daleth_{pn}, \exists \_ \Subset \mathbb{B}$ such that $\underline{¥} = \underline{\cup}$.

4.1. *Separation Axioms in Pythagorean m-Polar Fuzzy Neutrosophic Sets*

**Definition 4.36.** A P$m$FNTS $(\underline{X}, \daleth_{pn})$ is known as a *Pythagorean m-polar fuzzy neutrosophic $T_0$ space* (P$m$FN$T_0$S) if for every pair of distinct P$m$FN-points $\eth_1, \eth_2 \exists$ at any rate 1 P$m$FN-open set $\underline{\eth}$ including precisely one of the P$m$NF-points.

**Example 4.37.** Each discrete P$m$FNTS is a P$m$FN$T_0$S for $\exists$ a P$m$FN-open set $\{\eth_1\}$ that clearly contains $\eth_1$ but not $\eth_2$.

**Remark 4.38.** Each P$m$FN-subspace of a P$m$FN$T_0$S is P$m$FN$T_0$S means property of being a P$m$FN$T_0$S of any P$m$FNTS $(\underline{X}, \daleth_{pn})$ is innate.

**Definition 4.39.** A P$m$FNTS $(\underline{X}, \daleth_{pn})$ is *Pythagorean m-polar fuzzy neutrosophic $T_1$ space* (P$m$FN$T_1$S), *Pythagorean m-polar fuzzy Tychonoff space* or *Pythagorean m-polar fuzzy accessible space* if for any two unique P$m$FN-points $\eth_1, \eth_2$ of $(\underline{X}, \daleth_{pn}), \exists$ two P$m$FN-open sets $\underline{\eth}$ and $\Upsilon$ s.t. $\eth_1 \Subset \underline{\eth}, \eth_2 \not\Subset \underline{\eth}$ and $\eth_2 \Subset \Upsilon, \eth_1 \not\Subset \Upsilon$.

**Example 4.40.** Every discrete P$m$FNTS is a P$m$FN$T_1$S if $\eth_1$ and $\eth_2$ are two distinct P$m$FN-points then there are P$m$FN-open points $\{\eth_1\}$ and $\{\eth_2\}$ in $(\underline{X}, \daleth_{pn})$ s.t. $\eth_1 \Subset \{\eth_1\}$ whereas $\eth_2 \not\Subset \{\eth_1\}$.

**Theorem 4.41.** *The following assertions about a PmFNTS $(\underline{X}, \daleth_{pn})$ are equivalent:*

(1) *$(\underline{X}, \daleth_{pn})$ is a PmFN$T_1$S.*

(2) *Every PmFN singleton subset of $\underline{X}$ is PmFN-closed.*

(3) *Every PmFN-subset $\eth$ of $\underline{X}$ is the intersection of all its PmFN-open supersets.*

*Proof.* The proof is obvious. 0.1cm□

**Remark 4.42.** Every subspace of a P$m$FN$T_1$S is P$m$FN$T_1$S means property of being a P$m$FN$T_1$S of any P$m$FNTS $(\underline{X}, \beth_{pn})$ is innate.

**Definition 4.43.** A P$m$FNTS $(\underline{X}, \beth_{pn})$ is called a *Pythagorean m-polar fuzzy neutrosophic $T_2$ space* (P$m$FN$T_2$S), *Pythagorean m-polar fuzzy neutrosophic Hausdorff space* or *Pythagorean m-polar fuzzy neutrosophic separated space* if for any two unique P$m$FN-points $\eth_1$ & $\eth_2$ of $(\underline{X}, \beth_{pn}), \exists$ two P$m$FN-open sets $\underline{\eth}$ & $\Upsilon$ in such a way $\eth_1 \Subset \underline{\eth}, \eth_2 \Subset \Upsilon$ and $\underline{\eth} \Cap \Upsilon = \emptyset$.

**Example 4.44.** Consider the discrete P$m$FNTS $(\underline{X}, \beth_{pn})$. If $\eth_1$ and $\eth_2$ are two distinct P$m$FN-points in $\underline{X}$, then clearly $\{\eth_1\}$ and $\{\eth_2\}$ are disjoint P$m$FN-open sets such that $\eth_1 \Subset \{\eth_1\}$ and $\eth_2 \Subset \{\eth_2\}$. Thus, $(\underline{X}, \beth_{pn})$ is a P$m$FN$\beth_2^{\triangleright}$S.

**Theorem 4.45.** *A PmFNTS $(\underline{X}, \beth_{pn})$ is a PmFN$T_2$S iff for any two distinct PmFN-points $\eth_1$ and $\eth_2$, there are PmFN-closed sets $\underline{\eth}$ and $\Upsilon$ such that $\eth_1 \Subset \underline{\eth}, \eth_2 \not\Subset \underline{\eth}, \eth_1 \not\Subset \Upsilon, \eth_2 \Subset \Upsilon$ and $\underline{\eth} \Cup \Upsilon = \underline{X}_A$.*

*Proof.* Assume that $(\underline{X}, \beth_{pn})$ is a P$m$FN$T_2$S and let $\eth_1$ and $\eth_2$ be two distinct P$m$FN-points of $(\underline{X}, \beth_{pn})$. Then, by definition, there must exists two P$m$FN-open sets $\underline{\eth}$ and $\Upsilon$ such that $\eth_1 \Subset \underline{\eth}, \eth_2 \not\Subset \underline{\eth}$ and $\eth_1 \not\Subset \Upsilon, \eth_2 \Subset \Upsilon$ and $\underline{\eth} \Cap \Upsilon = \underline{\emptyset}$. But then, $\underline{\eth}^c \Cup \Upsilon^c = \underline{X}_A$ and $\eth_1 \not\Subset \underline{\eth}^c, \eth_2 \Subset \underline{\eth}^c, \eth_1 \Subset \Upsilon^c, \eth_2 \not\Subset \Upsilon^c$.
Conversely, assume that for any two distinct P$m$FN-points $\eth_1, \eth_2 \Subset (\underline{X}, \beth_{pn})$, there are P$m$FN-closed sets $\underline{\eth}$ and $\Upsilon$ such that $\eth_1 \Subset \underline{\eth}, \eth_2 \not\Subset \underline{\eth}, \eth_1 \not\Subset \Upsilon, \eth_2 \Subset \Upsilon$ and $\underline{\eth} \Cup \Upsilon = \underline{X}_A$. Then $\underline{\eth}^c$ and $\Upsilon^c$ are P$m$FN-open sets such that $\eth_1 \not\Subset \underline{\eth}^c, \eth_2 \Subset \underline{\eth}^c, \eth_1 \Subset \Upsilon^c, \eth_2 \not\Subset \Upsilon^c$ and $\underline{\eth}^c \Cap \Upsilon^c = \underline{X}_A{}^c = \underline{\emptyset}$. So, $(\underline{X}, \beth_{pn})$ is a P$m$FN$T_2$S. 0.1cm□

**Remark 4.46.** Each P$m$FN-subspace of a P$m$FN$T_2$S is also a P$m$FN$T_2$S means property of being a P$m$FN$T_2$S of any P$m$FNTS $(\underline{X}, \beth_{pn})$ is innate.

**Definition 4.47.** A P$m$FNTS $(\underline{X}, \beth_{pn})$ is called a *Pythagorean m-polar fuzzy neutrosophic regular space* (P$m$FN-regular space) if unspecified P$m$FN-closed set $\underline{\eth}$ & any P$m$FN-point $\eth_1 \not\Subset \underline{\eth}$ and here P$m$FN-open sets $\Upsilon$ & $\underline{\Upsilon}^*$ such that $\eth_1 \in \Upsilon, \underline{\eth} \subseteq \underline{\Upsilon}^*$ and $\Upsilon \Cap \underline{\Upsilon}^* = \underline{\emptyset}$.

**Definition 4.48.** A P$m$FNTS $(\underline{X}, \beth_{pn})$ is called *Pythagorean m-polar fuzzy neutrosophic $T_3$ space* (P$m$FN$T_3$S) if it is a P$m$FN regular $T_1$ space.

**Definition 4.49.** A P$m$FNTS $(\underline{X}, \underline{\beth}_{pn})$ is called *Pythagorean m-polar fuzzy neutrosophic normal space* if unspecified two P$m$FN-closed disjoint subsets $\underline{\eth}$ & $\Upsilon$ of $(\underline{X}, \underline{\beth}_{pn})$ and here P$m$FN-open sets $\underline{\Upsilon}^*$ and $\underline{\Upsilon}^\bullet$ such that $\underline{\eth} \subseteq \underline{\Upsilon}^*, \Upsilon \subseteq \underline{\Upsilon}^\bullet$ and $\underline{\Upsilon}^* \sqcap \underline{\Upsilon}^\bullet = \underline{\emptyset}$. A P$m$FN-normal $T_1$ space is called a *Pythagorean m-polar fuzzy neutrosophic $T_4$ space* (P$m$FN$T_4$S).

**Remark 4.50.** We have the following chain for different P$m$FNTSs studied above:

$$T_e \supseteq T_{e+1}$$

for $0 \leq e \leq 3$. The reverse chain, however, may not hold. The forthcoming Example 4.51 supports our claim.

**Example 4.51.** Let $(\underline{X}, \underline{\beth}_{pn})$ be a P$m$FNTS, where $\underline{X} = \{\hbar_1, \hbar_2\}, \underline{\beth}_{pn} = \{\underline{\emptyset}, \underline{B}, \underline{X}_A\}$. Then

Table 31. P3FNS $\underline{B}$

| $\underline{B}$ | | | |
|---|---|---|---|
| $\hbar_1$ | (0.000, 0.423, 0.801) | (0.167, 0.210, 0.562) | (0.472, 0.421, 0.301) |
| $\hbar_2$ | (0.162, 0.423, 0.004) | (0.000, 0.409, 0.210) | (0.100, 0.432, 0.720) |

is a P3FN$T_0$S but it is not a P3FN$T_1$S.

**Theorem 4.52.** *Each P$m$FN$T_4$S is a P$m$FN regular means each P$m$FN normal $T_1$ space is P$m$FN regular.*

*Proof.* Let $(\underline{X}, \underline{\beth}_{pn})$ be a P$m$FN$T_4$S. Let $\eth_1$ be a P$m$FN-point in $\underline{X}$. Then, by Theorem 4.41, $\{\eth_1\}$ is a closed P$m$FNS in $(\underline{X}, \underline{\beth}_{pn})$. Suppose that $\underline{\eth}$ be a P$m$FN-closed set not containing $\eth_1$. Since $(\underline{X}, \underline{\beth}_{pn})$ is P$m$FN normal, there are P$m$FN-open set namely $\Upsilon, \underline{\Upsilon}^*$ such that $\{\eth_1\} \subseteq \Upsilon, \underline{\eth} \subseteq \underline{\Upsilon}^*$ and $\Upsilon \sqcap \underline{\Upsilon}^* = \underline{\emptyset}$. But then, $\{\eth_1\} \Subset \Upsilon, \underline{\eth} \subseteq \underline{\Upsilon}^*$ and $\Upsilon \sqcap \underline{\Upsilon}^* \underline{\emptyset}$. So, $(\underline{X}, \underline{\beth}_{pn})$ is a Pythagorean $m$-polar fuzzy neutrosophic regular topological space. 0.1cm□

## 5. Intelligent Decision Making using P$m$FNS TOPSIS

In this section, we present an application of P$m$FNS in decision making.

**Case Study:**

A desert is a desolate region of land with hardly any rainfall and, as a result, unhealthy living conditions for flora and fauna. The absence of habitat reveals the ground's vulnerable surface to geomorphic activities. Around 33% of the world's land surface is sandy or semi-arid. The piece of land that attains fewer than 25 cm of rainfall per annum is considered a desert. Deserts are part of a broader class of regions named dry lands. Pakistan has five significant deserts comprising Cholistan, Katpana, Thar, Thal and Kharan deserts.

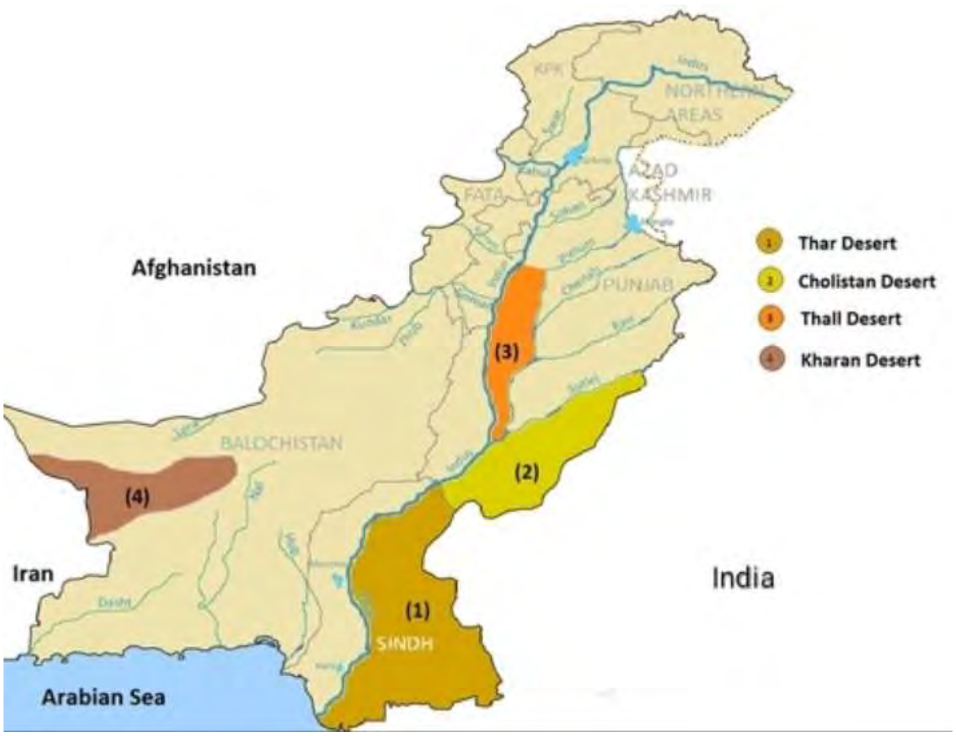


Figure 1. Deserts of Pakistan

About 85% of the Thar desert, also called the Great Indian Desert, is situated inside India, with the excess 15% in Pakistan. It covers around 170,000 $km^2$, and the leftover 30,000 $km^2$ of the desert is inside Pakistan. Thar desert is the world's seventeenth biggest desert, and the world's ninth biggest subtropical desert. During different periods of predominant breeze is the dry northeast storm. May and June are the most sweltering a long time of the year, with mercury ascending to $50^0$ C. In January, considered to be the coldest month there, the average minimum temperature drops down to $10^0$ C, and frost is frequent. Dust storms and dust-raising winds, often blow with a speed of 140 to 150 km per hour, are frequent in the months of May and June. The amount of annual rainfall in the desert is generally low, ranging from about less in the west to about 20 inches (500 mm) in the east or 4 inches (100 mm), mostly decreasing from July to September.

The desert of Kharan is situated in Balochistan. It makes a nature limit among Pakistan, Iran and Afghanistan. It is situated in Kharan region. The Kharan desert is a sandiest desert in Pakistan. It is particular from the remainder of the province's landscape because of its sandy nature and all the more even ter. The desert was utilized for atomic testing by the Pakistan military, making it the most renowned of the five deserts. In altitude these central deserts slope from about 1,000 m in the north to about 250 m on in the southwest. Maximum, average and minimum temperatures of kharan desert are $42^0$ C, $38^0$ C and $26^0$ C respectively. Average annual rainfall throughout these deserts is well under 100 mm. The desert includes areas of inland drainage and dry lakes.

The Cold Desert, otherwise called the Katpana Desert or Biama Nakpo, is a high-elevation desert situated close Skardu, northern Gilgit-Baltistan area of Pakistan controlled Jammu and Kashmir. The desert contains costs of huge sand rises that are once in a while shrouded in snow during winter. Situated at an elevation of 2,226 m (7,303 feet) above ocean level, the Katpana Desert is one of the most noteworthy deserts in the world. The desert actually extends from the Khaplu Valley to Nubra in Ladakh, yet the biggest desert area is found in Skardu and Shigar Valley. The part most visited is situated close Skardu Airport. Temperatures range from a maximum of $27^0$ C and a minimum (in October) $8^0$ C which can drop further to beneath $-17^0$ C in December and January. The temperature infrequently drops as low as $-25^0$ C.

The Thal Desert is situated in Bhakkar area of Pakistan between the Indus and Jhelum rivers. A huge canal-building venture is in progress to flood the land. Water system will make a large portion of the desert appropriate for cultivating. In the north of the Thal Desert there are salt reaches, in the east the Jhelum and Chenab streams and toward the west the Indus waterway. The maximum temperature is $34^0$ C and minimum temperature is $25^0$ C in Thal desert. The average annual temperature for Thal is $29^0$ C. It is dry for 207 days a year with an average humidity of 36%. The average annual rainfall varies from 385 mm in the north-east to 170 mm in the south. Approximately three-fourth of annual rainfall is received during monsoon.

Cholistan Desert is locally known as Rohi. It abuts the Thar Desert, stretching out over to Sindh and into India. Cholistan desert hosts an yearly Jeep rally, known as Cholistan Desert Jeep Rally which is the greatest engine game in Pakistan. Cholistan's atmosphere is described as a bone-dry and semi-dry Tropical desert, with exceptionally low yearly dampness. The mean temperature in Cholistan is $28.33^0$ C, with most smoking month being July with a mean temperature of $38.5^0$ C. Summer temperatures can outperform $46^0$ C and now and then ascents more than $50^0$ C during times of dry season. Winter temperatures infrequently dip to $0^0$ C. Normal precipitation in Cholistan is up to 180mm, with July and August being the wettest months, despite the fact that dry seasons are normal. Water is gathered occasionally in an arrangement of normal pools called Toba, or man made pools called Kund. Earth water is found at a profundity of 30-40 meters, yet is commonly bitter, and unacceptable for most plant development.

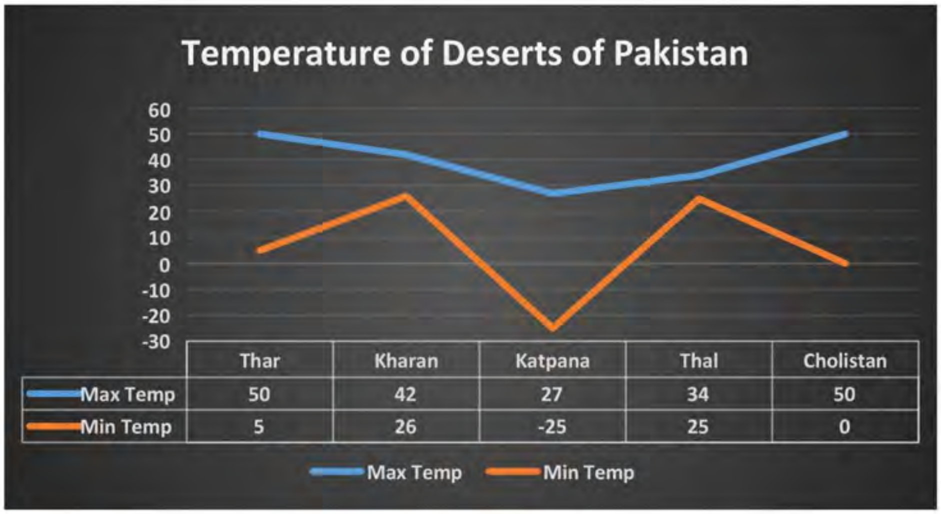


FIGURE 2. Temperature of Deserts of Pakistan

These deserts contains an extremely dry part, the Marusthali area in the west, and a semi desert locale in the east with less sand hills and somewhat more precipitation. For the most vital problem and the main hinderance, in the way of progress. Government considers that issue of lack of water in desert has to solve as early as possible.

The basic need-water, has greatly affected the lives of residents of desert. It can be said that water has not only changed their social life style but also economy has affected badly. Inadequate sanitary conditions have invited many diseases which can be said epidemic like cholera, typhoid etc. These disaster ruin the human race as well as their cattle.

Cultivation also wiped away due to scarcity of water. Indirectly water is the primary source of food also people face the horrible face of famine. Specially children, represent the reflection of poor humanity. Their body, without any health, you may say their skeletons cry for help or for water.

Scarcity of water has also a deep impact on the psyche of residents of desert. Their temperament, attitudes and behaviors indirectly affected by this vital problem. Tolerance, courtesy, desire, for progress, achievements, dreams and all ways leading to bright future are cover in mist. They cannot see or even have the eagerness for better living style. Their struggle only moves around the availability of water. So it is the need of the time that all the possible steps should be taken at all levels for the sake of humanity.

A city named Nagarparkar in Thar is consist upon 1 lac population people use under ground clean and clear water for the necessities of life but it is very hard to get it in summer.

In summary the level of underground water decreases at the lowest level and to get water becomes impossible by hand pump. For the last many years no proper planning has been made to provide water. In city water is brought far from areas. In this age of dearness to getting water is difficult. The fare of a cane is 20 to 25 rupees. The people are compelled to drink that kind of water which is jot acceptable to the animals of Lahore. Animals and human drink water from the same place there is no distinguish of camel, goat and the king of all races.

It is a hot issue, so a commission has established in which all the concerning problem experts were included. This commission visited the desert and collected all eye bared witness.

First of all they prepare a report in which they point out the problems facing towards water supply.

**Poor decision making:** The commission strongly condemned that decision making policies are not harmonized to the circumstances.

**Economically costs:** In Thar with boring a place of water is served 8 to 9 villages approximately water is available to 7 km distance. Government do not take solid steps only visits are arranged and due to lack of budgets, no attention is given for this reason people are deprived of water. It has also observed that which projects had passes in past they were very costly. Government could not afford them.

**Environmental and social problem:** Desert environment needs something special which can appropriate to its hottest environment and social settlement.

**Encouragement of local persons:** A reason which is also very important is that people do not have much facilities that they can bore or drill the land and can make it easily to get water because they are illiterate and cannot drive correct solution by correct strategy. It is also necessary to take help from the local persons and encouraged them to solve this problem with the help of government.

FIGURE 3. Environmental and social problem

For all these issues, they suggested some positive and skilled opinions.

(i) Government should take solid decisions. And the motto of these decisions should be welfare and progress because if the start is good then the end will be best.

(ii) Those projects should be of low cost and much beneficial.

(iii) It should be keep in mind that the trust of local persons is very necessary for their welfare because the negativity of being ignore has been kept its place in their minds.

(iv) Government should start small projects as they would be called tribal units or tribal beneficiary projects.

FIGURE 4. Lack of water

We clarify the procedure bit by bit as follows:

**Algorithm:**

Stage 1: Firstly analyze the issue to see that what we have and actually what we have need to do: Suppose that $\mathrm{R} = \{\sigma_i : i = 1, 2, \cdots, n\}$ is the finite aggregate of alternatives under consideration and $\mathrm{G} = \{g_j : j = 1, 2, \cdots, m\}$ is the family of captains. So the $(i, j)^{th}$ entry of the P$m$FNS matrix represents weight given by $j^{th}$ Captains to $i^{th}$ options.

Stage 2: Develop weighted parameter matrix P as

$$P = [w_{ij}]_{n\times m} = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1m} \\ w_{21} & w_{22} & \cdots & w_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & \cdots & w_{im} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nm} \end{pmatrix}$$

where $w_{ij}$ is the fuzzy weight given by the Captains $g_j$ to the options $\sigma_i$ by thinking about the phonetic entitle are given (for example) in Table 32.

Table 32. Phonetic terms for benefits of projects

| Phonetic Terms | Fuzzy Weights |
|---|---|
| Not fruitful (NF) | [0.00, 0.25] |
| Fruitful (F) | (0.25, 0.50] |
| More or less fruitful (MF) | (0.50, 0.75] |
| Extremely fruitful (EF) | (0.75, 1.00] |

Stage 3: Develop normalized weighted matrix

$$N = [\hat{w}_{ij}]_{n\times m} = \begin{pmatrix} \hat{w}_{11} & \hat{w}_{12} & \cdots & \hat{w}_{1m} \\ \hat{w}_{21} & \hat{w}_{22} & \cdots & \hat{w}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{w}_{i1} & \hat{w}_{i2} & \cdots & \hat{w}_{im} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{w}_{n1} & \hat{w}_{n2} & \cdots & \hat{w}_{nm} \end{pmatrix}$$

where $\hat{w}_{ij} = \frac{w_{ij}}{\sqrt{\sum_{i=1}^{n} w_{ij}^2}}$ and obtaining the weighted vector $W = (w_j : j = 1, 2, \cdots, m)$, where $w_j = \frac{\sum_{i=1}^{n} \hat{w}_{ij}}{n \sum_{k=1}^{m} \hat{w}_{ik}}$

Step 4: Develop P$m$FNS decision matrix $G_i = [\varsigma_{jk}^i]_{n\times m}$, where $\varsigma_{jk}^i = (\tau_{jk}^i, \upsilon_{jk}^i, \omega_{jk}^i)$. Then obtain the mean proportional matrix

$$X = \sqrt[n]{G_1 G_2 \cdots G_n} = [\dot{\varsigma}_{jk}]_{n\times m} = \left[\left(\sqrt[n]{\prod_{i=1}^{n} \tau_{jk}^i}, \sqrt[n]{\prod_{i=1}^{n} \upsilon_{jk}^i}, \sqrt[n]{\prod_{i=1}^{n} \omega_{jk}^i}\right)\right]_{n\times m}$$

Stage 5: Compute weighted P$m$FNS decision matrix $Y = [\ddot{\varsigma}_{jk}]_{n\times m}$, where $\ddot{\varsigma}_{jk} = w_k \times \dot{\varsigma}_{jk} = (\tau_{jk}, \upsilon_{jk}, \omega_{jk})$.

Stage 6: Get P$m$FNSV-PIS (P$m$FNS- valued positive ideal solution) and P$m$FNSV-NIS (P$m$FNS- valued negative ideal solution), by using

$$\begin{aligned} PmFNS-PIS &= \{\ddot{\varsigma}_1^+, \ddot{\varsigma}_2^+, \cdots, \ddot{\varsigma}_m^+\} \\ &= \{(\max_k \tau_{jk}, \min_k \upsilon_{jk}, \min_k \omega_{jk}) : k = 1, 2, \cdots, m\} \\ &= \{(\tau_k^+, \upsilon_k^+, \omega_k^+) : k = 1, 2, \cdots, m\} \end{aligned}$$

and

$$\begin{aligned} PmFNS-NIS &= \{\ddot{\varsigma}_1^-, \ddot{\varsigma}_2^-, \cdots, \ddot{\varsigma}_m^-\} \\ &= \{(\min_k \tau_{jk}, \max_k \upsilon_{jk}, \max_k \omega_{jk}) : k = 1, 2, \cdots, m\} \\ &= \{(\tau_k^-, \upsilon_k^-, \omega_k^-) : k = 1, 2, \cdots, m\} \end{aligned}$$

respectively.

Stage 7: Find P$m$FNS-Euclidean separations of every other option from P$m$FNS-PIS and P$m$FNS-NIS respectively, by making use of

$$g_j^+ = \sqrt{\sum_{k=1}^{m} (\tau_{jk} - \tau_k^+)^2 + (\upsilon_{jk} - \upsilon_k^+)^2 + (\omega_{jk} - \omega_k^+)^2}$$

$$g_j^- = \sqrt{\sum_{k=1}^{m} (\tau_{jk} - \tau_k^-)^2 + (\upsilon_{jk} - \upsilon_k^-)^2 + (\omega_{jk} - \omega_k^-)^2}$$

for $j = 1, 2, \cdots, n$.

Step 8: Compute the relative closeness using

$$C_j^* = \frac{g_j^-}{g_j^+ + g_j^-}$$

Stage 9: So as to get the inclination request of the other options, rank the options in descending (or ascending) order.

The procedural steps of above Algorithm are portrayed in Figure 5:

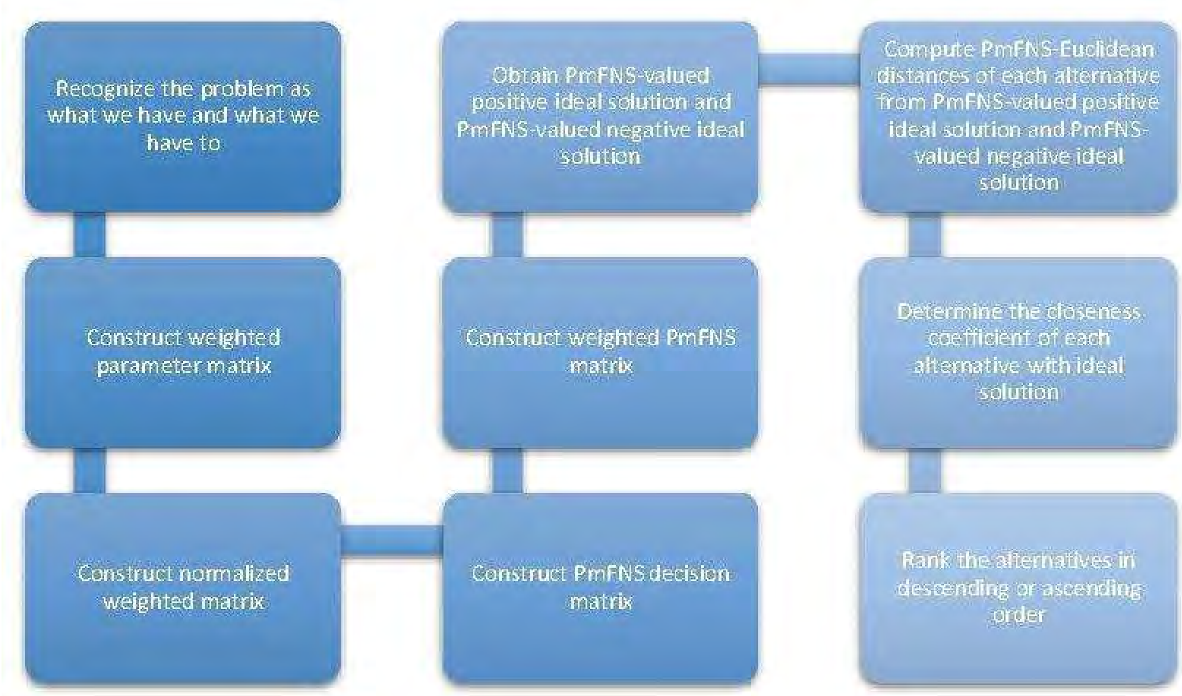


FIGURE 5. Flow chart of Algorithm

**Example 5.1.** Assume that experts wishes to determine the most vital problems and the main hinderance facing by desert. The experts establish a committee of four members.

Stage 1: Analyze the problem: Assume that R $= \{\sigma_i : i = 1, 2, \cdots, 4\}$ is the set of choices viable and G $= \{g_j : j = 1, 2, 3, 4\}$ is the family of experts, where

$$\begin{aligned} \sigma_1 &= \text{Poor decision making,} \\ \sigma_2 &= \text{Economic costs,} \\ \sigma_3 &= \text{Environmental and social problem, and} \\ \sigma_4 &= \text{Encouragement of local persons.} \end{aligned}$$

Stage 2: The weighted parameter matrix, by selecting phonetic terms from Table 32, is

$$\begin{aligned} P &= [w_{ij}]_{4\times 4} \\ &= \begin{pmatrix} \text{F} & \text{NF} & \text{MF} & \text{EF} \\ \text{NF} & \text{F} & \text{EF} & \text{MF} \\ \text{MF} & \text{NF} & \text{F} & \text{EF} \\ \text{EF} & \text{MF} & \text{NF} & \text{F} \end{pmatrix} \\ &= \begin{pmatrix} 0.50 & 0.25 & 0.75 & 1.00 \\ 0.25 & 0.50 & 1.00 & 0.75 \\ 0.75 & 0.25 & 0.50 & 1.00 \\ 1.00 & 0.75 & 0.25 & 0.50 \end{pmatrix} \end{aligned}$$

Where $w_{ij}$ is the weight given by the decision maker $g_j$ to the choices $\sigma_i$.

Stage 3: The normalized weighted matrix is

$$\begin{aligned} N &= [\hat{w}_{ij}]_{4\times 4} \\ &= \begin{pmatrix} 0.37 & 0.26 & 0.55 & 0.60 \\ 0.18 & 0.52 & 0.73 & 0.45 \\ 0.55 & 0.26 & 0.37 & 0.60 \\ 0.73 & 0.77 & 0.18 & 0.30 \end{pmatrix} \end{aligned}$$

and thus the weight vector is

$$W = (0.25, 0.24, 0.25, 0.26)$$

Stage 4: Suppose that the four experts give the following P$m$FNS matrix in which the $(i,j)^{th}$ elements shows the PFN $(\tau, \upsilon, \omega)$, where choices are showed by row-wise and the PFN assigned by experts are showed by column-wise.

$$G_1 = \begin{pmatrix} (0.61, 0.22, 0.39) & (0.73, 0.52, 0.11) & (0.66, 0.42, 0.33) & (0.36, 0.15, 0.49) \\ (0.38, 0.17, 0.50) & (0.48, 0.29, 0.30) & (0.61, 0.00, 0.18) & (0.46, 0.24, 0.17) \\ (0.54, 0.29, 0.32) & (0.46, 0.35, 0.45) & (0.24, 0.18, 0.59) & (0.78, 0.55, 0.12) \\ (0.08, 0.37, 0.88) & (1.00, 0.00, 0.00) & (0.34, 0.63, 0.35) & (0.69, 0.13, 0.04) \end{pmatrix}$$

$$G_2 = \begin{pmatrix} (0.52, 0.19, 0.22) & (0.39, 0.52, 0.35) & (0.43, 0.61, 0.50) & (0.66, 0.57, 0.14) \\ (0.43, 0.54, 0.29) & (0.48, 0.25, 0.40) & (0.76, 0.10, 0.22) & (0.45, 0.53, 0.41) \\ (0.24, 0.26, 0.30) & (0.37, 0.06, 0.19) & (0.00, 0.48, 0.71) & (0.33, 0.41, 0.28) \\ (0.36, 0.17, 0.29) & (0.62, 0.28, 0.00) & (0.05, 0.18, 0.77) & (0.23, 0.64, 0.59) \end{pmatrix}$$

$$G_3 = \begin{pmatrix} (0.54, 0.58, 0.38) & (1.00, 0.00, 0.00) & (0.52, 0.44, 0.39) & (0.23, 0.10, 0.11) \\ (0.30, 0.59, 0.20) & (0.52, 0.22, 0.33) & (0.13, 0.14, 0.04) & (0.51, 0.06, 0.44) \\ (0.41, 0.28, 0.51) & (0.29, 0.64, 0.39) & (0.78, 0.02, 0.16) & (0.31, 0.13, 0.64) \\ (0.57, 0.55, 0.37) & (0.36, 0.88, 0.14) & (0.40, 0.00, 0.53) & (0.05, 0.27, 0.77) \end{pmatrix}$$

$$G_4 = \begin{pmatrix} (0.37, 0.55, 0.30) & (0.43, 0.58, 0.19) & (0.35, 0.28, 0.44) & (0.59, 0.56, 0.17) \\ (0.35, 0.73, 0.12) & (0.41, 0.27, 0.39) & (0.67, 0.37, 0.21) & (0.64, 0.16, 0.20) \\ (0.00, 0.28, 0.72) & (0.58, 0.06, 0.41) & (0.40, 0.51, 0.31) & (0.35, 0.10, 0.57) \\ (0.47, 0.40, 0.26) & (0.44, 0.51, 0.38) & (0.44, 0.64, 0.26) & (0.28, 0.31, 0.60) \end{pmatrix}$$

Thus, the mean proportional matrix X is

$$\begin{aligned} X &= [\dot{\varsigma}_{jk}]_{4\times4} \\ &= \begin{pmatrix} (0.50, 0.34, 0.31) & (0.59, 0.00, 0.00) & (0.48, 0.42, 0.41) & (0.42, 0.26, 0.19) \\ (0.36, 0.45, 0.24) & (0.47, 0.26, 0.35) & (0.45, 0.00, 0.14) & (0.51, 0.19, 0.28) \\ (0.00, 0.28, 0.43) & (0.41, 0.17, 0.34) & (0.00, 0.17, 0.38) & (0.41, 0.23, 0.33) \\ (0.30, 0.34, 0.40) & (0.56, 0.00, 0.00) & (0.23, 0.00, 0.44) & (0.22, 0.29, 0.32) \end{pmatrix} \end{aligned}$$

where $\ddot{\varsigma}_{jk} = w_k \times \dot{\varsigma}_{jk}$

Stage 5: The weighted P$m$FN matrix is

$$\begin{aligned} Y &= [\ddot{\varsigma}_{jk}]_{4\times4} \\ &= \begin{pmatrix} (0.13, 0.09, 0.08) & (0.14, 0.00, 0.00) & (0.12, 0.11, 0.10) & (0.11, 0.07, 0.05) \\ (0.09, 0.11, 0.06) & (0.11, 0.06, 0.08) & (0.11, 0.00, 0.04) & (0.13, 0.05, 0.07) \\ (0.00, 0.07, 0.11) & (0.10, 0.04, 0.08) & (0.00, 0.04, 0.10) & (0.11, 0.06, 0.09) \\ (0.08, 0.09, 0.10) & (0.13, 0.00, 0.00) & (0.06, 0.00, 0.11) & (0.06, 0.08, 0.08) \end{pmatrix} \end{aligned}$$

Stage 6: Thus, P$m$FNS-PIS and P$m$FNS-NIS, are respectively

P$m$FNSV-PIS $= \{\ddot{\varsigma}_1^+, \cdots, \ddot{\varsigma}_4^+\}$

$= \{(0.13, 0.07, 0.06), (0.14, 0.00, 0.00), (0.12, 0.00, 0.04), (0.13, 0.05, 0.05)\}$

and

P$m$FNSV-NIS $= \{\ddot{\varsigma}_1^-, \cdots, \ddot{\varsigma}_4^-\}$

$= \{(0.00, 0.11, 0.11), (0.10, 0.06, 0.08), (0.00, 0.11, 0.11), (0.06, 0.08, 0.09)\}$

Stage 7 and 8: The Euclidean separation of every issue from P$m$FNS-PIS and P$m$FNS-NIS and corresponding relative coefficients of closeness are given in Table 33:

Table 33. Separation and coefficient of closeness of each issue

| Issue ($\ddot{\varsigma}_i$) | $g_i^+$ | $g_i^-$ | $C_i^*$ |
|---|---|---|---|
| $\ddot{\varsigma}_1$ | 0.13 | 0.22 | 0.63 |
| $\ddot{\varsigma}_2$ | 0.12 | 0.78 | 0.87 |
| $\ddot{\varsigma}_3$ | 0.23 | 0.10 | 0.30 |
| $\ddot{\varsigma}_4$ | 0.14 | 0.18 | 0.56 |

Stage 9: Thus, the preference ranking of the issues is

$$\ddot{\varsigma}_2 \succ \ddot{\varsigma}_1 \succ \ddot{\varsigma}_4 \succ \ddot{\varsigma}_3$$

This ranking is portrayed in Figure 6:

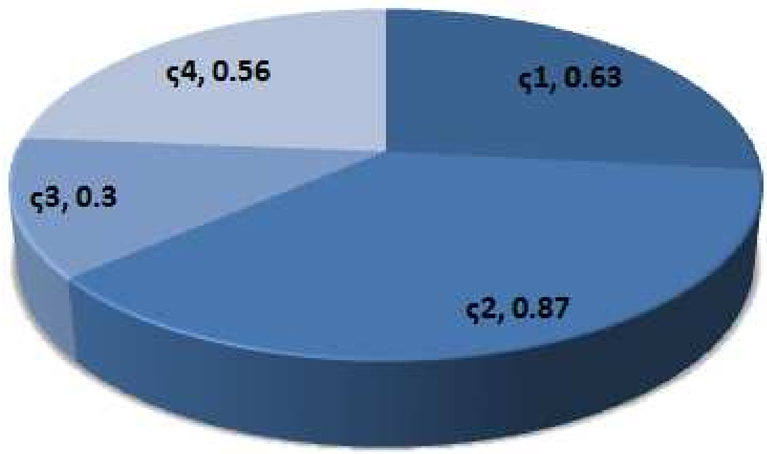


Figure 6. Ranking of alternatives

Hence, in view of above ranking, it may be concluded that poor decision making is the core issue.

## 6. Conclusion

We reviewed fuzzy set theory along with its tabular illustration and examples briefly. We established the axiomatic definitions of Pythagorean $m$-polar fuzzy neutrosophic set. We presented some fundamental properties of Pythagorean $m$-polar fuzzy neutrosophic topological space (P$m$FNTS) by numerous characteristics of crisp topology on the way to the P$m$FNTS. We defined Pythagorean $m$-polar fuzzy separation axioms. $T_0$, $T_1$, $T_2$, $T_3$ and $T_4$ spaces are modified in the aspects of P$m$FNS.

We presented example of decision making from real world situations based on TOPSIS accompanied by case study. We presented algorithm and flowcharts of method for comfort. We also showed 3D bar chart with application to make the contrast between different alternatives effectively.

These above mentioned concepts can be used in several real world difficulties such as in economics, business, robotics, medical sciences, water management, electoral systems, transportation problems and much more. We hope that this paper will gives new ideas to the researchers to promote research work in this field.

The notions presented in this article may be extended to define other sorts of topological structures like nano topology and pentapartitioned topology etc.

**Conflicts of Interest:** The authors declare that there is no conflict of interests.

**Authors' Contributions:** The authors contributed to each part of this paper equally.

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Received: Nov 22, 2021. Accepted: Feb 2, 2022

NSS
Neutrosophic Sets and Systems, Vol. 48, 2022
*University of New Mexico*

# Neutrosophic Design of the Exponential Model with Applications

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**Abstract:** Operations on neutrosophic numbers generalize operations on crisp numbers. In this way, the neutrosophic approach quantifies data ambiguity and enables the generalization of the existing statistical model. This study presents an extension of the conventional exponential distribution in a neutrosophic context. Neutrosophic generalization is restricted to characterize the properties of the neutrosophic exponential distribution (NED); however, related results can to other stochastic models for handling the situations involving uncertainties or vagueness in processing data. All essential features of the proposed NED, such as neutrosophic moments, neutrosophic distribution function, and other related quantities, are explored. The mathematical results in this work lay the groundwork for using the exponential distribution to produce drivers for other generalized models. The neutrosophic logic of the proposed model is illustrated with examples. The estimation technique for treating the imprecision in the unknown parameter is established. The performance of the estimator neutrosophic estimator has been evaluated through Monte Carlo simulation. Simulation findings reveal that a larger sample size provides reliable estimation results.

**Keywords:** Neutrosophic probability; neutrosophic distribution; exponential model; estimation

## 1. Introduction

Probability distributions are now an essential part of every scientific research. Several real-world random events are described by these probability models [1]. A basic statistical probability model is commonly applicable to problems encountered by researchers. One of the most common continuous distributions is the exponential distribution [2]. The exponential model is considerably connected with the Poisson distribution [3]. It is commonly utilized as a model to measure the time between events occurrence. Some examples of its application include measuring the time associated with obtaining a faulty component on an assembling line in an engineering framework, predicting the risk of a portfolio of financial assets on next default and calculating radioactive decay in physics [4]. It is also used to estimate the probability of a certain number of defaults occurring during a particular time period [5]. The exponential distribution is an adequate failure model for describing the failure patterns of many components and devices with constant hazard rates in reliability analysis [6]. In hydrology, the exponential distribution is frequently used to examine extreme values of yearly or monthly maximum river flow and total rainfall [7]. A DNA strand length between mutations or the distance between roads fatalities are examples of situations where exponential variables may also be used to describe the likelihood of events occurring at a constant rate per unit distance [8-9].

In this study, a novel generalization of the NED has been described with the primary goal of incorporating vague information about the study variables. The exponential distribution is considered a neutrosophic version because it is a versatile model that can reflect a wide range of distribution forms. This extension provides a broader and clear analysis of the studied variables under consideration. The neutrosophic extension of the exponential model paves the path for working with other classical probability models established for the precisely described datasets. This study presents the NED in a way that the conventional logic of the exponential model cannot handle the many applied data problems. This generalization is based on the notion of neutrosophy presented by Smarandache [10]. The analysis of false or true statements, but indeterminate, neutral, inconsistent, or something in between, is oriented by Neutrosophic logic [11]. Every area has its neutrosophic component, namely the indeterminacy part, on the mathematical side. Smarandache made the first effort to use the neutrosophic approach in statistics, precalculus, and calculus to cope with imprecision in study variables [12]. As a result, neutrosophic statistics have given rise to research topics that deal with the effect of indeterminacy in statistical modeling. Some recent literature has recently made the first step toward describing the neutrosophic principle of statistical modeling [13-16]. Neutrosophic measures probability and descriptive statistical are discussed in [17]. Neutrosophic decision-making applications in quality control seem to be very efficient [18]. Alhabib et al. first looked at the neutrosophic algebraic structures of probability distributions [19]. Some recent work on neutrosophical probabability distributions can be seen in [20-23]. Nevertheless, works focusing on neutrosophic statistics have always relied on the applications side of the neutrosophic logic, and algebraic structures of probability distributions have rarely been addressed.

The work is structured as follows: The NED and algebraic framework of the neutrosophic numbers are given in section 2. Mathematical properties of the proposed NED are provided in section 3. Section 4 demonstrates some examples of the NED. The estimation approach for the imprecise parameter of NED is established in section 5. A simulation study for demonstrating the performance of the NML estimator is carried out in section 6. A real application of the proposed model is given in section 7. Lastly, section 8 summarizes the research findings.

**2. Preliminaries**

All essential features of the proposed NED, such as moments, shape coefficients, and the moment generating function, are based on the algebraic framework of the neutrosophic numbers. Let $\mathrm{M} = (\mathrm{t_m}, \mathrm{i_m}, \mathrm{f_m})$ and $\mathrm{N} = (\mathrm{t_n}, \mathrm{i_n}, \mathrm{f_n})$ are two single-valued neutrosophic numbers with $\mathrm{t_m}, \mathrm{t_n}, \mathrm{i_m}, \mathrm{i_n}, \mathrm{f_m}, \mathrm{f_n} \in [0,1]$, $0 \leq \mathrm{t_m}, \mathrm{i_m}, \mathrm{f_m} \leq 3$ and $0 \leq \mathrm{t_n}, \mathrm{i_n}, \mathrm{f_n} \leq 3$ then the following operation are commonly employed in the framework of the neutrosophic algebra [16]:

$$\mathrm{M} \oplus \mathrm{N} = (\mathrm{t_m} + \mathrm{t_n} - \mathrm{t_m} \mathrm{t_n}, \mathrm{i_m} \mathrm{i_n} + \mathrm{f_m}\, \mathrm{f_n}) \quad (1)$$

$$\mathrm{M} \otimes \mathrm{N} = (\mathrm{t_m} \mathrm{t_n}, \mathrm{i_m} + \mathrm{i_n} - \mathrm{t_n}, \mathrm{f_m} + \mathrm{f_n} - \mathrm{f_m}\, \mathrm{f_n}) \quad (2)$$

$$\omega \mathrm{M} = (1 - (1 - \mathrm{t_m})^{\omega}, \mathrm{t_m}^{\omega}, \mathrm{f_m}^{\omega}); \quad (3)$$

$$\mathrm{M}^{\omega} = (\mathrm{t_m}^{\omega} - 1 - (1 - \mathrm{i_m})^{\omega}, 1 - (1 - \mathrm{f_m})^{\omega}), \quad (4)$$

where the scalar $\omega > 0, and\ \omega \in R$.

Equations (1), (2), (3) and (4) represent neutrosophic summation, neutrosophic multiplication, scalar multiplication and neutrosophic power respectively. Likewise, the single-valued neutrosophic operations can be extended to neutrosophic sets.

***Definition 2.1*** Neutrosophic data extends the classic data that contain some imprecise, vague or indeterminacy in some or all values. In general terms, it can be represented as:
$x = \text{constant} + I$,
where $I \in [u, l]$; for example, $7 + I$ where $I \in [3, 3.5]$.
***Definition 2.2*** The neutrosophic random variable W, which equals the distance between successive events in a Poisson process, follows the NED model with the following neutrosophic density function ($PDF_N$).

$$\varphi_N(w) = \theta_N \exp(-w\theta_N);\ w > 0,\ \text{and}\ z > 0, \quad (5)$$

where $\theta_N \in \{\theta_l, \theta_u\}$. Figure 1 shows the form of the distribution with neutrosophic parameter $\theta_N = \{0.25, 0.50\}, \{1.00, 1.50\}$ and $\{2.00, 2.50\}$ if the data are believed to be NED.

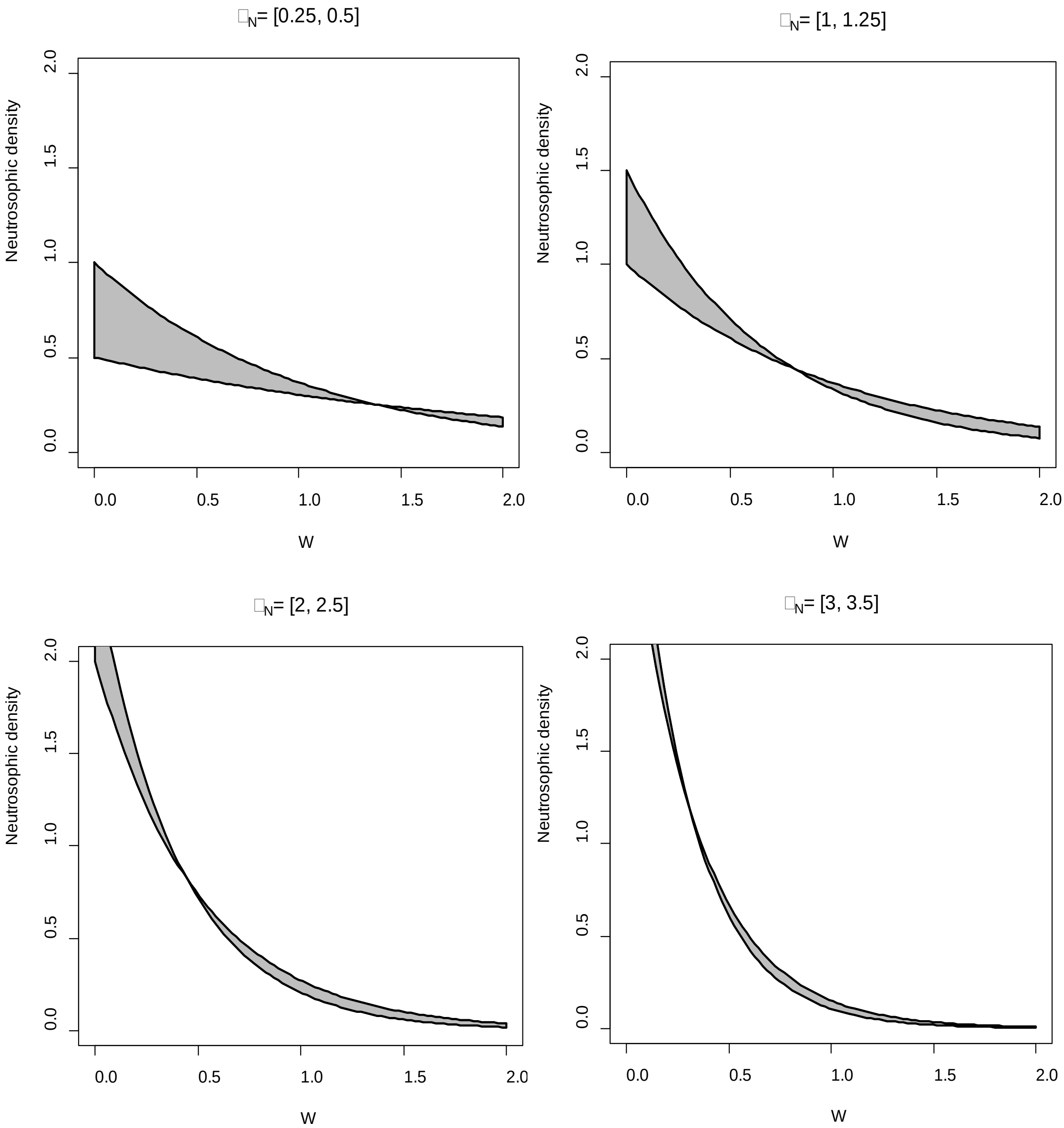


**Figure 1** Neutrosophic density graph of the NED

Figure 1 shows the neutrosophic area because of the indeterminate value of the failure rate parameter $\theta_N$. It is clearly demonstrated from Figure 1 that parameter settings may be changed to create a variety of neutrosophic exponential curves.

**3. Some useful functions of the proposed NED**

In this section, some widely used properties of the NED can be established in the form of the following theorems:

**Theorem 1**. Show that $r^{th}$ moment of the NED is $\frac{\Gamma(r+1)}{\theta_N^r}$

**Proof** By definition the $r^{th}$ moment of the NED can define as:

$$\mu'_{rN} = \int_0^\infty w^r \theta_N \exp(-w\theta_N) dw$$

$$= \int_0^\infty w^r \left[\theta_l \exp(-w\theta_l), \quad \theta_u \exp(-w\theta_u)\right] dw$$

$$= \left[\int_0^\infty w^r \theta_l \exp(-w\theta_l) dw, \ \int_0^\infty w^r \theta_u \exp(-w\theta_u) dw\right] \quad (6)$$

By substituting $y = w\theta_N$, we get from (6)

$$\int_0^\infty w^r \theta_l \exp(-w\theta_l) dw = \frac{\Gamma(r+1)}{\theta_l^r}$$

$$\int_0^\infty w^r \theta_l \exp(-w\theta_l) dw = \frac{\Gamma(r+1)}{\theta_u^r}$$

Thus (6) provides

$$= \left[\frac{\Gamma(r+1)}{\theta_l^r}, \quad \frac{\Gamma(r+1)}{\theta_u^r}\right]$$

Hence,

$$\mu'_{rn} = \frac{\Gamma(r+1)}{\theta_N^r} \qquad \text{where } r = 1, 2, 3, \quad (7)$$

Thus first four raw moments can be derived as:

$\mu'_{1N} = \frac{1}{\theta_N}$, $\mu'_{2N} = \frac{1}{2\theta_N^2}$, $\mu'_{3N} = \frac{1}{6\theta_N^3}$ and $\mu'_{4N} = \frac{1}{24\theta_N^4}$

**Theorem 2.** The distribution function $\Phi_N(w)$ of the NED is $1 - \exp(-w\theta_N)$.

**Proof** The result of the distribution function is obtained by solving the following expression:

$$\Phi_N(w) = \int_0^w \varphi_N(w) dw$$

$$= 1 - \exp(-w\theta_N) \quad (8)$$

Sketch of the CDF function of the proposed NED with neutrosophic parameter $\theta_N = \{1.5, 2\}$ is displayed in Figure 2.

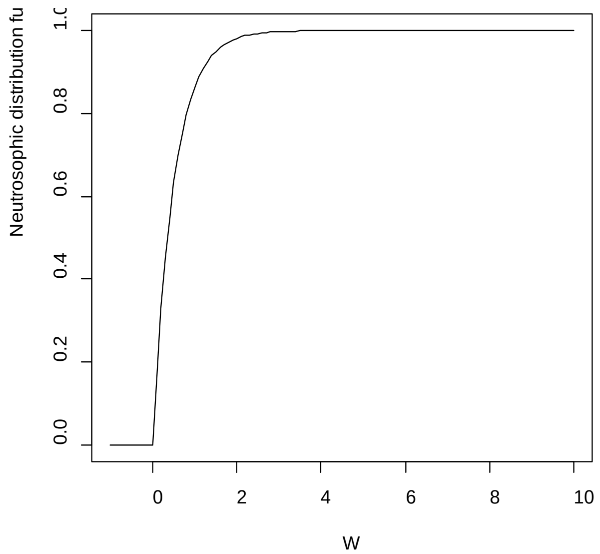


**Figure 2** CDF curve of the NED with $\theta_N = \{1.5, 2\}$

**Theorem 3.** The median of the NED is $\left[\frac{\ln(2)}{\theta_l}, \frac{\ln(2)}{\theta_u}\right]$.

**Proof** Neutrosophic median ($M_N$) is the solution of the following expression:

$$\int_0^{M_N} \Phi_N(w)dw = \left[\frac{1}{2}, \quad \frac{1}{2}\right]$$

$$\left[\int_0^{M_N} \Phi_l(w)dw, \quad \int_0^{M_N} \Phi_u(w)d\right] = \left[\frac{1}{2}, \frac{1}{2}\right] \quad (9)$$

where $\Phi_l(w) = 1 - \exp(w\theta_u)$ and $\Phi_u(w) = 1 - \exp(w\theta_u)$.

Analytical simplification of (9) implies:

$M_N\theta_l = \ln(2)$

$M_N\theta_u = \ln(2)$

Implying thereby $M_N = \left[\frac{\ln(2)}{\theta_u}, \frac{\ln(2)}{\theta_l}\right]$.

Theorem 4. First quantile ($Q_{IN}$) and the third quantile ($Q_{3N}$) of the NED are $\left[\frac{\ln(\frac{4}{3})}{\theta_u}, \frac{\ln(\frac{4}{3})}{\theta_l}\right]$ and $\left[\frac{\ln(4)}{\theta_u}, \frac{\ln(4)}{\theta_l}\right]$ respectively.

Proof The $Q_{IN}$ and $Q_{3N}$ by definition are corresponded to solutions such as:

$$\int_0^{Q_{IN}} \Phi_N(w)dw = \left[\frac{1}{4}, \quad \frac{1}{4}\right]$$

$$\int_0^{Q_{3N}} \Phi_N(w)dw = \left[\frac{3}{4}, \quad \frac{3}{4}\right]$$

Therefore following theorem 3, we can write:

$$Q_{IN} = \left[\frac{\ln(4)}{\theta_u}, \frac{\ln(4)}{\theta_l}\right] \text{ and } Q_{3N} = \left[\frac{\ln(\frac{4}{3})}{\theta_u}, \frac{\ln(\frac{4}{3})}{\theta_l}\right].$$

Theorem 5 The mean of the NED is $\frac{1}{\theta_N}$

Proof The neutrosophic mean of the NED is determined as:

$$\mu_N = \int_0^{\infty} \omega_N(w)dw$$

$$= \int_0^{\infty} [\omega_l(w), \omega_u(w)]dw$$

$$= \left[\int_0^{\infty} \exp(-w\theta_l)dw, \quad \int_0^{\infty} \exp(-w\theta_u)dw\right]$$

$$= \left[\frac{1}{\theta_u}, \quad \frac{1}{\theta_l}\right]$$

$$= \frac{1}{\theta_N}. \quad (10)$$

Theorem 6. The variance of the NED is $\frac{1}{\theta^2{}_N}$

Proof By definition variance is

$$\sigma_N{}^2(W) = E(W^2) - (\mu_N)^2 \quad (11)$$

where $\sigma_N{}^2(W)$ stands for neutrosophic variance

Now $E(W^2) = \int_0^{\infty} w^2 \varphi_N(w) dw$ (12)

Since $\varphi_N(w) = -\omega_N'(w)$

It follows:

$$E(W^2) = \frac{2}{\theta_N} \int_0^{\infty} \omega_N(w) dw$$

$$= \frac{2}{\theta_N} \int_0^{\infty} [\omega_l(w), \omega_u(w)] dw$$

$$= \frac{2}{\theta_N} \left[ \int_0^{\infty} \exp(-w\theta_l) dw, \quad \int_0^{\infty} \exp(-w\theta_u) dw \right]$$

$$= \frac{2}{\theta_N} \left[ \frac{1}{\theta_u}, \quad \frac{1}{\theta_l} \right]$$

$$= \left[ \frac{2}{\theta^2{}_u}, \quad \frac{2}{\theta^2{}_l} \right]$$

Thus (11) yields

$$\sigma_N{}^2(W) = \left[ \frac{2}{\theta^2{}_u}, \frac{2}{\theta^2{}_l} \right] - \left( \left[ \frac{1}{\theta_u}, \frac{1}{\theta_l} \right] \right)^2 \quad (13)$$

Simplifying (13) provides

$$\sigma_N{}^2(W) = \left[ \frac{1}{\theta^2{}_u}, \frac{1}{\theta^2{}_l} \right] \quad (14)$$

Likewise, the other properties of the NED can be established in a neutrosophic environment. Some applications of the proposed model are presented to understand the initial concepts derived for the NED.

**4. Illustrative Examples**

In this section the notion of the NED has been described with a series of examples in the area of applied statistics.

**Example1** Hits to certain website follow a Poisson process with an average of {2,4} hits per hour in a day. Let the time between two hits is denoted by the random variable W. Find the probability that waiting time is less than an hour.

**Solution** Poisson distribution is connected with the exponential distribution. The waiting time between Poisson events occurring follows the exponential distribution.

Using theorem 2 we can write:

$P(W < 1) = \Phi_N(1)$

$= 1 - \exp(-w\{2,4\})$

$= \{0.86, 0.98\}$

Thus chance to hit the website less than an hour is $\{86, 98\}\%$.

**Example 2** Failure mechanism of the alternators used in automobiles follows the NED for an average lifespan of $[8, 12]$ years. Mr. Adnan buys a six years old car with a functioning alternator to keep it for eight years. Determine the probability of the alternator failing during his possession.

**Solution** Let W denote the neutrosophic random variable that follows NED.

Given that $\mu_N = \left[\frac{1}{\theta_u}, \frac{1}{\theta_l}\right] = [8, 12]$ years

This implies $[\theta_l, \theta_u] = [0.083, 0.125]$

Now the required probability:

$P[W < 8] = \Phi_N(8)$

$= [0.079, 0.117]$

Thus the chance that the alternator fails during his ownership is approximated by $[8, 12]\%$.

**Example 3** Let an electrical device has a certain component whose failure time (in months) is determined by the random variable W that is nicely modelled by the NED with average time to failure equal to $\{5, 6\}$. What is the probability that the component would still be functional after 4 months?

**Solution** Using (1) we can write:

$$P(W > 6) = \int_{4}^{\infty} \{5, 6\}\exp(-w\{5, 6\})dw$$

$$= 1 - \int_{0}^{4} \{5, 6\}\exp(-w\{5, 6\})dw$$

Using the result given in the theorem 2 we can write:

$$= 1 - \Phi_N(4)$$

$$= \{0.48, 0.55\}$$

**5. Sample Estimation**

The method for estimating the parameter of the NED namely neutrosophic maximum likelihood estimation (NML) estimation has been introduced. Let we haves n sample $\{X_i, i = 1,2, .. n\}$ values are taken from the NED. The question is, which value of the neutrosophic parameter should be used for the observed sample?. This value can be determined by the likelihood function of the neutrosophic model. As neutrosophy exist in the parameter of the NED, therefore NML function of the NED is given by:

$$\varpi_N(w, \theta_N) = n\log\theta_N - \theta_N \sum_i^n w_i \quad (15)$$

The NML estimates namely $\hat{\theta}_L$ and $\hat{\theta}_U$ can be obtained by solving the following expression:

$$= \frac{\delta\varpi_N(w, \theta_N)}{\delta\theta_N}$$

Using the neutrosophic calculus [12], yielded

$$= \left[\frac{\delta\varpi_L(w,\theta_l)}{\delta\theta_u}, \frac{\delta\varpi_U(w,\theta_u)}{\delta\theta_l}\right] \quad (16)$$

where $\varpi_L(w, \theta_l) = n\log\theta_l - \theta_L \sum_i^n w_i$ and $\varpi_N(w, \theta_u) = n\log\theta_u - \theta_u \sum_i^n w_i$

Simplification of (15) provides:

$$\frac{\delta\varpi_N(w,\theta_N)}{\delta\theta_N} = \left[\frac{n}{\theta_l} - \sum_i^n w_i , \frac{n}{\theta_u} - \sum_i^n w_i\right] \quad (17)$$

Setting (17) equating to $[0, 0]$ provides:

$\hat{\theta}_l = \frac{n}{\sum_i^n w_i}$ and $\hat{\theta}_u = \frac{n}{\sum_i^n w_i}$

Thus

$\hat{\theta}_N = [\hat{\theta}_l, \hat{\theta}_u] = \frac{n}{\sum_i^n w_i}$ which is a single crisp value and coincides with the classical MLE.

However, if imprecision in the observed data ($\tilde{z}$) is considered then NML of the neutrosophic parameter would be modified as:

$$\hat{\theta}_N = [\hat{\theta}_l, \hat{\theta}_u] = \left[\frac{n}{A}, \frac{n}{B}\right] \quad (18)$$

where

$A = \min_{\tilde{w}} =$ sum ofthe minmum values of the neutrosophic dataset

$B = \max_{\tilde{w}} =$ sum of the maximum values of the neutrosophic dataset

**6 Simulation Analysis**

In this part, the performance of the NML estimator has been assessed in terms of the neutrosophic average biased ($AB_N$) and neutrosophic root mean square error ($RMS_N$) as defined below [21]:

$$AB_N = \frac{\sum_{j=1}^{N}(\hat{\theta}_{Nj} - \theta_N)}{N}$$

$$RMSE_N = \sqrt{\frac{\sum_{j=1}^{N}(\hat{\theta}_{Nj} - \theta_N)^2}{N}}$$

A Monte Carlo simulation is run in R software with various sample sizes and fixed value of the neutrosophic parameter $\theta_N = [0.5,\ 1.5]$. An imprecise dataset is generated using the NED with $\theta_N = [0.5,\ 1.5]$ and simulation analysis is replicated for a total of $N = 10000$ times with sample sizes of $n = 5, 15, 30,$ and 60, respectively. The performance measures of the NML estimator are then computed and given in Table 1.

Performance of NML estimate of the NED for simulated neutrosophic data

| | $AB_N$ | $RMSE_N$ |
|---|---|---|
| 5 | [0.124, 0.373] | [0.384, 1.143] |
| 15 | [0.035, 0.106] | [0.152, 0.457] |
| 30 | [0.017, 0.051] | [0.098, 0.296] |
| 60 | [0.008, 0.025] | [0.067, 0.201] |
| 150 | [0.003, 0.009] | [0.041, 0.125] |
| 300 | [0.002, 0.005] | [0.029, 0.087] |

It can be seen from the results, as the sample size $n$ goes up, the biases $AB_N$ and $RMSE_N$ decrease. Thus, the study concluded that the NML estimator provides reliable estimation with a larger sample size.

**7 Real Application**

In this section, real data has been used to illustrate the application of the proposed model. The data used for analysis is taken from the source [24]. Data contains the lifetime failures (in hours) of air conditioning instrument used in 720-Boeing planes. To check the adequacy of exponential model, an informal procedure of some necessary graphs have been used. The graphical diagnostic of the exponential model along with other candidate models to failure time data is displayed in Figure 3.

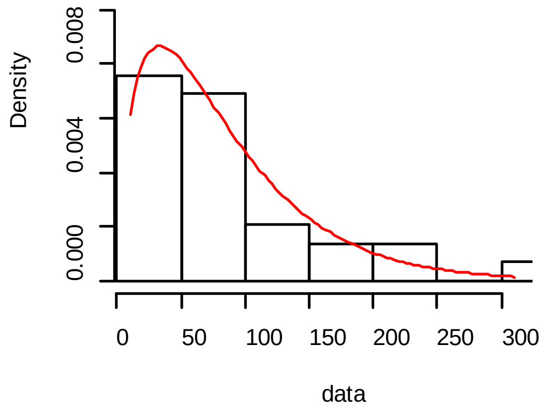


**Figure 3** Model fitting to failure time data using the candidate exponential family models

Figure 1 emphasizes the adequacy of the exponential distribution on life failures data. Theoretical lines in these necessary graphs from the exponential are shown with colored lines. Theoretical fits show the appropriateness of the exponential model among the predefined set of candidate probability models for the observed variable. Figure 1 describes that the exponential good fits the data at both tails and center of the empirical distribution. It has been assumed that all data values are not précised defined, and some values involve uncertainties and are given in the form of intervals. These uncertain observations are intentionally created according to the methodology defined in [25]. The indeterminate failure times data is given in Table 2.

**Table 2** The lifetime failures of air conditioning instrument used in 720-Boeing planes

| Failure times (in hours) | | | | |
|---|---|---|---|---|
| [89.40, 90.80] | [ 9.90, 10.02] | [59.12, 60.66] | [185.71, 186.66] | [ 60.80, 61.95] |
| [48.25, 49.21] | [13.05, 14.71] | [23.17, 24.45] | [55.36, 56.80] | [19.44, 20.25] |
| [78.29, 79.10] | [ 83.91, 84.18] | [ 43.33, 44.11] | [58.43, 59.11] | [28.28, 29.24] |
| [117.22, 118.90] | [24.12, 25.00] | [155.83, 156.07] | [309.10, 310.47] | [75.511, 76.43] |
| [ 25.51, 26.19] | [ 43.99, 44.70] | [22.82, 23.96] | [ 61.87, 62.64] | [129.82, 130.38] |
| [207.23, 208.68] | [ 69.28, 70.63] | [100.07, 101.48] | [207.97, 208.16] | |

The conventional exponential model cannot be used to analyze such data, as shown in Table 2. The values in Table 2 are provided in intervals because indeterminacies or exact values failure times are not recorded perfectly. On the contrary, the proposed exponential distribution can easily analyze such data. A descriptive summary of the failure times data rooted in the proposed model is shown in Table 3.

**Table 3** Descriptive summary of failure times data using the proposed model

| Descriptive Summary | |
|---|---|
| Estimated Rate parameter | [0.011, 0.012] |
| Estimated Mean | [82, 84] |
| Estimated Variance | [6888, 7051] |

The estimated values for rate, mean, and variance are in intervals due to indeterminacies in the observed data. Thus, the proposed model analyzes data more efficiently than the conventional model.

**8 Conclusions**

A new generalization of the classical exponential model, namely NED, has been presented in this work. The notion of neutrosophic theory has been utilized in order to quantify ambiguity in the absence of accurate distribution theory for analyzing data. The mathematical form of the proposed NED in a neutrosophic environment is thoroughly presented. The analytical expressions for the key properties of the proposed model, including neutrosophic moments, neutrosophic distribution function, and other related quantities, are derived. Some applicability examples of the NED mainly for the processing indeterminacies in data have been provided. An estimation approach of the maximum likelihood to estimate the parameter of the NED for dealing with imprecise data values is developed. To validate the performance of the neutrosophic estimator, a simulation study has been carried out. The simulation results demonstrate that indeterminate sample data with a larger size may be used to accurately estimate the unknown parameter of the proposed model.

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Received: Nov 29, 2021. Accepted: Feb 2, 2022

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# Single Valued Neutrosophic R-dynamic Vertex Coloring of Graphs

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**Abstract**. In 1998, Smarandache introduced the new theory - Neutrosophic sets. In order to achieve the best results in a current situation, policy makers must contend with uncertainty and unpredictability. The neutrosophic definition aids in the investigation of ambiguous or indeterminate values. Here, we have the amalgamated the theory of Single Valued Neutrosophic Vertex Coloring and $r$-dynamic coloring to introduce a new thought Single Valued Neutrosophic R-dynamic Vertex Coloring (SVNRVC) and have shown example. Further we have determined the Single Valued Neutrosophic R-dynamic chromatic number $\chi_R^v(G)$ for some graphs.

**Keywords:** Single Valued Neutrosophic Graph; Single Valued Neutrosophic Vertex Coloring; Single Valued Neutrosophic R-dynamic Vertex Coloring.

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## 1. Introduction

Graph Theory dates back to the year 1736 when the famous Mathematician Leonard Euler solved the Problem of Seven Bridges of Konigsberg. Graphs are mathematical structures made up of a set of vertices connected by edges. Many complex real-world problems can be successfully analysed using graphs as mathematical models. It can be used in a variety of fields such as chemical and physical sciences, networks, maps, sudoku, operations research, and so on. Graph coloring is a sub-discipline with graph theory. The famous Four Color Problem, posed by graduate student Francis Guthrie in 1852, inspired the problem of graph coloring. Is it possible to color the countries on the map with four or fewer colors so that any two countries sharing a border are colored differently? It was later on demonstrated by Appel and Haken in

1976. Graph coloring is the process of assigning colors to the elements of graph while keeping some constraints in mind.

Zadeh [26] proposed the theory of fuzzy sets way back in 1965, and ten years hence A. Rosenfeld [21] developed further work on fuzzy graph theory. Munoz et al. [24] first proposed the fuzzy chromatic number in 2004, and C. Eslahchi et al. [13] expanded it further in 2006. The idea of fuzzy total coloring was first suggested by S. Lavanya and R. Sattanathan [15] in 2009. In 2012 Arindam Dey and Anita Pal discussed fuzzy vetex coloring using $\alpha$-cut of fuzzy graphs in [6]. In a research paper published in 2014, the strong chromatic number of such graphs was addressed by Anjaly Kishore, M.S.Sunitha [4].

Intuitionistic fuzzy sets are used to deal with data on membership and non-membership values. In 1986, Kassimir T. Atanassov [5] proposed the theory of intuitionistic fuzzy sets, and in 1999, he proposed the notion of intuitionistic fuzzy graphs. In 2015, Ismail and Rifayathali [14] examined intuitionistic fuzzy graph coloring using $(\alpha, \beta)$ cuts, while Rifayathali et al. [17] in 2017 and 2018 published articles on intuitionistic fuzzy and strong intuitionistic fuzzy coloring.

The membership and non-membership principles are inadequate to determine the outcome of all real-time scenarios. Where the vagueness or indeterminacy qualities of a decision need to be weighed, intuitionistic fuzzy logic is inadequate to provide a solution. As a consequence of this condition, F. Smarandache devised a solution: "Neutrosophic logic." Neutrosophic logic is important in a number of real-world problems, including law, business, medicine, finance, information technology and so on. Thus in 1998 Smarandache [22] introduced the thought of Neutrosophic sets which is a generalized version of intuitionistic fuzzy set which includes three types of values: truth, indeterminacy and false membership values. In 2010, Wang et al. [25] investigated single valued neutrosophic sets. Dhavaseelan et al. [12] put forward and discussed the Strong Neutrosophic graphs in 2015, and Akram and Shahzadi [1–3] introduced and discussed the Single Valued Neutrosophic definition in 2016. Broumi et al. [7–11] built on their previous work in the areas of single-valued neutrosophic graphs. In their paper published in 2018, Dhavaseelan et al. [12] addressed single valued co-neutrosophic graphs. In 2018, Sinha et al. [23] widened the scope of the single-valued work for signed digraphs.

In the research articles [18, 19] published in 2019 Rohini et al. introduced the thought of single valued neutrosophic vertex, edge and total coloring of SVNG with examples. Further in [20] Rohini et al. have extended their work on single valued neutrosophic vertex coloring and put forward the new idea of single valued neutrosophic irregular vertex coloring.

The idea of $r$-dynamic coloring was put forward by Bruce Montgomery in [16]. The $r$-dynamic coloring of a graph is a proper coloring of the graph such that for each vertex $u$, the neighbors of the vertex $u$ receives $min\{r, d(v)\}$ different colors. Here we have integrated the thought of single valued neutrosophic vertex coloring and $r$-dynamic coloring to introduce the new idea of Single Valued Neutrosophic R-dynamic Vertex Coloring and have shown example. Further we have determined the Single Valued Neutrosophic R-dynamic chromatic number $\chi_R^v(G)$ for some graphs.

## 2. Preliminaries

**Definition 2.1.** [22] Assume S be a collection of points(objects). A **neutrosophic set** X in S is represented by truth membership function $t_X(s)$, an indeterminacy function $i_X(s)$ and a falsity membership (non-membership) function $f_X(s)$. $t_X(s)$, $i_X(s)$ and $f_X(s)$ are real standard or non-standard subsets of $]0^-, 1^+[$ which means $t_X(s) : S \to ]0^-, 1^+[$, $i_X(s) : S \to ]0^-, 1^+[$ and $f_X(s) : S \to ]0^-, 1^+[$. Also $0^- \leq t_X(s) + i_X(s) + f_X(s) \leq 3^+$.

**Definition 2.2.** [2] A **Single Valued Neutrosophic Graph (SNVG)** $G = (P, Q)$ is a pair where $P : N \to [0, 1]$ is a single valued neutrosophic set on N and $Q : N \times N \to [0, 1]$ is a single valued neutrosophic relation on N with the following properties:
$t_Q(uv) \leq min\{t_P(u), t_P(v)\}$
$i_Q(uv) \leq min\{i_P(u), i_P(v)\}$
$f_Q(uv) \leq max\{f_P(u), f_P(v)\}$
for all $u, v \in N$. The sets P and Q are said to be single valued neutrosophic vertex set and edge set of G respectively. The single valued neutrosophic relation Q is symmetric if it satisfies $t_Q(uv) = t_Q(vu)$, $i_Q(uv) = i_Q(vu)$ and $f_Q(uv) = f_Q(vu)$ for all $u, v \in N$.

**Definition 2.3.** [3] An SVNG $G = (P, Q)$ is called a **complete neutrosophic graph (CSVNG)** if it complies criteria below:
$t_Q(uv) = min\{t_P(u), t_P(v)\}$
$i_Q(uv) = min\{i_P(u), i_P(v)\}$
$f_Q(uv) = max\{f_P(u), f_P(v)\}$
for all $u, v \in P$.

**Definition 2.4.** [3] The **complement** of a SVNG $G = (P, Q)$ is a SNVG $G' = (P', Q')$ where
$i) P' = P$
$ii) t'_P(u) = t_P(u), i'_P(u) = i_P(u) \; and \; f'_P(u) = f_P(u)$
$$iii) t'_Q(uv) = \begin{matrix} min\{t_P(u), t_P(v)\} & if \quad t_Q(uv) = 0 \\ min\{t_P(u), t_P(v)\} - t_Q(uv) & if \quad t_Q(uv) > 0 \end{matrix}$$
$$iv) i'_Q(uv) = \begin{cases} min\{i_P(u), i_P(v)\} & if \quad i_Q(uv) = 0 \\ min\{i_P(u), i_P(v)\} - i_Q(uv) & if \quad i_Q(uv) > 0 \end{cases}$$
$$v) f'_Q(uv) = \begin{cases} max\{f_P(u), f_P(v)\} & if \quad f_Q(uv) = 0 \\ max\{f_P(u), f_P(v)\} - f_Q(uv) & if \quad f_Q(uv) > 0 \end{cases}$$
for all $u, v \in P$.

**Definition 2.5.** [3] An SVNG $G = (P, Q)$ is said to be a **strong neutrosophic graph (SSVNG)** if it complies criteria:
$t_Q(uv) = min\{t_P(u), t_P(v)\}$
$i_Q(uv) = min\{i_P(u), i_P(v)\}$
$f_Q(uv) = max\{f_P(u), f_P(v)\}$
for all $(u, v) \in Q$.

**Definition 2.6.** [18] The collection $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ of SVN fuzzy sets is termed as **k-Single Valued Neutrosophic Vertex Coloring(SVNVC)** of a SVNG $G = (P, Q)$ if the following criteria hold:

1. $\vee\gamma_j(u) = P, \forall u \in P$
2. $\gamma_j \wedge \gamma_h = 0$
3. For each incident vertices of the edge $uv$ of G, $min\{\gamma_j(t_P(u)), \gamma_j(t_P(v))\} = 0$, $min\{\gamma_j(i_P(u)), \gamma_j(i_P(v))\} = 0$ and $max\{\gamma_j(f_P(u)), \gamma_j(f_P(v))\} = 1, (1 \leq j \leq k)$.

This is indicated as $\chi^v(G)$ and is termed as the SVN chromatic number of the SVNG G.

