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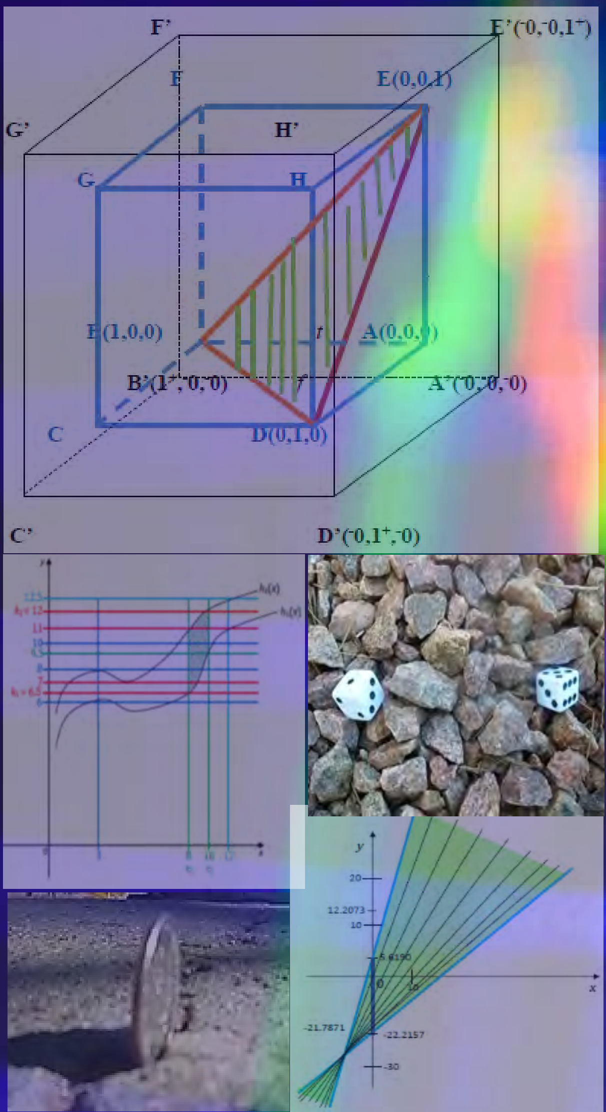

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Improved Definition of NonStandard Neutrosophic Logic and Introduction to Neutrosophic Hyperreals (Fifth version)

Prof. **Florentin Smarandache** [1]*, PhD, Postdoc

[1] Mathematics Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA

* Correspondence: smarand@unm.edu

Abstract: In the fifth version of our response-paper [26] to Imamura's criticism, we recall that NonStandard Neutrosophic Logic was never used by neutrosophic community in no application, that the quarter of century old neutrosophic operators (1995-1998) criticized by Imamura were never utilized since they were improved shortly after but he omits to tell their development, and that in real world applications we need to convert/approximate the NonStandard Analysis hyperreals, monads and binads to tiny intervals with the desired accuracy – otherwise they would be inapplicable.

We point out several errors and false statements by Imamura [21] with respect to the inf/sup of nonstandard subsets, also Imamura's "rigorous definition of neutrosophic logic" is wrong and the same for his definition of nonstandard unit interval, and we prove that there is not a total order on the set of hyperreals (because of the newly introduced Neutrosophic Hyperreals that are indeterminate), whence the Transfer Principle from R to R^* is questionable.

After his criticism, several response publications on theoretical nonstandard neutrosophics followed in the period 2018-2022. As such, I extended the NonStandard Analysis by adding the *left monad closed to the right*, *right monad closed to the left*, *pierced binad* (we introduced in 1998), and *unpierced binad* - all these in order to close the newly extended nonstandard space (R*) under nonstandard addition, nonstandard subtraction, nonstandard multiplication, nonstandard division, and nonstandard power operations [23, 24].

Improved definitions of NonStandard Unit Interval and NonStandard Neutrosophic Logic, together with NonStandard Neutrosophic Operators are presented.

Keywords: Neutrosophic Logic; NonStandard Analysis; NonStandard Neutrosophic Logic; Neutrosophic Operators; Neutrosophic Hyperreals

1. Introduction

I recall my first two answers to Imamura's 7th Nov. 2018 critics [1] about the NonStandard Neutrosophic Logic [20] on 24 Nov. 2018 (version 1) and 13 Feb. 2019 (version 2), and I update them after Imamura has published a third version [21] on a journal without even citing my previous response papers, nor making any comments or critics to them, although the paper was uploaded to arXiv shortly after him and also online at my UNM [20]. I find it as dishonest.

Surely, he can recall over and over again the first neutrosophic connectives, but he has to tell the whole story: they were never used in no application, and they were improved several times starting with the American researcher Ashbacher's neutrosophic connectives in 2002, Rivieccio in 2008, and Wang, Smarandache, Zhang, and Sunderraman in 2010. Version

The only reason I have added the nonstandard form to neutrosophic logic (and similarly to neutrosophic set and probability) was in order to make a distinction between *Relative Truth* (which is truth in some Worlds, according to Leibniz) and *Absolute Truth* (which is truth in all possible Words, according to Leibniz as well) that occur in philosophy.

Another possible reason may be when the neutrosophic degrees of truth, indeterminacy, or falsehood are infinitesimally determined, for example: the right monad (0.8^+) means a value strictly bigger than 0.8 but infinitely closer to 0.8. And similarly, the left monad ($^-0.8$) means a value strictly smaller than 0.8 but infinitely closer to 0.8. While the binad ($^-0.8^+$) means a value different from 0.8 but infinitely closer (from the right-hand side, or left-hand side) to 0.8. But they do not exist in our real world (the real set R), only in the hyperreal set R*, so we need to *convert / approximate* these hyperreal sets by tiny real intervals with the desired accuracy (ε), such as: $(0.8, 0.8+\varepsilon)$, $(0.8-\varepsilon, 0.8)$, or $(0.8-\varepsilon, 0.8)\cup(0.8, 0.8+\varepsilon)$ respectively [24].

Since the beginning of the neutrosophic field, many things have been developed and evolved, where better definitions, operators, descriptions, and applications of the neutrosophic logic have been defined. The same way happens in any scientific field: starting from some initial definitions and operations the community improves them little by little. The reader should check the last development of the neutroosphics - there are thousands of papers, books, and conference presentations online, check for example: http://fs.unm.edu/neutrosophy.htm. It is not fear to keep recalling the old definitions and operators since they have been improved in the meantime. The last development of the field should be revealed, not omitted.

The general definition of the neutrosophic set used in the last years.

Let *U* be a universe and a set *S* included in *U*. Then each element $x \in S$, denoted as

x(T(x), I(x), F(x)), has a degree of membership/truth T(x) with respect to *S*, degree of indeterminate-membership *I(x),* and degree of nonmembership *F(x),* where

T(x), I(x), F(x) are real subsets of [0, 1].

I was more prudent when I presented the sum of single valued standard neutrosophic components, saying:

Let T, I, F be single valued numbers, T, I, F ∈ [0, 1], such that $0 \leq T + I + F \leq 3$.

A friend alerted me: *"If T, I, F are numbers in [0, 1], of course their sum is between 0 and 3."* "Yes, I responded, I afford this tautology, because if I did not mention that the sum is up to *3*, readers would take for granted that the sum *T + I + F* is bounded by *1*, since that is in all logics and in probability!"

Similarly, for the Neutrosophic Logic, but instead of elements we have propositions (in the propositional logic).

2. Errors in Imamura's paper [21]:

2.1 Imamura's assertion, referring to the Neutrosophic components *T, I, F* as subsets, that:

"Subsets of $]^-0, 1^+[$" may have neither infima nor suprema" is false.

Counter-Examples of subsets that have both infima and suprema:

Let denote the nonstandard unit interval $U =]^-0, 1^+[$.

Let $M =]0.2^+, {}^-0.3[$, which is a subset of *U*, then

inf(M) = 0.2, sup(M) = 0.3.

In general, for any real numbers *a* and *b*, such that $0 \leq a < b \leq 1$, one has the corresponding nonstandard subset $S =]a^+, {}^-b[$ included in *U*, that has both exist: *inf(S) = a, sup(S) = b.*

As a particular and interesting case, one has: $]0^+, {}^-1[\subset]^-0, 1^+[$. In general, for any finite real numbers $a, b \in R$, $a < b$, the nonstandard subset $S =]a^+, {}^-b[$ included in R^*, has both: *inf(S) = a, sup(S) = b.* More generally, for any $x \in \{\overset{0}{a}, \overset{+}{a}, \overset{0+}{a}\}$ and any $y \in \{\overset{0}{b}, \overset{-}{b}, \overset{-0}{b}\}$ the nonstandard subset $]x, y[$ has

$\inf(x) = a$ and $\sup(y) = b$; even the subset $]\overset{0}{a},\overset{0}{b}[\equiv [a,b]$, which normally is standard, may become nonstandard if it contains inside at least one hyperreal. Of course, if at least one of x or y is hyperreal, then the subset]x, y[is nonstandard.

2.2 Imamura's *"rigorous definition of neutrosophic logic"* is wrong.

Let ***K*** be a nonarchimedean ordered field. The ordered field ***K*** is called nonarchimedean if it has nonzero infinitesimals.

He defined, for $x, y \in K$, x and y are said to be <u>infinitely close</u> (denoted by $x \approx y$) if x-y is infinitesimal. Then <u>x is roughly smaller than y</u> (denoted as $x \underset{\approx}{<} y$) if $x < y$ or $x \approx y$.

This is wrong. See the below Counter-Examples.

Let $\varepsilon > 0$ be a positive infinitesimal, also $x = 5 + \varepsilon$ and $y = 5 - \varepsilon$ be hyperreals.

Of course, $x \in (5^+)$, right monad of *5*, and $y \in ({}^-5)$, left monad of *5*.

$5 + \varepsilon$ is infinitely closer to 5, but above (strictly greater than) 5;

while $5 - \varepsilon$ is infinitely closer to 5, but below (strictly smaller than) 5.

Then $x - y = 2\varepsilon$, which is infinitesimal, and, because *x is infinitely close to y* ($x \approx y$), one has that x is roughly smaller than y (or $x \underset{\approx}{<} y$), according to Imamura's definition.

But this is false, since for $\varepsilon > 0$ clearly $5 + \varepsilon > 5 > 5 - \varepsilon$, whence $x > y$.

Therefore, x is not roughly smaller than y, but the opposite.

General Contra-Examples:

Let $\varepsilon > 0$ be a positive infinitesimal, and the real number $a \in R$.

Then for $x = a + \varepsilon$ and $y = a - \varepsilon$ we get the same wrong result $x < y$, according to Imamura.

Further on, for $x = a + \varepsilon$ and $y = a$, one gets the wrong result $x < y$.

And similarly, for $x = a$ and $y = a - \varepsilon$, one gets the wrong result $x < y$.

2.3 There exists no order between a and ${}^-a^+$ in R^*.

Let $a \in R$ be a real number, and ε be a positive or negative (we do not know exactly) infinitesimal.

Then $y = {}^-a^+$ is a hyperreal number of the form $y = a + \varepsilon$, where ε may be positive or negative infinitesimal.

Let $({}^-a^+)$ be the left-right binad [5] of a, defined as:

$({}^-a^+) = \{a \pm \varepsilon,$ where ε is a positive infinitesimal}.

Of course, ${}^-a^+ \in ({}^-a^+)$.

The transfer principle [21] states that R^* has the same first order properties as R.

But R^* has only a partial order, since there is no order between a and ${}^-a^+$ in R^*,

while R has a total order.

On has $\overset{-0}{a} \leq_N \overset{-0+}{a} \leq_N \overset{0+}{a}$, then $\overset{-0}{a} \leq_N a \leq_N \overset{0+}{a}$, whence $\overset{-0}{a} \leq_N \overset{0+}{a}$.

But, similar problems of non-order relationships are between $\overset{-0+}{a}$, $\overset{-0}{a}$ respectively and ${}^-a^+$.

Hence, the Transfer Principle from R to R^* is questionable…

3. Uselessness of Nonstandard Analysis in Neutrosophic Logic, Set, Probability. Statistics, *et al.*

Imamura's discussion [1] on the definition of neutrosphic logic is welcome, but it is useless, since from all neutrosophic papers and books published, from all conference presentations, and from all MSc and PhD theses defended around the world, etc. (more than two thousands) in the last two decades since the first neutrosophic research started *(1998-2022)*, and from thousands of neutrosophic researchers, not even a single one ever used the nonstandard form of neutrosophic logic, set, or probability and statistics in no occasion (extended researches or applications).

All researchers, with no exception, have used the *Standard Neutrosophic Set and Logic* [so no stance whatsoever of *Nonstandard Neutrosophic Set and Logic*], where the neutrosophic components *T, I, F* are real subsets of the standard unit interval *[0, 1].*

People don't even write "standard" since it is understood, because nonstandard was never used in no applications - it is unusable in real applications.

Even more, for simplifying the calculations, the majority of researchers have utilized the *Single-Valued Neutrosophic Set and Logic* {when *T, I, F* are single real numbers from *[0, 1]*}, on the second place was *Interval-Valued Neutrosophic Set and Logic* {when *T, I, F* are intervals included in *[0, 1]},* and on the third one the *Hesitant Neutrosophic Set and Logic* {when *T, I, F* were discrete finite subsets included in *[0, 1]*}.

In this direction, there have been published papers on single-valued "neutrosophic standard sets" [12, 13, 14], where the neutrosophic components are just *standard real numbers*, considering the particular case when $0 \le T + I + F \le 1$ (in the most general case $0 \le T + I + F \le 3$).

Actually, Imamura himself acknowledges on his paper [1], page 4, that:

> *"neutrosophic logic does not depend on transfer, so the use of non-standard analysis is not essential for this logic, and can be eliminated from its definition".*

Entire neutrosophic community has found out about this result and has ignored the nonstandard analysis from the beginning in the studies and applications of neutrosophic logic for two decades.

4. Applicability of Neutrosophic Logic et al. vs. Theoretical NonStandard Analysis

He wrote:

> *"we do not discuss the theoretical significance or the applications of neutrosophic logic"*

Why doesn't he discuss of the applications of neutrosophic logic? Because it has too many that brough its popularity among researchers [2], unlike the NonStandard Analysis that is a non-physical (idealistic, imaginary) object and it is hard to apply it in the real world.

Neutrosophic logic, set, measure, probability, statistics and so on were designed with the primordial goal of being applied in practical fields, such as:

Artificial Intelligence, Information Systems, Computer Science, Cybernetics, Theory Methods, Mathematical Algebraic Structures, Applied Mathematics, Automation, Control Systems, Big Data, Engineering, Electrical, Electronic, Philosophy, Social Science, Psychology, Biology, Biomedical, Engineering, Medical Informatics, Operational Research, Management Science, Imaging Science, Photographic Technology, Instruments, Instrumentation, Physics, Optics, Economics, Mechanics, Neurosciences, Radiology Nuclear, Medicine, Medical Imaging, Interdisciplinary Applications, Multidisciplinary Sciences etc. [2], while nonstandard analysis is mostly a pure mathematics.

Since 1990, when I emigrated from a political refugee camp in Turkey to America, working as a software engineer for Honeywell Inc., in Phoenix, Arizona State, I was advised by American coworkers to do theories that have *practical applications*, not pure-theories and abstractizations as "*art pour art*".

5. Theoretical Reason for the Nonstandard Form of Neutrosophic Logic

The only reason I have added the nonstandard form to neutrosophic logic (and similarly to neutrosophic set and probability) was in order to make a distinction between *Relative Truth* (which is truth in some Worlds, according to Leibniz) and *Absolute Truth* (which is truth in all possible Words, according to Leibniz as well) that occur in philosophy.

Another possible reason may be when the neutrosophic degrees of truth, indeterminacy, or falsehood are infinitesimally determined, for example a value infinitesimally bigger than *0.8* (or 0.8^+), or infinitesimally smaller than *0.8* (or ${}^-0.8$). But these can easily be overcome by roughly using interval neutrosophic values and depending on the desired accuracy, for example (0.80, 0.81) and (0.79, 0.80) respectively.

I wanted to get the neutrosophic logic as general as possible [6], extending all previous logics (Boolean, fuzzy, intuitionistic fuzzy logic, intuitionistic logic, paraconsistent logic, dialethism), and to have it able to deal with all kinds of logical propositions (including paradoxes, nonsensical propositions, etc.).

That's why in 2013 I extended the Neutrosophic Logic to Refined Neutrosophic Logic [from generalizations of 2-valued Boolean logic to fuzzy logic, also from the Kleene's and Lukasiewicz's and Bochvar's 3-symbol valued logics or Belnap's 4-symbol valued logic to the most general n-symbol or n-numerical valued refined neutrosophic logic, for any integer $n \geq 1$], the largest ever so far, when some or all neutrosophic components T, I, F were respectively split/refined into neutrosophic subcomponents: $T_1, T_2, \ldots; I_1, I_2, \ldots; F_1, F_2, \ldots;$ which were deduced from our everyday life [3].

6. From Paradoxism movement to Neutrosophy – generalization of Dialectics

I started first from *Paradoxism* (that I founded in 1980's as a movement based on antitheses, antinomies, paradoxes, contradictions in literature, arts, and sciences), then I introduced the *Neutrosophy* (as generalization of Dialectics (studied by Hegel and Marx) and of Yin Yang (Ancient Chinese Philosophy), neutrosophy is a branch of philosophy studying the dynamics of triads, inspired from our everyday life, triads that have the form:

<*A*>, its opposite <*antiA*>, and their neutrals <*neutA*>,

where <*A*> is any item or entity [4].

(Of course, we take into consideration only those triads that make sense in our real and scientific world.)

The Relative Truth neutrosophic value was marked as *1*, while the Absolute Truth neutrosophic value was marked as 1^+ (a tinny bigger than the Relative Truth's value):

$1^+ >_N 1$, where $>_N$ is a nonstandard inequality, meaning 1^+ is nonstandardly bigger than *1*.

Similarly for Relative Falsehood / Indeterminacy (which falsehood / indeterminacy in some Worlds), and Absolute Falsehood / Indeterminacy (which is falsehood / indeterminacy in all possible worlds).

7. Introduction to Nonstandard Analysis [15, 16]

An *infinitesimal number* is a number ε such that its absolute value $|\varepsilon| < 1/n$, for any non-null positive integer *n*. An infinitesimal is close to zero, and so small that it cannot be measured.

The infinitesimal is a number smaller, in absolute value, than anything positive nonzero.

Infinitesimals are used in calculus, but interpreted as tiny real numbers.

An *infinite number (ω)* is a number greater than anything:

$1 + 1 + 1 + \ldots + 1$ (for any finite number terms)

The infinites are reciprocals of infinitesimals.

The set of *hyperreals* (*non-standard reals*), denoted as R^*, is the extension of set of the real numbers, denoted as *R*, and it comprises the infinitesimals and the infinites, that may be represented on the *hyperreal number line*

$1/\varepsilon = \omega/1$.

The set of hyperreals satisfies the *transfer principle*, which states that the statements of first order in *R* are valid in R^* as well [according to the classical NonStandard Analysis]:

" 'Anything provable about a given superstructure V by passing to a nonstandard enlargement *V of V is also provable without doing so, and vice versa.' It is a result of Łoś' theorem and the completeness theorem for first-order predicate logic." [16]

A *monad* (*halo*) of an element $a \in R^*$, denoted by $\mu(a)$, is a subset of numbers infinitesimally close to a.

Let's denote by R_+^* the set of positive nonzero hyperreal numbers.

7.1. First Extension of NonStandard Analysis

We consider the left monad and right monad; afterwards we recall the *pierced binad* (Smarandache [5]) introduced in 1998:

Left Monad {that we denote, for simplicity, by (^-a)} is defined as:

$\mu(^-a) = (^-a) = \{a - x, x \in R_+^* \mid x \text{ is infinitesimal}\}$.

Right Monad {that we denote, for simplicity, by (a^+)} is defined as:

$\mu(a^+) = (a^+) = \{a + x, x \in R_+^* \mid x \text{ is infinitesimal}\}$.

The *Pierced Binad* {that we denote, for simplicity, by $(^-a^+)$} is defined as:

$\mu(^-a^+) = (^-a^+) = \{a - x, x \in R_+^* \mid x \text{ is infinitesimal}\} \cup \{a + x, x \in R_+^* \mid x \text{ is infinitesimal}\}$

$= \{a - x, x \in R^* \mid x \text{ is positive or negative infinitesimal}\}$.

7.1. Second Extension of Nonstandard Analysis [23]

For necessity of doing calculations that will be used in nonstandard neutrosophic logic in order to calculate the nonstandard neutrosophic logic operators (conjunction, disjunction, negation, implication, equivalence) and in order to have the Nonstandard Real MoBiNad Set closed under arithmetic operations, we extend now for the time: the left monad to the Left Monad Closed to the Right, the right monad to the Right Monad Closed to the Left; and the Pierced Binad to the Unpierced Binad, defined as follows (Smarandache, 2018-2019):

- Left Monad Closed to the Right

$$\mu\left(\overset{-0}{a}\right) = \left(\overset{-0}{a}\right) = \{a - x \mid x = 0, \text{ or } x \in R_+^* \text{ and } x \text{ is infinitesimal}\} = \mu(^-a) \cup \{a\}.$$

And by $x = \overset{-0}{a}$ we understand the **hyperreal** $x = a - \varepsilon$, or $x = a$, where ε is a positive infinitesimal. So, x is not clearly known, $x \in \{a - \varepsilon, a\}$.

- Right Monad Closed to the Left

$$\mu\left(\overset{0+}{a}\right) = \left(\overset{0+}{a}\right) = \{a + x \mid x = 0, \text{ or } x \in R_+^* \text{ and } x \text{ is infinitesimal}\} = \mu(a^+) \cup \{a\}.$$

And by $x = \overset{-0}{a}$ we understand the **hyperreal** $x = a + \varepsilon$, or $x = a$, where ε is a positive infinitesimal. So, x is not clearly known, $x \in \{a + \varepsilon, a\}$.

- Unpierced Binad

$$\mu\left(\overset{-0+}{a}\right) = \left(\overset{-0+}{a}\right) = \{a + x \mid x = 0, \text{ or } x \in R^* \text{ where } x \text{ is a positive or negative infinitesimal}\} =$$

$$= \mu\left(^-a\right) \cup \mu\left(a^+\right) \cup \{a\} = (^-a) \cup (a^+) \cup \{a\}.$$

And by $x = \overset{-0+}{a}$ we understand the **hyperreal** $x = a - \varepsilon$, or $x = a$, or $x = a + \varepsilon$, where ε is a positive infinitesimal. So, x is not clearly known, $x \in \{a - \varepsilon, a, a + \varepsilon\}$.

The left monad, left monad closed to the right, right monad, right monad closed to the left, the pierced binad, and the unpierced binad are subsets of R^*, while the above hyperreals are numbers from R^*.

Let's define a partial order on R^*.

8. Neutrosophic Strict Inequalities

We recall the neutrosophic strict inequality which is needed for the inequalities of nonstandard numbers.

Let *α, β* be elements in a partially ordered set *M*.

We have defined the neutrosophic strict inequality

$\alpha >_N \beta$

and read as

"α is neutrosophically greater than β"

if

α in general is greater than β,

or *α is approximately greater than β,*

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction (*situation when α is smaller than or equal to β*) α is greater than β.*

It means that in most of the cases, on the set *M, α is greater than β.*

And similarly for the opposite neutrosophic strict inequality $\alpha <_N \beta$.

9. Neutrosophic Equality

We have defined the neutrosophic inequality

$\alpha =_N \beta$

and read as

"α is neutrosophically equal to β"

if

α in general is equal to β,

or *α is approximately equal to β,*

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) or subject to some contradiction (situation when α is not equal to β) α is equal to β.

It means that in most of the cases, on the set *M, α is equal to β.*

10. Neutrosophic (Non-Strict) Inequalities

Combining the neutrosophic strict inequalities with neutrosophic equality, we get the $\geq_N$ and $\leq_N$ neutrosophic inequalities.

Let *α, β* be elements in a partially ordered set *M*.

The *neutrosophic (non-strict) inequality*

$\alpha \geq_N \beta$

and read as

"α is neutrosophically greater than or equal to β"

if

α in general is greater than or equal to β,

or *α is approximately greater than or equal to β,* or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction (*situation when α is smaller than β*) α is greater than or equal to β.*

It means that in most of the cases, on the set *M*, *α is greater than or equal to β.*

And similarly for the opposite neutrosophic (non-strict) inequality $\alpha \leq_N \beta$.

11. Neutrosophically Ordered Set

Let M be a set. (M, $<_N$) is called a neutrosophically ordered set if:

$\forall$ α, β ∈ M, one has: either $\alpha <_N \beta$, or $\alpha =_N \beta$, or $\alpha >_N \beta$.

12. Definition of Standard Part and Infinitesimal Part of a HyperReal Number

For each hyperreal (number) $h \in R^*$ one defines its <u>standard part</u>

st(h) be the real (standard) part of h, $st(h) \in R$,

and its <u>infinitesimal part</u>, that may be positive $(+\varepsilon)$, or zero (0), or negative $(-\varepsilon)$, and any combination of two or three of them in the case of Neutrosophic Hyperreals that have alternative (indeterminate) values as seen below, denoted as $in(h) \in R^*$.

Then $h = st(h) + in(h)$.

Two hyperreal numbers h_1 and h_2 are equal, if:

$st(h_1) = st(h_2)$ and $in(h_1) = in(h_2)$.

- Examples

Let ε be a positive infinitesimal, and the hyperreal numbers:

$$h_1 = 4 - \varepsilon \in ({}^{-}4)$$

$$h_2 = 4 + 0 \overset{def}{=} \overset{0}{4} \in R$$

$$h_3 = 4 + \varepsilon \in (4^+)$$

$h_4 = 4 - \{\varepsilon,$ or 0} = {4- ε , or 4-0} = {4- ε , or 4} $\in \begin{pmatrix} {}^{-0} \\ 4 \end{pmatrix}$

$h_5 = 4 + \{0,$ or ε } = {4+0, or 4+ ε } = {4, or 4+ ε } $\in \begin{pmatrix} {}^{0+} \\ 4 \end{pmatrix}$

$h_6 = 4 + \{-\varepsilon,$ or ε } = {4- ε , or 4+ ε } $\in \begin{pmatrix} {}^{-+} \\ 4 \end{pmatrix}$

$h_7 = 4 + \{-\varepsilon,$ or 0, or ε } = {4- ε , or 4+0, or 4+ ε }= {4- ε , or 4, or 4+ ε } $\in \begin{pmatrix} {}^{-0+} \\ 4 \end{pmatrix}$

Then, their standard parts are all the same:

$$st(h_1) = st(h_2) = ... = st(h_7) = 4$$

While their infinitesimal parts are different:

$$in(h_1) = -\varepsilon$$

$$in(h_2) = 0$$

$$in(h_3) = \varepsilon$$

13. Neutrosophic Hyperreal Numbers

The below cases are indeterminate, as in neutrosophy, that's why they are called *Neutrosophic Hyperreals*, introduced now for the first time:

$in(h_4) = \{-\varepsilon$, or 0}; one can also write that $in(h_4) \in \{-\varepsilon, 0\}$, because we are not sure if *in(h₄)*= $-\varepsilon$, or *in(h₄)*= 0.

$in(h_5) = \{\varepsilon$, or 0}; one can also write that $in(h_4) \in \{\varepsilon, 0\}$.

$in(h_6) = \{-\varepsilon,$ or ε}, or $in(h_6) \in \{-\varepsilon, \varepsilon\}$.

$in(h_7) = \{-\varepsilon,$ or 0, or ε}, or $in(h_6) \in \{-\varepsilon, 0, \varepsilon\}$.

14. Nonstandard Partial Order of Hyperreals

Let h_1 and h_2 be hyperreal numbers. Then $h_1 <_N h_2$ if:

either $st(h_1) < st(h_2)$, or $st(h_1) = st(h_2)$ and $in(h_1) <_N in(h_2)$.

By $in(h_1)$ we understand all possible infinitesimals of h_1, and similarly for $in(h_2)$.

This makes a partial order on the set of hyperreals R^*, because of the Neutrosophic Hyperreals that have indeterminate infinitesimal parts and cannot always be ordered.

15. Appurtenance of a Hyperreal number to a Nonstandard Set

We define for the first time the appurtenance of a hyperreal number (*h*) to a subset *S* of R^*, denoted as $\in_N$, or an approximate appurtenance (from a Neutrosophic point of view).

As seeing above, a hyperreal number may have one, two, or three infinitesimal parts - depending on its form.

Let's denote the standard part of *h* by *st(h)*, and its infinitesimal part(s) be $in(h) = in(h)_1$, $in(h)_2$, and $in(h)_3$. We construct three corresponding hyperreal numbers:

$h_1 = st(h) + in(h)_1$

$h_2 = st(h) + in(h)_2$

$h_3 = st(h) + in(h)_3$

If all three $h_1, h_2, h_3 \in_N S$, then $h \in_N S$. If at least one does not belong to *S*, then $h \notin_N S$.

(In the case when *h* has only one or two possible infinitesimals, of course we take only them.)

The appurtenance of a hyperreal number to a nonstandard set may be later extended if new forms of Neutrosophic Hyperreals are constructed in the meantime.

16. Notations and Approximations

Approximation is required with a desired accuracy, since the hyperreals, monads and binads do not exist in our real world. They are only very abstract concepts built in some imaginary math space.

That's why they must be approximated by real tiny sets.

As an example, let's assume that the truth-value (T) of a proposition (P), in the propositional logic, is the hyperreal T(P) = 0.7^+ that means, in nonstandard analysis, according to Imamura [22]:

> *"The interpretation of T(P) = 0.7^+ (right monad of 0.7 in your terminology):*
> *1. the truth value of P is strictly greater than and infinitely close to 0.7 (but its precise value is unknown);*
> *2. the truth value of P can be strictly greater than and infinitely close to 0.7;*
> *3. the truth value of P takes all hyperreals strictly greater than and infinitely close to 0.7 simultaneously."*

We prove by reductio ad absurdum that such a number does not exist in our real world. Let assume that 0.7^+ = w. Then w > 0.7, but on the set of continuous real numbers, in the interval (0.7, w] there exists a number v such that 0.7 < v < w, therefore v is closer to 0.7 than w, and thus w is not infinitely close to 0.7. Contradiction. Even Imamura acknowledges about 0.7^+ that "its value is unknown".

And because they do not exist in our real world, we need to approximate/convert them with a given accuracy to the real world, therefore, instead of 0.7^+ we may take for example the tiony interval (0.7, 0.7001) with four decimals, or (0.7, 0.7000001), etc.

In the same way one can prove that, for any real number $a \in R$, its left monad, left monad closed to the right, right monad, right monad closed to the left, pierced binad, and unpierced binad do not exist in our real world. They are just abstract concepts available in abstract/imaginary math spaces.

17. Nonstandard Unit Interval

Imamura cites my work:

"by "$_-a$" one signifies a monad, i.e., a set of hyper-real numbers in non-standard analysis:
$(_-a) = \{ a - x \in R_* \mid x$ is infinitesimal $\}$, and similarly "b_+" is a hyper monad:
$(b_+) = \{ b + x \in R_* \mid x$ is infinitesimal $\}$. ([5] p. 141; [6] p. 9)"

But these are inaccurate, because my exact definitions of monads, since my 1998 first world neutrosophic publication {see [5], page 9; and [6], pages 385 - 386}, were:

"$(^-a) = \{ a - x: x \in R_+{}^* \mid x$ is infinitesimal $\}$, and similarly "b+" is a hyper monad:
$(b^+) = \{ b + x: x \in R_+{}^* \mid x$ is infinitesimal $\}$"

Imamura says that:

"The correct definitions are the following:
$(_-a) = \{ a - x \in R_* \mid x$ is positive infinitesimal $\}$,
$(b_+) = \{ b + x \in R_* \mid x$ is positive infinitesimal $\}$."

I did not have a chance to see how my article was printed in *Proceedings of the 3rd Conference of the European Society for Fuzzy Logic and Technology* [7], that Imamura talks about, maybe there were some typos, but Imamura can check the *Multiple Valued Logic / An International Journal* [6], published in England in 2002 (ahead of the European Conference from 2003, that Imamura cites) by the prestigious Taylor & Francis Group Publishers, and clearly one sees that it is: $\boldsymbol{R_+^*}$ (so, x is a *positive* infinitesimal into the above formulas), therefore there is no error.

Then Imamura continues:

"Ambiguity of the definition of the nonstandard unit interval. Smarandache did not give any explicit definition of the notation $]^-0, 1^+[$ in [5] (or the notation ⊮$^-0, 1^+$⫣ in [6]). He only said:
Then, we call $]^-0, 1^+[$ a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, belong to the non-standard unit interval. ([5] p. 141; [6] p. 9)."

Concerning the notations I used for the nonstandard intervals, such as ⊩ ⫣ or] [, it was imperative to employ notations that are different from the classical *[ ]* or *( )* intervals, since the extremes of the nonstandard unit interval were unclear, vague with respect to the real set.

I thought it was easily understood that:

$]^-0, 1^+[\ = (^-0) \cup [0, 1] \cup (1^+)$.

Or, using the previous neutrosophic inequalities, we may write:

$]^-0, 1^+[\ = \{x \in R^*, {}^-0 \le_N x \le_N 1^+\}$.

Imamura says that:

"Here $^-0$ and 1^+ are particular real numbers defined in the previous paragraph:
$^-0 = 0 - \varepsilon$ and $1^+ = 1 + \varepsilon$, where ε is a fixed non-negative infinitesimal."

This is untrue, I never said that "ε is a *fixed* non-negative infinitesimal", ε was not fixed, I said that for any real numbers a and b {see again [5], page 9; and [6], pages 385 - 386}:

"$(^-a) = \{ a - x: x \in R_+{}^* \mid x$ is infinitesimal $\}$, $(b^+) = \{ b + x: x \in R_+{}^* \mid x$ is infinitesimal $\}$".

Therefore, once we replace $a = 0$ and $b = 1$, we get:

$$(^-0) = \{ 0 - x: x \in R_+{}^* \mid x \text{ is infinitesimal} \},$$

$$(1^+) = \{ 1 + x: x \in R_+{}^* \mid x \text{ is infinitesimal} \}.$$

Thinking out of box, inspired from the real world, was the first intent, i.e. allowing neutrosophic components (truth / indeterminacy / falsehood) values be outside of the classical (standard) unit real interval *[0, 1]* used in all previous (Boolean, multi-valued etc.) logics if needed in applications, so neutrosophic component values < *0* and > *1* had to occurs due to the Relative / Absolute stuff, with:

$$^{-}0 <_N 0 \quad \text{and} \quad 1^{+} >_N 1.$$

Later on, in 2007, I found plenty of cases and real applications in Standard Neutrosophic Logic and Set (therefore, not using the Nonstandard Neutrosophic Logic and Set), and it was thus possible the extension of the neutrosophic set *to Neutrosophic Overset (when some neutrosophic component is > 1), and to Neutrosophic Underset (when some neutrosophic component is < 0), and to Neutrosophic Offset (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and some neutrosophic component < 0). Then, similar extensions to respectively Neutrosophic Over/Under/Off Logic, Measure, Probability, Statistics etc.* [8, 17, 18, 19], extending the unit interval [0, 1] to

$$[\Psi, \Omega], \text{ with } \Psi \leq 0 < 1 \leq \Omega,$$

where Ψ, Ω are standard real numbers.

Imamura says, regarding the definition of neutrosophic logic that:

> *"In this logic, each proposition takes a value of the form (T, I, F), where T, I, F are subsets of the nonstandard unit interval]⁻0, 1⁺[and represent all possible values of Truthness, Indeterminacy and Falsity of the proposition, respectively."*

Unfortunately, this is not exactly how I defined it.

In my first book {see [5], p. 12; or [6] pp. 386 – 387} it is stated:

> *"Let T, I, F be real standard or non-standard subsets of]-0, 1+["*

meaning that T, I, F may also be "real standard" not only real non-standard.

In *The Free Online Dictionary of Computing*, 1999-07-29, edited by Denis Howe from England, it is written:

> *Neutrosophic Logic:*
>
> *<logic> (Or "Smarandache logic") A generalization of fuzzy logic based on Neutrosophy. A proposition is t true, i indeterminate, and f false, where t, i, and f are real values from the ranges T, I, F, with no restriction on T, I, F, or the sum*
>
> *n = t + i + f.*
>
> *Neutrosophic logic thus generalizes:*
>
> - *intuitionistic logic, which supports incomplete theories (for 0 < n < 100, 0 ≤ t,i,f ≤ 100);*
> - *fuzzy logic (for n = 100 and i = 0, and 0 ≤ t,i,f ≤ 100);*
> - *Boolean logic (for n=100 and i = 0, with t,f either 0 or 100);*
> - *multi-valued logic (for 0 ≤ t,i,f ≤ 100);*
> - *paraconsistent logic (for n > 100, with both t,f < 100);*
> - *dialetheism, which says that some contradictions are true (for t = f = 100 and i = 0; some paradoxes can be denoted this way).*
>
> *Compared with all other logics, neutrosophic logic introduces a percentage of "indeterminacy" - due to unexpected parameters hidden in some propositions. It also allows each component t,i,f to "boil over" 100 or "freeze" under 0. For example, in some tautologies t > 100, called "overtrue".*
>
> ["Neutrosophy / Neutrosophic probability, set, and logic", F. Smarandache, American Research Press, 1998].

As Denis Howe said in 1999, the neutrosophic components *t, i, f* are "*real values* from the ranges *T, I, F*", not nonstandard values or nonstandard intervals. And this was because nonstandard ones were not important for the neutrosophic logic (the Relative/Absolute plaid no role in technological and scientific applications and future theories).

18. Formal Notations

In my first version of the paper, I used informal notations. Let's see them improved.

Hyperreal Numbers are represented without parentheses ( ) around them:

$$^{-}a = \overset{-}{a} = a - \varepsilon$$

$\overset{0}{a} = a + 0$, which coincides with the real number *a.*

$$a^{+} = \overset{+}{a} = a + \varepsilon$$

Neutrosophic Hyperreal Numbers (that are indeterminate, alternative) are represented without braces, or with braces $\{\}$ around them for discrete sets that may have one, two, or three elements:

$\overset{-0}{a} = a - \varepsilon$, or $a + 0$ = $\{a - \varepsilon, \text{or } a + 0\}$

$\overset{+0}{a} = a + \varepsilon$, or $a + 0$ = $\{a + \varepsilon, \text{or } a + 0\}$

$\overset{-+}{a} = a - \varepsilon$, or $a + \varepsilon = \{a - \varepsilon, \text{or } a + \varepsilon\}$

$\overset{-0+}{a} = a - \varepsilon$, or $a + 0$, or $a + \varepsilon = \{a - \varepsilon, \text{or } a + 0, \text{or } a + \varepsilon\}$

For the monads and binads one just adds the parentheses around them:

Monad Sets: $a = \left(\overset{0}{a}\right), (^{-}a) = \left(\overset{-}{a}\right), (a^{+}) = \left(\overset{+}{a}\right)$

Binad Sets: $\left(\overset{-0}{a}\right), \left(\overset{0+}{a}\right), \left(\overset{-+}{a}\right), \left(\overset{-0+}{a}\right)$

19. Improved Definition of NonStandard Unit Interval

- **Formula of NonStandard Unit Interval**

$$]^{-}0,1^{+}[\equiv]\overset{-}{0},\overset{+}{1}[=\{a \in R^{*}, 0 \leq st(a) \leq 1\} = \{\overset{-}{a}, \overset{-0}{a}, \overset{0}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a}, \overset{+}{a}, a \in [0,1]\}.$$

Proof of the above formula

For $0 < st(a) < 1$ it does not matter what *in(a)* is, because $st(a) + in(a) \in_N]0,1[$, this being a nonstandard interval.

It is not necessarily to set any restriction on *in(a)* in this case, since $\overset{-}{a}$ is the smallest hyperreal, while $\overset{+}{a}$ is the greatest hyperreal in the set of seven types of hyperreals listed above.

Let ε be a positive infinitesimal, $\varepsilon \in R^{*}$.

Let *a* = *0*, and $\overset{m}{0}$ be any possible hyperreal number associated to 0.

For *st(* $\overset{m}{0}$ *) = 0*, the smallest *in(* $\overset{m}{0}$) may be $-\varepsilon$, whence $0 - \varepsilon = \overset{-}{0} \in_N]\overset{-}{0},\overset{+}{1}[$;

and if *in(* $\overset{m}{0}$) is bigger (i.e. *0*, or $+\varepsilon$), of course $0 + 0 = \overset{0}{0} \in_N]\overset{-}{0},\overset{+}{1}[$ and $0 + \varepsilon = \overset{+}{0} \in_N]\overset{-}{0},\overset{+}{1}[$.

Then also any other nonstandard version of the number *0*, such as: $\overset{-0}{0}, \overset{0+}{0}, \overset{-+}{0}, \overset{-0+}{0} \in_N]\overset{-}{0}, \overset{+}{1}[$.

Let $a = 1$, and $\overset{m}{1}$ be any possible hyperreal number associate to 1.

For $st(\overset{m}{1}) = 1$, the greatest $in(\overset{m}{1})$ may be $+\varepsilon$, whence $1+\varepsilon = \overset{+}{1} \in_N]\overset{-}{0}, \overset{+}{1}[$,

and if $in(\overset{m}{1})$ is smaller (i.e. *0*, or $-\varepsilon$), of course $1+0 = \overset{0}{1} \in_N]\overset{-}{0}, \overset{+}{1}[$ and $1-\varepsilon = \overset{-}{1} \in_N]\overset{-}{0}, \overset{+}{1}[$.

Then also any other nonstandard version of the number *1*, such as: $\overset{-0}{1}, \overset{0+}{1}, \overset{-+}{1}, \overset{-0+}{1} \in_N]\overset{-}{0}, \overset{+}{1}[$.

Remark:

This formula has to be updated if new types of hyperreals / monads / binads will be introduced

- Example of Inclusion of Nonstandard Sets

$$]\overset{+}{0}, \overset{-}{1}[\subset]0,1[\subset]\overset{-}{0}, \overset{+}{1}[$$

- Partial Ordering on the Set of Hyperreals

Let $a \in R$ be a real number. Then there is no order between *a* and $\overset{-+}{a}$, nor between *a* and $\overset{-0+}{a}$.
Some nonstandard inequalities involving hyperreals:

$\overset{-}{a} <_N \overset{0}{a} <_N \overset{+}{a}$

$\overset{-0}{a} \leq_N \overset{-+}{a} \leq_N \overset{0+}{a} \leq_N \overset{+}{a}$

$\overset{-}{a} \leq_N \overset{-0}{a} \leq_N \overset{-+}{a} \leq_N \overset{-0+}{a}$

$\overset{-}{a} \leq_N \overset{-+}{a} \leq_N \overset{+}{a}$

- Examples of Nonstandard Intervals

$$]\overset{-}{a}, \overset{0}{a}[= \{\overset{-}{a}, \overset{0}{a}, \overset{-0}{a}\}$$

$$]\overset{-}{a}, \overset{+}{a}[= \{\overset{-}{a}, \overset{0}{a}, \overset{+}{a}, \overset{-0}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a}\}$$

20. Improved Definition of NonStandard Neutrosophic Logic

In the nonstandard propositional calculus, a proposition P has degrees of truth (*T*), indeterminacy (*I*), and falsehood (*F*), such that *T, I, F* are nonstandard subsets of the nonstandard unit interval $]^{-}0,1^{+}[$, or $T, I, F \subseteq_N]^{-}0,1^{+}[$.

As a particular case one has when *T, I, F* are hyperreal or neutrosophic hyperreal numbers of the nonstandard unit interval $]^{-}0,1^{+}[$, or $T, I, F \in_N]^{-}0,1^{+}[$.

21. NonStandard Neutrosophic Operators

Since the Hyprereal Set *R** does not have a total order, in general we cannot use connectives (nonstandard conjunction, nonstandard disjunction, nonstandard negation, nonstandard implication, nonstandard equivalence, etc.) involving the operations of min/max or inf/sup, but we may use connectives involving addition, subtraction, scalar multiplication, multiplication, power, and division operations dealing with nonstandard subsets or hyprereals from the nonstandard unit interval $]^{-}0,1^{+}[$. See below operations with hyperreals, monads and binads.

For any nonstandard subsets or hyperreal numbers, $T_1, I_1, F_1, T_2, I_2, F_2$, from the nonstandard unit interval $]^{-}0,1^{+}[$ one has:

- NonStandard Neutrosophic Conjunction
  $(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2)$

- NonStandard Neutrosophic Disjunction
  $(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2)$

- NonStandard Neutrosophic Negation
  $\neg_N (T_1, I_1, F_1) = (F_1, 1^{+} - I_1, T_1)$

- NonStandard Neutrosophic Implication
  $(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, 1^{+} - I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, (1^{+} - I_1) \wedge_F I_2, T_1 \wedge_F F_2)$

- NonStandard Neutrosophic Equivalence
  $(T_1, I_1, F_1) \leftrightarrow_N (T_2, I_2, F_2)$ *means* $(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2)$ *and* $(T_2, I_2, F_2) \rightarrow_N (T_1, I_1, F_1)$

Example of Fuzzy Conjunction:
$A \wedge_F B = AB$

Example of Fuzzy Disjunction:
$A \vee_F B = A + B - AB$

More explanations about them follow.

22. Approximations of the NonStandard Logical Operators/Connectives ∧, ∨, , →, ↔

Imamura's critics of my first definition of the neutrosophic operators is history for over a quarter of century ago. He is attacking my paper with "errors… errors… paradoxes" etc., however my first operators were not kind of errors, but less accurate approximations of the aggregation with respect to the falsity component (F), but not with respect to the truth (T) and indeterminacy (I) ones that were correct.

The representations of sets of monads and binads by tiny intervals were also approximations ($\cong$) with a desired accuracy ($\varepsilon > 0$), from a classical (real) point of view, for the real number $a \in R$:

$$\left({}^{-}a\right) = \left(\overset{-}{a}\right) \cong (a-\varepsilon, a)$$

$$\left(a^{+}\right) = \left(\overset{+}{a}\right) \cong (a, a+\varepsilon)$$

$$\left({}^{-}a^{+}\right) = \left(\overset{-+}{a}\right) \cong (a-\varepsilon, a+\varepsilon)$$

$$\left(\overset{-0}{a}\right) \cong (a-\varepsilon, a]$$

$$\begin{pmatrix} ^{0+} \\ a \end{pmatrix} \cong [a, a+\varepsilon)$$

$$\begin{pmatrix} ^{-0+} \\ a \end{pmatrix} \cong (a-\varepsilon, a+\varepsilon)$$

And by language abuse one denotes:

$$\begin{pmatrix} ^{0} \\ a \end{pmatrix} = a = [a, a]$$

The representations of hyperreal numbers (*h = st(h) + in(h)*) by tiny numbers closed to their standard part (*st(h)*) were also approximations ($\cong$) with a desired accuracy (*ε > 0*), from a classical (real) point of view:

$\overset{-}{a} \cong a - \varepsilon$

$\overset{+}{a} \cong a + \varepsilon$

$\overset{-+}{a} \cong a - \varepsilon$, or $a + \varepsilon$

$\overset{-0}{a} \cong a - \varepsilon$, or *0*

$\overset{0+}{a} \cong 0$, or $a + \varepsilon$

$\overset{-0+}{a} \cong a - \varepsilon$, or *0*, or $a + \varepsilon$

$\overset{0}{a} = a$

All aggregations in fuzzy and fuzzy-extensions (that includes neutrosophic) logics and sets are *approximations* (not exact, as in classical logic), and they depend on each specific application and on the experts. Some experts/authors prefer ones, others prefer different operators.

It is NOT A UNIQUE operator of fuzzy/neutrosophic conjunction, as it is in classical logic, but a class of many neutrosophic operators, which is called neutrosophic t-norm; similarly for fuzzy/neutrosophic disjunction, called neutrosophic t-conorm, fuzzy/neutrosophic negation, fuzzy/neutrosophic implication, fuzzy/neutrosophic equivalence, etc.

All fuzzy, intuitionistic fuzzy, neutrosophic (and other fuzzy-extension) logic operators are *inferential approximations*, not written in stone. They are improved from application to application.

23. Operations with monads, binads, and hyperreals

In order to operate on them, it is easier to consider their real approximations to tiny intervals for the monads and binads, or to real numbers closed to the standard form of the hyperreal numbers, as in above section.

For **monads and binads**:

$\begin{pmatrix} ^{m_1,m_2,m_3} \\ a \end{pmatrix} \circ \begin{pmatrix} ^{m_1,m_2,m_3} \\ b \end{pmatrix} = \begin{pmatrix} ^{x_1,x_2,x_3} \\ a \circ b \end{pmatrix}$, where $\circ$ is any of the well-defined arithmetic operation (addition, subtraction, multiplication, scalar multiplication, power, root, division).

Where $m_1, m_2, m_3 \in \{-, 0, +\}$, but there are cases when some or all of the infinitesimal parts m_1, m_2, m_3 may be discarded for *a* or for *b* or for both, if one has only monads, or closed monads, or pierced binads. If such m_i is discarded, we consider it as $m_i = \phi$, for $i \in \{1, 2, 3\}$.

Always we do the classical operation $a \circ b$, but the problem is: what are the infinitesimals corresponding to the result $\left(\overset{x_1,x_2,x_3}{a \circ b}\right)$, i.e. what are $x_1, x_2, x_3 = ?$

Of course the infinitesimals $x_1, x_2, x_3 \in \{-, 0, +\}$, that represent respectively the left monad of $a \circ b$, just the real number $a \circ b$, or the right monad of $a \circ b$. To find them, we need to move from *R** to *R* using tiny approximations.

One gets the similar result for **hyperreal numbers** as for monads and binads:

$$\overset{m_1,m_2,m_3}{a} \circ \overset{m_1,m_2,m_3}{b} = \overset{x_1,x_2,x_3}{a \circ b}$$

- A Monad-Binad Example

Let $\varepsilon_1, \varepsilon_2 > 0$ be tiny real numbers.

Let's prove that:

$$\left(\overset{-}{a}\right) + \left(\overset{+}{b}\right) = \left(\overset{-0+}{a+b}\right)$$

We approximate the above monads by:

$$(a-\varepsilon_1, a) + (b, b+\varepsilon_2) = (a+b-\varepsilon_1, a+b+\varepsilon_2) \cong \left(\overset{-0+}{a+b}\right)$$

because, in the real interval $(a+b-\varepsilon_1, a+b+\varepsilon_2)$, one gets values smaller than *a+b* (whence the – on the top, standing for 'left monad of a+b'), equal to *a+b* (whence the 0 on the top, standing just for 'the real number a+b'), and greater than *a+b* (whence the + on the top, standing for 'right monad of a+b').

- Numerical example

$$\left(\overset{-}{2}\right) + \left(\overset{+}{3}\right) = \left(\overset{-0+}{2+3}\right) = \left(\overset{-0+}{5}\right)$$

because $\left(\overset{-}{2}\right) + \left(\overset{+}{3}\right) \cong (2-0.1, 2) + (3, 3+0.2) = (5-0.1, 5+0.2)$, and this interval is a little below 5, a little above 5, and also includes 5.

For hyperreal numbers the result is similar:

$\overset{-}{a} + \overset{+}{b} = \overset{-0+}{a+b}$ because

$\overset{-}{a} + \overset{+}{b} \cong a - \varepsilon_1 + b + \varepsilon_2 = a + b - \varepsilon_1 + \varepsilon_2$, where $\varepsilon_1, \varepsilon_2$ are any tiny positive numbers,

hence $a + b - \varepsilon_1 + \varepsilon_2$ can be less than *a+b*, equal to *a+b*, or greater than *a+b* by conveniently choosing the tiny positive numbers ε_1 and ε_2, as: $\varepsilon_1 > \varepsilon_2$, or $\varepsilon_1 = \varepsilon_2$, or $\varepsilon_1 < \varepsilon_2$ respectively.

- More Examples of NonStandard Operations

$$\left(\overset{-}{a}\right) + b = \left(\overset{-}{a+b}\right)$$

$$a + \left(\overset{+}{b}\right) = \left(\overset{+}{a+b}\right)$$

$$\left(\overset{-}{a}\right) + \left(\overset{-}{b}\right) = \left(\overset{-}{a+b}\right)$$

$$\left(\overset{+}{a}\right) + \left(\overset{+}{b}\right) = \left(\overset{+}{a+b}\right)$$

$$a + \left(\overset{-+}{b}\right) + b = \left(\overset{-+}{a+b}\right)$$

$$\left(\overset{-+}{a}\right)+\left(\overset{-+}{b}\right)=\left(\overset{-0+}{a+b}\right)$$

$$\left(\overset{-}{a}\right)+\left(\overset{-+}{b}\right)=\left(\overset{-0+}{a+b}\right)$$

$$8\div\left(\overset{+}{2}\right)=\left(\overset{-}{4}\right)$$

$$8\div\left(\overset{-}{2}\right)=\left(\overset{+}{4}\right)$$

$$8\div\left(\overset{-0+}{2}\right)=\left(\overset{-0+}{4}\right)$$

$$\sqrt{\left(\overset{-}{9}\right)}=\left(\overset{-}{3}\right)$$

$$\left(\overset{-}{11}\right)^2=\left(\overset{-}{121}\right)$$

$$\left(\overset{-}{6}\right)\times\left(\overset{+}{7}\right)=\left(\overset{-0+}{42}\right)$$

$$\left(\overset{-}{10}\right)-\left(\overset{+}{4}\right)=\left(\overset{-}{6}\right)$$

$$\left(\overset{+}{10}\right)-\left(\overset{-}{4}\right)=\left(\overset{+}{6}\right)$$

Etc.

24. NonStandard Neutrosophic Operators (revisited)

Let's denote:

$\wedge_F$, $\wedge_N$, $\wedge_P$ representing respectively the fuzzy conjunction, neutrosophic conjunction, and plithogenic conjunction; similarly

$\vee_F$, $\vee_N$, $\vee_P$ representing respectively the fuzzy disjunction, neutrosophic disjunction, and plithogenic disjunction,

$\neg_F, \neg_N, \neg_P$ representing respectively the fuzzy negation, neutrosophic negation, and plithogenic negation,

$\rightarrow_F$, $\rightarrow_N$, $\rightarrow_P$ representing respectively the fuzzy implication, neutrosophic implication, and plithogenic implication; and

$\leftrightarrow_F, \leftrightarrow_N, \leftrightarrow_P$ representing respectively the fuzzy equivalence, neutrosophic equivalence, and plithogenic equivalence.

I agree that my beginning neutrosophic operators (when I applied the same *fuzzy t-norm*, or the same *fuzzy t-conorm*, to all neutrosophic components *T, I, F*) were less accurate than others developed later by the neutrosophic community researchers. This was pointed out since 2002 partially corrected by Ashbacher [9] and confirmed in 2008 by Rivieccio [10] and fixed in 2010 by Wang, Smarandache, Zhang, and Sunderraman [25], much ahead of Imamura [1] in 2018. They observed that if on T_1 and T_2 one applies a *fuzzy t-norm*, on their opposites F_1 and F_2 one needs to apply the *fuzzy t-conorm* (the opposite of fuzzy t-norm), and reciprocally.

About inferring I_1 and I_2, some researchers combined them in the same directions as T_1 and T_2.

Then:

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \wedge_F I_2, F_1 \vee_F F_2),$$

$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \vee_F I_2, F_1 \wedge_F F_2),$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \vee_F I_2, T_1 \wedge_F F_2);$$

others combined I_1 and I_2 in the same direction as F_1 and F_2 (since both I and F are negatively qualitative neutrosophic components), the most used one:

$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2)$,

$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2)$,

$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \wedge_F I_2, T_1 \wedge_F F_2)$.

Now, applying the neutrosophic conjunction suggested by Imamura:

> *"This causes some counterintuitive phenomena. Let A be a (true) proposition with value ({ 1 } , { 0 } , { 0 }) and let B be a (false) proposition with value ({ 0 } , { 0 } , { 1 }).*
>
> *Usually we expect that the falsity of the conjunction A ∧ B is { 1 }. However, its actual falsity is { 0 }."*

we get:

$$(1, 0, 0) \wedge_N (0, 0, 1) = (0, 0, 1), \tag{50}$$

which is correct (so the falsity is *1*).

Even more, recently, in an extension of neutrosophic set to *plithogenic set* [11] (which is a set whose each element is characterized by many attribute values), the *degrees of contradiction c(,)* between the neutrosophic components *T, I, F* have been defined (in order to facilitate the design of the aggregation operators), as follows: $c(T, F) = 1$ (or *100%,* because they are totally opposite*),* $c(T, I) = c(F, I) = 0.5$ (or *50%,* because they are only half opposite), then:

$(T_1, I_1, F_1) \wedge_P (T_2, I_2, F_2) = (T_1 \wedge_F T_2, 0.5(I_1 \wedge_F I_2) + 0.5(I_1 \vee_F I_2), F_1 \vee_F F_2)$,

$(T_1, I_1, F_1) \vee_P (T_2, I_2, F_2) = (T_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), F_1 \wedge_F F_2)$.

$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = \neg_N (T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2)$

$= (F_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), T_1 \wedge_F F_2)$.

For NonStandard Neutrosophic Logic, one replace all the above neutrosophic components $T_1, I_1, F_1, T_2, I_2, F_2$ by hyperreal numbers, monads or binads from the nonstandard unit interval $]^-0, 1^+[$ and use the previous nonstandard operations.

25. Application of NonStandard Neutrosophic Logic

Assume two sources s_1 and s_2 provide information about the nonstandard truth value of a given proposition *P*:

$$s_1(P) = (T_1(P), I_1(P), F_1(P)) = \left(\overset{+}{1}, \overset{-+}{0.4}, \overset{-}{0.2}\right)$$

$$s_2(P) = (T_2(P), I_2(P), F_2(P)) = \left(\overset{0}{0.8}, \overset{+}{0.6}, \overset{-0}{0.3}\right)$$

Let's use the below *Fuzzy Conjunction:*

$$A \wedge_F B = A \cdot B$$

and *Fuzzy Disjunction*:

$$A \vee_F B = A + B - A \cdot B$$

We fusion the two sources (using the nonstandard neutrosophic conjunction):

$$s_1(P) \wedge_N s_2(P) = (T_1(P) \wedge_F T_2(P), I_1(P) \vee_F I_2(P), F_1(P) \vee_F F_2(P))$$

$$= (\overset{+}{1} \wedge_F \overset{0}{0.8}, \overset{-+}{0.4} \vee_F \overset{+}{0.6}, \overset{-}{0.2} \vee_F \overset{0-}{0.3}) = (\overset{+}{1} \times \overset{0}{0.8}, \overset{-+}{0.4} + \overset{+}{0.6} - \overset{-+}{0.4} \times \overset{+}{0.6}, \overset{-}{0.2} + \overset{-0}{0.3} - \overset{-}{0.2} \times \overset{-0}{0.3})$$

$$= (\overset{+}{0.8}, \overset{-0+}{1} - \overset{-+}{0.24}, \overset{-}{0.5} - \overset{-}{0.06}) = (\overset{+}{0.8}, \overset{-0+}{1} - \overset{-0+}{0.24}, \overset{-}{0.5} - \overset{-}{0.06}) = (\overset{+}{0.80}, \overset{-0+}{0.76}, \overset{-0+}{0.44}),$$

which means that with respect to the two fusioned sources, the nonstandard neutrosophic degree of truth of the proposition *P* is tinnily above *0.8,* its nonstandard neutrosophic degree of indeterminacy is tinnily below or above or equal to *0.76,* and similarly its nonstandard neutrosophic degree of falsity is tinnily below or above or equal to *0.44*.

Converting/approximating from hyperreal numbers to real numbers, with an accuracy ε = 0.001, one gets:

$$s_1(P) \wedge_N s_2(P) \cong \big((0.8, 0.8+0.001), (0.76-0.001, 0.76+0.001), (0.44-0.001, 0.44+0.001)\big)$$
$$= \big((0.800, 0.801), (0.759, 0.761), (0.439, 0.441)\big)$$

26. Open Statement

In general, the Transfer Principle, from a non-neutrosophic field to a corresponding neutrosophic field, does not work. This conjecture is motivated by the fact that each neutrosophic field may have various types of indeterminacies.

27. Conclusion

We thank very much Dr. Takura Imamura for his interest and critics of *Nonstandard Neutrosophic Logic*, which eventually helped in improving it. {In the history of mathematics, critics on nonstandard analysis, in general, have been made by Paul Halmos, Errett Bishop, Alain Connes and others.} We hope we'll have more dialogues on the subject in the future.

We introduced in this paper for the first time the Neutrosophic Hyperreals (that have an indeterminate form), and we improved the definitions of NonStandard Unit Interval and of NonStandard Neutrosophic Logic.

We pointed out several errors and false statements by Imamura [21] with respect to the inf/sup of nonstandard subsets, also Imamura's "rigorous definition of neutrosophic logic" is wrong and the same for his definition of nonstandard unit interval, and we proved that there is not a total order on the set of hyperreals (because of the newly introduced Neutrosophic Hyperreals that are indeterminate) therefore the transfer principle is questionable. We conjectured that: In general, the Transfer Principle, from a non-neutrosophic field to a corresponding neutrosophic field, does not work.

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Weighted aggregation operators of single-valued neutrosophic linguistic neutrosophic sets and their decision-making method

Wen-Hua Cui [1]*, Jun Ye [1,2]*, Jing-Jing Xue [1] and Ke-Li Hu [1]

[1] Department of Electrical Engineering and Automation, Shaoxing University, 312000 Shaoxing, Zhejiang, P.R. China; wenhuacui@usx.edu.cn (Wenhua Cui), yejun1@nbu.edu.cn(Jun Ye), xuejingjing@usx.edu.cn (Jingjing Xue), ancimoon@gmail.com (Keli Hu)

[2] Institute of Rock Mechanics, Ningbo University, 315211 Ningbo, Zhejiang, P.R. China;

* Correspondence: wenhuacui@usx.edu.cn; yejun1@nbu.edu.cn

Abstract: Multiple attribute decision-making (MADM) problems often contain quantitative and qualitative information that is inconsistent, uncertain, and incomplete. However, existing evaluation methods can only perform quantitative or qualitative processing on all attribute data, which easily leads to some information loss. In order to deal with MADM problems more effectively, this paper proposes a single-valued neutrosophic linguistic neutrosophic element (SvNLNE), which consists of a single-valued neutrosophic number for quantitative expression and a linguistic neutrosophic number for qualitative description. This paper also provides the fundamental operations of SvNLNEs, the SvNLNE score and accuracy functions for sorting the elements, and the SvNLNE weighted arithmetic averaging (SvNLNEWAA) and geometric averaging (SvNLNEWGA) operators for information aggregation. Finally, some MADM approaches are developed based on the SvNLNEWAA and SvNLNEWGA operators, and their application and rationality are further illustrated by an investment case in the SvNLNE setting.

Keywords: multi-attribute decision making; single-valued neutrosophic linguistic neutrosophic element; single-valued neutrosophic linguistic neutrosophic element weighted arithmetic averaging operator; single-valued neutrosophic linguistic neutrosophic element weighted geometric averaging operator

1. Introduction

In complex decision-making (DM) environments, conflicting quantitative and qualitative attribute data often need to be considered to optimize the selection of alternatives. Among them, quantitative information is usually expressed as numerical variables, while qualitative information is usually depicted as linguistic variables because linguistic items are more suitable to describe human cognition of objective things. In recent decades, many theories and methods on numerical and linguistic DM methods have been proposed for the DM problem. To handle uncertain and incomplete quantitative information, the fuzzy set [1] was firstly defined with a numerical membership degree. Then, the intuitionistic fuzzy set (IFS) [2] was presented by appending a numerical non-membership degree, and the interval-valued IFS [3] was represented by the interval-valued membership and non-membership degrees. Recently, for further comprehensive expression of the incomplete, uncertain and inconsistent data in DM problems, the simplified neutrosophic set (SNS) [4] that implied the definitions of single-valued neutrosophic set (SvNS) [5] and interval neutrosophic set (IvNS) [6] was put forward as a subclass of neutrosophic set (NS) [7] by constraining the membership degrees of truth, indeterminacy and falsity in the standard range of [0,1]. Since then, various aggregation operators and multi-attribute DM (MADM) methods of SvNSs/IvNSs/SNSs have been presented [4,8-10], and some extended neutrosophic sets, such as the neutrosophic cubic set (NCS) [11], the simplified neutrosophic indeterminate set (SNIS) [12], the neutrosophic Z-numbers [13] and the consistency neutrosophic set (CNS) [14], have also been proposed for specific applications. However, the numerical variable-based DM methods described above are more suitable for dealing with quantitative information than qualitative information. Thus, in terms of human thinking and expression habits, the linguistic neutrosophic number (LNN) [15], which is generalized from the concepts of linguistic variable (LV) [16], interval linguistic variable (ILV) [17] and linguistic intuitionistic fuzzy number (LIFN) [18], was proposed as a new branch of NS [7] to represent incomplete, indeterminate and inconsistent qualitative information using linguistic membership degrees of truth, falsity and uncertainty. Then, various aggregation operators and MADM methods of LNNs have been presented for linguistic DM problems [19-21]. Other extended sets, such as the linguistic neutrosophic uncertain number (LNUN) [22], have also been introduced to satisfy special applications. Unfortunately, theories and methods based on linguistic variables are more suitable for solving qualitative DM problems than quantitative DM problems.

In practical MADM applications, there is often quantitative and qualitative attribute information that needs to be evaluated together. However, existing DM methods can only make final decisions on a single type of information, but cannot handle multiple types of information. Especially for MADM problems with incomplete, inconsistent and indeterminate information, the single-valued neutrosophic number (SvNN) is only used for quantitative processing, while the LNN is only used for qualitative processing. Therefore, to overcome the limitations of existing DM approaches and better satisfy the preferences of the evaluators, this paper defines the single-valued neutrosophic linguistic neutrosophic set/element (SvNLNS/SvNLNE) as a combination of SvNN and LNN to uniformly describe quantitative and qualitative information and proposes the basic operational laws of SvNLNE. Then, this paper puts forward a SvNLNE weighted arithmetic averaging (SvNLNEWAA) operator and a SvNLNE weighted geometric averaging (SvNLNEWGA) operator, and further develops MADM approaches based on the presented operators in the SvNLNE setting.

In the construction of the paper, the preliminaries of SvNNs and LNNs are first reviewed in Section 2. The concepts, the fundamental operations, and the score and accuracy functions of SvNLNSs are put forward in Section 3. Then, two aggregation operators of SvNLNEWAA and SvNLNEWGA are presented and proved in Section 4. In Section 5, a new MADM method with SvNLNE information is developed by applying the proposed SvNLNEWAA and SvNLNEWGA operators. Finally, comparative analysis and conclusions are given in Sections 6 and 7, respectively.

2. Preliminaries of SvNNs and LNNs

This section introduces the concepts and operational relations of SvNNs and LNNs.

2.1 SvNNs

Definition 1 [5]. Set E as a fixed universal set. Then, a SvNS H in E can be given by

$$H = \{(\delta, \langle T(\delta), U(\delta), V(\delta)\rangle) \mid \delta \in E\},$$

Where $\langle T(\delta), U(\delta), V(\delta)\rangle$ is the SvNN for $\delta \in E$ satisfying the condition of $T(\delta), U(\delta), V(\delta) \in [0,1]$, and can simply be written as $h_\delta = \langle T_\delta, U_\delta, V_\delta\rangle$.

Definition 2 [5]. Assuming $h_{\delta 1} = \langle T_{\delta 1}, U_{\delta 1}, V_{\delta 1}\rangle$ and $h_{\delta 2} = \langle T_{\delta 2}, U_{\delta 2}, V_{\delta 2}\rangle$ are two SvNNs and $\sigma > 0$, there are the following relations:

(1) $h_{\delta1}\oplus h_{\delta2}=\langle T_{\delta1}+T_{\delta2}-T_{\delta1}T_{\delta2},U_{\delta1}U_{\delta2},V_{\delta1}V_{\delta2}\rangle$;

(2) $h_{\delta1}\otimes h_{\delta2}=\langle T_{\delta1}T_{\delta2},U_{\delta1}+U_{\delta2}-U_{\delta1}U_{\delta2},V_{\delta1}+V_{\delta2}-V_{\delta1}V_{\delta2}\rangle$;

(3) $\sigma h_{\delta1}=\left\langle 1-(1-T_{\delta1})^{\sigma},U_{\delta1}^{\sigma},V_{\delta1}^{\sigma}\right\rangle$;

(4) $h_{\delta1}^{\sigma}=\left\langle T_{\delta1}^{\sigma},1-(1-U_{\delta1})^{\sigma},1-(1-V_{\delta1})^{\sigma}\right\rangle$.

2.2 *LNNs*

Definition 3 [15]. Set E as a fixed universal set and $L = \{l_s \mid s = 0, 1, \cdots, r\}$ as a linguistic term set (LTS) whose odd cardinality is $r + 1$. Then, a linguistic neutrosophic set Z in E can be given by

$$Z=\left\{\left(\delta,\left\langle l_{\tau(\delta)},l_{u(\delta)},l_{v(\delta)}\right\rangle\right)\mid \delta\in E\right\},$$

where $\left\langle l_{\tau(\delta)},l_{u(\delta)},l_{v(\delta)}\right\rangle$ is a LNN for $\delta \in E$, containing the linguistic variables of truth, indeterminacy and falsity $l_{\tau(\delta)},l_{u(\delta)},l_{v(\delta)}\in L$. Then, the LNN $\left\langle l_{\tau(\delta)},l_{u(\delta)},l_{v(\delta)}\right\rangle$ can be simply denoted as $z_\delta=\left\langle l_{\tau_\delta},l_{u_\delta},l_{v_\delta}\right\rangle$.

Definition 4 [15]. Assuming $z_{\delta1}=\left\langle l_{\tau_{\delta1}},l_{u_{\delta1}},l_{v_{\delta1}}\right\rangle$ and $z_{\delta2}=\left\langle l_{\tau_{\delta2}},l_{u_{\delta2}},l_{v_{\delta2}}\right\rangle$ are two LNNs in L and $\sigma > 0$, there exist the following relations:

(1) $z_{\delta1}\oplus z_{\delta2}=\left\langle l_{\tau_{\delta1}+\tau_{\delta2}-\frac{\tau_{\delta1}\tau_{\delta2}}{r}},l_{\frac{u_{\delta1}u_{\delta2}}{r}},l_{\frac{v_{\delta1}v_{\delta2}}{r}}\right\rangle$;

(2) $z_{\delta1}\otimes z_{\delta2}=\left\langle l_{\frac{\tau_{\delta1}\tau_{\delta2}}{r}},l_{u_{\delta1}+u_{\delta2}-\frac{u_{\delta1}u_{\delta2}}{r}},l_{v_{\delta1}+v_{\delta2}-\frac{v_{\delta1}v_{\delta2}}{r}}\right\rangle$;

(3) $\sigma z_{\delta1}=\left\langle l_{r-r\left(1-\frac{\tau_{\delta1}}{r}\right)^{\sigma}},l_{r\left(\frac{u_{\delta1}}{r}\right)^{\sigma}},l_{r\left(\frac{v_{\delta1}}{r}\right)^{\sigma}}\right\rangle$;

(4) $z_{\delta1}^{\sigma}=\left\langle l_{r\left(\frac{\tau_{\delta1}}{r}\right)^{\sigma}},l_{r-r\left(1-\frac{u_{\delta1}}{r}\right)^{\sigma}},l_{r-r\left(1-\frac{v_{\delta1}}{r}\right)^{\sigma}}\right\rangle$.

3. SvNLNSs

Definition 5. Set E as a universal set and $L = \{l_s \mid s = 0, 1, \cdots, r\}$ as a LTS with an odd cardinality $r + 1$. Then, a SvNLNS H can be defined as

$$H=\left\{\left(\delta,\langle T(\delta),U(\delta),V(\delta)\rangle,\left\langle l_{\tau(\delta)},l_{u(\delta)},l_{v(\delta)}\right\rangle\right)\middle|\,\delta\in E\right\},$$

where $<T(\delta), U(\delta), V(\delta)>$ for $\delta \in E$ is a SvNN depicted independently by the truth, indeterminacy and falsity numerical variables $T(\delta), U(\delta), V(\delta) \in [0, 1]$, and $\left\langle l_{\tau(\delta)},l_{u(\delta)},l_{v(\delta)}\right\rangle$ for $\delta \in E$ is a LNN described independently by the truth, indeterminacy and falsity linguistic variables $l_{\tau(\delta)},l_{u(\delta)},l_{v(\delta)}\in L$ with $\tau(\delta),u(\delta),v(\delta) \in [0, r]$.

Then, the element $\left(\delta,\langle T(\delta),U(\delta),V(\delta)\rangle,\left\langle l_{\tau(\delta)},l_{u(\delta)},l_{v(\delta)}\right\rangle\right)$ of H can be simply represented by $\xi_\delta=\left(\langle T_\delta,U_\delta,V_\delta\rangle,\left\langle l_{\tau_\delta},l_{u_\delta},l_{v_\delta}\right\rangle\right)$ for T_δ, U_δ, $V_\delta \in [0, 1]$, $l_{\tau_\delta},l_{u_\delta},l_{v_\delta}\in L$, and τ_δ, u_δ, $v_\delta \in [0, r]$, called SvNLNE. It is obvious that SvNLNS is composed of SvNNs and LNNs in E.

Definition 6. Set $\xi_{\delta1}=\left(\langle T_{\delta1},U_{\delta1},V_{\delta1}\rangle,\left\langle l_{\tau_{\delta1}},l_{u_{\delta1}},l_{v_{\delta1}}\right\rangle\right)$ and $\xi_{\delta2}=\left(\langle T_{\delta2},U_{\delta2},V_{\delta2}\rangle,\left\langle l_{\tau_{\delta2}},l_{u_{\delta2}},l_{v_{\delta2}}\right\rangle\right)$ as two SvNLNEs. Then there exist the following relations:

(1) $\xi_{\delta1}\subseteq\xi_{\delta2}\Leftrightarrow T_{\delta1}\le T_{\delta2}$, $U_{\delta1}\ge U_{\delta2}$, $V_{\delta1}\ge V_{\delta2}$, $l_{\tau_{\delta1}}\le l_{\tau_{\delta2}}$, $l_{u_{\delta1}}\ge l_{u_{\delta2}}$, and $l_{v_{\delta1}}\ge l_{v_{\delta2}}$;

(2) $\xi_{\delta 1} = \xi_{\delta 2} \Leftrightarrow \xi_{\delta 1} \subseteq \xi_{\delta 2}$ and $\xi_{\delta 1} \supseteq \xi_{\delta 2}$, i.e., $T_{\delta 1} = T_{\delta 2}$, $U_{\delta 1} = U_{\delta 2}$, $V_{\delta 1} = V_{\delta 2}$, $l_{\tau_{\delta 1}} = l_{\tau_{\delta 2}}$, $l_{u_{\delta 1}} = l_{u_{\delta 2}}$, and $l_{v_{\delta 1}} = l_{v_{\delta 2}}$;

(3) $\xi_{\delta 1} \oplus \xi_{\delta 2} = \left(\left\langle T_{\delta 1} + T_{\delta 2} - T_{\delta 1}T_{\delta 2}, U_{\delta 1}U_{\delta 2}, V_{\delta 1}V_{\delta 2} \right\rangle, \left\langle l_{\tau_{\delta 1}+\tau_{\delta 2}-\frac{\tau_{\delta 1}\tau_{\delta 2}}{r}}, l_{\frac{u_{\delta 1}u_{\delta 2}}{r}}, l_{\frac{v_{\delta 1}v_{\delta 2}}{r}} \right\rangle \right)$;

(4) $\xi_{\delta 1} \otimes \xi_{\delta 2} = \left(\left\langle T_{\delta 1}T_{\delta 2}, U_{\delta 1} + U_{\delta 2} - U_{\delta 1}U_{\delta 2}, V_{\delta 1} + V_{\delta 2} - V_{\delta 1}V_{\delta 2} \right\rangle, \left\langle l_{\frac{\tau_{\delta 1}\tau_{\delta 2}}{r}}, l_{u_{\delta 1}+u_{\delta 2}-\frac{u_{\delta 1}u_{\delta 2}}{r}}, l_{v_{\delta 1}+v_{\delta 2}-\frac{v_{\delta 1}v_{\delta 2}}{r}} \right\rangle \right)$;

(5) $\sigma\xi_{\delta 1} = \left(\left\langle 1-(1-T_{\delta 1})^{\sigma}, U_{\delta 1}^{\sigma}, V_{\delta 1}^{\sigma} \right\rangle, \left\langle l_{r-r\left(1-\frac{\tau_{\delta 1}}{r}\right)^{\sigma}}, l_{r\left(\frac{u_{\delta 1}}{r}\right)^{\sigma}}, l_{r\left(\frac{v_{\delta 1}}{r}\right)^{\sigma}} \right\rangle \right)$ for σ > 0;

(6) $\xi_{\delta 1}^{\sigma} = \left(\left\langle T_{\delta 1}^{\sigma}, 1-(1-U_{\delta 1})^{\sigma}, 1-(1-V_{\delta 1})^{\sigma} \right\rangle, \left\langle l_{r\left(\frac{\tau_{\delta 1}}{r}\right)^{\sigma}}, l_{r-r\left(1-\frac{u_{\delta 1}}{r}\right)^{\sigma}}, l_{r-r\left(1-\frac{v_{\delta 1}}{r}\right)^{\sigma}} \right\rangle \right)$ for $\sigma > 0$.

To compare SvNLNEs, the score and accuracy functions for SvNLNEs and their sorting approaches are given by the definitions below.

Definition 7. Set ξ = (< T, U, V>, <l_τ, l_u, l_v>) as SvNLNE. Then, its score and accuracy functions are

$$F(\xi) = \frac{1}{2}\left(\frac{2+T-U-V}{3} + \frac{2r+\tau-u-v}{3r} \right) \text{ for } F(\xi) \in [0, 1], \quad (1)$$

$$G(\xi) = \frac{1}{2}\left[T - V + \frac{\tau - v}{r} \right] \text{ for } G(\xi) \in [-1, 1]. \quad (2)$$

Definition 8. Let $\xi_{\delta 1} = \left(\left\langle T_{\delta 1}, U_{\delta 1}, V_{\delta 1} \right\rangle, \left\langle l_{\tau_{\delta 1}}, l_{u_{\delta 1}}, l_{v_{\delta 1}} \right\rangle \right)$ and $\xi_{\delta 2} = \left(\left\langle T_{\delta 2}, U_{\delta 2}, V_{\delta 2} \right\rangle, \left\langle l_{\tau_{\delta 2}}, l_{u_{\delta 2}}, l_{v_{\delta 2}} \right\rangle \right)$ be two SvNLNEs, then based on the score and accuracy values of $F(\xi_{\delta\varsigma})$ and $G(\xi_{\delta\varsigma})$ (ς =1, 2), the ranking approaches are given below:

(1) If $F(\xi_{\delta 1}) > F(\xi_{\delta 2})$, then $\xi_{\delta 1} > \xi_{\delta 2}$;

(2) If $F(\xi_{\delta 1}) = F(\xi_{\delta 2})$ and $G(\xi_{\delta 1}) > G(\xi_{\delta 2})$, then $\xi_{\delta 1} > \xi_{\delta 2}$;

(3) If $F(\xi_{\delta 1}) = F(\xi_{\delta 2})$ and $G(\xi_{\delta 1}) = G(\xi_{\delta 2})$, then $\xi_{\delta 1} = \xi_{\delta 2}$.

4. Aggregation Operators of SvNLNEs

4.1 SvNLNEWAA Operator

Theorem 1. Set $\xi_\varsigma = \left(\left\langle T_\varsigma, U_\varsigma, V_\varsigma \right\rangle, \left\langle l_{\tau_\varsigma}, l_{u_\varsigma}, l_{v_\varsigma} \right\rangle \right)$ (ς = 1, 2, ⋯, η) as a collection of SvNLNEs. Then the SvNLNEWAA operator can be represented as

$$SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) = \sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi_\varsigma = \begin{pmatrix} \left\langle 1-\prod_{\varsigma=1}^{\eta}(1-T_\varsigma)^{\sigma_\varsigma}, \prod_{\varsigma=1}^{\eta} U_\varsigma^{\sigma_\varsigma}, \prod_{\varsigma=1}^{\eta} V_\varsigma^{\sigma_\varsigma}, \right\rangle, \\ \left\langle l_{r-r\prod_{\varsigma=1}^{\eta}(1-\frac{\tau_\varsigma}{r})^{\sigma_\varsigma}}, l_{r\prod_{\varsigma=1}^{\eta}(\frac{u_\varsigma}{r})^{\sigma_\varsigma}}, l_{r\prod_{\varsigma=1}^{\eta}(\frac{v_\varsigma}{r})^{\sigma_\varsigma}} \right\rangle \end{pmatrix}, \quad (3)$$

where $\sigma_\varsigma \in [0, 1]$ is the weight of ξ_ς (ς=1, 2, ⋯, η) with $\sum_{\varsigma=1}^{\eta} \sigma_\varsigma = 1$.

Proof. (1) It is straightforward that the theorem is valid when η = 1;

(2) When η = 2, from the relation (5) of Definition 6, we can obtain

$$\sigma_1\xi_1 = \left(\left\langle 1-(1-T_1)^{\sigma_1}, U_1^{\sigma_1}, V_1^{\sigma_1} \right\rangle, \left\langle l_{r-r\left(1-\frac{\tau_1}{r}\right)^{\sigma_1}}, l_{r\left(\frac{u_1}{r}\right)^{\sigma_1}}, l_{r\left(\frac{v_1}{r}\right)^{\sigma_1}} \right\rangle \right),$$

$$\sigma_2\xi_2=\left(\left\langle 1-(1-T_2)^{\sigma_2},U_2^{\sigma_2},V_2^{\sigma_2}\right\rangle,\left\langle l_{r-r\left(1-\frac{\tau_2}{r}\right)^{\sigma_2}},l_{r\left(\frac{u_2}{r}\right)^{\sigma_2}},l_{r\left(\frac{v_2}{r}\right)^{\sigma_2}}\right\rangle\right).$$

From the relation (3) of Definition 6, the SvNLNEWAA aggregation result is

$$SvNLNEWAA(\xi_1,\xi_2)=\sigma_1\xi_1\oplus\sigma_2\xi_2$$

$$=\left(\begin{array}{l}\left\langle 1-\left(1-T_1\right)^{\sigma_1}+1-\left(1-T_2\right)^{\sigma_2}-\left[1-\left(1-T_1\right)^{\sigma_1}\right]\left[1-\left(1-T_2\right)^{\sigma_2}\right],U_1^{\sigma_1}U_2^{\sigma_2},V_1^{\sigma_1}V_2^{\sigma_2}\right\rangle,\\ \left\langle l_{r-r\left(1-\frac{\tau_1}{r}\right)^{\sigma_1}+r-r\left(1-\frac{\tau_2}{r}\right)^{\sigma_2}-\frac{\left[r-r\left(1-\frac{\tau_1}{r}\right)^{\sigma_1}\right]\left[r-r\left(1-\frac{\tau_2}{r}\right)^{\sigma_2}\right]}{r}},l_{\frac{r\left(\frac{u_1}{r}\right)^{\sigma_1}r\left(\frac{u_2}{r}\right)^{\sigma_2}}{r}},l_{\frac{r\left(\frac{v_1}{r}\right)^{\sigma_1}r\left(\frac{v_2}{r}\right)^{\sigma_2}}{r}}\right\rangle\end{array}\right)$$

$$=\left(\left\langle 1-\prod_{\varsigma=1}^{2}\left(1-T_\varsigma\right)^{\sigma_\varsigma},\prod_{\varsigma=1}^{2}U_\varsigma^{\sigma_\varsigma},\prod_{\varsigma=1}^{2}V_\varsigma^{\sigma_\varsigma}\right\rangle,\left\langle l_{r-r\prod_{\varsigma=1}^{2}\left(1-\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{2}\left(\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{2}\left(\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\right).$$

(3) Let $\eta=\mu$, the aggregation result of SvNLNS is

$$SvNLNEWAA(\xi_1,\xi_2,\cdots,\xi_\mu)=\sum_{\varsigma=1}^{\mu}\sigma_\varsigma\xi_\varsigma$$

$$=\left(\left\langle 1-\prod_{\varsigma=1}^{\mu}\left(1-T_\varsigma\right)^{\sigma_\varsigma},\prod_{\varsigma=1}^{\mu}U_\varsigma^{\sigma_\varsigma},\prod_{\varsigma=1}^{\mu}V_\varsigma^{\sigma_\varsigma}\right\rangle,\left\langle l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{\mu}\left(\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{\mu}\left(\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\right).$$

(4) Let $\eta=\mu+1$, the aggregation result of SvNLNS is

$$SvNLNEWAA(\xi_1,\xi_2,\cdots,\xi_{\mu+1})=\sum_{\varsigma=1}^{\mu}\sigma_\varsigma\xi_\varsigma\oplus\sigma_{\mu+1}\xi_{\mu+1}$$

$$=\left(\begin{array}{l}\left\langle\begin{array}{l}1-\prod_{\varsigma=1}^{\mu}\left(1-T_\varsigma\right)^{\sigma_\varsigma}+1-\left(1-T_{\mu+1}\right)^{\sigma_{\mu+1}}-\left[1-\prod_{\varsigma=1}^{\mu}\left(1-T_\varsigma\right)^{\sigma_\varsigma}\right]\left[1-\left(1-T_{\mu+1}\right)^{\sigma_{\mu+1}}\right],\\ \left(\prod_{\varsigma=1}^{\mu}U_\varsigma^{\sigma_\varsigma}\right)U_{\mu+1}^{\sigma_{\mu+1}},\left(\prod_{\varsigma=1}^{\mu}V_\varsigma^{\sigma_\varsigma}\right)V_{\mu+1}^{\sigma_{\mu+1}}\end{array}\right\rangle,\\ \left\langle\begin{array}{l}l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}+r-r\left(1-\frac{\tau_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}-\frac{\left[r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}\right]\left[r-r\left(1-\frac{\tau_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}\right]}{r}},\\ l_{\frac{\left[r\prod_{\varsigma=1}^{\mu}\left(\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}\right]r\left(\frac{u_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}}{r}},l_{\frac{\left[r\prod_{\varsigma=1}^{\mu}\left(\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}\right]r\left(\frac{v_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}}{r}}\end{array}\right\rangle\end{array}\right)$$

$$=\left(\left\langle 1-\prod_{\varsigma=1}^{\mu+1}\left(1-T_\varsigma\right)^{\sigma_\varsigma},\prod_{\varsigma=1}^{\mu+1}U_\varsigma^{\sigma_\varsigma},\prod_{\varsigma=1}^{\mu+1}V_\varsigma^{\sigma_\varsigma}\right\rangle,\left\langle l_{r-r\prod_{\varsigma=1}^{\mu+1}\left(1-\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{\mu+1}\left(\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{\mu+1}\left(\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\right).$$

Thus, Theorem 1 is proved to be valid for any η.

Additionally, for the SvNLNE collection given by $\xi_\varsigma=\left(\left\langle T_\varsigma,U_\varsigma,V_\varsigma\right\rangle,\left\langle l_{\tau_\varsigma},l_{u_\varsigma},l_{v_\varsigma}\right\rangle\right)$ (ς = 1, 2, ⋯, η), the SvNLNEWAA operator implies some properties:

(1) Idempotency: There is $SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta)=\xi$ when $\xi_\varsigma=\xi$ is satisfied for ς = 1, 2, ⋯, η.

(2) Boundedness: Assume $\xi^- = \left(\left\langle \min_\varsigma(T_\varsigma), \max_\varsigma(U_\varsigma), \max_\varsigma(V_\varsigma) \right\rangle, \left\langle \min_\varsigma(l_{\tau_\varsigma}), \max_\varsigma(l_{u_\varsigma}), \max_\varsigma(l_{v_\varsigma}) \right\rangle \right)$ and $\xi^+ = \left(\left\langle \max_\varsigma(T_\varsigma), \min_\varsigma(U_\varsigma), \min_\varsigma(V_\varsigma) \right\rangle, \left\langle \max_\varsigma(l_{\tau_\varsigma}), \min_\varsigma(l_{u_\varsigma}), \min_\varsigma(l_{v_\varsigma}) \right\rangle \right)$ represent the minimum and maximum SvNLNEs for ς= 1, 2, …, η, then $\xi^- \le SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) \le \xi^+$.

(3) Monotonicity: There is $SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) \le SvNLNEWAA(\xi_1^*, \xi_2^*, \cdots, \xi_\eta^*)$ when the condition of $\xi_\varsigma \le \xi_\varsigma^*$ is satisfied for ς= 1, 2, ⋯, η.

Proof. (1) Suppose ξ=(<T, U, V>, <l_τ, l_u, l_v>). Since ξ_ς is equal to ξ for ς = 1, 2, ⋯, η, we can obtain

$$SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) = \sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi_\varsigma$$

$$= \left(\left\langle 1-(1-T)^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma}, U^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma}, V^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma} \right\rangle, \left\langle l_{r-r\left(1-\frac{\tau}{r}\right)^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma}}, l_{r\left(\frac{u}{r}\right)^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma}}, l_{r\left(\frac{v}{r}\right)^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma}} \right\rangle \right) = \left(\langle T, U, V \rangle, \langle l_\tau, l_u, l_v \rangle \right) = \xi .$$

(2) Because ξ^- is the minimum SvNLNE and ξ^+ is the maximum SvNLNE, $\xi^- \leqslant \xi_\varsigma \leqslant \xi^+$ can be obtained. Hence, $\sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi^- \le \sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi_\varsigma \le \sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi^+$. According to the property (1), $\sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi^- = \xi^-$ and

$\sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi^+ = \xi^+$. Thus, $\xi^- \le SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) \le \xi^+$.

(3) Since $\xi_\varsigma \le \xi_\varsigma^*$ for ς= 1, 2, …, η, there exists $\sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi_\varsigma \le \sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi_\varsigma^*$. Therefore,

$SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) \le SvNLNEWAA(\xi_1^*, \xi_2^*, \cdots, \xi_\eta^*)$.

The properties of the SvNLNEWAA operator are proved above. □

4.2 SvNLNEWGA Operator

Theorem 2. Set $\xi_\varsigma = \left(\langle T_\varsigma, U_\varsigma, V_\varsigma \rangle, \langle l_{\tau_\varsigma}, l_{u_\varsigma}, l_{v_\varsigma} \rangle \right)$ (ς = 1, 2, ⋯, η) as a cluster of SvNLNEs, then the SvNLNEWGA operator is

$$SvNLNEWGA(\xi_1, \xi_2, \cdots, \xi_\eta) = \prod_{\varsigma=1}^{\eta} \xi_\varsigma^{\sigma_\varsigma} = \begin{pmatrix} \left\langle \prod_{\varsigma=1}^{\eta} T_\varsigma^{\sigma_\varsigma}, 1-\prod_{\varsigma=1}^{\eta}(1-U_\varsigma)^{\sigma_\varsigma}, 1-\prod_{\varsigma=1}^{\eta}(1-V_\varsigma)^{\sigma_\varsigma} \right\rangle, \\ \left\langle l_{r\prod_{\varsigma=1}^{\eta}\left(\frac{T_\varsigma}{r}\right)^{\sigma_\varsigma}}, l_{r-r\prod_{\varsigma=1}^{\eta}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}}, l_{r-r\prod_{\varsigma=1}^{\eta}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}} \right\rangle \end{pmatrix}, \quad (4)$$

where $\sigma_\varsigma \in [0, 1]$ indicates the weight of ξ_ς with $\sum_{\varsigma=1}^{\eta} \sigma_\varsigma = 1$.

Proof. (1) When η = 1, the theorem 2 is obviously correct;
(2) When η = 2, from the relation (6) of Definition 6, we can obtain

$$\xi_1^{\sigma_1} = \left(\left\langle T_1^{\sigma_1}, 1-(1-U_1)^{\sigma_1}, 1-(1-V_1)^{\sigma_1} \right\rangle, \left\langle l_{r\left(\frac{\tau_1}{r}\right)^{\sigma_1}}, l_{r-r\left(1-\frac{u_1}{r}\right)^{\sigma_1}}, l_{r-r\left(1-\frac{v_1}{r}\right)^{\sigma_1}} \right\rangle \right),$$

$$\xi_2^{\sigma_2}=\left(\left\langle T_2^{\sigma_2},1-(1-U_2)^{\sigma_2},1-(1-V_2)^{\sigma_2}\right\rangle,\left\langle l_{r\left(\frac{\tau_2}{r}\right)^{\sigma_2}},l_{r-r\left(1-\frac{u_2}{r}\right)^{\sigma_2}},l_{r-r\left(1-\frac{v_2}{r}\right)^{\sigma_2}}\right\rangle\right).$$

From the relation (4) of Definition 6, the aggregation result is

$$SvNLNEWGA(\xi_1,\xi_2)=\xi_1^{\sigma_1}\otimes\xi_2^{\sigma_1}$$

$$=\left(\begin{array}{l}\left\langle\begin{array}{l}T_1^{\sigma_1}T_2^{\sigma_2},1-(1-U_1)^{\sigma_1}+1-(1-U_2)^{\sigma_2}-\left[1-(1-U_1)^{\sigma_1}\right]\left[1-(1-U_2)^{\sigma_2}\right],\\1-(1-V_1)^{\sigma_1}+1-(1-V_2)^{\sigma_2}-\left[1-(1-V_1)^{\sigma_1}\right]\left[1-(1-V_2)^{\sigma_2}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}l_{\frac{r\left(\frac{\tau_1}{r}\right)^{\sigma_1}r\left(\frac{\tau_2}{r}\right)^{\sigma_2}}{r}},l_{r-r\left(1-\frac{u_1}{r}\right)^{\sigma_1}+r-r\left(1-\frac{u_2}{r}\right)^{\sigma_2}-\frac{\left[r-r\left(1-\frac{u_1}{r}\right)^{\sigma_1}\right]\left[r-r\left(1-\frac{u_2}{r}\right)^{\sigma_2}\right]}{r}},\\ l_{r-r\left(1-\frac{v_1}{r}\right)^{\sigma_1}+r-r\left(1-\frac{v_2}{r}\right)^{\sigma_2}-\frac{\left[r-r\left(1-\frac{v_1}{r}\right)^{\sigma_1}\right]\left[r-r\left(1-\frac{v_2}{r}\right)^{\sigma_2}\right]}{r}}\end{array}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{l}\left\langle\prod_{\varsigma=1}^{2}T_\varsigma^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{2}(1-U_\varsigma)^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{2}(1-V_\varsigma)^{\sigma_\varsigma}\right\rangle,\\ \left\langle l_{r\prod_{\varsigma=1}^{2}\left(\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{2}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{2}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\end{array}\right).$$

(3) Let $\eta=\mu$, the aggregation result of SvNLNS is

$$SvNLNEWGA(\xi_1,\xi_2,\cdots,\xi_\mu)=\prod_{\varsigma=1}^{\mu}\xi_\varsigma^{\sigma_\varsigma}=\left(\begin{array}{l}\left\langle\prod_{\varsigma=1}^{\mu}T_\varsigma^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{\mu}(1-U_\varsigma)^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{\mu}(1-V_\varsigma)^{\sigma_\varsigma}\right\rangle,\\ \left\langle l_{r\prod_{\varsigma=1}^{\mu}\left(\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\end{array}\right).$$

(4) Let $\eta=\mu+1$, the aggregation result of SvNLNS is

$$SvNLNEWGA(\xi_1,\xi_2,\cdots,\xi_{\mu+1})=\prod_{\varsigma=1}^{\mu}\xi_\varsigma^{\sigma_\varsigma}\otimes\xi_{\mu+1}^{\sigma_{\mu+1}}$$

$$=\left(\begin{array}{l}\left\langle\begin{array}{l}\left(\prod_{\varsigma=1}^{\mu}T_\varsigma^{\sigma_\varsigma}\right)T_{\mu+1}^{\sigma_{\mu+1}},1-\prod_{\varsigma=1}^{\mu}(1-U_\varsigma)^{\sigma_\varsigma}+1-(1-U_{\mu+1})^{\sigma_{\mu+1}}-\left[1-\prod_{\varsigma=1}^{\mu}(1-U_\varsigma)^{\sigma_\varsigma}\right]\left[1-(1-U_{\mu+1})^{\sigma_{\mu+1}}\right],\\1-\prod_{\varsigma=1}^{\mu}(1-V_\varsigma)^{\sigma_\varsigma}+1-(1-V_{\mu+1})^{\sigma_{\mu+1}}-\left[1-\prod_{\varsigma=1}^{\mu}(1-V_\varsigma)^{\sigma_\varsigma}\right]\left[1-(1-V_{\mu+1})^{\sigma_{\mu+1}}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}l_{\frac{r\prod_{\varsigma=1}^{\mu}\left(\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}r\left(\frac{\tau_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}}{r}},l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}+r-r\left(1-\frac{u_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}-\frac{\left[r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}\right]\left[r-r\left(1-\frac{u_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}\right]}{r}},\\ l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}+r-r\left(1-\frac{v_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}-\frac{\left[r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}\right]\left[r-r\left(1-\frac{v_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}\right]}{r}}\end{array}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{l}\left\langle\prod_{\varsigma=1}^{\mu+1}T_\varsigma^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{\mu+1}(1-U_\varsigma)^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{\mu+1}(1-V_\varsigma)^{\sigma_\varsigma}\right\rangle,\\ \left\langle l_{r\prod_{\varsigma=1}^{\mu+1}\left(\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{\mu+1}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{\mu+1}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\end{array}\right).$$

Thus, Eq.(4) is proved to be valid for any η.

Additionally, for the group of SvNLNEs given by $\xi_\varsigma=\left(\left\langle T_\varsigma,U_\varsigma,V_\varsigma\right\rangle,\left\langle l_{\tau_\varsigma},l_{u_\varsigma},l_{v_\varsigma}\right\rangle\right)$ (ς = 1, 2, …, η), there are some properties of the SvNLNEWGA operator:

(1) Idempotency: There is $SvNLNEWGA(\xi_1, \xi_2, \cdots, \xi_\eta) = \xi$ when $\xi_\varsigma = \xi$ is satisfied for ς = 1, 2, ···, η.

(2) Boundedness: Assume $\xi^- = \left(\left\langle \min_\varsigma(T_\varsigma), \max_\varsigma(U_\varsigma), \max_\varsigma(V_\varsigma)\right\rangle, \left\langle \min_\varsigma(l_{\tau_\varsigma}), \max_\varsigma(l_{u_\varsigma}), \max_\varsigma(l_{v_\varsigma})\right\rangle\right)$ and $\xi^+ = \left(\left\langle \max_\varsigma(T_\varsigma), \min_\varsigma(U_\varsigma), \min_\varsigma(V_\varsigma)\right\rangle, \left\langle \max_\varsigma(l_{\tau_\varsigma}), \min_\varsigma(l_{u_\varsigma}), \min_\varsigma(l_{v_\varsigma})\right\rangle\right)$ represents the minimum and maximum SvNLNEs for ς = 1, 2, …, η, then $\xi^- \le SvNLNEWGA(\xi_1, \xi_2, \cdots, \xi_\eta) \le \xi^+$.

(3) Monotonicity: There is $SvNLNEWGA(\xi_1, \xi_2, \cdots, \xi_\eta) \le SvNLNEWGA(\xi_1^*, \xi_1^*, \cdots, \xi_\eta^*)$ when the condition of $\xi_\varsigma \le \xi_\varsigma^*$ (ς = 1, 2, ···, η) is satisfied.

Since the property proof of the SvNLNEWGA operator is similar to that of the SvNLNEWAA operator, it is omitted here. □

5. MADM Method in the SvNLNE Setting

In this section, by applying the SvNLNEWAA and SvNLNEWGA operators, a novel MADM method is developed to solve DM problems with quantitative and qualitative information.

For a complex DM problem, m alternatives (given by R= {R_1, R_2, R_3, ···, R_m}) need to be evaluated on η attributes (given by S = {s_1, s_2, ···, s_η}) in the SvNLNE setting, where the attribute types may be different. Assume each alternative is evaluated as a SvNLNE $\xi_{\iota\varsigma} = \left(\left\langle T_{\iota\varsigma}, U_{\iota\varsigma}, V_{\iota\varsigma}\right\rangle, \left\langle l_{\tau_{\iota\varsigma}}, l_{u_{\iota\varsigma}}, l_{v_{\iota\varsigma}}\right\rangle\right)$ with ι = 1, 2, ···, m and ς = 1, 2, ···, η. Then, all evaluated values can be further constructed as the SvNLNE decision matrix $E = (\xi_{\iota\varsigma})_{m\times\eta}$.

Then, MADM problems with SvNLNE information can be solved by the SvNLNEWAA and SvNLNEWGA operators along with the SvNLNE score and accuracy functions. Details about the new MADM method are given as below.

Step 1. Standardize the initial evaluation data in the SvNLNE format. For instance, a quantitative attribute data denoted by the SvNN ξ = (<T, U, V>) can be converted into the SvNLNE ξ'= (<T, U, V>, <$l_{T\times r}$, $l_{U\times r}$, $l_{V\times r}$>), and a qualitative attribute data given by the LNN ξ = (<l_τ, l_u, l_v>) can be transformed into the SvNLNE ξ'= (<τ/r, u/r, v/r>, <l_τ, l_u, l_v>). As a result, the initial decision matrix $E = (\xi_{\iota\varsigma})_{m\times\eta}$ can be standardized as $E' = \left(\xi'_{\iota\varsigma}\right)_{m\times\eta}$.

Step 2. Assume P = {σ_1, σ_2, ···, σ_η} is a weight vector that represents the importance of attributes S = {s_1, s_2, ···, s_η}, where σ_ς is the weight of $\xi'_{\iota\varsigma}$ (ς = 1, 2, ···, η) with $\sigma_\varsigma \in [0, 1]$ and $\sum_{\varsigma=1}^{\eta}\sigma_\varsigma = 1$. Then, by applying the SvNLNEWAA operator, the aggregation result of ξ'_ι (ι = 1, 2, ···, m) is

$$\xi'_\iota = SvNLNEWAA\left(\xi'_{\iota1}, \xi'_{\iota2}, \cdots, \xi'_{\iota\eta}\right) = \sum_{\varsigma=1}^{\eta}\sigma_\varsigma \xi'_{\iota\varsigma} = \begin{pmatrix} \left\langle 1-\prod_{\varsigma=1}^{\eta}\left(1-T_{\iota\varsigma}\right)^{\sigma_\varsigma}, \prod_{\varsigma=1}^{\eta}U_{\iota\varsigma}^{\sigma_\varsigma}, \prod_{\varsigma=1}^{\eta}V_{\iota\varsigma}^{\sigma_\varsigma}\right\rangle, \\ \left\langle l_{r-r\prod_{\varsigma=1}^{\eta}\left(1-\frac{\tau_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}}, l_{r\prod_{\varsigma=1}^{\eta}\left(\frac{u_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}}, l_{r\prod_{\varsigma=1}^{\eta}\left(\frac{v_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}}\right\rangle \end{pmatrix}.$$

Similarly, by applying the SvNLNEWGA operator, the aggregation result of ξ'_ι (ι = 1, 2, ···, m) is

$$\xi_\iota^{'} = SvNLNEWGA\left(\xi_{\iota 1}^{'}, \xi_{\iota 2}^{'}, \cdots, \xi_{\iota \eta}^{'}\right) = \prod_{\varsigma=1}^{\eta}\left(\xi_{\iota\varsigma}^{'}\right)^{\sigma_\varsigma} = \begin{pmatrix} \left\langle \prod_{\varsigma=1}^{\eta} T_{\iota\varsigma}^{\sigma_\varsigma}, 1-\prod_{\varsigma=1}^{\eta}\left(1-U_{\iota\varsigma}\right)^{\sigma_\varsigma}, 1-\prod_{\varsigma=1}^{\eta}\left(1-V_{\iota\varsigma}\right)^{\sigma_\varsigma} \right\rangle, \\ \left\langle l_{r\prod_{\varsigma=1}^{\eta}\left(\frac{\tau_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}}, l_{r-r\prod_{\varsigma=1}^{\eta}\left(1-\frac{u_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}}, l_{r-r\prod_{\varsigma=1}^{\eta}\left(1-\frac{v_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}} \right\rangle \end{pmatrix}.$$

Step 3: Get the score values of $F\left(\xi_\iota^{'}\right)$ (ι = 1, 2, ···, m) by Eq. (1) and the accuracy values of $G\left(\xi_\iota^{'}\right)$ (ι = 1, 2, ···, m) by Eq. (2) if necessary.

Step 4: Sort all the alternatives in descending order of the score and accuracy values, then the first one is optimal.

Step 5: End.

6. Example

To illustrate the application of the raised MADM method in the SvNLNE environment, an example of investment decision is given in this section. This example is adapted from references [4,15] and contains both quantitative and qualitative attributes.

A company needs to choose the best of the four alternatives $\vartheta = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ that are engaged in electronic devices, weapons, clothing, and construction, respectively. Then, some experts are asked to comprehensively assess the options by considering the attributes $E = \{\delta_1, \delta_2, \delta_3, \delta_4\}$, where δ_1 is the environmental impact, δ_2 is the growth, δ_3 is the risk, and δ_4 is the possible return rate of the investment. The evaluation data can be given in any form of LNN, SvNN, or SvNLNE according to the attribute characteristics and the preferences of the evaluators. Among them, the qualitative information will be evaluated from the LTS L = {l_0 = very low, l_1 = low, l_2 = slight low, l_3 = medium, l_4 = slight high, l_5 = high, l_6 = very high} with the odd cardinality r + 1 = 7. Suppose the original evaluation matrix $M = (\xi_{\iota\varsigma})_{m\times\eta}$ is established as

$$M = \begin{bmatrix} \left(\langle 0.7, 0.2, 0.2\rangle, \langle l_4, l_1, l_1\rangle\right) & \left(\langle l_5, l_1, l_1\rangle\right) & \left(\langle l_5, l_3, l_1\rangle\right) & \left(\langle 0.78, 0.1, 0.2\rangle\right) \\ \left(\langle 0.8, 0.1, 0.1\rangle, \langle l_5, l_2, l_2\rangle\right) & \left(\langle l_5, l_1, l_2\rangle\right) & \left(\langle l_6, l_2, l_1\rangle\right) & \left(\langle 0.8, 0.2, 0.3\rangle\right) \\ \left(\langle 0.75, 0.2, 0.1\rangle, \langle l_4, l_2, l_1\rangle\right) & \left(\langle l_4, l_2, l_1\rangle\right) & \left(\langle l_6, l_1, l_1\rangle\right) & \left(\langle 0.75, 0.1, 0.1\rangle\right) \\ \left(\langle 0.9, 0.1, 0.1\rangle, \langle l_5, l_1, l_1\rangle\right) & \left(\langle l_5, l_2, l_2\rangle\right) & \left(\langle l_4, l_2, l_3\rangle\right) & \left(\langle 0.81, 0.2, 0.1\rangle\right) \end{bmatrix}.$$

According to the information standardization rules of SvNLNE, the matrix M can be standardized as

$$M^{'} = \begin{bmatrix} \left(\langle 0.7, 0.2, 0.2\rangle, \langle l_4, l_1, l_1\rangle\right) & \left(\langle 0.83, 0.17, 0.17\rangle, \langle l_5, l_1, l_1\rangle\right) & \left(\langle 0.83, 0.5, 0.17\rangle, \langle l_5, l_3, l_1\rangle\right) & \left(\langle 0.78, 0.1, 0.2\rangle, \langle l_{4.68}, l_{0.6}, l_{1.2}\rangle\right) \\ \left(\langle 0.8, 0.1, 0.1\rangle, \langle l_5, l_2, l_2\rangle\right) & \left(\langle 0.83, 0.17, 0.33\rangle, \langle l_5, l_1, l_2\rangle\right) & \left(\langle 1, 0.33, 0.17\rangle, \langle l_6, l_2, l_1\rangle\right) & \left(\langle 0.8, 0.2, 0.3\rangle, \langle l_{4.8}, l_{1.2}, l_{1.8}\rangle\right) \\ \left(\langle 0.75, 0.2, 0.1\rangle, \langle l_4, l_2, l_1\rangle\right) & \left(\langle 0.67, 0.33, 0.17\rangle, \langle l_4, l_2, l_1\rangle\right) & \left(\langle 1, 0.17, 0.17\rangle, \langle l_6, l_1, l_1\rangle\right) & \left(\langle 0.75, 0.1, 0.1\rangle, \langle l_{4.5}, l_{0.6}, l_{0.6}\rangle\right) \\ \left(\langle 0.9, 0.1, 0.1\rangle, \langle l_5, l_1, l_1\rangle\right) & \left(\langle 0.83, 0.33, 0.33\rangle, \langle l_5, l_2, l_2\rangle\right) & \left(\langle 0.67, 0.33, 0.5\rangle, \langle l_4, l_2, l_3\rangle\right) & \left(\langle 0.81, 0.2, 0.1\rangle, \langle l_{4.86}, l_{1.2}, l_{0.6}\rangle\right) \end{bmatrix}.$$

Assuming the weight vector P = (0.25, 0.2, 0.25, 0.3) represents the attribute importance of E, the decision process using the SvNLNEWAA operator can be performed as below.

Step 1. By Eq. (3), the aggregated values of SvNLNEWAA for each alternative ϑ_ι (ι = 1, 2, 3, 4) can be obtained as

ξ_1= (<0.7902, 0.197, 0.1842>, <4.7075,1.1291,1.0562>),

ξ_2 = (<1,0.1842,0.201>, <6,1.4937,1.6295>),

ξ_3 = (<1,0.1719,0.1258>, <6,1.1719,0.8579>),

ξ_4 = (<0.8186,0.2116,0.1902>, <4.7631,1.4428,1.297>).

Step 2. By Eq. (1), the score values of $F(\xi_\iota)$ (ι = 1, 2, 3, 4) can be further obtained as

$F(\xi_1)$ =0.8049, $F(\xi_2)$ = 0.849, $F(\xi_3)$ = 0.894, $F(\xi_4)$ = 0.7923.

Step 3. Since $F(\xi_3) > F(\xi_2) > F(\xi_1) > F(\xi_4)$, the ranking of the four alternatives is $\vartheta_3 > \vartheta_2 > \vartheta_1 > \vartheta_4$. Therefore, ϑ_3 is the best choice.

Similarly, the decision steps using the SvNLNEWGA operator can be carried out as below.

Step 1'. By Eq. (4), the aggregated values of SvNLNEWGA for each alternative ϑ_ι (ι = 1, 2, 3, 4) can be obtained as

ξ_1= (<0.7821,0.2571,0.1852>, <4.6358,1.4967,1.0609>),

ξ_2 = (<0.8528,0.2064,0.229>, <5.1695,1.5823,1.7082>),

ξ_3 = (<0.7872,0.1927,0.1306>, <4.5859,1.3721,0.8832>),

ξ_4 = (<0.7966,0.241, 0.2683>, <4.6886,1.5327,1.6932>).

Step 2'. By Eq. (1), the score values of $F(\xi_\iota)$ (ι = 1, 2, 3, 4) are

$F(\xi_1)$ = 0.781, $F(\xi_2)$ = 0.7884, $F(\xi_3)$ = 0.8087, $F(\xi_4)$ = 0.7552.

Step 3'. Since $F(\xi_3) > F(\xi_2) > F(\xi_1) > F(\xi_4)$, the four alternatives are ranked as $\vartheta_3 > \vartheta_2 > \vartheta_1 > \vartheta_4$. Thus, ϑ_3 is also the best choice.

Obviously, the sorting results obtained by the above two operators are the same, and the best options are also the same. Thus, one can choose one of the two operators according to the actual needs.

Different from the existing MADM approaches, the MADM method proposed in this paper handles the incomplete, inconsistent and uncertain data in the form of SvNLNE instead of SvNN or LNN, and uses two novel aggregation operators of SvNLNEWAA and SvNLNEWGA. The SvNLNE composed of SvNN and LNN uses numerical and linguistic variables to represent the truth, uncertainty, and falsity membership degrees of fuzzy information. Hence, it can express mixed information of quantitative and qualitative attributes better than SvNN or LNN that can only depict quantitative or qualitative attribute information. Moreover, the proposed SvNLNEWAA and SvNLNEWGA operators can aggregate SvNNs and LNNs in addition to SvNLNEs, because SvNN and LNN are two special cases of SvNLNE when all attributes are quantitative or qualitative. And the proposed MADM method can handle DM problems in the SvNN and/or LNN setting, while the existing DM methods of SvNN and LNN cannot deal with DM problems under the SvNLNE environment.

All in all, SvNLNE is the further generalization of SvNN and LNN, and the MADM method based on the SvNLNEWAA and SvNLNEWGA operators offers a unified way for complex DM problems with both quantitative and qualitative attributes.

7. Conclusions

This paper originally defined the concept, fundamental operations, and score and accuracy functions of SvNLNE, and then developed the MADM method of the SvNLNE using the proposed SvNLNEWAA and SvNLNEWGA operators. Finally, an investment case proved that the proposed MADM method can effectively solve MADM problems with the SvNLNEs that contain mixed-type or single-type attribute information, overcoming the shortcomings of traditional methods that can only handle single-type attribute data. The research results of this paper enrich the neutrosophic theory and MADM methods.

The paper mainly contributes: (1) The presented SvNLNE can effectively express mixed quantitative and qualitative information for the first time; (2) The proposed SvNLNEWAA and SvNLNEWGA operators can aggregate the hybrid information of SvNN and LNN; (3) The proposed MADM approach of SvNLNS can effectively solve complex DM problems containing qualitative and quantitative attributes, which cannot be satisfactorily processed by existing methods.

Further research will concentrate on the similarity measures of SvNLNEs, the development of novel aggregation operators, and their applications such as pattern recognition and medical diagnosis in the SvNLNE environment.

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Neutrosophic SuperHyper Bi-Topological Spaces: Original Notions and New Insights

Huda E. Khalid[1*] , Gonca D. GÜNGÖR[2] , Muslim A. Noah Zainal[3]

[1]Telafer University, The Administration Assistant for the President of the Telafer University, Telafer, Iraq. ; https://orcid.org/0000-0002-0968-5611 , dr.huda-ismael@uotelafer.edu.iq

[2] Çankırı Karatekin University, Faculty of Sciences, Department of Mathematics, Çankırı, Turkey gncmatematik@hotmail.com.

[3] Çankırı Karatekin University, Faculty of Sciences, Department of Mathematics, MSc. Student, Çankırı, Turky, moslmnooh1993@gmail.com

*Correspondence: dr.huda-ismael@uotelafer.edu.iq

Abstract.

This manuscript comes as first attempt in building a new type of neutrosophic topological spaces, the aim is to shed the light on a new structure known as the n^{th}-power set $P^n(X)$ of a set, this new kind of sets enables authors to create and built new topology spaces called Neutrosophic SuperHyper Topological Spaces and Neutrosophic SuperHyper Bi-Topological Spaces , the n^{th}-power sets are the optimal representation for the applications in our real world. In this article, new concepts and theorems related to this new topologies have been discussed, which are pairwise neutrosophic open n^{th}-power set, pairwise neutrosophic closed n^{th}-power set, as well as, the closures and the interiors are defined with their properties. Many of relations for these concepts have been introduced.

Keywords: n^{th} -power set $P^n(X)$; Neutrosophic SuperHyper Topological Spaces (NSHTSs); Neutrosophic SuperHyper Bi-Topological Spaces (NSHBi-TSs).

Introduction.

The concepts of the neutrosophic n^{th}-power set of a set, SuperHyperGraph and Pliothogenic n-SuperHyperGraph, SuperHyperAlgebra, n-ary (classical-/Neutro-/Anti-) HyperAlgebra have been firstly introduced by the father of neutrosophic theory F. Smarandache in 2016 [4]. As the introduction for Neutrosophic SuperHyper Topological Spaces which is until yet is fathomless branch of science, in this section we recalling the fundamental definitions of the neutrosophic logic with preliminaries of related n^{th}-power set of a set. There is no doubt that the essential theory of neutrosophic was introduced and built by F. Smarandache in 1995 [5,6]. Any mathematician who tracking the trace of this knowledge will easily deduce that the neutrosophic theory was rapidly and broadly radiated through Neutrosophic Sets and Systems journal, and International Journal of

Neutrosophic Science, these two journals are very active and reputed journals indexed by dozens of repositories, encyclopedias, and identifications' websites especially Scopus database.
This manuscript has been organized as follow:

The authors present some basic preliminaries in section 1, while section 2 has been dedicated to submit a new structure of neutrosophic topology called Neutrosophic SuperHyper Topological Spaces, in this section and for the first time, this type of topology was discussed in details. The main core of this article is in section 3 which is contain definitions, theorems, and corollaries covered the new subject that introduced firstly in this paper which is named Neutrosophic SuperHyper Bi-Topological Spaces. The last section is the conclusion section.

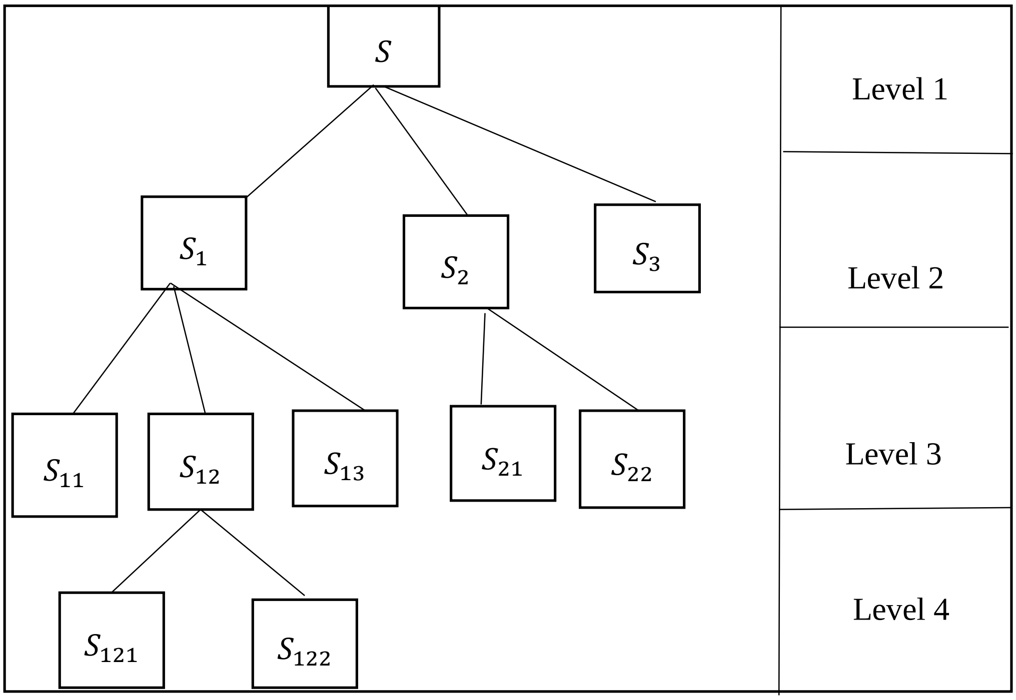
1. Preliminaries

1.1 System of Sub-System of Sub-Sub-System and so on [1]

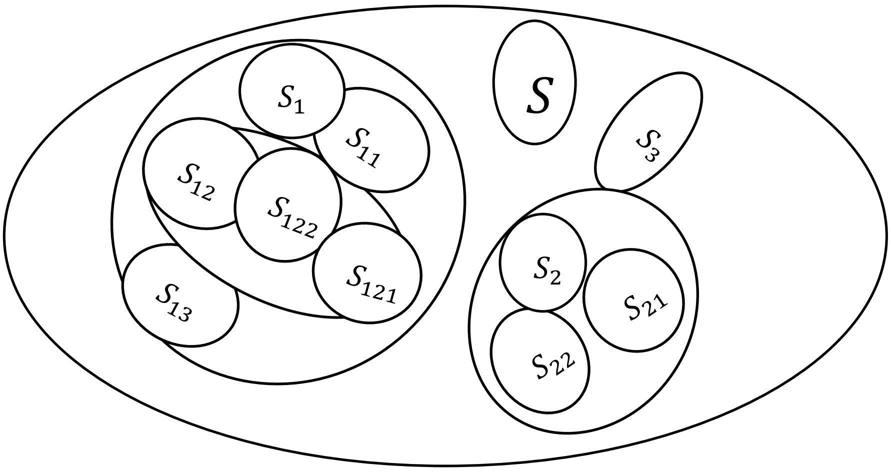
A system may be a set, space, organization, association, team, city, region, country, etc. One consider both: the static and dynamic systems.
With respect to various criteria, such as: political, religious, economic, military, educational, sportive, touristic, industrial, agricultural, etc.
A system S is made up of several sub-systems $S_1, S_2, \dots, S_p$, for integer $p \geq 1$; then each su-system S_i, for $i \in \{1,2,\dots,p\}$ is composed of many sub-sub-systems $S_{i1}, S_{i2}, \dots, S_{ip_i}$, for integer $p_i \geq 1$; then each sub-sub-systems S_{ij}, for $j \in \{1,2,\dots,p_i\}$ is composed sub-sub-sub-systems $S_{ij1}, S_{ij2}, \dots, S_{ijp_j}$, for integers p_j; and so on.

The following example of systems made of Sub-Sub-Sub-Systems (four levels)

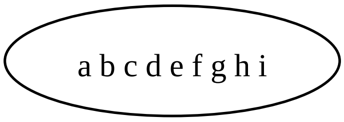
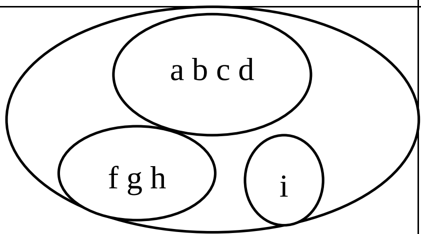
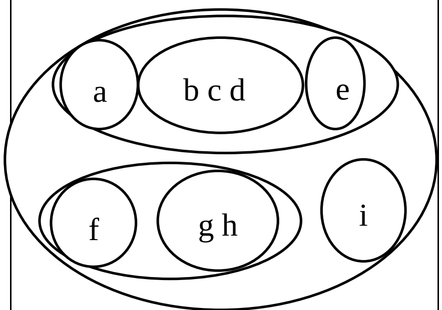
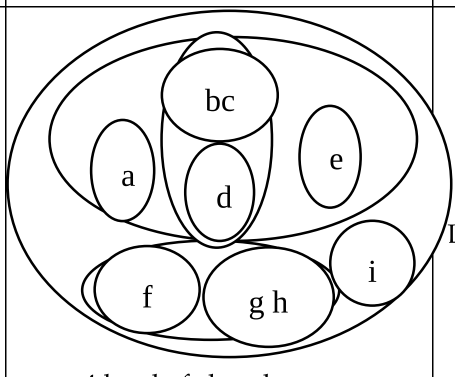
i) Using a Tree-Graph Representation, one has:



ii) Using a Geometric Representation, one has:



iii) Using an Algebraic Representation through pairs of braces {}, one has:

$P^0(S) = S = \{a, b, c, d, e, f, g, h, l\}$ 1 level of pairs of braces		Level 1
$P^1(S) = \{\{a, b, c, d, e\}, \{f, g, h\}, \{l\}\}$ 2 level of pairs of braces i.e. a pair of braces {} inside, another pair of braces {}, or {… {…} ...}	 2 level of closed curves	Level 2
$P^2(S) = P(P(S))$ $= \{\{\{a\}, \{b, c, d\}, \{e\}\}, \{\{f\}, \{g, h\}\}, \{l\}\}$ 3 levels of pairs of braces	 3 level of closed curves	Level 3
$P^3(S) = P(P^2(S))$ $= \{\{\{a\}, \{b, c\}, \{d\}, \{e\}\}, \{\{f\}, \{g, h\}\}, \{l\}\}$ 4 levels of pairs of braces	 4 level of closed curves	Level 4

1.2 Definition of nth -Power of a set [1]:

The n^{th} -Power of a set was firstly introduced by F. Smarandache at (2016) [4] by:
$P^n(S)$ as the n^{th} -PowerSet of the set S, for integer $n \geq 1$, is recursively defined as:
$P^2(S) = P(P(S))$, $P^3(S) = P\left(P(P(S))\right)$, …, $P^n(S) = P(P^{n-1}(S))$, where $P^0(S) = S$, $and\ P^1(S) = P(S)$, i.e. $P^0(S) \subset P^1(S) \subset P^2(S) \subset \cdots \subset P^{n-1}(S) \subset P^n(S)$.
The n^{th} -PowerSet of a Set is better reflect for our complex reality, since a set S (that may represent a group, a society, a country, a continent, etc.) of elements (such as: people, objects, and in general any items) is organized onto subsets $P(S)$, which on their turns are also organized onto subsets of subsets, and so on, that is our world.

1.3 Example

Suppose that the set of the grandparents represents the power set $P^2(S) = P(P(S))$, then the first offspring is the parents themselves which can be regarded as the power set $P(S)$, and the second offspring is the non-empty set $P^0(S) = S$, i.e. $S = P^0(S) \subset P^1(S) \subset P^2(S)$.
The following medical case study would be appropriate to demonstrate the importance of the power set concept:

There are many diseases and conditions that can be passed on through genes. Some of these diseases include Down syndrome, hemophilia, hypertension, sickle cell anemia, and cystic fibrosis. Most genetic diseases are a combination of mutations in multiple genes, often in combination with environmental factors. There are three groups of genetic diseases, each with their own causes: monogenetic diseases, multifactorial inherited diseases, and chromosomal abnormalities.

The couple of husband can be represented as PowerSet $P(S)$, it is important to know what $P(S)$ have inherited a genetic disease from their parents (i.e. represented the non-empty set $P^0(S) = S$ as grandparents) and to remember that the above mentioned genetic diseases can be passed on to their descendants (i.e. the offspring which is mathematically denoted by the power set $P^2(S) = P(P(S))$. If S & $P(S)$ are aware of possible diseases that can be inherited to $P(S)$& $P^2(S)$ respectively , contact a specialist and see what S & $P(S)$ can do to help $P(S)$& $P^2(S)$ and avoid serious problems later. By working together with the help of family and doctor, the health risks can be avoided instead of taking their toll later.

1.4 Neutrosophic HyperOperation and Neutrosophic HyperStructures [2]:

In the classical HyperOperation and classical HyperStructures, the empty-set $\emptyset$ does not belong to the power set, (i.e. $P_*(H) = P(H)/\{\emptyset\})$. Nonetheless, in the real world we encounter many situations when HyperOperation $\#$ is indeterminate, for example $a \# b = \emptyset$ (unknown, or undefined), or partially indeterminate, for example: $a \# b = \{\ [0.2, 0.3], \emptyset\}$. In our everyday life, there are many more operations and lows that have some degrees of indeterminacy (vagueness, unclearness, unknowingness, contradiction, etc.), than those that are totally determined. That's why in 2016 the scientists F. Smarandache have extended the classical HyperOperation to the Neutrosophic HyperOperation, by taking the whole power $P(H)$ (that includes the empty-set $\emptyset$ as well), instead of . $P_*(H)$(that does not include the empty-set $\emptyset$), as follow.

1.4.1 Definition of Neutrosophic HyperOperation:

Let U be a universe of discourse and H be a non-empty set, $H \subset U$.

A Neutrosophic Binary HyperOperation $\#_2$ is defined as follows:

$\#_2: H^2 \to P(H)$, where H is a discrete or continuous set, and $P(H)$ is the powerset of H that includes the empty-set $\emptyset$.

1.4.2 A Neutrosophic m-ary HyperOperation $\#_m$ is defined as:

$\#_m: H^m \to P(H)$, for integer $m \geq 1$. Similarly, for $m = 1$ one gets a Neutrosophic Unary HyperOperation.

2. Neutrosophic SuperHyper Topological Spaces

This section gives an original creativity neutrosophic mathematical structure for new notion named as Neutrosophic SuperHyper Topological Spaces (NSHTS) defined under a new kind of sets called neutrosophic n^{th}-power set $P^n(X)$.

2.1 Definition

Let X be a non-empty set, $P^n(X)$ is the neutrosophic n^{th}-power set of a set X, for integer $n \geq 1$. A Neutrosophic SuperHyper Topological space on $P^n(X)$ is a subfamily $\tau^{neutrotopo}$ of $N(P^n(X))$, and satisfying the following axioms:

1- The neutrosophic universal n^{th}-power set $1_{P^n(X)}$,and the neutrosophic empty n^{th}-power set $0_{P^n(X)}$ both are belonging to $\tau^{neutrotopo}$.

2- Any arbitrary (finite on infinite) union of members of $\tau^{neutrotopo}$ belong to $\tau^{neutrotopo}$.

3- $\tau^{neutrotopo}$ is closed under finite intersection of members of $\tau^{neutrotopo}$ (i.e. the intersection of any finite number of members of $\tau^{neutrotopo}$ belongs to $\tau^{neutrotopo}$).

Then $(\tau^{neutrotopo}, P^n(X))$ is called Neutrosophic SuperHyper Topological Spaces (NSHTS). Because of the definition of (NSHTS) via neutrosophic n^{th}-power open sets that commonly used in this manuscript, the family of neutrosophic sets $\tau^{neutrotopo}$ of the n^{th}-power sets are commonly called a (NSHTS) on the neutrosophic n^{th}-power sets $P^n(X)$.

A subpowerset $P^{m1}(C) \subseteq P^{m2}(X)$ for integers $m1 \leq m2$ is to be closed in $(\tau^{neutrotopo}, P^{n}(X))$ if its complement $P^{m2}(X)/P^{m1}(C)$ is an open set.

2.2 Numerical Example:

What is the difference between $P^1(x)$ & $P^2(x)$ in the structured of the Neutrosophic SuperHyper topological spaces $(\tau^{neutrotopo}, P^{n}(X))$, and how it effects on the distribution of the internal elements? take a look on the following example:

Suppose $X = \{a, b, c\}$ with the following

$$P^1(x) = \left\{ \{a, T = 0.3, I = 0.1, F = 0.6\}, \{b, c\} \begin{matrix} T = \{0.7, 0.4\} \\ I = \{0, 0.3\} \\ F = \{0.4, 0.3\} \end{matrix} \right\},$$

$$P^2(x) = \left\{ \{a, T = 0.3, I = 0.1, F = 0.6\}, \{\{b\}, \{c\}\} \begin{matrix} T = \{\{0.7\}, \{0.4\}\} \\ I = \{\{0\}, \{0.3\}\} \\ F = \{\{0.4\}, \{0.3\}\} \end{matrix} \right\}.$$

For more details, we can see that In $P^1(x)$ the element a affected by its membership functions $\{0.3, 0.1, 0.6\}$ directly, while the element(s) $\{b, c\}$ has (have) two kinds of affected (directed affect) and (indirect affect) as follow:

- The element b has a separate direct affect by its membership functions $\{0.7, 0, 0.4\}$, and the element c has a separate direct affect by its membership functions $\{0.4, 0.3, 0.3\}$.
- The structured element $\{b, c\}$ have common indirect affected by their membership functions $\{0.7, 0.4\}, \{0, 0.3\}, \{0.4, 0.3\}$.

This is a very harmonic with the previous example (1.3) stated in section one, by expressing the elements a, b as the parents (husband and wife), each one of them can affected separately by the inherited genes from their parents, also, they will crossing their parents' gene to their offspring mutually and their descendants will be affected directly by their parents and indirectly by their grandparents.

Then $(\tau^{neutrotopo}, P^{n}(X))$ is the Neutrosophic SuperHyper Topological spaces, where:

$$\tau^{neutrotopo} = \{0_{P^n(X)}, 1_{P^n(X)}, P^1(x), P^2(x)\}$$

2.3 Definition

Let $P^n(X)$ be a neutrosophic n^{th}-power set over a non-empty set X, the neutrosophic interior and the neutrosophic closure of $P^n(X)$ are respectively defined as:

$int^n\big(P^n(X)\big) = \cup \{P^m(X): P^m(X) \subseteq P^n(X), P^m(X) \in \tau^{neutrotopo}\}$, this means that for the same collection of the neutrosophic n^{th}-power set $P^n(X)$, all $P^m(X)$ given that $m \leq n$ regarded as interior for $P^n(X)$.

$cl^n\big(P^n(X)\big) = \cap \{P^h(X): P^n(X) \subseteq P^h(X), (P^h(X))^c \in \tau^{neutrotopo}\}$.

2.4 Definition

The following mathematical phrases are true for any two neutrosophic $n1^{th}$-power set $P^{n1}(Y1)$ and $n2^{th}$-power set $P^{n2}(Y2)$ on the neutrosophic n^{th}-power set $P^n(X)$, given that $n1, n2 \le n$, and that there is no restrictions on the relation between n_1 *and* n_2 :

1- $T_{P^{n1}(Y1)}(\{x\}) \le T_{P^{n2}(Y2)}(\{x\}), I_{P^{n1}(Y1)}(\{x\}) \le I_{P^{n2}(Y2)}(\{x\}), and\ F_{P^{n1}(Y1)}(\{x\}) \ge F_{P^{n2}(Y2)}(\{x\})$, for integers $n1, n2 \ge 1$, and for all $\{x\} \subseteq P^n(X)$ iff $P^{n1}(Y1) \subseteq P^{n2}(Y2)$.

2- $P^{n1}(Y1) \subseteq P^{n2}(Y2)$ and $P^{n2}(Y2) \subseteq P^{n1}(Y1)$ iff $P^{n1}(Y1) = P^{n2}(Y2)$, given that $n_1 = n_2$.

3- $P^{n1}(Y1) \cap P^{n2}(Y2) =$

$\{\langle\{x\}, \min\{T_{P^{n1}(Y1)}(\{x\}), T_{P^{n2}(Y2)}(\{x\})\}, \min\{I_{P^{n1}(Y1)}(\{x\}), I_{P^{n2}(Y2)}(\{x\})\}, \max\{F_{P^{n1}(Y1)}(\{x\}), F_{P^{n2}(Y2)}(\{x\})\}\rangle$
$: \{x\} \subseteq P^n(X)\}$

4- $P^{n1}(Y1) \cup P^{n2}(Y2) =$

$\{\langle\{x\}, \max\{T_{P^{n1}(Y1)}(\{x\}), T_{P^{n2}(Y2)}(\{x\})\}, \max\{I_{P^{n1}(Y1)}(\{x\}), I_{P^{n2}(Y2)}(\{x\})\}, \min\{F_{P^{n1}(Y1)}(\{x\}), F_{P^{n2}(Y2)}(\{x\})\}\rangle$
$: \{x\} \subseteq P^n(X)\}$

In general, the union or the intersection of any arbitrary members of neutrosophic n^{th}-power set $P^{ni}(X)_{i\in I}$ are defined by:

$\underset{i \in I}{\cap} P^{ni}(X) = \{\langle\{x\}, \inf\{T_{P^{ni}(\{x\})}\}, \inf\{I_{P^{ni}(\{x\})}\}, \sup\{F_{P^{ni}(\{x\})}\}\rangle : \{x\} \subseteq P^n(X)\}$,

$\underset{i \in I}{\cup} P^{ni}(X) = \{\langle\{x\}, \sup\{T_{P^{ni}(\{x\})}\}, \sup\{I_{P^{ni}(\{x\})}\}, \inf\{F_{P^{ni}(\{x\})}\}\rangle : \{x\} \subseteq P^n(X)\}$.

5- The neutrosophic n^{th}-power universal set $P^n(X)$ is denoted by $1_{P^n(X)}$,and it is exist if and only if the following conditions are holding together:

$T_{P^n(\{x\})} = 1_{P^n(X)}, I_{P^n(\{x\})} = 1_{P^n(X)}, and\ F_{P^n(\{x\})} = 0_{P^n(X)}$.

6- The neutrosophic n^{th}-power empty set $P^n(X)$ is denoted by $0_{P^n(X)}$, and it is exist if and only if the following conditions are holding together:

$T_{P^n(\{x\})} = 0_{P^n(X)}, I_{P^n(\{x\})} = 0_{P^n(X)}, and\ F_{P^n(\{x\})} = 1_{P^n(X)}$.

7- Let $P^{n1}(Y1) \subseteq P^{n2}(Y2)$, given that $n_1 \le n_2$, then the complementary of $P^{n1}(Y1)$ concerning to $P^{n2}(Y2)$ is defined as follow:

$P^{n1}(Y1) \setminus P^{n2}(Y2) = \{\langle |T_{P^{n1}(Y1)}(\{x\}) - T_{P^{n2}(Y2)}(\{x\})|, |I_{P^{n1}(Y1)}(\{x\}) - I_{P^{n2}(Y2)}(\{x\})|, 1_{P^n(X)} - |F_{P^{n1}(Y1)}(\{x\}) - F_{P^{n2}(Y2)}(\{x\})|\rangle\}$.

8- Clearly, the neutrosophic complement of $1_{P^n(X)}$ and $0_{P^n(X)}$ are defined as:

$(1_{P^n(X)})^c = \langle T_{P^n(\{x\})} = 0_{P^n(X)}, I_{P^n(\{x\})} = 0_{P^n(X)}, F_{P^n(\{x\})} = 1_{P^n(X)}\rangle = 0_{P^n(X)}$,
$(0_{P^n(X)})^c = \langle T_{P^n(\{x\})} = 1_{P^n(X)}, I_{P^n(\{x\})} = 1_{P^n(X)}, F_{P^n(\{x\})} = 0_{P^n(X)}\rangle = 1_{P^n(X)}$.

2.5 Proposition

Let $P^{n1}(X), P^{n2}(X), P^{n3}(X), and\ P^{n4}(X) \subseteq N(P^{n}(X))$ without any restrictions on the relations between $n_1, n_2, n_3, n_4, and\ n$, then the following mathematical statements are true:

i) Let $P^{n1}(X) \subseteq P^{n2}(X), and\ P^{n3}(X) \subseteq P^{n4}(X)$, given that $n_1 \leq n_2$, & $n_3 \leq n_4$, this implies that $P^{n1}(X) \cap P^{n3}(X) \subseteq P^{n2}(X) \cap P^{n4}(X)$,

ii) $(P^{n1}(X)^c)^c = P^{n1}(X)$, also if $P^{n2}(X)^c \subseteq P^{n1}(X)^c \Longrightarrow P^{n1}(X) \subseteq P^{n2}(X)$,

iii) $(P^{n1}(X) \cap P^{n2}(X))^c = P^{n1}(X)^c \cup P^{n2}(X)^c$,

iv) $(P^{n1}(X) \cup P^{n2}(X))^c = P^{n1}(X)^c \cap P^{n2}(X)^c$.

2.6 Definition

Let X be a non-empty set, $P^n(X)$ is the n^{th}-power neutrosophic set of a set X, for integer $n \geq 1$. If α, β, γ be real standard or non-standard subsets of $]^-0, 1^+[$, then the neutrosophic n^{th}-power set $P^n(x_{\alpha,\beta,\gamma})$ is called a neutrosophic n^{th}- power point, and it is defined by:

$$P^n\big(x_{\alpha,\beta,\gamma}(y)\big) = \begin{cases} \langle \alpha_{P^n(x)}, \beta_{P^n(x)}, \gamma_{P^n(x)} \rangle, & if\ P^n(x) = P^n(y) \\ \langle 0_{P^n(x)}, 0_{P^n(x)}, 1_{P^n(x)} \rangle, & if\ P^n(x) \neq P^n(y) \end{cases}$$

For $x, y \in X, and\ P^n\big(x_{\alpha,\beta,\gamma}\big), P^n(y) \subseteq P^n(X)$, here $P^n(y)$ is called the support of $P^n(x_{\alpha,\beta,\gamma})$.

2.7 Definition

Let $P^{n1}(X) \in N(P^n(X))$, the belonging operation of the neutrosophic n^{th}- power point $P^n(x_{\alpha,\beta,\gamma})$ to $P^{n1}(X)$ (i.e. $P^n(x_{\alpha,\beta,\gamma}) \in P^{n1}(X)$) is satisfied if and only if $T_{P^{n1}(\{x\})} \geq \alpha, I_{P^{n1}(\{x\})} \geq \beta, F_{P^{n1}(\{x\})} \leq \gamma$.

2.8 Definition

A sub-collection τ_n^* of neutrosophic n^{th}- power set $P^n(X)$ on a non-empty set X is said to be Neutrosophic SuperHyper Supra Topological Space on X if the n^{th}- power sets $0_{P^n(X)}, 1_{P^n(X)} \in \tau_n^*$, and $\overset{\infty}{\underset{i \in I}{\cup}} P^{ni}(X) \in \tau_n^*$ for $\{P^{ni}(X)\}_{i=1}^{\infty} \in \tau_n^*$. Then $(\tau_n^*, P^n(X))$ is called Neutrosophic SuperHyper Supra Topological Space on X.

3 Neutrosophic SuperHyper Bi-Topological Spaces

This section contains new concepts presents for the first time linking the concept of the neutrosophic n^{th}- power sets with the traditional neutrosophic bi-topological spaces.

3.3 Definition

Let $(\tau_1{}^{1stpair}, P^n(X)), and\ (\tau_2{}^{2ndpair}, P^n(X))$ be two different Neutrosophic SuperHyper topological spaces on X. Then $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ is called Neutrosophic SuperHyper Bi-Topological space (NSHBI-TS).

3.4 Definition

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be a (NSHBI-TS). A collection of a neutrosophic n^{th}- power set $N = \{\langle\{x\}: T_{P^n(\{x\})}, I_{P^n(\{x\})}, F_{P^n(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ over $P^n(X)$ is said to be a pairwise neutrosophic n^{th}- power open set in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ if there exist a neutrosophic n^{th}- power open set $N_1 = \{\langle\{x\}: T_{P^{n1}(\{x\})}, I_{P^{n1}(\{x\})}, F_{P^{n1}(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ in $\tau_1{}^{1stpair}$ and a neutrosophic n^{th}- power open set $N_2 = \{\langle\{x\}: T_{P^{n2}(\{x\})}, I_{P^{n2}(\{x\})}, F_{P^{n2}(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ in $\tau_2{}^{2ndpair}$ such that $N = N_1 \cup N_2 = \{\langle\{x\},\ T_{P^n(\{x\})} = \max\{T_{P^{n1}(\{x\})}, T_{P^{n2}(\{x\})}\},\ I_{P^n(\{x\})} = \max\{I_{P^{n1}(\{x\})}, I_{P^{n2}(\{x\})}\}, F_{P^n(\{x\})} = \min\{F_{P^{n1}(\{x\})}, F_{P^{n2}(\{x\})}\}\rangle : \{x\} \subseteq P^n(X)\}$.

3.5 Definition

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be a (SHNBI-TS). A collection of a neutrosophic n^{th}- power set $C = \{\langle\{x\}: T_{P^c(\{x\})}, I_{P^c(\{x\})}, F_{P^c(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ over $P^n(X)$ is said to be a pairwise neutrosophic n^{th}- power closed set in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ if its neutrosophic complement is a pairwise neutrosophic n^{th}- power open set in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$. Clearly, a neutrosophic n^{th}- power set C over $P^n(X)$ is a pairwise neutrosophic n^{th}- power closed set in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ if there exist a neutrosophic n^{th}- power closed set $C_1 = \{\langle\{x\}: T_{P^{c1}(\{x\})}, I_{P^{c1}(\{x\})}, F_{P^{c1}(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ in $(\tau_1{}^{1stpair})^c$,and a neutrosophic n^{th}- power closed set $C_2 = \{\langle\{x\}: T_{P^{c2}(\{x\})}, I_{P^{c2}(\{x\})}, F_{P^{c2}(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ in $(\tau_2{}^{2ndpair})^c$ such that $C = C_1 \cap C_2 = \{\langle\{x\},\ T_{P^c(\{x\})} = \min\{T_{P^{c1}(\{x\})}, T_{P^{c2}(\{x\})}\},$ $I_{P^c(\{x\})} = \min\{I_{P^{c1}(\{x\})}, I_{P^{c2}(\{x\})}\},\ F_{P^c(\{x\})} = \max\{F_{P^{c1}(\{x\})}, F_{P^{c2}(\{x\})}\}\rangle : \{x\} \subseteq P^n(X)\}$. Where $(\tau_i{}^{ipair})^c = \{(N)^c \subseteq N(P^n(X)) : N \subseteq \tau_i{}^{ipair}, i = 1st, 2nd$. The family of all pairwise neutrosophic n^{th}- power open/closed sets in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ is denoted by PN n^{th} POS in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ / PNn^{th}PCS in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$, respectively.

3.6 Theorem

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space. Then,

1. $0_{P^n(X)}, and\ 1_{P^n(X)}$ are pairwise neutrosophic n^{th}- power open/closed sets.
2. An arbitrary neutrosophic union of pairwise neutrosophic n^{th}- power open sets is a pairwise neutrosophic n^{th}- power open set.
3. An arbitrary neutrosophic intersection of pairwise neutrosophic n^{th}- power closed sets is a pairwise neutrosophic n^{th}- power closed set.

Proof:

1. Let $0_{P^{n1}(x)}$, $0_{P^{n2}(x)} \subseteq \tau_1{}^{1stpair}, \tau_2{}^{2ndpair}$ respectively, and $n_1 + n_2 = n$, since $0_{P^{n1}(x)} \cup 0_{P^{n2}(x)} = 0_{P^{n}(x)}$, hence $0_{P^{n}(x)}$ is a PN n^{th} POS in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$. Similarly, $1_{P^{n}(x)}$ is a PN n^{th} PCS in$(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$.
2. Suppose $\{N_i = \langle \{x\}: T_{P^{ni}(\{x\})}, I_{P^{ni}(\{x\})}, F_{P^{ni}(\{x\})} \rangle : i \in I\} \subseteq$ PNn^{th}POS in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$. Then each N_i is a pairwise neutrosophic n^{th} power open set for all $i \in I$, this implies that there exist $N_i^1 \in \tau_1{}^{1stpair}$ and $N_i^2 \in \tau_2{}^{2ndtpair}$ such that $N_i = N_i^1 \cup N_i^2$ for all $i \in I$ which implies that .

$$\underset{i \in I}{\cup N_i} = \underset{i \in I}{\cup} [N_i^1 \cup N_i^2] = \begin{bmatrix} \cup N_i^1 \\ i \in I \end{bmatrix} \cup \begin{bmatrix} \cup N_i^2 \\ i \in I \end{bmatrix}$$

Now, since $\tau_1{}^{1stpair}$ and $\tau_2{}^{2ndpair}$ are both Neutrosophic SuperHyper Topological Spaces on the neutrosophic n^{th} power set $P^n(X)$, then $\begin{bmatrix} \cup N_i^1 \\ i \in I \end{bmatrix} \subseteq \tau_1{}^{1stpair}$, and $\begin{bmatrix} \cup N_i^2 \\ i \in I \end{bmatrix} \subseteq \tau_2{}^{2ndpair}$. Therefore, $\underset{i \in I}{\cup N_i}$ is a pairwise neutrosophic n^{th}- power open set.

3. It is immediate from the definition (3.3).

3.7 Corollary

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space. Then, the family of all pairwise neutrosophic n^{th}- power open sets is a Neutrosophic SuperHyper Supra Topological Space (NSHSTS) on X. This (NSHSTS) is denoted by τ_{12}^{supra}.

3.8 Theorem

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space. Then,

1. Every $\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}$ - neutrosophic n^{th} - power open set is a pairwise neutrosophic n^{th}- power open set, i.e. $\tau_1{}^{1stpair} \cup \tau_2{}^{2ndpair} \subseteq \tau_{12}^{supra}$.
2. Every $\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}$ - neutrosophic n^{th} - power closed set is a pairwise neutrosophic n^{th}- power closed set, i.e. $(\tau_1{}^{1stpair})^c \cup (\tau_2{}^{2ndpair})^c \subseteq (\tau_{12}^{supra})^c$.
3. If $\tau_1{}^{1stpair} \subseteq \tau_2{}^{2ndpair}$, then $\tau_{12}^{supra} = \tau_2{}^{2ndpair}$ and $(\tau_{12}^{supra})^c = (\tau_2{}^{2ndpair})^c$.

Proof. Straightforward.

3.9 Definition

Let $({\tau_1}^{1stpair}, {\tau_2}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space, and $P^n(X) \in N(P^n(X))$.The pairwise neutrosophic closure of $P^n(X)$, denoted by $cl_p^n(P^n(X))$, is the neutrosophic intersection of all pairwise neutrosophic closed supra n^{th}- power sets of $P^n(X)$, i.e., $cl_p^n(P^n(X)) = \cap \{P^m(X) \in (\tau_{12}^{supra})^c : P^n(X) \subseteq P^m(X)\}$. It is clear that $cl_p^n(P^n(X))$ is the smallest pairwise neutrosophic n^{th}- power closed set containing $P^n(X)$.

3.10 Theorem

Let $({\tau_1}^{1stpair}, {\tau_2}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space, and $P^{n1}(X), P^{n2}(X) \in N(P^n(X))$, without restrictions on the relations between $n1, n2, n$. Then, the following mathematical statements are true:

1. $cl_p^n(0_{P^{ni}(x)}) = 0_{P^{ni}(x)}$, and $cl_p^n(1_{P^{ni}(x)}) = 1_{P^{ni}(x)}$, $i = 1,2$.
2. $P^n(X) \subseteq cl_p^n(P^n(X))$.
3. $P^n(X)$ is a pairwise neutrosophic n^{th} - power closed set if and only if $cl_p^n(P^n(X)) = P^n(X)$.
4. $P^{n1}(X) \subseteq P^{n2}(X) \Longrightarrow cl_p^n(P^{n1}(X)) \subseteq cl_p^n(P^{n2}(X))$.
5. $cl_p^n(P^{n1}(X)) \cup cl_p^n(P^{n2}(X)) \subseteq cl_p^n(P^{n1}(X) \cup P^{n2}(X))$.
6. $cl_p^n\ [\ cl_p^n(P^{ni}(X)] = cl_p^n(P^{ni}(X)$, i.e., $cl_p^n(P^{ni}(X)$ is a pairwise neutrosophic n^{th}-power closed set.

Proof. Straightforward.

3.11 Theorem

Let $({\tau_1}^{1stpair}, {\tau_2}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space, and $P^{n1}(X) \in N(P^n(X))$. Then,

$P^{n1}(x_{\alpha,\beta,\gamma}) \in cl_p^n(P^{n1}(X)) \Leftrightarrow U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \cap P^{n1}(X) \neq 0_{P^{n1}(x)}$, $\forall\, U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \in$

$\tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma}))$,

Where $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right)$ is any pairwise neutrosophic n^{th}- power open set contains $P^{n1}(x_{\alpha,\beta,\gamma})$, and $\tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma}))$ is the family of all pairwise neutrosophic supra n^{th}- power open set contains $P^{n1}(x_{\alpha,\beta,\gamma})$.

Proof:

Let $P^{n1}(x_{\alpha,\beta,\gamma}) \in cl_p^n(P^{n1}(X))$, and suppose that there exist $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \in \tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma}))$, such that $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \cap P^{n1}(X) = 0_{P^{n1}(x)}$. Then $P^{n1}(X) \subseteq (\,U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right))^c$, thus $cl_p^n(P^{n1}(X)) \subseteq$

$cl_p^n\left(U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right)\right)^c = (U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right))^c$ which implies that $cl_p^n(P^{n1}(X)) \cap U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) = 0_{P^{n1}(x)}$, this is a contradiction. hence $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \cap P^{n1}(X) \neq 0_{P^{n1}(x)}$.

Conversely, assume that $P^{n1}(x_{\alpha,\beta,\gamma}) \notin cl_p^n(P^{n1}(X))$, then $P^{n1}(x_{\alpha,\beta,\gamma}) \in (cl_p^n(P^{n1}(X)))^c$. Thus, $(cl_p^n(P^{n1}(X)))^c \in \tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma}))$, therefore, by hypothesis, $(cl_p^n(P^{n1}(X)))^c \cap P^{n1}(X) \neq 0_{P^{n1}(x)}$, this is a contradiction. Hence we get $P^{n1}(x_{\alpha,\beta,\gamma}) \in cl_p^n(P^{n1}(X))$.

3.12 Theorem

Let $({\tau_1}^{1stpair}, {\tau_2}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space. A neutrosophic n^{th}- power set $P^{n1}(X)$ over $P^n(X)$ is a pairwise neutrosophic n^{th}- power closed set if and only if $P^{n1}(X) = cl^n_{{\tau_1}^{1stpair}}(P^{n1}(X)) \cap cl^n_{{\tau_2}^{2ndpair}}(P^{n1}(X))$.

Proof:

Suppose that $P^{n1}(X)$ is a pairwise neutrosophic n^{th}-power closed set and $P^{n1}(x_{\alpha,\beta,\gamma}) \notin P^{n1}(X)$. Then $P^{n1}(x_{\alpha,\beta,\gamma}) \notin cl_p^n(P^{n1}(X))$. Thus, by theorem (3.9), there exists $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \in \tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma}))$ such that $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \cap P^{n1}(X) = 0_{P^{n1}(x)}$. Again, since $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \in \tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma}))$, then there exists $P^{m1}(X) \in {\tau_1}^{1stpair}$ and $P^{m2}(X) \in {\tau_2}^{2ndpair}$ such that $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) = P^{m1}(X) \cup P^{m2}(X)$. consequently, $\left(P^{m1}(X) \cup P^{m2}(X)\right) \cap P^{n1}(X) = 0_{P^{n1}(x)}$, this implies that $P^{m1}(X) \cap P^{n1}(X) = 0_{P^{n1}(x)}$, and $P^{m2}(X) \cap P^{n1}(X) = 0_{P^{n1}(x)}$. Since $P^{n1}(x_{\alpha,\beta,\gamma}) \in U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right)$, then either $P^{n1}(x_{\alpha,\beta,\gamma}) \in P^{m1}(X)$ or $P^{n1}(x_{\alpha,\beta,\gamma}) \in P^{m2}(X)$, this implies that either $P^{n1}(x_{\alpha,\beta,\gamma}) \notin cl^n_{{\tau_1}^{1stpair}}(P^{n1}(X))$ or $P^{n1}(x_{\alpha,\beta,\gamma}) \notin cl^n_{{\tau_2}^{2ndpair}}(P^{n1}(X))$. Therefore, $P^{n1}(x_{\alpha,\beta,\gamma}) \notin cl^n_{{\tau_1}^{1stpair}}(P^{n1}(X)) \cap cl^n_{{\tau_2}^{2ndpair}}(P^{n1}(X))$. Thus, $cl^n_{{\tau_1}^{1stpair}}(P^{n1}(X)) \cap cl^n_{{\tau_2}^{2ndpair}}(P^{n1}(X)) \subseteq P^{n1}(X)$. On the other hand, we have $P^{n1}(X) \subseteq cl^n_{{\tau_1}^{1stpair}}(P^{n1}(X)) \cap cl^n_{{\tau_2}^{2ndpair}}(P^{n1}(X))$. Hence, $P^{n1}(X) = cl^n_{{\tau_1}^{1stpair}}(P^{n1}(X)) \cap cl^n_{{\tau_2}^{2ndpair}}(P^{n1}(X))$.

Conversely, suppose that $P^{n1}(X) = cl^n_{{\tau_1}^{1stpair}}(P^{n1}(X)) \cap cl^n_{{\tau_2}^{2ndpair}}(P^{n1}(X))$. Since, $cl^n_{{\tau_1}^{1stpair}}(P^{n1}(X))$ is a neutrosophic n^{th} -power closed set in $({\tau_1}^{1stpair}, P^n(X))$, and $cl^n_{{\tau_2}^{2ndpair}}(P^{n1}(X))$ is a neutrosophic n^{th}-power closed set in $({\tau_2}^{2ndpair}, P^n(X))$, so, by definition (3.3), $cl^n_{{\tau_1}^{1stpair}}(P^{n1}(X)) \cap cl^n_{{\tau_2}^{2ndpair}}(P^{n1}(X))$ is a pairwise neutrosophic n^{th}- power closed set in $({\tau_1}^{1stpair}, {\tau_2}^{2ndpair}, P^n(X))$, consequently, $P^{n1}(X)$ is a pairwise neutrosophic n^{th}- power closed set.