**Example:** Consider the following SVNG $G_1 = (P, Q)$ with SVN vertex set $P = \{v_1, v_2, v_3, v_4\}$ and SVN edge $Q = \{v_jv_k | jk = 12, 13, 14, 23, 25, 34\}$ with

$$(t_P(v_j), i_P(v_j), f_P(v_j)) = \begin{cases} (0.2, 0.3, 0.7) & j = 1 \\ (0.7, 0.2, 0.8) & j = 2 \\ (0.6, 0.5, 0.9) & j = 3 \\ (0.5, 0.4, 0.6) & j = 4 \end{cases}$$

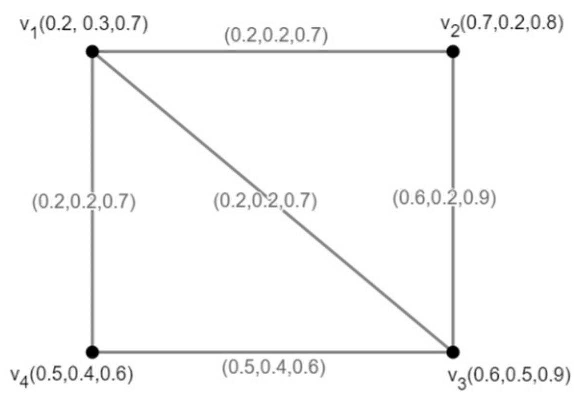


FIGURE 1. $G_1$

$$(t_Q(v_jv_k), i_Q(v_jv_k), f_Q(v_jv_k)) = \begin{cases} (0.2, 0.2, 0.7) & jk = 12, 13, 14 \\ (0.6, 0.2, 0.9) & jk = 23 \\ (0.5, 0.4, 0.6) & jk = 34 \end{cases}$$

Figure 1 depicts the SVNG $G_1$.

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ be collection of SVN fuzzy sets determined on P as below:

$$\gamma_1(v_j) = \begin{cases} (0.2, 0.3, 0.7) & for \quad j = 1 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_2(v_j) = \begin{cases} (0.7, 0.2, 0.8) & for \quad j = 2 \\ (0.5, 0.4, 0.6) & for \quad j = 4 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_3(v_j) = \begin{cases} (0.6, 0.5, 0.9) & for \quad j = 3 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ assures the criteria of SVNVC of the graph G. Any collection with points less than three points will not fulfill our definition. Hence $\chi^v(G_1) = 3$.

**Definition 2.7.** [20] The collection $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ of SVN fuzzy sets is called a **k-Single Valued Neutrosophic Irregular Vertex Coloring (SVNIVC)** of a SVNG $G = (P, Q)$ if the following criteria hold:

1. $\vee\, \gamma_j$ (u) $= P, \forall u \in P$

2.$\gamma_j \wedge \gamma_h = 0$

3. For each incident vertices of edge $uv$ of G, $min\{\gamma_j(t_P(u)), \gamma_j(t_P(v))\} = 0$, $min\{\gamma_j(i_P(u)), \gamma_j(i_P(v))\} = 0$ and $max\{\gamma_j(f_P(u)), \gamma_j(f_P(v))\} = 1, (1 \leq j \leq k)$.

4. All the vertices have different color codes.

This is depicted as $\chi^v{}_{ir}\,(G)$ and is termed as the SVNI chromatic number of the SVNG G.

**Definition 2.8.** [8] **Path** $P_n$ in a single valued neutrosophic graph $G = (P, Q)$ is an arrangement of distinct vertices $v_1, v_2, \cdots, v_n$ which complies the criteria $t_Q(v_{j-1}, v_j) > 0$, $i_Q(v_{j-1}, v_j) > 0$ and $f_Q(v_{j-1}, v_j) > 0$ for $2 \leq j \leq n$.

**Definition 2.9.** [8] A **cycle** $C_n$ in a single valued neutrosophic graph $G = (P, Q)$ is a sequence of distinct vertices $v_1, v_2, \cdots, v_n, v_1$ which satisfies the condition $t_Q(v_{i-1}, v_i) > 0$, $i_Q(v_{i-1}, v_i) > 0$ and $f_Q(v_{i-1}, v_i) > 0$ for $2 \leq i \leq n$ together with $t_Q(v_n, v_1) > 0$, $i_Q(v_n, v_1) > 0$ and $f_Q(v_n, v_1) > 0$.

## 3. Single Valued Neutrosophic R-dynamic Vertex Coloring

**Definition 3.1.** A family $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ of SVN fuzzy sets is termed as **k-Single Valued Neutrosophic R-dynamic Vertex Coloring (SVNRVC)** of a SVNG $G = (P, Q)$ if the following criteria hold:

1. $\vee\, \gamma_j$ (u) $= P, \forall u \in P$

2.$\gamma_j \wedge \gamma_h = 0$

3. For each incident vertices of edge $uv$ of G, $min\{\gamma_j(t_P(u)), \gamma_j(t_P(v))\} = 0$, $min\{\gamma_j(i_P(u)), \gamma_j(i_P(v))\} = 0$ and $max\{\gamma_j(f_P(u)), \gamma_j(f_P(v))\} = 1, (1 \leq j \leq k)$.

4. Every vertex $u$ with $m$ number of incident edges, the corresponding incident vertices of the vertex $u$ receives atleast $min\{R, m\}$ different members(colors) from the set $\Gamma$.

Here, $1 \leq R \leq M$ where $M$ represents the maximum number of incident edges of the vertices of SVNG G.
The least value of $k$ is the SVNRVC of SVNG G is denoted as $\chi_R^v(G)$, is called the Single Valued Neutrosophic R-dynamic chromatic number.

**Example:** Examine SVNG $G_2 = (P, Q)$ with SVN vertex set and edge set $P = \{v_1, v_2, \cdots, v_5\}$ and $Q = \{v_j v_k | jk = 12, 13, 14, 23, 25, 34, 35, 45\}$ respectively:

$$(t_P(v_j), i_P(v_j), f_P(v_j)) = \begin{cases} (0.4, 0.2, 0.7) & j = 1 \\ (0.6, 0.3, 0.4) & j = 2, 3 \\ (0.3, 0.1, 0.6) & j = 4 \\ (0.7, 0.4, 0.3) & j = 5 \end{cases}$$

$$(t_Q(v_j v_k), i_Q(v_j v_k), f_Q(v_j v_k)) = \begin{cases} (0.4, 0.2, 0.6) & jk = 12, 13 \\ (0.3, 0.1, 0.6) & jk = 14, 34, 45 \\ (0.6, 0.3, 0.4) & jk = 23, 25, 35 \end{cases}$$

Figure 2 depicts the SVNG $G_2$.
Here $M = 4$ so $1 \leq R \leq 4$

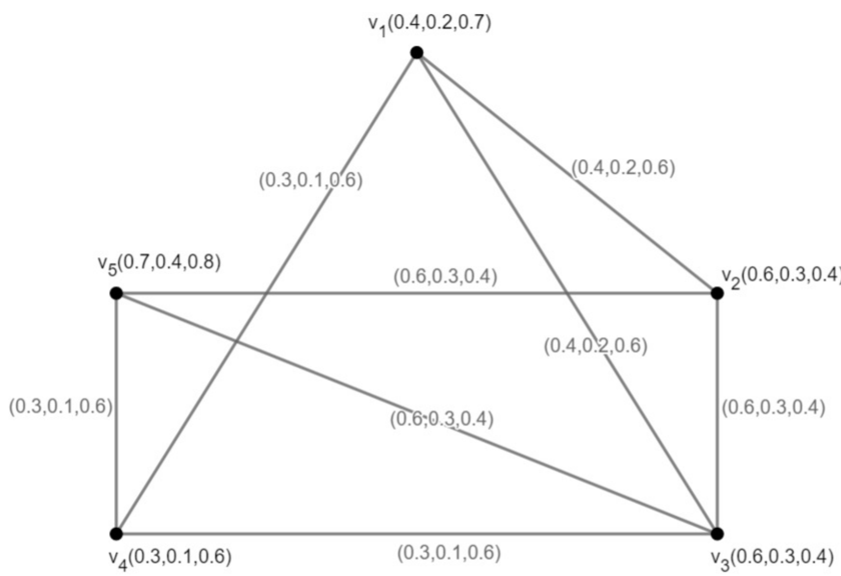


FIGURE 2. $G_2$

For $1 \leq R \leq 2$ let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ denote collection of SVN fuzzy sets determined on P as below:

$$\gamma_1(v_j) = \begin{cases} (0.4, 0.2, 0.7) & for \quad j = 1 \\ (0.7, 0.4, 0.8) & for \quad j = 5 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_2(v_j) = \begin{cases} (0.6, 0.3, 0.4) & for \quad j = 2 \\ (0.3, 0.1, 0.6) & for \quad j = 4 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_3(v_j) = \begin{cases} (0.6, 0.3, 0.4) & for \quad j = 3 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ assures criteria of SVNRVC of the graph G. Any collection with points lesser than three points will not fulfill our definition. Hence $\chi^v_R(G_2) = 3$ for $1 \leq R \leq 2$.

For $3 \leq R \leq 4$ let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ be collection of SVN fuzzy sets determined on P.

$$\gamma_1(v_j) = \begin{cases} (0.4, 0.2, 0.7) & for \quad j = 1 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_2(v_j) = \begin{cases} (0.6, 0.3, 0.4) & for \quad j = 2 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_3(v_j) = \begin{cases} (0.6, 0.3, 0.4) & for \quad j = 3 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_4(v_j) = \begin{cases} (0.3, 0.1, 0.6) & for \quad j = 4 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_5(v_j) = \begin{cases} (0.7, 0.4, 0.3) & for \quad j = 5 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ assures criteria of SVNRVC of the graph G. Any collection with points lesser than below five points will not fulfill our definition. Hence $\chi^v_R(G_2) = 5$ for $3 \leq R \leq 4$. Hence $\chi^v_R(G) = \begin{cases} 3 & for \quad 1 \leq R \leq 2 \\ 4 & for \quad 3 \leq R \leq 4 \end{cases}$

**Remark 3.2.** For any SVNG G we have $\chi^v(G) \leq \chi^v_R(G)$.

**Theorem 3.3.** *Let $n \geq 3$, $P_n$ be a path graph then* $\chi^v_R(P_n) = \begin{cases} 2 & for \quad R = 1 \\ 3 & for \quad R = 2 \end{cases}$

Proof:

For the path graph $P_n$, $1 \leq R \leq 2$.

Let $\Gamma = \{\gamma_1, \gamma_2\}$ be collection of fuzzy sets determined on vertices $V(P_n) = \{u_1, u_2, \cdots, u_n\}$ for $R = 1$ as follows:

$$\gamma_1(u_k) = \begin{cases} (t_P(u_k), i_P(u_k), f_P(u_k)) & for \quad k \text{ is odd} \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_2(u_k) = \begin{cases} (t_P(u_k), i_P(u_k), f_P(u_k)) & for \quad k \text{ is even} \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

Thus the family $\Gamma = \{\gamma_1, \gamma_2\}$ assures the conditions of SVNRVC of $P_n$. Any families with less than two points did not meet our criteria of the definition.

Thus $\chi^v_1(P_n) = 2$.

When $R = 2$, let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ be collection of fuzzy sets determined on vertices $V(P_n)$:

$$\gamma_1(u_k) = \begin{cases} (t_P(u_k), i_P(u_k), f_P(u_k)) & for \quad k \equiv 1(mod\ 3) \\ (0,0,1) & for \quad otherwise \end{cases}$$
$$\gamma_2(u_k) = \begin{cases} (t_P(u_k), i_P(u_k), f_P(u_k)) & for \quad k \equiv 2(mod\ 3) \\ (0,0,1) & for \quad otherwise \end{cases}$$
$$\gamma_3(u_k) = \begin{cases} (t_P(u_k), i_P(u_k), f_P(u_k)) & for \quad k \equiv 0(mod\ 3) \\ (0,0,1) & for \quad otherwise \end{cases}$$

Thus the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ assures the conditions of SVNRVC of $P_n$. Any families with less than three points did not meet our criteria of the definition.

Thus $\chi_2^v(P_n) = 3$.

Hence $\chi_R^v(P_n) = \begin{cases} 2 & for \quad R = 1 \\ 3 & for \quad R = 2 \end{cases}$

**Theorem 3.4.** *Let $k \geq 3$, $C_k$ be a cycle then* $\chi_R^v(C_k) = \begin{cases} 2 & for \quad R = 1 \quad and\ k\ is\ even \\ 3 & for \quad R = 1 \quad and\ k\ is\ odd \\ 5 & for \quad R = 2 \quad and\ k = 5 \\ 3 & for \quad R = 2 \quad and\ k = 3m \\ 4 & for \quad R = 2 \quad and\ otherwise \end{cases}$

Proof:

For a cycle $C_k$, $1 \leq R \leq 2$.

Let $\Gamma = \{\gamma_1, \gamma_2\}$ be collection of fuzzy sets determined on vertices $V(C_k) = \{c_1, c_2, \cdots, c_k\}$ for $R = 1$ and k is even as follows:

$$\gamma_1(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j\ is\ odd \\ (0,0,1) & for \quad otherwise \end{cases}$$
$$\gamma_2(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j\ is\ even \\ (0,0,1) & for \quad otherwise \end{cases}$$

Thus the family $\Gamma = \{\gamma_1, \gamma_2\}$ assures the conditions of SVNRVC of $C_k$. Any families with less than two points did not meet our criteria of the definition.

Thus $\chi_1^v(C_k) = 2$ when k is even.

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ be collection of fuzzy sets determined on vertices $V(C_k)$ for $R = 1$ and k is odd:

$$\gamma_1(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j \equiv 1(mod\ 2)\ but\ j \neq k \\ (0,0,1) & for \quad otherwise \end{cases}$$
$$\gamma_2(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j \equiv 0(mod\ 2) \\ (0,0,1) & for \quad otherwise \end{cases}$$
$$\gamma_3(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j = k \\ (0,0,1) & for \quad otherwise \end{cases}$$

Thus the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ assures the conditions of SVNRVC of $C_k$. Any families with less than three points did not meet our criteria of the definition.

Thus $\chi_1^v(C_k) = 3$ when k is odd.

For $R = 2$ and $n = 5$, let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ be a family of fuzzy sets determined on vertices $V(C_5)$:

$$\gamma_1(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j = 1 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_2(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j = 2 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_3(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j = 3 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_4(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j = 4 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_5(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j = 5 \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

Thus the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ assures the conditions of SVNRVC of $C_5$. Any families with less than five points did not meet our criteria of the definition.

Thus $\chi_2^v(C_5) = 3$.

For $R = 2$ and $k = 3m, m = 1, 2, \cdots$ let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ be a family of fuzzy sets determined on vertices:

$$\gamma_1(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j \equiv 1(mod\ 3) \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_2(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j \equiv 2(mod\ 3) \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_3(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j \equiv 0(mod\ 3) \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

Thus the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ assures the conditions of SVNRVC of $C_n$. Any families with less than three points did not meet our criteria of the definition.

Thus $\chi_1^v(C_k) = 3$ when $k = 3m$.

For $R = 2$ and otherwise let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be a family of fuzzy sets determined on vertices as follows:

When $k = 3m + 1, m = 1, 2, \cdots$.

$$\gamma_1(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j \equiv 1(mod\ 3)\ but\ j \neq k \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_2(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j \equiv 2(mod\ 3) \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_3(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j \equiv 0(mod\ 3) \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

$$\gamma_4(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j = k \\ (0, 0, 1) & for \quad otherwise \end{cases}$$

When $k = 3m + 2, m = 1, 2, \cdots$

$$\gamma_1(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j \equiv 1(mod\ 3)\ and\ j = k-2 \\ (0,0,1) & for \quad otherwise \end{cases}$$
$$\gamma_2(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j \equiv 2(mod\ 3)\ and\ j = k-3 \\ (0,0,1) & for \quad otherwise \end{cases}$$
$$\gamma_3(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j \equiv 0(mod\ 3)\ and\ j = k \\ (0,0,1) & for \quad otherwise \end{cases}$$
$$\gamma_4(c_j) = \begin{cases} (t_P(c_j), i_P(c_j), f_P(c_j)) & for \quad j = k-1, k-4 \\ (0,0,1) & for \quad otherwise \end{cases}$$

Thus the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ assures the conditions of SVNRVC of $C_n$. Any families with less than four points did not meet our criteria of the definition.

Thus $\chi_2^v(C_k) = 4$ when otherwise.

Hence $$\chi_R^v(C_k) = \begin{cases} 2 & for \quad R = 1 \quad and\ k\ is\ even \\ 3 & for \quad R = 1 \quad and\ k\ is\ odd \\ 5 & for \quad R = 2 \quad and\ k = 5 \\ 3 & for \quad R = 2 \quad and\ k = 3m \\ 4 & for \quad R = 2 \quad and\ otherwise \end{cases}$$

**Theorem 3.5.** *For the CSVNG with n vertices,* $\chi_R^v(K_n) = n$.

Proof:

For the CSVNG $M = n-1$ and hence $1 \leq R \leq n-1$. One can notice that all vertices are incident to one another. Let $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ be a family of fuzzy sets determined on vertices such that each set contains exactly one vertex with value $t_P(w), i_P(w), f_P(w) > 0$ and all the other vertices have the value $(0,0,1)$. By this the criteria of SVNRVC will be assured and hence $\chi_R^v(K_n) = n$.

## 4. Conclusion

We have the amalgamated the theory of Single Valued Neutrosophic Vertex Coloring and $r$-dynamic coloring to build a new thought Single Valued Neutrosophic R-dynamic Vertex Coloring (SVNRVC). We have defined the new coloring and provided examples. Further we have looked onto Single Valued Neutrosophic R-dynamic Chromatic Number of certain graphs.

**Funding:** There was no external funding for this work.

**Acknowledgment:** The authors gratefully acknowledge the referees for their thorough reading, insightful remarks and helpful suggestions that have enhanced the quality of this manuscript.

**Conflicts of Interest:** There are no conflicts of interest declared by the authors.

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Received: Nov 29, 2021. Accepted: Feb 2, 2022

Neutrosophic Sets and Systems, Vol. 48, 2022
University of New Mexico

# Tropezoidal Neutrosophic Deal with Logarithmic Demand with Shortage of Deteriorating Items

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**Abstract**: In this paper is to introduce trapezoidal neutrosophic deal with logarithmic demand model with shortage of deteriorating items. Order of enact customers can be place by the vendor depending on the stock accessibility. Logarithmic demand is related to several products, in this paper developed with the shortage of items at the beginning. Finally, valuable example is given to extract the optimum value and obtain effective valuable results.

**Keywords:** Inventory, Tropezoidal Neutrosophic number, deterioration, shortage, logarithmic demand.

## 1. INTRODUCTION

Initial stage of LPG Gas business starts with the Shortage against the booking and offers. With the incorporation of dual objects that is logarithmic demand and business starts with shortage. Many products of daily used in demand but LPG is the crucial. This will boost up retailers order in positive mode. Some products are huge need for people, like Milk, Oil, flour, beverages whose scarcity loss the customer's trust and received design. This scenario stimulates retailer to order intemperate quantity of items, despite of deterioration. So any uncertainty situation of decaying or due to deterioration is not negligible. The purpose is denied due to damage or spoiled items. Deterioration helps in managing several items due to virtue of modern advanced storage technics Deterioration factor is incorporated in this proposed model. For on-going successful business Inventory model demonstrate the real time problem.

Chakraborty et al. [3] focus on pentagonal neutrosophic numbers and their distinct properties. Smarandache [20] By considering the non-standard analysis, they implemented a neutrosophic set and a neutrosophic logic. Also, neutrosophic model of inventory without shortages is provides by Mullai et al. [14]. Chakraborty et al. [4] various types of triangular neutrosophic numbers, de-neutrosophication models and their applications have been clarified. Adaraniwon et al. [2], ignited the concept of An inventory model for delayed degradation of power demand goods, taking into account shortages and missed sales.

Burwell et al unraveled the issue emerging in trade by giving discounts and displayed financial qunatity size demonstrates with demand price dependent. Shin et al [18] discussed an optimal strategy or salable price and volume under vendor credit. Shula et al developed a three-factor order rate for new released product deteriorate couples of methods supported stable required rates and once launching the item in consumer use, it creates stable need.

Wen et al recommended a energetic estimating arrangement for offering a given commodities of indistinguishable biodegradable items over a restricted time skyline on the online. The deal closes either once the full stock oversubscribed out, or once released the launching date. Target of the vendor is to find a ultimate valuation reach that optimize to complete anticipated incomes.

Lin (2006) [13] the EOQ model designed reflects how the pattern of demand which is value, cash discount depends on market demand and product availability. They mention EOQ system that takes under consideration selling price relay directly on inflation and continuance. Occurrence and singularity of the optimum answer is unsolved during this article.

Karaaslan [7] formulated and analytically solved in multi-guidelines, Gaussian sole-rate neutrosophic numbers and their implementation. Smarandache [20] argued that Neutrosophic, probability of neutrosophic number, logic and set, unifying space of logic. Murugappan [15] mentioned the inventory model of neutrosophic variable, unit priced neutrosophic.

## 2. ASSUMPTIONS AND NOTATION

Assume that the shortages accumulated till time $t_1$ up to level $I_1(t_1)$ and order placed to the corporate seller at time $t_1$, therefore uncovered demand consummated and inventory meets up to level $I_2(t_1)$. The inventory level $I_2(t_1)$ is comfortable to meet the demand until time $T$. The optimal time $t_1$ are going to be resolve. $I_1(t_1)$ and $I_2(t_1)$, that optimize the overall inventory price. Inventory depletion is shown in Fig 1.

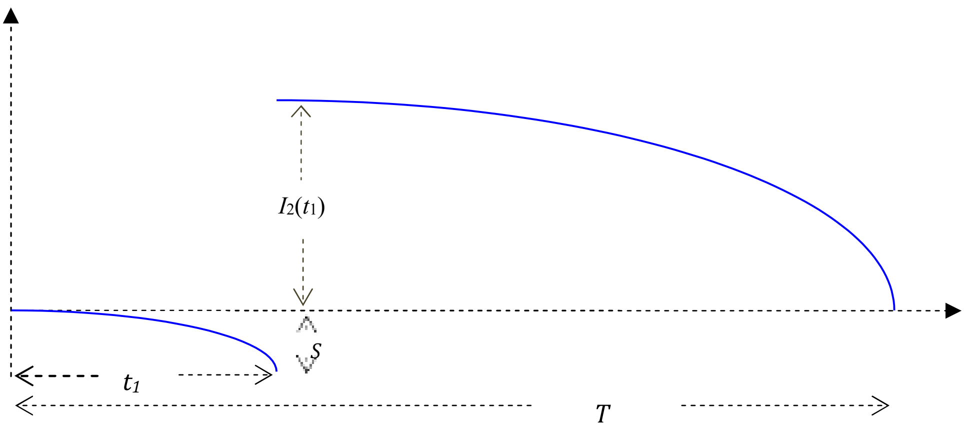


Figure: 1

The following symbols are used throughout this paper:

$D(t)$ : Demand rate is $D(t) = \alpha \log(\beta t)$, where $\alpha > 1$ and $\beta > 1$ are positive real values

.

$I_1(t)$ : Stock level at time t, $0 < t < t_1$

$I_2(t)$ : Stock level at time t, $t_1 < t < T$

Q : Total order Quantity per cycle

$\emptyset$ : Rate of deterioration $0 < \emptyset < 1$

$C_1$ : Holding cost per unit time

$C_2$ : Deterioration cost

$C_3$ : Shortage cost per unit time

$\widetilde{C_1}$ : Holding cost per unit time

$\widetilde{C_2}$ : Deterioration cost

$\widetilde{C_3}$ : Shortage cost per unit time

T : Span time

$t_1^*$ : Optimal time for accumulating shortage

$TF(t_1)$ : Optimal mean inventory price,

$H_C$ : Total holding price,

$D_C$ : Total deterioration cost.

$S_c$ : Total shortage units in the system,

## 3. MATHEMATICAL MODEL

**3.1 Definition: Neutrosophic Set: Smarandache[20]**

A collection of $\widetilde{Ns}$ in the universal discourse $X$, A symbolic notation by $x$, it is said to be neutrosophic set if $\widetilde{Ns} = \{\langle x;[\rho_{\widetilde{Ns}}(x), \sigma_{\widetilde{Ns}}(x), \tau_{\widetilde{Ns}}(x)]\rangle : x \in X\}$, where $\rho_{\widetilde{Ns}}(x): X \rightarrow [0,1]$ is called the real membership function, which addresses the level of confirmtion, $\sigma_{\widetilde{Ns}}(x): X \rightarrow [0,1]$ is called the dubiety membership, which denotes the degree of vagueness, and $\tau_{\widetilde{Ns}}(x): X \rightarrow [0,1]$ is called the falsehood membership, which demonstrates the level of scepticism on the decision taken by the decision maker $\rho_{S\widetilde{VTrNs}}(x)$, $\sigma_{S\widetilde{VTrNs}}(x)$, $\tau_{S\widetilde{VTrNs}}(x)$ The accompanying relationship reveals: $0 \leq \rho_{S\widetilde{VTrNs}}(x) + \sigma_{S\widetilde{VTrNs}}(x) + \tau_{S\widetilde{VTrNs}}(x) \leq 3$

**3.2 Definition: Single Valued Neutrosophic Set: Chakraborty [ 3]**

A set of Neutrosphic is $\widetilde{Ns}$ in the definition 3.1. is claimed to be a single-Valued neutrosophic set ($S\widetilde{VTrNs}$ ) if $x$ may be single valued independent variable. $S\widetilde{VTrNs} = \{\langle x;$ $[\rho_{S\widetilde{VTrNs}}(x), \sigma_{S\widetilde{VTrNs}}(x), \tau_{S\widetilde{VTrNs}}(x)]\rangle : x \in X\}$, where $\rho_{S\widetilde{VTrNs}}(x)$, $\sigma_{S\widetilde{VTrNs}}(x)$, $\tau_{S\widetilde{VTrNs}}(x)$ provided the method of accuracy, dubiety and falsehood memberships function respectively.

**Definition 3.2.1: (Neutro-normal.)**

If there exist three variable $\varphi_0$, $\chi_0$& $\psi_0$ , for which $\rho_{S\widetilde{VTrNs}}(\varphi_0) = 1$, $\rho_{S\widetilde{VTrNs}}(\chi_0) = 1$ &$\tau_{S\widetilde{VTrNs}}(\psi_0) = 1$, then the $S\widetilde{VTrNs}$ is called neutro-normal.

**Definition 3.2.2: (Neutro-convex.)**

$\widetilde{SVTrNs}$ is called Neutro-convex, which provides that $\widetilde{SVTrNs}$ is a member of a real value by satisfying the accompanying conditions:

i. $\rho_{\widetilde{SVTrNs}}\langle \vartheta\, \varphi_1 + (1-\vartheta)\varphi_2\rangle \geq min\langle\, \rho_{\widetilde{SVTrNs}}\,(\varphi_1),\ \ \rho_{\widetilde{SVTrNs}}\,(\varphi_2)\rangle$

ii. $\sigma_{\widetilde{SVTrNs}}\langle \vartheta\varphi + (\ 1-\vartheta)\varphi_2\rangle \leq\ max\langle \sigma_{\widetilde{SVTrNs}}\,(\varphi_1), \sigma_{\widetilde{SVTrNs}}\,(\varphi_2)\rangle$

iii. $\tau_{\widetilde{SVTrNs}}(\langle \vartheta\varphi_1 + (1-\vartheta)\varphi_2\rangle \leq max\langle \tau_{\widetilde{SVTrNs}}(\,(\varphi_1),\tau_{\widetilde{SVTrNs}}(\,(\varphi_2)\rangle$

where $\varphi_1$& $\in\mathbb{R}$ *and* $\vartheta\in$ [0, 1]

**Definition 3.3 ( Tropezoidal Single Valued Neutrosophic Number )**

Neutrosophic number with tropezoidal Single Valued ($\widetilde{\Omega}$) is defined a $\widetilde{\Omega}$=< $(\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3, \mathrm{r}_4\colon \Upsilon), (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4\colon \lambda), (\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3, \mathrm{q}_4\colon \eta)$ > , *where* $\mu$, $\vartheta$, $\zeta\in$ [0,1]. The real membership function $\rho_{\widetilde{\Omega}}$: R $\rightarrow$ [0, $\Upsilon$], the dubiety membership function $\sigma_{\widetilde{\Omega}}$: R $\rightarrow$ [$\lambda$, 1] and the falsehood membership function $\tau_{\widetilde{\Omega}}$: : R $\rightarrow$ [$\eta$, 1] are characterized as follows:

$$\pi_{\widetilde{\Omega}} = \begin{cases} \vartheta_{\widetilde{\Omega}l}(x), & r_1 \leq x < r_2 \\ \Upsilon, & r_2 \leq x < r_3 \\ \vartheta_{\widetilde{\Omega}r}(x), & r_3 < x \leq r_4 \\ 0, & otherwise \end{cases}$$

$$\theta_{\widetilde{\Omega}} = \begin{cases} \varepsilon_{\widetilde{\Omega}l}(x), & u_1 \leq x < u_2 \\ \lambda, & u_2 \leq x < u_3 \\ \varepsilon_{\widetilde{\Omega}r}(x), & u_3 < x \leq u_4 \\ 1, & otherwise \end{cases}$$

$$\eta_{\widetilde{\Omega}} = \begin{cases} \ell_{\widetilde{\Omega}l}(x), & q_1 \leq x < q_2 \\ \eta, & q_2 \leq x < q_3 \\ \ell_{\widetilde{\Omega}r}(x), & q_3 < x \leq q_4 \\ 1, & otherwise \end{cases}$$

**3.4 De-neutrosophication of Tropezoidal single Valued Neutrosophic number:**

This system, the expulsion region procedure executed to assess the de-neutrosophication worth of tropezoidal single esteemed neutrosophic number is $\widetilde{\Omega} =< (\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3, \mathrm{r}_4\colon \Upsilon), (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4\colon \lambda), (\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3, \mathrm{q}_4\colon \eta)$ > , de-neutrosophic form $\tilde{S}$ is provided as $TrneuD_{\widetilde{\Omega}} = \left(\frac{r_1+r_2+r_3+\mathrm{r}_4+u_1+u_2+u_3+\mathrm{u}_4+q_1+q_2+q_3+\mathrm{q}_4}{12}\right)$

$$\frac{dI_1(t)}{dt} = -\,\alpha\log(\beta t) \qquad 0 \leq t \leq t_1 \qquad I_1(0) = 0 \tag{1}$$

$$\frac{dI_2(t)}{dt} + \emptyset\, I_2(t) = -\,\alpha\log(\beta t) \qquad t_1 \leq t \leq T \tag{2}$$

Boundary values for above two differential equations are $I_1(0) = 0\,, I_2(\mathrm{T}) = 0$

On solving equation (1), we get

$$I_1(t) = \mathrm{A} - \int_0^t \alpha\log(\beta t)\,dt \quad with \quad I_1(0) = 0 \tag{3}$$

$$I_1(t) = \alpha t \circledcirc \alpha t\log(\beta t)$$

On solving equation (2), we get

$$I_2(t)e^{\emptyset t} = \mathrm{B} - \int_0^t \alpha\, e^{\emptyset t} \log(\beta t)\, dt \quad with \quad I_2(T) = 0 \tag{4}$$

Applying boundary condition $I_2(T) = 0$ , in the above equation, we get

$$I_2(t) = \alpha\left(T + \frac{\emptyset T^2}{2}\right)\log(\beta T) - \alpha\emptyset T t\log(\beta T) - \alpha t\log(\beta t) - \alpha(T-t) + \alpha\emptyset\left(Tt - \frac{T^2}{4} - \frac{3t^2}{4}\right) \tag{5}$$

Ordering cost $O_c$= A

Deteriorated cost ( $D_c$ ) in time $[t_1, T]$ is

$$D_c = C_1\left\{I_1(t) - \int_{t_1}^{T} \alpha \log(\beta t)\, dt\right\}$$

$$= C_1\left\{\alpha\left(T + \frac{\emptyset T^2}{2}\right)\log(\beta T) - \alpha\emptyset T\, t_1\log(\beta T) - \alpha \mathrm{T}\log(\beta T) + \alpha\emptyset\left(T\, t_1 - \frac{T^2}{4} - \frac{3\, t_1^2}{4}\right)\right\} \tag{6}$$

Holding cost $H_C$ , over time $[t_1, T]$

$$H_c = C_2\int_{t_1}^{T} I_2(t)\, dt$$

$$H_c = C_2\{(\mathrm{T} - t_1)\left(\alpha\left(T + \frac{\emptyset T^2}{2}\right)\log(\beta T) - aT\right) - \alpha\emptyset\frac{(T^3 - T\, t_1^2)}{2}\log(\beta T) - \alpha\frac{T^2}{2}\log(\beta T) + \frac{\alpha\, t_1^2}{2}\log(\beta\, t_1) + \frac{3\alpha}{4}(T^2 - t_1^2) + \frac{\emptyset\,\alpha}{4}(t_1^3 + t_1T^2 - 2T\, t_1^2) \tag{7}$$

Shortage cost $S_c$ over $[0, t_1]$ will be

$$S_c = C_3\int_0^{t_1} I_1(t)\, dt$$

$$S_c = C_3\left\{\frac{3\alpha\, t_1^2}{4} - \frac{\alpha\, t_1^2}{2}\log(\beta\, t_1)\right\} \tag{8}$$

Quantity remembering deficiency in trading will be $Q$

$$Q = I_1(t_1) + I_2(t_1)$$

$$= 2\alpha t_1 + \alpha\left(T + \frac{\emptyset T^2}{2}\right)\log(\beta T) - \alpha\emptyset T t_1\log(\beta T) - 2\alpha t_1\log(\beta t_1) - \alpha T + \alpha\emptyset\left(T t_1 - \frac{T^2}{4} - \frac{3t_1^2}{4}\right) \tag{9}$$

Total average inventory cost will be

$$TF(t_1) = \left[\frac{A + H_c + D_c + S_c}{T}\right]$$

$$TF(t_1) = \frac{1}{T}\left\{A + C_1\left\{\alpha\left(T + \frac{\emptyset T^2}{2}\right)\log(\beta T) - \alpha\emptyset T\, t_1 \log(\beta T) - \alpha \mathrm{T}\log(\beta T) + \alpha\emptyset\left(T\, t_1 - \frac{T^2}{4} - \frac{3\, t_1{}^2}{4}\right)\right\} + C_2\left\{(\mathrm{T} - t_1)\left(\alpha\left(T + \frac{\emptyset T^2}{2}\right)\log(\beta T) - aT\right) - \alpha\emptyset\frac{(T^3 - T\, t_1{}^2)}{2}\log(\beta T) - \alpha\frac{T^2}{2}\log(\beta T) + \frac{\alpha\, t_1{}^2}{2}\log(\beta\, t_1) + \frac{3\alpha}{4}(T^2 - t_1{}^2) + \frac{\emptyset\,\alpha}{4}(t_1{}^3 + t_1 T^2 - 2T\, t_1{}^2)\right\} + C_3\left\{\frac{3\alpha\, t_1{}^2}{4} - \frac{\alpha\, t_1{}^2}{2}\log(\beta\, t_1)\right\}\right\} \quad \textbf{(10)}$$

## 4. NUMERICAL EXAMPLE

To encapsulate this system, consider that various parameters are $\alpha$ = 20 units, $\beta$ = 0.2, $c_1$ = 1.4 per unit, $c_2$ = 2 per unit, $C_3$ = 2 per unit, $\emptyset$ = 0.01 and $T$ = 14 days . we get output parameters: $t_1$ = 4.675 days, optimal quantity $Q$ = 183units, total inventory cost $\mathrm{TF}(t_1) = 282$

### 5. Effect of Parameter Neutrosophication in the proposed inventory model

$$TF^{\widetilde{NS}}(t_1) = \frac{1}{T}\left\{A + \widetilde{C_1}\left\{\alpha\left(T + \frac{\emptyset T^2}{2}\right)\log(\beta T) - \alpha\emptyset\, T\, t_1\log(\beta T) - \alpha\mathrm{T}\log(\beta T) + \alpha\emptyset\left(T\, t_1 - \frac{T^2}{4} - \frac{3\, t_1{}^2}{4}\right)\right\} + \tilde{C}_2\left\{(\mathrm{T} - t_1)\left(\alpha\left(T + \frac{\emptyset T^2}{2}\right)\log(\beta T) - aT\right) - \alpha\emptyset\frac{(T^3 - T\, t_1{}^2)}{2}\log(\beta T) - \alpha\frac{T^2}{2}\log(\beta T) + \frac{\alpha\, t_1{}^2}{2}\log(\beta\, t_1) + \frac{3\alpha}{4}(T^2 - t_1{}^2) + \frac{\emptyset\,\alpha}{4}(t_1{}^3 + t_1 T^2 - 2T\, t_1{}^2)\right\} + \widetilde{C_3}\left\{\frac{3\alpha\, t_1{}^2}{4} - \frac{\alpha\, t_1{}^2}{2}\log(\beta\, t_1)\right\}\right\} \quad \textbf{(11)}$$

Here, holding cost $\widetilde{C_1}$, deterioition cost $\widetilde{C_2}$ and shortage cost $\widetilde{C_3}$ have been considered as tropezoidal neutrosophic fuzzy set. Thus, the parameters of neutrosophic numbers are:

$Then, \widetilde{C_1} = =<$ (1.5 ,2,2.5,3,3.5), (1,1.5,2,2.5,3), (2,2.5,3,3.5,4), 0.8,0.5,0.5 $>$,
$\widetilde{C_2}$ $=<$ (0.5,1.5,2.5,3.5), ( 0.3,1.3,2.2,3.2), ( 0.7,1.7,2.2,3.8)|0.8; 0.5; 0.5) $>$, *and*
$\widetilde{C_3}$ $=<$ (0.4,1.3,2.8,3.8), ( 0.6,1.5,2.5,3.5), ( 0.8,1.7,2.7,3.7)|0.8; 0.5; 0.5) $>$


**Figure 2**

## 6. SENSITIVE ANALYSIS

In this segment, investigate however the enter parameters change significantly the resultant parameters. The amendment in one parameter and maintain different parameters invariant. The bottom information area unit got consequently to the computative example.

**Table 1. Analysis of different parameter resulted.**

| *Changes of Parameter* | $b$ | $a$ | $\widetilde{C_1}$ | $\widetilde{C_2}$ | $\widetilde{C_3}$ | $\emptyset$ | $T$ | $t_1$ | $TF^{\widehat{NS}}$ | $Q$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.2 | 20 | 3.125 | 1.95 | 2.1 | 0.01 | **10** | 5.016 | 167.56 | 101 |
| | 0.2 | 20 | 3.125 | 1.95 | 2.1 | 0.01 | **11** | 4.875 | 166.11 | 110 |
| | 0.2 | 20 | 3.125 | 1.95 | 2.1 | 0.01 | **12** | 4.463 | 174.47 | 128 |
| | 0.2 | 20 | 3.125 | 1.95 | 2.1 | 0.01 | **13** | 3.858 | 195.58 | 143 |
| $T$ | 0.2 | 20 | 3.125 | 1.95 | 2.1 | 0.01 | **14** | 2.984 | 228.43 | 149 |
| | 0.2 | 20 | 3.125 | 1.95 | 2.1 | **0.01** | 14 | 3.35 | 326.99 | 152 |
| | 0.2 | 20 | 3.125 | 1.95 | 2.1 | 0.0125 | 14 | 4.591 | 339.96 | 163 |
| $\emptyset$ | 0.2 | 20 | 3.125 | 1.95 | 2.1 | 0.015 | 14 | 4.576 | 350.26 | 163 |
| | 0.2 | 20 | 3.125 | 1.95 | 2.1 | 0.0175 | 14 | 4.562 | 354.55 | 163 |
| | 0.2 | 20 | 3.125 | 1.95 | 2.1 | 0.02 | 14 | 4.548 | 367.85 | 163 |
| | 0.2 | 20 | 3.125 | 1.95 | 2.1 | 0.01 | 14 | 4.423 | 260.32 | 137 |
| | 0.2 | 20 | 3.125 | 1.95 | **2.3** | 0.01 | 14 | 4.855 | 383.29 | 154 |
| $\widetilde{C_1}$ | 0.2 | 20 | 3.125 | 1.95 | **2.5** | 0.01 | 14 | 4.934 | 393.59 | 157 |
| | 0.2 | 20 | 3.125 | 1.95 | **2.7** | 0.01 | 14 | 5.157 | 438.88 | 159 |
| | 0.2 | 20 | 3.125 | 1.95 | **2.9** | 0.01 | 14 | 5.248 | 454.18 | 162 |
| | 0.2 | 20 | 3.125 | **3** | 2.1 | 0.01 | 14 | 3.215 | 277.68 | 152 |
| | 0.2 | 20 | 3.125 | **5** | 2.1 | 0.01 | 14 | 4.111 | 296.61 | 161 |
| $\widetilde{C_2}$ | 0.2 | 20 | 3.125 | **6** | 2.1 | 0.01 | 14 | 4.309 | 308.94 | 162 |
| | 0.2 | 20 | 3.125 | **7** | 2.1 | 0.01 | 14 | 4.445 | 322.02 | 163 |
| | 0.2 | 20 | 3.125 | **9** | 2.1 | 0.01 | 14 | 4.62 | 349.36 | 163 |
| | 0.2 | 20 | **0.8** | 2 | 2 | 0.01 | 14 | 3.49 | 206.58 | 155 |
| | 0.2 | 20 | **0.9** | 2 | 2 | 0.01 | 14 | 3.246 | 217.26 | 152 |
| $\widetilde{C_3}$ | 0.2 | 20 | **1.2** | 2 | 2 | 0.01 | 14 | 2.998 | 252.45 | 139 |
| | 0.2 | 20 | **1.4** | 2 | 2 | 0.01 | 14 | 3.457 | 278.96 | 125 |
| | 0.2 | 20 | **1.5** | 2 | 2 | 0.01 | 14 | 3.02 | 293.58 | 113 |
| $a$ | 0.2 | **25** | 3.125 | 1.95 | 2.1 | 0.01 | 14 | 3.457 | 306.82 | 163 |
| | 0.2 | **30** | 3.125 | 1.95 | 2.1 | 0.01 | 14 | 3.455 | 334.69 | 201 |
| | 0.2 | **35** | 3.125 | 1.95 | 2.1 | 0.01 | 14 | 3.454 | 362.56 | 239 |
| | 0.2 | **40** | 3.125 | 1.95 | 2.1 | 0.01 | 14 | 3.453 | 390.43 | 278 |
| | 0.2 | **45** | 3.125 | 1.95 | 2.1 | 0.01 | 14 | 3.450 | 398.24 | 283 |

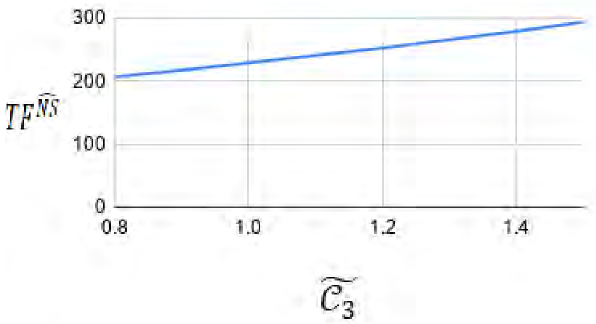


Figure 3

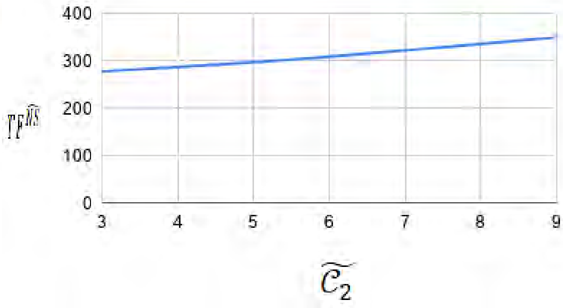


Figure 4

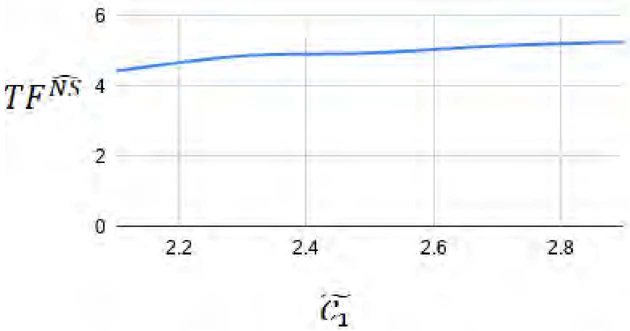


Figure 5

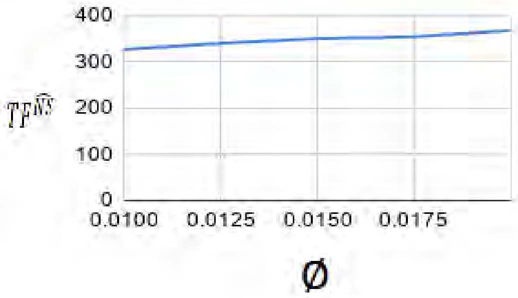


Figure 6

## 7. OBSERVATIONS

1) Table 1 provides the behavior of total price with the variation in cycle time ( T). From this table it is ascertained that because the worth of cycle time T increases, the total price of this model will increase.
2) Table 1 provides the variation in demand parameter ($a$ ),it's ascertained that increment in demand rate ($a$), total price of this model will increase.
3) Observe the behavior of total cost with the variation in deterioration rate (∅), and it's ascertained that with the increment in decay rate (∅), the total price of the supply chain will upwards.
4) The variation in parameter ($\widetilde{C_1}$) and it is ascertained that an increment in ($\widetilde{C_1}$) results an increment in total cost.
5) The variation in holding price ($\widetilde{C_2}$) is ascertained that the increment in($\widetilde{C_2}$), increase the total price of the logistics network.
6) The variation in shortage cost ($\widetilde{C_3}$) is ascertained that the increment in($\widetilde{C_3}$), increase the total price of the logistics network.

## 8. CONCLUSION

In this paper developed EOQ and total annual inventory cost in the crisp sense as well as neutrosophic sense. Shortage cost, holding cost, deterioration price are taken as trapezoidal neutrosophic set. This model discussed results and minimizing the entire inventory price. The expense for the demand is considered for progress of this model. Therefore this model is very successful in all situations. This model and demand pattern is applicable for patterned products, cosmetic products and backed items. The numerical example and sensitivity analysis is bestowed let's say this model and its important options. This model contains a more scope of extension with fractal method.

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# The Algebraic Structure of Normal Groups Associated with Q-Neutrosophic Soft Sets

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**Abstract:** In this paper, we define the notion of Q-neutrosophic normal soft groups and discuss several related structural characteristics and properties. Additionally, we discuss the relation between Q-neutrosophic normal soft groups and normal soft groups. Furthermore, we define the concept of Q-neutrosophic soft cosets and discuss several relevant attributes.

**Keywords:** Neutrosophic soft group, Neutrosophic soft set, Q-neutrosophic soft group, Q-neutrosophic soft set.

## 1 Introduction

Fuzzy sets were established by Zadeh [1] as a tool to deal with uncertain data. The idea of neutrosophic fuzzy sets, an extension of fuzzy sets, was introduced by Smarandache [2, 3] to handle indeterminate and uncertain situations. As another way to deal with uncertain information, Molodtsov [4] introduced the concept of soft sets. Various researchers around the world have extended fuzzy sets and soft sets in different directions in order to make them more appropriate to handle different types of information. However, in some cases the description of objects by fuzzy soft sets in terms of one dimensional membership function only is not adequate. This motivates Adam and Hassan [5–8] to define the Q-fuzzy soft sets and matrix as a way to deal with situations with a set of parameters and two-dimensional data. Q-neutrosophic soft sets (Q-NSSs) [9] were introduced as a new model that deals with two-dimensional uncertain data. It is a model that generalizes neutrosophic and Q-fuzzy sets simultaneously. Q-NSSs were further investigated and their basic operations and relations were discussed in [9, 10].

Different hybrid models of fuzzy sets and soft sets were utilized in different branches of mathematics, including algebra [11–13]. Bera and Mahapatra [14, 15] introduced neutrosophic soft groups and neutrosophic normal soft groups. This motivates Solairaju and Nagarajan [16] to introduce the new structure of Q-fuzzy groups which combine the concepts of Q-fuzzy sets and groups. Recently, Q-fuzzy sets were utilized to different algebraic structures, for example, Q-fuzzy normal subgroups [17], anti-Q-fuzzy normal subgroups [18]. Furthermore, Sarala and Suganya [19] utilized Q-fuzzy soft sets to establish Q-fuzzy soft rings.

In a particular view on the utilization of Q-NSSs to algebraic structures, Abu Qamar and Hassan [20] applied Q-NSS to group theory by introducing Q-neutrosophic soft groups, they examined numerous properties and basic attributes. Additionally, they characterized the thought of Q-level soft sets of a Q-neutrosophic soft set, which is a bridge between Q-neutrosophic soft groups and soft groups. Furthermore, rings and fields were studied under Q-neutrosophic soft settings in [21, 22].

In this paper, we provide a wider discussion on Q-NSGs, by defining the notions of Q-neutrosophic normal soft groups (Q-NNSGs) and Q-neutrosophic soft cosets. Also, we discuss the relation between Q-neutrosophic normal soft groups and normal soft groups. Further, we discuss several related structural characteristics and properties.

## 2 Preliminaries

In this section, we recall some basic definitions related to the work in this study.

**Definition 2.1** ( [9]). Let $X$ be a universal set, $Q$ be a nonempty set and $A \subseteq E$ be a set of parameters. Let $\mu^l QNS(X)$ be the set of all multi Q-NSs on $X$ with dimension $l = 1$. A pair $(\Gamma_Q, A)$ is called a Q-NSS over $X$, where $\Gamma_Q : A \to \mu^l QNS(X)$ is a mapping, such that $\Gamma_Q(e) = \phi$ if $e \notin A$.

**Definition 2.2** ( [10]). The union of two Q-neutrosophic soft sets $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ is the Q-neutrosophic soft set $(\Lambda_Q, C)$ written as $(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, C)$, where $C = A \cup B$ and for all $c \in C, (x, q) \in X \times Q$, the truth-membership, indeterminacy-membership and falsity-membership of $(\Lambda_Q, C)$ are as follows:

$$T_{\Lambda_{Q(c)}}(x,q) = \begin{cases} T_{\Gamma_{Q(c)}}(x,q) & \text{if } c \in A - B, \\ T_{\Psi_{Q(c)}}(x,q) & \text{if } c \in B - A, \\ \max\{T_{\Lambda_{Q(c)}}(x,q), T_{\Psi_{Q(c)}}(x,q)\} & \text{if } c \in A \cap B, \end{cases}$$

$$I_{\Lambda_{Q(c)}}(x,q) = \begin{cases} I_{\Gamma_{Q(c)}}(x,q) & \text{if } c \in A - B, \\ I_{\Psi_{Q(c)}}(x,q) & \text{if } c \in B - A, \\ \min\{I_{\Gamma_{Q(c)}}(x,q), I_{\Psi_{Q(c)}}(x,q)\} & \text{if } c \in A \cap B, \end{cases}$$

$$F_{\Lambda_{Q(c)}}(x,q) = \begin{cases} F_{\Gamma_{Q(c)}}(x,q) & \text{if } c \in A - B, \\ F_{\Psi_{Q(c)}}(x,q) & \text{if } c \in B - A, \\ \min\{F_{\Gamma_{Q(c)}}(x,q), F_{\Psi_{Q(c)}}(x,q)\} & \text{if } c \in A \cap B. \end{cases}$$

**Definition 2.3** ( [10]). The intersection of two Q-neutrosophic soft sets $(\Gamma_Q, A)$ and $(\Psi_Q, B)$ is the Q-neutrosophic soft set $(\Lambda_Q, C)$ written as $(\Gamma_Q, A) \cap (\Psi_Q, B) = (\Lambda_Q, C)$, where $C = A \cap B$ and for all $c \in C$ and $(x, q) \in X \times Q$ the truth-membership, indeterminacy-membership and falsity-membership of $(\Lambda_Q, C)$ are as follows:

$$\begin{aligned} T_{\Lambda_{Q(c)}}(x,q) &= \min\{T_{\Gamma_{Q(c)}}(x,q), T_{\Psi_{Q(c)}}(x,q)\}, \\ I_{\Lambda_{Q(c)}}(x,q) &= \max\{I_{\Gamma_{Q(c)}}(x,q), I_{\Psi_{Q(c)}}(x,q)\}, \\ F_{\Lambda_{Q(c)}}(x,q) &= \max\{F_{\Gamma_{Q(c)}}(x,q), F_{\Psi_{Q(c)}}(x,q)\}. \end{aligned}$$

**Definition 2.4.** [22] Let $G$ be a group and $(\Gamma_Q, A)$ be a Q-NSS over a group $G$. Then $(\Gamma_Q, A)$ is called a Q-neutrosophic soft group over $G$ if for all $x, y \in G$ and $e \in A$ it satisfies:

1. $T_{\Gamma_{Q(e)}}(xy, q) \geq \min\left\{T_{\Gamma_{Q(e)}}(x, q), T_{\Gamma_{Q(e)}}(y, q)\right\}, I_{\Gamma_{Q(e)}}(xy, q) \leq \max\left\{I_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(y, q)\right\}$ and $F_{\Gamma_{Q(e)}}(xy, q) \leq \max\left\{F_{\Gamma_{Q(e)}}(x, q), F_{\Gamma_{Q(e)}}(y, q)\right\}$.

2. $T_{\Gamma_{Q(e)}}(x^{-1}, q) \geq T_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(x^{-1}, q) \leq I_{\Gamma_{Q(e)}}(x, q)$ and $F_{\Gamma_{Q(e)}}(x^{-1}, q) \leq F_{\Gamma_{Q(e)}}(x, q)$.

# 3 Q-Neutrosophic Normal Soft Groups

In this section, we introduce the Q-NNSG and discuss several relevant structural properties.

**Definition 3.1.** A Q-NSG $(\Gamma_Q, A)$ over the group $G$ is called a Q-NNSG over $G$ if $\Gamma_Q(e)$ is a Q-neutrosophic normal subgroup of $G$ for each $e \in A$ i.e., for $x \in \Gamma_Q(e), y \in G, q \in Q$

$$\begin{aligned} T_{\Gamma_{Q(e)}}(yxy^{-1}, q) &\geq T_{\Gamma_{Q(e)}}(x, q), \\ I_{\Gamma_{Q(e)}}(yxy^{-1}, q) &\leq I_{\Gamma_{Q(e)}}(x, q), \\ F_{\Gamma_{Q(e)}}(yxy^{-1}, q) &\leq F_{\Gamma_{Q(e)}}(x, q). \end{aligned}$$

**Definition 3.2.** A Q-NSG $(\Gamma_Q, A)$ over the group $G$ is called abelian Q-NSG if $\forall x, y \in G, q \in Q, e \in A$ the following hold

$$\begin{aligned} T_{\Gamma_{Q(e)}}(xy, q) &= T_{\Gamma_{Q(e)}}(yx, q), \\ I_{\Gamma_{Q(e)}}(xy, q) &= I_{\Gamma_{Q(e)}}(yx, q), \\ F_{\Gamma_{Q(e)}}(xy, q) &= F_{\Gamma_{Q(e)}}(yx, q). \end{aligned}$$

**Example 3.3.** Let $G = (\mathbb{Z}, +)$ be a group and A=$\mathbb{N}$ be the parametric set. Define a Q-NSG $(\Gamma_Q, A)$ as follows: For $q \in Q, x \in \mathbb{Z}, m \in \mathbb{N}$

$$T_{\Gamma_{Q(m)}}(x, q) = \begin{cases} 0 & \text{if x is odd} \\ \frac{2}{3} & \text{if x is even,} \end{cases}$$

$$I_{\Gamma_{Q(m)}}(x, q) = \begin{cases} \frac{1}{n} & \text{if x is odd} \\ 0 & \text{if x is even,} \end{cases}$$

$$F_{\Gamma_{Q(m)}}(x, q) = \begin{cases} 1 - \frac{3}{n} & \text{if x is odd} \\ 0 & \text{if x is even.} \end{cases}$$

It is clear that $(\Gamma_Q, \mathbb{N})$ is a Q-NNSG over $G$.

**Proposition 3.4.** *Let $(\Gamma_Q, A)$ be a Q-NNSG over a group $G$. Then, $\forall x, y \in G, q \in Q$ and $e \in A$,*

1. $T_{\Gamma_{Q(e)}}(yxy^{-1}, q) = T_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(yxy^{-1}, q) = I_{\Gamma_{Q(e)}}(x, q), F_{\Gamma_{Q(e)}}(yxy^{-1}, q) = F_{\Gamma_{Q(e)}}(x, q)$.

2. $(\Gamma_Q, A)$ *is an abelian Q-NSG over $G$.*

*Proof.* 1.

$$\begin{aligned}
T_{\Gamma_{Q(e)}}(x,q) &= T_{\Gamma_{Q(e)}}\Big((y^{-1}y)x(y^{-1}y),q\Big)\\
&= T_{\Gamma_{Q(e)}}\Big(y^{-1}(yxy^{-1})y,q\Big)\\
&= T_{\Gamma_{Q(e)}}\Big(y^{-1}(yxy^{-1})(y^{-1})^{-1},q\Big)\\
&\geq T_{\Gamma_{Q(e)}}\big(yxy^{-1},q\big).
\end{aligned}$$

Now, from Definition 3.1 $T_{\Gamma_{Q(e)}}(yxy^{-1},q) = T_{\Gamma_{Q(e)}}(x,q)$.
In a similar manner we can show that $I_{\Gamma_{Q(e)}}(yxy^{-1},q) = I_{\Gamma_{Q(e)}}(x,q)$ and $F_{\Gamma_{Q(e)}}(yxy^{-1},q) = F_{\Gamma_{Q(e)}}(x,q)$.

2. $T_{\Gamma_{Q(e)}}(x,q) = T_{\Gamma_{Q(e)}}(yxy^{-1},q)$, this implies $T_{\Gamma_{Q(e)}}(xy,q) = T_{\Gamma_{Q(e)}}(yxyy^{-1},q) = T_{\Gamma_{Q(e)}}(yx,q)$. Similarly, we can show that $I_{\Gamma_{Q(e)}}(xy,q) = I_{\Gamma_{Q(e)}}(yx,q)$ and $F_{\Gamma_{Q(e)}}(xy,q) = F_{\Gamma_{Q(e)}}(yx,q)$. Hence, $(\Gamma_Q,A)$ is an abelian Q-NSG over $G$. □

**Theorem 3.5.** *Let $(\Gamma_Q,A)$ and $(\Psi_Q,B)$ be two Q-NNSG over a group $G$. Then, $(\Gamma_Q,A)\cap(\Psi_Q,B)$ is also a Q-NNSG over $G$.*

*Proof.* Let $(\Lambda_Q,C) = (\Gamma_Q,A)\cap(\Psi_Q,B)$. Then, for $x,y\in G, q\in Q, e\in C$

$$\begin{aligned}
T_{\Lambda_{Q(e)}}(yxy^{-1},q) &= \min\Big\{T_{\Gamma_{Q(e)}}(yxy^{-1},q), T_{\Psi_{Q(e)}}(yxy^{-1},q)\Big\}\\
&\geq \min\Big\{T_{\Gamma_{Q(e)}}(x,q), T_{\Psi_{Q(e)}}(x,q)\Big\}\\
&= T_{\Lambda_{Q(e)}}(x,q),
\end{aligned}$$

$$\begin{aligned}
I_{\Lambda_{Q(e)}}(yxy^{-1},q) &= \max\Big\{I_{\Gamma_{Q(e)}}(yxy^{-1},q), I_{\Psi_{Q(e)}}(yxy^{-1},q)\Big\}\\
&\leq \max\Big\{I_{\Gamma_{Q(e)}}(x,q), I_{\Psi_{Q(e)}}(x,q)\Big\}\\
&= I_{\Lambda_{Q(e)}}(x,q).
\end{aligned}$$

Similarly, we can show that $F_{\Lambda_{Q(e)}}(yxy^{-1},q) \leq F_{\Lambda_{Q(e)}}(x,q)$. This completes the proof. □

**Remark 3.6.** The union of two Q-NNSGs is not a Q-NNSG since the union is not a Q-NSG.

The next example illustrates the above remark.

**Example 3.7.** let $G = (\mathbb{Z},+)$ and $E = 2\mathbb{Z}$. Define the two Q-neutrosophic soft groups $(\Gamma_Q,E)$ and $(\Psi_Q,E)$ over $G$ as the following:
For $x,m\in\mathbb{Z}, q\in Q$

$$T_{\Gamma_{Q(2m)}}(x,q)=\begin{cases}0.50 & \text{if } x=4tm, \exists t\in\mathbb{Z},\\ 0 & \text{otherwise,}\end{cases}$$

$$I_{\Gamma_{Q(2m)}}(x,q)=\begin{cases}0 & \text{if } x=4tm, \exists t\in\mathbb{Z},\\ 0.25 & \text{otherwise,}\end{cases}$$

$$F_{\Gamma_{Q(2m)}}(x,q)=\begin{cases}0 & \text{if } x=4tm, \exists t\in\mathbb{Z},\\ 0.10 & \text{otherwise,}\end{cases}$$

and

$$T_{\Psi_{Q(2m)}}(x,q)=\begin{cases}0.67 & \text{if } x=6tm, \exists t\in\mathbb{Z},\\ 0 & \text{otherwise,}\end{cases}$$

$$I_{\Psi_{Q(2m)}}(x,q)=\begin{cases}0 & \text{if } x=6tm, \exists t\in\mathbb{Z},\\ 0.20 & \text{otherwise,}\end{cases}$$

$$F_{\Psi_{Q(2m)}}(x,q)=\begin{cases}0 & \text{if } x=6tm, \exists t\in\mathbb{Z},\\ 0.17 & \text{otherwise.}\end{cases}$$

Let $(\Gamma_Q,A)\cup(\Psi_Q,B)=(\Lambda_Q,E)$. For $m=3, x=12, y=18$ we have

$$T_{\Lambda_{Q(6)}}(12.18^{-1},q)=T_{\Lambda_{Q(6)}}(-6,q)=\max\left\{T_{\Gamma_{Q(6)}}(-6,q),T_{\Psi_{Q(6)}}(-6,q)\right\}=\max\{0,0\}=0$$

and

$$\begin{aligned}\min&\left\{T_{\Lambda_{Q(6)}}(12,q),T_{\Lambda_{Q(6)}}(18,q)\right\}\\ &=\min\left\{\max\left\{T_{\Gamma_{Q(6)}}(12,q),T_{\Psi_{Q(6)}}(12,q)\right\},\max\left\{T_{\Gamma_{Q(6)}}(18,q),T_{\Psi_{Q(6)}}(18,q)\right\}\right\}\\ &=\min\left\{\max\left\{0.50,0\right\},\max\left\{0,0.67\right\}\right\}\\ &=\min\left\{0.50,0.67\right\}=0.50.\end{aligned}$$

Hence, $T_{\Lambda_{Q(6)}}(12.18^{-1},q)=0<\min\left\{T_{\Lambda_{Q(6)}}(12,q),T_{\Lambda_{Q(6)}}(18,q)\right\}=0.50$; i.e. $(\Lambda_Q,E)=(\Gamma_Q,A)\cup(\Psi_Q,B)$ is not a Q-neutrosophic soft group.

**Theorem 3.8.** *Let $(\Gamma_Q,A)$ be a Q-NSS over G. Then, $(\Gamma_Q,A)$ is a Q-NNSG over G if and only if for all $\alpha,\beta,\gamma\in[0,1]$, the Q-level soft set $(\Gamma_Q,A)_{(\alpha,\beta,\gamma)}\neq\phi$ is a normal soft group over G.*

*Proof.* We only need to prove the normality. For $x\in(\Gamma_Q,A)_{(\alpha,\beta,\gamma)}, y\in G$ and $q\in Q$, we have

$$\begin{aligned}T_{\Gamma_{Q(e)}}(yxy^{-1},q)&=T_{\Gamma_{Q(e)}}(yy^{-1}x,q)=T_{\Gamma_{Q(e)}}(x,q)\geq\alpha,\\ I_{\Gamma_{Q(e)}}(yxy^{-1},q)&=I_{\Gamma_{Q(e)}}(yy^{-1}x,q)=I_{\Gamma_{Q(e)}}(x,q)\leq\beta,\\ F_{\Gamma_{Q(e)}}(yxy^{-1},q)&=F_{\Gamma_{Q(e)}}(yy^{-1}x,q)=F_{\Gamma_{Q(e)}}(x,q)\leq\gamma.\end{aligned}$$

It follows that $yxy^{-1} \in (\Gamma_Q, A)_{(\alpha,\beta,\gamma)}$, i.e. $(\Gamma_Q, A)_{(\alpha,\beta,\gamma)}$ is a Q-NNSG of $G$.

Conversely, assume that $(\Gamma_Q, A)$ is not a Q-NNSG over $G$. Then, there exists $e \in A$ such that $\Gamma_Q(e)$ is not a Q-NN subgroup of $G$. Then, there exists $x_1, y_1 \in G$ and $q \in Q$ such that

$$T_{\Gamma_{Q(e)}}(x_1y_1, q) < T_{\Gamma_{Q(e)}}(y_1x_1, q) \text{ or } T_{\Gamma_{Q(e)}}(x_1y_1, q) > T_{\Gamma_{Q(e)}}(y_1x_1, q) \text{ or}$$
$$I_{\Gamma_{Q(e)}}(x_1y_1, q) < I_{\Gamma_{Q(e)}}(y_1x_1, q) \text{ or } I_{\Gamma_{Q(e)}}(x_1y_1, q) > I_{\Gamma_{Q(e)}}(y_1x_1, q) \text{ or}$$
$$F_{\Gamma_{Q(e)}}(x_1y_1, q) < F_{\Gamma_{Q(e)}}(y_1x_1, q) \text{ or } F_{\Gamma_{Q(e)}}(x_1y_1, q) > F_{\Gamma_{Q(e)}}(y_1x_1, q).$$

In case $T_{\Gamma_{Q(e)}}(x_1y_1, q) < T_{\Gamma_{Q(e)}}(y_1x_1, q)$, there exists $\alpha \in [0, 1]$ such that $T_{\Gamma_{Q(e)}}(x_1y_1, q) < \alpha < T_{\Gamma_{Q(e)}}(y_1x_1, q)$. It follows that $x_1y_1 \notin \Gamma_{Q(e)_{(\alpha,\beta,\gamma)}}$, but for $I_{\Gamma_{Q(e)}}(x_1y_1, q) < \beta$ and $F_{\Gamma_{Q(e)}}(x_1y_1, q) < \gamma$, $x_1y_1 \notin \Gamma_{Q(e)_{(\alpha,\beta,\gamma)}}$ this contradicts with the fact that $(\Gamma_Q, A)_{(\alpha,\beta,\gamma)}$ is a normal soft group over $G$. In the other cases the proof can be obtained in a similar way. □

**Theorem 3.9.** *Let $(\Gamma_Q, A)$ be a Q-NNSG over $G$. Let*

$$(\Gamma_Q, A)|_{\acute{e}} = \Big\{x \in G : T_{\Gamma_{Q(e)}}(x, q) = T_{\Gamma_{Q(e)}}(\acute{e}, q), I_{\Gamma_{Q(e)}}(x, q) = I_{\Gamma_{Q(e)}}(\acute{e}, q), F_{\Gamma_{Q(e)}}(x, q) = F_{\Gamma_{Q(e)}}(\acute{e}, q), e \in A\Big\},$$

*where $\acute{e}$ is the unit element of $G$. Then, $(\Gamma_Q, A)|_{\acute{e}}$ is a normal soft group over $G$.*

*Proof.* For each $e \in A$ and $x, y \in (\Gamma_Q, A)|_{\acute{e}}, q \in Q$, we have

$$\begin{aligned} T_{\Gamma_{Q(e)}}(xy^{-1}, q) &\geq \min\Big\{T_{\Gamma_{Q(e)}}(x, q), T_{\Gamma_{Q(e)}}(y, q)\Big\} \\ &= \min\Big\{T_{\Gamma_{Q(e)}}(\acute{e}, q), T_{\Gamma_{Q(e)}}(\acute{e}, q)\Big\} \\ &= T_{\Gamma_{Q(e)}}(\acute{e}, q), \end{aligned}$$

$$\begin{aligned} I_{\Gamma_{Q(e)}}(xy^{-1}, q) &\leq \max\Big\{I_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(y, q)\Big\} \\ &= \max\Big\{I_{\Gamma_{Q(e)}}(\acute{e}, q), I_{\Gamma_{Q(e)}}(\acute{e}, q)\Big\} \\ &= I_{\Gamma_{Q(e)}}(\acute{e}, q). \end{aligned}$$

Similarly, we can show $F_{\Gamma_{Q(e)}}(xy^{-1}, q) \leq F_{\Gamma_{Q(e)}}(\acute{e}, q)$. Always, $T_{\Gamma_{Q(e)}}(\acute{e}, q) \geq T_{\Gamma_{Q(e)}}(xy^{-1}, q)$, $I_{\Gamma_{Q(e)}}(\acute{e}, q) \leq I_{\Gamma_{Q(e)}}(xy^{-1}, q)$ and $F_{\Gamma_{Q(e)}}(\acute{e}, q) \leq F_{\Gamma_{Q(e)}}(xy^{-1}, q)$. Therefore, $T_{\Gamma_{Q(e)}}(xy^{-1}, q) = T_{\Gamma_{Q(e)}}(\acute{e}, q)$, $I_{\Gamma_{Q(e)}}(xy^{-1}, q) = I_{\Gamma_{Q(e)}}(\acute{e}, q)$, $F_{\Gamma_{Q(e)}}(xy^{-1}, q) = F_{\Gamma_{Q(e)}}(\acute{e}, q)$ and $xy^{-1} \in (\Gamma_Q, A)|_{\acute{e}}$.

Next, let $x \in (\Gamma_Q, A)|_{\acute{e}}$ and $y \in G$. Then,

$$\begin{aligned} T_{\Gamma_{Q(e)}}(yxy^{-1}, q) &= T_{\Gamma_{Q(e)}}(x, q) = T_{\Gamma_{Q(e)}}(\acute{e}, q), \\ I_{\Gamma_{Q(e)}}(yxy^{-1}, q) &= I_{\Gamma_{Q(e)}}(x, q) = I_{\Gamma_{Q(e)}}(\acute{e}, q), \\ F_{\Gamma_{Q(e)}}(yxy^{-1}, q) &= F_{\Gamma_{Q(e)}}(x, q) = F_{\Gamma_{Q(e)}}(\acute{e}, q). \end{aligned}$$

Therefore, $yxy^{-1} \in (\Gamma_Q, A)|_{\acute{e}}$. Hence, $(\Gamma_Q, A)|_{\acute{e}}$ is a normal soft group over $G$. □

# 4 Q-Neutrosophic Soft Cosets

In this section, we present the Q-neutrosophic soft cosets with some related properties.

**Definition 4.1.** Let $(\Gamma_Q, A)$ be a Q-NSG over $G$ and $g \in G$ be a fixed element. Then, the set $g(\Gamma_Q, A) = \Big\{g\Gamma_Q(e) : e \in A\Big\}$ is called a left Q-neutrosophic soft coset of $(\Gamma_Q, A)$, where

$$\begin{aligned} g\Gamma_Q(e) &= \Big\{\Big\langle (x,q), T_{g\Gamma_Q(e)}(x,q), I_{g\Gamma_Q(e)}(x,q), F_{g\Gamma_Q(e)}(x,q)\Big\rangle : x \in G, q \in Q\Big\} \\ &= \Big\{\Big\langle (x,q), T_{\Gamma_Q(e)}(g^{-1}x,q), I_{\Gamma_Q(e)}(g^{-1}x,q), F_{\Gamma_Q(e)}(g^{-1}x,q)\Big\rangle : x \in G, q \in Q\Big\}. \end{aligned}$$

The right Q-neutrosophic soft coset of $(\Gamma_Q, A)$ in $G$ is $(\Gamma_Q, A)g = \Big\{\Gamma_Q(e)g : e \in A\Big\}$, where

$$\Gamma_Q(e)g = \Big\{\Big\langle (x,q), T_{\Gamma_Q(e)}(xg^{-1},q), I_{\Gamma_Q(e)}(xg^{-1},q), F_{\Gamma_Q(e)}(xg^{-1},q)\Big\rangle : x \in G, q \in Q\Big\}.$$

**Example 4.2.** Let $G$ be a classical group. Then, $(\Gamma_Q, A) = \Big\{\Gamma_Q(e) : e \in A\Big\}$, where

$$\Gamma_Q(e) = \Big\{\Big\langle (x,q), T_{\Gamma_Q(e)}(x,q), I_{\Gamma_Q(e)}(x,q), F_{\Gamma_Q(e)}(x,q)\Big\rangle : x \in G, q \in Q\Big\}$$

with $T_{\Gamma_Q(e)}(x,q) = T_{\Gamma_Q(e)}(\acute{e},q)$, $I_{\Gamma_Q(e)}(x,q) = I_{\Gamma_Q(e)}(\acute{e},q)$ and $F_{\Gamma_Q(e)}(x,q) = F_{\Gamma_Q(e)}(\acute{e},q)$; ($\acute{e}$ being the identity element in $G$ ) is a Q-NNSG of $G$. In that case, we can get a neutrosophic soft coset.

**Proposition 4.3.** $(\Gamma_Q, A)$ *is called a Q-NNSG over G if and only if the left and right Q-neutrosophic soft cosets are equal.*

*Proof.* Suppose that $(\Gamma_Q, A)$ is a Q-NNSG over $G$. Then,

$$\begin{aligned} g\Gamma_Q(e) &= \Big\{\Big\langle (x,q), T_{g\Gamma_Q(e)}(x,q), I_{g\Gamma_Q(e)}(x,q), F_{g\Gamma_Q(e)}(x,q)\Big\rangle : x \in G, q \in Q\Big\} \\ &= \Big\{\Big\langle (x,q), T_{\Gamma_Q(e)}(g^{-1}x,q), I_{\Gamma_Q(e)}(g^{-1}x,q), F_{\Gamma_Q(e)}(g^{-1}x,q)\Big\rangle : x \in G, q \in Q\Big\} \\ &= \Big\{\Big\langle (x,q), T_{\Gamma_Q(e)}(xg^{-1},q), I_{\Gamma_Q(e)}(xg^{-1},q), F_{\Gamma_Q(e)}(xg^{-1},q)\Big\rangle : x \in G, q \in Q\Big\} \\ &= \Big\{\Big\langle (x,q), T_{\Gamma_Q(e)g}(x,q), I_{\Gamma_Q(e)g}(x,q), F_{\Gamma_Q(e)g}(x,q)\Big\rangle : x \in G, q \in Q\Big\} \\ &= \Gamma_Q(e)g. \end{aligned}$$

Thus, $g(\Gamma_Q, A) = \{g\Gamma_Q(e) : e \in A\} = \{\Gamma_Q(e)g : e \in A\} = (\Gamma_Q, A)g$.

Next, suppose that $g(\Gamma_Q, A) = (\Gamma_Q, A)g$.
Then,

$$T_{g\Gamma_Q(e)}(x,q) = T_{\Gamma_Q(e)g}(x,q), I_{g\Gamma_Q(e)}(x,q) = I_{\Gamma_Q(e)g}(x,q) \text{ and } F_{g\Gamma_Q(e)}(x,q) = F_{\Gamma_Q(e)g}(x,q).$$

This implies,

$$T_{\Gamma_Q(e)}(g^{-1}x,q) = T_{\Gamma_Q(e)}(xg^{-1},q), I_{\Gamma_Q(e)}(g^{-1}x,q) = I_{\Gamma_Q(e)}(xg^{-1},q) \text{ and } F_{\Gamma_Q(e)}(g^{-1}x,q) = F_{\Gamma_Q(e)}(xg^{-1},q).$$

Thus,

$$T_{\Gamma_Q(e)}(xg^{-1},q) = T_{\Gamma_Q(e)}(g^{-1}x,q), I_{\Gamma_Q(e)}(xg^{-1},q) = I_{\Gamma_Q(e)}(g^{-1}x,q) \text{ and } F_{\Gamma_Q(e)}(xg^{-1},q) = F_{\Gamma_Q(e)}(g^{-1}x,q),$$

which implies

$$T_{\Gamma_Q(e)}(gxg^{-1},q) = T_{\Gamma_Q(e)}(x,q), I_{\Gamma_Q(e)}(gxg^{-1},q) = I_{\Gamma_Q(e)}(x,q), F_{\Gamma_Q(e)}(gxg^{-1},q) = F_{\Gamma_Q(e)}(x,q).$$

Thus, $(\Gamma_Q, A)$ is a Q-NNSG over $G$.
Therefore, if $(\Gamma_Q, A)$ is a Q-NNSG over $G$ then the left and right Q-neutrosophic soft cosets coincide. In this case, we call it Q-neutrosophic soft cosets instead of left or right Q-neutrosophic soft cosets separately. □

**Theorem 4.4.** *Let $(\Gamma_Q, A)$ be a Q-NNSG over the group $G$ and the set $\varsigma$ be the collection of all distinct Q-neutrosophic soft cosets of $(\Gamma_Q, A)$ in $G$. Then, $\varsigma$ is a group in classical sense under the operation of composition: $g_1(\Gamma_Q, A)y(\Gamma_Q, A) = (g_1y)(\Gamma_Q, A), \forall g_1, g_2 \in G$.*

*Proof.* First we show that the operation is well defined in the sense that if $g_1(\Gamma_Q, A) = \acute{g_1}(\Gamma_Q, A)$ and $g_2(\Gamma_Q, A) = \acute{g_2}(\Gamma_Q, A)$, then $g_1(\Gamma_Q, A)g_2(\Gamma_Q, A) = (\acute{g_1}\acute{g_2})(\Gamma_Q, A)$ for $g_1, g_2, \acute{g_1}, \acute{g_2} \in G$.
Now, $g_1(\Gamma_Q, A) = \acute{g_1}(\Gamma_Q, A)$ implies $g_1^{-1}\acute{g_1} = \Gamma_Q(e_1)$, $e_1 \in A$ and $g_2(\Gamma_Q, A) = \acute{g_2}(\Gamma_Q, A)$ implies $g_2^{-1}\acute{g_2} = \Gamma_Q(e_2)$, $e_2 \in A$.
We show, $(g_1g_2)(\Gamma_Q, A) = (\acute{g_1}\acute{g_2})(\Gamma_Q, A)$ i.e., $(g_1g_2)^{-1}(\acute{g_1}\acute{g_2}) \in G$. Now,

$$\begin{aligned}
(g_1g_2)^{-1}(\acute{g_1}\acute{g_2}) &= g_2^{-1}g_1^{-1}\acute{g_1}\acute{g_2} \\
&= g_2^{-1}\Gamma_Q(e_1)\acute{g_2} \\
&= g_2^{-1}g_2\Gamma_Q(e_1) \\
&= \Gamma_Q(e_2)\Gamma_Q(e_1) \\
&= \Gamma_Q(e_3) \in (\Gamma_Q, A), e_3 \in A.
\end{aligned}$$

Hence, the operation is well defined. Now,
1. the closure axiom is clearly satisfied.
2. $g_1(\Gamma_Q, A)[g_2(\Gamma_Q, A)g_3(\Gamma_Q, A)] = g_1(\Gamma_Q, A)(g_2g_3)(\Gamma_Q, A) = g_1(g_2g_3)(\Gamma_Q, A)$ and
$[g_1(\Gamma_Q, A)g_2(\Gamma_Q, A)]g_3(\Gamma_Q, A) = (g_1g_2)g_3(\Gamma_Q, A) = (g_1g_2)g_3(\Gamma_Q, A)$ for $g_1, g_2, g_3 \in G$. Now, $g_1(g_2g_3) = (g_1g_2)g_3$, since $G$ is a group.
3. $\acute{e}(\Gamma_Q, A)g_1(\Gamma_Q, A) = (\acute{e}g_1)(\Gamma_Q, A) = g_1(\Gamma_Q, A)$ and $g_1(\Gamma_Q, A)\acute{e}(\Gamma_Q, A) = (g_1\acute{e})(\Gamma_Q, A) = g_1(\Gamma_Q, A)$ for $\acute{e}$ being the unity in $G$.
4. $g_1^{-1}(\Gamma_Q, A)g_1(\Gamma_Q, A) = (g_1^{-1}g_1)(\Gamma_Q, A) = \acute{e}(\Gamma_Q, A) = (\Gamma_Q, A)$ and
$g_1(\Gamma_Q, A)g_1^{-1}(\Gamma_Q, A) = (g_1g_1^{-1})(\Gamma_Q, A) = \acute{e}(\Gamma_Q, A) = (\Gamma_Q, A)$.
Thus, $\varsigma$ is a group. This group is a called the quotient group of $G$ by $(\Gamma_Q, A)$ and is denoted by $G/(\Gamma_Q, A)$. □

**Theorem 4.5.** *Let $(\Gamma_Q, A)$ be a Q-NNSG over $G$. Then, there exists a natural homomorphism $\varphi : G \to G/(\Gamma_Q, A)$ defined by $\varphi(g) = g(\Gamma_Q, A), \forall g \in G$ in the classical sense.*

*Proof.* Let $\varphi : G \to G/(\Gamma_Q, A)$ be given by $\varphi(g) = g\Gamma_Q(e), \forall e \in A$. We show that $\varphi$ is a homomorphism i.e.