3.13 Corollary

Let $({\tau_1}^{1stpair}, {\tau_2}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space. Then, $cl_p^n(P^n(X)) = cl^n_{{\tau_1}^{1stpair}}(P^{n1}(X)) \cap cl^n_{{\tau_2}^{2ndpair}}(P^{n1}(X)), \forall\, P^{n1}(X) \in N(P^n(X))$.

4 Conclusion

The types of the topological spaces in neutrosophic theory are always changed depending upon the structure of the sets, in this article, the Neutrosophic SuperHyper Topological Spaces has been fathomed especially Neutrosophic SuperHyper Bi-Topological Spaces. The definitions of the neutrosophic interior and the neutrosophic closure of $P^n(X)$ have been presented. Also, the neutrosophic universal n^{th}-power set $P^n(X)$ and the neutrosophic empty n^{th}-power set $P^n(X)$ were discussed. The union and the intersection operations have been defined. As well as, the authors presented pairwise neutrosophic n^{th}-power open set, pairwise neutrosophic closed n^{th} - power set, many of theorems, propositions and examples to support the new notion.

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Some Basic Concepts of Neutrosophic Soft Block Matrices

Mamoni Dhar

[1] Science College, Kokrajhar

* Correspondence: mamonidhar@gmail.com

Abstract: The main focus of this article is to discuss the concept of neutrosophic sets, neutrosophic soft sets , neutrosophic soft matrices theory and neutrosophic soft block matrix which are very useful and applicable in various situations involving uncertainties and imprecisions. Here different types of neutrosophic soft block matrices are studied and some operations on it along with some associated properties are discussed in details.

Keywords: Neutrosophic sets; neutrosophic soft sets; neutrosophic soft matrix, neutrosophic soft block matrix

1. Introduction

In real life situations, most of the problems in economics, social science , environmental science and in many other cases information are vague, imprecise and insufficient. Fuzzy set [1], intuitionistic fuzzy set [2] etc are used as the tool to deal with such uncertainties.

Later on Molodtsov[3], pointed out that these theories have their own difficulties and as such the novel concept of soft set theory was initiated. The theory of soft set has rich potential for solving problems in economics, social science and medical science etc. Maji *.et.al* [4, 5] have studied the theory of fuzzy soft set. Maji. *et. al* [6], have extended the theory of fuzzy soft set to intuitionistic fuzzy soft sets.

Smarandhache [7], introduced the concept of neutrosiphic sets as a mathematical tool to deal with some situations involving impreciseness, inconsistencies and interminancy. It is expected that neutrosophic sets will produce more accurate result than those obtained by using fuzzy sets or intuitionistic fuzzy sets. Maji. *et. al* [8], have extended the theory of neutrosophic set to neutrosophic soft set. Maji. *et. al* [9] applied the theory of neutrosophic soft set in decision making process.

In recent years several mathematicians have used this concept in different mathematical structures, which can be seen in the works of Deli *et.al* [10-12]. Later this very concept has been modified by Deli and Broumi [13] in developing the basic of neutrosophic soft matrices and its successful utilization in decision making process. The concept of intuitionistic neutrosophic sets was developed by Broumi and Smarandache [14] and some of its properties were discussed. Bera and Mahapatra [15] studied some algebraic structure of neutrosophic soft set. Various decision making algorithms over neutrosophic soft set theory have been developed in the literature of neutrosophc set theory. Many researchers for example [16]have worked on applying neutrosophic sets invarious decision making processes. Many new development regarding neutrosophic sets and neutrosophic soft matrices are found in the works of [17-23] Neutrosophic soft block matrix is a neutrosophic soft matrix which is defined using smaller neutrosophic soft matrices. Some authors, as for example[24, 25] have discussed neutrosophic soft block matrices.

The main focus of this article is to deal with the concept of various types of neutrosophic soft block matrices and some operations on these matrices. The next section briefly introduces some definitions related to neutrosopphic set, neutrosophic soft set, neutrosophic soft matrix, neutrosophic soft block matrices and so on. Section 3 defines operations on neutrosophic soft matrices etc. Section 4 presents some special form of neutrosophic soft block matrices and some related properties.

2. Preliminaries (proposed work with more details)

Some basic definitions that are useful in subsequent sections of this article are discussed in this section.

Definition 2.1: Neutrosophic sets (Smarandache, 2005)

Let U be the universe of discourse, The neutrosophic set A on the universe of discourse U is defined as $A=\{<T_A(x), I_A(x), F_A(x)>: x \in U\}$, where the characteristic functions $T, I, F: U \rightarrow [0,1]$ and $^{-}0 \leq T+I+F \leq 3^{+}$; T,I,F are neutrosophic components which defines the degree of membership, the degree of interminancy and the degree of non membership respectively.

Definition 2.2: Neutrosophic soft set (Maji, 2013)

Let U be an initial universe set and E is the set of parameters. Suppose P (U) denotes the collection of all neutrosophic subsets of U. Let $A \subseteq E$. A pair (F, E) is called neutrosophic soft set over U where F is a mapping given by $F: E \rightarrow P(U)$

Definition 2.3: Neutrosophic soft Matrix (Deli.*et.al*, 2015)

Let $U=\{u_1, u_2, u_3, ...u_m\}$ be the universe set and $E=\{x_1, x_2, x_3, ...x_n\}$ be the set of parameters. Let $A \subseteq E$. The set (F, A) is a neutrosophic soft set over U. Then the subset of UXE is uniquely defined by $R_A=\{(u,e): e \in A, u \in F_A(e)\}$ which is a relation form of (F, E). Now the relation R_A is characterized by truth membership function $T_{R_A}: U \times E \rightarrow [0,1]$, interminancy membership function $I_{R_A}: U \times E \rightarrow [0,1]$ and falsity membership function $F_{R_A}: U \times E \rightarrow [0,1]$

where $T_{R_A}(u,e) \in [0,1]$, $I_{R_A}(u,e) \in [0,1]$ and $F_{R_A}(u,e) \in [0,1]$ indicates truthfulness, interminancy and falsity.

If

$(T_{A_{mn}}, I_{A_{mn}}, F_{A_{mn}})$=($T_A(u_m,e_n)$, $I_A(u_m,e_n)$, $F_A(u_m,e_n)$) and a_{ij}=($T_A(u_i,e_j)$, $I_A(u_i,e_j)$, $F_A(u_i,e_j)$) we can define a matrix $\tilde{A}_{ij} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

This is called neutrosophic soft matrix of order mxn corresponding to the neutrosophioc soft set

(F,E) over U.

Definition 2.4: Triangular neutrosophic soft matrix

Triangular neutrosophic soft matrix is a special type of square neutrosophic soft matrix. A square neutrosophic soft matrix is called lower triangular if all the entries $a_{ij} = (0,0,1)$ for $i < j$, i,j=1, 2, 3,…..,n and upper triangular if $a_{ij} = (0,0,1)$ for $i > j$, i,j=1, 2, 3, ……,n.

Definition 2.5: Toeplitz neutrosophic soft matrix

Toeplitz neutrosophic soft matrix is a square neutrosophic soft matrix of the form

$$\tilde{A} = \begin{bmatrix} (T^A{}_{11}, I^A{}_{11}, F_{11}{}^A) & (T^A{}_{12}, I^A{}_{12}, F_{12}{}^A) & (T^A{}_{13}, I^A{}_{13}, F_{13}{}^A) & (T^A{}_{14}, I^A{}_{14}, F_{14}{}^A) \\ (T^A{}_{21}, I^A{}_{21}, F_{21}{}^A) & (T^A{}_{11}, I^A{}_{11}, F_{11}{}^A) & (T^A{}_{21}, I^A{}_{21}, F_{21}{}^A) & (T^A{}_{13}, I^A{}_{13}, F_{13}{}^A) \\ (T^A{}_{31}, I^A{}_{31}, F_{31}{}^A) & (T^A{}_{21}, I^A{}_{21}, F_{21}{}^A) & (T^A{}_{11}, I^A{}_{11}, F_{11}{}^A) & (T^A{}_{21}, I^A{}_{21}, F_{21}{}^A) \\ (T^A{}_{41}, I^A{}_{41}, F_{41}{}^A) & (T^A{}_{31}, I^A{}_{31}, F_{31}{}^A) & (T^A{}_{21}, I^A{}_{21}, F_{21}{}^A) & (T^A{}_{11}, I^A{}_{11}, F_{11}{}^A) \end{bmatrix}$$

Definition 2.6: Zero neutrosophic soft matrix

Zero neutrosophic soft matrix is neutrosophic soft matrix in which all the entries are of the form (0,0,1).

Definition 2.7: Tridiagonal neutrosophic soft matrix

Neutrosophic soft tridiagonal matrix is another special neutrosophic soft matrix which has non zero entries in the lower diagonal, main diagonal and upper diagonal and all other entries being (0.0.1). That is a Neutrosophic soft tridiagonal matrix A has the form

$\tilde{A} = \begin{bmatrix} B_1 & C_1 & \dots & 0 \\ A_1 & B_2 & C_2 \dots & 0 \\ 0 & A_2 & B_3 & C_3 \\ 0 & 0 & A_3 & B_4 \end{bmatrix}$ where A_i, B_i, C_i are non zero entries in the lower, main and upper diagonal respectively.

Definition 2.8:Neutrosofic soft block matrix (Uma.*et.al*,2017 & Dhar, 2020)

A neutrosophic soft block matrix or a partitioned matrix is a neutrosophic soft matrix that is interpretated as having been broken into sections called blocks or submatrices. A neutrosophic soft block matrix can be visualized as the original neutrosophic soft matrix by drawing lines parallel to its rows and columns. These sub-matrices may be considered as the elements of the original matrices. Any neutrosophic soft matrix can be interpreted as a neutrosophic soft block matrix in one or more ways, with each interpretation defined by how its rows and columns are partitioned.

For example

$$\tilde{A}=\begin{bmatrix} (T^{A}_{11},I^{A}_{11},F_{11}^{A}) & (T^{A}_{12},I^{A}_{12},F_{12}^{A}) & : & (T^{A}_{13},I^{A}_{13},F_{13}^{A}) & (T^{A}_{14},I^{A}_{14},F_{14}^{A}) \\ \dots & \dots & & \dots & \dots \\ (T^{A}_{21},I^{A}_{21},F_{21}^{A}) & (T^{A}_{22},I^{A}_{22},F_{22}^{A}) & : & (T^{A}_{23},I^{A}_{23},F_{23}^{A}) & (T^{A}_{24},I^{A}_{24},F_{24}^{A}) \\ (T^{A}_{31},I^{A}_{31},F_{31}^{A}) & (T^{A}_{32},I^{A}_{32},F_{32}^{A}) & : & (T^{A}_{33},I^{A}_{33},F_{33}^{A}) & (T^{A}_{34},I^{A}_{34},F_{34}^{A}) \end{bmatrix}$$

The above neutrosophic soft matrix can be represented as

$$\tilde{A}=\begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix}$$

where

$$\tilde{P}_{11}=\begin{bmatrix} (T^{A}_{11},I^{A}_{11},F_{11}^{A}) & (T^{A}_{12},I^{A}_{12},F_{12}^{A}) \end{bmatrix}$$

$$\tilde{P}_{12}=\begin{bmatrix} (T^{A}_{13},I^{A}_{13},F^{A}_{13}) & (T^{A}_{14},I^{A}_{14},F^{A}_{14}) \end{bmatrix}$$

$$\tilde{P}_{21}=\begin{bmatrix} (T^{A}_{21},I^{A}_{21},F_{21}^{A}) & (T^{A}_{22},I^{A}_{22},F^{A}_{22}) \\ (T^{A}_{31},I^{A}_{31},F^{A}_{31}) & (T^{A}_{32},I^{A}_{32},F^{A}_{32}) \end{bmatrix}, \quad \tilde{P}_{22}=\begin{bmatrix} (T^{A}_{23},I^{A}_{23},F^{A}_{23}) & (T^{A}_{24},I^{A}_{24},F^{A}_{24}) \\ (T^{A}_{33},I^{A}_{33},F^{A}_{33}) & (T^{A}_{34},I^{A}_{34},F^{A}_{34}) \end{bmatrix}$$

Then

$\tilde{A}=\begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix}$ is an example of neutrosophic soft block matrix or neutrosophic soft partitioned matrix.

So the neutrosophic soft matrix A is partitioned by the dotted lines dividing the neutrosophic soft matrix into neutrosophic soft sub-matrices $\tilde{P}_{11}, \tilde{P}_{12}, \tilde{P}_{21}, \tilde{P}_{22}$. The neutrosophic soft matrix $\tilde{A}$ can be partitioned in several ways.

Definition 2.9: Square neutrosophic soft block matrix

If the number of rows and the number of columns of a neutrosophic soft blocks are equal then the matrix is said to be square neutrosophic soft block matrix.

$$\tilde{A}=\begin{bmatrix} (T_{11}^A,I_{11}^A,F_{11}^A) & (T_{12}^A,I_{12}^A,F_{12}^A) & : & (T_{13}^A,I_{13}^A,F_{13}^A) & (T_{14}^A,I_{14}^A,F_{14}^A) & : & (T_{15}^A,I_{15}^A,F_{15}^A) & (T_{16}^A,I_{16}^A,F_{16}^A) \\ (T_{21}^A,I_{21}^A,F_{21}^A) & (T_{22}^A,I_{22}^A,F_{22}^A) & : & (T_{23}^A,I_{23}^A,F_{23}^A) & (T_{24}^A,I_{24}^A,F_{24}^A) & : & (T_{25}^A,I_{25}^A,F_{25}^A) & (T_{25}^A,I_{25}^A,F_{25}^A) \\ \dots & \dots & .. & \dots & \dots & \dots & \dots & \dots \\ (T_{31}^A,I_{31}^A,F_{31}^A) & (T_{32}^A,I_{32}^A,F_{32}^A) & : & (T_{33}^A,I_{33}^A,F_{33}^A) & (T_{34}^A,I_{34}^A,F_{34}^A) & : & (T_{35}^A,I_{35}^A,F_{35}^A) & (T_{36}^A,I_{36}^A,F_{36}^A) \\ (T_{41}^A,I_{41}^A,F_{41}^A) & (T_{42}^A,I_{42}^A,F_{42}^A) & : & (T_{43}^A,I_{43}^A,F_{43}^A) & (T_{44}^A,I_{44}^A,F_{44}^A) & : & (T_{45}^A,I_{45}^A,F_{45}^A) & (T_{46}^A,I_{46}^A,F_{46}^A) \end{bmatrix}$$

or

$$\tilde{A}=\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \end{bmatrix}$$

is a square fuzzy block matrix since all $\tilde{A}_{ij}$'s are square blocks.

Definition 2.10: Rectangular neutrosophic soft block matrix

If the number of rows and the number of columns of blocks are unequal then the matrix is said to be rectangular neutrosophic soft block matrix .For example

$$\tilde{A}=\begin{bmatrix} (T^B{}_{11},I^B{}_{11},F_{11}{}^B) & (T^B{}_{12},I^B{}_{12},F_{12}{}^B) & : & (T^B{}_{13},I^B{}_{13},F_{13}{}^B) & (T^B{}_{14},I^B{}_{14},F_{14}{}^B) \\ \dots & \dots & & \dots & \dots \\ (T^B{}_{21},I^B{}_{21},F_{21}{}^B) & (T^B{}_{22},I^B{}_{22},F_{22}{}^B) & : & (T^B{}_{23},I^B{}_{23},F_{23}{}^B) & (T^B{}_{24},I^B{}_{24},F_{24}{}^B) \\ (T^B{}_{31},I^B{}_{31},F_{31}{}^B) & (T^B{}_{32},I^B{}_{32},F_{32}{}^B) & : & (T^B{}_{33},I^B{}_{33},F_{33}{}^B) & (T^B{}_{34},I^B{}_{34},F_{34}{}^B) \end{bmatrix}$$

Is a rectangular neutrosophic soft block matrix.

3. Operations on Neutrosophic Soft matrices

3.1 Addition of neutrosophic soft matrices

Let $\tilde{A}=[(T_{ij}^A,I_{ij}^A,F_{ij}^A)]$, $\tilde{B}=[(T_{ij}^B,I_{ij}^B,F_{ij}^B)]$ be two neutrosophic soft matrices. Then the max-min product of the two neutrosophic soft matricesA and B is denoted as $A+B$ is defined as

$\tilde{A}+\tilde{B}=[\max(T_{ij}^A,T_{ij}^B),\min(I_{ij}^A,I_{ij}^B),\min(F_{ij}^A,F_{ij}^B)]$ for all i and j.

3.2 Max-min product of neutrosophic soft matrices

Let $\tilde{A}=[(T_{ij}^A,I_{ij}^A,F_{ij}^A)]$, $\tilde{B}=[(T_{ij}^B,I_{ij}^B,F_{ij}^B)]$ be two neutrosophic soft matrices. Then the max-min product of the two neutrosophic soft matricesA and B is denoted as $\tilde{A}\tilde{B}$ is defined as

$\tilde{A}\tilde{B}=[\max\min(T_{ij}^A,T_{ij}^B),\min\max(I_{ij}^A,I_{ij}^B),\min\max(F_{ij}^A,F_{ij}^B)]$ for all i and j.

3.3 Transpose of neutrosophic soft matrices

Let $\tilde{A} = [(T_{ij}^A, I_{ij}^A, F_{ij}^A)]$ be two neutrosophic soft matrices. Then the transpose of this neutrosophic soft block matrix will be defined by denoted by $\tilde{A}^T$ and is defined by $\tilde{A}^T = [(T_{ji}^A, I_{ji}^A, F_{ji}^A)]$

3.4 Addition of neutrosophic soft block matrices

Let $\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & : & \tilde{A}_{12} \\ ..... & : & ..... \\ \tilde{A}_{21} & : & \tilde{A}_{22} \end{bmatrix}$ and $\tilde{B} = \begin{bmatrix} \tilde{B}_{11} & : & \tilde{B}_{12} \\ ..... & : & ..... \\ \tilde{B}_{21} & : & \tilde{B}_{22} \end{bmatrix}$ be two neutrosophic soft block matrices in which the corresponding blocks are conformable for addition, then the addition of two neutrosophic soft block matrices can be defined as

$$\tilde{A} + \tilde{B} = \begin{bmatrix} \tilde{A}_{11} + \tilde{B}_{11} & : & \tilde{A}_{12} + \tilde{B}_{12} \\ ..... & : & ..... \\ \tilde{A}_{21} + \tilde{B}_{21} & : & \tilde{A}_{22} + \tilde{B}_{22} \end{bmatrix}$$

3.5 Multiplication of neutrosophic soft block matrices

Let $\tilde{A}, \tilde{B}$ be two neutrosophic soft block matrices which can be represented by

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & : & \tilde{A}_{12} \\ ..... & : & ..... \\ \tilde{A}_{21} & : & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_{11} & : & \tilde{B}_{12} \\ ..... & : & ..... \\ \tilde{B}_{21} & : & \tilde{B}_{22} \end{bmatrix}$$

Then the product of two neutrosophic soft block matrices will be denoted by $\tilde{A}\tilde{B}$ and is defined by

$$\tilde{A}\tilde{B} = \begin{bmatrix} \tilde{A}_{11}\tilde{B}_{11} + \tilde{A}_{12}\tilde{B}_{21} & : & \tilde{A}_{11}\tilde{B}_{12} + \tilde{A}_{12}\tilde{B}_{22} \\ ..... & : & ..... \\ \tilde{A}_{21}\tilde{B}_{11} + \tilde{A}_{22}\tilde{B}_{21} & : & \tilde{A}_{21}\tilde{B}_{12} + \tilde{A}_{22}\tilde{B}_{22} \end{bmatrix}$$

provided that the blocks considered here are conformable for multiplication.

3.5 Transpose of a neutrosophic soft block matrix

Let $\tilde{A} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix}$

be a neutrosophic soft block matrix, then the transpose of that neutrosophic soft block matrix is defined as

$$\tilde{A}^T = \begin{bmatrix} \tilde{P}_{11}^{\ T} & \tilde{P}_{12}^{\ T} \\ \tilde{P}_{21}^{\ T} & \tilde{P}_{22}^{\ T} \end{bmatrix}$$

4. Some special types of neutrosophic soft block matrices

In this section, the intention is to discuss about various types of neutrosophic soft block matrices and the associative properties.

4.1 Neutrosophic soft block triangular matrix

Neutrosophic soft block triangular matrix is a special type of square neutrotrosophic soft matrix. Neutrosophic block triangular matrices can be of two forms such as upper triangular or lower triangular.

4.1.1 Neutrosophic Soft Block upper triangular matrix

Neutrosophic soft block upper triangular matrix is a neutrosophic soft matrix of the form $\tilde{A}=\begin{bmatrix}\tilde{X} & \tilde{Y}\\ O & \tilde{W}\end{bmatrix}$ where $\tilde{X}$, $\tilde{Y}$ and $\tilde{W}$ are square neutrosophic soft matrices.

4.1.2 Neutrosophic Soft Block lower triangular matrix

Neutrosophic soft block lower triangular matrix is a neutrosophic soft matrix of the form $\tilde{A}=\begin{bmatrix}\tilde{X} & O\\ \tilde{Y} & \tilde{W}\end{bmatrix}$ where $\tilde{X}$, $\tilde{Y}$ and $\tilde{W}$ are square neutrosophic soft matrices.

4.1.3 Properties of Neutrosophic Soft Block triangular matrix

- Addition of two neutrosophic soft block upper triangular matrices of same order results in a neutrosophic soft block upper triangular matrix.
- Product of two neutrosophic soft block upper triangular matrices is again a neutrosophic soft block upper triangular matrix.
- Addition of two neutrosophic soft block lower triangular matrices results in a neutrosophic soft block upper triangular matrix.
- Multiplication of two neutrosophic soft block lower triangular matrices of same order is again a neutrosophic soft block lower triangular matrix.

4.2 Neutrosophic soft block diagonal matrix

Neutrosophic soft block diagonal matrix is a square neutrosophic soft block matrix in which the main diagonal blocks are square neutrosophic soft matrices and all off diagonal blocks are zero neutrosophic soft matrices. Neutrosophic soft block diagonal matrix A has the following form

$$\tilde{A}=\begin{bmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A_{nn} \end{bmatrix}$$ where A_{ij} is a square neutrosophic soft block matrix for all i,j=1,2,3,…..,n.

4.3 Neutrosophic soft block quasidiagonal matrix

It is a neutrosophic soft block matrix whose diagonal blocks are square neutrosophic soft block matrices of different order and off diagonal blocks are zero neutrosophic soft block matrices. Thus

$$\tilde{A}=\begin{bmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & D_n \end{bmatrix}$$ is a quasidiagonal matrix whose diagonal blocks D_i, i=1,2,3,…..,n are square neutrosophic soft matrices of different orders.

4.4 Neutrosophic soft block tridiagonal matrix

Neutrosophic soft block tridiagonal matrix is another special neutrosophic soft block matrix which is just like the neutrosophic soft block diagonal matrix, a square neutrosophic soft matrix , having square neutrosophic soft matrices in the lower diagonal, main diagonal and upper diagonal with all other blocks being zero neutrosophic soft matrices. Neutrosophic soft block tridiagonal matrix A has the form

$$\tilde{A}=\begin{bmatrix} B_1 & C_1 & \dots & 0 \\ A_1 & B_2 & C_2\dots & 0 \\ 0 & A_2 & B_3 & C_3 \\ 0 & 0 & A_3 & B_4 \end{bmatrix}$$ where A_i, B_i, C_i are square neutrosophic soft block matrices in the lower diagonal, main diagonal and upper diagonal respectively.

4.4.1 Properties of Neutrosophic soft block tridiagonal matrix

- Sum of two neutrosophic soft block tridiagonal matrices of same order is again a neutrosophic soft block tridiagonal matrix.

Let $\tilde{A}=\begin{bmatrix} B_1 & C_1 & \dots & 0 \\ A_1 & B_2 & C_2\dots & 0 \\ 0 & A_2 & B_3 & C_3 \\ 0 & 0 & A_3 & B_4 \end{bmatrix}$ and $\tilde{B}=\begin{bmatrix} D_1 & E_1 & \dots & 0 \\ F_1 & D_2 & E_2\dots & 0 \\ 0 & F_2 & D_3 & E_3 \\ 0 & 0 & F_3 & D_4 \end{bmatrix}$

Then from the definition of addition of two neutrosopic soft block matrices it can be obtained that

$$\tilde{A}+\tilde{B}=\begin{bmatrix} B_1+D_1 & C_1+E_1 & \ldots & 0 \\ A_1+F_1 & B_2+D_2 & C_2+E_2\ldots & 0 \\ 0 & A_2+F_2 & B_3+D_3 & C_3+E_3 \\ 0 & 0 & A_3+F_3 & B_4+D_4 \end{bmatrix}$$

which is obviously a tridiagonal neutrosophic soft block matrix.

- Product of two neutrosophic soft block tridiagonal matrices is again a neutrosophic soft block up tridiagonal matrix.
- Transpose of neutrosophic soft block tridiagonal matrix is again a neutrosophic soft tridiagonal matrix.

Example: Let be $\tilde{A}$ neutrosophic soft block tridiagonal matrix $\tilde{A}=\begin{bmatrix} B_1 & C_1 & \ldots & 0 \\ A_1 & B_2 & C_2\ldots & 0 \\ 0 & A_2 & B_3 & C_3 \\ 0 & 0 & A_3 & B_4 \end{bmatrix}$ then

$\tilde{A}^T=\begin{bmatrix} B_1 & A_1 & \ldots & 0 \\ C_1 & B_2 & A_2\ldots & 0 \\ 0 & C_2 & B_3 & A_3 \\ 0 & 0 & C_3 & B_4 \end{bmatrix}$ which a neutrosophic soft block tridiagonal matrix is again

4.5 Neutrosophic soft block toeplitz matrix

Neutrosophic soft block tridiagonal matrix is another special neutrosophic soft block matrix, which contains blocks that are repeated down the diagonals of the matrix. The individual block elements of A_{ij} must also be Toeplitz matrices. Neutrosophic soft block toeplitz matrix $\tilde{A}$ has the form

$\tilde{A}=\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{11} & A_{12} & A_{13} \\ A_{31} & A_{21} & A_{11} & A_{12} \\ A_{41} & A_{31} & A_{21} & A_{11} \end{bmatrix}$ where A_{ij} are square neutrosophic soft block matrices respectively.

4.5.1 Properties of neutrosophic soft block toepliz matrix

- Addition of neutrosophic soft block toepliz matrices is again a neutrosophic soft block toepliz matrix provided the matrices are conformal for addition.

- Transpose of a neutrosophic soft block toepliz matrix is again a neutrosophic soft block toepliz matrix.

Example: If the above toepliz matrix A is considered then

$$\tilde{A}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{11} & A_{21} & A_{31} \\ A_{13} & A_{12} & A_{11} & A_{21} \\ A_{14} & A_{13} & A_{12} & A_{11} \end{bmatrix}$$

this is again a neutrosophic soft toplitz matrix.

8.1.3 Transportation

If $\tilde{A}, \tilde{B}$ be two neutrosophic soft block toeplitz matrices, then $(\tilde{A}+\tilde{B})^T = \tilde{A}^T + \tilde{B}^T$

Let $\tilde{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{11} & A_{12} & A_{13} \\ A_{31} & A_{21} & A_{11} & A_{12} \\ A_{41} & A_{31} & A_{21} & A_{11} \end{bmatrix}$ and $\tilde{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{11} & B_{12} & B_{13} \\ B_{31} & B_{21} & B_{11} & B_{12} \\ B_{41} & B_{31} & B_{21} & B_{11} \end{bmatrix}$

Then $\tilde{A}+\tilde{B} = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} & A_{14}+B_{14} \\ A_{21}+B_{21} & A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} \\ A_{31}+B_{31} & A_{21}+B_{21} & A_{11}+B_{11} & A_{12}+B_{12} \\ A_{41}+B_{41} & A_{31}+B_{31} & A_{21}+B_{21} & A_{11}+B_{11} \end{bmatrix}$

Then $\tilde{A}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{11} & A_{21} & A_{31} \\ A_{13} & A_{12} & A_{11} & A_{21} \\ A_{14} & A_{13} & A_{12} & A_{11} \end{bmatrix}$ and $\tilde{B}^T = \begin{bmatrix} B_{11} & B_{21} & B_{31} & B_{41} \\ B_{12} & B_{11} & B_{21} & B_{31} \\ B_{13} & B_{12} & B_{11} & B_{21} \\ B_{14} & B_{13} & B_{12} & B_{11} \end{bmatrix}$

$$(\tilde{A}+\tilde{B})^T = \begin{bmatrix} A_{11}+B_{11} & A_{21}+B_{21} & A_{31}+B_{31} & A_{41}+B_{41} \\ A_{12}+B_{21} & A_{11}+A_{11} & A_{21}+B_{21} & A_{31}+B_{31} \\ A_{13}+B_{13} & A_{12}+B_{12} & A_{11}+B_{11} & A_{21}+B_{21} \\ A_{14}+B_{14} & A_{13}+B_{13} & A_{12}+B_{12} & A_{11}+B_{11} \end{bmatrix} = \tilde{A}^T + \tilde{B}^T$$

4.6 Neutrosophic soft Block Circulant Matrix

A neutrosophic soft Block Circulant Matrix is a neutrosophic soft block matrix of the form

$$\tilde{A} = \begin{bmatrix} A_0 & A_1 & \ldots\ldots & A_{m-1} \\ A_{m-1} & A_0 & \ldots\ldots & A_{m-2} \\ \vdots & \vdots & \ldots\ldots & \vdots \\ A_1 & A_2 & \ldots\ldots & A_0 \end{bmatrix}$$

where A_i's are nxn arbitrary matrics.

4.6.1 **Properties of Neutrosophic soft Block Circulant Matrix**

If A and B be two neutrosophic block circulant matrices then A+B and AB is again a neutrosophic block circulant matrix. Again for block circulant matrices AB=BA.

4.7 Direct sum of neutrosophic soft block matrices.

If $\tilde{A}_{11},\tilde{A}_{22},\tilde{A}_{33},......,\tilde{A}_{rr}$ are square neutrosophic soft block matrices of order $m_1,m_2,m_3,......,m_r$ respectively.

Then $$diag(\tilde{A}_{11},\tilde{A}_{22},\tilde{A}_{33},......,\tilde{A}_{rr})=\begin{bmatrix}\tilde{A}_{11} & 0 & ... & 0\\ 0 & \tilde{A}_{22} & ... & 0\\ : & : & : & :\\ 0 & 0 & ... & \tilde{A}_{rr}\end{bmatrix}_{m_1+m_2+.......+m_r}$$

is called the direct sum of the square neutrosophic soft block matrix $\tilde{A}_{11},\tilde{A}_{22},\tilde{A}_{33},......,\tilde{A}_{rr}$ and it is expressed as $\tilde{A}_{11}\oplus\tilde{A}_{22}\oplus\tilde{A}_{33}\oplus......\oplus\tilde{A}_{rr}$ of order $m_1+m_2+m_3+......+m_r$.

4.7.1 Properties of direct sum

The following algebraic properties are hold by neutrosophic soft block matrices:

- **Commutativity:** Let $\tilde{A},\tilde{B}$ be two diagonal neutrosophic soft block matrices then

$$\tilde{A}\oplus\tilde{B}=\begin{bmatrix}\tilde{A} & : & 0\\ ..... & : & .....\\ 0 & : & \tilde{B}\end{bmatrix} \quad \text{and} \quad \tilde{B}\oplus\tilde{A}=\begin{bmatrix}\tilde{B} & : & 0\\ ..... & : & .....\\ 0 & : & \tilde{A}\end{bmatrix}$$

Thus $\tilde{A}\oplus\tilde{B}\neq\tilde{B}\oplus\tilde{A}$ and hence it can be concluded that the direct sum of two neutrosophic soft block matrices are not commutative.

- **Associativity:** Let $\tilde{A},\tilde{B},\tilde{C}$ be three square neutrosophic soft block matrices. Then as obtained above

$$\tilde{A}\oplus\tilde{B}=\begin{bmatrix}\tilde{A} & : & 0\\ ..... & : & .....\\ 0 & : & \tilde{B}\end{bmatrix}=\tilde{D}(say)$$

Therefore

$$(\tilde{A}\oplus\tilde{B})\oplus\tilde{C}=\tilde{D}\oplus\tilde{C}=\begin{bmatrix}\tilde{D} & 0\\ 0 & \tilde{C}\end{bmatrix}=\begin{bmatrix}\tilde{A} & 0 & 0\\ 0 & \tilde{B} & 0\\ 0 & 0 & \tilde{C}\end{bmatrix} \text{ where } \tilde{A}\oplus\tilde{B}=\tilde{D}$$

Again

$$\tilde{A}\oplus(\tilde{B}\oplus\tilde{C})=\tilde{A}\oplus\tilde{E}=\begin{bmatrix}\tilde{A} & 0\\ 0 & \tilde{E}\end{bmatrix}=\begin{bmatrix}\tilde{A} & 0 & 0\\ 0 & \tilde{B} & 0\\ 0 & 0 & \tilde{C}\end{bmatrix} \text{ where } \tilde{B}\oplus\tilde{C}=\tilde{E}$$

Hence associative laws hold for neutrosophic soft block matrices.

4.8 Mixed sum of neutrosophic soft block matrices

Let $\tilde{A},\tilde{B},\tilde{C},\tilde{D}$ be four neutrosophic soft block matrices which are conformable for addition. Then

By the definitions of addition and direct sum of neutrosophic soft block matrices it can be obtained that

$$\tilde{A}\oplus\tilde{C}=\begin{bmatrix}\tilde{A} & : & 0\\ ..... & : & .....\\ 0 & : & \tilde{C}\end{bmatrix},\ \tilde{B}\oplus\tilde{D}=\begin{bmatrix}\tilde{B} & : & 0\\ ..... & : & .....\\ 0 & : & \tilde{D}\end{bmatrix} \text{and } (\tilde{A}\oplus\tilde{C})+(\tilde{B}\oplus\tilde{D})=\begin{bmatrix}\tilde{A}+\tilde{B} & : & 0\\ ..... & : & .....\\ 0 & : & \tilde{C}+\tilde{D}\end{bmatrix}$$

Then the following result holds:

$$(\tilde{A}+\tilde{B})\oplus(\tilde{C}+\tilde{D})=(\tilde{A}\oplus\tilde{C})+(\tilde{B}\oplus\tilde{D})$$

4.9 Multiplication of direct sum of neutrosophic soft block matrices

If $\tilde{A},\tilde{B},\tilde{C},\tilde{D}$ be four neutrosophic soft block matrices which are conformable for addition and multiplication then the mixed multiplication of direct sum is

$$(\tilde{A}\oplus\tilde{B})(\tilde{C}\oplus\tilde{D})=(\tilde{A}\tilde{C})\oplus(\tilde{B}\tilde{D})$$

By the definition of direct sum and multiplication of neutrosophic soft block matrices

$$(\tilde{A}\oplus\tilde{B})(\tilde{C}\oplus\tilde{D})=\begin{bmatrix}\tilde{A} & 0\\ 0 & \tilde{B}\end{bmatrix}\begin{bmatrix}\tilde{C} & 0\\ 0 & \tilde{D}\end{bmatrix}$$

$$=\begin{bmatrix}\tilde{A}\tilde{C} & 0\\ 0 & \tilde{B}\tilde{D}\end{bmatrix}=\tilde{A}\tilde{C}\oplus\tilde{B}\tilde{D}$$

- **Transposition**

If $\tilde{A},\tilde{B}$ be two neutrosophic soft block matrices then the transportation of the direct sum of A and B is

$$(\tilde{A}\oplus\tilde{B})^T=\begin{bmatrix}\tilde{A}^T & 0\\ 0 & \tilde{B}^T\end{bmatrix}=\tilde{A}^T\oplus\tilde{B}^T$$

5. Results

In the process it is found that different types of neutrosophic soft block matrices which are discussed here behave in the same way as the block matrices that exist in the literature.

6. Applications

Neutrosophic soft matrices having been broken into sections called blocks or partitioned are useful for cutting down calculations in the cases of problems which involves neutrosophic soft matrices.

7. Conclusions

Different types of neutrosophic soft block matrices as triangular, tridiagonal, quasidiagonal, circulant, toepliz are discussed. Some operations on neutrosophic soft block matrices which are also discussed in this article gives a clear indication that such operations produces almost similar results to those of classical matrices. Future research will be in the direction of finding determinants of neutrosophic soft block matrices.

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University of New Mexico

LGU-Combined-Consciousness State Model

Shazia Rana [1], Muhammad Saeed [2], Florentin Smarandache [3*]

[1] University University of Management and Technology, Johar Town Campus, Lahore, 54000, Pakistan shaziaranams@gmail.com

[2] University University of Management and Technology, Johar Town Campus, Lahore, 54000, Pakistan muhammad.saeed@umt.edu.pk

[3] University of New Mexico, Mathematics, Physics, and Natural Science Division, 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu

Abstract

This article aims to introduce some modern algebraic structures as hyper super matrices. The classical algebra and matrices cannot process higher-dimensional information with several levels of ambiguity and uncertainty. Hence, it is necessary to establish such superalgebraic structures that can organize and classify the uncertain and incomplete information floating in parallel higher dimensions as facts, events, or realities. To achieve the desired goal, a particular construction of Hypersoft Matrix (HS-Matrix) and Subjectively Whole Hyper-SuperSoft Matrix (SWHSS-Matrix) is offered in a plithogenic Fuzzy environment initially, and some aggregation operators are formulated. A Local-Global-Universal Combined Consciousness State Ranking Model is formulated as an application. As the classification of non-physical phenomena like state of physical health or Consciousness has not yet been addressed in the area of decision making therefor the proposed model will open a new dimension of classification of the non-physical part of the universe in which one can select the most suitable possible reality from several parallel realities which would be useful in the field of artificial intelligence. This model classifies the accumulated states of matter bodies (subjects). And gives a possible description of the Combined-Consciousness State of a Universe. In addition, it offers a local ranking by observing the information through several angles of vision, just like a human mind does, and a universal ranking by classifying the accumulated states. Furthermore, the final Global Ranking is achieved by constructing a percentage frequency-matrix and an authenticity measure of the order is offered. A numerical example is constructed to describe SWHSS-Matrix and LGU-Ranking Model. Some pie graphs are used to describe the individual states, accumulated states, and the ultimate accumulated universal state of all given subjects (a Combined Conscious State of Universe).

Keywords: Subjectively-Whole-Hyper-Super-Soft-Matrix, Parallel-Dimensions, Attributive-Ranking, Local-Global-Universal-Ranking, Combined-Consciousness, Percentage-Frequency-Matrix, Pie-Graphs.

1. Introduction

As we know, the human brain has some factors of vagueness and precariousness in its judgments and inferences due to multiple opinions, and the complexity of the data, as attributes events, and information derived from its own environments. Scientists after taking into account this basic trait of the human mind start arguing the dire need for some different mathematics that could possibly handle this vagueness factor. Some of the following theories developed gradually. Fuzzy set theory by Zadeh (1965) [1] Intuitionistic fuzzy set (IFS) theory by k.Atanassov [2] [3]. The cloud of vagueness is further extended by F. Smarandache, [4][5][6]. Some more recent extensions and modernizations of the neutrosophic set are presented in [7] [8] [9] [10] [11] [12]. In 1999 Molodtsove [13] introduced Soft Set, a soft set is a parameterized representation of subsets in which one can express multiple attributes and subjects in a unique parameterized formulation. Some further extensions of the soft set were provided in [14] [15] [16]. Later, in 2018, F.Smarandache [17] [18] introduced another expanded version of Softest known as the Hypersoft-Set and the Plithogenic Hypersoft-Set. In these sets, he extended the function of the combination of attributes to multi attributes and sub-attributes. He presented the basic definitions and addressed many open problems of the development of new literature, such as aggregation operators and MADM techniques. We are going to answer some of the open issues raised by Smarandache, S.Rana and co-authors "[19] extended the Plithogenic Hyper-Soft Set to Plithogenic Whole-Hyper-Soft Set by accumulating the memberships and providing both exterior and interior states of the part of Universe/Event/Reality/Information (a combination of Attributes, Sub Attributes, Subjects represented). We represented the Plithogenic Fuzzy Hyper-Soft set and the Plithogenic Fuzzy Whole Hyper-Soft set in a novel form of matrices in the fuzzy environment named as Plithogenic Fuzzy Hyper-Soft Matrix (PFHS-Matrix) and Plithogenic Fuzzy Whole Hyper-Soft Matrix and some local operators were established. Furthermore. In the next phase, S.Rana and co-authors "[20] further dilated the Plithogenic Whole Hyper-Soft Set to Plithogenic Crisp/Fuzzy/Intuitionistic/Neutrosophic Subjective Hyper-Soft Set and represented them in the more dilated version of Soft-Matrix initially in the fuzzy environment termed as Plithogenic Subjective Hyper Super Soft-Matrix. Then developed a Local-Global Universal Subjective Ranking Model by using the new amplified expression of matrices. Some further literature on HyperSoft Set and Plithogency was established in [21-28]. In this article, in the first stage, we have further broadened those earlier introduced Plithogenic Fuzzy Whole Hyper Soft Set and Plithogenic Subjective Hyper-Soft Set to Plithogenic Attributive Subjectively Whole Hyper Soft Set (PASWHSS-Set) in the Fuzzy environment. we have formulated a new type of Matrix initially in a fuzzy environment named Plithogenic Subjectively Whole Hyper Super Soft Matrix (PSWHSS-Matrix). These advanced types of matrices are generated by the hybridization of hyper matrices and super matrices [29-32] These hypersoft matrices are sets/clusters of parallel layers of matrices representing clusters of parallel universes/ realities/ events/ information. These are such hyper-matrices (parallel layers of matrices) whose elements are also matrices. Thus, these matrices are tensors of rank three and four, respectively, having three and four indices of variations. Then later, we have formulated an LGU Combined-Consciousness State Ranking Model. The forte of this model is its classification of nonphysical phenomena. Thus, it will allow opening a new non-physical dimension of classification i.e. selecting one possibility out of multiple possibilities. Moreover, it offers a transparent ranking of attributes (states of subjects) and universes from micro-universe to macro-universe levels by observing them through numerous angles of vision in dissimilar environments of different ambiguity and hesitation levels. Furthermore, it will also furnish and formulate extreme and neutral values of these universes (sets of information, realities, events). This new model actually compacts the expanded Universe to a single lowest point. Finally, we have also anticipated producing a percentage authenticity measure of ranking, which is provided by using a frequency matrix. In the end, we have given an application of the Model using a numerical example. In this example, fuzzy linguistic scales are used to quantify the states of our subjects (bodies of matter known as individuals). The quantified states of subjects are attributes/sub-attributes known as individual fuzzy states or individual fuzzy memberships. Later, the aggregation operators are used to accumulate these states (subject-wise). The accumulated states are represented by fuzzy whole memberships. Initially, these states are accumulated at the local level using a single aggregation operator representing a viewpoint, and a local ordering of states would be achieved. The global ordering of states would be achieved through the use of multiple aggregation operators. By the further accumulation of the already accumulated states, the universal states of accumulation and the universal order would be reached.

Now the further query arises why we are specifically using hyper-Soft and Hyper-Super-Soft matrices for the expression of the Plithogenic Hyper-Soft Set and Plithogenic Attributive Hyper-Soft Set? The answer might be convincing that this Plithogenic Universe is so vast and expanded in its interior (having Fuzzy, Intuitionistic Fuzzy, Neutrosophic, environments with memberships non-memberships, and indeterminacies) and in its exterior (managing many attributes, sub-attributes, and sub-sub-attributes concerning to its subjects). Therefore to organize and classify such highly scattered information we need to formulate some super algebraic structures like these Matrices.

This article is organized into seven basic sections. After the (**section-1**) introduction, **Section 2** summarises some related preliminaries. In **Section 3** we introduce some fundamental new concepts and definitions of the Hypersoft set expression, the HS matrix, and the SWHSS matrix with examples in a plithogenic fuzzy environment. We use these new types of matrices to develop the LGU Combined-Consciousness State Ranking Model. While in **Section 4** some local aggregation operators such as disjunction operators, conjunction operators, averaging operators and compliment operators for PFHS matrices are formulated. **Section 5** describes the algorithm of the LGU Combined-Consciousness State Ranking Model in the plithogenic fuzzy environment In this Model, we would provide the classification of attributes (a non-physical phenomenon or states) at the local, Global and Universal levels. We offer the Universal ranking by classifying these already accumulated universal states. The Local Ranking is offered by observing the higher dimensional information through several angles of vision or states just like a human mind which possesses multiple layers of thought. These thoughts undergo and change their angles in order to achieve a precise or accurate status but before certain complex procedures of mind are applied upon them. Finally, mental thoughts hold their possibly best and desired status/angels depending upon certain complex procedures and environments. In order to learn the transparent Global Ranking, we have applied a Percentage-Frequency-Matrix by accumulating the states of the human mind (several angles of vision). Finally, to preserve transparency and accuracy, our model also provides the authenticity measure of the ordering. In **Section 6** Application of the LGU-Combined-Consciousness State Ranking Model is presented and final combined universal states are offered. In **Section 7** the flow of the model from individual states of subjects to their combined-universal states is described by pi graphs and some conclusions and open problems are discussed.

2 Preliminaries

This section, narrates some fundamental useful definitions of the hyper-soft set, Hyper matrices, and Super matrices.

Definition 2.1 [17] (Hyper-soft set)
Let U *be the initial universe of discourse* $P(U)$ *the power set of* U.
let $a_1, a_2, \ldots, a_n$ *for* $n \geq 1$ *be* n *distinct attributes, whose corresponding attribute-values are respectively the sets* $A_1, A_2, \ldots, A_n$ *with* $A_i \cap A_j = \varphi$ *for* $i \neq j$ *and* $i, j \in \{1,2,\ldots,n\}$.
Then the pair $(F, A_1 \times A \times \ldots \times A_n)$ *where,*

$$F: A_1 \times A \times \ldots \times A_n \rightarrow P(U),$$

is called a hyper-soft set over U;
Definition 2.2 [29] [30] (super-matrices)

A rectangular or square arrangements of numbers in rows and columns are known as matrices,or simply ordinary matrices, wheras a super-matrix is such matrix whose elements are matrices. These elements can be either scalars or matrices.

$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, *where*

$a_{11} = \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$, $a_{12} = \begin{bmatrix} 0 & 40 \\ 21 & -12 \end{bmatrix}$,

$a_{21} = \begin{bmatrix} 3 & -1 \\ 5 & 7 \\ -2 & 9 \end{bmatrix}$, $a_{22} = \begin{bmatrix} 4 & 12 \\ -17 & 6 \\ 3 & 7 \end{bmatrix}$ *a is a super-matrix.*

Note: The elements of super-matrices are considered as sub-matrices i.e. $a_{11}, a_{12}, a_{21}, a_{22}$ are submatrices of the super-matrix a.

Definition 2.3 [31] [32] (Hyper-matrices)

For $n_1, \ldots, n_d \in N$, a function $f: (n_1) \times \ldots \times (n_d) \rightarrow F$ is a hyper-matrix, or d-hyper-matrix. Often $a_{k_1 \ldots k_d}$ are used to denote the value $f(k_1 \ldots k_d)$ of f at $(k_1 \ldots k_d)$ and think of f (renamed as A) as specified by a d-dimensional table of values, writing $A = [a_{k_1 \ldots k_d}]_{k_1 \ldots k_d}^{n_1, \ldots, n_d}$

A 3-hypermatrix can be written on a (2-dimensional) piece of paper as a list of ordinary matrices, called slices. For example

$$A = \begin{bmatrix} a_{111} & a_{121} & a_{131} & . & a_{112} & a_{122} & a_{132} \\ a_{211} & a_{221} & a_{231} & . & a_{212} & a_{222} & a_{232} \\ a_{311} & a_{321} & a_{331} & . & a_{312} & a_{322} & a_{332} \end{bmatrix}$$

3. Plithogenic Fuzzy HS-Matrix and Plithogenic Fuzzy SWHSS-Matrix

This section, develops some literature about the plithogenic hypersoft set in the following manner.

1. We introduce some basic new beliefs and definitions of expression of hypersoft set and HS-Matrix with examples.
2. We introduce novel HS-matrix as SWHSS-Matrix in plithogenic Fuzzy environment.
3. We portray the compact and expanded expressions of HS-Mtricx and SWHSS-Matrix.

To develop an understanding of the literature, we give some new definitions.

Definition 3.1 (Plithogenic Fuzzy HyperSoft-Set (PFHS-Set)): Let U_F be the initial universe of discourse $P(U_F)$ the power set of U_F. A_j^k is a combination of attributes/Sub-Attributes for some $j = 1,2,3, \ldots, N$ Attributes, $k = 1,2,3, \ldots, L$ Sub-Attributes and x_i $i = 1,2,3, \ldots,$ M are subjects under consideration then $(F_F, A_1^k, A_2^k \ldots, A_3^k)$ is PFHS-Set represented by plithogenic fuzzy memberships $\mu_{A_j^k}(x_i)$.

where, $F_F: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \rightarrow, P(U_F)$ is a mapping from a complex cross product of the attributes to the power set $P(U_F)$. This PFHS-Set is represented as

$$\mathrm{F} = \left\{ \begin{array}{l} x_1 \left(\mu_{A_j^k}(x_1) \right), \\ x_2 \left(\mu_{A_j^k}(x_2) \right), \\ . \\ . \\ . \\ x_{\mathrm{M}} \left(\mu_{A_j^k}(x_M) \right) \end{array} \right\}$$

Definition 3.2 (Plithogenic Fuzzy HyperSoft-Matrix (PFHS-Matrix)):

Let U_F be the Fuzzy universe of discourse, $P(U_F)$ be the power set of U_F, A_j^k is a combination of attributes/sub-attributes for some $j = 1,2,3, \ldots, N$ attributes, $k = 1,2,3, \ldots, L$ sub-attributes and x_i $i =$

$1,2,3,\ldots, M$ are subjects under consideration then PFHS-Matrix, $F_{ij}^{k} = \left[\mu_{A_j^k}(x_i)\right]$ is a mapping $F_F: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \rightarrow P(U_F)$, from a complex cross product of the attributes to the power set $P(U_F)$., Where $\mu_{A_j^k}(x_i) \in [0,1]$ are fuzzy memberships s.t $\mu_{A_j^k}(x_i) + \upsilon_{A_j^k}(x_i) = 1$. These Fuzzy memberships $\mu_{A_j^k}(x_i)$ are the elements of PFHS-Matrix and are assigned for the Part of Universe/Reality/Event/Information, by decision-makers or concerned bodies through the linguistic scales. For further details, see ref. [28-31]. we may call these memberships the individual fuzzy memberships.

We may write F_{ij}^k simply as F.The compact form of PFHS-Matrix, is

$$F = \left[\mu_{A_j^k}(x_i)\right] \tag{3.1}$$

And an expanded form of PFHS-Matrix, is

$$\begin{array}{c} \quad A_1^k \quad A_2^k \; . \; . \; . \; A_N^k \\ F = \begin{array}{c} x_1 \\ x_2 \\ . \\ . \\ . \\ x_M \end{array} \begin{bmatrix} \mu_{A_1^k}(x_1) & \mu_{A_2^k}(x_1) & . & . & . & \mu_{A_N^k}(x_1) \\ \mu_{A_1^k}(x_2) & \mu_{A_2^k}(x_2) & . & . & . & \mu_{A_N}(x_2) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \mu_{A_1^k}(x_M) & \mu_{A_2^k}(x_M) & . & . & . & \mu_{A_N^k}(x_M) \end{bmatrix} \end{array} \tag{3.2}$$

Example 1:

Consider the mapping F defined as,

$F_F: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \rightarrow, P(U_F)$

(taking some specific numeric values of A_j^k)

Consider $T = \{x_1, x_2, x_3\}$, is a subset of powerset $P(U_F)$ and x_i subjects for $i = 1,2,3$, are x_1, x_2, x_3. The associated states of these subjects are A_j^k Attributes/Sub-Attributes for $j = 1,2,3,4$ and $k = 1,2,3$. To represent these states some fuzzy memberships would be assigned by the Concerned body, through the five-point linguistic scale (see ref. [28-31]) T

The set representation of information is described as PFHS-Set as,

$$F_\alpha(A_1^3, A_2^1, A_3^1, A_4^2) = \begin{Bmatrix} x_1(0.3,0.6,0.5,0.5), \\ x_2(0.4,0.4,0.3,0.1), \\ x_3(0.6,0.3,0.4,0.7) \end{Bmatrix} \tag{3.3}$$

And further organized and expressed in one layer of PFHS-Matrix F_{ij}^α,

$$\begin{array}{c} A_1^3 \; A_2^1 \; A_3^1 \; A_4^2 \\ \mathrm{F} = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \begin{bmatrix} 0.3 & 0.6 & 0.5 & 0.5 \\ 0.4 & 0.4 & 0.3 & 0.1 \\ 0.6 & 0.3 & 0.4 & 0.7 \end{bmatrix} \end{array} \tag{3.4}$$

Where $A_1^3 \; A_2^1 \; A_3^1 \; A_4^2$ is a specific α combination of Attributes/Sub-Attributes representing states of subjects $x_1, x_{2,} x_3$. F_{ij}^α is representing a single layer out of multiple possible layers of PFHS-Matrix. For a more detailed description and applications, see [19]

Example 2. Consider layered representation $F = \left[\mu_{A_j^k}(x_i)\right]$ for $k = 1$, $j = 1,2,3,4$ and $i = 1,2,3$, i.e (first level-layer) and for $k = 2$, $j = 1,2,3,4$ and $i = 1,2,3$, i.e (second level-layer). let $T = \{x_1, x_2, x_3\}$ be Subjects in PFHS-Set associated to given attribute the PFHS-Set is represented through fuzzy memberships as described bellow,

$$F(A_1^1, A_2^1, A_3^1, A_4^1) = \begin{Bmatrix} x_1(0.3,0.6,0.3,0.5), \\ x_2(0.4,0.5,0.2,0.1), \\ x_3(0.6,0.2,0.3,0.7) \end{Bmatrix} \quad (3.5)$$

$$F(A_1^2, A_2^2, A_3^2, A_4^2) = \begin{Bmatrix} x_1(0.5,0.4,0.2,0.6) \\ , x_2(0.5,0.7,0.8,0.4), \\ x_3(0.7,0.6,0.5,0.9) \end{Bmatrix} \quad (3.6)$$

The matrix representation of this PFHS-Set F is described as PFHS-Matrix,

$$\boldsymbol{F} = \begin{bmatrix} \begin{bmatrix} 0.3 & 0.6 & 0.3 & 0.5 \\ 0.4 & 0.5 & 0.2 & 0.1 \\ 0.6 & 0.2 & 0.3 & 0.7 \end{bmatrix} \\ \begin{bmatrix} 0.5 & 0.4 & 0.2 & 0.6 \\ 0.5 & 0.7 & 0.8 & 0.4 \\ 0.7 & 0.6 & 0.5 & 0.9 \end{bmatrix} \end{bmatrix} \quad (3.8)$$

For further details, see ref.[20]

Definition 3.3 (Plithogenic Fuzzy Subjectively-Whole Hyper-Super-Soft-Matrix (PFSWHSS-Matrix)):

Let U_F be the primary universe of discourse, in the Fuzzy situation and $P(U_F)$ be the power set of U_F. Let $A_1^k, A_2^k, \dots, A_N^k$ are A_j^k N distinct attributes/subattributes for $= 1,2,\dots N$, $k = 1,2,\dots L$ is representing attribute values then **PFSWHSS-Matrix** is, $F \begin{bmatrix} \left[\mu_{A_j^k}(x_i)\right] \\ \left[\Omega^t_{A_j^k}(X)\right] \end{bmatrix}$ is mapping

$$F_F: A_1^k \times A_2^k \times \ldots \times A_N^k \to P(U_F)$$

we may use a compact notation of PFSWHSS-Matrix, F_{ij}^{kt}, This matrix is expressed by both individual fuzzy memberships $\mu_{A_j^k}(x_i)$ (individual fuzzy states of subjects regarding each attribute) and the aggregated fuzzy memberships $\Omega_{A_j^k}(\boldsymbol{X})$ (subject-wise aggregated states). In F_{ij}^{kt} $t = 1,2, \dots O$ is representing aggregation operators. In PFSWHSS-Matrix the fuzzy states (fuzzy memberships) of all given subjects are aggregated and then represented as for each attribute/sub-attribute. This PFSWHSS-Matrix handles not only a single combination of attributes/subattributes but rather multiple combinations of attributes/sub-attributes out of their complex cross products or in other words. This matrix F_{ij}^{kt}, has four indices of variation is a soft tensor of rank 4. We may write F_{ij}^{kt} as F for the simplification of notation. Four types of variation are presented in this PFSWHSS matrix. The first Variations on the index $i = 1,2,\dots M$ generate M rows of Matrix, the second variations on the index $j = 1,2,\dots N$ generate N columns, and the third variations on $k = 1,2,\dots L$ produces L combinations of rows and columns as parallel-layers of $M \times N$ matrices as hyperSoft Matrix. The fourth variation on $t = 1,2,\dots P$ describes $the\ P\ sets$ of Clusters.

The representation of PFSWHS-Matrix in a compact form is,

$$\mathrm{F} = \begin{bmatrix} \left[\mu_{A_j^k}(x_i)\right] \\ \left[\Omega^t_{A_j^k}(X)\right] \end{bmatrix}, \quad (3.9)$$

$\mathrm{F} = \begin{bmatrix} \left[\mu_{A_j^1}(x_i)\right] \\ \left[\Omega^t_{A_j^1}(X)\right] \end{bmatrix}$ represents a single Layer of SWHSS-Matrix for $k = 1$ i.e an α universe.

$\mathrm{F} = \begin{bmatrix} \left[\mu_{A_j^1}(x_i)\right] \\ \left[\Omega^t_{A_j^1}(X)\right] \end{bmatrix}$ represents a single Layer of SWHSS-Matrix for $k = 2$ i.e an β universe.

The representation of PFSWHS-Matrix in an expanded form is,

$$\mathbf{F}=\begin{bmatrix}
\begin{bmatrix}\begin{bmatrix}\mu_{A_1^1}(x_1) & \mu_{A_2^1}(x_1) & \cdots & \mu_{A_N^1}(x_1)\\ \mu_{A_1^1}(x_2) & \mu_{A_2^1}(x_2) & \cdots & \mu_{A_N^1}(x_2)\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{A_1^1}(x_M) & \mu_{A_2^1}(x_M) & \cdots & \mu_{A_N^1}(x_M)\end{bmatrix}\\ \begin{bmatrix}\boldsymbol{\Omega}^1_{A_1^1}(\boldsymbol{X}) & \boldsymbol{\Omega}^1_{A_2^1}(\boldsymbol{X}) & \cdots & \boldsymbol{\Omega}^1_{A_N^1}(\boldsymbol{X})\end{bmatrix}\end{bmatrix}\\
\begin{bmatrix}\begin{bmatrix}\mu_{A_1^2}(x_1) & \mu_{A_2^2}(x_1) & \cdots & \mu_{A_N^2}(x_1)\\ \mu_{A_1^2}(x_2) & \mu_{A_2^2}(x_2) & \cdots & \mu_{A_N^2}(x_2)\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{A_1^2}(x_M) & \mu_{A_2^2}(x_M) & \cdots & \mu_{A_N^2}(x_M)\end{bmatrix}\\ \begin{bmatrix}\boldsymbol{\Omega}^1_{A_1^2}(\boldsymbol{X}) & \boldsymbol{\Omega}^1_{A_2^2}(\boldsymbol{X}) & \cdots & \boldsymbol{\Omega}^1_{A_N^2}(\boldsymbol{X})\end{bmatrix}\end{bmatrix}\\
\vdots\\
\begin{bmatrix}\begin{bmatrix}\mu_{A_1^L}(x_1) & \mu_{A_2^L}(x_1) & \cdots & \mu_{A_N^k}(x_1)\\ \mu_{A_1^L}(x_2) & \mu_{A_2^L}(x_2) & \cdots & \mu_{A_N^L}(x_2)\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{A_1^L}(x_M) & \mu_{A^L}(x_M) & \cdots & \mu_{A_N^L}(x_M)\end{bmatrix}\\ \begin{bmatrix}\boldsymbol{\Omega}^1_{A_1^L}(\boldsymbol{X}) & \boldsymbol{\Omega}^1_{A^L}(\boldsymbol{X}) & \cdots & \boldsymbol{\Omega}^1_{A_N^L}(\boldsymbol{X})\end{bmatrix}\end{bmatrix}\\
\vdots\\
\begin{bmatrix}\begin{bmatrix}\mu_{A_1^1}(x_1) & \mu_{A_2^1}(x_1) & \cdots & \mu_{A_N^1}(x_1)\\ \mu_{A_1^1}(x_2) & \mu_{A_2^1}(x_2) & \cdots & \mu_{A_N^1}(x_2)\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{A_1^1}(x_M) & \mu_{A_2^1}(x_M) & \cdots & \mu_{A_N^1}(x_M)\end{bmatrix}\\ \begin{bmatrix}\boldsymbol{\Omega}^2_{A_1^1}(\boldsymbol{X}) & \boldsymbol{\Omega}^2_{A_2^1}(\boldsymbol{X}) & \cdots & \boldsymbol{\Omega}^2_{A_N^1}(\boldsymbol{X})\end{bmatrix}\end{bmatrix}\\
\begin{bmatrix}\begin{bmatrix}\mu_{A_1^2}(x_1) & \mu_{A_2^2}(x_1) & \cdots & \mu_{A_N^2}(x_1)\\ \mu_{A_1^2}(x_2) & \mu_{A_2^2}(x_2) & \cdots & \mu_{A_N^2}(x_2)\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{A_1^2}(x_M) & \mu_{A_2^2}(x_M) & \cdots & \mu_{A_N^2}(x_M)\end{bmatrix}\\ \begin{bmatrix}\boldsymbol{\Omega}^2_{A_1^2}(\boldsymbol{X}) & \boldsymbol{\Omega}^2_{A_2^2}(\boldsymbol{X}) & \cdots & \boldsymbol{\Omega}^2_{A_N^2}(\boldsymbol{X})\end{bmatrix}\end{bmatrix}\\
\vdots\\
\begin{bmatrix}\begin{bmatrix}\mu_{A_1^L}(x_1) & \mu_{A_2^L}(x_1) & \cdots & \mu_{A_N^k}(x_1)\\ \mu_{A_1^L}(x_2) & \mu_{A_2^L}(x_2) & \cdots & \mu_{A_N^L}(x_2)\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{A_1^L}(x_M) & \mu_{A^L}(x_M) & \cdots & \mu_{A_N^L}(x_M)\end{bmatrix}\\ \begin{bmatrix}\boldsymbol{\Omega}^2_{A_1^L}(\boldsymbol{X}) & \boldsymbol{\Omega}^2_{A^L}(\boldsymbol{X}) & \cdots & \boldsymbol{\Omega}^2_{A_N^L}(\boldsymbol{X})\end{bmatrix}\end{bmatrix}
\end{bmatrix} \tag{3.10}$$

This PFSWHS-Matrix exhibits both internal and subjective external states of the universe. The internal state of the universe, event, or reality is reflected by individual fuzzy memberships $\mu_{A_j^k}(x_i)$ whilst the Subjectively exterior state of the universe, event, or reality is reflected through Subjectively aggregated memberships $\Omega_{A_j^k}(X)$ that is accumulated specifically for all given subjects at each attributive/sub-attributive level. Therefore the PFSWHSS-Matrix would provide an attributive classification (non-physical classification) through a subject-wise accumulation of states. The subjective aggregation is applied to fuzzy memberships $\mu_{A_j^k}(x_i)$ at the index i, i.e at each specific sub-attributive level by applying several suitable aggregation operators. In the next section-4 for the construction of this PFSWHSS-Matrix, we have formulated some aggregation operators. The application of these operators and SWHSS-Matrix as LGU Combined-Consciousness State Ranking Model is presented in Section-5, whereas the application of this Whole Model is described in Sec-6, where the faculty ranking Model is represented.

4 Local aggregation operators for the Construction of SWHSS-Matrix

This section describes Local aggregation operators like disjunction operators, conjunction operators, Averaging operators, and Compliment-operator for PFHS-Matrix. By applying these local operators on the **PFHS-Matrix** the **SWHSS-Matrix** would be constructed. By utilizing Local disjunction, Local conjunction, and Local averaging operators, we would develop a combined (whole) memberships $\Omega^t_{A_j^k}(X)$ for **PFSWSS-Matrix** that would be presented in the last row-matrix of the. SWHSS-Matrix
The general mathematical expression for SWHSS-Matrix $\boldsymbol{F}$ in the plithogenic fuzzy environment is given below.

$\boldsymbol{F} = \begin{bmatrix} \left[\boldsymbol{\mu}_{A_j^k}(\boldsymbol{x_i})\right] \\ \left[\boldsymbol{\Omega}^t_{A_j^k}(\boldsymbol{X})\right] \end{bmatrix}$ In this Matrix the last row of cumulative memberships $\boldsymbol{\Omega}^t_{A_j^k}(\boldsymbol{X})$ is framed by using three local operators, $t = 1$ is used for the Max-operator $t = 2$ for Min-operator, and $t = 3$ for the $averaging$-operator. Furthermore, $t = 4$ is representing Compliment-operator**.**

In SHWHS-Matrix

$\boldsymbol{F}_{S_t} = \left[\left[\boldsymbol{\mu}_{A_j^k}(\boldsymbol{x_i})\right] \quad \left[\boldsymbol{\Omega}^t_{A^k}(\boldsymbol{x_i})\right]\right]$ the last column of cumulative memberships $\Omega_{A^k}(x_i)$ are obtained by using three local operators, $t = 1$ used for the Max-operator $t = 2$ is used to portray the Min-operator, and $t = 3$ is used for the $averaging$-operator. Furthermore, $t\ =\ 4\, represents\ the\ compliment.$

These four operators are described as follows:

4.1 Local-Disjunction-Operator for the construction of SWHSS-Matrix:

$$\cup_{\boldsymbol{i}}\left(\boldsymbol{\mu}_{A_j^k}(\boldsymbol{x_i})\right) = \underset{i}{Max}\left(\mu_{A_j^k}(x_i)\right) = \Omega^1_{A_j^k}(X)\text{, for some } k = l \tag{4.1}$$

This Max-operator reflects the optimal state of mind of the decision-maker.

4.2 Local-Conjunction Operator for construction of SWHSS-Matrix:

$$\cap_i\left(\mu_{A_j^k}(x_i)\right) = \underset{i}{Min}\left(\mu_{A_j^k}(x_i)\right) = \Omega^2_{A_j^k}(X)\text{, for some } k = l \tag{4.2}$$

This Min-operator reflects the pessimistic state of mind of the decision-maker.

4.3 Local-Averaging-Operator for construction of SWHSS-Matrix:

$$\Gamma_i\left(\mu_{A_j^k}(x_i)\right) = \frac{\sum_i\left(\mu_{A_j^k}(x_i)\right)}{M} = \Omega^3_{A_j^k}(X)\text{, for some } k = l \tag{4.3}$$

This averaging operator reflects the neutral state of mind of the decision-maker.

4.4 Local Compliment for the construction of SWHSS-Matrix:

$$C_{loc}(F)=\begin{Bmatrix} Max_i\left(1-\mu_{A_j^k}(x_i)\right) \\ Min_i\left(1-\mu_{A_j^k}(x_i)\right) \\ \sum_{i=1}^{M}\frac{\left(1-\mu_{A_j^k}(x_i)\right)}{M} \end{Bmatrix}, \quad \text{for some } k=l \tag{4.4}$$

5. Algorithm of LGU Combined-Consciousness State Ranking Model

This section, utilizes the local operators built in the previous section for the formulation of the LGU Combined-Consciousness State Ranking Model in the Fuzzy environment.

In this model, we would provide the classification of attributes (a nonphysical phenomenon) at the local, Global, and Universal levels. We have called this Model the LGU Combined-Consciousness State Ranking Model. Some specialties of this LGU Combined-Consciousness State Ranking Model are mentioned to describe why this model would be preferred over previously developed MADM models

1. The first and most important feature of this model is that it provides a ranking of the non-physical states of the universe. As we know, the classification of non-physical phenomena has not yet been addressed in the area of decision-making. This model will open a new dimension of classification of the non-physical part of the universe / event / reality / information, in which one can choose a possible reality from several parallel realities that would be useful in the field of artificial intelligence.
2. The second peculiarity of this model is that it offers the classification of attributes by looking at them from multiple angles of visions. For example, the choice of the $max\ operator$ is an expression of an optimistic perspective. In contrast to this, the choice of the Min-operator is an expression of the pessimistic point of view and the choice of the $average$-operator is an expression of a neutral point of view. The combination of all operators in one model offers a transparent decision that is made from multiple perspectives
3. This model has the potential to offer a classification of attributes in numerous environments such as Fuzzy, Intuitionistic, Neutrosophic, or any other suitable environment required. Each environment has its own ambiguity or hesitation level. By choosing a particular environment, this model would be expanded to work on any level of uncertainty, hesitation, or ambiguity.
4. This attributive/state ranking model offers the ranking from micro-universe to macro-universe stages i.e. from inner smaller cell to outer larger universe.
5. Primarily, this Model delivers the internal ranking of attributes (states of subjects) named "Local Attributive ranking" (ranking of states) (classification of attributes/states of micro-universe)
6. On the next stage, this Model offers an exterior classification of states named "Global Attributive Ranking."
7. On a further extended level this Model offers the 3rd type of attributive ranking named "Universal Combined-Consciousness State Ranking (Classification of attributes of the macro-universe)
8. This model also offers extreme values, as extreme behaviors, and neutral values, as neutral behavior of universes that would be helpful to find the optimal and neutral states of all kinds of universes/realities/events/information from their micro- to macro levels.
9. At the final level, it provides a precise measure of the authenticity of classification by using the frequency matrix.

Initially, we consider the case of the PFSWHSS-Matrix to rank the given attributes or states of subjects. These subjects with their all attributes/sub attributes are considered to be one universe.

Later, we can generalize this Model into Plithogenic Intuitionistic, Plithogenic Neutrosophic, and other multiple useful required environments agreeing the state of mind of the decision-makers.

The Algorithm of the LGU Combined-Consciousness State Ranking Model is described below,

Step 1. Construction of Universe: Consider the fuzzy universe of discourse $U_F = \{x_i\}$ $i = 1,2,3,\ldots, M$. Consider some attributes/sub-attributes and subjects need to be classified where attributes/sub-attributes are A_j^k $j = 1,2,3,\ldots, N$ and $k = 1,2,\ldots, L$ represents numeric values of attributes A_j (parallel level layers), and concerned subjects are $T = \{x_i\} \subset U_F$ where i can take some values from 1 to M such that Define mappings F and G such that,

$$F: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \rightarrow P(U) \text{ For some fixed } k \text{ (leve-1)} \tag{5.1}$$

$$G: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \rightarrow P(U), \text{ For some different fixed } k \text{ (level-2)} \tag{5.2}$$

Step 2. Construction of PFHS-Matrix: Write the data or information (fuzzy-memberships) of PFHS-Set in the form of PFHS-Matrix $B = \left[\mu_{A_j^k}(x_i)\right]$. If there are some non-favorable attributes in the given Information, we may replace their memberships $(\mu_{A_j^k}(x_i))$ by non-membership $(1 - \mu_{A_j^k}(x_i))$ while the neutral and favorable attributes would be displayed by their fuzzy memberships.

Step 3. Construction of PFSWHSS-Matrix: By using local aggregation operators constructed in Sec. -4 formulate PFSWHSS-Matrix given as,

$$B_{\mathbf{A}_t} = \begin{bmatrix} \left[\mu_{A_j^k}(x_i)\right] \\ \left[\Omega^t_{\mathbf{A}_j^k}(X)\right] \end{bmatrix}. \tag{5.3}$$

Step 4. Local Attributive Ranking: The Local Attributive Ranking **is** the ranking of the accumulated states of matter bodies (subjects) that would be acquired by considering cumulative memberships $\Omega^t_{\mathbf{A}_j^k}(X)$ of the last row of each layer of $B_{\mathbf{A}_t}$.

The higher the membership value, the better the attribute / sub-attribute that corresponds to this membership. At this stage, the attributive classification of all layers or a selected layer would be provided according to the required situation. In addition, the process would eventually stop when the transparent local attributive ranking is obtained. If there are some ties or ambiguities in the local attributive ranking that would be eliminated in the next step of the global ranking, a more transparent ranking would be observed.

Step 5. Global Attributive Ranking: Final global attribute ordering would be provided by using the Frequency Matrix, "F_{ij}" and the percentage frequencies Matrix f_{ij}^* by combining the states of mind of the decision-makers.

$$F_{ij} = \begin{matrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_N \end{matrix} \begin{bmatrix} f_{11} & f_{12} & \cdot & \cdot & \cdot & f_{1N} \\ f_{21} & f_{22} & \cdot & \cdot & \cdot & f_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{M1} & f_{M2} & \cdot & \cdot & \cdot & f_{NN} \end{bmatrix} \tag{5.4)a}$$

$$F_{ij}^* = \begin{matrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_N \end{matrix} \begin{bmatrix} f_{11}^* & f_{12}^* & \cdot & \cdot & \cdot & f_{1N}^* \\ f_{21}^* & f_{22}^* & \cdot & \cdot & \cdot & f_{2N}^* \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{N1}^* & f_{N2}^* & \cdot & \cdot & \cdot & f_{NN}^* \end{bmatrix} \tag{5.4)b}$$

Where, f_{ij}^{*} is the percentage frequency measure

$$f_{ij}^{*}=\frac{(f_{ij})}{\sum_{i}(f_{ij})}\times 100 \quad (5.4)c$$

In F_{ij} the values of the first column signify the frequency with which the 1st position is achieved, which is reached by some specific attributes. The elements of the column 2 represent the frequency of acquiring the second position and so on. Similarly the elements of F_{ij}^{*} represent the percentage frequencies. To find out which attribute would be assigned the first position we consider the entries in the first column of F_{ij}^{*} the attribute corresponds to the highest value of the first column attains the first position and then we delete this column of the first position and the row associated with this attribute. This reduces the dimension of the matrix. Then, for the second position, add the remaining percentage frequencies of the first position into the next percentage frequencies of the second column and then look for the highest percentage frequency in the second column for the decision of the second position.

Once the second position is determined, we delete the corresponding column and row of that position and continue the practice until the final position is allocated.

This Percentage Frequency Matrix has a great potential to handle ties.

Step 6. **Authenticity measurement of the Global ranking:** In the last step, we can check the authenticity by means of ratios.

Percentage authenticity measure of j $_{th}$ selected positions for i_{th} Attribute,

$$\frac{\text{Highest frequency of } j_{th} \text{ position}}{\text{Total frequency of } j_{th} \text{position}}\times 100$$

$$f_{ij}^{\smile}=\frac{\max_{i}(f_{ij}^{*})}{\sum_{i}(f_{ij})}\times 100 \quad (5.5)$$

Step 7. **Final Universal States (Combined Consciousness States) and Ranking:**

The final universal states (Combined Consciousness states) of Universes as final accumulated fuzzy memberships $\boldsymbol{\Omega}_{kt}$ are provided by using the disjunction operator, ($t=1$) the conjunction operator, ($t=2$), and the average operator ($t=3$) on already cumulative memberships of the last row of SWHSS-Matrix $B_{\mathbf{A}_t}$, These accumulated fuzzy memberships $\boldsymbol{\Omega}_{kt}$ represent the final Universal State or the Combined Consciousness State of the universe.

For a fixed k and t the universe with the greatest cumulative membership would be considered the better universe, and further order of the universes would be observed by arranging the $\boldsymbol{\Omega}_{kt}$ in descending order. To get the final ranking of the universal states and to obtain extreme and neutral accumulated states of the Universe/Reality/Event/Information, we would proceed as

Taking $t=1,2,3$ respectively on $\boldsymbol{\Omega}_{kt}$ we would obtain the following extreme and neutral values.

$$\boldsymbol{\Omega}_{k1}=\max_{j}\Omega^{1}_{\mathbf{A}_j^k}(X) \quad (5.6)$$

$$\boldsymbol{\Omega}_{k2}=\min_{j}\left[\Omega^{2}_{\mathbf{A}_j^k}(X)\right] \quad (5.7)$$

$$\boldsymbol{\Omega}_{k3}=\frac{\sum_{j}\left[\Omega^{3}_{\mathbf{A}_j^k}(X)\right]}{N} \quad (5.8)$$

At this level $\boldsymbol{\Omega}_{k1}$ and $\boldsymbol{\Omega}_{k2}$ would give the extreme (lowest and highest) states and $\boldsymbol{\Omega}_{k3}$ would give the neutral states of Universe/Reality/Event/Information as accumulated fuzzy memberships.

The local order of the universes is obtained by arranging these cumulative memberships in descending order, and the global order is offered by using the same scenario of the frequency matrix (step-5).

5. Application of LGU Combined-Consciousness State Ranking Model

Numerical Example:

To achieve the purpose of non-physical classification, initially, we first develop two PFHS-Sets with α-Combination and β-Combination of attributes, i.e., for α and β universes. Then we represent it as PFHS-Matrix B, which consists of two layers that represent the mappings F and G that are used to parameterize a combination of attributes/subattributes. By assuming different or specific numerical values of k, consider α-Combination of attributes are parameterized by mapping F and β-Combination of attributes by mapping G. The overall LGU Combined-Consciousness State Ranking is described by following the steps in the algorithm described in Section -5.

Step 1. Construction of the Universe: Consider U be the set in five candidates of the mathematics department and out of these five only three have participated in consciousness quantification and classification experiment. let T be a set of these three candidates (subjects), T = {Peter, Aina, kitty}, ($T \subset U$). The elements of T are our subjects. The states of these subjects are A_j^k attributes quantified through the fuzzy linguistic scales. The classification of these attributes is required.

These A_j^kattributes are organized in the following manner:

A_1^k = Intelligence level with numeric values, $k = 1,2$ s.t
A_1^1 = very intelligent, A_1^2 = moderate intelligent
A_2^k = Fous, with numeric values, $k = 1,2$ s.t
A_2^1 = Strong focus A_2^2 = Weak focus
A_3^k = Observation with numeric values, $k = 1,2$ s.t
A_3^1 = Strong observation , A_3^2 = weak observation
A_4^k = Expression with numeric values $k = 1,2\ st$
$A_4^1 =$ Strong expression, A_4^2 = Weak expression

F and G be the plithogenic fuzzy parameterizations of the combination of their states (attributes) such that
$F: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \to P(U)$ (choosing some of the numeric values of A_j^k, $k = 1,2,\ldots,L$
$G: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \to P(U)$ (choosing some other numeric values of A_j^k, $k = 1,2,\ldots,L$

Let these candidates of set T are our $x_i$subjects, $i = 1,2,3$, and their states are attributed/sub-attribute represented A_j^k $j = 1,2,3,4$ and $k = 1,2$. We are looking for the best-reflected attribute among the given Combination of attributes (case of the local universe). The local universe of subjects and attributes for first level $k = 1$ is described as

$T = \{\text{Peter}, \text{Aina}, \text{kitty}\} = \{x_1, x_2, x_3\}$ where x_1, x_2, x_3 represent x_i subjects under consideration, initially, we represent the combination of states of the first level for $k = 1$(combination of attributes that are parametrized by mapping F)

1. Intelligence: $j = 1$, $k = 1$ (very intelligent)
2. Focus: $j = 2$, $k = 1$ (strong focus)
3. Observation: $j = 3$, $k = 1$ (strong observation)
4. Expression: $j = 4$, $k = 1$ (strong expression)

Now fuzzy memberships (fuzzy parameterization) are assigned by using fuzzy linguistic scales for details see ref. [33-36].

Let the Function F represents the fuzzy parameterization of the given combination of states/attributes s.t.,

$$F(A_1^1, A_2^1, A_3^1, A_4^1) = \{x_1(0.3,0.7,0.4,0.5), x_2(0.4,0.5,0.4,0.1), x_3(0.6,0.2,0.5,0.7)\} \qquad (6.1)$$

let us name the combination of attributes $A_1^1, A_2^1, A_3^1, A_4^1$ as α Combination representing the first level for $k = 1$

Consider some other combination of states described for $k = 2$ These states are parametrized by mapping G s.t $G: A_1^k \times A_2^k \times A_3^k \times A_4^k \to P(U)$

The local universe of subjects and attributes for second-level $k = 2$ is described below

1. Intelligence $j = 1$, $k = 2$ (moderate intelligent)
2. Focus: $j = 2$, $k = 2$ (weak focus)
3. Observation: $j = 3$, $k = 2$ (weak observation)
4. Expression: $j = 3$, $k = 2$ (weak expression)

Let the function be G represent the fuzzy parametrization of the given combination of states/attributes s.t,

$$G(A_1^2, A_2^2, A_3^2, A_4^2) = \{x_1(0.5,0.0,0.2,0.6), x_2(0.6,0.7,0.8,0.5), x_3(0.4,0.7,0.5,0.9)\} \tag{6.2}$$

let us name the combination of attributes $A_1^1, A_2^1, A_3^1, A_4^1$ as β Combination representing the second level for $k = 2$

Step 2. Construction of PFHS-Matrix:

The first layer of PFHS-Matrix $B = \left[\mu_{A_j^k}(x_i)\right]$ is constructed by using the parametrized states given in Eq. 6.1 for α combination (first level layer of PFHS-Matrix, k =1) and The second layer of PFHS-Matrix is constructed by using the parametrized states given in Eq. 6.2 for β combination (second level layer of PFHS-Matrix, k =2) and this information would be displayed in PFHS-Matrix as shown below.

$$B = \begin{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.5 & 0.7 \end{bmatrix} \\ \begin{bmatrix} 0.5 & 0.0 & 0.2 & 0.6 \\ 0.6 & 0.7 & 0.8 & 0.5 \\ 0.4 & 0.7 & 0.5 & 0.9 \end{bmatrix} \end{bmatrix} \tag{6.3}$$

Step 3. Construction of PFSWHSS-Matrix:

The PFSWHSS-Matrix $B_{\mathbf{A}_t}$ is constructed by using Eqs. (3.10), (4.1), (4.2), and (4.3) for information of (6.3)

$$B_{\mathbf{A}_t} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.5 & 0.7 \\ [\mathbf{0.6} & \mathbf{0.7} & \mathbf{0.5} & \mathbf{0.7}] \end{bmatrix} \\ \begin{bmatrix} 0.5 & 0.0 & 0.2 & 0.6 \\ 0.6 & 0.7 & 0.8 & 0.5 \\ 0.4 & 0.7 & 0.5 & 0.9 \\ [\mathbf{0.6} & \mathbf{0.7} & \mathbf{0.8} & \mathbf{0.9}] \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.5 & 0.7 \\ [\mathbf{0.3} & \mathbf{0.2} & \mathbf{0.4} & \mathbf{0.1}] \end{bmatrix} \\ \begin{bmatrix} 0.5 & 0.0 & 0.2 & 0.6 \\ 0.6 & 0.7 & 0.8 & 0.5 \\ 0.4 & 0.7 & 0.5 & 0.9 \\ [\mathbf{0.4} & \mathbf{0.0} & \mathbf{0.2} & \mathbf{0.5}] \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.5 & 0.7 \\ [\mathbf{0.43} & \mathbf{0.46} & \mathbf{0.43} & \mathbf{0.43}] \end{bmatrix} \\ \begin{bmatrix} 0.5 & 0.0 & 0.2 & 0.6 \\ 0.6 & 0.7 & 0.8 & 0.5 \\ 0.4 & 0.7 & 0.5 & 0.9 \\ [\mathbf{0.5} & \mathbf{0.46} & \mathbf{0.5} & \mathbf{0.66}] \end{bmatrix} \end{bmatrix} \end{bmatrix} \tag{6.4}$$

Step 4. Local Attributive/States Ranking: $B_{\mathbf{A}_{1\alpha}}$ provides The local order of states/attributes for α Combination of attributes or α-universe i.e the first level-layer is obtained by observing the whole memberships of (6.4) for first-level $k = 1$ and first aggregation operator ($t = 1$). See Eq.4.1 We observe here a tie between A_2^1 ($\Omega^1_{\mathbf{A}_2^1}(X) = 0.7$) and A_4^1 ($\Omega^1_{\mathbf{A}_4^1}(X) = 0.7$) which would be removed in the next step of the Global States ranking using the Frequency-Matrix F_{ij}.

$$A_2^1 = A_4^1 \succ A_1^1 \succ A_3^1 \tag{6.5}$$

$B_{\mathbf{A}_{1\beta}}$ provides The local ordering of attributes for β Combination of attributes or β-Universe (second level-layer obtained for $k = 2$) See Eq. 6.4 by using the first operator $t = 1$ (eq. 4.1)

$$A_4^2 \succ A_3^2 \succ A_2^2 \succ A_1^2 \tag{6.6}$$

$B_{\mathbf{A}_{2\alpha}}$ provides the local ordering of attributes for α-Combination of attributes (α-Universe) by using the second operator $t = 2$ Eqs 6.4 and (4.2)

$$A_3^1 \succ A_1^1 \succ A_2^1 \succ A_4^1 \tag{6.7}$$

Similarly

$B_{\mathbf{A}_{2\beta}}$ provides the local ordering of attributes for β Combination of attributes (β-Universe) by using the second operator $t = 2$ Eqs 6.4 and (4.2)

$$A_4^2 \succ A_1^2 \succ A_3^2 \succ A_2^2 \tag{6.8}$$

$B_{\mathbf{A}_{3\alpha}}$ provides the local ordering of attributes for α Combination of attributes (α-Universe) by using the third operator $t = 3$ Eqs 6.4 and (4.3)

$$A_2^1 \succ A_1^1 = A_3^1 = A_4^1 \tag{6.9}$$

$B_{\mathbf{A}_{3\beta}}$ provides the local ordering of attributes for β Combination of attributes (β-Universe) by using the third operator ($t = 3$) Eqs 6.4 and (4.3)

$$A_4^2 \succ A_1^2 = A_3^2 \succ A_2^2 \tag{6.10}$$

Step 5. Global States/Attributive Ranking:

The frequency matrix F_{ij} provides a final global ordering of attributes. In the frequency matrix F_{ij}^{α}, which is a square matrix of frequencies of positions for first level-layer α-Universe, the columns of F_{ij}^{α} represents frequencies of positions, i.e., the entries of the first column represent the frequencies of attaining the first position by some attributes while a row of F_{ij} represents the attributes. The F_{ij}^{α} is constructed from Eq. (6.5), (6.7), (6.9), and (5.4)a, (5.4)b, (5.4)c

$$F_{ij}^{\alpha} = \begin{array}{c} \alpha \\ A_1^1 \\ A_2^1 \\ A_3^1 \\ A_4^1 \end{array} \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \tag{6.11}$$

$$F_{ij}^{*\alpha} = \begin{array}{c} \alpha \\ A_1^1 \\ A_2^1 \\ A_3^1 \\ A_4^1 \end{array} \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ 0 & 100 & 0 & 0 \\ 66.7 & 0 & 33.3 & 0 \\ 33.3 & 33.3 & 33.3 & 0 \\ 33.3 & 33.3 & 0 & 33.3 \end{bmatrix} \tag{6.11a}$$

The Global States ranking of attributes obtained from $F_{ij}^{*\alpha}$ is given below.

$$A_2^1 \succ A_1^1 \succ A_3^1 \succ A_4^1 \tag{6.12}$$

The F_{ij}^{β} is constructed from Eq. (6.6), (6.8), (6.10), and (5.4)a, (5.4)b, (5.4)c

$$F_{ij}^{\beta}=\begin{matrix} \beta \\ A_1^2 \\ A_2^2 \\ A_3^2 \\ A_4^2 \end{matrix}\begin{bmatrix} p_1 & p_2 & p_3 & p_3 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \tag{6.13}$$

$$F_{ij}^{*\beta}=\begin{matrix} \beta \\ A_1^2 \\ A_2^2 \\ A_3^2 \\ A_4^2 \end{matrix}\begin{bmatrix} p_1 & p_2 & p_3 & p_3 \\ 0 & 66.7 & 0.0 & 33.3 \\ 0 & 0.0 & 33.3 & 66.7 \\ 0 & 33.3 & 66.7 & 0 \\ 100 & 0 & 0 & 0 \end{bmatrix} \tag{6.13}$$

The Global States ranking of attributes obtained from $F_{ij}^{*\beta}$ is given below.

$$A_4^2 \succ A_1^2 \succ A_3^2 \succ A_2^2 \tag{6.14}$$

It is observed that the ties of local ranking are removed in the final global ranking

Step 6. Authenticity measurement of the Global States Ranking:

Percentage authenticity measure for first level α-universe is obtained by using Eq. (5.5) and (6.11)a

Percentage authenticity of the first position for $A_2^1 = 66.7\%$

Percentage authenticity of the second position for $A_1^1 = 60\%$

Percentage authenticity of the third position for $A_3^1 = 50\%$

Percentage authenticity of the fourth position for $A_4^1 = 100\%$

Percentage authenticity measure for first level β-universe is obtained by using (5.5) and (6.13)a

Percentage authenticity of the first position for $A_4^2 = 100\%$

Percentage authenticity of the second position for $A_1^2 = 66.7\%$

Percentage authenticity of the third position for $A_3^2 = 66.67\%$

Percentage authenticity of the fourth position for $A_2^2 = 66.7\%$

Step 7. Final Universal States (Combined Consciousness States) and Ranking:

we provide the final ordering of the universe by using all three aggregation operators.

Maximum Combined Consciousness States (Universal Memberships) of α and β universes:

taking $k = 1,2$ for α and β universes and fixing $t = 1$ (Max-operator) using Eqs. (6.4) and (5.6)

$$\mathbf{\Omega}_{11} = 0.7,\ \mathbf{\Omega}_{21} = 0.9 \tag{6.15}$$

We can see by using operator $t = 1$, β universe is better than α universe.

Minimum Combined consciousness States (Universal Memberships) of α and β universes:

Taking $k = 1,2$ for α and β universes and fixing $t = 2$ minimum universal memberships of all given Attributes with respect to subjects, are obtained using Eqs. (6.4) and (5.7) respectively.

$$\mathbf{\Omega}_{12} = 0.1,\ \mathbf{\Omega}_{22} = 0.0 \tag{6.16}$$

We observe by using the operator $t = 2$, β universe is better than α universe.

Neutral Combined Consciousness States (Universal Memberships) of α and β universes:

similarly, taking $k = 1,2$ for α and β universes and fixing $t = 3$, we can provide average universal memberships of all given subjects with respect to attributes, using Eqs. (6.4) and (5.7)

$$\mathbf{\Omega}_{13} = 0.437,\ \mathbf{\Omega}_{23} = 0.53 \tag{6.17}$$

The Universal States ordering: By applying the frequency matrix analysis (Eqs. 6.15, 6.16, 6.17, and (5.4)a, (5.4)b, (5.4)c The ranking of the states of the universes is

$$\beta(\text{universe}) \succ \alpha(\text{universe}) \tag{6.18}$$

7. Pie graphs of the LGU Combined-Consciousness State Ranking Model

7.1 Pie graphs of the LGU Combined-Consciousness State Ranking Model for the α-Universe

The pie graphs (Fig1-Fig 4) present the individual states (fuzzy memberships) of 3 subjects considering one attribute at a time for the α-Universe (for aggregation purposes, we use the averaging operator (t = 3)

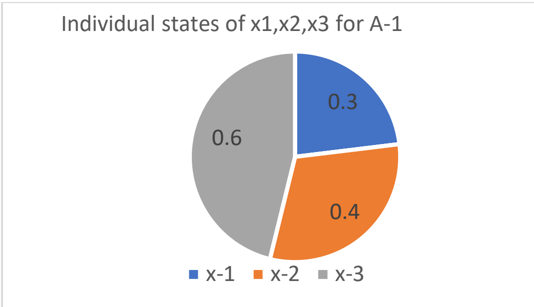

Figure 1a (Individual states of A-1)

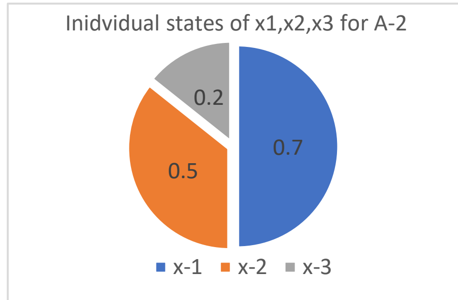

Figure 2a (Individual states of A-2)

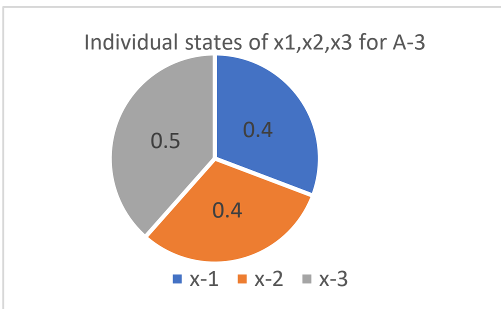

Figure 3a (Individual states of A-3)

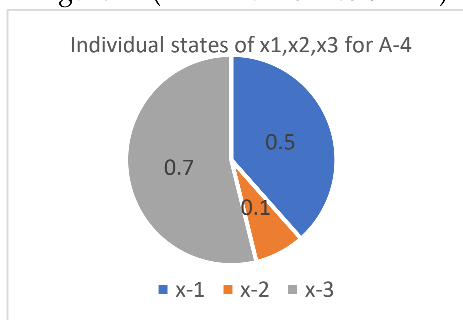

Figure 4a (Individual states of A-4)

Fig. 5 represents the aggregated states of the three subjects (α-Universe first level of aggregation) represented for each attribute.

Fig 6 is representing the aggregated state of the whole universe that is obtained by aggregating the previous aggregated states of fig 5 by using the averaging operator (α-Universe second level of aggregation)

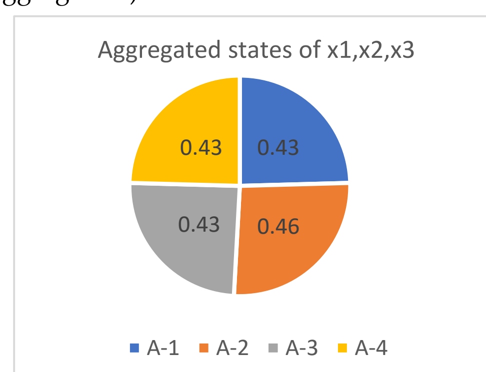

Figure 5a (Aggregated states)

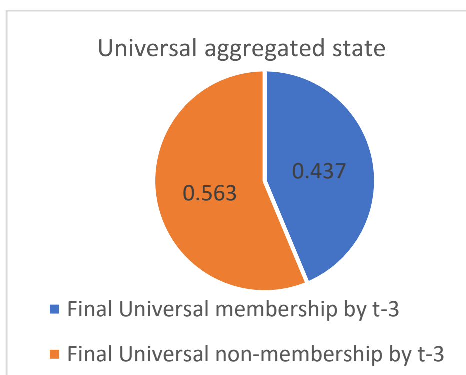

Figure 6a (Universal states)

7.2 Pie graphs of the LGU Combined-Consciousness State Ranking Model for the β-Universe

(Fig1b-Fig 4b) pie graphs are presenting the individual states (fuzzy memberships) of 3 subjects by considering one attribute at a time for the β-Universe (The aggregation operator used is the averaging operator (t = 3)

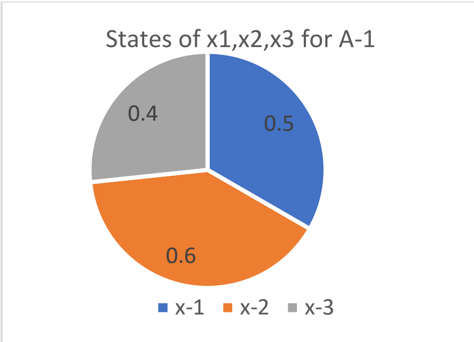

Figure 1b (Aggregated States for A-1)

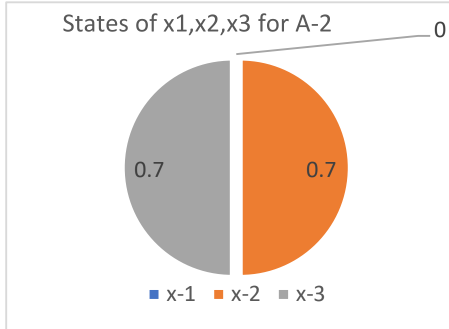

Fig 2b (Aggregated States for A-2)

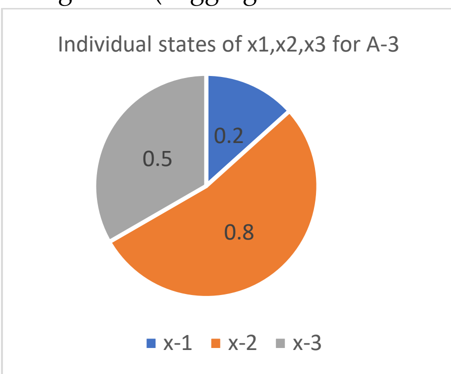

Figure 3b (Aggregated States for A3)

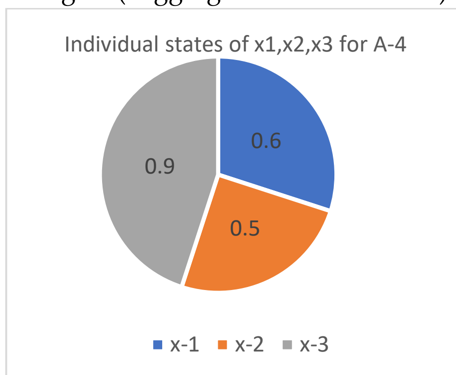

Figure 4b(Aggregated States for A-4)

Fig 5b is representing aggregated states of the three subjects (β-Universe first level of aggregation) represented for each attribute.

Fig. 6b represents the aggregated state of the entire -Universe that is obtained by aggregating the previous aggregated states of Fig. 5b by using the averaging operator (β-Universe, the second level of aggregation)

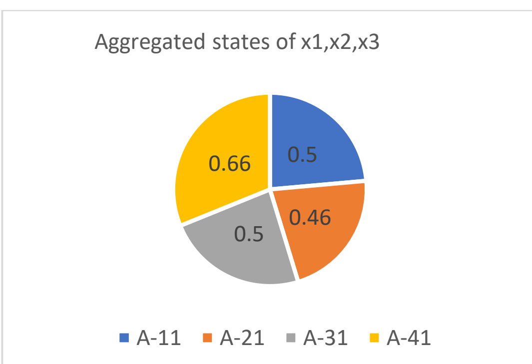

Figure 5b (Aggregated states)

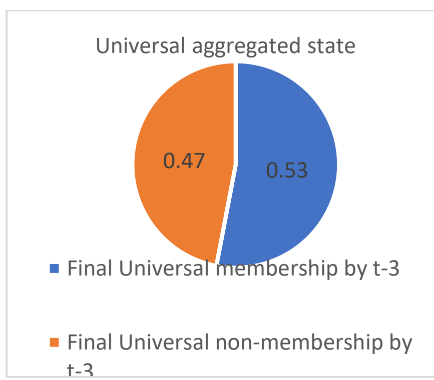

Figure 6b (Aggregated states)

8. Conclusion :

1. We have observed the final global ordering obtained in Eq. (6.12) is the most frequently observed local ordering in all these ranking orders, which is also observed the same in the local ordering of β universe in Eq. (6.14) which shows the final global State ranking is most transparent and authentic Ranking.
2. Expressions (6.15), (6.16), (6.17) provide the highest, lowest, and average states of universes, through final accumulative memberships.
3. The Ordering of universes shows that on the Global Universal level, β universe is better than α universe.

4. these results of local and global ordering are also verified by the pie graphs

1. Local ordering: we can observe local orders using these novel plithogenic hyper-supersoft matrices and local operators. Each operator reflects the state of mind of the decision-maker; for example, the Max operator reflects the optimal state of mind, the Min operator reflects the Passimistic state of mind, and the Average operator reflects the neutral state of mind.

2. Global ordering: We can provide a global order by combining the results of all three rankings using the frequency matrix. These three rank orders are obtained from three aggregation operators that represent three states of the human mind. The ranking at the levels of global states will be transparent and impartial, taking into account three different states of the human mind

3. Universal ordering: We can compare the universes by applying $the\ max\ operator\ (t = 1)$, $the\ min\ operator\ (t = 2)$ and $the\ average\ operator\ (t = 3)$ on cumulative memberships of the last row for each universe. The universe with the largest cumulative membership would be better, and further, a local ordering of the universes is obtained by arranging these cumulative memberships in descending order and the global ordering is offered by using the same scenario of the frequency matrix

4. Extreme Universal Memberships: We can also find out the extreme values of these universes and can observe these attributes in the large universe made up of several smaller parallel universes. We can choose from among all universes the best-reflected attribute that is best in most universes.

5. local and global ordering inside the universe: In this article, our focus is on the non-physical states of the subjects or universe. Local and global ordering We have offered a local and global ordering of states of subjects (Attributes, Sub-attributes) within a universe.

6. local and global ordering of the Universe: Furthermore, a local and global ordering of states of the Universe is offered. The state of the universe is obtained by accumulating the states of all subjects of the given universe.

7. Combined Consciousness of the Universe: The state of the universe is presented by the accumulated states of all its subjects. In this ranking model, the accumulated states of all subjects as a Combined Consciousness of the universe is offered in the form of universal memberships.

9. Open problems:

Now, let us list some of the open problems that could be addressed in future research.

• In this article, we developed the LGU Combined Consciousness State Ranking Model in the plithogenic fuzzy environment.

This model can be extended to other environments, such as intuitionistic environment, neutrosophic environment, or any other mixed environment according to the required conditions or states of mind of the decision-makers.

• In addition, some other local operators can be used in the construction of the model according to the requirements of the relevant authorities.

• Attributive and subjective ranking models can be constructed using the literature developed in this article.

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Location Selection of Low-carbon Logistics Park Based on the Neutrosophic Numbers Multiple Attribute Decision Making

Xinrui Xu[1], Dexue Deng[1*], Cun Wei[2]

[1]School of Civil Engineering, Chongqing University of Arts and Sciences, Chongqing 402160, China(20090013@cqwu.edu.cn; 20100017@cqwu.edu.cn)

[2]School of Statistics, Southwestern University of Finance and Economics, Chengdu, 611130, P.R. China(weicun1990@163.com)

*Corresponding author, E-mail: 20090013@cqwu.edu.cn

Abstract: With the continuous advancement of science and technology, the decision-making environment faced by managers is increasingly complex, which puts forward higher requirements for managers. Due to the complexity of the multi-attribute decision-making problem, it is difficult for the decision-maker to make the correct choice. In addition, due to the influence of the educational background and limited knowledge of the managers, it is impossible to evaluate with simple statistical data. In essence, the location problem of low-carbon logistics parks is a MADM problem. Therefore, this paper establishes an optimization model to solve the multiple-attribute decision-making (MADM) problem with alternative preference and single-valued neutrosophic (SVN) information. Considering that the information of weight is unknown, a scientific model is built based on the minimum deviation method deriving the criterion weight. Furthermore, the above models and methods are extended to interval neutrosophic sets (INs). To verify the validity of the modified model, a numerical case for low-carbon logistics park site selection is taken as an example. Through the case study, we found that the method has strong operability and can make the most of the available information.

Keywords: Multiple-attribute decision-making (MADM); single-valued neutrosophic sets (SVNSs); interval neutrosophic sets (INs); preference information; low-carbon logistics park site selection

1. Introduction

Decision makers (DMs) often have difficulty expressing their preferences accurately when presented with inaccurate, uncertain expression when solving MADM issues[1-5]. Fuzzy Sets

(FSs) [6]are considered to solve the MADM problems [7, 8]. Intuitionistic fuzzy sets (IFS)[9], as an extension, have subsequently been widely used in solving MCDM problems. Since IFSs consider both membership and non-membership, they are more flexible and practical than traditional FSs[10-14]. In some practical cases, membership, non-membership and hesitation of an element of IFSs may not be a specific figure. Therefore, it is extended to IVFS [15-20]. To represent uncertainty and inconsistency in information, Smarandache [21] introduced neutrosophic sets (NS) as an alternative to IFSs and IVIFSs. To facilitate practical application, the SVNS [22] and INS [23] were proposed as subclasses of NS, and then Ye [24] introduce`ed SNS, SVNS and INS. According to the literature. NS is a generalization of FS, IFS, and IVIFS. In practice, SNS (SVNS and INS) are ideal for the expression of incomplete and uncertain information in practical applications. In recent years, SNSs (ins) and SVNSs (SVNSs) is the ideal choice for expressing incomplete, uncertain and inconsistent information. Sahin and Kucuk [25] gave the introduction of entropy measure of SVNSs. And correlation coefficient of SVNSs and the method of using SVNSs for decision making are introduced based on the preliminary knowledges. In a time-neutral environment, Broumi and Smarandache [26] built the correlation coefficients of INSs. While Zhang, Wang and Chen [27] developed INNWA operator and INNWG operator. Furthermore, Ye [28-32] introduced a similarity measure between SVNSs and INSs. Ye [33] examined interval-neutral MADM methods based on probability degree sequencing and ordered weighted aggregate operators. Ye and Jun [34] proposes an interval-neutral MADM with confidence information. Peng, Wang, Zhang and Chen [35] studied a transcendent approach to MADM problems with simplified neutral sets. With interval-value neutral sets, Zhang, Wang and Chen [36] devised a transcendent method to solve the MADM issues. In their paper, Tian, Zhang, Wang, Wang and Chen [37] examined the use of interval neutral set cross entropy. The SVNNWBM operator was proposed by Liu and Wang [38] using the Bonferroni mean, the WBM, and the normalized WBM. Liu, Chu, Li and Chen [39] combined Hamacher operator and generalized operator into NS, proposed the GNNHWA operator, GNNHOWA operator and GNNHHA operator. Zhao, Du and Guan [40] extended the GWA operator to work in line with the IVNS data. Liu and Wang [41] further proposed INPOWA operator. In their study, the preferred weighted average operator and priority weighted geometric operator for SNN [42] were then defined. Ye [43] proposed INWEA operator and DUNWEA operator based on exponential algorithm. Li, Liu and Chen [44] proposed some Heronian mean operators with SVNSs.

Knowledge explosion, information torrent, rapid technological change, rapid social change, rapid economic development, and so on, this is an era of change, but also an era of development. With the popularization of the Internet and the rapid development of the e-commerce industry, the way of shopping has gradually shifted from offline to online[45, 46]. Online shopping has become an indispensable part of contemporary people's lives, and the accompanying logistics system is an important part of it. support. The rapid development of SF Express, "Three Links and One Delivery" (Zhongtong, Shentong, Yuantong, Yunda Express), JD Logistics and other niche express delivery has driven economic growth, but their extensive logistics operations have also caused great damage to the environment. Influence. Logistics systems include a variety of activities, such as supplier production, transportation and distribution, that consume energy and emit carbon[47-50]. In the context of global warming and environmental deterioration, it is extremely urgent to develop low-carbon logistics[51-53]. Government departments have formulated a series of plans to implement them. The primary task of the logistics industry system from the perspective of low carbon is to carry out reasonable planning of logistics activities, build logistics parks and solve the problem of site selection of logistics parks[54-56]. The location of a low-carbon logistics park depends on factors such as the economic development of a certain place, market demand, low-carbon attributes of logistics and transportation routes, and whether carbon emissions meet environmental requirements[57-59]. In essence, the location problem of low-carbon logistics parks is a MADM problem. During the process of single valued neutral MADM with alternative preference information. The weights are not completely known or completely unknown. Nevertheless, none of the above methods are suitable for dealing with this situation. To overcome this limitation, it is necessary to find methods based on the minimum deviation method. The aim of this manuscript is to establish a method based on the least deviation method. We will introduce SVNSs in the next section of this paper. In Section 3, we build the MADM model under SVNSs, where the information about criterion weight is not completely known, and the attribute value and preference value of options are SVNNs. In Section 4, There is no complete information about criterion weight, and the attribute value and preference value are expressed as INNs. In Section 5, illustrative examples for low-carbon logistics park site selection are indicated. In Section 6, we summarize the full text.

2. Preliminaries

Definition 1[60]. Assume W be a set with an element in a fixed set W, which is denoted by ϖ. A NSs ν in W is defined by the function of truth-membership $\pi_\nu(\varpi)$, indeterminacy-membership $\vartheta_\nu(\varpi)$ and a falsity-membership function $\sigma_\nu(\varpi)$. The functions $\pi_\nu(\varpi)$, $\vartheta_\nu(\varpi)$ and $\sigma_\nu(\varpi)$ are real standard or nonstandard subsets of $]^-0,1^+[$, that's, $\pi_\nu(\varpi):W\to]^-0,1^+[$, $\vartheta_\nu(\varpi):W\to]^-0,1^+[$ and $\vartheta_\nu(\varpi):W\to]^-0,1^+[$. There is no restriction of $\pi_\nu(\varpi)$, $\vartheta_\nu(\varpi)$ and $\sigma_\nu(\varpi)$, so $0^-\le\sup\pi_\nu(\varpi)+\sup\vartheta_\nu(\varpi)+\sup\sigma_\nu(\varpi)\le 3^+$.

Definition 2[22]. Let W be a collection in fixed set W, denoted by ϖ. A SVNSs ν in W is defined as follows:

$$\nu=\{(\varpi,\pi_\nu(\varpi),\vartheta_\nu(\varpi),\sigma_\nu(\varpi))W\} \tag{1}$$

Where $\pi_\nu(x)$, $\vartheta_\nu(\varpi)$ and $\sigma_\nu(\varpi)$ are in the value of $[0,1]$, that is, $\pi_\nu(\varpi):W\to[0,1]$, $\vartheta_\nu(\varpi):W\to[0,1]$ and $\sigma_\nu(\varpi):W\to[0,1]$. And the sum of $\pi_\nu(\varpi)$, $\vartheta_\nu(\varpi)$ and $\sigma_\nu(\varpi)$ meets the condition $0\le\pi_\nu(\varpi)+\vartheta_\nu(\varpi)+\sigma_\nu(\varpi)\le 3$. Then a simplification of ν is represented by $\nu=\{(\varpi,\pi_\nu(\varpi),\vartheta_\nu(\varpi),\sigma_\nu(\varpi))|\varpi\in W\}$, which is a SVNS.

For a SVNS $\{(\varpi,\pi_\nu(\varpi),\vartheta_\nu(\varpi),\sigma_\nu(\varpi))|\varpi\in W\}$, the ordered triple components $(\pi_\nu(\varpi),\vartheta_\nu(\varpi),\sigma_\nu(\varpi))$, are defined as a SVNN, and each SVNN can be expressed as $\nu=(\pi_\nu,\vartheta_\nu,\sigma_\nu)$, where $\pi_\nu\in[0,1],\vartheta_\nu\in[0,1],\sigma_\nu\in[0,1]$, and $0\le\pi_\nu+\vartheta_\nu+\sigma_\nu\le 3$.

Definition 3[61]. Set $\nu=(\pi_\nu,\vartheta_\nu,\sigma_\nu)$ be a SVNN, a score function ψ is represented:

$$\psi(\nu)=\frac{(2+\pi_\nu-\vartheta_\nu-\sigma_\nu)}{3},\ \psi(\nu)\in[0,1] \tag{2}$$

Definition 4[61]. Set $\nu=(\pi_\nu,\vartheta_\nu,\sigma_\nu)$ be a SVNN, an accuracy function χ is represented:

$$\chi(\nu)=\pi_\nu-\sigma_\nu,\ \chi(\nu)\in[-1,1]\ . \tag{3}$$

Definition 5[61]. Let $\nu=(\pi_\nu,\vartheta_\nu,\sigma_\nu)$ and $\mu=(\pi_\mu,\vartheta_\mu,\sigma_\mu)$ be two SVNNs, $\psi(\nu)=\frac{(2+\pi_\nu-\vartheta_\nu-\sigma_\nu)}{3}$ and $\psi(\mu)=\frac{(2+\pi_\mu-\vartheta_\mu-\sigma_\mu)}{3}$ be the scores function, and let

$\chi(\nu)=\pi_\nu-\sigma_\nu$ and $\chi(\mu)=\pi_\mu-\sigma_\mu$ be the accuracy degrees, then if $\psi(\nu)<\psi(\nu)$, then $\nu<\mu$; if $\psi(\nu)=\psi(\mu)$, then

(1) if $\psi(\nu)=\psi(\nu)$, then $\nu=\mu$; (2) if $\psi(\nu)<\psi(\mu)$, then $\nu<\mu$.

Definition 6[61]. Let $\nu=(\pi_\nu,\vartheta_\nu,\sigma_\nu)$ and $\mu=(\pi_\mu,\vartheta_\mu,\sigma_\mu)$ be two SVNNs, and some basic operations are defined:

(1) $\nu\oplus\mu=(\pi_\nu+\pi_\mu-\pi_\nu\pi_\mu,\vartheta_\nu\vartheta_\mu,\sigma_\nu\sigma_\mu)$;

(2) $\nu\otimes\mu=(\pi_\nu\pi_\mu,\vartheta_\nu+\vartheta_\mu-\vartheta_\nu\vartheta_\mu,\sigma_\nu+\sigma_\mu-\sigma_\nu\sigma_\mu)$;

(3) $\lambda\nu=\left(1-(1-\pi_\nu)^\lambda,(\vartheta_\nu)^\lambda,(\sigma_\nu)^\lambda\right),\lambda>0$;

(4) $(\nu)^\lambda=\left((\pi_\nu)^\lambda,(\vartheta_\nu)^\lambda,1-(1-\sigma_\nu)^\lambda\right),\lambda>0$.

Based on Definition 6, the following properties are derived.

Theorem 1[22]. Let $\nu=(\pi_\nu,\vartheta_\nu,\sigma_\nu)$ and $\mu=(\pi_\mu,\vartheta_\mu,\sigma_\mu)$ be two SVNNs, $\lambda,\lambda_1,\lambda_2>0$, then

(1) $\nu\oplus\mu=\mu\oplus\nu$;

(2) $\nu\otimes\mu=\mu\otimes\nu$;

(3) $\lambda(\nu\oplus\mu)=\lambda\nu\oplus\lambda\mu$;

(4) $(\nu\otimes\mu)^\lambda=(\nu)^\lambda\otimes(\mu)^\lambda$;

(5) $\lambda_1\nu\oplus\lambda_2\nu=(\lambda_1+\lambda_2)\nu$;

(6) $(\nu)^{\lambda_1}\otimes(\nu)^{\lambda_2}=(\nu)^{(\lambda_1+\lambda_2)}$;

(7) $\left((\nu)^{\lambda_1}\right)^{\lambda_2}=(\nu)^{\lambda_1\lambda_2}$.

Definition 7[61]. Let $\nu_\alpha=(\pi_\alpha,\vartheta_\alpha,\sigma_\alpha)(\alpha=1,2,\cdots,\phi)$ be a collection of SVNNs, and let SVNWA: $Q^\phi\to Q$, if

$$\begin{aligned}\text{SVNWA}_\gamma(\nu_1,\nu_2,\cdots,\nu_\phi)&=\overset{\phi}{\underset{\alpha=1}{\oplus}}(\gamma_\alpha\nu_\alpha)\\&=\left(1-\prod_{\alpha=1}^{\phi}(1-\pi_\alpha)^{\gamma_\alpha},\prod_{\alpha=1}^{\phi}(\vartheta_\alpha)^{\gamma_\alpha},\prod_{\alpha=1}^{\phi}(\sigma_\alpha)^{\gamma_\alpha}\right)\end{aligned} \tag{4}$$

where $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_\phi)^T$ be the weight of $\nu_\alpha(\alpha=1,2,\cdots,\phi)$, and $\gamma_\alpha>0$, $\sum_{\alpha=1}^{\phi}\gamma_\alpha=1$, then SVNWA is called SVNWA operator.

Definition 8[25]. Let $\nu=\left(\pi_\nu,\vartheta_\nu,\sigma_\nu\right)$ and $\mu=\left(\pi_\mu,\vartheta_\mu,\sigma_\mu\right)$ be two SVNNs, then the Hamming distance is defined:

$$d\left(\nu,\mu\right)=\frac{1}{3}\left(\left|\pi_\nu-\pi_\mu\right|+\left|\vartheta_\nu-\vartheta_\mu\right|+\left|\sigma_\nu-\sigma_\mu\right|\right) \tag{5}$$

3. Models for SVN MADM issues

(1) Let $\varepsilon=\left\{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\phi\right\}$ be a discrete set of alternatives;(2) Let $\varsigma=\left\{\varsigma_1,\varsigma_2,\cdots,\varsigma_\varphi\right\}$ be a set of attributes;(3) Let $\nu=\left(\nu_1,\nu_2,\cdots,\nu_\phi\right)$ be subjective preference and $\nu_\alpha=\left(\pi_{\nu_\alpha},\vartheta_{\nu_\alpha},\sigma_{\nu_\alpha}\right)$ are SVSNs, which is subjective preference on alternative $\varepsilon_\alpha\left(\alpha=1,2,\cdots,\phi\right)$.(4) The criterion weights is incompletely known. Let $\gamma=\left(\gamma_1,\gamma_2,\cdots,\gamma_\varphi\right)\in R$ be the weight of attributes, where $\gamma_\beta\ge 0$, $\beta=1,2,\cdots,\varphi$, $\sum_{\beta=1}^{\varphi}\gamma_\beta=1$, R is a set of criterion weight, constructed by the following forms [62, 63], for $\alpha\ne\beta$: **Form 1.** A weak sequence: $\gamma_\alpha\ge\gamma_\beta$; **Form 2.** A strict sequence: $\gamma_\alpha-\gamma_\beta\ge\ell_\alpha$, $\ell_\alpha>0$;**Form 3.** A sequence of differences: $\gamma_\alpha-\gamma_\beta\ge\gamma_o-\gamma_\rho$, for $\beta\ne o\ne\rho$;**Form 4.** A sequence with multiples: $\gamma_\alpha\ge\eta_\alpha\gamma_\beta$, $0\le\eta_\alpha\le 1$; **Form 5.** An interval form: $\ell_\alpha\le\gamma_\alpha\le\ell_\alpha+\varepsilon_i, 0\le\ell_\alpha<\ell_\alpha+\varepsilon_\alpha\le 1$. Suppose that $V=\left(\nu_{\alpha\beta}\right)_{\phi\times\varphi}=\left(\pi_{\alpha\beta},\vartheta_{\alpha\beta},\sigma_{\alpha\beta}\right)_{\phi\times\varphi}$ is SVN decision matrix, $\pi_{\alpha\beta}\in[0,1],\vartheta_{\alpha\beta}\in[0,1],\sigma_{\alpha\beta}\in[0,1]$, $0\le\pi_{\alpha\beta}+\vartheta_{\alpha\beta}+\sigma_{\alpha\beta}\le 3$, $\alpha=1,2,\cdots,\phi$, $\beta=1,2,\cdots,\varphi$.

Definition 9[61]. Let $V=\left(\nu_{\alpha\beta}\right)_{\phi\times\varphi}=\left(\pi_{\alpha\beta},\vartheta_{\alpha\beta},\sigma_{\alpha\beta}\right)_{\phi\times\varphi}$ is the SVN matrix, $\nu_\alpha=\left(\nu_{\alpha 1},\nu_{\alpha 2},\cdots,\nu_{\alpha\varphi}\right)$ be the attribute values for alternative ε_α, $\alpha=1,2,\cdots,\phi$, then we call

$$\begin{aligned}\nu_\alpha&=\left(\pi_\alpha,\vartheta_\alpha,\sigma_\alpha\right)=\text{SVNWA}_\gamma\left(\nu_{\alpha 1},\nu_{\alpha 2},\cdots,\nu_{\alpha\varphi}\right)\\&=\mathop{\oplus}_{\beta=1}^{\varphi}\left(\gamma_\beta\nu_{\alpha\beta}\right)=\left(1-\prod_{\beta=1}^{\varphi}\left(1-\pi_{\alpha\beta}\right)^{\gamma_\beta},\prod_{\beta=1}^{\varphi}\left(\vartheta_{\alpha\beta}\right)^{\gamma_\beta},\prod_{\beta=1}^{\varphi}\left(\sigma_{\alpha\beta}\right)^{\gamma_\beta}\right)\end{aligned} \tag{6}$$

the overall value of ε_α, where $\gamma=\left(\gamma_1,\gamma_2,\cdots,\gamma_\beta\right)^T$ is the criterion attributes.

On the premise that the attribute weight is known, we can aggregate the weighted values into a total value through Eq. (6). According to the overall attribute value, we can make a final ranking of possible solutions, and finally choose the most suitable solution.

If the attribute weight information of the decision model is unknown, to reflect the subjective preference and objective information of the DM at the same time, the optimization decision model is established. However, there are some differences between DM's subjective preference and objective information. To make decision-making more scientific and reasonable, the selection of attribute weight vector should minimize the total deviation between objective information and DM subjective preference.

For $\chi_\beta \in \chi$, the deviation of alternative ε_α to DM's subjective preference is defined as follows:

$$K_{\alpha\beta}(\gamma) = \kappa\left(\nu_{\alpha\beta}, \nu_\alpha\right)\gamma_\beta, \alpha = 1, 2, \cdots, \phi, \beta = 1, 2, \cdots, \varphi. \quad (7)$$

Let $K_\alpha(\gamma) = \sum_{\beta=1}^{\varphi} K_{\alpha\beta}(\gamma) = \sum_{\beta=1}^{\varphi} \kappa\left(\nu_{\alpha\beta}, \nu_\alpha\right)\gamma_\beta, \alpha = 1, 2, \cdots, \phi$

Then $K_\alpha(\gamma)$ denote the deviation of ε_α to DM's subjective preference value ν_α.

According to the above analysis, we must select the criterion weight vector to minimize all deviations of possible solutions. To this end, we establish a linear programming model:

$$(\text{M-1})\begin{cases} \min K(\gamma) = \sum_{\alpha=1}^{\phi}\sum_{\beta=1}^{\varphi} K_{\alpha\beta}(\gamma) = \sum_{\alpha=1}^{\phi}\sum_{\beta=1}^{\varphi} \kappa\left(\nu_{\alpha\beta}, \nu_\alpha\right)\gamma_\beta \\ \text{Subject to} \quad \sum_{\beta=1}^{\varphi}\gamma_\beta = 1, \gamma_\beta \geq 0, \beta = 1, 2, \cdots, \varphi \end{cases}$$

where $\kappa\left(\nu_{\alpha\beta}, \nu_\alpha\right) = \frac{1}{3}\left(\left|\pi_{\alpha\beta} - \pi_{\nu_\alpha}\right| + \left|\vartheta_{\alpha\beta} - \vartheta_{\nu_\alpha}\right| + \left|\sigma_{\alpha\beta} - \sigma_{\nu_\alpha}\right|\right)$.

If the attribute weight information is completely unknown, another programming model is established:

$$(\text{M-2})\begin{cases} \min K(\gamma) = \sum_{\alpha=1}^{\phi}\sum_{\beta=1}^{\varphi} K_{\alpha\beta}(\gamma) \\ = \frac{1}{3}\sum_{\alpha=1}^{\phi}\sum_{\beta=1}^{\varphi}\left(\left|\pi_{\alpha\beta} - \pi_{\nu_\alpha}\right| + \left|\vartheta_{\alpha\beta} - \vartheta_{\nu_\alpha}\right| + \left|\sigma_{\alpha\beta} - \sigma_{\nu_\alpha}\right|\right)\gamma_\beta \\ s.t. \ \sum_{\beta=1}^{\varphi}\gamma_\beta^2 = 1, \gamma_\beta \geq 0, \beta = 1, 2, \cdots, \varphi \end{cases}$$

The Lagrange function is constructed as follows:

$$L(\gamma,\lambda)=\frac{1}{3}\sum_{\alpha=1}^{\phi}\sum_{\beta=1}^{\varphi}\left(\left|\pi_{\alpha\varepsilon}-\pi_{\nu_\alpha}\right|+\left|\vartheta_{\alpha\beta}-\vartheta_{\nu_\alpha}\right|+\left|\sigma_{\alpha\beta}-\sigma_{\nu_\alpha}\right|\right)\gamma_\beta+\frac{\lambda}{6}\left(\sum_{\beta=1}^{\varphi}\gamma_\beta^2-1\right) \quad (8)$$

where λ is the Lagrange multiplier.

Differentiating Eq. (8) and setting these partial derivatives equal to zero:

$$\begin{cases}\dfrac{\partial L}{\partial \gamma_\beta}=\sum_{\alpha=1}^{m\phi}\left(\left|\pi_{\alpha\beta}-\pi_{\nu_\alpha}\right|+\left|\vartheta_{\alpha\beta}-\vartheta_{\nu_\alpha}\right|+\left|\sigma_{\alpha\beta}-\sigma_{\nu_\alpha}\right|\right)+\lambda\gamma_\beta=0\\ \dfrac{\partial L}{\partial \lambda}=\sum_{\beta=1}^{\varphi}\gamma_\beta^2-1=0\end{cases} \quad (9)$$

By solving Eq. (9), we get the attribute weights:

$$\gamma_\beta^*=\frac{\sum_{\alpha=1}^{\phi}\left(\left|\pi_{\alpha\beta}-\pi_{\nu_\alpha}\right|+\left|\vartheta_{\alpha\beta}-\vartheta_{\nu_\alpha}\right|+\left|\sigma_{\alpha\beta}-\sigma_{\nu_\alpha}\right|\right)}{\sqrt{\sum_{\beta=1}^{\varphi}\left[\left(\left|\pi_{\alpha\beta}-\pi_{\nu_\alpha}\right|+\left|\vartheta_{\alpha\beta}-\vartheta_{\nu_\alpha}\right|+\left|\sigma_{\alpha\beta}-\sigma_{\sigma_\alpha}\right|\right)\right]^2}} \quad (10)$$

By standardizing $\gamma_\beta^*\left(\beta=1,2,\cdots,\varphi\right)$ be a unit, we have

$$\gamma_\beta=\frac{\sum_{\alpha=1}^{\phi}\left(\left|\pi_{\alpha\beta}-\pi_{\nu_\alpha}\right|+\left|\vartheta_{\alpha\beta}-\vartheta_{\nu_\alpha}\right|+\left|\sigma_{\alpha\beta}-\sigma_{\nu_\alpha}\right|\right)}{\sum_{\alpha=1}^{\phi}\sum_{\beta=1}^{\varphi}\left(\left|\pi_{\alpha\beta}-\pi_{\nu_\alpha}\right|+\left|\vartheta_{\alpha\beta}-\vartheta_{\nu_\alpha}\right|+\left|\sigma_{\alpha\beta}-\sigma_{\nu_\alpha}\right|\right)} \quad (11)$$

We propose a practical method to solve MADM with alternative preference and SVNs.

(Procedure one)

Step 1. Let $V=\left(\nu_{\alpha\beta}\right)_{\phi\times\varphi}=\left(\pi_{\alpha\beta},\vartheta_{\alpha\beta},\sigma_{\alpha\beta}\right)_{\phi\times\varphi}$ be a SVN matrix, $\gamma=\left(\gamma_1,\gamma_2,\cdots,\gamma_\varphi\right)$ be the criterion weight, where $\gamma_\beta\in\left[0,1\right]$, $\beta=1,2,\cdots,\varphi$, γ is a set of the known weight information. Let $\nu=\left(\nu_1,\nu_2,\cdots,\nu_\phi\right)$ be subjective preference, $\nu_\alpha=\left(\pi_{\nu_\alpha},\vartheta_{\nu_\alpha},\sigma_{\nu_\alpha}\right)$ are SVNNs, which are subjective preference values on alternatives $\varepsilon_\alpha\left(\alpha=1,2,\cdots,\phi\right)$.

Step 2. By solving the model (M-1), the partially known index values of the weight information are obtained.

Step 3. Utilize the weight $\gamma=\left(\gamma_1,\gamma_2,\cdots,\gamma_\varphi\right)$ and Eq. (7), we obtain the $\tilde{\nu}_\alpha$ of $\varepsilon_\alpha\left(\alpha=1,2,\cdots,\phi\right)$.

Step 4. Obtain the scores $\psi(\nu_\alpha)$ of $\nu_\alpha(\alpha=1,2,\cdots,\phi)$ to rank all the solutions $\varepsilon_\alpha(\alpha=1,2,\cdots,\phi)$. Then we calculate the $\chi(\nu_\alpha)$ and $\chi(\nu_\beta)$, and then rank the alternatives through $\chi(\nu_\alpha)$ and $\chi(\nu_\beta)$.

Step 5. Rank solutions $\varepsilon_\alpha(\alpha=1,2,\cdots,\phi)$ and select the best one through $\psi(\nu_\alpha)$ and $\chi(\nu_\alpha)$ $(\alpha=1,2,\cdots,\phi)$.

Step 6. End.

4. Models for INS MADM problems

Definition 10[23]. Let W be a collection with element in fix set W, denoted by ϖ. An INSs $\tilde{\nu}$ in W is defined as follows:

$$\tilde{\nu}=\left\{\left(\varpi,\pi_{\tilde{\nu}}(\varpi),\vartheta_{\tilde{\nu}}(\varpi),\sigma_{\tilde{\nu}}(\varpi)\right)\middle|\varpi\in W\right\} \tag{12}$$

where $\pi_{\tilde{\nu}}(\varpi)$, $\vartheta_{\tilde{\nu}}(\varpi)$ and $\sigma_{\tilde{\nu}}(\varpi)$, which are interval values in the value of $[0,1]$, that is, $\pi_\nu(\varpi)\subseteq[0,1], \vartheta_\nu(\varpi)\subseteq[0,1]$ and $\sigma_\nu(\varpi)\subseteq[0,1]$. $0\le\sup\left(\pi_{\tilde{\nu}}(\varpi)\right)+\sup\left(\vartheta_{\tilde{\nu}}(\varpi)\right)+\sup\left(\sigma_{\tilde{\nu}}(\varpi)\right)\le 3$.

For a INSs $\left\{\left(\varpi,\pi_{\tilde{\nu}}(\varpi),\vartheta_{\tilde{\nu}}(\varpi),\sigma_{\tilde{\nu}}(\varpi)\right)\middle|\varpi\in W\right\}$, the ordered triple components $\left(\pi_{\tilde{\nu}}(\varpi),\vartheta_{\tilde{\nu}}(\varpi),\sigma_{\tilde{\nu}}(\varpi)\right)$, are described as an INNs, and each INN can be expressed as $\tilde{\nu}=\left(\tilde{\pi}_{\tilde{\nu}},\tilde{\vartheta}_{\tilde{\nu}},\tilde{\sigma}_{\tilde{\nu}}\right)=\left(\left[\tilde{\pi}_{\tilde{\nu}}^X,\tilde{\pi}_{\tilde{\nu}}^Y\right],\left[\tilde{\vartheta}_{\tilde{\nu}}^X,\tilde{\vartheta}_{\tilde{\nu}}^Y\right],\left[\tilde{\sigma}_{\tilde{\nu}}^X,\tilde{\sigma}_{\tilde{\nu}}^Y\right]\right)$, where $\left[\tilde{\pi}_{\tilde{\nu}}^X,\tilde{\pi}_{\tilde{\nu}}^Y\right]\subseteq[0,1],\left[\tilde{\vartheta}_{\tilde{\nu}}^X,\tilde{\vartheta}_{\tilde{\nu}}^Y\right]\subseteq[0,1],\left[\tilde{\sigma}_{\tilde{\nu}}^X,\tilde{\sigma}_{\tilde{\nu}}^Y\right]\subseteq[0,1]$, and $0\le\tilde{\pi}_{\tilde{\nu}}^Y+\tilde{\vartheta}_{\tilde{\nu}}^Y+\tilde{\sigma}_{\tilde{\nu}}^Y\le 3$.

Definition 11[64]. Let $\tilde{\nu}=\left(\left[\tilde{\pi}_{\tilde{\nu}}^X,\tilde{\pi}_{\tilde{\nu}}^Y\right],\left[\tilde{\vartheta}_{\tilde{\nu}}^X,\tilde{\vartheta}_{\tilde{\nu}}^Y\right],\left[\tilde{\sigma}_{\tilde{\nu}}^X,\tilde{\sigma}_{\tilde{\nu}}^Y\right]\right)$ be an INN, a score function ψ is represented:

$$\psi(\tilde{\nu})=\frac{\left(2+\tilde{\pi}_{\tilde{\nu}}^X-\tilde{\vartheta}_{\tilde{\nu}}^X-\tilde{\sigma}_{\tilde{\nu}}^X\right)+\left(2+\tilde{\pi}_{\tilde{\nu}}^Y-\tilde{\vartheta}_{\tilde{\nu}}^Y-\tilde{\sigma}_{\tilde{\nu}}^Y\right)}{6},\ \psi(\tilde{\nu})\in[0,1]. \tag{13}$$

Definition 12[64]. Let $\tilde{\nu}=\left(\left[\tilde{\pi}_{\tilde{\nu}}^X,\tilde{\pi}_{\tilde{\nu}}^Y\right],\left[\tilde{\vartheta}_{\tilde{\nu}}^X,\tilde{\vartheta}_{\tilde{\nu}}^Y\right],\left[\tilde{\sigma}_{\tilde{\nu}}^X,\tilde{\sigma}_{\tilde{\nu}}^Y\right]\right)$ be an INN, an accuracy function χ is represented:

$$\chi(\tilde{\nu})=\frac{\left(\tilde{\pi}_{\tilde{\nu}}^X+\tilde{\pi}_{\tilde{\nu}}^Y\right)-\left(\tilde{\sigma}_{\tilde{\nu}}^X+\tilde{\sigma}_{\tilde{\nu}}^Y\right)}{2},\ \chi(\tilde{\nu})\in[-1,1]. \tag{14}$$

Tang [64] gave an order relation between two INNs.

Definition 13[64]. Let $\tilde{\nu}=\left(\left[\tilde{\pi}_{\tilde{\nu}}^{X},\tilde{\pi}_{\tilde{\nu}}^{Y}\right],\left[\tilde{\vartheta}_{\tilde{\nu}}^{X},\tilde{\vartheta}_{\tilde{\nu}}^{Y}\right],\left[\tilde{\sigma}_{\tilde{\nu}}^{X},\tilde{\sigma}_{\tilde{\nu}}^{Y}\right]\right)$ and $\tilde{\mu}=\left(\left[\tilde{\pi}_{\tilde{\mu}}^{X},\tilde{\pi}_{\tilde{\mu}}^{Y}\right],\left[\tilde{\vartheta}_{\tilde{\mu}}^{X},\tilde{\vartheta}_{\tilde{\mu}}^{Y}\right],\left[\tilde{\sigma}_{\tilde{\mu}}^{X},\tilde{\sigma}_{\tilde{\mu}}^{Y}\right]\right)$ be two INNs, $\psi(\tilde{\nu})=\frac{\left(2+\tilde{\pi}_{\tilde{\nu}}^{X}-\tilde{\vartheta}_{\tilde{\nu}}^{X}-\tilde{\sigma}_{\tilde{\nu}}^{X}\right)+\left(2+\tilde{\pi}_{\tilde{\nu}}^{Y}-\tilde{\vartheta}_{\tilde{\nu}}^{Y}-\tilde{\sigma}_{\tilde{\nu}}^{Y}\right)}{6}$ and $\psi(\tilde{\mu})=\frac{\left(2+\tilde{\pi}_{\tilde{\mu}}^{X}-\tilde{\vartheta}_{\tilde{\mu}}^{X}-\tilde{\sigma}_{\tilde{\mu}}^{X}\right)+\left(2+\tilde{\pi}_{\tilde{\mu}}^{Y}-\tilde{\vartheta}_{\tilde{\mu}}^{Y}-\tilde{\sigma}_{\tilde{\mu}}^{Y}\right)}{6}$ be the scores of $\tilde{\nu}$ and $\tilde{\mu}$, respectively, and let $\chi(\tilde{\nu})=\frac{\left(\tilde{\pi}_{\tilde{\nu}}^{X}+\tilde{\pi}_{\tilde{\nu}}^{Y}\right)-\left(\tilde{\sigma}_{\tilde{\nu}}^{X}+\tilde{\sigma}_{\tilde{\nu}}^{Y}\right)}{2}$ and $\chi(\tilde{\mu})=\frac{\left(\tilde{\pi}_{\tilde{\mu}}^{X}+\tilde{\pi}_{\tilde{\mu}}^{Y}\right)-\left(\tilde{\sigma}_{\tilde{\mu}}^{X}+\tilde{\sigma}_{\tilde{\mu}}^{Y}\right)}{2}$ be the accuracy degrees of $\tilde{\nu}$ and $\tilde{\mu}$, then if $\psi(\tilde{\nu})<\psi(\tilde{\mu})$, then $\tilde{\nu}<\tilde{\mu}$; if $\psi(\tilde{\nu})=\psi(\tilde{\mu})$, then

(2) if $\psi(\tilde{\nu})=\psi(\tilde{\mu})$, then $\tilde{\nu}=\tilde{\mu}$; (2) if $\chi(\tilde{\nu})<\chi(\tilde{\mu})$, then $\tilde{\mu}<\tilde{\mu}$.

Definition 14[27]. Let $\tilde{\nu}_1=\left(\left[\tilde{\pi}_1^{X},\tilde{\pi}_1^{Y}\right],\left[\tilde{\vartheta}_1^{X},\tilde{\vartheta}_1^{Y}\right],\left[\tilde{\sigma}_1^{X},\tilde{\sigma}_1^{Y}\right]\right)$ and $\tilde{\nu}_2=\left(\left[\tilde{\pi}_2^{X},\tilde{\pi}_2^{Y}\right],\left[\tilde{\vartheta}_2^{X},\tilde{\vartheta}_2^{Y}\right],\left[\tilde{\sigma}_2^{X},\tilde{\sigma}_2^{Y}\right]\right)$ be two INNs, and some basic operations are defined:

(1) $\tilde{\nu}_1\oplus\tilde{\nu}_2=\begin{pmatrix}\left[\tilde{\pi}_1^{X}+\tilde{\pi}_1^{X}-\tilde{\pi}_1^{X}\tilde{\pi}_1^{X},\tilde{\pi}_1^{Y}+\tilde{\pi}_1^{Y}-\tilde{\pi}_1^{Y}\tilde{\pi}_1^{Y}\right],\\ \left[\tilde{\vartheta}_1^{X}\tilde{\vartheta}_2^{X},\tilde{\vartheta}_1^{Y}\tilde{\vartheta}_2^{Y}\right],\left[\tilde{\sigma}_1^{X}\tilde{\sigma}_2^{X},\tilde{\sigma}_1^{Y}\tilde{\sigma}_2^{Y}\right]\end{pmatrix}$;

(2) $\tilde{\nu}_1\otimes\tilde{\nu}_2=\begin{pmatrix}\left[\tilde{\pi}_1^{X}\tilde{\pi}_2^{X},\tilde{\pi}_1^{Y}\tilde{\pi}_2^{Y}\right],\\ \left[\tilde{\vartheta}_1^{X}+\tilde{\vartheta}_1^{X}-\tilde{\vartheta}_1^{X}\tilde{\vartheta}_1^{X},\tilde{\vartheta}_1^{Y}+\tilde{\vartheta}_1^{Y}-\tilde{\vartheta}_1^{Y}\tilde{\vartheta}_1^{Y}\right],\\ \left[\tilde{\sigma}_1^{X}+\tilde{\sigma}_1^{X}-\tilde{\sigma}_1^{X}\tilde{\sigma}_1^{X},\tilde{\sigma}_1^{Y}+\tilde{\sigma}_1^{Y}-\tilde{\sigma}_1^{Y}\tilde{\sigma}_1^{Y}\right]\end{pmatrix}$;

(3) $\lambda\tilde{\nu}_1=\begin{pmatrix}\left[1-\left(1-\tilde{\pi}_1^{X}\right)^{\lambda},1-\left(1-\tilde{\pi}_1^{Y}\right)^{\lambda}\right],\\ \left[\left(\tilde{\vartheta}_1^{X}\right)^{\lambda},\left(\tilde{\vartheta}_1^{Y}\right)^{\lambda}\right],\left[\left(\tilde{\sigma}_1^{X}\right)^{\lambda},\left(\tilde{\sigma}_1^{Y}\right)^{\lambda}\right]\end{pmatrix},\lambda>0;$

(4) $\left(\tilde{\nu}_1\right)^{\lambda}=\begin{pmatrix}\left[\left(\tilde{\pi}_1^{X}\right)^{\lambda},\left(\tilde{\pi}_1^{Y}\right)^{\lambda}\right],\left[\left(\tilde{\vartheta}_1^{X}\right)^{\lambda},\left(\tilde{\vartheta}_1^{Y}\right)^{\lambda}\right],\\ \left[1-\left(1-\tilde{\sigma}_1^{X}\right)^{\lambda},1-\left(1-\tilde{\sigma}_1^{Y}\right)^{\lambda}\right]\end{pmatrix},\lambda>0.$

Theorem 2[27]. Let $\tilde{\nu}_1=\left(\left[\tilde{\pi}_1^{X},\tilde{\pi}_1^{Y}\right],\left[\tilde{\vartheta}_1^{X},\tilde{\vartheta}_1^{Y}\right],\left[\tilde{\sigma}_1^{X},\tilde{\sigma}_1^{Y}\right]\right)$ and $\tilde{\nu}_2=\left(\left[\tilde{\pi}_2^{X},\tilde{\pi}_2^{Y}\right],\left[\tilde{\vartheta}_2^{X},\tilde{\vartheta}_2^{Y}\right],\left[\tilde{\sigma}_2^{X},\tilde{\sigma}_2^{Y}\right]\right)$ be two INNs, $\lambda,\lambda_1,\lambda_2>0$, then

(1) $\tilde{v}_1 \oplus \tilde{v}_2 = \tilde{v}_2 \oplus \tilde{v}_1$;

(2) $\tilde{v}_1 \otimes \tilde{v}_2 = \tilde{v}_2 \otimes \tilde{v}_1$;

(3) $\lambda\left(\tilde{v}_1 \oplus \tilde{v}_2\right) = \lambda \tilde{v}_1 \oplus \lambda \tilde{v}_2$;

(4) $\left(\tilde{v}_1 \otimes \tilde{v}_2\right)^{\lambda} = \left(\tilde{v}_1\right)^{\lambda} \otimes \left(\tilde{v}_2\right)^{\lambda}$;

(5) $\lambda_1\tilde{v}_1 \oplus \lambda_2\tilde{v}_1 = \left(\lambda_1 + \lambda_2\right)\tilde{v}_1$;

(6) $\left(\tilde{v}_1\right)^{\lambda_1} \otimes \left(\tilde{v}_1\right)^{\lambda_2} = \left(\tilde{v}_1\right)^{\left(\lambda_1+\lambda_2\right)}$;

(7) $\left(\left(\tilde{v}_1\right)^{\lambda_1}\right)^{\lambda_2} = \left(\tilde{v}_1\right)^{\lambda_1\lambda_2}$.

Definition 15[27]. Let $\tilde{v}_{\beta} = \left(\left[\tilde{\pi}_{\beta}^{X}, \tilde{\pi}_{\beta}^{Y}\right], \left[\tilde{\vartheta}_{\beta}^{X}, \tilde{\vartheta}_{\beta}^{Y}\right], \left[\tilde{\sigma}_{\beta}^{X}, \tilde{\sigma}_{\beta}^{Y}\right]\right)\left(\beta = 1, 2, \cdots, \varphi\right)$ $\left(\beta = 1, 2, \cdots, \varphi\right)$ be a collection of INNs, and let INWA: $Q^{\varphi} \rightarrow Q$, if

$$\mathrm{INWA}_{\gamma}\left(\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_{\varphi}\right) = \overset{\varphi}{\underset{\beta=1}{\oplus}}\left(\gamma_{\beta}\tilde{v}_{\beta}\right)$$
$$= \left(\begin{array}{l}\left[1 - \prod_{\beta=1}^{\varphi}\left(1 - \tilde{\pi}_{\beta}^{X}\right)^{\gamma_{\beta}}, 1 - \prod_{\beta=1}^{\varphi}\left(1 - \tilde{\pi}_{\beta}^{Y}\right)^{\gamma_{\beta}}\right], \\ \left[\prod_{\beta=1}^{\varphi}\left(\tilde{\vartheta}_{\beta}^{X}\right)^{\gamma_{\beta}}, \prod_{\beta=1}^{\varphi}\left(\tilde{\vartheta}_{\beta}^{Y}\right)^{\gamma_{\beta}}\right], \left[\prod_{\beta=1}^{\varphi}\left(\tilde{\sigma}_{\beta}^{X}\right)^{\gamma_{\beta}}, \prod_{\beta=1}^{\varphi}\left(\tilde{\sigma}_{\beta}^{Y}\right)^{\gamma_{\beta}}\right]\end{array}\right) \tag{15}$$

where $\gamma = \left(\gamma_1, \gamma_2, \cdots, \gamma_{\varphi}\right)^{T}$ be the criterion weight, and $\gamma_{\beta} > 0$, $\sum_{\beta=1}^{n}\gamma_{\beta} = 1$, then INWA is called INWA operator.

Definition 16[64]. Let $\tilde{v}_1 = \left(\left[\tilde{\pi}_1^{X}, \tilde{\pi}_1^{Y}\right], \left[\tilde{\vartheta}_1^{X}, \tilde{\vartheta}_1^{Y}\right], \left[\tilde{\sigma}_1^{X}, \tilde{\sigma}_1^{Y}\right]\right)$ and $\tilde{v}_2 = \left(\left[\tilde{\pi}_2^{X}, \tilde{\pi}_2^{Y}\right], \left[\tilde{\vartheta}_2^{X}, \tilde{\vartheta}_2^{Y}\right], \left[\tilde{\sigma}_2^{X}, \tilde{\sigma}_2^{Y}\right]\right)$ be two INNs, then the normalized Hamming distance between $\tilde{v}_1 = \left(\left[\tilde{\pi}_1^{X}, \tilde{\pi}_1^{Y}\right], \left[\tilde{\vartheta}_1^{X}, \tilde{\vartheta}_1^{Y}\right], \left[\tilde{\sigma}_1^{X}, \tilde{\sigma}_1^{Y}\right]\right)$ and $\tilde{v}_2 = \left(\left[\tilde{\pi}_2^{X}, \tilde{\pi}_2^{Y}\right], \left[\tilde{\vartheta}_2^{X}, \tilde{\vartheta}_2^{Y}\right], \left[\tilde{\sigma}_2^{X}, \tilde{\sigma}_2^{Y}\right]\right)$ is defined:

$$\kappa\left(\tilde{v}_1, \tilde{v}_2\right) = \frac{1}{6}\left(\begin{array}{l}\left|\tilde{\pi}_1^{X} - \tilde{\pi}_2^{X}\right| + \left|\tilde{\pi}_1^{Y} - \tilde{\pi}_2^{Y}\right| + \left|\tilde{\vartheta}_1^{X} - \tilde{\vartheta}_2^{X}\right| + \\ \left|\tilde{\vartheta}_1^{Y} - \tilde{\vartheta}_2^{Y}\right| + \left|\tilde{\sigma}_1^{X} - \tilde{\sigma}_2^{X}\right| + \left|\tilde{\sigma}_1^{Y} - \tilde{\sigma}_2^{Y}\right|\end{array}\right) \tag{16}$$

Let ε, ς and γ be presented as in section 3. Suppose that $\tilde{V} = \left(\tilde{v}_{\alpha\beta}\right)_{\phi\times\varphi} = \left(\left[\tilde{\pi}_{\alpha\beta}^{X}, \tilde{\pi}_{\alpha\beta}^{Y}\right], \left[\tilde{\vartheta}_{\alpha\beta}^{X}, \tilde{\vartheta}_{\alpha\beta}^{Y}\right], \left[\tilde{\sigma}_{\alpha\beta}^{X}, \tilde{\sigma}_{\alpha\beta}^{Y}\right]\right)_{\phi\times\varphi}$ is the INN matrix,

$\left[\tilde{\pi}_{\alpha\beta}^{X},\tilde{\pi}_{\alpha\beta}^{Y}\right]\subseteq[0,1]$, $\left[\tilde{\vartheta}_{\alpha\beta}^{X},\tilde{\vartheta}_{\alpha\beta}^{Y}\right]\subseteq[0,1]$, $\left[\tilde{\sigma}_{\alpha\beta}^{X},\tilde{\sigma}_{\alpha\beta}^{Y}\right]\subseteq[0,1]$, $0\le\tilde{\pi}_{\alpha\beta}^{Y}+\tilde{\vartheta}_{\alpha\beta}^{Y}+\tilde{\sigma}_{\alpha\beta}^{Y}\le 3$, $\alpha=1,2,\cdots,\phi$, $\beta=1,2,\cdots,\varphi$. The subjective preference information on alternatives is known, and $\tilde{v}_{\alpha}=\left(\left[\tilde{\pi}_{\tilde{v}_\alpha}^{X},\tilde{\pi}_{\tilde{v}_\alpha}^{Y}\right],\left[\tilde{\vartheta}_{\tilde{v}_\alpha}^{X},\tilde{\vartheta}_{\tilde{v}_\alpha}^{Y}\right],\left[\tilde{\sigma}_{\tilde{v}_\alpha}^{X},\tilde{\sigma}_{\tilde{v}_\alpha}^{Y}\right]\right)$ are INNs, which is subjective preference values on alternative $\varepsilon_{\alpha}\left(\alpha=1,2,\cdots,\phi\right)$.

Definition 17[27]. Let $\tilde{V}=\left(\tilde{v}_{\alpha\beta}\right)_{\phi\times\varphi}=\left(\left[\tilde{\pi}_{\alpha\beta}^{X},\tilde{\pi}_{\alpha\beta}^{Y}\right],\left[\tilde{\vartheta}_{\alpha\beta}^{X},\tilde{\vartheta}_{\alpha\beta}^{Y}\right],\left[\tilde{\sigma}_{\alpha\beta}^{X},\tilde{\sigma}_{\alpha\beta}^{Y}\right]\right)_{\phi\times\varphi}$ is the INN matrix, $\tilde{v}_{\alpha}=\left(\tilde{v}_{\alpha 1},\tilde{v}_{\alpha 2},\cdots,\tilde{v}_{\alpha\varphi}\right)$ be the vector of attribute values for ε_{α}, $\alpha=1,2,\cdots,\phi$, then we call

$$\begin{aligned}\tilde{v}_{\alpha}&=\left(\left[\tilde{\pi}_{\alpha}^{X},\tilde{\pi}_{\alpha}^{Y}\right],\left[\tilde{\vartheta}_{\alpha}^{X},\tilde{\vartheta}_{\alpha}^{Y}\right],\left[\tilde{\sigma}_{\alpha}^{X},\tilde{\sigma}_{\alpha}^{Y}\right]\right)\\&=\mathrm{INWA}_{\gamma}\left(\tilde{v}_{\alpha 1},\tilde{v}_{\alpha 2},\cdots,\tilde{v}_{\alpha\varphi}\right)=\mathop{\oplus}_{\beta=1}^{\varphi}\left(\gamma_{\beta}\tilde{v}_{\alpha\beta}\right)\\&=\left(\begin{array}{l}\left[1-\prod_{\beta=1}^{\varphi}\left(1-\tilde{\pi}_{\alpha\beta}^{X}\right)^{\gamma_\beta},1-\prod_{\beta=1}^{\varphi}\left(1-\tilde{\pi}_{\alpha\beta}^{Y}\right)^{\gamma_\beta}\right],\\\left[\prod_{\beta=1}^{\varphi}\left(\tilde{\vartheta}_{\alpha\beta}^{X}\right)^{\gamma_\beta},\prod_{\beta=1}^{\varphi}\left(\tilde{\vartheta}_{\alpha\beta}^{Y}\right)^{\gamma_\beta}\right],\left[\prod_{\beta=1}^{\varphi}\left(\tilde{\sigma}_{\alpha\beta}^{X}\right)^{\gamma_\beta},\prod_{\beta=1}^{\varphi}\left(\tilde{\sigma}_{\alpha\beta}^{Y}\right)^{\gamma_\beta}\right]\end{array}\right)\end{aligned}\tag{17}$$

the overall value of ε, where $\gamma=\left(\gamma_1,\gamma_2,\cdots,\gamma_{\varphi}\right)^T$ is the criterion weight.

When the attribute weight information is completely known, aggregate all the weighted attribute values corresponding to each alternative into a whole using Eq. (17).

If the decision model is difficult to obtain attribute weight, sometimes the criterion weight information is completely unknown. To reflect the subjective preference and objective information of decision-makers, an optimization model is established to obtain the weight of attributes. However, there are some differences between DM's subjective preference and objective information. To make the decision more reasonable, the selection of criterion weight vector is to minimize the total deviation between objective information and DM subjective preference.

The least deviation method was used to calculate the difference between DM's subjective preference and objective information. For the $\varsigma_{\beta}\in\varsigma$, the deviation of alternative ε_{α} to DM's subjective preference is described as follows:

$$K_{\alpha\beta}\left(\gamma\right)=\kappa\left(\tilde{v}_{\alpha\beta},\tilde{v}_{\alpha}\right)\gamma_{\beta},\alpha=1,2,\cdots,\phi,\beta=1,2,\cdots,\varphi.\tag{18}$$

Let $K_{\alpha}(\gamma)=\sum_{\beta=1}^{\varphi}K_{\alpha\beta}(\gamma)=\sum_{\beta=1}^{\varphi}\kappa(\tilde{v}_{\alpha\beta},\tilde{v}_{\alpha})\gamma_{\beta},\alpha=1,2,\cdots,\phi$ Based on the above analysis, we must choose weights to minimize all deviations from all alternatives. To this end, we establish a linear programming model:

$$(\text{M-3})\begin{cases}\min K(\gamma)=\sum_{\alpha=1}^{\phi}\sum_{\beta=1}^{\varphi}K_{\alpha\beta}(\gamma)=\sum_{\alpha=1}^{\phi}\sum_{\beta=1}^{\varphi}\kappa(\tilde{v}_{\alpha\beta},\tilde{v}_{\alpha})\gamma_{\beta}\\ \text{Subject to}\quad \sum_{\beta=1}^{\varphi}\gamma_{\beta}=1,\gamma_{\beta}\geq 0,\beta=1,2,\cdots,\varphi\end{cases}$$

where $\kappa(\tilde{v}_{\alpha\beta},\tilde{v}_{\alpha})=\frac{1}{6}\begin{pmatrix}\left|\tilde{\pi}_{\alpha\beta}^{X}-\tilde{\pi}_{\tilde{v}_{\alpha}}^{X}\right|+\left|\tilde{\pi}_{\alpha\beta}^{Y}-\tilde{\pi}_{\tilde{v}_{\alpha}}^{Y}\right|+\left|\tilde{\vartheta}_{\alpha\beta}^{X}-\tilde{\vartheta}_{\tilde{v}_{\alpha}}^{X}\right|\\ +\left|\tilde{\vartheta}_{\alpha\beta}^{Y}-\tilde{\vartheta}_{\tilde{v}_{\alpha}}^{Y}\right|+\left|\tilde{\sigma}_{\alpha\beta}^{X}-\tilde{\sigma}_{\tilde{v}_{\alpha}}^{X}\right|+\left|\tilde{\sigma}_{\alpha\beta}^{Y}-\tilde{\sigma}_{\tilde{v}_{\alpha}}^{Y}\right|\end{pmatrix}$.

By solving the model (M-3), we get the $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_{\varphi})$, which is used as attributes weight.