$\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2), \forall g_1, g_2 \in G$, i.e., $(g_1g_2)\Gamma_Q(e) = g_1\Gamma_Q(e)g_2\Gamma_Q(e)$. Now, for $x \in G, q \in Q$

$$\begin{aligned}(g_1\Gamma_Q(e))(x,q) &= \left\langle T_{g_1\Gamma_Q(e)}(x,q), I_{g_1\Gamma_Q(e)}(x,q), F_{g_1\Gamma_Q(e)}(x,q)\right\rangle \\ &= \left\langle T_{\Gamma_Q(e)}(g_1^{-1}x,q), I_{\Gamma_Q(e)}(g_1^{-1}x,q), F_{\Gamma_Q(e)}(g_1^{-1}x,q)\right\rangle,\end{aligned}$$

$$\begin{aligned}(g_2\Gamma_Q(e))(x,q) &= \left\langle T_{g_2\Gamma_Q(e)}(x,q), I_{g_2\Gamma_Q(e)}(x,q), F_{g_2\Gamma_Q(e)}(x,q)\right\rangle \\ &= \left\langle T_{\Gamma_Q(e)}(g_2^{-1}x,q), I_{\Gamma_Q(e)}(g_2^{-1}x,q), F_{\Gamma_Q(e)}(g_2^{-1}x,q)\right\rangle,\end{aligned}$$

$$(g_1g_2\Gamma_Q(e))(x,q) = \left\langle T_{\Gamma_Q(e)}((g_1g_2)^{-1}x,q), I_{\Gamma_Q(e)}((g_1g_2)^{-1}x,q), F_{\Gamma_Q(e)}((g_1g_2)^{-1}x,q)\right\rangle.$$

Then,

$$\begin{aligned}[(g_1\Gamma_Q(e))(g_2\Gamma_Q(e))](x,q) &= \Big\langle \min\left\{T_{g_1\Gamma_Q(e)}(x,q), T_{g_2\Gamma_Q(e)}(x,q)\right\}, \\ &\quad \max\left\{I_{g_1\Gamma_Q(e)}(x,q), I_{g_2\Gamma_Q(e)}(x,q)\right\}, \\ &\quad \max\left\{F_{g_1\Gamma_Q(e)}(x,q), F_{g_2\Gamma_Q(e)}(x,q)\right\}\Big\rangle \\ &= \Big\langle \min\left\{T_{\Gamma_Q(e)}((g_1^{-1}x,q), T_{\Gamma_Q(e)}((g_2^{-1}x,q)\right\}, \\ &\quad \max\left\{I_{\Gamma_Q(e)}((g_1^{-1}x,q), I_{\Gamma_Q(e)}((g_2^{-1}x,q)\right\}, \\ &\quad \max\left\{F_{\Gamma_Q(e)}((g_1^{-1}x,q), F_{\Gamma_Q(e)}((g_2^{-1}x,q)\right\}\Big\rangle\end{aligned}$$

Further,

$$\begin{aligned}T_{\Gamma_Q(e)}((g_1g_2)^{-1}x,q) &= T_{\Gamma_Q(e)}(g_2^{-1}g_1^{-1}x,q) \\ &= T_{\Gamma_Q(e)}(g_2^{-1}g_1^{-1}xg_2^{-1}g_2,q) \\ &= T_{\Gamma_Q(e)}(g_1^{-1}xg_2,q) \\ &\geq \min\left\{T_{\Gamma_Q(e)}(g_1x,q), T_{\Gamma_Q(e)}(g_2x,q)\right\}.\end{aligned}$$

Hence, $T_{\Gamma_Q(e)}((g_1g_2)^{-1}x,q) = \min\left\{T_{\Gamma_Q(e)}((g_1^{-1}x,q), T_{\Gamma_Q(e)}((g_2^{-1}x,q)\right\}$, similarly, $I_{\Gamma_Q(e)}((g_1g_2)^{-1}x,q) = \max\left\{I_{\Gamma_Q(e)}((g_1^{-1}x,q), I_{\Gamma_Q(e)}((g_2^{-1}x,q)\right\}$ and $F_{\Gamma_Q(e)}((g_1g_2)^{-1}x,q) = \max\left\{F_{\Gamma_Q(e)}((g_1^{-1}x,q), F_{\Gamma_Q(e)}((g_2^{-1}x,q)\right\}$. This shows that, $[(g_1\Gamma_Q(e))(g_2\Gamma_Q(e))](x,q) = [(g_1g_2)\Gamma_Q(e)](x,q)$ which implies, $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$. □

# 5 Conclusion

We have introduced the notions of Q-neutrosophic normal soft groups and Q-neutrosophic soft cosets. We have discussed several related structural characteristics and properties. For future research, we can extend these topics to hyperalgebra. Also, these topics may be discussed using t-norm and s-norm. We intend to further explore the applications of the algebraic structure to different extensions of fuzzy sets in order to provide a significant addition to existing theories for handling uncertainties, especially in the area of soft sets [23–25].

**Funding:** This research received no external funding.
**Conflicts of Interest:** The authors declare no conflict of interest.

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Received: Nov 30, 2021. Accepted: Feb 2, 2022

# Generalized Open Sets in Neutrosophic Soft Bitopological Spaces

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**Abstract**. In this study, some generalized neutrosophic soft open sets are defined in neutrosophic soft bitopological spaces. Also, some theorems related to the subject have been given with their proofs and supported with examples for a better understanding of the subject.

**Keywords:** Neutrosophic set; Neutrosophic soft set; Neutrosophic soft bitopological space; Generalized neutrosophic soft closed (open) set.

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## 1. **Introduction**

Neutrosophy was defined by Smarandache in 2013 for the first time [15]. After that, this topic became very popular in the scientific world, and many studies have been done in this area to date. Salama and Alblowi developed topological structure on neutrosophic sets in 2012 [14]. The concept of neutrosophic bitopological space was defined in 2019 by Ozturk and Alkan [13]. Then in 2020, neutrosophic interior, closure and boundary were defined in neutrosophic bitopological spaces by Mwchahary and Bhimraj [12]. Some generalized open sets were defined in neutrosophic bitopological spaces [5,6]. In 2013, neutrosophic soft set was defined by Maji [10]. The concept of neutrosophic soft topological space was defined in 2017 by Bera and Nirmal [2]. Neutrosophic soft bitopological space was defined in [4].
In this study some generalized open sets are defined in neutrosophic soft bitopological spaces.

## 2. **Preliminiaries**

[1] Let $X$ be a space of points. A neutrosophic set (NS) $A$ in $X$ is characterized by a falsity-membership function $F$, a indeterminacy-membership function $I$ and a truth-membership

function $T$ where $F, I, T : X \to [0,1]$, $0 \leq T(x)+I(x)+F(x) \leq 3$ . The set of all neutrosophic set in $X$ is denoted by $N^X$.

**Definition 2.1.** [14] Let $B, D \in N^X$. Then

(1) Subset: $D \subset B$ if $T_D(z) \leq T_B(z), I_D(z) \leq I_B(z), F_D(z) \geq F_B(z)$ for all $z \in X$ .

(2) Equality: $D = B$ if $D \subset B$ and $B \subset D$ .

(3) Intersection:

$$D \cap B = \{< z, min\{T_D(z), T_B(z)\}, max\{I_D(z), I_B(z)\}, max\{F_D(z), F_B(z)\} >: z \in X\}$$

(4) Union:

$$D \cup B = \{< z, max\{T_D(z), T_B(z)\}, min\{I_D(x), I_B(x)\}, min\{F_D(z), F_B(z)\} >: z \in X\}$$

The intersection and the union of a collection of NSs $\{D_i\} \in I$ are defined by:

$$\bigcap_{i \in I} D_i = \{< z, \inf\{T_i(z)\}, \sup\{I_{D_i}(z)\}, \sup\{F_{D_i}(z)\} >: z \in X\}$$

$$\bigcup_{i \in I} D_i = \{< z, \sup\{T_{D_i}(z)\}, \inf\{I_{D_i}(z)\}, \inf\{F_{D_i}(z)\} >: z \in X\}$$

(5) The neutrosophic set defined as $T_D(z) = 1, I_D(z) = 1$ and $F_D(z) = 0$ for all $z \in X$ is called the universal NS denoted by $1_X$ . Also the neutrosophic set defined as $T_D(z) = 0, I_D(z) = 0$ and $F_D(z) = 1$ for all $z \in X$ is called the empty NS denoted by $0_X$ .

(6) Difference: $D/B = \{< z, T_D(z) - T_B(z), I_D(z) - I_B(z), F_D(z) - F_B(z) >: z \in X\}$

(7) Complement: $D^c = 1_X/D$

Clearly, the complements of $1_X$ and $0_X$ are defined:

$$(1_X)^c = 1_X/1_X = \{< z, 0, 1, 1 >: z \in X\} = 0_X$$

$$(0_X)^c = 1_X/0_X = \{< z, 1, 0, 0 >: z \in X\} = 1_X$$

**Proposition 2.2.** *Let $D_1, D_2, D_3, D_4 \in N(X)$. Then the followings hold:*

(1) $D_1 \cap D_3 \subset D_2 \cap D_4$ *and* $D_1 \cup D_3 \subset D_2 \cup D_4$ *if* $D_1 \subset D_2$ *and* $D_3 \subset D_4$

(2) $(D_1^c)^c = D_1$ *and* $D_1 \subset D_2$ *if* $D_2^c \subset D_1^c$

(3) $(D_1 \cap D_2)^c = D_1^c \cup D_2^c$ *and* $(D_1 \cup D_2)^c = D_1^c \cap D_2^c$

**Definition 2.3.** Let $\Gamma^n \subset N(Y)$. Then $\Gamma^n$ is named a neutrosophic topology (NT) on $Y$ if the following conditions hold;

(1) $0_X$ and $1_X$ are belong to $\Gamma^n$.

(2) Union of any number of NSs in $\Gamma^n$ is again belong to $\Gamma^n$.

(3) Intersection of any two NSs in $\Gamma^n$ is belong to $\Gamma^n$.

Then the pair $(Y, \Gamma^n)$ is named neutrosophic topology on $Y$.

2.1. *Neutrosophic Soft Sets*

**Definition 2.4.** Let $U$ be an initial universe set and $E$ be a set of parameters. Then the pair $(H, E)$ is called as neutrosophic soft set $(NSS)$ over $U$, where $H$ is a mapping from $E$ to $N(U)$.

The set of all $NSS$ over $U$ is denoted by $NSS(U, E)$. A neutrosophic set $(H, E)$ can be written as: $(H, E) = \{(e, \{< x, T_H(x), I_H(x), F_H(x) >: x \in X\}) : e \in E\}$.

**Definition 2.5.** Let $X$ be an initial universe set and $E$ be a set of parameters. Then the neutrosophic soft set $x^e_{(\alpha,\beta,\gamma)}$ defined as

$$x^e_{(\alpha,\beta,\gamma)}(e')(y) = \begin{cases} (\alpha, \beta, \gamma) & \text{if } e = e' \text{ and } y = y' \\ (0, 0, 1) & \text{if } e \neq e' \text{ and } y \neq y' \end{cases}$$

for all $x \in X, 0 < \alpha, \beta, \gamma \leq 1, e \in E$ , is called a neutrosophic soft point.

**Definition 2.6.** [2] Let $(H, E), (G, E) \in NSS(U, E)$. Then for all $x \in U$

(1) Subset: $(H, E) \subset (G, E)$ if $T_{H(e)}(x) \leq T_{G(e)}(x), I_{H(e)}(x) \leq I_{G(e)}(x)$ and $F_{H(e)}(x) \geq F_{G(e)}(x)$ for all $e \in E$ .

(2) Equality: $(H, E) = (G, E)$ if $(H, E) \subset (G, E)$ and $(G, E) \subset (H, E)$ .

(3) Intersection:

$$(H, E) \cap (G, E) = \{(e, \{< x, min\{T_{H(e)}(x), T_{G(e)}(x)\}, max\{I_{H(e)}(x), I_{G(e)}(x)\}, max\{F_{H(e)}(x), F_{G(e)}(x)\} >\} : e \in E\}.$$

(4) Union:

$$(H, E) \cup (G, E) = \{(e, \{< x, max\{T_{H(e)}(x),\ T_{G(e)}(x)\}, min\{I_{H(e)}(x), I_{G(e)}(x)\}, min\{F_{H(e)}(x), F_{G(e)}(x)\} >\} : e \in E\}.$$

The intersection and the union of a collection of $\{(H_i, E)\} \subset NSS(U, E)$ are defined by:

$$\bigcap_{i \in I}(H_i, E) = \left\{\left(e, \{< x, \inf\{T_{H_{i(e)}}(x)\}, \sup\{I_{H_i(e)}(x)\}, \sup\{F_{H_i(e)}(x)\} >\}\right) : e \in E\right\}$$

$$\bigcup_{i \in I}(H_i, E) = \left\{\left(e, \{< x, \sup\{T_{H_{i(e)}}(x)\}, \inf\{I_{H_{i(e)}}(x)\}, \inf\{F_{H_{i(e)}}(x)\} >\}\right) : e \in E\right\}$$

(5) The $NSS$ defined as $T_{H(e)}(x) = 1, I_{H(e)}(x) = 0$ and $F_{H(e)}(x) = 0$ , for all $e \in E$ and $x \in U$ is called the universal $NSS$ denoted by $1_{(U,E)}$ . Also the neutrosophic set defined as $T_{H(e)}(x) = 0$ $I_{H(e)}(x) = 1$ and $F_{H(e)}(x) = 1$ for all $e \in E$ and $x \in U$ is called the empty $NSS$ denoted by $0_{(U;E)}$ .

(6) Complement:

$$(H, E)^c = 1_{(X,E)}/(H, E) = \{(e, \{< x, F_{H(e)}(x), 1 - I_{H(e)}(x), T_{H(e)}(x) >\} : e \in E\}$$

Clearly, the complements of $1_{(X,E)}$ and $0_{(X,E)}$ are defined:

$$(1_{(X,E)})^c = 1_{(X,E)}/1_{(X,E)} = \{(e, \{< x, 0, 1, 1 >\} : e \in E\} = 0_{(X,E)}$$

$$(0_{(X,E)})^c = 1_{(X,E)}/0_{(X,E)} = \{(e, \{< x, 1, 0, 0 >\} : e \in E\} = 1_{(X,E)}$$

**Definition 2.7.** [1] Let $\tau \subset NSS(Y,E)$ . Then $\tau$ is called as a neutrosophic soft topology on $Y$ if the following conditions hold:

$NST_1$) $0_{(Y,E)}, 1_{(Y,E)} \in \tau$

$NST_2$) Union of any number of $NSS$s in $\tau$ is belong to $\tau$.

$NST_3$) Intersection of finite number of $NSS$s in $\tau$ is belong to $\tau$.

Then $(Y,E,\tau)$ is called as neutrosophic soft topological space. Any element of $\tau$ is called as $\tau$-neutrosophic soft open ($\tau$-NSO) set. A $NSS$ is called as $\tau$-neutrosophic soft closed ($\tau$-NSC) if the complement of the set is $\tau$-NSO. The set of all neutrosophic soft closed sets is denoted by $(\tau)^c$.

**Definition 2.8.** [1] Let $(Y,E,\tau)$ be a neutrosophic soft topological space and $(M,E) \in NSS(Y,E)$. Then the intersection of all $\tau$-NSC sets containing $(M,E)$ is called as closure of $(M,E)$ and denoted by $cl_\tau(M,E)$, i.e. $cl_\tau(M,E) = \bigcap\{(N,E) \in (\tau)^c : (M,E) \subset (N,E)\}$

**Theorem 2.9.** *[1] Let $(Y,E,\tau)$ be a neutrosophic soft topological space and $(M,E),(N,E) \in NSS(Y,E)$. Then*

$cl_1$) $(M,E) \subset cl_\tau(M,E)$

$cl_2$) $(M,E) \subset (N,E)$ *then* $cl_\tau(M,E) \subset cl_\tau(N,E)$

$cl_3$) $cl_\tau((M,E) \cap (N,E)) \subset cl_\tau(M,E) \cap cl_\tau(N,E)$

$cl_4$) $cl_\tau((M,E) \cup (N,E)) = cl_\tau(M,E) \cup cl_\tau(N,E)$

**Definition 2.10.** Let $\tau \subset NSS(Y,E)$. Then $\tau$ is called as a neutrosophic soft supra topology on $Y$ if it satisfies just $NST_1$) and $NST_2$).

**Definition 2.11.** Let $(Y,E,\tau)$ be a neutrosophic soft topological space and $(M,E) \in NSS(Y,E)$. Then the union of all $\tau$-NSO sets subset of $(M,E)$ is called as interior of $(M,E)$ and denoted by $int_\tau(M,E)$, i.e. $int_\tau(M,E) = \bigcup\{(N,E) \in \tau : (N,E) \subset (M,E)\}$

**Theorem 2.12.** *[1] Let $(Y,E,\tau)$ be a neutrosophic soft topological space and $(M,E),(N,E) \in NSS(Y,E)$. Then*

$int_1$) $int_\tau(M,E) \subset (M,E)$

$int_2$) $(M,E) \subset (N,E)$ *then* $int_\tau(M,E) \subset int_\tau(N,E)$

$int_3$) $int_\tau((M,E) \cap (N,E)) = int_\tau(M,E) \cap int_\tau(N,E)$

$int_4$) $int_\tau(M,E) \cup int_\tau(N,E) \subset int_\tau((M,E) \cup (N,E))$

## 3. **Neutrosophic Soft Bitopological Space**

**Definition 3.1.** If $(Y, \tau_1, E)$ and $(Y, \tau_2, E)$ are two neutrosophic soft topological space, then $(Y, E, \tau_1, \tau_2)$ is named as neutrosophic soft bitopological space. The sets belong to $\tau_i$ are called as neutrosophic soft $\tau_i$-open set for $i = 1, 2$.

**Definition 3.2.** An operator $C : NSS(X, E) \to NSS(X, E)$ is called a neutrosophic soft supra closure operator if it satisfies the following conditions for all $(N, E), (M, E) \in NSS(X, E)$,

$C_1)$ $C(0_{(X,E)}) = 0_{(X,E)}$
$C_2)$ $(N, E) \subset C(N, E)$
$C_3)$ $C(N, E) \cup C(M, E) \subset C(N \cup M)$
$C_4)$ $C(C(N, E)) = C(N, E)$.

**Theorem 3.3.** *Let $(X, E, \tau_1, \tau_2)$ be a neutrosophic soft bitopological space. Then, the operator $cl_{12} : NSS(X, E) \to NSS(X, E)$ defined as $cl_{12}(N, E) = cl_{\tau_1}(N, E) \cap cl_{\tau_1}(N, E)$ is a neutrosophic soft supra closure operator on $(X, E)$ and induces the supra neutrosophic soft topology $\tau_{12} = \{(M, E) \in NSS(X, E) : cl_{12}((M, E)^c) = (M, E)^c\}$.*

*Proof.* First let prove that $cl_{12}$ is a neutrosophic soft supra closure operator.

$C_1)$ $cl_{12}(0_{(X,E)}) = cl_{\tau_1}(0_{(X,E)}) \cap cl_{\tau_2}(0_{(X,E)}) = 0_{(X,E)} \cap 0_{(X,E)} = 0_{(X,E)}$

$C_2)$ $(N, E) \subset cl_{\tau_1}(N, E)$ and $(N, E) \subset cl_{\tau_2}(N, E)$. Then $(N, E) \subset cl_{\tau_1}(N, E) \cap cl_{\tau_2}(N, E) = cl_{12}(N, E)$.

$C_3)$

$$\begin{aligned} cl_{12}(N, E) \cup cl_{12}(M, E) &= [cl_{\tau_1}(N, E) \cap cl_{\tau_2}(N, E)] \cup [cl_{\tau_1}(M, E) \cap cl_{\tau_2}(M, E)] \\ &= cl_{\tau_1}[(N, E) \cup (M, E)] \cap [cl_{\tau_2}(N, E) \cup cl_{\tau_1}(M, E)] \\ &\quad \cap [cl_{\tau_1}(N, E) \cup cl_{\tau_2}(M, E)] \cap cl_{\tau_2}[(N, E) \cup (M, E)] \\ &\subset cl_{\tau_1}[(N, E) \cup (M, E)] \cap cl_{\tau_2}[(N, E) \cup (M, E)] \\ &= cl_{12}[(N, E) \cup (M, E)]. \end{aligned}$$

$C_4)$ From $C_3$, $cl_{12}(N, E) \subset cl_{12}(cl_{12}(N, E))$. Also

$$\begin{aligned} cl_{12}(cl_{12}(N, E)) &= cl_{12}(cl_{\tau_1}(N, E) \cap cl_{\tau_2}(N, E)) \\ &= cl_{\tau_1}((cl_{\tau_1}(N, E) \cap cl_{\tau_2}(N, E)) \cap cl_{\tau_2}((cl_{\tau_1}(N, E) \cap cl_{\tau_2}(N, E)) \\ &\subset cl_{\tau_1}((cl_{\tau_1}(N, E)) \cap cl_{\tau_1}(cl_{\tau_2}(N, E)) \cap cl_{\tau_2}(cl_{\tau_1}(N, E)) \cap cl_{\tau_2}(cl_{\tau_2}(N, E)) \\ &\subset cl_{\tau_1}(N, E) \cap cl_{\tau_2}(N, E) = cl_{12}(N, E). \end{aligned}$$

Therefore $cl_{12}(N, E) = cl_{12}(cl_{12}(N, E))$.

Now let prove that $\tau_{12}$ is a neutrosophic soft supra topology.

$NST_1)$ Since $cl_{12}\left((1_{(X,E)})^c\right) = cl_{12}(0_{(X,E)}) = 0_{(X,E)}$, then $0_{(X,E)} \in \tau_{12}$. Also $cl_{12}\left((0_{(X,E)})^c\right) = cl_{12}(1_{(X,E)}) \subset 1_{(X,E)}$ and from $(C_2)$, $1_{(X,E)} \subset cl_{12}(1_{(X,E)})$. Therefore $0_{(X,E)} \in \tau_{12}$.

$NST_2)$ Let $(N_i, E) \in \tau_{12}$. Then $cl_{12}\left((N_i, E)^c\right) = (N_i, E)^c$.

$$\begin{aligned} cl_{12}\left(\bigcup_{i\in I}(N_i,E)\right)^c &= cl_{\tau_1}\left(\bigcup_{i\in I}(N_i,E)\right)^c \cap cl_{\tau_2}\left(\bigcup_{i\in I}(N_i,E)\right)^c \\ &= cl_{\tau_1}\left(\bigcap_{i\in I}(N_i,E)^c\right) \cap cl_{\tau_2}\left(\bigcap_{i\in I}(N_i,E)^c\right) \\ &\subset \bigcap_{i\in I}(cl_{\tau_1}(N_i,E)^c) \cap \bigcap_{i\in I}(cl_{\tau_2}(N_i,E)^c) \\ &= \bigcap_{i\in I}(cl_{\tau_1}(N_i,E)^c \cap cl_{\tau_2}(N_i,E)^c) \\ &= \bigcap_{i\in I}(cl_{12}(N_i,E)^c) = \bigcap_{i\in I}(N_i,E)^c = \left(\bigcup_{i\in I}(N_i,E)\right)^c. \end{aligned}$$

Also from $(C_2)$, $\left(\bigcup_{i\in I}(N_i,E)\right)^c \subset cl_{12}\left(\bigcup_{i\in I}(N_i,E)\right)^c$. Therefore $cl_{12}\left(\bigcup_{i\in I}(N_i,E)\right)^c = \left(\bigcup_{i\in I}(N_i,E)\right)^c$, then $\bigcup_{i\in I}(N_i,E) \in \tau_{12}$.

Consequently $\tau_{12}$ is a neutrosophic soft supra topology on $(X,E)$. □

**Theorem 3.4.** *Let $(X,E,\tau_1,\tau_2)$ be a neutrosophic soft bitopological space and $(N,E) \in NSS(X,E)$. Then $(N,E) \in \tau_{12}$ if and only if there exists a $\tau_1$-NSC set $(N_1,E)$ and $\tau_2$-NSC set $(N_2,E)$ such that $(N,E) = (N_1,E) \cap (N_2,E)$.*

*Proof.* If we take $(N_1,E) = cl_{\tau_1}(N,E)$ and $(N_2,E) = cl_{\tau_2}(N,E)$, then proof is clear. □

**Theorem 3.5.** *Let $(X,E,\tau_1,\tau_2)$ be a neutrosophic soft bitopological space and $(M,E),(N,E) \in NSS(X,E)$. Then*

1) *if $(M,E) \subset (N,E)$ then $cl_{12}(M,E) \subset cl_{12}(N,E)$.*
2) *$cl_{12}\left((M,E) \cap (N,E)\right) \subset cl_{12}(M,E) \cap cl_{12}(N,E)$.*

*Proof.* For any $(M,E),(N,E) \in NSS(X,E)$,

1) Let $(M,E) \subset (N,E)$. Then $cl_{\tau_1}(M,E) \subset cl_{\tau_1}(N,E)$ and $cl_{\tau_2}(M,E) \subset cl_{\tau_2}(N,E)$. Therefore $cl_{\tau_1}(M,E) \cap cl_{\tau_2}(M,E) \subset cl_{\tau_1}(N,E) \cap cl_{\tau_2}(N,E)$. So $cl_{12}(M,E) \subset cl_{12}(N,E)$.
2) $(M,E) \cap (N,E) \subset (M,E)$ and $(M,E) \cap (N,E) \subset (N,E)$. Then from (1), $cl_{12}((M,E) \cap (N,E)) \subset cl_{12}(M,E)$ and $cl_{12}((M,E) \cap (N,E)) \subset cl_{12}(N,E)$. Therefore $cl_{12}\left((M,E) \cap (N,E)\right) \subset cl_{12}(M,E) \cap cl_{12}(N,E)$.

. □

**Remark 3.6.** Let $(X, E, \tau_1, \tau_2)$ be a neutrosophic soft bitopological space. Then $cl_{12}(M, E) \cap cl_{12}(N, E) \neq cl_{12}((M, E) \cap (N, E))$, in general.

**Example 3.7.** Let the neutrosophic soft bitopological space $(X, U, \tau_1, \tau_2)$ be defined as $X = \{x_1, x_2, x_3\}$, $U = \{e_1, e_2\}$, $\tau_1 = \{0_{(X,U)}, 1_{(X,U)}, (A, U), (B, U), (C, U), (D, U)\}$, $\tau_2 = \{0_{(X,U)}, 1_{(X,U)}, (D, U), (F, U), (G, U), (H, U)\}$ where the tabular representations of NSSs are as follows:

$(A, U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.2, 0.3, 0.8 >$ | $< 0.9, 0.1, 0.3 >$ |
| $x_2$ | $< 0.1, 0.5, 0.4 >$ | $< 0.4, 0.4, 0.4 >$ |
| $x_3$ | $< 0.8, 0.1, 0.5 >$ | $< 0.2, 0.8, 0.1 >$ |

$(B, U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.1, 0.3, 0.8 >$ | $< 0.3, 0.1, 0.7 >$ |
| $x_2$ | $< 0.1, 0.1, 0.4 >$ | $< 0.1, 0.2, 0.5 >$ |
| $x_3$ | $< 0.3, 0.1, 0.5 >$ | $< 0.2, 0.1, 0.3 >$ |

$(C, U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.2, 0.3, 0.4 >$ | $< 0.9, 0.1, 0.3 >$ |
| $x_2$ | $< 0.2, 0.1, 0.3 >$ | $< 0.4, 0.2, 0.4 >$ |
| $x_3$ | $< 0.8, 0.1, 0.5 >$ | $< 0.6, 0.1, 0.1 >$ |

$(D, U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.1, 0.3, 0.4 >$ | $< 0.3, 0.2, 0.7 >$ |
| $x_2$ | $< 0.2, 0.1, 0.3 >$ | $< 0.1, 0.2, 0.5 >$ |
| $x_3$ | $< 0.3, 0.7, 0.8 >$ | $< 0.6, 0.1, 0.3 >$ |

$(F, U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.7, 0.1, 0.1 >$ | $< 0.2, 0.5, 0.5 >$ |
| $x_2$ | $< 0.9, 0.5, 0.3 >$ | $< 0.3, 0.8, 0.1 >$ |
| $x_3$ | $< 0.1, 0.8, 0.1 >$ | $< 0.8, 0.2, 0.7 >$ |

$(G,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $<0.1, 0.1, 0.4>$ | $<0.2, 0.2, 0.7>$ |
| $x_2$ | $<0.2, 0.1, 0.3>$ | $<0.1, 0.2, 0.5>$ |
| $x_3$ | $<0.1, 0.7, 0.8>$ | $<0.6, 0.1, 0.7>$ |

$(H,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $<0.7, 0.1, 0.1>$ | $<0.3, 0.2, 0.5>$ |
| $x_2$ | $<0.9, 0.1, 0.3>$ | $<0.3, 0.2, 0.1>$ |
| $x_3$ | $<0.3, 0.7, 0.1>$ | $<0.8, 0.1, 0.3>$ |

Let two NSSs $(X_1, U)$ and $(X_2, U)$ are defined as

$(X_1,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $<0.8, 0.5, 0.1>$ | $<0.7, 0.1, 0.3>$ |
| $x_2$ | $<0.5, 0.9, 0.1>$ | $<0.8, 0.1, 0.1>$ |
| $x_3$ | $<0.5, 0.8, 0.2>$ | $<0.5, 0.9, 0.2>$ |

$(X_2,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $<0.9, 0.7, 0.1>$ | $<0.9, 0.9, 0.1>$ |
| $x_2$ | $<0.4, 0.5, 0.1>$ | $<0.5, 0.8, 0.1>$ |
| $x_3$ | $<0.7, 0.9, 0.3>$ | $<0.3, 0.9, 0.1>$ |

Then $cl_{12}((X_1, U) \cap (X_2, U)) = (B, U)^c$ and $cl_{12}(X_1, U) = cl_{12}(X_2, U) = 1_{(X,U)}$. So $cl_{12}(X_1, U) \cap cl_{12}(X_2, U) \not\subset cl_{12}((X_1, U) \cap (X_2, U))$

**Definition 3.8.** An operator $I : NSS(X, E) \to NSS(X, E)$ is called a neutrosophic soft supra interior operator if it satisfies the following conditions for all $(N, E), (M, E) \in NSS(X, E)$,

- $I_1)$ $I(0_{(X,E)}) = 0_{(X,E)}$
- $I_2)$ $I(N, E) \subset (N, E)$
- $I_3)$ $I(N, E) \cap I(M, E) \subset I(N \cap M)$
- $I_4)$ $I(I(N, E)) = I(N, E)$.

**Theorem 3.9.** *Let $(X, E, \tau_1, \tau_2)$ be a neutrosophic soft bitopological space. Then, the operator $int_{12} : NSS(X, E) \to NSS(X, E)$ defined as $int_{12}(N, E) = int_{\tau_1}(N, E) \cup int_{\tau_1}(N, E)$ is a neutrosophic soft supra interior operator on $(X, E)$ and induces the supra neutrosophic soft topology $\tau_{12} = \{(M, E) \in NSS(X, E) : int_{12}(M, E) = (M, E)\}$.*

*Proof.* First let prove that $int_{12}$ is a neutrosophic soft supra interior operator.

$I_1)$ $int_{12}(0_{(X,E)}) = int_{\tau_1}(0_{(X,E)}) \cup int_{\tau_2}(0_{(X,E)}) = 0_{(X,E)} \cup 0_{(X,E)} = 0_{(X,E)}$

$I_2)$ $int_{\tau_1}(N,E) \subset (N,E)$ and $int_{\tau_2}(N,E) \subset (N,E)$. Then $int_{\tau_1}(N,E) \cup int_{\tau_2}(N,E) \subset (N,E)$. Therefore $int_{\tau_{12}}(N,E) \subset (N,E)$

$I_3)$

$$\begin{aligned} int_{12}(N,E) \cap int_{12}(M,E) &= [int_{\tau_1}(N,E) \cup int_{\tau_2}(N,E)] \cap [int_{\tau_1}(M,E) \cup int_{\tau_2}(M,E)] \\ &= int_{\tau_1}[(N,E) \cap (M,E)] \cup [int_{\tau_2}(N,E) \cap int_{\tau_1}(M,E)] \\ &\quad \cup [int_{\tau_1}(N,E) \cap int_{\tau_2}(M,E)] \cup int_{\tau_2}[(N,E) \cap (M,E)] \\ &= int_{\tau_{12}}[(N,E) \cap (M,E)] \cup [int_{\tau_2}(N,E) \cap int_{\tau_1}(M,E)] \\ &\quad \cup [int_{\tau_1}(N,E) \cap int_{\tau_2}(M,E)] \\ &\subset int_{12}[(N,E) \cap (M,E)]. \end{aligned}$$

$I_4)$ From $(I_3)$, $int_{12}(int_{12}(N,E)) \subset int_{12}(N,E)$. Also

$$\begin{aligned} int_{12}(N,E) &= int_{\tau_1}(N,E) \cup int_{\tau_2}(N,E)) \\ &= int_{\tau_1}(int_{\tau_1}(N,E)) \cup int_{\tau_2}(int_{\tau_2}(N,E)) \\ &\subset int_{\tau_1}(int_{\tau_1}(N,E)) \cup int_{\tau_1}(int_{\tau_2}(N,E)) \cup int_{\tau_2}(int_{\tau_1}(N,E)) \cup int_{\tau_2}(int_{\tau_2}(N,E)) \\ &\subset int_{\tau_1}(int_{\tau_1}(N,E)) \cup int_{\tau_2}(N,E)) \cup int_{\tau_2}(int_{\tau_1}(N,E)) \cup int_{\tau_2}(N,E)) \\ &= int_{\tau_1}(int_{12}(N,E)) \cup int_{\tau_2}(int_{12}(N,E)) \\ &= int_{12}(int_{12}(N,E)). \end{aligned}$$

Therefore $int_{12}(N,E) = int_{12}(int_{12}(N,E))$.

Now let prove that $\tau_{12}$ is a neutrosophic soft supra topology.

$NST_1)$ From $(I_1)$, $int_{12}(0_{(X,E)}) = 0_{(X,E)}$, then $0_{(X,E)} \in \tau_{12}$. Also $int_{12}((1_{(X,E)})) = int_{\tau_1}(1_{(X,E)}) \cup int_{\tau_2}(1_{(X,E)}) = 1_{(X,E)}$. Therefore $1_{(X,E)} \in \tau_{12}$.

$NST_2$) Let $(N_i, E) \in \tau_{12}$. Then $int_{12}((N_i, E)) = (N_i, E)$.

$$\begin{aligned}
\bigcup_{i\in I}(N_i, E) &= \bigcup_{i\in I} int_{12}(N_i, E) \\
&= \bigcup_{i\in I}(int_1(N_i, E) \cup int_2(N_i, E)) \\
&= \left(\bigcup_{i\in I} int_1(N_i, E)\right) \cup \left(\bigcup_{i\in I} int_2(N_i, E)\right) \\
&\subset int_1\left(\bigcup_{i\in I}(N_i, E)\right) \cup int_2\left(\bigcup_{i\in I}(N_i, E)\right) \\
&= int_{12}\left(\bigcup_{i\in I}(N_i, E)\right).
\end{aligned}$$

Also from $(I_2)$,

$$int_{12}\left(\bigcup_{i\in I}(N_i, E)\right) \subset \left(\bigcup_{i\in I}(N_i, E)\right).$$

Therefore $int_{12}\left(\bigcup_{i\in I}(N_i, E)\right) = \left(\bigcup_{i\in I}(N_i, E)\right)$, then $\bigcup_{i\in I}(N_i, E) \in \tau_{12}$.

Consequently $\tau_{12}$ is a neutrosophic soft supra topology on $(X, E)$. □

**Theorem 3.10.** *Let $(X, E, \tau_1, \tau_2)$ be a neutrosophic soft bitopological space and $(N, E) \in NSS(X, E)$. Then $(N, E) \in \tau_{12}$ if and only if there exists a $\tau_1$-NSO set $(N_1, E)$ and $\tau_2$-NSO set $(N_2, E)$ such that $(N, E) = (N_1, E) \cup (N_2, E)$.*

*Proof.* If we take $(N_1, E) = int_{\tau_1}(N, E)$ and $(N_2, E) = int_{\tau_2}(N, E)$, then proof is clear. □

**Theorem 3.11.** *Let $(X, E, \tau_1, \tau_2)$ be a neutrosophic soft bitopological space and $(M, E), (N, E) \in NSS(X, E)$. Then*

1) *if $(M, E) \subset (N, E)$ then $int_{12}(M, E) \subset int_{12}(N, E)$.*
2) *$int_{12}(M, E) \cup int_{12}(N, E) \subset int_{12}((M, E) \cup (N, E))$.*

*Proof.* For any $(M, E), (N, E) \in NSS(X, E)$,

1) Let $(M, E) \subset (N, E)$. Then $int_{\tau_1}(M, E) \subset int_{\tau_1}(N, E)$ and $int_{\tau_2}(M, E) \subset int_{\tau_2}(N, E)$. Therefore $int_{\tau_1}(M, E) \cap int_{\tau_2}(M, E) \subset int_{\tau_1}(N, E) \cap int_{\tau_2}(N, E)$. So $int_{12}(M, E) \subset int_{12}(N, E)$.

2) $(M, E) \subset (M, E) \cup (N, E)$ and $(N, E) \subset (M, E) \cup (N, E)$. Then from (1), $int_{12}(M, E) \subset int_{12}((M, E) \cup (N, E))$ and $int_{12}(N, E) \subset int_{12}((M, E) \cup (N, E))$. Therefore $int_{12}(M, E) \cup int_{12}(N, E) \subset int_{12}((M, E) \cup (N, E))$. □

**Remark 3.12.** Let $(X, E, \tau_1, \tau_2)$ be a neutrosophic soft bitopological space. Then $int_{12}(M, E) \cup int_{12}(N, E) \neq int_{12}((M, E) \cup (N, E))$, in general.

**Proposition 3.13.** *Let $(X, E, \tau_1, \tau_2)$ be a neutrosophic soft bitopological space and $(N, E) \in NSS(X, E)$. Then*

1) $\tau_1, \tau_2 \subset \tau_{12}$
2) $cl_{12}(N, E) = (int_{12}(N, E)^c)^c$
3) $int_{12}(N, E) = (cl_{12}(N, E)^c)^c$

*Proof.* 1) Let $(N, E) \in \tau_1$. Then $(N, E) = int_{\tau_1}(N, E)$. Therefore

$$int_{12}(N, E) = int_{\tau_1}(N, E) \cup int_{\tau_2}(N, E) = (N, E)$$

So $\tau_1 \subset \tau_{12}$. Similar for $\tau_2 \subset \tau_{12}$.

2)

$$\begin{aligned} cl_{12}(N, E) = cl_{\tau_1}(N, E) \cap cl_{\tau_2}(N, E) &= \left(\bigcap_{j \in I}(F_j^1, E)\right) \cap \left(\bigcap_{j \in J}(F_j^2, E)\right) \\ &= \left[\bigcup_{j \in I}(F_j^1, E)^c\right]^c \cap \left[\bigcup_{j \in J}(F_j^2, E)^c\right]^c \\ &= \left[\left(\bigcup_{j \in I}(F_j^1, E)^c\right) \cup \left(\bigcup_{j \in J}(F_j^2, E)^c\right)\right]^c \\ &= [int_{\tau_1}(N, E)^c \cup int_{\tau_2}(N, E)^c]^c = (int_{12}(N, E)^c)^c \end{aligned}$$

where $(N, E) \subset (F_j^i, E)$, $(F_j^i, E)^c \in \tau_i$ for all $j \in I, J$ and $i = 1, 2$.

3)

$$\begin{aligned} int_{12}(N, E) = int_{\tau_1}(N, E) \cup int_{\tau_2}(N, E) &= \left(\bigcup_{j \in I}(U_j^1, E)\right) \cup \left(\bigcup_{j \in J}(U_j^2, E)\right) \\ &= \left[\bigcap_{j \in I}(U_j^1, E)^c\right]^c \cup \left[\bigcap_{j \in J}(U_j^2, E)^c\right]^c \\ &= \left[\left(\bigcap_{j \in I}(U_j^1, E)^c\right) \cap \left(\bigcap_{j \in J}(U_j^2, E)^c\right)\right]^c \\ &= [cl_{\tau_1}(N, E)^c \cap cl_{\tau_2}(N, E)^c]^c = (cl_{12}(N, E)^c)^c \end{aligned}$$

where $(U_j^i, E) \subset (N, E)$, $(U_j^i, E) \in \tau_i$ for all $j \in I, J$ and $i = 1, 2$.

. □

## 4. Some Generalized Open Sets in Neutrosophic Soft Bitopological Spaces

Throughout this section, $i, j = 1, 2$ and $i \neq j$.

**Definition 4.1.** Let $(X, E, \tau_1, \tau_2)$ be a neutrosophic soft bitopological space and $(N, E) \in NSS(X, E)$. Then $(N, E)$ is called as

1) $ij-$ neutrosophic soft preopen $(ij - NSPO)$ if $(N, E) \subset int_{\tau_i}\left(cl_{\tau_j}(N, E)\right)$
2) $ij-$ neutrosophic soft semi-open $(ij - NSSO)$ if $(N, E) \subset cl_{\tau_j}\left(int_{\tau_i}(N, E)\right)$
3) $ij-$ neutrosophic soft b-open $(ij - NSbO)$ if $(N, E) \subset cl_{\tau_i}\left(int_{\tau_j}(N, E)\right) \cup int_{\tau_j}\left(cl_{\tau_i}(N, E)\right)$.
4) $ij-$ neutrosophic soft $\beta$-open $(ij - NS\beta O)$ if $(N, E) \subset cl_{\tau_j}\left(int_{\tau_i}(cl_{\tau_j}(N, E))\right)$.

**Example 4.2.** Let the neutrosophic soft bitopological space $(X, U, \tau_1, \tau_2)$ be defined as $X = \{x_1, x_2, x_3\}$, $U = \{e_1, e_2\}$, $\tau_1 = \{0_{(X,U)}, 1_{(X,U)}, (A, U), (B, U), (C, U), (D, U)\}$, $\tau_2 = \{0_{(X,U)}, 1_{(X,U)}, (E, U), (F, U), (G, U), (H, U)\}$ where the tabular representations of NSSs are as follows:

$(A, U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.2, 0.1, 0.9 >$ | $< 0.6, 0.1, 0.7 >$ |
| $x_2$ | $< 0.1, 0.8, 0.4 >$ | $< 0.1, 0.1, 0.8 >$ |
| $x_3$ | $< 0.3, 0.4, 0.8 >$ | $< 0.5, 0.1, 0.4 >$ |

$(B, U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.1, 0.3, 0.4 >$ | $< 0.2, 0.7, 0.8 >$ |
| $x_2$ | $< 0.2, 0.1, 0.5 >$ | $< 0.5, 0.6, 0.7 >$ |
| $x_3$ | $< 0.3, 0.3, 0.7 >$ | $< 0.1, 0.8, 0.8 >$ |

$(C, U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.2, 0.1, 0.4 >$ | $< 0.6, 0.1, 0.7 >$ |
| $x_2$ | $< 0.2, 0.1, 0.4 >$ | $< 0.5, 0.1, 0.7 >$ |
| $x_3$ | $< 0.3, 0.3, 0.7 >$ | $< 0.5, 0.1, 0.4 >$ |

$(D, U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.1, 0.3, 0.9 >$ | $< 0.2, 0.7, 0.8 >$ |
| $x_2$ | $< 0.1, 0.8, 0.5 >$ | $< 0.1, 0.6, 0.8 >$ |
| $x_3$ | $< 0.3, 0.4, 0.8 >$ | $< 0.1, 0.8, 0.8 >$ |

$(E,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.2, 0.6, 0.3 >$ | $< 0.1, 0.1, 0.9 >$ |
| $x_2$ | $< 0.3, 0.7, 0.4 >$ | $< 0.3, 0.4, 0.6 >$ |
| $x_3$ | $< 0.3, 0.5, 0.8 >$ | $< 0.6, 0.1, 0.8 >$ |

$(F,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.1, 0.1, 0.8 >$ | $< 0.1, 0.8, 0.9 >$ |
| $x_2$ | $< 0.3, 0.2, 0.5 >$ | $< 0.3, 0.5, 0.7 >$ |
| $x_3$ | $< 0.7, 0.5, 0.6 >$ | $< 0.1, 0.7, 0.7 >$ |

$(G,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.2, 0.1, 0.3 >$ | $< 0.1, 0.1, 0.9 >$ |
| $x_2$ | $< 0.3, 0.2, 0.4 >$ | $< 0.3, 0.4, 0.6 >$ |
| $x_3$ | $< 0.7, 0.5, 0.6 >$ | $< 0.6, 0.1, 0.7 >$ |

$(H,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.1, 0.6, 0.8 >$ | $< 0.1, 0.8, 0.9 >$ |
| $x_2$ | $< 0.3, 0.7, 0.5 >$ | $< 0.3, 0.5, 0.7 >$ |
| $x_3$ | $< 0.3, 0.5, 0.8 >$ | $< 0.1, 0.7, 0.8 >$ |

Let an NSSs $(W,U)$ is defined as

$(W,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.3, 0.7, 0.2 >$ | $< 0.8, 0.2, 0.3 >$ |
| $x_2$ | $< 0.4, 0.8, 0.3 >$ | $< 0.6, 0.4, 0.6 >$ |
| $x_3$ | $< 0.5, 0.5, 0.7 >$ | $< 0.7, 0.2, 0.3 >$ |

Then $int_{\tau_1}(W,U) = (A,U)$, $int_{\tau_2}(W,U) = (E,U)$.

$(W,U) \subset cl_{\tau_2}(A,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.8, 0.9, 0.1 >$ | $< 0.9, 0.2, 0.1 >$ |
| $x_2$ | $< 0.5, 0.8, 0.3 >$ | $< 0.7, 0.5, 0.3 >$ |
| $x_3$ | $< 0.6, 0.5, 0.7 >$ | $< 0.7, 0.3, 0.1 >$ |

Then $(W,U)$ is a $12-NSSO$ set.

$(W,U) \subset cl_{\tau_1}(E,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.4, 0.7, 0.1 >$ | $< 0.8, 0.3, 0.2 >$ |
| $x_2$ | $< 0.5, 0.9, 0.2 >$ | $< 0.7, 0.4, 0.5 >$ |
| $x_3$ | $< 0.7, 0.7, 0.3 >$ | $< 0.8, 0.2, 0.1 >$ |

Then $(W,U)$ is also a $21 - NSSO$ set.

$cl_{\tau_1}(W,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.4, 0.9, 0.2 >$ | $< 0.7, 0.9, 0.6 >$ |
| $x_2$ | $< 0.4, 0.9, 0.2 >$ | $< 0.7, 0.9, 0.5 >$ |
| $x_3$ | $< 0.7, 0.7, 0.3 >$ | $< 0.4, 0.9, 0.5 >$ |

$cl_{\tau_1}(W,U) =$

| $X$ | $e_1$ | $e_2$ |
|---|---|---|
| $x_1$ | $< 0.4, 0.9, 0.2 >$ | $< 0.7, 0.9, 0.6 >$ |
| $x_2$ | $< 0.4, 0.9, 0.2 >$ | $< 0.7, 0.9, 0.5 >$ |
| $x_3$ | $< 0.7, 0.7, 0.3 >$ | $< 0.4, 0.9, 0.5 >$ |

**Definition 4.3.** Let $(X, E, \tau_1, \tau_2)$ be a neutrosophic soft bitopological space and $(N, E) \in NSS(X, E)$. Then $(N, E)$ is called as

1) $ij-$ neutrosophic soft preclosed $(ij - NSPC)$ if $(N, E)^c$ is a $ij - NSPO$ set. Equivalently $(N, E)$ is called as $ij - NSPC$ if $(N, E) \supset cl_{\tau_i}\left(int_{\tau_j}(N, E)\right)$
2) $ij-$ neutrosophic soft semi-closed $(ij - NSSC)$ if $(N, E)^c$ is a $ij - NSSO$ set. Equivalently $(N, E)$ is called as $ij - NSSC$ if $(N, E) \supset int_{\tau_j}\left(cl_{\tau_i}(N, E)\right)$
3) $ij-$ neutrosophic soft b-closed $(ij - NSbC)$ $(N, E)^c$ is a $ij - NSbC$ set. Equivalently $(N, E)$ is called as $ij - NSbC$ if $(N, E) \supset int_{\tau_i}\left(cl_{\tau_j}(N, E)\right) \cap cl_{\tau_j}\left(int_{\tau_i}(N, E)\right)$.
4) $ij-$ neutrosophic soft $\beta$-closed $(ij - NS\beta C)$ $(N, E)^c$ is a $ij - NS\beta O$ set. Equivalently $(N, E)$ is called as $ij - NS\beta C$ if $(N, E) \supset int_{\tau_j}\left(cl_{\tau_i}(int_{\tau_j}(N, E))\right)$.

**Theorem 4.4.** *Let $(X, E, \tau_1, \tau_2)$ be a neutrosophic soft bitopological space and $(N, E) \in NSS(X, E)$. If $(N, E) \in \tau_j^c$ and $ij - NSPO$ then $(N, E)$ is a $ij - NSSO$ set.*

*Proof.* Let $(N, E) \in \tau_j^c$ and $ij - NSPO$. Then $(N, E) = cl_{\tau_j}(N, E)$ and $(N, E) \subset int_{\tau_i}\left(cl_{\tau_j}(N, E)\right)$. Therefore $(N, E) \subset int_{\tau_i}(N, E) \subset cl_{\tau_j}\left(int_{\tau_i}(N, E)\right)$. □

**Theorem 4.5.** *Let $(X, E, \tau_1, \tau_2)$ be a neutrosophic soft bitopological space and $(N, E) \in NSS(X, E)$. If $(N, E) \in \tau_j^c$ and $ij - NSPO$ then $(N, E)$ is a $ij - NSSO$ set.*

*Proof.* Let $(N,E)$ be $ij-NSPO$. Then $(N,E) \subset int_{\tau_i}\left(cl_{\tau_j}(N,E)\right)$. Since $(N,E) \in \tau_j^c$, then $(N,E) = cl_{\tau_j}(N,E)$. Therefore $(N,E) \subset int_{\tau_i}(N,E) \subset cl_{\tau_j}(int_{\tau_i}(N,E))$. □

**Theorem 4.6.** *Let $(X,E,\tau_1,\tau_2)$ be a neutrosophic soft bitopological space and $(N,E),(M,E) \in NSS(X,E)$. If $(N,E)$ is $ij-NSPO$ and $(M,E) \in \tau_1 \cap \tau_2$ then $(N,E) \cup (M,E)$ is $ij-NSPO$.*

*Proof.* Let $(N,E)$ is $ij-NSPO$ and $(M,E) \in \tau_1 \cap \tau_2$. Then $(N,E) \subset int_{\tau_i}\left(cl_{\tau_j}(N,E)\right)$ and $int_{\tau_i}(M,E) = (M,E)$. So

$$\begin{aligned}(N,E)\cup(M,E) &\subset int_{\tau_i}\left(cl_{\tau_j}(N,E)\right) \cup int_{\tau_i}(M,E)\\ &\subset int_{\tau_i}\left(cl_{\tau_j}(N,E)\cup(M,E)\right) \subset int_{\tau_i}\left(cl_{\tau_j}(N,E)\cup cl_{\tau_j}(M,E)\right)\\ &= int_{\tau_i}\left(cl_{\tau_j}\left((N,E)\cup(M,E)\right)\right).\end{aligned}$$

Therefore $(N,E)\cup(M,E)$ is $ij-NSPO$. □

**Theorem 4.7.** *Let $(X,E,\tau_1,\tau_2)$ be a neutrosophic soft bitopological space. Then*

1) *Every $ij-NSPO$ set is $ji-NSbO$.*
2) *Every $ij-NSSO$ set is $ji-NSbO$.*
3) *Every $ij-NSSO$ set is $ij-NS\beta O$.*

*Proof.*

1) Let $(N,E) \in NSS(X,E)$ be $ij-NSPO$ set. Then $(N,E) \subset int_{\tau_i}\left(cl_{\tau_j}(N,E)\right) \subset cl_{\tau_j}(int_{\tau_i}(N,E)) \cup int_{\tau_i}\left(cl_{\tau_j}(N,E)\right)$.
2) Let $(N,E)$ be a $ij-NSSO$ set. Then $(N,E) \subset cl_{\tau_j}(int_{\tau_i}(N,E)) \subset cl_{\tau_j}(int_{\tau_i}(N,E)) \cup int_{\tau_i}\left(cl_{\tau_j}(N,E)\right)$.
3) Let $(N,E)$ be a $ij-NSSO$ set. Then since $(N,E) \subset cl_{\tau_j}(N,E)$,

$$(N,E) \subset cl_{\tau_j}(int_{\tau_i}(N,E)) \subset cl_{\tau_j}\left(int_{\tau_i}(cl_{\tau_j}(N,E))\right).$$

□

**Theorem 4.8.** *Let $(X,E,\tau_1,\tau_2)$ be a neutrosophic soft bitopological space. Then*

1) *Union of any $ij-NSPO$ set is $ij-NSPO$.*
2) *Union of any $ij-NSSO$ set is $ij-NSSO$.*
3) *Union of any $ij-NSbO$ set is $ij-NSbO$.*
4) *Union of any $ij-NS\beta O$ set is $ij-NS\beta O$.*
5) *Intersection of any $ij-NSPC$ set is $ij-NSPC$.*
6) *Intersection of any $ij-NSSO$ set is $ij-NSSO$.*

7) *Intersection of any* $ij-NSbO$ *set is* $ij-NSbC$.

8) *Intersection of any* $ij-NS\beta O$ *set is* $ij-NS\beta C$.

*Proof.*

1) Let $(N_k, E)$ be $ij-NSPO$ set in $(X, E, \tau_1, \tau_2)$ for all $k \in I$. Then

$$\bigcup_{k\in I}(N_k,E) \subset \bigcup_{k\in I} int_{\tau_i}\left(cl_{\tau_j}(N_k,E)\right) \subset int_{\tau_i}\left(\bigcup_{k\in I}\left(cl_{\tau_j}(N_k,E)\right)\right) = int_{\tau_i}\left(cl_{\tau_j}\left(\bigcup_{k\in I}(N_k,E)\right)\right)$$

2) Let $(N_k, E)$ be $ij-NSSO$ set in $(X, E, \tau_1, \tau_2)$ for all $k \in I$. Then

$$\bigcup_{k\in I}(N_k,E) \subset \bigcup_{k\in I} cl_{\tau_j}\left(int_{\tau_i}(N_k,E)\right) = cl_{\tau_j}\left(\bigcup_{k\in I}\left(int_{\tau_i}(N_k,E)\right)\right) \subset cl_{\tau_j}\left(int_{\tau_i}\left(\bigcup_{k\in I}(N_k,E)\right)\right)$$

3) Let $(N_k, E)$ be $ij-NSbO$ set in $(X, E, \tau_1, \tau_2)$ for all $k \in I$. Then

$$\begin{aligned}\bigcup_{k\in I}(N_k,E) &\subset \bigcup_{k\in I}\left(cl_{\tau_i}\left(int_{\tau_j}(N_k,E)\right)\cup int_{\tau_j}\left(cl_{\tau_i}(N_k,E)\right)\right)\\ &= \left(\bigcup_{k\in I} cl_{\tau_i}\left(int_{\tau_j}(N_k,E)\right)\right)\cup\left(\bigcup_{k\in I} int_{\tau_j}\left(cl_{\tau_i}(N_k,E)\right)\right)\\ &\subset cl_{\tau_i}\left(int_{\tau_j}\left(\bigcup_{k\in I}(N_k,E)\right)\right)\cup int_{\tau_j}\left(cl_{\tau_i}\left(\bigcup_{k\in I}(N_k,E)\right)\right)\end{aligned}$$

4) Let $(N_k, E)$ be $ij-NS\beta O$ set in $(X, E, \tau_1, \tau_2)$ for all $k \in I$. Then

$$\bigcup_{k\in I}(N_k,E) \subset \bigcup_{k\in I} cl_{\tau_j}\left(int_{\tau_i}(cl_{\tau_j}(N_k,E))\right) = cl_{\tau_j}\left(\bigcup_{k\in I} int_{\tau_i}(cl_{\tau_j}(N_k,E))\right) \subset cl_{\tau_j}\left(int_{\tau_i}(cl_{\tau_j}\left(\bigcup_{k\in I}(N_k,E)\right))\right)$$

The rest of the theorem can be proved easily by taking the complement of 1-4. □

## 5. Conclusion

In this paper, we defined neutrosophic soft supra closure operator in a neutrosphic soft bitopological space and investigated some properties of it. Then we obtained a neutrosophic soft topology with this closure operator. Also we defined neutrosophic soft supra interior operator and obtained a neutrosophic soft topology with this interior operator. In the section 4, we defined some new generalized open sets in neutrosophic soft bitopological spaces such as $ij-$ neutrosophic soft preopen, $ij-$ neutrosophic soft semi-open, $ij-$ neutrosophic soft b-open, $ij-$ neutrosophic soft $\beta$-open set. We examined the relationships between these newly defined open sets.

**Received: date / Accepted: date**

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Received: Nov 26, 2021. Accepted: Feb 3, 2022

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# Vaccine distribution technique under QSVN environment using different aggregation operator

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**Abstract**. In this article some Dombi operations on Quadripartitioned single valued neutrosophic (QSVN) set are studied. Later on some QSVN weighted Dombi operators i.e. QSVNWDA and QSVNWDG operators are introduced and their properties are studied. Finally a vaccine distribution technique is solved with the help of QSVNWDA operator and QSVNWDG operator.

**Keywords:**QSVN set; Dombi operation, QSVN weighted Dombi arithmetic (QSVNWDA) operator; QSVN weighted Dombi geometric (QSVNWDG) operator; MADM.

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## 1. **Introduction**

To deal with the inconsistent and uncertain data in a more powerful way, Smarandache introduced Neutrosophic set (NS) theory [2]. Gradually many developments on NS structure have been made by a couple of researchers and applied it to different branches of science [3–12]. An extension of SVN set i.e. QSVN set was further restudied in [13]. Based on QSVN set R.Chatterjee et. al introduced the idea of $\mathcal{QNN}$ number in 2009 [16]. On contrary Dombi [1] presented the operations of Dombi $T$-norms (DT) and $T$-conorms (DTC) in 1982. Both norms has a huge operational flexibility as a parameter. Many researchers extended the idea of Dombi norms to IFS [15], NS [14] theories and applied to different MADM problems [17–21]. In this paper we have applied Dombi norms on $\mathcal{QNN}$. Vaccine distribution in India will be a very difficult task for Government of India in the upcoming years. To overcome this difficulty we have defined a model method of vaccine distribution under QSVN environment using different aggregation operator. In Section 2 we have discussed some preliminary theories which will be used throughout the rest of the article. We have defined some order relations on $\mathcal{QNN}$ in Section 3. In Section 4 QSVNWDA and QSVNWDG operators are defined and their properties are studied. A MADM problem is solved on the basis of QSVNWDA and QSVNWDG operators in Section 5. Section 6 winds up the article.

## 2. **Some basics**

**Definition 2.1.** [13] A QSVN set $A$ over a set $X \neq \phi$ characterizes each element $x$ in $X$ by a truth-membership function $A_t$, a contradiction membership function $A_c$, an ignorance-membership function $A_u$ and a falsity membership function $A_f$ s.t. for each $x \in X$, $A_t(x), A_c(x), A_u(x), A_f(x) \in [0,1]$ and $0 \leq A_t(x) + A_c(x) + A_u(x) + A_f(x) \leq 4$.

**Definition 2.2.** [16] An element $\beta = \langle \beta_t, \beta_c, \beta_u, \beta_f \rangle \in [0,1]^4$ is said to be a QNN number. We express the collection of QNN numbers as $\mathcal{QNN}$.

**Definition 2.3.** [16] Consider $\mu, \nu, \omega \in \mathcal{QNN}$ and $i, j, k \in \mathbb{N}$. Then the following basic operations hold on $\mathcal{QNN}$:

(i) $\mu \bigoplus \nu = \langle \mu_t + \nu_t - \mu_t\nu_t, \mu_c + \nu_c - \mu_c\nu_c, \mu_u\nu_u, \mu_f\nu_f \rangle$,
(ii) $\mu \bigodot \nu = \langle \mu_t\nu_t, \mu_c\nu_c, \mu_u + \nu_u - \mu_u\nu_u, \mu_f + \nu_f - \mu_f\nu_f \rangle$,
(iii) $(\mu)^k = \langle (\mu_t)^k, (\mu_c)^k, 1 - (1 - \mu_u)^k, 1 - (1 - \mu_f)^k \rangle$,
(iv) $k\mu = \langle 1 - (1 - \mu_t)^k, 1 - (1 - \mu_c)^k, (\mu_u)^k, (\mu_f)^k \rangle$,

Both the above operations are commutative and associative on $\mathcal{QNN}$.

### 2.1. *DT and DTC*

**Definition 2.4.** [1] Suppose $r, s \in \mathbb{R}$. Then DT $(D(r,s))$ and DTC $(\overline{D}(r,s))$ between $r$ and $s$ are defined respectively as below:

$$D(r,s) = \frac{1}{1 + \{(\frac{1-r}{r})^{\varrho} + (\frac{1-s}{s})^{\varrho}\}^{\frac{1}{\varrho}}}$$
$$\overline{D}(r,s) = \frac{1}{1 + \{(\frac{r}{1-r})^{\varrho} + (\frac{s}{1-s})^{\varrho}\}^{\frac{1}{\varrho}}},$$

where $\varrho \geq 1$ and $(p,q) \in [0,1] \times [0,1]$.

## 3. **Order relations on $\mathcal{QNN}$**

In this section we will first define some order relations of QNN based on newly introduced score functions and accuracy functions on $\mathcal{QNN}$.

**Definition 3.1.** The score function $S(\beta) : \mathcal{QNN} \to \mathbb{R}$ of $\beta = \langle \beta_t, \beta_c, \beta_u, \beta_f \rangle \in \mathcal{QNN}$ is defined as

$$S(\beta) = \frac{2 + \beta_t + \beta_c - \beta_u - \beta_f}{4}$$

The corresponding accuracy functions $H_i : \mathcal{QNN} \to \mathbb{R}, i = 1, 2, 3$ are defined as follows:

$$\begin{aligned} H_1(\beta) &= (\beta_t + \beta_c) - (\beta_u + \beta_f) \\ H_2(\beta) &= \frac{\beta_t - \beta_c}{2} \\ H_3(\beta) &= \frac{\beta_u - \beta_f}{2}. \end{aligned}$$

**Remark 3.2.** Now for any $\beta \in \mathcal{QNN}$, it follows that

(i) $0 \leq S(\beta) \leq 1$.
(ii) $-2 \leq H_1(\beta) \leq 2$.
(iii) $-0.5 \leq H_2(\beta) \leq 0.5$.
(iv) $-0.5 \leq H_3(\beta) \leq 0.5$.

**Definition 3.3.** Suppose $\beta, \gamma \in \mathcal{QNN}$. We define the order relation between any two $\beta, \gamma \in \mathcal{QNN}$ as follows:

(i) If $S(\beta) < S(\gamma)$, then $\beta \leq \gamma$.
(ii) If $S(\beta) = S(\gamma)$, then
  (a) $H_1(\beta) < H_1(\gamma) \Rightarrow \beta \leq \gamma$ else if
  (b) $H_1(\beta) = H_1(\gamma)$ with $H_2(\beta) < H_2(\gamma) \Rightarrow \beta \leq \gamma$ else if
  (c) $H_1(\beta) = H_1(\gamma), H_2(\beta) = H_2(\gamma)$ with $H_3(\beta) < H_3(\gamma) \Rightarrow \beta \leq \gamma$ else if
  (d) $H_1(\beta) = H_1(\gamma), H_2(\beta) = H_2(\gamma)$ and $H_3(\beta) = H_3(\gamma) \Rightarrow \beta = \gamma$.

Here $\beta \leq \gamma$ denotes $\beta$ proceeds $\gamma$.

### 3.1. *Some QSVN Dombi operations*

In this section we have discussed some QSVN Dombi operations [22].

**Definition 3.4.** Suppose $\alpha = \langle m_1, n_1, p_1, q_1 \rangle \in \mathcal{QNN}$ and $\beta = \langle m_2, n_2, p_2, q_1 \rangle \in \mathcal{QNN}$, $\varrho \geq 1$ and $k > 0$. Then the DT and DTC operations on $\mathcal{QNN}$ are defined as below:

(i)

$$\alpha \oplus \beta = \left\langle 1 - \frac{1}{1 + \left( (\frac{m_1}{1-m_1})^\varrho + (\frac{m_2}{1-m_2})^\varrho \right)^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left( (\frac{n_1}{1-n_1})^\varrho + (\frac{n_2}{1-n_2})^\varrho \right)^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left( (\frac{1-p_1}{p_1})^\varrho + (\frac{1-p_2}{p_2})^\varrho \right)^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left( (\frac{1-q_1}{q_1})^\varrho + (\frac{1-q_2}{q_2})^\varrho \right)^{\frac{1}{\varrho}}} \right\rangle$$

(ii)

$$\alpha \odot \beta = \left\langle 1 - \frac{1}{1 + \left( (\frac{1-m_1}{m_1})^\varrho + (\frac{1-m_2}{m_2})^\varrho \right)^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left( (\frac{1-n_1}{n_1})^\varrho + (\frac{1-n_2}{n_2})^\varrho \right)^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left( (\frac{p_1}{1-p_1})^\varrho + (\frac{p_2}{1-p_2})^\varrho \right)^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left( (\frac{q_1}{1-q_1})^\varrho + (\frac{q_2}{1-q_2})^\varrho \right)^{\frac{1}{\varrho}}} \right\rangle$$

(iii) $k\alpha=\left\langle 1-\frac{1}{1+\left(k(\frac{m_1}{1-m_1})^{\varrho}\right)^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left(k(\frac{n_1}{1-n_1})^{\varrho}\right)^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left(k(\frac{1-p_1}{p_1})^{\varrho}\right)^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left(k(\frac{1-q_1}{q_1})^{\varrho}\right)^{\frac{1}{\varrho}}}\right\rangle$,

(iv) $\alpha^k=\left\langle 1-\frac{1}{1+\left(k(\frac{1-m_1}{m_1})^{\varrho}\right)^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left(k(\frac{1-n_1}{n_1})^{\varrho}\right)^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left(k(\frac{p_1}{1-p_1})^{\varrho}\right)^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left(k(\frac{q_1}{1-q_1})^{\varrho}\right)^{\frac{1}{\varrho}}}\right\rangle$.

## 4. Dombi weighted aggregation operators on $\mathcal{QNN}$

**Definition 4.1.** Let $\beta_j = \langle m_j, n_j, p_j, q_j\rangle$ $(j = 1, 2, \ldots, l)$ be a collection on $\mathcal{QNN}$. A QSVN weighted Dombi arithmetic (QSVNWDA) operator of dimension $l$ is a function $f : \mathcal{QNN}^l \rightarrow \mathcal{QNN}$ defined by:

$$f(\beta_1, \beta_2, \ldots, \beta_l) = \bigoplus_{j=1}^{l} \omega_j \beta_j$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_l)^T$ is the weight vector, $\omega_j$ is attached with $\beta_j, j = 1, 2, \ldots, l$ with $0 \leq \omega_j \leq 1$ and $\sum_{j=1}^{l} \omega_j = 1$.

**Theorem 4.2.** *Suppose $\beta_j = \langle m_j, n_j, p_j, q_j\rangle$ $(j = 1, 2, \ldots, l)$ be a collection on $\mathcal{QNN}$ along weight vector $\omega$. Then*

$$f(\beta_1, \beta_2, \ldots, \beta_l) = \bigoplus_{j=1}^{l} \omega_j \beta_j$$

$$=\left\langle 1-\frac{1}{1+\left\{\sum_{j=1}^{l}\omega_j\left(\frac{m_j}{1-m_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{j=1}^{l}\omega_j\left(\frac{n_j}{1-n_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{j=1}^{l}\omega_j\left(\frac{1-p_j}{p_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{j=1}^{l}\omega_j\left(\frac{1-q_j}{q_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\right\rangle.$$

*Proof.* Here $\omega_1 \in \omega$ and $\beta_1 \in \mathcal{QNN}$. Now we have $\omega_1\beta_1 = \left\langle 1-\frac{1}{1+\left\{\left(\frac{m_1}{1-m_1}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\left(\frac{n_1}{1-n_1}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\left(\frac{1-p_1}{p_1}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\left(\frac{1-q_1}{q_1}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\right\rangle$. Hence the above equation trivially holds for $l = 1$. In a parallel way $\omega_2 \in \omega$ and $\beta_2 \in \mathcal{QNN}$ and we have $\omega_2\beta_2 = \left\langle 1-\frac{1}{1+\left\{\left(\frac{m_2}{1-m_2}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\left(\frac{n_2}{1-n_2}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\left(\frac{1-p_2}{p_2}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\left(\frac{1-q_2}{q_2}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\right\rangle$.
Therefore

$$f(\beta_1, \beta_2) = \omega_1\beta_1 \bigoplus \omega_2\beta_2$$

$$=\left\langle 1-\frac{1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{m_j}{1-m_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{n_j}{1-n_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{1-p_j}{p_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{1-q_j}{q_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\right\rangle.$$

Hence the equation is valid for $l = 1, 2$. We assume that the equation is valid for $l = s$ i.e.

$$f(\beta_1, \beta_2, \ldots, \beta_s) = \bigoplus_{j=1}^{s} \omega_j \beta_j$$

$$=\left\langle 1-\frac{1}{1+\left\{\sum_{j=1}^{s}\omega_j\left(\frac{m_j}{1-m_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{j=1}^{s}\omega_j\left(\frac{n_j}{1-n_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{j=1}^{s}\omega_j\left(\frac{1-p_j}{p_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1-\frac{1}{1+\left\{\sum_{j=1}^{s}\omega_j\left(\frac{1-q_j}{q_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\right\rangle.$$

Finally for $l = s+1$, one can easily see that

$$f(\beta_1, \beta_2, \ldots, \beta_s) = \bigoplus_{j=1}^{s} \omega_j \beta_j \bigoplus \omega_{s+1}\beta_{s+1}$$

$$= \left\langle 1 - \frac{1}{1+\left\{\sum_{j=1}^{s} \omega_j \left(\frac{m_j}{1-m_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{j=1}^{s} \omega_j \left(\frac{n_j}{1-n_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{j=1}^{s} \omega_j \left(\frac{1-p_j}{p_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{j=1}^{s} \omega_j \left(\frac{1-q_j}{q_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right\rangle \oplus \omega_{s+1}\beta_{s+1}$$

$$= \left\langle 1 - \frac{1}{1+\left\{\sum_{j=1}^{s+1} \omega_j \left(\frac{m_j}{1-m_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{j=1}^{s+1} \omega_j \left(\frac{n_j}{1-n_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{j=1}^{s+1} \omega_j \left(\frac{1-p_j}{p_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{j=1}^{s+1} \omega_j \left(\frac{1-q_j}{q_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right\rangle.$$

Thus in general the equation

$$f(\beta_1, \beta_2, \ldots, \beta_l) = \bigoplus_{j=1}^{l} \omega_j \beta_j$$

$$= \left\langle 1 - \frac{1}{1+\left\{\sum_{j=1}^{l} \omega_j \left(\frac{m_j}{1-m_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{j=1}^{l} \omega_j \left(\frac{n_j}{1-n_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{j=1}^{l} \omega_j \left(\frac{1-p_j}{p_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1+\left\{\sum_{j=1}^{l} \omega_j \left(\frac{1-q_j}{q_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}} \right\rangle.$$

holds $\forall l \in \mathbb{N}$. □

**Theorem 4.3.** *The QSVNWDA operator $f$ satisfies the following properties:*

(i) *Consistency:* $f(\beta_1, \beta_2, \ldots, \beta_l) \in \mathcal{QNN}$.

(ii) *Idempotency:* $f(\beta, l \text{ times } \ldots, \beta) = \beta$.

(iii) *Commutativity:* $f(\beta_1, \beta_2, \ldots, \beta_l) = f(\beta_l, \beta_{l-1}, \ldots, \beta_1)$.

(iv) $f(\beta_{\pi(1)}, \beta_{\pi(2)}, \ldots, \beta_{\pi(l)}) = f(\beta_1, \beta_2, \ldots, \beta_l)$ *where $\pi$ is a permutation on* $\{1, 2, \ldots, l\}$.

*Proof.* The proof of consistency and commutativity properties of QSVNWDA operator is quite easy. We now proceed to prove the part (ii). Since $\sum_{j=1}^{l} \omega_j = 1$, thus

$$f(\beta, l \text{ times} \ldots, \beta) = \bigoplus_{j=1}^{l} \omega_j \beta_j = (\sum_{j=1}^{l} \omega_j)\beta = \beta.$$

Finally consider$\pi$ as a permutation on $\{1, 2, \ldots, l\}$. Now due to additive commutativity in $\mathcal{QNN}$

$$f(\beta_{\pi(1)}, \beta_{\pi(2)}, \ldots, \beta_{\pi(l)}) = \bigoplus_{j=1}^{l} \omega(\beta_{\pi(j)})\beta_{\pi(j)} = \bigoplus_{j=1}^{l} \omega(\beta_j)\beta_j \quad = f(\beta_1, \beta_2, \ldots, \beta_l).$$

Hence we are done. □

**Theorem 4.4.** *Consider $\beta_j = \langle m_j, n_j, p_j, q_j \rangle, j = 1, 2, \ldots, l$ and $\gamma_j = \langle \widetilde{m_j}, \widetilde{n_j}, \widetilde{p_j}, \widetilde{q_j} \rangle, j = 1, 2, \ldots, l)$ are two collections on $\mathcal{QNN}$ such that $m_j \leq \widetilde{m_j}, n_j \leq \widetilde{n_j}, p_j \geq \widetilde{p_j}, q_j \geq \widetilde{q_j} \forall j$. Then $f(\beta_1, \beta_2, \ldots, \beta_l) \leq f(\gamma_1, \gamma_2, \ldots, \gamma_l)$.*

*Proof.* Here,

$$f(\beta_1,\beta_2,\ldots,\beta_l)=\bigoplus_{j=1}^{l}\omega_j\beta_j$$

$$=\left\langle 1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{m_j}{1-m_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}},1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{n_j}{1-n_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}},1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{1-p_j}{p_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}},1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{1-q_j}{q_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\right\rangle.$$

$$f(\gamma_1,\gamma_2,\ldots,\gamma_l)=\bigoplus_{j=1}^{l}\omega_j\gamma_j$$

$$=\left\langle 1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{\widetilde{m_j}}{1-\widetilde{m_j}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}},1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{\widetilde{n_j}}{1-\widetilde{n_j}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}},1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{1-\widetilde{p_j}}{\widetilde{p_j}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}},1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{1-\widetilde{q_j}}{\widetilde{q_j}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}\right\rangle.$$

Firstly we suppose that $m_j<\widetilde{m_j}, n_j<\widetilde{n_j}, p_j>\widetilde{p_j}, q_j>\widetilde{q_j}\forall j\in\{1,\ldots,l\}$. Then

$$1-m_j>1-\widetilde{m_j}\forall j\in\{1,\ldots,l\}$$
$$\Rightarrow\sum_{j=1}^{l}\omega_j\left(\frac{m_j}{1-m_j}\right)<\sum_{j=1}^{l}\omega_j\left(\frac{\widetilde{m_j}}{1-\widetilde{m_j}}\right)$$
$$\Rightarrow 1+\left\{\sum_{j=1}^{l}\omega_j\left(\frac{m_j}{1-m_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}<1+\left\{\sum_{j=1}^{l}\omega_j\left(\frac{\widetilde{m_j}}{1-\widetilde{m_j}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}$$
$$\Rightarrow\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{m_j}{1-m_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}>\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{\widetilde{m_j}}{1-\widetilde{m_j}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}$$
$$\Rightarrow 1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{m_j}{1-m_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}<1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{\widetilde{m_j}}{1-\widetilde{m_j}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}.$$

In a similar way we have

$$1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{n_j}{1-n_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}<1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{\widetilde{n_j}}{1-\widetilde{n_j}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}.$$

Conversely we can easily see that

$$1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{1-p_j}{p_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}>1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{1-\widetilde{p_j}}{\widetilde{p_j}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}.$$

$$1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{1-q_j}{q_j}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}>1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\omega_j\left(\frac{1-\widetilde{q_j}}{\widetilde{q_j}}\right)^{\varrho}\right\}^{\frac{1}{\varrho}}}.$$

Combining all the above we get

$$S(f(\beta_1,\beta_2,\ldots,\beta_l))<S(f(\gamma_1,\gamma_2,\ldots,\gamma_l)).$$

Hence $f(\beta_1, \beta_2, \ldots, \beta_l) < f(\gamma_1, \gamma_2, \ldots, \gamma_l)$. Now if $m_j = \widetilde{m_j}, n_j = \widetilde{n_j}, p_j = \widetilde{p_j}, q_j = \widetilde{q_j}$ $\forall j \in \{1, \ldots, l\}$. Then all the equalities as well as the score functions become equal i.e. $S(f(\beta_1, \beta_2, \ldots, \beta_l)) = S(f(\gamma_1, \gamma_2, \ldots, \gamma_l))$. Finally $f(\beta_1, \beta_2, \ldots, \beta_l) \leq f(\gamma_1, \gamma_2, \ldots, \gamma_l)$. □

**Theorem 4.5.** *Consider a collection of* $\beta_j = \langle m_j, n_j, p_j, q_j \rangle, j = 1, 2, \ldots, l$ *in* $\mathcal{QNN}$. *Then*

$$\underline{\beta} \leq f(\beta_1, \beta_2, \ldots, \beta_l) \leq \overline{\beta}, \textit{where}$$
$$\underline{\beta} = \langle \min_j(m_j), \min_j(n_j), \min_j(p_j), \min_j(q_j) \rangle = \langle \underline{m_j}, \underline{n_j}, \underline{p_j}, \underline{q_j} \rangle \textit{ and}$$
$$\overline{\beta} = \langle \max_j(m_j), \max_j(n_j), \max_j(p_j), \max_j(q_j) \rangle = \langle \overline{m_j}, \overline{n_j}, \overline{p_j}, \overline{q_j} \rangle.$$

*Proof.* From Definition of $\mathcal{QNN}$ we have $\forall j = \{1, 2, \ldots, l\}$,

$$\underline{m_j} \leq m_j, \underline{n_j} \leq n_j, \underline{p_j} \geq p_j, \underline{q_j} \geq q_j \text{ and}$$
$$m_j \leq \overline{m_j}, n_j \leq \overline{n_j}, p_j \geq \overline{p_j}, q_j \geq \overline{q_j} \text{ and}$$

Then

$$f(\underline{\beta}, l\ times, \underline{\beta}) \leq f(\beta_1, \beta_2, \ldots, \beta_l) \leq f(\overline{\beta}, l\ times, \overline{\beta}), i.e$$
$$\underline{\beta} \leq f(\beta_1, \beta_2, \ldots, \beta_l) \leq \overline{\beta}.$$

□

**Definition 4.6.** Suppose $\beta_j = \langle m_j, n_j, p_j, q_j \rangle$, $(j = 1, 2, \ldots, l)$ be a collection on $\mathcal{QNN}$. Then from Definition 4.1 a QSVNWDA operator $f$ of dimension $l$ can be written as follows

$$f(\beta_1, \beta_2, \ldots, \beta_l) = \bigoplus_{j=1}^{l} \omega_j \beta_j$$

Now if $\omega_j = \frac{1}{l}$ $\forall\, j \in \{1, 2, \ldots, l\}$ then

$$f(\beta_1, \beta_2, \ldots, \beta_l) = \frac{1}{l} \bigoplus_{j=1}^{l} \beta_j.$$

In that case $f(\beta_1, \beta_2, \ldots, \beta_l)$ is called average QSVNWDA operator i.e.QSVNWADA operator of $\beta_j = \langle m_j, n_j, p_j, q_j \rangle$ $(j = 1, 2, \ldots, l)$.