If the information about criterion weights is completely unknown, we build another programming model:

$$(\text{M-4})\begin{cases}\min K(\gamma)=\sum_{\alpha=1}^{\phi}\sum_{\beta=1}^{\varphi}K_{\alpha\beta}(\gamma)\\ =\frac{1}{6}\sum_{\alpha=1}^{\phi}\sum_{\beta=1}^{\varphi}\begin{pmatrix}\left|\tilde{\pi}_{\alpha\beta}^{X}-\tilde{\pi}_{\tilde{v}_{\alpha}}^{X}\right|+\left|\tilde{\pi}_{\alpha\beta}^{Y}-\tilde{\pi}_{\tilde{v}_{\alpha}}^{Y}\right|+\left|\tilde{\vartheta}_{\alpha\beta}^{X}-\tilde{\vartheta}_{\tilde{v}_{\alpha}}^{X}\right|+\\ \left|\tilde{\vartheta}_{\alpha\beta}^{Y}-\tilde{\vartheta}_{\tilde{v}_{\alpha}}^{Y}\right|+\left|\tilde{\sigma}_{\alpha\beta}^{X}-\tilde{\sigma}_{\tilde{v}_{\alpha}}^{X}\right|+\left|\tilde{\sigma}_{\alpha\beta}^{Y}-\tilde{\sigma}_{\tilde{v}_{\alpha}}^{Y}\right|\end{pmatrix}\gamma_{\beta}\\ s.t.\ \sum_{\beta=1}^{\varphi}\gamma_{\beta}^{2}=1,\gamma_{\beta}\geq 0,\beta=1,2,\cdots,\varphi\end{cases}$$

To solve this model, we build the Lagrange function:

$$L(\gamma,\lambda)=\frac{1}{6}\sum_{\alpha=1}^{\phi}\sum_{\beta=1}^{\varphi}\begin{pmatrix}\left|\tilde{\pi}_{\alpha\beta}^{X}-\tilde{\pi}_{\tilde{v}_{\alpha}}^{X}\right|+\left|\tilde{\pi}_{\alpha\beta}^{Y}-\tilde{\pi}_{\tilde{v}_{\alpha}}^{Y}\right|+\left|\tilde{\vartheta}_{\alpha\beta}^{X}-\tilde{\vartheta}_{\tilde{v}_{\alpha}}^{X}\right|+\\ \left|\tilde{\vartheta}_{\alpha\beta}^{Y}-\tilde{\vartheta}_{\tilde{v}_{\alpha}}^{Y}\right|+\left|\tilde{\sigma}_{\alpha\beta}^{X}-\tilde{\sigma}_{\tilde{v}_{\alpha}}^{X}\right|+\left|\tilde{\sigma}_{\alpha\beta}^{Y}-\tilde{\sigma}_{\tilde{v}_{\alpha}}^{Y}\right|\end{pmatrix}\gamma_{\beta}+\frac{\lambda}{12}\left(\sum_{\beta=1}^{\varphi}\gamma_{\beta}^{2}-1\right)\quad(19)$$

where λ is the Lagrange multiplier.

Differentiating Eq. (19) with respect to $\gamma_{\beta}(\beta=1,2,\cdots,\varphi)$ and λ, and setting these partial derivatives equal to zero,

$$\begin{cases}\frac{\partial L}{\partial\gamma_{\beta}}=\sum_{\alpha=1}^{\phi}\begin{pmatrix}\left|\tilde{\pi}_{\alpha\beta}^{X}-\tilde{\pi}_{\tilde{v}_{\alpha}}^{X}\right|+\left|\tilde{\pi}_{\alpha\beta}^{Y}-\tilde{\pi}_{\tilde{v}_{\alpha}}^{Y}\right|+\left|\tilde{\vartheta}_{\alpha\beta}^{X}-\tilde{\vartheta}_{\tilde{v}_{\alpha}}^{X}\right|\\ +\left|\tilde{\vartheta}_{\alpha\beta}^{Y}-\tilde{\vartheta}_{\tilde{v}_{\alpha}}^{Y}\right|+\left|\tilde{\sigma}_{\alpha\beta}^{X}-\tilde{\sigma}_{\tilde{v}_{\alpha}}^{X}\right|+\left|\tilde{\sigma}_{\alpha\beta}^{Y}-\tilde{\sigma}_{\tilde{v}_{\alpha}}^{Y}\right|\end{pmatrix}+\lambda\gamma_{\beta}=0\\ \frac{\partial L}{\partial\lambda}=\sum_{\beta=1}^{\varphi}\gamma_{\beta}^{2}-1=0\end{cases}\quad(20)$$

By solving Eq. (20), we get the attribute weights:

$$\gamma_\beta^* = \frac{\sum_{\alpha=1}^{\phi}\begin{pmatrix}\left|\tilde{\pi}_{\alpha\beta}^X-\tilde{\pi}_{\tilde{v}_\alpha}^X\right|+\left|\tilde{\pi}_{\alpha\beta}^Y-\tilde{\pi}_{\tilde{v}_\alpha}^Y\right|+\left|\tilde{\vartheta}_{\alpha\beta}^X-\tilde{\vartheta}_{\tilde{v}_\alpha}^X\right|+\\ \left|\tilde{\vartheta}_{\alpha\beta}^Y-\tilde{\vartheta}_{\tilde{v}_\alpha}^Y\right|+\left|\tilde{\sigma}_{\alpha\beta}^X-\tilde{\sigma}_{\tilde{v}_\alpha}^X\right|+\left|\tilde{\sigma}_{\alpha\beta}^Y-\tilde{\sigma}_{\tilde{v}_\alpha}^Y\right|\end{pmatrix}}{\sqrt{\sum_{\beta=1}^{\varphi}\left[\begin{pmatrix}\left|\tilde{\pi}_{\alpha\beta}^X-\tilde{\pi}_{\tilde{v}_\alpha}^X\right|+\left|\tilde{\pi}_{\alpha\beta}^Y-\tilde{\pi}_{\tilde{v}_\alpha}^Y\right|+\left|\tilde{\vartheta}_{\alpha\beta}^X-\tilde{\vartheta}_{\tilde{v}_\alpha}^X\right|+\\ \left|\tilde{\vartheta}_{\alpha\beta}^Y-\tilde{\vartheta}_{\tilde{v}_\alpha}^Y\right|+\left|\tilde{\sigma}_{\alpha\beta}^X-\tilde{\sigma}_{\tilde{v}_\alpha}^X\right|+\left|\tilde{\sigma}_{\alpha\beta}^Y-\tilde{\sigma}_{\tilde{v}_\alpha}^Y\right|\end{pmatrix}\right]^2}} \tag{21}$$

By normalizing $\gamma_\beta^*\left(\beta=1,2,\cdots,\varphi\right)$ be a unit, we have

$$w_j = \frac{\sum_{\alpha=1}^{\phi}\begin{pmatrix}\left|\tilde{\pi}_{\alpha\beta}^X-\tilde{\pi}_{\tilde{v}_\alpha}^X\right|+\left|\tilde{\pi}_{\alpha\beta}^Y-\tilde{\pi}_{\tilde{v}_\alpha}^Y\right|+\left|\tilde{\vartheta}_{\alpha\beta}^X-\tilde{\vartheta}_{\tilde{v}_\alpha}^X\right|+\\ \left|\tilde{\vartheta}_{\alpha\beta}^Y-\tilde{\vartheta}_{\tilde{v}_\alpha}^Y\right|+\left|\tilde{\sigma}_{\alpha\beta}^X-\tilde{\sigma}_{\tilde{v}_\alpha}^X\right|+\left|\tilde{\sigma}_{\alpha\beta}^Y-\tilde{\sigma}_{\tilde{v}_\alpha}^Y\right|\end{pmatrix}}{\sum_{\beta=1}^{\varphi}\sum_{\alpha=1}^{\phi}\begin{pmatrix}\left|\tilde{\pi}_{\alpha\beta}^X-\tilde{\pi}_{\tilde{v}_\alpha}^X\right|+\left|\tilde{\pi}_{\alpha\beta}^Y-\tilde{\pi}_{\tilde{v}_\alpha}^Y\right|+\left|\tilde{\vartheta}_{\alpha\beta}^X-\tilde{\vartheta}_{\tilde{v}_\alpha}^X\right|+\\ \left|\tilde{\vartheta}_{\alpha\beta}^Y-\tilde{\vartheta}_{\tilde{v}_\alpha}^Y\right|+\left|\tilde{\sigma}_{\alpha\beta}^X-\tilde{\sigma}_{\tilde{v}_\alpha}^X\right|+\left|\tilde{\sigma}_{\alpha\beta}^Y-\tilde{\sigma}_{\tilde{v}_\alpha}^Y\right|\end{pmatrix}} \tag{22}$$

Based on the above model, a practical method is proposed to solve INN-MADM with alternative preference information. The method includes the following steps:

(Procedure two)

Step 1. Let $\tilde{V}=\left(\tilde{v}_{\alpha\beta}\right)_{\phi\times\varphi}=\left(\left[\tilde{\pi}_{\alpha\beta}^X,\tilde{\pi}_{\alpha\beta}^Y\right],\left[\tilde{\vartheta}_{\alpha\beta}^X,\tilde{\vartheta}_{\alpha\beta}^Y\right],\left[\tilde{\sigma}_{\alpha\phi}^X,\tilde{\sigma}_{\alpha\beta}^Y\right]\right)_{\phi\times\varphi}$ be an INN matrix, where $\tilde{v}_{\alpha\beta}=\left[\tilde{\pi}_{\alpha\beta}^X,\tilde{\pi}_{\alpha\beta}^Y\right],\left[\tilde{\vartheta}_{\alpha\beta}^X,\tilde{\vartheta}_{\alpha\beta}^Y\right],\left[\tilde{\sigma}_{\alpha\beta}^X,\tilde{\sigma}_{\alpha\beta}^Y\right]$, for $\varepsilon_\alpha\in\varepsilon$ with respect to $\varsigma_\beta\in\varsigma$, $\gamma=\left(\gamma_1,\gamma_2,\cdots,\gamma_\varphi\right)$ be the weight of attributes, where $\gamma_\beta\in\left[0,1\right]$, $\beta=1,2,\cdots,\varphi$, which is constructed by the forms 1-5. Let $\tilde{v}=\left(\tilde{v}_1,\tilde{v}_2,\cdots,v_\phi\right)$ be subjective preference, $\tilde{v}_\alpha=\left(\left[\tilde{\pi}_{\tilde{v}_\alpha}^X,\tilde{\pi}_{\tilde{v}_\alpha}^Y\right],\left[\tilde{\vartheta}_{\tilde{v}_\alpha}^X,\tilde{\vartheta}_{\tilde{v}_\alpha}^Y\right],\left[\tilde{\sigma}_{\tilde{v}_\alpha}^X,\tilde{\sigma}_{\tilde{v}_\alpha}^Y\right]\right)$ are INNs, which are subjective preference on alternatives $\varepsilon_\alpha\left(\alpha=1,2,\cdots,\phi\right)$.

Step 2. By solving the model (M-3), the partially known index values of the weight is obtained. If the criterion weight is unknown, then we can obtain the criterion weights by Eq. (22).

Step 3. Utilize $\gamma=\left(\gamma_1,\gamma_2,\cdots,\gamma_\varphi\right)$ and Eq. (17), we obtain the $\tilde{v}_\alpha$ of $\varepsilon_\alpha\left(\alpha=1,2,\cdots,\phi\right)$.

Step 4. Compute out the scores $\psi\left(\tilde{v}_\alpha\right)$ of $\tilde{v}_\alpha\left(\alpha=1,2,\cdots,\phi\right)$ to rank all the alternatives $\varepsilon_\alpha\left(\alpha=1,2,\cdots,\phi\right)$ then rank the alternatives ε_α and ε_β through $\chi\left(\tilde{v}_\alpha\right)$ and $\chi\left(\tilde{v}_\beta\right)$.

Step 5. Rank all the alternatives $\varepsilon_\alpha\left(\alpha=1,2,\cdots,m\right)$ and select the best one(s) through $\psi\left(\tilde{v}_\alpha\right)$ and $\chi\left(\tilde{v}_\alpha\right)\left(\alpha=1,2,\cdots,\phi\right)$.

Step 6. End.

5. Case Study

With the expansion of the e-commerce Internet, online shopping has been enthusiastically sought after by people, and the logistics industry has also risen rapidly. Logistics promotes economic growth and is increasingly prominent in the national economic status. However, as an indispensable part of the logistics industry, logistics parks have many difficult problems. Usually occupies a large scale, and the construction investment cost is high. Once completed, it is not easy to relocate, and today's environmental problems are becoming more and more serious. The basic criteria for planning and building a low-carbon logistics park It is "low energy consumption, high efficiency". The location problem of low-carbon logistics parks can be regarded as a MADM problem. Generally, multiple decision-makers give corresponding evaluations to a limited number of alternatives under the influence of different factors, and use scientific decision-making methods to evaluate the relevant ones. The evaluation information is processed, so as to sort the different alternatives and make a reasonable choice. In this section, we apply the constructed model to a real-world example, taking the low-carbon logistics park site selection as an example. Through market research, a panel of five possible low-carbon logistics park sites $\varepsilon_\alpha\left(\alpha=1,2,3,4,5\right)$ was selected. The experts selected four indexes to evaluate five low-carbon logistics park sites: ① ς_1 is transportation and warehousing investments; ② ς_2 is regional goods material turnover; ③ ς_3 is land use; ④ ς_4 is degree of environmental protection. Five possible low-carbon logistics park sites $\varepsilon_\alpha\left(\alpha=1,2,3,4,5\right)$ will use the SVNNs by the decision maker under the above four attributes, as listed in the following matrix.

$$\tilde{V}=\begin{bmatrix}(0.5,0.8,0.1) & (0.6,0.3,0.3) & (0.3,0.6,0.1) & (0.5,0.7,0.2)\\(0.7,0.2,0.1) & (0.7,0.2,0.2) & (0.7,0.2,0.4) & (0.8,0.2,0.1)\\(0.6,0.7,0.2) & (0.5,0.7,0.3) & (0.5,0.3,0.1) & (0.6,0.3,0.2)\\(0.8,0.1,0.3) & (0.6,0.3,0.4) & (0.3,0.4,0.2) & (0.5,0.6,0.1)\\(0.6,0.4,0.4) & (0.4,0.8,0.1) & (0.7,0.6,0.1) & (0.5,0.8,0.2)\end{bmatrix}$$

DMs' subjective preference value on alternative:

$$\tilde{v}_1 = (0.6, 0.5, 0.2), \tilde{v}_2 = (0.7, 0.2, 0.1), \tilde{v}_3 = (0.3, 0.4, 0.3)$$
$$\tilde{v}_4 = (0.9, 0.3, 0.2), \tilde{v}_5 = (0.5, 0.6, 0.4)$$

Next, developed method is used to select the best location of low-carbon logistics park.

Case 1: Criterion weight information is known as follows:

$$\gamma = \{0.18 \le \gamma_1 \le 0.23, 0.20 \le \gamma_2 \le 0.24, 0.25 \le \gamma_3 \le 0.30,$$
$$0.25 \le \gamma_4 \le 0.33, \gamma_\beta \ge 0, \beta = 1, 2, 3, 4, \sum_{\beta=1}^{4} \gamma_\beta = 1\}$$

Step 1. The single-objective programming model is obtained as follows:

$$\min K(\gamma) = 0.6333\gamma_1 + 0.6667\gamma_2 + 0.8333\gamma_3 + 0.7000\gamma_4$$

Solving this model, we get the weight of attributes: $\gamma = (0.2300\ \ 0.2400\ \ 0.2500\ \ 0.2800\ )^T$

Step 2. Utilize the weight $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_\varphi)$ and by Eq. (6), we obtain the $\tilde{v}_\alpha$ of the low-carbon logistics park site $\varepsilon_\alpha\ (\alpha = 1, 2, \cdots, \phi)$.

$$\tilde{v}_1 = (0.4845, 0.5667, 0.1581), \tilde{v}_2 = (0.7322, 0.2000, 0.1670)$$
$$\tilde{v}_3 = (0.5538, 0.4468, 0.1854), \tilde{v}_4 = (0.5824, 0.3040, 0.2135)$$
$$\tilde{v}_5 = (0.5633, 0.6348, 0.1670)$$

Step 3. Calculate the scores $\psi(\tilde{v}_\alpha)$ of $\tilde{v}_\alpha\ (\alpha = 1, 2, \cdots, \phi)$

$$\psi(\tilde{v}_1) = 0.5866, \psi(\tilde{v}_2) = 0.7884, \psi(\tilde{v}_3) = 0.6406$$
$$\psi(\tilde{v}_4) = 0.6883, \psi(\tilde{v}_5) = 0.5872$$

Step 4. Rank all the low-carbon logistics park sites $\varepsilon_\alpha\ (\alpha = 1, 2, 3, 4, 5)$ through $\psi(\tilde{v}_\alpha)$ $(\alpha = 1, 2, \cdots, 5): \varepsilon_2 \succ \varepsilon_4 \succ \varepsilon_3 \succ \varepsilon_5 \succ \varepsilon_1$, and thus the most desirable low-carbon logistics park site is ε_5.

Case 2: When the weight is unknown, we use another method to get the optimal location of low-carbon logistics park.

Step 1. Get the weight of attributes:

$$\gamma = (0.2235\ \ 0.2353\ \ 0.2941\ \ 0.2471\ )^T$$

Step 2. Utilize the $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_\varphi)$ and Eq. (6), we obtain the overall values $\tilde{v}_\alpha$ of the low-carbon logistics park site $\varepsilon_\alpha\ (\alpha = 1, 2, \cdots, \phi)$.

$$\tilde{v}_1 = (0.4762, 0.5647, 0.1537), \tilde{v}_2 = (0.7286, 0.2000, 0.1770)$$
$$\tilde{v}_3 = (0.5498, 0.4425, 0.1794), \tilde{v}_4 = (0.5732, 0.3031, 0.2172)$$
$$\tilde{v}_5 = (0.5727, 0.6296, 0.1618)$$

Step 3. Compute out the scores $\psi(\tilde{v}_\alpha)(\alpha = 1, 2, \cdots, \phi)$ of $\tilde{v}_\alpha\ (\alpha = 1, 2, \cdots, \phi)$.

$$\psi(\tilde{v}_1) = 0.5860, \psi(\tilde{v}_2) = 0.7839, \psi(\tilde{v}_3) = 0.6426$$
$$\psi(\tilde{v}_4) = 0.6843, \psi(\tilde{v}_5) = 0.5938$$

Step 4. Rank all the solutions through $\psi(\tilde{v}_\alpha): \varepsilon_2 \succ \varepsilon_4 \succ \varepsilon_3 \succ \varepsilon_5 \succ \varepsilon_1$, and thus the most desirable low-carbon logistics park site is ε_2.

If the five possible low-carbon logistics park sites $\varepsilon_\alpha\ (\alpha = 1, 2, 3, 4, 5)$ are to be evaluated using the INNs, as listed in the following matrix.

$$\tilde{V} = \begin{bmatrix} ([0.5, 0.6], [0.8, 0.9], [0.1, 0.2]) & ([0.6, 0.7], [0.3, 0.4], [0.3, 0.4]) \\ ([0.7, 0.9], [0.2, 0.3], [0.1, 0.2]) & ([0.7, 0.8], [0.1, 0.2], [0.2, 0.3]) \\ ([0.6, 0.7], [0.7, 0.8], [0.2, 0.3]) & ([0.5, 0.6], [0.7, 0.8], [0.3, 0.4]) \\ ([0.8, 0.9], [0.1, 0.2], [0.3, 0.4]) & ([0.6, 0.7], [0.3, 0.4], [0.4, 0.5]) \\ ([0.6, 0.7], [0.4, 0.5], [0.4, 0.5]) & ([0.4, 0.5], [0.8, 0.9], [0.1, 0.2]) \end{bmatrix}$$
$$\begin{bmatrix} ([0.3, 0.4], [0.6, 0.7], [0.1, 0.2]) & ([0.5, 0.6], [0.7, 0.8], [0.1, 0.2]) \\ ([0.7, 0.9], [0.2, 0.3], [0.4, 0.5]) & ([0.8, 0.9], [0.2, 0.3], [0.1, 0.2]) \\ ([0.5, 0.6], [0.3, 0.4], [0.1, 0.2]) & ([0.6, 0.7], [0.3, 0.4], [0.2, 0.3]) \\ ([0.3, 0.4], [0.4, 0.5], [0.2, 0.3]) & ([0.5, 0.6], [0.6, 0.7], [0.1, 0.2]) \\ ([0.7, 0.8], [0.6, 0.7], [0.1, 0.2]) & ([0.5, 0.6], [0.8, 0.9], [0.2, 0.3]) \end{bmatrix}$$

DM's subjective preference value on alternative is:

$$\tilde{v}_1 = ([0.6, 0.7], [0.5, 0.6], [0.2, 0.3]), \tilde{v}_2 = ([0.7, 0.8], [0.2, 0.3], [0.1, 0.2])$$
$$\tilde{v}_3 = ([0.3, 0.4], [0.4, 0.5], [0.3, 0.4]), \tilde{v}_4 = ([0.9, 1.0], [0.3, 0.4], [0.2, 0.3])$$
$$\tilde{v}_5 = ([0.5, 0.6], [0.6, 0.7], [0.4, 0.5])$$

Case 1: The attribute weights are partially known,

$$\gamma = \{0.18 \le \gamma_1 \le 0.23, 0.20 \le \lambda_2 \le 0.24, 0.25 \le \gamma_3 \le 0.30,$$
$$0.25 \le \gamma_4 \le 0.33, \gamma_\beta \ge 0, \beta = 1, 2, 3, 4, \sum_{\beta=1}^{4} \gamma_\beta = 1\}$$

Step 1. Establish the ingle-objective programming model:

$$\min K(\gamma) = 0.6500\gamma_1 + 0.7000\gamma_2 + 0.8500\gamma_3 + 0.7333\gamma_4$$

Solving this model, we get the weight of attributes:

$$\gamma = \left(0.2300 \;\; 0.2400 \;\; 0.2500 \;\; 0.2800\right)^T$$

Step 2. Utilize the $\gamma = \left(\gamma_1, \gamma_2, \cdots, \gamma_\varphi\right)$ and Eq. (17), we obtain the $\tilde{v}_\alpha$ of the low-carbon logistics park sites $\varepsilon_\alpha \left(\alpha = 1, 2, \cdots, \phi\right)$.

$$\begin{aligned}
\tilde{v}_1 &= \left(\left[0.4845, 0.5168\right], \left[0.5667, 0.6731\right], \left[0.1302, 0.2362\right]\right) \\
\tilde{v}_2 &= \left(\left[0.7322, 0.8556\right], \left[0.1693, 0.2722\right], \left[0.1670, 0.2772\right]\right) \\
\tilde{v}_3 &= \left(\left[0.5538, 0.6323\right], \left[0.4468, 0.5540\right], \left[0.1854, 0.2905\right]\right) \\
\tilde{v}_4 &= \left(\left[0.5824, 0.6960\right], \left[0.3040, 0.4218\right], \left[0.2135, 0.3234\right]\right) \\
\tilde{v}_5 &= \left(\left[0.5633, 0.4059\right], \left[0.6348, 0.7383\right], \left[0.1670, 0.2766\right]\right)
\end{aligned}$$

Step 3. Calculate the scores $\psi\left(\tilde{\gamma}_\alpha\right)$ of $\varepsilon_\alpha \left(\alpha = 1, 2, \cdots, \phi\right)$

$$\begin{aligned}
&\psi\left(\tilde{v}_1\right) = 0.5658, \psi\left(\tilde{v}_2\right) = 0.7837, \psi\left(\tilde{v}_3\right) = 0.6183 \\
&\psi\left(\tilde{v}_4\right) = 0.6693, \psi\left(\tilde{v}_5\right) = 0.5254
\end{aligned}$$

Step 4. Rank all the low-carbon logistics park sites $\varepsilon_\alpha \left(\alpha = 1, 2, 3, 4, 5\right)$ through scores $\psi\left(\tilde{v}_\alpha\right)$ $\left(\alpha = 1, 2, \cdots, 5\right)$ of $\tilde{v}_\alpha \left(\alpha = 1, 2, \cdots, \phi\right)$: $\varepsilon_2 \succ \varepsilon_4 \succ \varepsilon_3 \succ \varepsilon_1 \succ \varepsilon_5$, and most desirable alternative is ε_2.

Case 2: If attribute weights are completely unknown, we utilize an alternative approach to obtain the best low-carbon logistics park sites.

Step 1. Utilize the Eq. (22) to get the weight of attributes:

$$\gamma = \left(0.2216 \;\; 0.2386 \;\; 0.2898 \;\; 0.2500\right)^T$$

Step 2. Utilize the $\gamma = \left(\gamma_1, \gamma_2, \cdots, \gamma_\varphi\right)$ and Eq. (17), we obtain the $\tilde{v}_\alpha$ of the low-carbon logistics park site $\varepsilon_\alpha \left(\alpha = 1, 2, \cdots, \phi\right)$.

$$\begin{aligned}
\tilde{v}_1 &= \left(\left[0.4774, 0.5169\right], \left[0.5633, 0.6696\right], \left[0.1300, 0.2360\right]\right) \\
\tilde{v}_2 &= \left(\left[0.7289, 0.8546\right], \left[0.1695, 0.2723\right], \left[0.1763, 0.2873\right]\right) \\
\tilde{v}_3 &= \left(\left[0.5499, 0.6328\right], \left[0.4431, 0.5503\right], \left[0.1802, 0.2857\right]\right) \\
\tilde{v}_4 &= \left(\left[0.5734, 0.6960\right], \left[0.3040, 0.4209\right], \left[0.2171, 0.3264\right]\right) \\
\tilde{v}_5 &= \left(\left[0.5714, 0.4078\right], \left[0.6312, 0.7346\right], \left[0.1617, 0.2712\right]\right)
\end{aligned}$$

Step 3. Calculate the scores of $\tilde{v}_\alpha \left(\alpha = 1, 2, \cdots, \phi\right)$.

$$\psi\left(\tilde{v}_1\right) = 0.5659, \psi\left(\tilde{v}_2\right) = 0.7797, \psi\left(\tilde{v}_3\right) = 0.6206,$$
$$\psi\left(\tilde{v}_4\right) = 0.6668, \psi\left(\tilde{v}_5\right) = 0.5301$$

Step 4. Rank all the low-carbon logistics park sites $\varepsilon_\alpha \left(\alpha = 1, 2, 3, 4, 5\right)$ through $\psi\left(\tilde{v}_\alpha\right)$: $\varepsilon_2 \succ \varepsilon_4 \succ \varepsilon_3 \succ \varepsilon_1 \succ \varepsilon_5$, and thus the most optimal low-carbon logistics park site is ε_2.

6. Conclusion

Under the background of increasingly standardized logistics market and increasingly fierce market competition, there is an increasing demand for establishing and improving logistics functions and information-based logistics centers. In order to respond to the new needs of economic and social development and advocate the concept of green, low-carbon and sustainable development, low-carbon logistics is the only way for the development of the logistics industry. The planning and construction of logistics parks are considered to be an important part of promoting the development of modern logistics. In the planning process of the logistics park, the layout and location function are the important basic parts that affect the overall development of the logistics park. Choosing a reasonable location is particularly important for building a logistics center. One of the most important parts of logistics park planning is the quantitative optimization of the logistics park location problem. In recent years, the location theory has developed rapidly, and there are many types of locations. The rapid development of the location theory of logistics parks is attributed to the informatization of today's science and technology, which provides a powerful tool for feasibility analysis and rational decision-making. The logistics park location problem is also regarded as a MADM problem. In this manuscript, we studied the SVN-MADM problem with alternative preference information. In the fuzzy background, the weight information of indicators is often uncertain, and based on this, the minimum deviation method is used to determine the weight of indicators. On the other hand, in the process of MADM, in order to obtain comprehensive evaluation information, The SVNWA operator is used to aggregate all decision information. Calculate the value of the scoring function and the accuracy function and rank the alternatives. On the basis of guaranteeing the validity, the calculation steps are relatively simple, thus realizing the operability. Furthermore, the above models and methods are extended to INNs. Finally, illustrative examples for low-carbon logistics park site selection demonstrates the extension of the model from theory to practical application. The constructed models and methods can be applied to other MADM problems, such as investment risk assessment, selection of commodity suppliers, selection of factory locations, etc. In the future research, we shall continue to focus on the detailed research of decision-making methods and aggregation operators by fusing TODIM method[65-67], QUALIFLEX method [68-71], ARAS method [72-75], WASPAS method [76-79], Maclaurin symmetric mean (MSM) [80], Muirhead mean (MM) [81-84] and power average (PA) [85, 86] operators to Neutrosophic numbers and propose some new MAGDM methods.

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Towards Intelligent Road Traffic Management Based on Neutrosophic Logic: A Brief Review

Asma Ait Ouallane [1,*], Said Broumi[1], Endalkachew Teshome Ayele[2], Assia Bakali[1,3], Ayoub Bahnasse[1,4] and Mohamed Talea[1]

[1] TI Lab, Faculty of Science Ben M'sik, Hassan II University of Casablanca, BP 7955 Casablanca, Morocco; asmaaaitouallane@gmail.com, broumisaid78@gmail.com, taleamohamed@yahoo.fr
[2] Department of Mathematics, Arbaminch University, Arbaminch, Ethiopia; endalkachewteshome83@yahoo.com
[3] Royal Naval School, Boulevard Sour Jdid, B.P 16303 Casablanca, Morocco; assiabakali@yahoo.fr
[4] ENSAM Casablanca, University Hassan II of Casablanca, Morocco; a.bahnasse@gmail.com

* Correspondence: asmaaaitouallane@gmail.com

Abstract: Road traffic management has been a serious concern in the transportation sector for many years now. The explosion of the number of cars along with the inability of creating new high-capacity road infrastructures in big cities makes mitigating this danger a problem for the scientific research community. Traffic congestion contributes to increased pollution, economic loss, and a general deterioration in the quality of life. As a result, researchers are being asked to cope with the complexity of establishing effective and smooth traffic flow. However, as in traffic congestion control, real-world decision-making problems are always fraught with uncertainty and indeterminacy. The neutrosophic environment has been applied successfully to deal with these problems and recently, researchers tried to use various neutrosophic approaches to tackle the traffic congestion problem. This paper provides a brief overview of the most recently used neutrosophic techniques to handle traffic congestion and transportation problems in general. The aim of the investigation is to summarize the available neutrosophic traffic flow problems and their progress to enable future researchers to differentiate the major problems to be manipulated and identify conditions to be optimized.

Keywords: Road traffic control; Intelligent Traffic Management System; Neutrosophic environment; Neutrosophic logic; Neutrosophic approaches.

1. Introduction

Transport researchers have long worked to improve traffic management on urban roads. Congestion is a critical issue affecting negatively road users and traffic controllers. Despite the important attempts and research that have been made to minimize traffic congestion, this serious problem continues to worsen [1]. The direct reason for this is the slow development of transportation systems and road capacity as well as the explosive growth of urban and rural population rates, which causes an increase in vehicle demand and hence the vehicles' number on the roads. Thus, traffic congestion is a serious matter that should be urgently addressed in order to offer a safe and healthy environment for people [1].

In order to manage the traffic flow, the road traffic management system takes various real-time decisions [3]. As existing practical situation traffic flow parameters involve vagueness due to several uncontrolled factors so that the developed models unable to tackle such conditions [4]. For instance, the number of vehicles in a specific lane in real-time is always unknown precisely. Furthermore, the non-recurrent traffic congestion sources, which are special incidents that happen suddenly as shown in Figure 1, cannot be accurately managed. This gives rise for the use of fuzzy logic controllers in traffic management endeavor. The concept of fuzzy set, which is initialized by Zadeh in 1965 [5], has been widely applied in problems that include uncertainty and vagueness since it imitates human perception and thinking based on linguistic information.

Neutrosophic concept was initialized by Florentin Smarandache in 1995 as an extension of the fuzzy logic and its derivatives. It goes beyond the fuzzy set and fuzzy logic by expressing the false membership information and beyond the intuitionistic fuzzy set and intuitionistic fuzzy logic by handling the indeterminacy of information. Neutrosophic logic is a logic in which each proposition is estimated to have a degree of truth (T), a degree of indeterminacy (neutrality) (I), and a degree of falsity (F). It can then handle the uncertainty and impreciseness related to the road traffic flow that the fuzzy logic may fail to properly address.

The forthcoming part of the study is arranged as follows. In Section 2, an introduction of the basic concepts that we focus on in this paper is provided. In Section 3, some of the available methods in the scientific literature that tackle road traffic problems based on the Neutrosophic sets are presented. In Section 4, a comparative analysis is provided for the different presented methods. Finally, section 5, introduces the main challenges and future perspectives and concludes our brief review.

Figure 1: Non-recurrent traffic congestion sources.

2. Basic concepts

This section introduces some of the fundamental principles covered in this paper as shown in Figure 2.

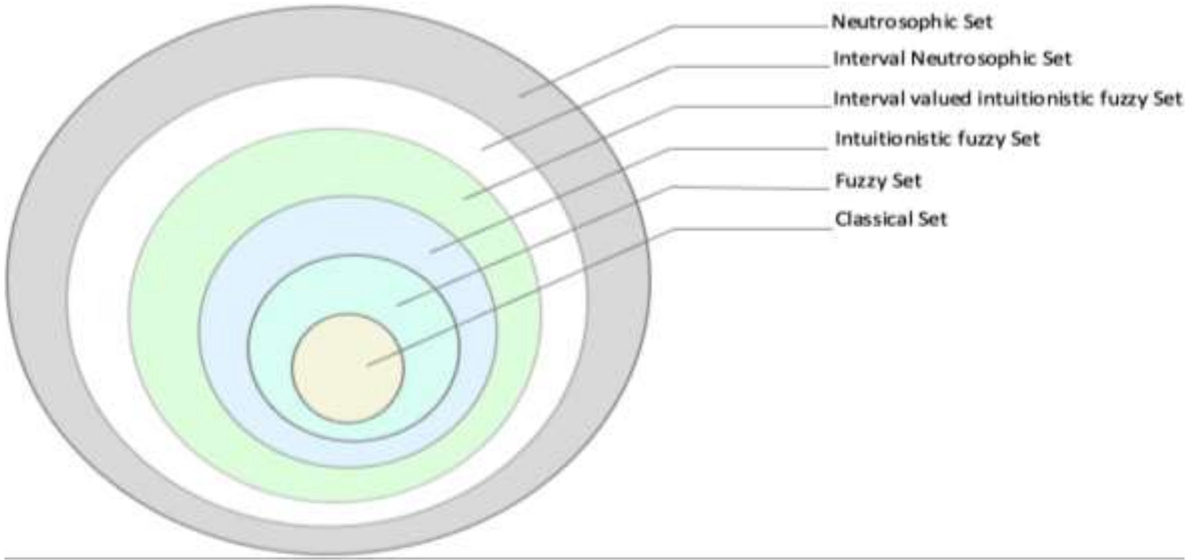

Figure 2: Relationship between classical, fuzzy, intuitionistic fuzzy, interval-valued intuitionistic fuzzy, interval neutrosophic sets, and neutrosophic sets.

2.1 Fuzzy set (FS)

When we encounter vagueness in our daily life activities fuzzy theory is the right tool to overcome it. It is often applicable transportation engineering and planning. In classical set theory if an element belongs to a set its membership degree is simply 1 and if it does not belong to a set its membership degree is 0. In contrast, in the theory of vagueness the degree to which the element belongs to a set is not clearly known, instead we use values in the interval [0,1]. This type of set is called fuzzy sets. So, a fuzzy set is identified by its membership degree alone.

2.2 Intuitionistic fuzzy set (IFS)

Let X be the universe of discourse. A set $A \in X$ that can be written in the form $A = \{\langle X, \mu_A(x), \upsilon_A(x), \rangle;\ x \in X$ is called an intuitionistic fuzzy set where, $\mu_A(x)$, $\upsilon_A(x)$ are degree of acceptance and degree of rejection of the element x in A respectively are each subsets of [0,1] such that, $0 \leq \mu_A(x) + \upsilon_A(x) \leq 1$.In addition, for A in X , $\pi_A(x) = 1 - \mu_A(x) - \upsilon_A(x)$ is called the intuitionistic fuzzy set index or the degree of indeterminacy of $x \in X$ and for every $x \in X$, $0 \leq \pi_A \leq 1$.

2.3 Neutrosophic logic

In Neutrosophic logic each statement has a truth degree (T), an indeterminacy degree (neutrality) (I), and a falsity degree(F), where $T, I, F \subseteq [0, 1]$ and $0 \leq T + I + F \leq 3$. The degrees T, I, F are nondependent to each other.

2.4 Single valued neutrosophic set (SVNS)

If in a set A every member of A has a degree of belongingness ($\mu_A(x)$), a degree of indeterminacy ($\upsilon_A(x)$) and a degree of non-belongingness ($\omega_A(x)$), with $\mu_A(x), \upsilon_A(x), \omega_A(x) \in [0,1]$,then the set is a single valued neutrosophic set and $x \equiv x(\mu_A(x), \upsilon_A(x), \omega_A(x))$ is a single valued neutrosophic element of A such that we have the following relations between the three degrees

$$0 \leq \mu_A(x), \upsilon_A(x), \omega_A(x) \leq 1 \text{ and}$$

$$0 \leq \mu_A (x)+\upsilon_A (x) + \omega_A (x) \leq 3 \ \forall x \in X$$

2.5 Interval valued neutrosophic set

If in a set A every member of A has a degree of belongingness $\mu_A (x)$, a degree of indeterminacy $\upsilon_A (x)$ and a degree of non-belongingness $\omega_A (x)$, with $\mu_A (x), \upsilon_A (x), \omega_A (x)$ are all elements of the closed interval $[0,1]$, and $\mu_A (x) = [\underline{\mu}_A(x), \overline{\mu}_A(x)]$, $\upsilon_A (x) = [\underline{\upsilon}_A(x), \overline{\upsilon}_A(x)]$, $\omega_A (x) = [\underline{\omega}_A(x), \overline{\omega}_A(x)]$ are respectively upper and lower degree of belongingness , upper and lower degree of indeterminacy and upper and lower degree non-belongingness , then A is an interval valued neutrosophic set.

2.1 Soft Set

Neutrosophic sets may be combined with other types of sets to get another hybrid structure which can be applicable in transport engineering. One of such type of set is a soft set which is initiated for the first time by Molodtsov in 1999 and defined as follows.
Let X be a universe of discourse and P be a set of parameters. Let $P(X)$ denote the power set of X and $A \subseteq P$. A combination (F, A) is called a soft set over X, where F is a mapping given by F: A→P (X). In other words, a soft set over U is a parameterized family of subsets of the universe X. For $e \in A$, F (e) may be considered as the set of e-approximate elements of the soft set (F, A). Clearly, a soft set is not a set in a classical sense.

3. The Application of Neutrosophic Theory in Intelligent Traffic Management Systems

This section outlines some of the suggested Neutrosophic logic-based solutions for managing traffic flow and transportation problems in general. Table 1 summarizes these approaches.

Jun Ye introduced in [6] the neutrosophic linear equations, the neutrosophic matrix and the neutrosophic matrix operations relying on the Neutrosophic Numbers concept. Then, he chose the traffic flow case study to apply the neutrosophic linear equations system in a real scenario and demonstrate its efficiency in handling the indeterminacy problem of a real environment.

For traffic management, El Bendadi et al. suggested in [7] two clustering strategies namely, Credal C-Means clustering (CCM) and Neutrosophic C-Means clustering (NCM). When overlapping items are found, both proposed techniques have a comparable propensity to construct a novel cluster that determines the imprecision object. The indeterminacy cluster is interpreted differently by each approach. The CCM algorithm forms a number of meta-cluster that is proportionate to the number of singleton clusters, while the NCM approach represents all indeterminate items with a single indeterminacy cluster.

In [8], Muhammad Akram created a traffic-monitoring road network model based on the notion of bipolar neutrosophic planar graphs. The suggested approach may be used to compute and track the yearly accident proportion. The overall number of accidents can be reduced by monitoring and installing additional security measures.

Nagarajan et al. studied in [9] a triangular interval type-2 Schweizer, Sklar weighted arithmetic (TIT2SSWA) operator, a triangular interval type-2 Schweizer and Sklar weighted geometric (TIT2SSWG) operator based on Schweizer and Sklar triangular norms. Afterward, the validity of these operators was examined based on a numerical example, and then an interval neutrosophic Schweizer and Sklar weighted arithmetic (INSSWA) and interval neutrosophic Schweizer and Sklar weighted geometric (INSSWG) operators were proposed in order to extend these operators to an interval neutrosophic environment. Moreover, a new traffic flow approach is introduced based on the presented operators as well as an improved score function. The score function was used to analyze the traffic flow based on TIT2SSWA and INSSWA operators as well as TIT2SSWG and INSSWG operators. Both used methods identified the same intersection as the more congested one.
In another paper [10], Nagarajan et al. used the Gauss Jordan method to examine the flow of traffic in a neutrosophic environment and under various indeterminacy ranges.
In another paper, Nagarajan et al. proposed [11] Dombi Single valued Neutrosophic Graph and Dombi Interval-valued Neutrosophic Graph. Furthermore, the Cartesian product and composition of the suggested graphs were extracted and then verified with the numerical example. The Neutrosophic Controllers' importance and their use in managing traffic are theoretically emphasized. It has been pointed out that the triangular norms T Norm and T-Conorm can be utilized rather than minimum and maximum operations in control systems like traffic management systems. Finally, the pros and cons of some fuzzy logic methods and neutrosophic logic methods have been discussed.
Finally, In another paper, Nagarajan et al. [12] examined the traffic flow control in a neutrosophic environment under diverse ranges of indeterminacy and then proposed a road traffic study based on Crisp, Fuzzy, and Neutrosophic.

Phillip Smith introduced in [13] a Multiple Attribute Decision-Making (MADM) method for picking out sustainable public transportation systems under uncertainty, which means using incomplete information involving single-valued neutrosophic sets (SVNSs) which means in turn a generalization of a classical set, a fuzzy set, and an intuitionistic fuzzy set. In the context of the Public Transit Sustainable Mobility Analysis Tool (PTSMAT) SVNSs and SVNS connectives are demonstrated and used with a composite (multiple attributes) sustainability index. The results of the presented case study of PTSMAT for the UBC Corridor study in Vancouver, Canada are identical to those of the original study despite the fact that neutrosophic formalism opens a wide range of possibilities for recognition of uncertainty in sustainability assessment. The results of the presented case study of PTSMAT for the UBC Corridor study in Vancouver, Canada are similar to those obtained in the original study despite the fact that to recognize the uncertainty in sustainability assessment, neutrosophic formalism opens a wide range of possibilities.

In [14], R. Sujatha et al. used Fuzzy Cognitive Map and Induced Fuzzy Cognitive Map to examine road traffic flow patterns at a congested intersection in Chennai, India's biggest city.

A new emergency transport model that simulates emergency transport from the logistics center to each incident area as well as between incident locations was created by Lin Lu and Xiaochun Luo in [15]. The emergency transshipment problem was transformed into a multiattribute decision-making problem using the SVNS concept in indeterminate and uncertain circumstances. The suggested technique was applied in an emergency operation scenario to rank and select effective transportation routes.

In developing countries, to control traffic flow at signaled crossroads, The fixed-time traffic light control method is used. However, this method does not allow congested intersections to identify their level of congestion and therefore allows vehicles to cross the intersection. To deal with this challenge, road managers must set their opinions and create an intelligent automated decision-making system to replace them. The manager's decision process might be analyzed utilizing the approach of Interval-Valued Neutrosophic Soft Set (IVNSS) theory to take advantage of fuzziness in traffic flow and determine efficient timings and optimal phase change. Enalkachew Teshome Ayele et al. [16] suggested an IVNSS traffic management system that can ameliorate traffic congestion control. It evaluates the different phases and timings of the traffic light based on the real-time traffic density at the intersection rather than a fixed phase and duration.

Under neutrosophic statistics, Muhammad Aslam created a control chart for neutrosophic exponentially weighted moving average (NEWMA) using recurrent sampling in [17]. To track traffic collisions on the highway (RTC), the author employed a NEWMA chart. The proposed NEWMA chart goes beyond the previously proposed control charts for tracking the RTC, according to a simulated study and a real-world example. According to the comparative study, the presented NEWMA chart might be utilized to successfully regulate RTC. The new chart will allow changes in accidents and injuries to be detected faster than previous charts.

In [18], Rayees et al. identified four different kinds of Plithogenic hypersoft sets (PHSS) relying on the application-specific features number used, the type of alternatives, or the degree of attribute value appurtenance. These four PHSS categories cover the fuzzy and neutrosophic situations that may have neutrosophic applications in symmetry. They then proposed a new multi-criteria decision-making (MCDM) technique based on PHSS (TOPSIS) as an extension of the method for order preference by similarity to an ideal solution. Uncertainty complicates a variety of real-world MCDM scenarios, necessitating the division of each selection criterion or attribute into attribute values and the independent evaluation of all options against each attribute value. The suggested PHSS-based TOPSIS may be utilized to tackle real MCDM challenges that are precisely defined by the PHSS notion depending on the provided criteria. The proposed PHSS-based TOPSIS resolves a parking space Choosing issue in a fuzzy neutrosophic environment in a real-world application, and it is verified by comparing it to fuzzy TOPSIS.

In [19], Simic et al. expanded the CRITIC and MABAC approaches to type-2 neutrosophic sets for the selection of public transportation pricing systems, and Pamucar et al. proposed in [20] a hybrid model that comprised fuzzy FUCOM and neutrosophic fuzzy MARCOS for assessing alternative fuel vehicles for sustainable road transportation in the United States.

For controlling road accidents and injuries when the smoothing constant is uncertain Muhammad Aslam and Mohammed Albassam [21] proposed an S2N NEWMA control chart to track road accidents and injuries by employing repeated sampling. The tables and control chart figures are generated using the neutrosophic Monte Carlo simulation. This chart identifies changes in accidents and injuries quicker than prior charts, lowering and pinpointing the causes of traffic accidents and injuries.

In [22], M. Abdel-Basset et al. stated that autonomous vehicles play an important role in the intelligent transportation system; nonetheless, these vehicles pose a number of risks. As a result, a novel hybrid model is proposed for recognizing these hazards. This process contains uncertainty and foggy data. The neutrosophic hypothesis is used to deal with uncertainty. The neutrosophic theory provides three membership functions: true, indeterminacy, and false (T, I, F). In this study, the concept of MCDM is combined with neutrosophic theory since autonomous vehicles have several contradictory criteria. First, the Analytic Hierarchy Process defines the weights of criteria (AHP). Second, to assess the dangers of autonomous cars, approaches such as Multi-Attributive Border Approximation Area Comparison (MABAC) and Preference Ranking Organization Method for Enrichment Evaluations II are utilized (PROMETHEE II). In the case study, ten distinct choices were used. An understanding and a sensitivity analysis of this process in an uncertain environment are given to demonstrate the robustness of the suggested model.

In [23], F. Xiao et al. introduced a method that ameliorates the multi-valued neutrosophic MULTIMOORA method relying on prospect theory. The suggested approach is utilized to select a suitable subway building scheme. Firstly, Multi-valued neutrosophic sets (MVNNs) were utilized to offer evaluations of subway building. Secondly, the IGMVNWHM operator is added, which takes into account the inputs interactions. Thirdly, a new distance measure between two MVNNs is determined. The fourth approach is an IMVN-PT-MULTIMOORA technique.

Nasrullah Khan et al. presented in this article [24] neutrosophic multiple dependent state sampling control chart for the neutrosophic EWMA statistic. The control chart coefficients were set by the neutrosophic statistical interval method for different process settings. The neutrosophic average run lengths and the neutrosophic standard deviation have been estimated by the Monte Carlo simulation to verify the efficiency of the suggested chart. A comparison of this chart with existing charts has been done. As result, this chart is comparatively robust in monitoring the incomplete, and unclear quality characteristics. However, the production process should adhere to the normal distribution, which represents a limitation of this study. The presented chart could be used in the chemical, packing, and electronic industries.

In [25], Fayed et al. introduced a robust occupancy detection system that relies on a novel fusion approach for merging heterogeneous sensor data that significantly enhances occupancy detection efficiency. The suggested method is suitable for use in traffic management.

Table 1. An overview of the most Neutrosophic approaches that deal with the problem of road traffic congestion

Year	Ref	Scope	Contributions and Methods used	Topics
2017	Jun Ye [6]	Traffic Flow	A traffic flow problem application example is	▪ Neutrosophic Numbers and Their Operational Laws.

			offered to demonstrate the application and efficacy of employing the system of neutrosophic linear equations to solve the indeterminate traffic flow problem.	▪ Neutrosophic Linear Equations and Neutrosophic Matrices. ▪ A Neutrosophic Linear Equations System Solving. ▪ A Traffic Flow Problem Application.
2018	El Bendadi et al. [7]	Road Safety	The Credal C-means (CCM) and Neutrosophic C-means (NCM) algorithms describe the credal clustering and neutrosophic clustering respectively. To demonstrate their behavior and efficacy, real-world road safety data were tested and their results compared.	▪ CCM working Principle. ▪ NCM working Principle. ▪ Comparison of the CCM and NCM algorithms for different datasets based on different criteria namely, error rate, imprecision rate, intra class inertia.
	Muhammad Akram [8]	Traffic Monitoring	Some applications of bipolar neutrosophic graphs were described.	▪ Bipolar Neutrosophic Graphs. ▪ Applications to MCDM. ▪ Bipolar Neutrosophic Planar Graphs. ▪ Applications of Neutrosophic Planar Graphs. ▪ Bipolar Neutrosophic Line Graphs. ▪ Application of Bipolar Neutrosophic Line Graphs.
2019	Nagarajan et al. [9]	Traffic Flow	To control traffic flow that has been analyzed by determining the intersection with more vehicles, an improved score function for interval neutrosophic numbers (INNs) is proposed.	▪ Basic concepts of a traffic control system, fuzzy logic' role, output methods from fuzzy linguistic terms and structure of the fuzzy control system. ▪ Operational laws. ▪ Neutrosophic perspective. ▪ Traffic flow using proposed operators.
	Nagarajan et al.[10]	Traffic Flow Control	MATLAB is used to investigate traffic flow control in a neutrosophic environment using Gauss Jordan method.	▪ Bipolar Neutrosophic Line Graphs. ▪ Basic concept: Single Valued Neutrosophic Set, Gauss Jordan Method.

				▪ Description of the proposed methodology
	Nagarajan et al.[11]	Traffic Control	Dombi Single valued Neutrosophic Graph and Dombi Interval valued Neutrosophic Graph have been suggested. As well as the theoretical significance of Neutrosophic Controllers and their application in traffic control management.	▪ Basic Concepts: Graph, Fuzzy Graph, Dombi Fuzzy Graph, Single Valued Neutrosophic Graph, Interval Valued Neutrosophic Graph, Triangular Norms, Dombi Triangular Norms, Hamacher Triangular Norms, Dombi and Hamacher Triangular Norms Special Cases, Standard Products of graphs, Neutrosophic Controllers. ▪ Proposed Dombi Interval Valued Neutrosophic Graph. ▪ Traffic Control Comparison based on divers types of set and Graph theory.
	Nagarajan et al.[12]	Traffic Flow	The Jordan approach is used in this study to evaluate traffic flow control in a neutrosophic environment.	▪ Neutrosophic number. ▪ Application: at analyzing the traffic flow
	Phillip Smith [13]	Transportation Sustainability Assessment	A multi-attribute decision-making method for selecting sustainable public transportation systems in the uncertainty, represented by SVNSs and their connectives.	▪ Neutrosophic sets. ▪ Single-valued neutrosophic averages. ▪ Score functions ▪ Cross-entropy. ▪ Application to sustainable transport.
	R Sujatha et al. [14]	Crowded junction in Chennai	Some traffic congestion causes are unknown and indeterminate, Thus, Neutrosophic Cognitive Maps is employed in this paper to identify a solution.	▪ Fundamental concepts of Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps. ▪ Description of the traffic congestion problem. ▪ Comparison of expert' opinion.
2020	Lin Lu and Xiaochun Luo [15]	Emergency Transportation Problem	In confusing and uncertain environments, the SVNS is used to turn the emergency transshipment problem into a	▪ Methods: The Basic Concept of Single-Valued Neutrosophic Set.

			multiattribute decision-making problem.	▪ A new emergency transport model is presented.
	Enalkachew Teshome Ayele [16]	Traffic light Control	To manage both phase change and green time extension / termination based on the traffic circumstances at any time, an algorithm is proposed in this paper.	▪ Preliminary concepts: Soft Set, Single valued neutrosophic set, Interval Valued Neutrosophic Set. ▪ The proposed two stage IVNSS traffic light control model and its verification.
	Muhammad Aslam [17]	Road traffic crashes monitoring	The suggested Neutrosophic Exponentially Weighted Moving Average (NEWMA) chart is used to monitor traffic accidents.	▪ Neutrosophic EWMA chart using repetitive sampling. ▪ Comparative study based on Road Traffic Crashes simulation data. ▪ Using real-time data to monitor road traffic accidents.
	Muhammad Rayees Ahmad [18]	Solve a parking problem	In a real-world application, the suggested Plithogenic fuzzy hypersoft set (PHSS)-based TOPSIS solves a parking place selection problem in a fuzzy neutrosophic environment.	▪ The Four Classifications of PHSS. ▪ The Proposed PHSS-Based TOPSIS Applied to a Parking Issue.
2021	Simic et al. [19]	Public transportation pricing system selection	The public transport services pricing is a complicated problem that authorities must handle since numerous elements must be observed when deciding on a pricing scheme. A two-stage hybrid MCDM model based on type-2 neutrosophic numbers (T2NNs) is presented to offer researchers and practitioners a simple and flexible decision-making tool.	–
	Pamucar et al. [20]	Assessment of alternative fuel vehicles for sustainable	The goal of this research is to create a multi-criteria decision-making (MCDM) framework that combine fuzzy FUCOM and	▪ AFV assessment methodology. ▪ Case study in the New Jersey.

	road transportation	neutrosophic fuzzy MARCOS for prioritizing various Alternative Fuel Vehicles (AFVs) for sustainable transportation.	
Muhammad Aslam and Mohammed Albassam [21]	Reducing and identifying the causes of traffic accidents and injuries	The use of a neutrosophic statistical approach for road safety.	▪ The Proposed S2N −NEW M A Chart. ▪ The Proposed Control Chart
M. Abdel-Basset et al. [22]	Risk Management in Autonomous Vehicles	To represent and handle uncertainty and incomplete risk information consistently and reliably, the proposed model combines the single-valued neutrosophic sets, the AHP, MABAC, and PROMETHEE II methodologies.	▪ Neutrosophic linguistic information. ▪ Suggested hybrid MCDM approach.
Fei Xiao et al. [23]	Traffic flow and its application in a multi-valued way	This paper improves the multi-valued neutrosophic MULTIMOORA method.	▪ Preliminaries: Multi-valued neutrosophic sets (MVNNs), Heronian Mean (HM) operators, The MULTIMOORA method, Prospect theory. ▪ IGMVNWHM operator, Distance measure between two MVNNs and IMVN-PT-MULTIMOORA method. ▪ Solution framework for MVN-MCGDM problem.
Nasrullah Khan [24]	Tracking Traffic Accidents and Injuries	The Use of Neutrosophic Exponentially Weighted Moving Average Statistics in Tracking Road Accidents and Injuries	▪ Methodology of the Proposed Chart. ▪ The Proposed NEWMA X-Bar Control Chart Based on Multiple Dependent State Sampling. ▪ The Proposed Neutrosophic Control Chart Simulation Study.

2022	Noha S. Fayed [25]	Improving occupancy Detection system.	The suggested approach addresses sensor data uncertainty using Neutrosophy. It also enhances reliability by combining data from various sensors. Training and testing time is decreased since it only employs one feature created by fusing input from many sensors.	▪ The efficient occupancy detection system and its evaluation.

4. Comparative Analysis

Table 2 below gives a comparative analysis of the different neutrosophic methods used recently in the literature for traffic management and transportation system improvement in general, in order to understand each method's key role, advantages, and limitations.

Table 2. Comparison of different neutrosophic methods used for traffic management.

Types	Advantages	Limitations
Neutrosophic Sets	▪ In neutrosophic theory, we use neutrosophic numbers -> $a + Ib$ where $a, b \in R$. ▪ Addresses uncertainty as well as uncertainty caused by unpredictable environmental disturbances ▪ The Neutrosophic set presents the degrees of membership, indeterminacy, and non-membership of the element $x \in$ S. For instance: μ(0.5,0.1,0.4) ∈ S means probability of 50% $'x'$ belong to the set S 10% $'x'$ is not in S and 40% is undecided. ▪ The operations are entirely different.	▪ Calculations errors can't be rounded up and down.
Interval Valued Neutrosophic Sets	▪ Adaptability and flexibility. ▪ handles more uncertainty and indeterminacy. ▪ Calculations errors can be rounded up and down. ▪ Can handle problems with one number or a group of numbers in the real unit interval.	▪ Can't handle criterion incomplete weight information.

Neutrosophic Graphs	▪ An optimized output is possible if the paths and the terminal points are uncertain.	▪ Can't deal with more uncertainty.
Interval Valued Neutrosophic Graphs	▪ Can address additional uncertainty discovered in terminal points (vertices) and paths (edges).	▪ Can't handle incomplete criterion weight information.
Dombi Neutrosophic Graphs	▪ Can handle indeterminacy.	▪ Can't handle uncertainty for interval values.
Dombi Interval Valued Neutrosophic Graphs	▪ Can handle uncertainty well for interval values.	▪ Can't handle incomplete criterion weight information.
Type 2 fuzzy and interval neutrosophic	▪ Based on a rule that fully accepts uncertainties. ▪ Adaptability.	▪ The membership functions are fuzzy thus computational complexity is high.
Single valued neutrosophic sets (SVNSs)	▪ Can handle uncertain and inconsistent information.	▪ Not flexible and practical than interval valued neutrosophic sets
Neutrosophic Cognitive Maps	▪ Provide the ability to treat the relation between two vertices as indeterminate.	▪ No comparative work has been done with respect to the existing models in relation to waiting time ▪ Applicability for other types of traffic junction is not clear
Neutrosophic Markov	▪ Can handle the occurred indeterminacy in a system. ▪ The neutrosophic Markov chain's equilibrium state demonstrates the ability of traffic states transitions accurately in order to predict the traffic.	▪ Applicable only for T-shaped traffic junction ▪ Applicability for other types of traffic junction is not clear

Interval valued neutrosophic soft sets	▪ Regarding the stability of traffic states, verification of ergodicity can be achieved in a minimum of steps. ▪ Applied parameterization tools in which others techniques lack ▪ The method is verified with numerical example	▪ The model is not validated. ▪ Other parameters like pedestrian movements and emission of pollutants are not considered

5. Conclusion and Future Challenges

For many years, road traffic control has been a major problem in the transportation field. Traffic jam adds to increased pollution and an overall decline in life quality. Real-world decision-making challenges, such as controlling traffic congestion, are always vague and indeterminate. Therefore, the neutrosophic environment has been effectively used to address these problems, and lately, researchers attempted to employ several neutrosophic techniques to address transportation problems.

In this paper, we have conducted a brief review that deal with the use of neutrosophic logic in the field of traffic control. The review concentrated on several methods for describing and optimizing traffic flow. The review looked at several traffic management approaches in a neutrosophic environment and analyzed the benefits and limitations of the offered models. Many research conducted comparisons with real data sets and demonstrated the benefits of using neutrosophic sets and neutrosophic logic.

According to the literature review, there are still unresolved concerns and issues that need to be addressed in future investigations. The issues include (i) controlling a large number of junctions at the same time to maintain uninterrupted traffic flow, especially during traffic jams, (ii) A comparative study between the developed models and the existing models should be made to test the efficiency of the developed model with respect to the average vehicle delay which is the major measure of effectiveness for the flow of traffic at traffic junction. (iii) The theory of neutrosophic sets is currently advancing quickly. However, there is a problem in determining membership, falsity and indeterminacy degrees in in traffic flow parameters. The nature of determining those degrees is extremely individual. The cause of these challenges might be the theory's parameterization tool's inadequacy. (v) No approach for analyzing the stability of neutrosophic controller systems has yet been created. (vi) The majority of neutrosophic logic-based results that deliver increased performance are simulation-based.

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Irresolute and its Contra Functions in Generalized Neutrosophic Topological Spaces

Santhi P [1], Yuvarani A [2] and Vijaya S [3]*

[1] PG & Research Department of Mathematics, The Standard Fireworks Rajaratnam College for Women, Madurai Kamaraj University, Sivakasi, Tamil Nadu, India, saayphd.11@gmail.com

[2] PG & Research Department of Mathematics, The American College, Madurai Kamaraj University, Madurai, Tamil Nadu, India, yuvamaths2003@gmail.com

[3] PG & Research Department of Mathematics, Thiagarajar College, Madurai Kamaraj University, Madurai, Tamil Nadu, India, viviphd.11@gmail.com

* Correspondence: viviphd.11@gmail.com

Abstract: The intention to study the idea of Generalized Topological Spaces by means of Neutrosophic sets leads to develop this article. In this write up we launch new ideas on λ_N-Topological Spaces. We study some of its characteristics and behaviours of λ_N-α-irresolute function, λ_N-semi-irresolute function and λ_N-pre-irresolute function. Also we discuss the above for contra λ_N-irresolute functions and derived some relations between them.

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Keywords: λ_N-α-irresolute function, λ_N-semi-irresolute function, λ_N-pre-irresolute function, contra λ_N-α-irresolute function, contra λ_N-semi-irresolute function, contra λ_N-pre-irresolute function.

1. Introduction

Zadeh [16] initiated fuzzy set theory in 1965 that deals with uncertainty in real life situations. Chang [2] designed fuzzy topology that gave a special note to the field of topology in 1968. Attanassov [1] in 1983, see the sights of intuitionistic fuzzy sets by considering both membership and non-membership of the elements. In 1997, Coker [4] worked on Intuitionistic fuzzy sets by extending the concepts of fuzziness and found a place for Intuitionistic fuzzy topological space.

Smarandache [5] to [7] & [14] introduced Neutrosophic set which is a generalization of fuzzy set and intuitionistic fuzzy set. This is a strong tool to discuss about the existence of incomplete, indeterminate and inconsistent information in the real life situation. Smarandache focused his observations en route for the degree of indeterminacy that directed into Neutrosophic Sets (NS). Soon after, Salama and Albowi [10] familiarized Neutrosophic Topological Spaces (NTS). Further, Salama, Smarandache and Valeri Kromov

[11] presented the continuous (Cts) functions in NTS. In [3], irresolute functions was introduced and analysed by Crossley and Hildebrand in Topological Spaces. Further, Vijaya [13] and Santhi [12] investigated the properties of λ-α-irresolute function and contra λ-α-irresolute function in Generalized Topological Spaces. In addition to that, properties of α-irresolute function and contra α-irresolute function in Nano Topological Spaces was look over by Yuvarani and et. al., [15]. By keeping all these works as a motivation, in 2020, Raksha Ben, Hari Siva Annam [8] & [9] contrived λ_N-Topological Space and deliberated its properties.

In this disquisition, we explore our perception of λ_N-α-irresolute function, λ_N-semi-irresolute function, λ_N-pre-irresolute function, contra λ_N-α-irresolute function, contra λ_N-semi-irresolute function, contra λ_N-pre-irresolute function and we have scrutinized about some of their basic properties. At every place the novel notions have been validated with apposite paradigms.

2. Prerequisites

2.1. Definition [10]

Let Ω be a non-empty fixed set. A NS, E = {⟨ ω, $M_E(\omega)$, $I_E(\omega)$, $N_E(\omega)$⟩ : ω ∈ Ω} where $M_E(\omega)$, $I_E(\omega)$ and $N_E(\omega)$ represents the degree of membership, indeterminacy and non-membership functions respectively of every element ω ∈ Ω.

2.2. Remark [10]

A NS, E can be recognized as a structured triple E = {⟨ ω, $M_E(\omega)$, $I_E(\omega)$, $N_E(\omega)$⟩ : ω ∈ Ω} in $]^-0, 1^+[$ on Ω.

2.3. Remark [10]

The NS, 0_N and 1_N in Ω is defined as

(P_1) 0_N = {⟨ ω, 0, 0, 1⟩ : ω ∈ Ω}

(P_2) 0_N = {⟨ ω, 0, 1, 1⟩ : ω ∈ Ω}

(P_3) 0_N = {⟨ ω, 0, 1, 0⟩ : ω ∈ Ω}

(P_4) 0_N = {⟨ ω, 0, 0, 0⟩ : ω ∈ Ω}

(P_5) 1_N = {⟨ ω, 1, 0, 0⟩ : ω ∈ Ω}

(P_6) 1_N = {⟨ ω, 1, 0, 1⟩ : ω ∈ Ω}

(P_7) 1_N = {⟨ ω, 1, 1, 0⟩ : ω ∈ Ω}

(P_8) 1_N = {⟨ ω, 1, 1, 1⟩ : ω ∈ Ω}

2.4. Definition [10]

If E = {⟨ $M_E(\omega)$, $I_E(\omega)$, $N_E(\omega)$⟩ }, then the complement of E on Ω is

(P_9) E′ = {⟨ ω, 1 − $M_E(\omega)$, 1 - $I_E(\omega)$ and 1 - $N_E(\omega)$⟩ : ω ∈ Ω}

(P_{10}) E′ = {⟨ ω, $N_E(\omega)$, $I_E(\omega)$ and $M_E(\omega)$⟩ : ω ∈ Ω}

(P_{11}) E′ = {⟨ ω, $N_E(\omega)$, 1 - $I_E(\omega)$ and $M_E(\omega)$⟩ : ω ∈ Ω}

2.5. Definition [10]

Let Ω be a non-empty set and let $E = \{\langle \omega, M_E(\omega), I_E(\omega), N_E(\omega)\rangle : \omega \in \Omega\}$ and $F = \{\langle \omega, M_F(\omega), I_F(\omega), N_F(\omega)\rangle : \omega \in \Omega\}$. Then

(i) $E \subseteq F \Rightarrow M_E(\omega) \leq M_F(\omega), I_E(\omega) \leq I_F(\omega), N_E(\omega) \geq N_F(\omega), \forall\ \omega \in \Omega$

(ii) $E \subseteq F \Rightarrow M_E(\omega) \leq M_F(\omega), I_E(\omega) \geq I_F(\omega), N_E(\omega) \geq N_F(\omega), \forall\ \omega \in \Omega$

2.6. Definition [10]

Let Ω be a non-empty set and $E = \{\langle \omega, M_E(\omega), I_E(\omega), N_E(\omega)\rangle : \omega \in \Omega\}$, $F = \{\langle \omega, M_F(\omega), I_F(\omega), N_F(\omega)\rangle : \omega \in \Omega\}$ are NSs. Then,

(P_{12}) $E \cap F = \langle \omega, M_E(\omega) \wedge M_F(\omega), I_E(\omega) \vee I_F(\omega), N_E(\omega) \vee N_F(\omega)\rangle$

(P_{13}) $E \cap F = \langle \omega, M_E(\omega) \wedge M_F(\omega), I_E(\omega) \wedge I_F(\omega), N_E(\omega) \vee N_F(\omega)\rangle$

(P_{14}) $E \cup F = \langle \omega, M_E(\omega) \vee M_F(\omega), I_E(\omega) \wedge I_F(\omega), N_E(\omega) \wedge N_F(\omega)\rangle$

(P_{15}) $E \cup F = \langle \omega, M_E(\omega) \vee M_F(\omega), I_E(\omega) \vee I_F(\omega), N_E(\omega) \wedge N_F(\omega)\rangle$

2.7. Definition [9]

Let $\Omega \neq \phi$. A family of Neutrosophic subsets of Ω is λ_N-topology if it satisfies

(Δ_1) $0_N \in \lambda_N$ (Δ_2) $E_1 \cup E_2 \in \lambda_N$ for any $E_1, E_2 \in \lambda_N$.

2.8. Remark [9]

Members of λ_N-topology are λ_N-Open Sets (λ_N-OS) and their complements are λ_N-Closed Sets (λ_N-CS).

2.9. Definition [9]

Let (Ω, λ_N) be a λ_N-TS and $E = \{\langle \omega, M_E(\omega), I_E(\omega), N_E(\omega)\rangle\}$ be a NS in Ω. Then

λ_N-Closure (E) = $\bigcap$ {F: $E \subseteq F$, F is λ_N-CS}

λ_N-Interior (E) = $\bigcup$ {G: $G \subseteq E$, G is λ_N-OS}

2.10. Definition [8]

A NS, E in λ_N-TS is said to be

(i) λ_N-Semi-Open Set (λ_N-SOS) if $E \subseteq \lambda_N$-Cl(λ_N-Int(E)),

(ii) λ_N-Pre-Open Set (λ_N-POS) if $E \subseteq \lambda_N$-Int(λ_N-Cl(E)),

(iii) λ_N-α-Open Set (λ_N-αOS) if $E \subseteq \lambda_N$-Int(λ_N-Cl(λ_N-Int(E))).

2.11. Lemma [8]

Every λ_N- αOS is λ_N-SOS and λ_N-POS.

2.12. Definition [8]

Let the function h: $(\Omega_1, \tau_1) \rightarrow (\Omega_2, \tau_2)$ is defined to be λ_N-Cts (resp. λ_N-SCts, λ_N-PCts, λ_N-αCts) if the inverse image of λ_N-CS in (Ω_2, τ_2) is a λ_N-CS (resp. λ_N-SCS, λ_N-PCS, λ_N-αCS) in (Ω_1, τ_1).

3. λ_N-Irresolute Functions

3.1. Definition

Let (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs. Then h: $\Omega_1 \rightarrow \Omega_2$ is said to be a λ_N-α-irresolute function (resp. λ_N-semi-irresolute, λ_N-pre-irresolute) if the inverse image of every λ_N-αOS (resp. λ_N-SOS, λ_N-POS) in (Ω_2, τ_2) is an λ_N-αOS (resp. λ_N-SOS, λ_N-POS) in (Ω_1, τ_1).

3.2. Example

Let h: $(\Omega_1, \tau_1) \to (\Omega_2, \tau_2)$ be defined as h(p) = s and h(q) = r, where $\Omega_1 = \{p, q\}$ and $\Omega_2 = \{r, s\}$, $\tau_1 = \{0_N, A, B\}$, $\tau_2 = \{0_N, C, D\}$.

(i) A = ⟨ (0.2, 0.8, 0.9), (0.1, 0.7, 0.8) ⟩, B = ⟨ (0.3, 0.5, 0.6), (0.4, 0.6, 0.7) ⟩,
C = ⟨ (0.1, 0.7, 0.8), (0.2, 0.8, 0.9) ⟩, D = ⟨ (0.4, 0.6, 0.7), (0.3, 0.5, 0.6) ⟩,
G = ⟨ (0.3, 0.7, 0.8), (0.2, 0.6, 0.7) ⟩, H = ⟨ (0.2, 0.6, 0.7), (0.3, 0.7, 0.8) ⟩.

Here $\{0_N, A, B, G\}$ and $\{0_N, C, D, H\}$ are λ_N-αOS of (Ω_1, τ_1) and (Ω_2, τ_2) respectively. Hence, h is a λ_N-α-irresolute function.

(ii) A = ⟨ (0.3, 0.7, 0.8), (0.2, 0.6, 0.8) ⟩, B = ⟨ (0.4, 0.6, 0.7), (0.5, 0.5, 0.6) ⟩,
C = ⟨ (0.5, 0.5, 0.6), (0.4, 0.6, 0.7) >, D = ⟨ (0.2, 0.6, 0.8), (0.3, 0.7, 0.8) ⟩,
G = ⟨ (0.3, 0.7, 0.8), (0.4, 0.5, 0.7) >, H = ⟨ (0.4, 0.5, 0.7), (0.3, 0.7, 0.8) ⟩.

Here $\{0_N, A, B, G\}$ and $\{0_N, C, D, H\}$ are λ_N-SOS of (Ω_1, τ_1) and (Ω_2, τ_2) respectively. Therefore, h is a λ_N-semi-irresolute function.

(iii) A = ⟨ (0.3, 0.8, 0.9), (0.4, 0.7, 0.6) ⟩, B = ⟨ (0.4, 0.6, 0.7), (0.5, 0.6, 0.6) ⟩,
C = ⟨ (0.5, 0.6, 0.6), (0.4, 0.6, 0.7) ⟩, D = ⟨ (0.4, 0.7, 0.6), (0.3, 0.8, 0.9) ⟩,
G = ⟨ (0.2, 0.9, 0.9), (0.3, 0.8, 0.9) ⟩, H = ⟨ (0.3, 0.7, 0.8), (0.5, 0.5, 0.6) ⟩,
I = ⟨ (0.3, 0.8, 0.9), (0.2, 0.9, 0.9) ⟩, J = ⟨ (0.5, 0.5, 0.6), (0.3, 0.7, 0.8) ⟩.

Here $\{0_N, A, B, G, H\}$ and $\{0_N, C, D, I, J\}$ are λ_N-POS of (Ω_1, τ_1) and (Ω_2, τ_2) respectively and so h is a λ_N-pre-irresolute function.

3.3. Theorem

Let (Ω, τ) be a λ_N-TS and $E \subseteq \Omega$. Then E is λ_N-αOS iff it is λ_N-SOS and λ_N-POS.

Proof:

If E is λ_N-αOS, then by Lemma 2.11, E is λ_N-SOS and λ_N-POS. Conversely if E is λ_N-SOS and λ_N-POS, then $E \subseteq \lambda_N\text{-Cl}(\lambda_N\text{-Int}(E))$ and $E \subseteq \lambda_N\text{-Int}(\lambda_N\text{-Cl}(E))$. Therefore $\lambda_N\text{-Int}(\lambda_N\text{-Cl}(E)) \subseteq \lambda_N\text{-Int}(\lambda_N\text{-Cl}(\lambda_N\text{-Cl}(\lambda_N\text{-Int}(E)))) = \lambda_N\text{-Int}(\lambda_N\text{-Cl}(\lambda_N\text{-Int}(E)))$. That is $\lambda_N\text{-Int}(\lambda_N\text{-Cl}(E)) \subseteq \lambda_N\text{-Int}(\lambda_N\text{-Cl}(\lambda_N\text{-Int}(E)))$. Also $E \subseteq \lambda_N\text{-Int}(\lambda_N\text{-Cl}(E)) \subseteq \lambda_N\text{-Int}(\lambda_N\text{-Cl}(\lambda_N\text{-Int}(E)))$ implies $E \subseteq \lambda_N\text{-Int}(\lambda_N\text{-Cl}(\lambda_N\text{-Int}(E)))$. Thus E is λ_N-αOS.

3.4. Theorem

Let h: $\Omega_1 \to \Omega_2$ be a function, where (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs. Then the succeeding are equivalent.

(i) h is λ_N-α-irresolute.

(ii) $h^{-1}(E)$ is λ_N-αCS in (Ω_1, τ_1), for every λ_N-αCS E in (Ω_2, τ_2).

(iii) $h(\lambda_N\text{-}\alpha\text{Cl}(E)) \subseteq \lambda_N\text{-}\alpha\text{Cl}(h(E))\ \forall\ E \subseteq \Omega_1$.

(iv) $\lambda_N\text{-}\alpha\text{Cl}(h^{-1}(E)) \subseteq h^{-1}(\lambda_N\text{-}\alpha\text{Cl}(E))\ \forall\ E \subseteq \Omega_2$.

(v) $h^{-1}(\lambda_N\text{-}\alpha\text{Int}(E)) \subseteq \lambda_N\text{-}\alpha\text{Int}(h^{-1}(E))\ \forall\ E \subseteq \Omega_2$.

(vi) h is λ_N-α-irresolute for every $\omega \in (\Omega_1, \tau_1)$.

Proof

(i) implies (ii) It is obvious.

(ii) implies (iii) Let E $\subseteq$ Ω_1. In that case, λ_N-αCl(h(E)) is a λ_N-αCS of (Ω_2, τ_2). By (ii), h^{-1}(λ_N-αCl(h(E))) is a λ_N-αCS in (Ω_1, τ_1), and λ_N-αCl(E) $\subseteq$ λ_N-αCl(h^{-1}h(E)) $\subseteq$ λ_N-αCl(h^{-1}(λ_N-αCl(h(E)))) = h^{-1}(λ_N-αCl(h(E))). So h(λ_N-αCl(E)) $\subseteq$ λ_N-αCl(h(E)).

(iii) implies (iv) Let E $\subseteq$ Ω_2. By (iii), h(λ_N-αCl(h^{-1}(E))) $\subseteq$ λ_N-αCl(hh^{-1}(E)) $\subseteq$ λ_N-αCl(E). So λ_N-αCl(h^{-1}(E)) $\subseteq$ h^{-1}(λ_N-αCl(E)).

(iv) implies (v) Let E $\subseteq$ Ω_2. By (iv), h^{-1}(λ_N-αCl(Ω_2-E)) $\supseteq$ λ_N-αCl(h^{-1}(Ω_2-E)) = λ_N-αCl(Ω_1- h^{-1}(E)). Since Ω_1–λ_N-αCl(Ω_1–E) = λ_N-αInt(E), subsequently h^{-1}(λ_N-αInt(E)) = h^{-1}(Ω_2-λ_N-αCl(Ω_2-E)) = Ω_1 - h^{-1}(λ_N-αCl(Ω_2-E)) $\subseteq$ Ω_1 - λ_N-αCl(Ω_1 – h^{-1}(E)) = λ_N-αInt(h^{-1}(E)).

(v) implies (vi) Let E be any λ_N-αOS of (Ω_2, τ_2), subsequently E = λ_N-αInt(E). By (v), h^{-1}(E) = h^{-1}(λ_N-αInt(E)) $\subseteq$ λ_N-αInt(h^{-1}(E)) $\subseteq$ h^{-1}(E). So, h^{-1}(E) = λ_N-αInt(h^{-1}(E)). Thus, h^{-1}(E) is a λ_N-αOS of (Ω_1, τ_1). Therefore, h is λ_N-α-irresolute.

(i) implies (vi) Let h be λ_N-α-irresolute, $\omega \in$ (Ω_1, τ_1) and any λ_N-αOS E of (Ω_2, τ_2), $\ni$ h(ω) $\subseteq$ E. Then $\omega \in h^{-1}$(E) = λ_N-αInt(h^{-1}(E)). Let F = h^{-1}(E) followed by F is a λ_N-αOS of (Ω_1, τ_1) and so h(F) = hh^{-1}(E) $\subseteq$ E. Thus, h is λ_N-α-irresolute for each $\omega \in$ (Ω_1, τ_1).

(vi) implies (i) Let E be a λ_N-αOS of (Ω_2, τ_2), $\omega \in h^{-1}$(E). Then h(ω) $\in$ E. By hypothesis there exists a λ_N-αOS F of (Ω_1, τ_1) $\ni$ $\omega \in$ F and h(F) $\subseteq$ E. Thus $\omega \in F \subseteq h^{-1}$(h(F)) $\subseteq h^{-1}$(E) and ω $\in$ F = λ_N-αInt(F) $\subseteq$ λ_N-αInt(h^{-1}(E)) $\Rightarrow$ h^{-1}(E) $\subseteq$ λ_N-αInt(h^{-1}(E)). Hence h^{-1}(E) = λ_N-αInt(f^{-1}(E)). Thus, h is λ_N-α-irresolute.

3.5. Theorem

Let h: $\Omega_1 \rightarrow \Omega_2$ be a bijective function, where (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs. Then h is λ_N-α-irresolute iff λ_N-αInt(h(E)) $\subseteq$ h(λ_N-αInt(E)) $\forall$ E $\subseteq$ Ω_1.

Proof

Let E $\subseteq$ Ω_1. By Theorem 3.4 and since h is bijective, h^{-1}(λ_N-αInt(h(E))) $\subseteq$ λ_N-αInt(h^{-1}(h(E))) = λ_N-αInt(E). So, hh^{-1}(λ_N-αInt(h(E))) $\subseteq$ h(λ_N-αInt(E)). Consequently λ_N-αInt(h(E)) $\subseteq$ h(λ_N-αInt(E)).

Conversely, let E be a λ_N-αOS of (Ω_2, τ_2). Then E = λ_N-αInt(E). By hypothesis, h(λ_N-αInt(h^{-1}(E))) $\supseteq$ λ_N-αInt(h(h^{-1}(E))) = λ_N-αInt(E) = E implies h^{-1}h(λ_N-αInt(h^{-1}(E))) $\supseteq$ h^{-1}(E). Since h is bijective, λ_N-αInt(h^{-1}(E)) = h^{-1}h(λ_N-αInt(h^{-1}(E))) $\supseteq$ h^{-1}(E).

Hence h^{-1}(E) = λ_N-αInt(h^{-1}(E)). So h^{-1}(E) is λ_N-αOS of (Ω_1, τ_1). Thus, h is λ_N-α-irresolute.

3.6. Lemma

Let (Ω, τ) be a λ_N-TS and E $\subset$ Ω. Then λ_N-αInt(E) = E $\bigcap$ λ_N-Int((λ_N-Cl(λ_N-Int(E))), λ_N-αCl(E) = E $\bigcup$ λ_N-Cl(λ_N-Int(λ_N-Cl(E))).

3.7. Lemma

Let (Ω, τ) be a λ_N-TS, then

(i) λ_N-αCl(E) $\subseteq$ λ_N-Cl(E) $\forall$ E $\subseteq$ Ω.

(ii) λ_N-Cl(E) = λ_N-αCl(E) $\forall$ E $\subseteq$ Ω where E is λ_N-αOS.

Proof

(i) Let E $\subseteq$ Ω. Since λ_N-Int(E) $\subseteq$ λ_N-αInt(E), U–λ_N-Int(E) $\supseteq$ U-λ_N-αInt(E). Hence λ_N-αCl(E) $\subseteq$ λ_N-Cl(E).

(ii) Let E be any λ_N-αOS of (Ω, τ), then E $\subseteq$ λ_N-Int(λ_N-Cl(λ_N-Int(E))). Then λ_N-Cl(E) $\subseteq$ λ_N-Cl(λ_N-Int(λ_N-Cl(λ_N-Int(E)))) = λ_N-Cl(λ_N-Int(E)) $\subseteq$ λ_N-Cl(λ_N-Int(λ_N-Cl(E))). So, λ_N-Cl(E) $\subseteq$ E $\cup$ λ_N-Cl(λ_N-Int(λ_N-Cl(E))). By Lemma 3.6, λ_N-Cl(E) $\subseteq$ λ_N-αCl(E). By (i), λ_N-αCl(E) $\subseteq$ λ_N-Cl(E), therefore λ_N-Cl(E) = λ_N-αCl(E).

3.8. Theorem

Let h: $\Omega_1 \to \Omega_2$ be a λ_N-α-irresolute function, where (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs. Then λ_N-Cl(h^{-1}(E)) $\subseteq$ h^{-1}(λ_N-Cl(E)) for every λ_N-OS E of Ω_2.

Proof

Let E be any λ_N-OS of Ω_2. Since h is λ_N-α-irresolute and by Lemma 3.7, λ_N-αCl(h^{-1}(E)) = λ_N-Cl(h^{-1}(E)). By Theorem 3.4, λ_N-αCl(h^{-1}(E)) $\subseteq$ h^{-1}(λ_N-αCl(E)) and by Lemma 3.7, h^{-1}(λ_N-αCl(E)) $\subseteq$ h^{-1}(λ_N-Cl(E)). Then λ_N-αCl(h^{-1}(E)) $\subseteq$ h^{-1}(λ_N-Cl(E)). Therefore λ_N-Cl(h^{-1}(E)) $\subseteq$ h^{-1}(λ_N-Cl(E)).

3.9. Theorem

Let (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs and h: $\Omega_1 \to \Omega_2$ is λ_N-semi-irresolute iff h^{-1}(E) is λ_N-SCS in Ω_1, $\forall$ λ_N-SCS E of Ω_2.

Proof

If h is λ_N-semi-irresolute, then for every λ_N-SOS F of Ω_2, h^{-1}(F) is λ_N-SOS in Ω_1. If E is any λ_N-SCS of Ω_2, then Ω_2 – E is λ_N-SOS. As a consequence, h^{-1}(Ω_2 – E) is λ_N-SOS but h^{-1}(Ω_2 – E) = Ω_1–h^{-1}(E) so that h^{-1}(E) is λ_N-SCS in Ω_1.

Conversely, if, for all λ_N-SCS E of Ω_2, h^{-1}(E) is λ_N-SCS in Ω_1 and if F is any λ_N-SOS of Ω_2, then Ω_2–F is λ_N-SCS. Also h^{-1}(Ω_2–F) = Ω_1–h^{-1}(F) is λ_N-SCS in Ω_1. Accordingly h^{-1}(F) is λ_N-SOS in Ω_1. As a result, h is λ_N-semi-irresolute.

3.10. Theorem

If h_1: (Ω_1, τ_1) $\to$ (Ω_2, τ_2) is λ_N-semi-irresolute and h_2: (Ω_2, τ_2) $\to$ (Ω_3, τ_3) is λ_N-semi-irresolute, then $h_2 \circ h_1$: (Ω_1, τ_1) $\to$ (Ω_3, τ_3) is λ_N-semi-irresolute.

Proof

If E $\subseteq$ Ω_3 is λ_N-SOS, then h_2^{-1}(E) is λ_N-SOS in Ω_2 because h_2 is λ_N-semi-irresolute. Consequently since h_1 is λ_N-semi-irresolute, h_1^{-1}(h_2^{-1}(E)) = ($h_2 \circ h_1$)$^{-1}$(E) is λ_N-SOS in Ω_1. Hence $h_2 \circ h_1$ is λ_N-semi-irresolute.

3.11. Example ($h_2 \circ h_1$ is λ_N-semi-irresolute $\not\Rightarrow$ h_1 & h_2 is λ_N-semi-irresolute)

Let h_1: (Ω_1, τ_1) $\to$ (Ω_2, τ_2) be defined by h_1(p) = s, h_1(q) = r and h_2: (Ω_2, τ_2) $\to$ (Ω_3, τ_3) be defined by h_2(r) = u and h_2(s) = v where Ω_1 = {p, q}, Ω_2 = {r, s} and Ω_3 = {u, v}. Let τ_1 = {0_N, A, B},

τ_2 = {0_N, C, D} and τ_3 = {0_N, E, F}. Now, {0_N, A, B, G}, {0_N, C, D, H} and {0_N, E, F, I} are λ_N-SOS of (Ω_1, τ_1), (Ω_2, τ_2) and (Ω_3, τ_3) respectively, where

A = ⟨ (0.3, 0.7, 0.8), (0.2, 0.6, 0.8) ⟩, B = ⟨ (0.4, 0.6, 0.7), (0.5, 0.5, 0.6) ⟩,

C = ⟨ (0.8, 0.4, 0.2), (0.8, 0.3, 0.3) ⟩, D = ⟨ (0.6, 0.5, 0.5), (0.7, 0.4, 0.4) ⟩,

E = ⟨ (0.2, 0.6, 0.8), (0.3, 0.7, 0.8) ⟩, F = ⟨ (0.5, 0.5, 0.6), (0.4, 0.6, 0.7) ⟩,

G = ⟨ (0.3, 0.7, 0.8), (0.4, 0.5, 0.7) ⟩, H = ⟨ (0.7, 0.5, 0.4), (0.8, 0.3, 0.3) ⟩,

I = ⟨ (0.4, 0.5, 0.7), (0.3, 0.7, 0.8) ⟩.

Here, $h_2 o h_1$: $\Omega_1 \rightarrow \Omega_3$ defined by $h_2 \circ h_1$ (p) = v and $h_2 \circ h_1$(q) = u is λ_N-semi-irresolute, but h_1 and h_2 are not λ_N-semi-irresolute.

3.12. Corollary

Let (Ω_1, τ_1), (Ω_2, τ_2) and (Ω_3, τ_3) be λ_N-TSs. If h_1: $\Omega_1 \rightarrow \Omega_2$ and h_2: $\Omega_2 \rightarrow \Omega_3$ are λ_N-α-irresolute then $h_2 \circ h_1$: $\Omega_1 \rightarrow \Omega_3$ is λ_N-α-irresolute.

Proof

Let E is λ_N-αOS in (Ω_3, τ_3). Since h_2 is λ_N-α-irresolute, h_2^{-1}(E) is λ_N-αOS in (Ω_2, τ_2). Also since h_1 is λ_N-α-irresolute, h_1^{-1} (h_2^{-1}(E)) = ($h_2 \circ h_1$)$^{-1}$(E) is λ_N-αOS in (Ω_1, τ_1). Therefore $h_2 \circ h_1$ is λ_N-α-irresolute.

3.13. Corollary

If h_1: (Ω_1, τ_1) $\rightarrow$ (Ω_2, τ_2) is λ_N-α-irresolute (resp. λ_N-semi-irresolute, λ_N-pre-irresolute) and h_2: (Ω_2, τ_2) $\rightarrow$ (Ω_3, τ_3) is λ_N-αCts (resp. λ_N-SCts, λ_N-PCts) then $h_2 \circ h_1$: (Ω_1, τ_1) $\rightarrow$ (Ω_3, τ_3) is λ_N-αCts (resp. λ_N-SCts, λ_N-PCts).

Proof

Let E is λ_N-OS in (Ω_3, τ_3). Since h_2 is λ_N-αCts (resp. λ_N-SCts, λ_N-PCts), h_2^{-1}(E) is λ_N-αOS (resp. λ_N-SOS, λ_N-POS) in (Ω_2, τ_2). Also since h_1 is λ_N-α-irresolute (resp. λ_N-semi-irresolute, λ_N-pre-irresolute), h_1^{-1} (h_2^{-1}(E)) = ($h_2 \circ h_1$)$^{-1}$(E) is λ_N-αOS (resp. λ_N-SOS, λ_N-POS) in (Ω_1, τ_1). Therefore $h_2 \circ h_1$ is λ_N-αCts (resp. λ_N-SCts, λ_N-PCts).

3.14. Theorem

Let (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs. If h: $\Omega_1 \rightarrow \Omega_2$ is λ_N-semi-irresolute and λ_N-pre-irresolute then h is λ_N-α-irresolute.

Proof

Let E is λ_N-αOS in (Ω_2, τ_2), then by Theorem 3.3, E is λ_N-SOS and λ_N-POS. Since h is λ_N-semi-irresolute and λ_N-pre-irresolute, h^{-1}(E) is λ_N-SOS and λ_N-POS. Therefore h^{-1}(E) is λ_N-αOS. Hence h is λ_N-α-irresolute.

3.15. Theorem

Let (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs. A function h: $\Omega_1 \rightarrow \Omega_2$ is λ_N-αCts iff it is λ_N-SCts and λ_N-PCts.

Proof

It is clear from Theorem 3.3.

4. Contra λ_N-Irresolute Functions

4.1. Definition

Let (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs. Then h: $\Omega_1 \to \Omega_2$ is said to be contra λ_N-α-irresolute (resp. contra λ_N-semi-irresolute, contra λ_N-pre-irresolute) if the inverse image of every λ_N-αOS (resp. λ_N-SOS, λ_N-POS) in (Ω_2, τ_2) is a λ_N-αCS (resp. λ_N-SCS, λ_N-PCS) in (Ω_1, τ_1).

4.2. Example

(i) Let h: $(\Omega_1, \tau_1) \to (\Omega_2, \tau_2)$ be defined as h(s) = u and h(t) = v, where Ω_1 = {s, t} and Ω_2 = {u, v}, $\tau_1 = \{0_N, A, B\}$, $\tau_2 = \{0_N, C, D\}$.

A = ⟨ (0.2, 0.8, 0.9), (0.1, 0.7, 0.8) ⟩, B = ⟨ (0.3, 0.5, 0.6), (0.4, 0.6, 0.7) ⟩,

C = ⟨ (0.8, 0.3, 0.1), (0.9, 0.2, 0.2) ⟩, D = ⟨ (0.7, 0.4, 0.4), (0.6, 0.5, 0.3) ⟩,

G = ⟨ (0.3, 0.7, 0.8), (0.2, 0.6, 0.7) ⟩, H = ⟨ (0.7, 0.4, 0.2), (0.8, 0.3, 0.3) ⟩.

Here, {A′, B′, G′, 1_N} are λ_N-αCS of (Ω_1, τ_1) and {0_N, C, D, H} are λ_N-αOS of (Ω_2, τ_2). Consequently, h is contra λ_N-α-irresolute function.

(ii) Let h: $(\Omega_1, \tau_1) \to (\Omega_2, \tau_2)$ be defined as h(p) = v, h(q) = w and h(r) = u, where Ω_1 = {p, q, r} and Ω_2 = {u, v, w}, $\tau_1 = \{0_N, A, B\}$, $\tau_2 = \{0_N, C, D\}$.

A = ⟨ (0.2, 0.6, 0.8), (0.1, 0.7, 0.9), (0.2, 0.8, 0.9) ⟩, B = ⟨ (0.3, 0.4, 0.7), (0.2, 0.5, 0.8), (0.4, 0.6, 0.7) ⟩,

C = ⟨ (0.9, 0.3, 0.1), (0.9, 0.2, 0.2), (0.8, 0.4, 0.2) ⟩, D = ⟨ (0.8, 0.5, 0.2), (0.7, 0.4, 0.4), (0.7, 0.6, 0.3) ⟩,

G = ⟨ (0.3, 0.5, 0.7), (0.2, 0.6, 0.9), (0.3, 0.7, 0.8) ⟩, H = ⟨ (0.9, 0.4, 0.2), (0.8, 0.3, 0.3), (0.7, 0.5, 0.3) ⟩.

Here, {A′, B′, G′, 1_N} are λ_N-SCS of (Ω_1, τ_1) and {0_N, C, D, H} are λ_N-SOS of (Ω_2, τ_2). Hence h is contra λ_N-semi-irresolute function.

(iii) Let h : $(\Omega_1, \tau_1) \to (\Omega_2, \tau_2)$ be defined as h(p) = w, h(q) = u and h(r) = v, where Ω_1 = {p, q, r} and Ω_2 = {u, v, w}, $\tau_1 = \{0_N, A, B\}$, $\tau_2 = \{0_N, C, D\}$.

A = ⟨ (0.2, 0.7, 0.7), (0.3, 0.7, 0.8), (0.1, 0.8, 0.8) ⟩, B = ⟨ (0.3, 0.7, 0.6), (0.4, 0.6, 0.7), (0.2, 0.7, 0.8) ⟩,

C = ⟨ (0.9, 0.1, 0.1), (0.8, 0.2, 0.2), (0.8, 0.3, 0.2) ⟩, D = ⟨ (0.8, 0.3, 0.2), (0.6, 0.3, 0.3), (0.7, 0.4, 0.4) ⟩,

G = ⟨ (0.2, 0.8, 0.8), (0.2, 0.7, 0.8), (0.1, 0.9, 0.9) ⟩, H = ⟨ 0.8, 0.2, 0.1), (0.7, 0.3, 0.2), (0.8, 0.3, 0.3) ⟩.

Here, {A′, B′, G′, 1_N} are λ_N-PCS of (Ω_1, τ_1) and {0_N, C, D, H} are λ_N-POS of (Ω_2, τ_2). That's why h is contra λ_N-pre-irresolute function.

4.3. Theorem

Let (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs. Then h: $\Omega_1 \to \Omega_2$ is contra λ_N-α-irresolute iff for every λ_N-αCS E of Ω_2, $h^{-1}(E)$ is λ_N-αOS in Ω_1.

Proof

If h is contra λ_N-α-irresolute, then for each λ_N-αOS F of Ω_2, $h^{-1}(F)$ is λ_N-αCS in Ω_1. If E is any λ_N-αCS of Ω_2, then $\Omega_2 - E$ is λ_N-αOS. Thus $h^{-1}(\Omega_2 - E)$ is λ_N-αCS but $h^{-1}(\Omega_2 - E) = \Omega_1 - h^{-1}(E)$ so that $h^{-1}(E)$ is λ_N-αOS in Ω_1.

Conversely, if, for all λ_N-αCS E of Ω_2, $h^{-1}(E)$ is λ_N-αOS in Ω_1 and if F is any λ_N-αOS of Ω_2, then $\Omega_2 - F$ is λ_N-αCS. Also, $h^{-1}(\Omega_2 - F) = \Omega_1 - h^{-1}(F)$ is λ_N-αOS. Thus $h^{-1}(F)$ is λ_N-αCS in Ω_1. Hence h is contra λ_N-α-irresolute.

4.4. Corollary

Let (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs. Then h: $\Omega_1 \to \Omega_2$ is contra λ_N-semi-irresolute (contra λ_N-pre-irresolute) iff for every λ_N-SCS (λ_N-PCS) E of Ω_2, $h^{-1}(E)$ is λ_N-SOS (λ_N-POS) in Ω_1.

Proof

If h is contra λ_N-semi-irresolute (contra λ_N-pre-irresolute), then for each λ_N-SOS (λ_N-POS) F of Ω_2, $h^{-1}(F)$ is λ_N-SCS (λ_N-PCS) in Ω_1. If E is any λ_N-SCS (λ_N-PCS) of Ω_2, then $\Omega_2 - E$ is λ_N-SOS (λ_N-POS). Thus $h^{-1}(\Omega_2 - E)$ is λ_N-SCS (λ_N-PCS) but $h^{-1}(\Omega_2 - E) = \Omega_1 - h^{-1}(E)$ so that $h^{-1}(E)$ is λ_N-SOS (λ_N-POS) in Ω_1.

Conversely, if, for all λ_N-SCS (λ_N-PCS) E of Ω_2, $h^{-1}(E)$ is λ_N-SOS (λ_N-POS) in Ω_1 and if F is any λ_N-SOS (λ_N-POS) of Ω_2, then $\Omega_2 - F$ is λ_N-SCS (λ_N-PCS). Also, $h^{-1}(\Omega_2 - F) = \Omega_1 - h^{-1}(F)$ is λ_N-SOS (λ_N-POS). Thus $h^{-1}(F)$ is λ_N-SCS (λ_N-PCS) in Ω_1. Hence h is contra λ_N-semi-irresolute (contra λ_N-pre-irresolute).

4.5. Theorem

Let (Ω_1, τ_1), (Ω_2, τ_2) and (Ω_3, τ_3) be λ_N-TSs. If h_1: $\Omega_1 \to \Omega_2$ and h_2: $\Omega_2 \to \Omega_3$ are contra λ_N-semi-irresolute functions, then $h_2 \circ h_1$: $\Omega_1 \to \Omega_3$ is λ_N-semi-irresolute.

Proof

If $E \subseteq Z$ is λ_N-SOS, then $h_2^{-1}(E)$ is λ_N-SCS in Ω_2 because h_2 is contra λ_N-semi-irresolute. Consequently, since h_1 is contra λ_N-semi-irresolute, $h_1^{-1}(h_2^{-1}(E)) = (h_2 \circ h_1)^{-1}(E)$ is λ_N-SOS in Ω_1. Hence $h_2 o h_1$ is λ_N-semi-irresolute.

4.6. Example ($h_2 \circ h_1$ is λ_N-semi-irresolute $\not\Rightarrow$ h_1 & h_2 is contra λ_N-semi-irresolute)

Let h_1: $(\Omega_1, \tau_1) \to (\Omega_2, \tau_2)$ be defined by $h_1(l) = q$, $h_1(m) = r$, $h_1(n) = p$ and h_2: $(\Omega_2, \tau_2) \to (\Omega_3, \tau_3)$ be defined by $h_2(p) = v$, $h_2(q) = w$ and $h_2(r) = u$ where $\Omega_1 = \{l, m, n\}$, $\Omega_2 = \{p, q, r\}$ and $\Omega_3 = \{u, v, w\}$. Let $\tau_1 = \{0_N, A, B\}$, $\tau_2 = \{0_N, C, D\}$ and $\tau_3 = \{0_N, E, F\}$. Here, $\{0_N, A, B, G\}$, $\{0_N, C, D, H\}$ and $\{0_N, E, F, I\}$ are λ_N-SOS of (Ω_1, τ_1), (Ω_2, τ_2) and (Ω_3, τ_3) where

A = $\langle$ (0.2, 0.6, 0.8), (0.1, 0.7, 0.9), (0.2, 0.8, 0.9) $\rangle$, B = $\langle$ (0.3, 0.4, 0.7), (0.2, 0.5, 0.8), (0.4, 0.6, 0.7) $\rangle$,

C = $\langle$ 0.2, 0.8, 0.9), (0.2, 0.6, 0.8), (0.1, 0.7, 0.9) $\rangle$, D = $\langle$ 0.4, 0.6, 0.7), (0.3, 0.4, 0.7), (0.2, 0.5, 0.8) $\rangle$,

E = $\langle$ (0.1, 0.7, 0.9), (0.2, 0.8, 0.9), (0.2, 0.6, 0.8) $\rangle$, F = $\langle$ (0.2, 0.5, 0.8), (0.4, 0.6, 0.7), (0.3, 0.4, 0.7) $\rangle$,

G = $\langle$ (0.3, 0.5, 0.7), (0.2, 0.6, 0.9), (0.3, 0.7, 0.8) $\rangle$, H = $\langle$ (0.3, 0.7, 0.8), (0.3, 0.5, 0.7), (0.2, 0.6, 0.9) $\rangle$,

I = $\langle$ (0.2, 0.6, 0.9), (0.3, 0.7, 0.8), (0.3, 0.5, 0.7) $\rangle$.

Here, $h_2 \circ h_1$: $\Omega_1 \to \Omega_3$ which is defined by $h_2 \circ h_1(l) = w$, $h_2 \circ h_1(m) = u$ and $h_2 \circ h_1(n) = v$ is λ_N-semi-irresolute , but h_1 and h_2 are not contra λ_N-semi-irresolute.

4.7. Corollary

Let (Ω_1, τ_1), (Ω_2, τ_2) and (Ω_3, τ_3) be λ_N-TSs. If h_1: $\Omega_1 \to \Omega_2$ and h_2: $\Omega_2 \to \Omega_3$ are contra λ_N-α-irresolute (contra λ_N-pre-irresolute) functions, then $h_2 \circ h_1$: $\Omega_1 \to \Omega_3$ is a λ_N-α-irresolute (λ_N-pre-irresolute) function.

4.8. Theorem

Let (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs. If h: $\Omega_1 \rightarrow \Omega_2$ is contra λ_N-α-irresolute, then it is contra λ_N-αCts.

Proof

Let E be any λ_N-OS in Ω_2. Then E is λ_N-αOS in Ω_2. Since h is contra λ_N-α-irresolute, $h^{-1}(E)$ is a λ_N-αCS in Ω_1. It shows that h is contra λ_N-αCts function.

4.9. Theorem

Let (Ω_1, τ_1), (Ω_2, τ_2) and (Ω_3, τ_3) be λ_N-TSs. If h_1: $\Omega_1 \rightarrow \Omega_2$ is contra λ_N-α-irresolute and h_2: $\Omega_2 \rightarrow \Omega_3$ is contra λ_N-αCts, then $h_2 \circ h_1$: $\Omega_1 \rightarrow \Omega_3$ is λ_N-αCts.

Proof

Let $E \subseteq \Omega_3$ is λ_N-OS. Since h_2 is contra λ_N-αCts, $h_2^{-1}(E)$ is λ_N-αCS in Ω_2. Consequently, since h_1 is contra λ_N-α-irresolute, $h_1^{-1}(h_2^{-1}(E)) = (h_2 \circ h_1)^{-1}(E)$ is λ_N-αOS in Ω_1, by Theorem 4.3. Hence $h_2 \circ h_1$ is λ_N-αCts.

4.10. Corollary

Let (Ω_1, τ_1), (Ω_2, τ_2) and (Ω_3, τ_3) be λ_N-TSs, and h_1: $\Omega_1 \rightarrow \Omega_2$ and h_2: $\Omega_2 \rightarrow \Omega_3$ be two functions. Then if h_1 is contra λ_N-semi-irresolute (contra λ_N-pre-irresolute) and h_2 is contra λ_N-SCts (contra λ_N-PCts), then $h_2 \circ h_1$: $\Omega_1 \rightarrow \Omega_3$ is λ_N-SCts (λ_N-PCts).

4.11. Theorem

Let (Ω_1, τ_1) and (Ω_2, τ_2) be λ_N-TSs. If h: $\Omega_1 \rightarrow \Omega_2$ is contra λ_N-semi-irresolute and contra λ_N-pre-irresolute, then h is contra λ_N-α-irresolute.

Proof

Let E is λ_N-αOS in (Ω_2, τ_2), then by Theorem 3.3, E is λ_N-SOS and λ_N-POS. Since h is contra λ_N-semi-irresolute and contra λ_N-pre-irresolute, $h^{-1}(E)$ is λ_N-SCS and λ_N-PCS. Therefore $h^{-1}(E)$ is λ_N-αCS. Hence h is contra λ_N-α-irresolute.

5. Conclusion

In this confab, we instigated λ_N-α-irresolute function, λ_N-semi-irresolute function and λ_N-pre-irresolute function on λ_N-TS. Subsequently, we have analyzed its various properties. Followed by this, the new postulations of contra λ_N-α-irresolute function, contra λ_N-semi-irresolute function and contra λ_N-pre-irresolute function were put forth on λ_N-TS and their features were probed along with illustrations.

λ_N-TS idea can be further developed and extended in the actual life applications such as medical field, robotics, machine learning, neural networks, natural image sensing, speech recognition, and so on.

In future, it provokes to apply these perceptions in further extensions of λ_N-TS such as almost continuity and its unique characteristics in G_N-TSs along with some separation axioms related to G_N-TSs. Also, this concept may be extended to Intuitionistic Fuzzy and Neutrosophic Fixed Point Theory.

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Minimal Structures and Grill in Neutrosophic Topological Spaces

Gour Pal[1], Runu Dhar[2], and Binod Chandra Tripathy[3,*]

[1]Department of Mathematics, Dasaratha Deb Memorial College, Khowai, 799201, Tripura, India
[1]E-mail: gourpal74@gmail.com
[2]Department of Applied Mathematics, Maharaja Bir Bikram University, Agartala, 799004, Tripura, India
[2]E-mail: runu.dhar@gmail.com
[3]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India
[3]E-mail: tripathybc@yahoo.com, tripathybc@rediffmail.com
***Correspondence:** tripathybc@yahoo.com

Abstract: The main objective of this article is to introduce the notion of minimal structure (in short M-structure) and grill in neutrosophic topological space (in short N-T-space). Besides, we establish its relation with some existing notions on different types of open sets in N-T-space, and investigate some basic properties of the class of M-structure and grill in NT-space. Further, we furnish some suitable examples of M-structure and grill via NT-space.

Keywords: *Neutrosophic Set*; *M-structure*; *Neutrosophic Topology*; *Neutrosophic Grill Topology*.

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1. Introduction: The concept of fuzzy set (in short F-set) theory was grounded by Zadeh [29] in 1965. Uncertainty plays an important role in our everyday life problems. Zadeh [29] associated the membership value with the elements to control the uncertainty. It was not sufficient to control uncertainty, so Atanaosv [3] added non-membership value along with the membership value, and introduced the notion of intuitionistic fuzzy set (in short IF-set). Still it was difficult to handle some real world problems under uncertainty, in particular for problems on decision making. In order to overcome this difficulty, Smarandache [24] considered the elements with truth-membership, indeterminacy-membership and false-membership values, and grounded the idea of neutrosophic set (in short N-set) theory in 1998. Till now, the concept of N-set has been applied in many branches of science and technology.

The notions of N-T-space was first grounded by Salama and Alblowi [22], followed by Salama and Alblowi [23], Iswaraya and Bageerathi [15], who studied the concept of neutrosophic semi-open set (in short NSO-set) and neutrosophic semi-closed set (in short NSC-set). Arokiarani et al. [2] studied some new notations and mappings via N-T-spaces. Iswarya and Bageerathi [15] established the notion of neutrosophic semi-open sets via neutrosophic topological space. Afterwards, Rao and Srinivasa [21] introduced and studied neutrosophic pre-open set (in short NPO-set) and neutrosophic pre-closed set (in short NPC-set) via N-T-space. Das and Pramanik [7] grounded the notion of generalized neutrosophicx *b*-open set via neutrosophic topological space. Later on, Das and Pramanik [8] introduced the notion of neutrosophic ϕ-open set and neutrosophic ϕ-continuous function via neutrosophic topological space. Recently, Das and Tripathy [11] presented the notion of neutrosophic simply *b*-open set via neutrosophic topological space. Thereafter, Das and Tripathy [13] introduced the idea of pairwise neutrosophic *b*-open sets via N-T-space. Parimala et al. [19] introduced the notion of neutrosophic nano ideal topological space. Later on Parimala et al. [20] grounded the idea of neutrosophic $\alpha\psi$-homomorphism via neutrosophic topological spaces.

Makai et al. [16] introduced and studied the concept of minimal structure (in short M-structure) via topological space. It is found to have useful applications and the notion was investigated by Madok [17]. The notion of minimal structure in a fuzzy topological space was introduce by Alimohammady and Roohi [1] and further investigated by Tripathy and Debnath [27] and others.

In this article, we introduce the idea of minimal structure and grill via N-T-spaces. We establish its relation with some existing notions on several types of open sets via N-T-spaces. Besides, we investigate some basic properties of the class of minimal structures and grill via N-T-spaces. Further, we furnish some suitable examples on minimal structures and grill via N-T-spaces.

The rest of the paper is divided into following sections:

In section 2, we provide some definitions and results those are very useful for the preparation of the main results of this article. In section 3, we introduce the concept of M-structure and gril via NT-spaces, and proved some basic results on them. In section 4, we introduce an operator $(\ )^*$ on M-structure via NT-spaces, and established several interesting results based on it. Finally, in section 5, we conclude the work done in this article.

2. Preliminaries & Definitions:

In this section, we provide some existing results on neutrosophic set and neutrosophic topology those are relevant to main results of this article.

Definition 2.1.[24] Suppose that Ĝ be a fixed set. Then *W*, an N-set over Ĝ is a set contains triplet having truth, indeterminacy and false membership values that can be characterized independently, denoted by T_W, I_W, F_W in the unit interval [0, 1].

We denote the N-set *W* as follows:

$$W = \{(r, T_W(r), I_W(r), F_W(r)) : r \in \hat{G}\}, \text{ where } T_W(r), I_W(r), F_W(r) \in [0, 1], \text{ for all } r \in \hat{G}.$$

Since, no restriction on the values of $T_W(r)$, $I_W(r)$ and $F_W(r)$ is imposed, so we have

$$0 \le T_W(r) + I_W(r) + F_W(r) \le 3, \text{ for all } r \in \hat{G}.$$

Example 2.1. Suppose that Ĝ={b, c} be a fixed set. Clearly, W={(b,0.4,0.8,0.7), (c,0.2,0.8,0.7)} is an N-set over Ĝ.

Definition 2.2.[24] Suppose that W = {(r, $T_W(r)$, $I_W(r)$, $F_W(r)$): $r \in$Ĝ} be an N-set over Ĝ. Then, W^c i.e., the complement of W is defined by W^c = {(r, 1-$T_W(r)$, 1-$I_W(r)$, 1-$F_W(r)$): $r \in$Ĝ}.

Example 2.2. Suppose that Ĝ={b, c} be a fixed set. Let W={(b,0.5,0.5,0.7), (c,0.5,0.7,0.8)} be an N-set over Ĝ. Then, the complement of W is W^c={(b,0.5,0.5,0.3), (c,0.5,0.3,0.2)}.

Definition 2.3.[24] An N-set W = {(r, $T_W(r)$, $I_W(r)$, $F_W(r)$): $r \in$Ĝ} is called a subset of an N-set L = {(r, $T_L(r)$, $I_L(r)$, $F_L(r)$): $r \in$Ĝ} (i.e., $W \subseteq L$) if and only if $T_W(r) \leq T_L(r)$, $I_W(r) \geq I_L(r)$, $F_W(r) \geq F_L(r)$, for each $r \in$Ĝ.

Example 2.3. Suppose that Ĝ={b, c} be a fixed set. Let M={(b,0.5,0.5,0.7), (c,0.5,0.7,0.8)} and W={(b,0.7,0.2,0.5), (c,0.9,0.5,0.3)} be two N-sets over Ĝ. Clearly, $M \subseteq W$.

Definition 2.4.[24] Assume that W = {(r, $T_W(r)$, $I_W(r)$, $F_W(r)$): $r \in$Ĝ} and L={(r, $T_L(r)$, $I_L(r)$, $F_L(r)$): $r \in$Ĝ} be any two N-sets over a fixed set Ĝ. Then, their intersection and union are defined as follows:

$$W \cap L = \{(r, T_N(r) \wedge T_L(r), I_N(r) \vee I_L(r), F_N(r) \vee F_L(r)): r \in \hat{G}\},$$
$$W \cup L = \{(r, T_N(r) \vee T_L(r), I_N(r) \wedge I_L(r), F_N(r) \wedge F_L(r)): r \in \hat{G}\}.$$

Example 2.4. Suppose that Ĝ={b, c} be a fixed set. Let W={(b,0.5,0.5,0.7), (c,0.5,0.7,0.8)} and M= {(b,0.7,0.2,0.5), (c,0.9,0.5,0.3)} be two N-sets over Ĝ. Then, $W \cup M$={(b,0.7,0.2,0.5), (c,0.9,0.5,0.3)} and $W \cap M$={(b,0.5,0.5,0.7), (c,0.5,0.7,0.8)}.

Definition 2.5.[22] The null N-set (0_N) and absolute N-set (1_N) over Ĝ are represented as follows:

(i) 0_N ={(r, 0, 0, 1) : $r \in$Ĝ};
(ii) 0_N ={(r, 0, 1, 0) : $r \in$Ĝ};
(iii) 0_N ={(r, 0, 1, 1) : $r \in$Ĝ};
(iv) 0_N ={(r, 0, 0, 0) : $r \in$Ĝ};
(v) 1_N ={(r, 1, 0, 1) : $r \in$Ĝ};
(vi) 1_N ={(r, 1, 1, 0) : $r \in$Ĝ};
(vii) 1_N ={(r, 1, 0, 0) : $r \in$Ĝ};
($viii$) 1_N ={(r, 1, 1, 1) : $r \in$Ĝ}.

Clearly, $0_N \subseteq 1_N$. We have, for any N-set W, $0_N \subseteq W \subseteq 1_N$.

Throughout the article, we will use 0_N ={(r, 0, 1, 1) : $r \in$Ĝ} and 1_N ={(r, 1, 0, 0) : $r \in$Ĝ}.

Definition 2.6.[22] Assume that Ĝ be a fixed set. Then τ, a family of N-sets over Ĝ is called an N-T on Ĝ if the following condition holds:

(*i*) 0_N, $1_N \in \tau$;

(*ii*) $W_1, W_2 \in \tau \Rightarrow W_1 \cap W_2 \in \tau$;

(*iii*) $\{W_i : i \in \Delta\} \subseteq \tau \Rightarrow \cup_{i\in\Delta} W_i \in \tau$.

Then, the pair (Ĝ, τ) is called an N-T-space. If $W \in \tau$, then *W* is called an neutrosophic open set (in short N-O-set) in (Ĝ, τ), and the complement of *W* is called an neutrosophic closed set (in short N-C-set) in (Ĝ, τ).

Example 2.5. Suppose that *W*, *E* and *Z* be three N-sets over a fixed set Ĝ={*p*, *q*} such that:

W={(*p*,0.7,0.5,0.7), (*q*,0.5,0.1,0.5): *p*, *q*∈Ĝ};

E={(*p*,0.6,0.9,0.8), (*q*,0.5,0.3,0.8): *p*, *q*∈Ĝ};

Z={(*p*,0.5,1.0,0.8), (*q*,0.4,0.4,0.9): *p*, *q*∈Ĝ}.

Then, the collection τ={0_N, 1_N, *W*, *E*, *Z*} forms a neutrosophic topology on Ĝ. Here, 0_N, 1_N, *W*, *E*, *Z* are NOSs in (Ĝ, τ), and their complements 1_N, 0_N, W^c={(*p*,0.3,0.5,0.3), (*q*,0.5,0.9,0.5): *p*, *q*∈ Ĝ}, E^c={(*p*,0.4,0.1, 0.2), (*q*,0.5,0.7,0.2): *p*, *q*∈ Ĝ} and *Z*={(*p*,0.5,0.0,0.2), (*q*,0.6,0.6,0.1): *p*, *q*∈ Ĝ} are NCSs in (Ĝ, τ).

The neutrosophic interior and neutrosophic closure of an N-set are defined as follows:

Definition 2.7.[22] Assume that τ be an N-T on Ĝ. Suppose that *W* be an N-set over Ĝ. Then,

(*i*) neutrosophic interior (in short N_{int}) of *W* is the union of all N-O-sets in (Ĝ, τ) those are contained in *W*, i.e., $N_{int}(W) = \cup\{E : E$ is an N-O-set in Ĝ such that $E \subseteq W\}$;

(*ii*) neutrosophic closure (in short N_{cl}) of *W* is the intersection of all N-C-sets in (Ĝ, τ) those containing *W*, i.e., $N_{cl}(W) = \cap\{F : F$ is an N-C-set in Ĝ such that $W \subseteq F\}$.

Clearly $N_{int}(W)$ is the largest N-O-set contained in *W*, and $N_{cl}(W)$ is the smallest N-C-set containing *W*.

Example 2.6. Suppose that (Ĝ, τ) be an NT-space as shown in **Example 2.5.** Suppose that *U*={(*p*,0.5,0.5,0.7), (*q*,0.5,0.7,0.8)} be an N-set over Ĝ. Then, $N_{int}(U)=0_N$ and $N_{cl}(U)$={(*p*,0.5,0.0,0.2), (*q*,0.6,0.6,0.1)}.

Proposition 2.1.[22] For any N-set *B* in (Ĝ, τ), we have the following:

(*i*) $N_{int}(B^c) = (N_{cl}(B))^c$;

(*ii*) $N_{cl}(B^c) = (N_{int}(B))^c$.

Definition 2.2.[21] Suppose that (Ĝ, τ) be an N-T-space, and *W* be an N-set over Ĝ. Then, *W* is called

(*i*) NSO-set if and only if $W \subseteq N_{cl}(N_{int}(W))$;

(*ii*) NPO-set if and only if $W \subseteq N_{int}(N_{cl}(W))$.

The collection of all NSO sets and NPO sets in (Ĝ, τ) are denoted by NSO(τ) and NPO(τ).

Example 2.7. Suppose that Ĝ={*a*, *b*} be a fixed set. Clearly, (Ĝ, τ) is an NT-space, where τ={0_N, 1_N, {(*a*, 0.3,0.3,0.4), (*b*,0.4,0.4,0.3): *a*, *b*∈Ĝ}, {(*a*,0.4,0.1,0.4), (*b*,0.5,0.3,0.1): *a*, *b*∈Ĝ}}. Then, the N-set *Q*={(*a*,0.6, 0.1,0.4), (*b*,0.9,0.2,0.1): *a*, *b*∈Ĝ} is an NSO set, and *P*={(*a*,0.3,0.2,0.9), (*b*,0.3,0.3,0.4): *a*, *b*∈Ĝ} is an NPO set in (Ĝ, τ).

Definition 2.8.[2] Assume that (Ĝ, τ) be an N-T-space. Then *W*, an N-set over Ĝ is called an neutrosophic *b*-open set (in short N-*b*-O-set) in (Ĝ, τ) if and only if $W \subseteq N_{int}(N_{cl}(W)) \cup N_{cl}(N_{int}(W))$.

An N-set *G* is called an neutrosophic *b*-closed set (in short N-*b*-C-set) in (Ĝ, τ) if and only if its complement is an N-*b*-O-set in (Ĝ, τ). The collection of all neutrosophic *b*-open sets in (Ĝ, τ) is denoted by N-*b*-O(τ).

Example 2.8. Suppose that (Ĝ, τ) be an NT-space as shown in **Example 2.7**. Then, the neutrosophic set *P*={(*a*,0.3,0.2,0.9), (*b*,0.3,0.3,0.4): *a*, *b*∈Ĝ} is an neutrosophic *b*-open set in (Ĝ, τ).

3. Minimal Structure in Neutrosophic Topological Space

In this section, we procure the notions of M-structure and grill in N-T-space.

Definition 3.1. A family *M* of neutrosophic subsets of Ĝ i.e., M ⊂ P(Ĝ), where P(Ĝ) is the collection of all N-sets defined over Ĝ is said to be a M-structure on Ĝ if 0_N and 1_N belong to *M*. By (Ĝ, *M*), we denote the neutrosophic minimal space (in short N-M-space). The members of *M* are called neutrosophic minimal-open (in short N-m-O) subset of Ĝ.

Example 3.1. Let *W*, *E* and *Z* be three neutrosophic sets over a fixed set Ĝ={*b*, *c*} such that:

W={(*b*,0.7,0.5,0.7), (*c*,0.5,0.1,0.5): *b*, *c*∈Ĝ};

E={(*b*,0.6,0.9,0.8), (*c*,0.5,0.3,0.8): *b*, *c*∈Ĝ};

Z={(*b*,0.5,1.0,0.8), (*c*,0.4,0.4,0.9): *b*, *c*∈Ĝ}.

Clearly, the collection M={0_N, 1_N, *W*, *E*, *Z*} forms a neutrosophic minimal structure on Ĝ, and the pair (Ĝ, M) is a neutrosophic minimal structure space.

Definition 3.2. The complement of N-m-O set *W* is an neutrosophic m-closed set (in short N-m-C set) in (Ĝ, *M*).

Example 3.2. Let us consider a neutrosophic minimal structure space as shown in Example 3.1. Here, 0_N, 1_N, *W*, *E*, *Z* are neutrosophic minimal open sets in (Ĝ, M), and $(0_N)^c = 1_N$, $(1_N)^c = 0_N$, W^c =

{(*b*,0.3,0.5,0.3), (*c*,0.5,0.9,0.5)}, E^c={(*b*,0.4,0.1,0.2), (*c*, 0.5,0.7,0.2)}, Z^c = {(*b*,0.5,0.0,0.2), (*c*,0.6,0.6,0.1)} are neutrosophic minimal closed sets in (Ĝ, M).

Example 3.3. From the above definitions, it is clear that NPO-sets, NSO-sets, N-α-O-sets, N-*b*-O-sets are N-m-O sets.

Example 3.4. Let *W*, *E* and *Z* be three neutrosophic sets over a non-empty set Ĝ ={*b*, *c*} such that:

W={(*b*,0.7,0.5,0.7), (*c*,0.5,0.5,0.1) : *b*, $c \in X$};

E={(*b*,0.6,0.8,0.9), (*c*,0.5,0.8,0.3) : *b*, $c \in X$};

Z={(*b*,0.5,0.8,1.0), (*c*,0.4,0.9,0.4) : *b*, $c \in X$}.

Here, the family τ={0_N, 1_N, *W*, *E*, *Z*} forms a neutrosophic topology on *X*, and so (Ĝ, τ) is a neutrosophic topological space. Suppose M = τ ∪NPO(τ)∪NSO(τ)∪N-*b*-O(τ), then (Ĝ, M) is a neutrosophic minimal structure. Now, from the above it is clear that, every neutrosophic pre-open sets, neutrosophic semi-open sets, neutrosophic *b*-open sets in (Ĝ, τ) are neutrosophic *m*-open sets in (Ĝ, M). Further, it is also seen that, every neutrosophic *m*-open set in (Ĝ, M) is also a neutrosophic pre-open set, neutrosophic semi-open set, neutrosophic *b*-open set in (Ĝ, τ).

Remark 3.1. We can define neutrosophic minimal interior (in short N_{mint}), neutrosophic minimal closure (in short N_{mcl}) etc. in an N-M-space as we have define in the previous section.

Example 3.5. Suppose that (Ĝ, M) be a neutrosophic minimal structure space as shown in Example 3.1. Then, the neutrosophic minimal interior of *D*={(*b*,0.2,0.6,0.4), (*c*,0.4,0.9,0.7)} is $N_{m\text{-}int}(D)$={(*b*,0,1,1), (*c*,0,1,1)}, and the neutrosophic minimal closure of *D*={(*b*,0.2,0.6,0.4), (*c*,0.4,0.9,0.7)} is $N_{m\text{-}cl}(D)$={(*b*,0.3, 0.5,0.3), (*c*,0.5,0.9,0.5)} respectively.

In view of the definitions given in this article, we state the following result without proof.

Theorem 3.1. Suppose that (Ĝ, *M*) be an N-M-space. Then, for any N-sets *S* and *R* over Ĝ, the following holds:

(*i*) $(N_{mcl}(S))^c = N_{mcl}(S^c)$ and $(N_{mint}(S))^c = N_{mcl}(S^c)$.

(*ii*) $N_{mcl}(S)=S$ if and only if *S* is an N-m-C set in (Ĝ, *M*).

(*iii*) $N_{mint}(S)=S$ if and only if *S* is an N-m-O set in (Ĝ, *M*).

(*iv*) $S \subseteq R \Rightarrow N_{mcl}(S) \subseteq N_{mcl}(R)$ and $N_{mint}(S) \subseteq N_{mint}(R)$.

(*v*) $S \subseteq N_{mcl}(S)$ and $N_{mint}(S) \subseteq S$.

(*vi*) $N_{mcl}(N_{mcl}(S))=N_{mcl}(S)$ and $N_{mint}(N_{mint}(S))=N_{mint}(S)$.

Theorem 3.2. Assume that (Ĝ, *M*) be an N-M-space. Suppose that *M* satisfies the property *B*. Then, for an neutrosophic subset *S* of Ĝ, the followings hold:

(*i*) $S \in M$ if and only if $N_{mint}(S)=A$.

(*ii*) *S* is N-m-C set if and only if $N_{mcl}(S)=S$.

(*iii*) $N_{mint}(S) \in M$ and $N_{mcl}(S)$ is an N-m-C set.

Definition 3.3. An N-M-structure *M* on a non-empty set Ĝ is said to be have property *B* if the union of only family of neutrosophic subsets belonging *M* belongs to *M*.

Example 3.6. Suppose that *R*, *E* and *Y* be three neutrosophic sets over a fixed set Ĝ={*b*, *c*} such that:

R={(*b*,0.8,0.5,0.8), (*c*,0.6,0.1,0.6): *b*, *c*∈Ĝ};

E={(*b*,0.7,0.9,0.9), (*c*,0.6,0.3,0.9): *b*, *c*∈Ĝ};

Y={(*b*,0.6,1.0,0.9), (*c*,0.5,0.4,1.0): *b*, *c*∈Ĝ}.

Here, the collection M={0_N, 1_N, *R*, *E*, *Y*} forms a N-M-structure on Ĝ, and so the pair (Ĝ, M) is a neutrosophic minimal structure space. Clearly, the N-M-structure on Ĝ satisfied the property *B* which was stated in Definition 3.3.

Definition 3.4. An N-M-structure (Ĝ, *M*) satisfies the property *J* if the finite intersection of N-m-O sets is an N-m-O set.

Example 3.7. Let us consider a N-M-structure M on Ĝ as shown in Example 3.6. Clearly, the N-M-structure M on Ĝ satisfied the property *j* which was stated in Definition 3.4.

Remark 3.2. If a N-M-structure M on Ĝ is a neutrosophic topology on Ĝ, then M satisfied the property *j* which was stated in Definition 3.4.

Definition 3.5. A family *G* ($0_N \notin G$) of N-sets over Ĝ is called a grill on Ĝ if *G* satisfies the following condition:

(*i*) $S \in G$ and $S \subseteq R \Rightarrow R \in G$;

(*ii*) *S*, $R \subseteq$ Ĝ and $S \cup R \in G \Rightarrow S \in G$ or $R \in G$.

Example 3.8. Suppose that Ĝ = {*b*, *c*} be a fixed set. Then, the collection M={1_N, {(*b*,0.9,0.0,0.0), (*c*,0.9,0.0,0.0)}, {(*b*,0.8,0.0,0.0), (*c*,0.8,0.0,0.0)}} forms a grill on Ĝ.

Remark 3.3. Since $0_N \notin G$, so *G* is not a N-M-structure on Ĝ. An N-M-structure with a grill is called as a neutrosophic grill minimal space (in short N-G-M-space), denoted by (Ĝ, *M*, *G*).

4. ( )*-Operator on Neutrosophic Minimal Structure:

Definition 4.1. Suppose that (Ĝ, M, G) be an N-G-M-space. A function $(\ )^{*m}$: P(Ĝ) → P(Ĝ) called as neutrosophic minimal local function (N-M-L-function) is defined by

$$(\ )^{*m} = \{x \in \hat{G}: S \cap U \in M, \text{ for all } U \in M(x)\}.$$

Now we discuss about some properties of the neutrosophic minimal local function $()^{*m}$ in (Ĝ, M, G).

Definition 4.2. Assume that (Ĝ, M, G) be a N-G-M-space. Then, the boundary of a N-set S over Ĝ is defined by $(\partial S)^{*m} = (S)^{*m} \cap (S^c)^{*m}$.

We state the following two results without prove in view of the above definitions.

Proposition 4.1. Suppose that (Ĝ, M, G) be a N-G-M-space. Then, the following holds:

(i) $(0_N)^{*m} = 0_N$;

(ii) $(S)^{*m} = 0_N$, if $S \notin G$;

(iii) $(S)^{*mP} \subseteq (S)^{*mQ}$, where P and Q are neutrosophic grill on Ĝ with $P \subseteq Q$.

Proposition 4.2. Suppose that P(Ĝ) be the collection of all neutrosophic sets defined over Ĝ. Assume that (Ĝ, M, G) be a N-G-M-space. Then, for $S \in$ P(Ĝ),

(i) $(S)^{*m} = S \cup (\partial S)^{*m}$;

(ii) $(S)^{*m} = \cap_{n \in \Delta} F_n$, where $\{F_n\}_{n \in \Delta}$ is the collection of all $(\ )^{*m}$-closed sets in (Ĝ, M, G).

Theorem 4.1. Assume that (Ĝ, M, G) be a N-G-M-space. Then, the following holds:

(i) S, $R \in$ P(Ĝ) and $S \subseteq R \Rightarrow (S)^{*m} \subseteq (R)^{*m}$;

(ii) For $S \subseteq \hat{G}$, $N_{mcl}(S)^{*m} \subseteq (S)^{*m}$;

(iii) For $S \subseteq \hat{G}$, $(S)^{*m}$ is a N-m-C set;

(iv) For $S \subseteq \hat{G}$, $(S)^{*m} \subseteq N_{mcl}(S)$;

(v) For $S \subseteq \hat{G}$, $[(S)^{*m}]^{*m} \subseteq (S)^{*m}$.

Proof. (i) Assume that $S \subseteq R$ and $x \in (S)^{*m}$. Then, for all $U \in M$, we have by definition, that $U \cap S \in G$. Thus by definition of neutrosophic grill we have $U \cap R \in G$. Hence, $x \in (R)^{*m}$. Therefore, $(S)^{*m} \subseteq (R)^{*m}$.

(ii) Assume that $x \notin N_{mcl}(S)$, for some $S \in \hat{G}$. Then by a known result there exists an $U_x \in M$ such that $U_x \cap S = 0_N \notin G$. Therefore, $x \notin (S)^{*m}$. Hence, we have $(S)^{*m} \subseteq N_{mcl}(S)$.

(iii) Assume that $x \in N_{mcl}(S)^{*m}$ and $U \in M(x)$, then $U \cap (S)^{*m} \neq 0_N$. Next, let $y \in U \cap (S)^{*m}$. Then, we have $y \in U$ and $y \in (S)^{*m}$. Therefore, $U \cap S \in G$, which implies $x \in (S)^{*m}$.

Thus, we have $N_{mcl}(S)^{*m} \subseteq (S)^{*m}$.

(iv) Suppose that $S \in$ P(Ĝ). Then, we have $(S)^{*m} \subseteq N_{mcl}(S)^{*m}$ and $N_{mcl}(S)^{*m} = (S)^{*m}$. Thus, we have for any $(S)^{*m} = N_{mcl}(S)^{*m}$, since $N_{mcl}(S)^{*m}$ is an N-m-C set, so $(S)^{*m}$ is an N-m-C set.

(*v*) Suppose that $S\in P(\hat{G})$. Then from (*iv*) we have $N_{mcl}(S)^{*m} \subseteq (S)^{*m}$. Further on considering $(S)^{*m}$ is place of S, from (vi) we have $((S)^{*m})^{*m} \subseteq N_{mcl}(S)^{*m}$. Hence, from these two inclusion we have $((S)^{*m})^{*m} \subseteq (S)^{*m}$.

Theorem 4.2. Assume that $(\hat{G}, M, G)$ be a N-G-M-space. Suppose that $(\hat{G}, M)$ satisfies the property *J*. Then, the following holds:

(*i*) $(S\cup R)^{*m} = (S)^{*m} \cup (R)^{*m}$, for $S, R \subseteq M$;

(*ii*) $(S\cap R)^{*m} \subseteq (S)^{*m} \cap (R)^{*m}$, for $S, R \subseteq M$.

Proof. (*i*) We have $S \subseteq S\cup R$, $R \subseteq S\cup R$. Thus, we have $(S)^{*m} \subseteq (S\cup R)^{*m}$ and $(R)^{*m} \subseteq (S\cup R)^{*m}$. This implies, $(S)^{*m} \cup (R)^{*m} \subseteq (S\cup R)^{*m}$ (1)

Suppose that $x \notin (S)^{*m}\cup(R)^{*m}$. Therefore there exists $U_1, U_2 \in M(x)$ such that $U_1\cap S\notin G$, $U_2\cap R\notin G$. This implies, $(U_1\cap S) \cup (U_2\cap R)\notin G$.

Now, $U_1, U_2 \in M(x) \Rightarrow U_1\cap U_2\in M(x)$ and $(S\cup R) \cap (U_1\cap U_2) \subseteq (U_1\cap S) \cup (U_2\cap R)\notin G$.

Therefore, $x\notin(S\cup R)^{*m}$. Thus, we have $(S\cup R)^{*m} \subseteq (S)^{*m} \cup (R)^{*m}$ (2)

From (1) and (2) we have, $(S\cup R)^{*m} = (S)^{*m}\cup (R)^{*m}$.

(*ii*) We have $S\cap R \subseteq S$ and $S\cap R \subseteq R$. This implies, $(S\cap R)^{*m}\subseteq (S)^{*m}$ and $(S\cap R)^{*m}\subseteq (R)^{*m}$. Therefore, $(S\cap R)^{*m} \subseteq (S)^{*m} \cap (R)^{*m}$.

Theorem 4.3. Suppose that $(\hat{G}, M, G)$ be a N-G-M-space. Assume that $(\hat{G}, M)$ satisfies the property J. Then, the following holds:

(*i*) For $W\subseteq M$ and $S\subseteq\hat{G}$, $W\cap(S)^{*m} = W\cap(W\cap S)^{*m}$;

(*ii*) For $S, R \subseteq \hat{G}$, $[(S)^{*m} \setminus (R)^{*m}] = [(S\setminus R)^{*m} \setminus (R)^{*m}]$;

(*iii*) For $S, R \subseteq \hat{G}$, with $R\notin G$. $(S\cup R)^{*m} = (S)^{*m} = (S\setminus R)^{*m}$.

Proof. (*i*) It is known that $(W\cap S) \subseteq S$.

Now, $(W\cap S) \subseteq S$

$\Rightarrow (W\cap S)^{*m} \subseteq (S)^{*m}$

$\Rightarrow W\cap(W\cap(S))^{*m} \subseteq W\cap(S))^{*m}$ (3)

Assume that $x\in W\cap(S)^{*m}$ and $V\in M(x)$.

Then, we have $W\cap V\in M(x)$ and $x\in(S)^{*m}$ implies $(W\cap V) \cap S\in G$.

Thus, $(W\cap S) \cap V\in G$. Thus, we have $x\in(W\cap S)^{*m}$, which implies $x\in W\cap(W\cap S)^{*m}$.

Hence, $(W\cap S)^{*m} \subseteq W\cap(W\cap S)^{*m}$ (4)

From (3) and (4), we have $W\cap(S)^{*m} = W\cap(W\cap S)^{*m}$.

(*ii*) We have,

$(S)^{*m} =[(S\setminus R) \cup (S\cap R)]^{*m}$

$= (S\setminus R)^{*m}\cup (S\cap R)^{*m}$ [by part (*i*)]

$\subseteq[(S\setminus R)^{*m}\cup (R)^{*m}]$.

Thus, we have $[(S)^{*m} \setminus (R)^{*m}] \subseteq [(S \setminus R)^{*m} \cup (R)^{*m}]$.

We have, $S \setminus R \subseteq S \Rightarrow (S \setminus R)^{*m} \subseteq (S)^{*m}$. This implies, $[(S \setminus R)^{*m} \setminus (R)^{*m}] \subseteq [(S)^{*m} \setminus (R)^{*m}]$.

Hence, we have $[(S)^{*m} \setminus (R)^{*m}] = [(S \setminus R)^{*m} \setminus (R)^{*m}]$.

(*iii*) By Theorem 4.4 (*i*), we have for $S, R \subseteq \hat{G}$, $(S \cup R)^{*m} = (S)^{*m} \cup (R)^{*m}$.

Further the earlier result we have $R \notin G$ implies $(R)^{*m} = 0_N$, so $(S)^{*m} \cup (R)^{*m} = (S)^{*m}$.

We have, $S \setminus R \subseteq S \Rightarrow (S \setminus R)^{*m} \subseteq (S)^{*m}$, by part (*iii*) we have,

$[(S)^{*m} \setminus (R)^{*m}] \subseteq (S \setminus R)^{*m}$, since $(R)^{*m} = 0_N$ implies $(S)^{*m} \subseteq (S \setminus R)^{*m}$.

Thus, we have $(S)^{*m} = (S \setminus R)^{*m}$.

Definition 4.3. Suppose that $(\hat{G}, M, G)$ be an N-G-M-space. Then, the mapping N_{clmG}: $P(\hat{G}) \rightarrow P(\hat{G})$ is define by $N_{clmG}(S) = S \cup (S)^{*m}$, for all $S \in P(\hat{G})$.

Theorem 4.4. The mapping N_{clmG}: $P(\hat{G}) \rightarrow P(\hat{G})$ satisfies the Kuratowski closure axioms.

Proof. We have, $N_{clmG}(0_N) = 0_N \cup (0_N)^{*m} = 0_N \cup 0_N = 0_N$. By the definition of N_{clmG}, we have for all $S \in P(\hat{G})$, $N_{clmG}(S) = S \cup (S)^{*m} \supseteq S$.

Further, we have

$$N_{clmG}(S \cup R) = (S \cup R) \cup (S \cup R)^{*m}$$

$$= ((S \cup R) \cup ((S)^{*m} \cup (R)^{*m}) \quad \text{[by Theorem 4.3]}$$

$$= (S \cup (S)^{*m}) \cup (R \cup (R)^{*m}) = N_{clmG}(S) \cup N_{clmG}(R).$$

Suppose that $S \in P(\hat{G})$. Then, we have

$$N_{clmG}(N_{clmG}(S)) = N_{clmG}(S \cup (S)^{*m})$$

$$= [(S \cup (S)^{*m})] \cup [(S \cup (S)^{*m}]^{*m}$$

$$= [(S \cup (S)^{*m})] \cup [(S)^{*m} \cup ((S)^{*m})^{*m}] \quad \text{[by Theorem 4.3]}$$

$$= S \cup (S)^{*m}$$

Hence, the mapping N_{clmG} satisfies the Kuratowski closure axioms.

5. Conclusions: In this article, we have introduced the notion of minimal structure and grill via N-T-spaces. Besides, we have established its relation with some existing notions on several types of open sets via N-T-spaces, and investigated some basic properties of the class of minimal structures and grill via N-T-spaces. Further, we have furnished some suitable examples on minimal structures and grill via N-T-spaces. In the future, it is hoped that the notion of minimal structures and grill on N-T-spaces can also be applied in neutrosophic supra topological space [5], quadripartitioned neutrosophic topological space [6], pentapartitioned neutrosophic topological space [12], neutrosophic bitopological space [18], neutrosophic tri-topological space [10], neutrosophic soft topological space [9], neutrosophic multiset topological space [14], multiset mixed topological space [4], etc.

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Interval quadripartitioned neutrosophic sets

Surapati Pramanik *

Nandalal Ghosh B. T. College, Department of Mathematics, Panpur, Narayanpur, Dist- North 24 Parganas, West Bengal, India-743126, *e-mail:surapati.math@gmail.com; sura_pati@yahoo.co.in; mob no:+919477035544

Abstract

Quadripartitioned neutrosophic set is a mathematical tool, which is the extension of neutrosophic set and n-valued neutrosophic refined logic for dealing with real-life problems. A generalization of the notion of quadripartitioned neutrosophic set is introduced. The new notion is called the Interval Quadripartitioned Neutrosophic set (IQNS). The interval quadripartitioned neutrosophic set is developed by combining the quadripartitioned neutrosophic set and interval neutrosophic set. Several set theoretic operations of IQNSs, namely, inclusion, complement, and intersection are defined. Various properties of set-theoretic operators of IQNS are established.

Keywords: Neutrosophic set, Single valued neutrosophic set, Interval neutrosophic set, quadripartitioned neutrosophic set, Interval quadripartitioned neutrosophic set

1. Introduction

Chatterjee et al. [1] defined Quadripartitioned Single Valued Neutrosophic Set (QSVNS) by utilizing the concept of Single Valued Neutrosophic Set (SVNS) [2], four valued logic [3] and n- valued refined logic [4] that involves degrees of truth, falsity, unknown and contradiction membership. Chatterjee et al. [5] investigated interval-valued possibility quadripartitioned single valued neutrosophic soft sets by generalizing the possibility intuitionistic fuzzy soft set [6].

No investigation regarding Interval Quadripartitioned Neutrosophic Set (IQNS) is reported in the literature. The motivation of the present work comes from the works of Chatterjee et al. [1, 5]. The notion of IQNS is developed by combining the concept of QSVNS and Interval Neutrosophic Set (INS) [7]. The proposed structure is a generalization of existing theories of INS and QSVNS.

The organization of the remainder of the paper is presented in table 1.

Table 1. Outline of the paper

Section	Content
2	Some preliminary results.
3	The concept of IQNS and set-theoretic operations over IQNS are introduced.
4	Conclusion and scope of further research are presented.

2. Preliminary

Definition 2.1. Assume that a set W is fixed. An NS [8] H over W is defined as:

$H = \{w, (T_H(w), I_H(w), F_H(w)) : w \in W\}$ where $T_H, I_H, F_H : W \to]^-0, 1^+[$ and

$^-0 \le T_H(w) + I_H(w) + F_H(w) \le 3^+$.

Definition 2.2. Assume that a set W is fixed. An SVNS [2] H over W is defined as:

$H = \{w, (T_H(w), I_H(w), F_H(w)) : w \in W\}$ where $T_H, I_H, F_H : W \to [0,1]$ and

$0 \le T_H(w) + I_H(w) + F_H(w) \le 3$.

Definition 2.3. Let a set W be fixed. An INS [7] H over W is defined as:

$H = \{(w, (T_H(w), I_H(w), F_H(w)) : w \in W\}$

where for each $w \in W$, $T_H(w), I_H(w), F_H(w) \subseteq [0,1]$ are the degrees of membership functions of truth, indeterminacy, and falsity and

$T_H(w) = [\inf T_H(w), \sup T_H(w)], I_H(w) = [\inf I_H(w), \sup I_H(w)]), F_H(w) = [\inf F_H(w), \sup F_H(w)]$ and

$0 \le \sup T_H(w) + \sup I_H(w) + \sup F_H(w) \le 3.$

H can be expressed as:

$H = \{w, ([\inf T_H(w), \sup T_H(w)], [\inf I_H(w), \sup I_H(w)]), [\inf F_H(w), \sup F_H(w)]) : w \in W\}$

2.4. Let a set W be fixed. A QSVNS [1] H over W is defined as:

H= {(w, $T_H(w)$, $C_H(w)$, $U_H(w)$, $F_H(w)$): w∈W}, where for each point $w \in W$, $T_H(w)$, $C_H(w)$, $U_H(w)$, $F_H(w)$ $\to [0,1]$ are the degrees of membership functions of truth, contradiction, ignorance, and falsity and

$0 \le \sup T_H(w) + \sup C_H(w) + \sup U_H(w) + \sup F_H(w) \le 4$.

3. The Basic Theory of IQNSs

Definition 3.1. IQNS

Let *W* be a fixed set. Then, an IQNS over *W* is denoted by H and defined as follows:

H= {(w, $T_H(w)$, $C_H(w)$, $U_H(w)$, $F_H(w)$): w∈W}, where for each point $w \in W$, $T_H(w)$, $C_H(w)$, $U_H(w)$, $F_H(w)$ $\subseteq [0, 1]$ are the degrees of membership functions of truth, contradiction, ignorance, and falsity and $T_H(w)$= [inf T_H(w, sup $T_H(w)$] , $C_H(w)$ =[inf $C_H(w)$, sup $C_H(w)$] $U_{H,}(w)$ = [inf $U_H(w)$, sup $U_H(w)$], $F_H(w)$ = [inf $F_H(w)$, sup $F_H(w)$] $\subseteq [0, 1]$ and

$0 \le \sup T_H(w) + \sup C_H(w) + \sup U_H(w) + \sup F_H(w) \le 4$.

An IQNS in R^1 is illustrated in Figure 1.

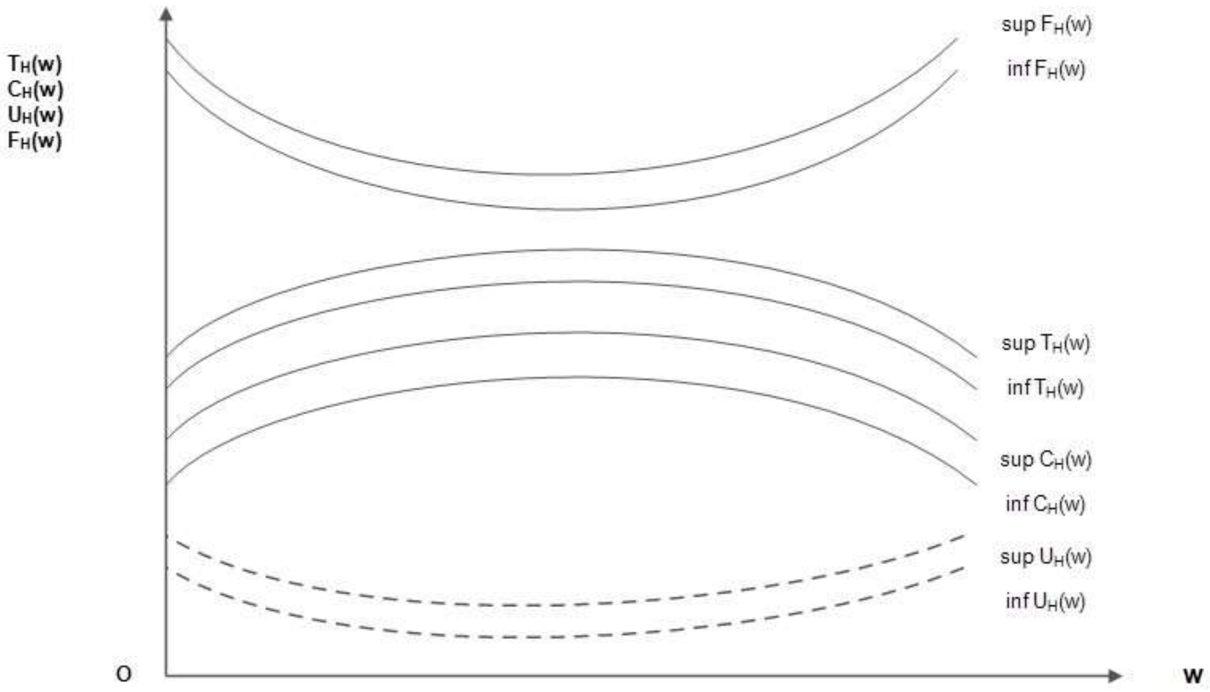

Figure 1. Illustration of an IQNS in R^1

Example 3.1. Suppose that W = [w_1, w_2, w_3], where w_1, w_2, and w_3 present respectively the capability, trustworthiness, and price. The values of w_1, w_2, and w_3 are in [0, 1]. They are obtained from the questionnaire of some domain experts, their option could be degree of truth (good), degree of contradiction, degree of ignorance, and degree of false (poor). H_1 is an IQNS of *W* defined by

H_1 = {[0.5, 0.7], [0.15, 0.2], [0.2, 0.4], [0.2, 0.3]}/w_1 + {[0.55, 0.85], [0.25, 0.35], [0.15, 0.25], [0.2, 0.3]}/w2+ [0.65, 0.85], [0.2, 0.35], [0.1, 0.25], [0.15, 0.25]}/ w_3

H_2 is an IPNS of *W* defined by

H_2 = {[0.6, 0.8], [0.1, 0.2], [0.1, 0.25], [0.15, 0.3]}/w_1 + {[0.6, 0.9], [0.25, 0.3], [0.1, 0.2], [0.1, 0.3]}/w2+ [0.5, 0.7], [0.1, 0.2], [0.15, 0.25], [0.1; 0.2]}/ w_3

Definition 3.2. An IQNS H is said to be empty (null) denoted by $\hat{0}$ iff

inf $T_H(w)$= sup $T_H(w)$ = 0, inf $C_H(w)$= sup $C_H(w)$ = 0, , inf $U_H(w)$ = sup $U_H(w)$= 1, inf $F_H(w)$= sup $F_H(w)$ = 1 ,

$$\hat{0} = \{[0,0],[0,0],[1,1],[1,1]\}$$

Definition 3.3. An IQNS H is said to be unity *denoted by* $\hat{1}$ iff

inf $T_H(w)$ = sup $T_H(w)$ = 1, inf $C_H(w)$ = sup $C_H(w)$ = 1, inf $U_H(w)$ = sup $U_H(w)$ = 0,
inf $F_H(w)$= sup $F_H(w)$ = 0

$\hat{1} = \{[1,1],[1,1],[0,0],[0,0]\}$

Also, we have $\underline{0} = \langle 0,0,1,1\rangle \, and \, \underline{1} = \langle 1,1,0,0\rangle$

Definition 3.4. (Containment) Let H_1 and H_2 be any two IQNS over W, H_1 is said to be contained in H_2, denoted by $H_1 \subseteq H_2$ iff

for any $w \in W$,

$$\begin{aligned}
&\inf T_{H_1}(w) \le \inf T_{H_2}(w), \sup T_{H_1}(w) \le \sup T_{H_2}(w),\\
&\inf C_{H_1}(w) \le \inf C_{H_2}(w), \sup C_{H_1}(w) \le \sup C_{H_2}(w),\\
&\inf U_{H_1}(w) \ge \inf U_{H_2}(w), \sup U_{H_1}(w) \ge \sup U_{H_2}(w),\\
&\inf F_{H_1}(w) \ge \inf F_{H_2}(w), \sup F_{H_1}(w) \ge \sup F_{H_2}(w),
\end{aligned}$$

Definition 3.5. Any two IQNSs H_1 and H_2 are equal iff $H_1 \subseteq H_2$ and $H_1 \supseteq H_2$

Definition 3.6. (Complement) Let $H = \{(w, T_H(w), C_H(w), U_H(w), F_H(w)) : w \in W\}$ be an IQNS. The complement of H is denoted by H' and defined as:

$$T_{H'}(w) = F_H(w),\ C_{H'}(w) = U_H(w),\ U_{H'}(w) = C_H(w), F_{H'}(w) = T_H(w)$$

$$H' = \{(w, [\inf F_H(w), \sup F_H(w)], [\inf U_H(w), \sup U_H(w)], [\inf C_H(w), \sup C_H(w)], [\inf T_H(w), \sup T_H(w)]) : w \in W\}$$

Example 3.2 . Consider an IQNS H of the form:

H = {[0.35, 0.75], [0.2, 0.25], [0.2, 0.3], [0.2, 0.4]}/w_1 + {[0.55, 0.85], [0.2, 0.3], [0.15, 0.25], [0.2, 0.35]}/w2+ [0.75, 0.85], [0.15, 0.25], [0.15, 0.25], [0.1, 0.25]}/ w_3

Then, complement of

H' == {[0.2, 0.4] , [0.2, 0.3], [0.2, 0.25], [0.35, 0.75]}/w_1 + {[0.2, 0.35], [0.15, 0.25], [0.2, 0.3], [0.55, 0.85]}/w2+ [0.1, 0.25], [0.15, 0.25], [0.15, 0.25], [0.75, 0.85]}/ w_3

Definition 3.7.(Intersection)

The intersection of any two IQNSs H_1 and H_2 is an IQNS, denoted as H_3 and presented as:

$H_3 = H_1 \cap H_2$

$$\{(w, [\inf T_{H_3}(w), \sup T_{H_3}(w)], [\inf C_{H_3}(w), \sup C_{H_3}(w)], [\inf U_{H_3}(w), \sup U_{H_3} w)], [\inf F_{H_3}(w), \sup F_{H_3}(w)]) : w \in W\}.$$

$$= \begin{aligned}
&\{(w, [\min(\inf T_{H_1}(w), \inf T_{H_2}(w)), \min(\sup T_{H_1}(w), \sup T_{H_2}(w))],\\
&[\min(\inf C_{H_1}(w), \inf C_{H_2}(w)), \min(\sup C_{H_1}(w), \sup C_{H_2}(w))],\\
&[\max(\inf U_{H_1}(w), \inf U_{H_2}(w))], \max(\sup U_{H_1}(w), \sup U_{H_2}(w))],\\
&[\max(\inf F_{H_1}(w), \inf F_{H_2}(w)), \max(\sup F_{H_1}(w), \sup F_{H_2}(w))]) : w \in W\}
\end{aligned}$$

Example 3.3. Let H_1 and H_2 be the IQNSs defined in Example 3.1.