**Definition 4.7.** Let $\beta_j = \langle m_j, n_j, p_j, q_j \rangle, j = 1, 2, \ldots, l)$ be a collection on $\mathcal{QNN}$. A quadripartioned single valued neutrosophic weighted Dombi geometric (QSVNWDG) operator of dimension $l$ is a function $g : \mathcal{QNN}^l \rightarrow \mathcal{QNN}$ defined by:

$$g(\beta_1, \beta_2, \ldots, \beta_l) = \bigodot_{j=1}^{l} \beta_j^{\omega_j}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_l)^T$ is the weight vector, $\omega_j$ is attached with $\beta_j, j = 1, 2, \ldots, l$ with $0 \leq \omega_j \leq 1$ and $\sum_{j=1}^{l} \omega_j = 1$.

**Theorem 4.8.** *Suppose $\beta_j = \langle m_j, n_j, p_j, q_j \rangle$ $(j = 1, 2, \ldots, l)$ be a collection on $\mathcal{QNN}$ along weight vector $\omega$. Then*

$$g(\beta_1, \beta_2, \ldots, \beta_l) = \bigoplus_{j=1}^{l} \beta_j^{\omega_j}$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \omega_j \left( \frac{1-m_j}{m_j} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \omega_j \left( \frac{1-n_j}{n_j} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \omega_j \left( \frac{p_j}{1-p_j} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \omega_j \left( \frac{q_j}{1-q_j} \right)^{\varrho} \right\}^{\frac{1}{\varrho}}} \right\rangle.$$

*Proof.* We have omitted it due to similarity with Theorem 4.2. □

**Theorem 4.9.** *The QSVNWDG operator $g$ satisfies properties as defined below:*

(i) *Consistency: $g(\beta_1, \beta_2, \ldots, \beta_l) \in \mathcal{QNN}$.*
(ii) *Idempotency: $g(\beta, l \text{ times } \ldots, \beta) = \beta$.*
(iii) *Commutativity: $g(\beta_1, \beta_2, \ldots, \beta_l) = g(\beta_l, \beta_{l-1}, \ldots, \beta_1)$.*
(iv) *$g(\beta_{\pi(1)}, \beta_{\pi(2)}, \ldots, \beta_{\pi(l)}) = g(\beta_1, \beta_2, \ldots, \beta_l)$ where $\pi$ is a permutation on $\{1, 2, \ldots, l\}$.*

*Proof.* We have omitted it due to similarity with Theorem 4.3. □

**Theorem 4.10.** *Consider $\beta_j = \langle m_j, n_j, p_j, q_j \rangle, j = 1, 2, \ldots, l)$ and $\gamma_j = \langle \widetilde{m_j}, \widetilde{n_j}, \widetilde{p_j}, \widetilde{q_j} \rangle$ $(j = 1, 2, \ldots, l)$ are two collections on $\mathcal{QNN}$ such that $m_j \leq \widetilde{m_j}, n_j \leq \widetilde{n_j}, p_j \geq \widetilde{p_j}, q_j \geq \widetilde{q_j} \forall j$. Then $g(\beta_1, \beta_2, \ldots, \beta_l) \leq g(\gamma_1, \gamma_2, \ldots, \gamma_l)$.*

*Proof.* Here the proof is similar with Theorem 4.4, hence we have omitted it. □

**Theorem 4.11.** *Consider a collection of $\beta_j = \langle m_j, n_j, p_j, q_j \rangle, j = 1, 2, \ldots, l$ in $\mathcal{QNN}$. Then*

$$\underline{\beta} \leq g(\beta_1, \beta_2, \ldots, \beta_l) \leq \overline{\beta}, \text{ where}$$
$$\underline{\beta} = \langle \min_j(m_j), \min_j(n_j), \min_j(p_j), \min_j(q_j) \rangle = \langle \underline{m_j}, \underline{n_j}, \underline{p_j}, \underline{q_j} \rangle \text{ and}$$
$$\overline{\beta} = \langle \max_j(m_j), \max_j(n_j), \max_j(p_j), \max_j(q_j) \rangle = \langle \overline{m_j}, \overline{n_j}, \overline{p_j}, \overline{q_j} \rangle.$$

*Proof.* Again proof is not done due to its similarity with Theorem 4.5. □

**Definition 4.12.** Suppose $\beta_j = \langle m_j, n_j, p_j, q_j \rangle, j = 1, 2, \ldots, l)$ be a collection on $\mathcal{QNN}$. Then from Definition 4.7 a QSVNWDG operator $g$ of dimension $l$ can be written as follows

$$g(\beta_1, \beta_2, \ldots, \beta_l) = \bigodot_{j=1}^{l} \beta_j^{\omega_j}$$

Now if $\omega_j = \frac{1}{l}\ \forall\ j \in \{1, 2, \ldots, l\}$ then

$$g(\beta_1, \beta_2, \ldots, \beta_l) = (\bigodot_{j=1}^{l} \beta_j)^{\frac{1}{l}}.$$

In that case $g(\beta_1, \beta_2, \ldots, \beta_l)$ is called average QSVNWDG operator i.e.QSVNWADG operator of $\beta_j = \langle m_j, n_j, p_j, q_j \rangle$ $(j = 1, 2, \ldots, l)$.

## 5. An application in MADM of QSVNWDA and QSVNWDG operator

Without an application in real life it is very tough to realize the utility of any operator. A reader can not get any interest if the operators cannot be used properly in MADM technique. For this reason we proposed a model with the help of QSVNWDA and QSVNWDG operator. Suppose Govt. of India want to distribute the Covid-19 vaccine in a smooth manner so that every Indian will get the vaccine. Now Govt of India has 4 vaccine $v_i, i = 1, 2, 3, 4$ in hand where $v_1$: the co-vaxin from Bharat Bio-tech, $v_2$: Sputnik-V from Russia, $v_3$: Astrazeneca vaccine from Oxford university, $v_4$: Pfizer vaccine from USA with equal storage. However there are four attributes $C_j, j =, 2, 3, 4$ which are to be considered for choosing a particular vaccine from the above list i.e. $(C_1)$ : the cost of the vaccine, $(C_2)$ : the effectiveness of a vaccine in human body, $(C_3)$: the rate of production of a vaccine $(C_4)$: the risk factor of a particular vaccine. In order to get a suitable vaccine $V_i$ after consideration of all attributes $C_j$ we have represented these MADM problems in the form of a decision making matrix $D(v_{ij})$ on $\mathcal{QNN}$ as following:

$$D(v_{ij}) = \begin{bmatrix} \langle 0.4, 0.5, 0.2, 0.6 \rangle & \langle 0.5, 0.5, 0.8, 0.1 \rangle & \langle 0.2, 0.6, 0.3, 0.2 \rangle & \langle 0.6, 0.5, 0.6, 0.7 \rangle \\ \langle 0.4, 0.2, 0.7, 0.6 \rangle & \langle 0.8, 0.5, 0.3, 0.4 \rangle & \langle 0.4, 0.1, 0.1, 0.1 \rangle & \langle 0.6, 0.6, 0.5, 0.5 \rangle \\ \langle 0.4, 0.4, 0.4, 0.5 \rangle & \langle 0.3, 0.6, 0.1, 0.4 \rangle & \langle 0.9, 0.2, 0.7, 0.3 \rangle & \langle 0.4, 0.2, 0.1, 0.1 \rangle \\ \langle 0.1, 0.1, 0.6, 0.3 \rangle & \langle 0.5, 0.3, 0.4, 0.2 \rangle & \langle 0.4, 0.8, 0.3, 0.2 \rangle & \langle 0.4, 0.5, 0.1, 0.5 \rangle \end{bmatrix}.$$

Here we consider the weight vector as $(0.25, 0.25, 0.25, 0.25)$ since every vaccine has equal stock.

**Case-I:** We now consider the QSVNWDA operator to face the MADM problem. In that case we consider $\varrho = 1$ and derive the collection of QSVNs say $v_i$ to find suitable vaccine among $V_i(i = 1, 2, 3, 4)$ by the help of Definition 4.1 as follows:

$$v_1 = \langle 0.460, 0.529, 0.644, 0.779 \rangle$$

$$v_2 = \langle 0.630, 0.417, 0.761, 0.753 \rangle$$

$$v_3 = \langle 0.319, 0.400, 0.833, 0.775 \rangle$$

$$v_1 = \langle 0.164, 0.341, 0.192, 0.538 \rangle.$$

Based on the Definition 3.1 we have:

$$S(v_1) = 0.392, S(v_2) = 0.384, S(v_3) = 0.3801, S(v_4) = 0.3624.$$

Hence the priority order of vaccine is $v_1 > v_2 > v_3 > v_4$. .

**Case-II:** Now We consider the QSVNWDG operator to face our problem. We again consider $\varrho = 1$ and derive the collective QSVNs $v_i$ with the help of Definition 4.7 as follows:

$$v_1 = \langle 0.5102, 0.4782, 0.607, 0.512 \rangle$$
$$v_2 = \langle 0.657, 0.785, 0.492, 0.4503 \rangle$$
$$v_3 = \langle 0.576, 0.718, 0.446, 0.355 \rangle$$
$$v_1 = \langle 0.739, 0.758, 0.403, 0.628 \rangle.$$

Again based on the Definition 3.1 we get:

$$S(v_1) = 0.935, S(v_2) = 0.625, S(v_3) = 0.6231, S(v_4) = 0.616.$$

Therefore the priority order of vaccine is $v_4 < v_3 < v_2 < v_1$. To find the more effect of the quantity $\varrho$ in the QSVNWDA and QSVNWDG operator we take the value of $\varrho$ in an increasing order starting from 0.2 to 1 with an increment 0.2. Our results are given in the following tables:

Table of QSVNWDA operator

| $\varrho$ | $S(v_1), S(v_2), S(v_3), S(v_4)$ | *Order of priority* |
|---|---|---|
| 0.2 | $0.627, 0.606, 0.598, 0.377$ | |
| 0.4 | $0.635, 0.621, 0.612, 0.396$ | |
| 0.6 | $0.676, 0.648, 0.639, 0.404$ | $v_4 < v_3 < v_2 < v_1$ |
| 0.8 | $0.695, 0.664, 0.652, 0.418$ | |
| 1.0 | $0.692, 0.678, 0.664, 0.444$ | |

Table of QSVNWDG operator

| $\varrho$ | $S(v_4), S(v_3), S(v_2), S(v_1)$ | *Order of priority* |
|---|---|---|
| 0.2 | $0.429, 0.492, 0.541, 0.568$ | |
| 0.4 | $0.417, 0.476, 0.525, 0.549$ | |
| 0.6 | $0.411, 0.449, 0.484, 0.502$ | $v_4 < v_3 < v_2 < v_1$ |
| 0.8 | $0.426, 0.442, 0.474, 0.491$ | |
| 1.0 | $0.394, 0.410, 0.439, 0.462$ | |

In both of the above cases we have seen that in respect of the values of $\varrho$, the order of priority of the vaccines remains always same for an individual operator. Thus the MADM of finding suitable vaccine using the QSVNWDA operator as well as QSVNWDG operator gives us a flexibility of choosing the value of $\varrho$. Thus the Govt of India will choose the vaccine $v_1$ in topmost priority.The above procedure help our Govt to choose a multi-solution based on the current situation at that time.

## 6. Conclusion

In this article two aggregation operators i.e. QSVNWDA and QSVNWDG operator based on Dombi operations on $\mathcal{QNN}$ sets are introduced. We have studied the properties of QSVNWDA and QSVNWDG operators. Finally we have solved a MADM problems using QSVNWDA and QSVNWDG operators. In solving MADM problems we have utilized the score functions of $\mathcal{QNN}$ to finding the order of priority of different parameters. Also we have seen that different large values of $\varrho$ may effect the score functions. In future we will develop more advanced type of QSVNWDA operator and QSVNWDG operator on $\mathcal{QNN}$ and will apply them to real life MADM problems.

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Received: Nov 29, 2021. Accepted: Feb 4, 2022

NSS

Neutrosophic Sets and Systems, Vol. 48, 2022

University of New Mexico

# $(\alpha, \beta)$ Neutrosophic Subbisemiring of Bisemiring

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**Abstract**. We introduce the notion of neutrosophic subbisemiring(shortly NSBS), level sets of NSBS and neutrosophic normal subbisemiring(NNSBS) of a bisemiring. The concept of neutrosophic subbisemiring is a new generalization of fuzzy subbisemiring over bisemiring. We interact the theory for $(\alpha, \beta)$ NSBS and NNSBS over bisemiring. Let $A$ be the neutrosophic subset in $\mathbb{S}$, we show that $\tilde{\varpi} = (\varpi_A^T, \varpi_A^I, \varpi_A^F)$ is an NSBS of $\mathbb{S}$ if and only if all non empty level set $\tilde{\varpi}^{(t,s)}$ is a subbisemiring of $\mathbb{S}$ for $t, s \in [0, 1]$. Let $A$ be the NSBS of a bisemiring $\mathbb{S}$ and $V$ be the strongest neutrosophic relation of $\mathbb{S}$, we observe that $A$ is an NSBS of $\mathbb{S}$ if and only if $V$ is an NSBS of $\mathbb{S} \times \mathbb{S}$. Let $A_1, A_2, ..., A_n$ be the family of $NSBS^s$ of $\mathbb{S}_1, \mathbb{S}_2, ..., \mathbb{S}_n$ respectively. We show that $A_1 \times A_2 \times ... \times A_n$ is an NSBS of $\mathbb{S}_1 \times \mathbb{S}_2 \times ... \times \mathbb{S}_n$. The homomorphic image of NSBS is an NSBS. The homomorphic preimage of NSBS is an NSBS. Examples are provided to illustrate our results.

**Keywords:** Neutrosophic subbisemiring; Neutrosophic bisemiring; Homomorphism; Normal.

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## 1. Introduction

The study of semirings was opened by the Dedekind in interaction with ideals of commutative rings. In 1934, semiring was studied by Vandever. It was basically the generalization of rings and distributive lattices. In 1950, However the developments of the theory in semirings had been taking place. The classic article of 1965, Zadeh proposed fuzzy set theory [15]. According to this definition a fuzzy set is a function described by a membership value . It takes degrees in real unit interval. But, later it has been seen that this definition is inadequate by considering not only the degree of membership but also the degree of non-membership. Neutrosophic set is a generalization of the fuzzy set and intuitionistic fuzzy set, where the truth-membership, indeterminacy-membership, and falsity-membership are represented independently. Atanassov [4] described a set that is called an intuitionistic fuzzy set to handle mentioned ambiguity. Since this set has some problems in applications, Smarandache [14] introduced neutrosophy to deal with the problems that involves indeterminate and inconsistent information. Arulmozhi interact the theory for various algebraic structures such semirings and ternary semirings [2,3]. A semiring $(S,+,\cdot)$ is a non-empty set in which $(S,+)$ and $(S,\cdot)$ are semigroups such that " $\cdot$ " is distributive over " $+$ " [6]. In 1993, J. Ahsan, K. Saifullah, and F. Khan [1] introduced the notion of fuzzy semirings. In 2001, M.K Sen and S. Ghosh were introduced in bisemirings. A bisemiring $(\mathbb{S},+,\circ,\times)$ is an algebraic structure in which $(\mathbb{S},+,\circ)$ and $(\mathbb{S},\circ,\times)$ are semirings in which $(\mathbb{S},+),(\mathbb{S},\circ)$ and $(\mathbb{S},\times)$ are semigroups such that (i) $x\circ(y+z)=(x\circ y)+(x\circ z)$, (ii) $(y+z)\circ x=(y\circ x)+(z\circ x)$ (iii) $x\times(y\circ z)=(x\times y)\circ(x\times z)$ and (iv) $(y\circ z)\times x=(y\times x)\circ(z\times x)$, $\forall\ x,y,z\in\mathbb{S}$ [13]. A non-empty subset $A$ of a bisemiring $(\mathbb{S},+,\circ,\times)$ is a subbisemiring if and only if $x+y\in A$, $x\circ y\in A$ and $x\times y\in A$ for all $x,y\in A$ [5]. Palanikumar et al. discussed various ideal structure of subbisemiring theory [7]- [12].

## 2. Preliminaries

**Definition 2.1.** [14] A neutrosophic set $A$ in a universe $U$ is an object having the form $A=\big\{\langle x,\varpi_A^T(x),\varpi_A^I(x),\varpi_A^F(x)\rangle : x\in X\big\}$, where $\varpi_A^T(x),\varpi_A^I(x),\varpi_A^F(x) : X\rightarrow[0,1]$ represents the truth-membership function , the indeterminacy membership function and the falsity-membership function respectively. For simplicity the symbol $\langle\varpi_A^T,\varpi_A^I,\varpi_A^F\rangle$ is used for the neutrosophic set $A=\big\{\langle x,\varpi_A^T(x),\varpi_A^I(x),\varpi_A^F(x)\rangle : x\in X\big\}$.

**Definition 2.2.** [14] Let $A=\{x,\varpi_A^T(x),\varpi_A^I(x),\varpi_A^F(x)\}$ and $B=\{x,\varpi_B^T(x),\varpi_B^I(x),\varpi_B^F(x)\}$ be the two neutrosophic set of a set $X$. Then
(i) $A\cap B=\Big\{\Big(x,min\{\varpi_A^T(x),\varpi_B^T(x)\},min\{\varpi_A^I(x),\varpi_B^I(x)\},max\{\varpi_A^F(x),\varpi_B^F(x)\}\Big)\Big|x\in X\Big\}$.
(ii) $A\cup B=\Big\{\Big(x,max\{\varpi_A^T(x),\varpi_B^T(x)\},max\{\varpi_A^I(x),\varpi_B^I(x)\},min\{\varpi_A^F(x),\varpi_B^F(x)\}\Big)\Big|x\in X\Big\}$.

**Definition 2.3.** [14] For any neutrosophic set $A=\{x,\varpi_A^T(x),\varpi_A^I(x),\varpi_A^F(x)\}$ of a set $X$, we defined a $(\alpha,\beta)$-cut of as the crisp subset $\{x\in X|\varpi_A^T(x)\geq\alpha,\varpi_A^I(x)\geq\alpha,\varpi_A^F(x)\leq\beta\}$ of $X$.

**Definition 2.4.** [14] Let $A$ and $B$ be be two neutrosophic subsets of $S$. The Cartesian product of $A$ and $B$ denoted by $A\times B$ is defined as $A\times B=\{\varpi_{A\times B}^T(x,y),\varpi_{A\times B}^I(x,y),\varpi_{A\times B}^F(x,y)|\ for\ all\ x,y\in S\}$, where

$$\left\{\begin{array}{l}\varpi_{A\times B}^T(x,y)=\min\{\varpi_A^T(x),\varpi_B^T(y)\}\\ \varpi_{A\times B}^I(x,y)=\frac{\varpi_A^I(x)+\varpi_B^I(y)}{2}\\ \varpi_{A\times B}^F(x,y)=\max\{\varpi_A^F(x),\varpi_B^F(y)\}\end{array}\right\}.$$

**Definition 2.5.** [8] A fuzzy subset $A$ of a bisemiring $(S,\diamond_1,\diamond_2,\diamond_3)$ is said to be a fuzzy subbisemiring of $S$ if

$$\left\{\begin{array}{l}\varpi_A(x\diamond_1 y)\geq\min\{\varpi_A(x),\varpi_A(y)\}\\ \varpi_A(x\diamond_2 y)\geq\min\{\varpi_A(x),\varpi_A(y)\}\\ \varpi_A(x\diamond_3 y)\geq\min\{\varpi_A(x),\varpi_A(y)\}\end{array}\right\}$$

for all $x,y\in S$.

**Definition 2.6.** [8] A fuzzy subset $A$ of a bisemiring $(S, \diamond_1, \diamond_2, \diamond_3)$ is said to be a fuzzy normal subbisemiring of $S$ if it satisfies the following conditions:

$$\begin{Bmatrix} \varpi_A(x \diamond_1 y) = \varpi_A(y \diamond_1 x) \\ \varpi_A(x \diamond_2 y) = \varpi_A(y \diamond_2 x) \\ \varpi_A(x \diamond_3 y) = \varpi_A(y \diamond_3 x) \end{Bmatrix}$$

for all $x, y \in S$.

**Definition 2.7.** [5] Let $(S, +, \cdot, \times)$ and $(T, \boxplus, \circ, \otimes)$ be two bisemirings. A function $\phi : S \to T$ is said to be a homomorphism if it satisfies the following conditions:

$$\begin{Bmatrix} \phi(x + y) = \phi(x) \boxplus \phi(y) \\ \phi(x \cdot y) = \phi(x) \circ \phi(y) \\ \phi(x \times y) = \phi(x) \otimes \phi(y) \end{Bmatrix}$$

for all $x, y \in S$.

## 3. **Neutrosophic Subbisemiring**

In what follows, let $\mathbb{S}$ denote a bisemiring unless otherwise stated. Here NSBS stands for neutrosophic subbisemiring.

**Definition 3.1.** A neutrosophic subset $A$ of $\mathbb{S}$ is said to be an NSBS of $\mathbb{S}$ if it satisfies the following conditions:

$$\begin{Bmatrix} \varpi^T_A(x \diamond_1 y) \geq \min\{\varpi^T_A(x), \varpi^T_A(y)\} \\ \varpi^T_A(x \diamond_2 y) \geq \min\{\varpi^T_A(x), \varpi^T_A(y)\} \\ \varpi^T_A(x \diamond_3 y) \geq \min\{\varpi^T_A(x), \varpi^T_A(y)\} \end{Bmatrix} \quad \begin{Bmatrix} \varpi^I_A(x \diamond_1 y) \geq \frac{\varpi^I_A(x)+\varpi^I_A(y)}{2} \\ OR \\ \varpi^I_A(x \diamond_2 y) \geq \frac{\varpi^I_A(x)+\varpi^I_A(y)}{2} \\ OR \\ \varpi^I_A(x \diamond_3 y) \geq \frac{\varpi^I_A(x)+\varpi^I_A(y)}{2} \end{Bmatrix}$$

$$\begin{Bmatrix} \varpi^F_A(x \diamond_1 y) \leq \max\{\varpi^F_A(x), \varpi^F_A(y)\} \\ \varpi^F_A(x \diamond_2 y) \leq \max\{\varpi^F_A(x), \varpi^F_A(y)\} \\ \varpi^F_A(x \diamond_3 y) \leq \max\{\varpi^F_A(x), \varpi^F_A(y)\} \end{Bmatrix}$$

for all $x, y \in \mathbb{S}$.

**Example 3.2.** Let $\mathbb{S} = \{n_1, n_2, n_3, n_4\}$ be the bisemiring with the following Cayley table:

| $\diamond_1$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|
| $n_1$ | $n_1$ | $n_1$ | $n_1$ | $n_1$ |
| $n_2$ | $n_1$ | $n_2$ | $n_1$ | $n_2$ |
| $n_3$ | $n_1$ | $n_1$ | $n_3$ | $n_3$ |
| $n_4$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |

| $\diamond_2$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|
| $n_1$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
| $n_2$ | $n_2$ | $n_2$ | $n_4$ | $n_4$ |
| $n_3$ | $n_3$ | $n_4$ | $n_3$ | $n_4$ |
| $n_4$ | $n_4$ | $n_4$ | $n_4$ | $n_4$ |

| $\diamond_3$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|
| $n_1$ | $n_1$ | $n_1$ | $n_1$ | $n_1$ |
| $n_2$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
| $n_3$ | $n_4$ | $n_4$ | $n_4$ | $n_4$ |
| $n_4$ | $n_4$ | $n_4$ | $n_4$ | $n_4$ |

| | $n = n_1$ | $n = n_2$ | $n = n_3$ | $n = n_4$ |
|---|---|---|---|---|
| $\varpi^T_A(n)$ | 0.7 | 0.6 | 0.3 | 0.5 |
| $\varpi^I_A(n)$ | 0.4 | 0.3 | 0.1 | 0.2 |
| $\varpi^F_A(n)$ | 0.5 | 0.6 | 0.9 | 0.8 |

Clearly, $A$ is an NSBS of $\mathbb{S}$.

**Theorem 3.3.** *The intersection of a family of $NSBS^s$ of $\mathbb{S}$ is an NSBS of $\mathbb{S}$.*

**Proof.** Let $\{V_i : i \in I\}$ be a family of $NSBS^s$ of $\mathbb{S}$ and $A = \bigcap_{i\in I} V_i$.
Let $x$ and $y$ in $\mathbb{S}$. Then

$$\begin{aligned}\varpi^T_A(x \diamond_1 y) &= \inf_{i\in I} \varpi^T_{V_i}(x \diamond_1 y)\\ &\geq \inf_{i\in I} \min\{\varpi^T_{V_i}(x), \varpi^T_{V_i}(y)\}\\ &= \min\left\{\inf_{i\in I} \varpi^T_{V_i}(x), \inf_{i\in I} \varpi^T_{V_i}(y)\right\}\\ &= \min\{\varpi^T_A(x), \varpi^T_A(y)\}.\end{aligned}$$

Similarly, $\varpi^T_A(x \diamond_2 y) \geq \min\{\varpi^T_A(x), \varpi^T_A(y)\}$, $\varpi^T_A(x \diamond_3 y) \geq \min\{\varpi^T_A(x), \varpi^T_A(y)\}$. Now,

$$\begin{aligned}\varpi^I_A(x \diamond_1 y) &= \inf_{i\in I} \varpi^I_{V_i}(x \diamond_1 y)\\ &\geq \inf_{i\in I} \frac{\varpi^I_{V_i}(x) + \varpi^I_{V_i}(y)}{2}\\ &= \frac{\inf_{i\in I} \varpi^I_{V_i}(x) + \inf_{i\in I} \varpi^I_{V_i}(y)}{2}\\ &= \frac{\varpi^I_A(x) + \varpi^I_A(y)}{2}.\end{aligned}$$

Similarly, $\varpi^I_A(x \diamond_2 y) \geq \frac{\varpi^I_A(x)+\varpi^I_A(y)}{2}$ and $\varpi^I_A(x \diamond_3 y) \geq \frac{\varpi^I_A(x)+\varpi^I_A(y)}{2}$. Now,

$$\begin{aligned}\varpi^F_A(x \diamond_1 y) &= \sup_{i\in I} \varpi^F_{V_i}(x \diamond_1 y)\\ &\leq \sup_{i\in I} \max\{\varpi^F_{V_i}(x), \varpi^F_{V_i}(y)\}\\ &= \max\left\{\sup_{i\in I} \varpi^F_{V_i}(x), \sup_{i\in I} \varpi^F_{V_i}(y)\right\}\\ &= \max\{\varpi^F_A(x), \varpi^F_A(y)\}.\end{aligned}$$

Similarly, $\varpi^F_A(x \diamond_2 y) \leq \max\{\varpi^F_A(x), \varpi^F_A(y)\}$, $\varpi^F_A(x \diamond_3 y) \leq \max\{\varpi^F_A(x), \varpi^F_A(y)\}$. Hence $A$ is an NSBS of $\mathbb{S}$.

**Theorem 3.4.** *If $A$ and $B$ are any two $NSBS^s$ of $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively, then $A \times B$ is an NSBS of $\mathbb{S}_1 \times \mathbb{S}_2$.*

**Proof.** Let $A$ and $B$ be two $NSBS^s$ of $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively. Let $x_1, x_2 \in \mathbb{S}_1$ and $y_1, y_2 \in \mathbb{S}_2$. Then $(x_1, y_1)$ and $(x_2, y_2)$ are in $\mathbb{S}_1 \times \mathbb{S}_2$. Now

$$\begin{aligned}
\varpi^T_{A\times B}[(x_1, y_1) \diamond_1 (x_2, y_2)] &= \varpi^T_{A\times B}(x_1 \diamond_1 x_2, y_1 \diamond_1 y_2) \\
&= \min\{\varpi^T_A(x_1 \diamond_1 x_2), \varpi^T_B(y_1 \diamond_1 y_2)\} \\
&\geq \min\{\min\{\varpi^T_A(x_1), \varpi^T_A(x_2)\}, \min\{\varpi^T_B(y_1), \varpi^T_B(y_2)\}\} \\
&= \min\{\min\{\varpi^T_A(x_1), \varpi^T_B(y_1)\}, \min\{\varpi^T_A(x_2), \varpi^T_B(y_2)\}\} \\
&= \min\{\varpi^T_{A\times B}(x_1, y_1), \varpi^T_{A\times B}(x_2, y_2)\}.
\end{aligned}$$

Also $\varpi^T_{A\times B}[(x_1, y_1) \diamond_2 (x_2, y_2)] \geq \min\{\varpi^T_{A\times B}(x_1, y_1), \varpi^T_{A\times B}(x_2, y_2)\}$,
$\varpi^T_{A\times B}[(x_1, y_1) \diamond_3 (x_2, y_2)] \geq \min\{\varpi^T_{A\times B}(x_1, y_1), \varpi^T_{A\times B}(x_2, y_2)\}$. Now,

$$\begin{aligned}
\varpi^I_{A\times B}[(x_1, y_1) \diamond_1 (x_2, y_2)] &= \varpi^I_{A\times B}(x_1 \diamond_1 x_2, y_1 \diamond_1 y_2) \\
&= \frac{\varpi^I_A(x_1 \diamond_1 x_2) + \varpi^I_B(y_1 \diamond_1 y_2)}{2} \\
&\geq \frac{1}{2}\left[\frac{\varpi^I_A(x_1) + \varpi^I_A(x_2)}{2} + \frac{\varpi^I_B(y_1) + \varpi^I_B(y_2)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\varpi^I_A(x_1) + \varpi^I_B(y_1)}{2} + \frac{\varpi^I_A(x_2) + \varpi^I_B(y_2)}{2}\right] \\
&= \frac{1}{2}\Big[\varpi^I_{A\times B}(x_1, y_1) + \varpi^I_{A\times B}(x_2, y_2)\Big].
\end{aligned}$$

Also $\varpi^I_{A\times B}[(x_1, y_1) \diamond_2 (x_2, y_2)] \geq \frac{1}{2}\Big[\varpi^I_{A\times B}(x_1, y_1) + \varpi^I_{A\times B}(x_2, y_2)\Big]$ and
$\varpi^I_{A\times B}[(x_1, y_1) \diamond_3 (x_2, y_2)] \geq \frac{1}{2}\Big[\varpi^I_{A\times B}(x_1, y_1) + \varpi^I_{A\times B}(x_2, y_2)\Big]$. Now,

$$\begin{aligned}
\varpi^F_{A\times B}[(x_1, y_1) \diamond_1 (x_2, y_2)] &= \varpi^F_{A\times B}(x_1 \diamond_1 x_2, y_1 \diamond_1 y_2) \\
&= \max\{\varpi^F_A(x_1 \diamond_1 x_2), \varpi^F_B(y_1 \diamond_1 y_2)\} \\
&\leq \max\{\max\{\varpi^F_A(x_1), \varpi^F_A(x_2)\}, \max\{\varpi^F_B(y_1), \varpi^F_B(y_2)\}\} \\
&= \max\{\max\{\varpi^F_A(x_1), \varpi^F_B(y_1)\}, \max\{\varpi^F_A(x_2), \varpi^F_B(y_2)\}\} \\
&= \max\{\varpi^F_{A\times B}(x_1, y_1), \varpi^F_{A\times B}(x_2, y_2)\}.
\end{aligned}$$

Also $\varpi^F_{A\times B}[(x_1, y_1) \diamond_2 (x_2, y_2)] \leq \max\{\varpi^F_{A\times B}(x_1, y_1), \varpi^F_{A\times B}(x_2, y_2)\}$,
$\varpi^F_{A\times B}[(x_1, y_1) \diamond_3 (x_2, y_2)] \leq \max\{\varpi^F_{A\times B}(x_1, y_1), \varpi^F_{A\times B}(x_2, y_2)\}$. Hence $A \times B$ is an NSBS of $\mathbb{S}$.

**Corollary 3.5.** *If $A_1, A_2, ..., A_n$ are the family of $NSBS^s$ of $\mathbb{S}_1, \mathbb{S}_2, ..., \mathbb{S}_n$ respectively, then $A_1 \times A_2 \times ... \times A_n$ is an NSBS of $\mathbb{S}_1 \times \mathbb{S}_2 \times ... \times \mathbb{S}_n$.*

**Definition 3.6.** Let $A$ be a neutrosophic subset in $\mathbb{S}$, the strongest neutrosophic relation on $\mathbb{S}$, that is a neutrosophic relation on $A$ is $V$ given by

$$\left\{\begin{array}{l} \varpi_V^T(x,y) = min\{\varpi_A^T(x), \varpi_A^T(y)\} \\ \varpi_V^I(x,y) = \frac{\varpi_A^I(x)+\varpi_A^I(y)}{2} \\ \varpi_V^F(x,y) = max\{\varpi_A^F(x), \varpi_A^F(y)\} \end{array}\right\}.$$

**Theorem 3.7.** *Let $A$ be the NSBS of $\mathbb{S}$ and $V$ be the strongest neutrosophic relation of $\mathbb{S}$. Then $A$ is an NSBS of $\mathbb{S}$ if and only if $V$ is an NSBS of $\mathbb{S} \times \mathbb{S}$.*

**Proof.** Let $A$ be the NSBS of $\mathbb{S}$ and $V$ be the strongest neutrosophic relation of $\mathbb{S}$. Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $\mathbb{S} \times \mathbb{S}$. We have

$$\begin{aligned} \varpi_V^T(x \diamond_1 y) &= \varpi_V^T[((x_1, x_2) \diamond_1 (y_1, y_2)] \\ &= \varpi_V^T(x_1 \diamond_1 y_1, x_2 \diamond_1 y_2) \\ &= \min\{\varpi_A^T(x_1 \diamond_1 y_1), \varpi_A^T(x_2 \diamond_1 y_2)\} \\ &\geq \min\{\min\{\varpi_A^T(x_1), \varpi_A^T(y_1)\}, \min\{\varpi_A^T(x_2), \varpi_A^T(y_2)\}\} \\ &= \min\{\min\{\varpi_A^T(x_1), \varpi_A^T(x_2)\}, \min\{\varpi_A^T(y_1), \varpi_A^T(y_2)\}\} \\ &= \min\{\varpi_V^T(x_1, x_2), \varpi_V^T(y_1, y_2)\} \\ &= \min\{\varpi_V^T(x), \varpi_V^T(y)\}. \end{aligned}$$

Also, $\varpi_V^T(x \diamond_2 y) \geq \min\{\varpi_V^T(x), \varpi_V^T(y)\}, \varpi_V^T(x \diamond_3 y) \geq \min\{\varpi_V^T(x), \varpi_V^T(y)\}$.
Now,

$$\begin{aligned} \varpi_V^I(x \diamond_1 y) &= \varpi_V^I[((x_1, x_2) \diamond_1 (y_1, y_2)] \\ &= \varpi_V^I(x_1 \diamond_1 y_1, x_2 \diamond_1 y_2) \\ &= \frac{\varpi_A^I(x_1 \diamond_1 y_1) + \varpi_A^I(x_2 \diamond_1 y_2)}{2} \\ &\geq \frac{1}{2}\left[\frac{\varpi_A^I(x_1) + \varpi_A^I(y_1)}{2} + \frac{\varpi_A^I(x_2) + \varpi_A^I(y_2)}{2}\right] \\ &= \frac{1}{2}\left[\frac{\varpi_A^I(x_1) + \varpi_A^I(x_2)}{2} + \frac{\varpi_A^I(y_1) + \varpi_A^I(y_2)}{2}\right] \\ &= \frac{\varpi_V^I(x_1, x_2) + \varpi_V^I(y_1, y_2)}{2} \\ &= \frac{\varpi_V^I(x) + \varpi_V^I(y)}{2}. \end{aligned}$$

Also, $\varpi_V^I(x \diamond_2 y) \geq \frac{\varpi_V^I(x)+\varpi_V^I(y)}{2}$ and $\varpi_V^I(x \diamond_3 y) \geq \frac{\varpi_V^I(x)+\varpi_V^I(y)}{2}$.
Similarly, $\varpi_V^F(x \diamond_1 y) \leq \max\{\varpi_V^F(x), \varpi_V^F(y)\}, \varpi_V^F(x \diamond_2 y) \leq \max\{\varpi_V^F(x), \varpi_V^F(y)\}$ and $\varpi_V^F(x \diamond_3 y) \leq \max\{\varpi_V^F(x), \varpi_V^F(y)\}$. Hence $V$ is an NSBS of $\mathbb{S} \times \mathbb{S}$.

Conversely assume that $V$ is an NSBS of $\mathbb{S}\times\mathbb{S}$, then for any $x=(x_1,x_2)$ and $y=(y_1,y_2)$ are in $\mathbb{S}\times\mathbb{S}$. We have

$$\begin{aligned}\min\{\varpi_A^T(x_1 \diamond_1 y_1), \varpi_A^T(x_2 \diamond_1 y_2)\} &= \varpi_V^T(x_1 \diamond_1 y_1, x_2 \diamond_1 y_2)\\ &= \varpi_V^T[(x_1,x_2)\diamond_1 (y_1,y_2)]\\ &= \varpi_V^T(x\diamond_1 y)\\ &\geq \min\{\varpi_V^T(x), \varpi_V^T(y)\}\\ &= \min\{\varpi_V^T(x_1,x_2)\}, \varpi_V^T(y_1,y_2)\}\\ &= \min\{\min\{\varpi_A^T(x_1), \varpi_A^T(x_2)\}, \min\{\varpi_A^T(y_1), \varpi_A^T(y_2)\}\}.\end{aligned}$$

If $\varpi_A^T(x_1 \diamond_1 y_1) \leq \varpi_A^T(x_2 \diamond_1 y_2)$, then $\varpi_A^T(x_1) \leq \varpi_A^T(x_2)$ and $\varpi_A^T(y_1) \leq \varpi_A^T(y_2)$. We get $\varpi_A^T(x_1 \diamond_1 y_1) \geq \min\{\varpi_A^T(x_1), \varpi_A^T(y_1)\}$ for all $x_1, y_1 \in \mathbb{S}$, and
$\min\{\varpi_A^T(x_1 \diamond_2 y_1), \varpi_A^T(x_2 \diamond_2 y_2)\} \geq \min\{\min\{\varpi_A^T(x_1), \varpi_A^T(x_2)\}, \min\{\varpi_A^T(y_1), \varpi_A^T(y_2)\}\}$
If $\varpi_A^T(x_1 \diamond_2 y_1) \leq \varpi_A^T(x_2 \diamond_2 y_2)$, then $\varpi_A^T(x_1 \diamond_2 y_1) \geq \min\{\varpi_A^T(x_1), \varpi_A^T(y_1)\}$.
$\min\{\varpi_A^T(x_1 \diamond_3 y_1), \varpi_A^T(x_2 \diamond_3 y_2)\} \geq \min\{\min\{\varpi_A^T(x_1), \varpi_A^T(x_2)\}, \min\{\varpi_A^T(y_1), \varpi_A^T(y_2)\}\}$.
If $\varpi_A^T(x_1 \diamond_3 y_1) \leq \varpi_A^T(x_2 \diamond_3 y_2)$, then $\varpi_A^T(x_1 \diamond_3 y_1) \geq \min\{\varpi_A^T(x_1), \varpi_A^T(y_1)\}$.
Now,

$$\begin{aligned}\frac{1}{2}\Big[\varpi_A^I(x_1 \diamond_1 y_1) + \varpi_A^I(x_2 \diamond_1 y_2)\Big] &= \varpi_V^I(x_1 \diamond_1 y_1, x_2 \diamond_1 y_2)\\ &= \varpi_V^I[(x_1,x_2)\diamond_1 (y_1,y_2)]\\ &= \varpi_V^I(x\diamond_1 y)\\ &\geq \frac{\varpi_V^I(x)+\varpi_V^I(y)}{2}\\ &= \frac{\varpi_V^I(x_1,x_2)+\varpi_V^I(y_1,y_2)}{2}\\ &= \frac{1}{2}\left[\frac{\varpi_A^I(x_1)+\varpi_A^I(x_2)}{2} + \frac{\varpi_A^I(y_1)+\varpi_A^I(y_2)}{2}\right].\end{aligned}$$

If $\varpi_A^I(x_1 \diamond_1 y_1) \leq \varpi_A^I(x_2 \diamond_1 y_2)$, then $\varpi_A^I(x_1) \leq \varpi_A^I(x_2)$ and $\varpi_A^I(y_1) \leq \varpi_A^I(y_2)$.
We get, $\varpi_A^I(x_1 \diamond_1 y_1) \geq \frac{\varpi_A^I(x_1)+\varpi_A^I(y_1)}{2}$.
Similarly, $\varpi_A^I(x_1 \diamond_2 y_1) \geq \frac{\varpi_A^I(x_1)+\varpi_A^I(y_1)}{2}$ and $\varpi_A^I(x_1 \diamond_3 y_1) \geq \frac{\varpi_A^I(x_1)+\varpi_A^I(y_1)}{2}$.
Similarly to prove that
$\max\{\varpi_A^F(x_1 \diamond_1 y_1), \varpi_A^F(x_2 \diamond_1 y_2)\} \leq \max\{\max\{\varpi_A^F(x_1), \varpi_A^F(x_2)\}, \max\{\varpi_A^F(y_1), \varpi_A^F(y_2)\}\}$.
If $\varpi_A^F(x_1 \diamond_1 y_1) \geq \varpi_A^F(x_2 \diamond_1 y_2)$, then $\varpi_A^F(x_1) \geq \varpi_A^F(x_2)$ and $\varpi_A^F(y_1) \geq \varpi_A^F(y_2)$.
We get, $\varpi_A^F(x_1 \diamond_1 y_1) \leq \max\{\varpi_A^F(x_1), \varpi_A^F(y_1)\}$.
$\max\{\varpi_A^F(x_1 \diamond_2 y_1), \varpi_A^F(x_2 \diamond_2 y_2)\} \leq \max\{\max\{\varpi_A^F(x_1), \varpi_A^F(x_2)\}, \max\{\varpi_A^F(y_1), \varpi_A^F(y_2)\}\}$.
If $\varpi_A^F(x_1 \diamond_2 y_1) \geq \varpi_A^F(x_2 \diamond_2 y_2)$, then $\varpi_A^F(x_1 \diamond_2 y_1) \leq \max\{\varpi_A^F(x_1), \varpi_A^F(y_1)\}$.
$\max\{\varpi_A^F(x_1 \diamond_3 y_1), \varpi_A^F(x_2 \diamond_3 y_2)\} \leq \max\{\max\{\varpi_A^F(x_1), \varpi_A^F(x_2)\}, \max\{\varpi_A^F(y_1), \varpi_A^F(y_2)\}\}$

If $\varpi_A^F(x_1 \diamond_3 y_1) \geq \varpi_A^F(x_2 \diamond_3 y_2)$, then $\varpi_A^F(x_1 \diamond_3 y_1) \leq \max\{\varpi_A^F(x_1), \varpi_A^F(y_1)\}$.
Hence $A$ is an NSBS of $\mathbb{S}$.

**Theorem 3.8.** *Let $A$ be a neutrosophic subset in $\mathbb{S}$. Then $\tilde{\varpi} = (\varpi_A^T, \varpi_A^I, \varpi_A^F)$ is an NSBS of $\mathbb{S}$ if and only if all non empty level set $\tilde{\varpi}^{(t,s)}$ is a subbisemiring of $\mathbb{S}$ for $t, s \in [0,1]$.*

**Proof.** Assume that $\tilde{\varpi}$ is an NSBS of $\mathbb{S}$. For each $t, s \in [0,1]$ and $a_1, a_2 \in \tilde{\varpi}^{(t,s)}$. We have $\varpi_A^T(a_1) \geq t, \varpi_A^T(a_2) \geq t$ and $\varpi_A^I(a_1) \geq t, \varpi_A^I(a_2) \geq t$ and $\varpi_A^F(a_1) \leq s, \varpi_A^F(a_2) \leq s$. Now, $\varpi_A^T(a_1 \diamond_1 a_2) \geq \min\{\varpi_A^T(a_1), \varpi_A^T(a_2)\} \geq t$ and $\varpi_A^I(a_1 \diamond_1 a_2) \geq \frac{\varpi_A^I(a_1)+\varpi_A^I(a_2)}{2} \geq \frac{t+t}{2} = t$ and $\varpi_A^F(a_1 \diamond_1 a_2) \leq \max\{\varpi_A^F(a_1), \varpi_A^F(a_2)\} \leq s$. This implies that $a_1 \diamond_1 a_2 \in \tilde{\varpi}^{(t,s)}$. Similarly, $a_1 \diamond_2 a_2 \in \tilde{\varpi}^{(t,s)}$ and $a_1 \diamond_3 a_2 \in \tilde{\varpi}^{(t,s)}$. Therefore $\tilde{\varpi}^{(t,s)}$ is a subbisemiring of $\mathbb{S}$ for each $t, s \in [0,1]$.

Conversely, assume that $\tilde{\varpi}^{(t,s)}$ is a subbisemiring of $\mathbb{S}$ for each $t, s \in [0,1]$. Suppose if there exist $a_1, a_2 \in \mathbb{S}$ such that $\varpi_A^T(a_1 \diamond_1 a_2) < \min\{\varpi_A^T(a_1), \varpi_A^T(a_2)\}$, $\varpi_A^I(a_1 \diamond_1 a_2) < \frac{\varpi_A^I(a_1)+\varpi_A^I(a_2)}{2}$ and $\varpi_A^F(a_1 \diamond_1 a_2) > \max\{\varpi_A^F(a_1), \varpi_A^F(a_2)\}$. Select $t, s \in [0,1]$ such that $\varpi_A^T(a_1 \diamond_1 a_2) < t \leq \min\{\varpi_A^T(a_1), \varpi_A^T(a_2)\}$ and $\varpi_A^I(a_1 \diamond_1 a_2) < t \leq \frac{\varpi_A^I(a_1)+\varpi_A^I(a_2)}{2}$ and $\varpi_A^F(a_1 \diamond_1 a_2) > s \geq \max\{\varpi_A^F(a_1), \varpi_A^F(a_2)\}$. Then $a_1, a_2 \in \tilde{\varpi}^{(t,s)}$, but $a_1 \diamond_1 a_2 \notin \tilde{\varpi}^{(t,s)}$. This contradicts to that $\tilde{\varpi}^{(t,s)}$ is a subbisemiring of $\mathbb{S}$. Hence $\varpi_A^T(a_1 \diamond_1 a_2) \geq \min\{\varpi_A^T(a_1), \varpi_A^T(a_2)\}$, $\varpi_A^I(a_1 \diamond_1 a_2) \geq \frac{\varpi_A^I(a_1)+\varpi_A^I(a_2)}{2}$ and $\varpi_A^F(a_1 \diamond_1 a_2) \leq \max\{\varpi_A^F(a_1), \varpi_A^F(a_2)\}$. Similarly, $\diamond_2$ and $\diamond_3$ cases. Hence $\tilde{\varpi} = (\varpi_A^T, \varpi_A^I, \varpi_A^F)$ is an NSBS of $\mathbb{S}$.

**Definition 3.9.** Let $A$ be any NSBS of $\mathbb{S}$ and $a \in \mathbb{S}$. Then the pseudo neutrosophic coset $(aA)^p$ is defined by

$$\begin{Bmatrix} ((a\varpi_A^T)^p)(x) = p(a)\varpi_A^T(x) \\ ((a\varpi_A^I)^p)(x) = p(a)\varpi_A^I(x) \\ ((a\varpi_A^F)^p)(x) = p(a)\varpi_A^F(x) \end{Bmatrix}$$

for every $x \in \mathbb{S}$ and for some $p \in P$.

**Theorem 3.10.** *Let $A$ be any NSBS of $\mathbb{S}$, then the pseudo neutrosophic coset $(aA)^p$ is an NSBS of $\mathbb{S}$, for every $a \in \mathbb{S}$.*

**Proof.** Let $A$ be any NSBS of $\mathbb{S}$ and for every $x, y \in \mathbb{S}$. Now, $((a\varpi_A^T)^p)(x \diamond_1 y) = p(a)\ \varpi_A^T(x \diamond_1 y) \geq p(a)\ min\{\varpi_A^T(x), \varpi_A^T(y)\} = min\{p(a)\ \varpi_A^T(x), p(a)\ \varpi_A^T(y)\} = min\{((a\varpi_A^T)^p)(x), ((a\varpi_A^T)^p)(y)\}$. Thus, $((a\varpi_A^T)^p)(x \diamond_1 y) \geq \min\{((a\varpi_A^T)^p)(x), ((a\varpi_A^T)^p)(y)\}$. Now, $((a\varpi_A^I)^p)(x \diamond_1 y) = p(a)\ \varpi_A^I(x \diamond_1 y) \geq p(a)\left[\frac{\varpi_A^I(x)+\varpi_A^I(y)}{2}\right] = \frac{p(a)\ \varpi_A^I(x)+p(a)\ \varpi_A^I(y)}{2} = \frac{((a\varpi_A^I)^p)(x)+((a\varpi_A^I)^p)(y)}{2}$. Thus, $((a\varpi_A^I)^p)(x \diamond_1 y) \geq \frac{((a\varpi_A^I)^p)(x)+((a\varpi_A^I)^p)(y)}{2}$. Now, $((a\varpi_A^F)^p)(x \diamond_1 y) = p(a)\ \varpi_A^F(x \diamond_1 y) \leq p(a)\ \max\{\varpi_A^F(x), \varpi_A^F(y)\} = \max\{p(a)\ \varpi_A^F(x), p(a)\ \varpi_A^F(y)\} =$

$\max\{((a\varpi_A^F)^p)(x), ((a\varpi_A^F)^p)(y)\}$. Thus, $((a\varpi_A^F)^p)(x \diamond_1 y) \leq \max\{((a\varpi_A^F)^p)(x), ((a\varpi_A^F)^p)(y)\}$. Similarly, $\diamond_2$ and $\diamond_3$ cases. Hence $(aA)^p$ is an NSBS of $\mathbb{S}$.

**Definition 3.11.** Let $(\mathbb{S}_1, \boxplus_1, \boxplus_2, \boxplus_3)$ and $(\mathbb{S}_2, \boxdot_1, \boxdot_2, \boxdot_3)$ be any two bisemirings. Let $\Delta : \mathbb{S}_1 \to \mathbb{S}_2$ be any function and $A$ be any NSBS in $\mathbb{S}_1$, $V$ be any NSBS in $\Delta(\mathbb{S}_1) = \mathbb{S}_2$. If $\varpi_A = [\varpi_A^T, \varpi_A^I, \varpi_A^F]$ is a neutrosophic set in $\mathbb{S}_1$, then $\varpi_V$ is a neutrosophic set in $\mathbb{S}_2$, defined by

$$\varpi_V^T(y) = \begin{cases} \sup \varpi_A^T(x) & if \ \ x \in \Delta^{-1}y \\ 0 & otherwise \end{cases} \qquad \varpi_V^I(y) = \begin{cases} \sup \varpi_A^I(x) & if \ \ x \in \Delta^{-1}y \\ 0 & otherwise \end{cases}$$

$$\varpi_V^F(y) = \begin{cases} \inf \varpi_A^F(x) & if \ \ x \in \Delta^{-1}y \\ 1 & otherwise \end{cases}$$

for all $x \in \mathbb{S}_1$ and $y \in \mathbb{S}_2$ is called the image of $\varpi_A$ under $\Delta$.
Similarly, If $\varpi_V = [\varpi_V^T, \varpi_V^I, \varpi_V^F]$ is a neutrosophic set in $\mathbb{S}_2$, then neutrosophic set $\varpi_A = \Delta \circ \varpi_V$ in $\mathbb{S}_1$ [ie, the neutrosophic set defined by $\varpi_A(x) = \varpi_V(\Delta(x))$] is called the preimage of $\varpi_V$ under $\Delta$.

**Theorem 3.12.** *Let $(\mathbb{S}_1, \boxplus_1, \boxplus_2, \boxplus_3)$ and $(\mathbb{S}_2, \boxdot_1, \boxdot_2, \boxdot_3)$ be any two bisemirings. The homomorphic image of NSBS of $\mathbb{S}_1$ is an NSBS of $\mathbb{S}_2$.*

**Proof.** Let $\Delta : \mathbb{S}_1 \to \mathbb{S}_2$ be any homomorphism. Then $\Delta(x \boxplus_1 y) = \Delta(x) \boxdot_1 \Delta(y), \Delta(x \boxplus_2 y) = \Delta(x) \boxdot_2 \Delta(y)$ and $\Delta(x \boxplus_3 y) = \Delta(x) \boxdot_3 \Delta(y)$ for all $x, y \in \mathbb{S}_1$. Let $V = \Delta(A)$, $A$ is any NSBS of $\mathbb{S}_1$. Let $\Delta(x), \Delta(y) \in \mathbb{S}_2$. Let $x \in \Delta^{-1}(\Delta(x))$ and $y \in \Delta^{-1}(\Delta(y))$ be such that $\varpi_A^T(x) = \sup_{z \in \Delta^{-1}(\Delta(x))} \varpi_A^T(z)$ and $\varpi_A^T(y) = \sup_{z \in \Delta^{-1}(\Delta(y))} \varpi_A^T(z)$. Now,

$$\begin{aligned} \varpi_V^T(\Delta(x) \boxdot_1 \Delta(y)) &= \sup_{z' \in \Delta^{-1}(\Delta(x) \boxdot_1 \Delta(y))} \varpi_A^T(z') \\ &= \sup_{z' \in \Delta^{-1}(\Delta(x \boxplus_1 y)} \varpi_A^T(z') \\ &= \varpi_A^T(x \boxplus_1 y) \\ &\geq \min\{\varpi_A^T(x), \varpi_A^T(y)\} \\ &= \min\{\varpi_V^T\Delta(x), \varpi_V^T\Delta(y)\}. \end{aligned}$$

Thus, $\varpi_V^T(\Delta(x) \boxdot_1 \Delta(y)) \geq \min\{\varpi_V^T\Delta(x), \varpi_V^T\Delta(y)\}$.
Similarly, $\varpi_V^T(\Delta(x) \boxdot_2 \Delta(y)) \geq \min\{\varpi_V^T\Delta(x), \varpi_V^T\Delta(y)\}$ and
$\varpi_V^T(\Delta(x) \boxdot_3 \Delta(y)) \geq \min\{\varpi_V^T\Delta(x), \varpi_V^T\Delta(y)\}$.
Let $x \in \Delta^{-1}(\Delta(x))$ and $y \in \Delta^{-1}(\Delta(y))$ be such that $\varpi_A^I(x) = \sup_{z \in \Delta^{-1}(\Delta(x))} \varpi_A^I(z)$ and

$\varpi_A^I(y) = \sup_{z\in\Delta^{-1}(\Delta(y))} \varpi_A^I(z)$. Now,

$$\begin{aligned}
\varpi_V^I(\Delta(x) \boxdot_1 \Delta(y)) &= \sup_{z^{'}\in\Delta^{-1}(\Delta(x)\boxdot_1\Delta(y))} \varpi_A^I(z^{'}) \\
&= \sup_{z^{'}\in\Delta^{-1}(\Delta(x\boxplus_1 y)} \varpi_A^I(z^{'}) \\
&= \varpi_A^I(x \boxplus_1 y) \\
&\geq \frac{\varpi_A^I(x) + \varpi_A^I(y)}{2} \\
&= \frac{\varpi_V^I\Delta(x) + \varpi_V^I\Delta(y)}{2}.
\end{aligned}$$

Thus, $\varpi_V^I(\Delta(x) \boxdot_1 \Delta(y)) \geq \frac{\varpi_V^I\Delta(x)+\varpi_V^I\Delta(y)}{2}$.
Similarly, $\varpi_V^I(\Delta(x) \boxdot_2 \Delta(y)) \geq \frac{\varpi_V^I\Delta(x)+\varpi_V^I\Delta(y)}{2}$ and $\varpi_V^I(\Delta(x) \boxdot_3 \Delta(y)) \geq \frac{\varpi_V^I\Delta(x)+\varpi_V^I\Delta(y)}{2}$.
Let $\Delta(x), \Delta(y) \in \mathbb{S}_2$. Let $x \in \Delta^{-1}(\Delta(x))$ and $y \in \Delta^{-1}(\Delta(y))$ be such that $\varpi_A^F(x) = \inf_{z\in\Delta^{-1}(\Delta(x))} \varpi_A^F(z)$ and $\varpi_A^F(y) = \inf_{z\in\Delta^{-1}(\Delta(y))} \varpi_A^F(z)$. Now,

$$\begin{aligned}
\varpi_V^F(\Delta(x) \boxdot_1 \Delta(y)) &= \inf_{z^{'}\in\Delta^{-1}(\Delta(x)\boxdot_1\Delta(y))} \varpi_A^F(z^{'}) \\
&= \inf_{z^{'}\in\Delta^{-1}(\Delta(x\boxplus_1 y)} \varpi_A^F(z^{'}) \\
&= \varpi_A^F(x \boxplus_1 y) \\
&\leq \max\{\varpi_A^F(x), \varpi_A^F(y)\} \\
&= \max\{\varpi_V^F\Delta(x), \varpi_V^F\Delta(y)\}.
\end{aligned}$$

Thus, $\varpi_V^F(\Delta(x) \boxdot_1 \Delta(y)) \leq \max\{\varpi_V^F\Delta(x), \varpi_V^F\Delta(y)\}$.
Similarly, $\varpi_V^F(\Delta(x) \boxdot_2 \Delta(y)) \leq \max\{\varpi_V^F\Delta(x), \varpi_V^F\Delta(y)\}$ and $\varpi_V^F(\Delta(x) \boxdot_3 \Delta(y)) \leq \max\{\varpi_V^F\Delta(x), \varpi_V^F\Delta(y)\}$. Hence $V$ is an NSBS of $\mathbb{S}_2$.

**Theorem 3.13.** *Let $(\mathbb{S}_1, \boxplus_1, \boxplus_2, \boxplus_3)$ and $(\mathbb{S}_2, \boxdot_1, \boxdot_2, \boxdot_3)$ be any two bisemirings. The homomorphic preimage of NSBS of $\mathbb{S}_2$ is an NSBS of $\mathbb{S}_1$.*

**Proof.** Let $\Delta : \mathbb{S}_1 \to \mathbb{S}_2$ be any homomorphism. Then $\Delta(x \boxplus_1 y) = \Delta(x) \boxdot_1 \Delta(y), \Delta(x \boxplus_2 y) = \Delta(x) \boxdot_2 \Delta(y)$ and $\Delta(x \boxplus_3 y) = \Delta(x) \boxdot_3 \Delta(y)$ for all $x, y \in \mathbb{S}_1$. Let $V = \Delta(A)$, where $V$ is any NSBS of $\mathbb{S}_2$. Let $x, y \in \mathbb{S}_1$. Now, $\varpi_A^T(x \boxplus_1 y) = \varpi_V^T(\Delta(x \boxplus_1 y)) = \varpi_V^T(\Delta(x) \boxdot_1 \Delta(y)) \geq \min\{\varpi_V^T\Delta(x), \varpi_V^T\Delta(y)\} = \min\{\varpi_A^T(x), \varpi_A^T(y)\}$. Thus, $\varpi_A^T(x \boxplus_1 y) \geq \min\{\varpi_A^T(x), \varpi_A^T(y)\}$. Now, $\varpi_A^I(x \boxplus_1 y) = \varpi_V^I(\Delta(x \boxplus_1 y)) = \varpi_V^I(\Delta(x) \boxdot_1 \Delta(y)) \geq \frac{\varpi_V^I\Delta(x)+\varpi_V^I\Delta(y)}{2} = \frac{\varpi_A^I(x)+\varpi_A^I(y)}{2}$. Thus, $\varpi_A^I(x \boxplus_1 y) \geq \frac{\varpi_A^I(x)+\varpi_A^I(y)}{2}$. Now, $\varpi_A^F(x \boxplus_1 y) = \varpi_V^F(\Delta(x \boxplus_1 y)) = \varpi_V^F(\Delta(x) \boxdot_1 \Delta(y)) \leq \max\{\varpi_V^F\Delta(x), \varpi_V^F\Delta(y)\} = \max\{\varpi_A^F(x), \varpi_A^F(y)\}$. Thus, $\varpi_A^F(x \boxplus_1 y) \leq \max\{\varpi_A^F(x), \varpi_A^F(y)\}$. Similarly to prove two other operations, hence $A$ is an NSBS of $\mathbb{S}_1$.

**Theorem 3.14.** *Let $(\mathbb{S}_1, \boxplus_1, \boxplus_2, \boxplus_3)$ and $(\mathbb{S}_2, \boxdot_1, \boxdot_2, \boxdot_3)$ be any two bisemirings. If $\Delta : \mathbb{S}_1 \to \mathbb{S}_2$ is a homomorphism, then $\Delta(A_{(t,s)})$ is a level subbisemiring of NSBS $V$ of $\mathbb{S}_2$.*

**Proof.** Let $\Delta : \mathbb{S}_1 \to \mathbb{S}_2$ be any homomorphism. Then $\Delta(x \boxplus_1 y) = \Delta(x) \boxdot_1 \Delta(y), \Delta(x \boxplus_2 y) = \Delta(x) \boxdot_2 \Delta(y)$ and $\Delta(x \boxplus_3 y) = \Delta(x) \boxdot_3 \Delta(y)$ for all $x, y \in \mathbb{S}_1$. Let $V = \Delta(A)$, $A$ is an NSBS of $\mathbb{S}_1$. By Theorem 3.12, $V$ is an NSBS of $\mathbb{S}_2$. Let $A_{(t,s)}$ be any level subbisemiring of $A$. Suppose that $x, y \in A_{(t,s)}$. Then $\Delta(x \boxplus_1 y), \Delta(x \boxplus_2 y)$ and $\Delta(x \boxplus_3 y) \in A_{(t,s)}$. Now, $\varpi^T_V(\Delta(x)) = \varpi^T_A(x) \geq t, \varpi^T_V(\Delta(y)) = \varpi^T_A(y) \geq t$. Thus, $\varpi^T_V(\Delta(x) \boxdot_1 \Delta(y)) \geq \varpi^T_A(x \boxplus_1 y) \geq t$. Now, $\varpi^I_V(\Delta(x)) = \varpi^I_A(x) \geq t, \varpi^I_V(\Delta(y)) = \varpi^I_A(y) \geq t$. Thus, $\varpi^I_V(\Delta(x) \boxdot_1 \Delta(y)) \geq \varpi^I_A(x \boxplus_1 y) \geq t$. Now, $\varpi^F_V(\Delta(x)) = \varpi^F_A(x) \leq s, \varpi^F_V(\Delta(y)) = \varpi^F_A(y) \leq s$. Thus, $\varpi^F_V(\Delta(x) \boxdot_1 \Delta(y)) \leq \varpi^F_A(x \boxplus_1 y) \leq s$, for all $\Delta(x), \Delta(y) \in \mathbb{S}_2$. Similarly to prove other operations, hence $\Delta(A_{(t,s)})$ is a level subbisemiring of NSBS $V$ of $\mathbb{S}_2$.

**Theorem 3.15.** *Let $(\mathbb{S}_1, \boxplus_1, \boxplus_2, \boxplus_3)$ and $(\mathbb{S}_2, \boxdot_1, \boxdot_2, \boxdot_3)$ be any two bisemirings. If $\Delta : \mathbb{S}_1 \to \mathbb{S}_2$ is any homomorphism, then $A_{(t,s)}$ is a level subbisemiring of NSBS $A$ of $\mathbb{S}_1$.*

**Proof.** Let $\Delta : \mathbb{S}_1 \to \mathbb{S}_2$ be any homomorphism. Then $\Delta(x \boxplus_1 y) = \Delta(x) \boxdot_1 \Delta(y), \Delta(x \boxplus_2 y) = \Delta(x) \boxdot_2 \Delta(y)$ and $\Delta(x \boxplus_3 y) = \Delta(x) \boxdot_3 \Delta(y)$ for all $x, y \in \mathbb{S}_1$. Let $V = \Delta(A)$, $V$ is an NSBS of $\mathbb{S}_2$. By Theorem 3.13, $A$ is an NSBS of $\mathbb{S}_1$. Let $\Delta(A_{(t,s)})$ be a level subbisemiring of $V$. Suppose that $\Delta(x), \Delta(y) \in \Delta(A_{(t,s)})$. Then $\Delta(x \boxplus_1 y), \Delta(x \boxplus_2 y)$ and $\Delta(x \boxplus_3 y) \in \Delta(A_{(t,s)})$. Now, $\varpi^T_A(x) = \varpi^T_V(\Delta(x)) \geq t, \varpi^T_A(y) = \varpi^T_V(\Delta(y)) \geq t$. Thus, $\varpi^T_A(x \boxplus_1 y) \geq \min\{\varpi^T_A(x), \varpi^T_A(y)\} \geq t$. Now, $\varpi^I_A(x) = \varpi^I_V(\Delta(x)) \geq t, \varpi^I_A(y) = \varpi^I_V(\Delta(y)) \geq t$. Thus, $\varpi^I_A(x \boxplus_1 y) \geq \frac{\varpi^I_A(x)+\varpi^I_A(y)}{2} \geq t$. Now, $\varpi^F_A(x) = \varpi^F_V(\Delta(x)) \leq s, \varpi^F_A(y) = \varpi^F_V(\Delta(y)) \leq s$. Thus, $\varpi^F_A(x \boxplus_1 y) = \varpi^F_V(\Delta(x) \boxdot_1 \Delta(y)) \leq \max\{\varpi^F_A(x), \varpi^F_A(y)\} \leq s$, for all $x, y \in \mathbb{S}_1$. Similarly to prove other two operations, hence $A_{(t,s)}$ is a level subbisemiring of NSBS $A$ of $\mathbb{S}_1$.

## 4. $(\alpha, \beta)$- neutrosophic Subbisemiring

In this section, we discuss about $(\alpha, \beta)$- neutrosophic subbisemiring. In what follows that, $(\alpha, \beta) \in [0, 1]$ be such that $0 \leq \alpha < \beta \leq 1$.

**Definition 4.1.** Let $A$ be any neutrosophic subset of $\mathbb{S}$ is called a $(\alpha, \beta)$- NSBS of $\mathbb{S}$ if it satisfies the following conditions:

$$\left\{\begin{matrix} \max\{\varpi^T_A(x \diamond_1 y), \alpha\} \geq \min\{\varpi^T_A(x), \varpi^T_A(y), \beta\} \\ \max\{\varpi^T_A(x \diamond_2 y), \alpha\} \geq \min\{\varpi^T_A(x), \varpi^T_A(y), \beta\} \\ \max\{\varpi^T_A(x \diamond_3 y), \alpha\} \geq \min\{\varpi^T_A(x), \varpi^T_A(y), \beta\} \end{matrix}\right\}$$

$$\left\{\begin{array}{c}\max\{\varpi^I_A(x\diamond_1 y),\alpha\}\geq\min\left\{\frac{\varpi^I_A(x)+\varpi^I_A(y)}{2},\beta\right\}\\ \text{OR}\\ \max\{\varpi^I_A(x\diamond_2 y),\alpha\}\geq\min\left\{\frac{\varpi^I_A(x)+\varpi^I_A(y)}{2},\beta\right\}\\ \text{OR}\\ \max\{\varpi^I_A(x\diamond_3 y),\alpha\}\geq\min\left\{\frac{\varpi^I_A(x)+\varpi^I_A(y)}{2},\beta\right\}\end{array}\right\}$$

$$\left\{\begin{array}{c}\min\{\varpi^F_A(x\diamond_1 y),\alpha\}\leq\max\{\varpi^F_A(x),\varpi^F_A(y),\beta\}\\ \min\{\varpi^F_A(x\diamond_2 y),\alpha\}\leq\max\{\varpi^F_A(x),\varpi^F_A(y),\beta\}\\ \min\{\varpi^F_A(x\diamond_3 y),\alpha\}\leq\max\{\varpi^F_A(x),\varpi^F_A(y),\beta\}\end{array}\right\}$$

for all $x, y \in \mathbb{S}$.

**Example 4.2.** By the Example 3.2,

| | $n = n_1$ | $n = n_2$ | $n = n_3$ | $n = n_4$ |
|---|---|---|---|---|
| $\varpi^T_A(n)$ | 0.80 | 0.75 | 0.55 | 0.70 |
| $\varpi^I_A(n)$ | 0.75 | 0.70 | 0.62 | 0.65 |
| $\varpi^F_A(n)$ | 0.35 | 0.65 | 0.80 | 0.70 |

Clearly, $A$ is a $(0.45, 0.60)$ NSBS of $\mathbb{S}$.

**Theorem 4.3.** *The intersection of a family of $(\alpha, \beta)$ $NSBS^s$ of $\mathbb{S}$ is a $(\alpha, \beta)$ NSBS of $\mathbb{S}$.*

**Proof.** Let $\{V_i : i \in I\}$ be a family of $(\alpha, \beta)$ $NSBS^s$ of $\mathbb{S}$ and $A = \bigcap_{i\in I} V_i$.
Let $x$ and $y$ in $\mathbb{S}$. Now,

$$\begin{aligned}\max\{\varpi^T_A(x\diamond_1 y),\alpha\} &= \inf_{i\in I}\ \max\{\varpi^T_{V_i}(x\diamond_1 y),\alpha\}\\ &\geq \inf_{i\in I}\ \min\{\varpi^T_{V_i}(x),\varpi^T_{V_i}(y),\beta\}\\ &= \min\left\{\inf_{i\in I}\ \varpi^T_{V_i}(x),\inf_{i\in I}\ \varpi^T_{V_i}(y),\beta\right\}\\ &= \min\{\varpi^T_A(x),\varpi^T_A(y),\beta\}.\end{aligned}$$

Similarly, $\max\{\varpi^T_A(x\diamond_2 y),\alpha\}\geq\min\{\varpi^T_A(x),\varpi^T_A(y),\beta\}$ and
$\max\{\varpi^T_A(x\diamond_3 y),\alpha\}\geq\min\{\varpi^T_A(x),\varpi^T_A(y),\beta\}$. Now,

$$\begin{aligned}\max\{\varpi^I_A(x\diamond_1 y),\alpha\} &= \inf_{i\in I}\ \max\{\varpi^I_{V_i}(x\diamond_1 y),\alpha\}\\ &\geq \inf_{i\in I}\ \min\left\{\frac{\varpi^I_{V_i}(x)+\varpi^I_{V_i}(y)}{2},\beta\right\}\\ &= \min\left\{\frac{\inf_{i\in I}\ \varpi^I_{V_i}(x)+\inf_{i\in I}\ \varpi^I_{V_i}(y)}{2},\beta\right\}\\ &= \min\left\{\frac{\varpi^I_A(x)+\varpi^I_A(y)}{2},\beta\right\}.\end{aligned}$$

Similarly, $\max\{\varpi^I_A(x \diamond_2 y), \alpha\} \geq \min\left\{\frac{\varpi^I_A(x)+\varpi^I_A(y)}{2}, \beta\right\}$ and $\max\{\varpi^I_A(x \diamond_3 y), \alpha\} \geq \min\left\{\frac{\varpi^I_A(x)+\varpi^I_A(y)}{2}, \beta\right\}$. Now,

$$\begin{aligned}\min\{\varpi^F_A(x \diamond_1 y), \alpha\} &= \sup_{i\in I} \min\{\varpi^F_{V_i}(x \diamond_1 y), \alpha\}\\ &\leq \sup_{i\in I} \max\{\varpi^F_{V_i}(x), \varpi^F_{V_i}(y), \beta\}\\ &= \max\left\{\sup_{i\in I} \varpi^F_{V_i}(x), \sup_{i\in I} \varpi^F_{V_i}(y), \beta\right\}\\ &= \max\{\varpi^F_A(x), \varpi^F_A(y), \beta\}.\end{aligned}$$

Similarly, $\min\{\varpi^F_A(x \diamond_2 y), \alpha\} \leq \max\{\varpi^F_A(x), \varpi^F_A(y), \beta\}$ and $\min\{\varpi^F_A(x \diamond_3 y), \alpha\} \leq \max\{\varpi^F_A(x), \varpi^F_A(y), \beta\}$. Hence, $A$ is a $(\alpha, \beta)$ NSBS of $\mathbb{S}$.

**Theorem 4.4.** *If $A$ and $B$ are any two $(\alpha, \beta)$ $NSBS^s$ of $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively, then $A \times B$ is a $(\alpha, \beta)$ NSBS of $\mathbb{S}_1 \times \mathbb{S}_2$.*

**Proof.** Let $A$ and $B$ be two $(\alpha, \beta)$ $NSBS^s$ of $\mathbb{S}_1$ and $\mathbb{S}_2$ respectively. Let $x_1, x_2 \in \mathbb{S}_1$ and $y_1, y_2 \in \mathbb{S}_2$. Then $(x_1, y_1)$ and $(x_2, y_2)$ are in $\mathbb{S}_1 \times \mathbb{S}_2$. Now

$$\begin{aligned}\max\left\{\varpi^T_{A\times B}[(x_1, y_1) \diamond_1 (x_2, y_2)], \alpha\right\} &= \max\left\{\varpi^T_{A\times B}(x_1 \diamond_1 x_2, y_1 \diamond_1 y_2), \alpha\right\}\\ &= \min\left\{\max\{\varpi^T_A(x_1 \diamond_1 x_2), \alpha\}, \max\{\varpi^T_B(y_1 \diamond_1 y_2), \alpha\}\right\}\\ &\geq \min\left\{\min\{\varpi^T_A(x_1), \varpi^T_A(x_2), \beta\}, \min\{\varpi^T_B(y_1), \varpi^T_B(y_2), \beta\}\right\}\\ &= \min\left\{\{\min\{\varpi^T_A(x_1), \varpi^T_B(y_1)\}, \min\{\varpi^T_A(x_2), \varpi^T_B(y_2)\}\}, \beta\right\}\\ &= \min\left\{\varpi^T_{A\times B}(x_1, y_1), \varpi^T_{A\times B}(x_2, y_2), \beta\right\}.\end{aligned}$$

Also, $\max\left\{\varpi^T_{A\times B}[(x_1, y_1) \diamond_2 (x_2, y_2)], \alpha\right\} \geq \min\left\{\varpi^T_{A\times B}(x_1, y_1), \varpi^T_{A\times B}(x_2, y_2), \beta\right\}$ and $\max\left\{\varpi^T_{A\times B}[(x_1, y_1) \diamond_3 (x_2, y_2)], \alpha\right\} \geq \min\left\{\varpi^T_{A\times B}(x_1, y_1), \varpi^T_{A\times B}(x_2, y_2), \beta\right\}$.

Now, $\max\left\{\varpi^I_{A\times B}[(x_1, y_1) \diamond_1 (x_2, y_2)], \alpha\right\}$

$$\begin{aligned}&= \max\left\{\varpi^I_{A\times B}(x_1 \diamond_1 x_2, y_1 \diamond_1 y_2), \alpha\right\}\\ &= \min\left\{\frac{1}{2}\left[\max\left\{\varpi^I_A(x_1 \diamond_1 x_2), \alpha\right\} + \max\left\{\varpi^I_B(y_1 \diamond_1 y_2), \alpha\right\}\right]\right\}\\ &\geq \min\left\{\frac{1}{2}\left[\min\left\{\frac{\varpi^I_A(x_1) + \varpi^I_A(x_2)}{2}, \beta\right\} + \min\left\{\frac{\varpi^I_B(y_1) + \varpi^I_B(y_2)}{2}, \beta\right\}\right]\right\}\\ &= \min\left\{\frac{1}{2}\left[\frac{\varpi^I_A(x_1) + \varpi^I_B(y_1)}{2} + \frac{\varpi^I_A(x_2) + \varpi^I_B(y_2)}{2}\right], \beta\right\}\\ &= \min\left\{\frac{\varpi^I_{A\times B}(x_1, y_1) + \varpi^I_{A\times B}(x_2, y_2)}{2}, \beta\right\}.\end{aligned}$$

Also, $\max\left\{\varpi^I_{A\times B}[(x_1,y_1)\diamond_2(x_2,y_2)],\alpha\right\}\geq\min\left\{\frac{\varpi^I_{A\times B}(x_1,y_1)+\varpi^I_{A\times B}(x_2,y_2)}{2},\beta\right\}$ and

$\max\left\{\varpi^I_{A\times B}[(x_1,y_1)\diamond_3(x_2,y_2)],\alpha\right\}\geq\min\left\{\frac{\varpi^I_{A\times B}(x_1,y_1)+\varpi^I_{A\times B}(x_2,y_2)}{2},\beta\right\}$.

Similarly,

$$\begin{aligned}\min\left\{\varpi^F_{A\times B}[(x_1,y_1)\diamond_1(x_2,y_2)],\alpha\right\}&=\min\left\{\varpi^F_{A\times B}(x_1\diamond_1 x_2,y_1\diamond_1 y_2),\alpha\right\}\\&=\max\left\{\min\{\varpi^F_A(x_1\diamond_1 x_2),\alpha\},\min\{\varpi^F_B(y_1\diamond_1 y_2),\alpha\}\right\}\\&\leq\max\left\{\max\{\varpi^F_A(x_1),\varpi^F_A(x_2),\beta\},\max\{\varpi^F_B(y_1),\varpi^F_B(y_2),\beta\}\right\}\\&=\max\left\{\{\max\{\varpi^F_A(x_1),\varpi^F_B(y_1)\},\max\{\varpi^F_A(x_2),\varpi^F_B(y_2)\}\},\beta\right\}\\&=\max\left\{\varpi^F_{A\times B}(x_1,y_1),\varpi^F_{A\times B}(x_2,y_2),\beta\right\}.\end{aligned}$$

Also, $\min\left\{\varpi^F_{A\times B}[(x_1,y_1)\diamond_2(x_2,y_2)],\alpha\right\}\leq\max\left\{\varpi^F_{A\times B}(x_1,y_1),\varpi^F_{A\times B}(x_2,y_2),\beta\right\}$,

$\min\left\{\varpi^F_{A\times B}[(x_1,y_1)\diamond_3(x_2,y_2)],\alpha\right\}\leq\max\left\{\varpi^F_{A\times B}(x_1,y_1),\varpi^F_{A\times B}(x_2,y_2),\beta\right\}$.

Hence $A\times B$ is a $(\alpha,\beta)$ NSBS of $\mathbb{S}_1\times\mathbb{S}_2$.