Then, $H_1 \cap H_2$ ={[0.5, 0.7], [0.1, 0.2], [0.2, 0.4], [0.2, 0.3]}/w_1 + {[0.55, 0.85], [0.25, 0.3], [0.15, 0.25], [0.2, 0.3]}/w2+ [0.5, 0.7], [0.1, 0.2], [0.15, 0.25], [0.15, 0.25]}/ w_3

Definition 3.8. (**Union**) The union of any two IQNSs H_1 and H_2 is an IQNS denoted as H_3, and presented as:

$H_3 = H_1 \cup H_2$
$\{(w,[\inf T_{H_3}(w), \sup T_{H_3}(w)],[\inf C_{H_3}(w), \sup C_{H_3}(w)],[\inf U_{H_3}(w), \sup U_{H_3} w)],$
$[\inf F_{H_3}(w), \sup F_{H_3}(w)]) : w \in W\}.\}.$
$= \{(w,[\max(\inf T_{H_1}(w), \inf T_{H_2}(w)), \max(\sup T_{H_1}(w), \sup T_{H_2}(w))],[\max(\inf C_{H_1}(w), \inf C_{H_2}(w)), \max(\sup C_{H_1}(w), \sup C_{H_2}(w))],$
$[\min(\inf U_{D_1}(w), \inf U_{D_2}(w)), \min(\sup U_{D_1}(w), \sup U_{D_2}(w))],[\min(\inf F_{D_1}(w), \inf F_{D_2}(w)), \min(\sup F_{D_1}(w), \sup F_{D_2}(w))]) : w \in W\}.$

Example 3. 4. Let H_1 and H_2 be the IQNSs in example 3.1. Then

$H_1 \cup H_2$ ={[0.6, 0.8], [0.15, 0.2], [0.1, 0.25], [0.15, 0.3]}/w1 + {[0.6, 0.9], [0.25, 0.35], [0.1, 0.2], [0.1, 0.3]}/w2+ [0.65, 0.85], [0.2, 0.35], [0.1, 0.25], [0.1, 0.2]}/ w3

Theorem 3.1 Let H_1 and H_2 be any two IQNSs over W *defined by*
$H_i = \{(w, T_{H_i}(w), C_{H_i}(w), G_{H_i}(w), U_{H_i}(w), F_{H_i}(w))]) : w \in W\}, i = 1, 2,$ and
$T_{H_i}(w), C_{H_i}(w), G_{H_i}(w), U_{H_i}(w), F_{H_i}(w))] \subseteq [0,1], i = 1, 2.$

Then
(a)$H_1 \cup H_2 = H_2 \cup H_1$
(b)$H_1 \cap H_2 = H_2 \cap H_1$

Proof. (a):

$H_1 \cup H_2 = \{(w,[\max(\inf T_{H_1}(w), \inf T_{H_2}(w)), \max(\sup T_{H_1}(w), \sup T_{H_2}(w))],$
$[\max(\inf C_{H_1}(w), \inf C_{H_2}(w)), \max(\sup C_{H_1}(w), \sup C_{H_2}(w))],$
$[\min(\inf U_{H_1}(w), \inf U_{H_2}(w)), \min(\sup U_{H_1}(w), \sup U_{H_2}(w))],$
$[\min(\inf F_{H_1}(w), \inf F_{H_2}(w)), \min(\sup F_{H_1}(w), \sup F_{H_2}(w))]) : w \in W\}$
$= \{w,[\max(\inf T_{H_2}(w), \inf T_{H_1}(w)), \max(\sup T_2(w), \sup T_{H_1}(w))],[\max(\inf C_{H_2}(w), \inf C_{H_1}(w)), \max(\sup C_{H_2}(w), \sup C_{H_1}(w))],$
$[\min(\inf U_{H_2}(w), \inf U_{H_1}(w)), \min(\sup U_{H_2}(w), \sup U_{H_1}(w))],[\min(\inf F_{H_2}(w), \inf F_{H_1}(w)), \min(\sup F_{H_2}(w), \sup F_{H_1}(w))] : w \in W\}$
$= H_2 \cup H_1$

Proof. (b):

$H_1 \cap H_2 = \{(w,[\min(\inf T_{H_1}(w), \inf T_{H_2}(w)), \min(\sup T_{H_1}(w), \sup T_{H_2}(w))],$
$[\min(\inf C_{H_1}(w), \inf C_{H_2}(w)), \min(\sup C_{H_1}(w), \sup C_{H_2}(w))],$
$[\max(\inf U_{H_1}(w), \inf U_{H_2}(w)), \max(\sup U_{H_1}(w), \sup U_{H_2}(w))],$
$[\max(\inf F_{H_1}(w), \inf F_{H_2}(w)), \max(\sup F_{H_1}(w), \sup F_{H_2}(w))]) : w \in W\}.$
$= \{(w,[\min(\inf T_{H_2}(w), \inf T_{H_1}(w)), \min(\sup T_{H_2}(w), \sup T_{H_1}(w))],$
$[\min(\inf C_{H_2}(w), \inf C_{H_1}(w)), \min(\sup C_{H_2}(w), \sup C_{H_1}(w))],$
$[\max(\inf U_{H_2}(w), \inf U_{H_1}(w)), \max(\sup U_{H_2}(w), \sup U_{H_1}(w))],$
$[\max(\inf F_{H_2}(w), \inf F_{H_1}(w)), \max(\sup F_{H_2}(w), \sup F_{H_1}(w))]) : \forall w \in W\}.$
$= H_2 \cap H_1$

Theorem 3.2. For any two IPNS, H_1, and H_2:

(a) $H_1 \cup (H_1 \cap H_2) = H_1$
(b) $H_1 \cap (H_1 \cup H_2) = H_1$

Proof .(a):

$H_1 \cup (H_1 \cap H_2) =$
$\{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [[\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$\cup \{(w, [\min(\inf T_{H_1}(w), \inf T_{H_2}(w)), \min(\sup T_{H_1}(w), \sup T_{H_2}(w))],$
$[\min(\inf C_{H_1}(w), \inf C_{H_2}(w)), \min(\sup C_{H_1}(w), \sup C_{H_2}(w))],$
$[\max(\inf U_{H_1}(w), \inf U_{H_2}(w)), \max(\sup U_{H_1}(w), \sup U_{H_2}(w))],$
$[\max(\inf F_{H_1}(w), \inf F_{H_2}(w)), \max(\sup F_{H_1}(w), \sup F_{H_2}(w))]) : w \in W\}.$
$= \{w, ([\max(\inf T_{H_1}(w), \min(\inf T_{H_1}(w), \inf T_{H_2}(w))), \max(\sup T_{H_1}(w), \min(\sup T_{H_1}(w), \sup T_{H_2}(w)))],$
$[\max(\inf C_{H_1}(w), \min(\inf C_{H_1}(w), \inf C_{H_2}(w))), \max(\sup C_{H_1}(w), \min(\sup C_{H_1}(w), \sup C_{H_2}(w)))]),$
$[\min(\inf (U_{H_1}(w), \max(\inf U_{H_1}(w), \inf U_{H_2}(w))), \min(\sup U_{H_1}(w), \max(\sup U_{H_1}(w), \sup U_{H_2}(w)))],$
$[\min(\inf (F_{H_1}(w), \max(\inf F_{H_1}(w), \inf F_{H_2}(w))), \min(\sup F_{H_1}(w), \max(\sup F_{H_1}(w), \sup F_{H_2}(w)))]) : w \in W\}$
$= \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [[\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$= H_1$

Proof (b):

$H_1 \cap (H_1 \cup H_2)$
$= \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]),$
$[\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\} \cap$
$\{(w, [\max(\inf T_{H_1}(w), \inf T_{H_2}(w)), \max(\sup T_{H_1}(w), \sup T_{H_2}(w))],$
$[\max(\inf C_{H_1}(w), \inf C_{H_2}(w)), \max(\sup C_{H_1}(w), \sup C_{H_2}(w))],$
$[\min(\inf U_{H_1}(w), \inf U_{H_2}(w)), \min(\sup U_{H_1}(w), \sup U_{H_2}(w))],$
$[\min(\inf F_{H_1}(w), \inf F_{H_2}(w)), \min(\sup F_{H_1}(w), \sup F_{H_2}(w))]) : w \in W\}$
$= \{w, [\min(\inf T_{H_1}(w), \max(\inf T_{H_1}(w), \inf T_{H_2}(w))), \min(\sup T_{H_1}(w), \max(\sup T_{H_1}(w), \sup T_{H_2}(w)))],$
$[\min(\inf C_{H_1}(w), \max(\inf C_{H_1}(w), \inf C_{H_2}(w))), \min(\sup C_{H_1}(w), \max(\sup C_{H_1}(w), \sup C_{H_2}(w)))],$
$[\max(\inf U_{H_1}(w), \min(\inf U_{H_1}(w), \inf U_{H_2}(w))), \max(\sup U_{H_1}(w), \min(\sup U_{H_1}(w), \sup U_{H_2}(w)))],$
$[\max(\inf F_{H_1}(w), \min(\inf F_{H_1}(w), \inf F_{H_2}(w))), \max(\sup F_{H_1}(w), \min(\sup F_{H_1}(w), \sup F_{H_2}(w)))]$
$= \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [[\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$= H_1$

Theorem 3.3. For any IPNS H_1**:**

(a) $H_1 \cup H_1 = H_1$

(b) $H_1 \cap H_1 = H_1$

Proof. (a):

$H_1 \cup H_1 = \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)],$
$[\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\} \cup \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]),$
$[\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$= \{w, ([\max(\inf T_{H_1}(w), \inf T_{H_1}(w)), \max(\sup T_{H_1}(w), \sup T_{H_1}(w))],$
$[\max(\inf C_{H_1}(w), (\inf C_{H_1}(w))), \max(\sup T_{H_1}(w), \sup C_{H_1}(w)]),$
$[\min(\inf U_{H_1}(w), \inf U_{H_1}(w)), (\min(\sup U_{H_1}(w), (\sup U_{H_1}(w))],$
$[\min(\inf F_{H_1}(w), \inf F_{H_1}(w)), \min(\sup F_{H_1}(w), \sup F_{H_1}(w))]) : w \in W\}$
$= \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$= H_1$

Proof. (b):

$H_1 \cap H_1 = \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)],$
$[\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\} \cap \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]),$
$[\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$

$\{(w, [\min(\inf T_{H_1}(w), \inf T_{H_1}(w)), \min(\sup T_{H_1}(w), \sup T_{H_1}(w))],$
$[\min(\inf C_{H_1}(w), \inf C_{H_1}(w)), \min(\sup C_{H_1}(w), \sup C_{H_1}(w))],$
$[\max(\inf U_{H_1}(w), \inf U_{H_1}(w)), \max(\sup U_{H_1}(w), \sup U_{H_1}(w))],$
$[\max(\inf F_{H_1}(w), \inf F_{H_1}(w)), \max(\sup F_{H_1}(w), \sup F_{H_2}(w))]) : w \in W\}.$
$== \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$= H_1$

Theorem 3.4 For any IQNS H_1,

$(a) H_1 \cap \hat{0} = \hat{0}$
$(b) H_1 \cup \hat{1} = \hat{1}$

Proof. (a):

$H_1 \cap \hat{0}$
$= \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$\cap \{[0,0], [0,0], [1,1], [1,1], [1,1]\}$
$= \{(w, [\min(\inf T_{H_1}(w), 0), \min(\sup T_{H_1}(w), 0)],$
$[\min(\inf C_{H_1}(w), 0), \min(\sup C_{H_1}(w), 0)],$
$[\max(\inf U_{H_1}(w), 1), \max(\sup U_{H_1}(w), 1)],$
$[\max(\inf F_{H_1}(w), 1), \max(\sup F_{H_1}(w), 1]) : w \in W\}$
$= \{(w, [0,0)], [0,0)], [1,1], [1,1]) : w \in W\}$
$= \hat{0}$

Proof. (b):

$H_1 \cup \hat{1}$
$= \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$\cup \{[1,1], [1,1], [0,0], [0,0]\}$
$= \{(w, [\max(\inf T_{H_1}(w), 1), \max(\sup T_{H_1}(w), 1))], [\max(\inf C_{H_1}(w), 1), \max(\sup C_{H_1}(w), 1)], [\min(\inf U_{H_1}(w), 0), \min(\sup U_{H_1}(w), 0)],$
$[\min(\inf F_{H_1}(w), 0)), \min(\sup F_{H_1}(w), 0)]) : w \in W\}$
$= \{w([1,1], [1,1], [0,0], [0,0], [0,0]) : w \in W\}$
$= \hat{1}$

Theorem 3.5. For any IQNS H_1,

$(a) H_1 \cup \hat{0} = H_1$
$(b) H_1 \cap \hat{1} = H_1$

Proof. (a):

$H_1 \cup \hat{0}$
$= \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$\cup \{[0,0], [0,0], [1,1], [1,1]\}$
$= \{(w, [\max(\inf T_{H_1}(w), 0), \max(\sup T_{H_1}(w), 0))], [\max(\inf C_{H_1}(w), 0), \max(\sup C_{H_1}(w), 0)],$
$[\min(\inf U_{H_1}(w), 1), \min(\sup U_{H_1}(w), 1)], [\min(\inf F_{H_1}(w), 1)), \min(\sup F_{H_1}(w), 1)]) : w \in W\}$
$= \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$= H_1$

$H_1 \cap \hat{1}$
$= \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$\cap \{[1,1, [1,1, [0,0], [0,0]\}$
$= \{(w, [\min(\inf T_{H_1}(w), 1), \min(\sup T_{H_1}(w), 1)], [\min(\inf C_{H_1}(w), 1), \min(\sup C_{H_1}(w), 1)],$
$[\max(\inf U_{H_1}(w), 0), \max(\sup U_{H_1}(w), 0)])[\max(\inf F_{H_1}(w), 0), \max(\sup F_{H_1}(w), 0)]) : w \in W\}$
$= \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W$
$= H_1$

Theorem 3.6. For any IQNS H_1, $(H_1')' = H_1$

Let $H_1 = \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$H_1' = \{w, ([\inf F_{H_1}(w), \sup F_{H_1}(w)], [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf T_{H_1}(w), \sup T_{H_1}(w)]) : w \in W\}$
$\therefore (H_1')' = \{w, ([\inf T_{H_1}(w), \sup T_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf F_{H_1}(w), \sup F_{H_1}(w)]) : w \in W\}$
$= H_1$

Theorem 3.7. For any two IQNSs, H_1 and H_2:

(a) $(H_1 \cup H_2)' = H_1' \cap H_2'$
(b) $(H_1 \cap H_2)' = H_1' \cup H_2'$

Proof. (a):

$H_1 \cup H_2 = \{(w, [\max(\inf T_{H_1}(w), \inf T_{H_2}(w)), \max(\sup T_{H_1}(w), \sup T_{H_2}(w))],$
$[\max(\inf C_{H_1}(w), \inf C_{H_2}(w)), \max(\sup C_{H_1}(w), \sup C_{H_2}(w))],$
$[\min(\inf U_{H_1}(w), \inf U_{H_2}(w)), \min(\sup U_{H_1}(w), \sup U_{H_2}(w))],$
$[\min(\inf F_{H_1}(w), \inf F_{H_2}(w)), \min(\sup F_{H_1}(w), \sup F_{H_2}(w))],$

$\therefore (H_1 \cup H_2)' = \{(w, [\min(\inf F_{H_1}(w), \inf F_{H_2}(w)), \min(\sup F_{H_1}(w), \sup F_{H_2}(w))],$
$[\min(\inf U_{H_1}(w), \inf U_{H_2}(w)), \min(\sup U_{H_1}(w), \sup U_{H_2}(w))],$
$[\max(\inf C_{H_1}(w), \inf C_{H_2}(w)), \max(\sup C_{H_1}(w), \sup C_{H_2}(w))],$
$[\max(\inf F_{H_1}(w), \inf T_{H_2}(w)), \max(\sup T_{H_1}(w), \sup T_{H_2}(w))]) : w \in W\}$ (1)

$$H_1' \cap H_2' = \{w, ([\inf F_{H_1}(w), \sup F_{H_1}(w)], [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf T_{H_1}(w), \sup T_{H_1}(w)]) : w \in W\}$$
$$\bigcap \{w, ([\inf F_{H_2}(w), \sup F_{H_2}(w)], [\inf U_{H_2}(w), \sup U_{H_2}(w)], [\inf C_{H_2}(w), \sup C_{H_2}(w)]), [\inf T_{H_2}(w), \sup T_{H_2}(w)]) : w \in W\}$$
$$= \{(w, [\min(\inf F_{H_1}(w), \inf F_{H_2}(w)), \min(\sup F_{H_1}(w), \sup F_{H_2}(w))],$$
$$[\min(\inf U_{H_1}(w), \inf U_{H_2}(w)), \min(\sup U_{H_1}(w), \sup U_{H_2}(w))],$$
$$[\max(\inf C_{H_1}(w), \inf C_{H_2}(w)), \max(\sup C_{H_1}(w), \sup C_{H_2}(w))],$$
$$[\max(\inf F_{H_1}(w), \inf T_{H_2}(w)), \max(\sup T_{H_1}(w), \sup T_{H_2}(w))]) : w \in W\} \qquad (2)$$

Therefore from (1) and (2), $(H_1 \cup H_2)' = H_1' \cap H_2'$

Proof. (b):

$$(H_1 \cap H_2) = \{(w, [\min(\inf T_{H_1}(w), \inf T_{H_2}(w)), \min(\sup T_{H_1}(w), \sup T_{H_2}(w))],$$
$$[\min(\inf C_{H_1}(w), \inf C_{H_2}(w)), \min(\sup C_{H_1}(w), \sup C_{H_2}(w))],$$
$$[\max(\inf U_{H_1}(w), \inf U_{H_2}(w)), \max(\sup U_{H_1}(w), \sup U_{H_2}(w))],$$
$$[\max(\inf F_{H_1}(w), \inf F_{H_2}(w)), \max(\sup F_{H_1}(w), \sup F_{H_2}(w))]) : w \in W\}'.$$
$$\therefore (H_1 \cap H_2)' = \{(w, [\max(\inf F_{H_1}(w), \inf F_{H_2}(w)), \max(\sup F_{H_1}(w), \sup F_{H_2}(w))]),$$
$$[\max(\inf U_{H_1}(w), \inf U_{H_2}(w)), \max(\sup U_{H_1}(w), \sup U_{H_2}(w))],$$
$$[\min(\inf C_{H_1}(w), \inf C_{H_2}(w)), \min(\sup C_{H_1}(w), \sup C_{H_2}(w))],$$
$$[\min(\inf T_{H_1}(w), \inf T_{H_2}(w)), \min(\sup T_{H_1}(w), \sup T_{H_2}(w))]) : w \in W\} \qquad (3)$$

Now

$$H_1' \cup H_2' = \{w, ([\inf F_{H_1}(w), \sup F_{H_1}(w)], [\inf U_{H_1}(w), \sup U_{H_1}(w)], [\inf C_{H_1}(w), \sup C_{H_1}(w)]), [\inf T_{H_1}(w), \sup T_{H_1}(w)]) : w \in W\} \cup$$
$$\{w, ([\inf F_{H_2}(w), \sup F_{H_2}(w)], [\inf U_{H_2}(w), \sup U_{H_2}(w)], [\inf C_{H_2}(w), \sup C_{H_2}(w)]), [\inf T_{H_2}(w), \sup T_{H_2}(w)]) : w \in W\}$$
$$= \{w, ([\max(\inf F_{H_1}(w), \inf F_{H_2}(w)), \max(\sup F_{H_1}(w)], \sup F_{H_2}(w)], [\max(\inf U_{H_1}(w), \inf U_{H_2}(w)), \max(\sup U_{H_1}(w), \sup U_{H_2}(w))],$$
$$[\min(\inf C_{H_1}(w), \inf C_{H_2}(w)), \min(\sup C_{H_1}(w), \sup C_{H_2}(w))], [\min(\inf T_{H_1}(w), \inf T_{H_2}(w)), \min(\sup T_{H_1}(w), \sup T_{H_2}(w))]) : w \in W\} \quad (4)$$

Therefore, from (3) and (4), $(H_1 \cap H_2)' = H_1' \cup H_2'$

Theorem 3.9. For any two IPNS H_1, H_2,

$$H_1 \subseteq H_2 \Leftrightarrow H_2' \subseteq H_1'.$$

Proof.

$$H_1 \subseteq H_2 \Leftrightarrow$$
$$\inf T_{H_1}(w) \le \inf T_{H_2}(w), \sup T_{H_1}(w) \le \sup T_{H_2}(w),$$
$$\inf C_{H_1}(w) \le \inf C_{H_2}(w), \sup C_{H_1}(w) \le \sup C_{H_2}(w),$$
$$\inf U_{H_1}(w) \ge \inf U_{H_2}(w), \sup U_{H_1}(w) \ge \sup U_{H_2}(w),$$
$$\inf F_{H_1}(w) \ge \inf F_{H_2}(w), \sup F_{H_1}(w) \ge \sup F_{H_2}(w),$$
$$\Leftrightarrow$$
$$\inf F_{H_2}(w) \le \inf F_{H_1}(w), \sup F_{H_2}(w) \le \sup F_{H_1}(w),$$
$$\inf U_{H_2}(w) \le \inf U_{H_1}(w), \sup U_{H_2}(w) \le \sup U_{H_1}(w),$$
$$\inf C_{H_2}(w) \le \inf C_{H_1}(w), \sup C_{H_2}(w) \le \sup C_{H_1}(w)$$
$$\inf T_{H_2} \ge \inf T_{H_1}, \ \sup T_{H_2}(w) \ge \sup T_{H_1}(w)$$
$$\Leftrightarrow$$
$$H_2' \subseteq H_1'$$

Note: Proposed IQNS can also be called as Interval Quadripartitioned Single Valued Neutrosophic Set (IQSVNS).

4. Conclusions

In this paper, the notion of IQNS is introduced by combining the QSVNS and INS. The notion of inclusion, complement, intersection, union of IQNSs are defined. Some of the properties of IQNSs, are established. In the future, the logic system based on the truth-value based IQNSs will be investigated and the theory can be used to solve real-life problems in the areas such as information fusion, bioinformatics, web intelligence, etc. Further it is hoped that the proposed IQNS is applicable in neutrosophic decision making [9-11]and graph theory dealing with uncertainty [12-14], etc.

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Lie-Algebra of Single-Valued Pentapartitioned Neutrosophic Set

Suman Das[1], Rakhal Das[2], Bimal Shil[3,*], and Binod Chandra Tripathy[4]

[1]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.

[1]E-mail: sumandas18842@gmail.com, dr.suman1995@yahoo.com

[2]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.

[2]E-mail: rakhaldas95@gmail.com, rakhal.mathematics@tripurauniv.in

[3]Department of Statistics, Tripura University, Agartala, 799022, Tripura, India.

[3]E-mail: bimalshil738@gmail.com

[4]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.

[4]E-mail: tripathybc@gmail.com, tripathybc@yahoo.com

***Correspondence:** bimalshil738@gmail.com

Abstract:

In this article, we procure the concept of single-valued pentapartitioned neutrosophic Lie (in short SVPN-Lie) algebra under single-valued pentapartitioned neutrosophic set (in short SVPN-set) environment. Besides, we study the notion of SVPN-Lie ideal of SVPN-Lie algebra, and produce several interesting results on SVPN-Lie algebra and SVPN-Lie ideal.

Keywords: *Lie-ideal*; *Lie-algebra*; *Neutrosophic Set*; *SVPN-set*; *SVPN-Lie ideal*.

1. Introduction:

In nineteenth century, Sophus Lie grounded the concept of Lie groups. Sophus Lie also discovered the notion of Lie algebra. Thereafter, Humphreys [30] introduced the concept of representation theory of Lie algebra in 1972. In 2003, Coelho and Nunes [10] proposed an application of Lie algebra to mobile robot control. Till now, the concept of Lie theory has been applied in mathematics, physics, continuum mechanics, cosmology and life sciences. The problems of computer vision can also be solved by using the idea of Lie algebra. In 1965, Zadeh [40] grounded the notion of Fuzzy Set (in short FS) theory. Afterwards, Yehia [38] presented the concept of Fuzzy-Lie ideals and Fuzzy-Lie sub-algebra of Lie algebra in 1996. Later on, Yehia [39] also studied the adjoint representation of Fuzzy-Lie algebra. In 1998, Kim and Lee [31] further studied the Fuzzy-Lie ideals and Fuzzy-Lie sub-algebra. The notion of anti-Fuzzy-Lie ideals of Lie algebra was studied by Akram [1]. Later on, Akram [4] studied the concept of generalized Fuzzy-Lie sub-algebra in 2008. The concept of Fuzzy-Lie ideals of Lie algebra with the interval-valued membership function was studied by Akram [5]. In 1986, Atanassov [8] grounded the idea of Intuitionistic Fuzzy

Set (in short IFS) theory by introducing the idea of non-membership of a mathematical expression. Afterwards, Akram and Shum [7] grounded the concept of Lie algebra on IFSs. The notion of Intuitionistic (S, T)-Fuzzy-Lie ideals was studied by Akram [2]. In 2008, Akram [3] further established several results on Intuitionistic Fuzzy-Lie ideals of Lie algebra.

In 1998, Smarandache [36] grounded the idea of neutrosophic set (in short NS) by introducing the indeterminacy membership function of mathematical expression. Later on, Wang et al. [37] defined single-valued neutrosophic set (in short SVNS) as a generalization of FS and IFS. In 2020, Das et al. [14] proposed a multi-criteria decision making algorithm via SVNS environment. Thereafter, Akram et al. [6] introduced the concept of Lie-Algebra on SVNSs in 2019. Afterwards, Das and Hassan [15] grounded the notion of *d*-ideals on NS. In 2016, Chatterjee et al. [9] presented the idea of single-valued quadripartitioned neutrosophic set (in short SVQN-set) be extending the notion of SVNS. Later on, Mallick and Pramanik [33] grounded the notion of SVPN-set by splitting the indeterminacy membership function into three different membership functions namely contradiction, ignorance and unknown membership functions. Recently, Das et al. [13] studied the concept of *Q*-Ideals on SVPN-sets.

In this article, we procure the idea of SVPN-Lie ideal of SVPN-Lie algebra. Further, we produce several interesting results on SVPN-Lie algebra and SVPN-Lie ideal.

Research gap: No investigation on SVPN-Lie algebra and SVPN-Lie ideal has been reported in the recent literature.

Motivation: To explore the unexplored research, we introduce the notion of SVPN-Lie algebra and SVPN-Lie ideal.

The remaining part of this article has been organized as follows:

In section-2, we recall some basic definitions and results on SVNS, Lie algebra, Lie ideal, SVN-Lie algebra, SVN-Lie ideal and SVPN-set those are useful for the preparation of the main results of this article. Section-3 introduces the idea of SVPN-Lie algebra and SVPN-Lie ideal. In this section, we also formulate several interesting results on them. Section-4 represents the concluding remarks on the work done in this article.

2. Some Relevant Results:

Definition 2.1.[30] Assume that Ω be a field, and L be a vector space on Ω. Consider an operation L×L → L defined by (a, b) → [a, b], for all a, b ∈ L. Then, L is called Lie algebra if the following properties hold:

(*i*) [a, b] is a bilinear,

(*ii*) [a, a] = 0, for all a ∈ L,

(*iii*) [[a, b], c] + [[b, c], a] + [[c, a], b] = 0, for all a, b, c ∈ L.

Definition 2.2.[40] A Fuzzy Set (in short FS) W over a universe of discourse Π is defined as follows:

$$W = \{(\eta, T_W(\eta)) : \eta \in \Pi\},$$

where $T_W(\eta)$ is the truth-membership value of each $\eta \in \Pi$ such that $0 \leq T_W(\eta) \leq 1$.

Definition 2.3.[38] A FS $W = \{(\eta, T_W(\eta)) : \eta \in L\}$ is called a Fuzzy Lie ideal (in short F-L-Ideal) of a Lie algebra L if and only if the following three conditions hold:

(*i*) $T_W(q + r) \geq \min \{ T_W(q), T_W(r)\}$;

(*ii*) $T_W(\alpha q) \geq T_W(q)$;

(*iii*) $T_W([q, r]) \geq T_W(q)$, for all $q, r \in L$, and $\alpha \in \Omega$.

Definition 2.4.[8] An intuitionistic fuzzy set (in short IFS) W over a fixed set Π is defined as follows:

$$W = \{(\eta, T_W(\eta), I_W(\eta)) : \eta \in \Pi\},$$

where T_W, I_W are the membership and non-membership functions from W to [0, 1], and so $0 \leq T_W(\eta) + I_W(\eta) \leq 2$, for all $\eta \in \Pi$.

Definition 2.5.[7] An IFS $W = \{(q, T_W(q), I_W(q)) : q \in L\}$ on Lie algebra L is called an Intuitionistic Fuzzy Lie (in short IF-Lie) algebra if the following condition holds:

(*i*) $T_W(q + r) \geq \min \{T_W(q), T_W(r)\}$ and $I_W(q + r) \leq \max \{I_W(q), I_W(r)\}$;

(*ii*) $T_W(\alpha q) \geq T_W(q)$ and $I_W(\alpha q) \leq I_W(q)$;

(*iii*) $T_W([q, r]) \geq \min \{T_W(q), T_W(r)\}$ and $I_W([q, r]) \leq \max \{I_W(q), I_W(r)\}$, for all $q, r \in L$, and $\alpha \in \Omega$.

Definition 2.6.[37] An Single-Valued Neutrosophic Set (in short SVNS) W over Π is defined as follows:

$$W = \{(\eta, T_W(\eta), I_W(\eta), F_W(\eta)) : \eta \in \Pi\},$$

where T_W, I_W, F_W are truth, indeterminacy and falsity membership mappings from W to [0, 1], and so $0 \leq T_W(\eta) + I_W(\eta) + F_W(\eta) \leq 3$, for all $\eta \in \Pi$.

Definition 2.7.[37] Assume that $Y = \{(c, T_Y(c), I_Y(c), F_Y(c)) : c \in \Pi\}$ be an SVNS over Π. Then, the sets $W(T_Y,\alpha)=\{c\in\Pi:T_Y(c)\geq\alpha\}$, $W(I_Y,\alpha)=\{c\in\Pi:I_Y(c)\leq\alpha\}$, $W(F_Y,\alpha)=\{c\in\Pi:F_Y(c)\leq\alpha\}$ are respectively called T-level α-cut, I-level α-cut, F-level α-cut of Y.

Definition 2.8.[6] An SVNS $W=\{(q, T_W(q), I_W(q), F_W(q)) : q\in L\}$ over a Lie algebra L is called an Single-Valued Neutrosophic Lie (in short SVN-Lie) algebra if the following condition holds:

(*i*) $T_W(q + r) \geq \min \{T_W(q), T_W(r)\}$, $I_W(q + r) \geq \min \{I_W(q), I_W(r)\}$ and $F_W(q + r) \leq \max \{F_W(q), F_W(r)\}$;

(*ii*) $T_W(\alpha q) \geq T_W(q)$, $I_W(\alpha q) \geq I_W(q)$ and $F_W(\alpha q) \leq F_W(q)$;

(*iii*) $T_W([q, r]) \geq \min \{T_W(q), T_W(r)\}$, $I_W([q, r]) \geq \min \{I_W(q), I_W(r)\}$ and $F_W([q, r]) \leq \max \{F_W(q), F_W(r)\}$, for all $q, r \in L$, and $\alpha \in \Omega$.

Example 2.1. Suppose that F = R be the set of all real number. Suppose that $L = R^3 = \{(a, b, c) : a, b, c \in R\}$ be the set of all three-dimensional real vectors. Then, L forms a Lie algebra. We define

$$R^3 \times R^3 \to R^3$$

$$[a, b] \to a \times b,$$

where '×' is the usual cross product. Now, we define an SVNS $N = (T_N, I_N, F_N) : R^3 \to [0, 1] \times [0, 1] \times [0, 1]$ by

$$T_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.4, & a = b = 0, c \neq 0, \\ 0.1, & a \neq b \neq c \neq 0 \end{cases}$$

$$I_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.4, & a = b = 0, c \neq 0, \\ 0.1, & a \neq b \neq c \neq 0 \end{cases}$$

$$\text{and } F_N(a, b, c) = \begin{cases} 0.1, & a = b = c = 0 \\ 0.4, & a = b = 0, c \neq 0. \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

Then, N = (T_N, I_N, F_N) is an SVN-Lie algebra of L.

Definition 2.9.[6] Suppose that L be a Lie algebra over a field Ω. An SVNS W={(q, $T_W(q)$, $I_W(q)$, $F_W(q)$):q∈L} on L is called an SVN-Lie ideal if the following conditions hold:

(i) $T_W(r+q) \geq \min \{T_W(r), T_W(q)\}$, $I_W(r+q) \geq \min \{I_W(r), I_W(q)\}$ and $F_W(r+q) \leq \max \{F_W(r), F_W(q)\}$;

(ii) $T_W(\alpha q) \geq T_W(q)$, $I_W(\alpha q) \geq I_W(q)$ and $F_W(\alpha q) \leq F_W(q)$;

(iii) $T_W([r, q]) \geq T_W(r)$, $I_W([r, q]) \geq I_W(r)$ and $F_W([r, q]) \leq F_W(r)$, for all r, q ∈L.

Example 2.2. Suppose that F = R be the set of all real number. Suppose that L = R^3 = {(a, b, c) : a, b, c ∈ R} be the set of all three-dimensional real vectors which forms a Lie algebra. Now, we define an SVNS N = (T_N, I_N, F_N) : $R^3 \rightarrow [0, 1] \times [0, 1] \times [0, 1]$ ('×' is the usual cross product) by

$$T_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.9, & a = b = 0, c \neq 0, \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

$$I_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.9, & a = b = 0, c \neq 0, \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

$$\text{and } F_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.9, & a = b = 0, c \neq 0. \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

Then, N is an SVN-Lie ideal of L.

Remark 2.1. Every SVN-Lie algebra may not be an SVN-Lie ideal. This follows from the following example.

Example 2.3. Let us consider an SVNS N = (T_N, I_N, F_N) over the field L = R as defined in Example 2.1. Then, the SVNS N = {($T_N(x, y, z)$, $I_N(x, y, z)$, $F_N(x, y, z)$) : (x, y, z) ∈ R^3} is an SVN-Lie algebra of L, but it is not an SVN-Lie ideal of L, because

$T_N([(1, 0, 0), (0, 0, 1)]) = T_N(0,-1, 0) = 0.1 \ngeq 0.4$ i.e., $T_N([(1, 0, 0), (0, 0, 1)]) \ngeq T_N(0, 0, 1)$,

$I_N([(1, 0, 0), (0, 0, 1)]) = C_N(0,-1, 0) = 0.1 \ngeq 0.4$ i.e., $C_N([(1, 0, 0), (0, 0, 1)]) \ngeq C_N(0, 0, 1)$,

$F_N([(1, 0, 0), (0, 0, 1)]) = F_N(0,-1, 0) = 0.9 \nleq 0.4$ i.e., $F_N([(1, 0, 0), (0, 0, 1)]) \nleq F_N(0, 0, 1)$.

Remark 2.2.[6] Let W={(q, $T_W(q)$, $I_W(q)$, $F_W(q)$):q∈L} be an SVN-Lie algebra on a Lie algebra L. Then,

(i) $T_W(0) \geq T_W(q)$, $I_W(0) \geq I_W(q)$, $F_W(0) \leq F_W(q)$;

(ii) $T_W(-q) \geq T_W(q)$, $I_W(-q) \geq I_W(q)$, $F_W(-q) \leq F_W(q)$, for all q ∈L.

Definition 2.10.[9] Suppose that Π be a universal set. Then, an Single-Valued Quadripartitioned Neutrosophic Set (in short SVQN-set) W over Π is defined as follows:

$$W = \{(\eta, T_W(\eta), C_W(\eta), G_W(\eta), F_W(\eta)) : \eta \in \Pi\},$$

where $T_W(\eta)$, $C_W(\eta)$, $G_W(\eta)$ and $F_W(\eta)$ ($\in[0, 1]$) are the truth, contradiction, ignorance and false membership values of each $\eta\in\Pi$. So, $0 \leq T_W(\eta) + C_W(\eta) + G_W(\eta) + F_W(\eta) \leq 4$, for all $\eta\in\Pi$.

Definition 2.11.[9] Assume that $W = \{(\eta, T_W(\eta), C_W(\eta), G_W(\eta), F_W(\eta)) : \eta \in \Pi\}$ and $E = \{(\eta, T_E(\eta), C_E(\eta), G_E(\eta), F_E(\eta)) : \eta \in \Pi\}$ be two SVQN-sets over a fixed set Π. Then,

(i) $W\subseteq E$ if and only if $T_W(\eta) \leq T_E(\eta)$, $C_W(\eta) \leq C_E(\eta)$, $G_W(\eta) \geq G_E(\eta)$, $F_W(\eta) \geq F_E(\eta)$, $\forall\eta\in\Pi$,

(ii) $W\cup E = \{(\eta, \max\{T_W(\eta), T_E(\eta)\}, \max\{C_W(\eta), C_E(\eta)\}, \min\{G_W(\eta), G_E(\eta)\}, \min\{F_W(\eta), F_E(\eta)\}) : \eta\in\Pi\}$,

(iii) $W\cap E = \{(\eta, \min\{T_W(\eta), T_E(\eta)\}, \min\{C_W(\eta), C_E(\eta)\}, \max\{G_W(\eta), G_E(\eta)\}, \max\{F_W(\eta), F_E(\eta)\}) : \eta\in\Pi\}$,

(iv) $W^c = \{(\eta, F_W(\eta), G_W(\eta), C_W(\eta), T_W(\eta)) : \eta\in\Pi\}$.

Definition 2.12.[33] Suppose that Π be a fixed set. Then, an Single-Valued Pentapartitioned Neutrosophic Set (in short SVPN-set) W over Π is defined by:

$$W = \{(\eta, T_W(\eta), C_W(\eta), G_W(\eta), U_W(\eta), F_W(\eta)) : \eta \in \Pi\},$$

where $T_W(\eta)$, $C_W(\eta)$, $G_W(\eta)$, $U_W(\eta)$ and $F_W(\eta)$ ($\in[0, 1]$) are the truth, contradiction, ignorance, unknown and false membership values of each $\eta\in\Pi$. So, $0 \leq T_W(\eta)+C_W(\eta)+G_W(\eta)+U_W(\eta)+F_W(\eta)\leq 4$, for all $\eta\in\Pi$.

Definition 2.13.[33] Assume that $W=\{(\eta, T_W(\eta), C_W(\eta), G_W(\eta), U_W(\eta), F_W(\eta)) : \eta\in\Pi\}$ and $E=\{(\eta, T_E(\eta), C_E(\eta), G_E(\eta), U_E(\eta), F_E(\eta)) : \eta\in\Pi\}$ be two SVPN-sets over a fixed set Π. Then,

(i) $W\subseteq E$ if and only if $T_W(\eta)\leq T_E(\eta)$, $C_W(\eta)\leq C_E(\eta)$, $G_W(\eta)\geq G_E(\eta)$, $U_W(\eta)\geq U_E(\eta)$, $F_W(\eta)\geq F_E(\eta)$, $\forall\eta\in\Pi$.

(ii) $W\cup E = \{(\eta, \max\{T_W(\eta), T_E(\eta)\}, \max\{C_W(\eta), C_E(\eta)\}, \min\{G_W(\eta), G_E(\eta)\}, \min\{U_W(\eta), U_E(\eta)\}, \min\{F_W(\eta), F_E(\eta)\}) : \eta\in\Pi\}$.

(iii) $W\cap E = \{(\eta, \min\{T_W(\eta), T_E(\eta)\}, \min\{C_W(\eta), C_E(\eta)\}, \max\{G_W(\eta), G_E(\eta)\}, \max\{U_W(\eta), U_E(\eta)\}, \max\{F_W(\eta), F_E(\eta)\}) : \eta\in\Pi\}$.

(iv) $W^c = \{(\eta, F_W(\eta), U_W(\eta), 1-G_W(\eta), C_W(\eta), T_W(\eta)) : \eta\in\Pi\}$.

3. SVPN-Lie Ideal of SVPN-Lie Algebra:

In this section, we procure the notion of SVPN-Lie ideal of SVPN-Lie algebra. Besides, we study different properties of SVPN-Lie ideal, and formulate several results on it.

Definition 3.1. Let L be a Lie algebra on a field Ω. Then, an SVPN-set $W = \{(\eta, T_W(\eta), C_W(\eta), G_W(\eta), U_W(\eta), F_W(\eta)) : \eta \in L\}$ over L is called an SVPN-Lie algebra if the following conditions hold:

(i) $T_W(\eta+\delta) \geq \min\{T_W(\eta), T_W(\delta)\}$, $C_W(\eta+\delta) \geq \min\{C_W(\eta), C_W(\delta)\}$, $G_W(\eta+\delta) \leq \max\{G_W(\eta), G_W(\delta)\}$, $U_W(\eta+\delta) \leq \max\{U_W(\eta), U_W(\delta)\}$ and $F_W(\eta+\delta) \leq \max\{F_W(\eta), F_W(\delta)\}$;

(ii) $T_W(\alpha\eta) \geq T_W(\eta)$, $C_W(\alpha\eta) \geq C_W(\eta)$, $G_W(\alpha\eta) \leq G_W(\eta)$, $U_W(\alpha\eta) \leq U_W(\eta)$ and $F_W(\alpha\eta) \leq F_W(\eta)$;

(iii) $T_W([\eta, \delta]) \geq \min\{T_W(\eta), T_W(\delta)\}$, $C_W([\eta, \delta]) \geq \min\{C_W(\eta), C_W(\delta)\}$, $G_W([\eta, \delta]) \leq \max\{G_W(\eta), G_W(\delta)\}$, $U_W([\eta, \delta]) \leq \max\{U_W(\eta), U_W(\delta)\}$ and $F_W([\eta, \delta]) \leq \max\{F_W(\eta), F_W(\delta)\}$, for all $\eta, \delta \in L$, and $\alpha \in \Omega$.

Example 3.1. Suppose that F = R be the set of all real number. Suppose that L = R^3 = {(a, b, c): a, b, c ∈R} be the set of all three-dimensional real vectors. Then, L forms a Lie algebra. We define

$$R^3 \times R^3 \rightarrow R^3$$

$$[a, b] \rightarrow a \times b,$$

where '×' is the usual cross product. Now, we define an SVPN-set N = (T_N, C_N, G_N, U_N, F_N) : R^3 → [0, 1] × [0, 1] × [0, 1] by

$$T_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.4, & a = b = 0, c \neq 0, \\ 0.1, & a \neq b \neq c \neq 0 \end{cases}$$

$$C_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.4, & a = b = 0, c \neq 0, \\ 0.1, & a \neq b \neq c \neq 0 \end{cases}$$

$$G_N(a, b, c) = \begin{cases} 0.1, & a = b = c = 0 \\ 0.4, & a = b = 0, c \neq 0, \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

$$U_N(a, b, c) = \begin{cases} 0.1, & a = b = c = 0 \\ 0.4, & a = b = 0, c \neq 0, \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

$$\text{and } F_N(a, b, c) = \begin{cases} 0.1, & a = b = c = 0 \\ 0.4, & a = b = 0, c \neq 0. \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

Then, N = (T_N, C_N, G_N, U_N, F_N) is an SVPN-Lie algebra of L.

Definition 3.2. Let L be a Lie algebra on a field Ω. Then, an SVPN-set W = {(η, T_W(η), C_W(η), G_W(η), U_W(η), F_W(η)) : η∈Π} over L is called an SVPN-Lie ideal if the following condition holds:

(i) $T_W(\eta+\delta) \geq \min \{T_W(\eta), T_W(\delta)\}$, $C_W(\eta+\delta) \geq \min \{C_W(\eta), C_W(\delta)\}$, $G_W(\eta+\delta) \leq \max \{G_W(\eta), G_W(\delta)\}$, $U_W(\eta+\delta) \leq \max \{U_W(\eta), U_W(\delta)\}$ and $F_W(\eta+\delta) \leq \max \{F_W(\eta), F_W(\delta)\}$;

(ii) $T_W(\alpha\eta) \geq T_W(\eta)$, $C_W(\alpha\eta) \geq C_W(\eta)$, $G_W(\alpha\eta) \leq G_W(\eta)$, $U_W(\alpha\eta) \leq U_W(\eta)$ and $F_W(\alpha\eta) \leq F_W(\eta)$;

(iii) $T_W([\eta, \delta]) \geq T_W(\eta)$, $C_W([\eta, \delta]) \geq C_W(\eta)$, $G_W([\eta, \delta]) \leq G_W(\eta)$, $U_W([\eta, \delta]) \leq U_W(\eta)$ and $F_W([\eta, \delta]) \leq F_W(\eta)$, for all η, δ∈L, and α∈Ω.

Example 3.2. Suppose that F = R be the set of all real number. Suppose that L = R^3 = {(a, b, c) : a, b, c ∈ R} be the set of all three-dimensional real vectors which forms a Lie algebra. Now, we define an SVNS N = (T_N, C_N, G_N, U_N, F_N) : R^3 → [0, 1] × [0, 1] × [0, 1] ('×' is the usual cross product) by

$$T_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.9, & a = b = 0, c \neq 0, \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

$$C_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.9, & a = b = 0, c \neq 0, \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

$$G_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.9, & a = b = 0, c \neq 0, \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

$$U_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.9, & a = b = 0, c \neq 0 \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

$$\text{and } F_N(a, b, c) = \begin{cases} 0.9, & a = b = c = 0 \\ 0.9, & a = b = 0, c \neq 0. \\ 0.9, & a \neq b \neq c \neq 0 \end{cases}$$

Then, N = (T_N, C_N, G_N, U_N, F_N) is an SVPN-Lie ideal of L.

Remark 3.1. Every SVPN-Lie algebra may not be an SVPN-Lie ideal. This follows from the following example.

Example 3.3. Let N = (T_N, C_N, G_N, U_N, F_N) be an SVPN-set over the field L = R as defined in Example 3.1. Then, the SVPN-set N = {(T_N(x, y, z), C_N(x, y, z), G_N(x, y, z), U_N(x, y, z), F_N(x, y, z)) : (x, y, z) ∈ R^3} is an SVPN-Lie algebra of L, but it is not an SVPN-Lie ideal of L, because

T_N([(1, 0, 0), (0, 0, 1)]) = T_N(0,–1, 0) = 0.1 $\ngeq$ 0.4 i.e., T_N([(1, 0, 0), (0, 0, 1)]) $\ngeq$ T_N(0, 0, 1),

C_N([(1, 0, 0), (0, 0, 1)]) = C_N(0,–1, 0) = 0.1 $\ngeq$ 0.4 i.e., C_N([(1, 0, 0), (0, 0, 1)]) $\ngeq$ C_N(0, 0, 1),

G_N([(1, 0, 0), (0, 0, 1)]) = G_N(0,–1, 0) = 0.9 $\nleq$ 0.4 i.e., G_N([(1, 0, 0), (0, 0, 1)]) $\nleq$ G_N(0, 0, 1),

U_N([(1, 0, 0), (0, 0, 1)]) = U_N(0,–1, 0) = 0.9 $\nleq$ 0.4 i.e., U_N([(1, 0, 0), (0, 0, 1)]) $\nleq$ U_N(0, 0, 1)

F_N([(1, 0, 0), (0, 0, 1)]) = F_N(0,–1, 0) = 0.9 $\nleq$ 0.4 i.e., F_N([(1, 0, 0), (0, 0, 1)]) $\nleq$ F_N(0, 0, 1).

Theorem 3.1. Suppose that {W_i: i ∈ Δ} be the family of SVPN-Lie ideals on a Lie-Algebra L. Then, their intersection $\cap W_i$ = {(η, $\wedge T_{N_i}(\eta)$, $\wedge C_{N_i}(\eta)$, $\vee G_{N_i}(\eta)$, $\vee U_{N_i}(\eta)$, $\vee F_{N_i}(\eta)$) : η∈L} is also an SVPN-Lie ideal of L.

Proof. Suppose that {W_i : i ∈ Δ} be the family of SVPN-Lie ideals on a Lie-Algebra L. It is known that, $\cap W_i$ = {(η, $\wedge T_{N_i}(\eta)$, $\wedge C_{N_i}(\eta)$, $\vee G_{N_i}(\eta)$, $\vee U_{N_i}(\eta)$, $\vee F_{N_i}(\eta)$) : η∈L}.

Now,

(i) $\wedge T_{N_i}(\eta+\delta) = \min\{T_{N_i}(\eta+\delta) : i\in\Delta\} \geq \min\{\min\{T_{N_i}(\eta), T_{N_i}(\delta)\} : i\in\Delta\} \geq \min\{\wedge T_{N_i}(\eta), \wedge T_{N_i}(\delta)\}$,

$\wedge C_{N_i}(\eta+\delta) = \min\{C_{N_i}(\eta+\delta) : i\in\Delta\} \geq \min\{\min\{C_{N_i}(\eta), C_{N_i}(\delta)\} : i\in\Delta\} \geq \min\{\wedge C_{N_i}(\eta), \wedge C_{N_i}(\delta)\}$,

$\vee G_{N_i}(\eta+\delta) = \max\{G_{N_i}(\eta+\delta) : i\in\Delta\} \leq \max\{\max\{G_{N_i}(\eta), G_{N_i}(\delta)\} : i\in\Delta\} \leq \max\{\vee G_{N_i}(\eta), \vee G_{N_i}(\delta)\}$,

$\vee U_{N_i}(\eta+\delta) = \max\{U_{N_i}(\eta+\delta) : i\in\Delta\} \leq \max\{\max\{U_{N_i}(\eta), U_{N_i}(\delta)\} : i\in\Delta\} \leq \max\{\vee U_{N_i}(\eta), \vee U_{N_i}(\delta)\}$,

$\vee F_{N_i}(\eta+\delta) = \max\{F_{N_i}(\eta+\delta) : i\in\Delta\} \leq \max\{\max\{F_{N_i}(\eta), F_{N_i}(\delta)\} : i\in\Delta\} \leq \max\{\vee F_{N_i}(\eta), \vee F_{N_i}(\delta)\}$.

(ii) $\wedge T_{N_i}(\alpha\eta) = \min\{T_{N_i}(\alpha\eta) : i\in\Delta\} \geq \min\{T_{N_i}(\eta) : i\in\Delta\} \geq \wedge T_{N_i}(\eta)$,

$\wedge C_{N_i}(\alpha\eta) = \min\{C_{N_i}(\alpha\eta) : i\in\Delta\} \geq \min\{C_{N_i}(\eta) : i\in\Delta\} \geq \wedge C_{N_i}(\eta)$,

$\vee G_{N_i}(\alpha\eta) = \max\{G_{N_i}(\alpha\eta) : i\in\Delta\} \leq \max\{G_{N_i}(\eta) : i\in\Delta\} \leq \vee G_{N_i}(\eta)$,

$\vee U_{N_i}(\alpha\eta) = \max\{U_{N_i}(\alpha\eta) : i\in\Delta\} \leq \max\{U_{N_i}(\eta) : i\in\Delta\} \leq \vee U_{N_i}(\eta)$,

$\vee F_{N_i}(\alpha\eta) = \max\{F_{N_i}(\alpha\eta) : i\in\Delta\} \leq \max\{F_{N_i}(\eta) : i\in\Delta\} \leq \vee F_{N_i}(\eta)$.

(iii) $\wedge T_{N_i}([\eta, \delta]) = \min\{T_{N_i}([\eta, \delta]) : i\in\Delta\} \geq \min\{T_{N_i}(\eta) : i\in\Delta\} \geq \wedge T_{N_i}(\eta)$,

$\wedge C_{N_i}([\eta, \delta]) = \min\{C_{N_i}([\eta, \delta]) : i\in\Delta\} \geq \min\{C_{N_i}(\eta) : i\in\Delta\} \geq \wedge C_{N_i}(\eta)$,

$\vee G_{N_i}([\eta, \delta]) = \max\{G_{N_i}([\eta, \delta]) : i\in\Delta\} \leq \max\{G_{N_i}(\eta) : i\in\Delta\} \leq \vee G_{N_i}(\eta)$,

$\vee U_{N_i}([\eta, \delta]) = \max\{U_{N_i}([\eta, \delta]) : i\in\Delta\} \leq \max\{U_{N_i}(\eta) : i\in\Delta\} \leq \vee U_{N_i}(\eta)$,

$\vee F_{N_i}([\eta, \delta]) = \max\{F_{N_i}([\eta, \delta]) : i\in\Delta\} \leq \max\{F_{N_i}(\eta) : i\in\Delta\} \leq \vee F_{N_i}(\eta)$.

Therefore, $\cap W_i = \{(\eta, \wedge T_{N_i}(\eta), \wedge C_{N_i}(\eta), \vee G_{N_i}(\eta), \vee U_{N_i}(\eta), \vee F_{N_i}(\eta)) : \eta\in L\}$ is an SVPN-Lie ideal of L.

Theorem 3.2. Assume that $W = \{(\eta, T_W(\eta), C_W(\eta), G_W(\eta), U_W(\eta), F_W(\eta)) : \eta\in L\}$ be an SVPN-Lie algebra on a Lie algebra L. Then,

(i) $T_W(0) \geq T_W(\delta)$, $C_W(0) \geq C_W(\delta)$, $G_W(0) \leq G_W(\delta)$, $U_W(0) \leq U_W(\delta)$, $F_W(0) \leq F_W(\delta)$;

(ii) $T_W(-\delta) \geq T_W(\delta)$, $C_W(-\delta) \geq C_W(\delta)$, $G_W(-\delta) \leq G_W(\delta)$, $U_W(-\delta) \leq U_W(\delta)$, $F_W(-\delta) \leq F_W(\delta)$, for all $\delta \in L$.

Proof. The proof is so easy, so omitted.

Lemma 3.1. Every SVPN-Lie ideal is also an SVPN-Lie algebra.

Theorem 3.3. Suppose that $W = \{(\delta, T_W(\delta), C_W(\delta), G_W(\delta), U_W(\delta), F_W(\delta)) : \delta\in L\}$ be an SVPN-Lie ideal of a Lie-Algebra L. Then, the following holds:

(i) $T_W(0) \geq T_W(\delta)$, $C_W(0) \geq C_W(\delta)$, $G_W(0) \leq G_W(\delta)$, $U_W(0) \leq U_W(\delta)$, $F_W(0) \leq F_W(\delta)$;

(ii) $T_W([\delta, \eta]) \geq \max\{T_W(\delta), T_W(\eta)\}$; $C_W([\delta, \eta]) \geq \max\{C_W(\delta), C_W(\eta)\}$; $G_W([\delta, \eta]) \leq \min\{G_W(\delta), G_W(\eta)\}$; $U_W([\delta, \eta]) \leq \min\{U_W(\delta), U_W(\eta)\}$; $F_W([\delta, \eta]) \leq \min\{F_W(\delta), F_W(\eta)\}$;

(iii) $T_W([\delta, \eta]) = T_W(-[\delta, \delta]) = T_W([\eta, \delta])$; $C_W([\delta, \eta]) = C_W(-[\delta, \delta]) = C_W([\eta, \delta])$; $G_W([\delta, \eta]) = G_W(-[\delta, \delta]) = G_W([\eta, \delta])$; $U_W([\delta, \eta]) = U_W(-[\delta, \delta]) = U_W([\eta, \delta])$; $F_W([\delta, \eta]) = F_W(-[\delta, \delta]) = F_W([\eta, \delta])$, for all $\delta, \eta \in L$.

Proof. The proofs are straightforward, so omitted.

Definition 3.3. Assume that $W = \{(\eta, T_W(\eta), C_W(\eta), G_W(\eta), U_W(\eta), F_W(\eta)) : \eta\in L\}$ be an SVPN-set over a Lie-Algebra L. Suppose that $\alpha, \beta, \gamma, \delta, \lambda \in [0, 1]$. Then, the sets $L(T_W, \alpha) = \{\eta\in L : T_W(\eta)\geq\alpha\}$, $L(C_W, \beta) = \{\eta\in L : C_W(\eta)\geq\beta\}$, $L(G_W, \gamma) = \{\eta\in L : G_W(\eta)\leq\gamma\}$, $L(U_W, \delta) = \{\eta\in L : U_W(\eta)\leq\delta\}$, $L(F_W, \lambda) = \{\eta\in L : F_W(\eta)\leq\lambda\}$ are called T-level α-cut, C-level β-cut, G-level γ-cut, U-level δ-cut and F-level λ-cut of W respectively.

Definition 3.4. Suppose that L be a Lie-Algebra. Assume that $W = \{(\eta, T_W(\eta), C_W(\eta), G_W(\eta), U_W(\eta), F_W(\eta)) : \eta\in L\}$ be an SVPN-set over L. Suppose that $\alpha, \beta, \gamma, \delta, \lambda \in [0, 1]$. Then, $(\alpha, \beta, \gamma, \delta, \lambda)$-level subset of W is defined by:

$$L(\alpha, \beta, \gamma, \delta, \lambda) = \{\eta\in L : T_W(\eta) \geq \alpha, C_W(\eta) \geq \beta, G_W(\eta) \leq \gamma, U_W(\eta) \leq \delta, F_W(\eta) \leq \delta\}.$$

Remark 3.2. Suppose that L be a Lie-Algebra. If $W = \{(\eta, T_W(\eta), C_W(\eta), G_W(\eta), U_W(\eta), F_W(\eta)) : \eta\in L\}$ be an SVPN-set over L, then $L(\alpha, \beta, \gamma, \delta, \lambda) = L(T_W, \alpha) \cap L(C_W, \beta) \cap L(G_W, \gamma) \cap L(U_W, \delta) \cap L(F_W, \lambda)$.

Proposition 3.1. Suppose that L be a Lie-Algebra. An SVPN-set $W = \{(\eta, T_W(\eta), C_W(\eta), G_W(\eta), U_W(\eta), F_W(\eta)) : \eta\in L\}$ is an SVPN-Lie ideal of L if and only if $L(\alpha, \beta, \gamma, \delta, \lambda)$ is a Lie-Ideal of L for every $\alpha, \beta, \gamma, \delta, \lambda \in [0, 1]$.

Proof. The proof is straightforward, so omitted.

Theorem 3.4. Let L be a Lie-Algebra. Assume that $W = \{(\eta, T_W(\eta), C_W(\eta), G_W(\eta), U_W(\eta), F_W(\eta)) : \eta\in L\}$ be an SVPN-Lie ideal of L. Let $\alpha_1, \beta_1, \gamma_1, \delta_1, \lambda_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \lambda_2 \in [0, 1]$. Then, $L(\alpha_1, \beta_1, \gamma_1, \delta_1, \lambda_1) = L(\alpha_2, \beta_2, \gamma_2, \delta_2, \lambda_2)$ if and only if $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$, $\lambda_1 = \lambda_2$.

Proof. Suppose that L be a Lie-Algebra. Let W = {(η, T_W(η), C_W(η), G_W(η), U_W(η), F_W(η)) : η∈L} be an SVPN-Lie ideal of L. Let α_1, β_1, γ_1, δ_1, λ_1, α_2, β_2, γ_2, δ_2, λ_2 ∈ [0, 1] such that L(α_1, β_1, γ_1, δ_1, λ_1) = L(α_2, β_2, γ_2, δ_2, λ_2). Therefore, {η∈L : $T_W(\eta) \geq \alpha_1$, $C_W(\eta) \geq \beta_1$, $G_W(\eta) \leq \gamma_1$, $U_W(\eta) \leq \delta_1$, $F_W(\eta) \leq \lambda_1$} = {η∈L : $T_W(\eta) \geq \alpha_2$, $C_W(\eta) \geq \beta_2$, $G_W(\eta) \leq \gamma_2$, $U_W(\eta) \leq \delta_2$, $F_W(\eta) \leq \lambda_2$}. This is possible only when $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$, $\lambda_1 = \lambda_2$. Therefore, L(α_1, β_1, γ_1, δ_1, λ_1) = L(α_2, β_2, γ_2, δ_2, λ_2) implies $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$, $\lambda_1 = \lambda_2$. Conversely, let $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$, $\lambda_1 = \lambda_2$.

Now, L(α_1, β_1, γ_1, δ_1, λ_1)

= {η∈L : $T_W(\eta) \geq \alpha_1$, $C_W(\eta) \geq \beta_1$, $G_W(\eta) \leq \gamma_1$, $U_W(\eta) \leq \delta_1$, $F_W(\eta) \leq \lambda_1$}

= {η∈L : $T_W(\eta) \geq \alpha_2$, $C_W(\eta) \geq \beta_2$, $G_W(\eta) \leq \gamma_2$, $U_W(\eta) \leq \delta_2$, $F_W(\eta) \leq \lambda_2$}

= L(α_2, β_2, γ_2, δ_2, λ_2)

Therefore, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$, $\lambda_1 = \lambda_2$ implies L(α_1, β_1, γ_1, δ_1, λ_1) = L(α_2, β_2, γ_2, δ_2, λ_2).

Definition 3.5. Assume that L_1 and L_2 be two Lie-Algebras on a common field Ω. Suppose that f be a bijective mapping from L_1 to L_2. If M = {(η, T_M(η), C_M(η), G_M(η), U_M(η), F_M(η)) : η∈L} be an SVPN-set in L_2, then f^{-1}(M) defined by f^{-1}(M) = {(η, f^{-1}(T_M(η)), f^{-1}(C_M(η)), f^{-1}(G_M(η)), f^{-1}(U_M(η)), f^{-1}(F_M(η))) : η∈L} is also an SVPN-set in L_1.

Theorem 3.5. Assume that L_1 and L_2 be two Lie-Algebras on a common field Ω. Suppose that f be an onto homomorphism from L_1 to L_2. If M = {(η, T_M(η), C_M(η), G_M(η), U_M(η), F_M(η)) : η∈L} is an SVPN-Lie ideal of L_2, then f^{-1}(M) = {(η, f^{-1}(T_M(η)), f^{-1}(C_M(η)), f^{-1}(G_M(η)), f^{-1}(U_M(η)), f^{-1}(F_M(η))) : η∈L} is also an SVPN-Lie ideal of L_1.

Proof. The proof is so easy, so omitted.

Proposition 3.2. Suppose that L_1 and L_2 be two Lie-Algebras. Let f be an epimorphism from L_1 to L_2. If M = {(η, T_M(η), C_M(η), G_M(η), U_M(η), F_M(η)) : η∈L} be an SVPN-Lie ideal of L_2, then $f^{-1}(M^c) = (f^{-1}(M))^c$ is also an SVPN-Lie ideal of L_1.

Proof. The proof is straightforward, so omitted.

Theorem 3.6. Suppose that L_1 and L_2 be two Lie-Algebras. Let f be an epimorphism from L_1 to L_2. If M = {(η, T_M(η), C_M(η), G_M(η), U_M(η), F_M(η)) : η∈L} be an SVPN-Lie ideal of L_2, then f^{-1}(M) = {(η, f^{-1}(T_M(η)), f^{-1}(C_M(η)), f^{-1}(G_M(η)), f^{-1}(U_M(η)), f^{-1}(F_M(η))) : η∈L} is also an SVPN-Lie ideal of L_1.

Proof. The proof is directly holds from Definitions 3.2 and Definition 3.5.

Definition 3.6. Let us consider two Lie-Algebras L_1 and L_2. Let f be a mapping from a L_1 to L_2. If W = {(η, T_W(η), C_W(η), G_W(η), U_W(η), F_W(η)) : η∈L} be an SVPN-set in L_1, then the image of W = {(η, T_W(η), C_W(η), G_W(η), U_W(η), F_W(η)) : η∈L} under f denoted by f(W) is an SVPN-set in L_2, defined as follows:

$$f(T_W)(r)=\begin{cases}\max_{u\in f^{-1}(r)} T_W(u), & \text{if } f^{-1}(r) \neq \emptyset \\ 0, & \text{otherwise}\end{cases}, \text{ for each } r\in L_2$$

$$f(C_W)(r)=\begin{cases}\max_{u\in f^{-1}(r)} C_W(u), & \text{if } f^{-1}(r) \neq \emptyset \\ 0, & \text{otherwise}\end{cases}, \text{ for each } r\in L_2$$

$$f(G_W)(r)=\begin{cases}\min_{u\in f^{-1}(r)} G_W(u), & \text{if } f^{-1}(r) \neq \emptyset \\ 1, & \text{otherwise}\end{cases}, \text{ for each } r\in L_2,$$

$f(U_W)(r)=\begin{cases}\min_{u\in f^{-1}(r)} U_W(u), & \text{if } f^{-1}(r)\neq \emptyset\\ 1, & \text{otherwise}\end{cases}$, for each r∈ L_2,

$f(F_W)(r)=\begin{cases}\min_{u\in f^{-1}(r)} F_W(u), & \text{if } f^{-1}(r)\neq \emptyset\\ 1, & \text{otherwise}\end{cases}$, for each r∈ L_2.

Theorem 3.7. Let us consider two Lie-Algebras L_1 and L_2. Suppose that f be an epimorphism from L_1 to L_2. If W = {(η, T_W(η), C_W(η), G_W(η), U_W(η), F_W(η)) : η∈L} is an SVPN-Lie ideal in L_1, then the image of W = {(η, T_W(η), C_W(η), G_W(η), U_W(η), F_W(η)) : η∈L} i.e., f(W) is also an SVPN-Lie ideal in L_2.

Proof. The proof is directly holds from Definition 3.2 and Definition 3.6.

Definition 3.7. Let us consider two Lie-Algebras L_1 and L_2. Suppose that f be an onto homomorphism from L_1 to L_2. Let M = {(η, T_M(η), C_M(η), G_M(η), U_M(η), F_M(η)) : η∈L} be an SVPN-set in L_2. Then, we define $L^f = \{(\eta, T^f_M(\eta), C^f_M(\eta), G^f_M(\eta), U^f_M(\eta), F^f_M(\eta)) : \eta\in L_1\}$ in L_1 by $T^f_M(\eta) = T_M(f(\eta))$, $C^f_M(\eta) = C_M(f(\eta))$, $G^f_M(\eta) = G_M(f(\eta))$, $U^f_M(\eta) = U_M(f(\eta))$, $F^f_M(\eta) = F_M(f(\eta))$, for all η∈$L_1$. Clearly, L^f is an SVPN-set in L_1.

Theorem 3.8. Suppose that L_1 and L_2 be two Lie-Algebras on a common field Ω. Assume that f be an onto homomorphism from L_1 to L_2. If M = {(η, T_M(η), C_M(η), G_M(η), U_M(η), F_M(η)) : η∈L_2} is an SVPN-Lie ideal of L_2, then $L^f = \{(\eta, T^f_M(\eta), C^f_M(\eta), G^f_M(\eta), U^f_M(\eta), F^f_M(\eta)) : \eta\in L_1\}$ is also an SVPN-Lie ideal of L_1.

Proof. Suppose that L_1 and L_2 be two Lie-Algebras on a common field Ω. Assume that η, δ ∈ L_1 and a∈ Ω. Then, we have

(i) $T^f_N(\eta+\delta)$

$=T_N(f(\eta+\delta))$

$=T_N(f(\eta)+f(\delta))$

$\geq\min\{T_N(f(\eta)), T_N(f(\delta))\}$

$=\min\{T^f_N(\eta), T^f_N(\delta)\}$,

$C^f_N(\eta+\delta)$

$=C_N(f(\eta+\delta))$

$=C_N(f(\eta)+f(\delta))$

$\geq\min\{C_N(f(\eta)), C_N(f(\delta))\}$

$=\min\{C^f_N(\eta), C^f_N(\delta)\}$,

$G^f_N(\eta+\delta)$

$=G_N(f(\eta+\delta))$

$=G_N(f(\eta)+f(\delta))$

$\leq\max\{G_N(f(\eta)), G_N(f(\delta))\}$

$=\max\{G^f_N(\eta), G^f_N(\delta)\}$,

$U^f_N(\eta+\delta)$

$=U_N(f(\eta+\delta))$

$=U_N(f(\eta)+f(\delta))$

$\le\max\{U_N(f(\eta)), U_N(f(\delta))\}$

$=\max\{U_N^f(\eta), U_N^f(\delta)\}$,

$F_N^f(\eta+\delta)$

$=F_N(f(\eta+\delta))$

$=F_N(f(\eta)+f(\delta))$

$\le\max\{F_N(f(\eta)), F_N(f(\delta))\}$

$=\max\{F_N^f(\eta), F_N^f(\delta)\}$,

(ii) $T_N^f(a\eta) = T_N(f(a\eta)) = T_N(af(\eta)) \ge T_N(f(\eta)) = T_N^f(\eta)$,

$C_N^f(a\eta) = C_N(f(a\eta)) = C_N(af(\eta)) \ge C_N(f(\eta)) = C_N^f(\eta)$,

$G_N^f(a\eta) = G_N(af(\eta)) = G_N(af(\eta)) \le G_N(f(\eta)) = G_N^f(\eta)$,

$U_N^f(a\eta) = U_N(af(\eta)) = U_N(af(\eta)) \le U_N(f(\eta)) = U_N^f(\eta)$,

$F_N^f(a\eta) = F_N(af(\eta)) = F_N(af(\eta)) \le F_N(f(\eta)) = F_N^f(\eta)$.

(iii) $T_N^f([\eta,\delta]) = T_N(f([\eta,\delta])) = T_N(f(\eta), f(\delta)]) \ge T_N(f(\eta)) = T_N^f(\eta)$,

$C_N^f([\eta,\delta]) = C_N(f([\eta,\delta])) = C_N(f(\eta), f(\delta)]) \ge C_N(f(\eta)) = C_N^f(\eta)$,

$G_N^f([\eta,\delta]) = G_N(f([\eta,\delta])) = G_N(f(\eta), f(\delta)]) \le G_N(f(\eta)) = G_N^f(\eta)$,

$U_N^f([\eta,\delta]) = U_N(f([\eta,\delta])) = U_N(f(\eta), f(\delta)]) \le U_N(f(\eta)) = U_N^f(\eta)$,

$F_N^f([\eta,\delta]) = F_N(f([\eta,\delta])) = F_N(f(\eta), f(\delta)]) \le F_N(f(\eta)) = F_N^f(\eta)$.

Therefore, $L^f = \{(\eta, T_M^f(\eta), C_M^f(\eta), G_M^f(\eta), U_M^f(\eta), F_M^f(\eta)) : \eta\in L_1\}$ satisfies all the conditions for being an SVPN-Lie ideal of L_1. Hence, L^f is an SVPN-Lie ideal of L_1.

Theorem 3.9. Assume that L_1 and L_2 be two Lie-Algebras on a common field Ω. Suppose that f be an onto homomorphism from L_1 to L_2. Then, $L^f=\{(w, T_M^f(w), C_M^f(w), G_M^f(w), U_M^f(w), F_M^f(w)): w\in L_1\}$ is an SVPN-Lie ideal of L_1 iff $M=\{(w, T_M(w), C_M(w), G_M(w), U_M(w), F_M(w)) : w\in L\}$ is an SVPN-Lie ideal of L_2.

Proof. The sufficiency of this theorem directly follows from the previous theorem.

Now, we just need to prove the necessity part of this theorem. Since, the mapping f is a onto mapping, so for any w, q ∈ L_2 there are w_1, q_1 ∈ L_2 such that $w = f(w_1), q = f(q_1)$. Therefore, $T_N(w) = T_N^f(w_1)$, $T_N(q) = T_N^f(q_1)$, $C_N(w) = C_N^f(w_1)$, $C_N(q) = C_N^f(q_1)$, $G_N(w) = G_N^f(w_1)$, $G_N(q) = G_N^f(q_1)$, $U_N(w) = U_N^f(w_1)$, $U_N(q) = U_N^f(q_1)$, $F_N(w) = F_N^f(w_1)$, $F_N(q) = F_N^f(q_1)$.

Now,

(i) $T_N(w + q)$

$= T_N(f(w_1) + f(q_1))$

$= T_N(f(w_1 + q_1))$

$= T_N^f(w_1 + q_1)$

$\ge \min\{T_N^f(w_1), T_N^f(q_1)\}$

$= \min\{T_N(w), T_N(q)\}$,

$C_N(w + q)$

$= C_N(f(w_1) + f(q_1))$

$= C_N(f(w_1 + q_1))$

$= C_N^f(w_1 + q_1)$

$\geq \min\{C_N^f(w_1), C_N^f(q_1)\}$

$= \min\{C_N(w), C_N(q)\},$

$G_N(w + q)$

$= G_N(f(w_1) + f(q_1))$

$= G_N(f(w_1 + q_1))$

$= G_N^f(w_1 + q_1)$

$\leq \max\{G_N^f(w_1), G_N^f(q_1)\}$

$= \max\{G_N(w), G_N(q)\},$

$U_N(w + q)$

$= U_N(f(w_1) + f(q_1))$

$= U_N(f(w_1 + q_1))$

$= U_N^f(w_1 + q_1)$

$\leq \max\{U_N^f(w_1), U_N^f(q_1)\}$

$= \max\{U_N(w), U_N(q)\},$

$F_N(w + q)$

$= F_N(f(w_1) + f(q_1))$

$= F_N(f(w_1 + q_1))$

$= F_N^f(w_1 + q_1)$

$\leq \max\{F_N^f(w_1), F_N^f(q_1)\}$

$= \max\{F_N(w), F_N(q)\}.$

(ii) $T_N(\alpha w)=T_N(\alpha f(w_1))=T_N(f(\alpha w_1))=T_N^f(f(\alpha w_1))\geq T_N^f(w_1)=T_N(w),$

$C_N(\alpha w)=C_N(\alpha f(w_1))=C_N(f(\alpha w_1))=C_N^f(f(\alpha w_1))\geq C_N^f(w_1)=C_N(w),$

$G_N(\alpha w)=G_N(\alpha f(w_1))=G_N(f(\alpha w_1))=G_N^f(f(\alpha w_1))\leq G_N^f(w_1)=G_N(w),$

$U_N(\alpha w)=U_N(\alpha f(w_1))=U_N(f(\alpha w_1))=U_N^f(f(\alpha w_1))\leq U_N^f(w_1)=U_N(w),$

$F_N(\alpha w)=F_N(\alpha f(w_1))=F_N(f(\alpha w_1))=F_N^f(f(\alpha w_1))\leq F_N^f(w_1)=F_N(w).$

(iii) $T_N([w,q])=T_N([f(w_1),f(q_1)])=T_N(f([w_1,q_1]))=T_N^f([w_1,q_1])\geq T_N(w_1)=T_N(w),$

$C_N([w,q])=C_N([f(w_1),f(q_1)])=C_N(f([w_1,q_1]))=C_N^f([w_1,q_1])\geq C_N(w_1)=C_N(w),$

$G_N([w,q])=G_N([f(w_1),f(q_1)])=G_N(f([w_1,q_1]))=G_N^f([w_1,q_1])\leq G_N(w_1)=G_N(w),$

$U_N([w,q])=U_N([f(w_1),f(q_1)])=U_N(f([w_1,q_1]))=U_N^f([w_1,q_1])\leq U_N(w_1)=U_N(w),$

$F_N([w,q])=F_N([f(w_1),f(q_1)])=F_N(f([w_1,q_1]))=F_N^f([w_1,q_1])\leq F_N(w_1)=F_N(w),$

Therefore, $L^f = \{(w, T_M^f(w), C_M^f(w), G_M^f(w), U_M^f(w), F_M^f(w)) : w \in L_1\}$ satisfies all the conditions for being an SVPN-Lie ideal of L_2.

Novelty:

Conclusions:

In this article, we introduced the notion of SVPN-Lie ideal of SVPN-Lie algebra. Besides, we formulated several interesting results on SVPN-Lie ideal and SVPN-Lie algebra. Further, we furnish few illustrative examples.

In the future, we hope that based on the current study many new notions namely single-valued pentapartitioned neutrosophic anti-Lie ideal, single-valued pentapartitioned neutrosophic Lie topology can also be introduce.

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A Neutrosophic Compromise Programming Technique to Solve Multi-Objective Assignment Problem with T2TpFNs

Murshid Kamal[1], Prabhjot Kaur[2], Irfan Ali[3],* and Aquil Ahmed[4]
[1] Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh- 202002, India.
Email: murshidamu@gmail.com
[2] University Institute of Engineering and Technology, Panjab University, Chandigarh, India.
Email: prabh.g3@gmail.com
[3] Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh- 202002, India.
Email: irfii.st@amu.ac.in
[4] Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh- 202002, India.
Email: aquilstat@gmail.com
* Correspondence: irfii.st@amu.ac.in

Abstract: Multi-objective assignment problems (MOAPs) emerge in a wide range of real-world scenarios, from everyday activities to large-scale industrial operations. In this study, a MOAP with fuzzy parameters is investigated, and the fuzziness is represented by a Type-2 fuzzy logic system. Because the T2FLS is more efficient in dealing with the uncertainty of a decision-making process, the current problem's many parameters are represented by Type-2 trapezoidal fuzzy numbers (T2TpFNs). T2TpFNs are first reduced to Type-1 fuzzy numbers, then to crisp numbers. Finally, the neutrosophic compromise programming technique (NCPT) is applied to produce a problem compromise solution. A numerical problem is used to demonstrate the validity and applicability of the NCPT for the current MOAP. Furthermore, a comparison of NCPT to other techniques such as FPT and IFPT shows its superiority.

Keywords: Multi-objective Optimization; Assignment Problem; Type-2 Fuzzy Logic; Neutrosophic Programming; Fuzzy Goal Programming; Intuitionistic Fuzzy Programming.

1. Introduction

An (AP) is a combinatorial discrete optimization decision making problem arising in operations research and project management. It is an indispensable part of human resource project management, one of the main project management areas. It includes selection, development and management/control of the project team. In literature, the assignment problem has also been called the maximum weight matching problem. It has a wide range of applications in many real-life projects related to, for instance, education [16], production planning in telecommunication [67], rail transport [70] and medicine [74]. A classical assignment problem deals with allocating n tasks to n agents so that each agent is assigned to a single task and only one agent performs each task to optimize a pre-defined objective. This may involve maximizing efficiency or minimizing assignment cost or execution time of the tasks.

Generally, a cost-minimizing assignment problem (CMAP) aims to find an assignment schedule that minimizes the total assignment cost. A time minimizing assignment problem (TMAP), also known as a bottleneck assignment problem, focuses on minimizing the overall execution time of all the tasks. The first polynomial-time algorithm, viz., Hungarian algorithm for solving a CMAP, was proposed by Kuhn [33] in 1955. Later, Ravindran and Ramaswamy [60] used the Hungarian approach to solve a single objective bottleneck assignment problem. Various researchers have discussed a number of variants of the CMAP as well as the TMAP [11,47,53,66,69,76]. Bogomolnaia and Moulin [11] discussed a random assignment problem with a unique solution in which probabilistic serial assignment has been characterized by efficiency in an ordinal sense and envy-freeness. Maxon and Bhadury [47] discussed a multi-period assignment problem with repetitive tasks and tried to integrate a human aspect into their analysis. Nuass [53] suggested an optimizing and heuristics approach for solving generalized assignment problems. Sasaki [66] discussed axiomatic characterizations like consistency and monotonicity of the core of assignment problems in his research. Sourd [69] addressed a persistent assignment problem to solve scheduling problems with periodic cost functions. Vatow and Orden [76] discussed a personnel assignment problem. A number of books are also available in the literature that discuss assignment problems and their variants thoroughly [12,22,51,72].

While making strategic planning decisions in many real-life situations related to economics, science and engineering, often there is a suggested need to optimize more than one objective simultaneously. It gives rise to multiobjective optimization problems (MOOPs). In MOOPs, the multiple objectives are mostly conflicting in nature, and therefore, a single optimal solution may or may not exist. One has to search for trade-off/compromising solution(s) that involves a loss in one of the objective values in return for the gains in the others. It is easy to determine the superiority of a solution over the others in a single objective optimization problem, but in a MOOP, compromising solutions' consistency is determined by the concept of dominance. Therefore, these compromising solutions form the so-called Pareto frontier of the problem and are called Pareto optimal solutions that give rise to non-dominated points of the problem in its criteria space. Likewise, depending upon various market segments in this competitive world, a business industry might choose a strategy to assign various jobs to various agents in such a way that some objectives are optimized simultaneously. These objectives may either involve minimizing total assignment cost or that of the overall execution time or both at a time. For instance, many business firms either follow low-cost strategies or follow better responsiveness and customer service rules. Assignment problems in which both these factors are taken into account become time-cost trade-off problems as the solution providing the lowest cost may not provide the least time as well. Such problems fall in the category of bi-objective/multiobjective assignment problems. These problems have been investigated intensively in literature by many researchers [1,6,7,19,23,30,48,55,57,75,77]. Adiche et al. [1] proposed a hybrid algorithm for solving MOAPs. Bao et al. [6] studied the 0-1 programming method to transform and solve a MOAP by transforming it to a single objective assignment problem (SOAP). Geetha et al. [19] discussed the cost-time trade-off in a multicriteria assignment problem, whereas Hammadi [23] solved a MOAP using a tabu search algorithm. Yadaiah et al. [77] discussed an assignment problem with multiple objectives viz., time-cost-quality using the Hungarian algorithm. Furthermore, in several real-world optimization issues, the decision-makers are not always able to assign precise values to the problem's many parameters.

Only a vague information may be available based on abrupt changes in the environmental conditions, sudden breakdown of machinery, changes in government policies like complete or partial lockdown in the concerned region (specifically, in the epidemic/pandemic scenario like Covid-19) that may result in sudden shortage of products with high demand or an increase in demand of the newly launched products etc. This vagueness may also be based on past experiences and knowledge about the related situations. Thus, there is uncertainty in the values of parameters which may be very large as well. The theory behind fuzzy techniques is based on the notion of relative graded membership, inspired by human perception and cognition processes. It can deal with information arising from cognition and computational perception that is partially true, imprecise or without sharp boundaries. In 1965, Lotfi A. Zadeh[80] published his first famous research paper on fuzzy sets. Since then, various computational optimization techniques based on fuzzy logic have been developed for pattern recognition and identifying, optimizing, controlling, and developing intelligent decision-making systems. It can also provide an effective means for conflict resolution of multiple criteria and assess the available options in a better way. Later, Zadeh [81] also discussed the concept of a linguistic variable and its application to approximate reasoning.

Assignment problems performed in turbulent times (e.g., economic crisis, pandemic, risks etc.) may also have complex parameter estimation that leads to the discussion of these problems in a fuzzy environment. Researchers have thoroughly discussed various SOAPs/MOAPs and their variants under fuzziness [9,13,14,17,25,26,33,37,38,39,40,41,42,43,44,49,59,61,65,71]. Biswas and Pramanik [9] discussed a MOAP in the context of military affairs with fuzzy costs as trapezoidal fuzzy numbers. To transform their problem into a crisp single objective assignment problem, they applied Yager's ranking method. Chen [13] proposed a fuzzy assignment paradigm that treated all individuals as having the same abilities. De and Yadav [14] proposed an algorithm to solve a MOAP with exponential (nonlinear) membership using an interactive fuzzy goal programming approach whereas Feng and Yang [17] discussed a bi-objective assignment problem and constructed a chance-constrained goal programming model for the problem. Huang et al. [25] discussed a fuzzy multicriteria decision-making approach for solving a bi-objective personnel assignment problem whereas Huang and Zhang [26] developed a mathematical model for a fuzzy assignment problem (FAP) with a set of qualification constraints. Then, they designed a tabu search algorithm based on fuzzy simulation to solve the problem. Kagade and Bajaj [31] solved a MOAP with cost coefficients of the objective functions as interval values. Li et al. [40] discussed FAPs and presented a metric uncertainty model of concentrated quantification value. The convergence of the solution algorithm developed by combining genetic algorithm and assignment problems has been analyzed using Markov chain theory. Lin and Wen [41] also considered an FAP with assignment costs as fuzzy numbers and proposed a methodology that reduces the problem, either to a linear fractional programming problem or to a bottleneck assignment problem. They used a labelling algorithm to solve the related linear fractional programming problem. Lin et al. [42] studied an FAP and performed advanced sensitivity analysis viz., Type II and Type III sensitivity analysis. Type II sensitivity analysis determined the range of perturbation so that the optimal solution remains optimal whereas Type III sensitivity analysis determined the range for which the rate at which the optimal value function changes remains unchanged. Liu and Gao [43] designed a genetic algorithm to solve the fuzzy weighted balance equilibrium multi-job assignment problem whereas Liu and Li [44] presented a fuzzy quadratic assignment problem with three penalty costs and developed a hybrid genetic algorithm to solve the problem. Mukherjee and Basu [49] proposed a fuzzy ranking method for solving assignment problems with fuzzy costs. Pramanik and Biswas [59] studied a MOAP in which time, costs and inefficiency were represented by generalized trapezoidal fuzzy numbers and developed a priority-based fuzzy goal programming method. A traffic assignment based on fuzzy choices has been discussed by Ridwan [61]. Sakawa et al. [65] used interactive fuzzy programming for the linear and linear fractional programming workforce and production assignment problems. Tada and Ishii [71] also discussed a bi-objective FAP. For some other fuzzy models of the assignment problem and its variants, one may refer to the works of Gupta and Mehlavat [21], Jose and Kuriakose [28], Majumdar and Bhunia[46], Mukherjee and Basu [50], Nirmala and Anju [54], Pandian and Kavitha [56]and Thorani and Shankar [73], Yang and Liu [78], Ye and Xu [79].

Generally, in fuzzy optimization theory, Type-1 fuzzy set (T1FS) is employed that represents the uncertainty of the parameters by the membership functions which are crisp numbers lying in the interval [0, 1]. From the beginning, one of the major issues with the T1FS is that it cannot handle the uncertainty of the parameters efficiently, specifically, in situations where there is further uncertainty associated with the membership functions of the parameters. There is a need to depict such uncertainties by fuzzy sets that have blur boundaries. Then, a Type-2 fuzzy set (T2FS) came into existence. Membership functions of T2FS are three dimensional that allow some additional degrees of freedom to manage these uncertainties in a better way. In recent years, researchers have discussed various decision-making problems using T2FS [15,20,27,29,34,35,36,45,52]. The problem studied in this paper is a MOAP with fuzzy parameters, represented by T2TpFNs. Firstly, a two-stage defuzzification process is used to convert these T2TpFNs to equivalent crisp values and then, the neutrosophic logic is applied to solve the problem. The definition of neutrosophic logic and the related literature review is provided in the next subsection.

1.1 Literature Review on Neutrosophic Logic

As mentioned in the previous section, the theory behind fuzzy techniques is based on the notion of relative graded membership, i.e., the degree of belongingness of a parameter in an interval or a fuzzy set. Nevertheless, sometimes it is important to discuss the non-belongingness or non-membership of that parameter to cater a more realistic scenario. Atanassov [5] proposed a generalization of fuzzy sets viz, intuitionistic fuzzy logic that incorporates both the aforementioned factors. In this approach, two different real numbers representing the degree of truth and degree of falsehood are associated with each parameter. However, a half-true expression in this logic is not always half false; there may be some hesitation degree as well. Many researchers have developed a number of intuitionistic fuzzy programming approaches which gained significant popularity among the existing multiobjective optimization techniques. Angelov [3,4] first discussed optimization in an intuitionistic fuzzy environment. Later on, various researchers discussed this technique to study assignment problems as well. Jose and Kuriakose [28] presented an algorithm for solving an assignment model in an intuitionistic fuzzy context. Mukherjee and Basu [50] solved an intuitionistic fuzzy assignment problem using similarity measures and score functions. Roy et al. [63] presented a new approach for solving intuitionistic fuzzy multiobjective transportation problems in which supply, demand and transportation costs are considered as intuitionistic fuzzy numbers. But certain real-world situations involve another factor called indeterminacy. In such problems, the indeterministic feature of ambiguous data plays an essential role in making a rational decision outside the reach of intuitionistic fuzzy set theory. Each membership function of the neutrosophic set is precisely quantified and independent. One obtains better and more refined results whenever the optimization is carried out in a neutrosophic or generalized neutrosophic setting. Many researchers have applied neutrosophic logic to solve various multiobjective optimization problems [2,18,32,58,62,64,82]. Aggarwal et al. [2] thoroughly discussed neutrosophic modelling and control. Freen et al. [18] discussed multiobjective nonlinear four-valued refined neutrosophic optimization. Kamal et al. [32] considered a multiobjective nonlinear selective maintenance allocation of system reliability and used a neutrosophic fuzzy goal programming approach to get the optimal solution.

Pintu and Tapan [58] presented a multiobjective nonlinear programming problem based on the neutrosophic optimization technique and discussed its application in the Riser Design problem. Rizk-Allah [62] also discussed a multiobjective transportation model under a neutrosophic environment. Şahin and Muhammed [64] studied a multicriteria neutrosophic group decision-making method based on TOPSIS for supplier selection. Zhang et al. [82] discussed neutrosophic interval sets and their applications in multicriteria decision-making problems. Next subsection discusses the motivation behind the present study.

1.2 Study Motivation

This paper aims to present an efficient algorithmic solution procedure based on neutrosophic logic for a MOAP with conflicting objectives viz., assignment cost and execution time in which T2TpFNs are used to represent these parameters. Using the output processor of T2FS these T2TpFNs are initially reduced to Type-1 fuzzy numbers and then to crisp numbers. The proposed solution procedure is named as Neutrosophic compromise programming technique (NCPT). The selection of T2FS for the present study is due to the fact that its membership functions allow some additional degrees of freedom to manage the uncertainties/vagueness in the parameters (here, time and cost) in a better way. However, the advantage of neutrosophic logic, as mentioned in the previous subsection, is that it offers a neutral perspective to decide the best possible compromise solution(s) of a MOOP. It is shown that NCPT is the best solution technique for dealing for dealing with inaccurate, missing, and inconsistent information of the present MOAP when compared to the available solution techniques viz., fuzzy and Intuitionistic fuzzy programming techniques. This comparison has been done with the help of a numerical problem. LINGO software, created by LINDO Systems Inc., is used for all calculation-based frameworks.

The rest of the paper is structured as follows: In Section 2, mathematical statement of the present MOAP is given. It explains the basic as well as the fuzzy model of the problem viz., "Model 1" and "Model 2", respectively. Section 3 discusses some basic mathematical preliminaries related to fuzzy, intuitionistic fuzzy and neutrosophic sets. Section 4 discusses the defuzzification process of T2TpFNs. In Section 5, three different solution techniques that are applied to the present MOAP have been discussed in detail. In Section 6, some real-world applications of the present MOAP are given. The efficacy of the proposed NCPT solution technique for a MOAP instance is addressed in Section 7. Section 8 discusses the performance and outcome of the proposed solution technique. It also provides its comparative study with the other two solution techniques. Advantage of using the NCPT solution technique instead of other commonly used techniques has been addressed in Section 9. Section 10 provides conclusion and the future aspects of the present study.

2. Mathematical Statement of MOAP

Nomenclature

Indices:

i **-** Index for *n* workers, (*i=1, 2,…, n*)

j **-** Index for *n* tasks, (*j=1,2,…, n*)

Decision Variable:

x_{ij} -Binary variable that takes the values 1 and 0 if j^{th} taskis assigned and not assigned to i^{th} worker, respectively. Equivalently,

$$x_{ij} = \begin{cases} 1, & if \quad j^{th} \ task \ is assigned \ to \ i^{th} \ worker \\ 0, & otherwise \end{cases}$$

Parameters:

c_{ij} - Assignment cost of j^{th} task to the i^{th} worker

t_{ij} -execution time when i^{th} worker performs j^{th} task

Model 1:

The mathematical formulation of a MOAP with the above-mentioned parameters is as follows:

$$\left.\begin{aligned} & Min\ Z_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \\ & Min\ Z_2 = \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}x_{ij} \\ & \text{Subject to} \\ & \qquad \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,2,\dots,n \\ & \qquad \sum_{i=1}^{m} x_{ij} = 1, \quad j = 1,2,\dots,n \\ & \qquad x_{ij} = 0 \text{ or } 1, \qquad i = 1,2,\dots,n,\ j = 1,2,\dots,n. \end{aligned}\right\}$$

In Model 1, time ($\tilde{\tilde{t}}_{ij}$) and cost ($\tilde{\tilde{c}}_{ij}$) parameters are assumed to be T2TpFNs.

Model 2:

$$\left.\begin{array}{l} Min\ Z_1 = \sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{\tilde{c}}_{ij}x_{ij} \\ Min\ Z_2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{\tilde{t}}_{ij}x_{ij} \\ \text{Subject to} \\ \text{Model}(1) \end{array}\right\}$$

3. Mathematical Preliminaries

Some basic definitions of fuzzy, intuitionistic fuzzy and neutrosophic set are discussed.

Definition 3.1 Fuzzy Set or Type I Fuzzy Set (T1FS) [10]

A fuzzy set $\tilde{C}$ is defined on the set Y of real numbers. Its membership function $\mu_{\tilde{C}}(y)$ can be characterized as:

$$\mu_{\tilde{C}} : Y \in [0, 1];\ 0 \le \mu_{\tilde{C}}(y) \le 1,$$

Thus, a T1FS can be defined as: $\tilde{C} = \{(y, \mu_{\tilde{C}}(y) : y \in Y\}$.

Definition 3.2 Defuzzification of T1FS [10]

Defuzzification is a process of transforming a fuzzy inference into a crisp output. For a Type-1 fuzzy number (T1FN) also, there exists an associated crisp quantity which is called defuzzified form of that T1FN. Let $\tilde{C} = (c_1, c_2, c_3, c_4)$ be a Type-1 Trapezoidal Fuzzy Number (T1TpFN). Using probability density function, defuzzified value of $\tilde{C}$ can be computed as:

$$V(\tilde{C}) = \frac{1}{3}\left(c_1 + c_2 + c_3 + c_4 + \frac{(c_1c_2 - c_3c_4)}{c_3 + c_4 - c_1 - c_2}\right)$$

Definition 3.3 Type-2 Fuzzy Set (T2FS) [10]

Generalization of interval-valued fuzzy sets is known as T2FS, if the intervals are fuzzy. A T2FScan be expressed in four TIFS. . That means four membership functions of a T2FS are T1FSs, which depict the uncertainty of T2FS in a justified manner. Therefore, a membership function of T2FS is of the form $\mu_{\tilde{C}} : Y \to \chi([0, 1])$ where $\chi([0, 1])$ denotes the set of all T1FSs defined on the interval [0, 1].

Definition 3.4 Type-1 Trapezoidal Fuzzy Number (T1TpFN) [10]

A T1TpFN $\tilde{C} = (c_1, c_2, c_3, c_4)$ on Y with the membership function can be defined as:

$$\mu_{\tilde{C}}(y) = \begin{cases} \dfrac{y - c_1}{c_2 - c_1} & \text{if } c_1 \le y \le c_2 \\ 1 & \text{if } c_2 \le y \le c_3 \\ \dfrac{c_4 - y}{c_4 - c_3} & \text{if } c_3 \le y \le c_4 \\ 0 & \text{if } y < c_1 \ or\ y > c_4 \end{cases}$$

Definition 3.5 Type-2 Trapezoidal Fuzzy Number (T2TpFN) [10]

A T2TpFN $\tilde{\tilde{C}}$ can be expressed in four T1TpFNs: $\tilde{\tilde{C}} = (\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4)$

Where, $\tilde{C}_1 = (c_1^L, c_0^L, c_0^R, c_1^R)$, $\tilde{C}_2 = (c_2^L, c_0^L, c_0^R, c_2^R)$, $\tilde{C}_3 = (c_3^L, c_0^L, c_0^R, c_3^R)$ and $\tilde{C}_4 = (c_4^L, c_0^L, c_0^R, c_4^R)$ represent T1TpFNs.

Now, primary membership functions $\mu_{\tilde{\tilde{C}}}(y)$ of $\tilde{\tilde{C}}$ can be defined as:

$$\mu_{\tilde{\tilde{\lambda}}}(y) = \begin{cases} (\ \mu_{1j}^L(y),\ \ \mu_{2j}^L(y),\ \ \mu_{3j}^L(y),\ \ \mu_{4j}^L(y)\ ) & \text{if } c_{j+1}^L \le y \le c_j^L \\ (\ \hat{\mu}_{1j}^R(y),\ \ \hat{\mu}_{2j}^R(y),\ \ \hat{\mu}_{3j}^R(y),\ \ \hat{\mu}_{4j}^R(y)\ ) & \text{if } c_j^R \le y \le c_{j+1}^R \\ 0 & \text{otherwise} \end{cases}$$

where

$$\hat{\mu}_{ij}^L(y) = \begin{cases} \dfrac{y - c_i^L}{c_0^L - c_i^L} & i > j \\ 0 & i \le j \end{cases} \quad ;\ (j = 0, 1, 2, 3;\ i = 1, 2, 3, 4\ )$$

$$\hat{\mu}_{ij}^R(y) = \begin{cases} \dfrac{c_i^R - y}{c_i^R - c_0^R} & i > j \\ 0 & i \le j \end{cases} \quad ;\ (j = 0, 1, 2, 3;\ i = 1, 2, 3, 4\ )$$

Secondary membership function $\mu_{\tilde{\tilde{C}}}(y)$ of $\tilde{\tilde{C}}$ can be defined as:

$$\begin{cases} \dfrac{\mu - \mu_{1j}^L(y)}{\mu_{2j}^L(y) - \mu_{1j}^L(y)} & \text{if } \mu_{1j}^L(y) \le \mu \le \mu_{2j}^L(y) \\ 1 & \text{if } \mu_{2j}^L(y) \le \mu \le \mu_{3j}^L(y) \\ \dfrac{\mu_{4j}^L(y) - \mu}{\mu_{4j}^L(y) - \mu_{3j}^L(y)} & \text{if } \mu_{3j}^L(y) \le \mu \le \mu_{4j}^L(y) \\ 0 & \text{otherwise} \end{cases} \quad ; \quad c_4^L \le y \le c_0^L$$

and

$$\begin{cases} \dfrac{\mu - \hat{\mu}_{1j}^R(y)}{\hat{\mu}_{2j}^R(y) - \hat{\mu}_{1j}^R(y)} & \text{if } \hat{\mu}_{1j}^R(y) \le \mu \le \hat{\mu}_{2j}^R(y) \\ 1 & \text{if } \hat{\mu}_{2j}^R(y) \le \mu \le \hat{\mu}_{3j}^R(y) \\ \dfrac{\hat{\mu}_{4j}^R(y) - \mu}{\hat{\mu}_{4j}^R(y) - \hat{\mu}_{3j}^R(y)} & \text{if } \hat{\mu}_{3j}^R(y) \le \mu \le \hat{\mu}_{4j}^R(y) \\ 0 & \text{otherwise} \end{cases} \quad ; \quad y_0^R \le y \le y_4^R$$

4. Defuzzification Technique of a T2TpFN

Since $\tilde{\tilde{c}}_{ij}$ and $\tilde{\tilde{t}}_{ij}$ in model 1 are assumed T2TpFN, therefore under this section, the defuzzification process of T2TpFNs is discussed. From definition 3.5, T2TpFN can be defined by four T1TpFNs and for each point of the universe of discourse of the T2TpFN, a T1TpFN corresponds as a secondary membership function. Therefore, a technique that defuzzifies a T1TpFN would be sufficient to provide a defuzzified value of the T2TpFN. The present defuzzification technique is divided into two stages. Stage-1 reduces T2TpFN into its equivalent T1TpFNs; however, Stage-2 defuzzifies these T1TpFNs to get the crisp values of the associated T2TpFN.

Stage 1.

Let $\tilde{\tilde{C}} = (\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4)$ be a T2TpFN where $\tilde{C}_i = (c_i^L, c_0^L, c_0^R, c_i^R)$; $i = 1, 2, 3, 4$ denotes a T1TpFN. The membership function of a T1TpFN $\tilde{C}_i$ can be defined as:

$$\mu_{\tilde{C}_i}(y) = \begin{cases} \dfrac{y - c_i^L}{c_0^L - c_i^L} & \text{if } c_i^L \le y \le c_0^L \\ 1 & \text{if } c_0^L \le y \le c_0^R \\ \dfrac{c_i^R - y}{c_i^R - c_0^R} & \text{if } c_0^R \le y \le c_i^R \\ 0 & \text{otherwise} \end{cases}$$

Then, the probability density function $f_{\tilde{C}_i}(y)$ corresponding to the T1TpFN $\tilde{C}_i$ can be stated as:

$$f_{\tilde{C}_i}(y) = \begin{cases} \dfrac{2(y - c_i^L)}{(c_0^L - c_i^L)(c_0^R + c_i^R - c_0^L - c_i^L)} & \text{if } c_i^L \le y \le c_0^L \\ \dfrac{2}{(c_0^R + c_i^R - c_0^L - c_i^L)} & \text{if } c_0^L \le y \le c_0^R \\ \dfrac{2(c_i^R - y)}{(c_i^R - c_0^R)(c_0^R + c_i^R - c_0^L - c_i^L)} & \text{if } c_0^R \le y \le c_i^R \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1, 2, 3, 4.)$$

Now, calculate the expected value $E(Y_{\tilde{C}_i})$ of $Y_{\tilde{C}_i}$ as $E(Y_{\tilde{C}_i}) = \int_{-\infty}^{\infty} y\, f_{\tilde{C}_i}(y)\,\mathrm{d}y$. This value is noted as the defuzzified value $V(\tilde{C}_i)$ of the T1TpFN $\tilde{C}_i$ for all $i \in \{1, 2, 3, 4\}$ i.e.,

$$V(\tilde{C}_i) = E(Y_{\tilde{C}_i}) = \left[\int_{c_i^L}^{c_0^L} \frac{2(y - c_i^L)y}{(c_0^L - c_i^L)(c_0^R + c_i^R - c_0^L - c_i^L)} \mathrm{d}y + \int_{c_0^L}^{c_0^R} \frac{2}{(c_0^R + c_i^R - c_0^L - c_i^L)} y\mathrm{d}y + \int_{c_0^R}^{c_i^R} \frac{2(c_i^R - y)y}{(c_i^R - c_0^R)(c_0^R + c_i^R - c_0^L - c_i^L)} \mathrm{d}y \right]$$

$$= \frac{1}{3(c_0^R + c_i^R - c_0^L - c_i^L)} \left[(c_0^R)^2 + (c_i^R)^2 - (c_0^L)^2 - (c_i^L)^2 + c_0^R c_i^R - c_0^L c_i^L \right]$$

$$= \frac{1}{3}\left(c_0^L + c_i^L + c_0^R + c_i^R + \frac{c_0^L c_i^L - c_0^R c_i^R}{c_0^R + c_i^R - c_i^L - c_0^L}\right) \quad \text{for all } i = 1, 2, 3, 4.$$

Thus, the T2TpFN based on the defuzzified values $V(\tilde{C}_i)$ of the T1TpFNs $\tilde{C}_i$ for all i's, can be defined as: $\tilde{\tilde{C}} = (V(\tilde{C}_1), V(\tilde{C}_2), V(\tilde{C}_3), V(\tilde{C}_4))$

$$=\begin{pmatrix}\frac{1}{3}\left(c_1^L+c_0^L+c_0^R+c_1^R+\frac{c_1^Lc_0^L-c_0^Rc_1^R}{c_0^R+c_1^R-c_1^L-c_0^L}\right),\frac{1}{3}\left(c_2^L+c_0^L+c_0^R+c_2^R+\frac{c_2^Lc_0^L-c_0^Rc_2^R}{c_0^R+c_2^R-c_2^L-c_0^L}\right),\\ \frac{1}{3}\left(c_3^L+c_0^L+c_0^R+c_3^R+\frac{c_3^Lc_0^L-c_0^Rc_3^R}{c_0^R+c_3^R-c_3^L-c_0^L}\right),\ \frac{1}{3}\left(c_4^L+c_0^L+c_0^R+c_4^R+\frac{c_4^Lc_0^L-c_0^Rc_4^R}{c_0^R+c_4^R-c_4^L-c_0^L}\right)\end{pmatrix}$$

Stage 2.

T1TpFNs are further defuzzified at this stage to generate the final defuzzified version of the T2TpFN as follows: $DV\left(\tilde{\tilde{C}}\right)=\frac{1}{3}\left(V(\tilde{C}_1)+V(\tilde{C}_2)+V(\tilde{C}_3)+V(\tilde{C}_4)+\frac{(V(\tilde{C}_1)V(\tilde{C}_2)-V(\tilde{C}_3)V(\tilde{C}_4))}{(V(\tilde{C}_3)+V(\tilde{C}_4)-V(\tilde{C}_1)\text{-}V(\tilde{C}_2))}\right)$

$$=\frac{1}{9}\left(\sum_{i=1}^{4}(c_i^L+c_i^R+c_0^L+c_0^R+\frac{c_i^Lc_0^L-c_0^Rc_i^R}{c_0^R+c_i^R-c_i^L-c_0^L})\right.$$

$$\left.+\frac{\prod_{i=1}^{2}\left(c_i^L+c_0^L+c_0^R+c_i^R+\frac{c_i^Lc_0^L-c_0^Rc_i^R}{c_0^R+c_i^R-c_i^L-c_0^L}\right)-\prod_{j=3}^{4}\left(c_j^L+c_0^L+c_0^R+c_j^R+\frac{c_j^Lc_0^L-c_0^Rc_j^R}{c_0^R+c_j^R-c_j^L-c_0^L}\right)}{\sum_{j=3}^{4}\left(c_j^L+c_j^R+\frac{c_j^Lc_0^L-c_0^Rc_j^R}{c_0^R+c_j^R-c_j^L-c_0^L}\right)-\sum_{i=1}^{2}\left(c_i^L+c_i^R+\frac{c_i^Lc_0^L-c_0^Rc_i^R}{c_0^R+c_i^R-c_i^L-c_0^L}\right)}\right)$$

The same procedure can be followed for T2TpFNs $\tilde{\tilde{t}}_{ij}$ to obtain their crisp values.

After the above defuzzification procedure, the resultant MOAP model finally takes the form

Model 3:

$$\left.\begin{aligned}
&Min\ Z_1=\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{\tilde{c}}_{ij}x_{ij}=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix}V(\tilde{c}_1)+V(\tilde{c}_2)+V(\tilde{c}_3)+V(\tilde{c}_4)\\ +\frac{(V(\tilde{c}_1)V(\tilde{c}_2)-V(\tilde{c}_3)V(\tilde{c}_4))}{(V(\tilde{c}_3)+V(\tilde{c}_4)-V(\tilde{c}_1)-V(\tilde{c}_2))}\end{pmatrix}x_{ij}\\
&Min\ Z_2=\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{\tilde{t}}_{ij}x_{ij}=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix}V(\tilde{t}_1)+V(\tilde{t}_2)+V(\tilde{t}_3)+V(\tilde{t}_4)\\ +\frac{(V(\tilde{t}_1)V(\tilde{t}_2)-V(\tilde{t}_3)V(\tilde{t}_4))}{(V(\tilde{t}_3)+V(\tilde{t}_3)-V(\tilde{t}_1)-V(\tilde{t}_2))}\end{pmatrix}x_{ij}\\
&\text{Subject to;}\\
&\sum_{j=1}^{n}x_{ij}=1,\quad i=1,2,...,n\\
&\sum_{i=1}^{m}x_{ij}=1,\quad j=1,2,...,n\\
&x_{ij}=0\text{ or }1,\quad i=1,2,...,n,\ j=1,2,...,n.
\end{aligned}\right\}$$

5. Methodology

In this section, we discuss three different solution techniques viz.,

(i) Neutrosophic compromise programming technique
(ii) Fuzzy programming technique
(iii) Intuitionistic fuzzy programming.

The method of transforming a multiobjective optimization problem into a related single-objective optimization problem is also discussed for all the suggested approaches.

5.1 Neutrosophic Compromise Programming Technique (NCPT)

The extended version of the fuzzy and intuitionistic fuzzy sets has been classified as a neutrosophic set (NS) (defined below) with an additional membership function called indeterminacy. In some specific real-life decision-making problems, there are many cases in which decision-makers have indeterminacy or unbiased reasoning in decision-making. The principles of indeterminacy often lie between those of Truth and Lies. Literally, neutrosophic means neutral thought or awareness of indeterminacy, therefore, a NS has three distinct membership features viz., truth, indeterminacy and falsehood. On the other hand, in a fuzzy set, we maximize the degree of membership function which indicates that the element belongs to that set. In contrast, in an intuitionistic fuzzy set, two types of membership functions viz., the degree of membership (also known as the degree of truth) and the degree of non-membership (also known as the degree of falsehood) of an element, are considered. To be more specific, an NS maximizes the degree of truth and indeterminacy while decreasing the degree of falsehood. A NS represents a major touchstone in a decision-making process where the decision-maker can be entirely satisfied (with truth), partly satisfied (with indeterminacy) and dissatisfied (with falsehood). In any decision-making problem, these factors increase the strength of making the right decision or achieving an optimal solution. Since for MOOPs with conflicting objectives, the challenge of finding the best solution using classical approaches is a significantly complicated issue, the NCPT would be a useful technique for achieving the best compromise solution due to its aforementioned features.

Definition 5.1 Neutrosophic Set [63]

Let Y be the universe of discourse and $y \in Y$. A neutrosophic set (NS) P over Y is the set of triplets consisting of a truth membership function $T_P(y)$, indeterminacy membership function $I_P(y)$ and a false membership function $F_P(y)$, for $y \in Y$. Mathematically;

$$P = \{< y, T_P(y), I_P(y), F_P(y) > | \, y \in Y\}$$

Here, $T_P(y)$, $I_P(y)$ and $F_P(y)$ are real non-standard or standard functions with range $]0^-, 1^+[$, i.e., $T_p(y): Y \to]0^-, 1^+[$, $I_p(y): Y \to]0^-, 1^+[$ and $F_p(y): Y \to]0^-, 1^+[$. Assume that

$$0^- \le \sup T_p(y) + \sup I_p(y) + \sup F_p(y) \le 3^+$$

Now, the general formulation of a MOOP can be defined as:

$$\text{Minimize } \{Z_1(x), Z_2(x), ..., Z_L(x)\}$$
$$\text{Subject to}$$
$$g_m(x) \le b_m(x), \quad m = 1,2,..., M$$
$$x \ge 0, \; l = 1,2,..., L$$

where, $Z_l(x); l = 1,2,3,..., L$ denotes the *lth* objective function, $g_m(x); m = 1,2,3,..., M$ denotes the constraints and x denotes the decision variables. In 1970, Bellman and Zadeh [8] introduced the definitions of fuzzy decision (D), fuzzy goal (G) and fuzzy constraint (C) that are useful for solving any real-life optimization problems under uncertainty. Consequently, a fuzzy decision set is described as: $D = G \cap C$

On the same lines, a neutrosophic decision set D_N, with neutrosophic goal set G_L and neutrosophic constraints C_m can be defined as follows:

$$D_N = \{(\cap_{l=1}^{L} G_L) \cap (\cap_{m=1}^{M} C_m) = (x, T_D(x), I_D(x), F_D(x))\}$$

where

$$\begin{Bmatrix} T_D(x) = \min\begin{bmatrix} T_{G_1}(x), T_{G_2}(x), ..., T_{G_L}(x) \\ T_{C_1}(x), T_{C_2}(x), ..., T_{C_M}(x) \end{bmatrix} & \forall x \in X, \\ I_D(x) = \max\begin{bmatrix} I_{G_1}(x), I_{G_2}(x), ..., I_{G_L}(x) \\ I_{C_1}(x), I_{C_2}(x), ..., I_{C_M}(x) \end{bmatrix} & \forall x \in X, \\ F_D(x) = \max\begin{bmatrix} F_{G_1}(x), F_{G_2}(x), ..., F_{G_L}(x) \\ F_{C_1}(x), F_{C_2}(x), ..., F_{C_M}(x) \end{bmatrix} & \forall x \in X, \end{Bmatrix}$$

Here $T_D(x)$, $I_D(x)$ and $F_D(x)$ are the truth, indeterminacy and false membership functions, respectively, defined under the neutrosophic decision D_N.

To find the compromise solutions for a multiobjective decision making optimization issue, membership functions are created for each objective function and the lower and upper bounds are calculated as L_i and U_i respectively, by solving them individually under the stated constraints:

$$U_l = \max_l \{Z_l(X)\} \text{ and } L_l = \min_l \{Z_l(X)\} \text{ for all } l = 1,2,...,L \tag{1}$$

Further, upper and lower bounds for l^{th} objectives under the NS can be determined as follows:

$$U_l^T = U_l, \qquad L_l^T = L_l \text{ for truth membership function} \tag{2}$$

$$U_l^I = L_l^T + a_l, \qquad L_l^I = L_l^T \text{ for indeterminacy membership function} \tag{3}$$

$$U_l^F = U_l^T, \qquad L_l^F = L_l^T + b_l \text{ for false membership function} \tag{4}$$

where a_l and b_l are predetermined real values assigned by the decision-makers that lie in the interval (0, 1). Further, the linear membership function $T_l(Z_l(x))$ of truth, $I_l(Z_l(x))$ of indeterminacy and $F_l(Z_l(x))$ of falsity under the neutrosophic environment can be constructed as follows:

$$T_l(Z_l(x)) = \begin{cases} 1 & if\ Z_l(x) \le L_l^T \\ \dfrac{U_l^T - Z_l(x)}{U_j^T - L_j^T} & if\ L_l^T \le Z_l(x) \le U_l^T \\ 0 & if\ Z_l(x) \ge U_l^T \end{cases} \tag{5}$$

$$I_l((Z_l(x)) = \begin{cases} 1 & if\ Z_l(x) \le L_l^I \\ \dfrac{U_l^I - Z_l(x)}{U_j^I - L_j^I} & if\ L_l^I \le Z_l(x) \le U_l^I \\ 0 & if\ Z_l(x) \ge U_l^I \end{cases} \tag{6}$$

$$F_l((Z_l(x)) = \begin{cases} 0 & if\ Z_l(x) \le L_l^F \\ \dfrac{Z_l(x) - L_l^F}{U_j^F - L_j^F} & if\ L_l^F \le Z_l(x) \le U_l^F \\ 1 & if\ Z_l(x) \ge U_l^F \end{cases} \tag{7}$$

It should be noted here that $U_l^{(.)} \ne L_l^{(.)}$, $\forall\ l = 1,2,...,L$.

If $U_l^{(\cdot)} = L_l^{(\cdot)}, \forall\ l = 1,2,...,L$, the membership value will be assumed to be 1.

Since the development of achievement functions helps to achieve the highest level or degree of satisfaction based on the priorities of the decision-makers, we also define a specific achievement variable for each membership function. The decision-maker may establish a target in a decision-making process to attain the maximum possible degree of satisfaction for the truth and indeterminacy membership functions while minimizing the degree of untruth as much as possible. After considering the linear membership of truth, indeterminacy and falsehood under neutrosophic nature, the mathematical expression of the neutrosophic compromise programming problem is given as

P1:

$$\left.\begin{array}{l} \text{Max } \min_{l=1,2,...,L}\ T_l(Z_l(x)) \\ \text{Max } \min_{l=1,2,...,L}\ I_l(Z_l(x)) \\ \text{Min } \max_{l=1,2,...,L}\ F_l(Z_l(x)) \\ \text{Subject to} \\ T_l(Z_l(x)) \geq I_l(Z_l(x)) \\ T_l(Z_l(x)) \geq F_l(Z_l(x)) \\ 0 \leq T_l(Z_l(x)) + I_l(Z_l(x)) + F_l(Z_l(x)) \leq 3 \end{array}\right\}$$

By using auxiliary parameters, the above problem **P1** can be transformed into a new problem, say, **P2** as follows

P2:

$$\left.\begin{array}{l} \text{Max } \alpha \\ \text{Max } \beta \\ \text{Min } \gamma \\ \text{Subject to} \\ T_l(Z_l(x)) \geq \alpha \\ I_l(Z_l(x)) \geq \beta \\ F_l(Z_l(x)) \leq \gamma \\ \alpha \geq \beta,\ \ \alpha \geq \gamma,\ \ 0 \leq \gamma + \beta + \gamma \leq 3 \\ \gamma, \beta, \gamma \in [0,1] \end{array}\right\}$$

Here α , β and γ are the auxiliary variables for the truth, indeterminacy and false membership functions, respectively. Further, the above problem **P2** can be expressed in the purest form as the problem **P3** as follows

P3:

$$\left.\begin{array}{l}\text{Max }\phi(x)=\alpha+\beta-\gamma\\ \text{Subject to;}\\ Z_l(x)+(U_l^T-L_l^T)\alpha\le U_l^T\\ Z_l(x)+(U_l^I-L_l^I)\beta\le U_l^I\\ Z_l(x)-(U_l^F-L_l^F)\gamma\le L_l^F\\ \alpha\ge\beta,\ \alpha\ge\gamma,\ \ 0\le\gamma+\beta+\gamma\le 3\\ \gamma,\beta,\gamma\in[0,1]\end{array}\right\}$$

Based on the above formulations of a neutrosophic compromise programming technique, Model 2 of the present MOAP can be presented as a neutrosophic programming model in the following manner:

Model 4:

$$\text{Max }\phi(x)=\alpha+\beta-\gamma$$

Subject to;

$$\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix}V(\tilde{c}_1)+V(\tilde{c}_2)+V(\tilde{c}_3)+V(\tilde{c}_4)\\ +\dfrac{(V(\tilde{c}_1)V(\tilde{c}_2)-V(\tilde{c}_3)V(\tilde{c}_4))}{(V(\tilde{c}_3)+V(\tilde{c}_4)-V(\tilde{c}_1)-V(\tilde{c}_2))}\end{pmatrix}x_{ij}\right\}+(U_1^T-L_1^T)\alpha\le U_1^T$$

$$\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix}V(\tilde{c}_1)+V(\tilde{c}_2)+V(\tilde{c}_3)+V(\tilde{c}_4)\\ +\dfrac{(V(\tilde{c}_1)V(\tilde{c}_2)-V(\tilde{c}_3)V(\tilde{c}_4))}{(V(\tilde{c}_3)+V(\tilde{c}_4)-V(\tilde{c}_1)-V(\tilde{c}_2))}\end{pmatrix}x_{ij}\right\}+(U_1^I-L_1^I)\beta\le U_1^I$$

$$\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix}V(\tilde{c}_1)+V(\tilde{c}_2)+V(\tilde{c}_3)+V(\tilde{c}_4)\\ +\dfrac{(V(\tilde{c}_1)V(\tilde{c}_2)-V(\tilde{c}_3)V(\tilde{c}_4))}{(V(\tilde{c}_3)+V(\tilde{c}_4)-V(\tilde{c}_1)-V(\tilde{c}_2))}\end{pmatrix}x_{ij}\right\}-(U_1^F-L_1^F)\gamma\le L_1^F$$

$$\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix}V(\tilde{t}_1)+V(\tilde{t}_2)+V(\tilde{t}_3)+V(\tilde{t}_4)\\ +\dfrac{(V(\tilde{t}_1)V(\tilde{t}_2)-V(\tilde{t}_3)V(\tilde{t}_4))}{(V(\tilde{t}_3)+V(\tilde{t}_3)-V(\tilde{t}_1)-V(\tilde{t}_2))}\end{pmatrix}x_{ij}\right\}+(U_2^T-L_2^T)\alpha\le U_2^T$$

$$\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix}V(\tilde{t}_1)+V(\tilde{t}_2)+V(\tilde{t}_3)+V(\tilde{t}_4)\\ +\dfrac{(V(\tilde{t}_1)V(\tilde{t}_2)-V(\tilde{t}_3)V(\tilde{t}_4))}{(V(\tilde{t}_3)+V(\tilde{t}_3)-V(\tilde{t}_1)-V(\tilde{t}_2))}\end{pmatrix}x_{ij}\right\}+(U_2^I-L_2^I)\beta\le U_2^I$$

$$\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix}V(\tilde{t}_1)+V(\tilde{t}_2)+V(\tilde{t}_3)+V(\tilde{t}_4)\\ +\dfrac{(V(\tilde{t}_1)V(\tilde{t}_2)-V(\tilde{t}_3)V(\tilde{t}_4))}{(V(\tilde{t}_3)+V(\tilde{t}_3)-V(\tilde{t}_1)-V(\tilde{t}_2))}\end{pmatrix}x_{ij}\right\}-(U_2^F-L_2^F)\gamma\le L_2^F$$

$$\sum_{j=1}^{n}x_{ij}=1,\quad i=1,2,\ldots,n,\ \sum_{i=1}^{m}x_{ij}=1,\quad j=1,2,\ldots,n,$$

$$\alpha\ge\beta,\ \ \alpha\ge\gamma,\ \ 0\le\gamma+\beta+\gamma\le 3$$

$$\gamma,\beta,\gamma\in[0,1],\ x_{ij0}=0\text{ or }1,$$

The following steps will be followed to discuss the present MOAP using NCPT.

Step 1. Formulate a MOAP under an uncertain environment as given by Model 2.
Step 2. Convert each fuzzy parameter of this problem into a crisp number using the defuzzification method discussed in Section 4.
Step 3. Calculate the best and worst solutions corresponding to each objective function under the given set of constraints using optimization software LINGO and create a payoff matrix (refer to Table 5).
Step 4. Determine the upper U_l and lower L_l bound, respectively, of each objectives using equation (1).
Step 5. With the help of these U_l and L_l values, find the upper and lower bound for all the membership functions (truth, indeterminacy and falsehood) using equations (2)-(4).
Step 6. Construct the linear membership function for the truth, indeterminacy and falsehood using equations (5)-(7).
Step 7. Construct the neutrosophic problem as problem P_2 and transform it into problem P_3.
Step 8. Solve the MOAP model as Model 4 and obtain the compromise solution using the Optimization Software Packages LINGO 16.0.

5.2 Fuzzy Programming Technique (FPT)

The problems involving undefined and imprecise parameters with multiple objectives are known to be typical mathematical problems. The fuzzy programming technique (FPT) is an effective and versatile solution technique for such a problem. Zimmermann [83] developed it in 1978, specifically to tackle MOOPs. A fuzzy programming model aims to optimize multiple objectives simultaneously, by reducing deviations from the goal features. Fuzzy programming needs the decision-makers to set a level of expectation for each target which is challenging as several uncertainties must also be considered in nature.
The general mathematical formulation of a fuzzy programming problem with *l* objectives and *j* constraints, with *i* decision variables, can be described as:

$$\left.\begin{array}{l} \text{Maximize } \lambda \\ \text{Subject to:} \\ \lambda \le \mu_l(x), \quad \forall l \\ g_j(x) \le 0, \quad j = 1,2,\ldots,n \\ x_i \ge 0, \quad i = 1,2,\ldots,m \end{array}\right\}$$

The following steps of the fuzzy programming technique can solve the MOAP given by Model 2.
Step 1. Find the optimal value of each objective function of the MOAP subject to the given set of constraints by ignoring all other objectives (use the optimization software LINGO).
Step 2. Calculate the best U_l and worst L_l values for each objective function separately and create a payoff matrix (Table 5).
Step 3. Define the membership function for each objective using equations (8) and (9) given below (refer [78]).
Membership function $\mu_l(Z_l(x))$ for *l*th objective function of minimization type

$$\mu_l(Z_l(x)) = \begin{cases} 1 & if \quad Z_l(x) \le L_l \\ \dfrac{U_l - Z_l(x)}{U_l - L_l} & if \quad L_l \le Z_l(x) \le U_l \\ 0 & if \quad Z_l(x) \ge U_l \end{cases} \tag{8}$$

Membership function for *lth* objective function of the maximization type

$$\mu_l(Z_l(x)) = \begin{cases} 1 & if \quad Z_l(x) \geq U_l \\ \dfrac{Z_l(x) - L_l}{U_l - L_l} & if \quad L_l \leq Z_l(x) \leq U_l \\ 0 & if \quad Z_l(x) \leq L_l \end{cases} \tag{9}$$

where L_l and U_l are the lower and upper bounds of the objective functions.
Finally, the MOAP can be defined as a fuzzy programming model as

Model 5:

$$\text{Maximize } \lambda$$
$$\text{subject to:}$$
$$\lambda \leq \mu_1 \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{3} \begin{pmatrix} V(\tilde{c}_1) + V(\tilde{c}_2) + V(\tilde{c}_3) + V(\tilde{c}_4) \\ + \dfrac{(V(\tilde{c}_1)V(\tilde{c}_2) - V(\tilde{c}_3)V(\tilde{c}_4))}{(V(\tilde{c}_3) + V(\tilde{c}_4) - V(\tilde{c}_1) - V(\tilde{c}_2))} \end{pmatrix} x_{ij} \right\}$$
$$\lambda \leq \mu_2 \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{3} \begin{pmatrix} V(\tilde{t}_1) + V(\tilde{t}_2) + V(\tilde{t}_3) + V(\tilde{t}_4) \\ + \dfrac{(V(\tilde{t}_1)V(\tilde{t}_2) - V(\tilde{t}_3)V(\tilde{t}_4))}{(V(\tilde{t}_3) + V(\tilde{t}_3) - V(\tilde{t}_1) - V(\tilde{t}_2))} \end{pmatrix} x_{ij} \right\}$$
$$\text{Model}(1)$$

Step 4. Solve this crisp MOAP above and obtain the compromise solution using the Optimization Software Package LINGO 16.0.

5.3 Intuitionistic Fuzzy Programming Technique (IFPT)

The intuitionistic fuzzy set theory is an alternative for defining a fuzzy set if the available knowledge is insufficient to describe an imprecise theory using a traditional fuzzy set. The degree of membership and non-membership for the objective functions and their limitations are concurrent and taken into account in such a way that the sum of both is either less than or equal to one.
The general mathematical formulation of a MOOP in the context of intuitionistic fuzzy programming is as follows:

$$\left. \begin{array}{ll} \text{Maximize } \alpha - \beta & \\ \text{Subject to} & \\ \mu_l(Z_l(x)) \geq \alpha, \ \chi_l(Z_l(x)) \leq \beta, \forall\, l & \\ \alpha + \beta \leq 1, & \alpha \geq \beta, \ \beta \geq 0, \\ g_k(x) \leq 0, & k = 1,2,\ldots,K \\ x_i \geq 0, & i = 1,2,\ldots,m \end{array} \right\} \tag{10}$$

where $\mu_l(Z_l(x))$ *and* $\chi_l(Z_l(x))$ are the membership and non-membership functions of the *lth* objective and α , β are their aspiration levels.
The following steps explain finding a compromise solution to the problem given by (10) using IFPT.
Step 1.Find the optimal value of each objective function of the MOOP subject to the given set of constraints by ignoring all other objectives, using the optimization software LINGO.
Step 2. Calculate the best U_l and worst L_l values for each objective function separately and create a payoff matrix (Table 5).

Step 3. Construct the membership and non-membership functions $\mu_l(Z_l(x))$ *and* $\chi_l(Z_l(x))$, respectively, of l*th* objective function, for all values of l, using equations (11) and (12) given as

$$\mu_l(Z_l(x)) = \begin{cases} 1 & if \quad Z_l(x) \le L_l \\ \dfrac{U_l - Z_l(x)}{U_l - L_l} & if \quad L_l \le Z_l(x) \le U_l \\ 0 & if \quad Z_l(x) \ge U_l \end{cases} \tag{11}$$

and

$$\chi_l(Z_l(x)) = \begin{cases} 0 & if \quad Z_l(x) \le L_l \\ \dfrac{Z_l(x) - L_l}{U_l - L_l} & if \quad L_l \le Z_l(x) \le U_l \\ 1 & if \quad Z_l(x) \ge U_l \end{cases} \tag{12}$$

Now, Model 2 of the present MOAP can be defined using IFPT as follows:

Model 6:

$$\left.\begin{array}{l}
\text{Maximize } \ (\alpha - \beta) \\
\text{Subject to;} \\
\mu_1\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix} V(\tilde{c}_1)+V(\tilde{c}_2)+V(\tilde{c}_3)+V(\tilde{c}_4) \\ +\dfrac{(V(\tilde{c}_1)V(\tilde{c}_2)-V(\tilde{c}_3)V(\tilde{c}_4))}{(V(\tilde{c}_3)+V(\tilde{c}_4)-V(\tilde{c}_1)-V(\tilde{c}_2))}\end{pmatrix}x_{ij}\right\} \ge \alpha, \\
\mu_2\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix} V(\tilde{t}_1)+V(\tilde{t}_2)+V(\tilde{t}_3)+V(\tilde{t}_4) \\ +\dfrac{(V(\tilde{t}_1)V(\tilde{t}_2)-V(\tilde{t}_3)V(\tilde{t}_4))}{(V(\tilde{t}_3)+V(\tilde{t}_3)-V(\tilde{t}_1)-V(\tilde{t}_2))}\end{pmatrix}x_{ij}\right\} \ge \alpha \\
\chi_1\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix} V(\tilde{c}_1)+V(\tilde{c}_2)+V(\tilde{c}_3)+V(\tilde{c}_4) \\ +\dfrac{(V(\tilde{c}_1)V(\tilde{c}_2)-V(\tilde{c}_3)V(\tilde{c}_4))}{(V(\tilde{c}_3)+V(\tilde{c}_4)-V(\tilde{c}_1)-V(\tilde{c}_2))}\end{pmatrix}x_{ij}\right\} \ge \beta \\
\chi_2\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{3}\begin{pmatrix} V(\tilde{t}_1)+V(\tilde{t}_2)+V(\tilde{t}_3)+V(\tilde{t}_4) \\ +\dfrac{(V(\tilde{t}_1)V(\tilde{t}_2)-V(\tilde{t}_3)V(\tilde{t}_4))}{(V(\tilde{t}_3)+V(\tilde{t}_3)-V(\tilde{t}_1)-V(\tilde{t}_2))}\end{pmatrix}x_{ij}\right\} \le \beta \\
\text{Model}(1), \\
\alpha + \beta \le 1, \quad \alpha \ge \beta, \ \beta \ge 0, \ x_{ij} = 0 \,\text{or}\, 1
\end{array}\right\}$$

Step 4. Solve this crisp model of the present MOAP by using the Optimization Software Packages LINGO 16.0 and obtain a compromise solution.

A flow chart of the proposed optimization procedure using all the techniques mentioned above is given in Figure 1.

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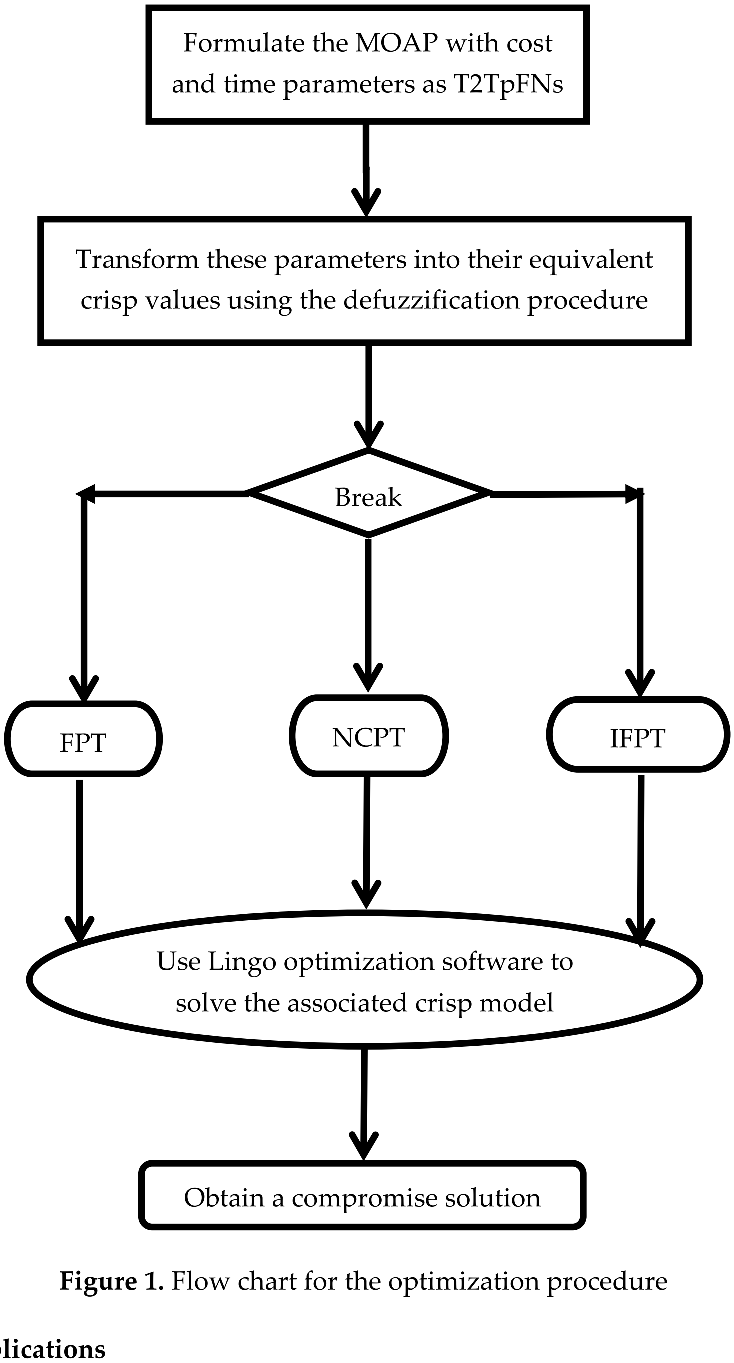

Figure 1. Flow chart for the optimization procedure

6. Real-World Applications

The present MOAP aims to minimize execution time and assignment cost, simultaneously. It finds its applications in many business scenarios where the quickest possible delivery of its product is as important as its financial budget. Generally, a quick mode of transportation may result in high transportation charges which mean that the objectives are conflicting in nature. So, the objective is to find such an assignment schedule that provides the best compromising solution to the problem. There may be many managerial implications of the present problem, but to quote some of them, consider the following real-life scenarios

(1) In an FMCG (fast-moving consumable goods) industry, due to the limited shelf life of the goods, it is important to deliver the products to the destinations as soon as possible. However, at the same time, the supply chain management team of the industry works to minimize the logistics cost. Therefore, it is important to find a way of transporting goods to minimize both objectives, simultaneously.

(2) In the commercial industry, road transportation is an extremely methodical way of hauling goods among various locations to improve the efficiency and growth of a business. Therefore, the use of heavy goods vehicles (HGVs) is an indispensable part of any business. Consider an industrial project of manufacturing some HGVs in minimum time and budget. For manufacturing various parts of an HGV in terms of both execution time and cost, quotations from various manufacturing units are taken. Then, an assignment schedule is looked for so that all the parts are produced in the minimum time and in the minimum budget so that a cost-efficient HGV is manufactured well in time. There are numerous other real-world situations of this kind that may give rise to the present MOAP.

7. Numerical Illustration

Consider an industrial manufacturing problem that uses third party operations. The product that the industry manufactures requires four major semi-finished parts. These semi-finished parts are finished and assembled to form the final product by the industry itself. All of these parts can be manufactured by any of the four different third party manufacturing units, which have imprecise values of the manufacturing time and cost corresponding to each part. The industry's objective is to assign the task of manufacturing four semi-finished parts to four third party manufacturing units so that all the parts are manufactured in the minimum time and with the least financial burden.

Here, the first objective Z_1 denotes the total manufacturing cost (in $), and the second objective Z_2 denotes the total manufacturing time (in minutes) of all the four semi-finished parts. Table 1 shows the key attributes of the problem. The imprecise manufacturing costs and times quoted by all the third party manufacturing units for manufacturing each semi-finished part are given as T2TpFNs in Table 2 and Table 3, respectively. The two-phase defuzzification process (discussed in Section 4) is used to achieve a crisp value of each of these imprecise T2TpFNs. The crisp values corresponding to Stage 1 and Stage 2 of the defuzzification process are summarized in Table 4. Table 5 provides the best and worst values of both the objective functions, achieved by solving each of them individually under a given set of constraints.

Table 1. Main attributes of the problem

Number of third party manufacturing units (i)	*4*
Number of tasks (j)	*4*

Table 2. Imprecise manufacturing costs as T2TpFNs

Z_1	Task 1	Task 2	Task 3	Task 4
Manufacturing unit 1	[(38,40,42,46); (35,40,42,48); (32,40,42,48); (31,40,42,55)]	[(43,45,46,49); (41,45,46,54); (38,45,46,56); (36,45,46,59)]	[(51,53,55,58); (49,53,55,60); (46,53,55,64); (44,53,55,67)]	[(65,67,69,72); (62,67,69,74); (60,67,69,78); (59,67,69,80)]
Manufacturing unit 2	[(35,37,39,43); (32,37,39,45);	[(69,71,73,76); (67,71,73,80);	[(66,68,70,74); (62,68,70,77);	[(77,79,82,86); (74,79,82,89);

	(29,37,39,49); (28,37,39,54)]	(65,71,73,83); (62,71,73,85)]	(60,68,70,81); (57,68,70,85)]	(72,79,82,94); (68,79,82,97)]
Manufacturing unit 3	[(89,91,94,98); (87,91,94,102); (85,91,94,106); (83,91,94,109)]	[(83,85,86,88); (82,85,86,91); (80,85,86,94); (77,85,86,98)]	[(96,98,100,104); (94,98,100,107); (91,98,100,110); (88,98,100,114)]	[(61,63,64,67); (58,63,64,71); (56,63,64,75); (53,63,64,79)]
Manufacturing unit 4	[(58,60,63,67); (56,60,63,71); (53,60,63,74); (51,60,63,78)]	[(35,38,40,43); (33,38,40,44); (32,38,40,45); (30,38,40,49)]	[(56,58,60,64); (54,58,60,68); (51,58,60,70); (49,58,60,74)]	[(73,75,77,81); (70,75,77,84); (68,75,77,87); (65,75,77,89)]

Table 3. Imprecise manufacturing times asT2TpFNs

Z_2	Task 1	Task 2	Task 3	Task 4
Manufacturing unit 1	[(218,220,222,225); (216,220,222,227); (213,220,222,231); (210,220,222,234)]	[(242,245,246,249); (240,245,246,252); (237,245,246,255); (234,245,246,259)]	[(211,209,215,218); (209,213,215,220); (206,213,215,224); (203,213,215,227)]	[(225,227,229,233); (224,227,229,235); (221,227,229,239); (217,227,229,244)]
Manufacturing unit 2	[(262,264,266,270); (260,264,266,273); (257,264,266,275); (254,264,266,276)]	[(250,252,254,257); (248,252,254,260); (245,252,254,264); (241,252,254,267)]	[(231,233,234,237); (228,233,234,240); 226,233,234,244); 223,233,234,247)]	[(255,257,259,262); (252,257,259,264); (249,257,259,267); (247,257,259,270)]
Manufacturing unit 3	[(278,280,281,284); (275,280,281,286); (273,280,281,289); (270,280,281,293)]	[(283,285,287,290); (280,285,287,292); (277,285,287,294); (274,285,287,298)]	[(295,297,299,303); (292,297,299,306); (290,297,299,309); (287,297,299,314)]	[(288,290,292,295); (285,290,292,298); (283,290,292,301); (280,290,292,303)]
Manufacturing unit 4	[(242,244,246,249); (240,244,246,253); (238,244,246,257); (236,244,246,261)]	[(285,287,289,292); (283,287,289,295); (281,287,289,297); (279,287,289,303)]	[(257,259,261,265); (255,259,261,268); (253,259,261,270); (251,259,261,274)]	[(273,275,277,282); (271,275,277,285); (269,275,277,288); (267,275,277,303)]

Table 4. Crisp values of the manufacturing costs and times obtained by the two-stage defuzzification process

Z_1 (Cost)			Z_2 (Time)		
c_{ij}	$V(\tilde{\tilde{c}}_{ij})$	$DV(\tilde{\tilde{c}}_{ij})$	t_{ij}	$V(\tilde{\tilde{t}}_{ij})$	$DV(\tilde{\tilde{t}}_{ij})$
c_{11}	(41.60,41.31,41.63,42.28)	41.73	t_{11}	(221.29,221.30,221.63,221.64)	221.46
c_{12}	(45.80,46.78,46.47,46.47)	46.35	t_{12}	(245.50,245.80,245.82,246.15)	245.82
c_{13}	(54.29,54.30,54.63,54.96)	54.55	t_{13}	(214.29,214.30,214.63,214.64)	214.46
c_{14}	(68.29,68.00,68.63,68.95)	68.46	t_{14}	(228.60,228.92,229.26,229.60)	229.09
c_{21}	(38.60,38.11,38.63,39.92)	38.87	t_{21}	(265.60,265.93,265.63,265.00)	265.51
c_{22}	(72.29,72.93,73.26,72.96)	72.83	t_{22}	(253.29,253.61,253.95,253.64)	253.64
c_{23}	(69.60,69.31,69.95,70.28)	69.78	t_{23}	(233.80,233.82,233.47,234.48)	235.95
c_{24}	(81.08,81.11,82.06,81.77)	81.50	t_{24}	(258.29,258.00,258.01,258.32)	258.20

c_{31}	(93.08,93.60,94.37,94.71)	93.93	t_{31}	(280.80,280.50,280.82,281.15)	280.81
c_{32}	(86.31,86.56,85.79,85.81)	86.12	t_{32}	(286.29,286.00,285.68,286.09)	286.31
c_{33}	(99.60,99.43,99.95,100.28)	99.80	t_{32}	(298.60,298.00,298.62,299.60)	298.94
c_{34}	(63.80,64.14,64.80,65.32)	64.52	t_{34}	(291.29,291.31,291.63,291.32)	291.41
c_{41}	(62.08,62.72,62.75,63.40)	62.73	t_{41}	(244.07,244.91,243.31,243.45)	243.96
c_{42}	(39.00,38.88,38.68,39.20)	38.79	t_{42}	(287.62,287.59,287.09,289.01)	288.11
c_{43}	(59.60,60.25,59.63,59.92)	59.94	t_{43}	(259.97,260.32,260.74,261.01)	260.50
c_{44}	(76.60,76.62,76.95,76.64)	76.72	t_{44}	(275.98,276.08,276.58,276.96)	276.40

The T2TpFN defuzzification process is divided into two stages. In stage I, the defuzzification technique transforms T2TpFN to T1TpFN, and in stage II, the T1TpFNs were again used to obtain the defuzzified value of T2TpFN.

Now, using the above available data in Table 2 and 3, the MOAP (Model 2) with Type 2 fuzzy parameters can be described as follows:

Stage I.

$Min\ Z_1 =$

$$\begin{pmatrix} \frac{1}{3}\left(38+40+42+46+\frac{38\times40-42\times46}{38+40-42-46}\right), & \frac{1}{3}\left(35+40+42+48+\frac{35\times40-42\times48}{35+40-42-48}\right) \\ \frac{1}{3}\left(32+40+42+46+\frac{32\times40-42\times46}{32+40-42-46}\right), & \frac{1}{3}\left(31+40+42+55+\frac{31\times40-42\times55}{31+40-42-55}\right) \end{pmatrix} x_{11}$$

$$+\begin{pmatrix} \frac{1}{3}\left(43+45+46+49+\frac{43\times45-46\times49}{43+45-46-49}\right), & \frac{1}{3}\left(41+45+46+54+\frac{41\times45-46\times54}{41+45-46-54}\right) \\ \frac{1}{3}\left(38+45+46+56+\frac{38\times45-46\times56}{38+45-46-56}\right), & \frac{1}{3}\left(36+45+46+59+\frac{36\times45-46\times59}{36+45-46-59}\right) \end{pmatrix} x_{12}$$

$$+\begin{pmatrix} \frac{1}{3}\left(51+53+55+58+\frac{51\times53-55\times58}{51+53-55-58}\right), & \frac{1}{3}\left(49+53+55+60+\frac{49\times53-55\times60}{49+53-55-60}\right) \\ \frac{1}{3}\left(46+53+55+64+\frac{46\times53-55\times64}{46+53-55-64}\right), & \frac{1}{3}\left(44+53+55+67+\frac{44\times53-55\times67}{44+53-55-67}\right) \end{pmatrix} x_{13}$$

$$+\begin{pmatrix} \frac{1}{3}\left(65+67+69+72+\frac{65\times67-69\times72}{65+67-69-72}\right), & \frac{1}{3}\left(62+67+69+74+\frac{62\times67-69\times74}{62+67-69-74}\right) \\ \frac{1}{3}\left(60+67+69+78+\frac{60\times67-69\times78}{60+67-69-78}\right), & \frac{1}{3}\left(59+67+69+80+\frac{59\times67-69\times80}{59+67-69-80}\right) \end{pmatrix} x_{14}$$

$$+\begin{pmatrix} \frac{1}{3}\left(35+37+39+43+\frac{35\times37-39\times43}{35+37-39-43}\right), & \frac{1}{3}\left(32+37+39+45+\frac{32\times37-39\times45}{32+37-39-45}\right) \\ \frac{1}{3}\left(29+37+39+49+\frac{29\times37-39\times49}{29+37-39-49}\right), & \frac{1}{3}\left(28+37+39+49+\frac{28\times37-39\times54}{28+37-39-54}\right) \end{pmatrix} x_{21}$$

$$+\begin{pmatrix} \frac{1}{3}\left(69+71+73+76+\frac{69\times71-73\times76}{69+71-73-76}\right), & \frac{1}{3}\left(67+71+73+80+\frac{67\times71-73\times80}{67+71-73-80}\right) \\ \frac{1}{3}\left(65+71+73+83+\frac{65\times71-73\times83}{65+71-73-83}\right), & \frac{1}{3}\left(62+71+73+85+\frac{62\times71-73\times85}{62+71-73-85}\right) \end{pmatrix} x_{22}$$

$$+\begin{pmatrix} \frac{1}{3}\left(66+68+70+74+\frac{66\times68-70\times74}{66+68-70-74}\right), & \frac{1}{3}\left(62+68+70+77+\frac{62\times68-70\times77}{62+68-70-77}\right) \\ \frac{1}{3}\left(60+68+70+81+\frac{60\times68-70\times81}{60+68-70-81}\right), & \frac{1}{3}\left(57+68+70+85+\frac{57\times68-70\times85}{57+68-70-85}\right) \end{pmatrix} x_{23}$$

$$+\begin{pmatrix} \frac{1}{3}\left(77+79+82+86+\frac{77\times79-82\times86}{77+79-82-86}\right), & \frac{1}{3}\left(74+79+82+89+\frac{74\times79-82\times89}{74+79-82-89}\right) \\ \frac{1}{3}\left(72+79+82+94+\frac{72\times79-82\times94}{72+79-82-94}\right), & \frac{1}{3}\left(68+79+82+97+\frac{68\times79-82\times97}{68+79-82-97}\right) \end{pmatrix} x_{24}$$

$$+\begin{pmatrix} \frac{1}{3}\left(89+91+94+98+\frac{89\times91-94\times98}{89+91-94-98}\right), & \frac{1}{3}\left(87+91+94+102+\frac{87\times91-94\times102}{87+91-94-102}\right) \\ \frac{1}{3}\left(85+91+94+106+\frac{85\times91-94\times106}{85+91-94-106}\right), & \frac{1}{3}\left(83+91+94+109+\frac{83\times91-94\times109}{83+91-94-109}\right) \end{pmatrix} x_{31}$$

$$+\begin{pmatrix} \frac{1}{3}\left(83+85+86+88+\frac{83\times85-86\times88}{83+85-86-88}\right), & \frac{1}{3}\left(82+85+86+91+\frac{82\times85-86\times91}{82+85-86-91}\right) \\ \frac{1}{3}\left(80+85+86+94+\frac{80\times85-86\times94}{80+85-86-94}\right), & \frac{1}{3}\left(77+85+86+98+\frac{77\times85-86\times98}{77+85-86-98}\right) \end{pmatrix} x_{32}$$

$$+\left(\begin{array}{l}\frac{1}{3}\left(96+98+100+104+\frac{96\times 98-100\times 104}{96+98-100-104}\right),\ \frac{1}{3}\left(94+98+100+107+\frac{94\times 98-100\times 107}{94+98-100-107}\right)\\ \frac{1}{3}\left(91+98+100+110+\frac{91\times 98-100\times 110}{91+98-100-110}\right),\ \frac{1}{3}\left(88+98+100+114+\frac{88\times 98-100\times 114}{88+98-100-114}\right)\end{array}\right)x_{33}$$

$$+\left(\begin{array}{l}\frac{1}{3}\left(61+63+64+67+\frac{61\times 63-64\times 67}{61+63-64-67}\right),\ \frac{1}{3}\left(58+63+64+71+\frac{58\times 63-64\times 71}{58+63-64-71}\right)\\ \frac{1}{3}\left(56+63+64+75+\frac{56\times 63-64\times 75}{56+63-64-75}\right),\ \frac{1}{3}\left(53+63+64+79+\frac{53\times 63-64\times 79}{53+63-64-79}\right)\end{array}\right)x_{34}$$

$$+\left(\begin{array}{l}\frac{1}{3}\left(58+60+63+67+\frac{58\times 60-63\times 67}{58+60-63-67}\right),\ \frac{1}{3}\left(56+60+63+71+\frac{56\times 60-63\times 71}{56+60-63-71}\right)\\ \frac{1}{3}\left(53+60+63+74+\frac{53\times 60-63\times 74}{51+60-63-74}\right),\ \frac{1}{3}\left(51+60+63+78+\frac{51\times 60-63\times 78}{51+60-63-78}\right)\end{array}\right)x_{41}$$

$$+\left(\begin{array}{l}\frac{1}{3}\left(35+38+40+43+\frac{35\times 38-40\times 43}{35+38-40-43}\right),\ \frac{1}{3}\left(33+38+40+44+\frac{33\times 38-40\times 44}{33+38-40-44}\right)\\ \frac{1}{3}\left(32+38+40+45+\frac{32\times 38-40\times 45}{32+38-40-45}\right),\ \frac{1}{3}\left(30+38+40+49+\frac{30\times 38-40\times 49}{30+38-40-49}\right)\end{array}\right)x_{42}$$

$$+\left(\begin{array}{l}\frac{1}{3}\left(56+58+60+64+\frac{56\times 58-60\times 64}{56+58-60-64}\right),\ \frac{1}{3}\left(54+58+60+68+\frac{54\times 58-60\times 68}{54+58-60-68}\right)\\ \frac{1}{3}\left(51+58+60+70+\frac{51\times 58-60\times 70}{51+58-60-70}\right),\ \frac{1}{3}\left(49+58+60+74+\frac{49\times 58-60\times 74}{49+58-60-74}\right)\end{array}\right)x_{43}$$

$$+\left(\begin{array}{l}\frac{1}{3}\left(73+75+77+81+\frac{73\times 75-77\times 81}{73+75-77-81}\right),\ \frac{1}{3}\left(70+75+77+84+\frac{70\times 75-77\times 84}{70+75-77-84}\right)\\ \frac{1}{3}\left(68+75+77+87+\frac{68\times 75-77\times 87}{68+75-77-87}\right),\ \frac{1}{3}\left(65+75+77+89+\frac{65\times 75-77\times 89}{65+75-77-89}\right)\end{array}\right)x_{44}$$

$Min\ Z_2 =$

$$\left(\begin{array}{l}\frac{1}{3}\left(218+220+222+225+\frac{218\times 220-222\times 225}{218+220-222-225}\right),\ \frac{1}{3}\left(216+220+222+227+\frac{216\times 220-222\times 227}{216+220-222-227}\right)\\ \frac{1}{3}\left(213+220+222+231+\frac{213\times 220-222\times 231}{213+220-222-231}\right),\ \frac{1}{3}\left(210+220+222+234+\frac{210\times 220-222\times 234}{210+220-222-234}\right)\end{array}\right)x_{11}$$

$$+\left(\begin{array}{l}\frac{1}{3}\left(242+245+246+249+\frac{242\times 245-246\times 249}{242+245-246-249}\right),\ \frac{1}{3}\left(240+245+246+252+\frac{240\times 245-246\times 252}{240+245-246-252}\right)\\ \frac{1}{3}\left(237+245+246+255+\frac{237\times 245-246\times 255}{237+245-246-255}\right),\ \frac{1}{3}\left(234+245+246+259+\frac{234\times 245-246\times 259}{234+245-246-259}\right)\end{array}\right)x_{12}$$

$$+\left(\begin{array}{l}\frac{1}{3}\left(211+209+215+218+\frac{211\times 209-215\times 218}{211+209-215-218}\right),\ \frac{1}{3}\left(209+213+215+220+\frac{209\times 213-215\times 220}{209+213-215-220}\right)\\ \frac{1}{3}\left(206+213+215+224+\frac{206\times 213-215\times 224}{206+213-215-224}\right),\ \frac{1}{3}\left(203+213+215+227+\frac{203\times 213-215\times 227}{203+213-215-227}\right)\end{array}\right)x_{13}$$

$$+\left(\begin{array}{l}\frac{1}{3}\left(225+227+229+233+\frac{225\times 227-229\times 233}{225+227-229-233}\right),\ \frac{1}{3}\left(224+227+229+235+\frac{224\times 227-229\times 235}{224+227-229-235}\right)\\ \frac{1}{3}\left(221+227+229+239+\frac{221\times 227-229\times 239}{221+227-229-239}\right),\ \frac{1}{3}\left(217+227+229+244+\frac{217\times 227-229\times 244}{217+227-229-244}\right)\end{array}\right)x_{14}$$

$$+\begin{pmatrix} \frac{1}{3}\left(262+264+266+270+\frac{262\times 264-266\times 270}{262+264-266-270}\right), & \frac{1}{3}\left(260+264+266+273+\frac{260\times 264-266\times 273}{260+264-266-273}\right) \\ \frac{1}{3}\left(257+264+266+275+\frac{257\times 264-266\times 275}{257+264-266-275}\right), & \frac{1}{3}\left(254+264+266+276+\frac{254\times 264-266\times 276}{254+264-266-276}\right) \end{pmatrix} x_{21}$$

$$+\begin{pmatrix} \frac{1}{3}\left(250+252+254+257+\frac{250\times 252-254\times 257}{250+252-254-257}\right), & \frac{1}{3}\left(248+252+254+260+\frac{248\times 252-254\times 260}{248+252-254-260}\right) \\ \frac{1}{3}\left(245+252+254+264+\frac{245\times 252-254\times 264}{245+252-254-264}\right), & \frac{1}{3}\left(241+252+254+267+\frac{241\times 252-254\times 267}{241+252-254-267}\right) \end{pmatrix} x_{22}$$

$$+\begin{pmatrix} \frac{1}{3}\left(231+233+234+237+\frac{231\times 233-234\times 237}{231+233-234-237}\right), & \frac{1}{3}\left(228+233+234+240+\frac{228\times 233-234\times 240}{228+233-234-240}\right) \\ \frac{1}{3}\left(226+233+234+244+\frac{226\times 233-234\times 244}{226+233-234-244}\right), & \frac{1}{3}\left(223+233+234+247+\frac{223\times 233-234\times 247}{223+233-234-247}\right) \end{pmatrix} x_{23}$$

$$+\begin{pmatrix} \frac{1}{3}\left(255+257+259+262+\frac{255\times 257-259\times 262}{255+257-259-262}\right), & \frac{1}{3}\left(252+257+259+264+\frac{252\times 257-259\times 264}{252+257-259-264}\right) \\ \frac{1}{3}\left(249+257+259+267+\frac{249\times 257-259\times 267}{249+257-259-267}\right), & \frac{1}{3}\left(247+257+259+270+\frac{247\times 257-259\times 270}{247+257-259-270}\right) \end{pmatrix} x_{24}$$

$$+\begin{pmatrix} \frac{1}{3}\left(278+280+281+284+\frac{278\times 280-281\times 284}{278+280-281-284}\right), & \frac{1}{3}\left(275+280+281+286+\frac{275\times 280-281\times 286}{275+280-281-286}\right) \\ \frac{1}{3}\left(273+280+281+289+\frac{273\times 280-281\times 289}{273+280-281-289}\right), & \frac{1}{3}\left(270+280+281+293+\frac{270\times 280-281\times 293}{270+280-281-293}\right) \end{pmatrix} x_{31}$$

$$+\begin{pmatrix} \frac{1}{3}\left(283+285+287+290+\frac{283\times 285-287\times 290}{283+285-287-290}\right), & \frac{1}{3}\left(280+285+287+292+\frac{280\times 285-287\times 292}{280+285-287-292}\right) \\ \frac{1}{3}\left(277+285+287+294+\frac{277\times 285-287\times 294}{277+285-287-294}\right), & \frac{1}{3}\left(274+285+287+298+\frac{274\times 285-287\times 298}{274+285-287-298}\right) \end{pmatrix} x_{32}$$

$$+\begin{pmatrix} \frac{1}{3}\left(295+297+299+303+\frac{295\times 297-299\times 303}{295+297-299-303}\right), & \frac{1}{3}\left(292+297+299+306+\frac{292\times 297-299\times 306}{292+297-299-306}\right) \\ \frac{1}{3}\left(290+297+299+309+\frac{290\times 297-299\times 309}{295+297-299-309}\right), & \frac{1}{3}\left(287+297+299+314+\frac{287\times 297-299\times 314}{287+297-299-314}\right) \end{pmatrix} x_{33}$$

$$+\begin{pmatrix} \frac{1}{3}\left(288+290+292+295+\frac{288\times 290-292\times 295}{288+290-292-295}\right), & \frac{1}{3}\left(285+290+292+298+\frac{285\times 290-292\times 298}{285+290-292-298}\right) \\ \frac{1}{3}\left(283+290+292+301+\frac{283\times 290-292\times 301}{283+290-292-301}\right), & \frac{1}{3}\left(280+290+292+303+\frac{280\times 290-292\times 303}{280+290-292-303}\right) \end{pmatrix} x_{34}$$

$$+\begin{pmatrix} \frac{1}{3}\left(242+244+246+249+\frac{242\times 244-246\times 249}{242+244-246-249}\right), & \frac{1}{3}\left(240+244+246+253+\frac{240\times 244-246\times 253}{240+244-246-253}\right) \\ \frac{1}{3}\left(238+244+246+257+\frac{238\times 244-246\times 257}{238+244-246-257}\right), & \frac{1}{3}\left(236+244+246+261+\frac{236\times 244-246\times 261}{236+244-246-261}\right) \end{pmatrix} x_{41}$$

$$+\begin{pmatrix} \frac{1}{3}\left(285+287+289+292+\frac{285\times 287-289\times 292}{285+287-289-292}\right), & \frac{1}{3}\left(283+287+289+295+\frac{283\times 287-289\times 295}{283+287-289-295}\right) \\ \frac{1}{3}\left(281+287+289+297+\frac{281\times 287-289\times 297}{281+287-289-297}\right), & \frac{1}{3}\left(279+287+289+303+\frac{279\times 287-289\times 303}{279+287-289-303}\right) \end{pmatrix} x_{42}$$

$$+\begin{pmatrix} \frac{1}{3}\left(257+259+261+265+\frac{257\times 259-261\times 265}{257+259-261-265}\right), & \frac{1}{3}\left(255+259+261+268+\frac{255\times 259-261\times 268}{255+259-261-268}\right) \\ \frac{1}{3}\left(253+259+261+270+\frac{253\times 259-261\times 270}{253+259-261-270}\right), & \frac{1}{3}\left(251+259+261+274+\frac{251\times 259-261\times 274}{251+259-261-274}\right) \end{pmatrix} x_{43}$$

$$+\begin{pmatrix} \frac{1}{3}\left(273+275+277+282+\frac{273\times275-277\times282}{273+275-277-282}\right), \frac{1}{3}\left(271+275+277+288+\frac{271\times275-277\times285}{271+275-277-285}\right) \\ \frac{1}{3}\left(269+275+277+288+\frac{269\times275-277\times288}{269+275-277-288}\right), \frac{1}{3}\left(267+275+277+303+\frac{267\times275-277\times303}{267+275-277-303}\right) \end{pmatrix} x_{44}$$

Subjec to;

$$x_{11}+x_{12}+x_{13}+x_{14}=1$$
$$x_{21}+x_{22}+x_{23}+x_{24}=1$$
$$x_{31}+x_{32}+x_{33}+x_{34}=1$$
$$x_{41}+x_{42}+x_{43}+x_{44}=1$$
$$x_{11}+x_{21}+x_{31}+x_{41}=1$$
$$x_{12}+x_{222}+x_{32}+x_{42}=1$$
$$x_{13}+x_{23}+x_{33}+x_{43}=1$$
$$x_{14}+x_{24}+x_{34}+x_{44}=1$$

Stage II.