**Corollary 4.5.** *If $A_1,A_2,...,A_n$ are the family of $(\alpha,\beta)$ $NSBS^s$ of $\mathbb{S}_1,\mathbb{S}_2,...,\mathbb{S}_n$ respectively, then $A_1\times A_2\times...\times A_n$ is a $(\alpha,\beta)$ NSBS of $\mathbb{S}_1\times\mathbb{S}_2\times...\times\mathbb{S}_n$.*

**Definition 4.6.** Let $A$ be a $(\alpha,\beta)$ neutrosophic subset in $\mathbb{S}$, the strongest $(\alpha,\beta)$ neutrosophic relation on $\mathbb{S}$, that is a $(\alpha,\beta)$ neutrosophic relation on $A$ is $V$ given by

$$\left\{\begin{array}{l}\max\{\varpi^T_V(x,y),\alpha\}=\min\{\varpi^T_A(x),\varpi^T_A(y),\beta\}\\\max\{\varpi^I_V(x,y),\alpha\}=\min\{\varpi^I_A(x),\varpi^I_A(y),\beta\}\\\min\{\varpi^F_V(x,y),\alpha\}=\max\{\varpi^F_A(x),\varpi^F_A(y),\beta\}\end{array}\right\}.$$

**Theorem 4.7.** *Let $A$ be a $(\alpha,\beta)$ NSBS of $\mathbb{S}$ and $V$ be the strongest $(\alpha,\beta)$ neutrosophic relation of $\mathbb{S}$. Then $A$ is a $(\alpha,\beta)$ NSBS of $\mathbb{S}$ if and only if $V$ is a $(\alpha,\beta)$ NSBS of $\mathbb{S}\times\mathbb{S}$.*

**Theorem 4.8.** *Let $(\mathbb{S}_1,\boxplus_1,\boxplus_2,\boxplus_3)$ and $(\mathbb{S}_2,\boxdot_1,\boxdot_2,\boxdot_3)$ be any two bisemirings. The homomorphic image of $(\alpha,\beta)$ NSBS of $\mathbb{S}_1$ is a $(\alpha,\beta)$ NSBS of $\mathbb{S}_2$.*

**Proof.** Let $\Delta:\mathbb{S}_1\to\mathbb{S}_2$ be any homomorphism. Then $\Delta(x\boxplus_1 y)=\Delta(x)\boxdot_1\Delta(y),\Delta(x\boxplus_2 y)=\Delta(x)\boxdot_2\Delta(y)$ and $\Delta(x\boxplus_3 y)=\Delta(x)\boxdot_3\Delta(y)$ for all $x,y\in\mathbb{S}_1$. Let $V=\Delta(A)$, $A$ is any $(\alpha,\beta)$ NSBS of $\mathbb{S}_1$. Let $\Delta(x),\Delta(y)\in\mathbb{S}_2$. Let $x\in\Delta^{-1}(\Delta(x))$ and $y\in\Delta^{-1}(\Delta(y))$ be such that $\varpi^T_A(x)=\sup_{z\in\Delta^{-1}(\Delta(x))}\varpi^T_A(z)$ and $\varpi^T_A(y)=\sup_{z\in\Delta^{-1}(\Delta(y))}\varpi^T_A(z)$. Now,

$$\begin{aligned}\max\left[\varpi^T_V(\Delta(x)\boxdot_1\Delta(y))\ ,\alpha\right]&=\max\left[\sup_{z'\in\Delta^{-1}(\Delta(x)\boxdot_1\Delta(y))}\varpi^T_A(z')\ ,\alpha\right]\\&=\max\left[\sup_{z'\in\Delta^{-1}(\Delta(x\boxplus_1 y)}\varpi^T_A(z')\ ,\alpha\right]\end{aligned}$$

$$= \max\left[\varpi^T_A(x \boxplus_1 y), \alpha\right]$$
$$\geq \min\left\{\varpi^T_A(x), \varpi^T_A(y), \beta\right\}$$
$$= \min\left\{\varpi^T_V\Delta(x), \varpi^T_V\Delta(y), \beta\right\}.$$

Thus, $\max\left[\varpi^T_V(\Delta(x) \boxdot_1 \Delta(y))\ , \alpha\right] \geq \min\left\{\varpi^T_V\Delta(x), \varpi^T_V\Delta(y), \beta\right\}$.
Similarly, $\max\left[\varpi^T_V(\Delta(x) \boxdot_2 \Delta(y))\ , \alpha\right] \geq \min\left\{\varpi^T_V\Delta(x), \varpi^T_V\Delta(y), \beta\right\}$ and
$\max\left[\varpi^T_V(\Delta(x) \boxdot_3 \Delta(y))\ , \alpha\right] \geq \min\left\{\varpi^T_V\Delta(x), \varpi^T_V\Delta(y), \beta\right\}$.
Let $\Delta(x), \Delta(y) \in \mathbb{S}_2$. Let $x \in \Delta^{-1}(\Delta(x))$ and $y \in \Delta^{-1}(\Delta(y))$ be such that $\varpi^I_A(x) = \sup_{z\in\Delta^{-1}(\Delta(x))} \varpi^I_A(z)$ and $\varpi^I_A(y) = \sup_{z\in\Delta^{-1}(\Delta(y))} \varpi^I_A(z)$. Now,

$$\max\left[\varpi^I_V(\Delta(x) \boxdot_1 \Delta(y))\ , \alpha\right] = \max\left[\sup_{z'\in\Delta^{-1}(\Delta(x)\boxdot_1\Delta(y))} \varpi^I_A(z')\ , \alpha\right]$$
$$= \max\left[\sup_{z'\in\Delta^{-1}(\Delta(x\boxplus_1 y)} \varpi^I_A(z')\ , \alpha\right]$$
$$= \max\left[\varpi^I_A(x \boxplus_1 y), \alpha\right]$$
$$\geq \min\left\{\frac{\varpi^I_A(x) + \varpi^I_A(y)}{2}, \beta\right\}$$
$$= \min\left\{\frac{\varpi^I_V\Delta(x) + \varpi^I_V\Delta(y)}{2}, \beta\right\}$$

Thus, $\max\left[\varpi^I_V(\Delta(x) \boxdot_1 \Delta(y))\ , \alpha\right] \geq \min\left\{\frac{\varpi^I_V\Delta(x)+\varpi^I_V\Delta(y)}{2}, \beta\right\}$.
Similarly, $\max\left[\varpi^I_V(\Delta(x) \boxdot_2 \Delta(y))\ , \alpha\right] \geq \min\left\{\frac{\varpi^I_V\Delta(x)+\varpi^I_V\Delta(y)}{2}, \beta\right\}$ and
$\max\left[\varpi^I_V(\Delta(x) \boxdot_3 \Delta(y))\ , \alpha\right] \geq \min\left\{\frac{\varpi^I_V\Delta(x)+\varpi^I_V\Delta(y)}{2}, \beta\right\}$.
Let $x \in \Delta^{-1}(\Delta(x))$ and $y \in \Delta^{-1}(\Delta(y))$ be such that $\varpi^F_A(x) = \inf_{z\in\Delta^{-1}(\Delta(x))} \varpi^F_A(z)$ and $\varpi^F_A(y) = \inf_{z\in\Delta^{-1}(\Delta(y))} \varpi^F_A(z)$. Now,

$$\min\left[\varpi^F_V(\Delta(x) \boxdot_1 \Delta(y))\ , \alpha\right] = \min\left[\inf_{z'\in\Delta^{-1}(\Delta(x)\boxdot_1\Delta(y))} \varpi^F_A(z')\ , \alpha\right]$$
$$= \min\left[\inf_{z'\in\Delta^{-1}(\Delta(x\boxplus_1 y)} \varpi^F_A(z')\ , \alpha\right]$$
$$= \min\left[\varpi^F_A(x \boxplus_1 y), \alpha\right]$$
$$\leq \max\left\{\varpi^F_A(x), \varpi^F_A(y), \beta\right\}$$
$$= \max\left\{\varpi^F_V\Delta(x), \varpi^F_V\Delta(y), \beta\right\}.$$

Thus, $\min\left[\varpi_V^F(\Delta(x)\boxdot_1\Delta(y))\ ,\alpha\right]\leq\max\left\{\varpi_V^F\Delta(x),\varpi_V^F\Delta(y),\beta\right\}$.
Similarly, $\min\left[\varpi_V^F(\Delta(x)\boxdot_2\Delta(y))\ ,\alpha\right]\leq\max\left\{\varpi_V^F\Delta(x),\varpi_V^F\Delta(y),\beta\right\}$ and
$\min\left[\varpi_V^F(\Delta(x)\boxdot_3\Delta(y))\ ,\alpha\right]\leq\max\left\{\varpi_V^F\Delta(x),\varpi_V^F\Delta(y),\beta\right\}$. Hence $V$ is a $(\alpha,\beta)$ NSBS of $\mathbb{S}_2$.

**Theorem 4.9.** *Let $(\mathbb{S}_1,\boxplus_1,\boxplus_2,\boxplus_3)$ and $(\mathbb{S}_2,\boxdot_1,\boxdot_2,\boxdot_3)$ be any two bisemirings. The homomorphic preimage of $(\alpha,\beta)$ NSBS of $\mathbb{S}_2$ is a $(\alpha,\beta)$ NSBS of $\mathbb{S}_1$.*

**Proof.** Let $\Delta:\mathbb{S}_1\to\mathbb{S}_2$ be any homomrphism. Then $\Delta(x\boxplus_1 y)=\Delta(x)\boxdot_1\Delta(y),\Delta(x\boxplus_2 y)=\Delta(x)\boxdot_2\Delta(y)$ and $\Delta(x\boxplus_3 y)=\Delta(x)\boxdot_3\Delta(y)$ for all $x,y\in\mathbb{S}_1$. Let $V=\Delta(A)$, where $V$ is any $(\alpha,\beta)$ NSBS of $\mathbb{S}_2$. Let $x,y\in\mathbb{S}_1$. Then $\max\{\varpi_A^T(x\boxplus_1 y),\alpha\}=\max\{\varpi_V^T(\Delta(x\boxplus_1 y)),\alpha\}=\max\{\varpi_V^T(\Delta(x)\boxdot_1\Delta(y)),\alpha\}\geq\min\{\varpi_V^T\Delta(x),\varpi_V^T\Delta(y),\beta\}=\min\{\varpi_A^T(x),\varpi_A^T(y),\beta\}$. Thus, $\max\{\varpi_A^T(x\boxplus_1 y),\alpha\}\geq\min\{\varpi_A^T(x),\varpi_A^T(y),\beta\}$. Now, $\max\{\varpi_A^I(x\boxplus_1 y),\alpha\}=\max\{\varpi_V^I(\Delta(x\boxplus_1 y)),\alpha\}=\max\{\varpi_V^I(\Delta(x)\boxdot_1\Delta(y)),\alpha\}\geq\min\{\varpi_V^I\Delta(x),\varpi_V^I\Delta(y),\beta\}=\min\{\varpi_A^I(x),\varpi_A^I(y),\beta\}$. Thus, $\max\{\varpi_A^I(x\boxplus_1 y),\alpha\}\geq\min\{\varpi_A^I(x),\varpi_A^I(y),\beta\}$. Now, $\min\{\varpi_A^F(x\boxplus_1 y),\alpha\}=\min\{\varpi_V^F(\Delta(x\boxplus_1 y)),\alpha\}=\min\{\varpi_V^F(\Delta(x)\boxdot_1\Delta(y)),\alpha\}\leq\max\{\varpi_V^F\Delta(x),\varpi_V^F\Delta(y),\beta\}=\max\{\varpi_A^F(x),\varpi_A^F(y),\beta\}$. Thus, $\min\{\varpi_A^F(x\boxplus_1 y),\alpha\}\leq\max\{\varpi_A^F(x),\varpi_A^F(y),\beta\}$. Similarly to prove other two operations, hence $A$ is a $(\alpha,\beta)$ NSBS of $\mathbb{S}_1$.

## 5. $(\alpha,\beta)$ neutrosophic Normal Subbisemiring

In this section, we interact the theory for $(\alpha,\beta)$- neutrosophic normal subbisemiring. Here NNSBS stands for neutrosophic normal subbisemiring.

**Definition 5.1.** Let $A$ be any neutrosophic subset of $\mathbb{S}$ is said to be a NNSBS of $\mathbb{S}$ if it satisfies the following conditions:

$$\begin{Bmatrix}\varpi_A^T(x\diamond_1 y)=\varpi_A^T(y\diamond_1 x)\\ \varpi_A^T(x\diamond_2 y)=\varpi_A^T(y\diamond_2 x)\\ \varpi_A^T(x\diamond_3 y)=\varpi_A^T(y\diamond_3 x)\end{Bmatrix}\quad\begin{Bmatrix}\varpi_A^I(x\diamond_1 y)=\varpi_A^I(y\diamond_1 x)\\ \text{OR}\\ \varpi_A^I(x\diamond_2 y)=\varpi_A^I(y\diamond_2 x)\\ \text{OR}\\ \varpi_A^I(x\diamond_3 y)=\varpi_A^I(y\diamond_3 x)\end{Bmatrix}$$

$$\begin{Bmatrix}\varpi_A^F(x\diamond_1 y)=\varpi_A^F(y\diamond_1 x)\\ \varpi_A^F(x\diamond_2 y)=\varpi_A^F(y\diamond_2 x)\\ \varpi_A^F(x\diamond_3 y)=\varpi_A^F(y\diamond_3 x)\end{Bmatrix}$$

for all $x,y\in\mathbb{S}$.

**Theorem 5.2.** *(i) The intersection of a family of NNSBS s of $\mathbb{S}$ is a $NNSBS^s$ of $\mathbb{S}$.*
*(ii) The intersection of a family of $(\alpha,\beta)$ NNSBS of $\mathbb{S}$ is a $(\alpha,\beta)$ NNSBS s of $\mathbb{S}$.*

**Proof.** Proof follows from Theorem 3.3 and Theorem 4.3.

**Theorem 5.3.** *(i) If $A_1, A_2, ..., A_n$ are the family of $NNSBS^s$ of $\mathbb{S}_1, \mathbb{S}_2, ..., \mathbb{S}_n$ respectively, then $A_1 \times A_2 \times ... \times A_n$ is a NNSBS of $\mathbb{S}_1 \times \mathbb{S}_2 \times ... \times \mathbb{S}_n$.*
*(ii) If $A_1, A_2, ..., A_n$ are the family of $(\alpha, \beta)$ $NNSBS^s$ of $\mathbb{S}_1, \mathbb{S}_2, ..., \mathbb{S}_n$ respectively, then $A_1 \times A_2 \times ... \times A_n$ is a $(\alpha, \beta)$ NNSBS of $\mathbb{S}_1 \times \mathbb{S}_2 \times ... \times \mathbb{S}_n$.*

**Proof.** Proof follows from Theorem 3.4 and Theorem 4.4.

**Theorem 5.4.** *(i) Let $A$ be any NNSBS of $\mathbb{S}$ and $V$ be the strongest neutrosophic relation of $\mathbb{S}$. Then $A$ is a NNSBS of $\mathbb{S}$ if and only if $V$ is a NNSBS of $\mathbb{S} \times \mathbb{S}$.*
*(ii) Let $A$ be any $(\alpha, \beta)$ NNSBS of $\mathbb{S}$ and $V$ be the strongest $(\alpha, \beta)$ neutrosophic relation of $\mathbb{S}$. Then $A$ is a $(\alpha, \beta)$ NNSBS of $\mathbb{S}$ if and only if $V$ is a $(\alpha, \beta)$ NNSBS of $\mathbb{S} \times \mathbb{S}$.*

**Proof.** Proof follows from Theorem 3.7.

**Theorem 5.5.** *Let $(\mathbb{S}_1, \boxplus_1, \boxplus_2, \boxplus_3)$ and $(\mathbb{S}_2, \boxdot_1, \boxdot_2, \boxdot_3)$ be any two bisemirings.*
*(i) The homomorphic image of any NNSBS of $\mathbb{S}_1$ is a NNSBS of $\mathbb{S}_2$.*
*(ii) The homomorphic image of any $(\alpha, \beta)$ NNSBS of $\mathbb{S}_1$ is a $(\alpha, \beta)$ NNSBS of $\mathbb{S}_2$.*

**Proof.** Proof follows from Theorem 3.12 and Theorem 4.8.

**Theorem 5.6.** *Let $(\mathbb{S}_1, \boxplus_1, \boxplus_2, \boxplus_3)$ and $(\mathbb{S}_2, \boxdot_1, \boxdot_2, \boxdot_3)$ be any two bisemirings.*
*(i) The homomorphic preimage of any NNSBS of $\mathbb{S}_2$ is a NNSBS of $\mathbb{S}_1$.*
*(ii) The homomorphic preimage of any $(\alpha, \beta)$ NNSBS of $\mathbb{S}_2$ is a $(\alpha, \beta)$ NNSBS of $\mathbb{S}_1$.*

**Proof.** Proof follows from Theorem 3.13 and Theorem 4.9.

**Acknowledgments:** The authors would like to thank the Editor-InChief and the anonymous referees for their various suggestions and helpful comments that have led to the improved in the quality and clarity version of the paper.

**Conflicts of Interest:** The author declares no conflict of interest.

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Received: Nov 19, 2021. Accepted: Feb 7, 2022

NSS

Neutrosophic Sets and Systems, Vol. 48, 2022

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# ClassicalBalanced, AntiBalanced and NeutroBalanced functions

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**Abstract**. In this paper, we extend the concepts of Neutrosophy to Boolean function and define ClassicalBalanced, AntiBalanced and NeutroBalanced functions. We consider functions of the form $f(x) = Tr(x^d)$, where the exponent $d$ may be Gold exponent, Kasami exponent, Welch exponent or any arbitrary positive integer. We, for different values of d, examine nature of these functions with respect to the above stated three categories.

**Keywords:** Balanced function; Neutrosophy; AntiBalanced function; NeutroBalanced function; Cryptography.

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## 1. Introduction

In an algebraic structure, the axioms are valid and the operations are defined everywhere. We cannot do much mathematics just on sets. We need some sort of algebraic structures for analysis. In real life situations when we require to combine the elements of a particular domain in a certain manner, it may happen the combination is not meaningful for certain pairs. It may be undefined, indeterminate or multivalued. In such situation, we cannot have an algebraic structure and we are left with no option but to modify the combining operations.

What if we have the theoretical platform to deal with such operation the way they are. This line of thinking lead to evolution of Neutrosophy. The history of Neutrosophy is dated back to 1998 when Florentin Smarandache propounde the notion of Neutrosophy in [3]. However, the research in this area gained momentum in last couple of years. Some recent works may be found in [8–11].

Neutrosophic structures have been defined on Algebra [1–7], groups [14–16, 20, 21] and ring [12, 13] and their properties have been explored. We define neutrosophic structure on a finite field. The main reasons of choosing finite fields are:

(1) This direction remains by and large unexplored.
(2) Fields are the richest structure. The finiteness is considered to make it more computer friendly. When it is not possible to produce a rigorous logic for proving an assertion, computational tools may be utilised to establish the assertion.
(3) Finite fields are widely used in cryptography. We try to translate the concepts of Boolean function to the Neutrosophic scenario. This may lead to application of these function to cryptography.

We define three types of functions viz., ClassicalBalanced, AntiBalanced and NeutroBalanced functions. The details can be found in the subsequent sections. The paper is structured in the following manner.

In the next section we discuss preliminaries required to comprehend the paper. In section 3 we introduce three neutrosophic functions as mentioned above. In the fourth section we prove some results related to the defined function. Finally, in section 5, we conclude the paper.

## 2. **Preliminaries**

**Definition 2.1.** [4]

(i) A classical operation is an operation well defined for all the set's elements while a Neutro Operation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set. An AntiOperation is an operation that is outer definedfor all the set's elements.
(ii) A NeutroAlgebra is an algebra that has at least one Neutro Operation or one NeutroAxiom ( axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no AntiOperation or AntiAxiom. An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

The study and analysis of cryptographic and combinatorial properties with respect to Boolean functions has been an important branch of cryptography. Boolean functions play a significant role in the construction of components used in symmetric ciphers, and cryptographic properties of such functions are of great interest. Boolean functions used in cryptographic applications provide security of a cipher against different kinds of attacks.

Over the prime field, $\mathbb{F}_2$ the $n$-dimensional vector space can be denoted as $\mathbb{F}_2^n$. One can identify this vector space $\mathbb{F}_2^n$ over $\mathbb{F}_2$ with the finite field $\mathbb{F}_{2^n}$ of $2^n$ elements, which is basically extension of the finite field $\mathbb{F}_2 = \{0, 1\}$ using some irreducible polynomial of degree $n$ with coeffiecients either 0 or 1.

A Boolean function in $n$ variables is an arbitrary function from $\mathbb{F}_2^n \rightarrow \mathbb{F}_2$, where $\mathbb{F} = \{0, 1\}$ is a Boolean domain and $n$ is a non-negative integer. It is called Boolean in honor of the British mathematician and philosopher George Boole $(1815 - 1864)$.

The vectorial Boolean function is of the form $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ with range of the function being $\mathbb{F}_2^m$, where $m > 1$. It is also called an $(n, m)$-function. If $m = n$ then it is called as $(n, n)$-function. For vectorial Boolean functions we use uppercase letters, whereas Boolean functions are denoted with lowercase letters.

The finite field $\mathbb{F}_{2^n}$ of order $2^n$ is also denoted as $GF(2^n)$, which is due to French mathematician Evariste Galois (1811-1832). Usually, $\mathbb{F}_{2^n}^*$ is the denotion used to represent the collection of all nonzero elements in the field $\mathbb{F}_{2^n}$. With respect to multiplication, $\mathbb{F}_{2^n}^*$ acts as a cyclic group with order of the group being $2^n - 1$. For basic and recent results on finite fields, permutation polynomials, balanced functions and trace functions we refer [19, 22–24, 27].

The trace representation is very useful in defining and analyzing various properties of Boolean functions.

**Definition 1.** *[25] If $c$ is an element of $K = GF(q^n)$, its trace relative to the subfield $\mathbb{F} = GF(q)$ is defined as follows:*

$$Tr_F^K(c) = c + c^q + c^{q^2} + \ldots + c^{q^{n-1}}.$$

The values of trace functions fall into the prime field $\mathbb{F}_2$ is the most important property of the trace. Since $\mathbb{F}_2^n$ is isomorphic with $\mathbb{F}_{2^n}$, trace function can also be viewed as a Boolean function in $n$ variables. In case of the base field $\mathbb{F}_2$, we use the notation $Tr$ for trace.

Consider the finite field $GF(9) = (00, 10, 11, -11, 0-1, -10, -1-1, 1-1, 01)$. Trace is tabulated as below.

| x | 00 | 10 | 11 | -11 | 0-1 | -10 | -1-1 | 1-1 | 01 |
|---|---|---|---|---|---|---|---|---|---|
| Trace(x) | 0 | 1 | 0 | 1 | 1 | -1 | 0 | -1 | -1 |

A Boolean function $f$ is said to be balanced if the output column of its truth table has same number of zero's and one's. As balanced functions would give outputs with the balanced number of zero and one, which appears more random. Hence, these functions avoid statistical dependencies between the input and output of the stream cipher, which prevents distinguishing attacks and statistical analysis [26, 27]. From cryptographic point of view, balanced functions are very important. In case of an unbalanced function, the input and output variables have considerable dependence on each other, which may cause susceptible cryptanalysis attacks.

In [17–19], one can find the construction of many such power function $f : \mathbb{F}_{2^n} \mapsto \mathbb{F}_2$ with trace representation. Some well known examples of Boolean functions $f(x) = Tr(x^d)$, where $d$ is given by the table 1 are as follows

Table 1. Boolean functions

| | Exponent "d " | Conditions |
|---|---|---|
| Gold function | $2^i + 1$ | $\gcd(i, n) = 1$ |
| Kasami function | $2^{2i} - 2^i + 1$ | $\gcd(i, n) = 1$ |
| Welch function | $2^t + 3$ | $n = 2t + 1$ |

The terminology of balancedness always comes with the idea of the measurement of different conditions. The balanced characteristic of a function is defined with the classification of its co-domain. Here in the next section, we present three types of balanced Neutrosophic functions.

## 3. ClassicalBalanced, AntiBalanced and NeutroBalanced functions

Let $\psi$ be a Neutrosophic functions defined on $\mathbb{F}_3^n$ to $K$, where $K$ is some arbitrary set. Note that there will be three partitions of the domains say $P_0$, $P_1$, $P_2$ such that $\psi$ is defined on $P_0$, not defined on $P_1$ and indeterminate on $P_2$. The Neutrosophic function $\psi$ induces a generalised Boolean function $f$ on $\mathbb{F}_3^n \to \mathbb{F}_3$ as

$$f(x) = i \text{ when } x \in P_i.$$

It can be seen easily, every function $f : \mathbb{F}_3^n \to \mathbb{F}_3$ induces a neutrosophic function $\mathbb{F}_3^n \to K$. Thus there are one to one correspondance between Neutrosophic functions $\mathbb{F}_3^n \to K$ and the generalised Boolean function from $\mathbb{F}_3^n \to \mathbb{F}_3$. We can therefore, identify a Neutrosophic function from $\mathbb{F}_3^n \to K$ by a generalised Boolean function from $\mathbb{F}_3^n \to \mathbb{F}_3$. With this identification, we proceed further and define neutrosophic functions.

Note that any $f : \mathbb{F}_3^n \to \mathbb{F}_3$ can be given as $f(x) = Tr(h(x))$, where $h$ is a function defined on $\mathbb{F}_3^n$. We are now fully equipped to define ClassicalBalanced, AntiBalanced and NeutroBalanced functions.

**Definition 3.1.** A function is said to be ClassicalBalanced function if it takes equal number of 1's, 0's and -1's.

**Example 3.2.** Over $\mathbb{F}_{3^5}$, the function $f(x) = Tr(x^9)$ is a ClassicalBalanced function.

**Definition 3.3.** A function is said to be AntiBalanced function if number of 1's, 0's, and -1's are all distinct from each other.

**Example 3.4.** Over $\mathbb{F}_{3^2}$, the function $f(x) = Tr(x^8)$ is a AntiBalanced function.

**Definition 3.5.** A function is said to be NeutroBalanced function if number of 1's, 0's, and -1's are not same but exactly two of them are equal.

**Example 3.6.** Over $\mathbb{F}_{3^4}$, the function $f(x) = Tr(x^{14})$ is a NeutroBalanced function.

## 4. **Some Special types of Neutrosophic functions**

Composition of two functions is an intrinsic approach in the upcoming results to construct ClassicalBalanced, AntiBalanced and NeutroBalanced function. Trace of finite field is a common choice for one of the compositions of two functions. Here in the next two propositions we present the necessary and sufficient conditions for a composition of two functions to be ClassicalBalanced, AntiBalanced and NeutroBalanced function.

**Proposition 1.** *Let $f : \mathbb{F}_{3^n} \to \mathbb{F}_3$ be a Boolean function and $h$ be any bijection on $f : \mathbb{F}_{3^n}$. Then $f$ is ClasicalBalanced, AntiBalanced or NeutroBalanced if and only if the composition map $fh$ is ClasicalBalanced, AntiBalanced or NeutroBalanced respectively.*

*Proof.* The proof is obvious. □

**Proposition 2.** *The exponential map $x \to x^a, a \in \mathbb{Z}$ on $\mathbb{F}_{3^n}$ is a bijection if and only if* $\gcd(a, 3^n - 1) = 1$.

*Proof.* If map $x \to x^a, a \in \mathbb{Z}$ on $\mathbb{F}_{3^n}$ is a bijection then the proof is obvious. Now let

$$\gcd(a, 3^n - 1) = 1$$

and $x_1(\neq 0)$, $x_2(\neq 0) \in \mathbb{F}_3^n$ and $g$ be a generator of non zero elements of $\mathbb{F}_3^n$. Let if

$$f(x_1) = f(x_2),$$

then

$$x_1^a = x_2^a,$$
$$\implies (g^{u_1})^a = (g^{u_2})^a,$$
$$\implies (g^{u_1 - u_2})^a = 1,$$

$$\implies 3^n - 1 | (u_1 - u_2)(a)$$

or

$$(u_1 - u_2)(a) = 0 \mod 3^n - 1. \quad (1)$$

Now if $\gcd(a, 3^n - 1) = 1$ then,

$$(u_1 - u_2) = 0 \mod 3^n - 1. \quad (2)$$

Now since, $1 \le u_1, u_2 \le 3^n - 1$ therefore

$$u_1 - u_2 \le 3^n - 1. \quad (3)$$

From (2) and (3), $u_1 = u_2$, which implies that, $x_1 = x_2$. Hence the result is proved. □

In the next theorem we prove the ClassicalBalanced property of Trace function over finite field $\mathbb{F}_{3^n}$.

**Theorem 4.1.** *A function of the form*

$$f(x) = Tr(x)$$

*is a ClassicalBalanced function over the finite field* $\mathbb{F}_{3^n}$.

*Proof.* We have

$$f(x) = Tr(x) = x + x^3 + x^9 + \ldots + x^{3^{n-1}}. \quad (4)$$

This is the absolute trace mapping the elements of $\mathbb{F}_{3^n}$ to the prime field $\mathbb{F}_3$. Therefore,

$$\begin{aligned} & f^{-1}(\mathbb{F}_3) = \mathbb{F}_{3^n}, \\ \implies & f^{-1}(0) \cup f^{-1}(1) \cup f^{-1}(2) = \mathbb{F}_{3^n}, \\ \implies & |f^{-1}(0)| + |f^{-1}(1)| + |f^{-1}(2)| = 3^n. \quad (5) \end{aligned}$$

Let $|f^{-1}(0)| = \alpha_1, |f^{-1}(1)| = \alpha_2$ and $|f^{-1}(2)| = \alpha_3$. Then from (5)

$$\alpha_1 + \alpha_2 + \alpha_3 = 3^n. \quad (6)$$

Now from (4) we have

$$\alpha_1 = |\{x | x + x^3 + x^9 + \ldots + x^{3^{n-1}} = 0\}|, \quad (7)$$

$$\alpha_2 = |\{x | x + x^3 + x^9 + \ldots + x^{3^{n-1}} - 1 = 0\}| \quad (8)$$

*and*

$$\alpha_3 = |\{x | x + x^3 + x^9 + \ldots + x^{3^{n-1}} - 2 = 0\}|. \quad (9)$$

All equations (7), (8) and (9) has a polynomial of degree $3^{n-1}$. So, each can have at most $3^{n-1}$ roots and we conclude that

$$0 \le \alpha_1, \alpha_2, \alpha_3 \le 3^{n-1}. \quad (10)$$

It is clear that (6) and (10) hold together if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 3^{n-1}$. Thus our assertion is proved. □

Before start of the proof for next theorem here we prove a lemma.

**Lemma 4.2.** *Let $i$ and $n$ are positive integers, If $< n, i >= 1$ then $< 3^n - 1, 2^i + 1 >= 1$.*

*Proof.* Here $n$ and $i$ are positive integers therefore let $p(\neq 2, 3)$ be any prime number such that $p|(3^n - 1)$, then we can write $3^n = 1 \mod p$, which implies that $p - 1|n$ or

$$n = 0 \mod (p-1). \tag{11}$$

Now if $p|2^i + 1$, then $2^i = -1 \mod p$ which implies that $\frac{p-1}{2}|i$, consequently $p - 1|2i$ or

$$2i = 0 \mod (p-1). \tag{12}$$

Now combining (11) and (12), we can write $< n, 2i >= p - 1$. Therefore if $< n, i >= 1$ then at max gcd of $n$ and $2i$ will be 2 but since $p \neq 2\, or\, 3$, hence there does not exist any prime $p \neq 2\, or\, 3$ such that $p|(3^n - 1)$ and $p|(2^i + 1)$. Hence

$$< 3^n - 1, 2^i + 1 >= 1.$$

□

Now in the next theorem we present the bijective condition for an exponent function on $\mathbb{F}_3^n$.

**Theorem 4.3.** *Let $f : \mathbb{F}_{3^n} \mapsto \mathbb{F}_{3^n}$ be a function defined as $f(x) = x^{2^i+1}$. If $< i, n >= 1$ for any positive integer $i$, then $f$ is a bijective function.*

*Proof.* The proof follow from the lemma 4.2 and proposition 2. □

**Corollary 4.4.** *A function of the form*

$$f(x) = Tr(x^{2^i+1}),$$

*for any positive integer $i$, is a ClassicalBalanced function with $\gcd(i, n) = 1$ over the finite field $\mathbb{F}_{3^n}$.*

*Proof.* Theorem 4.1 and theorem 4.3 follows the proof of this corollary. □

If $\gcd(i, n) \neq 1$, then the functions $f(x) = Tr(x^{2^i+1})$ cannot be a ClassicalBalanced functions, which we can observe from the following examples.

**Example 4.5.** Over $\mathbb{F}_{3^4}$, the function $f(x) = Tr(x^{2^2+1})$ is not a ClassicalBalanced function.

**Example 4.6.** Over $\mathbb{F}_{3^5}$, the function $f(x) = Tr(x^{2^5+1})$ is not a ClassicalBalanced function.

Some more general results for AntiBalanced and NeutroBalanced functions are presented in following theorems.

**Theorem 4.7.** *A function of the form*

$$f(x) = Tr(x^{8a})$$

*is a AntiBalanced function for* $a \in \mathbb{N}$, $gcd(a,\, 3^2 - 1) = 1$ *over the finite field* $\mathbb{F}_{3^2}$.

*Proof.* Given function $f(x)$ is composition of two functions, $Tr(x) = x + x^3$ and $h(x) = x^a$, therefore,

$$f(x) = Tr(x^{8a}) = f_1 h(x) \tag{13}$$

where $f_1(x) = Tr(x^8)$ and $h(x) = x^a$. It is given that $\gcd(a, 3^2 - 1) = 1$, therefore from propositions 1 and 2, $f$ is AntiBalanced if and only if $f_1$ is AntiBalanced. We now show that $f_1$ is AntiBalanced. From the expression of trace and $f_1$,

$$f_1(x) = 0 \implies x^8 + x^{24} = 0$$

Similarly,

$$f_1(x) = 1 \implies x^8 + x^{24} - 1 = 0$$

and

$$f_1(x) = 2 \implies x^8 + x^{24} - 2 = 0.$$

It can be verified computationally or otherwise that the polynomials $x^8 + x^{24}$, $x^8 + x^{24} - 1$, $x^8 + x^{24} - 2$ has 1, 0 and 8 distinct roots in $\mathbb{F}_{3^2}$. Hence, $f_1$ is AntiBalanced. This proves our assertion as well. □

**Theorem 4.8.** *A function of the form*

$$f(x) = Tr(x^{2a})$$

*is a AntiBalanced function for* $a \in \mathbb{N}$ *with* $2a \not\equiv 0(\mod 13)$ *over the finite field* $\mathbb{F}_{3^3}$.

*Proof.* Let

$$f(x) = Tr(x^{2a}) = f_1 h(x) \tag{14}$$

where $f_1(x) = Tr(x^2)$ and $h(x) = x^a$. Now $2a \not\equiv 0(\mod 13)$ implies that $\gcd(a, 3^3 - 1) = 1$. Propositions 1 and 2 confirm that $f$ is AntiBalanced if and only if $f_1$ is AntiBalanced. Further we show that $f_1$ is AntiBalanced. From Trace $Tr(x) = x + x^3 + x^9$ over $\mathbb{F}_{3^3}$,

$$f_1(x) = 0 \implies x^2 + x^6 + x^{18} = 0.$$

Similarly,

$$f_1(x) = 1 \implies x^2 + x^6 + x^{18} - 1 = 0$$

and

$$f_1(x) = 2 \implies x^2 + x^6 + x^{18} - 2 = 0$$

After computational verification we found that the polynomials $x^2+x^6+x^{18}$, $x^2+x^6+x^{18}-1$, $x^2 + x^6 + x^{18} - 2$ has 9,6 and 12 distinct roots in $\mathbb{F}_{3^2}$. Hence, $f_1$ is AntiBalanced. □

**Theorem 4.9.** *A function of the form*

$$f(x) = Tr(x^{16a})$$

*is a AntiBalanced function for* $a \in \mathbb{N}$ *and* $16a \neq 0 \mod 80$ *over the finite field* $\mathbb{F}_{3^4}$.

*Proof.* Here $Tr(x) = x + x^3 + x^9 + x^{27}$ over $\mathbb{F}_{3^4}$ and

$$f(x) = Tr(x^{16a}) = f_1 h(x) \tag{15}$$

where $f_1(x) = Tr(x^{16})$ and $h(x) = x^a, \gcd(a, 3^4 - 1) = 1$. In view of propositions 1 and 2, $f$ is AntiBalanced if and only if $f_1$ is AntiBalanced. We now show that $f_1$ is AntiBalanced.

$$f_1(x) = 0 \implies x^2 + x^6 + x^{18} = 0$$

Similarly,

$$f_1(x) = 1 \implies x^2 + x^6 + x^{18} - 1 = 0$$

and

$$f_1(x) = 2 \implies x^2 + x^6 + x^{18} - 2 = 0$$

It is verified computationally or otherwise that the polynomials $x + x^3 + x^9 + x^{27}$, $x + x^3 + x^9 + x^{27} - 1$ and $x + x^3 + x^9 + x^{27} - 2$ has 1,16 and 64 distinct roots in $\mathbb{F}_{3^4}$ respectively. Hence, $f_1$ is AntiBalanced. This proves our assertion as well. □

**Theorem 4.10.** *A function of the form*

$$f(x) = Tr(x^{2a})$$

*is a AntiBalanced function for* $a \in \mathbb{N}$, $gcd(a,\ 3^5 - 1) = 1$ *over the finite field* $\mathbb{F}_{3^5}$.

*Proof.* Trace function in the finite field $\mathbb{F}_{3^5}$ is a polynomial $Tr(x) = x+x^3+x^9+x^{27}+x^{81} \in \mathbb{F}_3$ where $x \in \mathbb{F}_{3^5}$. Now given function

$$f(x) = Tr(x^{16a}) = f_1 h(x) \tag{16}$$

where $f_1(x) = Tr(x^2)$ and $h(x) = x^a$. From proposition 2, $\gcd(a, 3^5 - 1) = 1$ implies that $h$ is a bijectiona. Now we now show that $f_1$ is AntiBalanced.

$$f_1(x) = 0 \implies x + x^3 + x^9 + x^{27} + x^{81} = 0$$

Similarly,

$$f_1(x) = 1 \implies x + x^3 + x^9 + x^{27} + x^{81} - 1 = 0$$

and

$$f_1(x) = 2 \implies x + x^3 + x^9 + x^{27} + x^{81} - 2 = 0$$

We found from computation search that the polynomials $x + x^3 + x^9 + x^{27} + x^{81}$, $x + x^3 + x^9 + x^{27} + x^{81} - 1$ and $x + x^3 + x^9 + x^{27} + x^{81} - 2$ has 81,90 and 72 distinct roots in $\mathbb{F}_{3^5}$ respectively. Therefore, $f_1$ is AntiBalanced. From proposition 1, $f$ is AntiBalanced if and only if $f_1$ is AntiBalanced. Hence the theorem is proved. □

**Theorem 4.11.** *A function of the form*

$$f(x) = Tr(x^{2a})$$

*is a NeutroBalanced function for* $a \in \mathbb{N}$ *with* $2a \not\equiv 0 \pmod 8$ *over the finite field* $\mathbb{F}_{3^2}$.

*Proof.* The trace of $\mathbb{F}_{3^2}$ is $Tr(x) = x + x^3 \in \mathbb{F}_3$ where $x \in \mathbb{F}_{3^2}$. Given function $f(x)$ can be written as,

$$f(x) = Tr(x^{2a}) = f_1 h(x) \tag{17}$$

where $f_1(x) = Tr(x^2)$ and $h(x) = x^a$. Now from the given condition $\gcd(a, 3^2 - 1) = 1$ and proposition 2, $h$ is a bijective function. Now We show that $f_1$ is NeutroBalanced. Using $Tr(x)$ on $\mathbb{F}_{3^2}$, we can write

$$f_1(x) = 0 \implies x + x^3 = 0$$

Similarly,

$$f_1(x) = 1 \implies x + x^3 - 1 = 0$$

and

$$f_1(x) = 2 \implies x + x^3 - 2 = 0$$

Count of the roots of above three polynomials can settle the proof of Neutrobalanced property of $f$. It can be verified computationally or otherwise that the polynomials $x+x^3$, $x+x^3-1$ and $x + x^3 - 2$ has 5, 2 and 2 distinct roots in $\mathbb{F}_{3^2}$ respectively. Hence, $f_1$ is NeutroBalanced. It is already proved in proposition 1 that $f$ is NeutroBalanced if and only if $f_1$ is NeutroBalanced. Hence the theorem is proved □

**Theorem 4.12.** *A function of the form*

$$f(x) = Tr(x^{2a})$$

*is a NeutroBalanced function for* $a \in \mathbb{N}$ *with* $2a \not\equiv 0(\mod 16)$ *over the finite field* $\mathbb{F}_{3^4}$.

*Proof.* Here the trace function, $Tr(x)$, on the extension field $\mathbb{F}_{3^4}$ is

$$Tr(x) = x + x^3 + x^9 + x^{27} \in \mathbb{F}_3,$$

where $x \in \mathbb{F}_{3^4}$. Now given function

$$f(x) = Tr(x^{2a}) = f_1 h(x), \tag{18}$$

where $f_1(x) = Tr(x^2)$ and $h(x) = x^a$, $\gcd(a, 3^4 - 1) = 1$. In view of propositions 1 and 2, $f$ is NeutroBalanced if and only if $f_1$ is NeutroBalanced. We now show that $f_1$ is NeutroBalanced. From the expression of $Tr(x)$ on $\mathbb{F}_{3^4}$,

$$f_1(x) = 0 \implies x + x^3 + x^9 + x^{27} = 0$$

Similarly,

$$f_1(x) = 1 \implies x + x^3 + x^9 + x^{27} - 1 = 0$$

and

$$f_1(x) = 2 \implies x + x^3 + x^9 + x^{27} - 2 = 0$$

Now it can be observe from proposition 1 that $f$ is NeutroBalanced if enumeration of roots of any two polynomials from $x + x^3 + x^9 + x^{27}$, $x + x^3 + x^9 + x^{27} - 1$ and $x + x^3 + x^9 + x^{27} - 2$ are same. We found computationally that the polynomials $x + x^3 + x^9 + x^{27}$, $x + x^3 + x^9 + x^{27} - 1$ and $x + x^3 + x^9 + x^{27} - 2$ has 21, 30 and 30 distinct roots in $\mathbb{F}_{3^4}$ respectively. Hence, $f_1$ is NeutroBalanced. This proves our assertion as well. □

## 5. Conclusions

In this paper, we have defined ClassicalBalanced, AntiBalanced and NeutroBalanced functions. So far a function over finite field is classified into balanced function and unbalanced function. With this work it is a new approach to define a class of functions which lie between these two, which are called as NeutroBalanced functions. NeutroBalanced functions are defined with the logic of neutrosophy. NeutroBalanced functions may lead to a new direction with its application in point of view.

**Funding:** "Fourth author received funding from VGST/KSTePS and DST, GoK, India"

**Acknowledgments:** Fourth author thanks VGST/KSTePS and DST, GoK, India for sanctioning the project [VGST/GRD-897/2019-20/2020-21]. The authors are grateful to Prof. Dr. Florentin Smarandache from University of New Mexico for his valuable ideas and guidance.

**Conflicts of Interest:** "The authors declare no conflict of interest."

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Received: Nov 20, 2021. Accepted: Feb 4, 2022

NSS

Neutrosophic Sets and Systems, Vol. 48, 2022

University of New Mexico

# Bipolar neutrosophic graded soft sets and their topological spaces

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**Abstract**. Bipolar neutrosophic soft sets and their properties were discussed in several articles by many researchers. Soft sets are parametrized sets and bipolar neutrosophic soft sets are the fusion of bipolar neutrosophic sets and soft sets and there will be a number of decision variables or parameters. In many circumstances, the significance of each parameter is not equal. So the selection at the end might be unfit to the scenario. In this paper, we proposed Bipolar neutrosophic graded soft sets and their topological spaces. The proposed method fills that gap among the selection.

**Keywords:** Neutrosophy; Bipolar neutrosophic set; Soft set; Bipolar neutrosophic graded soft topology; Topology.

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## 1. Introduction

Exact solution not always exist for real life problems. Most of the scenarios interfere with some unwanted information called uncertainties. Due to uncertainty, one cannot conclude the problem with exact solution. In such cases, conventional methods are not efficient to deal with indeterminate. Fortunately, in recent years, there are many concepts were defined to deal such uncertainties. Neutrosophy is one of the technique which is suitable for problems with uncertainties. Neutrosophy is the extension of Intiutionistic fuzzy theory (originated from fuzzy theory). Neutrosophic sets are derived from Neutrosophy which is used in many decision making problems. Florentin Smarandache [6,8] was introduced this Neutrosophy concept. Neutrosophic set is a set of three memberships namely, Truth, Indeterminacy and False membership range in the non-standard interval $]^{-}0, 1^{+}[$. The non-standard intervals are only for theoretical purpose, but we prefer specific solution for real life problems; Single valued neutrosophic set (SVNS) is the set which is defined by Wang et al. [7] having variables ranges in the standard interval [0,1] instead of non-standard interval.

Majundar et al. [4, 5] proposed some notions on neutrosophic sets and single valued neutrosophic sets. Bipolar neutrosophic sets (BNS), extension of neutrosophic sets were defined by Deli et al. [2] in 2015 and similarity measures of bipolar neutrosophic sets were proposed by Uluay et al. [3] in 2016. In 2012, A.A.Salama et al. [25] extended fuzzy topology and intuitionistic fuzzy topology to neutrosophic topology. In 2017, Francisco Gal. [23] proposed the difference between intiutionistic fuzzy topology and neutrosophic topology and proved that they're not the same. In 2017, Tuhin Bera et al. [22] extended the neutrosophic topology concept to neutrosophic soft topology and proposed some of their properties. Syeda Tayyba et al. [24] proposed a decision making technique using bipolar neutrosophic soft topology. D. Molodtsov [12] introduced soft set theory in 1999. In 2014, Ridvan Sahin et al. [1] proposed some notions on neutrosophic soft sets. Ali et al. [11] introduced the concepts of bipolar neutrosophic soft sets in 2017. In 2019, Arulpandy P et al. [18] and Taha Yasin Ozturk et al. [21] proposed the new approaches on bipolar neutrosophic soft sets and some of their similarity and entropy measurements. Neutrosophic sets are widely used in decision making scenarios. In 2019, Arulpandy et al. [17] were proposed the representation of grayscale images and reduction of indeterminacy in bipolar neutrosophic domain which is very useful for image processing tasks. Also many articles were published in recent years about the applications of neutrsophy in engineering and medical fields [14–20].

In our study, the novel set and topology namely, Bipolar neutrosophic graded soft set (BNGS) and Bipolar neutrosophic soft topological space were proposed. This paper was organized as follows: Section 1 consists introduction and literature survey about the main topic. Section 2 consists some of the preliminaries required for the main topic. Section 3 deals with the proposed set namely, Bipolar neutrosophic graded soft sets and their properties with numerical examples. Section 4 deals with the proposed topology namely, Bipolar neutrosophic graded soft topological spaces with their properties and some propositions about the proposed topology. Finally, Section 5 concludes our study with future research goals.

## 2. Preliminaries

**Definition 2.1.** [8] Let X be a universal set and $x \in X$. A Neutrosophic set $N$ is defined by

$$N = \{\langle x, T_N(x), I_N(x), F_N(x)\rangle : x \in X\}$$

where $T_N(x), I_N(x), F_N(x)$ known as truth, indeterminacy and falsify membership values respectively.

Also, $T_N(x), I_N(x), F_N(x) : X \to ]^-0, 1^+[$ and it satisfies $^-0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3^+$.

**Example 2.2.** Let $X = \{x_1, x_2, x_3\}$ be the universe set consists attributes of machines. Also, $x_1, x_2$ and $x_3$ denotes reliability, performance and cost of a machine, respectively; $T_N(x), I_N(x)$ and $F_N(x)$ denotes the degree of good service, indeterminacy, degree of poor service respectively. The neutrosophic set $N$ is defined by

$N = \left\{ \langle x_1, 0.7, 0.2, 0.3\rangle, \langle x_2, 0.2, 0.4, 0.8\rangle, \langle x_3, 0.4, 0.4, 0.6\rangle \right\}$

where $^-0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3^+$

**Definition 2.3.** [7] Single valued neutrosophic set is the neutrosophic set with the membership range of standard interval [0,1]. It is very convenient while solving real life problems.

A single valued neutrosophic set $N$ is defined by

$$N = \{\langle x, T_N(x), I_N(x), F_N(x)\rangle : x \in X\}$$

where $T_N(x), I_N(x), F_N(x) : X \to [0, 1]$ such that $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3$.

**Definition 2.4.** $[9, 12]$ Let $X$ be a universe set. A Soft set is a pair $(f, E)$ such that

$$f : E \to P(x)$$

Where $P(x)$ is a power set of $X$. Soft set is a parameterized family of subsets of the universe set $X$.

**Example 2.5.** Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of computer systems and let $E = \{e_1, e_2, e_3\}$ be set of parameters. where $e_1$=Processor, $e_2$=Graphics and $e_3$=Storage.

suppose that

$$f(e_1) = \{x_2, x_3\}$$
$$f(e_2) = \{x_2, x_4\}$$
$$f(e_3) = \{x_1, x_3\}.$$

Then, $f(E) = \{f(e_1), f(e_2), f(e_3)\}$.

The set $f(E)$ is a soft set (parameterized family of subsets of $X$).

**Definition 2.6.** [10] A neutrosophic soft set $(f_N, E)$ over $X$ is defined by the set

$$(f_N, E) = \left\{\langle e, f_N(e)\rangle : e \in E, f_N(e) \in NS(X)\right\}$$

where $f_N : E \longrightarrow NS(x)$ such that $f_N(e) = \varphi$ if $e \notin A$.

Also, since $f_N(e)$ is a neutrosophic set over $X$ is defined by

$$f_N(e) = \left\{\left\langle x, T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x)\right\rangle : x \in X\right\}$$

where $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x)$ represents truth value of $x$ for the parameter $e$, indeterminate value of $x$ for the parameter $e$ and false value of $x$ for the parameter $e$.

**Example 2.7.** Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of computer systems under consideration. Let $E = \{e_1, e_2, e_3\}$ be set of parameters where $e_1, e_2, e_3$ represents Processor speed, Graphics index and Storage capacity, respectively.
Then we define

$$(f_N, E) = \{\langle e_1, f_N(e_1)\rangle, \langle e_2, f_N(e_2)\rangle, \langle e_3, f_N(e_3)\rangle\}$$

Here

$$f_N(e_1) = \left\{\langle x_1, 0.1, 0.4, 0.2\rangle, \langle x_2, 0.3, 0.5, 0.3\rangle, \langle x_3, 0.9, 0.2, 0.1\rangle, \langle x_4, 0.4, 0.5, 0.9\rangle\right\}$$

$$f_N(e_2) = \left\{\langle x_1, 0.2, 0.3, 0.4\rangle, \langle x_2, 0.3, 0.5, 0.6\rangle, \langle x_3, 0.4, 0.1, 0.7\rangle, \langle x_4, 0.9, 0.5, 0.6\rangle\right\}$$

$$f_N(e_3) = \left\{\langle x_1, 0.1, 0.5, 0.8\rangle, \langle x_2, 0.7, 0.5, 0.3\rangle, \langle x_3, 0.5, 0.7, 0.2\rangle, \langle x_4, 0.3, 0.5, 0.2\rangle\right\}$$

So that $(f_A, E)$ is a Neutrosophic soft set.

**Definition 2.8.** [2, 3] Let $X$ be the universe set and $\forall x \in X$. A bipolar neutrosophic set (BNS) $BN$ is defined by

$$BN = \left\{\langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)\rangle : x \in X\right\}$$

where

$$\text{positive membership-degrees}: T^+, I^+, F^+ : E \to [0, 1]$$
$$\text{negative membership-degrees}: T^-, I^-, F^- : E \to [-1, 0]$$

such that

$$0 \le T^+(x) + I^+(x) + F^+(x) \le 3 \text{ and } -3 \le T^-(x) + I^-(x) + F^-(x) \le 0.$$

**Example 2.9.** Let $X = \{x_1, x_2, x_3\}$ be the universe set. A bipolar neutrosophic set (BNS) is defined by

$$BN = \left\{\langle x_1, 0.1, 0.3, 0.4, -0.5, -0.3, -0.7\rangle, \langle x_2, 0.3, 0.5, 0.8, -0.7, -0.2, -0.7\rangle, \langle x_3, 0.4, 0.1, 0.7, -0.7, -0.2, -0.9\rangle\right\}$$

where $0 \le T^+(x) + I^+(x) + F^+(x) \le 3$ ; $-3 \le T^-(x) + I^-(x) + F^-(x) \le 0$.
Also $T^+(x), I^+(x), F^+(x) \to [0, 1]$ and $T^-(x), I^-(x), F^-(x) \to [-1, 0]$.

**Definition 2.10.** [Mumtaz Ali et al. version] [11] Let $X$ be a universe set and $E$ be set of parameters that are describing the elements of $X$. A bipolar neutrosophic soft set $B$ in $X$ is defined as:

$$B = \{(e, \{(x, T^{+}(x), I^{+}(x), F^{+}(x), T^{-}(x), I^{-}(x), F^{-}(x) : x \in X\} : e \in E\}$$

where $T^{+}, I^{+}, F^{+}$ ranges from $[0, 1]$ and $T^{-}, I^{-}, F^{-}$ ranges from $[-1, 0]$. The positive degrees $T^{+}(x), I^{+}(x), F^{+}(x)$, denotes the truth, indeterminate and false values of an element in the BNS set $\mathbb{B}$ and the negative degrees $T^{-}(x), I^{-}(x), F^{-}(x)$ denotes the truth, indeterminate and false values of an element in the BNS set $\mathbb{B}$.

**Definition 2.11.** [Arulpandy et al. version] [18] Let $X$ be the universe and $E$ be the parameter set. We define a set $A$ be a subset of $E$.

A bipolar neutrosophic soft set $B$ over $X$ is defined by

$$B{=}(f_A, E) = \left\{ \left\langle e, f_A(e) \right\rangle : e \in E, f_A(e) \in BNS(X) \right\}$$

Here

$$f_A(e) = \left\{ \left\langle x, u^{+}_{f_A(e)}(x), v^{+}_{f_A(e)}(x), w^{+}_{f_A(e)}(x), u^{-}_{f_A(e)}(x), v^{-}_{f_A(e)}(x), w^{-}_{f_A(e)}(x) \right\rangle : x \in X \right\}.$$

where $u^{+}_{f_A(e)}(x), v^{+}_{f_A(e)}(x), w^{+}_{f_A(e)}(x)$ denoted positive truth, indeterminate and false-membership values of $x$ for the parameter $e$, and similarly $u^{-}_{f_A(e)}(x), v^{-}_{f_A(e)}(x), w^{-}_{f_A(e)}(x)$ denoted positive truth, indeterminate and false-membership values of $x$ for the parameter $e$.

**Example 2.12.** Let $X = \{x_1, x_2, x_3, x_4\}$ be universe set and let $E = \{e_1, e_2, e_3\}$ be the parameter set.
Now, let $A = \{e_1, e_2\} \subseteq E$ and $B = \{e_3\} \subseteq E$ be two subsets of $E$.

Then we define

$$B_1 = (f_A, E) = \{\langle e, f_A(e)\rangle : e \in E, f_A(e) \in BNS(X)\}$$
$$B_2 = (g_B, E) = \{\langle e, g_B(e)\rangle : e \in E, g_B(e) \in BNS(X)\}$$

where,

$$f_A(e_1) = \Big\{ \langle x_1,\ 0.3, 0.5, 0.7, -0.6, -0.5, -0.3\rangle, \langle x_2,\ 0.6, 0.2, 0.7, -0.3, -0.4, -0.6\rangle, \langle x_3,\ 0.5, 0.6, 0.3, -0.3, -0.5, -0.3\rangle, \langle x_4,\ 0.4, 0.5, 0.2, -0.7, -0.3, -0.4\rangle \Big\}$$

$$f_A(e_2) = \Big\{ \langle x_1,\ 0.5, 0.4, 0.3, -0.5, -0.6, -0.5\rangle, \langle x_2,\ 0.6, 0.3, 0.4, -0.2, -0.4, -0.7\rangle, \langle x_3,\ 0.4, 0.5, 0.3, -0.4, -0.5, -0.8\rangle, \langle x_4,\ 0.7, 0.3, 0.2, -0.5, -0.6, -0.2\rangle \Big\}$$

$$g_B(e_3) = \Big\{ \langle x_1,\ 0.7, 0.4, 0.5, -0.5, -0.6, -0.4 \rangle, \langle x_2,\ 0.5, 0.6, 0.2, -0.3, -0.5, -0.5 \rangle,$$
$$\langle x_3,\ 0.3, 0.4, 0.2, -0.5, -0.5, -0.3 \rangle, \langle x_4,\ 0.4, 0.5, 0.5, -0.6, -0.4, -0.3 \rangle \Big\}$$

Then $B_1$ and $B_2$ are the bipolar neutrosophic soft sets. (parameterized bipolar neutrosophic sets over $X$).

## 3. Bipolar neutrosophic graded soft sets

In this section, we have extended the Bipolar neutrosophic soft set (BNSS), namely Bipolar neutrosophic graded soft sets (BNGS) by categorizing parameters as below.

**Definition 3.1.** Let $X$ be the universe set and $E$ be the parameter set which consists at least two parameters. We define the graded set $G = \{\mathcal{L}, \mathcal{M}, \mathcal{H}\}$ be subsets of the parameter set $E$ in such a way that low priority, medium priority and high priority parameters respectively .

A bipolar neutrosophic graded soft set $\mathbb{BNGS}$ over $X$ is defined by

$$\mathbb{BNGS}{=}(f_G, E) = \Big\{ \langle e, f_{\mathcal{L}}(e) \rangle, \langle e, f_{\mathcal{M}}(e) \rangle, \langle e, f_{\mathcal{H}}(e) \rangle : e \in E, f(e) \in BNS(X) \Big\}$$

Here

$$f_{\mathcal{L}}(e) = \Big\{ \Big\langle x,\ u^+_{f_{\mathcal{L}(e)}}(x), v^+_{f_{\mathcal{L}(e)}}(x), w^+_{f_{\mathcal{L}(e)}}(x), u^-_{f_{\mathcal{L}(e)}}(x), v^-_{f_{\mathcal{L}(e)}}(x), w^-_{f_{\mathcal{L}(e)}}(x) \Big\rangle : x \in X \Big\}$$
$$f_{\mathcal{M}}(e) = \Big\{ \Big\langle x,\ u^+_{f_{\mathcal{M}(e)}}(x), v^+_{f_{\mathcal{M}(e)}}(x), w^+_{f_{\mathcal{M}(e)}}(x), u^-_{f_{\mathcal{M}(e)}}(x), v^-_{f_{\mathcal{M}(e)}}(x), w^-_{f_{\mathcal{M}(e)}}(x) \Big\rangle : x \in X \Big\}$$
$$f_{\mathcal{H}}(e) = \Big\{ \Big\langle x,\ u^+_{f_{\mathcal{H}(e)}}(x), v^+_{f_{\mathcal{H}(e)}}(x), w^+_{f_{\mathcal{H}(e)}}(x), u^-_{f_{\mathcal{H}(e)}}(x), v^-_{f_{\mathcal{H}(e)}}(x), w^-_{f_{\mathcal{H}(e)}}(x) \Big\rangle : x \in X \Big\}.$$

where $u^+_{f_{\mathcal{L}(e)}}(x), v^+_{f_{\mathcal{L}(e)}}(x), w^+_{f_{\mathcal{L}(e)}}(x)$ represents positive truth, indeterminate and false values of $x$ and similarly $u^-_{f_{\mathcal{L}(e)}}(x), v^-_{f_{\mathcal{L}(e)}}(x), w^-_{f_{\mathcal{L}(e)}}$ represents negative truth, indeterminate and false values of $x$ for the graded parameter $e$ and so on.

**Example 3.2.** Let $X = \{x_1, x_2, x_3\}$ be a set variety computers (alternatives) and let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be set of parameters which represents 'Brand', 'Power consumption', 'Processor', 'Price' and 'Modern look', respectively.

People may have different opinions about their priorities. For example, we listed the possible preferences of peoples for the above case.

Old fashioned peoples/ Elders prefers only quality and durability and they won't bother about trendy look.

For them, the choices are

$\mathcal{L} = \{$Brand, Modern look$\}$, $\mathcal{M} = \{$Power consumption$\}$ and $\mathcal{H} = \{$Processor, Price$\}$

i.e. $\mathcal{L} = \{e_1, e_5\}$, $\mathcal{M} = \{e_2\}$ and $\mathcal{H} = \{e_3, e_4\}$

Modern peoples/ Students prefers good looking and latest technology with affordable price.

For them, the choices are

$\mathcal{L} = \{\text{Brand, Power consumption}\}$, $\mathcal{M} = \{\text{Price}\}$ and $\mathcal{H} = \{\text{Processor, Modern look}\}$
i.e. $\mathcal{L} = \{e_1, e_2\}$, $\mathcal{M} = \{e_4\}$ and $\mathcal{H} = \{e_3, e_5\}$
Professionals/ Office workers prefers quality best in class and they won't worry about budget. For them, the choices are
$\mathcal{L} = \{\text{Power consumption, Price}\}$, $\mathcal{M} = \{\text{Brand}\}$ and $\mathcal{H} = \{\text{Processor, Modern look}\}$
i.e. $\mathcal{L} = \{e_2, e_4\}$, $\mathcal{M} = \{e_1\}$ and $\mathcal{H} = \{e_3, e_5\}$

**Example 3.3.** Let $X = \{x_1, x_2, x_3\}$ be a set of alternatives and $E = \{e_1, e_2, e_3, e_4\}$ be the parameter set for $X$. The graded set $G$ be $G = \{\mathcal{L}, \mathcal{M}, \mathcal{H}\}$.
In this problem, the graded parameters are $\mathcal{L} = \{e_4\}, \mathcal{M} = \{e_1, e_2\}, \mathcal{H} = \{e_3\}$. Here we define two $\mathbb{BNGS}$s $\mathbb{B}_1$ and $\mathbb{B}_2$ as follows.

$$\begin{aligned}\mathbb{B}_1=(f_G, E) &= \left\{\langle e, f_{\mathcal{L}}(e)\rangle, \langle e, f_{\mathcal{M}}(e)\rangle, \langle e, f_{\mathcal{H}}(e)\rangle : e \in E, f(e) \in BNS(X)\right\} \\ &= \left\{\langle e_4, f_{\mathcal{L}}(e_4)\rangle, \langle e_1, f_{\mathcal{M}}(e_1)\rangle, \langle e_2, f_{\mathcal{M}}(e_2)\rangle \langle e_3, f_{\mathcal{H}}(e_3)\rangle\right\}\end{aligned}$$

Here,

$f_{\mathcal{L}}(e_4) =$
$\{\langle x_1,\ 0.3, 0.5, 0.4, -0.2, -0.5, -0.7\rangle, \langle x_2,\ 0.5, 0.4, 0.1, -0.3, -0.5, -0.2\rangle, \langle x_3,\ 0.1, 0.7, 0.3, -0.2, -0.4, -0.1\rangle\}$
$f_{\mathcal{M}}(e_1) =$
$\{\langle x_1,\ 0.1, 0.3, 0.2, -0.5, -0.2, -0.4\rangle, \langle x_2,\ 0.3, 0.7, 0.3, -0.2, -0.7, -0.5\rangle, \langle x_3,\ 0.7, 0.2, 0.4, -0.3, -0.4, -0.5\rangle\}$
$f_{\mathcal{M}}(e_2) =$
$\{\langle x_1,\ 0.4, 0.3, 0.1, -0.6, -0.7, -0.3\rangle, \langle x_2,\ 0.2, 0.5, 0.6, -0.3, -0.4, -0.7\rangle, \langle x_3,\ 0.4, 0.1, 0.7, -0.7, -0.1, -0.4\rangle\}$
$f_{\mathcal{H}}(e_3) =$
$\{\langle x_1,\ 0.3, 0.5, 0.7, -0.2, -0.1, -0.8\rangle, \langle x_2,\ 0.7, 0.2, 0.3, -0.1, -0.5, -0.7\rangle, \langle x_3,\ 0.2, 0.1, 0.8, -0.4, -0.5, -0.6\rangle\}$

Also,

$$\begin{aligned}\mathbb{B}_2=(f_G, E) &= \left\{\langle e, f_{\mathcal{L}}(e)\rangle, \langle e, f_{\mathcal{M}}(e)\rangle, \langle e, f_{\mathcal{H}}(e)\rangle : e \in E, f(e) \in BNS(X)\right\} \\ &= \left\{\langle e_4, f_{\mathcal{L}}(e_4)\rangle, \langle e_1, f_{\mathcal{M}}(e_1)\rangle, \langle e_2, f_{\mathcal{M}}(e_2)\rangle \langle e_3, f_{\mathcal{H}}(e_3)\rangle\right\}\end{aligned}$$

Here,

$f_{\mathcal{L}}(e_4) =$
$\{\langle x_1,\ 0.2, 0.4, 0.3, -0.3, -0.6, -0.8\rangle, \langle x_2,\ 0.6, 0.5, 0.2, -0.2, -0.4, -0.1\rangle, \langle x_3,\ 0.2, 0.6, 0.4, -0.1, -0.3, -0.2\rangle\}$
$f_{\mathcal{M}}(e_1) =$
$\{\langle x_1,\ 0.2, 0.4, 0.3, -0.4, -0.1, -0.3\rangle, \langle x_2,\ 0.4, 0.8, 0.4, -0.3, -0.6, -0.4\rangle, \langle x_3,\ 0.6, 0.3, 0.5, -0.2, -0.3, -0.4\rangle\}$
$f_{\mathcal{M}}(e_2) =$
$\{\langle x_1,\ 0.3, 0.2, 0.3, -0.5, -0.6, -0.2\rangle, \langle x_2,\ 0.3, 0.6, 0.7, -0.2, -0.3, -0.8\rangle, \langle x_3,\ 0.5, 0.2, 0.6, -0.6, -0.2, -0.5\rangle\}$

$f_{\mathcal{H}}(e_3) =$
$\{\langle x_1,\ 0.2, 0.5, 0.6, -0.3, -0.1, -0.7\rangle, \langle x_2,\ 0.6, 0.2, 0.7, -0.3, -0.2, -0.4\rangle, \langle x_3,\ 0.4, 0.7, 0.8, -0.3, -0.2, -0.5\rangle\}$

### 3.1. *Properties of BNGS*

Let $\{\mathbb{B}_i : i = 1, 2...n\}$ be set of all Bipolar neutrosophic graded soft sets defined as below.
$\mathbb{B}_i = \{\langle e, f_{L_i}(e)\rangle, \langle e, f_{M_i}(e)\rangle, \langle e, f_{H_i}(e)\rangle : e \in E, f(e) \in \mathbb{BNS}\}$.
For any $i = 1, 2$, $L \cup M \cup H = E$ and $L \cap M \cap H = \phi$

**Definition 3.4.** Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be two $\mathbb{BNGS}$s. Then their union $\mathbb{B}_1 \cup \mathbb{B}_2$ is defined as

$$\mathbb{B}_1 \cup \mathbb{B}_2 = \left\{\left\langle e, \cup_i f^{(i)}_{\mathcal{L}}(e)\right\rangle, \left\langle e, \cup_i f^{(i)}_{\mathcal{M}}(e)\right\rangle, \left\langle e, \cup_i f^{(i)}_{\mathcal{H}}(e)\right\rangle\right\}.$$

Here,

$$\bigcup_i f^{(i)}_{\mathcal{L}}(e) = \Big\{\langle x, max\left[u^+_{f_{L_i}(e)}(x)\right], min\left[v^+_{f_{L_i}(e)}(x)\right], min\left[w^+_{f_{L_i}(e)}(x)\right], min\left[u^-_{f_{L_i}(e)}(x)\right], max\left[v^-_{f_{L_i}(e)}(x)\right], max\left[w^-_{f_{L_i}(e)}(x)\right]\rangle\Big\}$$

$$\bigcup_i f^{(i)}_{\mathcal{M}}(e) = \Big\{\langle x, max\left[u^+_{M_{L_i}(e)}(x)\right], min\left[v^+_{M_{L_i}(e)}(x)\right], min\left[w^+_{M_{L_i}(e)}(x)\right], min\left[u^-_{M_{L_i}(e)}(x)\right], max\left[v^-_{M_{L_i}(e)}(x)\right], max\left[w^-_{M_{L_i}(e)}(x)\right]\rangle\Big\}$$

$$\bigcup_i f^{(i)}_{\mathcal{H}}(e) = \Big\{\langle x, max\left[u^+_{H_{L_i}(e)}(x)\right], min\left[v^+_{H_{L_i}(e)}(x)\right], min\left[w^+_{H_{L_i}(e)}(x)\right], min\left[u^-_{H_{L_i}(e)}(x)\right], max\left[v^-_{H_{L_i}(e)}(x)\right], max\left[w^-_{H_{L_i}(e)}(x)\right]\rangle\Big\}$$

**Example 3.5.** Consider the $\mathbb{BNGS}$ sets $\mathbb{B}_1$ and $\mathbb{B}_2$ defined in Example 3.3. Then their union is defined by
$\mathbb{B}_1 \cup \mathbb{B}_2 = \left\{\left\langle e_4, \cup_i f^{(i)}_{\mathcal{L}}(e_4)\right\rangle, \left\langle e_1, \cup_i f^{(i)}_{\mathcal{M}}(e_1)\right\rangle, \left\langle e_2, \cup_i f^{(i)}_{\mathcal{M}}(e_2)\right\rangle, \left\langle e_3, \cup_i f^{(i)}_{\mathcal{H}}(e_3)\right\rangle\right\}$ Here
$\cup_i f^{(i)}_{\mathcal{L}}(e_4) =$
$\{\langle x_1,\ 0.3, 0.4, 0.3, -0.3, -0.5, -0.7\rangle, \langle x_2,\ 0.6, 0.4, 0.1, -0.3, -0.4, -0.1\rangle, \langle x_3,\ 0.2, 0.6, 0.3, -0.2, -0.3, -0.1\rangle\}$
$\cup_i f^{(i)}_{\mathcal{M}}(e_1) =$
$\{\langle x_1,\ 0.2, 0.3, 0.2, -0.5, -0.1, -0.3\rangle, \langle x_2,\ 0.4, 0.7, 0.3, -0.3, -0.6, -0.4\rangle, \langle x_3,\ 0.7, 0.2, 0.4, -0.3, -0.3, -0.4\rangle\}$
$\cup_i f^{(i)}_{\mathcal{M}}(e_2) =$
$\{\langle x_1,\ 0.4, 0.2, 0.1, -0.6, -0.6, -0.2\rangle, \langle x_2,\ 0.3, 0.5, 0.6, -0.3, -0.3, -0.7\rangle, \langle x_3,\ 0.5, 0.1, 0.6, -0.7, -0.1, -0.4\rangle\}$
$\cup_i f^{(i)}_{\mathcal{H}}(e_3) =$
$\{\langle x_1,\ 0.3, 0.5, 0.6, -0.3, -0.1, -0.7\rangle, \langle x_2,\ 0.7, 0.2, 0.3, -0.3, -0.1, -0.7\rangle, \langle x_3,\ 0.4, 0.1, 0.8, -0.4, -0.2, -0.5\rangle\}$

**Definition 3.6.** Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be two $\mathbb{BNGS}$s. Then their intersection $\mathbb{B}_1 \cap \mathbb{B}_2$ is defined as

$$\mathbb{B}_1 \cap \mathbb{B}_2 = \left\{ \left\langle e, \cap_i f_{\mathcal{L}}^{(i)}(e) \right\rangle, \left\langle e, \cap_i f_{\mathcal{M}}^{(i)}(e) \right\rangle, \left\langle e, \cap_i f_{\mathcal{H}}^{(i)}(e) \right\rangle \right\}.$$

Here,

$$\bigcap_i f_{\mathcal{L}}^{(i)}(e) = \Big\{ \langle x, min\left[u^+_{f_{L_i}(e)}(x)\right], max\left[v^+_{f_{L_i}(e)}(x)\right], max\left[w^+_{f_{L_i}(e)}(x)\right], max\left[u^-_{f_{L_i}(e)}(x)\right], min\left[v^-_{f_{L_i}(e)}(x)\right], min\left[w^-_{f_{L_i}(e)}(x)\right] \rangle \Big\}$$

$$\bigcap_i f_{\mathcal{M}}^{(i)}(e) = \Big\{ \langle x, min\left[u^+_{M_{L_i}(e)}(x)\right], max\left[v^+_{M_{L_i}(e)}(x)\right], max\left[w^+_{M_{L_i}(e)}(x)\right], max\left[u^-_{M_{L_i}(e)}(x)\right], min\left[v^-_{M_{L_i}(e)}(x)\right], min\left[w^-_{M_{L_i}(e)}(x)\right] \rangle \Big\}$$

$$\bigcap_i f_{\mathcal{H}}^{(i)}(e) = \Big\{ \langle x, min\left[u^+_{H_{L_i}(e)}(x)\right], max\left[v^+_{H_{L_i}(e)}(x)\right], max\left[w^+_{H_{L_i}(e)}(x)\right], max\left[u^-_{H_{L_i}(e)}(x)\right], min\left[v^-_{H_{L_i}(e)}(x)\right], min\left[w^-_{H_{L_i}(e)}(x)\right] \rangle \Big\}$$

**Example 3.7.** Consider the $\mathbb{BNGS}$ sets $\mathbb{B}_1$ and $\mathbb{B}_2$ defined in Example 3.3. Then their intersection is defined by

$\mathbb{B}_1 \cap \mathbb{B}_2 = \left\{ \left\langle e_4, \cap_i f_{\mathcal{L}}^{(i)}(e_4) \right\rangle, \left\langle e_1, \cap_i f_{\mathcal{M}}^{(i)}(e_1) \right\rangle, \left\langle e_2, \cap_i f_{\mathcal{M}}^{(i)}(e_2) \right\rangle, \left\langle e_3, \cap_i f_{\mathcal{H}}^{(i)}(e_3) \right\rangle \right\}$ Here

$\cap_i f_{\mathcal{L}}^{(i)}(e_4) =$

$\{\langle x_1,\ 0.2, 0.5, 0.4, -0.2, -0.6, -0.8\rangle, \langle x_2,\ 0.5, 0.5, 0.2, -0.2, -0.5, -0.2\rangle, \langle x_3,\ 0.1, 0.7, 0.4, -0.1, -0.4, -0.2\rangle\}$

$\cap_i f_{\mathcal{M}}^{(i)}(e_1) =$

$\{\langle x_1,\ 0.1, 0.4, 0.3, -0.4, -0.2, -0.4\rangle, \langle x_2,\ 0.3, 0.8, 0.4, -0.2, -0.7, -0.5\rangle, \langle x_3,\ 0.6, 0.3, 0.5, -0.2, -0.4, -0.5\rangle\}$

$\cap_i f_{\mathcal{M}}^{(i)}(e_2) =$

$\{\langle x_1,\ 0.3, 0.3, 0.3, -0.5, -0.7, -0.3\rangle, \langle x_2,\ 0.2, 0.6, 0.7, -0.2, -0.4, -0.8\rangle, \langle x_3,\ 0.4, 0.2, 0.7, -0.6, -0.2, -0.5\rangle\}$

$\cap_i f_{\mathcal{H}}^{(i)}(e_3) =$

$\{\langle x_1,\ 0.2, 0.5, 0.7, -0.2, -0.3, -0.8\rangle, \langle x_2,\ 0.6, 0.2, 0.7, -0.1, -0.5, -0.7\rangle, \langle x_3,\ 0.2, 0.7, 0.8, -0.3, -0.5, -0.6\rangle\}$

**Remark 3.8.** Suppose $\mathbb{B}_1$ and $\mathbb{B}_2$ are two $\mathbb{BNGS}$s with unequal number of graded parameters (i.e. the cardinality of $L_1$ and $L_2$ are not equal and so on.).

Let the universal parameter set $E = \{e_1, e_2, e_3, e_4\}$. We define two $\mathbb{BNGS}$s as follows.

$\mathbb{B}_1 = \{\langle e, f_{L_1}(e)\rangle, \langle e, f_{M_1}(e)\rangle, \langle e, f_{H_1}(e)\rangle : e \in E\}$

where $L_1 = \{e_1\}, M_1 = \{e_2\}, H_1 = \{e_3, e_4\}$.

$\mathbb{B}_2 = \{\langle e, f_{L_2}(e)\rangle, \langle e, f_{M_2}(e)\rangle, \langle e, f_{H_2}(e)\rangle : e \in E\}$

where $L_2 = \{e_1, e_2\}, H_2 = \{e_3, e_4\}$.

Then their union $\mathbb{B}_1 \cup \mathbb{B}_2$ is defined as

$$\mathbb{B}_1 \cup \mathbb{B}_2 = \left\{ \langle e, f_{L_1 \cup L_2}(e) \rangle , \langle e, f_{M_1 \cup \phi}(e) \rangle , \langle e, f_{H_1 \cup H_2}(e) \rangle \right\}.$$

Also, the intersection $\mathbb{B}_1 \cap \mathbb{B}_2$ is defined as

$$\mathbb{B}_1 \cap \mathbb{B}_2 = \left\{ \langle e, f_{L_1 \cap L_2}(e) \rangle , \langle e, f_{M_1 \cap \phi}(e) \rangle , \langle e, f_{H_1 \cap H_2}(e) \rangle \right\}.$$

**Definition 3.9.** Let $\mathbb{B}$ be a $\mathbb{BNGS}$. Then the complement of $\mathbb{B}$ is defined as

$$\mathbb{B}^c = \left\{ \langle e, f_L^c(e) \rangle , \langle e, f_M^c(e) \rangle , \langle e, f_H^c(e) \rangle \right\}.$$

Here,

$$\begin{aligned} f_L^c(e) =& \left\{ w_{f_L}^+(e), 1 - v_{f_L}^+(e), u_{f_L}^+(e), w_{f_L}^-(e), -1 - v_{f_L}^-(e), u_{f_L}^-(e) \right\} \\ f_M^c(e) =& \left\{ w_{f_M}^+(e), 1 - v_{f_M}^+(e), u_{f_M}^+(e), w_{f_M}^-(e), -1 - v_{f_M}^-(e), u_{f_M}^-(e) \right\} \\ f_H^c(e) =& \left\{ w_{f_H}^+(e), 1 - v_{f_H}^+(e), u_{f_H}^+(e), w_{f_H}^-(e), -1 - v_{f_H}^-(e), u_{f_H}^-(e) \right\} \end{aligned}$$

**Example 3.10.** Consider the $\mathbb{BNGS}$ set $\mathbb{B}_1$ in Example 3.3. Then the complement is defined by

$\mathbb{B}^c = \left\{ \langle e, f_{\mathcal{L}}^c(e_4) \rangle , \langle e, f_{\mathcal{M}}^c(e_1) \rangle , \langle e, f_{\mathcal{M}}^c(e_2) \rangle , \langle e, f_{\mathcal{H}}^c(e_3) \rangle \right\}$. Here

$f_{\mathcal{L}}^c(e_4) =$

$\left\{ \langle x_1,\ 0.4, 0.5, 0.3, -0.7, -0.5, -0.2 \rangle , \langle x_2,\ 0.1, 0.6, 0.5, -0.2, -0.5, -0.3 \rangle , \langle x_3,\ 0.3, 0.3, 0.1, -0.1, -0.6, -0.2 \rangle \right\}$

$f_{\mathcal{M}}^c(e_1) =$

$\left\{ \langle x_1,\ 0.2, 0.7, 0.1, -0.4, -0.8, -0.5 \rangle , \langle x_2,\ 0.3, 0.3, 0.3, -0.5, -0.3, -0.2 \rangle , \langle x_3,\ 0.4, 0.8, 0.7, -0.5, -0.6, -0.3 \rangle \right\}$

$f_{\mathcal{M}}^c(e_2) =$

$\left\{ \langle x_1,\ 0.1, 0.7, 0.4, -0.3, -0.3, -0.6 \rangle , \langle x_2,\ 0.6, 0.5, 0.2, -0.7, -0.6, -0.3 \rangle , \langle x_3,\ 0.7, 0.9, 0.4, -0.4, -0.9, -0.7 \rangle \right\}$

$f_{\mathcal{H}}^c(e_3) =$

$\left\{ \langle x_1,\ 0.7, 0.5, 0.3, -0.8, -0.9, -0.2 \rangle , \langle x_2,\ 0.3, 0.8, 0.7, -0.7, -0.5, -0.1 \rangle , \langle x_3,\ 0.8, 0.9, 0.2, -0.6, -0.5, -0.4 \rangle \right\}$

**Definition 3.11.** Let $\phi_{\mathbb{B}}$ be a null $\mathbb{BNGS}$ and is defined as

$$\phi_{\mathbb{B}} = \left\{ \langle e_i, f_\phi(e_i) \rangle : e \in E \right\}$$

Here $f_\phi(e_i) = \left\{ \langle x_i, 0, 1, 1, 0, -1, -1 \rangle : x \in X \right\}$

**Definition 3.12.** Let $1_{\mathbb{B}}$ be a complete $\mathbb{BNGS}$ and is defined as

$$1_{\mathbb{B}} = \left\{ \langle e_i, f_C(e_i) \rangle : e \in E \right\}$$

Here $f_C(e_i) = \left\{ \langle x_i, 1, 0, 0, -1, 0, 0 \rangle : x \in X \right\}$

**Proposition 3.13.** *For any* $\mathbb{BNGS}$ *set,*

*(i).* $\mathbb{B} \cup \phi_{\mathbb{B}} = \mathbb{B}$

*(ii).* $\mathbb{B} \cup 1_{\mathbb{B}} = 1_{\mathbb{B}}$

*(iii).* $\mathbb{B} \cap \phi_{\mathbb{B}} = \phi_{\mathbb{B}}$

*(iv).* $\mathbb{B} \cap 1_{\mathbb{B}} = \mathbb{B}$

*Proof.* By the definition of union and intersection of $\mathbb{BNGS}$s, results are obvious. □

**Proposition 3.14.** *For any three $\mathbb{BNGS}$ sets $\mathbb{B}_1, \mathbb{B}_2$ and $\mathbb{B}_3$, the following relations are hold.*

*(i).* $\mathbb{B}_1 \cup \mathbb{B}_2 = \mathbb{B}_2 \cup \mathbb{B}_1$

*(ii).* $\mathbb{B}_1 \cap \mathbb{B}_2 = \mathbb{B}_2 \cap \mathbb{B}_1$

*(iii).* $\mathbb{B}_1 \cup (\mathbb{B}_2 \cup \mathbb{B}_3) = (\mathbb{B}_1 \cup \mathbb{B}_2) \cup \mathbb{B}_3$

*(iv).* $\mathbb{B}_1 \cap (\mathbb{B}_2 \cap \mathbb{B}_3) = (\mathbb{B}_1 \cap \mathbb{B}_2) \cap \mathbb{B}_3$

*Proof.* It is obvious. □

**Proposition 3.15.** *For any two $\mathbb{BNGS}$ sets $\mathbb{B}_1$ and $\mathbb{B}_2$, the following conditions are hold.[De'Morgans law]*

*(i).* $(\mathbb{B}_1 \cup \mathbb{B}_2)^c = (\mathbb{B}_1)^c \cap (\mathbb{B}_2)^c$

*(ii).* $(\mathbb{B}_1 \cap \mathbb{B}_2)^c = (\mathbb{B}_1)^c \cup (\mathbb{B}_2)^c$

*Proof.* It is obvious. □

**Proposition 3.16.** *For any three $\mathbb{BNGS}$ sets $\mathbb{B}_1, \mathbb{B}_2$ and $\mathbb{B}_3$, the following relations are hold.[Distributive law]*

*(i).* $\mathbb{B}_1 \cap (\mathbb{B}_2 \cup \mathbb{B}_3) = (\mathbb{B}_1 \cap \mathbb{B}_2) \cup (\mathbb{B}_1 \cap \mathbb{B}_3)$

*(ii).* $\mathbb{B}_1 \cup (\mathbb{B}_2 \cap \mathbb{B}_3) = (\mathbb{B}_1 \cup \mathbb{B}_2) \cap (\mathbb{B}_1 \cup \mathbb{B}_3)$

*Proof.* It is obvious. □

## 4. Bipolar neutrosophic graded soft topological space

Let $X$ be a universal set which consists alternatives and $\mathbb{BNGS}(x)$ be the collection of all $\mathbb{BNGS}$s in $X$. Then the collection $\tau_{\mathbb{B}}$ containing all $\mathbb{BNGS}$s is called $\mathbb{BNGS}$-topology if it holds the following conditions.