$Min\ Z_1 =$

$$\frac{1}{3}\left(41.60+41.31+41.63+42.28+\frac{41.60\times41.31-41.63\times42.28}{41.60+41.31-41.63-42.28}\right)x_{11}$$
$$+\frac{1}{3}\left(45.80+46.78+46.47+46.49+\frac{45.80\times46.78-46.47\times46.49}{45.80+46.78-46.47-46.49}\right)x_{12}$$
$$+\frac{1}{3}\left(54.29+54.30+54.63+54.96+\frac{54.29\times54.30-54.63\times54.96}{54.29+54.30-54.63-54.96}\right)x_{13}$$
$$+\frac{1}{3}\left(68.29+68.00+68.63+68.95+\frac{68.29\times68.00-68.63\times68.95}{68.29+68.00-68.63-68.95}\right)x_{14}$$
$$+\frac{1}{3}\left(38.60+38.11+38.63+39.92+\frac{38.60\times38.11-38.63\times39.92}{38.60+38.11-38.63-39.92}\right)x_{21}$$
$$+\frac{1}{3}\left(72.29+72.93+73.26+72.96+\frac{72.29\times72.93-73.26\times72.96}{72.29+72.93-73.26-72.96}\right)x_{22}$$
$$+\frac{1}{3}\left(69.60+69.31+69.95+70.28+\frac{69.60\times69.31-69.95\times70.28}{69.60+69.31-69.95-70.28}\right)x_{23}$$
$$+\frac{1}{3}\left(81.08+81.11+82.06+81.77+\frac{81.08\times81.11-82.06\times81.77}{81.08+81.11-82.06-81.77}\right)x_{24}$$
$$+\frac{1}{3}\left(93.08+93.60+94.37+94.71+\frac{893.08\times93.60-94.37\times94.71}{93.08+93.60-94.37-94.71}\right)x_{31}$$
$$+\frac{1}{3}\left(86.31+86.56+85.79+85.81+\frac{86.31\times86.56-85.79\times85.81}{86.31+86.56-85.79-85.81}\right)x_{32}$$
$$+\frac{1}{3}\left(99.60+99.43+99.95+100.28+\frac{99.60\times99.43-99.95\times100.28}{99.60+99.43-99.95-100.28}\right)x_{33}$$
$$+\frac{1}{3}\left(63.80+64.14+64.80+65.32+\frac{63.80\times64.14+64.80\times65.32}{63.80+64.14-64.80-65.32}\right)x_{34}$$
$$+\frac{1}{3}\left(62.08+62.72+62.75+63.40+\frac{62.08\times62.72-62.75\times63.40}{62.08+62.72-62.75-63.40}\right)x_{41}$$
$$+\frac{1}{3}\left(39.00+38.88+38.68+39.20+\frac{39.00\times38.88-38.68\times39.20}{39.00+38.88-38.68-39.20}\right)x_{42}$$
$$+\frac{1}{3}\left(59.60+60.28+59.63+59.92+\frac{59.60\times60.28-59.63\times59.92}{59.60+60.28-59.63-59.92}\right)x_{43}$$

$$+\frac{1}{3}\left(76.60+76.62+76.95+76.64+\frac{76.60\times 76.62-76.95\times 76.64}{76.60+76.62-76.95-76.64}\right)x_{44}$$

$Min\ Z_2 =$

$$\frac{1}{3}\left(221.29+221.30+221.63+221.64+\frac{221.29\times 221.30-221.63\times 221.64}{221.29+221.30-221.63-221.64}\right)x_{11}$$

$$+\frac{1}{3}\left(245.50+245.80+245.82+246.15+\frac{245.50\times 245.80-245.82\times 246.15}{245.50+245.80-245.82-246.15}\right)x_{12}$$

$$+\frac{1}{3}\left(214.29+214.30+214.63+214.64+\frac{214.29\times 214.30-214.63\times 214.64}{214.29+214.30-214.63214.64}\right)x_{13}$$

$$+\frac{1}{3}\left(228.60+228.92+229.26+229.60+\frac{228.60\times 228.92-229.26\times 229.60}{228.60+228.92-229.26-229.60}\right)x_{14}$$

$$+\frac{1}{3}\left(265.60+265.93+265.63+265.00+\frac{265.60\times 265.93-265.63\times 265.00}{265.60+265.93-265.63-265.00}\right)x_{21}$$

$$+\frac{1}{3}\left(253.29+253.61+253.95+253.64+\frac{253.29\times 253.61-253.95\times 253.64}{253.29+253.61-253.95-253.64}\right)x_{22}$$

$$+\frac{1}{3}\left(233.80+233.82+233.47+234.48+\frac{233.80\times 233.82-233.47\times 234.48}{233.80+233.82-233.47-234.48}\right)x_{23}$$

$$+\frac{1}{3}\left(258.29+258.00+258.01+258.32+\frac{258.29\times 258.00-258.01\times 258.32}{258.29+258.00-258.01-258.32}\right)x_{24}$$

$$+\frac{1}{3}\left(280.80+280.50+280.82+281.15+\frac{280.80\times 280.50-280.82\times 281.15}{280.80+280.50-280.82-281.15}\right)x_{31}$$

$$+\frac{1}{3}\left(286.29+286.00+285.68+286.09+\frac{286.29\times 286.00-285.68\times 286.09}{286.29+286.00-285.68-286.09}\right)x_{32}$$

$$+\frac{1}{3}\left(298.60+298.00+298.62+299.60+\frac{298.60\times 298.00-298.62\times 299.60}{298.60+298.00-298.62-299.60}\right)x_{33}$$

$$+\frac{1}{3}\left(291.29+291.31+291.63+291.60+\frac{291.29\times 291.31-291.63\times 291.60}{291.29+291.31-291.63-291.60}\right)x_{34}$$

$$+\frac{1}{3}\left(244.07+244.91+243.31+243.45+\frac{244.07\times 244.91-243.31\times 243.45}{244.07+244.91-243.31-243.45}\right)x_{41}$$

$$+\frac{1}{3}\left(287.62+287.59+287.09+289.01+\frac{287.62\times 287.59-287.09\times 289.01}{287.62+287.59-287.09-289.01}\right)x_{42}$$

$$+\frac{1}{3}\left(259.97+260.32+260.74+260.01+\frac{259.97\times 260.32+260.74\times 260.01}{259.97+260.32-260.74-260.01}\right)x_{43}$$

$$+\frac{1}{3}\left(275.98+276.08+276.58+276.96+\frac{275.98\times 276.08-276.58\times 276.96}{275.98+276.08-276.58-276.96}\right)x_{44}$$

Subject to;

$$x_{11}+x_{12}+x_{13}+x_{14}=1$$
$$x_{21}+x_{22}+x_{23}+x_{24}=1$$
$$x_{31}+x_{32}+x_{33}+x_{34}=1$$
$$x_{41}+x_{42}+x_{43}+x_{44}=1$$
$$x_{11}+x_{21}+x_{31}+x_{41}=1$$
$$x_{12}+x_{222}+x_{32}+x_{42}=1$$
$$x_{13}+x_{23}+x_{33}+x_{43}=1$$
$$x_{14}+x_{24}+x_{34}+x_{44}=1$$

All parameters are in T2TpFNs and are translated to a crisp value using the procedure described above. The crisp value is presented in Table 4 for each objective function, repetitively. After using the crisp value the equivalent crisp MOAP can be defined as follows:

Using these crisp values of the manufacturing costs and manufacturing times which are obtained by using the two-stage defuzzification process, the present MOAP can be expressed as Model 7:
Model 7

$$Min\,Z_1 = 41.73x_{11} + 46.35x_{12} + 54.55x_{13} + 68.46x_{14} + 38.87x_{21} + 72.83x_{22} + 69.78x_{23} + 81.50x_{24} + 93.93x_{31} + 86.12x_{32} + 99.82x_{33} + 64.52x_{34} + 62.73x_{41} + 38.79x_{42} + 59.94x_{43} + 76.72x_{44}$$

$$Min\,Z_2 = 221.46x_{11} + 245.82x_{12} + 214.46x_{13} + 229.09x_{14} + 265.51x_{21} + 253.64x_{22} + 235.95x_{23} + 258.20x_{24} + 280.81x_{31} + 286.31x_{32} + 298.94x_{33} + 291.41x_{34} + 243.96x_{41} + 288.11x_{42} + 260.51x_{43} + 276.40x_{44}$$

$Subject\ \ to;$

$$x_{11} + x_{12} + x_{13} + x_{14} = 1; x_{21} + x_{22} + x_{23} + x_{24} = 1; x_{31} + x_{32} + x_{33} + x_{34} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1; x_{11} + x_{21} + x_{31} + x_{41} = 1; x_{12} + x_{222} + x_{32} + x_{42} = 1$$
$$x_{13} + x_{23} + x_{33} + x_{43} = 1; x_{14} + x_{24} + x_{34} + x_{44} = 1$$

Now, each objective function is minimized subject to the given set of constraints by ignoring the other objective function. This provides the minimum value of each objective function and the corresponding value (written as Max) of the other one. These values are depicted in Table 5, which is called the Payoff matrix.

Table 5. Payoff matrix

	Z_1 (Total manufacturing cost)	Z_2 (Total manufacturing time)
Max	287.07	1059.49
Min	196	995.31

Thus, the following inequalities hold for each objective function

$$196 \le Z_1 \le 287.07, \qquad 995.31 \le Z_2 \le 1059.49$$

8. Results and Discussion

The above MOAP is solved using three solution techniques viz., NCPT, FPT and IFPT. The best compromise solution obtained by each of these methods is given in Table 6.

1. While solving Model 7 using NCPT, we find each objective function's upper and lower bounds by solving them separately, subject to the given constraints. Then, we designed the linear membership functions for truth, indeterminacy and falsehood, respectively and maximized the truth and indeterminacy value and minimized the false value. Using Model 4 and LINGO 16.0 optimization software, we obtained the optimal solution of Model 7 as

$$x_{11} = 0, x_{12} = 0,\ x_{13} = 1, x_{14} = 0\ \ x_{21} = 0,\ x_{22} = 1, x_{23} = 0, x_{24} = 0\ \ x_{31} = 0,\ x_{32} = 0,\ x_{33} = 0, x_{34} = 1,$$
$$x_{41} = 0, x_{42} = 0, x_{43} = 0, x_{44} = 1,\ \gamma = 0.98546,\ Z_1 = 227.04,\quad Z_2 = 1003.05.$$

2. While solving Model 7 using FPT, we designed the linear membership functions of both the objectives and maximized them. Using Model 5 and LINGO 16.0 optimization software, we obtained the optimal solutions of Model 7 as

$$x_{11}=1,\ x_{12}=0,\ x_{13}=0, x_{14}=0\ x_{21}=0,\ x_{22}=1, x_{23}=0, x_{24}=0\ x_{31}=0,\ x_{32}=0,\ x_{33}=0, x_{34}=1,$$
$$x_{41}=0, x_{42}=0, x_{43}=1, x_{44}=0.\ \gamma=0.5, \quad Z_1=239.02, \quad Z_2=1027.02$$

3. While solving Model 7 using IFPT, we first designed the linear membership and non-membership functions and then maximized the membership function and minimized the non-membership function. Using Model 6 and LINGO 16.0 optimization software, we obtained the optimal solution of Model 7 as

$$x_{11}=1,\ x_{12}=0,\ x_{13}=0, x_{14}=0\ x_{21}=0,\ x_{22}=1, x_{23}=0, x_{24}=0\ x_{31}=0,\ x_{32}=0,\ x_{33}=0, x_{34}=1,$$
$$x_{41}=0, x_{42}=0, x_{43}=1, x_{44}=0.\ \gamma=0.5, \quad Z_1=239.02, \quad Z_2=1027.02$$

From Table 6, we can easily conclude that the optimal solution of the present MOAP derived from the technique NCPT is more desirable and therefore, NCPT is a more suitable technique than the FPT and IFPT. This is due to the same reason that the fuzzy and the intuitionistic fuzzy logics are based on the truth function only, however, in real-world decision-making problems, the decision may result in the form of agreement, disagreement or the state of being unsure. Since the concept of neutrosophy allows the decision-makers to consider all these aspects together, NCPT performed better than the other techniques for the present MOAPs. Thus, the main advantage of the present study on MOAPs over existing literature is to solve the problem by considering degrees of truthness, falsehood, and indeterminacy altogether which may help the decision-maker make a better and more realistic decision. From Table 6, it is concluded that the best compromise solution of the present MOAP given by NCPT, provides the total manufacturing cost as 224.04 $ and the total manufacturing time of all the semi-finished parts as 1003.05 mins. To be more precise, a graphical representation of the compromise optimal solutions of the present MOAP, extracted from different solution approaches is given in Figure 2.

Table 6. Optimal solutions obtained by NCPT, FPT and IFPT

Objective functions / **Decision variables**	**NCPT**	**FPT**	**IFPT**
Min Z_1	227.04	239.02	239.02
Min Z_2	1003.05	1027.02	1027.02
x_{11}	0	1	1
x_{12}	0	0	0
x_{13}	1	0	0
x_{14}	0	0	0
x_{21}	0	0	0
x_{22}	1	1	1
x_{23}	0	0	0
x_{24}	0	0	0

x_{31}	0	0	0
x_{32}	0	0	0
x_{33}	0	0	0
x_{34}	1	1	1
x_{41}	0	0	0
x_{42}	0	0	0
x_{43}	0	1	1
x_{44}	1	0	0

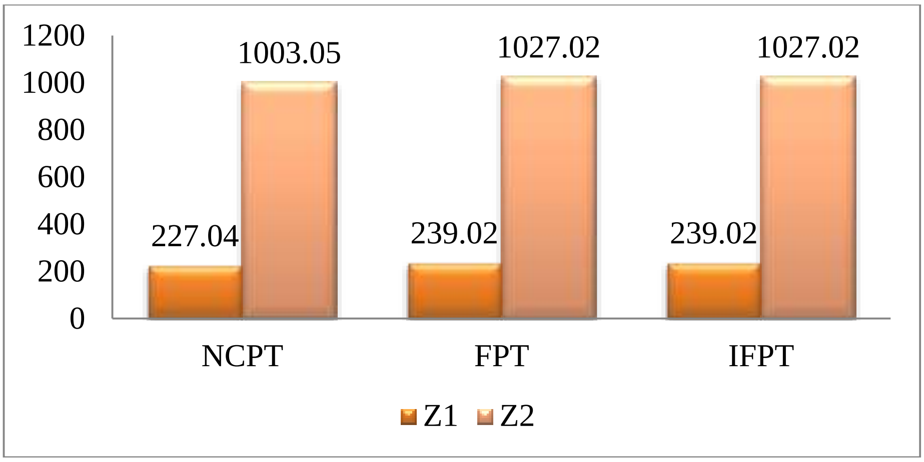

Figure 2. Comparison of objective values obtained from NCPT, FPT & IFPT

9. Advantage and Comparison of the Proposed Work with Some Existing Ones in Literature

The present problem is a MOAP with conflicting objectives which is discussed under fuzziness. This formulation of an assignment problem caters to a more realistic scenario arising in various commercial situations with vague information.

Further, in the study of MOAP under uncertainty, most of the authors like Biswas and Pramanik [9], Huang and Zhang [25], Jose and Kuriakose [27], Lin and Wen [36], Liu and Goa [38], Majumdar and Bhunia[41] and Thorani and Shankar [68] have used the concept of Type 1 fuzzy set (TIFS) whose membership functions are expressed as absolute numbers. The T1FS, in general, cannot handle the vagueness of the parameters efficiently as its membership functions are crisp. In contrast to this, Type 2 fuzzy sets (T2FS) can model the uncertainties/vagueness of optimization problems more appropriately as its membership functions are also presented as fuzzy numbers. To be more precise, the membership functions of T1FS are two-dimensional whereas the membership functions of T2FS are three-dimensional. This additional degree of freedom makes it possible to model the vagueness/uncertainties of an optimization problem more efficiently. So, the formulation of the present problem with T2TpF parameters is another advantage of the present study.

Furthermore, De and Yadav [14], Mukherjee and Basu [44], Pramanik and Biswas [54] and Sakawa et al. [60] are some of the authors who discussed assignment problems in an uncertain environment and either used fuzzy programming techniques or used the intuitionistic fuzzy programming techniques. The disadvantage of these techniques is that they can only handle information in the context of membership and/or non-membership function of a parameter but not the information related to indeterminacy or inconsistency in the parameter values. The neutrosophic approach discussed in this paper overcomes this limitation. In its theory, indeterminacy is quantified directly while the truth, indeterminacy and falsehood membership functions are independent. Since the present MOAP under uncertainty with T2TpF parameters is discussed using neutrosophic logic, this may be considered as another advantage of the present problem over existing literature. The efficiency of this technique over the existing ones reflects in Table 6.

10. Conclusion and Future Aspects

The current paper uses neutrosophic logic to solve MOAP in an uncertain environment. T2TpFNs are used to represent all of the uncertain parameters of the MOAP. The model is then crisped using a two-stage defuzzification procedure that finds the crisp values of these T2TpFNs. This crisp model is solved by using three solution techniques viz., FPT, IFPT and NCPT. The primary goal of this work is to solve the MOAP utilising NCPT and demonstrate its superiority over the others techniques described above. A numerical demonstration is shown that clearly shows that the NCPT outperforms the other two solution strategies that are also capable of dealing with uncertainty.

The concept of neutrosophic may be included into a multiobjective transportation model in future study. A MOAP's stochastic model may also be explored and solved using NCPT. Fuzzy-random or fuzzy-stochastic variations of a multiobjective assignment or transportation issue are also possibilities. Furthermore, the NCPT may be used in a variety of domains such as management science, financial management, and decision-making science, among others.

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Some fundamental Operations for multi-Polar Interval-Valued Neutrosophic Soft Set and a Decision-Making Approach to Solve MCDM Problem

Rana Muhammad Zulqarnain[1], Aiyared Iampan[2], Imran Siddique[3*], Hamiden Abd El-Wahed Khalifa[4,5]

[1] Department of Mathematics, University of Management and Technology Lahore, Sialkot Campus, Pakistan. E-mail: ranazulqarnain7777@gmail.com

[2] Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand. E-mail: aiyared.ia@up.ac.th

[3] Department of Mathematics, University of Management and Technology Lahore, Pakistan. E-mail: imransiddique@umt.edu.pk

[4] Department of Mathematics, College of Science and Arts, Al-Badaya, Qassim University E-mail: Ha.Ahmed@qu.edu.sa

[5] Operations Research Department, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt, E-mail: hamiden@cu.edu.eg

Correspondence: E-mail: imransiddique@umt.edu.pk

Abstract:

The main purpose of this research is to propose an m-polar interval-valued neutrosophic soft set (mPIVNSSs) by merging the m-polar fuzzy set and interval-valued neutrosophic soft set. The mPIVNSSs is the most generalized form of interval-valued neutrosophic soft set. It can accommodate the truthiness, indeterminacy, and falsity in intervals form. We develop some fundamental operations for mPIVNSS such as AND Operator, OR Operator, Truth-favorite, and False-favorite Operators with their properties. The weighted aggregation operator for mPIVNSS is also established with its properties. Furthermore, the developed mPIVNSWA operator has demonstrated a novel decision-making methodology for mPIVNSS to solve the multi-criteria decision-making (MCDM) problem. Finally, the comparative analysis of the developed algorithm is given with the prevailing techniques.

Keywords: multipolar interval-valued neutrosophic set; multipolar interval-valued neutrosophic soft set; mPIWNSWA operator; MCDM.

1. Introduction

Uncertainty plays a dynamic role in many areas of life (such as modeling, medicine, engineering, etc.). However, people have raised a common problem: how do we express and use the concept of uncertainty in mathematical modeling. Many researchers plan and endorse different methods to solve the difficulties that involve hesitation. First, Zadeh proposed the idea of a Fuzzy Set (FS) [1] to solve uncertain complications. But in some cases, fuzzy sets cannot handle this situation. To overcome this situation, Turksen [2] proposed the idea of interval-valued fuzzy sets (IVFS). In some cases, we must consider the non-member value of the object, which neither FS nor IVFS can handle. Atanasov planned the Intuitionistic Fuzzy Set (IFS) [3] to overcome these problems. The ideas proposed by Atanassov only involve under-considered data and member and non-member values.

However, the IFS theory cannot handle the overall incompatibility and inaccurate information. To solve the problem of incompatibility and incorrect information, Smarandache [4] proposed the idea of NS. Molodtsov [5] proposed a general mathematical tool for solving uncertain environments, called soft sets (SS). Maggie et al. [6] Expanded the concept of SS and presented basic operations with ideal properties. Maggie et al. [7] A decision-making technique was established using the operations they developed and used for decision-making. Ali et al. [8] Expanded the concept of SS and developed some new operations using their characteristics. The author [9] proved De Morgan's law by using different operators on the SS theory. Çağman developed the concept of soft matrix and Enginoglu [10]. They also introduced some basic operations of soft matrices and studied their required properties.

Çağman and Enginoglu [11] extended the soft set (SS) concept with basic operations and attributes. They also established a decision-making (DM) technology to use the methods they developed to solve decision-making complexity. In [12], the authors proposed some new operations on soft matrices, such as soft difference product, soft finite-difference product, soft extended difference product, and soft extended difference product and their properties. Maji [13] put forward the idea of NSS with necessary operations and attributes. The concept of Possibility NSS was proposed by Karaaslan [14]. He also established a DM technology that uses the And product based on the possibility of NSS to solve the DM problem. Broumi [15] developed a generalized NSS with some operations and properties and applied the proposed concept to DM. Deli and Subas [16] extended the Single Valued neutrosophic number (SVNN) concept and provided a DM method to solve the MCDM problem. They also developed the idea of SVNN cut sets. Wang et al. [17] proposed the correlation coefficient (CC) of single-valued neutrosophic sets (SVNS) and constructed the DM method using the correlation measurement they developed. Ye [18] proposed the idea of a simplified neutrosophic set (NS), developed an aggregation operator (AO) for the simplified NS, and established a DM method to solve the MCDM problem using the AO he developed. Masooma et al. [19] combined multipolar fuzzy sets, and NS proposed multipolar neutrosophic sets and established various representations and operations based on examples. Zulqarnain et al. [20] introduced some AO and correlation coefficients for the interval value IFSS. They also extended the TOPSIS technology to solve the MADM problem with the relevant metrics they developed. Zulqarnain et al. [21] introduced Pythagorean fuzzy soft number (PFSN) operational laws. They developed AO using defined operational laws, such as Pythagorean fuzzy soft weighted average and geometric operators. They also planned a DM method to solve the MADM problem with the help of the provided operator. Zulqarnain et al. [22] planned the TOPSIS method in the PFSS environment based on the correlation coefficient. They also established a DM method to solve the MCGDM problem and used the developed method in green supply chain management.

Many mathematicians have developed various similarity measures, correlation coefficients, aggregation operators, and decision-making applications in the past few years. Garg [23] introduced a weighted cosine similarity measure for intuitionistic fuzzy sets. He also constructed the MCDM method based on his proposed technology and used the developed method for pattern recognition and medical diagnosis. Garg and Kumar [24] proposed some new similarity measures to measure the relative strength of IFS. They also formulated the number of connections for set pair analysis (SPA) and developed a new similarity measure based on the defined SPA. Ruan et al. [25] Some similarity measures have been developed for PFS by using exponential membership and non-membership and their attributes and relationships. Peng and Garg [26] proposed various PFS similarity measures with multiple parameters. Zulqarnain et al. [27, 28] offered the generalized TOPSIS and integrated TOPSIS models for NS and used their proposed techniques for supplier selection in the production industry. Said et al. [29] Established the concept of mPNSS with attributes and operators. They also developed a distance-based similarity measure and used the proposed similarity measure for decision-making and medical diagnosis.

1.1 Motivation

In this era, professionals believe that real life is moving towards multi-polarity. Therefore, there is no doubt that the multi-polarization of information has played a vital role in the prosperity of many scientific and technological fields. In neurobiology, multipolar neurons accumulate a lot of information from other neurons. The motivation for expanding and mixing this research work is gradually given throughout the manuscript. We prove that different hybrid structures containing fuzzy sets will be converted into mPIVNSS special permissions under any appropriate circumstances. The concept of the neutrosophic environment of the multipolar neutrosophic soft set is novel. We discuss the effectiveness, flexibility, quality, and advantages of planning work and algorithms. This research will be the most versatile form that can be used to incorporate data from the complications of daily life. In the future, current work may be extended to different types of hybrid structures and decision-making techniques in many areas of life.

The structure of the following paper is organized as follows: In Section 2, we reviewed some basic definitions used in subsequent sequels, such as NS, SS, NSS, multi-polar neutrosophic set, and interval value neutrosophic soft set. Section 3 puts forward the new idea of mPIVNSS by combining m-pole fuzzy sets (mPFS) with interval-valued neutral soft sets, their attributes, and operations. This section also developed Truth-Favorite, False-Favorite, AND, and OR operators. In Section 4, the multi-polar interval value Neutral Soft Weighted Aggregation (mPIVNSWA) operator was developed using its decision-making technique. Section 5 uses the developed decision-making method and gives a numerical example. Finally, in Section 6, a brief comparison between the method we developed and the existing technology. In addition, superiority, practicality, and flexibility are also introduced in the same section.

2. Preliminaries

This section recollects some basic concepts such as the neutrosophic set, soft set, neutrosophic soft set, and m-polar neutrosophic soft set used in the following sequel.

Definition 2.1 [4] Let $\boldsymbol{\mathcal{U}}$ be a universe and $\boldsymbol{\mathcal{A}}$ be an NS on $\boldsymbol{\mathcal{U}}$ is defined as $\boldsymbol{\mathcal{A}} = \{< \boldsymbol{u}, \boldsymbol{u}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{v}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{w}_{\mathcal{A}}(\boldsymbol{u}) > : \boldsymbol{u} \in \boldsymbol{\mathcal{U}}\}$,where $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}: \boldsymbol{\mathcal{U}} \to]\boldsymbol{0}^-, \boldsymbol{1}^+[$ and $\boldsymbol{0}^- \le \boldsymbol{u}_{\mathcal{A}}(\boldsymbol{u}) + \boldsymbol{v}_{\mathcal{A}}(\boldsymbol{u}) + \boldsymbol{w}_{\mathcal{A}}(\boldsymbol{u}) \le \boldsymbol{3}^+$.

Definition 2.2 [19] Let $\boldsymbol{\mathcal{U}}$ be the universal set and $\wp_{\mathfrak{R}}$ is said to multipolar neutrosophic set if $\wp_{\mathfrak{R}} = \{(\boldsymbol{u}, \boldsymbol{u}_\alpha(\boldsymbol{u}), \boldsymbol{v}_\alpha(\boldsymbol{u}), \boldsymbol{w}_\alpha(\boldsymbol{u})) : \boldsymbol{u} \in \boldsymbol{\mathcal{U}}, \boldsymbol{\alpha} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, \dots, \boldsymbol{m}\}$, where $\boldsymbol{u}_\alpha(\boldsymbol{u})$, $\boldsymbol{v}_\alpha(\boldsymbol{u})$, and $\boldsymbol{w}_\alpha(\boldsymbol{u})$ represents the truthiness, indeterminacy, and falsity respectively, "$\boldsymbol{u}_\alpha(\boldsymbol{u}), \boldsymbol{v}_\alpha(\boldsymbol{u}), \boldsymbol{w}_\alpha(\boldsymbol{u}) \subseteq [\boldsymbol{0}, \boldsymbol{1}]$ and $0 \le \boldsymbol{u}_\alpha(\boldsymbol{u}) + \boldsymbol{v}_\alpha(\boldsymbol{u}) + \boldsymbol{w}_\alpha(\boldsymbol{u}) \le 3$, for all $\boldsymbol{\alpha} = 1, 2, 3, \dots, \boldsymbol{m}$; and $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$.

Definition 2.3 [5] Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a soft set over $\mathcal{U}$, and its mapping is given as

$$\mathcal{F}: \mathcal{A} \to \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F}, \mathcal{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \; if \; e \ne \mathcal{A}\}"$$

2.4 Definition [5]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called an SS over $\mathcal{U}$, and its mapping is given as

$$\mathcal{F}: \mathcal{A} \to \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F}, \mathcal{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \text{ if } e \neq \mathcal{A}\}$$

Definition 2.5 [13] Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the set of neutrosophic sets over $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a neutrosophic soft set over $\mathcal{U}$ and its mapping is given as

$$\mathcal{F}: \mathcal{A} \to \mathcal{P}(\mathcal{U})$$

Definition 2.6 [30] Let $\boldsymbol{\mathcal{U}}$ be a universal set, then interval valued neutrosophic set can be expressed by the set $\boldsymbol{\mathcal{A}} = \{\boldsymbol{u}, (\boldsymbol{u}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{v}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{w}_{\mathcal{A}}(\boldsymbol{u})): \boldsymbol{u} \in \boldsymbol{\mathcal{U}}\}$, where $\boldsymbol{u}_{\mathcal{A}}$, $\boldsymbol{v}_{\mathcal{A}}$, and $\boldsymbol{w}_{\mathcal{A}}$ are truth, indeterminacy and falsity membership functions for $\boldsymbol{\mathcal{A}}$ respectively, $\boldsymbol{u}_{\mathcal{A}}$, $\boldsymbol{v}_{\mathcal{A}}$, and $\boldsymbol{w}_{\mathcal{A}} \subseteq [0, 1]$ for each $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$. Where

$\boldsymbol{u}_{\mathcal{A}}(\boldsymbol{u}) = [\boldsymbol{u}^{L}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{u}^{U}_{\mathcal{A}}(\boldsymbol{u})]$,

$\boldsymbol{v}_{\mathcal{A}}(\boldsymbol{u}) = [\boldsymbol{v}^{L}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{v}^{U}_{\mathcal{A}}(\boldsymbol{u})]$, and

$\boldsymbol{w}_{\mathcal{A}}(\boldsymbol{u}) = [\boldsymbol{w}^{L}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{w}^{U}_{\mathcal{A}}(\boldsymbol{u})]$

For each point $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$, $0 \leq \boldsymbol{u}_{\mathcal{A}}(\boldsymbol{u}) + \boldsymbol{v}_{\mathcal{A}}(\boldsymbol{u}) + \boldsymbol{w}_{\mathcal{A}}(\boldsymbol{u}) \leq 3$ and IVN($\boldsymbol{\mathcal{U}}$) represent the family of all interval valued neutrosophic sets on $\boldsymbol{\mathcal{U}}$.

Definition 2.7 [31] Let $\boldsymbol{\mathcal{U}}$ be a universe of discourse and $\boldsymbol{\mathcal{E}}$ be a set of attributes, and m-polar neutrosophic soft set (mPNSS) $\wp_{\mathfrak{R}}$ over $\boldsymbol{\mathcal{U}}$ defined as

$\wp_{\mathfrak{R}} = \{(\boldsymbol{e}, \{(\boldsymbol{u}, \boldsymbol{u}_{\alpha}(\boldsymbol{u}), \boldsymbol{v}_{\alpha}(\boldsymbol{u}), \boldsymbol{w}_{\alpha}(\boldsymbol{u})): \boldsymbol{u} \in \boldsymbol{\mathcal{U}}, \boldsymbol{\alpha} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \boldsymbol{m}\}): \boldsymbol{e} \in \boldsymbol{\mathcal{E}}\}$,

where $\boldsymbol{u}_{\alpha}(\boldsymbol{u})$, $\boldsymbol{v}_{\alpha}(\boldsymbol{u})$, and $\boldsymbol{w}_{\alpha}(\boldsymbol{u})$ represent the truthiness, indeterminacy, and falsity respectively, $\boldsymbol{u}_{\alpha}(\boldsymbol{u}), \boldsymbol{v}_{\alpha}(\boldsymbol{u}), \boldsymbol{w}_{\alpha}(\boldsymbol{u}) \subseteq [\mathbf{0}, \mathbf{1}]$ and $0 \leq \boldsymbol{u}_{\alpha}(\boldsymbol{u}) + \boldsymbol{v}_{\alpha}(\boldsymbol{u}) + \boldsymbol{w}_{\alpha}(\boldsymbol{u}) \leq 3$, for all $\boldsymbol{\alpha} = 1, 2, 3, \ldots, \boldsymbol{m}$; $\boldsymbol{e} \in \boldsymbol{\mathcal{E}}$ and $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$. Simply an m-polar neutrosophic number (mPNSN) can be expressed as $\wp = \{\langle \boldsymbol{u}_{\alpha}, \boldsymbol{v}_{\alpha}, \boldsymbol{w}_{\alpha} \rangle\}$, where $0 \leq \boldsymbol{u}_{\alpha} + \boldsymbol{v}_{\alpha} + \boldsymbol{w}_{\alpha} \leq 3$ and $\boldsymbol{\alpha} = 1, 2, 3, \ldots, \boldsymbol{m}$.

Definition 2.8 [32] Let $\boldsymbol{\mathcal{U}}$ be a universe of discourse and $\boldsymbol{\mathcal{E}}$ be a set of attributes, an IVNSS $\wp_{\mathfrak{R}}$ over $\boldsymbol{\mathcal{U}}$ defined as

$\wp_{\mathfrak{R}} = \{(e, \{(u, u_{\mathfrak{R}}(u), v_{\mathfrak{R}}(u), w_{\mathfrak{R}}(u)): u \in \mathcal{U}, \alpha = 1, 2, 3, \ldots, m\}): e \in \mathcal{E}\}$,

where $\boldsymbol{u}_{\mathfrak{R}}(\boldsymbol{u}) = [\boldsymbol{u}^{\ell}_{\mathfrak{R}}(\boldsymbol{u}), \boldsymbol{u}^{\mathfrak{u}}_{\mathfrak{R}}(\boldsymbol{u})]$, $\boldsymbol{v}_{\mathfrak{R}}(\boldsymbol{u}) = [\boldsymbol{v}^{\ell}_{\mathfrak{R}}(\boldsymbol{u}), \boldsymbol{v}^{\mathfrak{u}}_{\mathfrak{R}}(\boldsymbol{u})]$, $\boldsymbol{w}_{\mathfrak{R}}(\boldsymbol{u}) = [\boldsymbol{w}^{\ell}_{\mathfrak{R}}(\boldsymbol{u}), \boldsymbol{w}^{\mathfrak{u}}_{\mathfrak{R}}(\boldsymbol{u})]$, represents the interval truthiness, indeterminacy, and falsity respectively, $\boldsymbol{u}_{\mathfrak{R}}(\boldsymbol{u}), \boldsymbol{v}_{\mathfrak{R}}(\boldsymbol{u}), \boldsymbol{w}_{\mathfrak{R}}(\boldsymbol{u}) \subseteq [\mathbf{0}, \mathbf{1}]$ and $0 \leq \boldsymbol{u}^{\mathfrak{u}}_{\mathfrak{R}}(\boldsymbol{u}) + \boldsymbol{v}^{\mathfrak{u}}_{\mathfrak{R}}(\boldsymbol{u}) + \boldsymbol{w}^{\mathfrak{u}}_{\mathfrak{R}}(\boldsymbol{u}) \leq 3$, for each $\boldsymbol{e} \in \boldsymbol{\mathcal{E}}$ and $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$.

3. Multi-Polar Interval Valued Neutrosophic Soft Set with Aggregate Operators and Properties

The idea of m-pole fuzzy sets (mPFS) was proposed by Chen et al. [33] In 2014, able to deal with ambiguous data and ambiguous multipolar information. Smarandache [34] proposed a three-pole, multi-pole neutrosophic set and its graph in 2016. The membership degree of mPFS is in the interval [0,1]^m, representing the m criteria of the object, but mPFS cannot deal with uncertainty and false objects. NS is bargaining with a single choice criterion of true, false, and uncertainty. But it cannot deal with the multi-standard, multi-source, and multi-polar information fusion that may be selected. Deli et al. [31] Combining the concepts of m-polar neutrosophic set and SS, a new model of mPNSS was introduced. The developed mPNSS can handle m standards for each alternative. mPNSS extends the bipolar Zhongzhi soft set proposed by Ali et al. [35]. Deli [32] established IVNSS, which is a combination of IVNS[30] and SS[5]. We constructed some basic concepts of mPNSS and extended mPNSS to mPIVSS with various operations and attributes.

Definition 3.1 Let $\boldsymbol{\mathcal{U}}$ be a universe of discourse and $\boldsymbol{\mathcal{E}}$ be a set of attributes, then m-polar interval-valued neutrosophic soft set (mPIVNSS) $\wp_{\mathfrak{R}}$ over $\boldsymbol{\mathcal{U}}$ defined as

$\wp_{\mathfrak{R}} = \{(e, \{(u, u_{\alpha}(u), v_{\alpha}(u), w_{\alpha}(u)): u \in \mathcal{U}, \alpha = 1,2,3, \ldots, m\}): e \in \mathcal{E}\}$,

where $u_{\alpha}(u) = [u^{\ell}_{\alpha}(u), u^{\mathfrak{u}}_{\alpha}(u)]$, $v_{\alpha}(u) = [v^{\ell}_{\alpha}(u), v^{\mathfrak{u}}_{\alpha}(u)]$, $w_{\alpha}(u) = [w^{\ell}_{\alpha}(u), w^{\mathfrak{u}}_{\alpha}(u)]$, represent the interval truthiness, indeterminacy, and falsity respectively, $u_{\alpha}(u), v_{\alpha}(u), w_{\alpha}(u) \subseteq [0, 1]$ and $0 \leq$

$u_\alpha^{u}(u) + v_\alpha^{u}(u) + w_\alpha^{u}(u) \leq 3$ for all $\alpha = 1, 2, 3, \ldots, m$; $e \in \mathcal{E}$ and $u \in \mathcal{U}$. Simply an m-polar interval-valued neutrosophic soft number (mPIVNSN) can be expressed as $\wp = \{[u_\alpha^{\ell}(u), u_\alpha^{u}(u)], [v_\alpha^{\ell}(u), v_\alpha^{u}(u)], [w_\alpha^{\ell}(u), w_\alpha^{u}(u)]\}$, where $0 \leq u_\alpha^{u}(u) + v_\alpha^{u}(u) + w_\alpha^{u}(u) \leq 3$ and $\alpha = 1, 2, 3, \ldots, m$.

Definition 3.2 Let $\wp_{\mathfrak{R}}$ and $\wp_{\mathcal{L}}$ be two mPIVNSSs over $\mathcal{U}$. Then, $\wp_{\mathfrak{R}}$ is called an m-polar interval-valued neutrosophic soft subset of $\wp_{\mathcal{L}}$. If

$u_\alpha^{\ell\mathfrak{R}}(u) \leq u_\alpha^{\ell\mathcal{L}}(u)$, $u_\alpha^{u\mathfrak{R}}(u) \leq u_\alpha^{u\mathcal{L}}(u)$

$v_\alpha^{\ell\mathfrak{R}}(u) \geq v_\alpha^{\ell\mathcal{L}}(u)$, $v_\alpha^{u\mathfrak{R}}(u) \geq v_\alpha^{u\mathcal{L}}(u)$

$w_\alpha^{\ell\mathfrak{R}}(u) \geq w_\alpha^{\ell\mathcal{L}}(u)$, $w_\alpha^{u\mathfrak{R}}(u) \geq w_\alpha^{u\mathcal{L}}(u)$

for all $\alpha = 1, 2, 3, \cdots, m$; $e \in \mathcal{E}$ and $u \in \mathcal{U}$.

Definition 3.3 Let $\wp_{\mathfrak{R}}$ and $\wp_{\mathcal{L}}$ be two mPIVNSSs over $\mathcal{U}$. Then, $\wp_{\mathfrak{R}} = \wp_{\mathcal{L}}$, if

$u_\alpha^{\ell\mathfrak{R}}(u) \leq u_\alpha^{\ell\mathcal{L}}(u)$, $u_\alpha^{\ell\mathcal{L}}(u) \leq u_\alpha^{\ell\mathfrak{R}}(u)$ and $u_\alpha^{u\mathfrak{R}}(u) \leq u_\alpha^{u\mathcal{L}}(u)$, $u_\alpha^{u\mathcal{L}}(u) \leq u_\alpha^{u\mathfrak{R}}(u)$

$v_\alpha^{\ell\mathfrak{R}}(u) \geq v_\alpha^{\ell\mathcal{L}}(u)$, $v_\alpha^{\ell\mathcal{L}}(u) \geq v_\alpha^{\ell\mathfrak{R}}(u)$ and $v_\alpha^{u\mathfrak{R}}(u) \geq v_\alpha^{u\mathcal{L}}(u)$, $v_\alpha^{u\mathcal{L}}(u) \geq v_\alpha^{u\mathfrak{R}}(u)$

$w_\alpha^{\ell\mathfrak{R}}(u) \geq w_\alpha^{\ell\mathcal{L}}(u)$, $w_\alpha^{\ell\mathcal{L}}(u) \geq w_\alpha^{\ell\mathfrak{R}}(u)$ and $w_\alpha^{u\mathfrak{R}}(u) \geq w_\alpha^{u\mathcal{L}}(u)$, $w_\alpha^{u\mathcal{L}}(u) \geq w_\alpha^{u\mathfrak{R}}(u)$

for all $\alpha = 1, 2, 3, \ldots, m$; $e \in \mathcal{E}$ and $u \in \mathcal{U}$.

Definition 3.4 Let $\wp_{\mathfrak{R}}$ and $\wp_{\mathcal{L}}$ be two mPIVNSSs over $\mathcal{U}$. Then,

$$\wp_{\mathfrak{R}} \cup \wp_{\mathcal{L}} = \left\{\left(e, \left\{\begin{pmatrix} u, \left[sup\{u_\alpha^{\ell\mathfrak{R}}(u), u_\alpha^{\ell\mathcal{L}}(u)\}, sup\{u_\alpha^{u\mathfrak{R}}(u), u_\alpha^{u\mathcal{L}}(u)\}\right], \\ \left[inf\{v_\alpha^{\ell\mathfrak{R}}(u), v_\alpha^{\ell\mathcal{L}}(u)\}, inf\{v_\alpha^{u\mathfrak{R}}(u), v_\alpha^{u\mathcal{L}}(u)\}\right], \\ \left[inf\{w_\alpha^{\ell\mathfrak{R}}(u), w_\alpha^{\ell\mathcal{L}}(u)\}, inf\{w_\alpha^{u\mathfrak{R}}(u), w_\alpha^{u\mathcal{L}}(u)\}\right] \end{pmatrix} : u \in \mathcal{U}, \alpha = 1,2,3, \ldots, m\right\}\right) : e \in \mathcal{E}\right\}$$

Definition 3.5 Let $\wp_{\mathfrak{R}}$ and $\wp_{\mathcal{L}}$ be two mPIVNSSs over $\mathcal{U}$. Then,

$$\wp_{\mathfrak{R}} \cap \wp_{\mathcal{L}} = \left\{\left(e, \left\{\begin{pmatrix} u, \left[inf\{u_\alpha^{\ell\mathfrak{R}}(u), u_\alpha^{\ell\mathcal{L}}(u)\}, inf\{u_\alpha^{u\mathfrak{R}}(u), u_\alpha^{u\mathcal{L}}(u)\}\right], \\ \left[sup\{v_\alpha^{\ell\mathfrak{R}}(u), v_\alpha^{\ell\mathcal{L}}(u)\}, sup\{v_\alpha^{u\mathfrak{R}}(u), v_\alpha^{u\mathcal{L}}(u)\}\right], \\ \left[sup\{w_\alpha^{\ell\mathfrak{R}}(u), w_\alpha^{\ell\mathcal{L}}(u)\}, sup\{w_\alpha^{u\mathfrak{R}}(u), w_\alpha^{u\mathcal{L}}(u)\}\right] \end{pmatrix} : u \in \mathcal{U}, \alpha = 1,2,3, \ldots, m\right\}\right) : e \in \mathcal{E}\right\}$$

3.6 Definition

Let $\wp_{\mathfrak{R}}$ be an mPIVNSS over $\mathcal{U}$. Then, the complement of mPIVNSS is defined as follows:

$$\wp_{\mathfrak{R}}^{c} = \left\{\left(e, \left\{\left(u, [w_\alpha^{\ell}(u), w_\alpha^{u}(u)], [1 - v_\alpha^{u}(u), 1 - v_\alpha^{\ell}(u)], [u_\alpha^{\ell}(u), u_\alpha^{u}(u)]\right) : u \in \mathcal{U}, \alpha = 1, 2, 3, \ldots, m\right\}\right) : e \in \mathcal{E}\right\}$$

Proposition 3.7 Let $\wp_{\mathfrak{R}}$ and $\wp_{\mathcal{L}}$ be two mPIVNSSs over $\mathcal{U}$. Then,

1. $(\wp_{\mathfrak{R}} \cup \wp_{\mathcal{L}})^{C} = {\wp_{\mathfrak{R}}}^{C} \cap {\wp_{\mathcal{L}}}^{C}$
2. $(\wp_{\mathfrak{R}} \cap \wp_{\mathcal{L}})^{C} = {\wp_{\mathfrak{R}}}^{C} \cup {\wp_{\mathcal{L}}}^{C}$

Proof 1 As we know that

$$\wp_{\mathfrak{R}} = \left\{\left(e, \left\{\left(u, \left[u_\alpha^{\ell\mathfrak{R}}(u), u_\alpha^{u\mathfrak{R}}(u)\right], \left[v_\alpha^{\ell\mathfrak{R}}(u), v_\alpha^{u\mathfrak{R}}(u)\right], \left[w_\alpha^{\ell\mathfrak{R}}(u), w_\alpha^{u\mathfrak{R}}(u)\right]\right) : u \in \mathcal{U}, \alpha = 1,2,3, \ldots, m\right\}\right) : e \in \mathcal{E}\right\}$$

and

$$\wp_{\mathcal{L}} = \left\{\left(e, \left\{\left(u, [u_\alpha^{\ell\mathcal{L}}(u), u_\alpha^{u\mathcal{L}}(u)], [v_\alpha^{\ell\mathcal{L}}(u), v_\alpha^{u\mathcal{L}}(u)], [w_\alpha^{\ell\mathcal{L}}(u), w_\alpha^{u\mathcal{L}}(u)]\right) : u \in \mathcal{U}, \alpha = 1,2,3, \ldots, m\right\}\right) : e \in \mathcal{E}\right\}$$

Then

$$\wp_{\mathfrak{R}} \cup \wp_{\mathcal{L}} = \left\{\left(e, \left\{\begin{pmatrix} u, \left[sup\{u_\alpha^{\ell\mathfrak{R}}(u), u_\alpha^{\ell\mathcal{L}}(u)\}, sup\{u_\alpha^{u\mathfrak{R}}(u), u_\alpha^{u\mathcal{L}}(u)\}\right], \\ \left[inf\{v_\alpha^{\ell\mathfrak{R}}(u), v_\alpha^{\ell\mathcal{L}}(u)\}, inf\{v_\alpha^{u\mathfrak{R}}(u), v_\alpha^{u\mathcal{L}}(u)\}\right], \\ \left[inf\{w_\alpha^{\ell\mathfrak{R}}(u), w_\alpha^{\ell\mathcal{L}}(u)\}, inf\{w_\alpha^{u\mathfrak{R}}(u), w_\alpha^{u\mathcal{L}}(u)\}\right] \end{pmatrix} : u \in \mathcal{U}, \alpha = 1,2,3, \ldots, m\right\}\right) : e \in \mathcal{E}\right\}$$

we get

$(\wp_{\mathfrak{R}} \cup \wp_{\mathcal{L}})^c$

$$=\left\{\left(e,\left\{\left(\begin{array}{c} u,\left[inf\{w_{\alpha}^{\ell\mathfrak{R}}(u), w_{\alpha}^{\ell\mathcal{L}}(u)\}, inf\{w_{\alpha}^{u\mathfrak{R}}(u), w_{\alpha}^{u\mathcal{L}}(u)\}\right], \\ \left[1-inf\{v_{\alpha}^{u\mathfrak{R}}(u), v_{\alpha}^{u\mathcal{L}}(u)\}, 1-inf\{v_{\alpha}^{\ell\mathfrak{R}}(u), v_{\alpha}^{\ell\mathcal{L}}(u)\}\right], \\ \left[sup\{u_{\alpha}^{\ell\mathfrak{R}}(u), u_{\alpha}^{\ell\mathcal{L}}(u)\}, sup\{u_{\alpha}^{u\mathfrak{R}}(u), u_{\alpha}^{u\mathcal{L}}(u)\}\right] \end{array}\right): u \in \mathcal{U}, \alpha = 1,2,3,\dots,m\right\}\right): e \in \mathcal{E}\right\}$$

Now

$\wp_{\mathfrak{R}}{}^C =$

$$\left\{\left(e,\left\{\left(u,\left[w_{\alpha}^{\ell\mathfrak{R}}(u), w_{\alpha}^{u\mathfrak{R}}(u)\right], \left[1-v_{\alpha}^{u\mathfrak{R}}(u), 1-v_{\alpha}^{\ell\mathfrak{R}}(u)\right], \left[u_{\alpha}^{\ell\mathfrak{R}}(u), u_{\alpha}^{u\mathfrak{R}}(u)\right]\right): u \in \mathcal{U}, \alpha = 1,2,3,\dots,m\right\}\right): e \in \mathcal{E}\right\}$$

$\wp_{\mathcal{L}}{}^C =$

$$\left\{\left(e,\left\{\left(u,\left[w_{\alpha}^{\ell\mathcal{L}}(u), w_{\alpha}^{u\mathcal{L}}(u)\right], \left[1-v_{\alpha}^{u\mathcal{L}}(u), 1-v_{\alpha}^{\ell\mathcal{L}}(u)\right], \left[u_{\alpha}^{\ell\mathcal{L}}(u), u_{\alpha}^{u\mathcal{L}}(u)\right]\right): u \in \mathcal{U}, \alpha = 1,2,3,\dots,m\right\}\right): e \in \mathcal{E}\right\}$$

By using definition 3.5

$\wp_{\mathfrak{R}}{}^C \cap \wp_{\mathcal{L}}{}^C =$

$$\left\{\left(e,\left\{\left(\begin{array}{c} u,\left[inf\{w_{\alpha}^{\ell\mathfrak{R}}(u), w_{\alpha}^{\ell\mathcal{L}}(u)\}, inf\{w_{\alpha}^{u\mathfrak{R}}(u), w_{\alpha}^{u\mathcal{L}}(u)\}\right], \\ \left[inf\{1-v_{\alpha}^{u\mathfrak{R}}(u), 1-v_{\alpha}^{u\mathcal{L}}(u)\}, inf\{1-v_{\alpha}^{\ell\mathfrak{R}}(u), 1-v_{\alpha}^{\ell\mathcal{L}}(u)\}\right], \\ \left[sup\{u_{\alpha}^{\ell\mathfrak{R}}(u), u_{\alpha}^{\ell\mathcal{L}}(u)\}, sup\{u_{\alpha}^{u\mathfrak{R}}(u), u_{\alpha}^{u\mathcal{L}}(u)\}\right] \end{array}\right): u \in \mathcal{U}, \alpha = 1,2,3,\dots,m\right\}\right): e \in \mathcal{E}\right\}$$

$\wp_{\mathfrak{R}}{}^C \cap \wp_{\mathcal{L}}{}^C =$

$$\left\{\left(e,\left\{\left(\begin{array}{c} u,\left[inf\{w_{\alpha}^{\ell\mathfrak{R}}(u), w_{\alpha}^{\ell\mathcal{L}}(u)\}, inf\{w_{\alpha}^{u\mathfrak{R}}(u), w_{\alpha}^{u\mathcal{L}}(u)\}\right], \\ \left[1-inf\{v_{\alpha}^{u\mathfrak{R}}(u), v_{\alpha}^{u\mathcal{L}}(u)\}, 1-inf\{v_{\alpha}^{\ell\mathfrak{R}}(u), v_{\alpha}^{\ell\mathcal{L}}(u)\}\right], \\ \left[sup\{u_{\alpha}^{\ell\mathfrak{R}}(u), u_{\alpha}^{\ell\mathcal{L}}(u)\}, sup\{u_{\alpha}^{u\mathfrak{R}}(u), u_{\alpha}^{u\mathcal{L}}(u)\}\right] \end{array}\right): u \in \mathcal{U}, \alpha = 1,2,3,\dots,m\right\}\right): e \in \mathcal{E}\right\}$$

Hence

$(\wp_{\mathfrak{R}} \cup \wp_{\mathcal{L}})^c = \wp_{\mathfrak{R}}{}^C \cap \wp_{\mathcal{L}}{}^C$.

Proof 2 Similar to assertion 1.

Definition 3.8 Let $\wp_{\mathfrak{R}}$ and $\wp_{\mathcal{L}}$ be two mPIVNSSs over $\mathcal{U}$. Then, their extended union is defined as

$$u(\wp_{\mathfrak{R}} \cup_{\varepsilon} \wp_{\mathcal{L}}) = \begin{cases} [u_{\alpha}^{\ell\mathfrak{R}}(u), u_{\alpha}^{u\mathfrak{R}}(u)] & if\ e \in \mathfrak{R} - \mathcal{L} \\ [u_{\alpha}^{\ell\mathcal{L}}(u), u_{\alpha}^{u\mathcal{L}}(u)] & if\ e \in \mathcal{L} - \mathfrak{R} \\ \left[sup\{u_{\alpha}^{\ell\mathfrak{R}}(u), u_{\alpha}^{\ell\mathcal{L}}(u)\}, sup\{u_{\alpha}^{u\mathfrak{R}}(u), u_{\alpha}^{u\mathcal{L}}(u)\}\right] & if\ e \in \mathfrak{R} \cap \mathcal{L} \end{cases}$$

$$v(\wp_{\mathfrak{R}} \cup_{\varepsilon} \wp_{\mathcal{L}}) = \begin{cases} [v_{\alpha}^{\ell\mathfrak{R}}(u), v_{\alpha}^{u\mathfrak{R}}(u)] & if\ e \in \mathfrak{R} - \mathcal{L} \\ [v_{\alpha}^{\ell\mathcal{L}}(u), v_{\alpha}^{u\mathcal{L}}(u)] & if\ e \in \mathcal{L} - \mathfrak{R} \\ \left[inf\{v_{\alpha}^{\ell\mathfrak{R}}(u), v_{\alpha}^{\ell\mathcal{L}}(u)\}, inf\{v_{\alpha}^{u\mathfrak{R}}(u), v_{\alpha}^{u\mathcal{L}}(u)\}\right] & if\ e \in \mathfrak{R} \cap \mathcal{L} \end{cases}$$

$$w(\wp_{\mathfrak{R}} \cup_{\varepsilon} \wp_{\mathcal{L}}) = \begin{cases} [w_{\alpha}^{\ell\mathfrak{R}}(u), w_{\alpha}^{u\mathfrak{R}}(u)] & if\ e \in \mathfrak{R} - \mathcal{L} \\ [w_{\alpha}^{\ell\mathcal{L}}(u), w_{\alpha}^{u\mathcal{L}}(u)] & if\ e \in \mathcal{L} - \mathfrak{R} \\ \left[inf\{w_{\alpha}^{\ell\mathfrak{R}}(u), w_{\alpha}^{\ell\mathcal{L}}(u)\}, inf\{w_{\alpha}^{u\mathfrak{R}}(u), w_{\alpha}^{u\mathcal{L}}(u)\}\right] & if\ e \in \mathfrak{R} \cap \mathcal{L} \end{cases}$$

Example 3.9 Assume $\mathcal{U} = \{u_1, u_2\}$ be a universe of discourse and $E = \{e_1, e_2, e_3, e_4\}$ be a set of attributes and $\mathfrak{R} = \{e_1, e_2\} \subseteq E$ and $\mathcal{L} = \{e_2, e_3\} \subseteq E$. Consider 3-PIVNSSs $\wp_{\mathfrak{R}}$ and $\wp_{\mathcal{L}}$ over $\mathcal{U}$ can be represented as follows:

$$\wp_{\mathfrak{R}} = \left\{\begin{array}{l} \left(e_1,\left\{\begin{array}{l} (u_1, ([.5,.8],[.2,.5],[.1,.2]), ([.3,.5],[.1,.3],[.2,.4]), ([.6,.9],[.7,.8],[.8,1])) \\ (u_2, ([.2,.4],[.3,.4],[.1,.3]), ([.2,.5],[.1,.6],[.1,.3]), ([.8,1],[.6,.9],[.6,.7])) \end{array}\right\}\right), \\ \left(e_2,\left\{\begin{array}{l} (u_1, ([.3,.6],[.1,.6],[.3,.4]), ([0,.2],[.1,.4],[.3,.5]), ([.5,.9],[.3,.8],[.5,.8])) \\ (u_2, ([.2,.5],[.2,.3],[.5,.6]), ([.3,.5],[.1,.5],[.5,.8]), ([.6,.9],[.5,.8],[.6,.9])) \end{array}\right\}\right) \end{array}\right\}$$

and

$$\wp_{\mathcal{L}} = \left\{ \begin{array}{l} \left(e_1, \left\{ \begin{array}{l} (u_1, ([.4,.8],[.3,.6],[.2,.5]),([.2,.7],[.3,.4],[.4,.6]),([.7,.8],[.4,.9],[.5,1])) \\ (u_2, ([.1,.6],[.5,.7],[.1,.2]),([.3,.4],[.2,.5],[.2,.5]),([.5,.9],[.7,.8],[.4,.6])) \end{array} \right\} \right), \\ \left(e_2, \left\{ \begin{array}{l} (u_1, ([.2,.7],[.3,.5],[.2,.6]),([.1,.3],[.2,.5],[.2,.7]),([.4,.9],[.4,.7],[.5,.8])) \\ (u_2, ([.1,.6],[.1,.5],[.4,.8]),([.3,.6],[.3,.4],[1,1]),([.5,.9],[.3,.7],[.1,.8])) \end{array} \right\} \right) \end{array} \right\}$$

Then

$$\wp_{\mathfrak{R}} \cup_{\varepsilon} \wp_{\mathcal{L}} = \left\{ \begin{array}{l} \left(e_1, \left\{ \begin{array}{l} (u_1, ([.5,.8],[.2,.5],[.1,.2]),([.3,.5],[.1,.3],[.2,.4]),([.6,.9],[.7,.8],[.8,1])) \\ (u_2, ([.2,.4],[.3,.4],[.1,.3]),([.2,.5],[.1,.6],[.1,.3]),([.8,1],[.6,.9],[.6,.7])) \end{array} \right\} \right), \\ \left(e_2, \left\{ \begin{array}{l} (u_1, ([.4,.8],[.1,.6],[.2,.4]),([.2,.7],[.1,.4],[.3,.5]),([.7,.9],[.3,.8],[.5,.8])) \\ (u_2, ([.2,.6],[.2,.3],[.1,.2]),([.3,.5],[.1,.5],[.2,.5]),([.6,.9],[.5,.8],[.4,.6])) \end{array} \right\} \right), \\ \left(e_2, \left\{ \begin{array}{l} (u_1, ([.2,.7],[.3,.5],[.2,.6]),([.1,.3],[.2,.5],[.2,.7]),([.4,.9],[.4,.7],[.5,.8])) \\ (u_2, ([.1,.6],[.1,.5],[.4,.8]),([.3,.6],[.3,.4],[1,1]),([.5,.9],[.3,.7],[.1,.8])) \end{array} \right\} \right) \end{array} \right\}$$

Definition 3.10 Let $\wp_{\mathfrak{R}}$ and $\wp_{\mathcal{L}}$ be two mPIVNSSs over $\mathcal{U}$. Then, their extended intersection is defined as

$$u\ (\wp_{\mathfrak{R}} \cap_{\varepsilon} \wp_{\mathcal{L}}) = \begin{cases} [u_{\alpha}^{\ell\mathfrak{R}}(u), u_{\alpha}^{u\mathfrak{R}}(u)] & if\ e \in \mathfrak{R} - \mathcal{L} \\ [u_{\alpha}^{\ell\mathcal{L}}(u), u_{\alpha}^{u\mathcal{L}}(u)] & if\ e \in \mathcal{L} - \mathfrak{R} \\ [inf\{u_{\alpha}^{\ell\mathfrak{R}}(u), u_{\alpha}^{\ell\mathcal{L}}(u)\}, inf\{u_{\alpha}^{u\mathfrak{R}}(u), u_{\alpha}^{u\mathcal{L}}(u)\}] & if\ e \in \mathfrak{R} \cap \mathcal{L} \end{cases}$$

$$v\ (\wp_{\mathfrak{R}} \cap_{\varepsilon} \wp_{\mathcal{L}}) = \begin{cases} [v_{\alpha}^{\ell\mathfrak{R}}(u), v_{\alpha}^{u\mathfrak{R}}(u)] & if\ e \in \mathfrak{R} - \mathcal{L} \\ [v_{\alpha}^{\ell\mathcal{L}}(u), v_{\alpha}^{u\mathcal{L}}(u)] & if\ e \in \mathcal{L} - \mathfrak{R} \\ [sup\{v_{\alpha}^{\ell\mathfrak{R}}(u), v_{\alpha}^{\ell\mathcal{L}}(u)\}, sup\{v_{\alpha}^{u\mathfrak{R}}(u), v_{\alpha}^{u\mathcal{L}}(u)\}] & if\ e \in \mathfrak{R} \cap \mathcal{L} \end{cases}$$

$$w\ (\wp_{\mathfrak{R}} \cap_{\varepsilon} \wp_{\mathcal{L}}) = \begin{cases} [w_{\alpha}^{\ell\mathfrak{R}}(u), w_{\alpha}^{u\mathfrak{R}}(u)] & if\ e \in \mathfrak{R} - \mathcal{L} \\ [w_{\alpha}^{\ell\mathcal{L}}(u), w_{\alpha}^{u\mathcal{L}}(u)] & if\ e \in \mathcal{L} - \mathfrak{R} \\ [sup\{w_{\alpha}^{\ell\mathfrak{R}}(u), w_{\alpha}^{\ell\mathcal{L}}(u)\}, sup\{w_{\alpha}^{u\mathfrak{R}}(u), w_{\alpha}^{u\mathcal{L}}(u)\}] & if\ e \in \mathfrak{R} \cap \mathcal{L} \end{cases}$$

Remark 3.1 Generally, if $\wp_{\mathfrak{R}} \neq \wp_{\breve{0}}$ and $\wp_{\mathfrak{R}} \neq \wp_{\grave{\mathrm{E}}}$, then the law of contradiction $\wp_{\mathfrak{R}} \cap \wp_{\mathfrak{R}}{}^{C} = \wp_{\breve{0}}$ and the law of the excluded middle $\wp_{\mathfrak{R}} \cup \wp_{\mathfrak{R}}{}^{C} = \wp_{\grave{\mathrm{E}}}$ does not hold in mPIVNSS. But in classical set theory law of contradiction and excluded middle always hold.

Definition 3.11 Let $\wp_{\mathfrak{R}}$ be an mPIVNSS over $\mathcal{U}$. Then, Truth-Favorite operator on $\wp_{\mathfrak{R}}$ is denoted by $\widetilde{\Delta}\wp_{\mathfrak{R}}$ and defined as follow:

$\widetilde{\Delta}\wp_{\mathfrak{R}} =$

$$\left\{ \left(e, \left\{ \left(\begin{array}{c} u, [inf\{u_{\alpha}^{\ell\mathfrak{R}}(u) + v_{\alpha}^{\ell\mathfrak{R}}(u), 1\}, inf\{u_{\alpha}^{u\mathfrak{R}}(u) + v_{\alpha}^{u\mathfrak{R}}(u), 1\}], \\ [0,0],[0,0],\ldots,[0,0],[w_{\alpha}^{\ell\mathfrak{R}}(u), w_{\alpha}^{u\mathfrak{R}}(u)] \end{array} \right) : u \in \mathcal{U}, \alpha = 1,2,3,\ldots,m \right\} \right) : e \in \mathcal{E} \right\}$$

Proposition 3.12 Let $\wp_{\mathfrak{R}}$ and $\wp_{\mathcal{L}}$ be two mPIVNSSs over $\mathcal{U}$. Then,

1. $\widetilde{\Delta}\widetilde{\Delta}\wp_{\mathfrak{R}} = \widetilde{\Delta}\wp_{\mathfrak{R}}$
2. $\widetilde{\Delta}(\wp_{\mathfrak{R}} \cup \wp_{\mathcal{L}}) \subseteq \widetilde{\Delta}\wp_{\mathfrak{R}} \cup \widetilde{\Delta}\wp_{\mathcal{L}}$
3. $\widetilde{\Delta}(\wp_{\mathfrak{R}} \cap \wp_{\mathcal{L}}) \subseteq \widetilde{\Delta}\wp_{\mathfrak{R}} \cap \widetilde{\Delta}\wp_{\mathcal{L}}$

Proof of the above proposition is easily obtained by using definitions 3.4, 3.5, 3.11.

Definition 3.13 Let $\wp_{\mathfrak{R}}$ be an mPIVNSS over $\mathcal{U}$. Then, the False-Favorite operator on $\wp_{\mathfrak{R}}$ denoted by $\widetilde{\Delta}\wp_{\mathfrak{R}}$ and is defined as follows:

$\widetilde{\nabla}\wp_{\mathfrak{R}} =$

$$\left\{ \left(e, \left\{ \left(\begin{array}{c} u, [u_{\alpha}^{\ell\mathfrak{R}}(u), u_{\alpha}^{u\mathfrak{R}}(u)],[0,0],[0,0],\ldots,[0,0], \\ [inf\{w_{\alpha}^{\ell\mathfrak{R}}(u) + v_{\alpha}^{\ell\mathfrak{R}}(u), 1\}, inf\{w_{\alpha}^{u\mathfrak{R}}(u) + v_{\alpha}^{u\mathfrak{R}}(u), 1\}] \end{array} \right) : u \in \mathcal{U}, \alpha = 1,2,3,\ldots,m \right\} \right) : e \in \mathcal{E} \right\}$$

Proposition 3.14 Let $\wp_{\mathfrak{R}}$ and $\wp_{\mathcal{L}}$ be two mPIVNSSs over $\mathcal{U}$. Then,

1. $\tilde{\nabla}\tilde{\nabla}\wp_{\Re} = \tilde{\nabla}\wp_{\Re}$
2. $\tilde{\nabla}(\wp_{\Re} \cup \wp_{\mathcal{L}}) \subseteq \tilde{\nabla}\wp_{\Re} \cup \tilde{\nabla}\wp_{\mathcal{L}}$
3. $\tilde{\nabla}(\wp_{\Re} \cap \wp_{\mathcal{L}}) \subseteq \tilde{\nabla}\wp_{\Re} \cap \tilde{\nabla}\wp_{\mathcal{L}}$

Proof of the above proposition is easily obtained using definitions 3.4, 3.5, 3.13.

Definition 3.15 Let $\wp_{\Re}$ and $\wp_{\mathcal{L}}$ be two mPIVNSSs over $\mathcal{U}$. Then, their AND-Operator is represented by $\wp_{\Re} \wedge \wp_{\mathcal{L}}$ and defined as follows:

$\wp_{\Re} \wedge \wp_{\mathcal{L}} = \daleth_{\Re\times\mathcal{L}}$, where

$\daleth_{\Re\times\mathcal{L}}(x,y) = \wp_{\Re}(x) \cap \wp_{\mathcal{L}}(y)$ for all $(x,y) \in \Re \times \mathcal{L}$.

Definition 3.16 Let $\wp_{\Re}$ and $\wp_{\mathcal{L}}$ be two mPIVNSSs over $\mathcal{U}$. Then, their OR-Operator is represented by $\wp_{\Re} \vee \wp_{\mathcal{L}}$ and defined as follows:

$\wp_{\Re} \vee \wp_{\mathcal{L}} = \daleth_{\Re\times\mathcal{L}}$, where

$\daleth_{\Re\times\mathcal{L}}(x,y) = \wp_{\Re}(x) \cup \wp_{\mathcal{L}}(y)$ for all $(x,y) \in \Re \times \mathcal{L}$.

Example 3.17 Reconsider example 3.9

$$\wp_{\Re} = \left\{\begin{array}{l} \left(e_1, \left\{\begin{array}{l}(u_1, ([.5,.8],[.2,.5],[.1,.2]),([.3,.5],[.1,.3],[.2,.4]),([.6,.9],[.7,.8],[.8,1])) \\ (u_2, ([.2,.4],[.3,.4],[.1,.3]),([.2,.5],[.1,.6],[.1,.3]),([.8,1],[.6,.9],[.6,.7]))\end{array}\right\}\right), \\ \left(e_2, \left\{\begin{array}{l}(u_1, ([.3,.6],[.1,.6],[.3,.4]),([0,.2],[.1,.4],[.3,.5]),([.5,.9],[.3,.8],[.5,.8])) \\ (u_2, ([.2,.5],[.2,.3],[.5,.6]),([.3,.5],[.1,.5],[.5,.8]),([.6,.9],[.5,.8],[.6,.9]))\end{array}\right\}\right) \end{array}\right\}$$

and

$$\wp_{\mathcal{L}} = \left\{\begin{array}{l} \left(e_1, \left\{\begin{array}{l}(u_1, ([.4,.8],[.3,.6],[.2,.5]),([.2,.7],[.3,.4],[.4,.6]),([.7,.8],[.4,.9],[.5,1])) \\ (u_2, ([.1,.6],[.5,.7],[.1,.2]),([.3,.4],[.2,.5],[.2,.5]),([.5,.9],[.7,.8],[.4,.6]))\end{array}\right\}\right), \\ \left(e_2, \left\{\begin{array}{l}(u_1, ([.2,.7],[.3,.5],[.2,.6]),([.1,.3],[.2,.5],[.2,.7]),([.4,.9],[.4,.7],[.5,.8])) \\ (u_2, ([.1,.6],[.1,.5],[.4,.8]),([.3,.6],[.3,.4],[1,1]),([.5,.9],[.3,.7],[.1,.8]))\end{array}\right\}\right) \end{array}\right\}$$

$$\wp_{\Re} \wedge \wp_{\mathcal{L}} = \left\{\begin{array}{c} (e_1,e_2),(u_1,([.4,.8],[.3,.6],[.2,.5]),([.2,.5],[.3,.4],[.4,.6]),([.6,.8],[.7,.9],[.8,1]), \\ (u_2,([.1,.4],[.5,.7],[.1,.3]),([.2,.4],[.2,.6],[.2,.5]),([.5,.9],[.7,.9],[.6,.7]), \\ (e_1,e_3),(u_1,([.2,.7],[.3,.5],[.2,.6]),([.1,.3],[.2,.5],[.2,.7]),([.4,.9],[.7,.8],[.8,1]), \\ (u_2,([.1,.4],[.3,.5],[.4,.8]),([.2,.5],[.3,.6],[1,1]),([.5,.9],[.6,.9],[.6,.8]), \\ (e_2,e_2),(u_1,([.3,.6],[.1,.6],[.3,.4]),([0,.2],[.1,.4],[.3,.5]),([.5,.9],[.3,.8],[.5,.8]), \\ (u_2,([.2,.5],[.2,.3],[.5,.6]),([.3,.5],[.1,.5],[.5,.8]),([.6,.9],[.5,.8],[.6,.9])), \\ (e_2,e_3),(u_1,([.2,.6],[.1,.6],[.3,.6]),([0,.2],[.2,.5],[.3,.7]),([.4,.9],[.4,.8],[.5,.8]), \\ (u_2,([.1,.5],[.2,.5],[.5,.8]),([.3,.5],[.3,.5],[.5,.8]),([.5,.9],[.5,.9],[.6,.9])) \end{array}\right\}.$$

Proposition 3.18 Let $\wp_{\Re}$, $\wp_{\mathcal{L}}$, and $\wp_{\mathcal{H}}$ be three mPIVNSSs over $\mathcal{U}$. Then,

1. $\wp_{\Re} \vee \wp_{\mathcal{L}} = \wp_{\mathcal{L}} \vee \wp_{\Re}$
2. $\wp_{\Re} \wedge \wp_{\mathcal{L}} = \wp_{\mathcal{L}} \wedge \wp_{\Re}$
3. $\wp_{\Re} \vee (\wp_{\mathcal{L}} \vee \wp_{\mathcal{H}}) = (\wp_{\Re} \vee \wp_{\mathcal{L}}) \vee \wp_{\mathcal{H}}$
4. $\wp_{\Re} \wedge (\wp_{\mathcal{L}} \wedge \wp_{\mathcal{H}}) = (\wp_{\Re} \wedge \wp_{\mathcal{L}}) \wedge \wp_{\mathcal{H}}$
5. $(\wp_{\Re} \vee \wp_{\mathcal{L}})^c = \wp^c(\Re) \wedge \wp^c(\mathcal{L})$
6. $(\wp_{\Re} \wedge \wp_{\mathcal{L}})^c = \wp^c(\Re) \vee \wp^c(\mathcal{L})$

Proof We can prove easily by using definitions 3.15, 3.16.

4. Weighted Aggregation Operator for m-Polar Interval Valued Neutrosophic Soft set

Many mathematicians developed various methodologies to solve MCDM problems in the past few years, such as aggregation operators for different hybrid structures, CC, similarity measures, and decision-making applications.

Definition 4.1 Let $\wp_{\mathfrak{R}} = \langle [\boldsymbol{u}_\alpha^{\ell}(\boldsymbol{u}), \boldsymbol{u}_\alpha^{\mathrm{u}}(\boldsymbol{u})], [\boldsymbol{v}_\alpha^{\ell}(\boldsymbol{u}), \boldsymbol{v}_\alpha^{\mathrm{u}}(\boldsymbol{u})], [\boldsymbol{w}_\alpha^{\ell}(\boldsymbol{u}), \boldsymbol{w}_\alpha^{\mathrm{u}}(\boldsymbol{u})] \rangle$, $\wp_{\mathfrak{R}_1} = \langle [\boldsymbol{u}_\alpha^{\ell\mathfrak{R}_1}(\boldsymbol{u}), \boldsymbol{u}_\alpha^{\mathrm{u}\mathfrak{R}_1}(\boldsymbol{u})], [\boldsymbol{v}_\alpha^{\ell\mathfrak{R}_1}(\boldsymbol{u}), \boldsymbol{v}_\alpha^{\mathrm{u}\mathfrak{R}_1}(\boldsymbol{u})], [\boldsymbol{w}_\alpha^{\ell\mathfrak{R}_1}(\boldsymbol{u}), \boldsymbol{w}_\alpha^{\mathrm{u}\mathfrak{R}_1}(\boldsymbol{u})] \rangle$, and $\wp_{\mathfrak{R}_2} = \langle [\boldsymbol{u}_\alpha^{\ell\mathfrak{R}_2}(\boldsymbol{u}), \boldsymbol{u}_\alpha^{\mathrm{u}\mathfrak{R}_2}(\boldsymbol{u})], [\boldsymbol{v}_\alpha^{\ell\mathfrak{R}_2}(\boldsymbol{u}), \boldsymbol{v}_\alpha^{\mathrm{u}\mathfrak{R}_2}(\boldsymbol{u})], [\boldsymbol{w}_\alpha^{\ell\mathfrak{R}_2}(\boldsymbol{u}), \boldsymbol{w}_\alpha^{\mathrm{u}\mathfrak{R}_2}(\boldsymbol{u})] \rangle$ are three mPIVNSNs, the basic operators for mPIVNSNs are defined as when $\boldsymbol{\delta} > 0$

1. $\wp_{\mathfrak{R}_1} \oplus \wp_{\mathfrak{R}_2} =$

$$\left\langle \begin{array}{c} \left[u_\alpha^{\ell\mathfrak{R}_1}(u) + u_\alpha^{\ell\mathfrak{R}_2}(u) - u_\alpha^{\ell\mathfrak{R}_1}(u)u_\alpha^{\ell\mathfrak{R}_2}(u), u_\alpha^{\mathrm{u}\mathfrak{R}_1}(u) + u_\alpha^{\mathrm{u}\mathfrak{R}_2}(u) - u_\alpha^{\mathrm{u}\mathfrak{R}_1}(u)u_\alpha^{\mathrm{u}\mathfrak{R}_2}(u)\right], \\ \left[v_\alpha^{\ell\mathfrak{R}_1}(u)v_\alpha^{\ell\mathfrak{R}_2}(u), v_\alpha^{\mathrm{u}\mathfrak{R}_1}(u)\, v_\alpha^{\mathrm{u}\mathfrak{R}_2}(u)\right], \left[w_\alpha^{\ell\mathfrak{R}_1}(u)w_\alpha^{\ell\mathfrak{R}_2}(u), w_\alpha^{\mathrm{u}\mathfrak{R}_1}(u)\, w_\alpha^{\mathrm{u}\mathfrak{R}_2}(u)\right] \end{array} \right\rangle$$

2. $\wp_{\mathfrak{R}_1} \otimes \wp_{\mathfrak{R}_2} =$

$$\left\langle \begin{array}{c} \left[u_\alpha^{\ell\mathfrak{R}_1}(u)u_\alpha^{\ell\mathfrak{R}_2}(u), u_\alpha^{\mathrm{u}\mathfrak{R}_1}(u)u_\alpha^{\mathrm{u}\mathfrak{R}_2}(u)\right], \\ \left[v_\alpha^{\ell\mathfrak{R}_1}(u) + v_\alpha^{\ell\mathfrak{R}_2}(u) - v_\alpha^{\ell\mathfrak{R}_1}(u)v_\alpha^{\ell\mathfrak{R}_2}(u), v_\alpha^{\mathrm{u}\mathfrak{R}_1}(u) + v_\alpha^{\mathrm{u}\mathfrak{R}_2}(u) - v_\alpha^{\mathrm{u}\mathfrak{R}_1}(u)v_\alpha^{\mathrm{u}\mathfrak{R}_2}(u)\right], \\ \left[w_\alpha^{\ell\mathfrak{R}_1}(u) + w_\alpha^{\ell\mathfrak{R}_2}(u) - w_\alpha^{\ell\mathfrak{R}_1}(u)w_\alpha^{\ell\mathfrak{R}_2}(u), w_\alpha^{\mathrm{u}\mathfrak{R}_1}(u) + w_\alpha^{\mathrm{u}\mathfrak{R}_2}(u) - w_\alpha^{\mathrm{u}\mathfrak{R}_1}(u)w_\alpha^{\mathrm{u}\mathfrak{R}_2}(u)\right] \end{array} \right\rangle$$

3. $\delta\wp_{\mathfrak{R}} =$

$$\left\langle \left[1 - \left(1 - u_\alpha^{\ell\mathfrak{R}}(u)\right)^\delta, 1 - \left(1 - u_\alpha^{\mathrm{u}\mathfrak{R}}(u)\right)^\delta\right], \left[\left(v_\alpha^{\ell\mathfrak{R}}(u)\right)^\delta, \left(v_\alpha^{\mathrm{u}\mathfrak{R}}(u)\right)^\delta\right], \left[\left(w_\alpha^{\ell\mathfrak{R}}(u)\right)^\delta, \left(w_\alpha^{\mathrm{u}\mathfrak{R}}(u)\right)^\delta\right] \right\rangle$$

4. $(\wp_{\mathfrak{R}})^\delta = \left\langle \begin{array}{c} \left[\left(\boldsymbol{u}_\alpha^{\ell}(\boldsymbol{u})\right)^{\boldsymbol{\delta}}, \left(\boldsymbol{u}_\alpha^{\mathrm{u}}(\boldsymbol{u})\right)^{\boldsymbol{\delta}}\right], \left[\boldsymbol{1} - \left(\boldsymbol{1} - \boldsymbol{v}_\alpha^{\ell\mathfrak{R}}(\boldsymbol{u})\right)^{\boldsymbol{\delta}}, \boldsymbol{1} - \left(\boldsymbol{1} - \boldsymbol{v}_\alpha^{\mathrm{u}\mathfrak{R}}(\boldsymbol{u})\right)^{\boldsymbol{\delta}}\right], \\ \left[\boldsymbol{1} - \left(\boldsymbol{1} - \boldsymbol{w}_\alpha^{\ell\mathfrak{R}}(\boldsymbol{u})\right)^{\boldsymbol{\delta}}, \boldsymbol{1} - \left(\boldsymbol{1} - \boldsymbol{w}_\alpha^{\mathrm{u}\mathfrak{R}}(\boldsymbol{u})\right)^{\boldsymbol{\delta}}\right] \end{array} \right\rangle$

Definition 4.3 Let $\wp_{\mathfrak{R}_{e_{ij}}}$ = " $\left\langle \left[u_{\alpha_{ij}}^{\ell\mathfrak{R}}(u), u_{\alpha_{ij}}^{\mathrm{u}\mathfrak{R}}(u)\right], \left[v_{\alpha_{ij}}^{\ell\mathfrak{R}}(u), v_{\alpha_{ij}}^{\mathrm{u}\mathfrak{R}}(u)\right], \left[w_{\alpha_{ij}}^{\ell\mathfrak{R}}(u), w_{\alpha_{ij}}^{\mathrm{u}\mathfrak{R}}(u)\right] \right\rangle$ be a collection of mPIVNSNs, Ω_i and γ_j are weight vector for expert's and parameters respectively with given conditions $\Omega_i > 0$, $\sum_{i=1}^{n} \Omega_i = 1$, $\gamma_j > 0$, $\sum_{j=1}^{m} \gamma_j = 1$, where $(i = 1, 2, \dots, n, and\ j = 1, 2, \dots, m)$. Then mPIVNSWA operator defined as
mPIVNSWA: $\Delta^n \to \Delta$ defined as follows

$$mPIVNSWA\left(\wp_{\mathfrak{R}_{e_{11}}}, \wp_{\mathfrak{R}_{e_{12}}}, \dots, \wp_{\mathfrak{R}_{e_{nk}}}\right) = \oplus_{j=1}^{k} \gamma_j \left(\oplus_{i=1}^{n} \Omega_i \wp_{\mathfrak{R}_{e_{ij}}}\right). \quad (4.1)$$

Theorem 4.4 Let $\wp_{\mathfrak{R}_{e_{ij}}} = \left\langle \left[u_{\alpha_{ij}}^{\ell\mathfrak{R}}(u), u_{\alpha_{ij}}^{\mathrm{u}\mathfrak{R}}(u)\right], \left[v_{\alpha_{ij}}^{\ell\mathfrak{R}}(u), v_{\alpha_{ij}}^{\mathrm{u}\mathfrak{R}}(u)\right], \left[w_{\alpha_{ij}}^{\ell\mathfrak{R}}(u), w_{\alpha_{ij}}^{\mathrm{u}\mathfrak{R}}(u)\right] \right\rangle$ be a collection of mPIVNSNs, where $(i = 1, 2, \dots, n, and\ j = 1, 2, \dots, k)$, the aggregated value is also an interval-valued neutrosophic soft number, such as

$$mPIVNSWA\left(\wp_{\mathfrak{R}_{e_{11}}}, \wp_{\mathfrak{R}_{e_{12}}}, \dots, \wp_{\mathfrak{R}_{e_{nk}}}\right)$$

$$= \left\langle \begin{array}{c} \left[\boldsymbol{1} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\boldsymbol{1} - \boldsymbol{u}_{\alpha_{ij}}^{\ell\mathfrak{R}}(\boldsymbol{u})\right)^{\Omega_i}\right)^{\gamma_j}, \boldsymbol{1} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\boldsymbol{1} - \boldsymbol{u}_{\alpha_{ij}}^{\mathrm{u}\mathfrak{R}}(\boldsymbol{u})\right)^{\Omega_i}\right)^{\gamma_j}\right], \\ \left[\boldsymbol{1} - \left(\boldsymbol{1} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\boldsymbol{1} - \boldsymbol{v}_{\alpha_{ij}}^{\ell\mathfrak{R}}(\boldsymbol{u})\right)^{\Omega_i}\right)^{\gamma_j}\right), \boldsymbol{1} - \left(\boldsymbol{1} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\boldsymbol{1} - \boldsymbol{v}_{\alpha_{ij}}^{\mathrm{u}\mathfrak{R}}(\boldsymbol{u})\right)^{\Omega_i}\right)^{\gamma_j}\right)\right], \\ \left[\boldsymbol{1} - \left(\boldsymbol{1} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\boldsymbol{1} - \boldsymbol{w}_{\alpha_{ij}}^{\ell\mathfrak{R}}(\boldsymbol{u})\right)^{\Omega_i}\right)^{\gamma_j}\right), \boldsymbol{1} - \left(\boldsymbol{1} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\boldsymbol{1} - \boldsymbol{w}_{\alpha_{ij}}^{\mathrm{u}\mathfrak{R}}(\boldsymbol{u})\right)^{\Omega_i}\right)^{\gamma_j}\right)\right] \end{array} \right\rangle. \quad (4.2)$$

Definition 4.5 Let $\wp_{\mathfrak{R}} = \langle [u_\alpha^{\ell}(u), u_\alpha^{\mathrm{u}}(u)], [v_\alpha^{\ell}(u), v_\alpha^{\mathrm{u}}(u)], [w_\alpha^{\ell}(u), w_\alpha^{\mathrm{u}}(u)] \rangle$ be an mPIVNSN, then the score, accuracy, and certainty functions for an mPIVNSN respectively defined as follows:

1. $\mathbb{S}(\wp_{\mathfrak{R}}) = \frac{1}{6m}\left(u_\alpha^{\ell}(u) + u_\alpha^{\mathrm{u}}(u) + 1 - v_\alpha^{\ell}(u) + 1 - v_\alpha^{\mathrm{u}}(u) + 1 - w_\alpha^{\ell}(u) + 1 - w_\alpha^{\mathrm{u}}(u)\right)$

2. $\mathbb{A}(\wp_{\mathfrak{R}}) = \frac{1}{4m}\left(4 + u_\alpha^{\ell}(u) + u_\alpha^{\mathrm{u}}(u) - w_\alpha^{\ell}(u) - w_\alpha^{\mathrm{u}}(u)\right)$

3. $\mathbb{C}(\wp_{\mathfrak{R}}) = \frac{1}{2m}\left(2 + u_\alpha^{\ell}(u) + u_\alpha^{\mathrm{u}}(u)\right)$, where $\alpha = 1, 2, \cdots, m$.

Definition 4.6 Let $\wp_{\Re}$ and $\wp_{\Re_1}$ be two mPIVNSSs. Then, the comparison approach is presented as follows:

1. If $\mathbb{S}(\wp_{\Re}) > \mathbb{S}(\wp_{\Re_1})$, then $\wp_{\Re}$ is superior to $\wp_{\Re_1}$.
2. If $\mathbb{S}(\wp_{\Re}) = \mathbb{S}(\wp_{\Re_1})$ and $\mathbb{A}(\wp_{\Re}) > \mathbb{A}(\wp_{\Re_1})$, then $\wp_{\Re}$ is superior to $\wp_{\Re_1}$.
3. If $\mathbb{S}(\wp_{\Re}) = \mathbb{S}(\wp_{\Re_1})$, $\mathbb{A}(\wp_{\Re}) = \mathbb{A}(\wp_{\Re_1})$, and $\mathbb{C}(\wp_{\Re}) > \mathbb{C}(\wp_{\Re_1})$, then $\wp_{\Re}$ is superior to $\wp_{\Re_1}$.
4. If $\mathbb{S}(\wp_{\Re}) = \mathbb{S}(\wp_{\Re_1})$, $\mathbb{A}(\wp_{\Re}) > \mathbb{A}(\wp_{\Re_1})$, and $\mathbb{C}(\wp_{\Re}) = \mathbb{C}(\wp_{\Re_1})$, then $\wp_{\Re}$ is indifferent to $\wp_{\Re_1}$, can be denoted as $\wp_{\Re} \sim \wp_{\Re_1}$.

5. Decision-making approach based mPIVNSWA for mPIVNSS

Assume a set of "s" alternatives such as $\beta = \{\beta^1, \beta^2, \beta^3, \dots, \beta^s\}$ for assessment under the team of experts such as $\mathcal{U} = \{\mathcal{u}_1, \mathcal{u}_2, \mathcal{u}_3, \dots, \mathcal{u}_n\}$ with weights $\Omega = (\Omega_1, \Omega_1, \dots, \Omega_n)^T$ such that $\Omega_i > 0$, $\sum_{i=1}^{n} \Omega_i = 1$. Let $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$ be a set of attributes with weights $\gamma = (\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_m)^T$ be a weight vector for parameters such as $\gamma_j > 0$, $\sum_{j=1}^{m} \gamma_j = 1$. The team of experts $\{\mathcal{u}_i$: i = 1, 2,…, $n\}$ evaluate the alternatives $\{\beta^{(z)}$: z = 1, 2, …, $s\}$ under the considered parameters $\{e_j$: j = 1, 2, …, $m\}$ given in the form of mPIVNSNs $\mathcal{L}_{ij}^{(z)} = \left(\mathcal{u}_{\alpha_{ij}}^{(z)}, \mathcal{v}_{\alpha_{ij}}^{(z)}, \mathcal{w}_{\alpha_{ij}}^{(z)}\right)$, where $\mathcal{u}_{\alpha_{ij}}^{(z)} = \left[\mathcal{u}_{\alpha_{ij}}^{\ell}(u), \mathcal{u}_{\alpha_{ij}}^{\mathrm{u}}(u)\right]$, $\mathcal{v}_{\alpha_{ij}}^{(z)} = \left[\mathcal{v}_{\alpha_{ij}}^{\ell}(u), \mathcal{v}_{\alpha_{ij}}^{\mathrm{u}}(u)\right]$, and $\mathcal{w}_{\alpha_{ij}}^{(z)} = \left[\mathcal{w}_{\alpha_{ij}}^{\ell}(u), \mathcal{w}_{\alpha_{ij}}^{\mathrm{u}}(u)\right]$, here $0 \leq \mathcal{u}_{\alpha}^{\ell}(u), \mathcal{u}_{\alpha}^{\mathrm{u}}(u)$, $\mathcal{v}_{\alpha}^{\ell}(u), \mathcal{v}_{\alpha}^{\mathrm{u}}(u)$, $\mathcal{w}_{\alpha}^{\ell}(u), \mathcal{w}_{\alpha}^{\mathrm{u}}(u) \leq 1$ and $0 \leq \mathcal{u}_{\alpha_{ij}}^{\mathrm{u}}(u) + \mathcal{v}_{\alpha_{ij}}^{\mathrm{u}}(u) + \mathcal{w}_{\alpha_{ij}}^{\mathrm{u}}(u) \leq 3$. So $\Delta_k = \left(\left[\mathcal{u}_{\alpha_{ij}}^{\ell}(u), \mathcal{u}_{\alpha_{ij}}^{\mathrm{u}}(u)\right], \left[\mathcal{v}_{\alpha_{ij}}^{\ell}(u), \mathcal{v}_{\alpha_{ij}}^{\mathrm{u}}(u)\right], \left[\mathcal{w}_{\alpha_{ij}}^{\ell}(u), \mathcal{w}_{\alpha_{ij}}^{\mathrm{u}}(u)\right]\right)$ for all i, j. Experts give their preferences for each alternative in terms of mPIVNSNs by using the mPIVNSWA operator in the form of $\Delta_k = \left(\left[\mathcal{u}_{\alpha_{ij}}^{\ell}(u), \mathcal{u}_{\alpha_{ij}}^{\mathrm{u}}(u)\right], \left[\mathcal{v}_{\alpha_{ij}}^{\ell}(u), \mathcal{v}_{\alpha_{ij}}^{\mathrm{u}}(u)\right], \left[\mathcal{w}_{\alpha_{ij}}^{\ell}(u), \mathcal{w}_{\alpha_{ij}}^{\mathrm{u}}(u)\right]\right)$. Compute the score values for each alternative and analyze the ranking of the alternatives.

5.1 Algorithm for mPIVNSWA operator

Step 1. Develop the m-polar interval-valued neutrosophic soft matrix for each alternative.

Step 2. Aggregate the mPIVNSNs for each alternative into a collective decision matrix Δ_k by using the mPIVNSWA operator.

Step 3. Compute the score value for each alternative Δ_k, where $k = 1,2,\cdots,s$."

Step 4. Choose the best alternative $\beta^{(k)}$.

Step 5. Alternatives ranking.

A flow chart of the above-presented model is given in the following Figure 1.

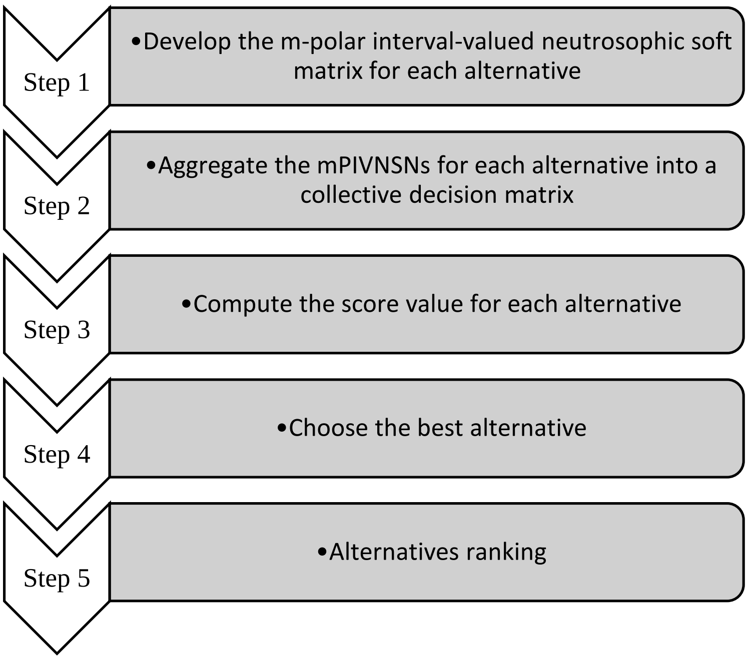

Figure 1: Flowchart of the proposed model

5.2. Application of the Proposed Model in Decision Making

This section utilized the developed approach based on the mPIVNSWA operator for decision-making.

5.2.1. Numerical Example

A university calls for the appointment of a vacant position of associate professor. For further assessment, four candidates (alternatives) chooses after preliminary review such as $\{\beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \beta^{(4)}\}$. The president of the institution {has hired a team of three experts$u_1, u_2, u_3\}$ with weights $(0.25, 0.30, 0.45)^T$ for final scrutiny. First of all, the group of experts decides the parameters for the selection of the candidate, such as e_1 = experience, e_2 = publications, and e_3 = research quality with weights $(0.35, 0.25, 0.40)^T$. Each expert gives preferences for each alternative in mPIVNSNs under the considered parameters. The developed methods to find the best alternative for the position of associate professor are presented in 5.1.

5.2.2. Applications of proposed approaches.

Assume $\{\beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \beta^{(4)}\}$ be a set of alternatives which are shortlisted for interview and $\mathcal{E}$ = $\{e_1$ = experience, e_2 = publications, e_3= research quality} be a set of parameters for the selection of associate professor. Let $\Re$ and $\mathcal{L} \subseteq \mathcal{E}$. Then we construct the 3-PIVNSS $\wp_{\Re}(e)$ according to the requirement of university management such as follows:

Step 1. The experts will evaluate the condition in the case of mPIVNSNs. There are just four alternatives; parameters and a summary of their scores given in Tables 2, 3, 4, 5.

Table 1. Construction of 3-PIVNSS of Alternatives According to Management Requirement

$\wp_{\Re}(e)$	e_1	e_2	e_3

u_1	([.3,.5],[.2,.4],[.2,.6]), ([.2,.3],[.5,.7],[.1,.3]), ([.5,.6],[.1,.3],[.4,.6])	([.2,.4],[.3,.5],[.3,.6]), ([.2,.3],[.2,.4],[.4,.5]), ([.4,.6],[.1,.3],[.2,.4])	([.6,.7],[.2,.3],[.3,.4]), ([.4,.5],[.5,.8],[.1,.2]), ([.1,.2],[.5,.8],[.2,.4])
u_2	([.5,.7],[.1,.2],[.4,.6]), ([.2,.4],[.3,.4],[.2,.5]) ([.6,.8],[.1,.2],[.3,.5])	([.5,.6],[.2,.3],[.3,.4]), ([.4,.6],[.4,.5],[.3,.5]), ([.3,.5],[.4,.5],[.1,.3])	([.5,.7],[.1,.2],[.5,.6]), ([.2,.4],[.5,.6],[.4,.6]), ([.2,.4],[.3,.4],[.2,.5])
u_3	([.4 .6],[.2,.3],[.1,.4]), ([.2,.5],[.2,.3],[.1,.6]), ([.3,.4],[.2,.5],[.5,.7])	([.3,.5],[.4,.5],[.1,.3]), ([.2,.4],[.7,.8],[.1,.2]), ([.1,.2],[.7,.8],[.2,.3])	([.2,.3],[.5,.7],[.1,.3]), ([.3,.4],[.2,.5],[.5,.7]), ([.2,.4],[.3,.5],[.3,.6])

Construct the 3-PIVNSS $\wp_{\mathcal{L}}^{(t)}(e)$ for each alternative according to experts, where t = 1, 2, 3, 4.

Table 2. Evaluation Report for Alternative $\beta^{(1)}$

$\wp_{\mathcal{L}}^{(1)}(e)$	e_1	e_2	e_3
u_1	([.2,.4],[.4,.5],[.3,.4]), ([.6,.7],[.1,.2],[.2,.3]), ([.3,.4],[.4,.5],[.2,.4])	([.3,.4],[.4,.5],[.2,.5]), ([.3,.6],[.2,.3],[.1,.2]), ([.4,.6],[.2,.3],[.4,.5])	([.2,.4],[.4,.6],[.1,.2]), ([.1,.3],[.6,.7],[.2,.3]), ([.4,.5],[.2,.5],[.2,.3])
u_2	([.5,.7],[.1,.2],[.2,.4]), ([.7,.8],[.1,.2],[.2,.4]) ([.1,.3],[.1,.5],[.2,.5])	([.1,.4],[.2,.4],[.1,.2]), ([.2,.5],[.2,.4],[.3,.5]), ([.3,.5],[.2,.4],[.4,.6])	([.5,.7],[.1,.2],[.5,.6]), ([.3,.5],[.3,.4],[.6,.7]), ([.2,.4],[.3,.4],[.2,.5])
u_3	([.4,.5],[.2,.5],[.1,.2]), ([.4,.7],[.1,.2],[.1,.2]), ([.3,.4],[.2,.5],[.5,.7])	([.6,.8],[.1,.2],[.1,.5]), ([.2,.4],[.7,.8],[.1,.2]), ([.5,.7],[.1,.2],[.2,.4])	([.5,.6],[.2,.3],[.4,.5]), ([.3,.4],[.4,.5],[.2,.4]), ([.2,.4],[.3,.5],[.3,.6])

Table 3. Evaluation Report for Alternative $\beta^{(2)}$

$\wp_{\mathcal{L}}^{(2)}(e)$	e_1	e_2	e_3
u_1	([.2,.4],[.4,.6],[.4,.5]), ([.2,.3],[.4,.6],[.3,.5]), ([.1,.2],[.6,.8],[.2,.5])	([.4,.5],[.2,.5],[.1,.2]), ([.2,.3],[.4,.6],[.3,.5]), ([.1,.2],[.6,.8],[.2,.5])	([.7,.8],[.1,.2],[.2,.3]), ([.1,.3],[.6,.7],[.2,.5]), ([.4,.5],[.2,.5],[.1,.2])
u_2	([.5,.7],[.1,.2],[.2,.4]), ([.1,.3],[.6,.7],[.2,.6]) ([.1,.4],[.2,.5],[.4,.6])	([.1,.4],[.2,.4],[.1,.2]), ([.1,.2],[.2,.5],[.4,.6]), ([.1,.4],[.2,.5],[.4,.6])	([.1,.4],[.2,.5],[.4,.6]), ([.3,.4],[.2,.6],[.4,.6]), ([.2,.4],[.3,.4],[.2,.5])
u_3	([.4,.5],[.2,.5],[.1,.2]), ([.1,.2],[.2,.5],[.4,.6]), ([.3,.5],[.3,.5],[.6,.7])	([.3,.5],[.3,.5],[.6,.7]), ([.1,.2],[.2,.5],[.4,.6]), ([.5,.7],[.1,.2],[.2,.4])	([.2,.4],[.4,.5],[.6,.8]), ([.3,.5],[.3,.5],[.6,.7]), ([.1,.2],[.2,.5],[.4,.6])

Table 4. Evaluation Report for Alternative $\beta^{(3)}$

$\wp_{\mathcal{L}}^{(3)}(e)$	e_1	e_2	e_3

u_1	([.6,.7], [.1,.2], [.3,.5]), ([.6,.8], [.1,.2], [.2,.3]), ([.6,.7], [.3,.5], [.1,.2])	([.7,.8], [.1,.2], [.2,.5]), ([.6,.7], [.1,.2], [.1,.2]), ([.5,.8], [.1,.2], [.2,.4])	([.1,.3], [.6,.7], [.2,.5]), ([.7,.8], [.1,.2], [.2,.3]), ([.5,.7], [.3,.4], [.2,.3])
u_2	([.5,.7], [.2,.5], [.2,.3]), ([.7,.8], [.3,.5], [.1,.3]) ([.4,.7], [.2,.3], [.3,.7])	([.5,.6], [.3,.4], [.1,.2]), ([.1,.2], [.2,.5], [.4,.6]), ([.4,.6], [.2,.3], [.1,.2])	([.1,.4], [.2,.5], [.4,.6]), ([.4,.6], [.2,.3], [.1,.2]]), ([.2,.4], [.3,.4], [.2,.5])
u_3	([.4,.6], [.2,.3], [.1,.2]), ([.1,.2], [.2,.5], [.4,.6]), ([.6,.8], [.1,.2], [.1,.3])	([.3,.5], [.3,.5], [.6,.7]), ([.6,.8], [.1,.2], [.1,.2]), ([.7,.8], [.1,.2], [.2,.4])	([.6,.8], [.3,.4], [.1,.2]), ([.5,.7], [.1,.2], [.4,.5]), ([.1,.2], [.2,.5], [.4,.6])

Table 5. Evaluation Report for Alternative $\beta^{(4)}$

$\wp_{\mathcal{L}}^{(4)}(e)$	e_1	e_2	e_3
u_1	([.3,.5], [.2,.4], [.1,.2]), ([.3,.6], [.1,.2], [.4,.7]), ([.4,.7], [.3,.4], [.2,.3])	([.7,.8], [.2,.4], [.3,.5]), ([.5,.7], [.3,.4], [.2,.4]), ([.4,.6], [.2,.5], [.3,.4])	([.2,.3], [.5,.7], [.2,.4]), ([.5,.7], [.2,.4], [.3,.5]), ([.4,.5], [.5,.7], [.2,.4])
u_2	([.4,.7], [.3,.5], [.2,.4]), ([.5,.8], [.3,.6], [.2,.3]) ([.4,.6], [.2,.3], [.3,.5])	([.5,.8], [.3,.4], [.2,.3]), ([.2,.4], [.2,.3], [.4,.5]), ([.3,.5], [.2,.3], [.3,.5])	([.2,.4], [.2,.3], [.3,.6]), ([.4,.6], [.2,.3], [.1,.2]]), ([.2,.4], [.3,.4], [.2,.5])
u_3	([.3,.5], [.3,.5], [.1,.2]), ([.1,.2], [.2,.5], [.4,.6]), ([.5,.7], [.2,.4], [.1,.3])	([.3,.5], [.4,.6], [.6,.7]), ([.5,.7], [.1,.2], [.4,.5]), ([.3,.5], [.2,.5], [.1,.3])	([.4,.6], [.3,.5], [.1,.2]), ([.6,.7], [.1,.2], [.3,.5]), ([.2,.5], [.2,.3], [.4,.6])

Step 2. The opinion of the experts for each alternative are aggregated by using equation 4.2. Hence, we get $\Delta_1 = \langle[.3144\ .5379], [.1819, .3711], [.2437, .3752]\rangle$, $\Delta_2 = \langle[.4569\ .6073], [.2813, .3947], [.2988, .4815]\rangle$, $\Delta_3 = \langle[.3303\ .4884], [.3018, .4429], [.4296, .5670]\rangle$, and $\Delta_4 = \langle[.3530\ .5200], [.2815, .4420], [.3546, .5037]\rangle$.

Step 3. Compute the score values for each alternative by using Definition 4.5 (1). $\mathbb{S}(\Delta_1) = .2045$, $\mathbb{S}(\Delta_2) = .2004$, $\mathbb{S}(\Delta_3) = .1709$, and $\mathbb{S}(\Delta_4) = .1828$.

Step 4. Therefore, the ranking of the alternatives is as follows $\mathbb{S}(\Delta_1) > \mathbb{S}(\Delta_2) > \mathbb{S}(\Delta_4) > \mathbb{S}(\Delta_3)$. So, $\beta^{(1)} > \beta^{(2)} > \beta^{(4)} > \beta^{(3)}$, hence, the alternative $\beta^{(1)}$ is the most suitable alternative for the position of associate professor.

6. Discussion and Comparative Analysis:

In the next section, we will discuss the proposed method's effectiveness, simplicity, flexibility, and good location. A brief comparative analysis of our proposed method and popular method.

6.1 Comparative Studies

This manuscript develops a new DM technology based on the mPIVNSWA operator using mPIVNSS. Compared with existing technologies, the developed method is more operative and provides appropriate results in MCDM problems. Through this scientific research and comparison, we realize that the results of the proposed method are more versatile than traditional methods. However, the DM process contains more information to deal with uncertain data than the current

DM method. Except that the hybrid structure of multiple FS becomes a particular case of mPIVNSS adds some appropriate conditions. Among them, the information related to the object can be displayed accurately and analytically, so mPIVNSS is an effective power tool to deal with inaccurate and uncertain information in the DM process. Therefore, our method is more suitable, flexible, and better than FS's unique and accessible hybrid structure.

Table 6: Comparative analysis between some existing techniques and the proposed approach

	Set	Truthiness	Indeterminacy	Falsity	Multi-polarity	Loss of information
Chen et al. [33]	mPFS	✓	×	×	✓	×
Xu et al. [38]	IFS	✓	×	✓	×	×
Zhang et al. [39]	IFS	✓	×	✓	×	✓
Talebi et al. [42]	mPIVIFS	✓	×	✓	✓	✓
Yager [40, 41]	PFS	✓	×	✓	×	×
Naeem et al. [43]	mPyFS	✓	×	✓	✓	×
Zhang et al. [44]	INSs	✓	✓	✓	×	×
Ali et al. [35]	BPNSS	✓	✓	✓	×	×
Proposed approach	mPIVNSS	✓	✓	✓	✓	×

7. Conclusion

This manuscript establishes a new hybrid structure, mPIVNSS, by combining two independent structure m-pole fuzzy sets and interval-valued neutrosophic soft sets. Several basic operations have been introduced for mPIVNSS, and their ideal characteristics have been discussed. In addition, we developed the algorithm of mPIVNSS and used the proposed algorithm to establish a neutrosophic weighted aggregation operator for m-polar interval-valued. A decision-making method was developed to solve the MCDM problem by using our mPIVNSWA operator. A comparative analysis was also carried out to prove the proposed method. Finally, the proposed technique shows higher stability and practicality for decision-makers in the decision-making process. Based on the results obtained, it can be concluded that this method is most suitable for solving the MCDM problem in today's life. We will apply this technique to other fields in future work, such as mathematical programming, cluster analysis, etc.

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A Decision-Making Approach Based on Correlation Coefficient For Generalized multi-Polar Neutrosophic Soft Set

Aiyared Iampan[1], Rana Muhammad Zulqarnain[2*], Imran Siddique[3], Hamiden Abd El-Wahed Khalifa[4, 5]

[1] Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand. E-mail: aiyared.ia@up.ac.th

[2] Department of Mathematics, University of Management and Technology Lahore, Sialkot Campus, Pakistan. E-mail: ranazulqarnain7777@gmail.com

[3] Department of Mathematics, University of Management and Technology Lahore, Pakistan. E-mail: imransiddique@umt.edu.pk

[4] Department of Mathematics, College of Science and Arts , Al-Badaya, Qassim University E-mail: Ha.Ahmed@qu.edu.sa

[5] Operations Research Department, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt, Email: hamiden@cu.edu.eg

Correspondence: E-mail: ranazulqarnain7777@gmail.com

Abstract:

The paper proposes the generalized version of the multipolar neutrosophic soft set. The neutrosophic soft set (NSS) is an advanced extension of the neutrosophic set, which accommodates the alternatives' parametrized values. This paper extends the NSS to generalized multipolar NSS and introduces some fundamental operations for generalized multipolar NSS with their necessary properties. We define the correlation coefficient (CC) and weighted correlation coefficient (WCC) for the generalized multi-polar neutrosophic soft set. Several desirable properties for CC and WCC and their relationship are derived. Finally, based on established correlation measures, a decision-making algorithm under the neutrosophic environment is stated to tackle uncertain and vague information. The applicability of the proposed algorithm is demonstrated through a case study of the decision-making problem. A comparative analysis with several existing studies reveals the effectiveness of the approach.

Keywords: multipolar neutrosophic set; generalized multipolar neutrosophic soft set; CC; WCC; MCDM.

1. Introduction

Uncertainty plays a dynamic role in many areas of life (such as modeling, medicine, engineering, etc.). However, people have raised a common question: how do we express and use the concept of uncertainty in mathematical modeling. Many researchers planned and endorsed different methods to resolve those difficulties that contain hesitation. First of all, Zadeh presented the idea of fuzzy sets (FS) [1] to resolve uncertain complications. But in some cases, fuzzy sets are unable to handle the situation. To overcome such scenarios, the idea of interval-valued fuzzy sets (IVFS) was presented by Turksen [2]. In some cases, we must consider the object's nonmembership value, which cannot be dealt with by FS nor by IVFS. To conquer such issues, Atanassov planned the intuitionistic fuzzy set

(IFS) [3]. The idea proposed by Atanassov involves only under-considered data and membership and non-membership values. However, the IFS theory cannot handle the overall incompatibility and inaccurate information. To solve the problem of incompatibility and incorrect information, Smarandache [4] proposed the idea of NS. Molodtsov [5] presented a general mathematical tool for addressing uncertain environments known as soft set (SS). Maji et al. [6] extended the concept of SS and proposed fundamental operations with their desirable properties. Maji et al. [7] established a decision-making technique utilizing their developed operations and used it for decision making. Ali et al. [8] extended the notion of SS and developed some new operations with their properties. The authors [9] proved De Morgan's law by using different operators for the SS theory.

Maji [10] offered the idea of a neutrosophic soft set (NSS) with necessary operations and properties. The concept of the possibility NSS was developed by Karaaslan [11]. He also established a DM technique utilizing And-product based on the possibility NSS to solve DM issues. Broumi [12] created the generalized NSS with some operations and properties and used the proposed concept for DM. Deli and Subas [13] extended the notion of single-valued Neutrosophic numbers (SVNNs) and offered a DM approach to solving MCDM problems. They also developed the idea of cut sets for SVNNs. Wang et al. [14] presented the correlation coefficient (CC) for SVNSs and constructed a DM approach utilizing their developed correlation measure. Ye [15] offered the idea of simplified NSs and developed the aggregation operators (AOs) for simplified NSs, and established a DM methodology to solve MCDM problems utilizing his developed AOs. Masooma et al. [16] proposed a multipolar neutrosophic set by combining the multipolar fuzzy set and NS. They also established various characterization and operations with examples. Zulqarnain et al. [17] introduced some AOs and correlation coefficients for interval-valued IFSS. They also extended the TOPSIS technique using their developed correlation measures to solve the MADM problem. Zulqarnain et al. [18] introduced operational laws for Pythagorean fuzzy soft numbers (PFSNs). They developed AOs such as Pythagorean fuzzy soft weighted average and geometric using defined operational laws for PFSNs. They also planned a DM approach to solve MADM problems with the help of presented operators. Zulqarnain et al. [19] planned the TOPSIS methodology in the PFSS environment based on the correlation coefficient. They also established a DM methodology to resolve the MCGDM concerns and utilized the developed approach in green supply chain management.

Many mathematicians have developed various similarity measures, correlation coefficients, aggregation operators, and decision-making applications in the past few years. Zulqarnain et al. [20, 21] introduced some novel AOs for PFSS based on Einstein norms. Siddique et al. [22] proposed the score matrix for PFSS. Peng and Garg [23] proposed various PFS similarity measures with multiple parameters. Zulqarnain et al. [24, 25] presented the generalized neutrosophic TOPSIS and an integrated neutrosophic TOPSIS model and used their proposed techniques for supplier selection in the production industry. Saeed et al. [26] established the concept of mPNSS with properties and operators. They also developed a distance-based similarity measure and used the proposed similarity measure for decision-making and medical diagnosis. Zulqarnain et al. [27] developed some novel AOs for PFSS considering the interaction. Zulqarnain et al. [34] presented the generalized multipolar NSS and introduced some information measures to solve decision-making problems. They also extended the concept of multipolar NSS to multipolar interval-valued NSS with basic operations and their desirable properties [35].

In this era, professionals believe that real life is moving toward multi-polarization. Therefore, there is no doubt that the multi-polarization of information has played an essential role in the prosperity of many fields of science and technology. In neurobiology, multipolar neurons accumulate a lot of info from other neurons. The motivation for expanding and mixing this research work is gradually given in the whole manuscript. We prove that under any appropriate circumstances, different hybrid structures containing fuzzy sets will be converted into special privileges of GmPNSS. The concept of the neutrosophic environment of multipolar neutrosophic soft sets is novel. We discuss the effectiveness, flexibility, quality, and advantages of planned work and algorithms. This research will be the most versatile form that can be used to merge data in daily life complications. In the future, current work may be extended to different types of hybrid structures and decision-making techniques in numerous fields of life.

The following research is organized: In section 2, we recollected some basic definitions used in the subsequent sequel, such as NS, SS, NSS, and multipolar neutrosophic set. In section 3, we proposed the generalized version of mPNSS with its operations and introduced the idea of CC and WCC with their properties. Furthermore, a decision-making approach has been established based on developed CC. Finally, we use the developed method for decision-making in section 4. We also presented the comparative study of our proposed similarity measures and CC with existing techniques in section 5.

2. Preliminaries

In this section, we recollect some basic concepts such as the neutrosophic set, soft set, neutrosophic soft set, and m-polar neutrosophic soft set used in the following sequel.

Definition 2.1 [4]

Let $\boldsymbol{\mathcal{U}}$ be a universe, and $\boldsymbol{\mathcal{A}}$ be an NS on $\boldsymbol{\mathcal{U}}$ is defined as $\boldsymbol{\mathcal{A}} = \{\boldsymbol{u}, (\boldsymbol{u}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{v}_{\mathcal{A}}(\boldsymbol{u}), \boldsymbol{w}_{\mathcal{A}}(\boldsymbol{u})) : \boldsymbol{u} \in \boldsymbol{\mathcal{U}}\}$, where $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}: \boldsymbol{\mathcal{U}} \to]\mathbf{0}^-, \mathbf{1}^+[$ and $\mathbf{0}^- \leq \boldsymbol{u}_{\mathcal{A}}(\boldsymbol{u}) + \boldsymbol{v}_{\mathcal{A}}(\boldsymbol{u}) + \boldsymbol{w}_{\mathcal{A}}(\boldsymbol{u}) \leq \mathbf{3}^+$.

Definition 2.2 [5]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a soft set over $\mathcal{U}$, and its mapping is given as

$$\mathcal{F}: \mathcal{A} \to \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F}, \mathcal{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \text{ } if \text{ } e \notin \mathcal{A}\}$$

Definition 2.3 [13]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the Neutrosophic values of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a Neutrosophic soft set over $\mathcal{U}$ and its mapping is given as

$$\mathcal{F}: \mathcal{A} \to \mathcal{P}(\mathcal{U})$$

Definition 2.4 [19]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$, then $\mathcal{F}_{\mathcal{E}}$ is said to multipolar neutrosophic set if

$\mathcal{F}_{\mathcal{E}} = \{u, (s_i \bullet u_e(u),\ s_i \bullet v_e(u), s_i \bullet w_e(u)): u \in \mathcal{U},\ e \in \mathcal{E}, i = 1, 2, 3, \ldots . m\}$, where $s_i \bullet u_{\mathcal{E}}$, $s_i \bullet v_{\mathcal{E}}, s_i \bullet w_{\mathcal{E}}$: $\mathcal{U} \rightarrow [0, 1]$, and $0 \leq s_i \bullet u_{\mathcal{E}}(u) + s_i \bullet v_{\mathcal{E}}(u) + s_i \bullet w_{\mathcal{E}}(u) \leq 3$; $i = 1, 2, 3, \ldots . m$. u_e, v_e, and w_e represent the truth, indeterminacy, and falsity of the considered alternative.

3. Basic Operations and Correlation Coefficient for Generalized Multi-Polar Neutrosophic Soft Set

In this section, we develop the concept of GmPNSS and introduce aggregate operators on GmPNSS with their properties.

Definition 3.1

Let $\boldsymbol{\mathcal{U}}$ and E are universal and set of attributes respectively, and $\boldsymbol{\mathcal{A}} \subseteq$ E, if there exists a mapping Φ such as

$$\Phi\colon \mathcal{A} \rightarrow GmPNSS^{\mathcal{U}}$$

Then (Φ, $\mathcal{A}$) is called GmPNSS over $\mathcal{U}$ defined as follows

$$\Upsilon_K = (\Phi,\ \mathcal{A}) = \left\{\left(e, \left(u, \Phi_{\mathcal{A}(e)}(u)\right)\right): e \in E, u \in \mathcal{U}\right\},$$

where $\Phi_{\mathcal{A}}(e) = \left\{u, \left(s_i \bullet u_{\mathcal{A}(e)}(u), s_i \bullet v_{\mathcal{A}(e)}(u),\ s_i \bullet w_{\mathcal{A}(e)}(u)\right): u \in \mathcal{U}, e \in E;\ i \in 1, 2, 3, \ldots, m\right\}$,

and

$0 \leq s_i \bullet u_{\mathcal{A}(e)}(u) + s_i \bullet v_{\mathcal{A}(e)}(u) + s_i \bullet w_{\mathcal{A}(e)}(u) \leq 3$ for all $i \in 1, 2, 3, \ldots, m$; $e \in E$ and $u \in \mathcal{U}$.

Definition 3.2

Let $\Upsilon_{\mathcal{A}}$ and Υ_B are two GmPNSS over $\mathcal{U}$, then $\Upsilon_{\mathcal{A}}$ is called a multi-polar neutrosophic soft subset of Υ_B. If

$s_i \bullet u_{\mathcal{A}(e)}(u) \leq s_i \bullet u_{B(e)}(u)$, $s_i \bullet v_{\mathcal{A}(e)}(u) \leq s_i \bullet v_{B(e)}(u)$ and $s_i \bullet w_{\mathcal{A}(e)}(u) \geq s_i \bullet w_{B(e)}(u)$

for all $i \in 1, 2, 3, \ldots, m$; $e \in E$ and $u \in \mathcal{U}$.

Definition 3.3

Let $\Upsilon_{\mathcal{A}}$ and Υ_B are two GmPNSS over $\mathcal{U}$, then $\Upsilon_{\mathcal{A}} = \Upsilon_B$, if

$$s_i \bullet u_{\mathcal{A}(e)}(u) \leq s_i \bullet u_{B(e)}(u),\ s_i \bullet u_{B(e)}(u) \leq s_i \bullet u_{\mathcal{A}(e)}(u)$$

$$s_i \bullet v_{\mathcal{A}(e)}(u) \leq s_i \bullet v_{B(e)}(u),\ s_i \bullet v_{B(e)}(u) \leq s_i \bullet v_{\mathcal{A}(e)}(u)$$

$$s_i \bullet w_{\mathcal{A}(e)}(u) \geq s_i \bullet w_{B(e)}(u),\ s_i \bullet w_{B(e)}(u) \geq s_i \bullet w_{\mathcal{A}(e)}(u)$$

for all $i \in 1, 2, 3, \ldots, m$; $e \in E$ and $u \in \mathcal{U}$.

Definition 3.4

Let $\boldsymbol{\mathcal{F}_{\breve{A}}} = \{\boldsymbol{u_k}, (\boldsymbol{s_i \bullet u_{\breve{A}}(u_k), s_i \bullet v_{\breve{A}}(u_k), s_i \bullet w_{\breve{A}}(u_k)}): \boldsymbol{u_k} \in \boldsymbol{\mathcal{U}};\ \boldsymbol{i} \in \boldsymbol{1, 2, 3, \ldots, m}\}$ and $\mathcal{G}_{\breve{B}} = \{u_k, (s_i \bullet u_{\breve{B}}(u_k), s_i \bullet v_{\breve{B}}(u_k), s_i \bullet w_{\breve{B}}(u_k)): u_k \in \mathcal{U};\ i \in 1, 2, 3, \ldots, m\}$ are two GmPNSS over a set of parameters $E = \{x_1,\ x_2,\ x_3,\ \ldots,\ x_n\}$. Then informational neutrosophic energies of two GmPNSS can be expressed as follows

$$\mathcal{E}_{GmPNSS}\ (\mathcal{F}_{\breve{A}}) = \sum_{j=1}^{z} \sum_{k=1}^{t} \left(\left(s_i \bullet u_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \bullet v_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \bullet w_{\breve{A}_j}(u_k)\right)^2\right)$$

$$\mathcal{E}_{GmPNSS}\ (\mathcal{G}_{\breve{B}}) = \sum_{j=1}^{z} \sum_{k=1}^{t} \left(\left(s_i \bullet u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k)\right)^2\right)$$

Definition 3.5

The correlation of two GmPNSS can be presented as follows

$\varsigma_{GmPNSS}(\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}}) =$

$$\sum_{j=1}^{z}\sum_{k=1}^{t}\left\{\begin{pmatrix} s_i \bullet u_{\breve{A}_j}(u_k) s_i \bullet u_{\breve{B}_j}(u_k) + s_i \bullet v_{\breve{A}_j}(u_k) s_i \bullet v_{\breve{B}_j}(u_k) + \\ s_i \bullet w_{\breve{A}_j}(u_k)\, s_i \bullet w_{\breve{B}_j}(u_k) \end{pmatrix} : i \in 1,2,3,\dots,m.\right\} \tag{3.1}$$

Definition 3.6

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$ are two GmPNSS, then the CC between them can be defined as follows

$$\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}}) = \frac{\varsigma_{GmPNSS}(\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}})}{\sqrt{\varepsilon_{GmPNSS}(\mathcal{F}_{\breve{A}},\mathcal{F}_{\breve{A}})\ .\ \varepsilon_{GmPNSS}(\mathcal{G}_{\breve{B}},\mathcal{G}_{\breve{B}})}} \tag{3.2}$$

$$\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}}) = \frac{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(s_i \bullet u_{\breve{A}_j}(u_k) s_i \bullet u_{\breve{B}_j}(u_k) + s_i \bullet v_{\breve{A}_j}(u_k) s_i \bullet v_{\breve{B}_j}(u_k) + s_i \bullet w_{\breve{A}_j}(u_k)\, s_i \bullet w_{\breve{B}_j}(u_k)\right)}{\sqrt{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \bullet u_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \bullet v_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \bullet w_{\breve{A}_j}(u_k)\right)^2\right)}\sqrt{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \bullet u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k)\right)^2\right)}}$$

Proposition 3.7

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$ are two GmPNSS, then the CC $\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}})$ between them satisfied the following properties

1. $0 \leq \mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}}) \leq 1$
2. $\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}}) = \mathcal{R}_{GmPNSS}\ (\mathcal{G}_{\breve{B}},\ \mathcal{F}_{\breve{A}})$
3. If $\mathcal{F}_{\breve{A}} = \mathcal{G}_{\breve{B}}$ i. e; $s_i \bullet u_{\breve{A}_j}(u_k) = s_i \bullet u_{\breve{B}_j}(u_k)$, $s_i \bullet v_{\breve{A}_j}(u_k) = s_i \bullet v_{\breve{B}_j}(u_k)$, and $s_i \bullet w_{\breve{A}_j}(u_k) = s_i \bullet w_{\breve{B}_j}(u_k)$ for all j, k, where $i \in 1,2,3,\dots,m$, then $\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}}) = 1$.

Proof 1

$\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}}) \geq 0$ is trivial, so we just need to prove that $\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}}) \leq 1$.

As we know that

$$\varsigma_{GmPNSS}\ (\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}}) = \sum_{j=1}^{z}\sum_{k=1}^{t}\begin{pmatrix} s_i \bullet u_{\breve{A}_j}(u_k) s_i \bullet u_{\breve{B}_j}(u_k) + s_i \bullet v_{\breve{A}_j}(u_k) s_i \bullet v_{\breve{B}_j}(u_k) + \\ s_i \bullet w_{\breve{A}_j}(u_k)\, s_i \bullet w_{\breve{B}_j}(u_k) \end{pmatrix}$$

$$= \sum_{j=1}^{z}\left(s_i \bullet u_{\breve{A}_j}(u_1) s_i \bullet u_{\breve{B}_j}(u_1) + s_i \bullet v_{\breve{A}_j}(u_1) s_i \bullet v_{\breve{B}_j}(u_1) + s_i \bullet w_{\breve{A}_j}(u_1)\, s_i \bullet w_{\breve{B}_j}(u_1)\right)$$
$$+ \sum_{j=1}^{z}\left(s_i \bullet u_{\breve{A}_j}(u_2) s_i \bullet u_{\breve{B}_j}(u_2) + s_i \bullet v_{\breve{A}_j}(u_2) s_i \bullet v_{\breve{B}_j}(u_2) + s_i \bullet w_{\breve{A}_j}(u_2)\, s_i \bullet w_{\breve{B}_j}(u_2)\right)$$
$$+$$
$$\vdots$$
$$+ \sum_{j=1}^{z}\left(s_i \bullet u_{\breve{A}_j}(u_t) s_i \bullet u_{\breve{B}_j}(u_t) + s_i \bullet v_{\breve{A}_j}(u_t) s_i \bullet v_{\breve{B}_j}(u_t) + s_i \bullet w_{\breve{A}_j}(u_t)\, s_i \bullet w_{\breve{B}_j}(u_t)\right)$$

$$=$$

$$\left\{\begin{matrix} \left(s_i \bullet u_{\breve{A}_1}(u_1) s_i \bullet u_{\breve{B}_1}(u_1) + s_i \bullet v_{\breve{A}_1}(u_1) s_i \bullet v_{\breve{B}_1}(u_1) + s_i \bullet w_{\breve{A}_1}(u_1)\, s_i \bullet w_{\breve{B}_1}(u_1)\right) + \\ \left(s_i \bullet u_{\breve{A}_2}(u_1) s_i \bullet u_{\breve{B}_2}(u_1) + s_i \bullet v_{\breve{A}_2}(u_1) s_i \bullet v_{\breve{B}_2}(u_1) + s_i \bullet w_{\breve{A}_2}(u_1)\, s_i \bullet w_{\breve{B}_2}(u_1)\right) + \cdots + \\ \left(s_i \bullet u_{\breve{A}_z}(u_1) s_i \bullet u_{\breve{B}_z}(u_1) + s_i \bullet v_{\breve{A}_z}(u_1) s_i \bullet v_{\breve{B}_z}(u_1) + s_i \bullet w_{\breve{A}_z}(u_1)\, s_i \bullet w_{\breve{B}_z}(u_1)\right) \end{matrix}\right\} +$$
$$\left\{\begin{matrix} \left(s_i \bullet u_{\breve{A}_1}(u_2) s_i \bullet u_{\breve{B}_1}(u_2) + s_i \bullet v_{\breve{A}_1}(u_2) s_i \bullet v_{\breve{B}_1}(u_2) + s_i \bullet w_{\breve{A}_1}(u_2)\, s_i \bullet w_{\breve{B}_1}(u_2)\right) + \\ \left(s_i \bullet u_{\breve{A}_2}(u_2) s_i \bullet u_{\breve{B}_2}(u_2) + s_i \bullet v_{\breve{A}_2}(u_2) s_i \bullet v_{\breve{B}_2}(u_2) + s_i \bullet w_{\breve{A}_2}(u_2)\, s_i \bullet w_{\breve{B}_2}(u_2)\right) + \cdots + \\ \left(s_i \bullet u_{\breve{A}_z}(u_2) s_i \bullet u_{\breve{B}_z}(u_2) + s_i \bullet v_{\breve{A}_z}(u_2) s_i \bullet v_{\breve{B}_z}(u_2) + s_i \bullet w_{\breve{A}_z}(u_2)\, s_i \bullet w_{\breve{B}_z}(u_2)\right) \end{matrix}\right\} + \cdots +$$
$$\left\{\begin{matrix} \left(s_i \bullet u_{\breve{A}_1}(u_k) s_i \bullet u_{\breve{B}_1}(u_k) + s_i \bullet v_{\breve{A}_1}(u_k) s_i \bullet v_{\breve{B}_1}(u_k) + s_i \bullet w_{\breve{A}_1}(u_k)\, s_i \bullet w_{\breve{B}_1}(u_k)\right) + \\ \left(s_i \bullet u_{\breve{A}_2}(u_k) s_i \bullet u_{\breve{B}_2}(u_k) + s_i \bullet v_{\breve{A}_2}(u_k) s_i \bullet v_{\breve{B}_2}(u_k) + s_i \bullet w_{\breve{A}_2}(u_k)\, s_i \bullet w_{\breve{B}_2}(u_k)\right) + \cdots + \\ \left(s_i \bullet u_{\breve{A}_z}(u_k) s_i \bullet u_{\breve{B}_z}(u_k) + s_i \bullet v_{\breve{A}_z}(u_k) s_i \bullet v_{\breve{B}_z}(u_k) + s_i \bullet w_{\breve{A}_z}(u_k)\, s_i \bullet w_{\breve{B}_z}(u_k)\right) \end{matrix}\right\}$$

$$= \sum_{j=1}^{z}\left(s_i \bullet u_{\breve{A}_j}(u_1) s_i \bullet u_{\breve{B}_j}(u_1) + s_i \bullet u_{\breve{A}_j}(u_2) s_i \bullet u_{\breve{B}_j}(u_2) + \cdots + s_i \bullet u_{\breve{A}_j}(u_t) s_i \bullet u_{\breve{B}_j}(u_t)\right)$$
$$+ \sum_{j=1}^{z}\left(s_i \bullet v_{\breve{A}_j}(u_1) s_i \bullet v_{\breve{B}_j}(u_1) + s_i \bullet v_{\breve{A}_j}(u_2) s_i \bullet v_{\breve{B}_j}(u_2) + \cdots + s_i \bullet v_{\breve{A}_j}(u_t) s_i \bullet v_{\breve{B}_j}(u_t)\right)$$
$$+ \sum_{j=1}^{z}\left(s_i \bullet w_{\breve{A}_j}(u_1) s_i \bullet w_{\breve{B}_j}(u_1) + s_i \bullet w_{\breve{A}_j}(u_2) s_i \bullet w_{\breve{B}_j}(u_2) + \cdots + s_i \bullet w_{\breve{A}_j}(u_t) s_i \bullet w_{\breve{B}_j}(u_t)\right)$$

By using Cauchy-Schwarz inequality, we get

$$(\varsigma_{GmPNSS}\ (\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}))^2 \le \begin{Bmatrix} \sum_{j=1}^{z}\left(\left(s_i \bullet u_{\breve{A}_j}(u_1)\right)^2 + \left(s_i \bullet u_{\breve{A}_j}(u_2)\right)^2 + \cdots + \left(s_i \bullet u_{\breve{A}_j}(u_t)\right)^2\right) + \\ \sum_{j=1}^{z}\left(\left(s_i \bullet v_{\breve{A}_j}(u_1)\right)^2 + \left(s_i \bullet v_{\breve{A}_j}(u_2)\right)^2 + \cdots + \left(s_i \bullet v_{\breve{A}_j}(u_t)\right)^2\right) + \cdots + \\ \sum_{j=1}^{z}\left(\left(s_i \bullet w_{\breve{A}_j}(u_1)\right)^2 + \left(s_i \bullet w_{\breve{A}_j}(u_2)\right)^2 + \cdots + \left(s_i \bullet w_{\breve{A}_j}(u_t)\right)^2\right) \end{Bmatrix} \times$$

$$\begin{Bmatrix} \sum_{j=1}^{z}\left(\left(s_i \bullet u_{\breve{B}_j}(u_1)\right)^2 + \left(s_i \bullet u_{\breve{B}_j}(u_2)\right)^2 + \cdots + \left(s_i \bullet u_{\breve{B}_j}(u_t)\right)^2\right) + \\ \sum_{j=1}^{z}\left(\left(s_i \bullet v_{\breve{B}_j}(u_1)\right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_2)\right)^2 + \cdots + \left(s_i \bullet v_{\breve{B}_j}(u_t)\right)^2\right) + \cdots + \\ \sum_{j=1}^{z}\left(\left(s_i \bullet w_{\breve{B}_j}(u_1)\right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_2)\right)^2 + \cdots + \left(s_i \bullet w_{\breve{B}_j}(u_t)\right)^2\right) \end{Bmatrix}$$

$$= \left\{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \bullet u_{\breve{A}_j}(u_t)\right)^2 + \left(s_i \bullet v_{\breve{A}_j}(u_t)\right)^2 + \left(s_i \bullet w_{\breve{A}_j}(u_t)\right)^2\right)\right\} \times \left\{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \bullet u_{\breve{B}_j}(u_t)\right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_t)\right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_t)\right)^2\right)\right\}$$

$$= \mathcal{E}_{GmPNSS}\ (\mathcal{F}_{\breve{A}}) \cdot \mathcal{E}_{GmPNSS}\ (\mathcal{G}_{\breve{B}})$$

Therefore, $(\varsigma_{GmPNSS}\ (\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}))^2 \le \mathcal{E}_{GmPNSS}\ (\mathcal{F}_{\breve{A}}) \cdot \mathcal{E}_{GmPNSS}\ (\mathcal{G}_{\breve{B}})$. Hence, by using Definition 3.6, we get

$\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) \le 1$, so $0 \le \mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) \le 1$.

Proof 2 The proof is obvious.

Proof 3

As we know that

$$\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \frac{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(s_i \bullet u_{\breve{B}_j}(u_k) s_i \bullet u_{\breve{B}_j}(u_k) + s_i \bullet v_{\breve{B}_j}(u_k) s_i \bullet v_{\breve{B}_j}(u_k) + s_i \bullet w_{\breve{B}_j}(u_k)\ s_i \bullet w_{\breve{B}_j}(u_k)\right)}{\sqrt{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \bullet u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k)\right)^2\right)}\sqrt{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \bullet u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k)\right)^2\right)}}$$

As we know that $s_i \bullet u_{\breve{A}_j}(u_k) = s_i \bullet u_{\breve{B}_j}(u_k)$, $s_i \bullet v_{\breve{A}_j}(u_k) = s_i \bullet v_{\breve{B}_j}(u_k)$, and $s_i \bullet w_{\breve{A}_j}(u_k) = s_i \bullet w_{\breve{B}_j}(u_k)$, for all j, k, so by using Definition 3.6, we have

$$\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \frac{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \bullet u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k)\right)^2\right)}{\sqrt{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \bullet u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k)\right)^2\right)}\sqrt{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \bullet u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k)\right)^2\right)}}$$

Hence

$\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = 1$.

Definition 3.8

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$ are two GmPNSS, then the CC between them also can be defined as follows

$$\mathcal{R}_{GmPNSS}^{1}\ (\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \frac{\varsigma_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})}{max\{\mathcal{E}_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{F}_{\breve{A}}), \mathcal{E}_{GmPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{G}_{\breve{B}})\}} \quad (3.3)$$

$$\mathcal{R}_{GmPNSS}^{1}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \frac{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(s_i \bullet u_{\breve{A}_j}(u_k) s_i \bullet u_{\breve{B}_j}(u_k) + s_i \bullet v_{\breve{A}_j}(u_k) s_i \bullet v_{\breve{B}_j}(u_k) + s_i \bullet w_{\breve{A}_j}(u_k)\ s_i \bullet w_{\breve{B}_j}(u_k)\right)}{max\begin{Bmatrix}\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \bullet u_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \bullet v_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \bullet w_{\breve{A}_j}(u_k)\right)^2\right), \\ \sum_{j=1}^{s}\sum_{k=1}^{t}\left(\left(s_i \bullet u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k)\right)^2\right)\end{Bmatrix}}$$

Proposition 3.9

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$ are two GmPNSS, then the CC $\mathcal{R}_{GmPNSS}^{1}\ (\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})$ between them satisfied the following properties.

1. $0 \le \mathcal{R}_{GmPNSS}^{1}\ (\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) \le 1$
2. $\mathcal{R}_{GmPNSS}^{1}\ (\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \mathcal{R}_{GmPNSS}^{1}\ (\mathcal{G}_{\breve{B}}, \mathcal{F}_{\breve{A}})$

3. If $\mathcal{F}_{\breve{A}} = \mathcal{G}_{\breve{B}}$ i. e; $s_i \bullet u_{\breve{A}_j}(u_k) = s_i \bullet u_{\breve{B}_j}(u_k)$, $s_i \bullet v_{\breve{A}_j}(u_k) = s_i \bullet v_{\breve{B}_j}(u_k)$, and $s_i \bullet w_{\breve{A}_j}(u_k) = s_i \bullet w_{\breve{B}_j}(u_k)$ for all j, k, where $i \in 1, 2, 3, \ldots, m$, then $\mathcal{R}^1_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = 1$.

Proof

We can prove easily according to Definition 3.7.

It is important to anticipate the weight of IVNSS for functional determinations. When a decision-maker alleviates a distinct weight for each object in the universe of discourse, the result of the purpose may be distinctive. So, it is necessary to consider the weights before making a decision. Let $\dot{\omega} = \{\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3, \ldots, \dot{\omega}_m\}$ be a weight vector for experts such as $\dot{\omega}_k > 0$, $\sum_{k=1}^{m} \dot{\omega}_k = 1$ and $\delta = \{\delta_1, \delta_2, \delta_3, \ldots, \delta_n\}$ be a weight vector for parameters such as $\delta_i > 0$, $\sum_{i=1}^{n} \delta_i = 1$. By extending definitions 3.6, 3.8, the notion of WCC has been developed in the following.

Definition 3.10

For two GmPNSS $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$, the WCC between them can be defined as follows

$$\mathcal{R}_{GWmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \frac{\varsigma_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})}{\sqrt{\varepsilon_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{F}_{\breve{A}})\, \varepsilon_{GmPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{G}_{\breve{B}})}} \tag{3.4}$$

$$\mathcal{R}_{GWmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \frac{\sum_{j=1}^{z} \delta_j \left(\sum_{k=1}^{t} \dot{\omega}_k \left(s_i \bullet u_{\breve{A}_j}(u_k) s_i \bullet u_{\breve{B}_j}(u_k) + s_i \bullet v_{\breve{A}_j}(u_k) s_i \bullet v_{\breve{B}_j}(u_k) + s_i \bullet w_{\breve{A}_j}(u_k)\, s_i \bullet w_{\breve{B}_j}(u_k) \right) \right)}{\left(\begin{array}{c} \sqrt{\sum_{j=1}^{z} \delta_j \left(\sum_{k=1}^{t} \dot{\omega}_k \left(\left(s_i \bullet u_{\breve{A}_j}(u_k) \right)^2 + \left(s_i \bullet v_{\breve{A}_j}(u_k) \right)^2 + \left(s_i \bullet w_{\breve{A}_j}(u_k) \right)^2 \right) \right)} \\ \sqrt{\sum_{j=1}^{z} \delta_j \left(\sum_{k=1}^{t} \dot{\omega}_k \left(\left(s_i \bullet u_{\breve{B}_j}(u_k) \right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k) \right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k) \right)^2 \right) \right)} \end{array} \right)}$$

Definition 3.11

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$ are two GmPNSS, then the WCC between them can be defined as follows

$$\mathcal{R}^1_{GWmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \frac{\varsigma_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})}{\max\{\varepsilon_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{F}_{\breve{A}}), \varepsilon_{GmPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{G}_{\breve{B}})\}} \tag{3.5}$$

$$\mathcal{R}^1_{GWmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \frac{\sum_{j=1}^{z} \delta_j \left(\sum_{k=1}^{t} \dot{\omega}_k \left(s_i \bullet u_{\breve{A}_j}(u_k) s_i \bullet u_{\breve{B}_j}(u_k) + s_i \bullet v_{\breve{A}_j}(u_k) s_i \bullet v_{\breve{B}_j}(u_k) + s_i \bullet w_{\breve{A}_j}(u_k)\, s_i \bullet w_{\breve{B}_j}(u_k) \right) \right)}{\max \left\{ \begin{array}{c} \sum_{j=1}^{z} \delta_j \left(\sum_{k=1}^{t} \dot{\omega}_k \left(\left(s_i \bullet u_{\breve{A}_j}(u_k) \right)^2 + \left(s_i \bullet v_{\breve{A}_j}(u_k) \right)^2 + \left(s_i \bullet w_{\breve{A}_j}(u_k) \right)^2 \right) \right), \\ \sum_{j=1}^{z} \delta_j \left(\sum_{k=1}^{t} \dot{\omega}_k \left(\left(s_i \bullet u_{\breve{B}_j}(u_k) \right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k) \right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k) \right)^2 \right) \right) \end{array} \right\}}$$

If we consider $\dot{\omega} = \{\frac{1}{t}, \frac{1}{t}, \ldots, \frac{1}{t}\}$ and $\delta = \{\frac{1}{z}, \frac{1}{z}, \ldots, \frac{1}{z}\}$, then $\mathcal{R}_{GWmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})$ and $\mathcal{R}^1_{GWmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})$ are reduced to $\mathcal{R}_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})$ and $\mathcal{R}^1_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})$ respectively defined in 3.6 and 3.8.

Proposition 3.12

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$ are two GmPNSS, then the CC $\mathcal{R}_{GWmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})$ between them satisfied the following properties

1. $0 \leq \mathcal{R}_{GWmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) \leq 1$
2. $\mathcal{R}_{GWmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \mathcal{R}_{GWmPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{F}_{\breve{A}})$
3. If $\mathcal{F}_{\breve{A}} = \mathcal{G}_{\breve{B}}$ i. e; $s_i \bullet u_{\breve{A}_j}(u_k) = s_i \bullet u_{\breve{B}_j}(u_k)$, $s_i \bullet v_{\breve{A}_j}(u_k) = s_i \bullet v_{\breve{B}_j}(u_k)$, and $s_i \bullet w_{\breve{A}_j}(u_k) = s_i \bullet w_{\breve{B}_j}(u_k)$ for all j, k, where $i \in 1, 2, 3, \ldots, m$, then $\mathcal{R}_{GWmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = 1$.

Proof

Similar to Proposition 3.7.

4. Application of Correlation Coefficient of GmPNSS for Decision Making

In this section, we proposed the algorithm for GmPNSS by using developed CC. We also used the proposed method for decision-making in real-life problems.

4.1. Algorithm for Correlation Coefficient of GmPNSS

Step 1. Pick out the set containing parameters.

Step 2. Construct the GmPNSS according to experts.

Step 3. Find the informational neutrosophic energies of any two GmPNSS.

Step 4. Calculate the correlation between two GmPNSS by using the following formula

$$\varsigma_{GmPNSS}(\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}}) = \sum_{j=1}^{z}\sum_{k=1}^{t}\begin{pmatrix} s_i \bullet u_{\breve{A}_j}(u_k) s_i \bullet u_{\breve{B}_j}(u_k) + s_i \bullet v_{\breve{A}_j}(u_k) s_i \bullet v_{\breve{B}_j}(u_k) + \\ s_i \bullet w_{\breve{A}_j}(u_k)\ s_i \bullet w_{\breve{B}_j}(u_k): i \in 1, 2, 3, \dots, m. \end{pmatrix}$$

Step 5. Calculate the CC between any two GmPNSS by using the following formula

$$\mathcal{R}_{GmPNSS}\ (\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}}) = \frac{\varsigma_{GmPNSS}(\mathcal{F}_{\breve{A}},\ \mathcal{G}_{\breve{B}})}{\sqrt{\varepsilon_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{F}_{\breve{A}}) \cdot \varepsilon_{GmPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{G}_{\breve{B}})}}$$

Step 6. Pick the most suitable alternate with a supreme correlation value

Step 7. Analyze the results.

The graphical representation of the proposed model is given in the following Figure 1.

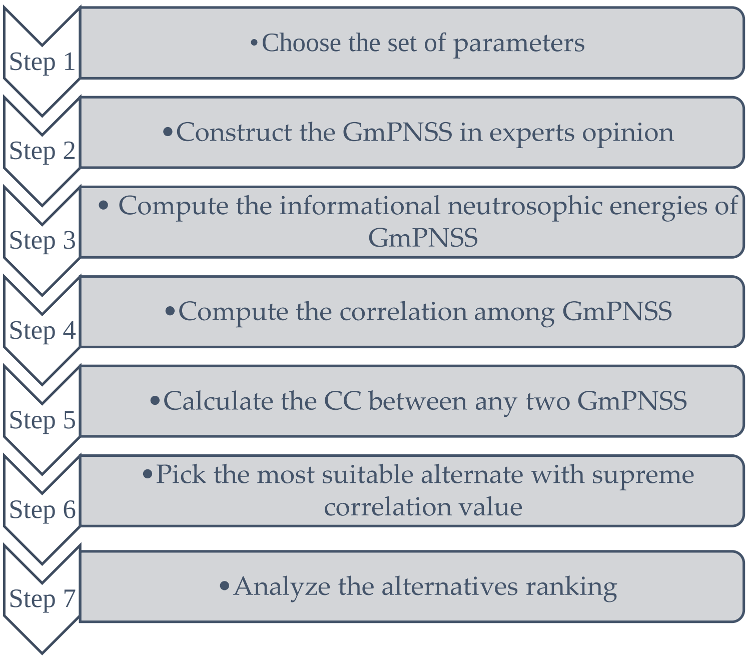

Figure 1: Flowchart of the proposed model

4.2. Problem Formulation and Application of CC for GmPNSS in Decision Making

Department of the scientific discipline of some universities U will have one scholarship for the post-doctorate position. Several scholars apply to get a scholarship but referable probabilistic along with CGPA (cumulative grade points average), simply four scholars call for enrolled for undervaluation such as $S = \{S_1, S_2, S_3, S_4\}$ be a set of selected scholars call for the interview. The president of the university hires a committee of four experts $X = \{X_1, X_2, X_3, X_4\}$ for the selection post-doctoral scholar. First of all, the committee decides the set of parameters such as $E = \{x_1, x_2, x_3\}$, where x_1, x_2, and x_3 represents the research papers, research quality, and communication skills for selecting post-doctoral scholars. The experts evaluate the scholars under defined parameters and forward the evaluation performa to the university's president. Finally, the university president scrutinizes the one best scholar based on the expert's evaluation for the post-doctoral scholarship.

4.3. Application of GmPNSS For Decision Making

Assume $S = \{S_1, S_2, S_3, S_4\}$ be a set of scholars who are shortlisted for interview and $E = \{x_1$ = research paper, x_2 = research quality, x_3 = interview$\}$ be a set of parameters for the selection of scholarship. Let $\mathcal{F}$ and $\mathcal{G} \subseteq E$. Then we construct the G3-PNSS $\Phi_{\mathcal{F}}(x)$ according to the requirement of the scientific discipline department.

Table 1. Construction of G3-PNSS of all Scholars According to Department Requirement

$\Phi_{\mathcal{F}}(x)$	x_1	x_2	x_3
X_1	(.82,.55,.63),(.55,.46,.28),(.43,.38,.60)	(.43,.68,.86),(.47,.67,.56),(.42,.51,.33)	(.73,.48,.53),(.87,.43,.77),(.76,.53,.62)
X_2	(.50,.62,.52),(.93,.57,.80),(.66,.48,.52)	(.77,.54,.81),(.75,.54,.72),(.53,.54,.69)	(.64,.48,.59),(.32,.58,.22),(.94,.64,.62)
X_3	(.29,.25,.41),(.73,.34,.32),(.64,.44,.56)	(.36,.45,.27),(.47,.65,.21),(.61,.37,.39)	(.57,.25,.41),(.72,.55,.29),(.64,.31,.34)
X_4	(.91,.50,.16),(.30,.24,.63),(.16,.55,.20)	(.69,.52,.61),(.37,.44,.23),(.46,.37,.29)	(.39,.35,.67),(.47,.24,.32),(.40,.71,.56)

Now we will construct the G3-PNSS $\varphi_{\mathcal{G}}^{t}$ according to four experts, where t = 1, 2, 3, 4.

Table 2. G3-PNSS Evaluation Report According to Experts of S_1

$\varphi_{\mathcal{G}}^{1}$	x_1	x_2	x_3
X_1	(.13,.15,.22),(.89,.78,.83),(.77,.82,.91)	(.91,.50,.16),(.30,.24,.63),(.16,.55,.20)	(.69,.52,.61),(.37,.44,.23),(.46,.37,.29)
X_2	(.79,.84,.93),(.36,.18,.26),(.21,.24,.16)	(.39,.35,.67),(.47,.24,.32),(.40,.71,.56)	(.76,.62,.41),(.36,.49,.79),(.53,.59,.91)
X_3	(.07,.23,.32),(.12,.18,.20),(.74,.79,.88)	(.70,.22,.11),(.67,.43,.53),(.41,.57,.49)	(.87,.58,.66),(.77,.22,.56),(.57,.33,.29)
X_4	(.23,.12,.17),(.25,.16,.22),(.14,.16,.18)	(.74,.62,.66),(.67,.41,.93),(.85,.67,.99)	(.27,.29,.61),(.71,.43,.21),(.47,.70,.89)

Table 3. G3-PNSS Evaluation Report According to Experts of S_2

$\varphi_{\mathcal{G}}^{2}$	x_1	x_2	x_3
X_1	(.16,.20,.27),(.83,.87,.89),(70,.75,.86)	(.91,.50,.16),(.30,.24,.63),(.16,.55,.20)	(.69,.52,.61),(.37,.44,.23),(.46,.37,.29)
X_2	(.13,.21,.24),(.18,.20,.20),(.70,.84,.90)	(.39,.35,.67),(.47,.24,.32),(.40,.71,.56)	(.76,.62,.41),(.36,.49,.79),(.53,.59,.91)
X_3	(.20,.16,.27),(.29,.17,.26),(.14,.15,.12)	(.70,.22,.11),(.67,.43,.53),(.41,.57,.49)	(.87,.58,.66),(.77,.22,.56),(.57,.33,.29)
X_4	(.88,.81,.90),(.40,.20,.26),(.22,.27,.17)	(.74,.62,.66),(.67,.41,.93),(.85,.67,.99)	(.27,.29,.61),(.71,.43,.21),(.47,.70,.89)

Table 4. G3-PNSS Evaluation Report According to Experts of S_3

$\varphi_{\mathcal{G}}^{3}$	x_1	x_2	x_3
X_1	(.77,.49,.61),(.71,.43,.21),(.47,.40,.69)	(.47,.59,.76),(.67,.62,.56),(.57,.43,.29)	(.70,.54,.61),(.79,.44,.63),(.61,.41,.51)
X_2	(.60,.32,.32),(.77,.49,.83),(.76,.32,.59)	(.76,.62,.61),(.56,.49,.79),(.53,.59,.81)	(.69,.62,.67),(.57,.74,.43),(.86,.47,.79)
X_3	(.60,.22,.21),(.67,.43,.53),(.49,.57,.49)	(.29,.72,.41),(.30,.66,.29),(.56,.32,.39)	(.74,.52,.66),(.67,.41,.93),(.85,.47,.59)
X_4	(.74,.26,.37),(.49,.41,.63),(.44,.35,.32)	(.41,.66,.51),(.39,.27,.36),(.41,.51,.21)	(.60,.16,.47),(.31,.17,.24),(.54,.35,.24)

Table 5. G3-PNSS Evaluation Report According to Experts of S_4

$\varphi_{\mathcal{G}}^{4}$	x_1	x_2	x_3

$\mathbf{X_1}$	(.23,.13,.22),(.31,.25,.43),(.19,.22,.27)	(.43,.68,.86),(.47,.67,.56),(.42,.51,.33)	(.82,.55,.63),(.55,.46,.28),(.43,.38,.60)
$\mathbf{X_2}$	(.10,.13,.11),(.91,.84,.69),(.31,.30,.28)	(.27,.29,.61),(.71,.43,.21),(.47,.70,.89)	(.50,.62,.52),(.93,.57,.80),(.66,.48,.52)
$\mathbf{X_3}$	(.70,.22,.11),(.67,.43,.53),(.41,.57,.49)	(.70,.22,.11),(.67,.43,.53),(.41,.57,.49)	(.36,.45,.27),(.47,.65,.21),(.61,.37,.39)
$\mathbf{X_4}$	(.45,.16,.27),(.91,.67,.23),(.64,.88,.10)	(.67,.81,.17),(.21,.54,.71),(.41,.54,.21)	(.20,.76,.47),(.39,.17,.46),(.41,.53,.22)

Solution by Using Developed Algorithm

Now, by using Tables 1, 2, 3, 4, and 5, we can find the correlation coefficient for each alternative by using equation 3.2 given as $\mathcal{R}_{GmPNSS}$ $(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}^1)$ = .8374, $\mathcal{R}_{GmPNSS}$ $(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}^2)$ = .7821, $\mathcal{R}_{GmPNSS}$ $(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}^3)$ = .9462, and $\mathcal{R}_{GmPNSS}$ $(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}^4)$ = .9422. This shows that $\mathcal{R}_{GmPNSS}$ $(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}^3) > \mathcal{R}_{GmPNSS}$ $(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}^4) > \mathcal{R}_{GmPNSS}$ $(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}^1) > \mathcal{R}_{GmPNSS}$ $(\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}^2)$. Hence S_3 is the best scholar for a postdoctoral position.