(1) $\phi_{\mathbb{B}}, 1_{\mathbb{B}} \in \tau_{\mathbb{B}}$

(2) $\bigcup_{i\in n} \mathbb{B}_i \in \tau_{\mathbb{B}}$ for each $\mathbb{B}_i \in \tau_{\mathbb{B}}$

(3) $\mathbb{B}_i \cap \mathbb{B}_j \in \tau_{\mathbb{B}}$ for any $\mathbb{B}_i, \mathbb{B}_j \in \tau_{\mathbb{B}}$

Then the pair $(X, \tau_{\mathbb{B}})$ is called $\mathbb{BNGS}$-topological space. The members of $\tau_{\mathbb{B}}$ are called open $\mathbb{BNGS}$s and their complements are called closed $\mathbb{BNGS}$s.

**Example 4.1.** Let $X = x_1, x_2$ be set of alternatives and $E = e_1, e_2, e_3$ be a parameter set. We define the graded parameter set as $G = \mathcal{L} = e_1, \mathcal{M} = e_2, \mathcal{H} = e_3$. Now let us define a topology on $(X, E)$ as follows.

$$\tau_{\mathcal{B}} = \left\{\phi_{\mathbb{B}}, 1_{\mathbb{B}}, \mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4\right\}$$

Here $\phi_{\mathbb{B}}, 1_{\mathbb{B}}$ are null and complete $\mathbb{BNGS}$ respectively. Also,

$$\mathbb{B}_1 = \left\{\left\langle e_1, f_{\mathcal{L}}^{(1)}(e_1)\right\rangle, \left\langle e_2, f_{\mathcal{M}}^{(1)}(e_2)\right\rangle, \left\langle e_3, f_{\mathcal{H}}^{(1)}(e_3)\right\rangle\right\}$$

$$= \left\{\left\langle e_1, \{\langle x_1,\ 1, 0, 1, -1, 0, 0\rangle, \langle x_2,\ 0.5, 0.2, 0.4, -0.5, -0.4, -0.3\rangle\}\right\rangle,\right.$$
$$\left\langle e_2, \{\langle x_1,\ 0.4, 0.6, 0.3, -0.4, -0.7, -0.2\rangle, \langle x_2,\ 0.7, 0.2, 0.1, -0.3, -0.5, -0.7\rangle\}\right\rangle,$$
$$\left.\left\langle e_3, \{\langle x_1,\ 0.5, 0.3, 0.7, -0.2, -0.4, -0.8\rangle, \langle x_2,\ 0.4, 0.3, 0.5, -0.1, -0.4, -0.6\rangle\}\right\rangle\right\}$$

$$\mathbb{B}_2 = \left\{\left\langle e_1, f_{\mathcal{L}}^{(2)}(e_1)\right\rangle, \left\langle e_2, f_{\mathcal{M}}^{(2)}(e_2)\right\rangle, \left\langle e_3, f_{\mathcal{H}}^{(2)}(e_3)\right\rangle\right\}$$

$$= \left\{\left\langle e_1, \{\langle x_1,\ 0.3, 0.1, 0.7, -0.5, -0.6, -0.3\rangle, \langle x_2,\ 0, 1, 1, -0.7, 0, -1\rangle\}\right\rangle,\right.$$
$$\left\langle e_2, \{\langle x_1,\ 0.2, 0.5, 0.7, -1, 0, -0.2\rangle, \langle x_2,\ 0.9, 0.1, 0.3, -0.1, -0.6, -0.3\rangle\}\right\rangle,$$
$$\left.\left\langle e_3, \{\langle x_1,\ 0.3, 0.5, 0.3, -0.2, 0, -0.4\rangle, \langle x_2,\ 0.7, 0.4, 0.1, -0.3, -0.5, -0.1\rangle\}\right\rangle\right\}$$

$$\mathbb{B}_3 = \left\{\left\langle e_1, f_{\mathcal{L}}^{(3)}(e_1)\right\rangle, \left\langle e_2, f_{\mathcal{M}}^{(3)}(e_2)\right\rangle, \left\langle e_3, f_{\mathcal{H}}^{(3)}(e_3)\right\rangle\right\}$$

$$= \left\{\left\langle e_1, \{\langle x_1,\ 1, 0, 0.7, -1, 0, 0\rangle, \langle x_2,\ 0.5, 0.2, 0.4, -0.7, 0, -0.3\rangle\}\right\rangle,\right.$$
$$\left\langle e_2, \{\langle x_1,\ 0.4, 0.5, 0.3, -1, 0, -0.2\rangle, \langle x_2,\ 0.9, 0.1, 0.1, -0.3, -0.5, -0.3\rangle\}\right\rangle,$$
$$\left.\left\langle e_3, \{\langle x_1,\ 0.5, 0.3, 0.3, -0.2, 0, -0.4\rangle, \langle x_2,\ 0.7, 0.3, 0.1, -0.3, -0.4, -0.1\rangle\}\right\rangle\right\}$$

$$\mathbb{B}_4 = \left\{\left\langle e_1, f_{\mathcal{L}}^{(4)}(e_1)\right\rangle, \left\langle e_2, f_{\mathcal{M}}^{(4)}(e_2)\right\rangle, \left\langle e_3, f_{\mathcal{H}}^{(4)}(e_3)\right\rangle\right\}$$

$$= \left\{\left\langle e_1, \{\langle x_1,\ 0.3, 0.1, 1, -0.5, -0.6, -0.3\rangle, \langle x_2,\ 0, 1, 1, -0.5, -0.4, -1\rangle\}\right\rangle,\right.$$
$$\left\langle e_2, \{\langle x_1,\ 0.2, 0.6, 0.7, -0.4, -0.7, -0.2\rangle, \langle x_2,\ 0.7, 0.2, 0.3, -0.1, -0.6, -0.7\rangle\}\right\rangle,$$
$$\left.\left\langle e_3, \{\langle x_1,\ 0.3, 0.5, 0.7, -0.2, -0.4, -0.8\rangle, \langle x_2,\ 0.4, 0.4, 0.5, -0.1, -0.5, -0.6\rangle\}\right\rangle\right\}$$

Here, $\mathbb{B}_1 \cup \mathbb{B}_2 = \mathbb{B}_3, \mathbb{B}_2 \cup \mathbb{B}_3 = \mathbb{B}_3, \mathbb{B}_1 \cup \mathbb{B}_3 = \mathbb{B}_3, \mathbb{B}_3 \cup \mathbb{B}_4 = \mathbb{B}_3$ and so on. Also, $\mathbb{B}_1 \cap \mathbb{B}_2 = \mathbb{B}_4, \mathbb{B}_2 \cap \mathbb{B}_3 = \mathbb{B}_4, \mathbb{B}_1 \cap \mathbb{B}_3 = \mathbb{B}_4, \mathbb{B}_3 \cap \mathbb{B}_4 = \mathbb{B}_4$ and so on.

The $\tau_{\mathbb{B}}$ satisfies all three conditions of topology. So $\tau_{\mathbb{B}}$ is a $\mathbb{BNGS}$-topology.

**Proposition 4.2.** *Let $(X, \tau_{\mathbb{B}})$ be an $\mathbb{BNGS}$. Then the following conditions hold.*

- *$\phi_{\mathbb{B}}$ and $1_{\mathbb{B}}$ are open $\mathbb{BNGS}$s.*

- *Union of any number of open $\mathbb{BNGS}$s is open.*
- *Intersection of finite number of closed $\mathbb{BNGS}$s is closed.*

**Definition 4.3.** Let $(X, \tau_{\mathbb{B}})$ and $(X, \tau'_{\mathbb{B}})$ be two $\mathbb{BNGS}$ in $X$. Two $\mathbb{BNGS}$'s are said to be Comparable if $\tau_{\mathbb{B}} \subseteq \tau'_{\mathbb{B}}$ or $\tau'_{\mathbb{B}} \subseteq \tau_{\mathbb{B}}$.
If $\tau_{\mathbb{B}} \subseteq \tau'_{\mathbb{B}}$ , then $\tau_{\mathbb{B}}$ is courser or weaker than $\tau'_{\mathbb{B}}$ . In other words, $\tau'_{\mathbb{B}}$ is stronger or finer than $\tau_{\mathbb{B}}$ and vice versa.

4.1. *Example*

**Proposition 4.4.** *Let $(X, \tau_{\mathbb{B}_1})$ and $(X, \tau_{\mathbb{B}_2})$ be two $\mathbb{BNGS}$-topological spaces over $(X, E)$. Suppose $\tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2} = \big\{B : B \in P(X)\big\}$. Then $\tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$ is also a $\mathbb{BNGS}$-topology.*

*Proof.* $B_1 \cap B_2$ must satisfy topology conditions in order to be a $\mathbb{BNGS}$-topology.

i). Clearly $\phi_{\mathbb{B}}, 1_{\mathbb{B}} \in \tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$.

ii). Let $B_1, B_2 \in \tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$

$\Rightarrow B_1, B_2 \in \mathbb{B}_1 and B_1, B_2 \in \mathbb{B}_2$

$\Rightarrow B_1 \cap B_2 \in \mathbb{B}_1 and B_1 \cap B_2 \in \mathbb{B}_2$

$\Rightarrow B_1 \cap B_2 \in \tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$

iii). Let $\{B_i\} \in \tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$

$\Rightarrow \{B_i\} \in \tau_{\mathbb{B}_1} and \{B_i\} \in \tau_{\mathbb{B}_2}$

$\Rightarrow \cup_i B_i \in \tau_{\mathbb{B}_1} and \cup_i B_i \in \tau_{\mathbb{B}_2}$

$\Rightarrow \cup_i B_i \in \tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$.

Hence $\tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$ in a $\mathbb{BNGS}$-topology. □

**Remark 4.5.** The union of any two $\mathbb{BNGS}$-topologies may or may not be a topology.
Since it may or may not satisfy the topology conditions.

i). $\phi_{\mathbb{B}}, 1_{\mathbb{B}} \in \tau_{\mathbb{B}_1} \cup \tau_{\mathbb{B}_2}$ always holds.

ii). For $B_1, B_2 \in \tau_{\mathbb{B}_1} \cup \tau_{\mathbb{B}_2}$, both $B_1$ and $B_2$ may or may not be in both $\tau_{\mathbb{B}_1}$ and $\tau_{\mathbb{B}_2}$.

**Definition 4.6.** Let $(X, \tau_{\mathbb{B}})$ be a $\mathbb{BNGS}$-topological space over $(X, E)$. Let $B \in \mathbb{BNGS}(X, E)$.
Then the interior of $B$ is defined by
$B^o = \bigcup \{N : N \text{is a bipolar neutrosophic graded soft open set and } N \in B\}$.
i.e. It is the union of all open $\mathbb{BNGS}$ open subsets of $B$.

**Proposition 4.7.** *Let $(X, \tau_{\mathbb{B}})$ be a $\mathbb{BNGS}$-topological space over $(X, E)$ and $B_1, B_2 \in \mathbb{BNGS}(X, E)$. Then*

*(i). $B_1^o \in B_1$ and $B_1^o$ is the largest open set.*

*(ii).* $B_1 \in B_2 \Rightarrow B_1^o \in B_2^o$.

*(iii).* $B_1^o$ *is an open* $\mathbb{BNGS}$. *i.e.* $B_1^o \in \tau_{\mathbb{B}}$.

*(iv).* $B_1$ *is* $\mathbb{BNGS}$ *soft open set if and only if* $B_1^o = B_1$.

*(v).* $(B_1^o)^o = B_1^o$.

*(vi).* $(\phi_{\mathbb{B}\not{k}})^o = \phi_{\mathbb{B}}$ *and* $(1_{\mathbb{B}})^o = 1_{\mathbb{B}}$.

*(vii).* $(B_1 \cap B_2)^o = B_1^o \cap B_2^o$.

*(viii).* $B_1^o \cup B_2^o \subset (B_1 \cup B_2)^o$.

*Proof.* (i) Since $B_1^o$ is the union of all open sets in $B_1$, $B_1^o$ is the largest open set which contained in $B_1$.

(ii) Let $B_1 \in B_2 \Rightarrow B_1^o \subset B_1 \subset B_2 \Rightarrow B_1^o \subset B_2$and also $B_2^o \subset B_2$.
But $B_2^o$ is the largest open set in $B_2$. Hence $B_1^o \subset B_2^o$.

(iii) By definition of $\mathbb{BNGS}$-toppology $\tau_{\mathbb{B}}$, it is obvious.

(iv) $B_1^o \subset B_1$ and let $B_1$ be bipolar neutrosophic graded soft open set.
$B_1 \subset B_1 \Rightarrow B_1 \subset \cap\{B_2 \in \tau_{\mathbb{B}} : B_2 \subset B_1\} = B_1^o$
$\Rightarrow B_1 \subset B_1^o \Rightarrow B_1 = B_1^o$. Conversely, let $B_1 = B_1^o$. Then $B_1 = B_1^o \in \tau_{\mathbb{B}} \Rightarrow B_1$ is open bipolar neutrosophic graded soft open set.

(v) If $B_1$ is an open $\mathbb{BNGS}$, then $B_1^o = B_1$. Clearly $B_1^o$ is an open $\mathbb{BNGS}$. Hence $(B_1^o)^o = B_1^o$.

(vi) Since $\phi_{\mathbb{B}}, 1_{\mathbb{B}} \in \tau_{\mathbb{B}}$. So they are open $\mathbb{BNGS}$. Hence it is obvious from $(iv)$.

(vii) $B_1 \cap B_2 \subset B_1$ and $B_1 \cap B_2 \subset B_2 \Rightarrow (B_1 \cap B_2)^o \subset B_1^o$ and $(B_1 \cap B_2)^o \subset B_2^o$
$\Rightarrow (B_1 \cap B_2)^o \subset B_1^o \cap B_2^o$.
Further, $B_1^o \subset B_1$ and $B_2^o \subset B_2$. Then $B_1^o \cap B_2^o \subset B_1 \cap B_2$. But $(B_1 \cap B_2)^o \subset B_1 \cap B_2$ and it is the largest open set. So $B_1^o \cap B_2^o \subset (B_1 \cap B_2)^o$.
Hence $(B_1 \cap B_2)^o = B_1^o \cap B_2^o$.

(viii) $B_1 \subset B_1 \cup B_2$ and $B_2 \subset B_1 \cup B_2$. $\Rightarrow B_1^o \subset (B_1 \cup B_2)^o$and $B_1^o \subset (B_1 \cup B_2)^o$
$\Rightarrow B_1^o \cup B_2^o \subset (B_1 \cup B_2)^o$.

□

**Definition 4.8.** Let $(X, \tau_{\mathbb{B}})$ be a $\mathbb{BNGS}$-topological space over $(X, E)$ and $B_1 \in \mathbb{BNGS}(X, E)$. Then the closure of $B$ is defined by
$\overline{B} = \bigcap\{N : N \text{ is bipolar neutrosophic graded soft closed set and } N \supset B\}$.
i.e. It is the intersection of all bipolar neutrosophic graded soft closed subsets of $B$.

**Proposition 4.9.** *Let* $(X, \tau_{\mathbb{B}})$ *be a* $\mathbb{BNGS}$*-topological space over* $(X, E)$ *and* $B_1, B_2 \in \mathbb{BNGS}(X, E)$. *Then*

*(i).* $B_1 \subset \overline{B_1}$ *and* $\overline{B_1}$ *is the smallest closed set.*

*(ii).* $B_1 \subset B_2 \Rightarrow \overline{B_1} \subset \overline{B_2}$.

*(iii).* $\overline{B_1}$ *is closed* $\mathbb{BNGS}$. *i.e.* $\overline{B_1} \in \tau^c_{\mathbb{B}}$.

*(iv).* $B_1$ *is* $\mathbb{BNGS}$ *closed set if and only if* $\overline{B_1} = B_1$.

*(v).* $\overline{\overline{B_1}} = \overline{B_1}$.

*(vi).* $\overline{\phi_{\mathbb{B}}} = \phi_{\mathbb{B}}$ *and* $\overline{1_{\mathbb{B}}} = 1_{\mathbb{B}}$.

*(vii).* $\overline{B_1 \cup B_2} = \overline{B_1} \cup \overline{B_2}$.

*(viii).* $\overline{B_1 \cup B_2} \subset \overline{B_1} \cap \overline{B_2}$.

*Proof.* (i) Since $\overline{B_1}$ is the intersection of all closed sets in $B_1$, $\overline{B_1}$ is the smallest closed set which contains $B_1$.

(ii) Let $B_1 \subset B_2$. Also $B_1 \subset \overline{B_1}$ and $B_2 \subset \overline{B_2} \Rightarrow B_1 \subset B_2 \subset \overline{B_2}$.
But $\overline{B_1}$ is the smallest set containing $B_1$. So $B_1 \subset \overline{B_1} \subset \overline{B_2}$. Hence $\overline{B_1} \subset \overline{B_2}$.

(iii) By definition of $\mathbb{BNGS}$-toppology $\tau_{\mathbb{B}}$ and $\overline{B_1}$, it is obvious.

(iv) $B_1 \subset \overline{B_1}$ and let $B_1$ be bipolar neutrosophic graded soft closed set. Then $B_1 \subset B_1$.
$\overline{B_1} = \bigcap \{B_2 \in \tau^c_{\mathbb{B}} : B_2 \supset B_1\} \subset \{B_1 \in \tau^c_{\mathbb{B}} : B_1 \supset B_1\} = B_1$
$\Rightarrow \overline{B_1} \subset B_1$
$\Rightarrow B_1 = \overline{B_1}$. Conversely, let $B_1 = \overline{B_1}$. Then $(\overline{B_1})^c \in \tau_{\mathbb{B}} \Rightarrow B_1^c \in \tau_{\mathbb{B}}$
$\Rightarrow B_1^c$ is open $\Rightarrow B_1$ is closed.

(v) If $N$ is closed $\mathbb{BNGS}$, then $N = \overline{N}$. But $\overline{N}$ is closed by default. Replacing $N$ by $\overline{B_1}$, we get $\overline{\overline{B_1}} = \overline{B_1}$.

(vi) Since $\phi_{\mathbb{B}}, 1_{\mathbb{B}} \in \tau_{\mathbb{B}}$ are both open and closed. So the result is obvious by $(iv)$.

(vii) $B_1 \subset B_1 \cup B_2$ and $B_2 \subset B_1 \cup B_2 \Rightarrow \overline{B_1} \subset \overline{B_1 \cup B_2}$ and $\overline{B_2} \subset \overline{B_1 \cup B_2}$
$\Rightarrow \overline{B_1} \cup \overline{B_2} \subset \overline{B_1 \cup B_2}$.

Also, $B_1 \subset \overline{B_1}$ and $B_2 \subset \overline{B_2}$. Then $B_1 \cup B_2 \subset \overline{B_1} \cup \overline{B_2}$.
But $B_1 \cup B_2 \subset \overline{B_1 \cup B_2} \subset \overline{B_1} \cup \overline{B_2}$.
Hence $\overline{B_1 \cup B_2} = \overline{B_1} \cup \overline{B_2}$.

(viii) $B_1 \cap B_2 \subset B_1$ and $B_1 \cap B_2 \subset B_2$. $\Rightarrow \overline{B_1 \cap B_2} \subset \overline{B_1}$and $\overline{B_1 \cap B_2} \subset \overline{B_2}$
$\Rightarrow \overline{B_1 \cap B_2} \subset \overline{B_1} \cap \overline{B_2}$.

□

**Definition 4.10.** Let $(X, \tau_{\mathbb{B}})$ be a $\mathbb{BNGS}$-topological space over $(X, E)$ and $B \in \mathbb{BNGS}(X, E)$. Then the boundary of $B$ is denoted by $Bd(B)$ and is defined by $Bd(B) = \overline{B} \cap \overline{B^c}$.

**Proposition 4.11.** *Let* $(X, \tau_{\mathbb{B}})$ *be a* $\mathbb{BNGS}$-*topological space over* $(X, E)$ *and* $B \in \mathbb{BNGS}(X, E)$. *Then* *(i).* $B^o \cap Bd(B) = \phi_{\mathbb{B}}$.

*(ii).* $\overline{B} = B^o \cup Bd(B)$.

*(iii).* $Bd(B) = \phi_{\mathbb{B}}$ *if and only if* $B$ *is both closed and open.*

*(iv).* $Bd(B) = \overline{B} \cap (B^o)^c$.

*Proof.* (i) $B^o \cap Bd(B) = B^o \cap (\overline{B} \cap \overline{B^c}) = B^o \cap (\overline{B} \cap (B^o)^c)$
$= B^o \cap (B^o)^c \cap \overline{B} = \phi_{\mathbb{B}} \cap \overline{B} = \phi_{\mathbb{B}}$.

(ii) $B^o \cup Bd(B) = B^o \cup (\overline{B} \cap \overline{B^c}) = B^o \cup (\overline{B} \cap (B^o)^c)$
$= (B^o \cup \overline{B}) \cap (B^o \cup (B^o)^c) = (B^o \cup \overline{B}) \cap 1_{\mathbb{B}}$
$= (B^o \cup \overline{B}) = \overline{B}$. [Since $B^o \subset B \subset \overline{B}$]

(iii) $Bd(B) = \overline{B} \cap \overline{B^c} = \phi_{\mathbb{B}}$
$\Rightarrow \overline{B} \cap (B^o)^c = \phi_{\mathbb{B}} \Rightarrow \overline{B} \cap ((B^o)^c)^c \neq \phi_{\mathbb{B}}$
$\Rightarrow \overline{B} \cap B^o \neq \phi_{\mathbb{B}} \Rightarrow \overline{B} \subset B^o$
$\Rightarrow B \subset \overline{B} \subset B^o \Rightarrow B \subset B^o$.
Also we know that $B^o \subset B$. Hence $B = B^o \Rightarrow B$ is open.
Further $\overline{B} \subset B^o \subset B \Rightarrow \overline{B} \subset B$, but we have $B \subset \overline{B}$
$\Rightarrow B = \overline{B} \Rightarrow B$ is closed.
Conversely, if $B$ is both open and closed, then $B = B^o$ and $B = \overline{B}$.
Now $Bd(B) = \overline{B} \cap \overline{B^c} = \overline{B} \cap (B^o)^c = B \cap B^c = \phi_{\mathbb{B}}$.

(iv) $Bd(B) = \overline{B} \cap \overline{B^c} = \overline{B} \cap (B^o)^c$.

□

## 5. Conclusion

Bipolar neutrosophic graded soft sets and some of their properties with real life examples were proposed in this paper. BNGS is the extension of bipolar neutrosophic soft set by categorizing parameter set. Further, we proposed bipolar neutrosophic graded soft topological spaces with their properties and some propositions about the BNGS-topology. In future, we will try to explore the real life applications and construct the algorithm based the BNGS set and their topological structure.

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Received: Nov 30, 2021. Accepted: Feb 3, 2022

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# Neutrosophic Cubic $\beta-$subalgebra

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**Abstract**. The main objective of this paper is to extend the notion of neutrosophic cubic sets to $\beta-$subalgebra. Some captivating results based on the P-union, P-intersection, R-union, R-intersection of neutrosophic cubic $\beta-$subalgebra have been explored. Further, the engrossing properties of the lower, upper level sets and homomorphism of neutrosophic cubic $\beta-$subalgebras were discussed.

**Keywords:** cubic set; Neutrosophic set; Neutrosophic cubic set; $\beta-$subalgebra; cubic $\beta-$subalgebra; neutrosophic cubic $\beta-$subalgebra.

## 1. Introduction

In 1965, Zadeh [30] initiated the concept of fuzzy sets which is a generalisation of the classical notion of a set. The notion of intuitionistic fuzzy set was proposed by Atanassov [4] whose elements have both membership and non-membership degrees. Biswas [5] introduced Rosenfeld's fuzzy subgroups with interval valued membership functions and studied some interesting properties. The idea of $\beta-$algebras has been presented by Neggers and Kim [23] which is a generlization of $BCK/BCI-$algebras where two operations have been used. Samarandache [27] proposed a generlization of intuitionistic fuzzy sets, known as neutrosophic set in which the distinction between the neutrosophic set and intuitionistic fuzzy set are emphasised. The notion of cubic sets introduced by Jun et al. [10, 11] and investigated the characteristics of cubic subgroups. Maji [16] applied the idea of soft set into neutrosophic sets and studied some compelling results.

The thought of fuzzy $\beta-$subalgebras originated by Ansari et al. [3] and relevant results have been examined. The attributes on intuitionistic fuzzy $\beta-$subalgebras were presented by Sujatha et al. [28]. Iqbal et al. [9] developed the idea of neutrosophic cubic subalgebras and ideals of $B$-algebras. The concept of neutrosophic cubic sets initiated by Jun et al. [12], [13], [14] and they have extended notion of neutrosophic subalgebras set to several types of $BCK/BCI-$algebras. Moreover, the applications of cubic interval valued intuitionistic fuzzy sets in $BCK/BCI-$algebras were provided. Hemavathi et al. [8] expressed the characteristics on interval valued intuitionistic fuzzy $\beta-$subalgebras. Made an approach on normed linear space using neutrosophic sets by Muralikrishna et al. [20] and examined the fascinating results.

The notion of $BMBJ$- neutrosophic aubalgebra in $BCK/BCI-$algebras presented by Bordbar et al. [6] and provided some engrossing results. Ajay et al. [1] discussed about neutrosophic cubic fuzzy dombi hamy mean operators with application to multi-criteria decision making. Akbar Razaei et al. [2] initiated the thought of neutrosophic triplet of $BI-$algebras and relevant results have been studied. Neutrosophic logic theory and applications were developed by Eman AboEIHamd et al. [7]. Some aspects on cubic fuzzy $\beta-$subalgebra of $\beta-$ algebra were discussed by Muralikrishna et al. [21]. Mohsin Khalid et al [17], [18], [19] interpreted the concept of translation and multiplication of neutrosophic cubic set and also introduced the notion of $T-MBJ$neutrosophic set under $M-$subalgebra. Moreover, the authors described the properties of $T-$neutrosophic cubic set on $BF-$algebra. Some special characteristics of neutrosophic vague binary $BCK/BCI-$algebra were discussed by Remya et al. [26]. Nanthini et al. [22] initiated the idea of interval valued neutrosophic topological spaces and relevant results have been examined.Diagnosing psychiatric disorder using neutrosophic soft set and its application presented deliberately by Veerappan Chinnadurai et al. [29]. Rajab Ali Borsooei et al. [25] intended to develop the polarity of generalized neutrosophic subalgebras in $BCK/BCI-$algebras. Johnson Awolola [15] introduced the concept of $\alpha-$level sets of neutrosophic set and investigated few of its associated properties. Prakasam Muralikrishna et al. [24] applied the concept of $\beta-$ideal into $MBJ-$neutrosophic set and investigated some engrossing results. With all these inspiration, this paper provides the study of neutrosophic cubic $\beta-$subalgebra. This work is organized into the following sections: Section 1 provides the introduction and section 2 presents the existing definitions required for this study. Section 3 deals the concept of neutrosophic cubic $\beta-$subalgebra, section 4 describes the characteristics on homomorphism of neutrosophic cubic $\beta-$subalgebra and section 5 gives the conclusion and future scope of the work.

## 2. Preliminaries

This section provides the necessary definitions and examples required for the work.

**Definition 2.1.** A fuzzy set in a universal set $X$ is defined as $\zeta : X \rightarrow [0, 1]$. For each element $x \in X$, $\zeta(x)$ is called the membership value of $x$.

**Definition 2.2.** If $\zeta_1$ and $\zeta_2$ are fuzzy sets in $X$, then the union of $\zeta_1$ and $\zeta_2$, denoted by $\zeta_1 \cup \zeta_2$ is defined by, $(\zeta_1 \cup \zeta_2)(x) = max\{\zeta_1(x), \zeta_2(x)\}\ \forall\ x \in X$.

**Definition 2.3.** If $\zeta_1$ and $\zeta_2$ are fuzzy sets in $X$, then the intersection of $\zeta_1$ and $\zeta_2$, denoted by $\zeta_1 \cap \zeta_2$ is defined by, $(\zeta_1 \cap \zeta_2)(x) = min\{\zeta_1(x), \zeta_2(x)\}\ \forall\ x \in X$.

**Definition 2.4.** Let $\zeta$ be a fuzzy set of $X$. for any $\delta \in [0, 1]$, the set $\zeta^\delta = \{x \in X/\zeta(x) \geq \delta\}$ is called an upper level subset of $\zeta$. The level subset $\zeta^\delta$ of a fuzzy set $\zeta$ is a crisp subset of the set $X$.

**Definition 2.5.** Let $\zeta$ be a fuzzy set of $X$. For $\delta \in [0, 1]$, the set $\zeta_\delta = \{x \in X : \zeta(x) \leq \delta\}$ is called a lower level subsets of $\zeta$.

**Definition 2.6.** The supremum property of the fuzzy set $\zeta$ for the subset $A$ in $X$ is defined as $\zeta(a_0) = \underset{a\in A}{Sup}\ \zeta(a)$ if there exist $a$, $a_0 \in A$.

**Definition 2.7.** Let $D[0, 1]$ denote the family of all closed sub intervals of $[0, 1]$. Consider two elements $D_1$,$D_2 \in D[0, 1]$. If $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$, then $rmax(D_1, D_2) = [max(a_1, a_2), max(b_1, b_2)]$ which is denoted by $D_1 \bigvee^r D_2$ and $rmin(D_1, D_2) = [min(a_1, a_2), min(b_1, b_2)]$ which is denoted by $D_1 \bigwedge^r D_2$.
Thus if $D_i = [a_i, b_i] \in D[0, 1]$ for i=1,2,3.... $rsup_i(D_i) = [sup_i(a_i), sup_i(b_i)]$, i.e. $\bigvee_i^r D_i = [\bigvee_i a_i, \bigvee_i b_i]$. Similarly $rinf_i(D_i) = [inf_i(a_i), inf_i(b_i)]$ i.e $\bigwedge_i^r D_i = [\bigwedge_i a_i, \bigwedge_i b_i]$. Now $D_1 \geq D_2$ iff $a_1 \geq a_2$ and $b_1 \geq b_2$. Similarly the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.

**Definition 2.8.** An interval valued fuzzy set $A$ defined on $X$ is given by $A$=$\{(x, [\zeta_A^L(x), \zeta_A^U(x)])\}\ \forall\ x \in X$ (briefly denoted by $A = [\zeta_A^L, \zeta_A^U]$), where $\zeta_A^L$ and $\zeta_A^U$ are two fuzzy sets in $X$ such that $\zeta_A^L(x) \leq \sigma_A^U(x)\ \forall\ x \in X$. Let $\overline{\zeta}_A(x) = [\zeta_A^L(x), \zeta_A^U(x)]\ \forall\ x \in X$ and let $D[0, 1]$ denotes the family of all closed sub intervals of $[0, 1]$. If $\zeta_A^L(x) = \zeta_A^U(x) = c$, say, where $0 \leq c \leq 1$, then $\overline{\zeta}_A(x) = [c, c]$ also for the sake of convenience, to belong to $D[0, 1]$. Thus $\overline{\zeta}_A(x) \in D[0, 1]\ \forall\ x \in X$, and therefore the interval valued fuzzy set $A$ is given by
$A = \{(x, \overline{\zeta}_A(x))\}\ \forall\ x \in X$, where $\overline{\zeta}_A : X \rightarrow D[0, 1]$.
Now let us define what is known as *refined mimimum*$(rmin)$ of two elements in $D[0, 1]$. Let us define the symbols " $\geq$ " , " $\leq$ ", and " $=$ " in case of two elements in $D[0, 1]$. Consider two elements $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0, 1]$. Then $rmin(D_1, D_2) =$

$[min\{a_1, a_2\}, min\{b_1, b_2\}]$; $D_1 \geq D_2$ if and only if $a_1 \geq a_2,\ b_1 \geq b_2$; Similarly , $D_1 \leq D_2$ and $D_1 = D_2$.

**Definition 2.9.** An Intuitionistic fuzzy set (IFS) in a nonempty set X is defined by A= $\{\langle x, \zeta_A(x), \eta_A(x)\rangle / x \in X\}$ where $\zeta_A : X \to [0,1]$ is a membership function of $A$ and $\eta_A : X \to [0,1]$ is a non membership function of $A$ satisfying $0 \leq \zeta_A(x) + \eta_A(x) \leq 1\ \forall\ x \in X$.

**Definition 2.10.** An intuitionistic fuzzy set $A$ is said to have sup-inf property if for any subset $T$ of $X$ there exists $x_0 \in T$ such that $\zeta_A(x_0) = \underset{x \in T}{Sup}\ \zeta_A(x)$ and $\eta_A(x_0) = \underset{x \in T}{inf}\ \eta_A(x)$.

**Definition 2.11.** A $\beta-$ algebra is a non-empty set $X$ with a constant 0 and two binary operations $+$ and $-$ satisfying the following axioms:
$(i)$ $x - 0 = x$
$(ii)$ $(0 - x) + x = 0$
$(iii)$ $(x - y) - z = x - (z + y)\ \forall\ x, y, z \in X$.

**Example 2.12.** The following Cayley table shows $(X = \{0, 1, 2, 3\}, +, -, 0)$ is a $\beta-$algebra.

**Table 1.** $\beta$-algebra

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 3 | 0 | 2 |
| 2 | 2 | 0 | 3 | 1 |
| 3 | 3 | 2 | 1 | 0 |

| − | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 1 | 2 | 0 |

**Definition 2.13.** A non empty subset $A$ of a $\beta-$algebra $(X, +, -, 0)$ is called a $\beta$-subalgebra of $X$, if
$(i)$ $x + y \in A$ and
$(ii)$ $x - y \in A\ \ \forall\ x, y \in A$.

**Definition 2.14.** Let $X$ be a non empty set. By a cubic set in $X$ we mean a structure

$$C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle :\ x \in\ X\}$$

in which $\overline{\zeta}_C$ is an interval valued fuzzy set in $X$ and $\eta_C$ is a fuzzy set in $X$.

**Definition 2.15.** Let $A = \{\langle x, \overline{\zeta}_A(x), \eta_A(x)\rangle :\ x \in\ X\}$ and $B = \{\langle x, \overline{\zeta}_B(x), \eta_B(x)\rangle :\ x \in\ X\}$ be two cubic sets on $X$, then the intersection of $A$ and $B$ denoted by $A \cap B$ is defined by $A \cap B = \{\langle x, \overline{\zeta}_{A\cap B}(x), \eta_{A\cap B}(x)\rangle\} = \{\langle x, rmin\{\zeta_A(x), \zeta_B(x)\}, max(\eta_A(x), \eta_B(x)), \rangle : x \in X\}$.

**Definition 2.16.** A cubic set $C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle\ :\ x \in X\}$ is said to have rsup-inf property if for any subset $T$ of $X$ there exists $x_0 \in T$ such that $\overline{\zeta}_C(x_0) = \underset{x \in T}{rsup}\ \overline{\zeta}_C(x)$ and $\eta_C(x_0) = \underset{x \in T}{inf}\ \eta_C(x)$.

**Definition 2.17.** Let $C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle : x \in X\}$ be a cubic set in $X$. Then the set $C$ is a cubic fuzzy $\beta-$ subalgebra if it satisfies the following conditions.
$(i)$ $\overline{\zeta}_C(x+y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$ & $\overline{\zeta}_C(x-y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$
$(ii)$ $\eta_C(x+y) \leq max\{\eta_C(x), \eta_C(y)\}$ & $\eta_C(x-y) \leq max\{\eta_C(x), \eta_C(y)\}$ $\forall\ x, y \in X$.

**Definition 2.18.** A neutrosophic set in $X$ is a structre of the form $\Omega = \{\langle x : \omega_T(x), \omega_I(x), \omega_F(x)\rangle / x \in X\}$. Where $\omega_T : X \to [0,1]$ is a truth membership function, $\omega_I : X \to [0,1]$ is a indeterminate membership function and $\omega_F : X \to [0,1]$ is a false membership function.

**Definition 2.19.** An interval neutrosophic set in $X$ is a structre of the form $\Delta = \{\langle x : \delta_T(x), \delta_I(x), \delta_F(x)\rangle / x \in X\}$ where $\delta_T, \delta_I, \delta_F$ are interval valued fuzzy sets in $X$, which are called an interval truth membership function, an interval indeterminate membership function and an interval false membership function respectively.

**Definition 2.20.** Let $X$ be a non-empty set. A neutrosophic cubic set is a pair $C = (\Delta, \omega)$ where $\Delta = \{x : \delta_T(x), \delta_I(x), \delta_F(x)/x \in X\}$ is interval valued neutrosophic set $\Omega = \{\langle x : \omega_T(x), \Omega_I(x), \omega_F(x)/x \in X\rangle\}$ is neutrosophic set.For our convenience, the neutrosophic cubic set will be denoted as $C = (\delta_{T,I,F}, \omega_{T,I,F}) = \{\langle x, \delta_{T,I,F}(x), \omega_{T,I,F}(x)\rangle\}$.

**Definition 2.21.** Let $f$ be a mapping from $X$ to $Y$. If $C = (\delta_{T,I,F}, \omega_{T,I,F})$ is neutrosophic cubic set of $X$. Then the image of $C$ under $f$ is denoted by $f(C)$ and is defined as $f(C) = \{\langle x, f_{rsup}(\delta_{T,I,F}), f_{inf}(\omega_{T,I,F})\rangle / x \in X\}$, where

$$f_{rsup}(\delta_{T,I,F})(y) = \begin{cases} \underset{x \in f^{-1}(y)}{rsup} (\delta_{T,I,F})(x) & : if\ f^{-1}(y) \neq \phi \\ [0,0] & : Otherwise \end{cases}$$

$$f_{inf}(\omega_{T,I,F})(y) = \begin{cases} \underset{x \in f^{-1}(y)}{inf} (\omega_{T,I,F})(x) & : if\ f^{-1}(y) \neq \phi \\ 1 & : Otherwise \end{cases}$$

**Definition 2.22.** Let $f$ be a mapping from $X$ to $Y$. If $C = (\delta_{T,I,F}, \omega_{T,I,F})$ is neutrosophic cubic set of $X$. Then the inverse image of $C$ is defined as $f^{-1}(C) = \{\langle x, f_{rsup}(\delta_{T,I,F}), f_{inf}(\omega_{T,I,F})\rangle / x \in X\}$, with $f^{-1}(\delta_{T,I,F}(x) = (\delta_{T,I,F}(f(x))$ and $f^{-1}(\omega_{T,I,F}(x) = (\omega_{T,I,F}(f(x))$.

**Definition 2.23.** For any $C_i = (\Delta_i, \Omega_i)$,where $\Delta_i = \{\langle x, \delta_{iT}(x), \delta_{iI}(x), \delta_{iF}(x)\rangle : x \in X\}$ and $\Omega_i = \{\langle x, \omega_{iT}(x), \omega_{iI}(x), \omega_{iF}(x)\rangle : x \in X\}$ for $i \in k$. $P-$ union, $P-$intersection & $R-$union,$R-$intersection is defined respectively by
$P-$union: $\bigcup_{i \in k} C_i = \left(\bigcup_{i \in k} \Delta_i, \bigvee_{i \in k} \Omega_i\right)$

$P-$intersection: $\bigcap_{i\in k} C_i = \left(\bigcap_{i\in k} \Delta_i, \bigwedge_{i\in k} \Omega_i\right)$
$R-$union: $\bigcup_{i\in k} C_i = \left(\bigcup_{i\in k} \Delta_i, \bigwedge_{i\in k} \Omega_i\right)$
$R-$intersection: $\bigcap_{i\in k} C_i = \left(\bigcap_{i\in k} \Delta_i, \bigvee_{i\in k} \Omega_i\right)$
where

$$\bigcup_{i\in k} \Delta_i = \left\{\langle x; \left(\bigcup_{i\in k} \delta_{iT}\right)(x), \left(\bigcup_{i\in k} \delta_{iI}\right)(x), \left(\bigcup_{i\in k} \delta_{iF}\right)(x)/x \in X\right\}$$

$$\bigvee_{i\in k} \Omega_i = \left\{\langle x; \left(\bigvee_{i\in k} \omega_{iT}\right)(x), \left(\bigvee_{i\in k} \omega_{iI}\right)(x), \left(\bigvee_{i\in k} \omega_{iF}\right)(x)/x \in X\right\}$$

$$\bigcap_{i\in k} \Delta_i = \left\{\langle x; \left(\bigcap_{i\in k} \delta_{iT}\right)(x), \left(\bigcap_{i\in k} \delta_{iI}\right)(x), \left(\bigcap_{i\in k} \delta_{iF}\right)(x)/x \in X\right\}$$

$$\bigwedge_{i\in k} \Omega_i = \left\{\langle x; \left(\bigwedge_{i\in k} \omega_{iT}\right)(x), \left(\bigwedge_{i\in k} \omega_{iI}\right)(x), \left(\bigwedge_{i\in k} \omega_{iF}\right)(x)/x \in X\right\}.$$

**Definition 2.24.** Let $C$ be a neutrosophic cubic set of $X$ where $C = (\delta_{T,I,F}, \omega_{T,I,F})$. For $[s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}] \in D[0,1]$ and $t_{T_1}, t_{T_1}, t_{F_1} \in [0,1]$,
the set $U(\delta_{T,I,F}/[s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}] = \{x \in X/\delta_T(x) \geq [s_{T_1}, s_{T_2}], \delta_T(x) \geq [s_{I_1}, s_{I_2}], \delta_T(x) \geq [s_{F_1}, s_{F_2}]\}$ is called upper $([s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}])$-level of $C$ and $L(\omega_{T,I,F}/(t_{T_1}, t_{T_1}, t_{F_1})) = \{x \in X/\omega_T(x) \leq t_{T_1}, \omega_I(x) \leq t_{I_1}, \omega_F(x) \leq t_{F_1}$ is called lower $(t_{T_1}, t_{T_1}, t_{F_1})$-level set of $A$.
For our convenience, we are introducing the new notion as
$U(\delta_{T,I,F}/[S_{T,I,F_1}, S_{T,I,F_2}] = \{x \in X/\delta_{T,I,F}(x) \geq [S_{T,I,F_1}, S_{T,I,F_2}]\}$ is called upper $[s_{T,I,F_1}, s_{T,I,F_2}]$-level set of $C$ and $L(\omega_{T,I,F}/[t_{T,I,F_1}, t_{T,I,F_2}] = \{x \in X/\omega_{T,I,F}(x) \leq [t_{T,I,F_1}, t_{T,I,F_2}]\}$ is called lower $t_{T,I,F_1}$-level set of $C$.

## 3. Neutrosophic Cubic $\beta-$ Subalgebra

This section introduces the notion of neutrosophic cubic $\beta-$ subalgebra and discusses some engrossing results.

**Definition 3.1.** $C = \{x, \Delta(x), \Omega(x)/x \in X\})$ be a neutrosophic cubic set in $X$. Then the set $C$ is a neutrosophic cubic $\beta-$subalgebra if it satisfies the following conditions:
$NS1:$
$\delta_T(x+y) \geq rmin\{\delta_T(x), \delta_T(y)\}$ & $\delta_T(x-y) \geq rmin\{\delta_T(x), \delta_T(y)\}$
$\delta_I(x+y) \geq rmin\{\delta_I(x), \delta_I(y)\}$ & $\delta_I(x-y) \geq rmin\{\delta_I(x), \delta_I(y)\}$
$\delta_F(x+y) \geq rmin\{\delta_F(x), \delta_F(y)\}$ & $\delta_F(x-y) \geq rmin\{\delta_F(x), \delta_F(y)\}$
$NS2$:
$\omega_T(x+y) \leq max\{\omega_T(x), \omega_T(y)\}$ & $\omega_T(x-y) \leq max\{\omega_T(x), \omega_T(y)\}$
$\omega_I(x+y) \leq max\{\omega_I(x), \omega_I(y)\}$ & $\omega_I(x-y) \leq max\{w_I(x), \omega_I(y)\}$
$\omega_F(x+y) \leq max\{\omega_F(x), \omega_F(y)\}$ & $\omega_F(x-y) \leq max\{\omega_F(x), \omega_F(y)\}$

For our convenience the neutrosophic cubic set will be denoted as
$C = (\delta_{T,I,F}, \omega_{T,I,F}) = \{\langle x, \delta_{T,I,F}(x), \omega_{T,I,F}(x)\rangle\}$ with conditions
$(i)\delta_{T,I,F}(x+y) \geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$ & $\delta_{T,I,F}(x-y) \geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$
$(ii)\omega_{T,I,F}(x+y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$ & $\omega_{T,I,F}(x-y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$.

**Example 3.2.** For the $\beta-$algebra $X$ in the example 2.6, the Cubic set $C = \{x, \Delta(x), \Omega(x)/x \in X\})$ on $X$ as follows.

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\delta_T$ | [0.4,0.6] | [0.3,0.7] | [0.4,0.6] | [0.3,0.7] |
| $\delta_I$ | [0.3,0.5] | [0.2,0.4] | [0.3,0.5] | [0.2,0.4] |
| $\delta_F$ | [0.2,0.3] | [0.1,0.2] | [0.2,0.3] | [0.1,0.2] |

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\omega_T$ | 0.2 | 0.4 | 0.2 | 0.4 |
| $\omega_I$ | 0.3 | 0.5 | 0.3 | 0.5 |
| $\omega_F$ | 0.4 | 0.6 | 0.4 | 0.6 |

is a neutrosophic cubic fuzzy $\beta-$sub algebra of $X$.

**Proposition 3.3.** *Let $C = \{\langle x, \delta_{T,I,F}(x), \omega_{T,I,F}(x)\rangle : x \in X\}$ is a neutrosophic cubic $\beta-$subalgebra of $X$. Then $\delta_{T,I,F}(0) \geq \delta_{T,I,F}(x)$ and $\omega_{T,I,F}(0) \leq \omega_{T,I,F}(x)\ \forall\ x \in X$. Thus $\delta_{T,I,F}(0)$ & $\omega_{T,I,F}(0)$ are upper bounds and lower bounds of $\delta_{T,I,F}(x)$ & $\omega_{T,I,F}(x)$ respectively.*
*proof: (1) For every $x \in X$,*

$$\begin{aligned}\delta_{T,I,F}(0) &= \delta_{T,I,F}(x-x)\\ &\geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(x)\}\\ &= \delta_{T,I,F}(x)\end{aligned}$$

$\therefore \delta_{T,I,F}(0) \geq \delta_{T,I,F}(x)$ *and*

$$\begin{aligned}\omega_{T,I,F}(0) &= \omega_{T,I,F}(x-x)\\ &\leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(x)\}\\ &= \omega_{T,I,F}(x)\end{aligned}$$

$\therefore \omega_{T,I,F}(0) \leq \omega_{T,I,F}(x)$.

**Theorem 3.4.** *Let $C = \{\langle x, \delta_{T,I,F}(x), \omega_{T,I,F}(x)\rangle : x \in X\}$ be a neutrosophic cubic $\beta-$subalgebra of $X$. If there exists a sequence $\{x_n\}$ of $X$ such that $\lim_{n\to\infty} \delta_{T,I,F}(x_n) = [1,1]$ and $\lim_{n\to\infty} \omega_{T,I,F}(x_n) = 0$. Then $\delta_{T,I,F}(x_n) = [1,1]$ and $\omega_{T,I,F}(x_n) = 0$.*
*Proof: By using Proposition 3.3, $\delta_{T,I,F}(0) \geq \delta_{T,I,F}(x)\ \forall x \in X$, then we have $\delta_{T,I,F}(0) \geq \delta_{T,I,F}(x_n)\ \forall n \in Z^+$. Consider, $[1,1] \geq \delta_{T,I,F}(0) \geq \lim_{n\to\infty} \delta_{T,I,F}(x_n) = [1,1]$ Hence, $\delta_{T,I,F}(0) = [1,1]$. Moreover using proposition 3.3, $\omega_{T,I,F}(0) \leq \omega_{T,I,F}(x)\ \forall x \in X$, then we have $\omega_{T,I,F}(0) \leq \omega_{T,I,F}(x_n)\ \forall n \in Z^+$ Consider, $0 \leq \omega_{T,I,F}(0) \leq \lim_{n\to\infty} \omega_{T,I,F}(x_n) = 0$. Hence, $\omega_{T,I,F}(0) = 0$.*

**Theorem 3.5.** *The $R-$intersection of any set of neutrosophic cubic $\beta-$subalgebras of $X$ is also a neutrosophic cubic $\beta-$subalgebra of $X$.*

*Proof: Let $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle / x \in X\}$ where $i \in k$ be a sets of neutrosophic cubic $\beta-$subalgebras of $X$ and $x, y \in X$. Then*

$$\begin{aligned}(\cap\delta_{iT,I,F})(x+y) &= rinf\ \delta_{iT,I,F}(x+y)\\ &\geq rinf\ \{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\}\}\\ &= rmin\{rinf\ \delta_{iT,I,F}(x), rinf\ \delta_{iT,I,F}(y)\}\\ &= rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \cap\delta_{iT,I,F}(x+y) \geq rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}$

*Similarly,*

$\delta_{iT,I,F}(x-y) \geq rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}$ *and*

$$\begin{aligned}(\vee\omega_{iT,I,F})(x+y) &= sup\ \omega_{iT,I,F}(x+y)\\ &\leq sup\ \{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\}\}\\ &= max\{sup\ \omega_{iT,I,F}(x), sup\ \omega_{iT,I,F}(y)\}\\ &= max\{\vee\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \vee\omega_{iT,I,F}(x+y) \leq max\{\vee\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}$

*In the same way, $\omega_{iT,I,F}(x-y) \leq max\{\vee\delta_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}$. Hence $R-$intersection of $C_i$ is a neutrosophic cubic $\beta-$subalgebra of $X$.*

**Theorem 3.6.** *The $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle\}/x \in X$ where $i \in k$ be a sets of neutrosophic cubic $\beta-$subalgebras of $X$. If $inf\{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y) = max\{inf\omega_{iT,I,F}(x), inf\omega_{iT,I,F}(y)\}\forall x \in X$. Then the $P-$intersection of $C_i$ is also a neutrosophic cubic $\beta-$subalgebra of $X$.*

*Proof: Let $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle / x \in X\}$ where $i \in k$ be a sets of neutrosophic cubic $\beta-$subalgebras of $X$ and $x, y \in X$. Then*

$$\begin{aligned}(\cap\delta_{iT,I,F})(x+y) &= rinf\ \delta_{iT,I,F}(x+y)\\ &\geq rinf\ \{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\}\}\\ &= rmin\{rinf\ \delta_{iT,I,F}(x), rinf\ \delta_{iT,I,F}(y)\}\\ &= rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \cap\delta_{iT,I,F}(x+y) \geq rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}$

*In the same manner,* $\delta_{iT,I,F}(x-y) \geq rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}$ *and*

$$\begin{aligned}(\wedge\omega_{iT,I,F})(x+y) &= inf\ \omega_{iT,I,F}(x+y)\\ &\leq inf\ \{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\}\}\\ &= max\{inf\omega_{iT,I,F}(x), inf\ \omega_{iT,I,F}(y)\}\\ &= max\{\wedge\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \wedge\omega_{iT,I,F}(x+y) \leq max\{\wedge\omega_{iT,I,F}(x), \wedge\omega_{iT,I,F}(y)\}$

*Similarly,* $\wedge\omega_{iT,I,F}(x-y) \leq max\{\wedge\omega_{iT,I,F}(x), \wedge\omega_{iT,I,F}(y)\}$. *Hence* $P-$*intersection of* $C_i$ *is a neutrosophic cubic* $\beta-$*subalgebra of* $X$.

**Theorem 3.7.** *The* $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle\}/x \in X$ *where* $i \in k$ *be a sets of neutrosophic cubic* $\beta-$*subalgebras of* $X$. *If* $sup\ \{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y) = rmin\{sup\ \delta_{iT,I,F}(x), sup\ \delta_{iT,I,F}(y)\}\forall x \in X$. *Then the* $P-$*union of* $C_i$ *is also a neutrosophic cubic* $\beta-$*subalgebra of* $X$.

*Proof: Let* $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle/x \in X\}$ *where* $i \in k$ *be a sets of neutrosophic cubic* $\beta-$*subalgebras of* $X$ *and* $x, y \in X$ *such that* $sup\{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y) = rmin\{sup\ \delta_{iT,I,F}(x), sup\ \delta_{iT,I,F}(y)\}\forall x \in X$. *Then for* $x, y \in X$,

$$\begin{aligned}(\cup\delta_{iT,I,F})(x+y) &= rsup\ \delta_{iT,I,F}(x+y)\\ &\geq rsup\ \{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\}\}\\ &= rmin\{rsup\ \delta_{iT,I,F}(x), rsup\ \delta_{iT,I,F}(y)\}\\ &= rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \cup\delta_{iT,I,F}(x+y) \geq rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}$

*Likewise,* $\cup\delta_{iT,I,F}(x-y) \geq rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}$ *and*

$$\begin{aligned}(\vee\omega_{iT,I,F})(x+y) &= sup\ \omega_{iT,I,F}(x+y)\\ &\leq sup\ \{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\}\}\\ &= max\{sup\ \omega_{iT,I,F}(x), sup\ \omega_{iT,I,F}(y)\}\\ &= max\{\vee\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \vee\omega_{iT,I,F}(x+y) \leq max\{\vee\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}$

*Similarly,* $\vee\omega_{iT,I,F}(x-y) \leq max\{\vee\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}$. *Hence* $P-$*union of* $C_i$ *is a neutrosophic cubic* $\beta-$*subalgebra of* $X$.

**Theorem 3.8.** *The* $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle\}/x \in X$ *where* $i \in k$ *be a sets of neutrosophic cubic* $\beta-$*subalgebras of* $X$. *If* $inf\ \{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\} =$

*$max\{inf\ \omega_{iT,I,F}(x), inf\ \omega_{iT,I,F}(y)\}$ & $sup\ \{rmin\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\} = rmin\{sup\ \omega_{iT,I,F}(x), sup\ \omega_{iT,I,F}(y)\} \forall x \in X$. Then the $R-$union of $C_i$ is also a neutrosophic cubic $\beta-$subalgebra of $X$.*

*Proof: Let $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle / x \in X\}$ where $i \in k$ be a sets of neutrosophic cubic $\beta-$subalgebras of $X$ such that $inf\{max\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\}$ &*
*$sup\{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\} = rmin\{sup\ \delta_{iT,I,F}(x), sup\ \delta_{iT,I,F}(y)\} \forall x \in X$. Then for $x, y \in X$,*

$$\begin{aligned}(\cup\delta_{iT,I,F})(x+y) &= rsup\ \delta_{iT,I,F}(x+y)\\ &\geq rsup\ \{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\}\}\\ &= rmin\{rsup\ \delta_{iT,I,F}(x), rsup\ \delta_{iT,I,F}(y)\}\\ &= rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \cup\delta_{iT,I,F}(x+y) \geq rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}$

*In the same way, $\cup\delta_{iT,I,F}(x-y) \geq rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}$ and*

$$\begin{aligned}(\wedge\omega_{iT,I,F})(x+y) &= inf\ \omega_{iT,I,F}(x+y)\\ &\leq inf\ \{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\}\}\\ &= max\{inf\ \omega_{iT,I,F}(x), inf\ \omega_{iT,I,F}(y)\}\\ &= max\{\wedge\omega_{iT,I,F}(x), \wedge\omega_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \wedge\omega_{iT,I,F}(x+y) \leq max\{\wedge\omega_{iT,I,F}(x), \wedge\omega_{iT,I,F}(y)\}$

*Similarly, $\wedge\omega_{iT,I,F}(x-y) \leq max\{\wedge\omega_{iT,I,F}(x), \wedge\omega_{iT,I,F}(y)\}$. Hence $R-$union of $C_i$ is a neutrosophic cubic $\beta-$subalgebra of $X$.*

**Theorem 3.9.** *Neutrosophic cubic set $C_i = \{\Delta_{T,I,F}, \Omega_{T,I,F}\}$of $X$ is a neutrosophic cubic $\beta-$subalgebra of $X$ if and ony if $\delta^L_{T,I,F}, \delta^U_{T,I,F}$ & $\omega_{T,I,F}$ are fuzzy subalgebras of $X$.*

*Proof: Let $\delta^L_{T,I,F}, \delta^U_{T,I,F}$ & $\omega_{T,I,F}$ are fuzzy subalgebras of $X$ and $x, y \in X$.Then*
$\delta^L_{T,I,F}(x+y) \geq min\{\delta^L_{T,I,F}(x), \delta^L_{T,I,F}(y)\}$
$\delta^U_{T,I,F}(x+y) \geq min\{\delta^U_{T,I,F}(x), \delta^U_{T,I,F}(y)\}$ *and*
$\omega_{T,I,F}(x+y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$
*Now*

$$\begin{aligned}\delta_{T,I,F}(x+y) &= [\delta^L_{T,I,F}(x+y), \delta^U_{T,I,F}(x+y)]\\ &\geq [min\{\delta^L_{T,I,F}(x), \delta^L_{T,I,F}(y)\}, min\{\delta^U_{T,I,F}(x), \delta^U_{T,I,F}(y)\}]\\ &\geq rmin\{[\delta^L_{T,I,F}(x), \delta^U_{T,I,F}(x)], [\delta^L_{T,I,F}(y), \delta^U_{T,I,F}(y)]\}\\ &= rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}\end{aligned}$$

$\therefore$ *C is neutrosophic cubic β−subalgebra of X.*

*Conversely, assume that C is neutrosophic cubic β−subalgebra of X. For any $x, y \in X$,*

$$\begin{aligned}[\delta^L_{T,I,F}(x+y), \delta^U_{T,I,F}(x+y)] &= \delta_{T,I,F}(x+y) \\ &\geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\} \\ &\geq rmin\{[\delta^L_{T,I,F}(x), \delta^U_{T,I,F}(x)], [\delta^L_{T,I,F}(y), \delta^U_{T,I,F}(y)]\}\end{aligned}$$

*Thus, $\delta^L_{T,I,F}(x+y) \geq min\{\delta^L_{T,I,F}(x), \delta^L_{T,I,F}(y)\}$, $\delta^U_{T,I,F}(x+y) \geq min\{\delta^U_{T,I,F}(x), \delta^U_{T,I,F}(y)\}$ and $\omega_{T,I,F}(x+y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$. Hence $\delta^L_{T,I,F}, \delta^U_{T,I,F}$ and $\omega_{T,I,F}$ are fuzzy subalgebra of X.*

**Remark 3.10.** The sets denoted by $I_{\delta_{T,I,F}}$ and $I_{\omega_{T,I,F}}$ are also subalgebra of $X$ which are defined as $I_{\delta_{T,I,F}} = \{x \in X/\delta_{T,I,F}(x) = \delta_{T,I,F}(0)\}$ and $I_{\omega_{T,I,F}} = \{x \in X/\omega_{T,I,F}(x) = \omega_{T,I,F}(0)\}$.

**Theorem 3.11.** *Let $C = (\delta_{T,I,F}, \omega_{T,I,F})$ be a neutrosophic cubic β−subalgebra of X. Then the sets $I_{\delta_{T,I,F}}$ and $I_{\omega_{T,I,F}}$ are also subalgebra of X.*

*Proof: Let $x, y \in I_{\delta_{T,I,F}}$.*

*Then $\delta_{T,I,F}(x) = \delta_{T,I,F}(0) = \delta_{T,I,F}(y)$. Consider*

$$\begin{aligned}\delta_{T,I,F}(x+y) &\geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\} \\ &\geq rmin\{\delta_{T,I,F}(0), \delta_{T,I,F}(0)\} \\ &= \delta_{T,I,F}(0)\end{aligned}$$

$\therefore \delta_{T,I,F}(x+y) \geq \delta_{T,I,F}(0)$. *By using proposition 3.3,* $\delta_{T,I,F}(0) \geq \delta_{T,I,F}(x+y)$

*Then we have $\delta_{T,I,F}(x+y) = \delta_{T,I,F}(0)$ or equivalently, $x+y \in I_{\delta_{T,I,F}}$*

*Similarly, $x-y \in I_{\delta_{T,I,F}}$.*

*Now, let $x, y \in I_{\delta_{T,I,F}}$. Then $\omega_{T,I,F}(x) = \omega_{T,I,F}(0) = \omega_{T,I,F}(y)$.*

*Consider*

$$\begin{aligned}\omega_{T,I,F}(x+y) &\leq max\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\} \\ &= max\{\omega_{T,I,F}(0), \omega_{T,I,F}(0)\} \\ &= \omega_{T,I,F}(0)\end{aligned}$$

$\therefore \omega_{T,I,F}(x+y) \leq \omega_{T,I,F}(0)$. *By using proposition 3.3,* $\omega_{T,I,F}(0) \leq \omega_{T,I,F}(x+y)$

*Then we have $\omega_{T,I,F}(x+y) = \omega_{T,I,F}(0)$ or equivalently, $x+y \in I_{\omega_{T,I,F}}$*

*Similarly,$x-y \in I_{\omega_{T,I,F}}$. Hence the sets $I_{\delta_{T,I,F}}$ and $I_{\omega_{T,I,F}}$ are β−subalgebras of X.*

**Theorem 3.12.** *Let P be a non empty subset of X and $C = (\delta_{T,I,F}, \omega_{T,I,F})$ be a neutrosophic cubic β−subalgebra of X defined by*

$$\delta_{T,I,F}(x) = \begin{cases} [\phi_{T,I,F_1}, \phi_{T,I,F_1}] : & if\ x \in P \\ [\psi_{T,I,F_1}, \psi_{T,I,F_1}] : & Otherwise \end{cases} \qquad \omega_{T,I,F}(x) = \begin{cases} \rho_{T,I,F} : & if\ x \in P \\ \epsilon_{T,I,F} : & Otherwise \end{cases}$$

*$\forall$ $[\phi_{T,I,F_1}, \phi_{T,I,F_2}], [\psi_{T,I,F_1}, \psi_{T,I,F_2}] \in D[0,1]$ and $\rho_{T,I,F}, \epsilon_{T,I,F} \in [0,1]$ with $[\phi_{T,I,F_1}, \phi_{T,I,F_2}] \geq [\psi_{T,I,F_1}, \psi_{T,I,F_2}]$ and $\rho_{T,I,F} \leq \epsilon_{T,I,F}$. Then $C$ is a neutrosophic cubic $\beta-$subalgebra of $X \Leftrightarrow P$ is a $\beta-$subalgebra of $X$.*

*Proof:* Let $C$ be a neutrosophic cubic $\beta-$subalgebra of $X$.
Let $x, y \in X$ such tat $x, y \in P$. Then

$$\begin{aligned} \delta_{T,I,F}(x+y) &\geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\} \\ &\geq rmin\{[\phi_{T,I,F_1}, \phi_{T,I,F_2}], [\phi_{T,I,F_1}, \phi_{T,I,F_2}]\} \\ &= [\phi_{T,I,F_1}, \phi_{T,I,F_2}] \end{aligned}$$

*and*

$$\begin{aligned} \omega_{T,I,F}(x+y) &\leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\} \\ &\leq max\{\rho_{T,I,F}, \rho_{T,I,F}\} \\ &= \rho_{T,I,F} \end{aligned}$$

*Therefore $x + y \in P$. Similarly, we have $x - y \in P$.*
*Hence $P$ is a $\beta-$subalgebra of $X$.*
*Conversely, suppose that $P$ is a $\beta-$subalgebra of $X$. Let $x, y \in X$.*
*Case(i): If $x, y \in P$ then $x + y \in P$ & $x - y \in P$*
*Thus* $\delta_{T,I,F}(x+y) = [\phi_{T,I,F_1}, \phi_{T,I,F_2}] = rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$
*Similarly,* $\delta_{T,I,F}(x-y) = rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$ *and*
$\omega_{T,I,F}(x+y) = \rho_{T,I,F} = max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$.
*In the same way,* $\omega_{T,I,F}(x-y) = max\{\omega_{T,I,F}(x), \delta_{T,I,F}(y)\}$
*Case (ii): if $x, y \notin B$,then*
$\delta_{T,I,F}(x+y) = [\psi_{T,I,F_1}, \psi_{T,I,F_2}] = rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$
*Similarly,* $\delta_{T,I,F}(x-y) = rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$ *and*
$\omega_{T,I,F}(x+y) = \epsilon_{T,I,F} = max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$.
*In the same way,* $\omega_{T,I,F}(x-y) = max\{\omega_{T,I,F}(x), \delta_{T,I,F}(y)\}$
*Hence $C$ is a neutrosophic cubic $\beta-$subalgebra of $X$.*

*Now,*

$$\begin{aligned} I_{\delta_{T,I,F}} &= \{x \in X, \delta_{T,I,F}(x) = \delta_{T,I,F}(0)\} \\ &= \{x \in X, \delta_{T,I,F}(x) = [\phi_{T,I,F_1}, \phi_{T,I,F_2}]\} \\ &= P \end{aligned}$$

$$\begin{aligned} I_{\omega_{T,I,F}} &= \{x \in X, \omega_{T,I,F}(x) = \omega_{T,I,F}(0)\} \\ &= \{x \in X, \omega_{T,I,F}(x) = \rho_{T,I,F}\} \\ &= P. \end{aligned}$$

**Theorem 3.13.** *If $C = (\delta_{T,I,F}, \omega_{T,I,F})$ be a neutrosophic cubic $\beta-$subalgebra of $X$ then the upper $[s_{T,I,F_1}, s_{T,I,F_2}]$-level and lower $t_{T,I,F_1}$-level set of $C$ are $\beta-$subalgebra of $X$.*
*proof: Let $x, y \in U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$, then $\delta_{T,I,F}(x) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$ and $\delta_{T,I,F}(y) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$. It follows that $\delta_{T,I,F}(x+y) \geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$*
*$\Rightarrow x + y \in U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$. Similarly, $x - y \in U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}]$.*
*Hence $U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}]$ is a $\beta-$subalgebra of $X$.*
*Let $x, y \in L(\omega_{T,I,F}/t_{T,I,F_1})$ then $\omega_{T,I,F}(x) \leq t_{T,I,F_1}$ and $\omega_{T,I,F}(y) \leq t_{T,I,F_1}$.*
*It follows that $\omega_{T,I,F}(x+y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y) \leq t_{T,I,F_1}\}$*
*$\Rightarrow x + y \in L(\omega_{T,I,F}/t_{T,I,F_1})$. Similarly, $x - y \in L(\omega_{T,I,F}/t_{T,I,F_1})$.*
*Hence $L(\omega_{T,I,F}/t_{T,I,F_1})$ is a $\beta-$subalgebra of $X$.*

**Theorem 3.14.** *Let $C = (\delta_{T,I,F}, \omega_{T,I,F})$ be a neutrosophic cubic set of $X$, such that the sets $U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\omega_{T,I,F}/t_{T,I,F_1})$ are $\beta-$subalgebra of $X$ for every $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$. Then $C = (\delta_{T,I,F}, \omega_{T,I,F})$ is neutrosophic cubic $\beta-$subalgebra of $X$.*
*proof: Let $U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\omega_{T,I,F}/t_{T,I,F_1})$ are $\beta-$subalgebra of $X$ for every $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$.*
*On the contrary, let $x_0, y_0 \in X$ be such that $\delta_{T,I,F}(x_0 + y_0) < rmin\{\delta_{T,I,F}(x_0), \delta_{T,I,F}(y_0)\}$*
*Let$\delta_{T,I,F}(x_0) = [\theta_1, \theta_2], \delta_{T,I,F}(y_0) = [\theta_3, \theta_4]$ and $\delta_{T,I,F}(x_0 + y_0) = [s_{T,I,F_1}, s_{T,I,F_2}]$. Then*
*$[s_{T,I,F_1}, s_{T,I,F_2}] < rmin\{[\theta_1, \theta_2], [\theta_3, \theta_4]\} = [min\{\theta_1, \theta_2\}, min\{\theta_3, \theta_4\}]$*
*So, $\delta_{T,I,F_1} < min\{[\theta_1, \theta_3]\}$ and $\delta_{T,I,F_2} < min\{[\theta_2, \theta_4]\}$*
*Let us consider,*

$$\begin{aligned} [\gamma_1, \gamma_2] &= (1/2)[\delta_{T,I,F}(x_0 + y_0) + rmin\{\delta_{T,I,F}(x_0), \delta_{T,I,F}(y_0)\}] \\ &= (1/2)[s_{T,I,F_1}, s_{T,I,F_2}] + min\{\theta_1, \theta_3\}, min\{\theta_3, \theta_4\} \\ &= (1/2)(s_{T,I,F_1} + min\{\theta_1, \theta_3\}), (1/2)(s_{T,I,F_1} + min\{\theta_2, \theta_4\}) \end{aligned}$$

$\therefore, min\{\theta_1, \theta_3\} > \gamma_1 = (1/2)(s_{T,I,F_1} + min\{\theta_1, \theta_3\}) > s_{T,I,F_1}$
*and* $\therefore, min\{\theta_2, \theta_4\} > \gamma_2 = (1/2)(s_{T,I,F_2} + min\{\theta_2, \theta_4\}) > s_{T,I,F_2}$

*Hence* $[min\{\theta_1,\theta_3\}, min\{\theta_2,\theta_4\}] > [\gamma_1,\gamma_2] > [s_{T,I,F_1}, s_{T,I,F_2}]$,*so that* $x_0 + y_0 \notin U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$ *which is a contradiction. Since* $\delta_{T,I,F}(x_0) = [\theta_1,\theta_2] \geq [min\{\theta_1,\theta_3\}, min\{\theta_2,\theta_4\}] > [\gamma_1,\gamma_2]$ *and* $\delta_{T,I,F}(y_0) = [\theta_3,\theta_4] \geq [min\{\theta_1,\theta_3\}, min\{\theta_2,\theta_4\}] > [\gamma_1,\gamma_2]$. $\Rightarrow x_0 + y_0 \in U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$. *Thus* $\delta_{T,I,F}(x+y) \geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\} \ \forall \ x,y \in X$.

*Similarly,*$\delta_{T,I,F}(x-y) \geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\} \ \forall \ x,y \in X$. *In the same way, we can prove* $\omega_{T,I,F}(x+y) = \omega_{T,I,F}(x+y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\} \ \forall \ x,y \in X$.

## 4. Homomorphism of Neutrosophic Cubic $\beta-$subalgebras

In this section, some of the interesting results on homomorphism of neutrosophic cubic $\beta-$subalgebra is being investigated.

**Theorem 4.1.** *Suppose that* $f : X \to Y$ *be a homomorphism from a* $\beta-$*algebra* $X$ *to* $Y$. *If* $C = (\delta_{T,I,F}\omega_{T,I,F})$ *is a neutrosophic cubic* $\beta-$*subalgebra of* $X$, *then the image* $f(C) = \{\langle x, f_{rsup}(\delta_{T,I,F}), f_{inf}(\omega_{T,I,F})\rangle / x \in X\}$ *of* $C$ *under* $f$ *is a neutrosophic cubic* $\beta-$*subalgebra of* $Y$.

*Proof: Let* $C = (\delta_{T,I,F}, \omega_{T,I,F})$ *be a neutrosophic cubic* $\beta-$*subalgebra of* $X$ *and let* $y_1, y_2 \in Y$. *We know that* $\{x_1 + x_2 / x_1 \in f^{-1}(y_1) \ \& \ x_2 \in f^{-1}(y_2)\} \subseteq \{x \in X / x \in f^{-1}(y_1 + y_2)\}$.*Now*

$$\begin{aligned} f_{rsup}(\delta_{T,I,F})(y_1+y_2) &= rsup\{\delta_{T,I,F}(x)/x \in f^{-1}(y_1+y_2)\} \\ &= rsup\{\delta_{T,I,F}(x_1+x_2)/x_1 \in f^{-1}(y_1) \ \& \ x_2 \in f^{-1}(y_2)\} \\ &\geq rsup\{rmin\{\delta_{T,I,F}(x_1), \delta_{T,I,F}(x_2)/x_1 \in f^{-1}(y_1) \ \& \ x_2 \in f^{-1}(y_2)\} \\ &= rmin\{rsup\{\delta_{T,I,F}(x_1)/x_1 \in f^{-1}(y_1), \delta_{T,I,F}(x_2)/x_2 \in f^{-1}(y_2)\} \end{aligned}$$

*In the same manner, we have*

$f_{rsup}(\delta_{T,I,F})(y_1 - y_2) \geq rmin\{rsup\{\delta_{T,I,F}(x_1)/x_1 \in f^{-1}(y_1), \delta_{T,I,F}(x_2)/x_2 \in f^{-1}(y_2)\}$. *Also,*

$$\begin{aligned} f_{inf}(\delta_{T,I,F})(y_1+y_2) &= inf\{\omega_{T,I,F}(x)/x \in f^{-1}(y_1+y_2)\} \\ &= inf\{\omega_{T,I,F}(x_1+x_2)/x_1 \in f^{-1}(y_1) \ \& \ x_2 \in f^{-1}(y_2)\} \\ &\leq inf\{max\{\omega_{T,I,F}(x_1), \omega_{T,I,F}(x_2)/x_1 \in f^{-1}(y_1) \ \& \ x_2 \in f^{-1}(y_2)\} \\ &= max\{inf\{\omega_{T,I,F}(x_1)/x_1 \in f^{-1}(y_1), \omega_{T,I,F}(x_2)/x_2 \in f^{-1}(y_2)\} \end{aligned}$$

*In the same way, we have*

$f_{inf}(\omega_{T,I,F})(y_1 - y_2) \leq max\{inf\{\omega_{T,I,F}(x_1)/x_1 \in f^{-1}(y_1), \omega_{T,I,F}(x_2)/x_2 \in f^{-1}(y_2)\}$.

**Theorem 4.2.** *Suppose that* $f : X \to Y$ *be a homomorphism of* $\beta-$*algebra. If* $C = (\delta_{T,I,F}, \omega_{T,I,F})$ *is a neutrosophic cubic* $\beta-$*subalgebra of* $Y$, *then the pre-image* $f^{-1}(C) = \{\langle x, f^{-1}(\delta_{T,I,F}), f^{-1}(\omega_{T,I,F})\rangle / x \in X\}$ *of* $C$ *under* $f$ *is a neutrosophic cubic* $\beta-$*subalgebra of* $X$.

*proof: Assume that* $C = (\delta_{T,I,F}\omega_{T,I,F})$ *is a neutrosophic cubic* $\beta-$*subalgebra of* $Y$ *and let* $x, y \in X$*. Then*

$$\begin{aligned} f^{-1}(\delta_{T,I,F})(x+y) &= \delta_{T,I,F}(f(x+y)) \\ &= \delta_{T,I,F}(f(x)+f(y)) \\ &\geq rmin\{\delta_{T,I,F}(f(x)), \delta_{T,I,F}(f(y))\} \\ &= rmin\{f^{-1}(\delta_{T,I,F})(x), f^{-1}(\delta_{T,I,F})(y)\} \end{aligned}$$

*Similarly,* $f^{-1}(\delta_{T,I,F})(x-y) \geq rmin\{f^{-1}(\delta_{T,I,F})(x), f^{-1}(\delta_{T,I,F})(y)\}$

$$\begin{aligned} f^{-1}(\omega_{T,I,F})(x+y) &= \omega_{T,I,F}(f(x+y)) \\ &= \omega_{T,I,F}(f(x)+f(y)) \\ &\leq max\{\omega_{T,I,F}(f(x)), \omega_{T,I,F}(f(y))\} \\ &= max\{f^{-1}(\omega_{T,I,F})(x), f^{-1}(\omega_{T,I,F})(y)\} \end{aligned}$$

*Similarly,* $f^{-1}(\omega_{T,I,F})(x-y) \leq rmin\{f^{-1}(\omega_{T,I,F})(x), f^{-1}(\omega_{T,I,F})(y)\}$
$\therefore$ $f^{-1}(C) = \{\langle x, f^{-1}(\delta_{T,I,F}), f^{-1}(\omega_{T,I,F})\rangle / x \in X\}$ *of* $C$ *under* $f$ *is a neutrosophic cubic* $\beta-$*subalgebra of* $X$*.*

## Acknowledgements

The authors are grateful to the editorial team for their precious comments in the refinement of this article.

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Received: Nov 3, 2021. Accepted: Feb 3, 2022

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# An Abstract Approach to $\mathcal{W}$-Structures Based on Hypersoft Set with Properties

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**Abstract**. Hypersoft set is an emerging knowledge of study which is projected to address the limitations of soft set for the entitlement of multi-argument approximate function. This function maps sub-parametric tuples to power set of universe. It emphasizes the partitioning of each attribute into its respective attribute-valued set that is missing in existing soft set-like structures. These features make it a completely new mathematical tool for solving problems dealing with uncertainties. In this study, classical concept of weak structures ($\mathcal{W}$-structures) is characterized under hypersoft set environment which will provide a conceptual framework for further characterization of respective topological spaces and other spaces of functional analysis. Some of its important properties and results are investigated. Moreover, new notions of hypersoft weak axioms $\mathcal{W}$-$\tau_0$, $\mathcal{W}$-$\tau_1$ and $\mathcal{W}$-$\tau_2$ are discussed with illustrative examples.

**Keywords:** Hypersoft set, Hypersoft $\mathcal{W}$-structure, Hypersoft $\mathcal{W}$-$\tau_0$, Hypersoft $\mathcal{W}$-$\tau_1$, Hypersoft $\mathcal{W}$-$\tau_2$.

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## 1. **Introduction**

Molodtsov [1] characterized soft set (SST) as a new parametrization tool to address the inadequacy of fuzzy-like structures. Later Maji et al. [2] and Pei et al. [3] extended the work and discussed some of its fundamentals and set-theoretic operations. Shabir et al. [4] applied soft set theory in topological spaces and introduced new notions of soft set topology, later modified by Min [5]. Zorlutuna et al [6], Cagman et al. [7], Roy et al. [8] discussed the properties of soft topology and proposed some modifications. Zakari et al. [9], Min et al. [11] developed a soft weak structure in support of the generalized soft topology. Al-Saadi et al. [10] investigated closed sets for soft weak structure. In many real life situations, distinct attributes are further partitioned in disjoint attribute-valued sets but existing SST is insufficient for dealing with such kind of attribute-valued sets. Hypersoft set (HS-set) [13] is developed to make the SST in line with attribute-valued sets to tackle real life scenarios. HS-set is an extension of SS-Theory as it transforms the single argument function into a multi-argument function. Certain elementary properties, aggregation operations, laws, relations and functions of HS-set, are investigated by [14, 15] for proper understanding and further utilization in different fields. Saeed et al. [16–21] discussed decision-making applications based on complex multi-fuzzy HS-set, mapping on Hs-calsses, neutrosophic HS-graphs and neutrosophic HS-mapping to medical diagnosis and other optimal selections. Rahman et al. [22] developed hybrids of HS-set with complex fuzzy set, complex intuitionistic fuzzy set and complex neutrosophic set. They [23] introduced the notions of convex and concave HS-sets with some properties. Decision-making applications for optimal object selection have been discussed by them under the environments of parameterization of HS-sets in fuzzy set-like structures, bijective HS-sets and complex fuzzy hypersoft in [24–27]. Saqlain et al. [28] investigated single and multi-valued neutrosophic HS-sets and discussed tangent similarity measure of single valued neutrosophic HS-sets. Zulqarnain et al. [29] characterized generalized aggregate operators on neutrosophic HS-sets and discussed their essential properties. Ihsan et al. [30,31] employed the concept of HS-sets in expert system and developed HS expert set and fuzzy HS expert set with application in decision-making. Kamacı et al. [32] extended this work to n-ary fuzzy expert set and discussed its properties. Ajay et al. [33] developed the notions of Alpha Open HS-sets and applied them in MCDM. Musa et al. [34] developed bipolar HS-set and discussed its properties and operations.

### 1.1. *Motivation*

In many daily-life decision-making problems, we encounter with some scenarios where each attribute is required to be further classified into its respective attribute-valued set. In order to tackle such scenarios, HS-set is projected which employs the cartesian product of disjoint attribute-valued sets as domain of approximate function ( i.e. multi-argument approximate function). The existing models [9–12] are insufficient to deal uncertainties with such kind of approximate function. Therefore, the main aim of this study is to generalize these models by developing HS-week structures. All the new proposed operations and properties are explained with the support of illustrated examples.

### 1.2. *Paper Layout*

The rest of paper is organized as:

Section 2: reviews some basic definitions to support the main results.

Section 3: characterizes HS $\mathcal{W}$-structures along with their important properties and results.

Section 4: summarizes the paper with future directions.

## 2. Preliminaries

In this section, definitions of soft sets , hypersoft sets and soft weak structures are reviewed.

**Definition 2.1.** [1]
A pair $(\psi, R)$ is called soft set over $\mathcal{U}$, where $\psi : R \rightarrow \mathbb{P}(\mathcal{U})$ and R be a subset of a set of attributes $\mathfrak{E}$.

**Definition 2.2.** [13]
Suppose $b_1, b_2, ......, b_n$, for $b \geq 1$, be n distinct traits, whose corresponding trait values are respectively the sets $\mathcal{Q}_1, \mathcal{Q}_2, ....., \mathcal{Q}_n$, with $\mathcal{Q}_r \bigcap \mathcal{Q}_s = \phi$, $i \neq j$, and r,s $\in \{1, 2, ...., n\}$. Then the pair $(\Psi, \mathcal{Q}_1 \times \mathcal{Q}_2 \times ..... \times \mathcal{Q}_n)$, where $\Psi : \mathcal{Q}_1 \times \mathcal{Q}_2 \times ..... \times \mathcal{Q}_n \rightarrow P(\mathcal{U})$ is called a Hypersoft Set over $\mathcal{U}$.

**Definition 2.3.** [12]
$s\mathcal{W}$ is collection of $(\psi, R)$ over $\mathcal{X}$. if

(i) $\phi, \mathcal{X} \in s\mathcal{W}$
(ii) $(\psi_a, R_1) \bigcap (\psi_b, R_2)$ in $s\mathcal{W} \in s\mathcal{W}$.

then $s\mathcal{W}$ is weak structure. $\mathcal{W}$-space is denoted by ($\mathcal{X}$, $s\mathcal{W}$, E). Elements of $s\mathcal{W}$ are $\mathcal{W}$-open and $(\psi, R)$ is soft $\mathcal{W}$-closed if $(\psi, R)^r \in s\mathcal{W}$.

## 3. Hypersoft $\mathcal{W}$-Structures

In this section, hypersoft $\mathcal{W}$-structures are characterized and some of their important properties and results are discussed.

**Definition 3.1. Hypersoft $\mathcal{W}$-Structure**
Suppose $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, ...., \mathcal{P}_m$ be disjoint attribute-valued sets corresponding to $m$ distinct attributes $p_1, p_2, p_3, ..., p_m$ respectively and $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3 \times .... \times \mathcal{P}_m$. A collection $\Omega_{\mathcal{W}}$ of HS-sets defined over $\mathcal{U}$ w.r.t $\mathcal{P}$ is called HS $w$-Structure if

(i) $\emptyset_{HS}, \mathcal{U}$ belong to $\Omega_{\mathcal{W}}$
(ii) $(\Psi_i, \mathcal{P}) \bigcap (\Psi_j, \mathcal{P})$ in $\Omega_{\mathcal{W}}$ belongs to $\Omega_{\mathcal{W}}$ $\forall\ i \neq j$

A HS set is said to be HS $\mathcal{W}$-open if it belongs to collection $\Omega_{\mathcal{W}}$ and if $(\Psi, \mathcal{P})^r \in \Omega_{\mathcal{W}}$ then HS $\mathcal{W}$-closed.

**Example 3.2.** Suppose $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ and $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4\}$ such that $\mathcal{P}_1 = \{p_{11}, p_{12}\}, \mathcal{P}_2 = \{p_{21}, p_{22}\}, \mathcal{P}_3 = \{p_{31}, p_{32}\}$.

Now $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$

$$\mathcal{P} = \left\{ \begin{array}{ll} q_1 = (p_{11}, p_{21}, p_{31}), & q_2 = (p_{11}, p_{21}, p_{32}), \\ q_3 = (p_{11}, p_{22}, p_{31}), & q_4 = (p_{11}, p_{22}, p_{32}), \\ q_5 = (p_{12}, p_{21}, p_{31}), & q_6 = (p_{11}, p_{21}, p_{32}), \\ q_7 = (p_{11}, p_{22}, p_{31}), & q_8 = (p_{11}, p_{22}, p_{32}) \end{array} \right\}$$

and

$$\Omega_{\mathcal{W}} = \{\emptyset_{HS}, \mathcal{U}, (\Psi_1, \mathcal{P}), (\Psi_2, \mathcal{P}), (\Psi_3, \mathcal{P})\},$$

$$(\Psi_1, \mathcal{P}) = \left\{ \begin{array}{ll} \Psi_1(q_1) = \{u_1, u_2, u_7, u_8\}, & \Psi_1(q_2) = \{u_1, u_3, u_6, u_8\}, \\ \Psi_1(q_4) = \{u_2, u_5, u_7, u_8\}, & \Psi_1(q_6) = \{u_1, u_3, u_5, u_7\}, \\ \Psi_1(q_7) = \{u_4, u_5, u_6, u_8\} & \end{array} \right\},$$

$$(\Psi_2, \mathcal{P}) = \left\{ \begin{array}{ll} \Psi_2(q_1) = \{u_1, u_2, u_3, u_7\}, & \Psi_2(q_3) = \{u_2, u_4, u_5, u_7\}, \\ \Psi_2(q_4) = \{u_1, u_5, u_7, u_8\}, & \Psi_2(q_7) = \{u_4, u_5, u_7, u_8\}, \\ \Psi_2(q_8) = \{u_2, u_5, u_4, u_8\} & \end{array} \right\},$$

$$(\Psi_3, \mathcal{P}) = \left\{ \Psi_3(q_1) = \{u_1, u_2, u_7\}, \quad \Psi_3(q_4) = \{u_5, u_7, u_8\}, \quad \Psi_3(q_7) = \{u_4, u_5, u_8\} \right\}.$$

$\Omega_{\mathcal{W}}$ is a HS $\mathcal{W}$-structure.

**Definition 3.3. Hypersoft $\mathcal{W}$-Interior**

The HS $\mathcal{W}$-$\mathcal{W}$-interior of $(\Psi, \mathcal{P})$, denoted by $(\Psi, \mathcal{P})^\circ$, is defined as

$$(\Psi, \mathcal{P})^\circ = \cup \{(\Psi_i, \mathcal{P}) : (\Psi_i, \mathcal{P}) \subseteq (\Psi, \mathcal{P}), (\Psi_i, \mathcal{P}) \in \Omega_{\mathcal{W}}\}.$$

**Remark 3.4.** If there exists a HS $\mathcal{W}$-open set $(\Psi_2, \mathcal{P})$ s.t $q \in (\Psi_2, \mathcal{P})$ is subset of $(\Psi_1, \mathcal{P})$, then q belongs to $(\Psi_1, \mathcal{P})^\circ$.

**Example 3.5.** Considering example 3.2, we have

$$(\Psi_1, \mathcal{P})^\circ = \{(\Psi_3, \mathcal{P})\}.$$

**Theorem 3.6.** *If $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$ belongs to $\Omega_{\mathcal{W}}$, then*

(i) $(\Psi, \mathcal{P})^\circ$ *is subset of* $(\Psi, \mathcal{P})$

(ii) *If* $(\Psi_1, \mathcal{P})$ *is subset of* $(\Psi_2, \mathcal{P})$ *then* $(\Psi_1, \mathcal{P})^\circ$ *is subset of* $(\Psi_2, \mathcal{P})^\circ$

(iii) *HS $\mathcal{W}$-interior of intersection of* $(\Psi_1, \mathcal{P})$ *and* $(\Psi_2, \mathcal{P})$ *is equal to intersection of HS $\mathcal{W}$-interior of* $(\Psi_1, \mathcal{P})$ *and HS $\mathcal{W}$-interior of* $(\Psi_2, \mathcal{P})$

(iv) $((\Psi, \mathcal{P})^\circ)^\circ$ *is equal to* $(\Psi, \mathcal{P})^\circ$

*Proof.* (i) is obvious.

(ii) Given $(\Psi_1, \mathcal{P})$ is subset of $(\Psi_2, \mathcal{P})$

From (i) $(\Psi_1, \mathcal{P})^\circ$ is subset of $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})^\circ$ is subset of $(\Psi_2, \mathcal{P})$.

implies $(\Psi_1, \mathcal{P})^\circ$ is subset of $(\Psi_2, \mathcal{P})$

but $(\Psi_2, \mathcal{P})^\circ$ is subset of $(\Psi_2, \mathcal{P})$.

Hence $(\Psi_1,\mathcal{P})^\circ$ is subset of $(\Psi_2,\mathcal{P})^\circ$

(iii) Since intersection of $(\Psi_1,\mathcal{P})$ and $(\Psi_2,\mathcal{P})$ is subset of $(\Psi_1,\mathcal{P})$, Intersection of $(\Psi_1,\mathcal{P})$ and $(\Psi_2,\mathcal{P})$ is subset of $(\Psi_2,\mathcal{P})$.

from (i) $(\Psi,\mathcal{P})^\circ$ is subset of $(\Psi,\mathcal{P})$ implies

HS $\mathcal{W}$-interior of intersection of $(\Psi_1,\mathcal{P})$ and $(\Psi_2,\mathcal{P})$ is subset of $(\Psi_1,\mathcal{P})^\circ$ and HS $\mathcal{W}$-interior of intersection of $(\Psi_1,\mathcal{P})$ and $(\Psi_2,\mathcal{P})$ is subset of $(\Psi_2,\mathcal{P})^\circ$.

So HS $\mathcal{W}$-interior of intersection of $(\Psi_1,\mathcal{P})$ and $(\Psi_2,\mathcal{P})$ is subset of intersection of HS $\mathcal{W}$-interior of $(\Psi_1,\mathcal{P})$ and HS $\mathcal{W}$-interior of $(\Psi_2,\mathcal{P})$.

Also intersection of HS $\mathcal{W}$-interior of $(\Psi_1,\mathcal{P})$ and HS $\mathcal{W}$-interior of $(\Psi_2,\mathcal{P})$ is subset of intersection of $(\Psi_1,\mathcal{P})$ and $(\Psi_2,\mathcal{P})$.

Therefore intersection of HS $\mathcal{W}$-interior of $(\Psi_1,\mathcal{P})$ and HS $\mathcal{W}$-interior of $(\Psi_2,\mathcal{P})$ is open subset of intersection of $(\Psi_1,\mathcal{P})$ and $(\Psi_2,\mathcal{P})$.

Hence intersection of HS $\mathcal{W}$-interior of $(\Psi_1,\mathcal{P})$ and HS $\mathcal{W}$-interior of $(\Psi_2,\mathcal{P})$ is subset of HS $\mathcal{W}$-interior of intersection of $(\Psi_1,\mathcal{P})$ and $(\Psi_2,\mathcal{P})$.

HS $\mathcal{W}$-interior of intersection of $(\Psi_1,\mathcal{P})$ and $(\Psi_2,\mathcal{P})$ is equal to intersection of HS $\mathcal{W}$-interior of $(\Psi_1,\mathcal{P})$ and HS $\mathcal{W}$-interior of $(\Psi_2,\mathcal{P})$.

(iv) From (i), it follows $((\Psi,\mathcal{P})^\circ)^\circ$ is subset of $(\Psi,\mathcal{P})^\circ$. For any HS $\mathcal{W}$-open set $(\Psi_1,\mathcal{P})$ s.t $((\Psi_1,\mathcal{P})$ is subset of $(\Psi,\mathcal{P})^\circ$,

$(\Psi_1,\mathcal{P})$ is equal to $(\Psi_1,\mathcal{P})^\circ$ is subset of $((\Psi,\mathcal{P})^\circ)^\circ$, so $(\Psi,\mathcal{P})^\circ \ddot{\subset} ((\Psi,\mathcal{P})^\circ)^\circ$ Consequently, we have

$((\Psi,\mathcal{P})^\circ)^\circ$ is equal to $(\Psi,\mathcal{P})^\circ$ □

**Definition 3.7. Hypersoft $\mathcal{W}$-exterior**

The HS $\mathcal{W}$-exterior of $(\Psi,\mathcal{P})$, denoted by $(\Psi,\mathcal{P})^\varepsilon$, is defined as

$$(\Psi,\mathcal{P})^\varepsilon = ((\Psi,\mathcal{P})^c)^\circ$$

**Example 3.8.** Consider the sets given in example 3.2, let we have a hypersoft set

$$(\Psi,\mathcal{P}) = \left\{ \begin{array}{ll} \Psi(q_1) = \{u_1,u_2,u_7,u_8\}, & \Psi(q_2) = \{u_1,u_3,u_6,u_8\}, \\ \Psi(q_4) = \{u_2,u_5,u_7,u_8\}, & \Psi(q_6) = \{u_1,u_3,u_5,u_7\}, \\ \Psi(q_7) = \{u_4,u_5,u_6,u_8\} & \end{array} \right\}$$

.

$$((\Psi,\mathcal{P}))^c = \left\{ \begin{array}{ll} \Psi(q_1) = \{u_3,u_4,u_5,u_6\}, & \Psi(q_2) = \{u_2,u_4,u_5,u_7\}, \\ \Psi(q_4) = \{u_1,u_3,u_4,u_6\}, & \Psi(q_6) = \{u_2,u_4,u_6,u_8\}, \\ \Psi(q_7) = \{u_1,u_2,u_3,u_7\} & \end{array} \right\}$$

.

$$(\Psi_4,\mathcal{P}) = \left\{ \begin{array}{ll} \Psi_4(q_1) = \{u_3,u_5,u_6\}, & \Psi_4(q_2) = \{u_2,u_5,u_7\}, \\ \Psi_4(q_4) = \{u_1,u_3,u_6\}, & \Psi_4(q_6) = \{u_2,u_4,u_6\}, \\ \Psi_4(q_7) = \{u_1,u_3,u_7\} & \end{array} \right\}$$

.

$$(\Psi, \mathcal{P})^{\varepsilon} = (\Psi_4, \mathcal{P})$$

**Definition 3.9. Hypersoft $\mathcal{W}$-boundry**

The HS $\mathcal{W}$-boundry of $(\Psi, \mathcal{P})$, denoted by $(\Psi, \mathcal{P})^b$, contains those HS sets which do not belongs to HS $\mathcal{W}$-interior and HS exterior.

**Example 3.10.** in example 3.2, we have

$$(\Psi, \mathcal{P})^b = \{(\Psi_1, \mathcal{P}), (\Psi_2, \mathcal{P})\}$$

**Definition 3.11. Hypersoft $\mathcal{W}$-Closure**

HS $\mathcal{W}$-closure of $(\Psi, \mathcal{P})$ is denoted by $(\Psi, \mathcal{P})^{\bullet}$, is defined as

$$(\Psi, \mathcal{P})^{\bullet} = \bigcap\{(\Psi_1, \mathcal{P}) : (\Psi, \mathcal{P}) \subseteq (\Psi_1, \mathcal{P}), (\Psi_1, \mathcal{P})^c \in \Omega_{\mathcal{W}}\}$$

.

**Example 3.12.** It is clear from example 3.2

$$(\Psi_3, \mathcal{P})^{\bullet} = \{(\Psi_1, \mathcal{P})\}$$

**Theorem 3.13.**

*If* $q \in (\Psi, \mathcal{P})^{\bullet}$, *then* $(\Psi_i, \mathcal{P}) \bigcap (\Psi, \mathcal{P}) \neq \emptyset \ \forall \ (\Psi_i, \mathcal{P}) \in \Omega_{\mathcal{W}}$ *s.t* $q \in (\Psi_i, \mathcal{P})$.

*Proof.* Suppose $q \in (\Psi, \mathcal{P})^{\bullet}$ then there exists $(\Psi_i, \mathcal{P}) \in \Omega_{\mathcal{W}}$ s.t $q \in (\Psi_i, \mathcal{P})$
and $(\Psi_i, \mathcal{P}) \bigcap (\Psi, \mathcal{P}) = \emptyset$
this implies $(\Psi, \mathcal{P}) \ddot{\subset} (\Psi_i, \mathcal{P})^c$ so $(\Psi, \mathcal{P})^{\bullet} \ddot{\subset} (\Psi_i, \mathcal{P})^c$ and $q \notin (\Psi, \mathcal{P})^{\bullet}$. So it is a contradiction. □

**Theorem 3.14.**

*If* $(\Psi_1, \mathcal{P})$ *and* $(\Psi_2, \mathcal{P})$ *are two HS sets then*

(i) $(\Psi, \mathcal{P})$ *is subset of* $(\Psi, \mathcal{P})^{\bullet}$
(ii) *if* $(\Psi_1, \mathcal{P})$ *is subset of* $(\Psi_2, \mathcal{P})$ *then* $(\Psi_1, \mathcal{P})^{\bullet}$ *is subset of* $(\Psi_2, \mathcal{P})^{\bullet}$
(iii) $(\Psi_1, \mathcal{P})^{\bullet} \cup (\Psi_2, \mathcal{P})^{\bullet} = ((\Psi_1, \mathcal{P}) \cup (\Psi_2, \mathcal{P}))^{\bullet}$
(iv) $((\Psi, \mathcal{P})^{\bullet})^{\bullet} = (\Psi, \mathcal{P})^{\bullet}$

*Proof.* (i) is obvious.
(ii) Since $(\Psi_1, \mathcal{P})$ is subset of $(\Psi_2, \mathcal{P})$
from (i) $(\Psi_1, \mathcal{P})$ is subset of $(\Psi_1, \mathcal{P})^{\bullet}$ and $(\Psi_2, \mathcal{P})$ is subset of $(\Psi_2, \mathcal{P})^{\bullet}$
then $(\Psi_1, \mathcal{P})$ *is subset of* $(\Psi_2, \mathcal{P})^{\bullet}$
but $(\Psi_1, \mathcal{P})$ *is subset of* $(\Psi_1, \mathcal{P})^{\bullet}$ implies $(\Psi_1, \mathcal{P})^{\bullet}$ *is subset of* $(\Psi_2, \mathcal{P})^{\bullet}$
(iii) Since $(\Psi_1, \mathcal{P})$ *is subset of* $(\Psi_1, \mathcal{P}) \cup (\Psi_2, \mathcal{P})$ , $(\Psi_2, \mathcal{P})$ is subset of $(\Psi_1, \mathcal{P}) \cup (\Psi_2, \mathcal{P})$

and $(\Psi,\mathcal{P})$ *is subset of* $(\Psi,\mathcal{P})^{\bullet}$ then $(\Psi_1,\mathcal{P})^{\bullet}$ *is subset of* $((\Psi_1,\mathcal{P})\cup(\Psi_2,\mathcal{P}))^{\bullet}$ and $(\Psi_2,\mathcal{P})^{\bullet}$ *is subset of* $((\Psi_1,\mathcal{P})\cup(\Psi_2,\mathcal{P}))^{\bullet}$,
$(\Psi_1,\mathcal{P})^{\bullet}\cup(\Psi_2,\mathcal{P})^{\bullet}$ *is subset of* $((\Psi_1,\mathcal{P})\cup(\Psi_2,\mathcal{P}))^{\bullet}$
also $(\Psi_1,\mathcal{P})\cup(\Psi_2,\mathcal{P}))^{\bullet}$ *is subset of* $(\Psi_1,\mathcal{P})^{\bullet}\cup(\Psi_2,\mathcal{P})^{\bullet}$ Hence

$$(\Psi_1,\mathcal{P})^{\bullet}\cup(\Psi_2,\mathcal{P})^{\bullet}=((\Psi_1,\mathcal{P})\cup(\Psi_2,\mathcal{P}))^{\bullet}$$

(iv) From (i), $(\Psi,\mathcal{P})$ is subset of $(\Psi,\mathcal{P})^{\bullet}$ then $(\Psi,\mathcal{P})^{\bullet}$ is subset of $((\Psi,\mathcal{P})^{\bullet})^{\bullet}$,
$((\Psi,\mathcal{P})^{\bullet})^{\bullet}=(\Psi,\mathcal{P})$ is subset of $(\Psi,\mathcal{P})^{\bullet}$, then $((\Psi,\mathcal{P})^{\bullet})^{\bullet}$ is subset of $(\Psi,\mathcal{P})^{\bullet}$
Consequently, we have
$((\Psi,\mathcal{P})^{\bullet})^{\bullet}=(\Psi,\mathcal{P})^{\bullet}$ □

**Remark 3.15.**

(i) if $(\Psi,\mathcal{P})\in\Omega_{\mathcal{W}}$ then $(\Psi,\mathcal{P})=((\Psi,\mathcal{P}))^{\circ}$
(ii) if $(\Psi,\mathcal{P})^{r}\in\Omega_{\mathcal{W}}$ then $(\Psi,\mathcal{P})=((\Psi,\mathcal{P}))^{\bullet}$

**Definition 3.16. Hypersoft $\mathcal{W}$-$\tau_0$**
If $u_1,u_2\in\mathcal{U}$ and $u_1\neq u_2$, $\exists$ a HS $\mathcal{W}$-open set $(\Psi,\mathcal{P})$ s.t $u_1\in(\Psi,\mathcal{P})$ and $u_2\notin(\Psi,\mathcal{P})$ or $u_1\notin(\Psi,\mathcal{P})$ and $u_2\in(\Psi,\mathcal{P})$ then $(\mathcal{U},\Omega_{\mathcal{W}},\mathcal{P})$ is called $\mathcal{W}$-$\tau_0$

**Example 3.17.** Suppose $\mathcal{U}=\{u_1,u_2\}$ then $\Omega_{\mathcal{W}}=\{\emptyset,\mathcal{U},(\Psi,\mathcal{P})\}$ where
$(\Psi,\mathcal{P})=\{\Psi_1(q_1)=\{u_1\}\}$ is $\mathcal{W}$-$\tau_0$.

**Theorem 3.18.**
*If $\mathcal{U}$ is a relative HS $\mathcal{W}$-$\tau_0$ space, then for each $u_1,u_2\in\mathcal{U}$ such that $u_1\neq u_2$, we have $(u_1,\mathcal{P})^{\bullet}\neq(u_2,\mathcal{P})^{\bullet}$.*

*Proof.* For every $u_1,u_2\in\mathcal{U}$ and $u_1\neq u_2$ $\exists$ a HS $(\Psi,\mathcal{P})\in\Omega_{\mathcal{W}}$ s.t $u_1\in(\Psi,\mathcal{P})$ and $u_2\in(\Psi,\mathcal{P})^{c}$.
Therefore $(\Psi,\mathcal{P})^{c}$ is a HS $\mathcal{W}$-closed set s.t $u_1\notin(\Psi,\mathcal{P})^{c}$ and $u_2\in(\Psi,\mathcal{P})^{c}$.
Since $(u_2,\mathcal{P})^{\bullet}\ddot{\subset}(\Psi,\mathcal{P})^{c}$ and $u_1\notin(u_2,\mathcal{P})^{\bullet}$ Thus $(u_1,\mathcal{P})^{\bullet}\neq(u_2,\mathcal{P})^{\bullet}$. □

**Definition 3.19. Hypersoft $\mathcal{W}$-$\tau_1$**
If for each $u_1,u_2\in\mathcal{U}$ s.t $u_1\neq u_2$, $\exists$ HS $\mathcal{W}$-open sets $(\Psi_1,\mathcal{P})$ and $(\Psi_2,\mathcal{P})$ s.t $u_1\in(\Psi_1,\mathcal{P})$ and $u_2\notin(\Psi_1,\mathcal{P})$ and $u_1\notin(\Psi_2,\mathcal{P})$ and $u_2\in(\Psi_2,\mathcal{P})$ then HS $\Omega_{\mathcal{W}}$ space is known as $\mathcal{W}$-$\tau_1$ .

**Example 3.20.** Suppose $\mathcal{U}=\{u_1,u_2\}$ then $\Omega_{\mathcal{W}}=\{\emptyset,\mathcal{U},(\Psi_1,\mathcal{P}),(\Psi_2,\mathcal{P})\}$ where
$(\Psi_1,\mathcal{P})=\{\Psi_1(q_1)=\{u_1\}\}$ and $(\Psi_2,\mathcal{P})=\{\Psi_2(q_1)=\{u_2\}\}$ is $\mathcal{W}$-$\tau_1$.

**Theorem 3.21.**
*A HS $\mathcal{W}$-space $(\mathcal{U},\Omega_{\mathcal{W}},\mathcal{P})$ is HS $\mathcal{W}$-$\tau_1$ if $(u,\mathcal{P})$ is HS $\mathcal{W}$-closed set for all $u\in\mathcal{U}$.*

*Proof.* suppose $u_1, u_2 \in \mathcal{U}$ and $u_1 \neq u_2$ $\exists$ HS $\mathcal{W}$-open sets $(u_1, \mathcal{P})^c$ and $(u_2, \mathcal{P})^c$ s.t $u_1 \in (u_1, \mathcal{P})^c$, $u_2 \in (u_1, \mathcal{P})^c$ and $u_2 \notin (u_2, \mathcal{P})^c$ , $u_1 \in (u_2, \mathcal{P})^c$, It prove that $\mathcal{U}$ is HS $\mathcal{W}$-$\tau_1$. □

**Definition 3.22. Hypersoft $\mathcal{W}$-$\tau_2$**
$\mathcal{W}$-$\tau_2$ if for each $u_1, u_2 \in \mathcal{U}$ s.t $u_1 \neq u_2$, $\exists$ HS $\mathcal{W}$-open sets $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$ then each $u_1 \in (\Psi_1, \mathcal{P})$, $u_2 \in (\Psi_2, \mathcal{P})$ and $(\Psi_1, \mathcal{P}) \bigcap (\Psi_2, \mathcal{P}) = \emptyset$

**Example 3.23.** Suppose $\mathcal{U} = \{u_1, u_2\}$ then $\Omega_{\mathcal{W}} = \{\emptyset, \mathcal{U}, (\Psi_1, \mathcal{P}), (\Psi_2, \mathcal{P})\}$ where $(\Psi_1, \mathcal{P}) = \{\Psi_1(q_1) = \{u_1\}\}$ and $(\Psi_2, \mathcal{P}) = \{\Psi_2(q_1) = \{u_2\}\}$ is $\mathcal{W}$-$\tau_2$.

## 4. Conclusions

In this study, weak structures are characterized under hypersoft set environment, and some of its essential properties and results are discussed. Moreover, some separation axioms like $\tau_0, \tau_1$, and $\tau_2$ are introduced with the help of weak structures on hypersoft set. Further study may include the development of :

(1) HS-compact spaces
(2) HS-connected spaces
(3) HS-normed spaces
(4) HS-Hilbert spaces
(5) HS-inner product spaces
(6) HS-metric spaces

with their applications in decision-making by using certain techniques like TOPSIS, MCDM etc.

**Conflicts of Interest:**
The authors declare no conflict of interest.

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Received: Nov 8, 2021. Accepted: Feb 3, 2022

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# MADM technique under QSVN environment using different prioritized operator

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**Abstract**. In this article the notion of two quadripartitioned single valued weighted dombi prioritized operators, namely, quadripartitioned single valued weighted dombi prioritized average (QSVNWDPA) operator and quadripartitioned single valued weighted dombi prioritized geometric ( QSVNWDPG) operator have been developed which are based on quadripartitioned single valued neutrosophic (QSVN) sets. Further some important properties of these two operators are studied. Finally a multi-attribute decision making (MADM) problem has been solved using QSVNWDPA operator and QSVNWDPG operator.

**Keywords:**Quadripartitioned single valued neutrosophic set; Aggregation operator; Dombi operator; Prioritized operator; QSVN weighted Dombi prioritized average operator; QSVN weighted Dombi prioritized geometric operator; Multi-attributive decision making.

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## 1. **Introduction**

Smarandache [3, 13, 20] introduced Neutrosophic set (NS) theory in which each element of this set is assigned with a truth value ($T$), a indeterminacy value ($I$) and a falsity value ($F$) which are independent of each other. Later many authors have introduced several types of generalizations of NS along with their various types of applications [4–12, 14, 15]. An extension of Neutrosophic set, called QSVN set, was further developed in [16] which was motivated by Belnap's four valued logic [1]. Here every element in a set have four values associated with it namely truth value $T$, a contradiction value $C$, an ignorance value $U$ and a falsity value $F$. Thus QSVN sets are equipped with better tool for solving various types of decision making problems in comparison with other types of neutrosophic sets. The idea of quadripartitioned neutrosophic numbers ($\mathcal{QNN}$ number) which are based on QSVN sets are introduced and studied along with some well known properties of $\mathcal{QNN}$-numbers in [19]. Currently the application of information aggregation operators in the area of multi-attribute decision making process has become a popular topic of research. Aggression operator based decision making methods are preferred than ordinary decision making methods because these operators readily combines data into one single entity from which one could easily make decisions. Several researchers have proposed new aggregation operators or have extended known operators to new settings. On contrary Dombi [2] presented the operations of Dombi $T$ -norms ($D_T$) and $T$-conorms ($\widehat{D_T}$) for fuzzy sets way back in 1982. Both the norms have wide applications as an operator as they have good advantage of flexibility to tackle the operational parameters. Also Dombi aggregation operators make the optimal outcomes more accurate and definite when used properly in any MADM problem. Many researchers extended the idea of Dombi norms together with Prioritized operator to IFS [18], NS [17, 21] theories and applied to different MADM problems. In this paper we have applied weighted Dombi Prioritized norms on $\mathcal{QNN}$ and applied them to solve a very relevant MADM problem. The rest of this paper is constructed as follows: In Section 2 we have discussed some basic theories which will be used throughout the rest of the article. We have defined some order relations on $\mathcal{QNN}$ in Section 3. In the next section some Dombi operations on $\mathcal{QNN}$ are defined. Section 5 introduces the QSVNWDPA and QSVNWDPG operators and studied their properties. Next a MADM problem is solved using QSVNWDA and QSVNWDG operators in section 6 along with sensitivity analysis of these two methods. Then Section 7 concludes the article.

## 2. Some Basics

For better understanding of this article we need some terminologies from literature of NS sets.

**Definition 2.1.** [3] A neutrosophic set (NS) $A$ in $Y \neq \phi$ is characterized by a truth-membership function $A_t$, an indeterminacy membership function $A_i$ and a falsity-membership function $A_f$. Here for each $y \in Y$, $A_t(y), A_i(y)$ and $A_f(y)$ are real non-standard elements of $]0^-, 1^+[$. $A$ can be written as:

$$A = \{(y, A_t(y), A_i(y), A_f(y)) : y \in Y, A_t(y), A_i(y), A_f(y) \in ]0^-, 1^+[\}.$$

**Definition 2.2.** [16] A QSVN set $M$ over a set $Y \neq \phi$ distinguishes each element $y$ in $Y$ by a truth-value $M_t$, a contradiction value $M_c$, an ignorance-value $M_u$ and a falsity value $M_f$ s.t. for each $y \in Y$, $M_t(y), M_c(y), M_u(y), M_f(y) \in [0, 1]$, $0 \leq M_t(y) + M_c(y) + M_u(y) + M_f(y) \leq 4$.

Based on QSVN set Prof. R. Chatterjee et. al. introduced the QSVN numbers together with some operations in their paper [19] in 2019.

**Definition 2.3.** [19] An QSVN element $\omega = \langle \omega_t, \omega_c, \omega_u, \omega_f \rangle \in [0,1]^4$ is said to be a QSVN number. We represent the set of QSVN numbers as $\mathcal{QNN}$.

**Definition 2.4.** [19] Consider $\varsigma, \tau, \upsilon \in \mathcal{QNN}$ and $k \in \mathbb{N}$. Then the following basic operations hold on $\mathcal{QNN}$:

(i) $\varsigma \bigoplus \tau = \langle \varsigma_t + \tau_t - \varsigma_t\tau_t, \varsigma_c + \tau_c - \varsigma_c\tau_c, \varsigma_u\tau_u, \varsigma_f\tau_f \rangle$,
(ii) $\varsigma \bigodot \tau = \langle \varsigma_t\tau_t, \varsigma_c\tau_c, \varsigma_u + \tau_u - \varsigma_u\tau_u, \varsigma_f + \tau_f - \varsigma_f\tau_f \rangle$,
(iii) $(\varsigma)^k = \langle (\varsigma_t)^k, (\varsigma_c)^k, 1-(1-\varsigma_u)^k, 1-(1-\varsigma_f)^k \rangle$,
(iv) $k\varsigma = \langle 1-(1-\varsigma_t)^k, 1-(1-\varsigma_c)^k, (\varsigma_u)^k, (\varsigma_f)^k \rangle$,
(v) $\varsigma \bigoplus \tau = \tau \bigoplus \varsigma$,
(vi) $(\varsigma \bigoplus \tau) \bigoplus \upsilon = \tau \bigoplus (\varsigma \bigoplus \upsilon)$,
(vii) $\varsigma \bigodot \tau = \tau \bigodot \varsigma$,
(viii) $(\varsigma \bigodot \tau) \bigodot \upsilon = \tau \bigodot (\varsigma \bigodot \upsilon)$,

### 2.1. *Dombi T-norm and T-conorm*

Dombi Operator was introduced by J. Dombi in 1982 in [2]. In 2008 Prof Yager firstly introduced the Prioritized aggregation operators in [4]. For convenience of the readers of this article we request you to follow the articles [2] and [4] respectively.

**Definition 2.5.** [2] Suppose $r, s \in \mathbb{R}$. The $D_T$ and $\widehat{D_T}$ between $r$ and $s$ are defined respectively as below:

$$D_T(r,s) = \frac{1}{1 + \{(\frac{1-r}{r})^\lambda + (\frac{1-s}{s})^\lambda\}^{\frac{1}{\lambda}}}$$
$$\widehat{D_T}(r,s) = \frac{1}{1 + \{(\frac{r}{1-r})^\lambda + (\frac{s}{1-s})^\lambda\}^{\frac{1}{\lambda}}},$$

$\lambda \geq 1$ and $(r,s) \in [0,1] \times [0,1]$.

## 3. **Order properties in $\mathcal{QNN}$**

Now we will discuss some order relations of $\mathcal{QNN}$.

**Definition 3.1.** The score function of $\omega = \langle \omega_t, \omega_c, \omega_u, \omega_f \rangle : \mathcal{QNN} \to [0,1]$ is defined as

$$S(\omega) = \frac{3 + \omega_t + \omega_c - \omega_u - \omega_f}{4}$$

We now define a few accuracy functions $A_i : \mathcal{QNN} \to [0,1], i = \infty, \in, \ni$ of $\omega = \langle \omega_t, \omega_c, \omega_u, \omega_f \rangle \in \mathcal{QNN}$ as follows:

$$\mathcal{A}_{\infty}(\omega) = \frac{(\omega_t + \omega_c) - (\omega_u + \omega_f)}{2}$$
$$\mathcal{A}_{\in}(\omega) = \frac{\omega_t - \omega_c}{4}$$
$$\mathcal{A}_{\ni}(\omega) = \frac{\omega_u - \omega_f}{4}.$$

**Remark 3.2.** From Definition 3.1, the following properties of score function and accuracy functions of a QSVN number $\omega \in \mathcal{QNN}$ can be obtained:

(i) $0 \leq S(\omega) \leq 1.25$.
(ii) $-1 \leq \mathcal{A}_{\infty}(\omega) \leq 1$.
(iii) $-0.25 \leq \mathcal{A}_{\in}(\omega) \leq 0.25$.
(iv) $-0.25 \leq \mathcal{A}_{\ni}(\omega) \leq 0.25$.

**Definition 3.3.** Suppose $\mu, \nu \in \mathcal{QNN}$. We define the order relation between any two $\mu, \nu \in \mathcal{QNN}$ as following:

(i) If $S(\mu) < S(\nu)$, then $\mu \leq \nu$.
(ii) If $S(\mu) = S(\nu)$, then
  (a) $\mathcal{A}_{\infty}(\mu) < \mathcal{A}_{\infty}(\nu) \Rightarrow \mu \leq \nu$ else if
  (b) $\mathcal{A}_{\infty}(\mu) = \mathcal{A}_{\infty}(\nu)$ with $\mathcal{A}_{\in}(\mu) < \mathcal{A}_{\in}(\nu) \Rightarrow \mu \leq \nu$ else if
  (c) $\mathcal{A}_{\infty}(\mu) = \mathcal{A}_{\infty}(\nu), \mathcal{A}_{\in}(\mu) = \mathcal{A}_{\in}(\nu)$ with $\mathcal{A}_{\ni}(\mu) < \mathcal{A}_{\ni}(\nu) \Rightarrow \mu \leq \nu$ else if
  (d) $\mathcal{A}_{\infty}(\mu) = \mathcal{A}_{\infty}(\nu), \mathcal{A}_{\in}(\mu) = \mathcal{A}_{\in}(\nu)$ and $\mathcal{A}_{\ni}(\mu) = \mathcal{A}_{\ni}(\nu) \Rightarrow \mu = \nu$.

Here $\mu \leq \nu$ denotes $\mu$ proceeds $\nu$.

## 4. Some QSVN Dombi operations

**Definition 4.1.** Let $\mu = \langle m_1, n_1, p_1, q_1 \rangle \in \mathcal{QNN}$ and $\nu = \langle m_2, n_2, p_2, q_1 \rangle \in \mathcal{QNN}$, $\lambda \geq 1$ and $k > 0$. Then the $D_T$ and $\widehat{D_T}$ operations on $\mathcal{QNN}$ are defined as below:

(i)
$$\mu \oplus \nu = \left\langle 1 - \frac{1}{1+\left((\frac{m_1}{1-m_1})^\lambda + (\frac{m_2}{1-m_2})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left((\frac{n_1}{1-n_1})^\lambda + (\frac{n_2}{1-n_2})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left((\frac{1-p_1}{p_1})^\lambda + (\frac{1-p_2}{p_2})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left((\frac{1-q_1}{q_1})^\lambda + (\frac{1-q_2}{q_2})^\lambda\right)^{\frac{1}{\lambda}}} \right\rangle$$

(ii)
$$\mu \odot \nu = \left\langle 1 - \frac{1}{1+\left((\frac{1-m_1}{m_1})^\lambda + (\frac{1-m_2}{m_2})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left((\frac{1-n_1}{n_1})^\lambda + (\frac{1-n_2}{n_2})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left((\frac{p_1}{1-p_1})^\lambda + (\frac{p_2}{1-p_2})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left((\frac{q_1}{1-q_1})^\lambda + (\frac{q_2}{1-q_2})^\lambda\right)^{\frac{1}{\lambda}}} \right\rangle$$

(iii)
$$k\mu = \left\langle 1 - \frac{1}{1+\left(k(\frac{m_1}{1-m_1})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(k(\frac{n_1}{1-n_1})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(k(\frac{1-p_1}{p_1})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(k(\frac{1-q_1}{q_1})^\lambda\right)^{\frac{1}{\lambda}}} \right\rangle,$$

(iv)
$$\mu^k = \left\langle 1 - \frac{1}{1+\left(k(\frac{1-m_1}{m_1})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(k(\frac{1-n_1}{n_1})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(k(\frac{p_1}{1-p_1})^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(k(\frac{q_1}{1-q_1})^\lambda\right)^{\frac{1}{\lambda}}} \right\rangle.$$

## 5. Dombi prioritized average operators on $\mathcal{QNN}$

**Definition 5.1.** Let $\gamma_j = \langle m_j, n_j, p_j, q_j \rangle$ $(j = 1, 2, \ldots, l)$ be a collection on $\mathcal{QNN}$. A QSVN Dombi prioritized average (QSVNDPA) operator of dimension $l$ is a function $s_1 : \mathcal{QNN}^l \to \mathcal{QNN}$ defined by:

$$s_1(\gamma_1, \gamma_2, \ldots, \gamma_l) = \bigoplus_{j=1}^{l} \left( \frac{T_j \gamma_j}{\sum_{j=1}^{l} T_j} \right)$$

where $T_j = \prod_{k=1}^{j-1} S(\gamma_k) \; \forall\, k,\; T_1 = 1$ and $S(\gamma_j) = \frac{3+m_j+n_j-p_j-q_j}{4}$

**Theorem 5.2.** *Suppose $\gamma_j = \langle m_j, n_j, p_j, q_j \rangle$ $\forall j \in \mathbb{N}$ be a collection on $\mathcal{QNN}$. Then*

$$s_1(\gamma_1, \gamma_2, \ldots, \gamma_l) = \bigoplus_{j=1}^{l} \left( \frac{T_j \gamma_j}{\sum_{j=1}^{l} T_j} \right)$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \frac{T_j \left( \frac{m_j}{1-m_j} \right)^{\lambda}}{\sum_{j=1}^{l} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \frac{T_j \left( \frac{n_j}{1-n_j} \right)^{\lambda}}{\sum_{j=1}^{l} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \frac{T_j \left( \frac{1-p_j}{p_j} \right)^{\lambda}}{\sum_{j=1}^{l} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \frac{T_j \left( \frac{1-q_j}{q_j} \right)^{\lambda}}{\sum_{j=1}^{l} T_j} \right\}^{\frac{1}{\lambda}}} \right\rangle.$$

*Proof.* Here $\gamma_1 \in \mathcal{QNN}$. Now we have $\frac{T_1 \gamma_1}{T_1} = \gamma_1$ $\left\langle 1 - \frac{1}{1 + \left\{ \left( \frac{m_1}{1-m_1} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \left( \frac{n_1}{1-n_1} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \left( \frac{1-p_1}{p_1} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \left( \frac{1-q_1}{q_1} \right)^{\lambda} \right\}^{\frac{1}{\lambda}}} \right\rangle$. Hence the above equation trivially holds for $l = 1$. In a parallel way for $\gamma_2 \in \mathcal{QNN}$, we have $\frac{T_2 \gamma_2}{T_1 + T_2} =$

$$\left\langle 1 - \frac{1}{1 + \left\{ \frac{T_2 \left( \frac{m_2}{1-m_2} \right)^{\lambda}}{T_1 + T_2} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \frac{T_2 \left( \frac{n_2}{1-n_2} \right)^{\lambda}}{T_1 + T_2} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \frac{T_2 \left( \frac{1-p_2}{p_2} \right)^{\lambda}}{T_1 + T_2} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \frac{T_2 \left( \frac{1-q_2}{q_2} \right)^{\lambda}}{T_1 + T_2} \right\}^{\frac{1}{\lambda}}} \right\rangle.$$

Therefore

$$s_1(\gamma_1, \gamma_2) = \bigoplus_{j=1}^{2} \left( \frac{T_j \gamma_j}{\sum_{j=1}^{2} T_j} \right)$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{2} \frac{T_j \left( \frac{m_j}{1-m_j} \right)^{\lambda}}{\sum_{j=1}^{2} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{2} \frac{T_j \left( \frac{n_j}{1-n_j} \right)^{\lambda}}{\sum_{j=1}^{2} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{2} \frac{T_j \left( \frac{1-p_j}{p_j} \right)^{\lambda}}{\sum_{j=1}^{2} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{2} \frac{T_j \left( \frac{1-q_j}{q_j} \right)^{\lambda}}{\sum_{j=1}^{2} T_j} \right\}^{\frac{1}{\lambda}}} \right\rangle.$$

Hence the equation is valid for $l = 1, 2$. We assume that the equation is valid for $l = s$ i.e.

$$s_1(\gamma_1, \gamma_2, \ldots, \gamma_s) = \bigoplus_{j=1}^{s} \left( \frac{T_j \gamma_j}{\sum_{j=1}^{s} T_j} \right)$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{m_j}{1-m_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{n_j}{1-n_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{1-p_j}{p_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{1-q_j}{q_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}} \right\rangle.$$

Finally for $l = s + 1$, one can easily see that

$$s_1(\gamma_1, \gamma_2, \ldots, \gamma_s) = \bigoplus_{j=1}^{s} \left( \frac{T_j \gamma_j}{\sum_{j=1}^{s} T_j} \right) \bigoplus \left( \frac{T_{s+1} \gamma_{s+1}}{\sum_{j=1}^{s+1} T_j} \right)$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{m_j}{1-m_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{n_j}{1-n_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{1-p_j}{p_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{1-q_j}{q_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}} \right\rangle \oplus \left( \frac{T_{s+1} \gamma_{s+1}}{\sum_{j=1}^{s+1} T_j} \right)$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s+1} \frac{T_j \left( \frac{m_j}{1-m_j} \right)^{\lambda}}{\sum_{j=1}^{s+1} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s+1} \frac{T_j \left( \frac{n_j}{1-n_j} \right)^{\lambda}}{\sum_{j=1}^{s+1} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s+1} \frac{T_j \left( \frac{1-p_j}{p_j} \right)^{\lambda}}{\sum_{j=1}^{s+1} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s+1} \frac{T_j \left( \frac{1-q_j}{q_j} \right)^{\lambda}}{\sum_{j=1}^{s+1} T_j} \right\}^{\frac{1}{\lambda}}} \right\rangle.$$

Finally the equation

$$s_1(\gamma_1, \gamma_2, \ldots, \gamma_s) = \bigoplus_{j=1}^{s} \left( \frac{T_j \gamma_j}{\sum_{j=1}^{s} T_j} \right)$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{m_j}{1-m_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{n_j}{1-n_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{1-p_j}{p_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{s} \frac{T_j \left( \frac{1-q_j}{q_j} \right)^{\lambda}}{\sum_{j=1}^{s} T_j} \right\}^{\frac{1}{\lambda}}} \right\rangle.$$

holds for all $s \in \mathbb{N}$. □

**Theorem 5.3.** *The QSVNDPA operator $s_1$ satisfies the following properties:*

(i) *Consistency:* $s_1(\gamma_1, \gamma_2, \ldots, \gamma_l) \in \mathcal{QNN}$.

(ii) *Idempotency:* $s_1(\gamma, l \text{ times } \ldots, \gamma) = \gamma$.

(iii) *Commutativity:* $s_1(\gamma_1, \gamma_2, \ldots, \gamma_l) = s_1(\gamma_l, \gamma_{l-1}, \ldots, \gamma_1)$.

(iv) $s_1(\gamma_{\pi(1)}, \gamma_{\pi(2)}, \ldots, \gamma_{\pi(l)}) = s_1(\gamma_1, \gamma_2, \ldots, \gamma_l)$ *where $\pi$ is a permutation on* $\{1, 2, \ldots, l\}$.

*Proof.* The basic two properties of QSVNDPA operator i.e. consistency and commutativity properties are quite easy. We will prove the property (ii) and (iv) respectively. If $\gamma_j = \gamma \; \forall \; j$

then

$$s_1(\gamma, l \text{ times} \ldots, \gamma) = \bigoplus_{j=1}^{s}\left(\frac{T_j\gamma_j}{\sum\limits_{j=1}^{s}T_j}\right) = \bigoplus_{j=1}^{s}\left(\frac{T_j}{\sum\limits_{j=1}^{s}T_j}\right)\gamma \quad = \gamma.$$

Finally consider $\pi$ as a permutation on $\{1, 2, \ldots, l\}$. Now due to additive commutativity in $\mathcal{QNN}$

$$s_1(\gamma_{\pi(1)}, \gamma_{\pi(2)}, \ldots, \gamma_{\pi(s)}) = \bigoplus_{j=1}^{s}\left(\frac{T_{\pi(j)}\gamma_{\pi(j)}}{\sum\limits_{j=1}^{s}T_{\pi(j)}}\right) = \bigoplus_{j=1}^{s}\left(\frac{T_j\gamma_j}{\sum\limits_{j=1}^{s}T_j}\right).$$

Hence we are done. □

**Theorem 5.4.** *Consider* $\gamma_j = \langle m_j, n_j, p_j, q_j\rangle$ $(j = 1, 2, \ldots, l)$ *and* $\delta_j = \langle \widetilde{m_j}, \widetilde{n_j}, \widetilde{p_j}, \widetilde{q_j}\rangle$ $(j = 1, 2, \ldots, l)$ *are two collections on* $\mathcal{QNN}$ *such that* $m_j \leq \widetilde{m_j}, n_j \leq \widetilde{n_j}, p_j \geq \widetilde{p_j}, q_j \geq \widetilde{q_j} \forall j$. *Then* $s_1(\gamma_1, \gamma_2, \ldots, \gamma_l) \leq s_1(\delta_1, \delta_2, \ldots, \delta_l)$.

*Proof.* Here,

$$s_1(\gamma_1, \gamma_2, \ldots, \gamma_l) = \bigoplus_{j=1}^{l}\left(\frac{T_j\gamma_j}{\sum\limits_{j=1}^{l}T_j}\right)$$

$$=\left\langle 1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\frac{T_j\left(\frac{m_j}{1-m_j}\right)^{\lambda}}{\sum\limits_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\frac{T_j\left(\frac{n_j}{1-n_j}\right)^{\lambda}}{\sum\limits_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\frac{T_j\left(\frac{1-p_j}{p_j}\right)^{\lambda}}{\sum\limits_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\frac{T_j\left(\frac{1-q_j}{q_j}\right)^{\lambda}}{\sum\limits_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}\right\rangle.$$

$$s_1(\delta_1, \delta_2, \ldots, \delta_l) = \bigoplus_{j=1}^{l}\left(\frac{T_j\delta_j}{\sum\limits_{j=1}^{l}T_j}\right)$$

$$=\left\langle 1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\frac{T_j\left(\frac{\widetilde{m_j}}{1-\widetilde{m_j}}\right)^{\lambda}}{\sum\limits_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\frac{T_j\left(\frac{\widetilde{n_j}}{1-\widetilde{n_j}}\right)^{\lambda}}{\sum\limits_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\frac{T_j\left(\frac{1-\widetilde{p_j}}{\widetilde{p_j}}\right)^{\lambda}}{\sum\limits_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left\{\sum\limits_{j=1}^{l}\frac{T_j\left(\frac{1-\widetilde{q_j}}{\widetilde{q_j}}\right)^{\lambda}}{\sum\limits_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}\right\rangle.$$

Firstly we consider that $m_j < \widetilde{m_j}, n_j < \widetilde{n_j}, p_j > \widetilde{p_j}, q_j > \widetilde{q_j} \forall j \in \{1, \ldots, l\}$. Then

$$1 - m_j > 1 - \widetilde{m_j} \quad \forall j \in \{1, \ldots, l\}$$
$$\Rightarrow \left(\frac{1-m_j}{m_j}\right) > \left(\frac{1-\widetilde{m_j}}{\widetilde{m_j}}\right)$$
$$\Rightarrow T_j\left(\frac{m_j}{1-m_j}\right)^{\lambda} < T_j\left(\frac{\widetilde{m_j}}{1-\widetilde{m_j}}\right)^{\lambda}$$

$$\Rightarrow \frac{1}{1+\left\{\sum_{j=1}^{l}\frac{T_j\left(\frac{m_j}{1-m_j}\right)^{\lambda}}{\sum_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}} > \frac{1}{1+\left\{\sum_{j=1}^{l}\frac{T_j\left(\frac{\widetilde{m_j}}{1-\widetilde{m_j}}\right)^{\lambda}}{\sum_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}$$

$$\Rightarrow 1-\frac{1}{1+\left\{\sum_{j=1}^{l}\frac{T_j\left(\frac{m_j}{1-m_j}\right)^{\lambda}}{\sum_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}} < 1-\frac{1}{1+\left\{\sum_{j=1}^{l}\frac{T_j\left(\frac{\widetilde{m_j}}{1-\widetilde{m_j}}\right)^{\lambda}}{\sum_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}$$

In a same way we can observe that

$$\Rightarrow \frac{1}{1+\left\{\sum_{j=1}^{l}\frac{T_j\left(\frac{n_j}{1-n_j}\right)^{\lambda}}{\sum_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}} > \frac{1}{1+\left\{\sum_{j=1}^{l}\frac{T_j\left(\frac{\widetilde{n_j}}{1-\widetilde{n_j}}\right)^{\lambda}}{\sum_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}$$

$$\Rightarrow \frac{1}{1+\left\{\sum_{j=1}^{l}\frac{T_j\left(\frac{1-p_j}{p_j}\right)^{\lambda}}{\sum_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}} > \frac{1}{1+\left\{\sum_{j=1}^{l}\frac{T_j\left(\frac{1-\widetilde{p_j}}{\widetilde{p_j}}\right)^{\lambda}}{\sum_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}$$

$$\Rightarrow \frac{1}{1+\left\{\sum_{j=1}^{l}\frac{T_j\left(\frac{1-q_j}{q_j}\right)^{\lambda}}{\sum_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}} > \frac{1}{1+\left\{\sum_{j=1}^{l}\frac{T_j\left(\frac{1-\widetilde{q_j}}{\widetilde{q_j}}\right)^{\lambda}}{\sum_{j=1}^{l}T_j}\right\}^{\frac{1}{\lambda}}}$$

Combining all the above we get $s_1(\gamma_1,\gamma_2,\ldots,\gamma_l) < s_1(\delta_1,\delta_2,\ldots,\delta_l)$. Now if $m_j = \widetilde{m_j}, n_j = \widetilde{n_j}, p_j = \widetilde{p_j}, q_j = \widetilde{q_j}\ \forall j \in \{1,\ldots,l\}$, then all the equalities as well as the score functions become equal. Finally $s_1(\gamma_1,\gamma_2,\ldots,\gamma_l) \leq s_1(\delta_1,\delta_2,\ldots,\delta_l)$. □

**Theorem 5.5.** *Consider a collection of* $\gamma_j = \langle m_j, n_j, p_j, q_j\rangle, j \in \mathbb{N}$ *in* $\mathcal{QNN}$. *Then*

$$\underline{\gamma} \leq s_1(\gamma_1,\gamma_2,\ldots,\gamma_l) \leq \overline{\gamma}, \textit{where}$$

$$\underline{\gamma} = \langle \min_j(m_j), \min_j(n_j), \min_j(p_j), \min_j(q_j)\rangle = \langle \underline{m_j}, \underline{n_j}, \underline{p_j}, \underline{q_j}\rangle \textit{ and}$$

$$\overline{\gamma} = \langle \max_j(m_j), \max_j(n_j), \max_j(p_j), \max_j(q_j)\rangle = \langle \overline{m_j}, \overline{n_j}, \overline{p_j}, \overline{q_j}\rangle.$$

*Proof.* From Definition of $\mathcal{QNN}$ we have $\forall j = \{1,2,\ldots,l\}$,

$$\underline{m_j} \leq m_j \leq \overline{m_j}, \underline{n_j} \leq n_j \leq \overline{n_j} \text{ and}$$

$$\underline{p_j} \geq p_j \geq \overline{p_j}, \underline{q_j} \geq q_j \geq \overline{q_j}.$$

Then

$$s_(\underline{\gamma}, l\ times, \underline{\gamma}) \leq s_(\gamma_1,\gamma_2,\ldots,\gamma_l) \leq s_(\overline{\gamma}, l\ times, \overline{\gamma}), i.e$$

$$\underline{\gamma} \leq s_(\gamma_1,\gamma_2,\ldots,\gamma_l) \leq \overline{\gamma}.$$

□

**Definition 5.6.** Consider the mass associated with $\gamma_j$ as $M_j > 0\ \forall\, j = 1, \ldots, l$, where $M = (M_1, M_2, \ldots, M_l)^T$ is the mass vector such that $\sum_{j=1}^{l} M_j = 1$. Then the QSVNWDPA (QSVN weighted DPA) can be defined as follows:

$$s_M(\gamma_1, \gamma_2, \ldots, \gamma_l) = \bigoplus_{j=1}^{l} \left( \frac{M_j T_j}{\sum_{j=1}^{l} M_j T_j} \gamma_j \right)$$

where $T_j = \prod_{k=1}^{j-1} S(\gamma_k)\ (k = 1, 2, \ldots, l), T_1 = 1$ and $S(\gamma_j) = \frac{3+m_j+n_j-p_j-q_j}{4}$

**Definition 5.7.** Suppose $\gamma_j = \langle m_j, n_j, p_j, q_j \rangle,\ (j = 1, 2, \ldots, l)$ be a collection on $\mathcal{QNN}$. Then a QSVNDPA operator $s_1$ of dimension $l$ can be written as follows

$$s_1(\gamma_1, \gamma_2, \ldots, \gamma_l) = \bigoplus_{j=1}^{l} \left( \frac{T_j \gamma_j}{\sum_{j=1}^{l} T_j} \right)$$

Now if $T_j = \frac{1}{l}\ \forall\, j$ then

$$s_1(\gamma_1, \gamma_2, \ldots, \gamma_l) = \frac{1}{l} \bigoplus_{j=1}^{l} \gamma_j.$$

is called average QSVNDPA operator of $\gamma_j = \langle m_j, n_j, p_j, q_j \rangle\ (j = 1, 2, \ldots, l)$.

**Definition 5.8.** Let $\gamma_j = \langle m_j, n_j, p_j, q_j \rangle\ (j = 1, 2, \ldots, l)$ be a collection on $\mathcal{QNN}$. A QSVN Dombi prioritized geometric (QSVNDPG) operator of dimension $l$ is a function $s_2 : \mathcal{QNN}^l \to \mathcal{QNN}$ defined by:

$$s_2(\gamma_1, \gamma_2, \ldots, \gamma_l) = \bigodot_{j=1}^{l} \gamma_j^{\frac{T_j}{\sum_{j=1}^{l} T_j}}$$

where $T_j = \prod_{k=1}^{j-1} S(\gamma_k)\ (k = 1, 2, \ldots, l),\ T_1 = 1$ and $S(\gamma_j) = \frac{3+m_j+n_j-p_j-q_j}{4}$.

**Theorem 5.9.** *Suppose $\gamma_j = \langle m_j, n_j, p_j, q_j \rangle\ (j = 1, 2, \ldots, l)$ be a collection on $\mathcal{QNN}$. Then*

$$s_2(\gamma_1, \gamma_2, \ldots, \gamma_l) = \bigodot_{j=1}^{l} \gamma_j^{\frac{T_j}{\sum_{j=1}^{l} T_j}}$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \frac{T_j \left( \frac{1-m_j}{m_j} \right)^{\lambda}}{\sum_{j=1}^{l} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \frac{T_j \left( \frac{1-n_j}{n_j} \right)^{\lambda}}{\sum_{j=1}^{l} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \frac{T_j \left( \frac{p_j}{1-p_j} \right)^{\lambda}}{\sum_{j=1}^{l} T_j} \right\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{l} \frac{T_j \left( \frac{q_j}{1-q_j} \right)^{\lambda}}{\sum_{j=1}^{l} T_j} \right\}^{\frac{1}{\lambda}}} \right\rangle.$$

*Proof.* The above theorem can be proved using the same proof procedure of Theorem 5.2. □

**Theorem 5.10.** *The QSVNDPG operator $s_2$ satisfies properties as defined below:*

(i) *Consistency:* $s_2(\gamma_1, \gamma_2, \ldots, \gamma_l) \in \mathcal{QNN}$.

(ii) *Idempotency:* $s_2(\gamma, l \text{ times } \ldots, \gamma) = \gamma$.

(iii) *Commutativity:* $s_2(\gamma_1, \gamma_2, \ldots, \gamma_l) = s_2(\gamma_l, \gamma_{l-1}, \ldots, \gamma_1)$.

(iv) $s_2(\gamma_{\pi(1)}, \gamma_{\pi(2)}, \ldots, \gamma_{\pi(l)}) = s_2(\gamma_1, \gamma_2, \ldots, \gamma_l)$ *where* $\pi$ *is a permutation on* $\{1, 2, \ldots, l\}$.

*Proof.* We have omitted it due to similarity with Theorem 5.3. □

**Theorem 5.11.** *Consider* $\gamma_j = \langle m_j, n_j, p_j, q_j \rangle$ $(j = 1, 2, \ldots, l)$ *and* $\widetilde{\gamma_j} = \langle \widetilde{m_j}, \widetilde{n_j}, \widetilde{p_j}, \widetilde{q_j} \rangle$ $(j = 1, 2, \ldots, l)$ *are two collections on* $\mathcal{QNN}$ *such that* $m_j \leq \widetilde{m_j}, n_j \leq \widetilde{n_j}, p_j \geq \widetilde{p_j}, q_j \geq \widetilde{q_j} \forall j$. *Then* $s_2(\gamma_1, \gamma_2, \ldots, \gamma_l) \leq s_2(\widetilde{\gamma_1}, \widetilde{\gamma_2}, \ldots, \widetilde{\gamma_l})$.

*Proof.* Here the proof is similar with Theorem 5.4, hence we have omitted it. □

**Theorem 5.12.** *Consider a collection of* $\gamma_j = \langle m_j, n_j, p_j, q_j \rangle, j = 1, 2, \ldots, l$ *in* $\mathcal{QNN}$. *Then*

$$\underline{\gamma} \leq s_2(\gamma_1, \gamma_2, \ldots, \gamma_l) \leq \overline{\gamma}, \textit{ where}$$
$$\underline{\gamma} = \langle \min_j(m_j), \min_j(n_j), \min_j(p_j), \min_j(q_j) \rangle = \langle \underline{m_j}, \underline{n_j}, \underline{p_j}, \underline{q_j} \rangle \textit{ and}$$
$$\overline{\gamma} = \langle \max_j(m_j), \max_j(n_j), \max_j(p_j), \max_j(q_j) \rangle = \langle \overline{m_j}, \overline{n_j}, \overline{p_j}, \overline{q_j} \rangle.$$

*Proof.* Again proof is not done due to its similarity with Theorem 5.5. □

**Definition 5.13.** Suppose $\gamma_j = \langle m_j, n_j, p_j, q_j \rangle$, $(j = 1, 2, \ldots, l)$ be a collection on $\mathcal{QNN}$. Then a QSVNDPG operator $s_2$ of dimension $l$ can be written as follows:

$$s_2(\gamma_1, \gamma_2, \ldots, \gamma_l) = \bigodot_{j=1}^{l} \gamma_j^{\frac{T_j}{\sum_{j=1}^{l} T_j}}.$$

If $T_j = \frac{1}{l}$ $\forall$ $j \in \{1, 2, \ldots, l\}$ then

$$s_2(\gamma_1, \gamma_2, \ldots, \gamma_l) = (\bigodot_{j=1}^{l} \gamma_j)^{\frac{1}{l}}.$$

is called average QSVNDPG operator of $\gamma_j = \langle m_j, n_j, p_j, q_j \rangle$ $(j = 1, 2, \ldots, l)$.

**Definition 5.14.** Consider the mass associated with $\gamma_j$ as $M_j > 0 \ \forall j = 1, \ldots, l$, where $M = (M_1, M_2, \ldots, M_l)^T$ is the mass vector such that $\sum_{j=1}^{l} M_j = 1$. Then the QSVNWDPG (SVN weighted DPG) can be defined as follows:

$$s_{\mathcal{M}}(\gamma_1, \gamma_2, \ldots, \gamma_l) = \bigodot_{j=1}^{l} \gamma_j^{\frac{M_j T_j}{\sum_{j=1}^{l} M_j T_j}},$$

where $T_j = \prod_{k=1}^{j-1} S(\gamma_k)(k = 1, 2, \ldots, l), T_1 = 1$ and $S(\gamma_j) = \frac{3+m_j+n_j-p_j-q_j}{4}$

## 6. An application in MADM of QSVNWDPA and QSVNWDPG operator

For smooth understanding of QSVN operators it is better to apply our operators in MADM problems. Without a real life application any researcher cannot get any interest of studying this. In this regard we have tried to formulate a real life problem with the help of QSVNWDPA and QSVNWDPG operator. Suppose Govt of India wants to stop the spread of the second wave of Covid-19 virus. For this reason Govt of India has 4 ways of lockdown process in their policy i.e. $L_1$: Complete lockdown , $L_2$: Statewise lockdown, $L_3$: District wise lockdown , $L_4$: Specific area wise lockdown. However there are four attributes $A_j, j =, 2, 3, 4$ which are to be considered for choosing a particular process i.e. $(A_1)$ : the economic growth of the country, $(A_2)$ : the migrant workers, $(A_3)$: The small industry $(A_4)$: The poor people. In order to get a suitable choice $L_i$ after consideration of all attributes $A_j$ we have represented these MADM problems in the form of a decision making matrix $D(l_{ij})$ on $\mathcal{QNN}$ as following:

$$D(l_{ij}) = \begin{bmatrix} \langle 0.4, 0.6, 0.2, 0.3 \rangle & \langle 0.4, 0.8, 0.7, 0.9 \rangle & \langle 0.5, 0.6, 0.4, 0.2 \rangle & \langle 0.1, 0.5, 0.2, 0.3 \rangle \\ \langle 0.7, 0.5, 0.7, 0.6 \rangle & \langle 0.2, 0.8, 0.3, 0.5 \rangle & \langle 0.6, 0.6, 0.1, 0.4 \rangle & \langle 0.3, 0.4, 0.5, 0.1 \rangle \\ \langle 0.8, 0.5, 0.4, 0.6 \rangle & \langle 0.3, 0.6, 0.1, 0.4 \rangle & \langle 0.2, 0.5, 0.5, 0.3 \rangle & \langle 0.6, 0.6, 0.2, 0.1 \rangle \\ \langle 0, 7, 0.1, 0.6, 0.9 \rangle & \langle 0.8, 0.3, 0.4, 0.6 \rangle & \langle 0.5, 0.2, 0.8, 0.6 \rangle & \langle 0.6, 0.4, 0.4, 0.9 \rangle \end{bmatrix}.$$

**Case-I:** Firstly we take the help of QSVNWDPA operator to find out a possible way out of our MADM. Here we take $\lambda = 1, M = (0.4, 0.3, 0.2, 0.1)$ and derive the collection of QSVNs say $L_i$ to find suitable way out among $L_i(i = 1, 2, 3, 4)$ by the help of Definition 5.1 as follows:

$$s_1(L_1) = \langle 0.473, 0.712, 0.731, 0.673 \rangle$$

$$s_1(L_2) = \langle 0.639, 0.682, 0.716, 0.6324 \rangle$$

$$s_1(L_3) = \langle 0.702, 0.578, 0.798, 0.654 \rangle$$

$$s_1(L_4) = \langle 0.806, 0.406, 0.171, 0.615 \rangle.$$

Based on the Definition 3.1 the scores are as follows:

$$S(L_1) = 0.69525, S(L_2) = 0.74335, S(L_3) = 0.70694, S(L_4) = 0.8567.$$

From above we have the priority order of lockdown process as $L_4 > L_2 > L_3 > L_1$. .

**Case-II:** Secondly we take the help of QSVNWDPG operator to find out a possible solution to our problem. Again we take $\lambda = 1, M = (0.4, 0.3, 0.2, 0.1)$ and derive the collective QSVNs $L_i$ with the help of Definition 5.8 as follows:

$$s_2(L_1) = \langle 0.616, 0.434, 0.568, 0.769\rangle$$

$$s_2(L_2) = \langle 0.649, 0.518, 0.625, 0.586\rangle$$

$$s_2(L_3) = \langle 0.631, 0.518, 0.461, 0.542\rangle$$

$$s_2(L_4) = \langle 0.676, 0.873, 0.735, 0.919\rangle.$$

Based on the Definition 3.1 the scores are as follows:

$$S(L_1) = 0.6779, S(L_2) = 0.7378, S(L_3) = 0.786, S(L_4) = 0.724.$$

According to obtained scores, the priority order of lockdown process is $L_3 > L_2 > L_4 > L_1$.

### 6.1. *Sensitivity analysis*

In this section we have done a sensitivity analysis based on our method. For this purpose we have change the value of our parameter $\lambda$ in an increasing manner starting from 0.2 to 1 with an increment 0.2. For both the operators i.e. QSVNWDPA and QSVNWDPG operator the following results are obtained. Tabular representation in case of QSVNWDPA operator

| $\lambda$ | $S(L_1), S(L_2), S(L_3), S(L_4)$ |
|---|---|
| 0.2 | 0.583, 0.643, 0.616, 0.677 |
| 0.4 | 0.549, 0.632, 0.607, 0.658 |
| 0.6 | 0.536, 0.617, 0.594, 0.634 |
| 0.8 | 0.493, 0.536, 0.511, 0.577 |
| 1.0 | 0.462, 0.514, 0.473, 0.543 |

Result: $L_4 > L_2 > L_3 > L_1$. Tabular representation in case of QSVNWDPG operator:

| $\lambda$ | $S(L_1), S(L_2), S(L_3), S(L_4)$ |
|---|---|
| 0.2 | 0.613, 0.649, 0.677, 0.627 |
| 0.4 | 0.553, 0.625, 0.652, 0.586 |
| 0.6 | 0.497, 0.531, 0.568, 0.519 |
| 0.8 | 0.468, 0.511, 0.547, 0.487 |
| 1.0 | 0.421, 0.473, 0.489, 0.445 |

Result: $L_3 > L_2 > L_4 > L_1$. Considering all the above cases we observed that the priority order of lockdown process remains unaltered irrespective of the values of $\lambda$. According to us that either specific area wise lockdown or district wise lock down will be the suitable process against the spread of corona virus second wave in India. But in all the above cases complete lockdown will not be proffered. The above procedure help our Government to choose a multi-solution based on the current situation at that time.

## 7. Conclusion

Benlap introduced the four valued logic in [1] and applied it in different areas. The QSVN sets are developed on Benlap's Model and they are very good in modeling uncertainty because they can single handedly tackle consistent, inconsistent, vague etc. information. Based on the QSVN set, $\mathcal{QNN}$ is introduced in 2019. In this article two prioritized aggregation operators i.e. QSVNWDPA and QSVNWDPG operator based on Dombi operations on $\mathcal{QNN}$ sets are studied. These aggregation operators are better than other available aggregation operators because they have combined effects of neutrosophy, four valued logic and the power of Dombi. We have also added weights in our operators to add flexibility in them. We have also shown the applicability of our operators by solving a MADM problem where we have utilized the score functions of $\mathcal{QNN}$ to finding the order of priority of different parameters. In future one can develop more advanced type of operators on $\mathcal{QNN}$ and apply them to solve real life MADM problems.

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Received: Nov 9, 2021. Accepted: Feb 3, 2022

NSS
Neutrosophic Sets and Systems, Vol. 48, 2022
University of New Mexico

# Norms and Delta-Equalities of Complex Neutrosophic Sets

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**Abstract:** The purpose of this paper is to put forward the basics results of complex fuzzy sets (CFSs) such as union, intersection, complement, product into complex neutrosophic sets because as the CFSs and complex intuitionistics sets does give the erroneous and inconvenient information about uncertainty and periodicity and also there are results related to different norms. Moreover we give some results about the distance measures of complex neutrosophic sets and define some notions.

**Keywords:** CFSs, complex neutrosophic sets, distance measures, delta-equalities.

## 1. Introduction

Lotfi A. Zadeh [19] introduced a fuzzy set (FS) in 1965. FS was designed to manipulate ambiguity, fuzziness, vagueness, crispness, and uncertainty in different aspects of life. It has great significance in the field of genetic algorithm in chemical industry.

Krassimir Atanassov [3] generalized the concept of L. A. Zadeh and introduced an intuitionistic fuzzy set (IFS) in which instead of the truth function of each element there is also the falsehood function. It indicates that statement can be true or false, yes or no, right or wrong, feasible or not.

De et al., [6] in 2001, use the idea of a fuzzy set for modeling in real life problems, like marketing, psychological investigations, and determination of diagnosis [16] etc. IFS has great significance in career determination. In IFS the concept of distance measure also introduced but there was a problem to deal when both the informations contain uncertainties of yes and no at a time and at a time neither yes nor no. Thus F. Smarandache [15] gave the solution of this problem by introducing new FS called a neutrosophic fuzzy set (NFS) which is a framework for unification of a FS and an IFS or it is a bridge between FS and IFS. Neutrosophy is the philosophys branch, in which we deal with the scope, nature, and origin of neutral along with ideational spectra. Neutrosophy has a great engineering application like in medicine, military, airspace, cybernetics etc. A neutrosophic set is that which contain truth function $T$, indeterministic function $I$ and falsehood function $F$.

A neutrosophic fuzzy set yields three type of chances like win, lose, draw or accept, reject, pending or positive, negative, zero etc. NS is the extension of some FSs like interval valued fuzzy sets (IVFSs) [16], conventional FSs [19], paradoxist sets [15] and IFS [3]. Wang et al., [17] gave more information about NS by presenting the single valued NS which has a lot of application in engineering and social problems and have additional benefit to interpret vagueness, crispness, and uncertainty. For more details about neutrosophic sets one can refer [1], [7], [9], [10], [11] and [14].

After that Ramot et al., [12] gave the idea of a complex fuzzy set (CFS) for handling problems having amplitude term where the complex mapping is used a instead of real valued mapping and is defined as

$$\mu_s(x) = r_s(x)e^{i\omega_s(x)} \ , \quad i = \sqrt{-1}$$

where amplitude term $r_s(x)$ and phase term $\omega_s(x)$ are the real valued function having the range $[0,1]$ , and the range of $\mu_s(x)$ is expanded to a circle of radius 1. In a CFS amplitude term conserve the crispness idea together with the phase term which declare the periodicity in a CFS. The phase term makes it different from conventional fuzzy set [19], IFSs [3], and cubic set because it gives constructive and destructive interference which concludes that a complex fuzzy set has wavelike character. G. Zhang et al., [12] defined several important properties in complex fuzzy sets like union, intersection, complement, product, some norms like quasi-triangular norm, s-norm, t-norm etc.

After this Alkouri and Saleh [2] extended a CFS into a complex intuitionistic set and it contains complex valued truth function together with the complex valued falsehood function. They differ the idea of a FS in a way such that an IFS have two phase terms instead of one. F. Smarandache introduced a complex neutrosophic set (CNS) which contains truth function $T$, indeterministic function $I$ and falsehood function $F$ having the range is extended to unit circle. CNSs contain amplitude terms together with the three phase terms and can work with information containing uncertainties, crispness and vagueness in periodicity.

Pappis [pappis] for the first time worked on the concept of proximity measure and approximately equal fuzzy set whose work was generalized by Hong and Hwang [hong]. Later on Cai [cai],[4] felt that both were using the same concept so he changed that approach and expressed as special measure is used for defining $\delta - equalities$. Two FSs A and B are called $\delta - equal$ if they are $1-\delta$ part away. Zhang et al. [18] used this concept of $\delta - equality$ for applications in signal processing which certify $\delta - equality$ of CFSs practically.

We are extending the work of G. Zhang et al., [18] from CFSs into complex neutrosophic sets and investigate some useful results.