5. Result Discussion and Comparative Analysis

In the subsequent section, we will debate the suggested method's effectiveness, simplicity, flexibility, and good position. A brief comparative analysis has been presented among our proposed and prevailing approaches.

5.1 Advantages and Superiority of Proposed Approach

This manuscript has developed a novel DM technique based on CC utilizing GmPNSS. The developed approach is more operative and delivers the appropriate results in MCDM problems comparative to existing techniques. Through this scientific research and comparison, we have realized that the suggested approach's outcomes are more general than conventional methods. However, compared to the current DM method, the DM process contains more information to deal with uncertain data. In addition to the fact that the hybrid structure of multiple FSs becomes a particular case of GmPNSS, add some appropriate conditions. Among them, the information associated with the object may be displayed precisely and analytically, so GmPNSS is an effective power tool to cope with imprecise and uncertain information in the DM process. Hence, our approach is more suitable, pliable, and better than FS's distinctive, accessible hybrid structures.

Table 6: Comparative analysis between some existing techniques and the proposed approach

	Set	Truthiness	Indeterminacy	Falsity	Multi-polarity	Loss of information
Chen et al. [28]	mPFS	✓	×	×	✓	×
Xu et al. [29]	IFS	✓	×	✓	×	×
Zhang et al. [30]	IFS	✓	×	✓	×	✓
Ali et al. [31]	BPNSS	✓	✓	✓	×	×
Proposed approach	GmPNSS	✓	✓	✓	✓	×

It turns out to be a contemporary problem. Why do we particularize novel algorithms according to the present novel structure? Several indications indicate that the recommended methodology can be exceptional compared to other existing methods. We remember that IFS, picture fuzzy set, FS, hesitant fuzzy set, NS, and different fuzzy sets have been restricted by the hybrid structure and cannot provide complete information regarding the situation. But our m-polarity model GmPNSS can be most suitable for MCDM because it can deal with truthiness, indeterminacy, and falsity. Due to the exaggerated multipolar neutrosophy, those three degrees have been independent of each other and furnish many information regarding alternative specifications. The similarity measures of other available hybrid structures are converted into a particular case of mPIVNSS. The comparative analysis with some prevailing techniques is listening above Table 6. Therefore, the model is more versatile and can quickly solve complications comparative to intuitionistic, neutrosophy, hesitation, image, and ambiguity substitution. The similarity measure established for GmPNSS becomes better than the existing similarity measure for MCDM.

5.2 Discussion

Chen et al. [28] multi-polar information of fuzzy sets deals with the membership value of the objects; the multi-polar fuzzy set cannot handle the circumstances when the objects have indeterminacy and falsity information. Xu et al. [29] and Zhang et al. [30] IFS only deal with the membership and non-membership values of the alternatives. These techniques cannot deal with the multi-polar information and indeterminacy of the alternative. Ali et al. [24] dealt with the truthiness, indeterminacy, and falsity grades for substitutes, but these techniques cannot manage multiple data. Instead, our established technique is an innovative method that can handle various information alternatives. But, on the other hand, the strategy we have progressed is about truth, indeterminacy, and the falsity of other options. So, the methodology we have offered is very efficient and will provide better outcomes for experts through additional information.

5.3 Comparative analysis

In this article, we propose a novel algorithm. First, an algorithm is proposed based on the correlation coefficient for GmPNSS. Next, utilize the developed algorithm to solve practical problems in real-life, that is, to select a postdoctoral position. The obtained results show that the proposed technique is effective and beneficial. Finally, the ranking of all alternatives using the existing methodologies gives the same final decision, that is, the position of "postdoctoral" is selected as S_3. All rankings are also calculated by applying existing methods with the same case study. The proposed method is also compared with other existing methods, Saeed et al. [26], Riaz et al. [16, 32], and Mohd Kamal et al. [33]. But these techniques cannot manage multiple data. Instead, our established technique is an innovative method that can handle various information alternatives. But, on the other hand, the strategy we have progressed is about truth, indeterminacy, and the falsity of alternatives. So, the methodology we have offered is very efficient and will provide better outcomes for experts through additional information.

Table 7. Comparison Between mPNSS and GmPNSS Techniques

Method	Alternative final Ranking	Optimal Choice
Riaz et al. [16]	$S_3 > S_2 > S_1 > S_4$	S_3
Saeed et al. [26]	$S_3 > S_4 > S_2 > S_1$	S_3
Riaz et al. [32]	$S_3 > S_2 > S_1 > S_4$	S_3
Mohd Kamal et al. [33]	$S_3 > S_4 > S_2 > S_1$	S_3
Proposed Approach	$S_3 > S_4 > S_1 > S_2$	S_3

6. Conclusion

In this manuscript, a novel hybrid structure has been established by GmPNSS by extending the mPNSS. We have developed the CC and WCC with their properties in the content of GmPNSS. A novel algorithm for GmPNSS utilizing our developed measure has been constructed to solve MCDM problems. A comparative analysis was also performed to demonstrate the proposed method.

Through comparative analysis, it is observed that the proposed technique exhibited higher steadiness and pragmatism for decision-makers in the DM process. Based on the results obtained, it is concluded that this method is most suitable for solving the MCDM problem in today's life. We will apply these techniques to other fields in future work, such as mathematical programming, cluster analysis, etc. This research article has pragmatic boundaries and can be immensely helpful in real-life dimensions: including the medical profession, pattern recognition, economics, etc.

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A Novel Approach by using Interval-Valued Trapezoidal Neutrosophic Numbers in Transportation Problem

Rajesh Kumar Saini* [1] Atul Sangal [2] Ashik Ahirwar[3]

[1, 3]Deptt. of Mathematical Sciences and Computer Applications, Bundelkhand University, Jhansi, India, prof.rksaini@bujhansi.ac.in*

[2]Deptt. of Management Studies, Sharda University, Greater Noida, Uttar Pradesh, India
atul.sangal@sharda.ac.in

* Correspondence: prof.rksaini@bujhansi.ac.in, Mob. 9412322576(091)

Abstract: In today's scenario transportation problem [TP] is the prominent area of optimization. In the present paper, a TP in a neutrosophic environment, known as a neutrosophic transportation problem [NTP] is introduced with interval-valued trapezoidal neutrosophic numbers [IVTrNeNs]. To maintain physical distance among the industrialists and researchers during the covid-19 pandemic, the interval-valued fuzzy numbers [IVFNs] in place of crisp numbers are very much essential to address the hesitation and uncertainty in real-life situations. IVTrNeN is the generalization of single-valued neutrosophic numbers [SVNeN], which are used as the cost, the demand, and the supply to transport the necessary equipment, medicines, food products, and other relevant items from one place to another to save the human lives in a covid-19 pandemic. A Neutrosophic set, which has uncertainty, inconsistency, and incompleteness information is the abstract principle of crisp, fuzzy, and intuitionistic fuzzy sets. Here we suggest some numerical problems for better execution of the neutrosophic transportation problem [NTP], to understand the practical applications of interval-valued neutrosophic numbers [IVNeNs]. In the last, we compare our results and a conclusion is given in support of our proposed result methodology with IVTrNeNs.

Keywords: Interval–valued trapezoidal neutrosophic number, De-neutrosophication, neutrosophic transportation problem.

1. Introduction

In the current scenario of covid-19, the role of a neutrosophic optimization technique in TP has fascinated awareness of their high efficiency, accuracy, and adaptability that gives high standard real-life outcomes. Neutrosophic optimization has been extremely searched in industrial, management, engineering, and health sectors. Zadeh in 1965 introduced the mathematical formula of fuzzy set FS [1] by which the researchers try to check the ambiguity or uncertainty in engineering, industrial, and management problems [2, 3]. In realistic problems, the FS was not perfect to observe the uncertainty and hesitation. To encompass this problem, Atanassov extended the FS and introduced a set with membership and non-membership degrees, called an intuitionistic fuzzy set IFS [4]. For more detailed applications of IFS, please (see [5-10] and references therein). Atanassov and Gargov generalized the IFS by introducing the interval-valued IFS to strengthen the attitude of grasp uncertainty and hesitation in IFS [11]. To solve the real-world problems with inconsistent information or contain indeterminacy in data the FS and IFS are not sufficient. To rectify such problems, Smarandache in 1988 introduced the neutrosophic set [NS] [12], by which the inconsistent information is in the form of truth-membership, indeterminacy-membership, and falsity-membership degrees respectively. For practical applications and some technical references in real-world problems, NSs are difficult to apply, so the notion of a single-valued neutrosophic set [SVNeS] was imported by Wang et. al. in 2010 [13]. The idea of SVNeS is more suitable and effective in solving many real-life problems of decision-makers that contain uncertainty in data by using fuzzy numbers and intuitionistic fuzzy numbers. Since in the real world, there exists stipulated and non-stipulated knowledge, so to overcome such problems Samaranache introduced the neutrosophic number [NN] [14, 15]. In 2016, Ye. proposed de-neutrosophic and possibility degree ranking methods for the application of NNs [16]. Samrandache in 2015, proposed the interval function to describe the stipulated and non-stipulated issues in real-world problems [17]. For more uncertain linear programming problems (see [18-29] and references therein).

In real-life optimization problems, the TP shows high execution and due to its clarity and minimum cost, it is a noted optimization technique in the current scenario. The basic theme of a TP is to find a direct connection between source and destination in minimum time with minimum cost. Hitchcock introduced the initial basic structure of TP and developed a special mathematical module for the basic results of TP by the simplex method [30]. For more recent development in fuzzy transportation problem [FTP] (see [31-47] and references therein).

The IFS theory can handle the problems of incomplete information but not the indeterminate and inconsistent information that exists in the transportation modal. The TP with inconsistent information or indeterminate data i.e. in fuzzy numbers or intuitionistic fuzzy numbers cannot be handled in the current structure. To resolve such issues, the NTP is the best option with indeterminacy and inconsistent information by truth, indeterminacy, and falsity membership degree function. Many researchers formulated efficient mathematical models in various uncertain environments. We proposed the NTP of type-4, with all entries such as cost, demand, and supply termed as IVTrNeNs, which include membership, indeterminacy, and non-memberships degree function. The more real-world developments in the field of neutrosophic optimization problems (see [48-63] and references therein).

For the solution of NTP, the first one will change it into a crisp transportation problem [CTP] by converting the cost, demand, and supply, which are in IVTrNeNs into crisp values with the help of the introduced ranking method. For unbalanced CTP or NTP, here we use Vogel's approximation method [VAM] and minimum row-column method [MRCM] to solve these by excel solver and then compare our results [46]. The paper is well organized in several sections such as the introduction of the present paper with some earlier research are given, the basics concepts of FS, IFS, and NS are discussed and reviewed, introduce the ranking function, score function, and de-neutrosophication to convert neutrosophic values into crisp values and vice-versa. Here we proposed CTP & NTP of type-4, their solution by existing and MRCM, comparison, and the conclusion for future aspects of research work.

2. Preliminaries

Definition 2.1 ([39]): A FS $\tilde{A}$ of a non empty set X is defined as $\tilde{A}=\{\langle x,\mu_{\tilde{A}}(x)\rangle / x\in X\}$ where $\mu_{\tilde{A}}(x):X\to[0,1]$ is the membership function.

Definition 2.2: A fuzzy number on the universal set ***R*** is a convex, normalized fuzzy set $\tilde{A}$, where the membership function $\mu_{\tilde{A}}(x):X\to[0,1]$ is continuous, strictly increasing on [a, b] and strictly decreasing on [c, d], $\mu_{\tilde{A}}(x)=1,$ for all $x\in[b,c]$, where $a\le b\le c\le d$ and $\mu_{\tilde{A}}(x)=0,$ for all $x\in(-\infty,a]\cup[d,\infty)$.

Definition 2.3 ([52]): A trapezoidal fuzzy number (TrFN) denoted as $\tilde{A}=(a,b,c,d)$, with its membership function $\mu_{\tilde{A}}(x)$ on ***R***, is given by

$$\mu_{\tilde{A}}(x)=\begin{cases}(x-a)/(b-a), & \text{for } a\le x<b\\ 1, & \text{for } b\le x<c\\ (d-x)/(d-c), & \text{for } c<x\le d\\ 0, & \text{otherwise}\end{cases}$$

If $b=c$ in TrFN $\tilde{A}=(a,b,c,d)$, then it becomes TFN $\tilde{A}=(a,b,c)$.

Definition 2.4: An IFS in a non-empty set X is denoted by $\tilde{A}^I$ and defined as $\tilde{A}^I=\left\{\left\langle x,\mu_{\tilde{A}^I},\nu_{\tilde{A}^I}\right\rangle : x\in X\right\}$, where $\mu_{\tilde{A}^I},\nu_{\tilde{A}^I} : X\rightarrow[0,1]$, are denoted as degree of membership and degree of non-membership functions respectively. The function $h(x)=1-\mu_{\tilde{A}^I}-\nu_{\tilde{A}^I}\le 1, \forall\, x\in X$ called the degree of hesitancy in $\tilde{A}^I$.

The single valued neutrosophic numbers [SVNN] introduced by Deli and Suba [64] in 2014.

Definition 2.5: A SVNS is denoted and defined as $\tilde{A}_N=\{x,T_{\tilde{A}_N}(x),I_{\tilde{A}_N}(x),F_{\tilde{A}_N}(x)/x\in X\}$, where for each generic point x in X, $T_{\tilde{A}_N}(x)$ called truth membership function, $I_{\tilde{A}_N}(x)$ called indeterminacy membership function and $F_{\tilde{A}_N}(x)$ called falsity membership function in [0,1] and $0\le T_{\tilde{A}_N}(x)+I_{\tilde{A}_N}(x)+F_{\tilde{A}_N}(x)\le 3$. For continuous SVNS $\tilde{A}_N=\int_{\tilde{A}_N}\left\langle T_{\tilde{A}_N}(x),I_{\tilde{A}_N}(x),F_{\tilde{A}_N}(x)\right\rangle/x,\ x\in X$. For discrete values, SVNS can be written as $\tilde{A}_N=\sum_{i=1}^{n}\left\langle T_{\tilde{A}_N}(x_i),I_{\tilde{A}_N}(x_i),F_{\tilde{A}_N}(x_i)\right\rangle/x_i,\ x_i\in X.$

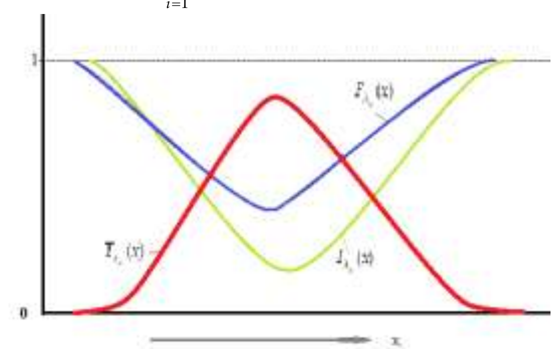

Fig. 1: Neutrosophic set

Definition 2.6 ([15]): Let x be a generic element of a non empty set X. A neutrosophic number $\tilde{A}_N$ in X is defined as $\tilde{A}_N=\{\langle x,T_{\tilde{A}_N}(x),I_{\tilde{A}_N}(x),F_{\tilde{A}_N}(x)\rangle/x\in X\}$, $\forall\ T_{\tilde{A}_N}(x)$, $I_{\tilde{A}_N}(x)$ and $F_{\tilde{A}_N}(x)$ belongs $]^-0,1^+[$ where $T_{\tilde{A}_N}:X\rightarrow]0^-,1^+[$, $I_{\tilde{A}_N}:X\rightarrow]0^-,1^+[$ and $F_{\tilde{A}_N}:X\rightarrow]0^-,1^+[$ are functions of truth-membership, indeterminacy membership and falsity-membership in $\tilde{A}_N$ respectively with $^-0\le T_{\tilde{A}_N}(x)+I_{\tilde{A}_N}(x)+F_{\tilde{A}_N}(x)\le 3^+$.

Definition 2.7 ([17]): Let X be a nonempty set. Then an interval-valued neutrosophic set [IVNS] $\tilde{A}_N^{IV}$ of X is defined as:

$$\tilde{A}_N^{IV}=\left\{\left\langle x;\left[T^L_{\tilde{A}_N},T^U_{\tilde{A}_N}\right],\left[I^L_{\tilde{A}_N},I^U_{\tilde{A}_N}\right],\left[F^L_{\tilde{A}_N},F^U_{\tilde{A}_N}\right]\right\rangle : x\in X\right\}$$

where $\left[T^L_{\tilde{A}_N}, T^U_{\tilde{A}_N}\right], \left[I^L_{\tilde{A}_N}, I^U_{\tilde{A}_N}\right]$ and $\left[F^L_{\tilde{A}_N}, F^U_{\tilde{A}_N}\right] \subset [0,1]$ $\forall x \in X$. $T^L_{\tilde{A}_N} = \inf\left(T_{\tilde{A}_N}\right)$, $T^U_{\tilde{A}_N} = \sup\left(T_{\tilde{A}_N}\right)$; $I^L_{\tilde{A}_N} = \inf\left(I_{\tilde{A}_N}\right)$, $I^U_{\tilde{A}_N} = \sup\left(I_{\tilde{A}_N}\right)$ and $F^L_{\tilde{A}_N} = \inf\left(F_{\tilde{A}_N}\right)$, $F^U_{\tilde{A}_N} = \sup\left(F_{\tilde{A}_N}\right)$.

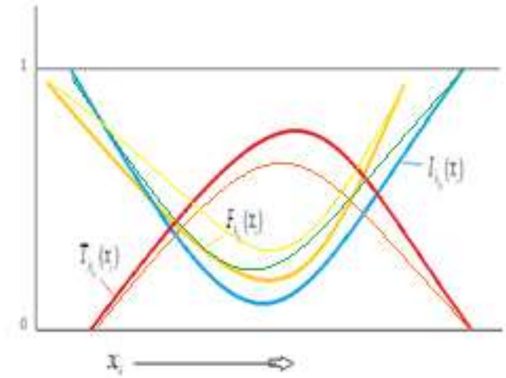

Fig. 2: Interval-valued neutrosophic set

Definition 2.8: Let $\tilde{A}^{IV}_N = \left\{\left\langle x; \left[T^L_{\tilde{A}_N}, T^U_{\tilde{A}_N}\right], \left[I^L_{\tilde{A}_N}, I^U_{\tilde{A}_N}\right], \left[F^L_{\tilde{A}_N}, F^U_{\tilde{A}_N}\right]\right\rangle : x \in X\right\}$ be IVNS, then

(i) $\tilde{A}^{IV}_N$ is empty if $T^L_{\tilde{A}_N} = T^U_{\tilde{A}_N} = 0, I^L_{\tilde{A}_N} = I^U_{\tilde{A}_N} = 1, F^L_{\tilde{A}_N} = F^U_{\tilde{A}_N} = 1$, $\forall x \in X$.

(ii) let $\underline{0} = \langle x, 0, 1, 1\rangle$ and $\langle x, 1, 0, 0\rangle$.

The interval-valued numbers and their operational properties are most valuable to survey for interval-valued neutrosophic numbers [IVNeNs]. Here we are given some impotent operations & facts about Interval valued numbers.

Definition 2.9 ([65]): An interval on R is defined as $A = [a^L, a^R] = \{a : a^L \le a \le a^R, a \in R\}$, where a^L in left limit and a^R is the right limit of *A*, it may also be defined as $A = \langle a_c, a_w \rangle = \{a : a_c - a_w \le a \le a_c + a_w, a \in R\}$, where $a_c = \frac{(a^R + a^L)}{2}$ in centre $a_w = \frac{(a^R - a^L)}{2}$ is width of *A*.

Definition 2.10 ([66, 67]): Let $A(x) = [a^L, a^U] = \{x : a^L \le x \le a^U\}$, then $A(x)$ is called an interval number. $A(x)$ is positive interval if $0 \le a^L \le x \le a^U$. Let $A(x) = [a^L, a^U]$ and $B(x) = [b^L, b^U]$ be two interval numbers, then the following operational properties are holds:

(i) $A = B \Leftrightarrow a^L = b^L,\ a^U = b^U$;

(ii) $A + B = [a^L + b^L,\ a^U + b^U]$; $A - B = [a^L - b^U,\ a^U - b^L]$;

(iii) $A \times B \Leftrightarrow [min\{a^L b^L,\ a^L b^U, a^U b^L,\ a^U b^U\}; \max\{a^L b^L,\ a^L b^U, a^U b^L,\ a^U b^U\}]$;

(iv) $\mu A = [\mu a^L, \mu a^U],\ \mu > 0$.

Definition 2.11 ([35]): The interval-valued trapezoidal neutrosophic number [IVTrNeN]) is a special case of NS on the real line **R.** Let $a_1, a_2, a_3, a_4 \in R$ such that $a_1 \le a_2 \le a_3 \le a_4$ then

$$\tilde{a}^{IV}_N = \left\langle (a_1, a_2, a_3, a_4); \left[u^L_{\tilde{a}^{IV}_N}, u^U_{\tilde{a}^{IV}_N}\right], \left[v^L_{\tilde{a}^{IV}_N}, v^U_{\tilde{a}^{IV}_N}\right], \left[w^L_{\tilde{a}^{IV}_N}, w^U_{\tilde{a}^{IV}_N}\right]\right\rangle,$$

is IVTrNeN, where $\left[u^L_{\tilde{a}^{IV}_N}, u^U_{\tilde{a}^{IV}_N}\right]$ are upper and lower bound of the truth-membership degree function $u_{\tilde{a}^{IV}_N}$, $\left[v^L_{\tilde{a}^{IV}_N}, v^U_{\tilde{a}^{IV}_N}\right]$ are upper and lower bound of the indeterminacy-membership degree function $v_{\tilde{a}^{IV}_N}$ and

$\left[w^L_{\tilde{a}^{IV}_N}, w^U_{\tilde{a}^{IV}_N}\right]$ are the upper and lower bound of the falsity-membership degree function $w_{\tilde{a}^{IV}_N}$ in [0,1] respectively, whose truth-membership $T_{\tilde{a}^{IV}_N}(x)$, indeterminacy-membership $I_{\tilde{a}^{IV}_N}(x)$, and a falsity-membership $F_{\tilde{a}^{IV}_N}(x)$ are defined as follows:

$$T_{\tilde{a}^{IV}_N}(x)=\begin{cases} u_{\tilde{a}^{IV}_N}\left(\dfrac{x-a_1}{a_2-a_1}\right), & \text{for } a_1\le x\le a_2,\\ u_{\tilde{a}^{IV}_N}, & \text{for } a_2\le x\le a_3,\\ u_{\tilde{a}^{IV}_N}\left(\dfrac{a_4-x}{a_4-a_3}\right), & \text{for } a_3\le x\le a_4,\\ 0, & \text{for } x<a_1 \text{ and } x>a_4.\end{cases} \quad I_{\tilde{a}^{IV}_N}(x)=\begin{cases} \dfrac{a_2-x+v_{\tilde{a}^{IV}_N}(x-a_1)}{a_2-a_1}, & \text{for } a_1\le x\le a_2,\\ v_{\tilde{a}^{IV}_N}, & \text{for } a_2\le x\le a_3,\\ \dfrac{x-a_3+v_{\tilde{a}^{IV}_N}(a_4-x)}{a_4-a_3}, & \text{for } a_3\le x\le a_4,\\ 1, & \text{for } x<a_1 \text{ and } x>a_4.\end{cases}$$

$$F_{\tilde{a}^{IV}_N}(x)=\begin{cases} \dfrac{a_2-x+w_{\tilde{a}^{IV}_N}(x-a_1)}{a_2-a_1}, & \text{for } a_1\le x\le a_2,\\ w_{\tilde{a}^{IV}_N}, & \text{for } a_2\le x\le a_3,\\ \dfrac{x-a_3+w_{\tilde{a}^{IV}_N}(a_4-x)}{a_4-a_3}, & \text{for } a_3\le x\le a_4,\\ 1, & \text{for } x<a_1 \text{ and } x>a_4.\end{cases}$$

when $a_1>0$, $\tilde{a}^{IV}_N=\left\langle (a_1,a_2,a_3,a_4);\left[u^L_{\tilde{a}^{IV}_N},u^U_{\tilde{a}^{IV}_N}\right],\left[v^L_{\tilde{a}^{IV}_N},v^U_{\tilde{a}^{IV}_N}\right],\left[w^L_{\tilde{a}^{IV}_N},w^U_{\tilde{a}^{IV}_N}\right]\right\rangle$, is called positive IVTrNeN, denoted by $\tilde{a}^{IV}_N>0$, and if $a_4\le 0$, then $\tilde{a}^{IV}_N$ becomes a negative IVTrNeN, denoted by $\tilde{a}^{IV}_N>0$. If $a_2=a_3$, then IVTrNeN is reduces interval-valued triangular neutrosophic number [IVTriNeN], denoted as $\tilde{a}^{IV}_N=\left\langle (a_1,a_2,a_3);\left[u^L_{\tilde{a}^{IV}_N},u^U_{\tilde{a}^{IV}_N}\right],\left[v^L_{\tilde{a}^{IV}_N},v^U_{\tilde{a}^{IV}_N}\right],\left[w^L_{\tilde{a}^{IV}_N},w^U_{\tilde{a}^{IV}_N}\right]\right\rangle$.

On the basis of [8, 41, 58], we will take here the twelve components of IVTrNeNs i.e. $\tilde{a}^{IV}_{N_1}=\left\langle\left[\left\{(a_1,b_1,c_1,d_1);u_{\tilde{a}^{IV}_{N_1}}\right\},\left\{(e_1,f_1,g_1,h_1);v_{\tilde{a}^{IV}_{N_1}}\right\},\left\{(l_1,m_1,n_1,p_1);w_{\tilde{a}^{IV}_{N_1}}\right\}\right]\right\rangle$ guided as $l_1\le e_1\le a_1\le m_1\le f_1\le b_1\le n_1\le g_1\le c_1\le p_1\le h_1\le d_1$.

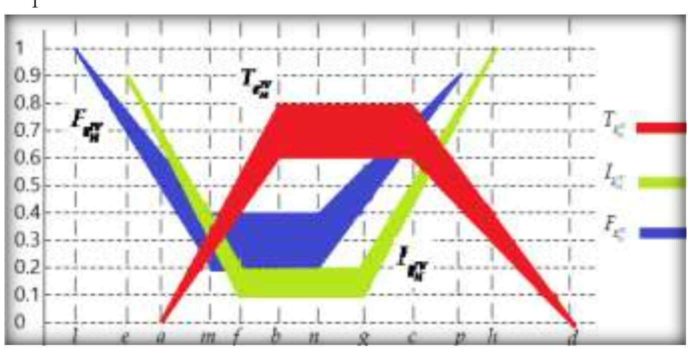

Fig. 3: Interval-valued trapezoidal neutrosophic number

2.1. Operational Laws on IVTrNeNs

Let $\tilde{a}^{IV}_{N_1}=\left\langle\left[\left\{(a_1,b_1,c_1,d_1);u_{\tilde{a}^{IV}_{N_1}}\right\},\left\{(e_1,f_1,g_1,h_1);v_{\tilde{a}^{IV}_{N_1}}\right\},\left\{(l_1,m_1,n_1,p_1);w_{\tilde{a}^{IV}_{N_1}}\right\}\right]\right\rangle$ and

$\tilde{a}_{N_2}^{IV}=\left\langle\left[\left\{(a_2,b_2,c_2,d_2);u_{\tilde{a}_{N_2}^{IV}}\right\},\left\{(e_2,f_2,g_2,h_2);v_{\tilde{a}_{N_2}^{IV}}\right\},\left\{(l_2,m_2,n_2,p_2);w_{\tilde{a}_{N_2}^{IV}}\right\}\right]\right\rangle$ be two IVTrNeNs with twelve components, where $u_{\tilde{a}_{N_1}^{IV}}=\left[u^L_{\tilde{a}_{N_1}^{LIV}},u^U_{\tilde{a}_{N_1}^{UIV}}\right]$; $u_{\tilde{a}_{N_2}^{IV}}=\left[u^L_{\tilde{a}_{N_2}^{LIV}},u^U_{\tilde{a}_{N_2}^{UIV}}\right]$; $v_{\tilde{a}_{N_1}^{IV}}=\left[v^L_{\tilde{a}_{N_1}^{LIV}},v^U_{\tilde{a}_{N_1}^{UIV}}\right]$; $v_{\tilde{a}_{N_2}^{IV}}=\left[v^L_{\tilde{a}_{N_2}^{LIV}},v^U_{\tilde{a}_{N_2}^{UIV}}\right]$; and $w_{\tilde{a}_{N_1}^{IV}}=\left[w^L_{\tilde{a}_{N_1}^{LIV}},w^U_{\tilde{a}_{N_1}^{UIV}}\right]$; $w_{\tilde{a}_{N_2}^{IV}}=\left[w^L_{\tilde{a}_{N_2}^{LIV}},w^U_{\tilde{a}_{N_2}^{UIV}}\right]$, then the following operations hold:

Addition of IVTrNeNs:

$$\tilde{a}_{N_1}^{IV}+\tilde{a}_{N_2}^{IV}=\left\langle\begin{bmatrix}\left\{(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2);u_{\tilde{a}_{N_1}^{IV}}\wedge u_{\tilde{a}_{N_2}^{IV}}\right\},\\ \left\{(e_1+e_2,f_1+f_2,g_1+g_2,h_1+h_2);v_{\tilde{a}_{N_1}^{IV}}\vee v_{\tilde{a}_{N_2}^{IV}}\right\},\\ \left\{(l_1+l_2,m_1+m_2,n_1+n_2,p_1+p_2);w_{\tilde{a}_{N_1}^{IV}}\vee w_{\tilde{a}_{N_2}^{IV}}\right\}\end{bmatrix}\right\rangle$$

Negative of IVTrNeN:

$$-\tilde{a}_{N_2}^{IV}=\left\langle\left[\left\{(-d_2,-c_2,-b_2,-a_2);u_{\tilde{a}_{N_2}^{IV}}\right\},\left\{(-h_2,-g_2,-f_2,-e_2);v_{\tilde{a}_{N_2}^{IV}}\right\},\left\{(-p_2,-n_2,-m_2,-l_2);w_{\tilde{a}_{N_2}^{IV}}\right\}\right]\right\rangle$$

Subtraction of IVTrNeNs:

$$\tilde{a}_{N_1}^{IV}-\tilde{a}_{N_2}^{IV}=\left\langle\begin{bmatrix}\left\{(a_1-d_2,b_1-c_2,c_1-b_2,d_1-a_2);u_{\tilde{a}_{N_1}^{IV}}\wedge u_{\tilde{a}_{N_2}^{IV}}\right\},\\ \left\{(e_1-h_2,f_1-g_2,g_1-f_2,h_1-e_2);v_{\tilde{a}_{N_1}^{IV}}\vee v_{\tilde{a}_{N_2}^{IV}}\right\},\\ \left\{(l_1-p_2,m_1-n_2,n_1-m_2,p_1-l_2);w_{\tilde{a}_{N_1}^{IV}}\vee w_{\tilde{a}_{N_2}^{IV}}\right\}\end{bmatrix}\right\rangle$$

Scalar multiplication of SVTrNeN:

$$\lambda.\tilde{a}_{N_1}^{IV}=\begin{cases}\left\langle\left\{(\lambda a_1,\lambda b_1,\lambda c_1,\lambda d_1);u_{\tilde{a}_{N_1}^{IV}}\right\},\left\{(\lambda e_1,\lambda f_1,\lambda g_1,\lambda h_1);v_{\tilde{a}_{N_1}^{IV}}\right\},\left\{(\lambda l_1,\lambda m_1,\lambda n_1,\lambda p_1);w_{\tilde{a}_{N_1}^{IV}}\right\}\right\rangle \text{ if } \lambda>0\\ \left\langle\left\{(\lambda d_1,\lambda c_1,\lambda b_1,\lambda a_1);u_{\tilde{a}_{N_1}^{IV}}\right\},\left\{(\lambda h_1,\lambda g_1,\lambda f_1,\lambda e_1);v_{\tilde{a}_{N_1}^{IV}}\right\},\left\{(\lambda p_1,\lambda n_1,\lambda m_1,\lambda l_1);w_{\tilde{a}_{N_1}^{IV}}\right\}\right\rangle \text{if } \lambda<0\end{cases}$$

Multiplication of IVTrNeNs:

$$\tilde{a}_{N_1}^{IV}.\tilde{a}_{N_2}^{IV}=\begin{cases}\left\langle\begin{matrix}(a_1.a_2,b_1.b_2,c_1.c_2,d_1.d_2);(e_1.e_2,f_1.f_2,g_1.g_2,h_1.h_2);(l_1.l_2,m_1.m_2,n_1.n_2,p_1.p_2)\\ \left[u_{\tilde{a}_{N_1}^{IV}}\wedge u_{\tilde{a}_{N_2}^{IV}}\right],\left[v_{\tilde{a}_{N_1}^{IV}}\vee v_{\tilde{a}_{N_2}^{IV}}\right],\left[w_{\tilde{a}_{N_1}^{IV}}\vee w_{\tilde{a}_{N_2}^{IV}}\right] \text{ if } d_1>0,d_2>0,h_1>0,h_2>0,p_1>0,p_2>0\end{matrix}\right\rangle,\\ \left\langle\begin{matrix}(a_1.d_2,b_1.c_2,c_1.b_2,d_1.a_2);(e_1.h_2,f_1.g_2,g_1.f_2,h_1.e_2);(l_1.p_2,m_1.n_2,n_1.m_2,p_1.l_2)\\ \left[u_{\tilde{a}_{N_1}^{IV}}\wedge u_{\tilde{a}_{N_2}^{IV}}\right],\left[v_{\tilde{a}_{N_1}^{IV}}\vee v_{\tilde{a}_{N_2}^{IV}}\right],\left[w_{\tilde{a}_{N_1}^{IV}}\vee w_{\tilde{a}_{N_2}^{IV}}\right] \text{ if } d_1<0,d_2>0,h_1<0,h_2>0,p_1<0,p_2>0\end{matrix}\right\rangle,\\ \left\langle\begin{matrix}(d_1.d_2,c_1.c_2,b_1.b_2,a_1.a_2);(h_1.h_2,g_1.g_2,f_1.f_2,e_1.e_2);(p_1.p_2,n_1.n_2,m_1.m_2,l_1.l_2)\\ \left[u_{\tilde{a}_{N_1}^{IV}}\wedge u_{\tilde{a}_{N_2}^{IV}}\right],\left[v_{\tilde{a}_{N_1}^{IV}}\vee v_{\tilde{a}_{N_2}^{IV}}\right],\left[w_{\tilde{a}_{N_1}^{IV}}\vee w_{\tilde{a}_{N_2}^{IV}}\right] \text{ if } d_1<0,d_2<0,h_1<0,h_2<0,p_1<0,p_2<0\end{matrix}\right\rangle,\end{cases}$$

Inverse of SVTrNeN:

$$(\tilde{a}_{N_1}^{IV})^{-1}=\frac{1}{\tilde{a}_{N_1}^{IV}}=\begin{cases}\left\langle\left(\frac{1}{d_1},\frac{1}{c_1},\frac{1}{b_1},\frac{1}{a_1}\right);\left(\frac{1}{h_1},\frac{1}{g_1},\frac{1}{f_1},\frac{1}{e_1}\right);\left(\frac{1}{p_1},\frac{1}{n_1},\frac{1}{m_1},\frac{1}{l_1}\right);u_{\tilde{a}_{N_1}^{IV}},v_{\tilde{a}_{N_1}^{IV}},w_{\tilde{a}_{N_1}^{IV}}\right\rangle,\\ \text{if } a_1>0,b_1>0,c_1>0,d_1>0,e_1>0,f_1>0,g_1>0,h_1>0,l_1>0,m_1>0,n_1>0,p_1>0,\\ \left\langle\left(\frac{1}{a_1},\frac{1}{b_1},\frac{1}{c_1},\frac{1}{d_1}\right);\left(\frac{1}{e_1},\frac{1}{f_1},\frac{1}{g_1},\frac{1}{h_1}\right);\left(\frac{1}{l_1},\frac{1}{m_1},\frac{1}{n_1},\frac{1}{p_1}\right);u_{\tilde{a}_{N_1}^{IV}},v_{\tilde{a}_{N_1}^{IV}},w_{\tilde{a}_{N_1}^{IV}}\right\rangle,\\ \text{if } a_1<0,b_1<0,c_1<0,d_1<0,e_1<0,f_1<0,g_1<0,h_1<0,l_1<0,m_1<0,n_1<0,p_1<0.\end{cases}$$

Division of SVTrNeNs:

$$\frac{\tilde{a}_{N_1}^{IV}}{\tilde{a}_{N_2}^{IV}}=\begin{cases}\left\langle\left(\frac{a_1}{d_1},\frac{b_1}{c_1},\frac{c_1}{b_1},\frac{d_1}{a_1}\right);\left(\frac{e_1}{h_1},\frac{f_1}{g_1},\frac{g_1}{f_1},\frac{h_1}{e_1}\right);\left(\frac{l_1}{p_1},\frac{m_1}{n_1},\frac{n_1}{m_1},\frac{p_1}{l_1}\right);\left[u_{\tilde{a}_{N_1}^{IV}}\wedge u_{\tilde{a}_{N_2}^{IV}}\right],\left[v_{\tilde{a}_{N_1}^{IV}}\vee v_{\tilde{a}_{N_2}^{IV}}\right],\left[w_{\tilde{a}_{N_1}^{IV}}\vee w_{\tilde{a}_{N_2}^{IV}}\right]\right\rangle, & \text{if } d_1>0,d_2>0,h_1>0,h_2>0,p_1>0,p_2>0,\\ \left\langle\left(\frac{d_2}{d_1},\frac{c_2}{c_1},\frac{b_2}{b_1},\frac{a_2}{a_1}\right);\left(\frac{h_2}{h_1},\frac{g_2}{g_1},\frac{f_2}{f_1},\frac{e_2}{e_1}\right);\left(\frac{p_2}{p_1},\frac{n_2}{n_1},\frac{m_2}{m_1},\frac{l_2}{l_1}\right);\left[u_{\tilde{a}_{N_1}^{IV}}\wedge u_{\tilde{a}_{N_2}^{IV}}\right],\left[v_{\tilde{a}_{N_1}^{IV}}\vee v_{\tilde{a}_{N_2}^{IV}}\right],\left[w_{\tilde{a}_{N_1}^{IV}}\vee w_{\tilde{a}_{N_2}^{IV}}\right]\right\rangle, & \text{if } d_1<0,d_2>0,h_1<0,h_2>0,p_1<0,p_2>0,\\ \left\langle\left(\frac{d_2}{a_1},\frac{c_2}{b_1},\frac{b_2}{c_1},\frac{a_2}{d_1}\right);\left(\frac{h_2}{e_1},\frac{g_2}{f_1},\frac{f_2}{g_1},\frac{e_2}{h_1}\right);\left(\frac{p_2}{l_1},\frac{n_2}{m_1},\frac{m_2}{n_1},\frac{l_2}{p_1}\right);\left[u_{\tilde{a}_{N_1}^{IV}}\wedge u_{\tilde{a}_{N_2}^{IV}}\right],\left[v_{\tilde{a}_{N_1}^{IV}}\vee v_{\tilde{a}_{N_2}^{IV}}\right],\left[w_{\tilde{a}_{N_1}^{IV}}\vee w_{\tilde{a}_{N_2}^{IV}}\right]\right\rangle, & \text{if } d_1<0,d_2<0,h_1<0,h_2<0,p_1<0,p_2<0\end{cases}$$

wher $u_{\tilde{a}_{N_1}^{IV}}\wedge u_{\tilde{a}_{N_2}^{IV}}=\left[\min\left(u^L_{\tilde{a}_{N_1}^{LIV}},u^L_{\tilde{a}_{N_2}^{LIV}}\right),\min\left(u^U_{\tilde{a}_{N_1}^{UIV}},u^U_{\tilde{a}_{N_2}^{UIV}}\right)\right]$, $v_{\tilde{a}_{N_1}^{IV}}\vee v_{\tilde{a}_{N_2}^{IV}}=\left[\max\left(v^L_{\tilde{a}_{N_1}^{LIV}},v^L_{\tilde{a}_{N_2}^{LIV}}\right),\max\left(v^U_{\tilde{a}_{N_1}^{UIV}},v^U_{\tilde{a}_{N_2}^{UIV}}\right)\right]$

and $w_{\tilde{a}_{N_1}^{IV}}\vee w_{\tilde{a}_{N_2}^{IV}}=\left[\max\left(w^L_{\tilde{a}_{N_1}^{LIV}},w^L_{\tilde{a}_{N_2}^{LIV}}\right),\max\left(w^U_{\tilde{a}_{N_1}^{UIV}},w^U_{\tilde{a}_{N_2}^{UIV}}\right)\right]$.

Example 2.1.1: let $\tilde{a}_{N_1}^{IV}=\left\langle\begin{matrix}(7,11,16,21);[0.6,0.8]\\(6,10,15,20);[0.3,0.4]\\(5,9,14,19);[0.4,0.6]\end{matrix}\right\rangle$ and $\tilde{a}_{N_2}^{IV}=\left\langle\begin{matrix}(6,11,13,20);[0.7,0.8]\\(5,10,12,18);[0.4,0.5]\\(3,8,11,16);[0.5,0.6]\end{matrix}\right\rangle$ be two IVTrNeNs,

then $\tilde{a}_{N_1}^{IV}+\tilde{a}_{N_2}^{IV}=\left\langle\begin{matrix}(13,22,29,41);[0.6,0.8]\\(11,20,27,38);[0.4,0.5]\\(8,17,25,35);[0.5,0.6]\end{matrix}\right\rangle$, $\tilde{a}_{N_1}^{IV}-\tilde{a}_{N_2}^{IV}=\left\langle\begin{matrix}(-13,-2,5,15);[0.6,0.8]\\(-13,-2,5,15);[0.4,0.5]\\(-11,-2,6,16);[0.5,0.6]\end{matrix}\right\rangle$

$$\tilde{a}_{N_1}^{IV}.\tilde{a}_{N_2}^{IV}=\left\langle\begin{matrix}(42,121,208,420);[0.6,0.8]\\(30,100,180,360);[0.4,0.5]\\(15,72,154,304);[0.5,0.6]\end{matrix}\right\rangle,\quad \frac{\tilde{a}_{N_1}^{IV}}{\tilde{a}_{N_2}^{IV}}=\left\langle\begin{matrix}(0.35,0.85,1.45,3.50);[0.6,0.8]\\(0.33,0.83,1.50,4.00);[0.4,0.5]\\(0.31,0.81,1.75,6.33);[0.5,0.6]\end{matrix}\right\rangle$$

$$5\tilde{a}_{N_1}^{IV}=\left\langle\begin{matrix}(35,55,80,105);[0.6,0.8]\\(30,50,75,100);[0.3,0.4]\\(25,45,70,95);[0.4,0.6]\end{matrix}\right\rangle$$

3. Score and Accuracy functions of IVTrNeNs

Definition 3.1: Sahin [69] used the score function concept to find comparison between two IVTrNeNs. Greater of score function value demonstrate the greater of IVTrNeN. According the base of [70] the score and accuracy functions of an IVTrNeN $\tilde{a}_N^{IV}$ can be defined as follows:

$$S(\tilde{a}_N^{IV})=\frac{1}{12}\begin{pmatrix}8+(a_1+b_1+c_1+d_1)-(e_1+f_1+g_1+h_1)\\-(l_1+m_1+n_1+p_1)\end{pmatrix}\times\left(2+u^L_{\tilde{a}_N^{LIV}}+u^U_{\tilde{a}_N^{UIV}}-v^L_{\tilde{a}_N^{LIV}}-v^U_{\tilde{a}_N^{UIV}}-w^L_{\tilde{a}_N^{LIV}}-w^U_{\tilde{a}_N^{UIV}}\right)$$

$S(\tilde{a}_N^{IV})\in[0,1]$. The accuracy function $A(\tilde{a}_N^{IV})\in[-1,1]$ is defined as:

$$A(\tilde{a}_N^{IV})=\frac{1}{4}\left(a_1+b_1+c_1+d_1-l_1-m_1-n_1-p_1\right)\times\left(2+u^L_{\tilde{a}_N^{LIV}}+u^U_{\tilde{a}_N^{UIV}}-v^L_{\tilde{a}_N^{LIV}}-v^U_{\tilde{a}_N^{UIV}}-w^L_{\tilde{a}_N^{LIV}}-w^U_{\tilde{a}_N^{UIV}}\right)$$

Definition 3.2 Let $\tilde{a}_{N_1}^{IV}$ and $\tilde{a}_{N_2}^{IV}$ be any two IVTrNeNs, then one has the following comparison:

(a) If $S(\tilde{a}_{N_1}^{IV})<S(\tilde{a}_{N_2}^{IV})\Rightarrow\ \tilde{a}_{N_1}^{IV}<\tilde{a}_{N_2}^{IV}$

(b) If $S(\tilde{a}_{N_1}^{IV})=S(\tilde{a}_{N_2}^{IV})$ with $A(\tilde{a}_{N_1}^{IV})<A(\tilde{a}_{N_2}^{IV})\Rightarrow\tilde{a}_{N_1}^{IV}<\tilde{a}_{N_2}^{IV}$, $A(\tilde{a}_{N_1}^{IV})>A(\tilde{a}_{N_2}^{IV})\Rightarrow\tilde{a}_{N_1}^{IV}>\tilde{a}_{N_2}^{IV}$ and

$A(\tilde{a}^{IV}_{N_1}) = A(\tilde{a}^{IV}_{N_2})$ then $\tilde{a}^{IV}_{N_1} = \tilde{a}^{IV}_{N_2}$.

Example 3.1. Let $\tilde{a}^{IV}_{N_1} = \langle(7,11,16,21),(6,10,15,20),(5,9,14,19);[0.6,0.8],[0.3,0.4],[0.4,0.6]\rangle$ and $\tilde{a}^{IV}_{N_2} = \langle(6,11,13,20),(5,10,12,18),(3,8,11,16);[0.7,0.8],[0.4,0.5],[0.5,0.6]\rangle$ be two SVTrNeNs, then the score and accuracy function $S(\tilde{a}^{IV}_{N_1}) = -4.95833$, $A(\tilde{a}^{IV}_{N_1}) = 5.1$ and $S(\tilde{a}^{IV}_{N_2}) = -9.375$, $A(\tilde{a}^{IV}_{N_2}) = 4.875$. Here $S(\tilde{a}^{IV}_{N_1}) > S(\tilde{a}^{IV}_{N_2})$ and $A(\tilde{a}^{IV}_{N_1}) > A(\tilde{a}^{IV}_{N_2})$ implies that $\tilde{a}^{IV}_{N_1} > \tilde{a}^{IV}_{N_2}$.

4. Neutrosophic Transportation Problem [NTP] and its Mathematical formulation

In a TP, if at least one parameter such as cost, demand, or supply is in form of neutrosophic numbers, then TP is termed as NTP. An NTP has neutrosophic availabilities and neutrosophic demand but the crisp cost is classified as NTP of type-1, if NTP has crisp availabilities and crisp demand but neutrosophic cost, is classified as NTP of type-2. If all the specifications of TP such as cost, demand, and availabilities are a combination of crisp, triangular, or trapezoidal neutrosophic numbers, then it is classified as NTP of type-3. In last if all the specifications of TP must be in neutrosophic numbers form, then TP is said to be NTP of type-4 or fully NTP.

4.1 Mathematical Formulation of NTP

In TP if uncertainty occurs in cost, demand or supply then it is more difficult to find the strict way and time. During the current scenario of covid-19, it is very important for transporting the drugs and medical equipment from one source to another destination in an unchallenging way. Keeping in mind for social distancing the IVTrNeS has a deep concern and special features. To maintain this type of impreciseness in cost to a transferred product from ith sources to jth destination or uncertainty in supply and demand, the decision-maker introduces NTP with IVTrNeNs. Here we discuss NTP of type-4 with IVTrNeNs in cost, supply and demand.

Let the cost and number of units and assumptions and constraints in NTP be defined as IVTrNeNs that are transported from ith sources to jth destination. In the formulation of NTP the following assumptions and constraints are required:

m	**total number of source point**
n	total number of destination point
i	table of source (for all m)
j	table of destination (for all n)
$\tilde{x}^{IV}_{ij}$	number of transported neutrosophic unites from i^{th} source to j^{th} destination
$\tilde{c}^{IV}_{ij}$	Neutrosophic cost of one unit transported from i^{th} source to j^{th} destination
$\tilde{a}^{IV}_{ij}$	available neutrosophic supply quantity from i^{th} source
$\tilde{b}^{IV}_{ij}$	required neutrosophic demand quantity to j^{th} destination
$c^{IV}_{ij} = [c^L_{ij}, c^U_{ij}]$	crisp cost of one unit quantity
$x^{IV}_{ij} = [x^L_{ij}, x^U_{ij}]$	number of transported crisp unites from i^{th} source to j^{th} destination
$a^{IV}_{i} = [a^L_{i}, a^U_{i}]$	available crisp supply quantity from i^{th} source
$b^{IV}_{j} = [b^L_{j}, b^U_{j}]$	required crisp demand quantity to j^{th} destination

For balance of NTP $\sum_{i=0}^{m}\tilde{a}_i^{IV}=\sum_{j=0}^{n}\tilde{a}_j^{IV}$ i.e. total supply is equal to total demand. The objective of this NTP model is to minimize the cost of transported product. The mathematical formulation of NTP with uncertain transported units, cost, demand and supply is as follows:

$$\text{Minimum } \tilde{Z}^{IV}=\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{x}_{ij}^{IV}\tilde{c}_{ij}^{IV}$$

$$\text{Subject to } \sum_{j=0}^{n}\tilde{x}_{ij}^{IV}\approx\tilde{a}_i^{IV},\ \forall i=1,2,3,\ldots,m(\text{sources}),$$

$$\sum_{j=0}^{n}\tilde{x}_{ij}^{IV}\approx\tilde{b}_j^{IV},\forall j=1,2,3,\ldots,n(\text{destination}),$$

$$\tilde{x}_{ij}^{IV}\ge\tilde{0},\ \forall i=1,2,3,\ldots,m, j=1,2,3,\ldots,n.$$

where

$$\tilde{c}_{ij}^{IV}=\begin{Bmatrix}\left(\tilde{c}(a)_{ij}^{IV},\tilde{c}(b)_{ij}^{IV},\tilde{c}(c)_{ij}^{IV},\tilde{c}(d)_{ij}^{IV}\right);\left[u_{\tilde{c}_{ij}^{IV}}^{L},u_{\tilde{c}_{ij}^{IV}}^{U}\right],\\ \left(\tilde{c}(e)_{ij}^{IV},\tilde{c}(f)_{ij}^{IV},\tilde{c}(g)_{ij}^{IV},\tilde{c}(h)_{ij}^{IV}\right)\left[v_{\tilde{c}_{ij}^{IV}}^{L},v_{\tilde{c}_{ij}^{IV}}^{U}\right],\\ \left(\tilde{c}(l)_{ij}^{IV},\tilde{c}(m)_{ij}^{IV},\tilde{c}(n)_{ij}^{IV},\tilde{c}(p)_{ij}^{IV}\right)\left[w_{\tilde{c}_{ij}^{IV}}^{L},w_{\tilde{c}_{ij}^{IV}}^{U}\right]\end{Bmatrix},\quad \tilde{x}_{ij}^{IV}=\left[\begin{Bmatrix}\left(\tilde{x}(a)_{ij}^{IV},\tilde{x}(b)_{ij}^{IV},\tilde{x}(c)_{ij}^{IV},\tilde{x}(d)_{ij}^{IV}\right);\left[u_{\tilde{c}_{ij}^{IV}}^{L},u_{\tilde{c}_{ij}^{IV}}^{U}\right],\\ \left(\tilde{x}(e)_{ij}^{IV},\tilde{x}(f)_{ij}^{IV},\tilde{x}(g)_{ij}^{IV},\tilde{x}(h)_{ij}^{IV}\right)\left[v_{\tilde{c}_{ij}^{IV}}^{L},v_{\tilde{c}_{ij}^{IV}}^{U}\right],\\ \left(\tilde{x}(l)_{ij}^{IV},\tilde{x}(m)_{ij}^{IV},\tilde{x}(n)_{ij}^{IV},\tilde{x}(p)_{ij}^{IV}\right)\left[w_{\tilde{c}_{ij}^{IV}}^{L},w_{\tilde{c}_{ij}^{IV}}^{U}\right]\end{Bmatrix}\right]^{t}$$

$$\tilde{a}_{i}^{IV}=\begin{Bmatrix}\left(\tilde{a}(a)_{ij}^{IV},\tilde{a}(b)_{ij}^{IV},\tilde{a}(c)_{ij}^{IV},\tilde{a}(d)_{ij}^{IV}\right);\left[u_{\tilde{c}_{ij}^{IV}}^{L},u_{\tilde{c}_{ij}^{IV}}^{U}\right],\\ \left(\tilde{a}(e)_{ij}^{IV},\tilde{a}(f)_{ij}^{IV},\tilde{a}(g)_{ij}^{IV},\tilde{a}(h)_{ij}^{IV}\right)\left[v_{\tilde{c}_{ij}^{IV}}^{L},v_{\tilde{c}_{ij}^{IV}}^{U}\right],\\ \left(\tilde{a}(l)_{ij}^{IV},\tilde{a}(m)_{ij}^{IV},\tilde{a}(n)_{ij}^{IV},\tilde{a}(p)_{ij}^{IV}\right)\left[w_{\tilde{c}_{ij}^{IV}}^{L},w_{\tilde{c}_{ij}^{IV}}^{U}\right]\end{Bmatrix},\quad \tilde{b}_{j}^{IV}=\left[\begin{Bmatrix}\left(\tilde{b}(a)_{ij}^{IV},\tilde{b}(b)_{ij}^{IV},\tilde{b}(c)_{ij}^{IV},\tilde{b}(d)_{ij}^{IV}\right);\left[u_{\tilde{c}_{ij}^{IV}}^{L},u_{\tilde{c}_{ij}^{IV}}^{U}\right],\\ \left(\tilde{b}(e)_{ij}^{IV},\tilde{b}(f)_{ij}^{IV},\tilde{b}(g)_{ij}^{IV},\tilde{b}(h)_{ij}^{IV}\right)\left[v_{\tilde{c}_{ij}^{IV}}^{L},v_{\tilde{c}_{ij}^{IV}}^{U}\right],\\ \left(\tilde{b}(l)_{ij}^{IV},\tilde{b}(m)_{ij}^{IV},\tilde{b}(n)_{ij}^{IV},\tilde{b}(p)_{ij}^{IV}\right)\left[w_{\tilde{c}_{ij}^{IV}}^{L},w_{\tilde{c}_{ij}^{IV}}^{U}\right]\end{Bmatrix}\right]^{t}$$

4.2. Steps for Balancing of NTP by Existing Method

The total transportation cost does not depends on the mode of transportation and distance, also the framework of the problem will be denoted by either crisp or IVTrNeNs. For solution of NTP, first we convert all IVTrNeNs into crisp values by using score function and so the NTP converted into simple TP. After balancing by existing method, the following steps are required for solution of NTP:

Step 4.2.1: To change the each neutrosophic cost $\tilde{c}_{ij}^{IV}$ neutrosophic supply $\tilde{a}_i^{IV}$ and neutrosophic demand $\tilde{b}_j^{IV}$ of NTP in cost matrix into crisp values by using score function $S(\tilde{a}_N^{IV})$.

Step 4.2.2: For balance TP, verify that the sum of demands is equal to the sum of supply i.e. If $\sum_{i=0}^{m}\tilde{a}_i^{IV}<\sum_{j=0}^{n}\tilde{b}_j^{IV}$ or $\sum_{i=0}^{m}\tilde{a}_i^{IV}>\sum_{j=0}^{n}\tilde{b}_j^{IV}\ \forall i,j$ the one can make sure to balance the TP, as $\sum_{i=0}^{m}\tilde{a}_i^{IV}=\sum_{j=0}^{n}\tilde{b}_j^{IV},\ \forall i,j$, by adding a row or column with zero entries in cost matrix.

Step 4.2.3: Verify that the sum of demands is greater than the supply in each row and the sum of supplies are greater than the demand in each column, if ok go on step 4.2.4, otherwise go on step 4.2.2

Step 4.2.4: Here we use the excel solver to solve the TP and obtained optimal solution.

4.3. Steps for Balancing of NTP by MRCM

For balance the unbalance NTP, we use minimum row column method [MRCM] introduced by Saini [45] as follows:

Step 4.3.1. Convert neutrosophic cost $\tilde{c}_{ij}^{IV}$ neutrosophic supply $\tilde{a}_{i}^{IV}$ and neutrosophic demand $\tilde{b}_{j}^{IV}$ of NTP in cost matrix to crisp values by using score function $S(\tilde{a}_N^{IV})$.

Step 4.3.2 If NTP is unbalance i.e. $\sum_{i=0}^{m}\tilde{a}_i^{IV} < \text{or} > \sum_{j=0}^{n}\tilde{b}_j^{IV}$, $\forall i, j$ than we find

$$\tilde{a}_{i(m+1)}^{IV} = \sum_{i=0}^{m}\tilde{a}_i^{IV} \text{ and } \tilde{b}_{j(n+1)}^{IV} = \sum_{j=0}^{n}\tilde{b}_j^{IV} \oplus \text{excess supply},$$

or

$$\tilde{b}_{j(n+1)}^{IV} = \sum_{j=0}^{n}\tilde{b}_j^{IV} \text{ and } \tilde{a}_{i(m+1)}^{IV} = \sum_{i=1}^{m}\tilde{a}_i^{IV} \oplus \text{excess demand}.$$

The unit transportation costs are taken as follows:

$$\tilde{c}_{i(n+1)}^{IV} = \min_{1\le j\le n} \tilde{c}_{ij}^{IV},\ 1 \le i \le m, \quad \tilde{c}_{(m+1)j}^{IV} = \min_{1\le i\le m} \tilde{c}_{ij}^{IV},\ 1 \le j \le n,$$

$$\tilde{c}_{ij}^{IV} = \tilde{c}_{ji}^{IV},\ 1 \le i \le m,\ 1 \le j \le n, \text{ and } \tilde{c}_{(m+1)(n+1)}^{IV} = 0.$$

Step 4.3.3 Obtain optimal solution of NTP by excel solver. Let the neutrosophic optimal solution obtained be $\tilde{x}_{ij}^{IV}$, $1 \le i \le m+1$, $1 \le j \le n+1$.

Step 4.3.4 By assuming $\tilde{\omega}_{m+1}^{'IV} = 0$ and using the relation $\tilde{\omega}_i^{'IV} \oplus \tilde{v}_j^{'IV} = \tilde{\sigma}_{ij}^{'IV}$ for basic variables, find the values of all the dual variables $\tilde{\omega}_i^{'IV}$, $1 \le i \le m$ and $\tilde{v}_j^{'IV}$, $1 \le j \le n+1$,

Step 4.3.5. According to MRCM, $\tilde{\omega}_i^{'IV} = \tilde{\omega}_i^{IV}$ and $\tilde{v}_j^{'IV} = \tilde{v}_j^{IV}$ for $1 \le i \le m, 1 \le j \le n$, obtain only central rank zero duals.

5. Numerical Example

Let us consider a NTP of type-4 with three container (sources) say M_1, M_2, M_3 in which medical equipment are initially stored and ready to transport in three different destinations (cities), say C_1, C_2, C_3 with unit transportation cost, demand and supply are as IVTrNeN. The input data of NTP with IVTrNeNs is given in table 1:

Table 1	M_1	M_2	M_3	*Supply*
C_1	$\langle$(7,12,21.5,28);[0.7,0.9], (4,10,17.5,25);[0.4,0.5], (2,8,15.5,22);[0.3,0.4]$\rangle$	$\langle$(7,11,15,19);[0.6,0.8] (5,8,12,15);[0.4,0.5] (3,5,7,11);[0.2,0.3]$\rangle$	$\langle$5,10,14,19.5);[0.5,0.6], (3,7.5,10,15);[0.4,0.6], (1,4,7,10);[0.3,0.4]$\rangle$	$\langle$(9,19,28,34);[0.7,0.8], (7,12,19,24);[0.4,0.5], (3,8,11,16);[0.2,0.4]$\rangle$
C_2	$\langle$(5,11,16.5,21);[0.6,0.7], (3,8,12,16);[0.3,0.5], (0,3,9.5,12);[0.2,0.4]$\rangle$	$\langle$(2,4,7.5,10);[0.6,0.7], (1,3,6.5,9);[0.3,0.5], (−1,2,5,7);[0.2,0.3]$\rangle$	$\langle$(3,6,11,16);[0.7,0.9], (1,5,9,14);[0.4,0.5], (−3,2,6,12);[0.3,0.4]$\rangle$	$\langle$8,14,25,35);[0.7,0.8], (4,10,18,28);[0.3,0.5], (1,8,14,22);[0.3,0.4]$\rangle$
C_3	$\langle$(6,14,21,28);[0.8,0.9], (4,11,18,25);[0.4,0.6], (2,8,15,22);[0.3,0.4]$\rangle$	$\langle$(4,8.5,14,17);[0.6,0.8], (2,6.5,11,15);[0.4,0.5], (−2,2,7,11);[0.2,0.3]$\rangle$	$\langle$(5,10,14,20);[0.7,0.9], (3,8,9,15);[0.3,0.5], (−1,5,7,12);[0.2,0.4]$\rangle$	$\langle$(14,22,30,39);[0.6,0.8], (12,18,25,34);[0.3,0.5], (8,15,23,31);[0.2,0.4]$\rangle$
Demand	$\langle$(10,20,28,35);[0.7,0.9], (4,12,22.5,29);[0.4,0.5], (-1,7,12,19);[0.3,0.4],$\rangle$	$\langle$6,12,23,33);[0.7,0.8], (4,10,19,28);[0.4,0.5], (2,8,15,24);[0.2,0.4]$\rangle$	$\langle$(12,18,25,33);[0.7,0.9], (9,16,23,30);[0.4,0.5], (5,14,20,27);[0.3,0.4]$\rangle$	

With the help of score function, the cost, demand and supply of NTP i.e. in IVTrNeNs are convert into the crisp numbers as follows:

$S(\tilde{c}_{11}^{IV}) = -4.58333$, $S(\tilde{c}_{12}^{IV}) = -1$, $S(\tilde{c}_{13}^{IV}) = -0.11667$, $S(\tilde{a}_{1}^{IV}) = -0.33333$, $S(\tilde{c}_{21}^{IV}) = -0.31667$, $S(\tilde{c}_{22}^{IV}) = -0.16667$, $S(\tilde{c}_{23}^{IV}) = -0.333333$, $S(\tilde{a}_{2}^{IV}) = -2.50$, $S(\tilde{c}_{31}^{IV}) = -4.66667$, $S(\tilde{c}_{32}^{IV}) = -0.16667$, $S(\tilde{c}_{33}^{IV}) = -0.183333$,

$S(\tilde{a}_{3}^{IV}) = -8.83334$, $S(\tilde{b}_{1}^{IV}) = -0.58333$, $S(\tilde{b}_{2}^{IV}) = -4.66667$, $S(\tilde{b}_{3}^{IV}) = -8.0$

The unbalance TP with crisp values shown in table 2:

Table 2	M_1	M_2	M_3	***Supply***
C_1	-4.58333	-1	-0.11667	***-0.33333***
C_2	-0.31667	-0.16667	-0.33333	***-2.50***
C_3	-4.66667	-0.16667	-0.18333	***-8.83334***
Demand	***-0.58333***	***-4.66667***	***-8.0***	

In table 2, $\sum_{i=0}^{m} \tilde{a}_i^{IV} = -11.6667$, $\sum_{j=0}^{n} \tilde{b}_j^{IV} = -13.25$, i.e. $\sum_{i=0}^{m} \tilde{b}_j^{IV} - \sum_{j=0}^{n} \tilde{a}_i^{IV} = 1.58300$, this shows that NTP is unbalanced. The balance TP and the solution of NTP in crisp form by excel solver shown in table 3 and table 4 respectively as follows:

Table 3	M_1	M_2	M_3	M_4	*Supply*
C_1	-4.58333	-1	-0.11667	0	***-0.33333***
C_2	-0.31667	-0.16667	-0.33333	0	***-2.50***
C_3	-4.66667	-0.16667	-0.18333	0	***-8.83334***
Demand	***-0.58333***	***-4.66667***	***-8.0***	***1.58300***	

Table 4	M_1	M_2	M_3	M_4	*Supply*
C_1	1.91667	-3.83333	-	1.583	***-0.33333***
C_2	-2.5	-	-	-	***-2.50***
C_3	-	-0.83334	-8	-	***-8.83334***
Demand	***-0.58333***	***-4.66667***	***-8.0***	***1.58300***	

The optimal solution of NTP in crisp form is $Z_{CTP} = -2.55419$. The solution of NTP with IVTrNeNs shown in table 5:

Table 5	M_1	M_2	M_3	M_4	*Supply*
C_1	$\left\langle \begin{matrix}(-25,-5,14,27);[0.7,0.9],\\(-24,-6,12.5,25);[0.3,0.5],\\(-23,-7,4,18);[0.3,0.4]\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(-71,-14,49,104);[0.7,0.8]\\(-62,-12,43.5,88);[0.3,0.5]\\(-60,-11,27,74);[0.2,0.4]\end{matrix} \right\rangle$	-	$\left\langle \begin{matrix}(-70,-21,33,80);[0.6,0.8],\\(-64,-24.5,24,69);[0.3,0.5],\\(-58,-16,19,63);[0.2,0.4]\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(9,19,28,34);[0.7,0.8],\\(7,12,19,24);[0.4,0.5],\\(3,8,11,16);[0.2,0.4]\end{matrix} \right\rangle$
C_2	$\left\langle \begin{matrix}8,14,25,35);[0.7,0.8],\\(4,10,18,28);[0.3,0.5],\\(1,8,14,22);[0.3,0.4]\end{matrix} \right\rangle$	-	-	-	$\left\langle \begin{matrix}8,14,25,35);[0.7,0.8],\\(4,10,18,28);[0.3,0.5],\\(1,8,14,22);[0.3,0.4]\end{matrix} \right\rangle$
C_3	-	$\left\langle \begin{matrix}(-19,-3,12,27);[0.7,0.9],\\(-18,-5,9,25);[0.3,0.5],\\(-19,-5,9,26);[0.2,0.4]\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(12,18,25,33);[0.7,0.9],\\(9,16,23,30);[0.4,0.5],\\(5,14,20,27);[0.3,0.4]\end{matrix} \right\rangle$	-	$\left\langle \begin{matrix}(14,22,30,39);[0.6,0.8],\\(12,18,25,34);[0.3,0.5],\\(8,15,23,31);[0.2,0.4]\end{matrix} \right\rangle$
Demand	$\left\langle \begin{matrix}(10,20,28,35);[0.7,0.9],\\(4,12,22.5,29);[0.4,0.5],\\(-1,7,12,19);[0.3,0.4],\end{matrix} \right\rangle$	$\left\langle \begin{matrix}6,12,23,33);[0.7,0.8],\\(4,10,19,28);[0.4,0.5],\\(2,8,15,24);[0.2,0.4]\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(12,18,25,33);[0.7,0.9],\\(9,16,23,30);[0.4,0.5],\\(5,14,20,27);[0.3,0.4]\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(-70,-21,33,80);[0.6,0.8],\\(-64,-24.5,24,69);[0.3,0.5],\\(-58,-16,19,63);[0.2,0.4]\end{matrix} \right\rangle$	

$$Z_{NTP}=\left\langle\begin{matrix}(7,12,21.5,28);[0.7,0.9],\\(4,10,17.5,25);[0.4,0.5],\\(2,8,15.5,22);[0.3,0.4]\end{matrix}\right\rangle.\left\langle\begin{matrix}(-25,-5,14,27);[0.7,0.9],\\(-24,-6,12.5,25);[0.3,0.5],\\(-23,-7,4,18);[0.3,0.4]\end{matrix}\right\rangle+\left\langle\begin{matrix}(7,11,15,19);[0.6,0.8]\\(5,8,12,15);[0.4,0.5]\\(3,5,7,11);[0.2,0.3]\end{matrix}\right\rangle.\left\langle\begin{matrix}(-71,-14,49,104);[0.7,0.8]\\(-62,-12,43.5,88);[0.3,0.5]\\(-60,-11,27,74);[0.2,0.4]\end{matrix}\right\rangle$$

$$+\left\langle\begin{matrix}(-70,-21,33,80);[0.6,0.8],\\(-64,-24.5,24,69);[0.3,0.5],\\(-58,-16,19,63);[0.2,0.4]\end{matrix}\right\rangle.\left\langle\begin{matrix}(0,0,0,0);[0.6,0.8],\\(0,0,0,0);[0.3,0.5],\\(0,0,0,0);[0.2,0.4]\end{matrix}\right\rangle+\left\langle\begin{matrix}(5,11,16.5,21);[0.6,0.7],\\(3,8,12,16);[0.3,0.5],\\(0,3,9.5,12);[0.2,0.4]\end{matrix}\right\rangle.\left\langle\begin{matrix}8,14,25,35);[0.7,0.8],\\(4,10,18,28);[0.3,0.5],\\(1,8,14,22);[0.3,0.4]\end{matrix}\right\rangle$$

$$+\left\langle\begin{matrix}(4,8.5,14,17);[0.6,0.8],\\(2,6.5,11,15);[0.4,0.5],\\(-2,2,7,11);[0.2,0.3]\end{matrix}\right\rangle.\left\langle\begin{matrix}(-19,-3,12,27);[0.7,0.9],\\(-18,-5,9,25);[0.3,0.5],\\(-19,-5,9,26);[0.2,0.4]\end{matrix}\right\rangle+\left\langle\begin{matrix}(5,10,14,20);[0.7,0.9],\\(3,8,9,15);[0.3,0.5],\\(-1,5,7,12);[0.2,0.4]\end{matrix}\right\rangle.\left\langle\begin{matrix}(12,18,25,33);[0.7,0.9],\\(9,16,23,30);[0.4,0.5],\\(5,14,20,27);[0.3,0.4]\end{matrix}\right\rangle$$

i.e. $$Z_{NTP}=\left\langle\begin{matrix}(-648,94.5,1966.5,4586);[0.6,0.9],\\(-403,19.5,1262.75,3218);[0.3,0.5],\\(-193,-27,587,2084);[0.2,0.3]\end{matrix}\right\rangle\approx-7.07292$$

Now we balance the unbalance CTP in table 2 by MRCM, the balance CTP with crisp numbers shown in table 6 as follows:

Table 6	M_1	M_2	M_3	M_4	*Supply*
C_1	-4.58333	-1	-0.11667	-4.58333	***-0.33333***
C_2	-0.31667	-0.16667	-0.33333	-0.33333	***-2.50***
C_3	-4.66667	-0.16667	-0.18333	-4.66667	***-8.83334***
C_4	-4.66667	-1	-0.33333	***0***	***-11.6667***
Demand	***-0.58333***	***-4.66667***	***-8.0***	***-10.0834***	

The solution of balance CTP as in table 7

Table 7	M_1	M_2	M_3	M_4	*Supply*
C_1	10.75001	-4.66667	-8	1.5833	***-0.33333***
C_2	-2.5	0	0	0	***-2.50***
C_3	-8.83334	0	0	0	***-8.83334***
C_4	0	0	0	-11.6667	***-11.6667***
Demand	***-0.58333***	***-4.66667***	***-8.0***	***-10.0834***	

The cost $Z_{CTP(MRCM)}=-8.91364$.

The solution of corresponding balanced NTP shown in table 8 as follows:

Table 8	M_1	M_2	M_3	M_4	*Supply*
C_1	$\left\langle\begin{matrix}(-64,-35,-8,13);[0.7,0.9],\\(-48,-31,-5.5,13);[0.4,0.5],\\(-54,-30,-11,10);[0.3,0.4]\end{matrix}\right\rangle$	$\left\langle\begin{matrix}6,12,23,33);[0.7,0.8],\\(4,10,19,28);[0.4,0.5],\\(2,8,15,24);[0.2,0.4]\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(12,18,25,33);[0.7,0.9],\\(9,16,23,30);[0.4,0.5],\\(5,14,20,27);[0.3,0.4]\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(-147,-49,61,157);[0.7,0.9],\\(-127,-46.5,46,132);[0.4,0.5],\\(-115,-33,36,120);[0.3,0.4]\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(9,19,28,34);[0.7,0.8],\\(7,12,19,24);[0.4,0.5],\\(3,8,11,16);[0.2,0.4]\end{matrix}\right\rangle$
C_2	$\left\langle\begin{matrix}8,14,25,35);[0.7,0.8],\\(4,10,18,28);[0.3,0.5],\\(1,8,14,22);[0.3,0.4]\end{matrix}\right\rangle$	-	-	-	$\left\langle\begin{matrix}8,14,25,35);[0.7,0.8],\\(4,10,18,28);[0.3,0.5],\\(1,8,14,22);[0.3,0.4]\end{matrix}\right\rangle$
C_3	$\left\langle\begin{matrix}(14,22,30,39);[0.6,0.8],\\(12,18,25,34);[0.3,0.5],\\(8,15,23,31);[0.2,0.4]\end{matrix}\right\rangle$	-	-	-	$\left\langle\begin{matrix}(14,22,30,39);[0.6,0.8],\\(12,18,25,34);[0.3,0.5],\\(8,15,23,31);[0.2,0.4]\end{matrix}\right\rangle$
C_4	-	-	-	$\left\langle\begin{matrix}(31,55,83,108);[0.6,0.8],\\(23,40,62,86);[0.3,0.5],\\(12,31,48,69);[0.2,0.4]\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(31,55,83,108);[0.6,0.8],\\(23,40,62,86);[0.3,0.5],\\(12,31,48,69);[0.2,0.4]\end{matrix}\right\rangle$
Demand	$\left\langle\begin{matrix}(10,20,28,35);[0.7,0.9],\\(4,12,22.5,29);[0.4,0.5],\\(-1,7,12,19);[0.3,0.4],\end{matrix}\right\rangle$	$\left\langle\begin{matrix}6,12,23,33);[0.7,0.8],\\(4,10,19,28);[0.4,0.5],\\(2,8,15,24);[0.2,0.4]\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(12,18,25,33);[0.7,0.9],\\(9,16,23,30);[0.4,0.5],\\(5,14,20,27);[0.3,0.4]\end{matrix}\right\rangle$	$\left\langle\begin{matrix}(-39,34,116,188);[0.7,0.9],\\(-41,15.5,86,155);[0.4,0.5],\\(-46,15,67,132);[0.3,0.4]\end{matrix}\right\rangle$	

$$Z_{NTP(MRCM)} = \left\langle \begin{matrix} (-64,-35,-8,13);[0.7,0.9], \\ (-48,-31,-5.5,13);[0.4,0.5], \\ (-54,-30,-11,10);[0.3,0.4] \end{matrix} \right\rangle . \left\langle \begin{matrix} (7,12,21.5,28);[0.7,0.9], \\ (4,10,17.5,25);[0.4,0.5], \\ (2,8,15.5,22);[0.3,0.4] \end{matrix} \right\rangle + \left\langle \begin{matrix} 6,12,23,33);[0.7,0.8], \\ (4,10,19,28);[0.4,0.5], \\ (2,8,15,24);[0.2,0.4] \end{matrix} \right\rangle . \left\langle \begin{matrix} (7,11,15,19);[0.6,0.8] \\ (5,8,12,15);[0.4,0.5] \\ (3,5,7,11);[0.2,0.3] \end{matrix} \right\rangle + \left\langle \begin{matrix} (12,18,25,33);[0.7,0.9], \\ (9,16,23,30);[0.4,0.5], \\ (5,14,20,27);[0.3,0.4] \end{matrix} \right\rangle . \left\langle \begin{matrix} 5,10,14,19.5);[0.5,0.6], \\ (3,7.5,10,15);[0.4,0.6], \\ (1,4,7,10);[0.3,0.4] \end{matrix} \right\rangle$$

$$+ \left\langle \begin{matrix} (-147,-49,61,157);[0.7,0.9], \\ (-127,-46.5,46,132);[0.4,0.5], \\ (-115,-33,36,120);[0.3,0.4] \end{matrix} \right\rangle . \left\langle \begin{matrix} (7,12,21.5,28);[0.7,0.9], \\ (4,10,17.5,25);[0.4,0.5], \\ (2,8,15.5,22);[0.3,0.4] \end{matrix} \right\rangle + \left\langle \begin{matrix} 8,14,25,35);[0.7,0.8], \\ (4,10,18,28);[0.3,0.5], \\ (1,8,14,22);[0.3,0.4] \end{matrix} \right\rangle . \left\langle \begin{matrix} (5,11,16.5,21);[0.6,0.7], \\ (3,8,12,16);[0.3,0.5], \\ (0,3,9.5,12);[0.2,0.4] \end{matrix} \right\rangle + \left\langle \begin{matrix} (14,22,30,39);[0.6,0.8], \\ (12,18,25,34);[0.3,0.5], \\ (8,15,23,31);[0.2,0.4] \end{matrix} \right\rangle . \left\langle \begin{matrix} (6,14,21,28);[0.8,0.9], \\ (4,11,18,25);[0.4,0.6], \\ (2,8,15,22);[0.3,0.4] \end{matrix} \right\rangle$$

$$+ \left\langle \begin{matrix} (31,55,83,108);[0.6,0.8], \\ (23,40,62,86);[0.3,0.5], \\ (12,31,48,69);[0.2,0.4] \end{matrix} \right\rangle . \left\langle \begin{matrix} (0,0,0,0);[0.6,0.8], \\ (0,0,0,0);[0.3,0.5], \\ (0,0,0,0);[0.2,0.4] \end{matrix} \right\rangle$$

$$Z_{NTP(MRCM)} = \left\langle \begin{matrix} (-1230,-234,3221,7857.5);[0.6,0.9], \\ (-593,-297,1832.75,5793);[0.3,0.5], \\ (-311,-264,1110.5,4340);[0.2,0.3] \end{matrix} \right\rangle \approx -13.810$$

6. Comparative Study

To maintain physical distance during Covid-19 pandemic, we introduced here some advanced version of neutrosophic numbers such as IVTrNeNs, which provides the better results in real life for uncertainty and hesitation in place of crisp numbers. For practical application of NTP type-4, the minimum cost of unbalanced CTP and NTP obtained by VAM and MRCM is summarized in table 9. It is also clear from the table 9, that minimum cost of unbalanced CTP and NTP obtained by using MRCM is far better than the existing method VAM. In figure 4, the bar graph represents the minimum cost of CTP and NTP and their comparison for better one.

Table 9: **Comparative Study**

Balance of CTP by existing method	**Balance of CTP by MRCM**
$Z_{CTP} = -2.55419$	$Z_{CTP(MRCM)} = -8.91364$
Balance of NTP by existing method	**Balance of NTP by MRCM**
$Z_{NTP} = \left\langle \begin{matrix} (-648,94.5,1966.5,4586);[0.6,0.9], \\ (-403,19.5,1262.75,3218);[0.3,0.5], \\ (-193,-27,587,2084);[0.2,0.3] \end{matrix} \right\rangle$	$Z_{NTP(MRCM)} = \left\langle \begin{matrix} (-1230,-234,3221,7857.5);[0.6,0.9], \\ (-593,-297,1832.75,5793);[0.3,0.5], \\ (-311,-264,1110.5,4340);[0.2,0.3] \end{matrix} \right\rangle$
≈ -7.07292	≈ -13.810

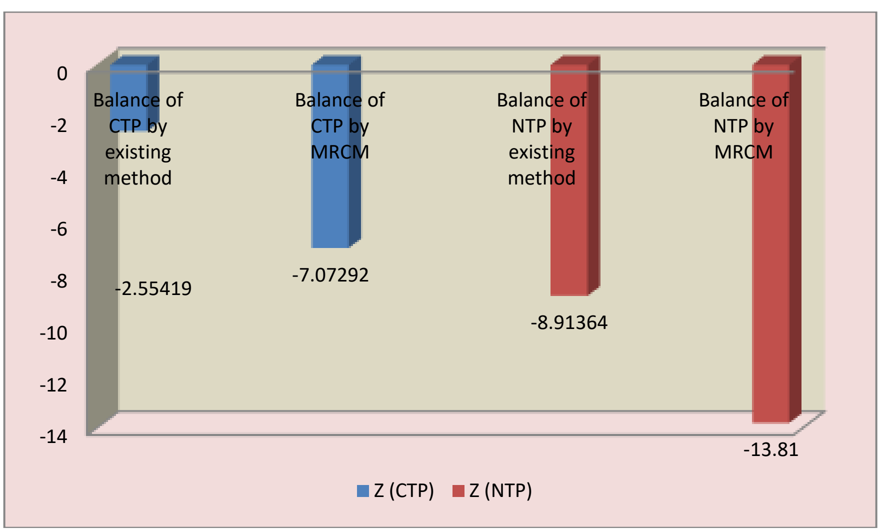

Figure 3: Comparison of results by chart

8. Result and discussion

In this present study the optimal transportation crisp cost and optimal transportation neutrosophic cost of unbalanced NTP using MRCM is minimum than the existing method in [30]. It is also verified that in de-neutrosophication, the crisp values before and after conversion from neutrosophic to crisp and crisp to neutrosophic are different. For the real life applications one can find the degree of result. The best of minimum neutrosophic cost of unbalanced NTP is

$$Z_{NTP(MRCM)} = \left\langle \begin{array}{l} (\text{-}1230, \text{-}234, 3221, 7857.5); [0.6, 0.9], \\ (\text{-}593, \text{-}297, 1832.75, 5793); [0.3, 0.5], \\ (\text{-}311, \text{-}264, 1110.5, 4340); [0.2, 0.3] \end{array} \right\rangle$$

i.e. total minimum transportation cost lies between -1230 to 7857.5 in the interval [0.6, 0.9] for level of truthfulness, -593 to 5793 in the interval [0.3, 0.5] for level of indeterminacy and -311 to 4340 in the interval [0.2, 0.3] for level of falsity. $u_{T_{\tilde{a}_N^{IV}}} \times 100$, $v_{I_{\tilde{a}_N^{IV}}} \times 100$, and $w_{F_{\tilde{a}_N^{IV}}} \times 100$ represents the degree of truthfulness, degree of indeterminacy and degree of falsity respectively. Thus

$$u_{T_{\tilde{a}_N^{IV}}}(x) = \begin{cases} \left(\dfrac{x+1230}{1230-234}\right)[0.6, 0.9], & \text{for } -1230 \le x \le -234, \\ [0.6, 0.9], & \text{for } -234 \le x \le 3221, \\ \left(\dfrac{7857.5-x}{7857.5-3221}\right)[0.6, 0.9], & \text{for } 3221 \le x \le 7857.5, \\ 0, & \text{for otherwise.} \end{cases}$$

$$v_{I_{\tilde{a}_N^{IV}}}(x) = \begin{cases} \dfrac{(-297-x)+(x+593)[0.3, 0.5]}{593-297}, & \text{for } -593 \le x \le -297, \\ [0.3, 0.5], & \text{for } -297 \le x \le 1832.75, \\ \dfrac{(x-1832.75)+(5793-x)[0.3, 0.5]}{5793-1832.75}, & \text{for } 1832.75 \le x \le 5793, \\ 0, & \text{for otherwise.} \end{cases}$$

$$w_{F_{\tilde{a}_N^{IV}}}(x) = \begin{cases} \dfrac{(-264-x)+(x+311)[0.2, 0.3]}{311-264}, & \text{for } -311 \le x \le -264, \\ [0.2, 0.3], & \text{for } -264 \le x \le 1110.5, \\ \dfrac{(x-1110.5)+(4340-x)[0.2, 0.3]}{4340-1110.5}, & \text{for } 1110.5 \le x \le 4340, \\ 0, & \text{for otherwise.} \end{cases}$$

where x denotes the total cost.

Table 10

$x \rightarrow$ Degree $\downarrow$	-500	0	2000	3000	4000	5000	7000
$u_{T_{\tilde{a}_N^{IV}}} \times 100$	[43.976, 5.964]	[60, 90]	[60, 90]	[60, 90]	[49.919, 74.879]	[36.978, 55.467]	[11.097, 16.645]
$v_{I_{\tilde{a}_N^{IV}}} \times 100$	[78.007, 84.290]	[30, 50]	[32.969, 52.124]	[50.632, 64.737]	[68.307, 77.362]	[85.983, 89.987]	-
$w_{F_{\tilde{a}_N^{IV}}} \times 100$	-	[20, 30]	[42.034, 49.280]	[66.806, 70.956]	[91.577, 92.630]	-	-

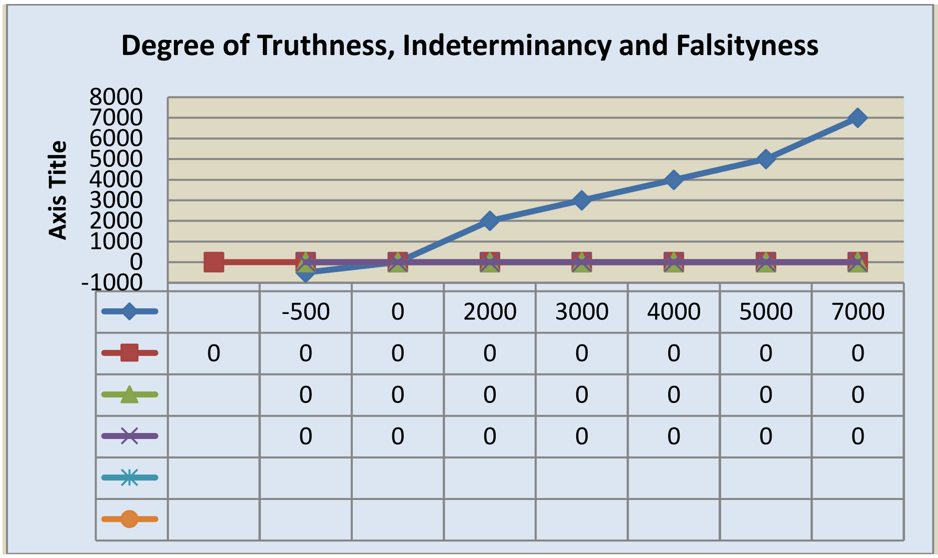

Figure 3: Degree of truthfulness, indeterminacy and falsity

The total neutrosophic cost from the range of -1230 to 7857.5 for truthfulness, -593 to 5793 for indeterminacy and -311to 4340 for falsity are concluded by degree of truthfulness, degree of indeterminacy and degree of falsity to schedule the transportation cost and budget allocation.

9. Conclusions and Novelty

Today in society, the concept of neutrosophic numbers is well linked to handling uncertainty or vagueness in applied mathematical modeling. The current research paper is the study of unbalanced CTP & NTP by introducing a new balancing approach MRCM to obtain an optimal solution where all parameters and values of TP are as IVTrNeNs. The proposed ranking function provides a more practical structure and considers the various characteristics of TP in a neutrosophic environment. Such a type of transportation problem with IVTrNeNs and their comparison between the two methods are not introduced earlier, and we hope that in the future, the proposed MRCM will be more applicable to the multilevel programming problem, unbalanced multi-attribute transportation problem, and multi-level assignment problems. The existing analysis will be a landmark for TP's with generalization by considering the pick value of truth, indeterminacy, and falsity functions and for schedule transportation cost and budget allocation for the total neutrosophic cost, that concluded by a degree of truthfulness, degree of indeterminacy, and degree of falsity.

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Operations on Neutrosophic Vague Soft Graphs

S. Satham Hussain [1,*], R. Jahir Hussain [1], Ghulam Muhiuddin [2] and P. Anitha [3]
[1]PG & Research Department of Mathematics, Jamal Mohamed College(Autonomous), Tiruchirappalli - 620 020, Tamil Nadu, India.
[2]Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk - 71491, Saudi Arabia.
[3]PG and Research Department of Mathematics, H.K.R.H College, Uthamapalayam - 625 533, Tamil Nadu, India.
*Correspondence: E-mail: sathamhussain5592@gmail.com (S. Satham Hussain)
E-mail: hssn_jhr@yahoo.com (R. Jahir Hussain), E-mail: chishtygm@gmail.com (Ghulam Muhiuddin)
E-mail: anieparthi@gmail.com (P. Anitha)

Abstract: This article concerns with the neutrosophic vague soft graphs for treating neutrosophic vague soft information by employing the theory of neutrosophic vague soft sets with graphs. Operations like Cartesian product, cross product, lexicographic product, strong product and composition of neutrosophic vague soft graphs are established. The proposed concepts are explained with examples.

Keywords: Cartesian product; Neutrosophic vague soft graph; Operations on neutrosophic vague soft graph.
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1 Introduction

Nowadays, the great success of neutrosophic sets in modelling natural phenomena is that its efficiency to hold incomplete data, and handling of indeterminate information. It is the base of neutrosophic logic, a multiple value logic that generalizes the fuzzy logic that carries with paradoxes, contradictions, antitheses, antinomies, invented by the author Smarandache [8, 21, 22]. For example, suppose there are hundred patients to check a pandemic during testing. In that time, there are thirty patients having positive, fifty will have negative, and twenty are undecided or yet to come. By employing the neutrosophic concepts it can be expressed as $x(0.3,0.2,0.5)$. Hence the neutrosophic field arises to hold the indeterminacy data more accurately. It generalizes many concepts from the philosophical viewpoint. The single-valued neutrosophic set is the generalization of intuitionistic fuzzy sets and is used expediently to deal with real-world problems, especially in decision support [23]. The computation of belief in that element (truth), they disbelieve in that element (falsehood) and the indeterminacy part of that element with the sum of these three components are strictly less than 1. Neutrosophic set and related notions have shown applications in many different fields. In the definition of single valued neutrosophic set, the indeterminacy value is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are defined completely independent with the sum of these values lies between 0 and 3 (see [21]-[23]). The indeterminacy function is considered as an individual term and each element x is characterized by a truth-membership function $\mathcal{T}_A(x)$, an indeterminacy membership function $\mathcal{I}_A(x)$ and a falsity-membership function $\mathcal{F}_A(x)$, each of that from the non-standard unit interval $]0^-,1^+[$. Despite the neutrosophic indeterminacy is independent of the truth and falsity-membership values, but it is more general than the hesitation margin of intuitionistic fuzzy sets. It's not sure whether the indeterminacy values relevant to a particular element correspond to hesitant values about its belonging or not belonging to it. In another way, if a person identifies an indeterminacy membership $\mathcal{I}_A(x)$ with a specific event x, it becomes difficult to understand whether the person's degree of uncertainty regarding the event's occurrence is $\mathcal{I}_A(x)$ or whether the person's degree of uncertainty regarding the event's non-occurrence is $\mathcal{I}_A(x)$. As a result, some authors prefer to model the indeterminacy's behaviour in the same way they similar to truth-membership, others may prefer to model it, in the same way, they similar to falsity-membership. Wang et al. [23] initiated the concept of a single-valued neutrosophic set and provide its various properties.

Molodtsov [20] successfully proposed a completely new theory namely soft set theory by using classical sets in 1999 and after that, there has been a rapid development of interest in soft sets and their various applications [7, 9]. This theory provides a parametrized point of view for uncertainty modelling and soft computing. Vague sets are considered as a particular case of context-dependent fuzzy sets. Vague sets are studied by Gau and Buehrer [24] as an extension of fuzzy set theory. Neutrosophic soft rough graphs with applications are established in [5]. Neutrosophic soft relations and neutrosophic refined relations with their properties are studied in [8, 18]. Recently, the generalization of neutrosophic graphs are developed in [25]. Also, neutrosophic soft graphs, neutrosophic graphs, co-neutrosophic graphs,single valued neutrosophic graphs are established in [2, 10, 12]. Neutrosophic vague set is first invented by the author in [6]. In [3], the authors studied the notion of neutrosophic vague soft expert set as a combination of soft expert set and neutrosophic vague set to the substantial improvement in decision making. Also, the neutrosophic vague soft set is studied by him [4] with application in decision making problems. In [2], the certain notions, including neutrosophic soft graphs, strong neutrosophic soft graphs, complete neutrosophic soft graphs are discussed. Neutrosophic vague graphs are introduced in [15]. Intuitionistic bipolar neutrosophic set and its application to graphs are established in [16]. Motivated by papers [2, 4, 6, 15, 17], we introduce the concept of operations on neutrosophic vague soft graphs. The major contributions of this work are as follows:

- In this paper, we present a novel frame work for handling neutrosophic vague soft information by combining the theory of neutrosophic vague soft sets with graphs.
- The operations on neutrosophic vague soft graphs are established and this manuscript makes the first attempt in this domain. Some basic definitions regarding to neutrosophic vague graphs are explained with example.
- Results on the Cartesian product, cross product, lexicographic product, strong product and composition of neutrosophic vague graph are illustrated with examples.
- The validity of the developed method is verifying in multi-attribute decision-making method based on neutrosophic vague soft graphs.
- Finally, a conclusion is elaborated with future direction.

The paper is organised as follows: Some elementary definitions and results are provided in Section 2. Operations on neutrosophic vague soft graphs with example are established in Section 3. In Section 4, the multi-attribute decision making method is solved for neutrosophic vague soft graphs, in that, the solving procedure is based on the score function S_{ij} [25]. Finally, the advantages and limitations of the proposed concepts are given.

2 Preliminaries

In this section, basic definitions and example are given.

Definition 2.1 *[20] Let $\mathcal{U}$ be the universe of discourse and $\mathcal{P}$ be the universe of all possible parameters related to the objects in $\mathcal{U}$. Each parameters are considered to be attributes, characteristics of objects in $\mathcal{U}$. The pair $(\mathcal{U},\mathcal{P})$ is also known as a soft universe. The power set of $\mathcal{U}$ is dentoted as $\rho(\mathcal{U})$*

Definition 2.2 *[20] A pair (F,A) is called soft set over $\mathcal{U}$, where $A\subseteq P$, F is a set- valued function $F\colon A\to\rho(\mathcal{U})$. In other words, a soft set over $\mathcal{U}$ is a parametrized family of subsets of $\mathcal{U}$.*

By means of parametrization, a soft set produces a series of approximate descriptions of a complicated object being perceived from various points of view. It is apparent that a soft set $F_A=(F,A)$ over a universe $\mathcal{U}$. For any parameter $\epsilon\in A$, the subset $F(\epsilon)\subseteq\mathcal{U}$ may be interpreted as the set of ϵ-approximate elements.

Definition 2.3 *[19] Let $\mathcal{U}$ be an initial universe and P be a set of parameters. Consider $A \subseteq P$. Let $p(\mathcal{U})$ denotes the set of all neutrosophic sets of $\mathcal{U}$. The collection of (F, A) is termed to be neutrosophic soft set over $\mathcal{U}$, where F is a mapping given by $F: A \to P(\mathcal{U})$.*

Definition 2.4 *[24] A vague set A on a non empty set X is a pair (T_A, F_A), where $T_A: X \to [0,1]$ and $F_A: X \to [0,1]$ are true membership and false membership functions, respectively, such that*

$$0 \le T_A(x) + F_A(y) \le 1 \text{ for any } x \in X.$$

Let X and Y be two non-empty sets. A vague relation R of X to Y is a vague set R on $X \times Y$ that is $R = (T_R, F_R)$, where $T_R: X \times Y \to [0,1], F_R: X \times Y \to [0,1]$ and satisfy the condition:

$$0 \le T_R(x,y) + F_R(x,y) \le 1 \text{ for any } x, y \in X.$$

Definition 2.5 *[7] Let $G^* = (V, E)$ be a graph. A pair $G = (J, K)$ is called a vague graph on G^*, where $J = (T_J, F_J)$ is a vague set on V and $K = (T_K, F_K)$ is a vague set on $E \subseteq V \times V$ such that for each $xy \in E$,*

$$T_K(xy) \le \min(T_J(x), T_J(y)) \text{ and } F_K(xy) \ge \max(T_J(x), F_J(y)).$$

Definition 2.6 *[21] Let X be a space of points (objects), with generic elements in X denoted by x. A single valued neutrosophic set A in X is characterised by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$ and falsity-membership-function $F_A(x)$,*

For each point x in X, $T_A(x), F_A(x), I_A(x) \in [0,1]$. Also

$$A = \{x, T_A(x), F_A(x), I_A(x)\} \text{ and } 0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Definition 2.7 *[1] A single valued neutrosophic graph is defined as a pair $G = (J, K)$ where*

(i) $V = \{v_1, v_2, .., v_n\}$ such that $T_J: V \to [0,1]$, $I_J: V \to [0,1]$ and $F_J: V \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and

$$0 \le T_J(v) + I_J(v) + F_J(v) \le 3,$$

(ii) $E \subseteq V \times V$ where $T_K: E \to [0,1]$, $I_K: E \to [0,1]$ and $F_K: E \to [0,1]$ are such that

$$T_K(uv) \le \min\{T_J(u), T_J(v)\},$$
$$I_K(uv) \le \min\{I_J(u), I_J(v)\},$$
$$F_K(uv) \le \max\{F_J(u), F_J(v)\},$$
$$\text{and } 0 \le T_K(uv) + I_K(uv) + F_K(uv) \le 3, \ \forall uv \in E.$$

Definition 2.8 *[21] A Neutrosophic set A is contained in another neutrosophic set B, (i.e) $A \subseteq B$ if $\forall x \in X, T_A(x) \le T_B(x), I_A(x) \ge I_B(x)$and $F_A(x) \ge F_B(x)$.*

Definition 2.9 *[6] A Neutrosophic Vague Set A_{NV} (NVS in short) on the universe of discourse X written as*

$$A_{NV} = \{\langle x, \hat{T}_{A_{NV}}(x), \hat{I}_{A_{NV}}(x), \hat{F}_{A_{NV}}(x)\rangle, x \in X\},$$

whose truth-membership, indeterminacy membership and falsity-membership functions are defined as

$$\hat{T}_{A_{NV}}(x) = [T^-(x), T^+(x)], \hat{I}_{A_{NV}}(x) = [I^-(x), I^+(x)] \text{and } \hat{F}_{A_{NV}}(x) = [F^-(x), F^+(x)],$$

where $T^+(x) = 1 - F^-(x), F^+(x) = 1 - T^-(x)$, and $0 \le T^-(x) + I^-(x) + F^-(x) \le 2$.

Definition 2.10 *[6] The complement of NVS A_{NV} is denoted by A^c_{NV} and it is defined by*

$$\hat{T}^c_{A_{NV}}(x) = [1 - T^+(x), 1 - T^-(x)],$$
$$\hat{I}^c_{A_{NV}}(x) = [1 - I^+(x), 1 - I^-(x)],$$
$$\hat{F}^c_{A_{NV}}(x) = [1 - F^+(x), 1 - F^-(x)].$$

Definition 2.11 *[6] Let A_{NV} and B_{NV} be two NVSs of the universe U. If for all $u_i \in U$,*

$$\hat{T}_{A_{NV}}(u_i) \le \hat{T}_{B_{NV}}(u_i), \hat{I}_{A_{NV}}(u_i) \ge \hat{I}_{B_{NV}}(u_i), \hat{F}_{A_{NV}}(u_i) \ge \hat{F}_{B_{NV}}(u_i).$$

Then, the NVSs, A_{NV} are included in B_{NV}, denoted by $A_{NV} \subseteq B_{NV}$ where $1 \le i \le n$.

Definition 2.12 *[6] The union of two NVSs A_{NV} and B_{NV} is an NVSs, C_{NV}, written as $C_{NV} = A_{NV} \cup B_{NV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by*

$$\hat{T}_{C_{NV}}(x) = [\max(T^-_{A_{NV}}(x), T^-_{B_{NV}}(x)), \max(T^+_{A_{NV}}(x), T^+_{B_{NV}}(x))]$$
$$\hat{I}_{C_{NV}}(x) = [\min(I^-_{A_{NV}}(x), I^-_{B_{NV}}(x)), \min(I^+_{A_{NV}}(x), I^+_{B_{NV}}(x))]$$
$$\hat{F}_{C_{NV}}(x) = [\min(F^-_{A_{NV}}(x), F^-_{B_{NV}}(x)), \min(F^+_{A_{NV}}(x), F^+_{B_{NV}}(x))].$$

Definition 2.13 *[6] The intersection of two NVSs, A_{NV} and B_{NV} is an NVSs C_{NV}, written as $C_{NV} = A_{NV} \cap B_{NV}$, whose truth-membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by*

$$\hat{T}_{C_{NV}}(x) = [\min(T^-_{A_{NV}}(x), T^-_{B_{NV}}(x)), \min(T^+_{A_{NV}}(x), T^+_{B_{NV}}(x))]$$
$$\hat{I}_{C_{NV}}(x) = [\max(I^-_{A_{NV}}(x), I^-_{B_{NV}}(x)), \max(I^+_{A_{NV}}(x), I^+_{B_{NV}}(x))]$$
$$\hat{F}_{C_{NV}}(x) = [\max(F^-_{A_{NV}}(x), F^-_{B_{NV}}(x)), \max(F^+_{A_{NV}}(x), F^+_{B_{NV}}(x))].$$

Definition 2.14 *[19] Let $\mathcal{U}$ be a universe, E a set of parameters and $A \subseteq E$. A collection of pairs (F, A) is called a neutrosophic vague soft set (NVSS) over $\mathcal{U}$ where F is a mapping given by $F: A \to NV(\mathcal{U})$ and $NV(\mathcal{U})$ denotes the set of all neutrosophic vague subsets of $\mathcal{U}$.*

Definition 2.15 *[2] A neutrosophic soft graph $G = (G^*, J, K, R)$ is an ordered four tuple if it satisfies the following conditions:*

- $G^* = (V, E)$ is a simple graph,
- R is a non-empty set of parameters,
- (J, R) is a neutrosophic soft set over V,
- (K, R) is a neutrosophic soft set over E,
- $(J(e), K(e))$ is a neutrosophic graph of G^*, that is,

$$T_{K(e)}(ab) \le \min\{T^-_{J(e)}(a), T^-_{J(e)}(b)\},$$
$$I_{K(e)}(ab) \le \min\{I^-_{J(e)}(a), I^-_{J(e)}(b)\},$$
$$F_{K(e)}(ab) \le \max\{F^-_{J(e)}(a), F^-_{J(e)}(b)\}$$

such that,

$$0 \le T_{K(e)}(ab) + I_{K(e)}(ab) + T_{K(e)}(ab) \le 3 \ \ \forall e \in R, a, b \in V.$$

For convenience, the neutrosophic graph $(J(e), K(e))$ is denoted by $H(e)$. A neutrosophic vague soft graph is a parametrized family of neutrosophic graphs.

Definition 2.16 *[15] Let $G^* = (R, S)$ be a graph. A pair $G = (A, B)$ is called a neutrosophic vague graph (NVG) on G^* or a neutrosophic vague graph where $A = (\hat{T}_A, \hat{I}_A, \hat{F}_A)$ is a neutrosophic vague set on R and $B = (\hat{T}_B, \hat{I}_B, \hat{F}_B)$ is a neutrosophic vague set $S \subseteq R \times R$ where*

(1)$R = \{v_1, v_2, \dots, v_n\}$ such that $T^-_A: R \to [0,1], I^-_A: R \to [0,1], F^-_A: R \to [0,1]$ satisfies the condition $F^-_A = [1 - T^+_A]$, and $T^+_A: R \to [0,1], I^+_A: R \to [0,1], F^+_A: R \to [0,1]$ which satisfies the condition $F^+_A = [1 - T^-_A]$, denote the degrees of truth membership, indeterminacy membership and falsity membership of the element $v_i \in R$, and

$$0 \le T^-_A(v_i) + I^-_A(v_i) + F^-_A(v_i) \le 2$$
$$0 \le T^+_A(v_i) + I^+_A(v_i) + F^+_A(v_i) \le 2.$$

(2) $S \subseteq R \times R$ where

$$T^-_B: R \times R \to [0,1], I^-_B: R \times R \to [0,1], F^-_B: R \times R \to [0,1]$$
$$T^+_B: R \times R \to [0,1], I^+_B: R \times R \to [0,1], F^+_B: R \times R \to [0,1]$$

denote the degrees of truth membership, indeterminacy membership and falsity membership of the element $v_i, v_j \in S$, respectively and such that,

$$0 \leq T_B^-(v_i v_j) + I_B^-(v_i v_j) + F_B^-(v_i v_j) \leq 2$$
$$0 \leq T_B^+(v_i v_j) + I_B^+(v_i v_j) + F_B^+(v_i v_j) \leq 2,$$

such that

$$T_B^-(v_i v_j) \leq \min\{T_A^-(v_i), T_A^-(v_j)\}$$
$$I_B^-(v_i v_j) \leq \min\{I_A^-(v_i), I_A^-(v_j)\}$$
$$F_B^-(v_i v_j) \leq \max\{F_A^-(v_i), F_A^-(v_j)\},$$

and similarly

$$T_B^+(v_i v_j) \leq \min\{T_A^+(v_i), T_A^+(v_j)\}$$
$$I_B^+(v_i v_j) \leq \min\{I_A^+(v_i), I_A^+(v_j)\}$$
$$F_B^+(v_i v_j) \leq \max\{F_A^+(v_i), F_A^+(v_j)\}.$$

Example 2.17 *Consider a neutrosophic vague graph* $G = (A, B)$ *such that* $A = \{a, b, c\}$ *and* $B = \{ab, bc, ca\}$ *are defined by*

$$\hat{a} = T[0.5,0.5], I[0.4,0.3], F[0.5,0.5], \qquad \hat{b} = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6],$$
$$\hat{c} = T[0.4,0.4], I[0.5,0.3], F[0.6,0.6]$$

where $\hat{a}, \hat{b}, \hat{c}$ are the neutrosophic vague sets on A. Now, $\hat{a} = (a^-, a^+), \hat{b} = (b^-, b^+), \hat{c} = (c^-, c^+)$.

$$a^- = (0.5,0.4,0.4), b^- = (0.4,0.7,0.4), c^- = (0.4,0.5,0.6)$$
$$a^+ = (0.5,0.3,0.5), b^+ = (0.6,0.3,0.6), c^+ = (0.4,0.3,0.6).$$

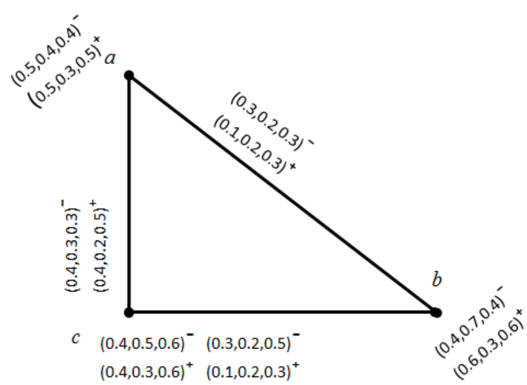


Figure 1 NEUTROSOPHIC VAGUE GRAPH

Definition 2.18 *A partial neutrosophic vague subgraph of netrosophic vague graph* $G = (A, B)$ *is a neutrosophic vague graph* $G^* = (V', E')$ *such that*

- $V' \subseteq V$ where $\hat{T}'_A(v_i) \leq \hat{T}_A(v_i)$, $\hat{I}'_A(v_i) \leq \hat{I}_A(v_i)$ and $\hat{F}'_A(v_i) \geq \hat{F}_A(v_i)$ for all $v_i \in V$.
- $E' \subseteq E$ where $\hat{T}'_B(v_i, v_j) \leq \hat{T}_B(v_i, v_j)$, $\hat{I}'_B(v_i, v_j) \leq \hat{I}_B(v_i, v_j)$ and $\hat{F}'_B(v_i, v_j) \geq \hat{F}_B(v_i, v_j)$ for all $(v_i, v_j) \in E$.

3 Operations on Neutrosophic Vague Soft Graphs

In this section, the results on operations of neutrosophic vague soft graphs with example are established.

Let $\mathcal{U}$ be an initial universe and P be the set of all parameters. $P(\mathcal{U})$ denotes the set of all neutrosophic vague soft sets of $\mathcal{U}$. Let A be a subset of P. A pair (F, A) is called a neutrosophic vague soft set over $\mathcal{U}$. Let $P(V)$ denotes the set of all neutrosophic vague sets of V and $P(E)$ denotes the set of all neutrosophic vague sets of E.

Definition 3.1 *A neutrosophic vague soft graph* $G=(G^*,J,K,R)$ *is an ordered four tuple if it satisfies the following conditions:*

- $G^*=(V,E)$ is a simple graph,
- R is a non-empty set of parameters,
- (J,R) is a neutrosophic vague soft set over V,
- (K,R) is a neutrosophic vague soft set over E,
- $(J(e),K(e))$ is a neutrosophic vague graph of G^*, that is,

$$T^-_{K(e)}(ab)\le \min\{T^-_{J(e)}(a),T^-_{J(e)}(b)\},$$
$$I^-_{K(e)}(ab)\le \min\{I^-_{J(e)}(a),I^-_{J(e)}(b)\},$$
$$F^-_{K(e)}(ab)\le \max\{F^-_{J(e)}(a),F^-_{J(e)}(b)\}$$
$$T^+_{K(e)}(ab)\le \min\{T^+_{J(e)}(a),T^+_{J(e)}(b)\},$$
$$I^+_{K(e)}(ab)\le \min\{I^+_{J(e)}(a),I^+_{J(e)}(b)\},$$
$$F^+_{K(e)}(ab)\le \max\{F^+_{J(e)}(a),F^+_{J(e)}(b)\}$$

such that,

$$0\le T^-_{K(e)}(ab)+I^-_{K(e)}(ab)+F^-_{K(e)}(ab)\le 2,$$
$$0\le T^+_{K(e)}(ab)+I^+_{K(e)}(ab)+F^+_{K(e)}(ab)\le 2,\quad \forall e\in R,\ a,b\in V.$$

For the convenience, the neutrosophic vague graph $(J(e),K(e))$ is denoted by $H(e)$. A neutrosophic vague soft graph is a parametrized family of neutrosophic vague graphs.

Definition 3.2 *Let* $G_1=(J_1,K_1,R)$ *and* $G_2=(J_2,K_2,S)$ *be two neutrosophic vague soft graphs of* G^*. *Then* G_1 *is neutrosophic vague soft subgraph of* G_2 *if*

- $R\subseteq S$.
- $H_1(e)$ partial neutrosophic vague subgraph of $H_2(e)$ for all $e\in R$.

Example 3.3 *Consider a simple graph* $G^*=(V,E)$ *such that* $V=\{v_1,v_2,v_3,v_4\}$ *and*

$$E=\{v_1v_2,v_1v_3,v_1v_4,v_2v_4,v_3v_4\}.$$

Let $R=\{e_1,e_2\}$ be a set of parameters and let (J,R) be a neutrosophic vague soft set over V with neutrosophic approximation function $J:R\to\rho(V)$ defined by

$$J(e_1)=\hat{v}_1=T[0.4,0.4],I[0.3,0.4],F[0.6,0.6],\quad \hat{v}_2=T[0.3,0.7],I[0.3,0.5],F[0.3,0.7],$$
$$\hat{v}_3=T[0.5,0.6],I[0.4,0.2],F[0.4,0.5],\quad \hat{v}_4=T[0.8,0.3],I[0.5,0.6],F[0.7,0.2]$$
$$J(e_1)=v_1^-=(0.4,0.3,0.6),v_2^-=(0.3,0.3,0.3),v_3^-=(0.5,0.4,0.4),v_4^-=(0.8,0.5,0.7)$$
$$J(e_1)=v_1^+=(0.4,0.4,0.6),v_2^+=(0.7,0.5,0.7),v_3^+=(0.6,0.2,0.5),v_4^+=(0.2,0.6,0.2).$$
$$J(e_2)=\hat{v}_1=T[0.5,0.4],I[0.4,0.5],F[0.6,0.5],\quad \hat{v}_2=T[0.4,0.6],I[0.5,0.6],F[0.4,0.6],$$
$$\hat{v}_3=T[0.6,0.6],I[0.4,0.4],F[0.4,0.4],\quad \hat{v}_4=T[0.7,0.3],I[0.6,0.3],F[0.7,0.3]$$
$$J(e_2)=v_1^-=(0.5,0.4,0.6),v_2^-=(0.3,0.3,0.3),v_3^-=(0.5,0.4,0.4),v_4^-=(0.8,0.5,0.7)$$
$$J(e_2)=v_1^+=(0.4,0.5,0.5),v_2^+=(0.7,0.5,0.7),v_3^+=(0.6,0.2,0.5),v_4^+=(0.3,0.6,0.2).$$

Let (K,R) be a neutrosophic vague soft set over E with neutrosophic approximation function $K:R\to\rho(E)$ defined by

$$K(e_1)=\{(v_1v_2)^-=(0.3,0.2,0.5)^-,(v_1v_2)^+=(0.3,0.3,0.6)^+,(v_1v_3)^-=(0.4,0.3,0.4)^-,(v_1v_3)^+$$
$$=(0.3,0.2,0.5)^+,(v_1v_4)^-=(0.3,0.3,0.5)^-,(v_1v_4)^+=(0.1,0.2,0.3)^+\}$$
$$K(e_2)=\{(v_1v_2)^-=(0.4,0.3,0.5)^-,(v_1v_2)^+=(0.3,0.1,0.4)^+,(v_1v_3)^-=(0.2,0.2,0.5)^-,(v_1v_3)^+$$
$$=(0.3,0.5,0.7)^+,(v_1v_4)^-=(0.4,0.3,0.6)^-,(v_1v_4)^+=(0.3,0.2,0.5)^+\}.$$

Clearly, $H(e_1)=(J(e_1),K(e_1))$ and $H(e_2)=(J(e_2),K(e_2))$ are neutrosophic vague graphs corresponding to the parameters e_1 and e_2 respectively as shown in Figure 2

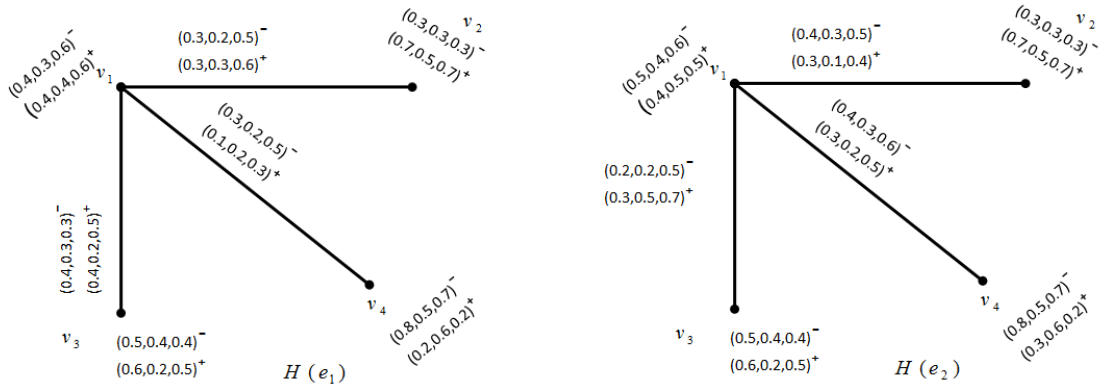


Figure 2: Neutrosophic vague soft graph

Definition 3.4 *The neutrosophic vague soft graph* $G_1 = (G^*, J_1, K_1, A)$ *is called spanning neutrosophic vague soft subgraph of* $G = (G^*, J, K, B)$ *if*

- $A \subset B$.
- $\hat{T}_{J_1(e)}(v) = \hat{T}_{J(e)}(v)$, $\hat{I}_{J_1(e)}(v) = \hat{I}_{J(e)}(v)$ $\hat{F}_{J_1(e)}(v) = \hat{F}_{J(e)}(v)$ for all $e \in A, v \in V$.

Definition 3.5 *Let* $G_1 = (J_1, K_1, R)$ *and* $G_2 = (J_2, K_2, S)$ *be two neutrosophic vague soft graphs of* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$*, respectively. The Cartesian product of* G_1 *and* G_2 *is* $G = G_1 \times G_2 = (J, K, R \times S)$*, where* $(J = J_1 \times J_2, R \times S)$ *is a neutrosophic vague soft set over* $V = V_1 \times V_2$, $(K = K_1 \times K_2, R \times S)$ *is a neutrsophic vague soft set over* $E = \{((u, v_1), (u, v_2)): u \in V_1, (v_1, v_2) \in E_2\} \cup \{((u_1, v), (u_2, v)): v \in V_2, (u_1, u_2) \in E_1\}$ *such that,*

$(i)\ \hat{T}_{J(a,b)}(u, v) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{J_2(b)}(v)$,
$\hat{I}_{J(a,b)}(u, v) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{J_2(b)}(v)$,
$\hat{F}_{J(a,b)}(u, v) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{J_2(b)}(v)$,
$\forall (u, v) \in V, (a, b) \in R \times S$.
$(ii) \hat{T}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{K_2(b)}(v_1, v_2)$,
$\hat{I}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{K_2(b)}(v_1, v_2)$,
$\hat{F}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{K_2(b)}(v_1, v_2)$,
$\forall u \in V_1, (v_1, v_2) \in E_2$.
$(iii)\ \hat{T}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{T}_{J_2(a)}(v) \wedge \hat{T}_{K_2(b)}(u_1, u_2)$,
$\hat{I}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{I}_{J_2(a)}(v) \wedge \hat{I}_{K_2(b)}(u_1, u_2)$,
$\hat{F}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{F}_{J_2(a)}(v) \vee \hat{F}_{K_2(b)}(u_1, u_2)$,
$\forall v \in V_2, (u_1, u_2) \in E_1$.

Theorem 3.6 *The Cartesian product of two neutrosophic vague soft graphs is a neutrosophic vague soft graph.*

Proof. Let $G_1 = (J_1, K_1, R)$ and $G_2 = (J_2, K_2, S)$ be two neutrosophic vague soft graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Let $G = G_1 \times G_2 = (J, K, R \times S)$ be the Cartesian product of G_1 and G_2. We claim that $G = (J, K, R \times S)$ is a neutrosophic vague soft graph and $(H, R \times S) = \{(J_1 \times J_2)(a_i, b_j), (K_1 \times K_2)(a_i, b_j)\}$ $\forall a_i \in R, b_j \in S$ for $i = 1,2, \ldots, m, j = 1,2, \ldots, n$ are neutrosophic vague graphs of G.

Consider,

$\hat{T}_{K_{(a_i, b_j)}}((u, v_1), (u, v_2)) = \min\{\hat{T}_{J_1(a_i)}(u), \hat{T}_{K_2(b_j)}(v_1, v_2)\}$
for $i = 1,2, .., m, j = 1,2, \ldots n$
$\leq \min\{\hat{T}_{J_1(a_i)}(u), \min\{\hat{T}_{J_2(b_j)}(v_1), \hat{T}_{J_2(b_j)}(v_2)\}\}$

$= \min\{\min\{\hat{T}_{J_1(a_i)}(u), \hat{T}_{J_2(b_j)}(v_1)\}, \min\{\hat{T}_{J_1(a_i)}(u), \hat{T}_{J_2(b_j)}(v_2)\}\}$
$\hat{T}_{K_{(a_i,b_j)}}((u, v_1), (u, v_2)) \leq \min\{(\hat{T}_{J_1(a_i)} \times \hat{T}_{J_2(b_j)})(u, v_1), (\hat{T}_{J_1(a_i)} \times \hat{T}_{J_2(b_j)})(u, v_2)\}$
for $i = 1,2, \dots m, j = 1,2, \dots, n$,

$\hat{I}_{K_{(a_i,b_j)}}((u, v_1), (u, v_2)) = \min\{\hat{I}_{J_1(a_i)}(u), \hat{I}_{K_2(b_j)}(v_1, v_2)\}$
for$i = 1,2, \dots, m, j = 1,2, \dots n$
$\leq \min\{\hat{I}_{J_1(a_i)}(u), \min\{\hat{I}_{J_2(b_j)}(v_1), \hat{I}_{J_2(b_j)}(v_2)\}\}$
$= \min\{\min\{\hat{I}_{J_1(a_i)}(u), \hat{I}_{J_2(b_j)}(v_1)\}, \min\{\hat{I}_{J_1(a_i)}(u), \hat{I}_{J_2(b_j)}(v_2)\}\}$
$\hat{I}_{K_{(a_i,b_j)}}((u, v_1), (u, v_2)) \leq \min\{(\hat{I}_{J_1(a_i)} \times \hat{I}_{J_2(b_j)})(u, v_1), (\hat{I}_{J_1(a_i)} \times \hat{I}_{J_2(b_j)})(u, v_2)\}$
for$i = 1,2, \dots m, j = 1,2, \dots, n$,

$\hat{F}_{K_{(a_i,b_j)}}((u, v_1), (u, v_2)) = \max\{\hat{F}_{J_1(a_i)}(u), \hat{I}_{K_2(b_j)}(v_1, v_2)\}$
for$i = 1,2, \dots, m, j = 1,2, \dots n$
$\leq \max\{\hat{F}_{J_1(a_i)}(u), \max\{\hat{F}_{J_2(b_j)}(v_1), \hat{F}_{J_2(b_j)}(v_2)\}\}$
$= \max\{\max\{\hat{F}_{J_1(a_i)}(u), \hat{F}_{J_2(b_j)}(v_1)\}, \max\{\hat{F}_{J_1(a_i)}(u), \hat{F}_{J_2(b_j)}(v_2)\}\}$
$\hat{F}_{K_{(a_i,b_j)}}((u, v_1), (u, v_2)) \leq \max\{(\hat{F}_{J_1(a_i)} \times \hat{F}_{J_2(b_j)})(u, v_1), (\hat{F}_{J_1(a_i)} \times \hat{F}_{J_2(b_j)})(u, v_2)\}$
for$i = 1,2, \dots m, j = 1,2, \dots, n$,

Similarly,

$\hat{T}_{K_{(a_i,b_j)}}((u_1, v), (u_2, v)) \leq \min\{(\hat{T}_{J_1(a_i)} \times \hat{T}_{J_2(b_j)})(u_1, v), (\hat{T}_{J_1(a_i)} \times \hat{T}_{J_2(b_j)})(u_2, v)\}$
for$i = 1,2, \dots m, j = 1,2, \dots, n$,

$\hat{I}_{K_{(a_i,b_j)}}((u_1, v), (u_2, v)) \leq \min\{(\hat{I}_{J_1(a_i)} \times \hat{I}_{J_2(b_j)})(u_1, v), (\hat{I}_{J_1(a_i)} \times \hat{I}_{J_2(b_j)})(u_2, v)\}$
for$i = 1,2, \dots m, j = 1,2, \dots, n$,

$\hat{F}_{K_{(a_i,b_j)}}((u_1, v), (u_2, v)) \leq \max\{(\hat{F}_{J_1(a_i)} \times \hat{F}_{J_2(b_j)})(u_1, v), (\hat{F}_{J_1(a_i)} \times \hat{F}_{J_2(b_j)})(u_2, v)\}$
for$i = 1,2, \dots m, j = 1,2, \dots, n$,

Hence $G = (J, K, R \times S)$ is a neutrosophic vague soft graph.

Definition 3.7 *The cross product of* G_1 *and* G_2 *is defined as a neutrosophic vague soft graphs of* $G = G_1 \odot G_2 = (J, K, R \times S)$, *where* $(J, R \times S)$ *is a neutrosophic vague soft set over* $V = V_1 \times V_2$, $(K, R \times S)$ *is a neutrsophic vague soft set over* $E = \{((u_1, v_1), (u_2, v_2)) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$ *such that*

$(i) \hat{T}_{J(a,b)}(u, v) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{J_2(b)}(v)$,
$\hat{I}_{J(a,b)}(u, v) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{J_2(b)}(v)$,
$\hat{F}_{J(a,b)}(u, v) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{J_2(b)}(v)$,
$\forall (u, v) \in V, (a, b) \in R \times S$.

$(ii) \hat{T}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{T}_{K_1(a)}(u_1, u_2) \wedge \hat{T}_{K_2(b)}(v_1, v_2)$,
$\hat{I}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{I}_{K_1(a)}(u_1, u_2) \wedge \hat{I}_{K_2(b)}(v_1, v_2)$,
$\hat{F}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{F}_{K_1(a)}(u_1, u_2) \vee \hat{F}_{K_2(b)}(v_1, v_2)$,
$\forall (u_1, u_2) \in E_1, (v_1, v_2) \in E_2$.

$H(a, b) = H_1(a) \odot H_2(b)$ for all $(a, b) \in R \times S$ are neutrosophic vague graphs of G.

Theorem 3.8 *The cross product of two neutrosophic vague soft graphs is a neutrosophic vague soft graph.*

Proof. Let $G_1 = (J_1, K_1, R)$ and $G_2 = (J_2, K_2, S)$ be two neutrosophic vague soft graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Let $G = G_1 \odot G_2 = (J, K, R \times S)$ be the cross product of G_1 and G_2. We claim that $G=(J,K,R\times S)$ is a neutrosophic vague soft graph and $(H,R\times S)=\{J_1\circledcirc J_2(a_i,b_j),K_1\circledcirc K_2(a_i,b_j)\}$ $\forall a_i\in R, b_j\in S$ for $i=1,2,\dots,m, j=1,2,\dots,n$ are neutrosophic vague graphs of G.

Consider,

$$\hat{T}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))=\min\{\hat{T}_{K_1(a_i)}(u_1,u_2),\hat{T}_{K_2(b_j)}(v_1,v_2)\}$$
for $i=1,2,..,m, j=1,2,\dots n$
$$\le\min\{\{\hat{T}_{J_1(a_i)}(u_1),\hat{T}_{J_1(a_i)}(u_2)\},\min\{\hat{T}_{J_2(b_j)}(v_1),\hat{T}_{J_2(b_j)}(v_2)\}\}$$
$$=\min\{\min\{\hat{T}_{J_1(a_i)}(u_1),\hat{T}_{J_2(b_j)}(v_1)\},\min\{\hat{T}_{J_1(a_i)}(u_2),\hat{T}_{J_2(b_j)}(v_2)\}\}$$
$$\hat{T}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))\le\min\{(\hat{T}_{J_1(a_i)}\circledcirc\hat{T}_{J_2(b_j)})(u_1,v_1),(\hat{T}_{J_1(a_i)}\circledcirc\hat{T}_{J_2(b_j)})(u_2,v_2)\}$$
for $i=1,2,\dots m, j=1,2,\dots,n,$

$$\hat{I}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))=\min\{\hat{I}_{K_1(a_i)}(u_1,u_2),\hat{I}_{K_2(b_j)}(v_1,v_2)\}$$
for $i=1,2,..,m, j=1,2,\dots n$
$$\le\min\{\{\hat{I}_{J_1(a_i)}(u_1),\hat{I}_{J_1(a_i)}(u_2)\},\min\{\hat{I}_{J_2(b_j)}(v_1),\hat{I}_{J_2(b_j)}(v_2)\}\}$$
$$=\min\{\min\{\hat{I}_{J_1(a_i)}(u_1),\hat{I}_{J_2(b_j)}(v_1)\},\min\{\hat{I}_{J_1(a_i)}(u_2),\hat{I}_{J_2(b_j)}(v_2)\}\}$$
$$\hat{I}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))\le\min\{(\hat{I}_{J_1(a_i)}\circledcirc\hat{I}_{J_2(b_j)})(u_1,v_1),(\hat{I}_{J_1(a_i)}\circledcirc\hat{I}_{J_2(b_j)})(u_2,v_2)\}$$
for $i=1,2,\dots m, j=1,2,\dots,n,$

$$\hat{F}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))=\min\{\hat{F}_{K_1(a_i)}(u_1,u_2),\hat{F}_{K_2(b_j)}(v_1,v_2)\}$$
for $i=1,2,..,m, j=1,2,\dots n$
$$\le\min\{\{\hat{F}_{J_1(a_i)}(u_1),\hat{F}_{J_1(a_i)}(u_2)\},\min\{\hat{F}_{J_2(b_j)}(v_1),\hat{F}_{J_2(b_j)}(v_2)\}\}$$
$$=\min\{\min\{\hat{F}_{J_1(a_i)}(u_1),\hat{F}_{J_2(b_j)}(v_1)\},\min\{\hat{F}_{J_1(a_i)}(u_2),\hat{F}_{J_2(b_j)}(v_2)\}\}$$
$$\hat{F}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))\le\min\{(\hat{F}_{J_1(a_i)}\circledcirc\hat{F}_{J_2(b_j)})(u_1,v_1),(\hat{F}_{J_1(a_i)}\circledcirc\hat{F}_{J_2(b_j)})(u_2,v_2)\}$$
for $i=1,2,\dots m, j=1,2,\dots,n,$

Hence $G=(J,K,R\times S)$ is a neutrosophic vague soft graph.

Definition 3.9 *The lexicographic product of G_1 and G_2 is defined as a neutrosophic vague soft graphs of $G=G_1\odot G_2=(J,K,R\times S)$, where $(J,R\times S)$ is a neutrosophic vague soft set over $V=V_1\times V_2$, $(K,R\times S)$ is a neutrosophic vague soft set over $E=\{((u,v_1),(u,v_2)):u\in V_1,(v_1,v_2)\in E_2\}\cup\{((u_1,v_1),(u_2,v_2)):(u_1,u_2)\in E_1,(v_1,v_2)\in E_2\}$ such that*

(i) $\hat{T}_{J(a,b)}(u,v)=\hat{T}_{J_1(a)}(u)\wedge\hat{T}_{J_2(b)}(v)$,
$\hat{I}_{J(a,b)}(u,v)=\hat{I}_{J_1(a)}(u)\wedge\hat{I}_{J_2(b)}(v)$,
$\hat{F}_{J(a,b)}(u,v)=\hat{F}_{J_1(a)}(u)\vee\hat{F}_{J_2(b)}(v), \forall(u,v)\in V,(a,b)\in R\times S.$

(ii) $\hat{T}_{K(a,b)}((u,v_1),(u,v_2))=\hat{T}_{J_1(a)}(u)\wedge\hat{T}_{K_2(b)}(v_1,v_2)$,
$\hat{I}_{K(a,b)}((u,v_1),(u,v_2))=\hat{I}_{J_1(a)}(u)\wedge\hat{I}_{K_2(b)}(v_1,v_2)$,
$\hat{F}_{K(a,b)}((u,v_1),(u,v_2))=\hat{F}_{J_1(a)}(u)\vee\hat{F}_{K_2(b)}(v_1,v_2), \forall u\in V_1,(v_1,v_2)\in E_2.$

(iii) $\hat{T}_{K(a,b)}((u_1,v_1),(u_2,v_2))=\hat{T}_{K_1(a)}(u_1,u_2)\wedge\hat{T}_{K_2(b)}(v_1,v_2)$,
$\hat{I}_{K(a,b)}((u_1,v_1),(u_2,v_2))=\hat{I}_{K_1(a)}(u_1,u_2)\wedge\hat{I}_{K_2(b)}(v_1,v_2)$,
$\hat{F}_{K(a,b)}((u_1,v_1),(u_2,v_2))=\hat{F}_{K_1(a)}(u_1,u_2)\vee\hat{F}_{K_2(b)}(v_1,v_2), \forall(u_1,u_2)\in E_1,(v_1,v_2)\in E_2.$

$H(a,b)=H_1(a)\odot H_2(b)$ for all $(a,b)\in R\times S$ are neutrosophic vague graphs of G.

Theorem 3.10 *The lexicographic product of two neutrosophic vague soft graphs is a neutrosophic vague soft graph.*

Proof. Similar to the proof of Theorem 3.8.

Definition 3.11 *The strong product of G_1 and G_2 is defined as a neutrosophic vague soft graphs of $G=G_1\otimes G_2=(J,K,R\times S)$, where $(J,R\times S)$ is a neutrosophic vague soft set over $V=V_1\times V_2$, $(K,R\times S)$ is a neutrsophic* *vague soft set over* $E = \{((u, v_1), (u, v_2)): (u) \in V_1, (v_1, v_2) \in E_2\} \cup \{((u_1, v), (u_2, v)): (v) \in V_2, (u_1, u_2) \in E_1\}, \cup$ $\{((u_1, v_1), (u_2, v_2)): (u_1, u_2) \in E_1, (v_1, v_2 \in E_2)\}$ *such that*

$(i)\quad \hat{T}_{J(a,b)}(u, v) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{J_2(b)}(v),$
$\hat{I}_{J(a,b)}(u, v) = \hat{I}_{J_1(a)}(u) \wedge \hat{T}_{J_2(b)}(v),$
$\hat{F}_{J(a,b)}(u, v) = \hat{F}_{J_1(a)}(u) \vee \hat{T}_{J_2(b)}(v), \forall (u, v) \in V, (a, b) \in R \times S.$
$(ii)\quad \hat{T}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{K_2(b)}(v_1, v_2),$
$\hat{I}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{K_2(b)}(v_1, v_2),$
$\hat{F}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{K_2(b)}(v_1, v_2), \forall u \in V_1, (v_1, v_2) \in E_2.$
$(iii)\quad \hat{T}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{T}_{J_2(b)}(v) \wedge \hat{T}_{K_2(a)}(u_1, u_2),$
$\hat{I}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{I}_{J_2(b)}(v) \wedge \hat{I}_{K_2(a)}(u_1, u_2),$
$\hat{F}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{F}_{J_2(b)}(v) \vee \hat{F}_{K_2(a)}(u_1, u_2), \forall v \in V_2, (u_1, u_2) \in E_1.$
$(iv)\quad \hat{T}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{T}_{K_1(a)}(u_1, u_2) \wedge \hat{T}_{K_2(b)}(v_1, v_2),$
$\hat{I}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{I}_{K_1(a)}(u_1, u_2) \wedge \hat{I}_{K_2(b)}(v_1, v_2),$
$\hat{F}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{F}_{K_1(a)}(u_1, u_2) \vee \hat{F}_{K_2(b)}(v_1, v_2), \forall (u_1, u_2) \in E_1, (v_1, v_2) \in E_2.$

$H(a, b) = H_1(a) \otimes H_2(b)$ for all $(a, b) \in R \times S$ are neutrosophic vague graphs of G.

Theorem 3.12 *The strong product of two neutrosophic vague soft graphs is a neutrosophic vague soft graph.*

Proof. Similar to the proof of Theorem 3.8.

Definition 3.13 *The composition of* G_1 *and* G_2 *is defined as a a neutrosophic vague soft graphs of* $G = G_1[G_2] = (J, K, R \times S)$*, where* $(J, R \times S)$ *is a neutrosophic vague soft set over* $V = V_1 \times V_2, (K, R \times S)$ *is a neutrosophic vague soft set over* $E = \{((u, v_1), (u, v_2)): u \in V_1, (v_1, v_2) \in E_2\} \cup \{((u_1, v), (u_2, v)): v \in V_2, (u_1, u_2) \in E_1\} \cup$ $\{((u_1, v_1), (u_2, v_2)): (u_1, u_2) \in E_1, (v_1 \neq v_2)\}$ *such that*

$(i)\hat{T}_{J(a,b)}(u, v) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{J_2(b)}(v),$
$\hat{I}_{J(a,b)}(u, v) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{J_2(b)}(v),$
$\hat{F}_{J(a,b)}(u, v) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{J_2(b)}(v),$
$\forall (u, v) \in V, (a, b) \in R \times S.$
$(ii)\hat{T}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{K_2(b)}(v_1, v_2),$
$\hat{I}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{K_2(b)}(v_1, v_2),$
$\hat{F}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{K_2(b)}(v_1, v_2),$
$\forall u \in V_1, (v_1, v_2) \in E_2.$
$(iii)\hat{T}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{T}_{J_2(b)}(v) \wedge \hat{T}_{K_2(a)}(u_1, u_2),$
$\hat{I}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{I}_{J_2(b)}(v) \wedge \hat{I}_{K_2(a)}(u_1, u_2),$
$\hat{F}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{F}_{J_2(b)}(v) \vee \hat{F}_{K_2(a)}(u_1, u_2),$
$\forall v \in V_2, (u_1, u_2) \in E_1.$
$(iv)\hat{T}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{T}_{K_1(a)}(u_1, u_2) \wedge \hat{T}_{J_2(a)}(v_1) \wedge \hat{T}_{J_2(b)}(v_2),$
$\hat{I}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{I}_{K_1(a)}(u_1, u_2) \wedge \hat{I}_{J_2(a)}(v_1) \wedge \hat{I}_{J_2(b)}(v_2),$
$\hat{F}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{F}_{K_1(a)}(u_1, u_2) \vee \hat{F}_{J_2(a)}(v_1) \vee \hat{F}_{J_2(b)}(v_2),$
$\forall (u_1, u_2) \in E_1, where v_1 \neq v_2.$

$H(a, b) = H_1(a)[H_2(b)]$ for all $(a, b) \in R \times S$ are neutrosophic vague graphs of G.

Theorem 3.14 *The composition product of two neutrosophic vague soft graphs is a neutrosophic vague soft graph.*

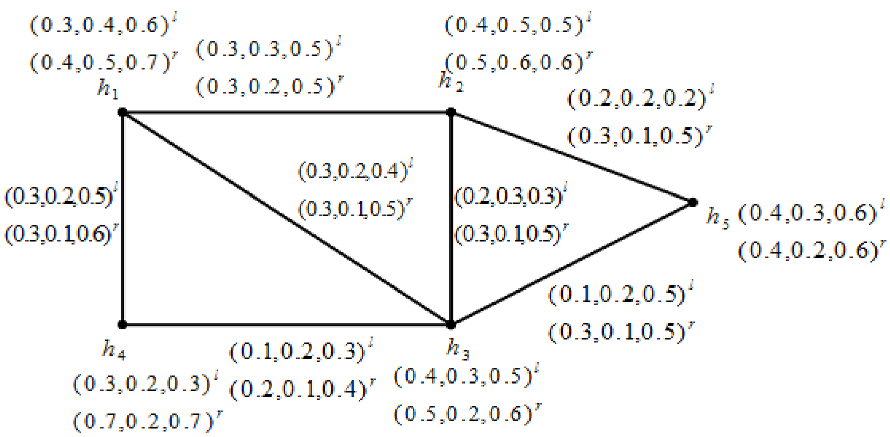
Proof. Similar to the proof of Theorem 3.8.

Application to Decision-making problem:

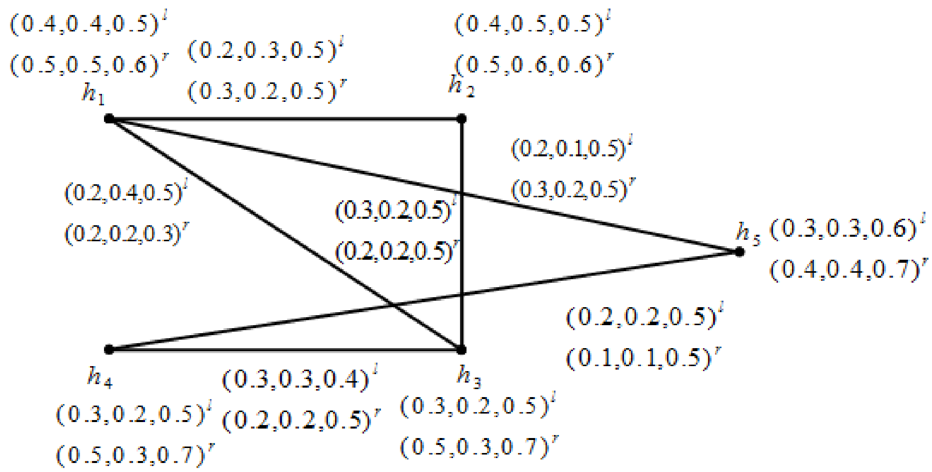
Neutrosophic vague soft set has several applications in decision making problems and used to deal with uncertainties from our different real-life problems. In this section we apply the concept of neutrosophic vague soft sets in a decision-making problem to its graphs and then construct an algorithm for the selection of optimal object based upon given set of information. Suppose that $V = \{h_1, h_2, h_3, h_4, h_5\}$ be the set of five institutions under consideration on which Mr. Z is going to join for his studies on the basis of wishing parameters with 0.5-degree risk value on his risk preferece, with the attributes set $A = \{e_1 = NIRF\ ranking, e_2 = IoE\ Institution, e_3 = University\}$.
(F, A) is the neutrosophic vague soft set on V which describe the value of the students based upon the given parameters $e_1 = NIRF\ ranking, e_2 = IoE\ Institution, e_3 = University,$ respectively.

$$F(e_1) = \{(h_1, (0.3,0.4,0.6)^l\ (0.4,0.5,0.7)^r), (h_2, (0.4,0.5,0.5)^l (0.5,0.6,0.6)^r), (h_3, (0.4,0.3,0.5)^l (0.5,0.4,0.6)^r)$$
$$(h_4, (0.3,0.2,0.3)^l (0.7,0.3,0.7)^r), (h_5, (0.4,0.3,0.5)^l (0.5,0.4,0.6)^r)\}$$
$$F(e_2) = \{(h_1, (0.4,0.4,0.5)^l\ (0.5,0.5,0.6)^r), (h_2, (0.4,0.5,0.5)^l (0.5,0.6,0.6)^r), (h_3, (0.3,0.2,0.5)^l (0.5,0.3,0.7)^r)$$
$$(h_4, (0.3,0.2,0.5)^l (0.5,0.3,0.7)^r), (h_5, (0.3,0.3,0.6)^l (0.4,0.4,0.7)^r)\}$$
$$F(e_3) = \{(h_1, (0.2,0.4,0.7)^l\ (0.3,0.5,0.8)^r), (h_2, (0.3,0.3,0.6)^l (0.4,0.4,0.7)^r), (h_3, (0.2,0.4,0.6)^l (0.4,0.4,0.8)^r)$$
$$(h_4, (0.2,0.3,0.6)^l (0.4,0.3,0.8)^r), (h_5, (0.3,0.4,0.6)^l (0.4,0.4,0.7)^r)\}$$

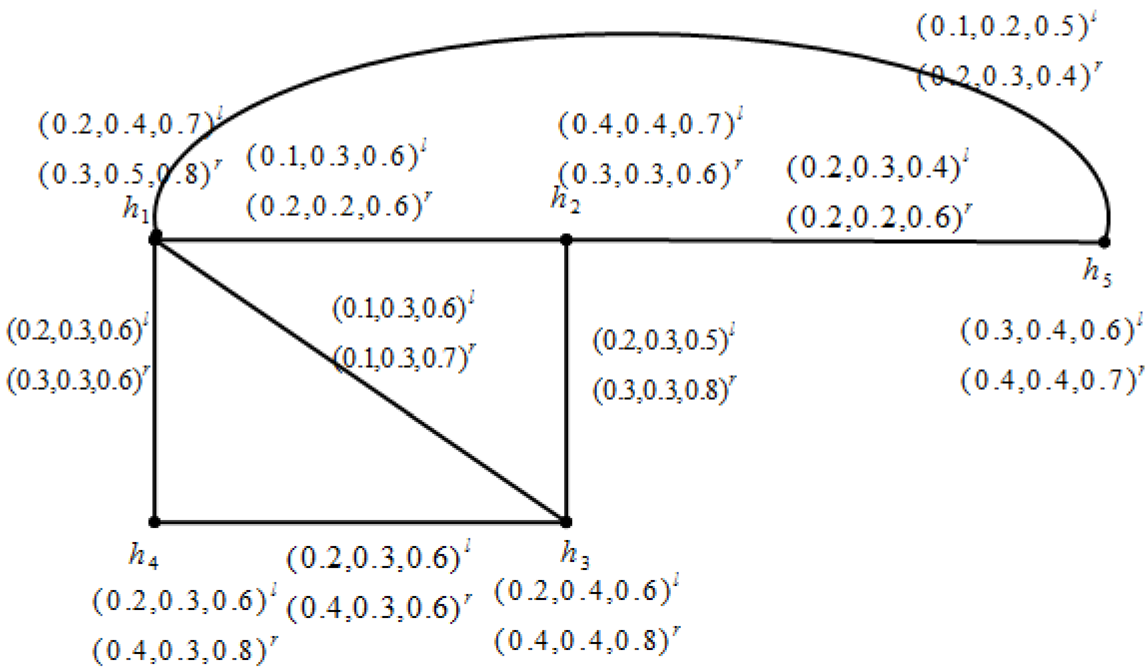
The neutrosophic vague soft graphs $G = (F, K, A)$ corresponding to the parameters e_i for $i = 1,2,3$ are shown in Figure 4.1



$H\ (e_1)$



H (e_2)



H (e_3)

Figure 4.1 Neutrosophic vague soft graphs

Table 1 Tabular representation of the Neutrosophic vague soft graph in example 4.1

	e_1	e_2	e_3
h_1	$(0.3,0.4,0.6)^l$ $(0.4,0.5,0.7)^r$	$(0.4,0.4,0.5)^l$ $(0.5,0.5,0.6)^r$	$(0.2,0.4,0.7)^l$ $(0.3,0.5,0.8)^r$
h_2	$(0.4,0.5,0.5)^l$ $(0.5,0.6,0.6)^r$	$(0.4,0.5,0.5)^l$ $(0.5,0.6,0.6)^r$	$(0.3,0.3,0.6)^l$ $(0.4,0.4,0.7)^r$
h_3	$(0.4,0.3,0.5)^l$ $(0.5,0.4,0.6)^r$	$(0.3,0.4,0.4)^l$ $(0.6,0.6,0.7)^r$	$(0.2,0.4,0.6)^l$ $(0.4,0.5,0.8)^r$
h_4	$(0.3,0.2,0.3)^l$ $(0.7,0.3,0.7)^r$	$(0.3,0.2,0.5)^l$ $(0.5,0.3,0.7)^r$	$(0.2,0.3,0.6)^l$ $(0.4,0.3,0.8)^r$
h_5	$(0.4,0.3,0.5)^l$ $(0.5,0.4,0.6)^r$	$(0.3,0.3,0.6)^l$ $(0.4,0.4,0.7)^r$	$(0.3,0.4,0.6)^l$ $(0.4,0.4,0.7)^r$

Table 2 The grade based on Neutrosophic vague soft graph in example 4.1

	e_1	e_2	e_3	G_{min}
h_1	2	2	1	1
h_2	2	2	2	2
h_3	2	2	1	1
h_4	3	2	2	2
h_5	2	2	2	2

Table 3. The resultant neutrosophic vague soft graphs in example 4.1
The score function S_{ij} based on neutrosophic vague soft graph

	e_1	e_2	e_3
h_2	$< 2,0.45,0.55,0.55,-0.65 >$	$< 2,0.45,0.55,0.55,0.55,-65 >$	$< 2,0.35,0.35,0.65,-0.65 >$
h_4	$< 3,0.35,0.25,0.55,-0.45 >$	$< 2,0.55,0.25,0.6,-0.50 >$	$< 2,0.3,0.3,0.7,-0.7 >$
h_5	$< 2,0.45,0.35,0.65,-0.55 >$	$< 2,0.35,0.35,0.65,-0.65 >$	$< 2,0.35,0.4,0.65,-0.75 >$

Table 4. Comparison table for Grade function and Score function, based on e_1

	h_2	h_4	h_5	h_k (min)
h_2	$< 0,0 >$	$< -1,-20 >$	$< 0,-10 >$	$< -1,-20 >$
h_4	$< 1,20 >$	$< 0,0 >$	$< 0,10 >$	$< 0,10 >$
h_5	$< 0,10 >$	$< -1,-10 >$	$< 0,0 >$	$< -1,-10 >$

Table 5. Comparison table for grade function and score function based on e_1 without h_2

	h_4	h_5	h_k(min)
h_4	$< 0,0 >$	$< 1,10 >$	$< 1,10 >$
h_5	$< -1,-10 >$	$< 0,0 >$	$< -1,-10 >$

We get h_2, h_4, h_5 attributes. Similarly, we can get h_4, h_2, h_5 under the attributes e_2 and h_2, h_5, h_4 under the attributes e_3.
Finally, compute the ranking of the research objects under all attributes. Suppose the decision maker assigns weights to each attribute, $a_1 = 0.2, a_2 = 0.3, a_3 = 0.1$. And we can get $A_3 > A_2 > A_1$ from table 6. We consider h_2 is the first superior object, h_4 is the second superior object and h_5 is the third superior object under the E. Therefore, Mr. Z will selected particular institution h_2.
Table 6. The ranking of the objects under all attributes.

	e_1 . 0.2	e_2 . 0.3	e_3 . 0.1	A_i
h_2	1	2	1	0.9
h_4	2	1	3	1
h_5	3	3	2	1.7

Advantages and Limitations:

1. The proposed application is more significant, since it has the method of solving based on the idea of probability in grade function.
2. The developed method is utilised for solving practical decision making problems containing vagueness.
3. The addressed graphs can be extended to the bipolar environment.
4.The challenging one is to handle the vagueness in the application viewpoint of big data. If the indeterminate membership function has the huge data, then it is difficult to handle. This leads to have a

massive calculation in the decision-making problems.

Conclusion

Vague sets and neutrosophic soft sets provide a powerful tool to represent the data with uncertain information and have fruitful applications. In this work, neutrosophic vague soft graphs have been developed. This helps the decision-makers more sufficient for taking their input best suit to their domain of reference. Hence, the proposed graphs and their operations have enough capabilities to address the related dependability on the imprecise information. Further, the authors will aim to develop this research to the isomorphic properties of the proposed concepts in future.

Compliance with ethical standards

Conflict of interest: The authors declare that they have no conflict of interest.

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Quadripartitioned Neutrosophic Graph Structures

S. Satham Hussain [1,*], Hossein Rashmonlou [2], Mofidnakhaei F [3], R Jahir Hussain [1], Sankar Sahoo [4] and Said Broumi [5]

[1]PG & Research Department of Mathematics, Jamal Mohamed College, Trichy, Tamilnadu, India ; sathamhusain5592@gamil.com, hssn_jhr@yahoo.com

[2]Department of Mathematics, University of Mazandarn, Babolsar, Iran.; h.rashmanlou@stu.umz.ac.ir

[3]Department of Physics, Sari Branch, Islamic Azad University, Sari, Iran; farshid.mofidnakhaei@gmail.com

[4]Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore, India.; ssahoovu@gmail.com

[5] Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, Casablanca, Morocco .; broumisaid78@gmail.com

*Correspondence: sathamhussain5592@gmail.com [1]

Abstract: The quadripartitioned neutrosophic set is the partition of indeterminacy function of the neutrosophic set into contradiction part and ignorance part. In this work, the concept of quadripartitioned neutrosophic graph structures and its properties are invented. The strong, tree, $\phi-$ permutation and $\phi-$ complement of quadripartitioned neutrosophic graph structure are investigated. The operations like Cartesian Product, cross product, lexicographic product, composition in graph structures and join operations are established.

Keywords: Quadripartitioned neutrosophic graph, quadripartitioned neutrosophic graph structure, ϕ permutation, ϕ complement, Operations

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1. Introduction

The intuitionistic fuzzy sets represent a novel component in the fuzzy sets, namely a non-membership function. However, some limits only allow for the storage of incomplete data when interpreting the degree of true and false membership functions, but the handling of indeterminate data is still possible. Can we look at an example where ten patients are being tested for a pandemic? Three patients will have a positive result, five will have a negative result, and two will be uncertain or have yet to be determined throughout that period. It can be stated as $x(0.3,0.2,0.5)$ using neutrosophic notions. Using the neutrosophic set, one can classify the environment as cold as truth, moderate as indeterminacy, and hot as false for a clear comprehension. As a result, the neutrosophic field emerges to hold the indeterminacy data. From a philosophical standpoint, it generalises the aforementioned sets. The single-valued neutrosophic set is a generalisation of intuitionistic fuzzy sets that can be utilised to solve real-world problems, particularly in decision support. The sum of the three components of belief in that element (truth), disbelief in that element (falsehood), and the indeterminacy part of that element is strictly less than 1. Smarandache [36, 38] and references therein propose neutrosophic sets as the foundation of neutrosophic logic, a multiple value logic that generalises fuzzy logic and deals with paradoxes, contradictions, antitheses, and antinomies.

In the situation of neutrosophic sets, indeterminacy is considered as a distinct concept, and each component is defined by a truth-membership function, an indeterminacy membership

function, and a falsity-membership function, all of which are obtained from the non-standard unit interval $]0^{-}, 1^{+}[$. Ignoring the fact that neutrosophic indeterminacy is independent of truth and falsity-membership values, it is more general than the hesitation margin of intuitionistic fuzzy sets. It is unclear whether the indeterminacy values relevant to a specific element correspond to hesitant values about its belonging or non-belonging to it. As a result, some authors prefer to model the indeterminacy's behaviour in the same way they similar to truth-membership, others may prefer to model it in the same way they similar to falsity-membership. Wang et al. [43] initiated the concept of a single valued neutrosophic set and provide its various properties. It has been widely applied in various fields, such as information fusion in which data are combined from different sensors [10], control theory [1], image processing [12], medical diagnosis [42], decision making [41], and graph theory [4, 8, 15-18, 25, 35], etc. When the indeterminacy portion of the netrosophic set is divided into two parts, we get four components: 'Contradiction' (both true and false) and 'Unknown' (neither true nor false), that is $\mathbb{T}, \mathbb{C}, \mathbb{U}$ and $\mathbb{F}$ which defines a new set called 'quadripartitioned single valued neutrosophic set', introduced by Chatterjee., et al. [11]. This study is completely based on "Belnap's four valued logic" [9] and Smarandache's "Four Numerical valued neutrosophic logic" [39]. By employing the concept of Quadripartitioned neutrosophic set, this paper presents the quadripartitioned neutrosophic graphs structure. Operations on single-valued neutrosophic graph structures are studied in [2, 6]. Motivated by the above mentioned works, to the best of authors' knowledge, there is no work reported on the concepts of quadripartioned single valued neutrosophic graphs with application. The major contributions in this work are foregrounded as follows:

1. The notions of Quadripartitioned Neutrosophic Graph Structure (QNGS) and its properties are introduced.
2. In addition, the complete, strong and complement of QNGS are defined.
3. Furthermore, the ϕ −permutation and ϕ −complement of QNGS are investigated. The proposed concepts are illustrated with examples.
4. The operations like Cartesian Product, cross product, lexicographic product, composition in graph structures and join operations are established.

2. Preliminaries

Definition 2.1 *A graph structure* $\mathfrak{G} = (\mathcal{P}, \mathfrak{R}_1, \mathfrak{R}_2, .. \mathfrak{R}_n)$ *consists of a non-empty set* $\mathcal{V}$ *together with relation* $\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_n$ *on* $\mathcal{P}$ *which are mutually disjoint such that each* $\mathfrak{R}_i$, $1 \leq i \leq n$, *is symmetric and irreflexive.*

Definition 2.2 *A neutrosophic set* $\mathcal{N}$ *on a universal set* $\mathcal{P}$ *is an object of the form*

$\mathcal{N} = \{(p, \mathfrak{T}_{\mathcal{N}}(p), \mathfrak{I}_{\mathcal{N}}(p), \mathfrak{F}_{\mathcal{N}}(p): p \in \mathcal{P})\}$, where $\mathcal{T}_{\mathcal{N}}, \mathcal{I}_{\mathcal{N}}, \mathcal{F}_{\mathcal{N}}: \mathcal{P} \to\ ]0^{-}, 1^{+}[$ and $0^{-} \leq \mathfrak{T}_{\mathcal{N}}(p), \mathfrak{I}_{\mathcal{N}}(p), \mathfrak{F}_{\mathcal{N}}(p) \leq 3^{+}$.

Definition 2.3 *A single valued neutrosophic set* $\mathcal{N}$ *on a universal set* $\mathcal{P}$ *is an object of the form*

$\mathcal{N} = \{(p, \mathfrak{T}_{\mathcal{N}}(p), \mathfrak{I}_{\mathcal{N}}(p), \mathfrak{F}_{\mathcal{N}}(p): p \in \mathcal{P})\}$, where $\mathcal{T}_{\mathcal{N}}, \mathcal{I}_{\mathcal{N}}, \mathcal{F}_{\mathcal{N}}: \mathcal{P} \to [0,1]$ and $0 \leq \mathfrak{T}_{\mathcal{N}}(p), \mathfrak{I}_{\mathcal{N}}(p), \mathfrak{F}_{\mathcal{N}}(p) \leq 3$.

Definition 2.4 *[3] A neutrosophic graph is defined as a pair* $G^* = (V, E)$ *where*

(i) $V = \{v_1, v_2, .., v_n\}$ such that $\mathcal{T}_A: V \to [0,1]$, $\mathcal{I}_A: V \to [0,1]$ and $\mathcal{F}_A: V \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and

$$0 \leq \mathcal{T}_A(v) + \mathcal{I}_A(v) + \mathcal{F}_A(v) \leq 3, \ \forall \quad v \in V.$$

(ii) $E \subseteq V \times V$ where $\mathcal{T}_B: E \to [0,1]$, $\mathcal{I}_B: E \to [0,1]$ and $\mathcal{F}_B: E \to [0,1]$ are such that

$$\mathcal{T}_B(uv) \leq \min\{T_A(u), T_A(v)\},$$
$$\mathcal{I}_B(uv) \leq \min\{I_A(u), I_A(v)\},$$
$$\mathcal{F}_B(uv) \leq \max\{F_A(u), F_A(v)\},$$
$$\forall u, v \in V.$$

For more details about the following definitions and results, see the article [11].

Definition 2.5 *Let $\mathcal{X}$ be a non-empty set. A quadripartitioned neutrosopohic set (QSVNS) $\mathcal{A}$ over $\mathcal{R}$ characterizes each elements x in $\mathcal{X}$ by a truth membership function $\mathcal{T}_{\mathcal{A}}$, a contradiction membership function $\mathcal{C}_{\mathcal{A}}$, an ignorance membership function $\mathcal{U}_{\mathcal{A}}$ and a false membership function $\mathcal{F}_{\mathcal{A}}$ such that for each $x \in \mathcal{R}, \mathcal{T}_{\mathcal{A}}, \mathcal{C}_{\mathcal{A}}, \mathcal{U}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}} \in [0,1]$ and $0 \leq \mathcal{T}_{\mathcal{A}}(r) + \mathcal{C}_{\mathcal{A}}(r) + \mathcal{U}_{\mathcal{A}}(r) + \mathcal{F}_{\mathcal{A}}(r) \leq 4$.*

Remark 2.6 *A QSVNS $\mathfrak{A}$, can be decomposed to yields two SVNS say, $\mathfrak{A}_t$ and $\mathfrak{A}_f$ where the respective membership functions of both these sets are defined as*

$$\mathcal{T}_{\mathfrak{A}_t}(r) = \mathcal{T}_{\mathfrak{A}}(r) = \mathcal{T}_{\mathfrak{A}_f}(r)$$
$$\mathcal{I}_{\mathfrak{A}_t}(r) = \mathcal{C}_{\mathfrak{A}}(r), \quad \mathcal{I}_{\mathfrak{A}_f}(r) = \mathcal{U}_{\mathfrak{A}}(r)$$
$$\mathcal{F}_{\mathfrak{A}_t}(r) = \mathcal{F}_{\mathfrak{A}}(r) = \mathcal{F}_{\mathfrak{A}_f}(r), \quad \forall r \in \mathcal{R}.$$

In this respect to needs to be stated that while performing set-theoretic operations over these SVNS, behavior of $\mathcal{I}_{\mathfrak{A}_t}$ is treated similar to that of $\mathcal{T}_{\mathfrak{A}_t}$ while the behavior of $\mathcal{I}_{\mathfrak{A}_f}$ is modeled in a way similar to that of $\mathcal{F}_{\mathfrak{A}_f}$.

Definition 2.7 *A QSVNS is said to be an absolute QSVNS, denoted by $\mathfrak{A}$, if its is membership values are respectively defined as $\mathcal{T}_{\mathfrak{A}}(r) = 1$, $\mathcal{C}_{\mathfrak{A}}(r) = 1$, $\mathcal{U}_{\mathfrak{A}}(r) = 0$ and $\mathcal{F}_{\mathfrak{A}}(r) = 0$.*

Definition 2.8 *Consider two QSVNS $\mathfrak{A}$ and $\mathfrak{B}$, over $\mathcal{R}$. $\mathfrak{A}$ is said to be contained in $\mathfrak{B}$, denoted by $\mathfrak{A} \subseteq \mathfrak{B}$ if, and only, if $\mathcal{T}_{\mathfrak{A}}(r) \leq \mathcal{T}_{\mathfrak{B}}(r)$, $\mathcal{C}_{\mathfrak{A}}(r) \leq \mathcal{C}_{\mathfrak{B}}(r)$, $\mathcal{U}_{\mathfrak{A}}(r) \geq \mathcal{U}_{\mathfrak{B}}(r)$ and $\mathcal{F}_{\mathfrak{A}}(r) \geq \mathcal{F}_{\mathfrak{B}}(r)$.*

Definition 2.9 *The complement of a QSVNS $\mathfrak{A}$, is denoted by $\mathfrak{A}^c$ and is defined as*

$$\mathfrak{A}^c = \sum_{i=1}^{n} \langle \mathcal{F}_{\mathfrak{A}}(r_i), \mathcal{U}_{\mathfrak{A}}(r_i), \mathcal{C}_{\mathfrak{A}}(r_i), \mathcal{T}_{\mathfrak{A}}(r_i) \rangle, \qquad \forall r_i \in \mathcal{R}.$$

i.e. $\mathcal{T}_{\mathfrak{A}^c}(r_i) = \mathcal{F}_{\mathfrak{A}}(r_i), \quad \mathcal{C}_{\mathfrak{A}^c}(r_i) = \mathcal{U}_{\mathfrak{A}}(r_i)$

$\mathcal{U}_{\mathfrak{A}^c}(r_i) = \mathcal{C}_{\mathfrak{A}}(r_i), \quad \mathcal{F}_{\mathfrak{A}^c}(r_i) = \mathcal{T}_{\mathfrak{A}}(r_i), \ \forall r_i \in \mathcal{R}.$

Definition 2.10 *The union of two QSVNS $\mathfrak{A}$ and $\mathfrak{B}$ is denoted by $\mathfrak{A} \cup \mathfrak{B}$ and is defined as*

$$\mathfrak{A} \cup \mathfrak{B} = \sum_{i=1}^{n} \langle \mathcal{T}_{\mathfrak{A}}(r_i) \vee \mathcal{T}_{\mathfrak{B}}(r_i), \mathcal{C}_{\mathfrak{A}}(r_i) \vee \mathcal{C}_{\mathfrak{B}}(r_i)$$
$$\mathcal{U}_{\mathfrak{A}}(r_i) \wedge \mathcal{U}_{\mathfrak{B}}(r_i), \mathcal{F}_{\mathfrak{A}}(r_i) \wedge \mathcal{F}_{\mathfrak{B}}(r_i) \rangle / \mathcal{R}.$$

Definition 2.11 *The intersection of two QSVNS $\mathfrak{A}$ and $\mathfrak{B}$ is denoted by $\mathfrak{A} \cap \mathfrak{B}$ and is defined as*

$$\mathfrak{A} \cap \mathfrak{B} = \sum_{i=1}^{n} \langle \mathcal{T}_{\mathfrak{A}}(r_i) \wedge \mathcal{T}_{\mathfrak{B}}(r_i), \mathcal{C}_{\mathfrak{A}}(r_i) \wedge \mathcal{C}_{\mathfrak{B}}(r_i)$$
$$\mathcal{U}_{\mathfrak{A}}(r_i) \vee \mathcal{U}_{\mathfrak{B}}(r_i), \mathcal{F}_{\mathfrak{A}}(r_i) \vee \mathcal{F}_{\mathfrak{B}}(r_i) \rangle / \mathcal{R}$$

3. Quadripartitioned Neutrosophic Graph structure

Definition 3.1 *Let $\mathbb{R}$ be a non-empty set and $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_n$ relation on $\mathbb{R}$. $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ is called a quadripartioned neutrosophic graph structure if*

$$\mathfrak{A} = \{n, \mathbb{T}_i(l), \mathbb{C}_i(l), \mathbb{U}_i(l), \mathbb{F}_i(l) : n \in \mathbb{R}\}$$

is a quadripartitioned neutrosophic set on $\mathbb{R}$ and

$$\mathfrak{B}_i = \{(k,l), \mathbb{T}(k,l), \mathbb{I}(k,l), \mathbb{U}(k,l), \mathbb{F}(k,l) : n \in \mathbb{E}_i\}$$

is a quadripartitioned neutrosophic set on $\mathbb{E}_i$ such that

$$\mathbb{T}_i(k,l) \leq \min\{\mathbb{T}(k), \mathbb{T}(l)\},$$
$$\mathbb{C}_i(k,l) \leq \min\{\mathbb{C}(k), \mathbb{C}(l)\},$$
$$\mathbb{U}_i(k,l) \leq \max\{\mathbb{U}(k), \mathbb{U}(l)\},$$
$$\mathbb{F}_i(k,l) \leq \max\{\mathbb{F}(k), \mathbb{F}(l)\},$$
$$\forall m, n \in \mathbb{R}.$$

$$0 \le \mathbb{T}_i(k,l) + \mathbb{C}_i(k,l) + \mathbb{U}_i(k,l) + \mathbb{F}_i(k,l) \le 4. \text{ for all } (k,l) \in \mathbb{E}_i$$

where $\mathbb{R}$ and $\mathbb{E}_i$ $(i = 1,2,\dots,n)$ are underlying vertex and underlying i-edge sets of $\mathbb{G}$, respectively.

Example 3.2 *Let* $\mathfrak{G}^* = (\mathbb{R}, \mathbb{E}_1, \mathbb{E}_2)$ *be a graph structure* $\mathfrak{G} = \{q_1, q_2, q_3, q_4, q_5, q_5, q_6\}$, $\mathbb{E}_1 = \{q_1q_6, q_2q_3, q_3q_4, q_4q_5\}$, $\mathbb{E}_2 = \{q_1q_2, q_5q_6, q_4q_6, q_1q_3\}$. *Now we can define quadripartitioned neutrosophic sets* $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ *on* $\mathbb{R}, \mathbb{E}_1, \mathbb{E}_2$ *respectively,*

Let $\mathfrak{A} = \{(q_1, 0.3,0.7,0.7,0.4), (q_2, 0.4,0.7,0.6,0.6), (q_3, 0.4,0.4,0.3,0.2)$ $(q_4, 0.5,0.6,0.7,0.4), (q_5, 0.3,0.4,0.7,0.8), (q_6, 0.4,0.3,0.4,0.3)\}$

$\mathfrak{B}_1 = \{(q_1q_6, 0.2,0.1,0.4,0.3), (q_2q_3, 0.3,0.4,0.5,0.5), (q_3q_4, 0.3,0.3,0.5,0.1), (q_4q_5, 0.3,0.4,0.7,0.4)\}$

$\mathfrak{B}_2 = \{(q_1q_2, 0.3,0.2,0.6,0.3), (q_5q_6, 0.2,0.3,0.3,0.2), (q_4q_6, 0.3,0. ,0.6,0.2), (q_1q_3, 0.2,0.2,0.2,0.2)\}$

By direct calculation, it is easy to show that $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ is a QNGS of $\mathfrak{G}^*$ is shown in figure 1

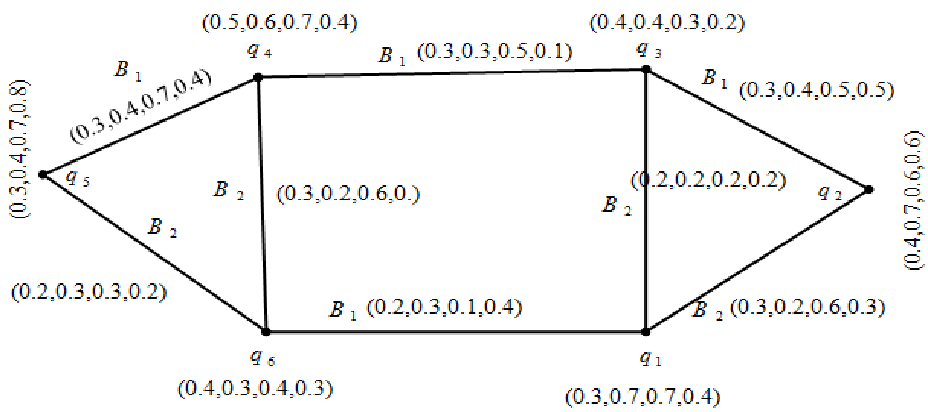

Figure 1: QUADRIPARTITIONED NEUTROSOPHIC GRAPH STRUCTURE

Definition 3.3 *Let* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, .., \mathfrak{B}_n)$ *be a QNGS of* $\mathfrak{G}^*$. *If* $\mathcal{H} = (\mathfrak{A}', \mathfrak{B}'_1, \mathfrak{B}'_2, \dots, \mathfrak{B}'_n)$ *is a QNGS of* $\mathfrak{G}^*$ *such that*

$$\mathbb{T}'(l) \le \mathbb{T}(l), \mathbb{C}'(l) \le \mathbb{C}(l), \mathbb{U}'(l) \ge \mathbb{U}(l), \mathbb{F}'(l) \ge \mathbb{F}(l)$$

for all $n \in \mathbb{R}$,

$$\mathbb{T}'_i(k,l) \le \mathbb{T}_i(k,l), \mathbb{C}'_i(k,l) \le \mathbb{C}_i(k,l), \mathbb{U}'_i(k,l) \ge \mathbb{U}_i(k,l), \mathbb{F}'_i(k,l) \ge \mathbb{F}_i(k,l)$$

for all $m,n \in \mathbb{E}_i$, where $i = 1,2,\dots,n$. Then $\mathcal{H}$ is called a quadripartitioned neutrosophic subgraph structure of QNGS $\mathbb{G}$.

Example 3.4 *Consider a graph structure* $\mathfrak{G}^* = (\mathbb{R}, \mathbb{E}_1, \mathbb{E}_2)$ *and let* $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ *be quadripartitioned neutrosophic subsets of* $(\mathbb{R}, \mathbb{E}_1, \mathbb{E}_2)$ *respectively, such that*

$\mathfrak{A} = \{(n_1, 0.8,0.6,0.5,0.4), (n_2, 0.7,0.6,0.5,0.4), (n_3, 0.6,0.8,0.4,0.4), (n_4, 0.5,0.5,0.3,0.4)\}$

$\mathfrak{B}_1 = \{(n_1n_2, 0.6,0.5,0.4,0.3), (n_2n_4, 0.3,0.3,0.4,0.3)\}$,

$\mathfrak{B}_2 = \{(n_3n_4, 0{,}4{,}0.3{,}0.3{,}0.3), (n_1n_4, 0.4{,}0.4{,}0.5{,}0.3)\}$

Direct calculations show that $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ is a QNGS of $\mathfrak{G}^*$ as presented in Figure 3.

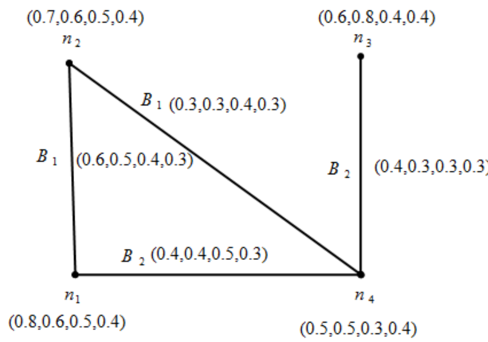

Figure 2: QUADRIPARTITIONED NEUTROSOPHIC GRAPH STRUCTURE

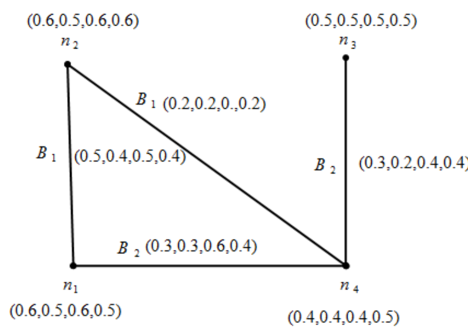

Figure 3: QUADRIPARTITIONED NEUTROSOPHIC SUBGRAPH STRUCTURE

Definition 3.5 *A QNGS* $\mathcal{H} = (\mathfrak{A}', \mathfrak{B}'_1, \mathfrak{B}'_2, \ldots, \mathfrak{B}'_n)$ *is called an induced subgraph structure of* $\mathfrak{G}$ *by a subset* $\mathcal{R}$ *of* $\mathcal{X}$ *if*

$$\mathbb{T}'(l) = \mathbb{T}(l), \mathbb{C}'(l) = \mathbb{C}(l), \mathbb{U}'(l) = \mathbb{U}(l), \mathbb{F}'(l) = \mathbb{F}(l)$$

for all $n \in \mathbb{E}$,

$$\mathbb{T}'_i(k,l) = \mathbb{T}_i(k,l), \mathbb{C}'_i(k,l) = \mathbb{C}_i(k,l), \mathbb{U}'_i(k,l) = \mathbb{U}_i(k,l), \mathbb{F}'_i(k,l) = \mathbb{F}_i(k,l)$$

for all $m, n \in \mathbb{E}_i$, where $i = 1,2,\ldots,n$.

Definition 3.6 *A QNGS* $\mathcal{H} = (\mathfrak{A}', \mathfrak{B}'_1, \mathfrak{B}'_2, \ldots, \mathfrak{B}'_n)$ *is said to be a spanning subgraph structure of* $\mathfrak{G}$ *when* $\mathfrak{A}' = \mathfrak{A}$ *and*

$$\mathbb{T}'_i(k,l) \leq \mathbb{T}_i(k,l), \mathbb{C}'_i(k,l) \leq \mathbb{C}_i(k,l), \mathbb{U}'_i(k,l) \geq \mathbb{U}_i(k,l), \mathbb{F}'_i(k,l) \geq \mathbb{F}_i(k,l)$$

$i = 1,2,\ldots,n$.

Definition 3.7 *Let* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *be an QNGS of* $\mathfrak{G}^*$. *Then* $kl \in \mathbb{E}_i$ *is called* $\mathfrak{B}_i$ *edge if* $\mathbb{T}_i(k,l) > 0$ *or* $\mathbb{C}_i(k,l) > 0$ *or* $\mathbb{U}_i(k,l) > 0$ *or* $\mathbb{F}_i(k,l) > 0$ *all the four conditions hold. Consequently, support of* $\mathfrak{B}_i$ *is defined as:*

$supp(\mathfrak{B}_i) = \{kl \in \mathfrak{B}_i : \mathbb{T}_i(k,l) > 0\} \cup \{kl \in \mathfrak{B}_i : \mathbb{C}_i(k,l) > 0\} \cup \{kl \in \mathfrak{B}_i : \mathbb{U}_i(k,l) > 0\} \cup \{kl \in \mathfrak{B}_i : \mathbb{F}_i(k,l) > 0\}, i = 1,2,\ldots,n.$

Definition 3.9 $\mathfrak{B}_i$ *−path in a QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *is a sequence of different nodes* $n_1, n_2, \ldots, n_m$ *(except choice that* $n_m = n_1$*) in* $\mathcal{X}$, *such that* $n_{j-1}n_j$ *is a quadripartitioned neutrosophic* $\mathfrak{B}_i$*-edge, for all* $j = 2, \ldots, m$.

Definition 3.10 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *is called* $\mathfrak{B}_i$*- strong for some* $i \in \{1,2,3,\ldots n\}$ *if*

$$\mathbb{T}_i(k,l) = \min\{\mathbb{T}(k), \mathbb{T}(l)\},$$
$$\mathbb{C}_i(k,l) = \min\{\mathbb{C}(k), \mathbb{C}(l)\},$$
$$\mathbb{U}_i(k,l) = \max\{\mathbb{U}(k), \mathbb{U}(l)\}, \text{and}$$
$$\mathbb{F}_i(k,l) = \max\{\mathbb{F}(k), \mathbb{F}(l)\}, \forall mn \in supp(\mathfrak{B}_i).$$

Further, QNGS $\mathfrak{G}$ is said to be strong if it is $\mathfrak{B}_i$- strong for all $i \in \{1,2,\ldots,n.\}$

Definition 3.11 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, .., \mathfrak{B}_n)$ *is said to be complete if* $\mathfrak{G}$ *is a strong QNGS,* $supp(\mathfrak{B}_i) \neq \phi$ *for all* $i = 1,2,\ldots,n$ *and for all pair of nodes* $k, l \in \mathcal{X}$, kl *is a* $\mathfrak{B}_i$ *edge for some* i.

Definition 3.12 *Let* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, .., \mathfrak{B}_n)$ *be a QNGS. Now truth strength, contradiction strength, ignorance strength and false strength of a* $\mathfrak{B}_i$*-path* $P_{\mathfrak{B}_i} = n_1, n_2, \ldots, n_m$ *are denoted by* $T.P_{\mathfrak{B}_i}, C.P_{\mathfrak{B}_i}, U.P_{\mathfrak{B}_i}$ *and* $F.P_{\mathfrak{B}_i}$, *respectively, and defined as*

$$T.P_{\mathfrak{B}_i} = \bigwedge_{j=2}^{m} [\mathbb{T}^P_{\mathfrak{B}_i}(n_{j-1}n_j)],$$
$$C.P_{\mathfrak{B}_i} = \bigwedge_{j=2}^{m} [\mathbb{C}^P_{\mathfrak{B}_i}(n_{j-1}n_j)],$$
$$U.P_{\mathfrak{B}_i} = \bigvee_{j=2}^{m} [\mathbb{U}^P_{\mathfrak{B}_i}(n_{j-1}n_j)],$$
$$F.P_{\mathfrak{B}_i} = \bigvee_{j=2}^{m} [\mathbb{F}^P_{\mathfrak{B}_i}(n_{j-1}n_j)].$$

Definition 3.13 *Suppose* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, .., \mathfrak{B}_n)$ *is a QNGS. Then*

1. $\mathfrak{B}_i$- truth strength of connectedness between m and n is defined as: $\mathbb{T}^{\infty}_{\mathfrak{B}_i}(kl) = \bigvee_{j\geq 1} \{\mathbb{T}^j_{\mathfrak{B}_i}(kl)\}$ such that $\mathbb{T}^j_{\mathfrak{B}_i}(kl) = (\mathbb{T}^{j-1}_{\mathfrak{B}_i} \circ \mathbb{T}^1_{\mathfrak{B}_i})(kl)$ for $j \geq 2$ and
$\mathbb{T}^2_{\mathfrak{B}_i}(kl) = (\mathbb{T}^1_{\mathfrak{B}_i} \circ \mathbb{T}^1_{\mathfrak{B}_i})(kl) = \bigvee_{z} (\mathbb{T}^1_{B_i}(mz) \wedge \mathbb{T}^1_{B_i}(zn))$.
2. $\mathfrak{B}_i$- contradiction strength of connectedness between m and n is defined as: $\mathbb{C}^{\infty}_{\mathfrak{B}_i}(kl) = \bigvee_{j\geq 1} \{\mathbb{C}^j_{\mathfrak{B}_i}(kl)\}$ such that $\mathbb{C}^j_{\mathfrak{B}_i}(kl) = (\mathbb{C}^1_{\mathfrak{B}_i} \circ \mathbb{C}^1_{\mathfrak{B}_i})(kl)$ for $j \geq 2$ and
$\mathbb{C}^2_{\mathfrak{B}_i}(kl) = (\mathbb{C}^1_{\mathfrak{B}_i} \circ \mathbb{C}^1_{\mathfrak{B}_i})(kl) = \vee_z (\mathbb{C}^1_{B_i}(mz) \wedge \mathbb{C}^1_{B_i}(zn))$.
3. $\mathfrak{B}_i$- ignorance strength of connectedness between m and n is defined as: $\mathbb{U}^{\infty}_{\mathfrak{B}_i}(kl) = \bigwedge_{j\geq 1} \{\mathbb{U}^j_{\mathfrak{B}_i}(kl)\}$ such that $\mathbb{U}^j_{\mathfrak{B}_i}(kl) = (\mathbb{U}^1_{\mathfrak{B}_i} \circ \mathbb{U}^1_{\mathfrak{B}_i})(kl)$ for $j \geq 2$ and
$\mathbb{U}^2_{\mathfrak{B}_i}(kl) = (\mathbb{U}^1_{\mathfrak{B}_i} \circ \mathbb{U}^1_{\mathfrak{B}_i})(kl) = \bigwedge_{z} (\mathbb{U}^1_{B_i}(mz) \vee \mathbb{U}^1_{B_i}(zn))$.
4. $\mathfrak{B}_i$- false strength of connectedness between m and n is defined as: $\mathbb{F}^{\infty}_{\mathfrak{B}_i}(kl) = \bigwedge_{j\geq 1} \{\mathbb{F}^j_{\mathfrak{B}_i}(kl)\}$ such that $\mathbb{F}^j_{\mathfrak{B}_i}(kl) = (\mathbb{F}^1_{\mathfrak{B}_i} \circ \mathbb{F}^1_{\mathfrak{B}_i})(kl)$ for $j \geq 2$ and
$\mathbb{F}^2_{\mathfrak{B}_i}(kl) = (\mathbb{F}^1_{\mathfrak{B}_i} \circ \mathbb{F}^1_{\mathfrak{B}_i})(kl) = \bigwedge_{z} (\mathbb{F}^1_{B_i}(mz) \vee \mathbb{F}^1_{B_i}(zn))$.

Definition 3.14 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *is a* $\mathfrak{B}_i$*-cycle if*

$$(supp(\mathfrak{A}), supp(\mathfrak{B}_1), supp(\mathfrak{B}_2), \ldots, supp(\mathfrak{B}_n)) \text{isa} \mathfrak{B}_i - \text{cycle}.$$

Definition 3.15 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *is a* $\mathfrak{B}_i$*-cycle (for some* i*) if* $\mathfrak{G}$ *is a* $\mathfrak{B}_i$*-cycle, no unique* $\mathfrak{B}_i$*-edge* kl *belongs to 1* $\mathfrak{G}$ *with*

$$\mathbb{T}_{B_i}(kl) = \min\{\mathbb{T}_{B_i}(rs) : rs \in \mathbb{E}_i = \text{supp}(\mathfrak{B}_i)\},$$
$$\mathbb{C}_{B_i}(kl) = \min\{\mathbb{C}_{B_i}(rs) : rs \in \mathbb{E}_i = \text{supp}(\mathfrak{B}_i)\},$$
$$\mathbb{U}_{B_i}(kl) = \max\{\mathbb{C}_{B_i}(rs) : rs \in \mathbb{E}_i = \text{supp}(\mathfrak{B}_i)\},$$
$$\mathbb{F}_{B_i}(kl) = \max\{\mathbb{F}_{B_i}(rs) : rs \in \mathbb{E}_i = \text{supp}(\mathfrak{B}_i)\}.$$

Definition 3.16 *Let* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *be a QNGS and* q *be a node in* $\mathfrak{G}$*. Let* $(\mathfrak{A}', \mathfrak{B}'_1, \mathfrak{B}'_2, \ldots, \mathfrak{B}'_n)$ *be a QNGS induced by* $\mathcal{X} \backslash \{q\}$ *such that, for all* $m \neq q$*,* $o \neq q$*,*

$$\mathbb{T}_{A'}(q) = \mathbb{C}_{A'}(q) = 0 = \mathbb{U}_{A'}(q) = \mathbb{F}_{A'}(q),$$
$$\mathbb{T}_{B'_i}(qm) = \mathbb{C}_{B'_i}(qm) = 0 = \mathbb{U}_{B'_i}(qm) = \mathbb{F}_{B'_i}(qm), \forall \quad \text{edges} \quad qm \in \mathfrak{G}$$
$$\mathbb{T}_{A'}(m) = \mathbb{T}_A(m), \mathbb{C}_{A'}(m) = \mathbb{C}_A(m), \mathbb{U}_{A'}(m) = \mathbb{U}_A(m), \mathbb{F}_{A'}(m) = \mathbb{F}_A(m),$$
$$\mathbb{T}_{B'_i}(mo) = \mathbb{T}_{B_i}(mo), \mathbb{C}_{B'_i}(mo) = \mathbb{C}_{B_i}(mo), \mathbb{U}_{B'_i}(mo) = \mathbb{U}_{B_i}(mo), \mathbb{F}_{B'_i}(mo) = \mathbb{F}_{B_i}(mo).$$

Now q is quadripartitioned neutrosophic $\mathfrak{B}_i$ cut vertex for some i if

$$\mathbb{T}^{\infty}_{B_i}(mo) > \mathbb{T}^{\infty}_{B'_i}(mo), \mathbb{C}^{\infty}_{B_i}(mo) > \mathbb{C}^{\infty}_{B'_i}(mo), \mathbb{U}^{\infty}_{B_i}(mo) > \mathbb{U}^{\infty}_{B'_i}(mo), \mathbb{F}^{\infty}_{B_i}(mo) > \mathbb{F}^{\infty}_{B'_i}(mo)$$

for some $m, o \in \mathcal{X} \backslash \{q\}$. Note that q is a

- $\mathfrak{B}_i$- $\mathbb{T}$ quadripartitioned neutrosophic cut node if $\mathbb{T}^{\infty}_{B_i}(mo) > \mathbb{T}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{C}$ quadripartitioned neutrosophic cut node if $\mathbb{C}^{\infty}_{B_i}(mo) > \mathbb{C}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{U}$ quadripartitioned neutrosophic cut node if $\mathbb{U}^{\infty}_{B_i}(mo) > \mathbb{U}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{F}$ quadripartitioned neutrosophic cut node if $\mathbb{F}^{\infty}_{B_i}(mo) > \mathbb{F}^{\infty}_{B'_i}(mo)$.

Definition 3.17 *Suppose* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *be a QNGS and* kl *be* $\mathfrak{B}_i$*-edge. Let* $(\mathfrak{A}', \mathfrak{B}'_1, \mathfrak{B}'_2, \ldots, \mathfrak{B}'_n)$ *be a quadripartitioned neutrosophic graph spanning subgraph structure of* $\mathfrak{G}$ *with for all lines* $kl \neq rs$*,*

$$\mathbb{T}_{B'_i}(kl) = \mathbb{C}_{B'_i}(kl) = 0 = \mathbb{U}_{B'_i}(kl) = \mathbb{F}_{B'_i}(kl),$$
$$\mathbb{T}_{B'_i}(rs) = \mathbb{T}_{B_i}(rs), \mathbb{C}_{B'_i}(rs) = \mathbb{C}_{B_i}(rs), \mathbb{U}_{B'_i}(rs) = \mathbb{U}_{B_i}(rs), \mathbb{F}_{B'_i}(rs) = \mathbb{F}_{B_i}(rs).$$

Then kl is quadripartitioned neutrosophic $\mathfrak{B}_i$-bridge if

$$\mathbb{T}^{\infty}_{B_i}(mo) > \mathbb{T}^{\infty}_{B'_i}(mo), \mathbb{C}^{\infty}_{B_i}(mo) > \mathbb{C}^{\infty}_{B'_i}(mo), \mathbb{U}^{\infty}_{B_i}(mo) > \mathbb{U}^{\infty}_{B'_i}(vw), \mathbb{F}^{\infty}_{B_i}(mo) > \mathbb{F}^{\infty}_{B'_i}(mo)$$

for some $m, o \in \mathcal{X}$. Note kl is a

- $\mathfrak{B}_i$- $\mathbb{T}$ quadripartitioned neutrosophic bridge if $\mathbb{T}^{\infty}_{B_i}(mo) > \mathbb{T}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{C}$ quadripartitioned neutrosophic bridge if $\mathbb{C}^{\infty}_{B_i}(mo) > \mathbb{C}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{U}$ quadripartitioned neutrosophic bridge if $\mathbb{U}^{\infty}_{B_i}(mo) > \mathbb{U}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{F}$ quadripartitioned neutrosophic bridge if $\mathbb{F}^{\infty}_{B_i}(mo) > \mathbb{F}^{\infty}_{B'_i}(mo)$.

Definition 3.18 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *is a* $\mathfrak{B}_i$ *tree if*

$$(supp(\mathfrak{A}), supp(\mathfrak{B}_i), supp(\mathfrak{B}_2, \ldots, supp(\mathfrak{B}_n))$$

is a $\mathfrak{B}_i$-tree. In otherwords, $\mathfrak{G}$ is a $\mathfrak{B}_i$-tree provided a subgraph of $\mathfrak{G}$ induced by $supp(\mathfrak{B}_i)$ produces a tree.

Definition 3.19 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *is a* $\mathfrak{B}_i$*- tree if* $\mathfrak{G}$ *has a quadripartitioned neutrosophic spanning subgraph structure* $\mathcal{H} = (\mathfrak{A}', \mathfrak{B}_1', \mathfrak{B}_2', \ldots, \mathfrak{B}_n')$ *with for every* $\mathfrak{B}_i$*-edges* kl *not belongs to* $\mathcal{H}$, $\mathcal{H}$ *is a* $\mathfrak{B}_i'$*-tree,*

$$\mathbb{T}_{B_i}^{\infty}(kl) < \mathbb{T}_{B_i'}^{\infty}(kl), \mathbb{C}_{B_i}^{\infty}(kl) < \mathbb{C}_{B_i'}^{\infty}(kl), \mathbb{U}_{B_i}^{\infty}(kl) > \mathbb{U}_{B_i'}^{\infty}(kl), \mathbb{F}_{B_i}^{\infty}(kl) > \mathbb{F}_{B_i'}^{\infty}(kl)$$

In particular, $\mathfrak{G}$ is a:

- $\mathfrak{B}_i$- $\mathbb{T}$ quadripartitioned neutrosophic tree if $\mathbb{T}_{B_i}^{\infty}(kl) < \mathbb{T}_{B_i'}^{\infty}(kl)$.
- $\mathfrak{B}_i$- $\mathbb{C}$ quadripartitioned neutrosophic tree if $\mathbb{C}_{B_i}^{\infty}(kl) < \mathbb{C}_{B_i'}^{\infty}(kl)$.
- $\mathfrak{B}_i$- $\mathbb{U}$ quadripartitioned neutrosophic tree if $\mathbb{U}_{B_i}^{\infty}(kl) > \mathbb{U}_{B_i'}^{\infty}(kl)$.
- $\mathfrak{B}_i$- $\mathbb{F}$ quadripartitioned neutrosophic bridge if $\mathbb{F}_{B_i}^{\infty}(kl) > \mathbb{F}_{B_i'}^{\infty}(kl)$.

Definition 3.20 *A QNGS* $\mathfrak{G}_1 = (\mathfrak{A}_1, \mathfrak{B}_{11}, \mathfrak{B}_{12}, \ldots, \mathfrak{B}_{1n})$ *of the graph structure* $\mathfrak{G}_1^* = (\mathbb{R}_1, \mathbb{E}_{11}, \mathbb{E}_{12}, \ldots, \mathbb{E}_{1n})$ *is isomorphic to QNGS* $\mathfrak{G}_2 = (\mathfrak{A}_2, \mathfrak{B}_{21}, \mathfrak{B}_{22}, \ldots, \mathfrak{B}_{2n})$ *of graph structure* $\mathfrak{G}_2^* = (\mathbb{R}_2, \mathbb{E}_{21}, \mathbb{E}_{22}, \ldots, \mathbb{E}_{2n})$ *if* $f: \mathbb{R}_1 \to \mathbb{R}_2$ *is a bijection and the conditions below are fulfilled:*

$$\mathbb{T}_{A_1}(k) = \mathbb{T}_{A_2}(f(k)), \mathbb{C}_{A_1}(k) = \mathbb{C}_{A_2}(f(k)), \mathbb{U}_{A_1}(k) = \mathbb{U}_{A_2}(f(k)), \mathbb{F}_{A_1}(k) = \mathbb{F}_{A_2}(f(k)),$$

for all $m \in \mathbb{R}_1$ and

$$\mathbb{T}_{B_{1i}}(kl) = \mathbb{T}_{B_{2\phi(i)}}(f(k)f(l)), \mathbb{C}_{B_{1i}}(kl) = \mathbb{C}_{B_{2\phi(i)}}(f(k)f(l)),$$
$$\mathbb{U}_{B_{1i}}(kl) = \mathbb{U}_{B_{2\phi(i)}}(f(k)f(l)), \mathbb{F}_{B_{1i}}(kl) = \mathbb{F}_{B_{2\phi(i)}}(f(k)f(l)),$$

for all $kl \in \mathbb{E}_{1i}$ and $i = 1,2,..,n$.

Definition 3.21 *A QNGS* $\mathfrak{G}_1 = (\mathfrak{A}_1, \mathfrak{B}_{11}, \mathfrak{B}_{12}, \ldots, \mathfrak{B}_{1n})$ *of the graph structure* $\mathfrak{G}_1^* = (\mathbb{R}_1, \mathbb{E}_{11}, \mathbb{E}_{12}, \ldots, \mathbb{E}_{1n})$ *is identical to QNGS* $\mathfrak{G}_2 = (\mathfrak{A}_2, \mathfrak{B}_{21}, \mathfrak{B}_{22}, \ldots, \mathfrak{B}_{2n})$ *of graph structure* $\mathfrak{G}_2^* = (\mathbb{R}_2, \mathbb{E}_{21}, \mathbb{E}_{22}, \ldots, \mathbb{E}_{2n})$ *if* $f: \mathbb{R}_1 \to \mathbb{R}_2$ *is a bijection and the conditions below are fulfilled:*

$$\mathbb{T}_{A_1}(k) = \mathbb{T}_{A_2}(f(k)), \mathbb{C}_{A_1}(k) = \mathbb{C}_{A_2}(f(k)), \mathbb{U}_{A_1}(k) = \mathbb{U}_{A_2}(f(k)), \mathbb{F}_{A_1}(k) = \mathbb{F}_{A_2}(f(k)),$$

for all $m \in \mathbb{R}_1$ and

$$\mathbb{T}_{B_{1i}}(kl) = \mathbb{T}_{B_{2i}}(f(k)f(l)), \mathbb{C}_{B_{1i}}(kl) = \mathbb{C}_{B_2}(f(k)f(l)),$$
$$\mathbb{U}_{B_{1i}}(kl) = \mathbb{U}_{B_{2i}}(f(k)f(l)), \mathbb{F}_{B_{1i}}(kl) = \mathbb{F}_{B_{2i}}(f(k)f(l)),$$

for all $kl \in \mathbb{E}_{1i}$ and $i = 1,2,..,n$.

Definition 3.22 *Let* $\mathfrak{G}_1 = (\mathfrak{A}_1, \mathfrak{B}_{11}, \mathfrak{B}_{12}, \ldots, \mathfrak{B}_{1n})$ *be a QNGS and* ϕ *−permutation on* $\mathfrak{B}_1, \mathfrak{B}_2, .., \mathfrak{B}_n$ *and on* $\{1,2,\ldots,n\}$ *defined by* $\phi(\mathfrak{B}_i) = \mathfrak{B}_j$ *if and only if* $\phi(i) = j$ *for every* i*. If* $kl \in \mathfrak{B}_i$ *for some* i *and*

$$\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) - \bigvee_{j \neq 1} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl)$$
$$\mathbb{C}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l) - \bigvee_{j \neq 1} \mathbb{C}_{\phi(\mathfrak{B}_j)}(kl)$$
$$\mathbb{U}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{U}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l) - \bigwedge_{j \neq 1} \mathbb{U}_{\phi(\mathbb{B}_j)}(kl)$$

$$\mathbb{F}_{\mathbb{B}_i^{\phi}}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) - \underset{j\neq 1}{\wedge} \mathbb{F}_{\phi(\mathbb{B}_j)}(kl), i = 1,2,\dots,n,$$

then $kl \in \mathfrak{B}_k^{\phi}$, where k is selected such that

$$\mathbb{T}_{\mathbb{B}_k^{\phi}}(kl) \geq \mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl),$$
$$\mathbb{C}_{\mathfrak{B}_k^{\phi}}(kl) \geq \mathbb{C}_{\mathfrak{B}_i^{\phi}}(kl),$$
$$\mathbb{U}_{\mathfrak{B}_k^{\phi}}(kl) \geq \mathbb{U}_{\mathfrak{B}_i^{\phi}}(kl),$$
$$\mathbb{F}_{\mathfrak{B}_k^{\phi}}(kl) \geq \mathbb{F}_{\mathfrak{B}_i^{\phi}}(kl).$$

then quadripartitined neutrosophic graph structure $(\mathfrak{A}, \mathfrak{B}_1^{\phi}, \mathfrak{B}_2^{\phi}, \dots, \mathfrak{B}_n^{\phi})$ is called ϕ- complement of $\mathfrak{G}$ and dentoted by $\mathfrak{G}^{\phi c}$

Proposition 3.23 *ϕ-complement of a QNGS $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n)$ is always a strong QNGS. Further, if $\phi(i) = k$, where $i, k \in \{1,2,\dots,n\}$ then for all $\mathfrak{B}_k$-edges in quadripartitioned neutrosophic graphic structure $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n)$ become $\mathfrak{B}_i^{\phi}$-edges in $(\mathfrak{A}^{\phi}, \mathfrak{B}_1^{\phi}, \mathfrak{B}_2^{\phi}, \dots, \mathfrak{B}_n^{\phi})$.*

Proof. We know that,

$$\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) - \underset{j\neq i}{\vee} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl),$$
$$\mathbb{C}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l) - \underset{j\neq i}{\vee} \mathbb{C}_{\phi(\mathbb{B}_j)}(kl),$$
$$\mathbb{U}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{U}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l) - \underset{j\neq i}{\wedge} \mathbb{U}_{\phi(\mathbb{B}_j)}(kl),$$
$$\mathbb{F}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) - \underset{j\neq i}{\wedge} \mathbb{F}_{\phi(\mathbb{B}_j)}(kl),$$

for $i \in 1,2,\dots,n$. Due to the expression of truthness in ϕ -complement, $\mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) \geq 0$, $\underset{j\neq i}{\vee} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl) \geq 0$ and $\mathbb{T}_{\mathfrak{B}_i}(kl) \leq \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l)$, for all $\mathfrak{B}_i$, now $\underset{j\neq i}{\vee} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl) \leq \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l)$

which implies that

$$\mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) - \underset{j\neq i}{\vee} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl) \geq 0$$

Hence, $\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl) \geq 0$ for every i. Further, $\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl)$ attains its maximum provided $\vee_{j\neq i} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl) \geq 0$ is zero. Clearly, when $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$ and kl is a $\mathfrak{B}_k$-edge then $\vee_{j\neq i} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl)$ gets zero value. So

$\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l)$, for some $(kl) \in \mathfrak{B}_k$, $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$

Similarly, we have

$\mathbb{C}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l)$, for some $(kl) \in \mathfrak{B}_k$, $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$

$\mathbb{U}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l)$, for some $(kl) \in \mathfrak{B}_k$, $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$

$\mathbb{F}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l)$, for some $(kl) \in \mathfrak{B}_k$, $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$.

Likewise, the expression of falsity in ϕ- complement:

$$\mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) \geq 0, \underset{j\neq i}{\wedge} \mathbb{F}_{\phi(\mathfrak{B}_j)}(kl) \geq 0 and \mathbb{F}_{\mathfrak{B}i}(kl) \leq \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) \forall \mathfrak{B}_i$$

Then

$$\underset{j\neq 1}{\wedge} \mathbb{F}_{\phi(\mathfrak{B}_j)}(kl) \leq \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l)$$

yields,

$$\mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) - \underset{j\neq i}{\wedge} \mathbb{F}_{\phi(\mathfrak{B}_j)}(kl) \geq 0$$

Therefore, $\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl)$ is non-negative for all i . Morevoer, $\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl)$ reaches its maximum when $\underset{j\neq i}{\wedge} \mathbb{F}_{\phi(\mathfrak{B}_j)}(kl)$ becomes zero. It is clear that when $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$ and kl is a $\mathfrak{B}_k$ edge then $\wedge_{j\neq i} \mathbb{F}_{\phi(\mathfrak{B}_j)}(kl)$ gets zero value. So

$$\mathbb{F}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) \quad for \quad (kl) \in \mathfrak{B}_k, \phi(\mathfrak{B}i) = \mathfrak{B}_k$$

Definition 3.24 *Let* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *be a QNGS and* ϕ *be a permutation on* $\{1,2,\ldots,n\}$ *then*

- If $\mathfrak{G}$ is isomorphic to $\mathfrak{G}^{\phi c}$, then $\mathfrak{G}$ is called self-complementary.
- If $\mathfrak{G}$ is identical to $\mathfrak{G}^{\phi c}$, then $\mathfrak{G}$ is called strong-self-complementary.

Definition 3.25 *Suppose* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *be a QNGS. Then*

- If $\mathfrak{G}$ is isomorphic to $\mathfrak{G}^{\phi c}$, for all permutation ϕ on $\{1,2,\ldots,n\}$, then $\mathfrak{G}$ is totally self complementary.
- If $\mathfrak{G}$ is identical to $\mathfrak{G}^{\phi c}$, for all permutation ϕ on $\{1,2,\ldots,n\}$, then $\mathfrak{G}$ is totally strong self complementary.

Remark 3.26 *All strong QNGSs are self complementary or totally self-complementary QNGSs.*

Theorem 3.27*A QNGSs is totally self-complementary if and only if it is strong QNGS.*

Proof. Consider a strong QNGS $\mathfrak{G}$ and permutation ϕ on $\{1,2,\ldots,n\}$. In the view of Proposition 3.22, ϕ-complement of a QNGS $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ is always a strong QNGS. Moreover, if $\phi(i) = k$, here $i, k \in \{1,2,\ldots,n\}$, then every $\mathfrak{B}_k$ lines in QNGSs $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ becomes $\mathfrak{B}_i^{\phi}$ -edges in $(\mathfrak{A}^{\phi}, \mathfrak{B}_1^{\phi}, \mathfrak{B}_2^{\phi}, \ldots, \mathfrak{B}_n^{\phi})$. It yields

$$\mathbb{T}_{\mathfrak{B}_k}(kl) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) = \mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl)$$

$$\mathbb{C}_{\mathfrak{B}_k}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l) = \mathbb{C}_{\mathfrak{B}_i^{\phi}}(kl)$$

$$\mathbb{U}_{\mathfrak{B}_k}(kl) = \mathbb{U}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l) = \mathbb{U}_{\mathfrak{B}_i^{\phi}}(kl)$$

$$\mathbb{F}_{\mathfrak{B}_k}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) = \mathbb{F}_{\mathfrak{B}_i^{\phi}}(kl)$$

Thus, in identity mapping $f: \mathcal{X} \to \mathcal{X}$, $\mathfrak{G}$ and $\mathfrak{G}^{\phi}$ are isomorphic with

$$\mathbb{T}_{\mathfrak{A}}(k) = \mathbb{T}_{\mathfrak{A}}(f(k)), \mathbb{C}_{\mathfrak{A}}(k) = \mathbb{C}_{\mathfrak{A}}(f(k)),$$

$$\mathbb{U}_{\mathfrak{A}}(k) = \mathbb{U}_{\mathfrak{A}}(f(k)), \mathbb{F}_{\mathfrak{A}}(k) = \mathbb{F}_{\mathfrak{A}}(f(k)),$$

$$\mathbb{T}_{\mathfrak{B}_k}(kl) = \mathbb{T}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{T}_{\mathfrak{B}_k^{\phi}}(kl),\ \mathbb{C}_{\mathfrak{B}_k}(kl) = \mathbb{C}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{C}_{\mathfrak{B}_k^{\phi}}(kl),$$

$$\mathbb{U}_{\mathfrak{B}_k}(kl) = \mathbb{U}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{U}_{\mathfrak{B}_k^{\phi}}(kl),\ \mathbb{F}_{\mathfrak{B}_k}(kl) = \mathbb{F}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{F}_{\mathfrak{B}_k^{\phi}}(kl),$$

for all $kl \in \mathcal{E}_k$, $\phi^{-1}(k) = i$ and $k = 1, , \ldots, n$. It holds for all permutation ϕ on $\{1,2,\ldots,n\}$. Thus, $\mathfrak{G}$ is totally self-complementary QNGS. Conversely, suppose for all permutation ϕ on $\{1,2,..,n\}$ $\mathfrak{G}$ is isomorphic to $\mathfrak{G}^{\phi}$. Then according to the definition of isomorphism of QNGSs and ϕ -complement of QNGS,

$$\mathbb{T}_{\mathfrak{B}_k}(kl) = \mathbb{T}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{T}_{\mathfrak{A}}(f(k)) \wedge \mathbb{T}_{\mathfrak{A}}(f(l)) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l)$$
$$\mathbb{C}_{\mathfrak{B}_k}(kl) = \mathbb{C}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{C}_{\mathfrak{A}}(f(k)) \wedge \mathbb{C}_{\mathfrak{A}}(f(l)) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l)$$
$$\mathbb{U}_{\mathfrak{B}_k}(kl) = \mathbb{U}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{U}_{\mathfrak{A}}(f(k)) \vee \mathbb{U}_{\mathfrak{A}}(f(l)) = \mathbb{U}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l)$$
$$\mathbb{F}_{\mathfrak{B}_k}(kl) = \mathbb{F}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{F}_{\mathfrak{A}}(f(k)) \vee \mathbb{F}_{\mathfrak{A}}(f(l)) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l).$$

for all $kl \in \mathcal{E}_k$ and $k = 1,2, . . , n$. Hence, $\mathfrak{G}$ is strong QNGS.

Remark 3.28 *All self-complementary QNGS is totally self-complementary.*

Theorem3.39*If* $\mathfrak{G}^* = (\mathcal{X}, \mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n)$ *is totally strong self-complementary QNGS and* $A = \langle \mathbb{T}_{\mathfrak{A}}, \mathbb{C}_{\mathfrak{A}}, \mathbb{U}_{\mathfrak{A}}, \mathbb{F}_{\mathfrak{A}} \rangle$ *is a quadripartitioned neutrosophic subset of* $\mathcal{X}$ *here* $\mathbb{T}_{\mathfrak{A}}, \mathbb{C}_{\mathfrak{A}}, \mathbb{U}_{\mathfrak{A}}, \mathbb{F}_{\mathfrak{A}}$ *are constant value functions, then a strong QNGS of* $\mathfrak{G}^*$ *with quadripartitioned neutrosophic node set* $\mathfrak{A}$ *is always a totally strong self-complementary QNGS.*

Proof. Let the four constants be $p, q, r, s \in [0,1]$, such that $\mathbb{T}_{\mathfrak{A}}(k) = p$ $\mathbb{C}_{\mathfrak{A}}(k) = q$, $\mathbb{U}_{\mathfrak{A}}(k) = r$, $\mathbb{F}_{\mathfrak{A}}(k) = s$ for all $m \in \mathcal{X}$. Because $\mathfrak{G}^*$ is totally self-complementary strong QNGS, hence there exists a bijection $f: \mathcal{X} \to \mathcal{X}$ for permutation ϕ^{-1} on $\{1,2,\dots,n\}$, with for any $\mathbb{E}_k$- edge $(kl), (f(k)f(l))$ [an $\mathbb{E}_i$-line in $\mathfrak{G}^*$] is an $\mathbb{E}_k$ line in $\mathfrak{G}^{*\phi^{-1}c}$. Thus, for all $\mathfrak{B}_k$- edge $(kl), (f(k)f(l))$ [an $\mathfrak{B}_i$-edge in $\mathfrak{G}$] is a $\mathfrak{B}_k^{\phi}$- edge in $\mathfrak{G}^{*\phi^{-1}c}$. Further, $\mathfrak{G}$ is strong QNGS. Hence

$$\mathbb{T}_{\mathfrak{A}}(k) = p = \mathbb{T}_{\mathfrak{A}}(f(k)), \mathbb{C}_{\mathfrak{A}}(k) = q = \mathbb{C}_{\mathfrak{A}}(f(k)),$$
$$\mathbb{U}_{\mathfrak{A}}(k) = r = \mathbb{U}_{\mathfrak{A}}(f(k)), \mathbb{F}_{\mathfrak{A}}(k) = s = \mathbb{F}_{\mathfrak{A}}(f(k)), \forall m \in \mathcal{X},$$

$$\mathbb{T}_{\mathfrak{B}_k}(kl) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) = \mathbb{T}_{\mathfrak{A}}(f(k)) \wedge \mathbb{T}_{\mathfrak{A}}(f(l)) = \mathbb{T}_{\mathfrak{B}_i^{\phi}}(f(k)f(l))$$
$$\mathbb{C}_{\mathfrak{B}_k}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l) = \mathbb{T}_{\mathfrak{A}}(f(k)) \wedge \mathbb{C}_{\mathfrak{A}}(f(l)) = \mathbb{C}_{\mathfrak{B}_i^{\phi}}(f(k)f(l))$$
$$\mathbb{U}_{\mathfrak{B}_k}(kl) = \mathbb{U}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l) = \mathbb{U}_{\mathfrak{A}}(f(k)) \vee \mathbb{U}_{\mathfrak{A}}(f(l)) = \mathbb{U}_{\mathfrak{B}_i^{\phi}}(f(k)f(l))$$
$$\mathbb{F}_{\mathfrak{B}_k}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) = \mathbb{F}_{\mathfrak{A}}(f(k)) \vee \mathbb{F}_{\mathfrak{A}}(f(l)) = \mathbb{F}_{\mathfrak{B}_i^{\phi}}(f(k)f(l)).$$

for every $kl \in \mathbb{E}_i$ and $i = 1,2,\dots n$. This leads to $\mathfrak{G}$ is self complementary strong QNGS. All permutation ϕ and ϕ^{-1} on $\{1,2,..,n\}$ fulfils the above arguments, hence $\mathfrak{G}$ is totally strong self-complementary QNGS. Converse of the theorem may not be true.

Definition 3.40 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \dots . \mathfrak{Q}_{1n)}$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \dots . \mathfrak{Q}_{2n)}$ *be QNGS. The Cartesian product of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *denoted by*

$$\mathfrak{G}_{n1} \times \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \times \mathfrak{Q}_2, \mathfrak{Q}_{11} \times \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \times \mathfrak{Q}_{22}, \dots, \mathfrak{Q}_{1n} \times \mathfrak{Q}_{2n}),$$

is defined by the following:

(i) $\mathbb{T}_{\mathfrak{Q}1\times\mathfrak{Q}2}(rs) = (\mathbb{T}_{\mathfrak{Q}1} \times \mathbb{T}_{\mathfrak{Q}2})(rs) = \mathbb{T}_{\mathfrak{Q}1}(r) \wedge \mathbb{T}_{\mathfrak{Q}2}(s)$.
$\mathbb{C}_{\mathfrak{Q}1\times\mathfrak{Q}2}(rs) = (\mathbb{C}_{\mathfrak{Q}1} \times \mathbb{C}_{\mathfrak{Q}2})(rs) = \mathbb{C}_{\mathfrak{Q}1}(r) \wedge \mathbb{C}_{\mathfrak{Q}2}(s)$.
$\mathbb{U}_{\mathfrak{Q}1\times\mathfrak{Q}2}(rs) = (\mathbb{U}_{\mathfrak{Q}1} \times \mathbb{U}_{\mathfrak{Q}2})(rs) = \mathbb{U}_{\mathfrak{Q}1}(r) \vee \mathbb{U}_{\mathfrak{Q}2}(s)$.
$\mathbb{F}_{\mathfrak{Q}1\times\mathfrak{Q}2}(rs) = (\mathbb{F}_{\mathfrak{Q}1} \times \mathbb{F}_{\mathfrak{Q}2})(rs) = \mathbb{F}_{\mathfrak{Q}1}(r) \vee \mathbb{F}_{\mathfrak{Q}2}(s)$.
$\forall rs \in S_1 \times S_2$.

(ii) $\mathbb{T}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \times \mathbb{T}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \times \mathbb{C}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \times \mathbb{U}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \times \mathbb{F}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\forall r \in S_1, s_1s_2 \in S_{2i}$,

(iii) $\mathbb{T}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(r_1s)(r_2s) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \times \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s)(r_2s) = \mathbb{T}_{Q2}(s) \wedge \mathbb{T}_{Q1i}(r_1r_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(r_1s)(r_2s) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \times \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s)(r_2s) = \mathbb{C}_{Q2}(s) \wedge \mathbb{C}_{Q1i}(r_1r_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(r_1s)(r_2s) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \times \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s)(r_2s) = \mathbb{U}_{Q2}(s) \vee \mathbb{U}_{Q1i}(r_1r_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(r_1s)(r_2s) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \times \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s)(r_2s) = \mathbb{F}_{Q2}(s) \vee \mathbb{F}_{Q1i}(r_1r_2)$
$\forall s \in S_2, r_1r_2 \in S_{1i}$.

Theorem 3.41 *The Cartesian product* $\mathfrak{G}_{n1} \times \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \times \mathfrak{Q}_2, \mathfrak{Q}_{11} \times \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \times \mathfrak{Q}_{22}, \dots, \mathfrak{Q}_{1n} \times \mathfrak{Q}_{2n})$ *of two QNGS of the GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is a QNGs of* $\mathfrak{G}_1 \times \mathfrak{G}_2$.

Proof. According to the definition of Cartesian product there are two cases:

Case1: when $r \in S_1, r_1r_2 \in S_{2i}$

$$\mathbb{T}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1s_2)$$
$$\leq \mathbb{T}_{Q1}(r) \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$$
$$= [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_2)]$$
$$= \mathbb{T}_{(Q_1\times Q_2)}(rs_1) \wedge \mathbb{T}_{(Q_1\times Q_2)}(rs_2).$$

$$\mathbb{C}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1s_2)$$
$$\leq \mathbb{C}_{Q1}(r) \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$$
$$= [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_2)]$$
$$= \mathbb{C}_{(Q_1\times Q_2)}(rs_1) \wedge \mathbb{C}_{(Q_1\times Q_2)}(rs_2).$$

$$\mathbb{U}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1s_2)$$
$$\leq \mathbb{U}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$$
$$= [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_2)]$$
$$= \mathbb{U}_{(Q_1\times Q_2)}(rs_1) \vee \mathbb{U}_{(Q_1\times Q_2)}(rs_2).$$

$$\mathbb{F}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1s_2)$$
$$\leq \mathbb{F}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$$
$$= [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_2)]$$
$$= \mathbb{F}_{(Q_1\times Q_2)}(rs_1) \vee \mathbb{F}_{(Q_1\times Q_2)}(rs_2).$$

for $rs_1, rs_2 \in S_1 \times S_2$.

Case 2: when $r \in S_2, s_1s_2 \in S_{1i}$

$$\mathbb{T}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1i}(r_1r_2)$$
$$\leq \mathbb{T}_{Q2}(r) \wedge [\mathbb{T}_{Q1}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$$
$$= [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_1)] \wedge [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_2)]$$
$$= \mathbb{T}_{(Q_1\times Q_2)}(s_1r) \wedge \mathbb{T}_{(Q_1\times Q_2)}(s_2r).$$

$$\mathbb{C}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1i}(r_1r_2)$$
$$\leq \mathbb{C}_{Q2}(r) \wedge [\mathbb{C}_{Q1}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$$
$$= [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_1)] \wedge [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_2)]$$
$$= \mathbb{C}_{(Q_1\times Q_2)}(s_1r) \wedge \mathbb{C}_{(Q_1\times Q_2)}(s_2r).$$

$$\mathbb{U}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{U}_{Q2}(r) \vee \mathbb{U}_{Q1i}(r_1r_2)$$
$$\leq \mathbb{U}_{Q2}(r) \vee [\mathbb{U}_{Q1}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$$
$$= [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_1)] \vee [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_2)]$$
$$= \mathbb{U}_{(Q_1\times Q_2)}(s_1r) \vee \mathbb{U}_{(Q_1\times Q_2)}(s_2r).$$

$$\mathbb{F}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{F}_{Q2}(r) \vee \mathbb{F}_{Q1i}(r_1r_2)$$
$$\leq \mathbb{F}_{Q2}(r) \vee [\mathbb{F}_{Q1}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$$
$$= [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_1)] \vee [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_2)]$$
$$= \mathbb{F}_{(Q_1\times Q_2)}(s_1r) \vee \mathbb{F}_{(Q_1\times Q_2)}(s_2r).$$

for $s_1r, s_2r \in S_1S_2$.

Hence Proved.

Definition 3.42 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots . \mathfrak{Q}_{1n)}$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots . \mathfrak{Q}_{2n)}$ *be QNGS. The cross product of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *denoted by*

$\mathfrak{G}_{n1} \star \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \star \mathfrak{Q}_2, \mathfrak{Q}_{11} \star \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \star \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \star \mathfrak{Q}_{2n})$,

is defined by the following:

(i) $\mathbb{T}_{\mathfrak{Q}1\star\mathfrak{Q}2}(rs) = (\mathbb{T}_{\mathfrak{Q}1} \star \mathbb{T}_{\mathfrak{Q}2})(rs) = \mathbb{T}_{\mathfrak{Q}1}(r) \wedge \mathbb{T}_{\mathfrak{Q}2}(s)$.
$\mathbb{C}_{\mathfrak{Q}1\star\mathfrak{Q}2}(rs) = (\mathbb{C}_{\mathfrak{Q}1} \star \mathbb{C}_{\mathfrak{Q}2})(rs) = \mathbb{C}_{\mathfrak{Q}1}(r) \wedge \mathbb{C}_{\mathfrak{Q}2}(s)$.
$\mathbb{U}_{\mathfrak{Q}1\star\mathfrak{Q}2}(rs) = (\mathbb{U}_{\mathfrak{Q}1} \star \mathbb{U}_{\mathfrak{Q}2})(rs) = \mathbb{U}_{\mathfrak{Q}1}(r) \vee \mathbb{U}_{\mathfrak{Q}2}(s)$.
$\mathbb{F}_{\mathfrak{Q}1\star\mathfrak{Q}2}(rs) = (\mathbb{F}_{\mathfrak{Q}1} \star \mathbb{F}_{\mathfrak{Q}2})(rs) = \mathbb{F}_{\mathfrak{Q}1}(r) \vee \mathbb{F}_{\mathfrak{Q}2}(s)$.
$\forall rs \in S_1 \star S_2$.

(ii) $\mathbb{T}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \star \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \star \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{C}_{Q1i}(r_1r_2) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \star \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{U}_{Q1i}(r_1r_2) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \star \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{F}_{Q1i}(r_1r_2) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\forall r_1r_2 \in S_{1i}, s_1s_2 \in S_{2i}$,

Theorem 3.43 *The cross product* $\mathfrak{G}_{n1} \star \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \star \mathfrak{Q}_2, \mathfrak{Q}_{11} \star \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \star \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \star \mathfrak{Q}_{2n})$ *of two QNGS of the GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is a QNGs of* $\mathfrak{G}_1 \star \mathfrak{G}_2$.

Proof. For all $r_1s_1, r_2s_2 \in S_1 \star S_2$

$\mathbb{T}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \star \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\leq [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$
$= [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r_2) \wedge \mathbb{T}_{Q2}(s_2)]$
$= \mathbb{T}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_1s_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_2s_2)$,

$\mathbb{C}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \star \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{C}_{Q1i}(r_1r_2) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\leq [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$
$= [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r_2) \wedge \mathbb{C}_{Q2}(s_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_1s_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_2s_2)$,

$\mathbb{U}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \star \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{U}_{Q1i}(r_1r_2) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\leq [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \vee [\mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$
$= [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r_2) \vee \mathbb{U}_{Q2}(s_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_1s_1) \vee \mathbb{U}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_2s_2)$,

$\mathbb{F}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \star \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{F}_{Q1i}(r_1r_2) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\leq [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$
$= [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r_2) \vee \mathbb{F}_{Q2}(s_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_1s_1) \vee \mathbb{F}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_2s_2)$,

for $i \in 1,2,\ldots,n$. This gives required result.

Definition 3.44 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots . \mathfrak{Q}_{1n)}$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots . \mathfrak{Q}_{2n)}$ *be QNGS. The lexicographic product of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *denoted by*

$\mathfrak{G}_{n1} \bullet \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \bullet \mathfrak{Q}_2, \mathfrak{Q}_{11} \bullet \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \bullet \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \bullet \mathfrak{Q}_{2n})$,
is defined by the following:

$(i)\quad \mathbb{T}_{\mathfrak{Q}1\bullet\mathfrak{Q}2}(rs) = (\mathbb{T}_{\mathfrak{Q}1} \bullet \mathbb{T}_{\mathfrak{Q}2})(rs) = \mathbb{T}_{\mathfrak{Q}1}(r) \wedge \mathbb{T}_{\mathfrak{Q}2}(s)$.
$\mathbb{C}_{\mathfrak{Q}1\bullet\mathfrak{Q}2}(rs) = (\mathbb{C}_{\mathfrak{Q}1} \bullet \mathbb{C}_{\mathfrak{Q}2})(rs) = \mathbb{C}_{\mathfrak{Q}1}(r) \wedge \mathbb{C}_{\mathfrak{Q}2}(s)$.
$\mathbb{U}_{\mathfrak{Q}1\bullet\mathfrak{Q}2}(rs) = (\mathbb{U}_{\mathfrak{Q}1} \bullet \mathbb{U}_{\mathfrak{Q}2})(rs) = \mathbb{U}_{\mathfrak{Q}1}(r) \vee \mathbb{U}_{\mathfrak{Q}2}(s)$.
$\mathbb{F}_{\mathfrak{Q}1\bullet\mathfrak{Q}2}(rs) = (\mathbb{F}_{\mathfrak{Q}1} \bullet \mathbb{F}_{\mathfrak{Q}2})(rs) = \mathbb{F}_{\mathfrak{Q}1}(r) \vee \mathbb{F}_{\mathfrak{Q}2}(s)$.
$\forall rs \in S_1 \bullet S_2$.

$(ii)\quad \mathbb{T}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \bullet \mathbb{T}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \bullet \mathbb{C}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \bullet \mathbb{U}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \bullet \mathbb{F}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\forall r \in S_1, s_1s_2 \in S_{2i}$,

$(iii)\quad \mathbb{T}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \bullet \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \bullet \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{C}_{Q1i}(r_1r_2) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \bullet \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{U}_{Q1i}(r_1r_2) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \bullet \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{F}_{Q1i}(r_1r_2) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\forall r_1r_2 \in S_{1i}, s_1s_2 \in S_{2i}$,

Theorem 3.45 *The lexicographic product* $\mathfrak{G}_{n1} \bullet \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \bullet \mathfrak{Q}_2, \mathfrak{Q}_{11} \bullet \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \star \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \bullet \mathfrak{Q}_{2n})$ *of two QNGS of the GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is a QNGs of* $\mathfrak{G}_1 \bullet \mathfrak{G}_2$.

Proof. According to the definition of lexicographic product there are two cases:
Case 1: when $r \in S_1, s_1s_2 \in S_{2i}$

$\mathbb{T}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\leq \mathbb{T}_{Q1}(r) \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$
$= [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_2)]$
$= \mathbb{T}_{(Q_1\bullet Q_2)}(rs_1) \wedge \mathbb{T}_{(Q_1\bullet Q_2)}(rs_2)$.

$\mathbb{C}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\leq \mathbb{C}_{Q1}(r) \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$
$= [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_2)]$
$= \mathbb{C}_{(Q_1\bullet Q_2)}(rs_1) \wedge \mathbb{C}_{(Q_1\bullet Q_2)}(rs_2)$.

$\mathbb{U}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\leq \mathbb{U}_{Q1}(r) \vee [\mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$
$= [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_2)]$
$= \mathbb{U}_{(Q_1\bullet Q_2)}(rs_1) \vee \mathbb{U}_{(Q_1\bullet Q_2)}(rs_2)$.

$\mathbb{F}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\leq \mathbb{F}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$
$= [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_2)]$
$= \mathbb{F}_{(Q_1\bullet Q_2)}(rs_1) \vee \mathbb{F}_{(Q_1\bullet Q_2)}(rs_2)$.
for $rs_1, rs_2 \in S_1 \bullet S_2$.

Case 2: For all $r_1s_1, r_2s_2 \in S_1 \bullet S_2$

$\mathbb{T}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \bullet \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\leq [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$

$= [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r_2) \wedge \mathbb{T}_{Q2}(s_2)]$
$= \mathbb{T}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_1s_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_2s_2),$

$\mathbb{C}_{(\mathfrak{Q}_{1i} \bullet \mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \bullet \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{C}_{Q1i}(r_1r_2) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\leq [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$
$= [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r_2) \wedge \mathbb{C}_{Q2}(s_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_1s_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_2s_2),$

$\mathbb{U}_{(\mathfrak{Q}_{1i} \bullet \mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \bullet \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{U}_{Q1i}(r_1r_2) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\leq [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \vee [\mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$
$= [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r_2) \vee \mathbb{U}_{Q2}(s_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_1s_1) \vee \mathbb{U}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_2s_2),$

$\mathbb{F}_{(\mathfrak{Q}_{1i} \bullet \mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \bullet \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{F}_{Q1i}(r_1r_2) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\leq [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$
$= [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r_2) \vee \mathbb{F}_{Q2}(s_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_1s_1) \vee \mathbb{F}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_2s_2),$

for $i \in 1,2,\ldots,n$. This gives required result.

Definition 3.46 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots. \mathfrak{Q}_{1n})$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots. \mathfrak{Q}_{2n})$ *be QNGS. The strong product of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *denoted by*

$\mathfrak{G}_{n1} \boxtimes \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2, \mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \boxtimes \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \boxtimes \mathfrak{Q}_{2n}),$

is defined by the following:

(i) $\mathbb{T}_{\mathfrak{Q}1 \boxtimes \mathfrak{Q}2}(rs) = (\mathbb{T}_{\mathfrak{Q}1} \boxtimes \mathbb{T}_{\mathfrak{Q}2})(rs) = \mathbb{T}_{\mathfrak{Q}1}(r) \wedge \mathbb{T}_{\mathfrak{Q}2}(s).$
$\mathbb{C}_{\mathfrak{Q}1 \boxtimes \mathfrak{Q}2}(rs) = (\mathbb{C}_{\mathfrak{Q}1} \boxtimes \mathbb{C}_{\mathfrak{Q}2})(rs) = \mathbb{C}_{\mathfrak{Q}1}(r) \wedge \mathbb{C}_{\mathfrak{Q}2}(s).$
$\mathbb{U}_{\mathfrak{Q}1 \boxtimes \mathfrak{Q}2}(rs) = (\mathbb{U}_{\mathfrak{Q}1} \boxtimes \mathbb{U}_{\mathfrak{Q}2})(rs) = \mathbb{U}_{\mathfrak{Q}1}(r) \vee \mathbb{U}_{\mathfrak{Q}2}(s).$
$\mathbb{F}_{\mathfrak{Q}1 \boxtimes \mathfrak{Q}2}(rs) = (\mathbb{F}_{\mathfrak{Q}1} \boxtimes \mathbb{F}_{\mathfrak{Q}2})(rs) = \mathbb{F}_{\mathfrak{Q}1}(r) \vee \mathbb{F}_{\mathfrak{Q}2}(s).$
$\forall rs \in S_1 \boxtimes S_2.$

(ii) $\mathbb{T}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{T}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{C}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{U}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{F}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\forall r \in S_1, s_1s_2 \in S_{2i},$

(iii) $\mathbb{T}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1s)(r_2s) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s)(r_2s) = \mathbb{T}_{Q2}(s) \wedge \mathbb{T}_{Q1i}(r_1r_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1s)(r_2s) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s)(r_2s) = \mathbb{C}_{Q2}(s) \wedge \mathbb{C}_{Q1i}(r_1r_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1s)(r_2s) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s)(r_2s) = \mathbb{U}_{Q2}(s) \vee \mathbb{U}_{Q1i}(r_1r_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1s)(r_2s) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s)(r_2s) = \mathbb{F}_{Q2}(s) \vee \mathbb{F}_{Q1i}(r_1r_2)$
$\forall s \in S_2, r_1r_2 \in S_{1i}.$

(iv) $\mathbb{T}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{C}_{Q1i}(r_1r_2) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{U}_{Q1i}(r_1r_2) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{F}_{Q1i}(r_1r_2) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\forall r_1r_2 \in S_{1i}, s_1s_2 \in S_{2i},$

Theorem 3.47 *The strong product* $\mathfrak{G}_{n1} \boxtimes \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2, \mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \boxtimes \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \boxtimes \mathfrak{Q}_{2n})$ *of two QNGS of the GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is a QNGs of* $\mathfrak{G}_1 \boxtimes \mathfrak{G}_2$.

Proof. According to the definition of strong product there are three cases:
Case1: when $r \in S_1, s_1s_2 \in S_{2i}$

$$\mathbb{T}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1s_2)$$
$$\leq \mathbb{T}_{Q1}(r) \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$$
$$= [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_2)]$$
$$= \mathbb{T}_{(Q_1\boxtimes Q_2)}(rs_1) \wedge \mathbb{T}_{(Q_1\boxtimes Q_2)}(rs_2).$$

$$\mathbb{C}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1s_2)$$
$$\leq \mathbb{C}_{Q1}(r) \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$$
$$= [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_2)]$$
$$= \mathbb{C}_{(Q_1\boxtimes Q_2)}(rs_1) \wedge \mathbb{C}_{(Q_1\boxtimes Q_2)}(rs_2).$$

$$\mathbb{U}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1s_2)$$
$$\leq \mathbb{U}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$$
$$= [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_2)]$$
$$= \mathbb{U}_{(Q_1\boxtimes Q_2)}(rs_1) \vee \mathbb{U}_{(Q_1\boxtimes Q_2)}(rs_2).$$

$$\mathbb{F}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1s_2)$$
$$\leq \mathbb{F}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$$
$$= [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_2)]$$
$$= \mathbb{F}_{(Q_1\boxtimes Q_2)}(rs_1) \vee \mathbb{F}_{(Q_1\boxtimes Q_2)}(rs_2).$$

for $rs_1, rs_2 \in S_1 \boxtimes S_2$.
Case 2: when $r \in S_2, s_1s_2 \in S_{1i}$

$$\mathbb{T}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1i}(s_1s_2)$$
$$\leq \mathbb{T}_{Q2}(r) \wedge [\mathbb{T}_{Q1}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$$
$$= [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_1)] \wedge [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_2)]$$
$$= \mathbb{T}_{(Q_1\boxtimes Q_2)}(s_1r) \wedge \mathbb{T}_{(Q_1\boxtimes Q_2)}(s_2r).$$

$$\mathbb{C}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1i}(s_1s_2)$$
$$\leq \mathbb{C}_{Q2}(r) \wedge [\mathbb{C}_{Q1}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$$
$$= [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_1)] \wedge [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_2)]$$
$$= \mathbb{C}_{(Q_1\boxtimes Q_2)}(s_1r) \wedge \mathbb{C}_{(Q_1\boxtimes Q_2)}(s_2r).$$

$$\mathbb{U}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{U}_{Q2}(r) \vee \mathbb{U}_{Q1i}(s_1s_2)$$
$$\leq \mathbb{U}_{Q2}(r) \vee [\mathbb{U}_{Q1}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$$
$$= [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_1)] \vee [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_2)]$$
$$= \mathbb{U}_{(Q_1\boxtimes Q_2)}(s_1r) \vee \mathbb{U}_{(Q_1\boxtimes Q_2)}(s_2r).$$

$$\mathbb{F}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{F}_{Q2}(r) \vee \mathbb{F}_{Q1i}(s_1s_2)$$
$$\leq \mathbb{F}_{Q2}(r) \vee [\mathbb{F}_{Q1}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$$
$$= [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_1)] \vee [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_2)]$$

$= \mathbb{F}_{(Q_1 \boxtimes Q_2)}(s_1 r) \vee \mathbb{F}_{(Q_1 \boxtimes Q_2)}(s_2 r).$

for $s_1 r, s_2 r \in S_1 \boxtimes S_2$.

Case 3: For all $r_1 r_2 \in S_{1i}, s_1 s_2 \in S_{2i}$

$$\mathbb{T}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{T}_{Q1i}(r_1 r_2) \wedge \mathbb{T}_{Q2i}(s_1 s_2)$$
$$\leq [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$$
$$= [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r_2) \wedge \mathbb{T}_{Q2}(s_2)]$$
$$= \mathbb{T}_{(\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2)}(r_1 s_1) \wedge \mathbb{T}_{(\mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_2)}(r_2 s_2),$$

$$\mathbb{C}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{C}_{Q1i}(r_1 r_2) \wedge \mathbb{C}_{Q2i}(s_1 s_2)$$
$$\leq [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$$
$$= [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r_2) \wedge \mathbb{C}_{Q2}(s_2)]$$
$$= \mathbb{C}_{(\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2)}(r_1 s_1) \wedge \mathbb{C}_{(\mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_2)}(r_2 s_2),$$

$$\mathbb{U}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{U}_{Q1i}(r_1 r_2) \vee \mathbb{U}_{Q2i}(s_1 s_2)$$
$$\leq [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \vee [\mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$$
$$= [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r_2) \vee \mathbb{U}_{Q2}(s_2)]$$
$$= \mathbb{U}_{(\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2)}(r_1 s_1) \vee \mathbb{U}_{(\mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_2)}(r_2 s_2),$$

$$\mathbb{F}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{F}_{Q1i}(r_1 r_2) \vee \mathbb{F}_{Q2i}(s_1 s_2)$$
$$\leq [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$$
$$= [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r_2) \vee \mathbb{F}_{Q2}(s_2)]$$
$$= \mathbb{F}_{(\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2)}(r_1 s_1) \vee \mathbb{F}_{(\mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_2)}(r_2 s_2),$$

for $i \in 1,2,\dots,n$. This gives required result.

Definition 3.48 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \dots. \mathfrak{Q}_{1n})$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \dots. \mathfrak{Q}_{2n})$ *be QNGS. The composition product of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *denoted by*

$\mathfrak{G}_{n1} \circ \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \circ \mathfrak{Q}_2, \mathfrak{Q}_{11} \circ \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \circ \mathfrak{Q}_{22}, \dots, \mathfrak{Q}_{1n} \circ \mathfrak{Q}_{2n})$,

is defined by the following:

(i) $\mathbb{T}_{\mathfrak{Q}1 \circ \mathfrak{Q}2}(rs) = (\mathbb{T}_{\mathfrak{Q}1} \circ \mathbb{T}_{\mathfrak{Q}2})(rs) = \mathbb{T}_{\mathfrak{Q}1}(r) \wedge \mathbb{T}_{\mathfrak{Q}2}(s).$
$\mathbb{C}_{\mathfrak{Q}1 \circ \mathfrak{Q}2}(rs) = (\mathbb{C}_{\mathfrak{Q}1} \circ \mathbb{C}_{\mathfrak{Q}2})(rs) = \mathbb{C}_{\mathfrak{Q}1}(r) \wedge \mathbb{C}_{\mathfrak{Q}2}(s).$
$\mathbb{U}_{\mathfrak{Q}1 \circ \mathfrak{Q}2}(rs) = (\mathbb{U}_{\mathfrak{Q}1} \circ \mathbb{U}_{\mathfrak{Q}2})(rs) = \mathbb{U}_{\mathfrak{Q}1}(r) \vee \mathbb{U}_{\mathfrak{Q}2}(s).$
$\mathbb{F}_{\mathfrak{Q}1 \circ \mathfrak{Q}2}(rs) = (\mathbb{F}_{\mathfrak{Q}1} \circ \mathbb{F}_{\mathfrak{Q}2})(rs) = \mathbb{F}_{\mathfrak{Q}1}(r) \vee \mathbb{F}_{\mathfrak{Q}2}(s).$
$\forall rs \in S_1 \circ S_2.$

(ii) $\mathbb{T}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \circ \mathbb{T}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1 s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \circ \mathbb{C}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1 s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \circ \mathbb{U}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1 s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \circ \mathbb{F}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1 s_2)$
$\forall r \in S_1, s_1 s_2 \in S_{2i},$

(iii) $\mathbb{T}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \circ \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{T}_{Q2}(s) \wedge \mathbb{T}_{Q1i}(r_1 r_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \circ \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{C}_{Q2}(s) \wedge \mathbb{C}_{Q1i}(r_1 r_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \circ \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{U}_{Q2}(s) \vee \mathbb{U}_{Q1i}(r_1 r_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \circ \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{F}_{Q2}(s) \vee \mathbb{F}_{Q1i}(r_1 r_2)$
$\forall s \in S_2, r_1 r_2 \in S_{1i}.$

$(iv)\quad \mathbb{T}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \circ \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \circ \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{C}_{Q1i}(r_1r_2) \wedge \mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \circ \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{U}_{Q1i}(r_1r_2) \vee \mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \circ \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{F}_{Q1i}(r_1r_2) \vee \mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)$
$\forall r_1r_2 \in S_{1i}, s_1s_2 \in S_{2i}$ such that $s_1 \neq s_2$.

Theorem 3.49 *The Composition product* $\mathfrak{G}_{n1} \circ \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \circ \mathfrak{Q}_2, \mathfrak{Q}_{11} \circ \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \circ \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \circ \mathfrak{Q}_{2n})$ *of two QNGS of the GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is a QNGs of* $\mathfrak{G}_1 \circ \mathfrak{G}_2$.

Proof. According to the definition of composition product there are three cases:
Case1: when $r \in S_1, s_1s_2 \in S_{2i}$

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1s_2)\\
&\le \mathbb{T}_{Q1}(r) \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]\\
&= [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_2)]\\
&= \mathbb{T}_{(Q_1\circ Q_2)}(rs_1) \wedge \mathbb{T}_{(Q_1\circ Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{C}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1s_2)\\
&\le \mathbb{C}_{Q1}(r) \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]\\
&= [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_2)]\\
&= \mathbb{C}_{(Q_1\circ Q_2)}(rs_1) \wedge \mathbb{C}_{(Q_1\circ Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{U}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1s_2)\\
&\le \mathbb{U}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]\\
&= [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_2)]\\
&= \mathbb{U}_{(Q_1\circ Q_2)}(rs_1) \vee \mathbb{U}_{(Q_1\circ Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{F}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1s_2)\\
&\le \mathbb{F}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]\\
&= [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_2)]\\
&= \mathbb{F}_{(Q_1\circ Q_2)}(rs_1) \vee \mathbb{F}_{(Q_1\circ Q_2)}(rs_2).
\end{aligned}$$

for $rs_1, rs_2 \in S_1 \circ S_2$.
Case 2: when $r \in S_2, s_1s_2 \in S_{1i}$

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{T}_{Q2}(s) \wedge \mathbb{T}_{Q1i}(s_1s_2)\\
&\le \mathbb{T}_{Q2}(r) \wedge [\mathbb{T}_{Q1}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]\\
&= [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_1)] \wedge [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_2)]\\
&= \mathbb{T}_{(Q_1\circ Q_2)}(s_1r) \wedge \mathbb{T}_{(Q_1\circ Q_2)}(s_2r).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{C}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{C}_{Q2}(s) \wedge \mathbb{C}_{Q1i}(s_1s_2)\\
&\le \mathbb{C}_{Q2}(r) \wedge [\mathbb{C}_{Q1}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]\\
&= [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_1)] \wedge [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_2)]\\
&= \mathbb{C}_{(Q_1\circ Q_2)}(s_1r) \wedge \mathbb{C}_{(Q_1\circ Q_2)}(s_2r).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{U}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{U}_{Q2}(s) \vee \mathbb{U}_{Q1i}(s_1s_2)\\
&\le \mathbb{U}_{Q2}(r) \vee [\mathbb{U}_{Q1}(s_1) \vee \mathbb{U}_{Q2}(s_2)]\\
&= [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_1)] \vee [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_2)]
\end{aligned}$$

$= \mathbb{U}_{(Q_1\circ Q_2)}(s_1r) \vee \mathbb{U}_{(Q_1\circ Q_2)}(s_2r).$

$\mathbb{F}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{F}_{Q2}(s) \vee \mathbb{F}_{Q1i}(s_1s_2)$
$\leq \mathbb{F}_{Q2}(r) \vee [\mathbb{F}_{Q1}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$
$= [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_1)] \vee [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_2)]$
$= \mathbb{F}_{(Q_1\circ Q_2)}(s_1r) \vee \mathbb{F}_{(Q_1\circ Q_2)}(s_2r).$

for $s_1r, s_2r \in S_1S_2$.

Case 3: For all $r_1r_2 \in S_{1i}, s_1s_2 \in S_2$ such that $s_1 \neq s_2$

$\mathbb{T}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \circ \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)$
$\leq [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \wedge \mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)$
$= [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r_2) \wedge \mathbb{T}_{Q2}(s_2)]$
$= \mathbb{T}_{(\mathfrak{Q}_1\circ\mathfrak{Q}_2)}(r_1s_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1\circ\mathfrak{Q}_2)}(r_2s_2)$

$\mathbb{C}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \circ \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{C}_{Q1i}(r_1r_2) \wedge \mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)$
$\leq [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \wedge \mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)$
$= [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r_2) \wedge \mathbb{C}_{Q2}(s_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1\circ\mathfrak{Q}_2)}(r_1s_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1\circ\mathfrak{Q}_2)}(r_2s_2)$

$\mathbb{U}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \circ \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{U}_{Q1i}(r_1r_2) \vee \mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)$
$\leq [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \vee \mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)$
$= [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r_2) \vee \mathbb{U}_{Q2}(s_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1\circ\mathfrak{Q}_2)}(r_1s_1) \vee \mathbb{U}_{(\mathfrak{Q}_1\circ\mathfrak{Q}_2)}(r_2s_2)$

$\mathbb{F}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \circ \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{F}_{Q1i}(r_1r_2) \vee \mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)$
$\leq [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \vee \mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)$
$= [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r_2) \vee \mathbb{F}_{Q2}(s_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1\circ\mathfrak{Q}_2)}(r_1s_1) \vee \mathbb{F}_{(\mathfrak{Q}_1\circ\mathfrak{Q}_2)}(r_2s_2)$

for $r_1s_1, r_2s_2 \in S_1 \circ S_2$. Hence proved.

Definition 3.50 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots. \mathfrak{Q}_{1n})$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots. \mathfrak{Q}_{2n})$ *be QNGS. The union of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *denoted by*

$$\mathfrak{G}_{n1} \cup \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \cup \mathfrak{Q}_2, \mathfrak{Q}_{11} \cup \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \cup \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \cup \mathfrak{Q}_{2n}),$$

is defined by the following:

(i) $\mathbb{T}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r) = (\mathbb{T}_{\mathfrak{Q}_1} \cup \mathbb{T}_{\mathfrak{Q}_2})(r) = \mathbb{T}_{Q1}(r) \vee \mathbb{T}_{Q2}(r)$
$\mathbb{U}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r) = (\mathbb{U}_{\mathfrak{Q}_1} \cup \mathbb{U}_{\mathfrak{Q}_2})(r) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(r)$
$\mathbb{C}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r) = (\mathbb{C}_{\mathfrak{Q}_1} \cup \mathbb{C}_{\mathfrak{Q}_2})(r) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(r)$
$\mathbb{F}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r) = (\mathbb{F}_{\mathfrak{Q}_1} \cup \mathbb{F}_{\mathfrak{Q}_2})(r) = \mathbb{F}_{Q1}(r) \wedge \mathbb{F}_{Q2}(r),$
$\forall r \in S_1 \cup S_2,$

(ii) $\mathbb{T}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \cup \mathbb{T}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{T}_{Q1i}(rs) \vee \mathbb{T}_{Q2i}(rs)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \cup \mathbb{C}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{C}_{Q1i}(rs) \vee \mathbb{C}_{Q2i}(rs)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \cup \mathbb{U}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{U}_{Q1i}(rs) \wedge \mathbb{U}_{Q2i}(rs)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \cup \mathbb{F}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{F}_{Q1i}(rs) \wedge \mathbb{F}_{Q2i}(rs),$

for all $(rs) \in S_{1i} \cup S_{2i}$.

Theorem 3.51 *The union* $\mathfrak{G}_{n1} \cup \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \cup \mathfrak{Q}_2, \mathfrak{Q}_{11} \cup \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \cup \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \cup \mathfrak{Q}_{2n})$ *of two QNGS of the*

GS $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is a QNGs of* $\mathfrak{G}_1 \cup \mathfrak{G}_2$.

Proof. Let $r_1r_2 \in S_{1i} \cup S_{2i}$. Here we consider two cases:

Case 1: when $r_1r_2 \in S_1$, then according to Definition 3.39, $\mathbb{T}_{\mathfrak{Q}_2}(r_1) = \mathbb{T}_{\mathfrak{Q}_2}(r_2) = \mathbb{T}_{\mathfrak{Q}_{2i}}(r_1r_2) = 0$

$$\mathbb{C}_{\mathfrak{Q}_2}(r_1) = \mathbb{C}_{\mathfrak{Q}_2}(r_2) = \mathbb{C}_{\mathfrak{Q}_{2i}}(r_1r_2) = 0$$

$$\mathbb{U}_{\mathfrak{Q}_2}(r_1) = \mathbb{U}_{\mathfrak{Q}_2}(r_2) = \mathbb{U}_{\mathfrak{Q}_{2i}}(r_1r_2) = 1$$

$\mathbb{F}_{\mathfrak{Q}_2}(r_1) = \mathbb{F}_{\mathfrak{Q}_2}(r_2) = \mathbb{F}_{\mathfrak{Q}_{2i}}(r_1r_2) = 1$, so

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{T}_{Q1i}(r_1r_2) \vee \mathbb{T}_{Q2i}(r_1r_2)\\
&= \mathbb{T}_{Q1i}(r_1r_2) \vee 0\\
&\leq [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \vee 0\\
&= [\mathbb{T}_{Q1}(r_1) \vee 0] \wedge [\mathbb{T}_{Q1}(r_2) \vee 0]\\
&= [\mathbb{T}_{Q1}(r_1) \vee \mathbb{T}_{Q2}(r_1)] \wedge [\mathbb{T}_{Q1}(r_2) \vee \mathbb{T}_{Q2}(r_2)]\\
&= \mathbb{T}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_2)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{C}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{C}_{Q1i}(r_1r_2) \vee \mathbb{C}_{Q2i}(r_1r_2)\\
&= \mathbb{C}_{Q1i}(r_1r_2) \vee 0\\
&\leq [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \vee 0\\
&= [\mathbb{C}_{Q1}(r_1) \vee 0] \wedge [\mathbb{C}_{Q1}(r_2) \vee 0]\\
&= [\mathbb{C}_{Q1}(r_1) \vee \mathbb{C}_{Q2}(r_1)] \wedge [\mathbb{C}_{Q1}(r_2) \vee \mathbb{C}_{Q2}(r_2)]\\
&= \mathbb{C}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_2)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{U}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{U}_{Q1i}(r_1r_2) \wedge \mathbb{U}_{Q2i}(r_1r_2)\\
&= \mathbb{U}_{Q1i}(r_1r_2) \vee 1\\
&\leq [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \wedge 1\\
&= [\mathbb{U}_{Q1}(r_1) \wedge 1] \vee [\mathbb{U}_{Q1}(r_2) \wedge 1]\\
&= [\mathbb{U}_{Q1}(r_1) \wedge \mathbb{U}_{Q2}(r_1)] \vee [\mathbb{U}_{Q1}(r_2) \wedge \mathbb{U}_{Q2}(r_2)]\\
&= \mathbb{U}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_1) \vee \mathbb{U}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_2)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{F}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{F}_{Q1i}(r_1r_2) \wedge \mathbb{F}_{Q2i}(r_1r_2)\\
&= \mathbb{F}_{Q1i}(r_1r_2) \vee 1\\
&\leq [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \wedge 1\\
&= [\mathbb{F}_{Q1}(r_1) \wedge 1] \vee [\mathbb{F}_{Q1}(r_2) \wedge 1]\\
&= [\mathbb{F}_{Q1}(r_1) \wedge \mathbb{F}_{Q2}(r_1)] \vee [\mathbb{F}_{Q1}(r_2) \wedge \mathbb{F}_{Q2}(r_2)]\\
&= \mathbb{F}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_1) \vee \mathbb{F}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_2)
\end{aligned}$$

For $r_1r_2 \in S_1 \cup S_2$.

Case 2: when $r_1r_2 \in S_2$, then according to Definition 3.39, $\mathbb{T}_{\mathfrak{Q}_1}(r_1) = \mathbb{T}_{\mathfrak{Q}_1}(r_2) = \mathbb{T}_{\mathfrak{Q}_{1i}}(r_1r_2) = 0$

$$\mathbb{C}_{\mathfrak{Q}_1}(r_1) = \mathbb{C}_{\mathfrak{Q}_1}(r_2) = \mathbb{C}_{\mathfrak{Q}_{1i}}(r_1r_2) = 0$$

$$\mathbb{U}_{\mathfrak{Q}_1}(r_1) = \mathbb{U}_{\mathfrak{Q}_1}(r_2) = \mathbb{U}_{\mathfrak{Q}_{1i}}(r_1r_2) = 1$$

$\mathbb{F}_{\mathfrak{Q}_1}(r_1) = \mathbb{F}_{\mathfrak{Q}_1}(r_2) = \mathbb{F}_{\mathfrak{Q}_{1i}}(r_1r_2) = 1$, so

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{T}_{Q1i}(r_1r_2) \vee \mathbb{T}_{Q2i}(r_1r_2)\\
&= \mathbb{T}_{Q2i}(r_1r_2) \vee 0\\
&\leq [\mathbb{T}_{Q2}(r_1) \wedge \mathbb{T}_{Q2}(r_2)] \vee 0\\
&= [\mathbb{T}_{Q2}(r_1) \vee 0] \wedge [\mathbb{T}_{Q2}(r_2) \vee 0]\\
&= [\mathbb{T}_{Q1}(r_1) \vee \mathbb{T}_{Q2}(r_1)] \wedge [\mathbb{T}_{Q1}(r_2) \vee \mathbb{T}_{Q2}(r_2)]\\
&= \mathbb{T}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_2)
\end{aligned}$$

$\mathbb{C}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{C}_{Q1i}(r_1r_2) \vee \mathbb{C}_{Q2i}(r_1r_2)$
$= \mathbb{C}_{Q2i}(r_1r_2) \vee 0$
$\leq [\mathbb{C}_{Q2}(r_1) \wedge \mathbb{C}_{Q2}(r_2)] \vee 0$
$= [\mathbb{C}_{Q2}(r_1) \vee 0] \wedge [\mathbb{C}_{Q2}(r_2) \vee 0]$
$= [\mathbb{C}_{Q1}(r_1) \vee \mathbb{C}_{Q2}(r_1)] \wedge [\mathbb{C}_{Q1}(r_2) \vee \mathbb{C}_{Q2}(r_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_2)$

$\mathbb{U}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{U}_{Q1i}(r_1r_2) \wedge \mathbb{U}_{Q2i}(r_1r_2)$
$= \mathbb{U}_{Q2i}(r_1r_2) \vee 1$
$\leq [\mathbb{U}_{Q2}(r_1) \vee \mathbb{U}_{Q2}(r_2)] \wedge 1$
$= [\mathbb{U}_{Q2}(r_1) \wedge 1] \vee [\mathbb{U}_{Q2}(r_2) \wedge 1]$
$= [\mathbb{U}_{Q1}(r_1) \wedge \mathbb{U}_{Q2}(r_1)] \vee [\mathbb{U}_{Q1}(r_2) \wedge \mathbb{U}_{Q2}(r_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_1) \vee \mathbb{U}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_2)$

$\mathbb{F}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{F}_{Q1i}(r_1r_2) \wedge \mathbb{F}_{Q2i}(r_1r_2)$
$= \mathbb{F}_{Q2i}(r_1r_2) \vee 1$
$\leq [\mathbb{F}_{Q2}(r_1) \vee \mathbb{F}_{Q2}(r_2)] \wedge 1$
$= [\mathbb{F}_{Q2}(r_1) \wedge 1] \vee [\mathbb{F}_{Q2}(r_2) \wedge 1]$
$= [\mathbb{F}_{Q1}(r_1) \wedge \mathbb{F}_{Q2}(r_1)] \vee [\mathbb{F}_{Q1}(r_2) \wedge \mathbb{F}_{Q2}(r_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_1) \vee \mathbb{F}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_2)$

For $r_1r_2 \in S_1 \cup S_2$. Hence Proved.

Theorem 3.52 *Let* $\mathfrak{G} = (S_1 \cup S_2, S_{11} \cup S_{21}, S_{12} \cup S_{22}, \ldots . S_{1n} \cup S_{2n})$ *to be union of two GSs* $\mathfrak{G}_1 = (S_1, S_{11}, S_{12}, \ldots, S_{1n})$ *and* $\mathfrak{G}_2 = (S_2, S_{21}, S_{22}, \ldots, S_{2n})$. *Then every QNGS* $\mathfrak{G} = (\mathfrak{Q}, \mathfrak{Q}_1, \mathfrak{Q}_2, \ldots . \mathfrak{Q}_n)$ *of* $\mathfrak{G}$ *is union of two QNGs* $\mathfrak{G}_{n1}$ *and* $\mathfrak{G}_{n2}$ *of GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$*, respectively,*

Proof. First we define $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_{1i}$ and $\mathfrak{Q}_{2i}$ for $i \in 1,2, \ldots, n$ as

$$\mathbb{T}_{Q1}(r) = \mathbb{T}_Q(r), \mathbb{C}_{Q1}(r) = \mathbb{C}_Q(r), \mathbb{U}_{Q1}(r) = \mathbb{U}_Q(r), \mathbb{F}_{Q1}(r) = \mathbb{F}_Q(r), \text{if} r \in S_1$$
$$\mathbb{T}_{Q2}(r) = \mathbb{T}_Q(r), \mathbb{C}_{Q1}(r) = \mathbb{C}_Q(r), \mathbb{U}_{Q1}(r) = \mathbb{U}_Q(r), \mathbb{F}_{Q1}(r) = \mathbb{F}_Q(r), \text{if} r \in S_2.$$

$$\mathbb{T}_{Q1i}(r_1r_2) = \mathbb{T}_{Qi}(r_1r_2), \mathbb{C}_{Q1i}(r_1r_2) = \mathbb{C}_{Qi}(r_1r_2), \mathbb{U}_{Q1i}(r_1r_2) = \mathbb{U}_{Qi}(r_1r_2), \mathbb{F}_{Q1i}(r_1r_2) = \mathbb{F}_{Qi}(r_1r_2),$$
if $r_1r_2 \in S_{1i}$,

$$\mathbb{T}_{Q2i}(r_1r_2) = \mathbb{T}_{Qi}(r_1r_2), \mathbb{C}_{Q2i}(r_1r_2) = \mathbb{C}_{Qi}(r_1r_2), \mathbb{U}_{Q2i}(r_1r_2) = \mathbb{U}_{Qi}(r_1r_2), \mathbb{F}_{Q2i}(r_1r_2) = \mathbb{F}_{Qi}(r_1r_2),$$
if $r_1r_2 \in S_{2i}$,

Then $\mathfrak{Q} = \mathfrak{Q}_1 \cup \mathfrak{Q}_2$ and $\mathfrak{Q}_i = \mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i}$, $i \in 1,2, \ldots, n$.

Now for $r_1r_2 \in S_{ki}, k = 1,2, i = 1,2, \ldots, n$

$\mathbb{T}_{Qki}(r_1r_2) = \mathbb{T}_{Q1i}(r_1r_2) \leq \mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2) = \mathbb{T}_{Qk}(r_1) \wedge \mathbb{T}_{Qk}(r_2)$
$\mathbb{C}_{Qki}(r_1r_2) = \mathbb{C}_{Q1i}(r_1r_2) \leq \mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2) = \mathbb{C}_{Qk}(r_1) \wedge \mathbb{C}_{Qk}(r_2)$
$\mathbb{U}_{Qki}(r_1r_2) = \mathbb{U}_{Q1i}(r_1r_2) \leq \mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2) = \mathbb{U}_{Qk}(r_1) \vee \mathbb{U}_{Qk}(r_2)$
$\mathbb{F}_{Qki}(r_1r_2) = \mathbb{F}_{Q1i}(r_1r_2) \leq \mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2) = \mathbb{F}_{Qk}(r_1) \vee \mathbb{F}_{Qk}(r_2)$.

i. e

$\mathfrak{G}_{nk} = (\mathfrak{Q}_k, \mathfrak{Q}_{k1}, \ldots . \mathfrak{Q}_{kn})$ isaQNGS of $\mathfrak{G}_k, k = 1,2$. Thus $\mathfrak{G}_{nk} = \mathfrak{Q}, \mathfrak{Q}_1, \mathfrak{Q}_3, \ldots ., \mathfrak{Q}_n$, a QNG of $\mathfrak{G} = \mathfrak{G}_1 \cup \mathfrak{G}_2$ is union of two QNGSs $\mathfrak{G}_{n1}$ and $\mathfrak{G}_{n2}$.

Definition 3.53 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots . \mathfrak{Q}_{1n})$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots . \mathfrak{Q}_{2n})$ *be QNGS and let* $S_1 \cap S_2 = \emptyset$. *The join of* $\mathfrak{G}_{n1}$ *and* $\mathfrak{G}_{n2}$*, denoted by*

$\mathfrak{G}_{n1} + \mathfrak{G}_{n2} = (\mathfrak{Q}_1 + \mathfrak{Q}_2, \mathfrak{Q}_{11} + \mathfrak{Q}_{21}, \mathfrak{Q}_{12} + \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} + \mathfrak{Q}_{2n})$,
is defined by following:

$(i)\quad \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r) = \mathbb{T}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r)$
$\mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r) = \mathbb{C}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r)$
$\mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r) = \mathbb{U}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r)$
$\mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r) = \mathbb{F}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r)$
$\forall \quad r \in S_1 \cup S_2$,

$(ii)\quad \mathbb{T}_{(\mathfrak{Q}_{11}+\mathfrak{Q}_{2i})}(rs) = \mathbb{T}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs)$
$\mathbb{C}_{(\mathfrak{Q}_{11}+\mathfrak{Q}_{2i})}(rs) = \mathbb{C}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs)$
$\mathbb{U}_{(\mathfrak{Q}_{11}+\mathfrak{Q}_{2i})}(rs) = \mathbb{U}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs)$
$\mathbb{F}_{(\mathfrak{Q}_{11}+\mathfrak{Q}_{2i})}(rs) = \mathbb{F}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs)$
$\forall \quad rs \in S_{1i} \cup S_{2i}$,

$(iii)\quad \mathbb{T}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(rs) = (\mathbb{T}_{\mathfrak{Q}_{1i}} + \mathbb{T}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(rs) = (\mathbb{C}_{\mathfrak{Q}_{1i}} + \mathbb{C}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(rs) = (\mathbb{U}_{\mathfrak{Q}_{1i}} + \mathbb{U}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(rs) = (\mathbb{F}_{\mathfrak{Q}_{1i}} + \mathbb{F}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s)$
$\forall \quad r \in S_1, s \in S_2$.

Theorem 3.54 *The join* $\mathfrak{G}_{n1} + \mathfrak{Q}_{n2} = (\mathfrak{Q}_1 + \mathfrak{Q}_2, \mathfrak{Q}_{11} + \mathfrak{Q}_{21}, \mathfrak{Q}_{12} + \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} + \mathfrak{Q}_{2n})$ *of two QNG of the GSS* $\mathfrak{G}$ *and* $\mathfrak{G}_2$ *is a QNG of* $\mathfrak{G}_1 + \mathfrak{G}_2$.

Proof. Let $r_1r_2 \in S_{1i} + S_{2i}$. Here we consider three cases:
Case 1: when $r_1r_2 \in S_1$, then according to definition 3.40

$$\mathbb{T}_{\mathfrak{Q}_2}(r_1) = \mathbb{T}_{\mathfrak{Q}_2}(r_2) = \mathbb{T}_{\mathfrak{Q}_{2i}}(r_1r_2) = 0$$
$$\mathbb{C}_{\mathfrak{Q}_2}(r_1) = \mathbb{C}_{\mathfrak{Q}_2}(r_2) = \mathbb{C}_{\mathfrak{Q}_{2i}}(r_1r_2) = 0$$
$$\mathbb{U}_{\mathfrak{Q}_2}(r_1) = \mathbb{U}_{\mathfrak{Q}_2}(r_2) = \mathbb{U}_{\mathfrak{Q}_{2i}}(r_1r_2) = 1$$

$\mathbb{F}_{\mathfrak{Q}_2}(r_1) = \mathbb{F}_{\mathfrak{Q}_2}(r_2) = \mathbb{F}_{\mathfrak{Q}_{2i}}(r_1r_2) = 1$, so

$\mathbb{T}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{T}_{Q1i}(r_1r_2) \vee \mathbb{T}_{Q2i}(r_1r_2)$
$= \mathbb{T}_{Q1i}(r_1r_2) \vee 0$
$\leq [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \vee 0$
$= [\mathbb{T}_{Q1}(r_1) \vee 0] \wedge [\mathbb{T}_{Q1}(r_2) \vee 0]$
$= [\mathbb{T}_{Q1}(r_1) \vee \mathbb{T}_{Q2}(r_1)] \wedge [\mathbb{T}_{Q1}(r_2) \vee \mathbb{T}_{Q2}(r_2)]$
$= \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{C}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{C}_{Q1i}(r_1r_2) \vee \mathbb{C}_{Q2i}(r_1r_2)$
$= \mathbb{C}_{Q1i}(r_1r_2) \vee 0$
$\leq [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \vee 0$
$= [\mathbb{C}_{Q1}(r_1) \vee 0] \wedge [\mathbb{C}_{Q1}(r_2) \vee 0]$
$= [\mathbb{C}_{Q1}(r_1) \vee \mathbb{C}_{Q2}(r_1)] \wedge [\mathbb{C}_{Q1}(r_2) \vee \mathbb{C}_{Q2}(r_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{U}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{U}_{Q1i}(r_1r_2) \wedge \mathbb{U}_{Q2i}(r_1r_2)$
$= \mathbb{U}_{Q1i}(r_1r_2) \vee 1$
$\leq [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \wedge 1$
$= [\mathbb{U}_{Q1}(r_1) \wedge 1] \vee [\mathbb{U}_{Q1}(r_2) \wedge 1]$

$= [\mathbb{U}_{Q1}(r_1) \wedge \mathbb{U}_{Q2}(r_1)] \vee [\mathbb{U}_{Q1}(r_2) \wedge \mathbb{U}_{Q2}(r_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{F}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{F}_{Q1i}(r_1r_2) \wedge \mathbb{F}_{Q2i}(r_1r_2)$
$= \mathbb{F}_{Q1i}(r_1r_2) \vee 1$
$\leq [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \wedge 1$
$= [\mathbb{F}_{Q1}(r_1) \wedge 1] \vee [\mathbb{F}_{Q1}(r_2) \wedge 1]$
$= [\mathbb{F}_{Q1}(r_1) \wedge \mathbb{F}_{Q2}(r_1)] \vee [\mathbb{F}_{Q1}(r_2) \wedge \mathbb{F}_{Q2}(r_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

For $r_1r_2 \in S_1 + S_2$.

Case 2: when $r_1r_2 \in S_2$, then according to Definition 3.40, $\mathbb{T}_{\mathfrak{Q}_1}(r_1) = \mathbb{T}_{\mathfrak{Q}_1}(r_2) = \mathbb{T}_{\mathfrak{Q}_{1i}}(r_1r_2) = 0$

$\mathbb{C}_{\mathfrak{Q}_1}(r_1) = \mathbb{C}_{\mathfrak{Q}_1}(r_2) = \mathbb{C}_{\mathfrak{Q}_{1i}}(r_1r_2) = 0$

$\mathbb{U}_{\mathfrak{Q}_1}(r_1) = \mathbb{U}_{\mathfrak{Q}_1}(r_2) = \mathbb{U}_{\mathfrak{Q}_{1i}}(r_1r_2) = 1$

$\mathbb{F}_{\mathfrak{Q}_1}(r_1) = \mathbb{F}_{\mathfrak{Q}_1}(r_2) = \mathbb{F}_{\mathfrak{Q}_{1i}}(r_1r_2) = 1$, so

$\mathbb{T}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{T}_{Q1i}(r_1r_2) \vee \mathbb{T}_{Q2i}(r_1r_2)$
$= \mathbb{T}_{Q2i}(r_1r_2) \vee 0$
$\leq [\mathbb{T}_{Q2}(r_1) \wedge \mathbb{T}_{Q2}(r_2)] \vee 0$
$= [\mathbb{T}_{Q2}(r_1) \vee 0] \wedge [\mathbb{T}_{Q2}(r_2) \vee 0]$
$= [\mathbb{T}_{Q1}(r_1) \vee \mathbb{T}_{Q2}(r_1)] \wedge [\mathbb{T}_{Q1}(r_2) \vee \mathbb{T}_{Q2}(r_2)]$
$= \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{C}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{C}_{Q1i}(r_1r_2) \vee \mathbb{C}_{Q2i}(r_1r_2)$
$= \mathbb{C}_{Q2i}(r_1r_2) \vee 0$
$\leq [\mathbb{C}_{Q2}(r_1) \wedge \mathbb{C}_{Q2}(r_2)] \vee 0$
$= [\mathbb{C}_{Q2}(r_1) \vee 0] \wedge [\mathbb{C}_{Q2}(r_2) \vee 0]$
$= [\mathbb{C}_{Q1}(r_1) \vee \mathbb{C}_{Q2}(r_1)] \wedge [\mathbb{C}_{Q1}(r_2) \vee \mathbb{C}_{Q2}(r_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{U}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{U}_{Q1i}(r_1r_2) \wedge \mathbb{U}_{Q2i}(r_1r_2)$
$= \mathbb{U}_{Q2i}(r_1r_2) \vee 1$
$\leq [\mathbb{U}_{Q2}(r_1) \vee \mathbb{U}_{Q2}(r_2)] \wedge 1$
$= [\mathbb{U}_{Q2}(r_1) \wedge 1] \vee [\mathbb{U}_{Q2}(r_2) \wedge 1]$
$= [\mathbb{U}_{Q1}(r_1) \wedge \mathbb{U}_{Q2}(r_1)] \vee [\mathbb{U}_{Q1}(r_2) \wedge \mathbb{U}_{Q2}(r_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{F}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{F}_{Q1i}(r_1r_2) \wedge \mathbb{F}_{Q2i}(r_1r_2)$
$= \mathbb{F}_{Q2i}(r_1r_2) \vee 1$
$\leq [\mathbb{F}_{Q2}(r_1) \vee \mathbb{F}_{Q2}(r_2)] \wedge 1$
$= [\mathbb{F}_{Q2}(r_1) \wedge 1] \vee [\mathbb{F}_{Q2}(r_2) \wedge 1]$
$= [\mathbb{F}_{Q1}(r_1) \wedge \mathbb{F}_{Q2}(r_1)] \vee [\mathbb{F}_{Q1}(r_2) \wedge \mathbb{F}_{Q2}(r_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

For $r_1r_2 \in S_1 + S_2$.

Csse3: $r_1 \in S_1, r_2 \in S_2$, then according to definition 3.42,

$\mathbb{T}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q2}(r_2)$
$= [\mathbb{T}_{Q1}(r_1) \vee 0] \wedge [\mathbb{T}_{Q2}(r_2) \vee 0]$
$= [\mathbb{T}_{Q1}(r_1) \vee \mathbb{T}_{Q2}(r_1)] \wedge [\mathbb{T}_{Q2}(r_2) \vee \mathbb{T}_{Q1}(r_2)]$

$= \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{C}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q2}(r_2)$
$= [\mathbb{C}_{Q1}(r_1) \vee 0] \wedge [\mathbb{C}_{Q2}(r_2) \vee 0]$
$= [\mathbb{C}_{Q1}(r_1) \vee \mathbb{C}_{Q2}(r_1)] \wedge [\mathbb{C}_{Q2}(r_2) \vee \mathbb{C}_{Q1}(r_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{U}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q2}(r_2)$
$= [\mathbb{U}_{Q1}(r_1) \wedge 0] \vee [\mathbb{U}_{Q2}(r_2) \wedge 0]$
$= [\mathbb{U}_{Q1}(r_1) \wedge \mathbb{U}_{Q2}(r_1)] \vee [\mathbb{U}_{Q2}(r_2) \wedge \mathbb{U}_{Q1}(r_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{F}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q2}(r_2)$
$= [\mathbb{F}_{Q1}(r_1) \wedge 0] \vee [\mathbb{F}_{Q2}(r_2) \wedge 0]$
$= [\mathbb{F}_{Q1}(r_1) \wedge \mathbb{F}_{Q2}(r_1)] \vee [\mathbb{F}_{Q2}(r_2) \wedge \mathbb{F}_{Q1}(r_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

For $r_1r_2 \in S_1 + S_2$. Hence proved.

Theorem 3.55 *Let* $\mathfrak{G} = (S_1 + S_2, S_{11} + S_{21}, S_{12} + S_{22}, \ldots. S_{1n} + S_{2n})$ *to be join of two GSs* $\mathfrak{G}_1 = (S_1, S_{11}, S_{12}, \ldots, S_{1n})$ *and* $\mathfrak{G}_2 = (S_2, S_{21}, S_{22}, \ldots, S_{2n})$*. Then every strong QNGS* $\mathfrak{G} = (\mathfrak{Q}, \mathfrak{Q}_1, \mathfrak{Q}_2, \ldots. \mathfrak{Q}_n)$ *of* $\mathfrak{G}$ *is join of two strong QNGs* $\mathfrak{G}_{n1}$ *and* $\mathfrak{G}_{n2}$ *of GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$*, respectively,*

Proof. First we define $\mathfrak{Q}_k$ and $\mathfrak{Q}_{ki}$ for $k = 1,2$ and $i = 1,2,\ldots,n$ as:

$\mathbb{T}_{\mathfrak{Q}_k}(r) = \mathbb{T}_Q(r), \mathbb{C}_{\mathfrak{Q}_k}(r) = \mathbb{C}_Q(r), \mathbb{U}_{\mathfrak{Q}_k}(r) = \mathbb{U}_Q(r), \mathbb{F}_{\mathfrak{Q}_k}(r) = \mathbb{F}_Q(r),$ if $r \in S_k$

$\mathbb{T}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{T}_{Q_i}(r_1r_2), \mathbb{C}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{C}_{Q_i}(r_1r_2), \mathbb{U}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{U}_{Q_i}(r_1r_2), \mathbb{F}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{F}_{Q_i}(r_1r_2),$ if $r_1r_2 \in S_{ki}$

Now for $r_1r_2 \in S_{ki}, k = 1,2, i = 1,2,\ldots,n$

$\mathbb{T}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{T}_{Q_i}(r_1r_2) = \mathbb{T}_Q(r_1) \wedge \mathbb{T}_Q(r_2) = \mathbb{T}_{\mathfrak{Q}_k}(r_1) \wedge \mathbb{T}_{\mathfrak{Q}_k}(r_2)$
$\mathbb{C}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{C}_{Q_i}(r_1r_2) = \mathbb{C}_Q(r_1) \wedge \mathbb{C}_Q(r_2) = \mathbb{C}_{\mathfrak{Q}_k}(r_1) \wedge \mathbb{C}_{\mathfrak{Q}_k}(r_2)$
$\mathbb{U}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{U}_{Q_i}(r_1r_2) = \mathbb{U}_Q(r_1) \vee \mathbb{U}_Q(r_2) = \mathbb{U}_{\mathfrak{Q}_k}(r_1) \vee \mathbb{U}_{\mathfrak{Q}_k}(r_2)$
$\mathbb{F}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{F}_{Q_i}(r_1r_2) = \mathbb{F}_Q(r_1) \vee \mathbb{F}_Q(r_2) = \mathbb{F}_{\mathfrak{Q}_k}(r_1) \vee \mathbb{F}_{\mathfrak{Q}_k}(r_2).$

(i.e) $\mathfrak{G}_{nk} = (\mathfrak{Q}_k, \mathfrak{Q}_{k1}, \mathfrak{Q}_{k2}, \ldots, \mathfrak{Q}_{kn})$ is a strong QNGS of $\mathfrak{G}_k, k = 1,2.$

Moreover, $\mathfrak{G}_n$ is join of $\mathfrak{G}_{n1}$ and $\mathfrak{G}_{n2}$ as shown: Using Definition 3.39 and 3.42, $Q = Q_1 \cup Q_2 = Q_1 + Q_2$ and $\mathfrak{Q}_i = \mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i} = \mathfrak{Q}_{1i} + \mathfrak{Q}_{2i}, \forall r_1r_2 \in S_{1i} \cap S_{2i}$.

when $r_1r_2 \in S_{1i} + S_{2i}$ $(S_{1i} \cup S_{2i})$, (i.e) $r_1 \in S_1$ and $r_2 \in S_2$

$\mathbb{T}_{\mathfrak{Q}_i}(r_1r_2) = \mathbb{T}_Q(r_1) \wedge \mathbb{T}_Q(r_2) = \mathbb{T}_{Qk}(r_1) \wedge \mathbb{T}_{Qk}(r_2) = \mathbb{T}_{\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i}}(r_1r_2)$
$\mathbb{C}_{\mathfrak{Q}_i}(r_1r_2) = \mathbb{C}_Q(r_1) \wedge \mathbb{C}_Q(r_2) = \mathbb{C}_{Qk}(r_1) \wedge \mathbb{C}_{Qk}(r_2) = \mathbb{C}_{\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i}}(r_1r_2)$
$\mathbb{U}_{\mathfrak{Q}_i}(r_1r_2) = \mathbb{U}_Q(r_1) \vee \mathbb{U}_Q(r_2) = \mathbb{U}_{Qk}(r_1) \vee \mathbb{U}_{Qk}(r_2) = \mathbb{U}_{\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i}}(r_1r_2)$
$\mathbb{F}_{\mathfrak{Q}_i}(r_1r_2) = \mathbb{F}_Q(r_1) \vee \mathbb{F}_Q(r_2) = \mathbb{F}_{Qk}(r_1) \vee \mathbb{F}_{Qk}(r_2) = \mathbb{F}_{\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i}}(r_1r_2)$

Calculation are similar when $r_1 \in S_2$, $r_2 \in S_1$. It is true when $i = 1,2,\ldots,n.$ Complete the proof.

4. Conclusions

In this work, the concept of quadripartitioned neutrosophic graph structure and its properties have been discussed. The strong, tree, $\phi-$permutation and $\phi-$complement of quadripartitioned neutrosophic graph structure have been studied. The operations like Cartesian Product, cross product, lexicographic product, composition in graph structures and join operations are established. In future, the authors will extend this proposed concept to some applications in decision making and bipolar environment. Wiener index of QNGSs will be studied based on [21, 22]. The proposed concepts are also extended to bipolar QNGSs, interval QNGSs, single valued neutrosophic quadripartitioned hypergraphs and in soft QNGSs.

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The Neutrosophic Axial Set theory

Mustafa Hasan Hadi[1,*] and L. A.A. Al-Swidi[2]

[1] Affiliation 1; Department of Mathematics, University of Babylon, Iraq; pure.mustafa.hassan@uobabylon.edu.iq
[2] Affiliation 2; Department of Mathematics, University of Babylon, Iraq; pure.leal.abd@uobabylon.edu.iq

* Correspondence: pure.mustafa.hassan@uobabylon.edu.iq

Abstract: We presented in this paper a new concept of sets that we launched it neutrosophic axial sets . These sets are considered as generalization of neutrosophic sets . The union relationships , intersection, union , belonging and other concepts were built on these sets , then we created two different concepts of points . Also we studied many important properties and basic theories about axial sets theory.

Keywords: neutrosophic sets; fuzzy sets; SƝĄ-points ; ƝĄ-sets ; union relationships.

1. Introduction

The neutrosophic sets [1] are the important and influential topic in human life in direct way. It's considered to be one of the applied and pure topics at the same time .

Also it contributes to quantum leaps in the field of electronics , software and other sciences as well as in various engineering branches . Where Salama, A., FlorentinSmarandache, and Valeri Kromov who were the researchers first to know these sets in [2,3]. Researchers and scientists have taken it upon them and solves to develop and work on it . On the other hand , these sets are considered to be a development to the second type of fuzzy sets , which the researcher Zadeh , L. A. know in 1965 [4] introduced. At the same time , the fuzzy sets are generalized into so-called the soft sets which were defined in [5- 7] and attributed to Molodtsov [8]. There are many researchers who worked in this field and remind them of [9- 12].

In 2019 Abdulsada, D.A., Al-Swidi, L.A.A. defined a new concept of sets and called it the center sets , for more information , you can review the papers [13- 15] , and the pillar of construction is proximity spaces by . A. Naimpally and . D. Warrack [16], where we combined the proximity space with the i-topological space by Al Talkany, A.Y.K.M., AL-Swidi, L.A.A [17] to produce the i-topological proximity space in 2020 [18, 19]. These ideas can be generalized on the topic of neutrosophic.

2. The Neutrosophic Axial Set theory

2.1. Definition Let X be any set , the set of the form $\text{ƝĄA} = \{ < A, A_1, A_2 > ;\ A \cap A_i = \emptyset , i = 1,2 \}$ is called neutrosophic axial set , where A be any subset of X , and the sets A_1, A_2 are called the parts of $< A, A_1, A_2 >$ For example , if we take $X = R$ the real numbers ,then $\text{ƝĄ}A = \{ < (1,2), A1, A2 > ;\ (1,2) \cap A_i = \emptyset , i = 1,2 \}$ where

$$A_i = \begin{cases} \emptyset \ \ or\ discret\ set\ in\ {}^{R}/_{(1,2)} \\ [2, x) & for\ x \geq 2 \\ (y, 1] & for\ y \leq 1 \end{cases}$$

2.2. Definition

I- Let X be any non-empty set , the neutrosophic point (ƝĄ-point) are of the forms $\text{ƝP}_A^{\emptyset} = < A, \emptyset, \emptyset >$, $\text{Ɲ}PA = < \emptyset, A, \emptyset >$ and

$\text{Ɲ}PA = < \emptyset, \emptyset, A >$, for any proper non-empty subset A of .

II- The singular neutrosophic point (SƝĄ-point) of any ƝĄ-set ƝĄA is denoted by $SA = < A, A_1, A_2 >$, where $A \cap A_i = \emptyset$ where $i = 1,2$, so $SA \in \text{ƝĄ}A$ (where $\in$ is the notion of classical belongs to) .

So we can claim the number of ƝĄ-points of any non-empty universal set X is $|X|.(|P(X)/\{\emptyset, X\}|$ where $|X|$ the number of elements of X.

Also from $|X|$ of above definition any ƝĄ-set is the classical union of its SƝĄ-points and any SƝĄ-point of any ƝĄ-set is neutrosophic set , but the converse is not true .

2.3. Definition

I- The empty ƝĄ-set with respect to the subset A of X is denoted by $\text{ƝĄ}_A^{\emptyset}$ is of the form $\text{ƝĄ}_A^{\emptyset} = \{ < \emptyset, A_1, A_2 > ;\ where\ A \cap A_i = \emptyset\ \ i = 1,2 \}$.

For example , if $X = \{ a, b, c \}$, then $\text{ƝĄ}_{\{a,b\}}^{\emptyset} = \{ < \emptyset, \emptyset, \emptyset > , < \emptyset, \{c\}, \{c\} > , < \emptyset, \{c\}, \emptyset > , < \emptyset, \emptyset, \{c\} > \}$.

II- The null ƝĄ-set with respect to a subset A of , which denoted by ƝƝĄA is of form $\text{ƝƝĄ}A = \{ < A, \emptyset, \emptyset > \}$.

2.4. Definition

I- The ƝĄ-sum between two SƝĄ-points SA and SB is denoted by the notion $\oplus_N$ which is defined by $S_A \oplus_N S_B = \{ < A \cup B, C_1 \cup D_1, C_2 \cup D_2 >$, where $C_i \cap A = \emptyset\ \ and\ D_i \cap B = \emptyset , i = 1,2 \}$. So from this definition we claim that every SƝĄ-point is ƝĄ-sum of two or more than two , but every ƝĄ-point is SƝĄ-point .

II- The SꞃAꞔ-point SA is called interlaced with respect to ꞃAꞔ-set ꞃAꞔB , if $SA \in$ ꞃAꞔB there exist $SB \in$ ꞃAꞔB such that $A_i \sqsubseteq B_i$, $S_A =< A, A_1, A_2 >$, $S_B =< B, B_1, B_2 >$ $i = 1, 2$ .If any ꞃAꞔ-point of the forms ꞃ$P_A^{\emptyset}$, ꞃPA and ꞃPA belong to ꞃAꞔ-set ꞃAꞔB , if A is a part of some SꞃAꞔ-point of ꞃAꞔB .So that we easily show that , $SB \in$ ꞃAꞔB iff $SB \in_{N}$ ꞃAꞔB .

III- The ꞃAꞔ-set ꞃAꞔA is said to be interlaced set with one of the ꞃAꞔ-set ꞃAꞔB which is denoted by ꞃAꞔ$A <_N$ ꞃAꞔB iff for each $S_A =< A, A_1, A_2 >$ $\in$ ꞃAꞔB there exist $SB =< B, B_1, B_2 > \in$ ꞃAꞔB with the condition $A_i \sqsubseteq B_i, i = 1, 2$. Clearly every ꞃAꞔ-set is interlaced set of ꞃAꞔ$_{\emptyset}$, also ꞃAꞔX is interlaced set of any ꞃAꞔ-set.

2.5. Note

If $A \sqsubset B$, then ꞃAꞔ$_B <_N$ ꞃAꞔ$_A$. Because , for every SꞃAꞔ-point$SB =< B, B_1, B_2 > \in$ꞃAꞔB that is $B \cap B_i = \emptyset$ and $A \sqsubset B$ imply that $A \cap B_i = \emptyset$ for $i = 1, 2$, thus $SB \in$ ꞃAꞔA , which satisfy the condition of interlaced set .Two ꞃAꞔ-sets ꞃAꞔ$_A$ and ꞃAꞔ$_B$are called intertwind sets which is denoted by ꞃAꞔ$_A \approx_N$ ꞃAꞔ$_B$ iff ꞃAꞔ$A <_N$ ꞃAꞔB and ꞃAꞔ$B <_N$ ꞃAꞔA .

2.6. Proposition

Let X be any sets and A, B are subsets of X. $A = B$ iff ꞃAꞔ$_A \approx_N$ ꞃAꞔ$_B$.

Proof .

Assume that $= B$, so by (Note 2.5) , we get that ꞃAꞔ$_A \approx_N$ ꞃAꞔ$_B$. Conversely , if possible that $A \neq B$.

Case 1 . If $A \cap B = \emptyset$, then each $S_A =< A, A_1, A_2 >$ $\in$ ꞃAꞔA and since each subset C of X with $C \cap B = \emptyset$ there is no SꞃAꞔ-points in ꞃAꞔ$_B$ which satisfy the condition of interlaced set , so ꞃAꞔA is not interlaced set of ꞃAꞔ$_B$. Similarly that ꞃAꞔ$_B$ is not interlaced set of ꞃAꞔA , which contradiction with ꞃAꞔ$_A \approx_N$ ꞃAꞔ$_B$.

Case 2. If $A \cap B \neq \emptyset$, that is there exist a point x in A and not in B or the point y in B but not in A , so $SB =< B, \{x\}, \{x\} > \in$ ꞃAꞔB imply that no SꞃAꞔ-points in ꞃAꞔA which satisfy the condition of interlaced set , hence ꞃAꞔB , similarly if we take $y \in B$ and y is not in A , which contradiction with ꞃAꞔ$_A \approx_N$ ꞃAꞔ$_B$. Therefore we get $= B$.

2.7.Definition

The ꞃAꞔ- complement of any ꞃAꞔ-set ꞃAꞔ$_A$ which is denoted by (ꞃAꞔ$_A$)c and of the form (ꞃAꞔ$_A$)$c =$ ꞃAꞔ$_{A^c}$. Now we give the notions of union , intersection of ꞃAꞔ - sets .

2. 8. Definition

The ƝĄ – union of two ƝĄ-sets $ƝĄ_A$ and $ƝĄ_B$, which is denoted by $ƝĄ_A \cup_N ƝĄ_B$ and is of the form $ƝĄ_A \cup_N ƝĄ_B = \{ < A \cup B, A_1 \cap B_1, A_2 \cap B_2 > ; \forall < A, A_1, A_2 > \in ƝĄ_A, < B, B_1, B_2 > \in ƝĄ_B \}$ .Also for the same away we defined that for any collection$\{ƝĄ_{A_i}; i \in II\}$ of ƝĄ - sets , the ƝĄ-union of this collection is of form $(\cup_{i\in II} A_i)_N = \{ < \cup_{i\in II} A_i, C_{j_1} \cap C_{j_2}, D_{j_1} \cap D_{j_2} >; \forall j_1, j_2 \in II\}$ where $S_{A_{j_1}} =< A_{j_1}, C_{j_1}, D_{j_1} >$and $S_{B_{j_2}} =< B_{j_2}, C_{j_2}, D_{j_2} >$.

The ƝĄ – intersection of two ƝĄ-sets $ƝĄ_A$ and $ƝĄ_B$, which is denoted by $ƝĄ_A \cap_N ƝĄ_B$ and is of the form $ƝĄ_A \cap_N ƝĄ_B = \{ < A \cap B, A_1 \cap B_1, A_2 \cap B_2 > ; \forall < A, A_1, A_2 > \in ƝĄ_A, < B, B_1, B_2 > \in ƝĄ_B \}$.
Also for the same away we defined that for any collection$\{ƝĄ_{A_i}; i \in II\}$ of ƝĄ – sets , the ƝĄ-intersection of this collection is of form $(\cap_{i\in II} A_i)_N = \{ < \cap_{i\in II} A_i, C_{i_1} \cap C_{i_2}, D_{i_1} \cap D_{i_2} >; \forall i_1, i_2 \in II\}$ where $S_{A_{i_1}} =< A_{i_1}, C_{i_1}, D_{i_1} >$and $S_{B_{i_2}} =< B_{i_2}, C_{i_2}, D_{i_2} >$.
The ƝĄ –participating of two ƝĄ-sets $ƝĄ_A$ and $ƝĄ_B$, which is denoted by $ƝĄ_A \Cap ƝĄ_B = ƝĄ_{A\cap B}$.

It is easy to show that the commutative and associative properties for the ƝĄ – union, ƝĄ – intersection and ƝĄ –participating are satisfied .

2.9. Proposition

For any set X and any subsets A, B , that is $ƝĄ_A \cup_N ƝĄ_B = ƝĄ_{A\cup B}$.

Proof .

For any SƝĄ-point $SD \in ƝĄ_A \cup_N ƝĄ_B$, which of the form $SD =< A \cap B, A_1 \cap B_1, A_2 \cap B_2 >$, with $S_A =< A, A_1, A_2 > \in ƝĄ_A$, $S_B =< B, B_1, B_2 > \in ƝĄ_B$. Thus $SD \in ƝĄ_{A\cup B}$, because that $(A \cup B) \cap (A_i \cap B_i) =(A \cap (A_i \cap B_i)) \cup(B \cap (A_i \cap B_i)) = \emptyset$.
Conversely , now let $S_{A\cup B} \in ƝĄ_{A\cup B}$, if possible that $S_{A\cup B} \notin ƝĄ_A \cup_N ƝĄ_B$, since $S_{A\cup B} =< A \cup B, D1, D2 >$ with $(A \cup B) \cap Di = \emptyset$, $for\ i = 1, 2$. But $\emptyset =(A \cup B) \cap Di = (A \cap Di) \cup (B \cap Di)$, imply that $A \cap Di = \emptyset$, $B \cap Di = \emptyset$, so we have $< A, D1, D2 > \in ƝĄ_A$ and $< B, D1, D2 > \in ƝĄ_B$, also $S_{A\cup B} =< A \cup B, D1, D2 > = < A \cup B, D1 \cap D1, D2 \cap D2 > \in ƝĄ_A \cup_N ƝĄ_B$, which is a contradiction .

From this proposition we can prove easily the following corollary.

2.10 .Corollary

1. $ƝĄ_A \cup_N ƝĄ_{A^c} = ƝĄX$.
2. $ƝĄ_A \cup_N ƝĄ_A = ƝĄ_A$.
3. $ƝĄ_A \cup_N ƝĄX = ƝĄX$.

2.11. Remark

For any set X and subset A of X we have $ꞐꜲ_A \cap_N ꞐꜲ_X = ꞐꞐꜲ_A = \{ ꞐP^{\emptyset}_A \}$, because $ꞐꜲ_A \cap_N ꞐꜲ_X = \{< A \cap X, \emptyset \cap A_1, \emptyset \cap A_2 >$; for each $< A, A_1, A_2 > \in ꞐꜲ_A\} = \{< A, \emptyset, \emptyset >\} = \{ ꞐP^{\emptyset}_A \} = ꞐꞐꜲ_A$.

2.12. Remark

Let X be any set with $ꞐꜲ_A$ and $ꞐꜲ_B$ are $ꞐꜲ$ − sets on X. If $ꞐꜲ_A <_N ꞐꜲ_B$, then $(ꞐꜲ_A \cap_N ꞐꜲ_B) <_N ꞐꜲ_A$ also $(ꞐꜲ_A \cap_N ꞐꜲ_B) <_N ꞐꜲ_B$ because for any $< A \cap B, C_1 \cap D_1, C_2 \cap D_2 > \in ꞐꜲ_A$ from the fact $< A, C1, C2 > \in ꞐꜲ_A$ and $A \cap C_i \cap D_i = \emptyset$, for $i = 1, 2$.from above (Remark 2.12.) and(Note 2.5.) we have the following proposition .

2. 13. proposition

1- $(ꞐꜲ_A \cap_N ꞐꜲ_B) <_N ꞐꜲ_{A\cap B}$.
2- $(ꞐꜲ_A \cap_N ꞐꜲ_B) \approx_N ꞐꜲ_A$.
3- If $ꞐꜲ_A <_N ꞐꜲ_B$ and $ꞐꜲ_B <_N ꞐꜲ_C$, then $ꞐꜲ_A <_N ꞐꜲ_C$.
4- If $\sqsubseteq B$, then $ꞐꜲ_A \cap_N ꞐꜲ_B \approx_N ꞐꜲ_B$.

Proof (4).

By (Note 2 . 5) and Remark (2 .12) we have $(ꞐꜲ_A \cap_N ꞐꜲ_B) <_N ꞐꜲ_B$. Now let us $< B, D_1, D_2 > \in ꞐꜲ_B$, so $B \cap D_i = \emptyset$, but $A \sqsubseteq B$, then $A \cap D_i = \emptyset$ for $i = 1,2$, then $< A, D_1, D_2 > = < A \cap B, D_1, D_2 > \in ꞐꜲ_A \cap_N ꞐꜲ_B$. So we get the result .

2. 14. Proposition

For any $ꞐꜲ$ − points $ꞐP_D, Ꞑp^D \in_N ꞐꜲ_A$ which satisfy that , there exist a part C of some $SꞐꜲ$ − point of $ꞐꜲ_B$ such that $D \sqsubseteq C$ iff $ꞐꜲ_A <_N ꞐꜲ_B$.

Proof.

Let $SA = < A, C1, C2 > \in ꞐꜲ_A$, then $ꞐꜲ$ − points $ꞐP_{C_1}$ and $ꞐP^{C_2}$ are in $ꞐꜲ_A$, so by assumption , there exist parts $D1$, $D2$ of $SꞐꜲ$ − points in $ꞐꜲ_B$ with $Ci \sqsubseteq Di$ and $B \cap D_i = \emptyset$, $i = 1, 2$, so $< B, D_1, D_2 > \in ꞐꜲ_B$, this imply that $ꞐꜲ_A <_N ꞐꜲ_B$.

Conversely , let $ꞐꜲ_A <_N ꞐꜲ_B$ and let $ꞐP_D \in_N ꞐꜲ_A$, then the $SꞐꜲ$ − point $< A, D, \emptyset >$, $< A, \emptyset, D >$ or $< A, D, D >$ are in $ꞐꜲ_A$, then there exist $< B, C1, \emptyset >$. Such that $D \sqsubseteq C1$ with $ꞐP_{C_1} \in ꞐꜲ_B$ or $< B, \emptyset, C2 >$ such that $D \sqsubseteq C2$ with $ꞐP^{C_2} \in ꞐꜲ_B$ or $< B, H1, H2 > \in ꞐꜲ_B$ with $D \sqsubseteq Hi$, so $ꞐP_{D_1}$ and $ꞐP^{D_2} \in ꞐꜲ_B$.

3. Conclusions

1. After an extensive study of these sets and spaces , we did this research establishing the basic structures for generalizing the neutrosophic sets , and under the name neutrosophicsets . Therefore , we can put the identification of the topological spaces on it , by taking a family of these ȠĄ – sets that achieve the following ; $ȠĄ_X$, $ȠĄ_\emptyset$ belong it , second is closed under the finite ȠĄ – intersection , finally is must be closed under ȠĄ – union for any subfamily of it .

2. Also we can study their properties and characteristics , as well as define the functions on there to give as a good suggestions to work . Then , we can modify the various open sets and further study can be continued with this concept. For example , we can modify in the papers [20- 28] .

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Sine Exponential Measure of Single Valued Neutrosophic Sets in Medical Diagnosis

R.Narmadhagnanam[1] and A. Edward Samuel [2]

[1] P.G. & Research Department of Mathematics,Government Arts College (Autonomous), Kumbakonam, Tamil Nadu, India.
E-mail: narmadhagnanam03@gmail.com

[2] P.G. & Research Department of Mathematics,Government Arts College (Autonomous), Kumbakonam, Tamil Nadu, India
E-mail: edwardsamuel.a@gacakum.ac.in

Abstract. The objective of the study is to find out the relationship between the disease and the symptoms seen with the patient and diagnose the disease that impacted the patient using single valued neutrosophic set. Innovative method [sine exponential measure] is devised in single valued neutrosophic set and some of its properties are discussed herein. Utilization of medical diagnosis was commenced with using prescribed procedures to identify a person suffering from the disease for a considerable period. The result showed that the proposed method was free from shortcomings that affect the existing methods and found to be more accurate in diagnosing the diseases. It was concluded that the technique adopted in this study were more reliable and easier to handle medical diagnosis problems.

Keywords: Sine Exponential; Neutrosophic; Medical Diagnosis; Single Valued Neutrosophic

1 Introduction

Kumbakonam is a thickly populated town. Although underground drainage system is available here, it is yet to cover all the houses in the town. So, open drainage system continues to be in practice in different places of the town. Further this town is racing fast towards total sanitation in all spheres. As a result, Kumbakonam continues to be a repository of all new kinds of diseases. This created an urge to carry out research in the medical field. By introducing innovative methods in the research, the diseases can be diagnosed instantly and infallibly.

Mathematical principles play a vital role in solving the real life problems in engineering, medical sciences, social sciences, economics and so on. These problems are having no definite data and they are mostly imprecise in character. We are therefore employing probability theory, fuzzy set theory, rough set theory *etc.*, in Mathematics to find solutions to these problems. In the same way, fuzzy logic techniques have been integrated with conventional clinical decision in healthcare industry. As clinicians find it hard to have a fool proof diagnosis, they are initiating certain steps without any guidance from the experts. Neutrosophic set which is a generalized set possesses all attributes necessary to encode medical knowledge base and capture medical inputs.

The law of average has been applied in Medical diagnosis combining the information of which most of them are quantifiable derived through various sources and the inconsistent data derived through intuitive thought and the whole process offers low intra and inter personal consistency. So contradictions, inconsistency, indeterminacy and fuzziness should be accepted as unavoidable as they are integrated in the behavior of biological systems as well as in their characterization. To model an expert doctor it is imperative that it should not disallow uncertainty as it would be then inapt to capture fuzzy or incomplete knowledge that might lead to the danger of fallacies due to misplaced precision. As medical diagnosis contains lots of uncertainties and increased volume of information available to physicians from new medical technologies, the process of classifying different sets of symptoms under a single name of disease becomes difficult.

In 1965, Fuzzy set theory was firstly given by Zadeh[1] which is applied in many real applications to handle uncertainty. Sometimes membership function itself is uncertain and hard to be defined by a crisp value. So the concept of interval valued fuzzy sets was proposed to capture the uncertainty of grade of membership. In 1986, Atanassov[2] introduced the intuitionistic fuzzy sets which consider both truth-membership and falsity-membership. Edward Samuel and Narmadhagnanam[3] proposed the tangent inverse distance and sine similarity measure of intuitionistic fuzzy sets and apply them in medical diagnosis.Kozae *et al* [4]applied intuitionistic fuzzy sets in corona covid-19 determination.Rajkalpana *et al* [5]applied intuitionistic fuzzy set and its operators in medical diagnosis.Shinoj and John [6] extended the concept of fuzzy multi sets by introducing intuitionistic fuzzy multi sets. Rajarajeswari and Uma [7,8] proposed few methods among intuitionistic fuzzy multi sets. Edward Samuel and Narmadhagnanam[9] proposed sine inverse distance of intuitionistic fuzzy multi sets and apply them in medical diagnosis Later on, intuitionistic fuzzy sets were extended to the interval valued intuitionistic fuzzy sets. Intuitionistic fuzzy sets and interval valued intuitionistic fuzzy sets can only handle incomplete information not the indeterminate information and inconsistent information which exists commonly in belief systems. So, Neutrosophic set (generalization of fuzzy sets, intuitionistic fuzzy sets and so on) defined by FlorentinSmarandache[10] has capability to deal with uncertainty, imprecise, incomplete and inconsistent information which exists in real world from philosophical point of view.In 1982, Pawlak[11] introduced the concept of rough set, as a formal tool for modeling and processing incomplete information in information systems. There are two basic elements in rough set theory, crisp set and equivalence relation, which constitute the mathematical basis of rough sets. The basic idea of rough set is based upon the approximation of sets by a pair of sets known as the lower approximation and the upper approximation of a set. Here, the lower and upper approximation operators are based on equivalence relation. Nanda and Majumdar [12] examined fuzzy rough sets. Broumi *et al* [13] introduced rough neutrosophic sets. SurapatiPramanik and KalyanMondal [14,15] introduced cosine and cotangent similarity measures of rough neutrosophic sets. Pramanik *et al* [16] introduced correlation coefficient of rough neutrosophic sets. Edward Samuel and Narmadhagnanam [17-20] proposed few methods among rough neutrosophic sets and applied it in medical diagnosis.Neutrosophic set is applied to different areas including decision making by many researchers[21-27]. Mohana and Mohanasundari[28] proposed similarity measures of single valued neutrosophic rough sets. Tuhin Bera and Nirmal KumarMahapatra[29] applied generalised single valued neutrosophic number in neutrosophic linear programming. Ulucay et al [30] proposed a new approach for multi-attribute decision-making problems in bipolar neutrosophic sets. Wang *et al*[31] proposed the single valued neutrosophic set.PinakiMajumdar and S.K.Samanta [32]proposed the similarity and entropy of neutrosophic sets. Jun Ye[33] proposed the cotangent similarity measure of single valued neutrosophic sets.

Broumi et al[34] proposed single valued (2N+1) sided polygonal neutrosophic numbers and single valued (2N) sided polygonal neutrosophic numbers. Li et al [35] Slope stability assessment method using the arctangent and tangent similarity measure of neutrosophic numbers. Edward Samuel and Narmadhagnanam [36,37] introduced cosine logarithmic distance and tangent inverse similarity measure among single valued neutrosophic sets and applied it in medical diagnosis.. Harish Garg and Nancy[38] proposed new distance measure of single valued neutrosophic sets. Chai *et al*[39]proposed new similarity measures of single valued neutrosophic sets. Shan Ye and Jun Ye [40] introduced the concept of single valued neutrosophic multi sets. Edward Samuel and Narmadhagnanam [41] introduced cosine exponential distance among single valued neutrosophic multi sets and applied it in medical diagnosis. In 2013[42], Florentin Smarandache extended the classical neutrosophic logic to n-valued refined neutrosophic logic, by refining each neutrosophic component T,I,F into respectively, $T_1,T_2,..,T_m,I_1,I_2,...,I_p$ and $F_1,F_2,..,F_r$. The concept of neutrosophic refined sets is a generalization of fuzzy multisets and intuitionistic fuzzy multi sets. In 2014,Broumi and Smarandache[43] extended the improved cosine similarity of single valued neutrosophic set proposed by Ye[44] to the case of neutrosophic refined sets. Edward Samuel and Narmadhagnanam [45-47] introduced few methods in neutrosophic refined sets and applied it in medical diagnosis. Broumi *et al* [48] generalise the concept of n-valued neutrosophic sets to the case of n-valued interval neutrosophic sets. Edward Samuel and Narmadhagnanam [49-51] introduced many methods in n-valued interval neutrosophic sets and applied it in medical diagnosis. The proposed method had more accuracy than the others and they could handle the limitations and drawbacks of the previous works well. This study discovers the relationship between the symptoms found within patients and set of diseases. This study will help the researcher to find out the diseases accurately that impacted the patients. The method employed is free from the limitations that are commonly found in other studies. Without such limitations, in this study a new theory on image processing, cluster analysis etc., has been developed.

Rest of the article is structured as follows. In Section 2, we briefly present the basic definitions. Section 3 deals with proposed definition and some of its properties. Sections 4, 5 and 6 deal with methodology,algorithm and case study related to medical diagnosis respectively. Conclusion is given in Section 7.

2 Preliminaries

2.1 Definition[52]

Let X be a Universe of discourse,with a generic element in X denoted by x, the neutrosophicset(NS) A is an object having the form

$A=\{\langle x:T_A(x),I_A(x),F_A(x)\rangle, x\in X\}$

where the functions define $T,I,F:X\to\,]^-0,1^+[$ respectively the degree of membership(or Truth), the degree of indeterminacy and the degree of non-membership(or Falsehood) ofthe element $x\in X$ to the set A with the condition

$^-0\le T_A(x)+I_A(x)+F_A(x)\le 3^+$

2.2 Definition[31]

Let X be a space of points (objects) with a generic element in X denoted by x. A single valued neutrosophic set A in X is characterized by truth membership function T_A, indeterminacy function I_A and falsity membership function F_A .For each point x in X,

$T_A(x),I_A(x),F_A(x)\in[0,1]$

When X is continuous, a SVNS A can be written as

$A=\int_x\langle T(x),I(x),F(x)\rangle/x, x\in X$

When X is discrete, a SVNS A can be written as

$A=\sum_{i=1}^{n}\langle T(x_i),I(x_i),F(x_i)\rangle/x_i, x_i\in X$

3 Proposed definition

3.1 Definition

Let $A=\sum_{i=1}^{n}\frac{x_i}{\langle T_A(x_i),I_A(x_i),F_A(x_i)\rangle}$ and $B=\sum_{i=1}^{n}\frac{x_i}{\langle T_B(x_i),I_B(x_i),F_B(x_i)\rangle}$ be two single valued neutrosophic sets in $X=\{x_1,x_2,...,x_n\}$, then the sine exponential measure is defined as

$$SEM_{SVNS}(A,B)=\frac{1}{n}\sum_{i=1}^{q}\left[\sum_{j=1}^{n}\sin e^{-\left[|T_A(x_i)-T_B(x_i)|+|I_A(x_i)-I_B(x_i)|+|F_A(x_i)-F_B(x_i)|\right]}\right] \quad (1)$$

Proposition 1

(i) $SEM_{SVNS}(A,B)>0$

(ii) $SEM_{SVNS}(A,B)=SEM_{SVNS}(B,A)$

(iii)If $A\subseteq B\subseteq C$ then $SEM_{SVNS}(A,C)\le SEM_{SVNS}(A,B)$ & $SEM_{SVNS}(A,C)\le SEM_{SVNS}(B,C)$

Proof

(i) We know that, the truth-membership function, indeterminacy –membership function and falsity–membership function in single valued neutrosophic sets are within $[0,1]$.

Hence $SEM_{SVNS}(A,B)>0$

(ii)We know that,

$$\left|T_A(x_i)-T_B(x_i)\right|=\left|T_B(x_i)-T_A(x_i)\right|$$
$$\left|I_A(x_i)-I_B(x_i)\right|=\left|I_B(x_i)-I_A(x_i)\right|$$
$$\left|F_A(x_i)-F_B(x_i)\right|=\left|F_B(x_i)-F_A(x_i)\right|$$

Hence $SEM_{SVNS}(A,B)=SEM_{SVNS}(B,A)$

(iii) We know that,

$$T_A(x_i)\le T_B(x_i)\le T_C(x_i)$$
$$I_A(x_i)\ge I_B(x_i)\ge I_C(x_i)$$
$$F_A(x_i)\ge F_B(x_i)\ge F_C(x_i)$$
$$\because A\subseteq B\subseteq C$$

Hence,

$$\left|T_A(x_i)-T_B(x_i)\right|\le\left|T_A(x_i)-T_C(x_i)\right|$$
$$\left|I_A(x_i)-I_B(x_i)\right|\le\left|I_A(x_i)-I_C(x_i)\right|$$
$$\left|F_A(x_i)-F_B(x_i)\right|\le\left|F_A(x_i)-F_C(x_i)\right|$$
$$\left|T_B(x_i)-T_C(x_i)\right|\le\left|T_A(x_i)-T_C(x_i)\right|$$
$$\left|I_B(x_i)-I_C(x_i)\right|\le\left|I_A(x_i)-I_C(x_i)\right|$$
$$\left|F_B(x_i)-F_C(x_i)\right|\le\left|F_A(x_i)-F_C(x_i)\right|$$

Here, the sine exponential measure is a decreasing function

$$\therefore\quad SEM_{SVNS}(A,C)\le SEM_{SVNS}(A,B)\ \&\ SEM_{SVNS}(A,C)\le SEM_{SVNS}(B,C)$$

4. Methodology

In this section, we present an application of single valued neutrosophic set in medical diagnosis. In a given pathology, Suppose S is a set of symptoms, D is a set of diseases and P is a set of patients and let Q be a single valued neutrosophic relation from the set of patients to the symptoms.i.e., $Q(P\to S)$ and R be a single valued neutrosophic relation from the set of symptoms to the diseases i.e., $R(S\to D)$ and then the methodology involves three main jobs:

1. Determination of symptoms.
2. Formulation of medical knowledge based on single valued neutrosophic sets.
3. Determination of diagnosis on the basis of new computation technique of single valued neutrosophic sets.

5. Algorithm

Step 1 : The symptoms of the patients are given to obtain the patient symptom relation Q and are noted in Table 1.

Step 2 : The medical knowledge relating the symptoms with the set of diseases under consideration are given to obtain the symptom - disease relation R and are noted in Table 2.

Step 3 : The Computation T (relation between patients and diseases) is found using (1) between Table 1 & Table 2 and are noted in Table 3

Step 4: Finally, we select the maximum value from Table 3 of each row for possibility of the patient affected with the respective disease and then we conclude that the patient P_k is suffering from the disease D_r.

6. Case study [53]

In this section, an example adapted from Gulfam Shahzadi,Muhammad Akram and Arsham Borumand Saied (An application of single valued neutrosophic sets in medical diagnosis) is used.

Let there be three patients P=(Ali,Hamza,Imran) and the set of symptoms S={Temperature,Insulin, Blood Pressure, Blood Platelets, Cough}.The Single valued neutrosophic relation $Q(P \rightarrow S)$ is given as in Table 1.Let the set of diseases D = {Diabetes, Dengue, Tuberculosis}.The Single valued neutrosophic relation$R(S \rightarrow D)$ is given as in Table 2.

Q	Temperature	Insulin	Blood Pressure	Blood Platelets	Cough
Ali	(0.8,0.1,0.1)	(0.2,0.2,0.6)	(0.4,0.2,0.4)	(0.8,0.1,0.1)	(0.3,0.3,0.4)
Hamza	(0.6,0.2,0.2)	(0.9,0.0,0.1)	(0.1,0.1,0.8)	(0.2,0.1,0.7)	(0.5,0.1,0.4)
Imran	(0.4,0.2,0.4)	(0.2,0.1,0.7)	(0.1,0.2,0.7)	(0.3,0.1,0.6)	(0.8,0.0,0.2)

Table 1:Patient-symptom relation(using step 1)

R	Diabetes	Dengue	Tuberculosis
Temperature	(0.2,0.0,0.8)	(0.9,0.0,0.1)	(0.6,0.2,0.2)
Insulin	(0.9,0.0,0.1)	(0.0,0.2,0.8)	(0.0,0.1,0.9)
Blood Pressure	(0.1,0.1,0.8)	(0.8,0.1,0.1)	(0.4,0.2,0.4)
Blood Platelets	(0.1,0.1,0.8)	(0.9,0.0,0.1)	(0.0,0.2,0.8)
Cough	(0.1,0.1,0.8)	(0.1,0.1,0.8)	(0.9,0.0,0.1)

Table 2: Symptom-Disease relation (Using step 2)

T	Diabetes	Dengue	Tuberculosis
Ali	0.3201	**0.5900**	0.4962
Hamza	**0.6287**	0.3217	0.4991
Imran	0.4547	0.3531	**0.6031**

Table 3: Sine exponential measure(Using step 3 and step 4)

7 Conclusion

Our propounded techniques are most decisive to hold the problems related to medical diagnosis competently. The proposed approaches can find more implementation in other areas such as decision making, cluster analysis etc.

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Solution of First Order Initial Value Problem using Analytical and Numerical Method in Neutrosophic Environment

Meghna Parikh[1], Manoj Sahni[2,*], Ritu Sahni[3]

[1]parikhmeghna2210@gmail.com, [2,*]manojsahani117@gmail.com, [3]ritusrivastava1981@gmail.com
[1,2,3]Department of Mathematics, School of Technology, Pandit Deendayal Energy University, Gandhinagar, Gujarat, India.

*Correspondence: manojsahani117@gmail.com

Abstract: In this paper, the solution of a first-order linear non-homogeneous fuzzy differential equation with an initial condition is described in a neutrosophic environment. For this purpose, using triangular neutrosophic numbers, the neutrosophic analytical method, and the fourth-order Runge-Kutta numerical method have been introduced for solving fuzzified first order differential equation. We also observed solutions at the (α,β,γ)-cut with varied time scales. In addition, the error between the analytical and numerical solution obtained on the (α,β,γ)-cut is evaluated and illustrated using tables with varying time. A good amount of agreement is seen using closed form and numerical solutions.

Keywords: Differential equation, Fuzzy set, Triangular Fuzzy numbers, Neutrosophic, Runge-Kutta 4th order.

1. Introduction

We are often faced with many ambiguous situations because of the limited, vague and uncertain knowledge available in our daily lives. It becomes impossible to depict and characterize any phenomenon in precise manner. In order to deal with these circumstances, Zadeh proposed fuzzy set theory in 1965 [1]. In numerous situations, we all employ intellectual terms such as "easy," "hard," "extremely easy," "very hard," and so on. These are ambiguous words, and the information derived from them differs in many ways. In fuzzy set theory, each intellectual word is assigned a membership grade, and all of these intellectual forms may then be simply fitted into the fuzzy environment. Basically, fuzzy set theory allows each element of a set "A" to have a specific degree of membership, represented by $\mu_A(x)$, which denotes that each element x of set A has a membership value lying in the closed interval [0, 1]. When we want to fit distinct intellectual words into a fuzzy set then we assign a numerical value to them between 0 and 1, and call them fuzzy numbers. Chang and Zadeh developed fuzzy numbers in 1972 [2], while Dubois and Prade studied generalization of fuzzy numbers in 1978 [3].
In practice, we normally consider the membership value, although this is insufficient. In such cases, the non-membership value must also be taken into account. But fuzzy sets are established solely for membership values, they do not take non-membership values into account. Atanassov presented the

intuitionistic fuzzy set (IFS) in 1986 [4-5], which is an extension of fuzzy sets that encompassed both situations. Because it contains information that belongs to the set as well as information that does not belong to the set, intuitionistic fuzzy sets are regarded as an extension of fuzzy sets.