Definition *CNS S is defined on a X, distinguished by degree of truth , indeterminate function and falsehood function respectively. The truth function, indeterminate function and falsehood function are defined as*

$$T_S(x) = p_S(x)e^{i\mu_S(x)}, I_S(x) = q_S(x)e^{i\nu_S(x)}, F_S(x) = r_S(x)e^{i\omega_S(x)},$$

*where* $p_S(x)$ *represents a FS and* $\mu_S(x)$ *is any real function. Similarly for indeterminacy and falsity* $q_S(x)$ *&* $\nu_S(x)$ and $r_S(x)$ $\omega_S(x)$, *such that*

$$0^- \le p_S(x) + q_S(x) + r_S(x) \le 3^+.$$

*CNS S is defined to be*

$S = \{(x, T_S(x) = a_T, I_S(x) = a_I, F_S(x) = a_I)?\, x \in X\}$, where

$$T_S: X \rightarrow \{a_T: a_T \in C, |a_T| \leq 1\},$$
$$I_S: X \rightarrow \{a_I: a_I \in C, |a_I| \leq 1\},$$
$$F_S: X \rightarrow \{a_F: a_F \in C, |a_F| \leq 1\},$$
$$\text{and } T_S(x) + I_S(x) + F_S(x) \leq 3^+.$$

Definition (1) *A function* $(0,1] \times (0,1] \rightarrow [0,1]$ *is a quasi-triangular norm* $T$ *if following holds:*

(*i*) $T(1,1) = 0$

(*ii*) $T(a, b) = T(b, a)$

(*iii*) $T(a, b) \leq T(c, d)$, *whenever,* $a \leq c, b \leq d$

(*iv*) $T(T(a, b), c) = T\big(a, (b, c)\big)$

(2) *A function* $(0,1] \times (0,1] \rightarrow [0,1]$ *is a triangular norm* $T$ *if it satisfies previous* $(i) - (iv)$ *conditions together with*

(*v*) $T(0,0) = 0$

(3) *A function* $(0,1] \times (0,1] \rightarrow [0,1]$ *is s-norm if it satisfies triangular norm's conditions together with*

(*vi*) $T(a, 0) = a$

(4) *A function* $(0,1] \times (0,1] \rightarrow [0,1]$ *is t-norm if it satisfies triangular norm's conditions together with*

(*vii*) $T(a, 1) = a.$

Definition *The union for CNSs is defined as: Assume*

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$

*be the complex neutrosophic sets on* $X$ *such that*

$$T_A(x) = p_A(x)e^{i\mu_A(x)}, I_A(x) = q_A(x)e^{i\nu_A(x)}, F_A(x) = r_A(x)e^{i\omega_A(x)},$$
$$T_B(x) = p_B(x)e^{i\mu_B(x)}, I_B(x) = q_B(x)e^{i\nu_B(x)}, F_B(x) = r_B(x)e^{i\omega_B(x)},$$

*be complex valued truth, indeterminate and falsehood functions respectively, then union of* $A$ *and* $B$ *be represented as*

$$A \cup B = \{x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x), x \in X\}$$

*where* $T_{A\cup B(x)}$, $I_{A\cup B(x)}$, $F_{A\cup B(x)}$ *are defined as*

$$T_{A\cup B}(x) = [p_A(x) \vee p_B(x)]e^{i\mu_{T_{A\cup B}}(x)},$$
$$I_{A\cup B}(x) = [q_A(x) \wedge q_B(x)]e^{i\nu_{I_{A\cup B}}(x)},$$
$$F_{A\cup B}(x) = [r_A(x) \wedge r_B(x)]e^{i\omega_{F_{A\cup B}}(x)},$$

*where* $\vee$ *represent the max operator and* $\wedge$ *represent min operator.*

Proposition. *The complex neutrosophic union is s-norm.*

Proof Here we prove only $(iii)\&(iv)$ properties because others are quite easy

$(iii)$ Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be the complex neutrosophic sets on $X$ such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)},$$

we suppose that

$$|p_A(x)| \le |p_B(x)|, |r_A(x)| \le |r_B(x)|, |q_A(x)| \le |q_B(x)|,$$
$$\mu_A(x) \le \mu_B(x), \nu_A(x) \le \nu_B(x), \omega_A(x) \le \omega_B(x), \text{ for all } x \in X.$$

Thus

$$|T_{A\cup C}(x)| = max(p_A(x), p_C(x)) \le max(p_B(x), p_C(x)) = |T_{B\cup C}(x)|, \text{ for all } x \in X.$$

Similarly

$$|I_{A\cup C}(x)| = max(q_A(x), q_C(x)) \le max\left(q_B(x), p_{qC}(x)\right) = |I_{B\cup C}(x)|, \text{ for all } x \in X, \text{ and}$$

$$|F_{A\cup C}(x)| = max(r_A(x), r_C(x)) \le max(r_B(x), r_C(x)) = |F_{B\cup C}(x)|\ , \text{ for all } x \in X.$$

Also

$$|\mu_{A\cup C}(x)| = max(\mu_A(x), \mu_C(x)) \le max(\mu_B(x), \mu_C(x)) = |\mu_{B\cup C}(x)|, \text{ for all } x \in X,$$

$$|\nu_{A\cup C}(x)| = max\left(\nu_A(x), \nu_C(x)\right) \leq max\left(\nu_C(x), \nu_C(x)\right) = |\nu_{B\cup C}(x)|, \text{ for all } x \in X,$$

$$|\omega_{A\cup C}(x)| = max\left(\omega_A(x), \omega_C(x)\right) \leq max\left(\omega_B(x), \omega_C(x)\right) = |\omega_{B\cup C}(x)|, \text{ for all } x \in X.$$

[illegible]

Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be the complex neutrosophic sets on $X$, such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)}.$$

Therefore

$$T_{(A\cup B)\cup C}(x) = p_{(A\cup B)\cup C}(x)e^{i\mu_{(A\cup B)\cup C}(x)}$$

$$= max\,[p_{A\cup B}(x), p_C(x)]e^{i\,max\,[\mu_{A\cup B}(x), \mu_C(x)]}$$

$$= max\left[max\left(p_A(x), p_B(x)\right), p_C(x)\right]e^{i\,max\,[max\,(\mu_A(x), \mu_B(x)), \mu_C(x)]}$$

$$= max\left[\left(p_A(x)\right), max\left(p_B(x), p_C(x)\right)\right]e^{i\,max[(\mu_A(x)), max\,(\mu_B(x), \mu_C(x))]}$$

$$= max\,[p_A(x), p_{B\cup C}(x)]e^{i\,max\,[\mu_A(x), \mu_{B\cup C}(x)]}$$

$$= p_{A\cup(B\cup C)}(x)e^{i\mu_{A\cup(B\cup C)}(x)} = T_{A\cup(B\cup C)}(x).$$

Following the same procedure we can prove for indeterminacy and falsehood functions.

Corollary *Let* $C_\alpha \in X, \alpha \in I$ *and*

$$T_{C_\alpha}(x) = p_{C_\alpha}(x)e^{i\mu_{C_\alpha}(x)}, I_{C_\alpha}(x) = q_{C_\alpha}e^{i\nu_{C_\alpha}(x)}, F_{C_\alpha}(x) = r_{C_\alpha}e^{i\omega_{C_\alpha}(x)}$$

*Then* $\bigcup_{\alpha\in I} C_\alpha \in X$. *Thus*

$$T \bigcup_{\alpha\in I} C_\alpha(x) = \sup_{\alpha\in I} p_{C_\alpha}(x) e^{i\sup_{\alpha\in I}\mu_{C_\alpha}(x)},$$

$$I \bigcup_{\alpha\in I} C_\alpha(x) = \inf_{\alpha\in I} q_{C_\alpha}(x) e^{i\sup_{\alpha\in I}\nu_{C_\alpha}(x)},$$

$$F \bigcup_{\alpha\in I} C_\alpha(x) = \inf_{\alpha\in I} r_{C_\alpha}(x) e^{i\sup_{\alpha\in I}\omega_{C_\alpha}(x)}.$$

Proof It is trivial.

Definition *The intersection of CNSs is defined as*

*Let*

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$

*be CNSs on* $X$ *such that*

$$T_A(x) = p_A(x)e^{i\mu_A(x)}, I_A(x) = q_A e^{i\nu_A(x)}, F_A(x) = r_A e^{i\omega_A(x)},$$
$$T_B(x) = p_B(x)e^{i\mu_B(x)}, I_B(x) = q_B e^{i\nu_B(x)}, F_B(x) = r_B e^{i\omega_B(x)},$$

*is represented as*

$$A \cap B = \{x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x), x \in X\},$$

*where* $T_{A\cap B(x)}$, $I_{A\cap B(x)}$, $F_{A\cap B(x)}$ *are defined as*

$$T_{A\cap B}(x) = [p_A(x) \wedge p_B(x)]e^{i\mu_{T_{A\cap B}}(x)},$$
$$I_{A\cap B}(x) = [q_A(x) \vee q_B(x)]e^{i\nu_{I_{A\cap B}}(x)},$$
$$F_{A\cap B}(x) = [r_A(x) \vee r_B(x)]e^{i\omega_{F_{A\cap B}}(x)},$$

*where* $\vee$ *is a maximum operator &* $\wedge$ *is a minimum operator.*

Proposition *If* $A$ *and* $B$ *are CNSs on* $X$. *Then* $\overline{A\cap B} = \overline{A} \cup \overline{B}$.

Proof For membership function

$$T_{\overline{A\cap B}}(x) = p_{\overline{A\cap B}}(x)e^{i\mu_{\overline{A\cap B}}(x)} = (1 - p_{A\cap B}(x))e^{i(2\pi - \mu_{A\cap B}(x))}$$

$$= \left(1 - \min(p_A(x), p_B(x))\right) e^{i(2\pi - \min(\mu_A(x), \mu_B(x)))}$$

$$= \max(1 - p_A(x), 1 - p_B(x))e^{i\max(2\pi - (\mu_A(x), 2\pi - \mu_B(x)))}$$

$$= max\left(p_{\overline{A}}(x), p_{\overline{B}}(x)e^{i\,max\left(2\pi-\left(\mu_A(x), 2\pi-\mu_B(x)\right)\right)}\right) = T_{\overline{A}\cup\overline{B}}(x).$$

For indeterminacy function

$$I_{\overline{A\cap B}}(x) = q_{\overline{A\cap B}}(x)e^{i\nu_{\overline{A\cap B}}(x)} = \left(1 - q_{A\cap B}(x)\right)e^{i\left(2\pi-\nu_{A\cap B}(x)\right)}$$

$$= \left(1 - max\left(p_A(x), p_B(x)\right)\right)e^{i\left(2\pi-min\left(\nu_A(x), \nu_B(x)\right)\right)}$$

$$= min\left(1 - q_A(x), 1 - q_B(x)\right)e^{i\,max\left(2\pi-\left(\nu_A(x), 2\pi-\nu_B(x)\right)\right)}$$

$$= min\left(q_{\overline{A}}(x), q_{\overline{B}}(x)e^{i\,max\left(2\pi-\left(\nu_A(x), 2\pi-\nu_B(x)\right)\right)}\right) = I_{\overline{A}\cup\overline{B}}(x).$$

Similarly we can prove for falsehood function.

Proposition *The complex neutrosophic intersection on* $X$ *is t-norm.*

Proof Here we prove only $(ii)$ & $(iv)$ properties because others are quite easy

$(ii)$ Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be the complex neutrosophic sets on $X$ such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)},$$

Now we suppose

$$|p_A(x)| \le |p_B(x)|, |q_A(x)| \le |q_B(x)|, |r_A(x)| \le |r_B(x)|,$$
$$\mu_A(x) \le \mu_B(x),\ \nu_A(x) \le \nu_B(x),\ \omega_A(x) \le \omega_B(x), \text{ for all } x \in X.$$

Thus

$$|T_{A\cap C}(x)| = min\left(p_A(x), p_C(x)\right) \le min\left(p_B(x), p_C(x)\right) = |T_{B\cap C}(x)|, \text{ for all } x \in X.$$

Similarly

$$|I_{A\cap C}(x)| = max\left(q_A(x), q_C(x)\right) \le max\left(q_B(x), p_C(x)\right) = |I_{B\cap C}(x)|, \text{ for all } x \in X,$$

$$|F_{A\cap C}(x)| = max\left(r_A(x), r_C(x)\right) \leq max\left(r_B(x), r_C(x)\right) = |F_{B\cap C}(x)|, \text{ for all } x \in X.$$

Likewise

$$|\mu_{A\cup C}(x)| = min\left(\mu_A(x), \mu_C(x)\right) \leq min\left(\mu_B(x), \mu_C(x)\right) = |\mu_{B\cap C}(x)|, \text{ for all } x \in X,$$

$$|\nu_{A\cap C}(x)| = min\left(\nu_A(x), \nu_C(x)\right) \leq min\left(\nu_C(x), \nu_C(x)\right) = |\nu_{B\cap C}(x)| \text{ for all } x \in X,$$

$$|\omega_{A\cap C}(x)| = min\left(\omega_A(x), \omega_C(x)\right) \leq min\left(\omega_B(x), \omega_C(x)\right) = |\omega_{B\cap C}(x)| \text{ for all } x \in X.$$

[illegible] Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be complex neutrosophic sets on $X$ such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)}.$$

Thus

$$T_{(A\cap B)\cup C}(x) = p_{(A\cap B)\cap C}(x)e^{i\mu_{(A\cap B)\cap C}(x)}$$

$$= min[p_{A\cap B}(x), p_C(x)]e^{i\,min[\mu_{A\cap B}(x), \mu_C(x)]}$$

$$= min\left[min\left(p_A(x), p_B(x)\right), p_C(x)\right]e^{i\,min[min(\mu_A(x), \mu_B(x)), \mu_C(x)]}$$

$$= min\left[\left(p_A(x)\right), min\left(p_B(x), p_C(x)\right)\right]e^{i\,min[(\mu_A(x)), min(\mu_B(x), \mu_C(x))]}$$

$$= min[p_A(x), p_{B\cap C}(x)]e^{i\,min[\mu_A(x), \mu_{B\cap C}(x)]}$$

$$= p_{A\cap(B\cap C)}(x)e^{i\mu_{A\cap(B\cap C)}(x)} = T_{A\cap(B\cap C)}(x).$$

Following the same procedure we can prove for indeterminacy and falsehood functions.

Corollary *Let* $C_\alpha \in X, \alpha \in I$ *and*

$$T_{C_\alpha}(x) = p_{C_\alpha}(x)e^{i\mu_{C_\alpha}(x)}, I_{C_\alpha}(x) = q_{C_\alpha}e^{i\nu_{C_\alpha}(x)}, F_{C_\alpha}(x) = r_{C_\alpha}e^{i\omega_{C_\alpha}(x)},$$

*Then* $\bigcup_{\alpha\in I} C_\alpha \in X$ *thus*

$$T_{\bigcup_{\alpha\in I} C_\alpha}(x) = \inf_{\alpha\in I} p_{C_\alpha}(x)e^{i\sup_{\alpha\in I}\mu_{C_\alpha}(x)},$$

$$I_{\bigcup_{\alpha\in I} C_\alpha}(x) = \sup_{\alpha\in I} q_{C_\alpha}(x)e^{i\sup_{\alpha\in I}\nu_{C_\alpha}(x)},$$

$$F_{\bigcup_{\alpha\in I} C_\alpha}(x) = \sup_{\alpha\in I} r_{C_\alpha}(x)e^{i\sup_{\alpha\in I}\omega_{C_\alpha}(x)}.$$

Proof It is trivial.

Corollary *Let* $C_{\alpha\beta} \in X, \alpha \in I_1, \beta \in I_2$ *and*

$$T_{C_{\alpha\beta}}(x) = p_{C_{\alpha\beta}}(x)e^{i\mu_{C_{\alpha\beta}}(x)}, I_{C_{\alpha\beta}}(x) = q_{C_{\alpha\beta}}e^{i\nu_{C_{\alpha\beta}}(x)}, F_{C_{\alpha\beta}}(x) = r_{C_{\alpha\beta}}e^{i\omega_{C_{\alpha\beta}}(x)},$$

*where* $I_1$ *and* $I_2$ *are arbitrary index sets. Then* $\bigcup_{\alpha\in I_1}\bigcap_{\alpha\in I_2} C_{\alpha\beta} \in X$, $\bigcap_{\alpha\in I_1}\bigcup_{\alpha\in I_2} C_{\alpha\beta} \in X$ *. Then*

$$T_{\bigcup_{\alpha\in I_1}\bigcap_{\alpha\in I_2} C_\alpha}(x) = \sup_{\alpha\in I_1}\inf_{\alpha\in I_2} p_{C_\alpha}(x)e^{i\sup_{\alpha\in I_1}\inf_{\alpha\in I_2}\mu_{C_\alpha}(x)},$$

$$I_{\bigcup_{\alpha\in I_1}\bigcap_{\alpha\in I_2} C_\alpha}(x) = \inf_{\alpha\in I_1}\sup_{\alpha\in I_2} q_{C_\alpha}(x)e^{i\sup_{\alpha\in I_1}\inf_{\alpha\in I_2}\nu_{C_\alpha}(x)},$$

$$F_{\bigcup_{\alpha\in I_1}\bigcap_{\alpha\in I_2} C_\alpha}(x) = \inf_{\alpha\in I_1}\sup_{\alpha\in I_2} r_{C_\alpha}(x)e^{i\sup_{\alpha\in I_1}\inf_{\alpha\in I_2}\omega_{C_\alpha}(x)}.$$

*Or*

$$T_{\bigcap_{\alpha\in I_1}\bigcup_{\alpha\in I_2} C_\alpha}(x) = \inf_{\alpha\in I_1}\sup_{\alpha\in I_2} p_{C_\alpha}(x).e^{i\inf_{\alpha\in I_1}\sup_{\alpha\in I_2}\mu_{C_\alpha}(x)},$$

$$I_{\bigcap_{\alpha\in I_1}\bigcup_{\alpha\in I_2} C_\alpha}(x) = \sup_{\alpha\in I_1}\inf_{\alpha\in I_2} q_{C_\alpha}(x).e^{i\inf_{\alpha\in I_1}\sup_{\alpha\in I_2}\nu_{C_\alpha}(x)},$$

$$F_{\bigcap_{\alpha\in I_1}\bigcup_{\alpha\in I_2} C_\alpha}(x) = \sup_{\alpha\in I_1}\inf_{\alpha\in I_2} r_{C_\alpha}(x).e^{i\inf_{\alpha\in I_1}\sup_{\alpha\in I_2}\omega_{C_\alpha}(x)}.$$

Proof It is trivial.

Definition *The product of CNSs is defined as*

*Let*

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$

*be complex valued NSs such that*

$$T_A(x) = p_A(x)e^{i\mu_A(x)}, I_A(x) = q_A(x)e^{i\nu_A(x)}, F_A(x) = r_A(x)e^{i\omega_A(x)},$$
$$T_B(x) = p_B(x)e^{i\mu_B(x)}, I_B(x) = q_B(x)e^{i\nu_B(x)}, F_B(x) = r_B(x)e^{i\omega_B(x)},$$

*is denoted as*

$$A \circ B = \{x, T_{A\circ B}(x), I_{A\circ B}(x), F_{A\circ B}(x), x \in X\},$$

*where* $T_{A\circ B(x)}$, $I_{A\circ B(x)}$, $F_{A\circ B(x)}$ *are defined as*

$$T_{A\circ B}(x) = p_{A\circ B}(x)e^{i\mu_{A\circ B}(x)} = [p_A(x).p_B(x)]e^{i2\pi\left(\frac{\mu_A(x)}{2\pi}.\frac{\mu_B(x)}{2\pi}\right)},$$

$$I_{A\circ B}(x) = q_{A\circ B}(x)e^{i\nu_{A\circ B}(x)} = [q_A(x).q_B(x)]e^{i2\pi\left(\frac{\nu_A(x)}{2\pi}.\frac{\nu_B(x)}{2\pi}\right)},$$

$$F_{A\circ B}(x) = r_{A\circ B}(x)e^{i\omega_{A\circ B}(x)} = [r_A(x).r_B(x)]e^{i2\pi\left(\frac{\omega_A(x)}{2\pi}.\frac{\omega_B(x)}{2\pi}\right)}.$$

Proposition *The complex neutrosophic product on* $X$ *is t-norm.*

Proof Here we prove only $(iii)\&(iv)$ properties because others are quite easy

**(iii)** Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be the CNSs on $X$ such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)},$$

Now, we suppose that

$$|p_A(x)| \leq |p_B(x)|, |q_A(x)| \leq |q_B(x)|, |r_A(x)| \leq |r_B(x)|,$$
$$\mu_A(x) \leq \mu_B(x),\ \nu_A(x) \leq \nu_B(x),\ \omega_A(x) \leq \omega_B(x), \text{ for all } x \in X.$$

Thus

$$|T_{A\circ C}(x)| = |p_A(x)|.|p_C(x)| \le |p_B(x)|.|p_C(x)| = |T_{B\circ C}(x)|, \text{ for all } x \in X.$$

Similarly

$$|I_{A\circ C}(x)| = |q_A(x)|.|q_C(x)| \le |q_B(x)|.|q_C(x)| = |I_{B\circ C}(x)|, \text{ for all } x \in X,$$
$$|F_{A\cap C}(x)| = |r_A(x)|.|r_C(x)| \le |r_B(x)|.|r_C(x)| = |F_{B\circ C}(x)|, \text{ for all } x \in X.$$

Likewise

$$|\mu_{A\circ C}(x)| = 2\pi\left(\frac{\mu_{A(x)}}{2\pi}.\frac{\mu_{C(x)}}{2\pi}\right) \le 2\pi\left(\frac{\mu_{B(x)}}{2\pi}.\frac{\mu_{C(x)}}{2\pi}\right) = |\mu_{B\circ C}(x)|, \text{ for all } x \in X,$$

$$|\nu_{A\circ C}(x)| = 2\pi\left(\frac{\nu_{A(x)}}{2\pi}.\frac{\nu_{C(x)}}{2\pi}\right) \le 2\pi\left(\frac{\nu_{B(x)}}{2\pi}.\frac{\nu_{C(x)}}{2\pi}\right) = |\nu_{B\circ C}(x)|, \text{ for all } x \in X,$$

$$|\omega_{A\circ C}(x)| = 2\pi\left(\frac{\omega_{A(x)}}{2\pi}.\frac{\omega_{C(x)}}{2\pi}\right) \le 2\pi\left(\frac{\omega_{B(x)}}{2\pi}.\frac{\omega_{C(x)}}{2\pi}\right) = |\omega_{B\circ C}(x)|, \text{ for all } x \in X.$$

(iv) Let

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$
$$C = \{x, T_C(x), I_C(x), F_C(x), x \in X\},$$

be complex neutrosophic sets on $X$ such that

$$T_A(x) = p_A(x)e^{i\mu_A(x)},\ T_B(x) = p_B(x)e^{i\mu_B(x)},\ T_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$I_A(x) = p_A(x)e^{i\mu_A(x)},\ I_B(x) = p_B(x)e^{i\mu_B(x)},\ I_C(x) = p_C(x)e^{i\mu_C(x)},$$
$$F_A(x) = p_A(x)e^{i\mu_A(x)},\ F_B(x) = p_B(x)e^{i\mu_B(x)},\ F_C(x) = p_C(x)e^{i\mu_C(x)}.$$

We have

$$T_{A\circ(B\circ C)}(x) = p_{A\circ(B\circ C)}(x).e^{i.\mu_{A\circ(B\circ C)}(x)}$$
$$= [p_A(x).p_{B\circ C}(x)].e^{i2\pi\left(\frac{\mu_{A(x)}}{2\pi}.\frac{\mu_{B\circ C(x)}}{2\pi}\right)}$$
$$= [p_A(x).(p_B(x).p_C(x))].e^{i2\pi\left(\frac{\mu_{A(x)}}{2\pi}.2\pi\frac{\left(\frac{\mu_{B(x)}}{2\pi}.\frac{\mu_{C(x)}}{2\pi}\right)}{2\pi}\right)}$$
$$= [(p_A(x).p_B(x)).p_C(x)].e^{i2\pi\left(2\pi\frac{\left(\frac{\mu_{A(x)}}{2\pi}.\frac{\mu_{B(x)}}{2\pi}\right)}{2\pi}.\frac{\mu_{C(x)}}{2\pi}\right)}$$
$$= [p_{A\circ B}(x).p_C(x)].e^{i2\pi\left(\frac{\mu_{A\circ B}(x)}{2\pi}.\frac{\mu_{C(x)}}{2\pi}\right)}$$
$$= p_{(A\circ B)\circ C}(x).e^{\mu_{(A\circ B)\circ C}} = T_{(A\circ B)\circ C}(x).$$

Following the same procedure we can prove for indeterminacy and falsehood functions.

Corollary *Let* $C_\alpha \in X, \alpha \in I$ *and*

$$T_{C_\alpha}(x) = p_{C_\alpha}(x)e^{i\mu_{C_\alpha}(x)}, I_{C_\alpha}(x) = q_{C_\alpha}e^{iv_{C_\alpha}(x)}, F_{C_\alpha}(x) = r_{C_\alpha}e^{i\omega_{C_\alpha}(x)},$$

*Then* $\underset{\alpha\in I}{\Pi} C_\alpha = C_1(x) \circ C_2(x) \circ \ldots \circ C_\alpha(x) \in X.$ *Thus*

$$T \underset{\alpha\in I}{\Pi} C_\alpha(x) = p_{C_1(x)} \cdot p_{C_2(x)} \cdots p_{C_\alpha(x)} e^{i2\pi\left(\frac{\mu_{C1}(x)}{2\pi} \cdot \frac{\mu_{C2}(x)}{2\pi} \cdots \frac{\mu_{C\alpha}(x)}{2\pi}\right)},$$

$$I \underset{\alpha\in I}{\Pi} C_\alpha(x) = q_{C_1(x)} \cdot q_{C_2(x)} \cdots q_{C_\alpha(x)} e^{i2\pi\left(\frac{v_{C1}(x)}{2\pi} \cdot \frac{v_{C2}(x)}{2\pi} \cdots \frac{v_{C\alpha}(x)}{2\pi}\right)},$$

$$F \underset{\alpha\in I}{\Pi} C_\alpha(x) = r_{C_1(x)} \cdot r_{C_2(x)} \cdots r_{C_\alpha(x)} e^{i2\pi\left(\frac{\omega_{C1}(x)}{2\pi} \cdot \frac{\omega_{C2}(x)}{2\pi} \cdots \frac{\omega_{C\alpha}(x)}{2\pi}\right)}.$$

Proof It is trivial.

Definition *Let* $An$ *be* $N$ *CNSs on* $X$ $(n = 1, 2, \ldots, N)$ *and*

$$T_{A_n}(x) = p_A(x)e^{i\mu_{A_n}(x)}, I_{A_n}(x) = q_{A_n}(x)e^{iv_{A_n}(x)}, F_{A_n}(x) = r_{A_n}(x)e^{i\mu_{\omega A_n}(x)},$$

*The Cartesian product of* $An$ *, denoted as* $A_1 \times A_2 \times \ldots \times A_N$, *defined as*

$$T_{A_1\times A_2\times\ldots\times A_N}(x) = p_{A_1\times A_2\times\ldots\times A_N}(x)e^{i\mu_{A_1\times A_2\times\ldots\times A_N}(x)}$$

$$= min\left(p_{A_1}(x_1), p_{A_2}(x_2), \ldots, p_{A_N}(x_N)\right)e^{i\, min\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_N}(x_N)\right)}.$$

*Similarly*

$$I_{A_1\times A_2\times\ldots\times A_N}(x) = q_{A_1\times A_2\times\ldots\times A_N}(x)e^{iv_{A_1\times A_2\times\ldots\times A_N}(x)}$$

$$= max\left(q_{A_1}(x_1), q_{A_2}(x_2), \ldots, q_{A_N}(x_N)\right)e^{i\, max\left(v_{A_1}(x_1), v_{A_2}(x_2), \ldots, v_{A_N}(x_N)\right)},$$

*and*

$$F_{A_1\times A_2\times\ldots\times A_N}(x) = r_{A_1\times A_2\times\ldots\times A_N}(x)e^{i\omega_{A_1\times A_2\times\ldots\times A_N}(x)}$$

$$= max\left(r_{A_1}(x_1), r_{A_2}(x_2), \ldots, r_{A_N}(x_N)\right)e^{i\, max\left(\omega_{A_1}(x_1), \omega_{A_2}(x_2), \ldots, \omega_{A_N}(x_N)\right)},$$

*where* $x = (x_1, x_2, \ldots, x_N) \in \underbrace{X \times X \times \ldots \times X}_{N}.$

**Delta-equalities of Complex Neutrosophic Sets**

Definition *The distance of CNS is a function* $d = CN \times CN \rightarrow [0,1]$ *such that for any* $A, B, C \in CN$

$(i)\, d(A, B) \geq 0$ if and only if $A = B$,

$(ii)\, d(A, B) = d(B, A)$,

$(iii)\, d(A, B) \leq d(A, C) + d(C, B)$,

*where* $d(A, B)$ *is defined as*

$$d(A,B) = max\begin{pmatrix} max\left(\sup_{x\in X}|p_A(x) - p_B(x)|, \sup_{x\in X}|q_A(x) - q_B(x)|, \sup_{x\in X}|r_A(x) - r_B(x)|\right), \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}|\mu_A(x) - \mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x) - \nu_B(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_B(x)| \end{pmatrix}. \end{pmatrix}$$

Definition *Let*

$$A = \{x, T_A(x), I_A(x), F_A(x), x \in X\},$$
$$B = \{x, T_B(x), I_B(x), F_B(x), x \in X\},$$

*be the complex neutrosophic sets on* $X$ *such that*

$$T_A(x) = p_A(x)e^{i\mu_A(x)}, I_A(x) = q_A(x)e^{i\nu_A(x)}, F_A(x) = r_A(x)e^{i\mu_{\omega_A}(x)},$$
$$T_B(x) = p_B(x)e^{i\mu_B(x)}, I_B(x) = q_B(x)e^{i\nu_B(x)}, F_B(x) = r_B(x)e^{i\omega_B(x)},$$

*be complex valued truth, indeterminate and falsehood functions respectively. Then* $A$ *and* $B$ *are said to be* $\delta - equal$ *if and only if* $d(A,B) \leq 1$ *where* $0 \leq \delta \leq 1$, *which is denoted by* $A = (\delta)B$.

Lemma *Let*

$$\delta_1 * \delta_2 = max(0, \delta_1 + \delta_2 - 1); 0 \leq \delta_1, \delta_2 \leq 1,$$

*then the following results hold,*

(1) $0 * \delta_1 = 0$; *for all* $\delta_1 \in [0,1]$,

(2) $1 * \delta_1 = \delta_1$; *for all* $\delta_1 \in [0,1]$,

(3) $0 \leq \delta_1 * \delta_2 \leq 1$; *for all* $\delta_1, \delta_2 \in [0,1]$,

(4) $\delta_1 \leq \delta_1' \Rightarrow \delta_1 * \delta_2 \leq \delta_1' * \delta_2$; *for all* $\delta_1, \delta_1', \delta_2 \in [0,1]$,

(5) $\delta_1 * \delta_2 = \delta_2 * \delta_1$; *for all* $\delta_1, \delta_2 \in [0,1]$,

(6) $(\delta_1 * \delta_2) * \delta_3 = \delta_2 * (\delta_1 * \delta_3)$; *for all* $\delta_1, \delta_2, \delta_3 \in [0,1]$.

Proof It is *trivial.*

Lemma *For the complex valued bounded function* $f$, $g$ *on a set* $X$. *We have*

$$\left|\sup_{x\in U} f(x) - \sup_{x\in U} g(x)\right| \leq \sup_{x\in U}|f(x) - g(x)|,$$

$$\left|\inf_{x\in U} f(x) - \inf_{x\in U} g(x)\right| \leq \inf_{x\in U}|f(x) - g(x)|.$$

Theorem *If* $A = (\delta_1)A^{'}$ *and* $B = (\delta_2)B^{'}$ *, then* $A \cup B = (min(\delta_1, \delta_2))A^{'} \cup B^{'}$ .

Proof

$$d(A, A^{'}) = max\begin{pmatrix} max\left(\sup_{x\in X}|p_A(x) - p_{A^{'}}(x)|, \sup_{x\in X}|q_A(x) - q_{A^{'}}(x)|, \sup_{x\in X}|r_A(x) - r_{A^{'}}(x)|\right), \\ max\begin{pmatrix}\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x) - \mu_{A^{'}}(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x) - \nu_{A^{'}}(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_{A^{'}}(x)|\end{pmatrix}\end{pmatrix}$$

$$\leq 1 - \delta_1$$

$$d(B, B^{'}) = max\begin{pmatrix} max\left(\sup_{x\in X}|p_B(x) - p_{B^{'}}(x)|, \sup_{x\in X}|q_B(x) - q_{B^{'}}(x)|, \sup_{x\in X}|r_B(x) - r_{B^{'}}(x)|\right), \\ max\begin{pmatrix}\frac{1}{2\pi}\sup_{x\in X}|\mu_B(x) - \mu_{B^{'}}(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x) - \nu_{B^{'}}(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}|\omega_B(x) - \omega_{B^{'}}(x)|\end{pmatrix}\end{pmatrix}$$

$$\leq 1 - \delta_2.$$

Therefore

$$\sup_{x\in X}|p_A(x) - p_{A^{'}}(x)| \leq 1 - \delta_1, \frac{1}{2\pi}\sup_{x\in X}|\mu_A(x) - \mu_{A^{'}}(x)| \leq 1 - \delta_1,$$

$$\sup_{x\in X}|q_A(x) - q_{A^{'}}(x)| \leq 1 - \delta_1, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x) - \nu_{A^{'}}(x)| \leq 1 - \delta_1,$$

$$\sup_{x\in X}|r_B(x) - r_{B^{'}}(x)| \leq 1 - \delta_1, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_{A^{'}}(x)| \leq 1 - \delta_1,$$

$$\sup_{x\in X}|p_B(x) - p_{B^{'}}(x)| \leq 1 - \delta_2, \frac{1}{2\pi}\sup_{x\in X}|\mu_B(x) - \mu_{B^{'}}(x)| \leq 1 - \delta_2,$$

$$\sup_{x\in X}|q_B(x) - q_{B^{'}}(x)| \leq 1 - \delta_2, \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x) - \nu_{B^{'}}(x)| \leq 1 - \delta_2,$$

$$\sup_{x\in X}|r_B(x) - r_{B^{'}}(x)| \leq 1 - \delta_2, \frac{1}{2\pi}\sup_{x\in X}|\omega_B(x) - \omega_{B^{'}}(x)| \leq 1 - \delta_2.$$

For membership function

$$\sup_{x\in X}|p_{A\cup B}(x) - p_{A^{'}\cup B^{'}}(x)| = \sup_{x\in X}\left|max(p_A(x), p_B(x)) - max\left(p_{A^{'}}(x), p_{B^{'}}(x)\right)\right|$$

$$= \begin{cases} \sup_{x\in X}|p_A(x) - p_{A'}(x)|, \text{ if } p_A(x) \geq p_B(x) \text{ and } p_{A'}(x) \geq p_{B'}(x) \\ \sup_{x\in X}|p_A(x) - p_{B'}(x)|, \text{ if } p_A(x) \geq p_B(x) \text{ and } p_{B'}(x) \geq p_{A'}(x) \\ \sup_{x\in X}|p_B(x) - p_{A'}(x)|, \text{ if } p_B(x) > p_A(x) \text{ and } p_{A'}(x) \geq p_{B'}x, \\ \sup_{x\in X}|p_B(x) - p_{B'}(x)|, \text{ if } p_B(x) > p_A(x) \text{ and } p_{B'}(x) \geq p_{A'}(x) \end{cases}$$

$$\leq \begin{cases} 1-\delta_1, \text{ if } p_A(x) \geq p_B(x) \text{ and } p_{A'}(x) \geq p_{B'}(x) \\ \sup_{x\in X}|p_A(x) - p_{B'}(x)|, \text{ if } p_A(x) \geq p_B(x) \text{ and } p_{B'}(x) \geq p_{A'}(x) \\ \sup_{x\in X}|p_B(x) - p_{A'}(x)|, \text{ if } p_B(x) > p_A(x) \text{ and } p_{A'}(x) \geq p_{B'}(x) \\ 1-\delta_2, \text{ if } p_B(x) > p_A(x) \text{ and } p_{B'}(x) \geq p_{A'}(x) \end{cases}$$

(i) Consider the case $p_A(x) \geq p_B(x)$ and $p_{A'}(x) > p_{B'}(x)$

(1) $p_A(x) \geq p_{B'}(x) \geq 0$, then $p_A(x) \geq p_{A'}(x) \geq p_A(x) \geq p_{B'}(x) \geq 0$ from $p_{B'}(x) \geq p_{A'}(x)$, therefore

$$\sup_{x\in X}|p_A(x) - p_{B'}(x)| = \sup_{x\in X}\left(p_A(x) - p_{B'}(x)\right) \leq \sup_{x\in X}\left(p_A(x) - p_{A'}(x)\right) \leq$$

$$\sup_{x\in X}|p_A(x) - p_{B'}(x)| \leq 1-\delta_1.$$

(2) $p_A(x) \geq p_{B'}(x) \leq 0$ then $p_{B'}(x) \geq p_B(x) \geq p_{B'}(x) \geq p_A(x) \geq 0$ from $p_B(x) \leq p_A(x)$, therefore

$$\sup_{x\in X}|p_A(x) - p_{B'}(x)| = \sup_{x\in X}\left(p_{B'}(x) - p_A(x)\right) \leq \sup_{x\in X}\left(p_{B'}(x) - p_B(x)\right) \leq$$

$$\sup_{x\in X}|p_{B'}(x) - p_B(x)| \leq 1-\delta_2.$$

Thus if
$p_A(x) \geq p_B(x) \text{ and } p_{B'}(x) > p_{A'}(x).$

We have

$$\sup_{x\in X}|p_A(x) - p_{B'}(x)| \leq max(1-\delta_1, 1-\delta_2) = 1 - min(\delta_1, \delta_2).$$

(ii) Similarly for the case

$$\sup_{x\in X}|p_B(x) - p_{A'}(x)| \leq max(1-\delta_1, 1-\delta_2) = 1 - min(\delta_1, \delta_2).$$

Now if $p_B(x) > p_A(x)$ and $p_{A'}(x) \geq p_{B'}(x)$, thus

$$\sup_{x\in X}\left|p_{A\cup B}(x) - p_{A'\cup B'}(x)\right| \leq max(1-\delta_1, 1-\delta_2) = 1 - min(\delta_1, \delta_2).$$

On same steps we can prove for indeterminacy function and falsehood function, likewise

$$\frac{1}{2\pi}\sup_{x\in X}\left|\mu_{A\cup B}(x) - \mu_{A'\cup B'}(x)\right| = \frac{1}{2\pi}\sup_{x\in X}\left|max\left(\mu_A(x), \mu_B(x)\right) - max\left(\mu_{A'}(x), \mu_{B'}(x)\right)\right|$$

$$= \begin{cases} \frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{A'}(x)\right|, \text{ if } \mu_A(x) \geq \mu_B(x) \text{ and } \mu_{A'}(x) \geq \mu_{B'}(x) \\ \frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{B'}(x)\right|, \text{ if } \mu_A(x) \geq \mu_B(x) \text{ and } \mu_{B'}(x) \geq \mu_{A'}(x) \\ \frac{1}{2\pi}\sup_{x\in X}\left|\mu_B(x) - \mu_{A'}(x)\right|, \text{ if } \mu_B(x) > \mu_A(x) \text{ and } \mu_{A'}(x) \geq \mu_{B'}(x) \\ \frac{1}{2\pi}\sup_{x\in X}\left|\mu_B(x) - \mu_{B'}(x)\right|, \text{ if } \mu_B(x) > \mu_A(x) \text{ and } \mu_{B'}(x) \geq \mu_{A'}(x) \end{cases}$$

$$\leq \begin{cases} 1-\delta_1, \text{ if } \mu_A(x) \geq \mu_B(x) \text{ and } \mu_{A'}(x) \geq \mu_{B'}(x) \\ \frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{B'}(x)\right|, \text{ if } \mu_A(x) \geq \mu_B(x) \text{ and } \mu_{B'}(x) \geq \mu_{A'}(x) \\ \frac{1}{2\pi}\sup_{x\in X}\left|\mu_B(x) - \mu_{A'}(x)\right|, \text{ if } \mu_B(x) > \mu_A(x) \text{ and } \mu_{A'}(x) \geq \mu_{B'}(x) \\ 1-\delta_2, \text{ if } \mu_B(x) > \mu_A(x) \text{ and } \mu_{B'}(x) \geq \mu_{A'}(x) \end{cases}$$

(i) Consider the case $\mu_A(x) \geq \mu_B(x)$ and $\mu_{A'}(x) > \mu_{B'}(x)$

(1) $\mu_A(x) \geq \mu_{B'}(x) \geq 0$ ,then $\mu_A(x) \geq \mu_{A'}(x) \geq \mu_A(x) \geq \mu_{B'}(x) \geq 0$ from $\mu_{B'}(x) > \mu_{A'}(x)$ ,

therefore

$$\frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{B'}(x)\right| = \frac{1}{2\pi}\sup_{x\in X}\left(\mu_A(x) - \mu_{B'}(x)\right) \leq \frac{1}{2\pi}\sup_{x\in X}\left(\mu_A(x) - \mu_{A'}(x)\right)$$
$$\leq \frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{B'}(x)\right| \leq 1 - \delta_1.$$

(2) $\mu_A(x) \geq \mu_{B'}(x) \leq 0$ then $\mu_{B'}(x) \geq \mu_B(x) \geq \mu_{B'}(x) \geq \mu_A(x) \geq 0$ from $\mu_B(x) \leq \mu_A(x)$ ,

therefore

$$\frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x) - \mu_{B'}(x)\right| = \frac{1}{2\pi}\sup_{x\in X}\left(\mu_{B'}(x) - \mu_A(x)\right) \leq \frac{1}{2\pi}\sup_{x\in X}\left(\mu_{B'}(x) - \mu_B(x)\right)$$

$$\leq \frac{1}{2\pi} \sup_{x \in X} \left| \mu_{B'}(x) - \mu_B(x) \right| \leq 1 - \delta_2.$$

Thus if

$\mu_A(x) \geq \mu_B(x)$ and $\mu_{B'}(x) > \mu_{A'}(x)$.

We have

$$\frac{1}{2\pi} \sup_{x \in X} \left| \mu_A(x) - \mu_{B'}(x) \right| \leq max(1 - \delta_1, 1 - \delta_2) = 1 - min(\delta_1, \delta_2).$$

Similarly for the case

$$\frac{1}{2\pi} \sup_{x \in X} \left| \mu_B(x) - \mu_{A'}(x) \right| \leq max(1 - \delta_1, 1 - \delta_2) = 1 - min(\delta_1, \delta_2).$$

Now if $\mu_B(x) > \mu_A(x)$ and $\mu_{A'}(x) \geq \mu_{B'}(x)$ thus

$$\frac{1}{2\pi} \sup_{x \in X} \left| \mu_{A \cup B}(x) - \mu_{A' \cup B'}(x) \right| \leq max(1 - \delta_1, 1 - \delta_2) = 1 - min(\delta_1, \delta_2).$$

On same steps we can prove for indeterminacy function and falsehood function, likewise

$$d(A \cup B, A' \cup B')$$

$$= max \left( \begin{array}{c} max \left( \begin{array}{c} \sup_{x \in X} \left| p_{A \cup B}(x) - p_{A' \cup B'} \right|, \sup_{x \in X} \left| q_{A \cup B}(x) - q_{A' \cup B'}(x) \right|, \\ \sup_{x \in X} \left| r_{A \cup B}(x) - r_{A' \cup B'}(x) \right| \end{array} \right), \\ max \left( \begin{array}{c} \frac{1}{2\pi} \sup_{x \in X} \left| \mu_{A \cup B}(x) - \mu_{A' \cup B'}(x) \right|, \frac{1}{2\pi} \sup_{x \in X} \left| \nu_{A \cup B}(x) - \nu_{A' \cup B'}(x) \right|, \\ \frac{1}{2\pi} \sup_{x \in X} \left| \omega_{A \cup B}(x) - \omega_{A' \cup B'}(x) \right| \end{array} \right) \end{array} \right)$$

$$\leq max(1 - \delta_1, 1 - \delta_2) = 1 - min(\delta_1, \delta_2).$$

Thus $A \cup B = (min(\delta_1, \delta_2)) A' \cup B'$.

Corollary

*If* $A_\alpha = (\delta_\alpha) B_\alpha, \alpha \in I$, *then* $\bigcup_{\alpha \in I} A_\alpha = \left( \inf_{\alpha \in I} (\delta_\alpha) \right) \bigcup_{\alpha \in I} B_\alpha$.

Proof Using lemma mod sup less or equal inf mod, we get

$$d\left( \bigcup_{\alpha \in I} A_\alpha, \bigcup_{\alpha \in I} B_\alpha \right)$$

$$= max\begin{pmatrix} max\begin{pmatrix} \sup_{x\in X}\left|p \bigcup_{\alpha\in I} A_\alpha(x) - p \bigcup_{\alpha\in I} B_\alpha\right|, \sup_{x\in X}\left|q \bigcup_{\alpha\in I} A_\alpha(x) - q \bigcup_{\alpha\in I} B_\alpha(x)\right|, \\ \sup_{x\in X}\left|r \bigcup_{\alpha\in I} A_\alpha(x) - r \bigcup_{\alpha\in I} B_\alpha(x)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|\mu \bigcup_{\alpha\in I} A_\alpha(x) - \mu \bigcup_{\alpha\in I} B_\alpha\right|, \frac{1}{2\pi}\sup_{x\in X}\left|\nu \bigcup_{\alpha\in I} A_\alpha(x) - \nu \bigcup_{\alpha\in I} B_\alpha(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\omega \bigcup_{\alpha\in I} A_\alpha(x) - \omega \bigcup_{\alpha\in I} B_\alpha(x)\right| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} \left( max \begin{matrix} \sup_{x\in X}\left|p\sup_{\alpha\in I} A_\alpha(x) - p\sup_{\alpha\in I} B_\alpha\right|, \sup_{x\in X}\left|q\sup_{\alpha\in I} A_\alpha(x) - q\sup_{\alpha\in I} B_\alpha(x)\right|, \\ \sup_{x\in X}\left|r\sup_{\alpha\in I} A_\alpha(x) - r\sup_{\alpha\in I} B_\alpha(x)\right| \end{matrix}\right), \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|\mu\sup_{\alpha\in I} A_\alpha(x) - \mu\sup_{\alpha\in I} B_\alpha\right|, \frac{1}{2\pi}\sup_{x\in X}\left|\nu\sup_{\alpha\in I} A_\alpha(x) - \nu\sup_{\alpha\in I} B_\alpha(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\omega\sup_{\alpha\in I} A_\alpha(x) - \omega\sup_{\alpha\in I} B_\alpha(x)\right| \end{pmatrix} \end{pmatrix}$$

$$\leq max\begin{pmatrix} \left( max \begin{matrix} \sup_{x\in X}\sup_{\alpha\in I}|pA_\alpha(x) - pB_\alpha(x)|, \sup_{x\in X}\inf_{\alpha\in I}|qA_\alpha(x) - qB_\alpha(x)|, \\ \sup_{x\in X}\inf_{\alpha\in I}|rA_\alpha(x) - rB_\alpha(x)| \end{matrix}\right), \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\sup_{\alpha\in I}|\mu A_\alpha(x) - \mu B_\alpha|, \frac{1}{2\pi}\sup_{x\in X}\sup_{\alpha\in I}|\nu A_\alpha(x) - \nu B_\alpha(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}\sup_{\alpha\in I}|\omega A_\alpha(x) - \omega B_\alpha(x)| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} \left( max \begin{matrix} \sup_{\alpha\in I}\sup_{x\in X}|pA_\alpha(x) - pB_\alpha(x)|, \inf_{\alpha\in I}\sup_{x\in X}|qA_\alpha(x) - qB_\alpha(x)|, \\ \inf_{\alpha\in I}\sup_{x\in X}|rA_\alpha(x) - rB_\alpha(x)| \end{matrix}\right), \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{\alpha\in I}\sup_{x\in X}|\mu A_\alpha(x) - \mu B_\alpha|, \frac{1}{2\pi}\sup_{\alpha\in I}\sup_{x\in X}|\nu A_\alpha(x) - \nu B_\alpha(x)|, \\ \frac{1}{2\pi}\sup_{\alpha\in I}\sup_{x\in X}|\omega A_\alpha(x) - \omega B_\alpha(x)| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} max\left(\sup_{\alpha\in I}(1-\delta_\alpha), \inf_{\alpha\in I}(1-\delta_\alpha), \inf(1-\delta_\alpha)\right), \\ max\left(\sup_{\alpha\in I}(1-\delta_\alpha), \sup_{\alpha\in I}(1-\delta_\alpha), \sup(1-\delta_\alpha)\right) \end{pmatrix}$$

$$= max\left(\sup_{\alpha\in I}(1-\delta_\alpha), \sup_{\alpha\in I}(1-\delta_\alpha)\right)$$

$$= \sup_{\alpha\in I}(1-\delta_\alpha) = 1 - \inf_{\alpha\in I}\delta_\alpha.$$

Theorem *If* $A = (\delta)B$, *then* $\overline{A} = (\delta)\overline{B}$.

Proof As

$$d(\overline{A},\overline{B})$$

$$= max\begin{pmatrix} max\left(\sup_{x\in X}\left|p_{\overline{A}}(x)-p_{\overline{B}}(x)\right|,\sup_{x\in X}\left|q_{\overline{A}}(x)-q_{\overline{B}}(x)\right|,\sup_{x\in X}\left|r_{\overline{A}}(x)-r_{\overline{B}}(x)\right|\right), \\ max\left(\frac{1}{2\pi}\sup_{x\in X}\left|\mu_{\overline{A}}(x)-\mu_{\overline{B}}(x)\right|,\frac{1}{2\pi}\sup_{x\in X}\left|\nu_{\overline{A}}(x)-\nu_{\overline{B}}(x)\right|,\frac{1}{2\pi}\sup_{x\in X}\left|\omega_{\overline{A}}(x)-\omega_{\overline{B}}(x)\right|\right)\end{pmatrix}$$

$$= max\begin{pmatrix} max\begin{pmatrix}\sup_{x\in X}\left|(1-p_A(x))-(1-p_B(x))\right|,\sup_{x\in X}\left|(1-q_A(x))-(1-q_B(x))\right|, \\ \sup_{x\in X}\left|(1-r_A(x))-(1-r_B(x))\right|\end{pmatrix}, \\ max\begin{pmatrix}\frac{1}{2\pi}\sup_{x\in X}\left|(2\pi-\mu_A(x))-(2\pi-\mu_B(x))\right|,\frac{1}{2\pi}\sup_{x\in X}\left|(2\pi-\nu_A(x))-(2\pi-\nu_B(x))\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|(2\pi-\omega_A(x))-(2\pi-\omega_B(x))\right|\end{pmatrix}\end{pmatrix}$$

$$= max\begin{pmatrix} max\left(\sup_{x\in X}\left|p_A(x)-p_B(x)\right|,\sup_{x\in X}\left|q_A(x)-q_B(x)\right|,\sup_{x\in X}\left|r_A(x)-r_B(x)\right|\right), \\ max\left(\frac{1}{2\pi}\sup_{x\in X}\left|\mu_A(x)-\mu_B(x)\right|,\frac{1}{2\pi}\sup_{x\in X}\left|\nu_A(x)-\nu_B(x)\right|,\frac{1}{2\pi}\sup_{x\in X}\left|\omega_A(x)-\omega_B(x)\right|\right)\end{pmatrix}$$

$$= d(A,B) \leq 1-\delta.$$

Theorem *If* $A = (\delta_1)A^{'}$ *and* $B = (\delta_2)B^{'}$ *, then* $A \cap B = (min(\delta_1,\delta_2))A^{'} \cap B^{'}$.

Proof By use of previous theorem a complement equals del b complement, we have

$\overline{A} = (\delta_1)\overline{A}^{'}$, $\bar{B} = (\delta_1)\bar{B}^{'}$ and

$\overline{A} \cup \overline{B} = min(\delta_1,\delta_2)\overline{A}^{'} \cup \bar{B}^{'}$.

Thus

$$A \cap B = \overline{\overline{A} \cup \bar{B}}$$

$$= \left(min(\delta_1,\delta_2)\right)\overline{\overline{A}^{'} \cup \bar{B}^{'}}$$

$$= \left(min(\delta_1,\delta_2)\right)A^{'} \cap B^{'}.$$

Corollary *If* $A_\alpha = (\delta_\alpha)B_\alpha, \alpha \in I,$ *where I is an index set, then* $\underset{\alpha\in I}{\cap} A_\alpha = \left(\underset{\alpha\in I}{inf}(\delta_\alpha)\right)\underset{\alpha\in I}{\cap} B_\alpha.$

Proof From above corollary union alpha equals inf union beta, we have

$$d\left(\underset{\alpha\in I}{\cup} A_\alpha, \underset{\alpha\in I}{\cup} B_\alpha\right) = 1 - \underset{\alpha\in I}{inf}\,\delta_\alpha, \text{ and}$$

$$\overline{A_\alpha} = (\delta_\alpha)\overline{B_\alpha}, \text{ for all } \alpha \in I, \text{ and}$$

$$\underset{\alpha\in I}{\cup}\overline{A_\alpha} = \left(\underset{\alpha\in I}{inf}(\delta_\alpha)\right)\underset{\alpha\in I}{\cup}\overline{B_\alpha}.$$

Thus

$$\underset{\alpha\in I}{\cap} A_\alpha = \overline{\underset{\alpha\in I}{\cup}\overline{A_\alpha}} = \left(\underset{\alpha\in I}{inf}(\delta_\alpha)\right)\overline{\underset{\alpha\in I}{\cup}\bar{B}_\alpha}$$

$$= \left(\underset{\alpha\in I}{inf}(\delta_\alpha)\right)\underset{\alpha\in I}{\cap} B.$$

Corollary *If* $A_{\alpha\beta} = (\delta_{\alpha\beta})B_{\alpha\beta}, \alpha \in I_1, \beta \in I_2,$ *where* $I_1$ *and* $I_2$ *are index sets, then*

$$\underset{\alpha\in I_1}{\cup}\underset{\alpha\in I_2}{\cap} A_{\alpha\beta} = \left(\underset{\alpha\in I_1}{inf}\underset{\alpha\in I_2}{inf}(\delta_{\alpha\beta})\right)\underset{\alpha\in I_1}{\cup}\underset{\alpha\in I_2}{\cap} B_{\alpha\beta},$$

$$\underset{\alpha\in I_1}{\cap}\underset{\alpha\in I_2}{\cup} A_{\alpha\beta} = \left(\underset{\alpha\in I_1}{inf}\underset{\alpha\in I_2}{inf}(\delta_{\alpha\beta})\right)\underset{\alpha\in I_1}{\cap}\underset{\alpha\in I_2}{\cup} B_{\alpha\beta}.$$

Proof By using corollary union alpha equals inf union beta and intersection alpha equals inf intersection beta we can easily prove it.

Theorem *If* $A = (\delta_1)A'$ *and* $B = (\delta_2)B'$ *, then* $A \circ B = (\delta_1 * \delta_2)A' \circ B'.$

Proof As $A = (\delta_1)A'$ and $B = (\delta_2)B'$, so we have

$$d(A, A') = max\begin{pmatrix} max\left(\underset{x\in X}{sup}|p_A(x) - p_{A'}(x)|, \underset{x\in X}{sup}|q_A(x) - q_{A'}(x)|, \underset{x\in X}{sup}|r_A(x) - r_{A'}(x)|\right), \\ max\begin{pmatrix} \frac{1}{2\pi}\underset{x\in X}{sup}|\mu_A(x) - \mu_{A'}(x)|, \frac{1}{2\pi}\underset{x\in X}{sup}|\nu_A(x) - \nu_{A'}(x)|, \\ \frac{1}{2\pi}\underset{x\in X}{sup}|\omega_A(x) - \omega_{A'}(x)| \end{pmatrix} \end{pmatrix}$$

$$\leq 1-\delta_1$$

$$d(B,B^{'}) = max\left(\begin{array}{c} max\left(\sup_{x\in X}|p_B(x)-p_{B^{'}}(x)|, \sup_{x\in X}|q_B(x)-q_{B^{'}}(x)|, \sup_{x\in X}|r_B(x)-r_{B^{'}}(x)|\right), \\ max\left(\begin{array}{c} \frac{1}{2\pi}\sup_{x\in X}|\mu_B(x)-\mu_{B^{'}}(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x)-\nu_{B^{'}}(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}|\omega_B(x)-\omega_{B^{'}}(x)| \end{array}\right) \end{array}\right)$$

$$\leq 1-\delta_2.$$

Therefore

$$\sup_{x\in X}|p_A(x)-p_{A^{'}}(x)| \leq 1-\delta_1, \frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_{A^{'}}(x)| \leq 1-\delta_1,$$

$$\sup_{x\in X}|q_A(x)-q_{A^{'}}(x)| \leq 1-\delta_1, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_{A^{'}}(x)| \leq 1-\delta_1,$$

$$\sup_{x\in X}|r_B(x)-r_{B^{'}}(x)| \leq 1-\delta_1, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_{A^{'}}(x)| \leq 1-\delta_1,$$

$$\sup_{x\in X}|p_B(x)-p_{B^{'}}(x)| \leq 1-\delta_2, \frac{1}{2\pi}\sup_{x\in X}|\mu_B(x)-\mu_{B^{'}}(x)| \leq 1-\delta_2,$$

$$\sup_{x\in X}|q_B(x)-q_{B^{'}}(x)| \leq 1-\delta_2, \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x)-\nu_{B^{'}}(x)| \leq 1-\delta_2,$$

$$\sup_{x\in X}|r_B(x)-r_{B^{'}}(x)| \leq 1-\delta_2, \frac{1}{2\pi}\sup_{x\in X}|\omega_B(x)-\omega_{B^{'}}(x)| \leq 1-\delta_2.$$

We have,

$$d(A\circ B, A^{'}\circ B^{'})$$

$$= max\left(\begin{array}{c} max\left(\sup_{x\in X}|p_{A\circ B}(x)-p_{A^{'}\circ B^{'}}(x)|, \sup_{x\in X}|q_{A\circ B}(x)-q_{A^{'}\circ B^{'}}(x)|, \sup_{x\in X}|r_{A\circ B}(x)-r_{A^{'}\circ B^{'}}(x)|\right), \\ max\left(\begin{array}{c} \frac{1}{2\pi}\sup_{x\in X}|\mu_{A\circ B}(x)-\mu_{A^{'}\circ B^{'}}(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_{A\circ B}(x)-\nu_{A^{'}\circ B^{'}}(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}|\omega_{A\circ B}(x)-\omega_{A^{'}\circ B^{'}}(x)| \end{array}\right) \end{array}\right)$$

$$= max\begin{pmatrix} max\begin{pmatrix} \sup_{x\in X}\left|p_A.p_B(x) - p_{A'}.p_{B'}(x)\right|, \sup_{x\in X}\left|q_A.q_B(x) - q_{A'}.q_{B'}(x)\right|, \\ \sup_{x\in X}\left|r_A.r_B(x) - r_{A'}.r_{B'}(x)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|2\pi\left(\frac{\mu_A}{2\pi}.\frac{\mu_B}{2\pi}\right) - 2\pi\left(\frac{\mu_{A'}}{2\pi}.\frac{\mu_{B'}}{2\pi}\right)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|2\pi\left(\frac{\nu_A}{2\pi}.\frac{\nu_B}{2\pi}\right) - 2\pi\left(\frac{\nu_{A'}}{2\pi}.\frac{\nu_{B'}}{2\pi}\right)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|2\pi\left(\frac{\omega_A}{2\pi}.\frac{\omega_B}{2\pi}\right) - 2\pi\left(\frac{\omega_{A'}}{2\pi}.\frac{\omega_{B'}}{2\pi}\right)\right| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} max\begin{pmatrix} \sup_{x\in X}\left|p_A.p_B(x) - p_A.p_{B'}(x) + p_A.p_{B'}(x) - p_{A'}.p_{B'}(x)\right|, \\ \sup_{x\in X}\left|q_A.q_B(x) - q_A.q_{B'}(x) + q_A.q_{B'}(x) - q_{A'}.q_{B'}(x)\right|, \\ \sup_{x\in X}\left|r_A.r_B(x) - r_A.r_B(x) + r_A.r_B(x) - r_{A'}.r_{B'}(x)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\mu_A(x).\mu_B(x)}{2\pi} - \frac{\mu_A(x).\mu_{B'}(x)}{2\pi} + \frac{\mu_A(x).\mu_{B'}(x)}{2\pi} - \frac{\mu_{A'}(x).\mu_{B'}(x)}{2\pi}\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\nu_A(x).\nu_B(x)}{2\pi} - \frac{\nu_A(x).\nu_{B'}(x)}{2\pi} + \frac{\nu_A(x).\nu_{B'}(x)}{2\pi} - \frac{\nu_{A'}(x).\nu_{B'}(x)}{2\pi}\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\omega_A(x).\omega_B(x)}{2\pi} - \frac{\omega_A(x).\omega_{B'}(x)}{2\pi} + \frac{\omega_A(x).\omega_{B'}(x)}{2\pi} - \frac{\omega_{A'}(x).\omega_{B'}(x)}{2\pi}\right| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} max\begin{pmatrix} \sup_{x\in X}\left|p_A(x)\left(p_B(x) - p_{B'}(x)\right) + p_{B'}(x)\left(p_A(x) - p_{A'}(x)\right)\right|, \\ \sup_{x\in X}\left|q_A(x)\left(q_B(x) - q_{B'}(x)\right) + q_{B'}(x)\left(q_A(x) - q_{A'}(x)\right)\right|, \\ \sup_{x\in X}\left|r_A(x)\left(r_B(x) - r_B(x)\right) + r_B(x)\left(r_A(x) - r_{A'}(x)\right)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\mu_A(x)}{2\pi}\left(\mu_B(x) - \mu_{B'}(x)\right) + \frac{\mu_{B'}(x)}{2\pi}\left(\mu_A(x) - \mu_{A'}(x)\right)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\nu_A(x)}{2\pi}\left(\nu_B(x) - \nu_{B'}(x)\right) + \frac{\nu_{B'}(x)}{2\pi}\left(\nu_A(x) - \nu_{A'}(x)\right)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\frac{\omega_A(x)}{2\pi}\left(\omega_B(x) - \omega_{B'}(x)\right) + \frac{\omega_{B'}(x)}{2\pi}\left(\omega_A(x) - \omega_{A'}(x)\right)\right| \end{pmatrix} \end{pmatrix}$$

$$\leq max\left(\begin{array}{c} max\left(\begin{array}{c}\left|\sup_{x\in X}\left(p_B(x)-p_{B'}(x)\right)+\sup_{x\in X}\left(p_A(x)-p_{A'}(x)\right)\right|,\\ \left|\sup_{x\in X}\left(q_B(x)-q_{B'}(x)\right)+\sup_{x\in X}\left(q_A(x)-q_{A'}(x)\right)\right|,\\ \left|\sup_{x\in X}\left(r_B(x)-r_B(x)\right)+\sup_{x\in X}\left(r_A(x)-r_{A'}(x)\right)\right|\end{array}\right),\\ max\left(\begin{array}{c}\frac{1}{2\pi}\left|\sup_{x\in X}\left(\mu_B(x)-\mu_{B'}(x)\right)+\sup_{x\in X}\left(\mu_A(x)-\mu_{A'}(x)\right)\right|,\\ \frac{1}{2\pi}\left|\sup_{x\in X}\left(\nu_B(x)-\nu_{B'}(x)\right)+\sup_{x\in X}\left(\nu_A(x)-\nu_{A'}(x)\right)\right|,\\ \frac{1}{2\pi}\left|\sup_{x\in X}\left(\omega_B(x)-\omega_{B'}(x)\right)+\sup_{x\in X}\left(\omega_A(x)-\omega_{A'}(x)\right)\right|\end{array}\right)\end{array}\right)$$

$$\leq max\left(\begin{array}{c}max\left((1-\delta_2)+(1-\delta_1),(1-\delta_2)+(1-\delta_1),(1-\delta_2)+(1-\delta_1)\right),\\ max\left((1-\delta_2)+(1-\delta_1),(1-\delta_2)+(1-\delta_1),(1-\delta_2)+(1-\delta_1)\right)\end{array}\right)$$

$$= max\left((1-\delta_2)+(1-\delta_1),(1-\delta_2)+(1-\delta_1)\right)$$

$$= 1-(\delta_1+\delta_2-1).$$

As $d\left(A\circ B, A'\circ B'\right)\leq 1,$ so $d\left(A\circ B, A'\circ B'\right)\leq 1-\delta_1 * \delta_2$ .

Corollary $A_\alpha = (\delta_\alpha)B_\alpha,$ [illegible] $\alpha\in I$ *, where I is an index set, then*

$$A_1\circ A_2\circ\ldots\circ A_\alpha = (\delta_1 * \delta_2 * \ldots \delta_\alpha)B_1\circ B_2\circ\ldots\circ B_\alpha\ .$$

Proof It follows from theorem AB equal delta AB.

Theorem *If* $A_n = (\delta_n)A_n',\quad n = 1,2,\ldots,N$ *then* $A_1\times A_2\times\ldots\times A_N = \left(\inf_{1\leq n\leq N}\delta_n\right)A_1'\times A_2'\times\ldots\times A_N'.$

Proof As $A_n = (\delta_n)A_n', n = 1,2,\ldots,N$ . Therefore

$$d\left(A_n, A_n'\right) = max\left(\begin{array}{c} max\left(\begin{array}{c}\sup_{x\in X}\left|p_{A_n}(x)-p_{A_n'}(x)\right|,\sup_{x\in X}\left|q_{A_n}(x)-q_{A_n'}(x)\right|,\\ \sup_{x\in X}\left|r_{A_n}(x)-r_{A_n'}(x)\right|\end{array}\right),\\ max\left(\begin{array}{c}\frac{1}{2\pi}\sup_{x\in X}\left|\mu_{A_n}(x)-\mu_{A_n'}(x)\right|,\frac{1}{2\pi}\sup_{x\in X}\left|\nu_{A_n}(x)-\nu_{A_n'}(x)\right|,\\ \frac{1}{2\pi}\sup_{x\in X}\left|\omega_{A_n}(x)-\omega_{A_n'}(x)\right|\end{array}\right)\end{array}\right)$$

$$\leq 1-\delta_n, \text{ for any } n = 1,2,\ldots,N.$$

Therefore

$$\sup_{x\in X}\left|p_{A_n}(x)-p_{A_n'}(x)\right|\leq 1-\delta_n, \frac{1}{2\pi}\sup_{x\in X}\left|\mu_{A_n}(x)-\mu_{A_n'}(x)\right|\leq 1-\delta_n,$$

$$\sup_{x\in X}\left|q_{A_n}(x)-q_{A_n^{'}}(x)\right|\leq 1-\delta_n, \frac{1}{2\pi}\sup_{x\in X}\left|\nu_{A_n}(x)-\nu_{A_n^{'}}(x)\right|\leq 1-\delta_n,$$

$$\sup_{x\in X}\left|r_{A_n}(x)-r_{A_n^{'}}(x)\right|\leq 1-\delta_n, \frac{1}{2\pi}\sup_{x\in X}\left|\omega_{A_n}(x)-\omega_{A_n^{'}}(x)\right|\leq 1-\delta_n.$$

Then by lemma mod sup less or equal inf mod

$$d\left(A_1\times A_2\times\ldots\times A_N, A_1^{'}\times A_2^{'}\times\ldots\times A_N^{'}\right)$$

$$= max\begin{pmatrix} max\begin{pmatrix} \sup_{x\in X\times X\ldots X}\left|p_{A_1\times A_2\times\ldots\times A_N}(x)-p_{A_1^{'}\times A_2^{'}\times\ldots\times A_N^{'}}(x)\right|, \\ \sup_{x\in X\times X\ldots X}\left|q_{A_1\times A_2\times\ldots\times A_N}(x)-q_{A_1^{'}\times A_2^{'}\times\ldots\times A_N^{'}}(x)\right|, \\ \sup_{x\in X\times X\ldots X}\left|r_{A_1\times A_2\times\ldots\times A_N}(x)-r_{A_1^{'}\times A_2^{'}\times\ldots\times A_N^{'}}(x)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\mu_{A_1\times A_2\times\ldots\times A_N}(x)-\mu_{A_1^{'}\times A_2^{'}\times\ldots\times A_N^{'}}(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\nu_{A_1\times A_2\times\ldots\times A_N}(x)-\nu_{A_1^{'}\times A_2^{'}\times\ldots\times A_N^{'}}(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\omega_{A_1\times A_2\times\ldots\times A_N}(x)-\omega_{A_1^{'}\times A_2^{'}\times\ldots\times A_N^{'}}(x)\right| \end{pmatrix} \end{pmatrix}$$

$$= max\begin{pmatrix} max\begin{pmatrix} \sup_{x\in X\times X\ldots X}\left|\min_{1\leq n\leq N}p_{A_n}(x)-\min_{1\leq n\leq N}p_{A_n^{'}}(x)\right|, \\ \sup_{x\in X\times X\ldots X}\left|\max_{1\leq n\leq N}q_{A_n}(x)-\max_{1\leq n\leq N}q_{A_n^{'}}(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\max_{1\leq n\leq N}r_{A_n}(x)-\max_{1\leq n\leq N}r_{A_n^{'}}(x)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\min_{1\leq n\leq N}\mu_{A_n}(x)-\min_{1\leq n\leq N}\mu_{A_n^{'}}(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X\times X\ldots X}\left|\max_{1\leq n\leq N}\nu_{A_n}(x)-\max_{1\leq n\leq N}\nu_{A_n^{'}}(x)\right|, \\ \sup_{x\in X\times X\ldots X}\left|\max_{1\leq n\leq N}\omega_{A_n}(x)-\max_{1\leq n\leq N}\omega_{A_n^{'}}(x)\right| \end{pmatrix} \end{pmatrix}$$

$$\leq max\begin{pmatrix} max\begin{pmatrix} \sup_{1\leq n\leq N}\sup_{x_n\in X_n}\left|\min_{1\leq n\leq N}p_{A_n}(x)-\min_{1\leq n\leq N}p_{A_n^{'}}(x)\right|, \\ \sup_{1\leq n\leq N}\sup_{x_n\in X_n}\left|\max_{1\leq n\leq N}q_{A_n}(x)-\max_{1\leq n\leq N}q_{A_n^{'}}(x)\right|, \\ \sup_{1\leq n\leq N}\sup_{x_n\in X_n}\left|\max_{1\leq n\leq N}r_{A_n}(x)-\max_{1\leq n\leq N}r_{A_n^{'}}(x)\right| \end{pmatrix}, \\ max\begin{pmatrix} \frac{1}{2\pi}\sup_{1\leq n\leq N}\sup_{x_n\in X_n}\left|\min_{1\leq n\leq N}\mu_{A_n}(x)-\min_{1\leq n\leq N}\mu_{A_n^{'}}(x)\right|, \\ \frac{1}{2\pi}\sup_{1\leq n\leq N}\sup_{x_n\in X_n}\left|\max_{1\leq n\leq N}\nu_{A_n}(x)-\max_{1\leq n\leq N}\nu_{A_n^{'}}(x)\right|, \\ \frac{1}{2\pi}\sup_{1\leq n\leq N}\sup_{x_n\in X_n}\left|\max_{1\leq n\leq N}\omega_{A_n}(x)-\max_{1\leq n\leq N}\omega_{A_n^{'}}(x)\right| \end{pmatrix} \end{pmatrix}$$

$$\leq max\left(\sup_{1\leq n\leq N}(1-\delta_n), \sup_{1\leq n\leq N}(1-\delta_n)\right)=1-\inf_{1\leq n\leq N}\delta_n.$$

Conclusion *We worked on basic operations of CNSs. First we discussed some properties like union, intersection, complement, Cartesian product and investigated some results related to norms. Moreover we worked on the distance measures which are used to defined* $\delta-$ *equalities of CNSs. Some results such as union, intersection, complement, product on* $\delta-$ *equality also presented. We hope that the theory developed in this paper can be used in computing, data analysis, socio economic problems, medical diagnosis and other problems related to Decision Analysis.*

**Acknowledgments.** This work is financially supported by the Higher Education Commission of Pakistan (Grant No: 7750/Federal/ NRPU/R&D/HEC/ 2017). The third author (S.Z. Song) was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812).

**Conflict of Interest.** The authors declare that there is no conflict of interest regarding the publication of this article.

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Received: Nov 19, 2021. Accepted: Feb 3, 2022

Neutrosophic Sets and Systems, Vol. 48, 2022
University of New Mexico

# Introduction to the n-SuperHyperGraph - the most general form of graph today

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**Abstract:** We recall and improve our 2019 and 2020 concepts of *n-SuperHyperGraph, Plithogenic n-SuperHyperGraph, n-Power Set of a Set*, and we present some application from the real world. The n-SuperHyperGraph is the most general form of graph today and it is able to describe the complex reality we live in, by using n-SuperVertices (groups of groups of groups etc.) and n-SuperHyperEdges (edges connecting groups of groups of groups etc.).

**Keywords:** n-SuperHyperGraph (n-SHG), n-SHG-vertex, n-SHG-edge, Plithogenic (Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, etc.) n-SuperHyperGraph, n-Power Set of a Set, MultiEdge, Loop, Indeterminate Vertex, Null Vertex, Indeterminate Edge, Null Edge, Neutrosophic Directed Graph

---

## 1. Definition of the n-SuperHyperGraph

Let $V = \{v_1, v_2, \ldots, v_m\}$, for $1 \leq m \leq \infty$, be a set of vertices, that contains Single Vertices (the classical ones), Indeterminate Vertices (unclear, vague, partially known), and Null Vertices (totally unknown, empty).

Let $P(V)$ pe the power of set $V$, that includes the empty set $\phi$ too.

Then $P^n(V)$ be the $n$-power set of the set $V$, defined in a recurent way, i.e.:

P(V), $P^2(V) = P(P(V))$, $P^3(V) = P(P^2(V)) = P\left(P(P(V))\right)$, …,

$P^n(V) = P(P^{n-1}(V))$, for $1 \leq n \leq \infty$, where by definition $P^0(V) \overset{def}{=} V$ .

Then, the **n-SuperHyperGraph (n-SHG)** is an ordered pair:

$$\text{n-SHG} = (G_n, E_n),$$

where $G_n \subseteq P^n(V)$, and $E_n \subseteq P^n(V)$, for $1 \leq n \leq \infty$.

$G_n$ is the set of vertices, and $E_n$ is the set of edges.

The set of vertices $G_n$ contains the following types of vertices:

- Singles Vertices (the classical ones);
- Indeterminate Vertices (unclear, vagues, partially unkwnown);
- Null Vertices (totally unknown, empty);

and:

- SuperVertex (or SubsetVertex), i.e. two ore more (single, indeterminate, or null) vertices put together as a group (organization).
- n-SuperVertex that is a collection of many vertices such that at least one is a $(n-1)$-SuperVertex and all other $r$-SuperVertices into the collection, if any, have the order $r \leq n-1$.
- The set of edges $E_n$ contains the following types of edges:
- Singles Edges (the classical ones);

- Indeterminate Edges (unclear, vagues, partially unkwnown);
- Null Edges (totally unknown, empty);

and:

- HyperEdge (connecting three or more single vertices);
- SuperEdge (connecting two vertices, at least one of them being a SuperVertex);
- $n$-SuperEdge (connecting two vertices, at least one being a $n$-SuperVertex, and the other of order $r$-SuperVertex, with $r \leq n$);
- SuperHyperEdge (connecting three or more vertices, at least one being a SuperVertex);
- $n$-SuperHyperEdge (connecting three or more vertices, at least one being a $n$-SuperVertex, and the other $r$-SuperVertices with $r \leq n$;
- MultiEdges (two or more edges connecting the same two vertices);
- Loop (and edge that connects an element with itself).

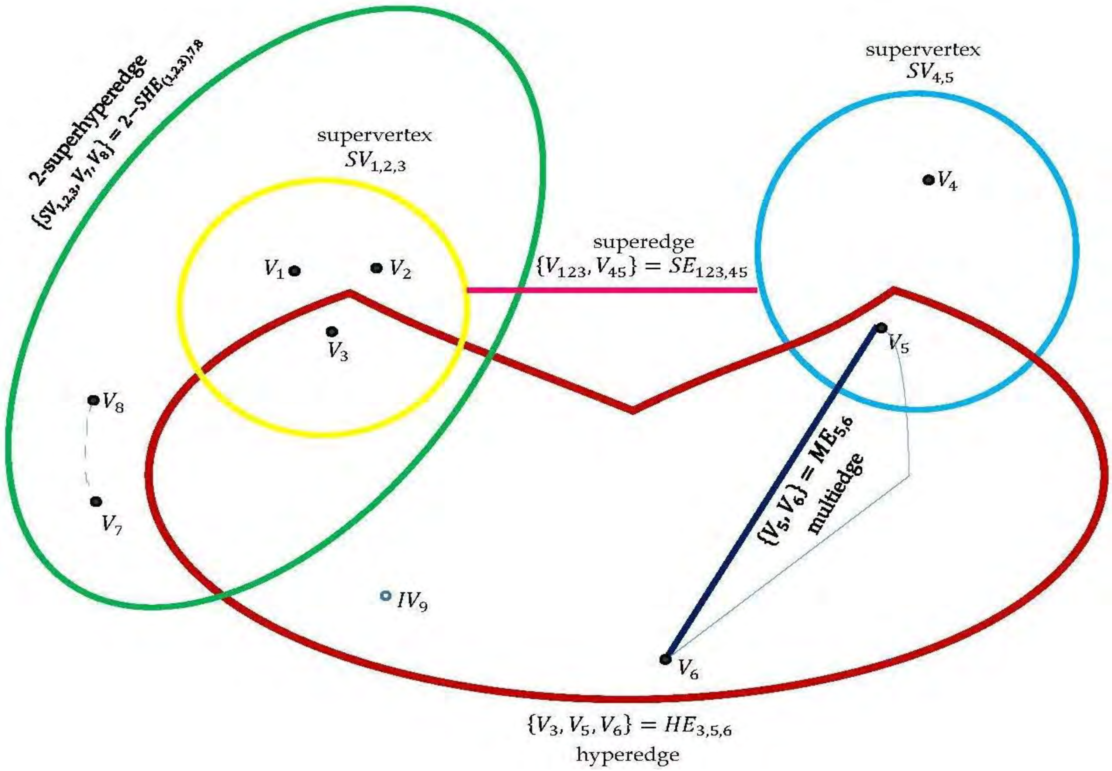
and:

- Directed Graph (classical one);
- Undirected Grpah (classical one);
- Neutrosophic Directed Graph (partially directed, partially undirected, partially indeterminate direction).

## 2. SuperHyperGraph

When *n = 1* we call the *1-SuperHyperGraph* simply **SuperHyperGraph**, because only the first power set of V is used, P(V).

## 3. Examples of 2-SuperHyperGraph, SuperVertex, IndeterminateVertex, SingleEdge, Indeterminate Edge, HyperEdge, SuperEdge, MultiEdge, 2-SuperHyperEdge [2]




*$IE_{7,8}$* is an Indeterminate Edge between single vertices $V_7$ and $V_8$, since the connecting curve is dotted;

$IV_9$ is an Indeterminate Vertex (since the dot is not filled in);

while $ME_{5,6}$ is a MultiEdge (double edge in this case) between single vertices $V_5$ and $V_6$.

## 4. Types of n-SuperHyperGraphs

The attributes values degrees of appurtenance of a vertex or an edge to the graph may be: crisp / fuzzy / intuitionistic fuzzy / picture fuzzy / spherical fuzzy / etc. / neutrosophic / refined neutrosophic / degrees with respect to each *n-SHG-vertex* and to each *n-SHG-edge* respectively.

For example, one has:

## 5. Plithogenic n-SuperHyperGraph

We recall the Plithogenic n-SuperHyperGraph.

A *Plithogenic n-SuperHyperGraph (n-PSHG)* is a n-SuperHyperGraph whose each *n-SHG-vertex* and each *n-SHG-edge* are characterized by many distinct attributes values $(a_1, a_2, \dots, a_p)$, $p \geq 1$.

Therefore one gets n-SHG-vertex($a_1, a_2, \dots, a_p$) and n-SHG-edge($a_1, a_2, \dots, a_p$).

## 6. Plithogenic Fuzzy-n-SHG-vertex $(a_1(t_1), a_2(t_2), \dots, a_p(t_p))$

and Fuzzy-n-SHG-edge($a_1(t_1), a_2(t_2), \dots, a_p(t_p)$);

## 7. Plithogenic Intuitionistic Fuzzy-n-SHG-vertex $(a_1(t_1, f_1), a_2(t_2, f_2), \dots, a_p(t_p, f_p))$

and Intuitionistic Fuzzy-n-SHG-edge($a_1(t_1, f_1), a_2(t_2, f_2), \dots, ap(t_p, f_p)$);

## 8. Plithogenic Neutrosophic-n-SHG-vertex $(a_1(t_1, i_1, f_1), a_2(t_2, i_2, f_2), \dots, a_p(t_p, i_p, f_p))$

and Neutrosophic-n-SHG-edge ($a_1(t_1, i_1, f_1), a_2(t_2, i_2, f_2), \dots, ap(t_p, i_p, f_p)$);

etc.

Whence in general we get:

## 9. The Plithogenic (Crisp / Fuzzy / Intuitionistic Fuzzy / Picture Fuzzy / Spherical Fuzzy / etc. / Neutrosophic / Refined Neutrosophic) n-SuperHyperGraph

## 10. Conclusions

The n-SuperHyperGraph is the most general for of graph today, designed in order to catch our complex real world.

First, the SuperVertex was introduced in 2019, then the SuperHyperGraph constructed on the power set P(V), and further on this was extended to the n-SuperHyperGraph built on the n-power set of the power set, $P^n(V)$, in order to overcome the complex groups of individuals and the sophisticated connections between them.

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Received: January 18, 2022. Accepted: February 10, 2022.

Neutrosophic Sets and Systems (NSS) is an academic journal, published quarterly online and on paper, that has been created for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics etc. and their applications in any field.

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ISSN (print): 2331-6055, ISSN (online): 2331-608X
Impact Factor: 1.739

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