In the real-life uncertainty, there is also the possibility of a different situation, known as indeterminacy. When the knowledge on which items belong to the set and do not belong to the set is insufficient, a neutral state condition known as indeterminacy arises. In order to comprehend this scenario in real life, Florentin Smarandache was the first to establish neutrosophic set theory which consider truth value, indeterminate value, and false value in 2006 [6]. In Neutrosophic set, grade of membership of Truth values (T), Indeterminate values (I) and False values (F) has been defined within the non-standard interval -]0,1[+. Non-standard intervals of the neutrosophic set theory works good in the concept of philosophy. In reality if we deal with engineering and science problems it is impossible to fit data in the non-standard interval. To solve such problems, Wang et al. created single-valued Neutrosophic sets by considering the unit interval [0,1] in its standard form in 2010 [7]. Furthermore, many researchers, including Aal SIA et al., Deli and Subas , and Chakraborty et al. have defined single-valued neutrosophic number [8-10]. Similarly, Ye defined Trapezoidal Neutrosophic numbers and its application in the field of decision-making [11]. Using this approach, lots of work is going on by considering several real-life issues (see for instance [12-20]). For example, Abdel-Basset et al. presented type-2 neutrosophic numbers for decision making problems, results on recent pandemic COVID-19, supply chain model, industrial and management problems. Similar applications and other generalization of the theory are discussed in the research articles, viz; [16-21]. Researchers must use certain methodologies, particularly differential equations, in order to initiate a discussion about modelling any phenomena and study its behaviour. In the modelling of any phenomenon, the data we receive is incomplete, imprecise, and uncertain., Kaleva proposed fuzzy differential equations to better grasp such ambiguity in real life in 1986 [22]. To develop the area of fuzzy differential equation some researchers extended the concept of calculus in fuzzy environment. Dubois & Prade, Goetschel & Voxman, Puri & Ralescu and others pioneered the fuzzy derivative and its extended theory [23-26]. They solved an initial value problem for a first order differential equation by employing the notion of fuzzy derivatives. Similarly, Buckley et al. proposed the solution of an nth order ordinary differential equation using fuzzy initial conditions [27-28]. The generalized Hukuhara differentiability for fuzzy-valued functions plays the most important role in the development of the fuzzy differential equation, which was presented by Bede and Seikkala [29-31]. There are many methods available in the literature to solve fuzzy differential equations, such as analytical, semi-analytical, and numerical methods, which have been used by various researchers, for example, Nieto et al. and Ghazanfari et al. who used Numerical method, namely Euler approximation and Runge-Kutta method of order 4 for solving first order linear fuzzy differential equations [32-33]. Lots of works have been done for the development of FDE (see for instance [34-39]). Because intuitionistic fuzzy set is a generalization of fuzzy set, fuzzy differential equation is likewise generalized to an intuitionistic environment. Researchers Ben et al. and many others have discussed analytical and numerical techniques for the solution of Intuitionistic fuzzy differential equation [40-41].

In this paper, we describe how to solve differential equations in a neutrosophic setting using calculus features of the neutrosophic set, which was discussed by Smarandache in 2015 [42]. He was first introduced neutrosophic derivative which is an extension of fuzzy derivative. Neutrosophic derivative has new type of the granular derivative (gr-derivative) which was introduced by Son et al. [43]. Also, he gave the gr-partial derivative of neutrosophic-valued several variable functions and investigated the *if and only if condition* for the existence of gr-derivative of neutrosophic-valued function. In the recent time, a lot of effort is done in the neutrosophic environment to describe many real-life occurrences using differential equations. For example, Sumanthi et al. has discussed the solution of neutrosophic differential equation using trapezoidal neutrosophic numbers, Parikh and Sahni discussed about the second order differential using Sumudu transform in neutrosophic environment, Moi discussed boundary value problem for second order differential equation in neutrosophic environment, and many other researchers discussed similar problems [44-47]. In this study, we addressed theory for the solution of first order differential equations using numerical approach, namely, Runge Kutta of 4th order in neutrosophic environment, which was inspired by these researches.

1.1 Motivation:

Our review of the literature revealed that there has been little research on Neutrosophic differential equations. Thus, there is a lot of scope for progress in this area. So, in order to proceed in this direction, we must first define the basic theory of first-order differential equations in a neutrosophic environment. As a result, the development of a technique for finding a solution to a differential equation, which has previously been done in a classical and fuzzy environment, has prompted us to consider similar forms of expansion in a neutrosophic environment.

1.2 Uniqueness of paper

This research article presents the theory of first order neutrosophic initial value problems in order to find a solution to a first order differential equation in a neutrosophic environment. The aims of this paper are as follows:

- To define fundamental preliminary concept in the neutrosophic environment.
- To offer an analytical approach for solving first-order differential equations using triangular neutrosophic numbers.
- To propose a numerical approach for solving first-order differential equations using triangular neutrosophic numbers.
- To solve the problem using both analytical and numerical methods, and to obtain the solution using neutrosophic triangular number.
- Interpret the solution obtained using analytical and numerical method and calculate the difference between them.

1.3 Structure of the paper

The structure of the paper is as follow: In Section 2, certain mathematical preliminaries are provided, which is relevant to our study. The development of a first order differential equation employing neutrosophic triangular numbers, lemma, and theorems is covered in Section 3, which includes both analytical and numerical theory. In Section 4, the neutrosophic initial-value problem is established and validated using a classical solution. Section 5 contains the results and discussions which is depicted graphically. Finally, a brief conclusion about this article has been given in Section 6.

2. Mathematical Preliminaries

Definition 2.1 Fuzzy set [3]: A membership function $\mu_A(x)$ is defined on an element x in the universal set X, for every $x \in X$, which can also be expressed as $\mu_A(x) \in [0,1]$. The fuzzy set A is defined as, $A = \{(x, \mu_A(x)) | \forall x \in X\}$.

Definition 2.2 α-level of Fuzzy set [3]: The α-level of set A is defined as $A_\alpha = \{\mu_A(x) \geq \alpha, \alpha \in [0,1]\}$, where x ϵ X. This set includes all the elements of X with membership values in A that are greater than or equal to α.

Definition 2.3 Intuitionistic fuzzy set (IFS) [2]: An Intuitionistic fuzzy set B over universal set of X is represented by $B = \{(x, \mu_B(x), v_B(x)) | \forall x \in X\}$, where value $\mu_B(x)$ represent membership value of x in B, and value $v_B(x)$ represent non-membership value of x in B.

Definition 2.4 α, β - level of Intuitionistic Fuzzy set [2]: For any Intuitionistic fuzzy set B with the α, β-level set which is defined as i.e., $B_{\alpha,\beta} = \{x: \mu_B(x) \geq \alpha, v_B(x) \leq \beta, \forall x \in X, \alpha, \beta \in [0,1]\}$ with $\alpha + \beta \leq 1$, where X is universal set.

Definition 2.5 Neutrosophic set (NS) [6]: A neutrosophic set defined as N = {$T_N(x)$, $I_N(x)$, $F_N(x)$: $|\forall x \in X$}, where $T_N(x)$, $I_N(x)$, $F_N(x)$ are from universal set X →] $^-0$, 1^+ [, which represents the truth membership grade ($T_N(x)$), indeterminacy membership grade ($I_N(x)$), and false membership grade ($F_N(x)$) of the element x ∈ X, with the condition $^-0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3^+$.

Definition 2.6 Single-Valued Neutrosophic Set (SVNS) [6]: Let N be any single-valued Neutrosophic Set which is defined as N= {$T_N(x)$, $I_N(x)$, $F_N(x)$: $\forall x \in X$}, where $T_N(x)$, $I_N(x)$, $F_N(x)$ are from universal set X → [0,1] represents the truth membership ($T_N(x)$), indeterminacy membership ($I_N(x)$), and false membership ($F_N(x)$) of the element x ∈ X, with the condition $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3$.

Definition 2.7 Neutrosophic Number [6]: A neutrosophic set N defined over the universal single valued set of real numbers R is said to be neutrosophic number if it has the following properties:

1) N is normal: if $\exists\, x_0 \in R$, such that $T_N(x_0) = 1$ ($I_N(x_0) = F_N(x_0) = 0$).
2) N is convex set for the truth function $T_N(x)$, i.e., $T_N(\mu x_1 + (1-\mu)x_2) \geq \min(T_N(x_1), T_N(x_2))$, $\forall x_1, x_2 \in$ R, μ ∈ [0,1].
3) N is concave set for the indeterminacy function ($I_N(x)$) and false function ($F_N(x)$), i.e., $I_N(\mu x_1 + (1-\mu)x_2) \geq \max(I_N(x_1), I_N(x_2))$, $\forall x_1, x_2 \in$ R, μ ∈ [0,1], $F_N(\mu x_1 + (1-\mu)x_2) \geq \max(F_N(x_1), F_N(x_2))$, $\forall x_1, x_2 \in$ R, μ ∈ [0,1].

Definition 2.8 (α, β, γ)-level of Neutrosophic set [46]: A neutrosophic set with (α, β, γ)-level of X is denoted by G (α,β,γ), where α,β,γ ∈ [0,1], and is defined as G(α,β,γ) ={$T_N(x)$, $I_N(x)$, $F_N(x)$: $\forall x \in X$, $T_N(x) \geq \alpha$, $I_N(x) \leq \beta$, $F_N(x) \leq \gamma$}, where $0 \leq \alpha + \beta + \gamma \leq 3$.

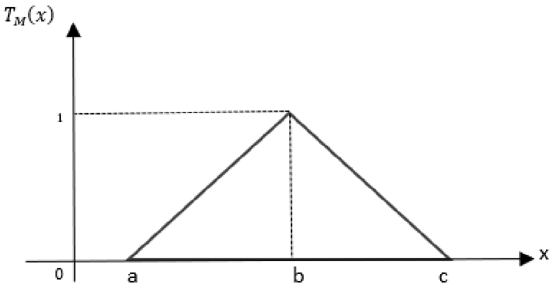
Definition 2.9 Triangular Neutrosophic Number [46]: Let N be a single valued neutrosophic set (SVNS) having Truth $T_N(x)$, Indeterminacy $I_N(x)$ and False $F_N(x)$ membership function over universal set X, then the triangular neutrosophic number is defined as

$$T_N(x) = \begin{cases} \left(\frac{x-a}{b-a}\right) & for\ a \le x < b \\ 1 & for\ x = b \\ \left(\frac{c-x}{c-b}\right) & for\ b < x \le c \\ 0 & otherwise \end{cases}$$

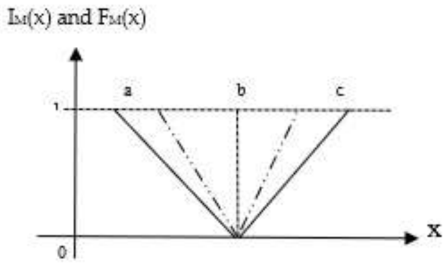
$$I_N(x) = \begin{cases} \left(\frac{b-x}{b-a}\right) & for\ a \le x < b \\ 0 & for\ x = b \\ \left(\frac{x-b}{c-b}\right) & for\ b < x \le c \\ 1 & otherwise \end{cases}$$

$$F_M(x) = \begin{cases} \left(\frac{b-x}{b-a}\right) & for\ a \le x < b \\ 0, & for\ x = b \\ \left(\frac{x-c}{c-b}\right) & for\ b < x \le c \\ 1 & otherwise \end{cases}$$

where $a \le b \le c$ and $a, b, c \in R$. Triangular neutrosophic number are denoted as $N_T\langle(a, b, c)\rangle$ where the truth membership function (T_N (x)) increases in a linear way for x ∈ $[a, b]$ and decrease in a linear form for x ∈ $[b, c]$ for I_N (x) and F_N (x) inverse behavior is seen from the truth membership for x ∈ $[a, b]$ and for x ∈ $[b, c]$ which is depicted in figure 1.



(a) T_M(x) as triangular form



(b) I_M(x) and F_M(x) as triangular form

Figure 1: Graph of Triangular Neutrosophic Number.

Definition 2.10 (α, β, γ)-cut of a Triangular Neutrosophic Number [46]: A Triangular neutrosophic set with (α, β, γ)-cut is denoted by $A_{TN(\alpha,\beta,\gamma)}$, where α, β, γ ∈ [0,1], and is defined as $A_{TN(\alpha,\beta,\gamma)}$ = {T_N(x), I_N(x), F_N(x): $T_N(x) \ge \alpha$, $I_N(x) \le \beta$, $F_N(x) \le \gamma$, x ∈ X }. Here $0 \le \alpha + \beta + \gamma \le 3$ and

$$A_{TN(\alpha,\beta,\gamma)} = \left[\left(a + \alpha(b-a)\right), \left(c - \alpha(c-b)\right)\right],$$
$$\left[\left(b - \beta(b-a)\right), \left(b + \beta(c-b)\right)\right],$$
$$\left[\left(b - \gamma(b-a)\right), \left(b + \gamma(c-b)\right)\right].$$

3. Ordinary Differential Equation of First Order with initial value in form of Triangular Neutrosophic numbers

Let us consider a linear non-homogeneous ordinary differential equation of first order,

$$\frac{dy(t)}{dt} = py(t) + \varepsilon\ , \quad y(t_0) = y_0 \qquad (1)$$

where y is a dependent variable, t is a independent variable, p and ε are constant and t_0 is the initial value of the parameter t.

Here, we consider initial value in the form of neutrosophic environment, so we have

$$y_T(t_0) = [a + \alpha(b-a), c - \alpha(c-b)]$$
$$y_I(t_0) = [b - \beta(b-a), b + \beta(c-b)]$$
$$y_F(t_0) = [b - \gamma(b-a), b + \gamma(c-b)]$$

where $y_T(t_0), y_I(t_0)$, and $y_F(t_0)$ represents truth, indeterminacy and false membership respectively.

Case 1: When the coefficient p in differential equation (1) is positive ($p > 0$), and taking (α, β, γ)-cut, the modified differential equation (1) in neutrosophic environment can be written as

$$\frac{d\left(\left[\underline{y_T}(t,\alpha),\overline{y_T}(t,\alpha)\right];\ \left[\underline{y_I}(t,\beta),\overline{y_I}(t,\beta)\right];\left[\underline{y_F}(t,\gamma),\overline{y_F}(t,\gamma)\right]\right)}{dt}$$
$$= p\left(\left[\underline{y_T}(t,\alpha),\overline{y_T}(t,\alpha)\right];\ \left[\underline{y_I}(t,\beta),\overline{y_I}(t,\beta)\right];\left[\underline{y_F}(t,\gamma),\overline{y_F}(t,\gamma)\right]\right) + [\varepsilon,\varepsilon];[\varepsilon,\varepsilon];[\varepsilon,\varepsilon]$$

with the initial condition

$$y(t_0,\ \alpha,\beta,\gamma) = \left(\left[\underline{y_T}(t_0,\alpha),\overline{y_T}(t_0,\alpha)\right];\ \left[\underline{y_I}(t_0,\beta),\overline{y_I}(t_0,\beta)\right];\left[\underline{y_F}(t_0,\gamma),\overline{y_F}(t_0,\gamma)\right]\right).$$

Solving equation (1) analytically in neutrosophic environment and using initial condition, we obtained the solution for T, I and F as

$$\underline{y_T}(t_0,\alpha) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + a + \alpha(b-a)\right)e^{p(t-t_0)} \tag{2}$$

$$\overline{y_T}(t_0,\alpha) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + c - \alpha(c-b)\right)e^{p(t-t_0)} \tag{3}$$

$$\underline{y_I}(t_0,\beta) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + b - \beta(b-a)\right)e^{p(t-t_0)} \tag{4}$$

$$\overline{y_I}(t_0,\beta) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + b + \beta(c-b)\right)e^{p(t-t_0)} \tag{5}$$

$$\underline{y_F}(t_0,\gamma) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + b - \gamma(b-a)\right)e^{p(t-t_0)} \tag{6}$$

$$\overline{y_F}(t_0,\gamma) = -\frac{\varepsilon}{p} + \left(\frac{\varepsilon}{p} + b + \gamma(c-b)\right)e^{p(t-t_0)} \tag{7}$$

where $\underline{y_T}(t_0,\alpha)$ and $\overline{y_T}(t_0,\alpha)$ represents solution in the form of lower and upper bound of truth value respectively. Similarly, $\underline{y_I}(t_0,\beta)$, $\overline{y_I}(t_0,\beta)$, $\underline{y_F}(t_0,\gamma)$ and $\overline{y_F}(t_0,\gamma)$ represents solution in the form of lower and upper bound of indeterminacy and false value respectively.

If $\varepsilon = 0$, then the solution of the differential equation reduced to,

$$\underline{y_T}(t_0,\alpha) = \big(a + \alpha(b-a)\big)e^{p(t-t_0)}$$
$$\overline{y_T}(t_0,\alpha) = \big(c - \alpha(c-b)\big)e^{p(t-t_0)}$$
$$\underline{y_I}(t_0,\beta) = \big(b - \beta(b-a)\big)e^{p(t-t_0)}$$
$$\overline{y_I}(t_0,\beta) = \big(b + \beta(c-b)\big)e^{p(t-t_0)}$$
$$\underline{y_F}(t_0,\gamma) = \big(b - \gamma(b-a)\big)e^{p(t-t_0)}$$
$$\overline{y_F}(t_0,\gamma) = \big(b + \gamma(c-b)\big)e^{p(t-t_0)}$$

Case: 2 When the coefficient p in differential equation (1) is negative (i.e $p = -m$ and $m > 0$), and taking (α, β, γ)-cut, the modified differential equation (1) in neutrosophic environment can be written as

$$\frac{d\left(\left[\underline{y_T}(t,\alpha),\overline{y_T}(t,\alpha)\right];\ \left[\underline{y_I}(t,\beta),\overline{y_I}(t,\beta)\right];\left[\underline{y_F}(t,\gamma),\overline{y_F}(t,\gamma)\right]\right)}{dt}$$

$$= p\left(\left[\underline{y_T}(t,\alpha),\overline{y_T}(t,\alpha)\right];\ \left[\underline{y_I}(t,\beta),\overline{y_I}(t,\beta)\right];\left[\underline{y_F}(t,\gamma),\overline{y_F}(t,\gamma)\right]\right) + [\varepsilon,\varepsilon];[\varepsilon,\varepsilon];[\varepsilon,\varepsilon]$$

with initial condition

$$y(t_0,\ \alpha,\beta,\gamma) = \left(\left[\underline{y_T}(t_0,\alpha),\overline{y_T}(t_0,\alpha)\right];\ \left[\underline{y_I}(t_0,\beta),\overline{y_I}(t_0,\beta)\right];\left[\underline{y_F}(t_0,\gamma),\overline{y_F}(t_0,\gamma)\right]\right)$$

where t_0 is the initial value of the parameter t.

Solving equation (1) analytically in neutrosophic environment and using initial condition, we obtained the solution for T, I and F as

$$\left[\underline{y_T}(t_0,\alpha),\overline{y}_T(t_0,\alpha)\right]$$

$$= \frac{1}{2}((a+\alpha(b-a)) - (c-\alpha(b-a))e^{m(t-t_0)} + \frac{1}{2}(((a+\alpha(b-a)) - (c-\alpha(b-a)) - \frac{2\varepsilon}{m})e^{-m(t-t_0)} + \frac{\varepsilon}{m} \quad (8)$$

$$\left[\underline{y_I}(t_0,\beta),\overline{y_I}(t_0,\beta)\right]$$

$$= \frac{1}{2}\left((b-\beta(b-a)) - (b+\beta(c-b))\right)e^{m(t-t_0)} + \frac{1}{2}\left((b-\beta(b-a)) + (b+\beta(c-b)) - \frac{2\varepsilon}{m}\right)e^{-m(t-t_0)} + \frac{\varepsilon}{m} \quad (9)$$

$$\left[\underline{y_F}(t_0,\gamma),\overline{y_F}(t_0,\gamma)\right] =$$

$$= \frac{1}{2}\left((b-\gamma(b-a)) - (b+\gamma(c-b))\right)e^{m(t-t_0)} + \frac{1}{2}\left((b-\gamma(b-a)) + (b+\gamma(c-b)) - \frac{2\varepsilon}{m}\right)e^{-m(t-t_0)} + \frac{\varepsilon}{m} \quad (10)$$

If $\varepsilon = 0$ in equation (1) then the solution of the differential equation from the above equation is given as,

$$\left[\underline{y_T}(t_0,\alpha),\overline{y_T}(t_0,\alpha)\right]$$

$$= \frac{1}{2}\left((a+\alpha(b-a)) - (c-\alpha(b-a))\right)e^{m(t-t_0)} + \frac{1}{2}\left((a+\alpha(b-a)) + (c-\alpha(b-a))\right)e^{-m(t-t_0)}$$

$\left[\underline{y_I}(t_0, \beta), \overline{y_I}(t_0, \beta)\right]$

$$= \frac{1}{2}\Big(\big(b - \beta(b - a)\big) - \big(b + \beta(c - b)\big)\Big)\, e^{m(t-t_0)}$$

$$+ \frac{1}{2}\Big(\big(b - \beta(b - a)\big) + \big(b + \beta(c - b)\big)\Big)\, e^{-m(t-t_0)}$$

$\left[\underline{y_F}(t_0, \gamma), \overline{y_F}(t_0, \gamma)\right]$

$$= \frac{1}{2}\Big(\big(b - \gamma(b - a)\big) - \big(b + \gamma(c - b)\big)\Big)\, e^{m(t-t_0)}$$

$$+ \frac{1}{2}\Big(\big(b - \gamma(b - a)\big) + \big(b + \gamma(c - b)\big)\Big)\, e^{-m(t-t_0)}$$

3.1 Solution of First Order Differential Equation with initial value in the form of Triangular Neutrosophic numbers using Runge-Kutta method of 4th order

Let us consider a linear non-homogeneous ordinary differential equation of first order

$$\frac{dy(t)}{dt} = py(t) + \varepsilon \ , \quad y(t_0) = y_0 \tag{11}$$

where y is dependent variable, t is independent variable, p and ε are constant and t_0 is the initial value of the parameter t.

Here, we consider initial value in form of neutrosophic environment and we have

$$y_T(t_0) = [a + \alpha(b - a), c - \alpha(c - b)]$$
$$y_I(t_0) = [b - \beta(b - a), b + \beta(c - b)]$$
$$y_F(t_0) = [b - \gamma(b - a), b + \gamma(c - b)]$$

where $y_T(t_0), y_I(t_0),$ and $y_F(t_0)$ represents truth, indeterminacy and false membership respectively. Fuzzy numerical solution of the given differential equation denoted as

$$y(t_n)_{T,I,F} = \left[\underline{y}(t_n)_T, \overline{y}(t_n)_T, \underline{y}(t_n)_I, \overline{y}(t_n)_I, \underline{y}(t_n)_F, \overline{y}(t_n)_F\right]$$

where $y(t_n)_T = \left[\underline{y}(t_n)_T, \overline{y}(t_n)_T\right], y(t_n)_I = \left[\underline{y}(t_n)_I, \overline{y}(t_n)_I\right], y(t_n)_F = \left[\underline{y}(t_n)_F, \overline{y}(t_n)_F\right]$

represents function of truth , indeterminacy and false membership respectively.

Solving equation (11) numerically in neutrosophic environment and using initial condition, we obtained the solution for T, I and F as

$$\underline{y}(t_{n+1})_T = \underline{y}(t_n)_T + \textstyle\sum_{j=1}^{4} P_j\, k_{j,1}(t_n, y(t_n)_T) \tag{12}$$
$$\overline{y}(t_{n+1})_T = \overline{y}(t_n)_T + \textstyle\sum_{j=1}^{4} P_j\, k_{j,2}(t_n, y(t_n)_T) \tag{13}$$
$$\underline{y}(t_{n+1})_I = \underline{y}(t_n)_I + \textstyle\sum_{j=1}^{4} P_j\, k_{j,1}(t_n, y(t_n)_I) \tag{14}$$
$$\overline{y}(t_{n+1})_I = \overline{y}(t_n)_I + \textstyle\sum_{j=1}^{4} P_j\, k_{j,2}(t_n, y(t_n)_I) \tag{15}$$
$$\underline{y}(t_{n+1})_F = \underline{y}(t_n)_F + \textstyle\sum_{j=1}^{4} P_j\, k_{j,1}(t_n, y(t_n)_F) \tag{16}$$
$$\overline{y}(t_{n+1})_F = \overline{y}(t_n)_F + \textstyle\sum_{j=1}^{4} P_j\, k_{j,2}(t_n, y(t_n)_F) \tag{17}$$

where the P_j's are constants. Then $k_{j,1}, k_{j,2}$ for j =1, 2, 3, 4 are defined as follow for truth, indeterminacy and false membership respectively:

First we obtained equation for the truth membership, which are denoted as $k_{j,1}(t_n, y(t_n)_T)$, $k_{j,2}(t_n, y(t_n)_T)$, where j =1, 2, 3, 4, and u is function which is defined as $\{u \in \left[\underline{y}'(t_n)_T, \overline{y}'(t_n)_T\right]\}$.

The coefficients $k_{1,1}(t_n, y(t_n)_T)$, $k_{4,2}(t_n, y(t_n)_T)$are defined as

$$k_{1,1}(t_n, y(t_n)_T) = \min h\{ y(t_n, u)/u\epsilon \; (p\underline{y}(t_n)_T + \varepsilon \,, p\overline{y}\,(t_n)_T + \varepsilon)\} \quad \text{(18-a)}$$

$$k_{1,2}(t_n, y(t_n)_T) = \max h\{ y(t_n, u)/u\epsilon \; (p\underline{y}(t_n)_T + \varepsilon \,, p\overline{y}\,(t_n)_T + \varepsilon)\} \quad \text{(18-b)}$$

$$k_{2,1}(t_n, y(t_n)_T) = \min h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \; ((q_{1,1}(t_n, y(t_n))), (q_{1,2}(t_n, y(t_n)))) \} \quad \text{(18-c)}$$

$$k_{2,2}(t_n, y(t_n)_T) = \max h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \; ((q_{1,1}(t_n, y(t_n))), (q_{1,2}(t_n, y(t_n))))\} \quad \text{(18-d)}$$

$$k_{3,1}(t_n, y(t_n)_T) = \min h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \; ((q_{2,1}(t_n, y(t_n))), (q_{2,2}(t_n, y(t_n))))\} \quad \text{(18-e)}$$

$$k_{3,2}(t_n, y(t_n)_T) = \max h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \; ((q_{2,1}(t_n, y(t_n))), (q_{2,2}(t_n, y(t_n))))\} \quad \text{(18-f)}$$

$$k_{4,1}(t_n, y(t_n)_T) = \min h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \; ((q_{3,1}(t_n, y(t_n))), (q_{3,2}(t_n, y(t_n))))\} \quad \text{(18-g)}$$

$$k_{4,2}(t_n, y(t_n)_T) = \max h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \; (((q_{3,1}(t_n, y(t_n))), (q_{3,2}(t_n, y(t_n))))\} \quad \text{(18-h)}$$

In above equations, we define $q_{j,1}(t_n, y(t_n)))$, $q_{j,2}(t_n, y(t_n)))$ for j =1, 2, 3 as follows,

$$q_{1,1}(t_n, y(t_n)) = \underline{y}(t_n)_T + \frac{h}{2} k_{1,1}(t_n, y(t_n)_T)$$

$$q_{1,2}(t_n, y(t_n)) = \overline{y}(t_n)_T + \frac{h}{2} k_{1,2}(t_n, y(t_n)_T)$$

$$q_{2,1}(t_n, y(t_n)) = \underline{y}(t_n)_T + \frac{h}{2} k_{2,1}(t_n, y(t_n)_T)$$

$$q_{2,2}(t_n, y(t_n)) = \overline{y}(t_n)_T + \frac{h}{2} k_{2,2}\,(t_n, y(t_n)_T)$$

$$q_{3,1}(t_n, y(t_n)) = \underline{y}(t_n)_T + \frac{h}{2} k_{3,1}(t_n, y(t_n)_T)$$

$$q_{3,2}(t_n, y(t_n)) = \overline{y}(t_n)_T + \frac{h}{2} k_{3,2}(t_n, y(t_n)_T)$$

Secondly, we obtained equation for the intederminancy membership which are denoted as $k_{j,1}(t_n, y(t_n)_I)$, $k_{j,2}(t_n, y(t_n)_I)$, where j = 1, 2, 3, 4, and u is function which is defined as $\{u \in \left[\underline{y}'(t_n)_I, \overline{y}'(t_n)_I\right]\}$. Thus,

$$k_{1,1}(t_n, y(t_n)_I) = \min h\{ y(t_n, u)/u\epsilon \; (p\underline{y}(t_n)_I + \varepsilon \,, p\overline{y}\,(t_n)_I + \varepsilon)\} \quad \text{(19-a)}$$

$$k_{1,2}(t_n, y(t_n)_I) = \max h\{ y(t_n, u)/u\epsilon \; (p\underline{y}(t_n)_I + \varepsilon \,, p\overline{y}\,(t_n)_I + \varepsilon)\} \quad \text{(19-b)}$$

$$k_{2,1}(t_n, y(t_n)_I) = \min h\{ y(t_n + \tfrac{h}{2}, u)/u\epsilon \; ((r_{1,1}(t_n, y(t_n))), (r_{1,2}(t_n, y(t_n))))\} \quad \text{(19-c)}$$

$$k_{2,2}(t_n, y(t_n)_I) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((r_{1,1}(t_n, y(t_n))), (r_{1,2}(t_n, y(t_n)))\} \quad (19\text{-d})$$

$$k_{3,1}(t_n, y(t_n)_I) = \min h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((r_{2,1}(t_n, y(t_n))), (r_{2,2}(t_n, y(t_n)))\} \quad (19\text{-e})$$

$$k_{3,2}(t_n, y(t_n)_I) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((r_{2,1}(t_n, y(t_n))), (r_{2,2}(t_n, y(t_n)))\} \quad (19\text{-f})$$

$$k_{4,1}(t_n, y(t_n)_I) = \min h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((r_{3,1}(t_n, y(t_n))), (r_{3,2}(t_n, y(t_n)))\} \quad (19\text{-g})$$

$$k_{4,2}(t_n, y(t_n)_I) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((r_{3,1}(t_n, y(t_n))), (r_{3,2}(t_n, y(t_n))) \quad (19\text{-h})$$

In above equations, we define $r_{j,1}(t_n, y(t_n))), r_{j,2}(t_n, y(t_n)))$ for j =1, 2, 3 as follows,

$$r_{1,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_I + \frac{h}{2} k_{1,1}(t_n, y(t_n)_I)$$

$$r_{1,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_I + \frac{h}{2} k_{1,1}(t_n, y(t_n)_I)$$

$$r_{1,2}\big(t_n, y(t_n)\big) = \overline{y}(t_n)_I + \frac{h}{2} k_{1,2}(t_n, y(t_n)_I)$$

$$r_{2,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_I + \frac{h}{2} k_{2,1}(t_n, y(t_n)_I)$$

$$r_{2,2}\big(t_n, y(t_n)\big) = \overline{y}(t_n)_I + \frac{h}{2} k_{2,2}\,(t_n, y(t_n)_I)$$

$$r_{3,1}\big(t_n, y(t_n)\big) = \underline{y}(t_n)_I + \frac{h}{2} k_{3,1}(t_n, y(t_n)_I)$$

$$r_{3,2}\big(t_n, y(t_n)\big) = \overline{y}(t_n)_I + \frac{h}{2} k_{3,2}(t_n, y(t_n)_I)$$

Lastly, we obtained equation for the false membership which are denoted as $k_{j,1}(t_n, y(t_n)_F)$, $k_{j,2}(t_n, y(t_n)_F)$, where j = 1, 2, 3, 4, and u is function which is defined as $\{u\ \epsilon\ \left[\underline{y'}(t_n)_F\ , \overline{y}\,'\ (t_n)_F\ \right]\}$.

Thus,

$$k_{1,1}(t_n, y(t_n)_F) = \min h\{\, y(t_n, u)/u\epsilon\ (\, p\underline{y}(t_n)_F + \varepsilon\ , p\overline{y}\ (t_n)_F + \varepsilon)\ \} \quad (20\text{-a})$$

$$k_{1,2}(t_n, y(t_n)_F) = \max h\{\, y(t_n, u)/u\epsilon\ \ (\, p\underline{y}(t_n)_F + \varepsilon\ , p\overline{y}\ (t_n)_F + \varepsilon)\ \} \quad (20\text{-b})$$

$$k_{2,1}(t_n, y(t_n)_F) = \min h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((s_{1,1}(t_n, y(t_n))), (s_{1,2}(t_n, y(t_n)))\} \quad (20\text{-c})$$

$$k_{2,2}(t_n, y(t_n)_F) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((s_{1,1}(t_n, y(t_n))), (s_{1,2}(t_n, y(t_n)))\} \quad (20\text{-d})$$

$$k_{3,1}(t_n, y(t_n)_F) = \min h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((s_{2,1}(t_n, y(t_n))), (s_{2,2}(t_n, y(t_n)))\} \quad (20\text{-e})$$

$$k_{3,2}(t_n, y(t_n)_F) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((s_{2,1}(t_n, y(t_n))), (s_{2,2}(t_n, y(t_n)))\} \quad (20\text{-f})$$

$$k_{4,1}(t_n, y(t_n)_F) = \min h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((s_{3,1}(t_n, y(t_n))), (s_{3,2}(t_n, y(t_n)))\ \} \quad (20\text{-g})$$

$$k_{4,2}(t_n, y(t_n)_F) = \max h\{\, y(t_n + \tfrac{h}{2}, u)/u\epsilon\ ((s_{3,1}(t_n, y(t_n))), (s_{3,2}(t_n, y(t_n)))\} \qquad (20\text{-h})$$

In the above equations, we define $s_{j,1}(t_n, y(t_n))), s_{j,2}(t_n, y(t_n)))$ for j =1, 2, 3 as follows,

$$s_{1,1}(t_n, y(t_n)) = \underline{y}(t_n)_F + \frac{h}{2}k_{1,1}(t_n, y(t_n)_F)$$

$$s_{1,1}(t_n, y(t_n)) = \underline{y}(t_n)_F + \frac{h}{2}k_{1,1}(t_n, y(t_n)_F)$$

$$s_{1,2}(t_n, y(t_n)) = \overline{y}(t_n)_F + \frac{h}{2}k_{1,2}(t_n, y(t_n)_F)$$

$$s_{2,1}(t_n, y(t_n)) = \underline{y}(t_n)_F + \frac{h}{2}k_{2,1}(t_n, y(t_n)_F)$$

$$s_{2,2}(t_n, y(t_n)) = \overline{y}(t_n)_F + \frac{h}{2}k_{2,2}\,(t_n, y(t_n)_F)$$

$$s_{3,1}(t_n, y(t_n)) = \underline{y}(t_n)_F + \frac{h}{2}k_{3,1}(t_n, y(t_n)_F)$$

$$s_{3,2}(t_n, y(t_n)) = \overline{y}(t_n)_F + \frac{h}{2}k_{3,2}(t_n, y(t_n)_F)$$

From the equations 18-(a to h), 19-(a to h) and 20-(a to h) we obtained the solution as follows,

$$\underline{y}(t_{n+1})_T = \underline{y}(t_n)_T + \tfrac{1}{6}\left[\, k_{1,2}(t_n, y(t_n)_T) + 2k_{2,2}(t_n, y(t_n)_T) + 2k_{3,2}(t_n, y(t_n)_T) + k_{4,2}(t_n, y(t_n)_T)\right] \quad (21)$$

$$\overline{y}(t_{n+1})_T = \overline{y}(t_n)_T + \tfrac{1}{6}\left[\, k_{1,2}(t_n, y(t_n)_T) + 2k_{2,2}(t_n, y(t_n)_T) + 2k_{3,2}(t_n, y(t_n)_T) + k_{4,2}(t_n, y(t_n)_T)\right] \quad (22)$$

$$\underline{y}(t_{n+1})_I = \underline{y}(t_n)_I + \tfrac{1}{6}\left[\, k_{1,1}(t_n, y(t_n)_I) + 2k_{2,1}(t_n, y(t_n)_I) + 2k_{3,1}(t_n, y(t_n)_I) + k_{4,1}(t_n, y(t_n)_I)\right] \quad (23)$$

$$\overline{y}(t_{n+1})_I = \overline{y}(t_n)_I + \tfrac{1}{6}\left[\, k_{1,2}(t_n, y(t_n)_I) + 2k_{2,2}(t_n, y(t_n)_I) + 2k_{3,2}(t_n, y(t_n)_I) + k_{4,2}(t_n, y(t_n)_I)\right] \quad (24)$$

$$\underline{y}(t_{n+1})_F = \underline{y}(t_n)_F + \tfrac{1}{6}\left[\, k_{1,1}(t_n, y(t_n)_F) + 2k_{2,1}(t_n, y(t_n)_F) + 2k_{3,1}(t_n, y(t_n)_F) + k_{4,1}(t_n, y(t_n)_F)\right] \quad (25)$$

$$\overline{y}(t_{n+1})_F = \overline{y}(t_n)_F + \tfrac{1}{6}\left[\, k_{1,2}(t_n, y(t_n)_F) + 2k_{2,2}(t_n, y(t_n)_F) + 2k_{3,2}(t_n, y(t_n)_F) + k_{4,2}(t_n, y(t_n)_F)\right] \quad (26)$$

where $y(t_{n+1})_T = [\underline{y}(t_{n+1})_T\,,\ \overline{y}(t_{n+1})_T]$ represent solution in the form of truth membership. Similarly, $y(t_{n+1})_I = [\underline{y}(t_{n+1})_I\,,\ \overline{y}(t_{n+1})_I]$, $y(t_{n+1})_F = [\underline{y}(t_{n+1})_F, \ \overline{y}(t_{n+1})_F$ represents solution for indeterminacy and false membership respectively.
The approximate solutions for $t_n, 0 \le t \le N$ are denoted by

$$y(t_n)_{T,I,F} = \left[\underline{y}(t_n)_T\,, \overline{y}\,(t_n)_T, \underline{y}(t_n)_I\,, \overline{y}\,(t_n)_I, \underline{y}(t_n)_F\,, \overline{y}\,(t_n)_F\right].$$

The solution is calculated using grid points $a = t_0 \le t_1 \le t_2 \ldots \le t_n = b$ and $h = \frac{b-a}{N} = t_{n+1} - t_n$

$$\underline{y}(t_{n+1})_T = \underline{y}(t_n)_T + \tfrac{1}{6}y[(t_n, y(t_n)_T)] \qquad (27)$$

$$\overline{y}(t_{n+1})_T = \overline{y}(t_n)_T + \frac{1}{6} y[(t_n, y(t_n)_T)] \quad (28)$$

$$\underline{y}(t_{n+1})_I = \underline{y}(t_n)_I + \frac{1}{6} y[(t_n, y(t_n)_I)] \quad (29)$$

$$\overline{y}(t_{n+1})_I = \overline{y}(t_n)_I + \frac{1}{6} y[(t_n, y(t_n)_I)] \quad (30)$$

$$\underline{y}(t_{n+1})_F = \underline{y}(t_n)_F + \frac{1}{6} y[(t_n, y(t_n)_F)] \quad (31)$$

$$\overline{y}(t_{n+1})_F = \overline{y}(t_n)_F + \frac{1}{6} y[(t_n, y(t_n)_F)] \quad (32)$$

where $y(t_n)_T = \left[\underline{y}(t_n)_T, \overline{y}(t_n)_T\right]$, $y(t_n)_I = \left[\underline{y}(t_n)_I, \overline{y}(t_n)_I\right]$, and $y(t_n)_F = \left[\underline{y}(t_n)_F, \overline{y}(t_n)_F\right]$ represents function of truth, indeterminacy, and false membership respectively.

4. Numerical Example:

In order to validate our development of theoretical approach we have performed numerical studies. In validation section, we summarize the results of these tests and compare the results with classical solution as well as fuzzy analytical solution and also discuss the error between them. So, for that we consider generalized Fuzzy initial value problem, which is $y'(t) = y(t)$, $y(0) = 1$ and we find the solution for y at t=1.

Solution: We apply classical method, analytical method and numerical method in an neutrosophic environment and then compare the solution as well as error between different method.

Method-1 Classical method

Given equation is $y'(t) = y(t), y(0) = 1$

Solving first order linear differntial eqution with initial condition we get following equation,

$$y(t) = e^t$$

For t=1, the solution of y(t) is 2.7183 upto four decimal places.

Method-2 Fuzzified Analytical method

Let us consider differential equation $y'(t) = y(t)$ with initial values for truth, indeterminancy,and false membership given in the form of tringular neutrosophic numbers,

$$y_T(0) = [\alpha, 2-\alpha], \quad y_I(0) = [1-0.5\beta, 1+0.5\beta], \quad y_F(0) = [1-0.25\gamma, 1+0.25\gamma]$$

Solving differential equation $y'(t) = y(t)$ with proposed fuzzified analytical theory (section 3 case 1 equations (2) to (7)), we get the following solution,

$$\underline{y}(t)_{T\alpha} = \alpha e^t \qquad \overline{y}(t)_{T\alpha} = (2-\alpha)e^t$$

$$\underline{y}(t)_{I\beta} = (1-0.5\beta)e^t \qquad \overline{y}(t)_{I\beta} = (1+0.5\beta)e^t$$

$$\underline{y}(t)_{F\gamma} = (1-0.25\gamma)e^t \qquad \overline{y}(t)_{F\gamma} = (1+0.25\gamma)e^t$$

Method-3 Fuzzy Numerical method

Let us consider differential equation $y'(t) = y(t)$ with initial values for truth, indeterminancy,and false membership which is in the form of tringular neutrosophic numbers as,

$$y(0)_T = [\alpha, 2-\alpha], y(0)_I = [1-0.5\beta, 1+0.5\beta], y(0)_F = [1-0.25\gamma, 1+0.25\gamma]$$

Solving differential equation $y'(t) = y(t)$ by proposed Runge kutta method of 4 th order (section 3.1 case -1 equations (27) to (32)), we get following solutions,

$$\underline{y}(t_1)_{T\alpha} = \alpha + \frac{1}{6}\left(\frac{41\alpha}{4}\right) \tag{33}$$

$$\overline{y}(t_1)_{T\alpha} = (2-\alpha) + \frac{1}{6}\left(\frac{82-41\alpha}{4}\right) \tag{34}$$

$$\underline{y}(t_1)_{I\beta} = (1-0.5\beta) + \frac{1}{6}(10.25 - 0.8541\beta) \tag{35}$$

$$\overline{y}(t_1)_{I\beta} = (1+0.5\beta) + \frac{1}{6}(10.25 + 0.8541\beta) \tag{36}$$

$$\underline{y}(t_1)_{F\gamma} = (1-0.25\gamma) + \frac{1}{6}(10.25 - 2.5625\gamma) \tag{37}$$

$$\overline{y}(t_1)_{F\gamma} = (1+0.25\gamma) + \frac{1}{6}(10.25 + 2.5625\gamma) \tag{38}$$

5. **Numerical observation**

Table :1 Solution of y(t) using RK 4th order at t=0.1 and h=0.1

$(\alpha,\beta,\gamma)-$ *cut*	Lower bound of Truth value at t=0.1 $\underline{y}(t_0)_{T\alpha}$	Upper bound of Truth value at t=0.1 $\overline{y}(t_0)_{T\alpha}$	Lower bound of Indeterminacy value at t=0.1 $\underline{y}(t_0)_{I\beta}$	Upper bound of Indeterminacy value at t=0.1 $\overline{y}(t_0)_{I\beta}$	Lower bound of Falsity value at t=0.1 $\underline{y}(t_0)_{F\gamma}$	Upper bound of Falsity value at t=0.1 $\overline{y}(t_0)_{F\gamma}$
0	0.0000000000	2.2103416667	1.1051708333	1.1051708333	1.1051708333	1.1051708333
0.2	0.2210341667	1.9893075000	0.9946537500	1.2156879167	1.0499122917	1.1604293750
0.4	0.4420683333	1.7682733333	0.8841366667	1.3262050000	0.9946537500	1.2156879167
0.6	0.6631025000	1.5472391667	0.7736195833	1.4367220833	0.9393952083	1.2709464583
0.8	0.8841366667	1.3262050000	0.6631025000	1.5472391667	0.8841366667	1.3262050000
1	1.1051708333	1.1051708333	0.5525854167	1.6577562500	0.8288781250	1.3814635417

Table :2 Solution of y(t) using RK 4th order at t=0.5 and h=0.1

$(\alpha,\beta,\gamma)-$ *cut*	Lower bound of Truth value at t=0.5 $\underline{y}(t_{0.5})_{T\alpha}$	Upper bound of Truth value at t=0.5 $\overline{y}(t_{0.5})_{T\alpha}$	Lower bound of Indeterminacy value at t=0.5 $\underline{y}(t_{0.5})_{I\beta}$	Upper bound of Indeterminacy value at t=0.5 $\overline{y}(t_{0.5})_{I\beta}$	Lower bound of Falsity value at t=0.5 $\underline{y}(t_{0.5})_{F\gamma}$	Upper bound of Falsity value at t=0.5 $\overline{y}(t_{0.5})_{F\gamma}$
0	0.0000000000	3.6442359242	1.8221179621	1.8221179621	1.8221179621	1.8221179621
0.2	0.3644235924	3.2798123318	1.6399061659	2.0043297583	1.7310120640	1.9132238602
0.4	0.5967296960	2.9153887393	1.4576943697	2.1865415545	1.6399061659	2.0043297583
0.6	1.0932707773	2.5509651469	1.2754825735	2.3687533507	1.5488002678	2.0954356564
0.8	1.1934593921	2.1865415545	1.0932707773	2.5509651469	1.4576943697	2.1865415545

1	1.8221179621	1.8221179621	0.9110589810	2.7331769431	1.3665884716	2.2776474526

Table :3 Solution of y(t) using RK 4th order at t=1 and h=0.1

$(\alpha,\beta,\gamma)-$ *cut*	Lower bound of Truth value at t=1 $\underline{y}(t_1)_{T\alpha}$	Upper bound of Truth value at t=1 $\overline{y}(t_1)_{T\alpha}$	Lower bound of Indeterminacy value at t=1 $\underline{y}(t_1)_{I\beta}$	Upper bound of Indeterminacy value at t=1 $\overline{y}(t_1)_{I\beta}$	Lower bound of Falsity value at t=1 $\underline{y}(t_1)_{F\gamma}$	Upper bound of Falsity value at t=1 $\overline{y}(t_1)_{F\gamma}$
0	0.0000000000	5.4365594883	2.7182797441	2.7182797441	2.7182797441	2.7182797441
0.2	0.4919202828	4.8929035394	2.4464517697	2.9901077185	2.5823657569	2.8541937313
0.4	0.8902158253	4.3492475906	2.1746237953	3.2619356930	2.4464517697	2.9901077185
0.6	1.6309678465	3.8055916418	1.9027958209	3.5337636674	2.3105377825	3.1260217058
0.8	1.7804316506	3.2619356930	1.6309678465	3.8055916418	2.1746237953	3.2619356930
1	2.7182797441	2.7182797441	1.3591398721	4.0774196162	2.0387098081	3.3978496802

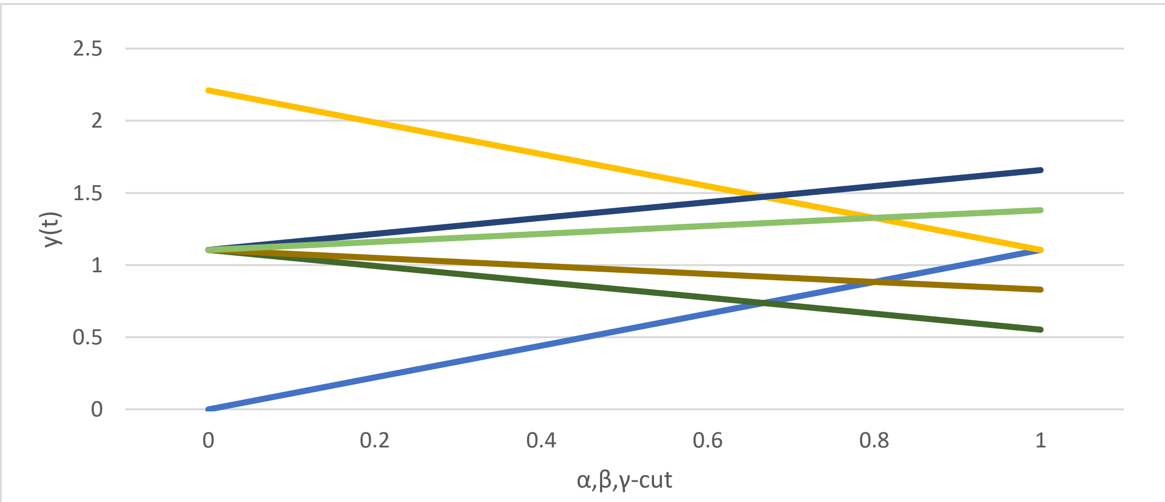


Figure 2: Solution of y(t) at t=0.1.

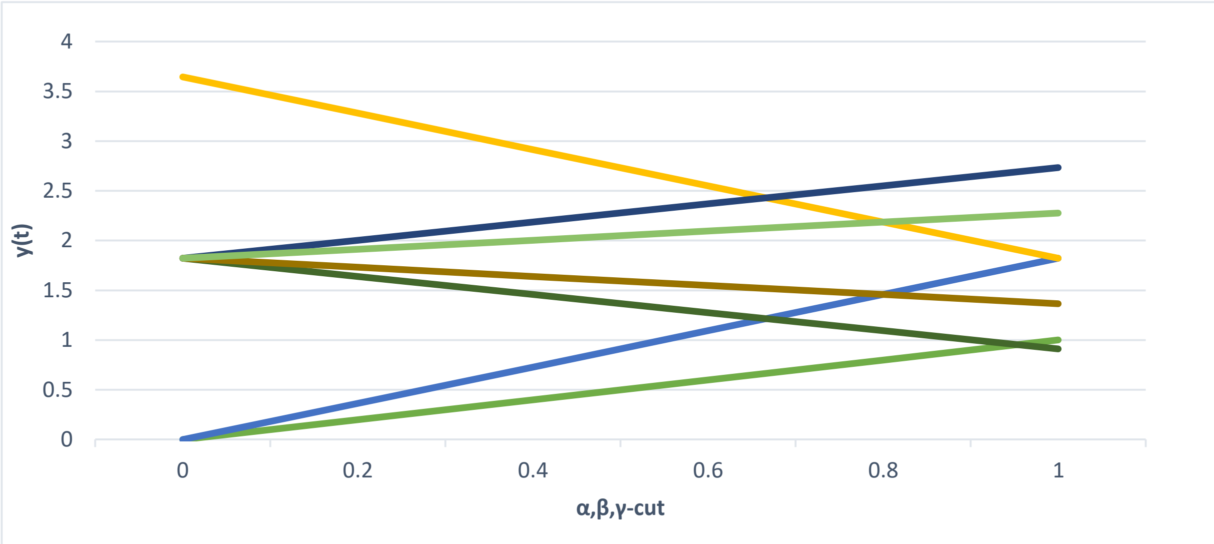


Figure 3: Solution of y(t) at t=0.5.

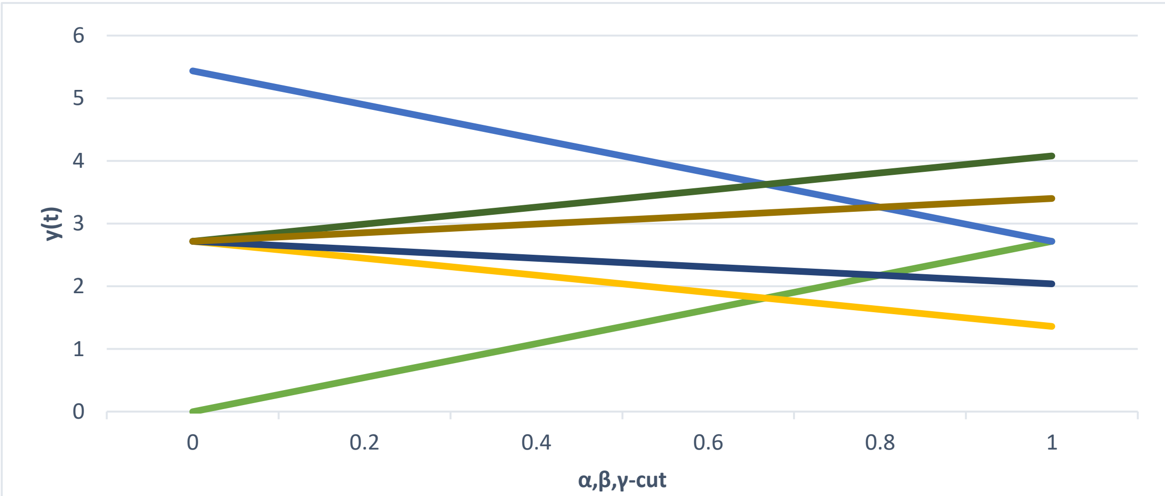


Figure 4: Solution of y(t) at t=1.

The results obtained from the calculation of equations (33) to (38) are shown in tables 1 to 3 respectively, for different (α, β,γ)-cut values with respect to the step size h= 0.2. It is clearly seen from the table 1 that the value of truth membership $y(t)_T = [\underline{y}(t_0)_{T\alpha}, \overline{y}(t_0)_{T\alpha}]$ for lower bound increases and the upper bound decreases. Similarly for indeterminacy, given by $y(t)_I = [\underline{y}(t_0)_{I\beta}, \overline{y}(t_0)_{I\beta}]$) and false membership ($y(t)_F = [\underline{y}(t_0)_{F\gamma}, \overline{y}(t_0)_{F\gamma}]$) for lower bound decreases and for the upper bound increases (depicted in the tables 2 and 3 respectively). In addition, from table 3, we observed that value of lower and upper bound of truth membership for (α, β,γ)-cut, when equal to 1 is 2.7182797441 and value of lower and upper bound for indeterminacy ($y(t)_I = [\underline{y}(t_0)_{I\beta}, \overline{y}(t_0)_{I\beta}]$) and false membership ($y(t)_F = [\underline{y}(t_0)_{F\gamma}, \overline{y}(t_0)_{F\gamma}]$) at (α, β,γ)-cut when equal to 0 are 2.7182797441, which match with exact solution. The graphs for various values for truth, indeterminacy and falsity with (α,β,γ)-cut are shown in figures 2, 3 and 4 respectively for different values of t (time). As the α-cut value increases and β,γ-cut values decrease solution approaches to the exact solution.

Table :4 Error Between RK 4th order and exact solution.

t(time)	Exact solution	Approximate solution by RK 4th order method where step size h=0.1	Error Between exact solution and solution find by RK 4th order
0.1	1.105170918	1.105170833	0.00000008467
0.2	1.221402758	1.221402571	0.00000018731
0.3	1.349858808	1.349858497	0.00000031052
0.4	1.491824698	1.491824240	0.00000045756
0.5	1.648721271	1.648720639	0.00000063210
0.6	1.822118800	1.822117962	0.00000083830
0.7	2.013752707	2.013751627	0.00000108087
0.8	2.225540928	2.225539563	0.00000136520
0.9	2.459603111	2.459601414	0.00000169738
1	2.718281828	2.718279744	0.00000208432

Furthermore, table 4 represents error between exact solution and solution obtained from Runge-Kutta 4th order. From the table 4, it is clearly seen that the exact solution at t=1 is 2.718281828 and on the other hand solution at t=1 is 2.718279744 using Runge-Kutta 4th order in neutrosophic environment for truth membership at (α, β,γ)-cut equal to 1 and the error between them is 0.00000208432.

6. Conclusion

In this paper, the first order ordinary differential equation using neutrosophic numbers with initial conditions have been solved. We have developed theory in a neutrosophic environment supplemented with an example showing the solution for first-order linear homogeneous differential equation both using analytical and numerical approach. For generalization, the (α,β,γ) - cut values are used for the neutrosophic numbers. Thus, to show the effectiveness of proposed method it has been applied to general example where the solution is given in terms of the truth, indeterminacy and falsity membership grade. We have shown the results in the form of tables for different (α,β,γ) - cut values and the graphs are also drawn. The results obtained are also discussed in details. Also, we have shown the growth of error between exact solution and approximate solution which are represented by tabulated values. This will promote the future study on higher order differential equations with neutrosophic numbers using numerical method which will help to decrease the error.

Conflict of interest

The Authors have no conflict of interest.

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On Some New Concepts of Weakly Neutrosophic Crisp Separation Axioms

Qays Hatem Imran[1*], Ali H. M. Al-Obaidi[2], Florentin Smarandache[3] and Md. Hanif PAGE[4]

[1]Department of Mathematics, College of Education for Pure Science, Al-Muthanna University, Samawah, Iraq.
E-mail: qays.imran@mu.edu.iq
[2]Department of Mathematics, College of Education for Pure Science, University of Babylon, Hillah, Iraq.
E-mail: aalobaidi@uobabylon.edu.iq
[3]Department of Mathematics, University of New Mexico 705 Gurley Ave. Gallup, NM 87301, USA.
E-mail: smarand@unm.edu
[4]Department of Mathematics, KLE Technological University, Hubballi-580031, Karnataka, India.
E-mail: mb_page@kletech.ac.in
*Correspondence: qays.imran@mu.edu.iq

Abstract: In this paper, we shall study some new concepts of weakly neutrosophic crisp separation axioms, which are called "neutrosophic crisp α-separation and neutrosophic crisp semi-α-separation axioms" such as neutrosophic crisp α-T_i and neutrosophic crisp semi-α-$T_i, \forall\, i = 0, 1, \dots, 4$. Moreover, we shall study the relationship between usual neutrosophic crisp separation axioms and these kinds of weakly neutrosophic crisp separation axioms.

Mathematics Subject Classification (2010): 03E72, 03F55, 54A40, 62C86.
Keywords: NC^{α}-T_i, $\mathrm{NC}^{S\alpha}$-$T_i, \forall\, i = 0, 1, \dots, 4$, NC^{α}-regular, $\mathrm{NC}^{S\alpha}$-regular, NC^{α}-normal and $\mathrm{NC}^{S\alpha}$-normal spaces.

1. Introduction

A. A. Salama et al. [1] give a concept of neutrosophic crisp topological space (briefly NCTS). A. A. Salama [2] provided some classes of neutrosophic crisp nearly open sets. A. H. M. Al-Obaidi et al. [3,4] give concepts of weakly neutrosophic crisp functions. Md. Hanif PAGE et al. [5] examined the view of neutrosophic generalized homeomorphism. Q. H. Imran et al. [6-8] established neutrosophic semi-α-open sets, new types of weakly neutrosophic crisp continuity and new concepts of neutrosophic crisp open sets. R. Dhavaseelan et al. [9] examined the view of neutrosophic α^m-continuity. R. K. Al-Hamido et al. [10] tendered the interpretation of neutrosophic crisp semi-α-closed sets. A. B. Al-Nafee et al. [11] demonstrated the principle of separation axioms in neutrosophic crisp topological spaces. R. K. Al-Hamido et al. [12] provided neutrosophic crisp semi separation axioms. The objective of this paper is to study some new concepts of weakly neutrosophic crisp separation axioms, which are called "neutrosophic crisp α-separation and neutrosophic crisp semi-α-separation axioms" such as neutrosophic crisp α-T_i and neutrosophic crisp semi-α-$T_i, \forall\, i = 0, 1, \dots, 4$. Moreover, we shall study the relationship between usual neutrosophic crisp separation axioms and these kinds of weakly neutrosophic crisp separation axioms.

2. Preliminaries

Throughout this paper, $(\mathcal{S}, \zeta)$ and $(\mathcal{J}, \eta)$ (or simply $\mathcal{S}$ and $\mathcal{J}$) always mean NCTSs. The complement of a neutrosophic crisp open set (briefly NC-OS) is called a neutrosophic crisp closed set (briefly NC-CS) in $(\mathcal{S},\zeta)$. For a NCS $\mathfrak{B}$ in a NCTS $(\mathcal{S},\zeta)$, $NCcl(\mathfrak{B})$, $NCint(\mathfrak{B})$ and $\mathfrak{B}^c$ denote the NC-closure of $\mathfrak{B}$, the NC-interior of $\mathfrak{B}$ and the NC-complement of $\mathfrak{B}$ respectively.

Definition 2.1 [1]:

For any nonempty under-consideration set $\mathcal{S}$, a neutrosophic crisp set (in short NCS) $\mathfrak{B}$ is an object holding the establish $\mathfrak{B} = \langle \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3 \rangle$ where $\mathfrak{B}_1, \mathfrak{B}_2$ and $\mathfrak{B}_3$ are mutually disjoint sets included in $\mathcal{S}$.

Definition 2.2:

A NC-subset $\mathfrak{B}$ of a NCTS $(\mathcal{S},\zeta)$ is said to be:

(i) neutrosophic crisp α-open set (in short NC^{α}-OS) [2] if $\mathfrak{B} \sqsubseteq NCint(NCcl(NCint(\mathfrak{B})))$. The family of all NC^{α}-OSs of $\mathcal{S}$ is denoted by $NC^{\alpha}O(\mathcal{S})$. The complement of NC^{α}-OS is called a neutrosophic crisp α-closed set (in short NC^{α}-CS). The family of all NC^{α}-CSs of $\mathcal{S}$ is denoted by $NC^{\alpha}C(\mathcal{S})$.

(ii) neutrosophic crisp semi-α-open set (in short $\mathrm{NC}^{\mathrm{S}\alpha}$-OS) [10] if there exists a NC^{α}-OS $\mathfrak{D}$ in $\mathcal{S}$ such that $\mathfrak{D} \sqsubseteq \mathfrak{B} \sqsubseteq NCcl(\mathfrak{D})$ or equivalently if $\mathfrak{B} \sqsubseteq NCcl(NCint(NCcl(NCint(\mathfrak{B}))))$. The family of all $\mathrm{NC}^{\mathrm{S}\alpha}$-OSs of $\mathcal{S}$ is denoted by $NC^{\mathrm{S}\alpha}O(\mathcal{S})$. The complement of $\mathrm{NC}^{\mathrm{S}\alpha}$-OS is called a neutrosophic crisp semi-α-closed set (in short $\mathrm{NC}^{\mathrm{S}\alpha}$-CS). The family of all $\mathrm{NC}^{\mathrm{S}\alpha}$-CSs of $\mathcal{S}$ is denoted by $NC^{\mathrm{S}\alpha}C(\mathcal{S})$.

Example 2.3:

Let $\mathcal{S} = \{k_1, k_2, k_3, k_4\}$. Then $\zeta = \{\emptyset_{\mathcal{N}}, \langle\{k_1\},\emptyset,\emptyset\rangle, \langle\{k_2\},\emptyset,\emptyset\rangle, \langle\{k_1,k_2\},\emptyset,\emptyset\rangle, \langle\{k_1,k_2,k_3\},\emptyset,\emptyset\rangle, \mathcal{S}_{\mathcal{N}}\}$ is a NCTS. The family of all NC^{α}-OSs of $\mathcal{S}$ is : $NC^{\alpha}O(\mathcal{S}) = \zeta \sqcup \langle\{k_1,k_2,k_4\},\emptyset,\emptyset\rangle$.

The family of all $\mathrm{NC}^{\mathrm{S}\alpha}$-OSs of $\mathcal{S}$ is : $NC^{\mathrm{S}\alpha}O(\mathcal{S}) = NC^{\alpha}O(\mathcal{S}) \sqcup \{\langle\{k_1,k_3\},\emptyset,\emptyset\rangle, \langle\{k_1,k_4\},\emptyset,\emptyset\rangle, \langle\{k_2,k_3\},\emptyset,\emptyset\rangle, \langle\{k_2,k_4\},\emptyset,\emptyset\rangle, \langle\{k_1,k_3,k_4\},\emptyset,\emptyset\rangle, \langle\{k_2,k_3,k_4\},\emptyset,\emptyset\rangle\}$.

Remark 2.4 [10,14]:

In a NCTS $(\mathcal{S},\zeta)$, then the following statements hold, and the opposite of each statement is not true:

(i) Every NC-OS (resp. NC-CS) is a NC^{α}-OS (resp. NC^{α}-CS) and $\mathrm{NC}^{\mathrm{S}\alpha}$-OS (resp. $\mathrm{NC}^{\mathrm{S}\alpha}$-CS).

(ii) Every NC^{α}-OS (resp. NC^{α}-CS) is a $\mathrm{NC}^{\mathrm{S}\alpha}$-OS (resp. $\mathrm{NC}^{\mathrm{S}\alpha}$-CS).

Definition 2.5:

(i) The NC^{α}-interior of a NCS $\mathfrak{B}$ of a NCTS $(\mathcal{S},\zeta)$ is the union of all NC^{α}-OSs contained in $\mathfrak{B}$ and is denoted by $NC^{\alpha}int(\mathfrak{B})$[3].

(ii) The $\mathrm{NC}^{\mathrm{S}\alpha}$-interior of a NCS $\mathfrak{B}$ of a NCTS $(\mathcal{S},\zeta)$ is the union of all $\mathrm{NC}^{\mathrm{S}\alpha}$-OSs contained in $\mathfrak{B}$ and is denoted by $NC^{\mathrm{S}\alpha}int(\mathfrak{B})$[10].

Definition 2.6:

(i) The NC^{α}-closure of a NCS $\mathfrak{B}$ of a NCTS $(\mathcal{S},\zeta)$ is the intersection of all NC^{α}-CSs containing $\mathfrak{B}$ and is denoted by $NC^{\alpha}cl(\mathfrak{B})$[3].

(ii) The $\mathrm{NC}^{\mathrm{S}\alpha}$-closure of a NCS $\mathfrak{B}$ of a NCTS $(\mathcal{S},\zeta)$ is the intersection of all $\mathrm{NC}^{\mathrm{S}\alpha}$-CSs containing $\mathfrak{B}$ and is denoted by $NC^{\mathrm{S}\alpha}cl(\mathfrak{B})$[10].

Theorem 2.7:

Let $(\mathcal{S},\zeta)$ and $(\mathcal{J},\eta)$ be two NCTSs. If $\mathfrak{B} \in NC^{\alpha}O(\mathcal{S})$(resp. $\mathfrak{B} \in NC^{\mathrm{S}\alpha}O(\mathcal{S})$), $\mathfrak{D} \in NC^{\alpha}O(\mathcal{J})$(resp. $\mathfrak{D} \in NC^{\mathrm{S}\alpha}O(\mathcal{J})$), then $\mathfrak{B} \times \mathfrak{D} \in NC^{\alpha}O(\mathcal{S} \times \mathcal{J})$ (resp. $\mathfrak{B} \times \mathfrak{D} \in NC^{\mathrm{S}\alpha}O(\mathcal{S} \times \mathcal{J})$).

Proof:

Since $\mathfrak{B} \sqsubseteq NCint(NCcl(NCint(\mathfrak{B})))$, $\mathfrak{D} \sqsubseteq NCint(NCcl(NCint(\mathfrak{D})))$.

Hence $\mathfrak{B} \times \mathfrak{D} \sqsubseteq NCint(NCcl(NCint(\mathfrak{B}))) \times NCint(NCcl(NCint(\mathfrak{D}))) = NCint(NCcl(NCint(\mathfrak{B} \times \mathfrak{D})))$.

Therefore $\mathfrak{B} \times \mathfrak{D} \sqsubseteq NCint(NCcl(NCint(\mathfrak{B} \times \mathfrak{D}))) \implies \mathfrak{B} \times \mathfrak{D} \in NC^{\alpha}O(\mathcal{S} \times \mathcal{I})$. The second case is similar. ▪

Corollary 2.8:

Let $(\mathcal{S}, \zeta)$ and $(\mathcal{I}, \eta)$ be two NCTSs. If $\mathfrak{B} \in NC^{\alpha}C(\mathcal{S})$(resp. $\mathfrak{B} \in NC^{S\alpha}C(\mathcal{S})$), $\mathfrak{D} \in NC^{\alpha}C(\mathcal{I})$(resp. $\mathfrak{D} \in NC^{S\alpha}C(\mathcal{I})$), then $\mathfrak{B} \times \mathfrak{D} \in NC^{\alpha}C(\mathcal{S} \times \mathcal{I})$ (resp. $\mathfrak{B} \times \mathfrak{D} \in NC^{S\alpha}C(\mathcal{S} \times \mathcal{I})$).

Proof:

The proof of this is similar to that of theorem (2.6). ▪

Proposition 2.9 [10]:

For any NC-subset $\mathfrak{B}$ of a NCTS $(\mathcal{S}, \zeta)$, then:

(i) $NCint(\mathfrak{B}) \sqsubseteq NC^{\alpha}int(\mathfrak{B}) \sqsubseteq NC^{S\alpha}int(\mathfrak{B}) \sqsubseteq NC^{S\alpha}cl(\mathfrak{B}) \sqsubseteq NC^{\alpha}cl(\mathfrak{B}) \sqsubseteq NCcl(\mathfrak{B})$.

(ii) $NCint(NC^{S\alpha}int(\mathfrak{B})) = NC^{S\alpha}int(NCint(\mathfrak{B})) = NCint(\mathfrak{B})$.

(iii) $NC^{\alpha}int(NC^{S\alpha}int(\mathfrak{B})) = NC^{S\alpha}int(NC^{\alpha}int(\mathfrak{B})) = NC^{\alpha}int(\mathfrak{B})$.

(iv) $NCcl(NC^{S\alpha}cl(\mathfrak{B})) = NC^{S\alpha}cl(NCcl(\mathfrak{B})) = NCcl(\mathfrak{B})$.

(v) $NC^{\alpha}cl(NC^{S\alpha}cl(\mathfrak{B})) = NC^{S\alpha}cl(NC^{\alpha}cl(\mathfrak{B})) = NC^{\alpha}cl(\mathfrak{B})$.

(vi) $NC^{S\alpha}cl(\mathfrak{B}) = \mathfrak{B} \sqcup NCint(NCcl(NCint(NCcl(\mathfrak{B}))))$.

(vii) $NC^{S\alpha}int(\mathfrak{B}) = \mathfrak{B} \sqcap NCcl(NCint(NCcl(NCint(\mathfrak{B}))))$.

(viii) $NCint(NCcl(\mathfrak{B})) \sqsubseteq NC^{S\alpha}int(NC^{S\alpha}cl(\mathfrak{B}))$

Definition 2.10 [1]:

Let $\rho: (\mathcal{S}, \zeta) \longrightarrow (\mathcal{I}, \eta)$ be a function, then ρ is said to be NC-continuous (in short NC-CF) iff $\forall$ $\mathfrak{B}$ NC-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a NC-OS in $\mathcal{S}$.

Definition 2.11 [13]:

Let $\rho: (\mathcal{S}, \zeta) \longrightarrow (\mathcal{I}, \eta)$ be a function, then ρ is said to be NC^{α}-continuous (in short NC^{α}-CF) iff $\forall$ $\mathfrak{B}$ NC-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a NC^{α}-OS in $\mathcal{S}$.

Definition 2.12 [10]:

Let $\rho: (\mathcal{S}, \zeta) \longrightarrow (\mathcal{I}, \eta)$ be a function, then ρ is said to be:

(i) NC^{α^*}-continuous (in short NC^{α^*}-CF) iff $\forall$ $\mathfrak{B}$ NC^{α}-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a NC^{α}-OS in $\mathcal{S}$.

(ii) $NC^{\alpha^{**}}$-continuous (in short $NC^{\alpha^{**}}$-CF) iff $\forall$ $\mathfrak{B}$ NC^{α}-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a NC-OS in $\mathcal{S}$.

Definition 2.13 [10]:

Let $\rho: (\mathcal{S}, \zeta) \longrightarrow (\mathcal{I}, \eta)$ be a function, then ρ is said to be:

(i) $NC^{S\alpha}$-continuous (in short $NC^{S\alpha}$-CF) iff $\forall$ $\mathfrak{B}$ NC-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a $NC^{S\alpha}$-OS in $\mathcal{S}$.

(ii) $NC^{S\alpha^*}$-continuous (in short $NC^{S\alpha^*}$-CF) iff $\forall$ $\mathfrak{B}$ $NC^{S\alpha}$-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a $NC^{S\alpha}$-OS in $\mathcal{S}$.

(iii) $NC^{S\alpha^{**}}$-continuous (in short $NC^{S\alpha^{**}}$-CF) iff $\forall$ $\mathfrak{B}$ $NC^{S\alpha}$-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a NC-OS in $\mathcal{S}$.

3. Some New Concepts of Weakly Neutrosophic Crisp Separation Axioms

Definition 3.1:

(i) A NCTS $(\mathcal{S}, \zeta)$ is said to be a NC^{α}-T_0-space if for each pair of distinct neutrosophic crisp points in $(\mathcal{S}, \zeta)$ there exists NC^{α}-OS of $(\mathcal{S}, \zeta)$ containing one neutrosophic crisp point but not the other.

(ii) A NCTS $(\mathcal{S}, \zeta)$ is said to be a $NC^{S\alpha}$-T_0-space if for each pair of distinct neutrosophic crisp points in $(\mathcal{S}, \zeta)$ there exists $NC^{S\alpha}$-OS of $(\mathcal{S}, \zeta)$ containing one neutrosophic crisp point but not the other.

Theorem 3.3:

A NCTS $(\mathcal{S},\zeta)$ is NC$^{\alpha}$-T_0-space (NC$^{S\alpha}$-T_0-space respectively) iff $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \neq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$ $(NC^{S\alpha}cl(<\{u\},\emptyset,\emptyset>) \neq NC^{S\alpha}cl(<\{v\},\emptyset,\emptyset>)$ receptively) for each $u \neq v$ in $\mathcal{S}$.

Proof:

$\Rightarrow$ Let $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \neq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$, $\forall u \neq v \in \mathcal{S}$. Hence $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \not\sqsubseteq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$ or $NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>) \not\sqsubseteq NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>)$. Suppose that $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \not\sqsubseteq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$ $\Rightarrow u \notin NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>) \Rightarrow u \in (NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>))^c$ but $(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>))^c$ is a NC$^{\alpha}$-OS and $v \notin (NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>))^c$. Therefore $\mathcal{S}$ is a NC$^{\alpha}$-T_0-space.

$\Leftarrow$ Let $\mathcal{S}$ be a NC$^{\alpha}$-T_0-space, $\forall u \neq v \in \mathcal{S}$. Hence there exists a NC$^{\alpha}$-OS $\mathfrak{B}$ in $\mathcal{S}$ such that $u \in \mathfrak{B}, v \notin \mathfrak{B}$ or $u \notin \mathfrak{B}, v \in \mathfrak{B}$. Then $\mathfrak{B}^c$ is a NC$^{\alpha}$-CS and $u \notin \mathfrak{B}^c, v \in \mathfrak{B}^c$. Therefore $u \notin NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$ (since $u \notin \mathfrak{B}^c$). Hence $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \not\sqsubseteq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$. The second case is similar. ▪

Theorem 3.4:

If $(\mathcal{S},\zeta)$ is a NC$^{\alpha}$-T_0-space (NC$^{S\alpha}$-T_0-space respectively), then $NC^{\alpha}int\big(NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>)\big) \sqcap NC^{\alpha}int\big(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)\big) = \emptyset_{\mathcal{N}}$ $\big(NC^{S\alpha}int\big(NC^{S\alpha}cl(<\{u\},\emptyset,\emptyset>)\big) \sqcap NC^{S\alpha}int\big(NC^{S\alpha}cl(<\{v\},\emptyset,\emptyset>)\big) = \emptyset_{\mathcal{N}}$ receptively), $\forall u \neq v$ in $\mathcal{S}$.

Proof:

Let $(\mathcal{S},\zeta)$ be a NC$^{\alpha}$-T_0-space. Then there exists a NC$^{\alpha}$-OS $\mathfrak{B}$ such that $u \in \mathfrak{B}, v \notin \mathfrak{B}$ or $u \notin \mathfrak{B}, v \in \mathfrak{B}$. If $u \in \mathfrak{B}, v \notin \mathfrak{B} \Rightarrow u \notin \mathfrak{B}^c, v \in \mathfrak{B}^c$. Thus $NC^{\alpha}int\big(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)\big) \sqsubseteq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>) \sqsubseteq \mathfrak{B}^c = NC^{\alpha}cl(\mathfrak{B}^c)$ (since $\mathfrak{B}^c$ is a NC$^{\alpha}$ -CS). Hence $NC^{\alpha}int\big(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)\big) \sqsubseteq \mathfrak{B}^c \Rightarrow NC^{\alpha}int\big(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)\big) \sqcap \mathfrak{B} = \emptyset_{\mathcal{N}}$. Therefore, $u \in \mathfrak{B} \sqsubseteq (NC^{\alpha}int\big(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)\big))^c$. Hence $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \sqsubseteq (NC^{\alpha}int\big(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)\big))^c \Rightarrow NC^{\alpha}int\big(NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>)\big)$ $\sqsubseteq \Big(NC^{\alpha}int\big(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)\big)\Big)^c \Rightarrow NC^{\alpha}int\big(NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>)\big) \sqcap NC^{\alpha}int\big(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)\big) = \emptyset_{\mathcal{N}}$. The second case is similar. ▪

Remark 3.5:

(i) Every NC-T_0-space is a NC$^{\alpha}$-T_0-space and NC$^{S\alpha}$-T_0-space.

(ii) Every NC$^{\alpha}$-T_0-space is a NC$^{S\alpha}$-T_0-space.

Remark 3.6:

(i) NC$^{\alpha}$-T_0 (NC$^{S\alpha}$-T_0 respectively) property is a NC$^{\alpha^*}$ (NC$^{S\alpha^*}$ respectively) topological property.

(ii) NC$^{\alpha}$-T_0 (NC$^{S\alpha}$-T_0 respectively) property is a NC$^{\alpha^{**}}$ (NC$^{S\alpha^{**}}$ respectively) topological property.

(iii) NC$^{\alpha}$-T_0 is a NC$^{\alpha}$-hereditary property.

Proposition 3.7:

(i) Let $(\mathcal{S},\zeta)$ and $(\mathcal{J},\eta)$ be NC$^{\alpha}$-T_0-spaces if and only if $\mathcal{S} \times \mathcal{J}$ is a NC$^{\alpha}$-T_0-space.

(ii) If $(\mathcal{S},\zeta)$ and $(\mathcal{J},\eta)$ are NC$^{S\alpha}$-T_0-spaces, then $\mathcal{S} \times \mathcal{J}$ is a NC$^{S\alpha}$-T_0-space.

Proof:

(i) $\Rightarrow$ Let $\mathcal{S}$ and $\mathcal{J}$ be NC$^{\alpha}$-T_0-spaces. Let $(u_1,v_1) \neq (u_2,v_2)$ in $\mathcal{S} \times \mathcal{J}$. Then $u_1 \neq u_2$ in $\mathcal{S} \Rightarrow$ there exists $\mathfrak{B}_1 \in NC^{\alpha}O(\mathcal{S})$ such that $u_1 \in \mathfrak{B}_1, u_2 \notin \mathfrak{B}_1$ or $u_1 \notin \mathfrak{B}_1, u_2 \in \mathfrak{B}_1$.

Also $v_1 \neq v_2$ in $\mathcal{J} \Rightarrow$ there exists $\mathfrak{B}_2 \in NC^{\alpha}O(\mathcal{J})$ such that $v_1 \in \mathfrak{B}_2, v_2 \notin \mathfrak{B}_2$ or $v_1 \notin \mathfrak{B}_2, v_2 \in \mathfrak{B}_2$.

Then $(u_1,v_1) \in \mathfrak{B}_1 \times \mathfrak{B}_2, (u_2,v_2) \notin \mathfrak{B}_1 \times \mathfrak{B}_2$ or $(u_1,v_1) \notin \mathfrak{B}_1 \times \mathfrak{B}_2, (u_2,v_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2$.

But $\mathfrak{B}_1 \times \mathfrak{B}_2 \in NC^{\alpha}O(\mathcal{S} \times \mathcal{J})$ (since by theorem (2.6)). Hence $\mathcal{S} \times \mathcal{J}$ is a NC$^{\alpha}$-T_0-space.

$\Leftarrow$ Let $\mathcal{S} \times \mathcal{I}$ be a NC^{α}-T_0-space, to prove that $\mathcal{S}$ and $\mathcal{I}$ are NC^{α}-T_0-spaces. Since $\mathcal{S} \times \mathcal{I}$ is a NC^{α}-T_0-space, then $\mathcal{S} \times < \{v_0\}, \emptyset, \emptyset >$ and $< \{u_0\}, \emptyset, \emptyset > \times \mathcal{I}$ are NC^{α}-T_0-spaces (since NC^{α}-T_0 property is a NC^{α}-hereditary). Hence $\mathcal{S}$ and $\mathcal{I}$ are NC^{α}-T_0-spaces. The proof (ii) is evident for others. ▪

Definition 3.8:

(i) A NCTS $(\mathcal{S}, \zeta)$ is said to be a NC^{α}-T_1-space if for each pair of distinct NC- points u and v of $\mathcal{S}$, there exist two NC^{α}-OSs $\mathfrak{B}$ and $\mathfrak{D}$ containing u and v respectively, such that $u \in \mathfrak{B}$, $v \in \mathfrak{D}$.

(ii) A NCTS $(\mathcal{S}, \zeta)$ is said to be a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space if for each pair of distinct NC-points u and v of $\mathcal{S}$, there exist two $\mathrm{NC}^{\mathrm{S}\alpha}$-OSs $\mathfrak{B}$ and $\mathfrak{D}$ containing u and v respectively, such that $u \in \mathfrak{B}$, $v \in \mathfrak{D}$.

Proposition 3.9:

A NCTS $(\mathcal{S}, \zeta)$ is NC^{α}-T_1-space ($\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space respectively) if and only if $< \{u\}, \emptyset, \emptyset >$ is a NC^{α}-CS ($\mathrm{NC}^{\mathrm{S}\alpha}$-CS respectively), $\forall u \in \mathcal{S}$.

Proof:

$\Rightarrow$ Let $\mathcal{S}$ be a NC^{α}-T_1-space. Let $w \in \mathcal{S}$, to prove that $< \{w\}, \emptyset, \emptyset >$ is a NC^{α}-CS. Let $u \in (< \{w\}, \emptyset, \emptyset >)^c \Rightarrow u \neq w$ in $\mathcal{S}$. Hence there exists a NC^{α}-OS $\mathfrak{B}$ such that $u \in \mathfrak{B}, w \notin \mathfrak{B}$ or $u \notin \mathfrak{B}, w \in \mathfrak{B}$. If $u \in \mathfrak{B}, w \notin \mathfrak{B} \Rightarrow u \in \mathfrak{B} \sqsubseteq (< \{w\}, \emptyset, \emptyset >)^c \Rightarrow (< \{w\}, \emptyset, \emptyset >)^c$ is a NC^{α}-OS $\Rightarrow < \{w\}, \emptyset, \emptyset >$ is a NC^{α}-CS.

$\Leftarrow$ Let $< \{w\}, \emptyset, \emptyset >$ be a NC^{α}-CS, $\forall w \in \mathcal{S}$, to prove that $\mathcal{S}$ is a NC^{α}-T_1-space. Let $u \neq v$ in $\mathcal{S}$. Hence $< \{u\}, \emptyset, \emptyset >, < \{v\}, \emptyset, \emptyset >$ are NC^{α}-CSs $\Rightarrow (< \{u\}, \emptyset, \emptyset >)^c, (< \{v\}, \emptyset, \emptyset >)^c$ are NC^{α}-OSs and $v \in (< \{u\}, \emptyset, \emptyset >)^c, u \notin (< \{u\}, \emptyset, \emptyset >)^c, u \in (< \{v\}, \emptyset, \emptyset >)^c, v \notin (< \{v\}, \emptyset, \emptyset >)^c$. Therefore $\mathcal{S}$ is a NC^{α}-T_1-space. The second case is similar. ▪

Remark 3.10:

(i) Every NC-T_1-space is a NC^{α}-T_1-space and $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space.

(ii) Every NC^{α}-T_1-space is a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space.

(iii) Every NC^{α}-T_1-space is a NC^{α}-T_0-space.

(iv) Every $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space is a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_0$-space.

Remark 3.11:

(i) NC^{α}-T_1 ($\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$ respectively) property is a NC^{α^*} ($\mathrm{NC}^{\mathrm{S}\alpha^*}$ respectively) topological property.

(ii) NC^{α}-T_1 ($\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$ respectively) property is a $\mathrm{NC}^{\alpha^{**}}$ ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$ respectively) topological property.

(iii) NC^{α}-T_1 property is a NC^{α}-hereditary property.

Proposition 3.12:

(i) Let $\mathcal{S}$ and $\mathcal{I}$ be NC^{α}-T_1-spaces if and only if $\mathcal{S} \times \mathcal{I}$ is a NC^{α}-T_1-space.

(ii) If $\mathcal{S}$ and $\mathcal{I}$ are $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-spaces, then $\mathcal{S} \times \mathcal{I}$ is a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space.

Proof:

The proof of this is similar to that of proposition (3.7). ▪

Definition 3.13:

(i) A NCTS $(\mathcal{S}, \zeta)$ is said to be a NC^{α}-T_2-space if for each pair of distinct NC-points u and v in $\mathcal{S}$, there exist two NC^{α}-OSs $\mathfrak{D}_1$ and $\mathfrak{D}_2$ such that $u \in \mathfrak{D}_1$, $v \in \mathfrak{D}_2$ and $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}}$.

(ii) A NCTS $(\mathcal{S}, \zeta)$ is said to be a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_2$-space if for each pair of distinct NC-points u and v in $\mathcal{S}$, there exist two $\mathrm{NC}^{\mathrm{S}\alpha}$-OSs $\mathfrak{D}_1$ and $\mathfrak{D}_2$ such that $u \in \mathfrak{D}_1$, $v \in \mathfrak{D}_2$ and $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}}$.

Proposition 3.14:

If $(\mathcal{S},\zeta)$ is a NC^{α}-T_2-space ($\mathrm{NC}^{S\alpha}$-T_2-space respectively), then $\mathfrak{B}=\{(u,v): u=v, u,v\in\mathcal{S}\}$ is a NC^{α}-CS ($\mathrm{NC}^{S\alpha}$-CS respectively).

Proof:

Let $\mathcal{S}$ be a NC^{α}-T_2-space, to prove that $\mathfrak{B}$ is a NC^{α}-CS. Let $(u,v)\in\mathfrak{B}^c=\mathcal{S}\times\mathcal{S}-\mathfrak{B}$. Hence $u\neq v$ in $\mathcal{S}$ $\Longrightarrow$ there exist $\mathfrak{D}_1,\mathfrak{D}_2\in NC^{\alpha}O(\mathcal{S})$ such that $u\in\mathfrak{D}_1$, $v\in\mathfrak{D}_2$ and $\mathfrak{D}_1\sqcap\mathfrak{D}_2=\emptyset_{\mathcal{N}}$ (since $\mathcal{S}$ is a NC^{α}-T_2-space). Hence $\mathfrak{D}_1\times\mathfrak{D}_2\in NC^{\alpha}O(\mathcal{S}\times\mathcal{S})$ by theorem (2.7) $(u,v)\in\mathfrak{D}_1\times\mathfrak{D}_2\sqsubseteq\mathfrak{B}^c$, hence $\mathfrak{B}^c$ is a NC^{α}-OS. Therefore $\mathfrak{B}$ is a NC^{α}-CS. The second case is similar. ▪

Remark 3.15:

(i) Every NC-T_2-space is a NC^{α}-T_2-space and $\mathrm{NC}^{S\alpha}$-T_2-space.

(ii) Every NC^{α}-T_2-space is a $\mathrm{NC}^{S\alpha}$-T_2-space.

(iii) Every NC^{α}-T_2-space is a NC^{α}-T_1-space.

(iv) Every $\mathrm{NC}^{S\alpha}$-T_2-space is a $\mathrm{NC}^{S\alpha}$-T_1-space.

Remark 3.16:

(i) NC^{α}-T_2 ($\mathrm{NC}^{S\alpha}$-T_2 respectively) property is a NC^{α^*} ($\mathrm{NC}^{S\alpha^*}$ respectively) topological property.

(ii) NC^{α}-T_2 ($\mathrm{NC}^{S\alpha}$-T_2respectively) property is a $\mathrm{NC}^{\alpha^{**}}$ ($\mathrm{NC}^{S\alpha^{**}}$ respectively) topological property.

(iii) NC^{α}-T_2 property is a NC^{α}-hereditary property.

Proposition 3.17:

(i) Let $\mathcal{S}$ and $\mathcal{I}$ be NC^{α}-T_2-spaces if and only if $\mathcal{S}\times\mathcal{I}$ is a NC^{α}-T_2-space.

(ii) If $\mathcal{S}$ and $\mathcal{I}$ are $\mathrm{NC}^{S\alpha}$-T_2-spaces, then $\mathcal{S}\times\mathcal{I}$ is a $\mathrm{NC}^{S\alpha}$-T_2-space.

Proof:

The proof of this is similar to that of proposition (3.12). ▪

Proposition 3.18:

(i) If $\rho,\mu:\mathcal{S}\longrightarrow\mathcal{I}$ are NC^{α^*}-CF and $\mathcal{I}$ is a NC^{α}-T_2-space, then the NC-set $\mathfrak{B}=\{u: u\in\mathcal{S},\rho(u)=\mu(u)\}$ is a NC^{α}-CS.

(ii) If $\rho,\mu:\mathcal{S}\longrightarrow\mathcal{I}$ are NC^{α}-CF and $\mathcal{I}$ is a NC-T_2-space, then the NC-set $\mathfrak{B}=\{u: u\in\mathcal{S},\rho(u)=\mu(u)\}$ is a NC^{α}-CS.

Proof:

(i) If $u\notin\mathfrak{B}\Longrightarrow u\in\mathfrak{B}^c\Longrightarrow\rho(u)\neq\mu(u)$ in $\mathcal{I}$. Hence there exist two NC^{α}-OSs $\mathfrak{D}_1$ and $\mathfrak{D}_2$ in $\mathcal{I}$ such that $\rho(u)\in\mathfrak{D}_1$, $\mu(u)\in\mathfrak{D}_2$ and $\mathfrak{D}_1\sqcap\mathfrak{D}_2=\emptyset_{\mathcal{N}}$ (since $\mathcal{I}$ is a NC^{α}-T_2-space). But $\rho^{-1}(\mathfrak{D}_1),\mu^{-1}(\mathfrak{D}_2)\in NC^{\alpha}O(\mathcal{S})$ (since ρ,μ are NC^{α^*}-CF). Therefore, $u\in\rho^{-1}(\mathfrak{D}_1)$ and $u\in\mu^{-1}(\mathfrak{D}_2)$. Hence $u\in\rho^{-1}(\mathfrak{D}_1)\sqcap\mu^{-1}(\mathfrak{D}_2)$. Let $\mathfrak{U}=\rho^{-1}(\mathfrak{D}_1)\sqcap\mu^{-1}(\mathfrak{D}_2)\in NC^{\alpha}O(\mathcal{S})$. To prove $\mathfrak{U}\sqsubseteq\mathfrak{B}^c$, i.e., $\mathfrak{U}\sqcap\mathfrak{B}=\emptyset_{\mathcal{N}}$.
Suppose that $\mathfrak{U}\sqcap\mathfrak{B}\neq\emptyset_{\mathcal{N}}\Longrightarrow\exists v\in\mathfrak{U}\sqcap\mathfrak{B}\Longrightarrow v\in\mathfrak{U}$ and $v\in\mathfrak{B}$, i.e., $v\in\rho^{-1}(\mathfrak{D}_1)$ and $v\in\mu^{-1}(\mathfrak{D}_2)$ and $v\in\mathfrak{B}$. Hence $\rho(v)\in\mathfrak{D}_1$, $\mu(v)\in\mathfrak{D}_2$ and $v\in\mathfrak{B}$. Therefore $\rho(v)=\mu(v)$ (since $v\in\mathfrak{B}$). Hence $\mathfrak{D}_1\sqcap\mathfrak{D}_2\neq\emptyset_{\mathcal{N}}$ which is a contradiction. Therefore $\mathfrak{U}\sqsubseteq\mathfrak{B}^c\Longrightarrow\mathfrak{B}^c\in NC^{\alpha}O(\mathcal{S})\Longrightarrow\mathfrak{B}$ is a NC^{α}-CS. The proof (ii) is evident for others. ▪

4. Some New Concepts of Weakly Neutrosophic Crisp Regularity

Definition 4.1:

Let $(\mathcal{S},\zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) NC^{α}-regular ($\mathrm{NC}^{S\alpha}$-regular respectively) if every $u\in\mathcal{S}$ and every Q NC-CS such that $u\notin Q$, there exist two NC$^{\alpha}$-OSs (NC$^{S\alpha}$-OSs respectively) $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}$, $\mathcal{Q} \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

(ii) NC$^{\alpha^*}$-regular (NC$^{S\alpha^*}$-regular respectively) if every $u \in \mathcal{S}$ and every $\mathcal{Q}$ NC$^{\alpha}$-CS (NC$^{S\alpha}$-CS respectively) such that $u \notin \mathcal{Q}$, there exist two NC$^{\alpha}$-OSs (NC$^{S\alpha}$-OSs respectively) $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}$, $\mathcal{Q} \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

(iii) NC$^{\alpha^{**}}$-regular (NC$^{S\alpha^{**}}$-regular respectively) if every $u \in \mathcal{S}$ and every $\mathcal{Q}$ NC$^{\alpha}$-CS (NC$^{S\alpha}$-CS respectively) such that $u \notin \mathcal{Q}$, there exist two NC-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}$, $\mathcal{Q} \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

Remark 4.2:

The following diagram shows the relation between the different types of weakly NC-regular and weakly NC$^{\alpha}$-regular (NC$^{S\alpha}$-regular respectively) spaces:

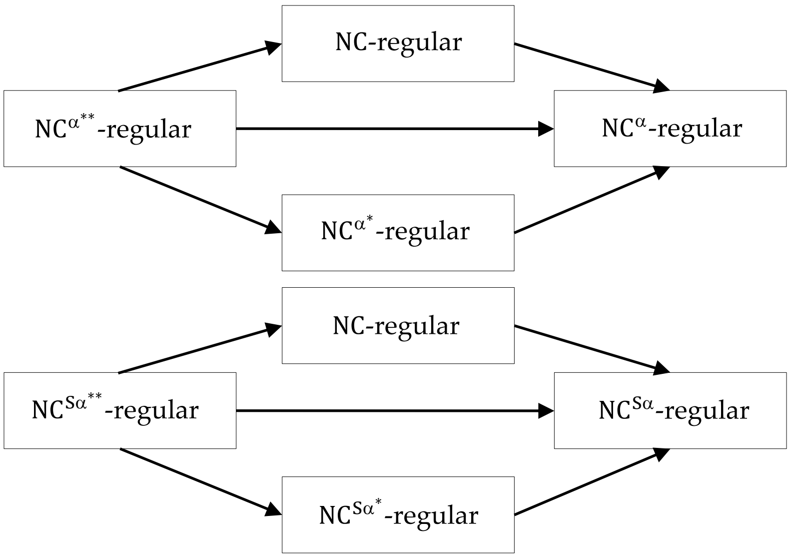


Fig. 4.1

Theorem 4.3:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a NC$^{\alpha}$-regular if and only if for each $\mathfrak{B}$ NC-OS containing u, there exists $\mathfrak{D}$ NC$^{\alpha}$-OS containing u such that $u \in \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(ii) $\mathcal{S}$ is a NC$^{\alpha^*}$-regular if and only if for each $\mathfrak{B}$ NC$^{\alpha}$-OS contains u, there exists $\mathfrak{D}$ NC$^{\alpha}$-OS contains u such that $u \in \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(iii) $\mathcal{S}$ is a NC$^{\alpha^{**}}$-regular if and only if for each $\mathfrak{B}$ NC$^{\alpha}$-OS contains u, there exists $\mathfrak{D}$ NC-OS contains u such that $u \in \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

Proof:

(i) $\Rightarrow$ Let $\mathcal{S}$ be a NC$^{\alpha}$-regular space and let $\mathfrak{B}$ be a NC-OS containing u. Hence $\mathfrak{B}^c$ is a NC-CS and $u \notin \mathfrak{B}^c$. Then there exist $\mathfrak{D}_1, \mathfrak{D}_2$ NC$^{\alpha}$-OSs in $\mathcal{S}$ such that $u \in \mathfrak{D}_1$, $\mathfrak{B}^c \sqsubseteq \mathfrak{D}_2$ and $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}}$ (since $\mathcal{S}$ is a NC$^{\alpha}$-regular space). Hence $u \in \mathfrak{D}_1 \sqsubseteq {\mathfrak{D}_2}^c \sqsubseteq \mathfrak{B}$ (since $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}} \Rightarrow \mathfrak{D}_1 \sqsubseteq {\mathfrak{D}_2}^c$). Therefore $u \in \mathfrak{D}_1 \sqsubseteq NC^{\alpha}cl(\mathfrak{D}_1) \sqsubseteq NC^{\alpha}cl({\mathfrak{D}_2}^c) \sqsubseteq NC^{\alpha}cl(\mathfrak{B})$. Therefore $u \in \mathfrak{D}_1 \sqsubseteq NC^{\alpha}cl(\mathfrak{D}_1) \sqsubseteq {\mathfrak{D}_2}^c \sqsubseteq \mathfrak{B}$. The implies that $u \in \mathfrak{D}_1 \sqsubseteq NC^{\alpha}cl(\mathfrak{D}_1) \sqsubseteq \mathfrak{B}$, where $\mathfrak{D}_1$ is a NC$^{\alpha}$-OS.

$\Leftarrow$ Let $\mathcal{Q}$ be a NC-CS such that $u \notin \mathcal{Q} \Longrightarrow \mathcal{Q}^c$ is a NC-OS contains u. Hence there exists $\mathfrak{D}$ NC^{α}-OS contains u such that $u \in \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathcal{Q}^c$. We get $\mathcal{Q} \sqsubseteq (NC^{\alpha}cl(\mathfrak{D}))^c$, so it is $(NC^{\alpha}cl(\mathfrak{D}))^c$ is a NC^{α}-OS and contains $\mathcal{Q}$. Now, to prove $\mathfrak{D}\sqcap(NC^{\alpha}cl(\mathfrak{D}))^c = \emptyset_{\mathcal{N}}$. Since $\mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D})$, but $NC^{\alpha}cl(\mathfrak{D})\sqcap(NC^{\alpha}cl(\mathfrak{D}))^c = \emptyset_{\mathcal{N}} \Longrightarrow \mathfrak{D}\sqcap(NC^{\alpha}cl(\mathfrak{D}))^c = \emptyset_{\mathcal{N}}$. Hence $\mathcal{S}$ is a NC^{α}-regular space. The proofs (ii), (iii) are evident for others. ▪

Theorem 4.4:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a $NC^{S\alpha^*}$-regular if and only if for each $\mathfrak{B}$ $NC^{S\alpha}$-OS contains u, there exists $\mathfrak{D}$ $NC^{S\alpha}$-OS contains u such that $u \in \mathfrak{D} \sqsubseteq NC^{S\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(ii) $\mathcal{S}$ is a $NC^{S\alpha^{**}}$-regular if and only if for each $\mathfrak{B}$ $NC^{S\alpha}$-OS contains u, there exists $\mathfrak{D}$ NC-OS contains u such that $u \in \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

Proof:

The proof of this is similar to that of theorem (4.3). ▪

Theorem 4.5:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a NC^{α^*}-regular if and only if $u \notin \mathcal{Q}$ where $\mathcal{Q}$ is a NC^{α}-CS, there exist two NC^{α}-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}, \mathcal{Q} \sqsubseteq \mathfrak{D}$ and $NC^{\alpha}cl(\mathfrak{B})\sqcap NC^{\alpha}cl(\mathfrak{D}) = \emptyset_{\mathcal{N}}$.

(ii) $\mathcal{S}$ is a $NC^{\alpha^{**}}$-regular if and only if for each $\mathcal{Q}$ NC^{α}-CS, such that $u \notin \mathcal{Q}$, there exist two NC-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}, \mathcal{Q} \sqsubseteq \mathfrak{D}$ and $NCcl(\mathfrak{B})\sqcap NCcl(\mathfrak{D}) = \emptyset_{\mathcal{N}}$.

Proof:

(i) Let $\mathcal{S}$ be a NC^{α^*}-regular space and let $\mathcal{Q}$ be a NC^{α}-CS, such that $u \notin \mathcal{Q}$. Then there exist two NC^{α}-OSs $\mathcal{U}$ and $\mathcal{V}$ such that $u \in \mathcal{U}, \mathcal{Q} \sqsubseteq \mathcal{V}$ and $\mathcal{U}\sqcap\mathcal{V} = \emptyset_{\mathcal{N}}$. Therefore $\mathcal{U}$ is a NC^{α}-OS containing u in $\mathcal{S}$, where $\mathcal{S}$ is a NC^{α^*}-regular space. Then there exists $\mathfrak{B}$ NC^{α}-OS containing u such that $u \in \mathfrak{B} \sqsubseteq NC^{\alpha}cl(\mathfrak{B}) \sqsubseteq \mathcal{U}$ (since by theorem (4.3) (ii)). Hence $NC^{\alpha}cl(\mathfrak{B}) \sqsubseteq \mathcal{U}$. Also, $\mathcal{Q} \sqsubseteq \mathcal{V} \sqsubseteq NC^{\alpha}cl(\mathcal{V})$, but $NC^{\alpha}cl(\mathcal{V}) \sqsubseteq (NC^{\alpha}cl(\mathcal{U}))^c$ (since $\mathcal{U}\sqcap\mathcal{V} = \emptyset_{\mathcal{N}} \Longrightarrow \mathcal{V} \sqsubseteq \mathcal{U}^c \Longrightarrow NC^{\alpha}cl(\mathcal{V}) \sqsubseteq NC^{\alpha}cl(\mathcal{U}^c)$). Hence $\mathcal{Q} \sqsubseteq \mathcal{V} \sqsubseteq NC^{\alpha}cl(\mathcal{V}) \sqsubseteq NC^{\alpha}cl(\mathcal{U}^c) = \mathcal{U}^c$ (since $\mathcal{U}^c$ is a NC^{α}-CS). Suppose that $\mathcal{V} = \mathfrak{D}$, hence $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathcal{U}^c \Longrightarrow NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathcal{U}^c$. Since $\mathcal{U}\sqcap\mathcal{U}^c = \emptyset_{\mathcal{N}}$, hence $NC^{\alpha}cl(\mathfrak{B})\sqcap NC^{\alpha}cl(\mathfrak{D}) = \emptyset_{\mathcal{N}}$ (since $NC^{\alpha}cl(\mathfrak{B}) \sqsubseteq \mathcal{U}$ and $NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathcal{U}^c$). The other side is clear. The proof (ii) is evident for others. ▪

Theorem 4.6:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a $NC^{S\alpha^*}$-regular if and only if $u \notin \mathcal{Q}$, where $\mathcal{Q}$ is a $NC^{S\alpha}$-CS, there exist two $NC^{S\alpha}$-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}, \mathcal{Q} \sqsubseteq \mathfrak{D}$ and $NC^{S\alpha}cl(\mathfrak{B})\sqcap NC^{S\alpha}cl(\mathfrak{D}) = \emptyset_{\mathcal{N}}$.

(ii) $\mathcal{S}$ is a $NC^{S\alpha^{**}}$-regular if and only if $u \notin \mathcal{Q}$, where $\mathcal{Q}$ is a $NC^{S\alpha}$-CS, there exist two NC-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}, \mathcal{Q} \sqsubseteq \mathfrak{D}$ and $NCcl(\mathfrak{B})\sqcap NCcl(\mathfrak{D}) = \emptyset_{\mathcal{N}}$.

Proof:

The proof of this is similar to that of theorem (4.5). ▪

Remark 4.7:

(i) NC^{α}-regular property is a $NC^{\alpha^{**}}$-topological property.

(ii) NC^{α^*}-regular property is a NC^{α^*}-topological property.

(iii) $NC^{\alpha^{**}}$-regular property is a $NC^{\alpha^{**}}$-topological property.

(iv) $NC^{S\alpha}$-regular property is a $NC^{S\alpha^{**}}$-topological property.

(v) $NC^{S\alpha^*}$-regular property is a $NC^{S\alpha^*}$-topological property.

(vi) $NC^{S\alpha^{**}}$-regular property is a $NC^{S\alpha^{**}}$-topological property.

Proposition 4.8:

(i) If $\mathcal{S} \times \mathcal{J}$ is a $NC^{\alpha^{**}}$-regular, then both $\mathcal{S}$ and $\mathcal{J}$ are $NC^{\alpha^{**}}$-regular spaces.

(ii) If $\mathcal{S} \times \mathcal{J}$ is a $NC^{S\alpha^{**}}$-regular, then both $\mathcal{S}$ and $\mathcal{J}$ are $NC^{S\alpha^{**}}$-regular spaces.

Proof:

(i) Suppose that $\mathcal{S} \times \mathcal{J}$ is a $NC^{\alpha^{**}}$-regular, to prove that $\mathcal{S}$ and $\mathcal{J}$ are $NC^{\alpha^{**}}$-regular spaces.

Let $\mathcal{U}$ and $\mathcal{V}$ be two NC^{α}-OSs in $\mathcal{S}$ and $\mathcal{J}$ containing u and v respectively. Hence $(u, v) \in \mathcal{U} \times \mathcal{V}$ where $\mathcal{U} \times \mathcal{V}$ is a NC^{α}-OS in $\mathcal{S} \times \mathcal{J}$ (by theorem (2.7)). Hence there exists NC-OS $\mathcal{K}$ in $\mathcal{S} \times \mathcal{J}$ such that $(u, v) \in \mathcal{K} \sqsubseteq NCcl(\mathcal{K}) \sqsubseteq \mathcal{U} \times \mathcal{V}$ (since $\mathcal{S} \times \mathcal{J}$ is a $NC^{\alpha^{**}}$-regular). Then there exist two NC-OSs $\mathfrak{B}$ and $\mathfrak{D}$ in $\mathcal{S}$ and $\mathcal{J}$ such that $(u, v) \in \mathfrak{B} \times \mathfrak{D} \sqsubseteq NCcl(\mathfrak{B} \times \mathfrak{D}) = NCcl(\mathfrak{B}) \times NCcl(\mathfrak{D}) \sqsubseteq \mathcal{U} \times \mathcal{V}$. Hence $u \in \mathfrak{B} \sqsubseteq NCcl(\mathfrak{B}) \sqsubseteq \mathcal{U} \Longrightarrow \mathcal{S}$ is a $NC^{\alpha^{**}}$-regular space. Also, $v \in \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathcal{V} \Longrightarrow \mathcal{J}$ is a $NC^{\alpha^{**}}$-regular space. The proof (ii) is evident for others. ▪

Theorem 4.9:

If $(\mathcal{S}, \zeta)$ is a $NC^{\alpha^{*}}$-regular ($NC^{\alpha^{**}}$-regular respectively), then $\zeta = NC^{\alpha}O(\mathcal{S})$.

Proof:

It is clear that $\zeta \sqsubseteq NC^{\alpha}O(\mathcal{S})$. Let $\mathfrak{B}$ be a NC^{α}-OS in $\mathcal{S}$ containing u. Then there exists a NC^{α}-OS $\mathfrak{D}$ containing u such that $u \in \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$ ($\mathcal{S}$ is a $NC^{\alpha^{*}}$-regular). Therefore $NC^{\alpha}int(\mathfrak{D}) \sqsubseteq NC^{\alpha}int(NC^{\alpha}cl(\mathfrak{D})) \sqsubseteq \mathfrak{B}$. Thus $u \in \mathfrak{D} \sqsubseteq NCcl(NCint(\mathfrak{D})) \sqsubseteq \mathfrak{B}$ (since by proposition (2.9)). Hence $\mathfrak{B}$ is a NC-OS $\Longrightarrow NC^{\alpha}O(\mathcal{S}) \sqsubseteq \zeta$. Therefore $\zeta = NC^{\alpha}O(\mathcal{S})$. ▪

Proposition 4.10:

(i) If $\rho: \mathcal{S} \longrightarrow \mathcal{J}$ is a NC^{α}-CF and $\mathcal{S}$ is a $NC^{\alpha^{*}}$-regular, then ρ is a NC-CF.

(ii) If $\rho: \mathcal{S} \longrightarrow \mathcal{J}$ is a NC^{α}-CF and $\mathcal{J}$ is a $NC^{\alpha^{*}}$-regular, then ρ is a $NC^{\alpha^{*}}$-CF.

(iii) If $\rho: \mathcal{S} \longrightarrow \mathcal{J}$ is a $NC^{\alpha^{*}}$-CF and $\mathcal{S}$ is a $NC^{\alpha^{*}}$-regular, then ρ is a $NC^{\alpha^{**}}$-CF.

Proof:

(i) Let $\rho: \mathcal{S} \longrightarrow \mathcal{J}$ be a NC^{α}-CF, to prove that ρ is a NC-CF.

Let $\mathfrak{B}$ be a NC-OS in $\mathcal{J}$, then $\rho^{-1}(\mathfrak{B})$ is a NC^{α}-OS in $\mathcal{S}$ (since ρ is a NC^{α}-CF). But $\mathcal{S}$ is a $NC^{\alpha^{*}}$-regular space (by hypothesis). Hence $\rho^{-1}(\mathfrak{B})$ is a NC^{α}-OS in $\mathcal{S}$ (since by theorem (4.9)). Therefore ρ is a NC-CF. The proofs (ii), (iii) are evident for others. ▪

Definition 4.11:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) NC^{α}-T_3-space if $\mathcal{S}$ is a NC^{α}-T_1-space and NC^{α}-regular space.

(ii) $NC^{\alpha^{*}}$-T_3-space if $\mathcal{S}$ is NC^{α}-T_1-space and $NC^{\alpha^{*}}$-regular space.

(iii) $NC^{\alpha^{**}}$-T_3-space if $\mathcal{S}$ is NC^{α}-T_1-space and $NC^{\alpha^{**}}$-regular space.

Definition 4.12:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) $NC^{S\alpha}$-T_3-space if $\mathcal{S}$ is a $NC^{S\alpha}$-T_1-space and $NC^{S\alpha}$-regular space.

(ii) $NC^{S\alpha^{*}}$-T_3-space if $\mathcal{S}$ is $NC^{S\alpha}$-T_1-space and $NC^{S\alpha^{*}}$-regular space.

(iii) $NC^{S\alpha^{**}}$-T_3-space if $\mathcal{S}$ is $NC^{S\alpha}$-T_1-space and $NC^{S\alpha^{**}}$-regular space.

Remark 4.13:

(i) $NC^{\alpha^{*}}$-T_3 ($NC^{S\alpha^{*}}$-T_3 respectively) property is a $NC^{\alpha^{*}}$ ($NC^{S\alpha^{*}}$ respectively) topological property.

(ii) $NC^{\alpha^{**}}$-T_3 ($NC^{S\alpha^{**}}$-T_3 respectively) property is a $NC^{\alpha^{**}}$ ($NC^{S\alpha^{**}}$ respectively) topological property.

Remark 4.14:

(i) Every NC-T_3-space is a NC^{α}-T_3-space and $\mathrm{NC}^{S\alpha}$-T_3-space.

(ii) Every NC^{α}-T_3-space is a $\mathrm{NC}^{S\alpha}$-T_3-space.

(iii) Every $\mathrm{NC}^{\alpha^{**}}$-$T_3$-space ($\mathrm{NC}^{S\alpha^{**}}$-$T_3$-space respectively) is a $\mathrm{NC}^{\alpha^{*}}$-T_3-space ($\mathrm{NC}^{S\alpha^{*}}$-$T_3$-space, respectively).

(iv) Every $\mathrm{NC}^{\alpha^{*}}$-T_3-space ($\mathrm{NC}^{S\alpha^{*}}$-$T_3$-space respectively) is a NC^{α}-T_2-space ($\mathrm{NC}^{S\alpha}$-T_2-space, respectively).

Proposition 4.15:

$\mathcal{S} \times \mathcal{I}$ is a $\mathrm{NC}^{\alpha^{**}}$-$T_3$-space if and only if both $\mathcal{S}$ and $\mathcal{I}$ are $\mathrm{NC}^{\alpha^{**}}$-$T_3$-spaces.

Proof:

Follow directly from proposition (3.12) part (i) and proposition (4.8) part (i). ▪

5. Some New Concepts of Weakly Neutrosophic Crisp Normality

Definition 5.1:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) NC^{α}-normal ($\mathrm{NC}^{S\alpha}$-normal respectively) if for every two NC-CSs $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that $\mathcal{Q}_1 \sqcap \mathcal{Q}_2 = \emptyset_{\mathcal{N}}$There exist two NC^{α}-OSs ($\mathrm{NC}^{S\alpha}$-OSs respectively) $\mathfrak{B}$ and $\mathfrak{D}$ such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{B}$ and $\mathcal{Q}_2 \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

(ii) $\mathrm{NC}^{\alpha^{*}}$-normal ($\mathrm{NC}^{S\alpha^{*}}$-normal respectively) if for every two NC^{α}-CSs ($\mathrm{NC}^{S\alpha}$-CSs respectively) $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that $\mathcal{Q}_1 \sqcap \mathcal{Q}_2 = \emptyset_{\mathcal{N}}$There exist two NC^{α}-OSs ($\mathrm{NC}^{S\alpha}$-OSs respectively) $\mathfrak{B}$ and $\mathfrak{D}$ such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{B}$ and $\mathcal{Q}_2 \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

(iii) $\mathrm{NC}^{\alpha^{**}}$-normal ($\mathrm{NC}^{S\alpha^{**}}$-normal respectively) if for every two NC^{α}-CSs ($\mathrm{NC}^{S\alpha}$-CSs respectively) $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that $\mathcal{Q}_1 \sqcap \mathcal{Q}_2 = \emptyset_{\mathcal{N}}$, there exist two NC-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{B}$ and $\mathcal{Q}_2 \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

Remark 5.2:

The following diagram shows the relation between the different types of weakly NC-normal and weakly NC^{α}-normal ($\mathrm{NC}^{S\alpha}$-normal respectively) spaces:

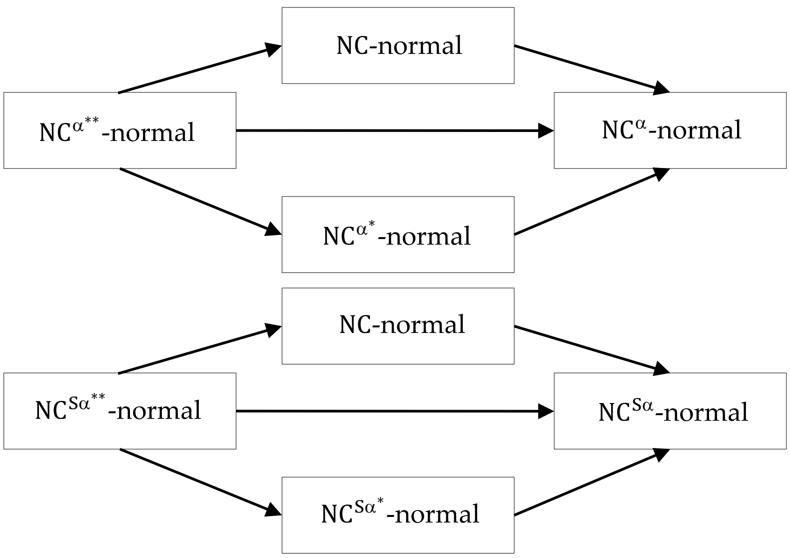


Fig. 5.1

Theorem 5.3:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a NC^{α}-normal space if and only if for every NC-CS $\mathcal{Q}$ and every NC-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists NC^{α}-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(ii) $\mathcal{S}$ is a NC^{α^*}-normal space if and only if for every NC^{α}-CS $\mathcal{Q}$ and every NC^{α}-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists NC^{α}-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(iii) $\mathcal{S}$ is a $NC^{\alpha^{**}}$-normal space if and only if for every NC^{α}-CS $\mathcal{Q}$ and every NC^{α}-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists NC-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

Proof:

(i) $\Rightarrow$ Let $\mathcal{S}$ be a NC^{α}-normal space. Let $\mathcal{Q} \sqsubseteq \mathfrak{B}$, where $\mathcal{Q}$ is a NC-CS and $\mathfrak{B}$ is a NC-OS $\Longrightarrow \mathcal{Q} \sqcap \mathfrak{B}^c = \emptyset_{\mathcal{N}}$, where $\mathfrak{B}^c$ is a NC-CS. Hence there exist two NC^{α}-OSs $\mathfrak{D}_1, \mathfrak{D}_2$ such that $\mathcal{Q} \sqsubseteq \mathfrak{D}_1$ and $\mathfrak{B}^c \sqsubseteq \mathfrak{D}_2$ and $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}}$ (since $\mathcal{S}$ is a NC^{α}-normal space). Therefore $\mathcal{Q} \sqsubseteq \mathfrak{D}_1 \sqsubseteq {\mathfrak{D}_2}^c \sqsubseteq \mathfrak{B} \Longrightarrow NC^{\alpha}cl(\mathcal{Q}) \sqsubseteq NC^{\alpha}cl(\mathfrak{D}_1) \sqsubseteq NC^{\alpha}cl({\mathfrak{D}_2}^c) = {\mathfrak{D}_2}^c \sqsubseteq \mathfrak{B}$. Hence $\mathcal{Q} \sqsubseteq \mathfrak{D}_1 \sqsubseteq NC^{\alpha}cl(\mathfrak{D}_1) \sqsubseteq \mathfrak{B}$, where $\mathfrak{D}_1$ is a NC^{α}-OS in $\mathcal{S}$.

$\Leftarrow$ To prove $\mathcal{S}$ is a NC^{α}-normal space. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be NC-CSs in $\mathcal{S}$ such that $\mathcal{Q}_1 \sqcap \mathcal{Q}_2 = \emptyset_{\mathcal{N}}$. Hence $\mathcal{Q}_1 \sqsubseteq {\mathcal{Q}_2}^c$, where ${\mathcal{Q}_2}^c$ is a NC-OS. Then there exists a NC^{α}-OS $\mathfrak{D}$ such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq {\mathcal{Q}_2}^c$ (by hypothesis). Hence $\mathcal{Q}_1 \sqsubseteq \mathfrak{D}$, $\mathcal{Q}_2 \sqsubseteq (NC^{\alpha}cl(\mathfrak{D}))^c$. On the other hand $NC^{\alpha}cl(\mathfrak{D}) \sqcap \left(NC^{\alpha}cl(\mathfrak{D})\right)^c = \emptyset_{\mathcal{N}}$. Hence $\mathfrak{D} \sqcap \left(NC^{\alpha}cl(\mathfrak{D})\right)^c = \emptyset_{\mathcal{N}}$ (since $\mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D})$). Therefore $\mathcal{S}$ is a NC^{α}-normal space. The proofs (ii), (iii) are evident for others. ▪

Theorem 5.4:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a $NC^{S\alpha}$-normal space if and only if for every NC-CS $\mathcal{Q}$ and every NC-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists $NC^{S\alpha}$-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NC^{S\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(ii) $\mathcal{S}$ is a $NC^{S\alpha^*}$-normal space if and only if for every $NC^{S\alpha}$-CS $\mathcal{Q}$ and every $\mathfrak{B}$ $NC^{S\alpha}$-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists NC^{α}-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NC^{S\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(iii) $\mathcal{S}$ is a $NC^{S\alpha^{**}}$-normal space if and only if for every $NC^{S\alpha}$-CS $\mathcal{Q}$ and every $NC^{S\alpha}$-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists NC-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

Proof:

(i) $\Rightarrow$ Let $\mathcal{S}$ be a $NC^{S\alpha}$-normal space. Let $\mathcal{Q} \sqsubseteq \mathfrak{B}$, where $\mathcal{Q}$ is a NC-CS and $\mathfrak{B}$ is a NC-OS $\Longrightarrow \mathcal{Q} \sqcap \mathfrak{B}^c = \emptyset_{\mathcal{N}}$, where $\mathfrak{B}^c$ is a NC-CS. Hence there exist two $NC^{S\alpha}$-OSs $\mathfrak{D}_1, \mathfrak{D}_2$ such that $\mathcal{Q} \sqsubseteq \mathfrak{D}_1$ and $\mathfrak{B}^c \sqsubseteq \mathfrak{D}_2$ and $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}}$ (since $\mathcal{S}$ is a $NC^{S\alpha}$-normal space). Therefore $\mathcal{Q} \sqsubseteq \mathfrak{D}_1 \sqsubseteq {\mathfrak{D}_2}^c \sqsubseteq \mathfrak{B} \Longrightarrow NC^{S\alpha}cl(\mathcal{Q}) \sqsubseteq NC^{S\alpha}cl(\mathfrak{D}_1) \sqsubseteq NC^{S\alpha}cl({\mathfrak{D}_2}^c) \sqsubseteq NC^{S\alpha}cl(\mathfrak{B})$. Hence $\mathcal{Q} \sqsubseteq \mathfrak{D}_1 \sqsubseteq NC^{S\alpha}cl(\mathfrak{D}_1) \sqsubseteq \mathfrak{B}$, where $\mathfrak{D}_1$ is a $NC^{S\alpha}$-OS in $\mathcal{S}$.

$\Leftarrow$ To prove $\mathcal{S}$ is a $NC^{S\alpha}$-normal space. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be NC-CSs in $\mathcal{S}$, such that $\mathcal{Q}_1 \sqcap \mathcal{Q}_2 = \emptyset_{\mathcal{N}}$. Hence $\mathcal{Q}_1 \sqsubseteq {\mathcal{Q}_2}^c$, where ${\mathcal{Q}_2}^c$ is a NC-OS. Then there exists a $NC^{S\alpha}$-OS $\mathfrak{D}$ such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{D} \sqsubseteq NC^{S\alpha}cl(\mathfrak{D}) \sqsubseteq {\mathcal{Q}_2}^c$ (by hypothesis). Hence $\mathcal{Q}_1 \sqsubseteq \mathfrak{D}$, $\mathcal{Q}_2 \sqsubseteq (NC^{S\alpha}cl(\mathfrak{D}))^c$. On the other hand $NC^{S\alpha}cl(\mathfrak{D}) \sqcap \left(NC^{S\alpha}cl(\mathfrak{D})\right)^c = \emptyset_{\mathcal{N}}$. Hence $\mathfrak{D} \sqcap \left(NC^{S\alpha}cl(\mathfrak{D})\right)^c = \emptyset_{\mathcal{N}}$ (since $\mathfrak{D} \sqsubseteq NC^{S\alpha}cl(\mathfrak{D})$). Therefore $\mathcal{S}$ is a $NC^{S\alpha}$-normal space. The proofs (ii), (iii) are evident for others. ▪

Remark 5.5:

(i) NC^{α}-normal property is a $NC^{\alpha^{**}}$-topological property.

(ii) NC^{α^*}-normal property is a NC^{α^*}-topological property.

(iii) $NC^{\alpha^{**}}$-normal property is a $NC^{\alpha^{**}}$-topological property.

(iv) $NC^{S\alpha}$-normal property is a $NC^{S\alpha^*}$-topological property.

(v) $NC^{S\alpha^*}$-normal property is a $NC^{S\alpha^*}$-topological property.

(vi) $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-normal property is a $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-topological property.

Proposition 5.6:

(i) If $\mathcal{S} \times \mathcal{J}$ is a $\mathrm{NC}^{\alpha^{**}}$-normal space, then both $\mathcal{S}$ and $\mathcal{J}$ are $\mathrm{NC}^{\alpha^{**}}$-normal spaces.

(ii) If $\mathcal{S} \times \mathcal{J}$ is a $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-normal space, then both $\mathcal{S}$ and $\mathcal{J}$ are $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-normal spaces.

Proof:

(i) Suppose that $\mathcal{S} \times \mathcal{J}$ is a $\mathrm{NC}^{\alpha^{**}}$-normal space, to prove that $\mathcal{S}$ and $\mathcal{J}$ are $\mathrm{NC}^{\alpha^{**}}$-normal spaces. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two NC^{α}-OSs in $\mathcal{S}$ and $\mathcal{J}$ respectively, such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{B}_1$ and $\mathcal{Q}_2 \sqsubseteq \mathfrak{B}_2$, where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are NC^{α}-CSs in $\mathcal{S}$ and $\mathcal{J}$ respectively. Hence $\mathcal{Q}_1 \times \mathcal{Q}_2 \sqsubseteq \mathfrak{B}_1 \times \mathfrak{B}_2$ where $\mathcal{Q}_1 \times \mathcal{Q}_2$ is a NC^{α}-CS and $\mathfrak{B}_1 \times \mathfrak{B}_2$ is a NC^{α}-OS in $\mathcal{S} \times \mathcal{J}$ (by theorem (2.7) and corollary (2.8)). But $\mathcal{S} \times \mathcal{J}$ is a $\mathrm{NC}^{\alpha^{**}}$-normal space. Then there exists a NC-OS say $\mathfrak{D}$ in $\mathcal{S} \times \mathcal{J}$ such that $\mathcal{Q}_1 \times \mathcal{Q}_2 \sqsubseteq \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathfrak{B}_1 \times \mathfrak{B}_2$. Then there exist NC-OSs $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathcal{S} \times \mathcal{J}$ such that $\mathcal{Q}_1 \times \mathcal{Q}_2 \sqsubseteq \mathcal{U}_1 \times \mathcal{U}_2 \sqsubseteq NCcl(\mathcal{U}_1 \times \mathcal{U}_2) = NCcl(\mathcal{U}_1) \times NCcl(\mathcal{U}_2) \sqsubseteq \mathfrak{B}_1 \times \mathfrak{B}_2$. Hence $\mathcal{Q}_1 \sqsubseteq \mathcal{U}_1 \sqsubseteq NCcl(\mathcal{U}_1) \sqsubseteq \mathfrak{B}_1 \Longrightarrow \mathcal{S}$ is a $\mathrm{NC}^{\alpha^{**}}$-normal space. Also, $\mathcal{Q}_2 \sqsubseteq \mathcal{U}_2 \sqsubseteq NCcl(\mathcal{U}_2) \sqsubseteq \mathfrak{B}_2 \Longrightarrow \mathcal{J}$ is a $\mathrm{NC}^{\alpha^{**}}$-normal space. The proof (ii) is evident for others. ▪

Definition 5.7:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) NC^{α}-T_4-space if $\mathcal{S}$ is a NC^{α}-T_1-space and NC^{α}-normal space.

(ii) $\mathrm{NC}^{\alpha^{*}}$-T_4-space if $\mathcal{S}$ is NC^{α}-T_1-space and $\mathrm{NC}^{\alpha^{*}}$-normal space.

(iii) $\mathrm{NC}^{\alpha^{**}}$-$T_4$-space if $\mathcal{S}$ is NC^{α}-T_1-space and $\mathrm{NC}^{\alpha^{**}}$-normal space.

Definition 5.8:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_4$-space if $\mathcal{S}$ is a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space and $\mathrm{NC}^{\mathrm{S}\alpha}$-normal space.

(ii) $\mathrm{NC}^{\mathrm{S}\alpha^{*}}$-$T_4$-space if $\mathcal{S}$ is $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space and $\mathrm{NC}^{\mathrm{S}\alpha^{*}}$-normal space.

(iii) $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-$T_4$-space if $\mathcal{S}$ is $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space and $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-normal space.

Remark 5.9:

(i) NC^{α}-T_4 ($\mathrm{NC}^{\mathrm{S}\alpha}$-$T_4$ respectively) property is a $\mathrm{NC}^{\alpha^{**}}$ ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$ respectively) topological property.

(ii) $\mathrm{NC}^{\alpha^{*}}$-T_4 ($\mathrm{NC}^{\mathrm{S}\alpha^{*}}$-$T_4$ respectively) property is a $\mathrm{NC}^{\alpha^{*}}$ ($\mathrm{NC}^{\mathrm{S}\alpha^{*}}$ respectively) topological property.

(iii) $\mathrm{NC}^{\alpha^{**}}$-$T_4$ ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-$T_4$ respectively) property is a $\mathrm{NC}^{\alpha^{**}}$ ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$ respectively) topological property.

Remark 5.10:

(i) Every NC-T_4-space is a NC^{α}-T_4-space and $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_4$-space.

(ii) Every NC^{α}-T_4-space is a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_4$-space.

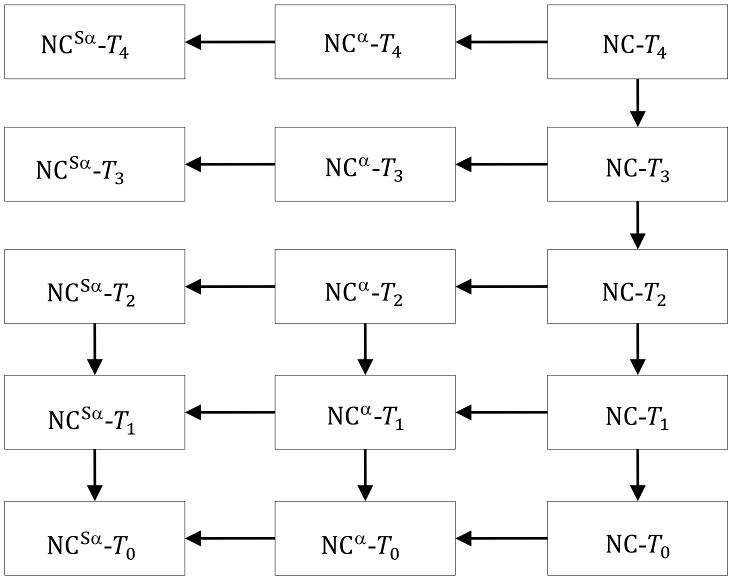
(iii) Every $\mathrm{NC}^{\alpha^{**}}$-$T_4$-space is a $\mathrm{NC}^{\alpha^{*}}$-T_4-space and $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_4$-space.

(iv) Every $\mathrm{NC}^{\alpha^{*}}$-T_4-space ($\mathrm{NC}^{\mathrm{S}\alpha^{*}}$-$T_4$-space respectively) is a $\mathrm{NC}^{\alpha^{*}}$-T_3-space ($\mathrm{NC}^{\mathrm{S}\alpha^{*}}$-$T_3$-space, respectively).

(v) Every $\mathrm{NC}^{\alpha^{**}}$-$T_4$-space ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-$T_4$-space respectively) is a $\mathrm{NC}^{\alpha^{**}}$-$T_3$-space ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-$T_3$-space, respectively).

Remark 5.11:

The following diagram explains the relationships between usual NC-separation axioms, NC^{α}-separation axioms and $\mathrm{NC}^{\mathrm{S}\alpha}$-separation axioms:



Also, we have the following diagram:

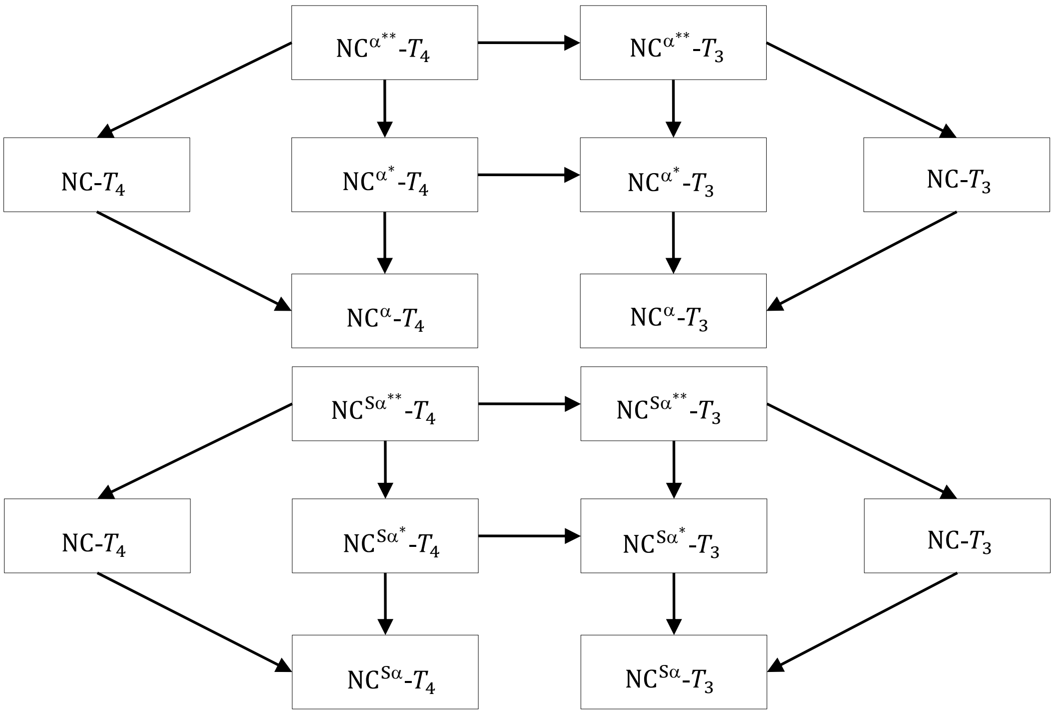


Fig. 5.2

6. Conclusions

We have provided some new concepts of weakly neutrosophic crisp separation axioms. Some characterizations have been provided to illustrate how far topological structures are conserved by the new neutrosophic crisp notion defined. Furthermore, some new concepts of weakly neutrosophic crisp regularity are also studied. The study demonstrated some new concepts of weakly neutrosophic crisp normality and proved some of their related attributes.

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Novel Heuristic for New Pentagonal Neutrosophic Travelling Salesman Problem

Souhail Dhouib *

OLID laboratory,

Higher Institute of Industrial Management,

University of Sfax, Tunisia

* Correspondence: dhouib.matrix@gmail.com.

Abstract: This paper presents a new variant of Travelling Salesman Problem (TSP) and its first resolution. In literature there is not any research work that has presented the TSP under pentagonal fuzzy neutrosophic environment yet. TSP is a critical issue for manufacturing companies where all cities need to be visited only once except the starting city with a minimal cost. In real life, information provided (cost, time … etc.) are generally uncertain, indeterminate or inconsistent that's why in this paper parameters of the TSP are presented as neutrosophic pentagonal fuzzy numbers. To solve this problem, the novel heuristic Dhouib-Matrix-TSP1 (DM-TSP1) is applied using a ranking function in order to transform the fuzzy set to crisp data and range function to select cities. To prove the efficiency of the proposed DM-TSP1 in solving the new variant of TSP, we create novel benchmark instances. Then, a stepwise application of DM-TSP1 is illustrated in details.

Keywords: Fuzzy Optimization Techniques; Neutrosophic Applications; Travelling Salesman Problem; Operational Research; Combinatorial optimization; Dhouib-Matrix Optimization Methods, Dhouib-Matrix-TSP1.

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1. Introduction

In real life, data of industrial companies are most of the time uncertain. That's why these data can be suitably presented by the neutrosophic concept in which the imprecision, the uncertainty and the indetermination are flexibly explored. This philosophy was firstly announced by Smarandache in [1] via three membership functions: Truth (T), Indeterminacy (I) and Falsity (F) with values belonging to]-0,1+[.

In literature, few research papers deal with combinatorial optimization under pentagonal neutrosophic number. In fact, Chakraborty *et al.* studied the Transportation Problem under single value pentagonal neutrosophic numbers for all parameters (supply, demand and transportation cost) in [2]. Also, Das and Chakraborty optimized the linear programming problem with pentagonal neutrosophic environment [3]. Radhika and Prakash considered the Assignment Problems with pentagonal neutrosophic number using a new magnitude ranking function for defuzzification [4]. Das unraveled the Transportation Problem where all parameters are presented using pentagonal neutrosophic numbers [5]. Then, Chakraborty hackled the networking problem in single valued pentagonal neutrosophic environment and introduced a new score function for defuzzification [6]. In addition, Chakraborty *et al.* studied the mobile communication system under pentagonal neutrosophic domain with multi-criteria group decision-making problem [7]. Besides, Chakraborty designed a job-sequencing model in pentagonal neutrosophic area [8]. Kane *et al.* involved the pentagonal and hexagonal fully fuzzy Transportation Problems [9].

To the best of our knowledge, in literature there is no research work which solved the Travelling Salesman Problem (TSP) under pentagonal fuzzy neutrosophic environment. In fact, many applications of TSP with five numbers of variable for each of the three components *T*, *I* and *F* can be found in real life such as the useful of pentagonal membership functions under multi objective environment. Also, the representation of verbal phrase with five different information and even the dynamic variation of the information with time can be presented by five membership functions.

From the above discussion carried on pentagonal neutrosophic problems, there are no current methods for solving pentagonal TSP under Neutrosophic condition. Thus, we generate new benchmark instances for pentagonal TSP with adapting the novel heuristic Dhouib-Matrix-TSP1 (DM-TSP1) to solve this problem. Correspondingly, this paper presents also the first application of DM-TSP1 on pentagonal domain.

Hence, this paper supports the theory and practical efficiency with several novel contributions which can be enumerated as follows:

- First resolution of TSP under pentagonal neutrosophic domain
- Introduce new instances for the pentagonal neutrosophic TSP
- Enhancing the novel heuristic DM-TSP1
- Step wise application of DM-TSP1

This paper is structured as follows: section 2 introduces one of the most important problems in operational research: the TSP. Section 3 presents the concept of the pentagonal neutrosophic environment. Section 4 presents the proposed novel heuristic DM-TSP1. Section 5 illustrates several numerical examples in order to clarify the application of optimization technique DM-TSP1. Finally, section 6 concludes the manuscript with the presentation of the future works.

2. The Travelling Salesman Problem

Materials and Methods should be described with sufficient details to allow others to replicate and build on published results. Please note that publication of your manuscript implicates that you must make all materials, data, computer code, and protocols associated with the publication available to readers. Please disclose at the submission stage any restrictions on the availability of materials or information. New methods and protocols should be described in detail while well-established methods can be briefly described and appropriately cited.

In real-world, the Travelling Salesman Problem (TSP) is extensively used. It deals with generating the shortest round between all nodes (cities, customers, suppliers, … etc.) namely the Hamiltonian cycle: each node is visited only once except the starting node which will be also the last visited node. The TSP is mathematically formulated as described in Equation 1.

Minimize:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij} x_{ij} \tag{1}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,...,n$$
$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,...,n \qquad (2)$$
$$x_{ij} = 0 \text{ or } 1, \ i = 1,...,n, \ \ j = 1,...,n$$

The binary variable x_{ij} is used to indicate either city i is connected to city j (then $x_{ij} = 1$) or city *i* and *j* are not connected ($x_{ij} = 0$). The parameter t represents the time (cost, distance, … etc.) while t_{ij} represents the time between city i and city j.

Panwar and Deep proposed the Grey Wolf metaheuristic for the symmetric TSP [10]. Gunduz and Aslan developed the stochastic Jaya algorithm in order to solve the TSP [11].

Mosayebi *et al.* generated a new type of hybrid TSP with Scheduling Problem in order to minimize the time of completion of the last job [12]. Wang and Han combined the Symbiotic Organisms Search with the Ant Colony Optimization algorithms to optimize the standard TSP [13]. Krishna *et al.* designed a new optimization method for the TSP namely the Spotted Hyena-based Rider Optimization by integrating the Rider Optimization with the Spotted Hyena Optimizer methods [14].

Çakir *et al.* integrated the Dijkstra algorithm with the Minimum Vertex Degree method in order to find the minimal transportation network [15]. Hu *et al.* designed a Bidirectional Graph Neural Network as an element of Deep Learning to solve the TSP [16]. Pandiri and Singh adapted the Artificial Bee Colony metaheuristic for the generalized covering TSP [17]. Luo *et al.* developed a Multi-start Tabu Search metaheuristic for Multi-visit TSP with Multi-drones [18]. Cavani *et al.* used the Branch-and-Cut technique for TSP with multiple drones for last-mile delivery [19]. Pereira *et al.* integrated the Branch-and-Cut with Valid Inequalities method for pickup and delivery TSP with multiple stacks [20]. Huerta *et al.* proposed a new spatial representation of nodes in TSP and used the Anytime Automatic Algorithm [21]. Hougardy *et al.* computed for the metric TSP an approximative ratio of the 2-Opt method [22]. Baniasadi *et al.* presented an application on two modern logistic problems with a description of how to transform the clustered generalized TSP to classical TSP [23]. Chen *et al.* introduced the Branch-and-Price algorithm for a multiple TSP [24].

3. Preliminaries

Several definitions and basic concepts are presented in this section in order to introduce the fuzzy and neutrosophic concepts.

Definition 1:

Let X be a space of points with its generic elements denoted by *x*. The neutrosophic set *N* has the form $N = \{\langle x : T_N(x), I_N(x), F_N(x)\rangle, x \in X\}$ where the functions T, I, F: $X \to]^-0,1^+[$ verifying the condition $^-0 \le T_N(x) + I_N(x) + F_N(x) \le 3^+$.

Definition 2:

Obviously the pentagonal neutrosophic number $N^N = \langle (t_1,t_2,t_3,t_4,t_5),(f_1, f_2, f_3, f_4, f_5),(i_1,i_2,i_3,i_4,i_5);(p,q,r)\rangle$, with $p,q,r \in [0,1]$, presents three membership functions: Truth (T), Indeterminacy (I) and Falsity (F) defined by:

$$T_N(x)=\begin{cases}0,\ x\le t_1\\ \dfrac{p(x-t_1)}{t_2-t_1},t_1\le x\le t_2\\ \dfrac{p(x-t_2)}{t_3-t_2},t_2\le x\le t_3\\ 1,x=t_3\\ \dfrac{p(t_4-x)}{t_4-t_3},t_3\le x\le t_4\\ \dfrac{p(t_5-x)}{t_5-t_4},t_4\le x\le t_5\\ 0,\ x\ge t_5\end{cases}$$

$$I_N(x)=\begin{cases}1,\ x\le i_1\\ \dfrac{i_2-x+q(x-i_1)}{i_2-i_1},i_1\le x\le i_2\\ \dfrac{i_3-x+q(x-i_2)}{i_3-i_2},i_2\le x\le i_3\\ q,x=i_3\\ \dfrac{x-i_3+q(i_4-x)}{i_4-i_3},i_3\le x\le i_4\\ \dfrac{x-i_4+q(i_5-x)}{i_5-i_4},i_4\le x\le i_5\\ 1,\ x\ge i_5\end{cases}$$

$$F_N(x)=\begin{cases}1,\ x\le f_1\\ \dfrac{f_2-x+r(x-f_1)}{f_2-f_1},\ f_1\le x\le f_2\\ \dfrac{f_3-x+r(x-f_2)}{f_3-f_2},\ f_2\le x\le f_3\\ r,x=f_3\\ \dfrac{x-f_3+r(f_4-x)}{f_4-f_3},\ f_3\le x\le f_4\\ \dfrac{x-f_4+r(f_5-x)}{f_5-f_4},\ f_4\le x\le f_5\\ 1,\ x\ge f_5\end{cases}$$

Here is an example of a graphical representation (see Figure 1) for a pentagonal neutrosophic number $N^N=\langle(2,4,8,10,11),(1,2,5,9,11),(3,7,9,12,13);1,0,0\rangle$.

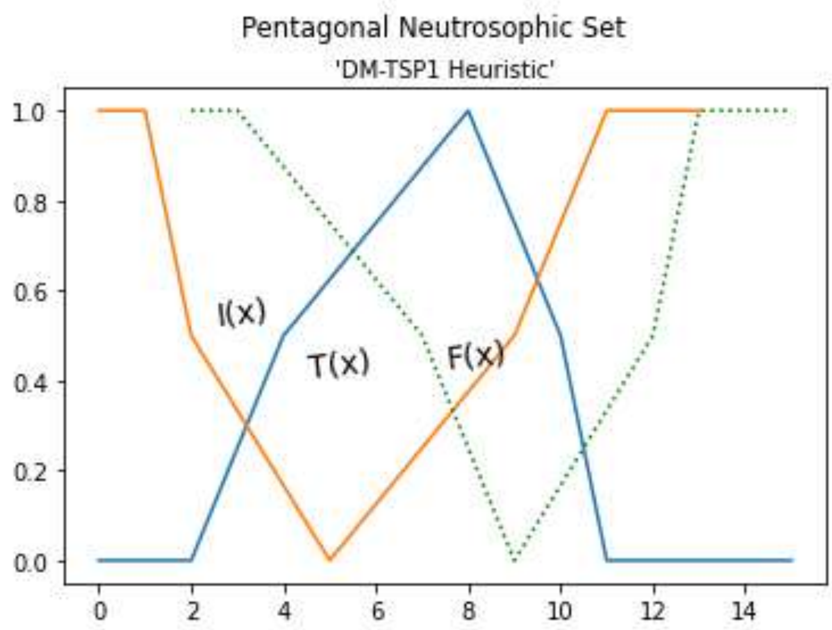

Figure 1. Graphical representation of the neutrosophic pentagonal fuzzy set.

Definition 3:

Let define $N^N = \langle (t_1, t_2, t_3, t_4, t_5); (f_1, f_2, f_3, f_4, f_5); (i_1, i_2, i_3, i_4, i_5) \rangle$ as a pentagonal neutrosophic number. From [2], the score and accuracy functions can be described as follows:

$$S(N^N) = \left(2 + \frac{t_1 + t_2 + t_3 + t_4 + t_5}{5} - \frac{i_1 + i_2 + i_3 + i_4 + i_5}{5} - \frac{f_1 + f_2 + f_3 + f_4 + f_5}{5} \right) \Big/ 3, \quad (3)$$

$$AC(N^N) = \left(\frac{t_1 + t_2 + t_3 + t_4 + t_5}{5} - \frac{f_1 + f_2 + f_3 + f_4 + f_5}{5} \right), \quad (4)$$

Definition 4:

To define the order between several pentagonal neutrosophic numbers, the score and accurate functions can be agreeably used. Let us assume two arbitrary pentagonal neutrosophic numbers N_a^N and N_b^N where:

$$N_a^N = \langle (t_{a1}, t_{a2}, t_{a3}, t_{a4}, t_{a5}), (i_{a1}, i_{a2}, i_{a3}, i_{a4}, i_{a5}), (f_{a1}, f_{a2}, f_{a3}, f_{a4}, f_{a5}) \rangle,$$

$$N_b^N = \langle (t_{b1}, t_{b2}, t_{b3}, t_{b4}, t_{b5}), (i_{b1}, i_{b2}, i_{b3}, i_{b4}, i_{b5}), (f_{b1}, f_{b2}, f_{b3}, f_{b4}, f_{b5}) \rangle.$$

The first step is to compute the score function for each number, so $S(N_a^N)$ and $S(N_b^N)$ verify:

1. if $S(N_a^N) > S(N_b^N)$, then $N_a^N > N_a^N$
2. if $S(N_a^N) < S(N_b^N)$, then $N_a^N < N_a^N$
3. if $S(N_a^N) = S(N_b^N)$, then:
   a. if $AC(N_a^N) > AC(N_b^N)$, then $N_a^N > N_a^N$
   b. if $AC(N_a^N) < AC(N_b^N)$, then $N_a^N < N_a^N$
   c. if $AC(N_a^N) = AC(N_b^N)$, then $N_a^N = N_a^N$

Here is a numerical example that illustrates the previous order definition between two pentagonal numbers:

$N_a^N = \langle(4,5,6,7,9),(1,3,5,6,7),(0,1,2,3,4)\rangle$ then $S(N_a^N) = 0.60$

$N_b^N = \langle(5,6,7,8,9),(0,1,2,3,4),(3,4,5,6,7)\rangle$ then $S(N_b^N) = 0.67$

if $S(N_a^N) < S(N_b^N)$ then $N_a^N < N_b^N$

4. The Novel Heuristic Dhouib-Matrix-TSP1 (DM-TSP1)

The novel deterministic heuristic Dhouib-matrix-TSP1 (DM-TSP1) was firstly designed by Dhouib in order to rapidly find an initial basic feasible solution for the TSP [25]. Then, it was followed by a stochastic version entitled Dhouib-Matrix-TSP2 in [26]. Also, an application of those two methods on automobile industry was presented in [27]. Furthermore, an application of these methods on TSP under uncertain environment was illustrated with triangular fuzzy numbers in [28], trapezoidal fuzzy numbers in [29] and octagonal fuzzy numbers in [30]. Moreover, the TSP was solved with DM-TSP1 under intuitionistic environment in [31] and with neutrosophic area in [32,33,34]. Besides, a new metaheuristic entitled Dhouib-Matrix-3 (DM3) was invented in [35] and a novel multi-start metaheuristic namely Dhouib-Matrix-4 (DM4) was introduced in [36].

The heuristic DM-TSP1 is composed of four steps (see Figure 2) where step 1 and step 4 are executed once. Nevertheless, step 2 and step 3 are repeated n iterations (n is the number of cities). DM-TSP1 is characterized by its flexibility to use different descriptive statistical metrics (Dhouib, 2021g). In this current research work, we will use the range (max-min) metric.

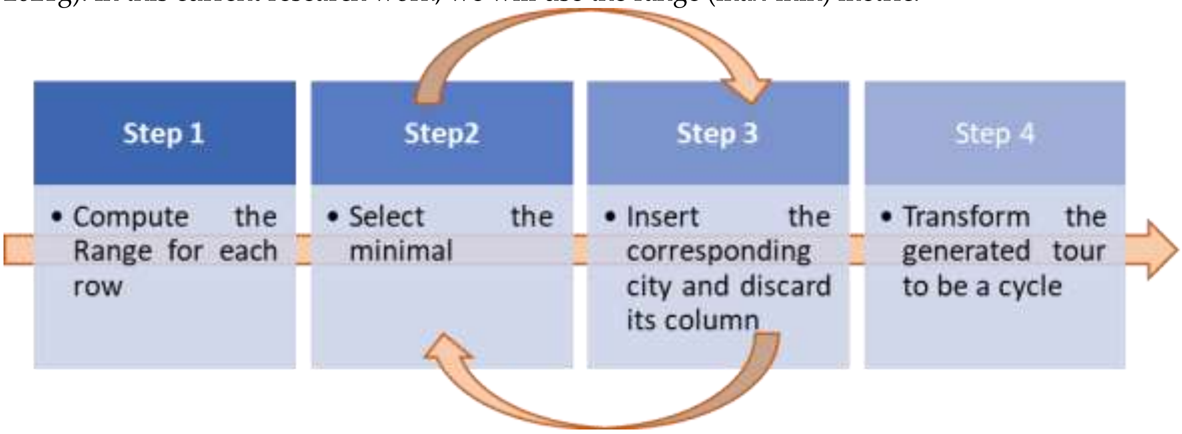

Figure 2. The flowchart of the proposed DM-TSP1.

In this paper, all parameters of the TSP are presented as pentagonal neutrosophic number. So, the function described in Equation 3 is used to convert these numbers into crisp numbers. Besides, the four steps can be started:

Step1: Compute the range function (max-min) for each row and write it on the right-hand side of the matrix. Next, find the minimal range and select its row. Then, select the smallest element in this row which will specify the two first cities x and y to be inserted in the list *List-cities {x, y}*. Finally, discard the respected columns of city x and city y.

Step 2: Find the minimal element for city x and for city y and select the smallest distance which will indicate city z.

Step 3: Add city z to the list *List-cities* and discard its column. Next, if there is no column go to step 4 otherwise go to step 2.

Step 4: Modify the realizable solution in *List-cities* in order to generate a cycle (the starting city in the cycle has to be also the last one). First, to ensure that the starting city will be at the first position, translate all the cities (one by one) before the starting one at the end of the list. Second, duplicate the starting city at the last position.

5. Application of DM-TSP1 heuristic in Neutrosophic Pentagonal Travelling Salesman Problem

This section will describe the stepwise application of the novel heuristic DM-TSP1 on pentagonal neutrosophic TSP. We generate two new instances because in literature there is not any work that solved this problem under pentagonal neutrosophic environment.

5.1. Illustration example 1

Let consider a travel salesman who needs to generate a Hamiltonian cycle between 5 cities namely A, B, C, D and E (each city is visited only once except the starting city which will also be the last visited city). The estimated time between all cities is presented as pentagonal neutrosophic number as presented in table 1.

Table 1. The pentagonal neutrosophic time between 5 cities.

	A	B	C	D	E
A	∞	<5,7,8,9,11; 2,3,4,5,6; 0,1,2,3,4>	<11,13,14,15,16; 2,4,6,8,10; 1,3,5,7,9>	<7,8,9,10,11; 2,3,4,5,6; 0,2,3,4,5>	<11,12,13,14,15; 3,4,6,8,9; 1,2,3,4,5>
B	<5,7,8,9,11; 2,3,4,5,6; 0,1,2,3,4>	∞	<10,11,12,13,14; 1,2,3,4,5; 6,7,8,9,10>	<8,9,10,11,13; 3,5,6,8,9; 1,2,4,5,6>	<5,7,8,9,14; 3,4,5,6,7; 0,1,2,3,4>
C	<11,13,14,15,16; 2,4,6,8,10; 1,3,5,7,9>	<10,11,12,13,14; 1,2,3,4,5; 6,7,8,9,10>	∞	<5,9,11,13,14; 4,6,7,8,9; 1,2,3,4,5>	<7,8,9,14,15; 0,1,2,3,4; 2,3,4,5,6>
D	<7,8,9,10,11; 2,3,4,5,6; 0,2,3,4,5>	<8,9,10,11,13; 3,5,6,8,9; 1,2,4,5,6>	<5,9,11,13,14; 4,6,7,8,9; 1,2,3,4,5>	∞	<8,9,14,15,16 1,2,3,5,7; 2,5,6,7,8>
E	<11,12,13,14,15; 3,4,6,8,9; 1,2,3,4,5>	<5,7,8,9,14; 3,4,5,6,7; 0,1,2,3,4>	<7,8,9,14,15; 0,1,2,3,4; 2,3,4,5,6>	<8,9,14,15,16 1,2,3,5,7; 2,5,6,7,8>	∞

At first, convert the pentagonal neutrosophic number to crisp number through to Equation (3). Figure 3 depicts the generated crisp time matrix.

$$\begin{pmatrix} \infty & 1.33 & 1.60 & 1.40 & 2.00 \\ 1.33 & \infty & 1.00 & 0.80 & 1.20 \\ 1.60 & 1.00 & \infty & 0.87 & 2.20 \\ 1.40 & 0.80 & 0.87 & \infty & 1.73 \\ 2.00 & 1.20 & 2.20 & 1.73 & \infty \end{pmatrix}$$

Figure 3. The crisp time matrix.

Now, DM-TSP1 can start by computing the range function (max-min) for each row (See figure 4).

$$\begin{pmatrix} \infty & 1.33 & 1.60 & 1.40 & 2.00 \\ 1.33 & \infty & 1.00 & \boxed{0.80} & 1.20 \\ 1.60 & 1.00 & \infty & 0.87 & 2.20 \\ 1.40 & 0.80 & 0.87 & \infty & 1.73 \\ 2.00 & 1.20 & 2.20 & 1.73 & \infty \end{pmatrix} \begin{matrix} 0.67 \\ 0.53 \\ 1.33 \\ 0.93 \\ 1.00 \end{matrix}$$

Figure 4. Compute the range (max-min) of each row.

Apparently, the minimal range is 0.53, so we look for the minimal element in the second row: it is 0.80 at position d_{24}. Thus, city 2 and city 4 are inserted in *List-cities* {2-4} and their corresponding columns are discarded (see figure 5).

$$\begin{pmatrix} \infty & 1.33 & 1.60 & 1.40 & 2.00 \\ 1.33 & \infty & 1.00 & 0.80 & 1.20 \\ 1.60 & 1.00 & \infty & 0.87 & 2.20 \\ 1.40 & 0.80 & 0.87 & \infty & 1.73 \\ 2.00 & 1.20 & 2.20 & 1.73 & \infty \end{pmatrix}$$

Figure 5. Discard columns 2 and 4.

Besides, find the smallest element between row 2 and row 4 which is 0.87 at position d_{43}. Then, insert city 3 at the last position (after city 4) in *List-cities* {2-4-3} and discarded its corresponding column (see Figure 6).

$$\begin{pmatrix} \infty & 1.33 & 1.60 & 1.40 & 2.00 \\ 1.33 & \infty & 1.00 & 0.80 & 1.20 \\ 1.60 & 1.00 & \infty & 0.87 & 2.20 \\ 1.40 & 0.80 & 0.87 & \infty & 1.73 \\ 2.00 & 1.20 & 2.20 & 1.73 & \infty \end{pmatrix}$$

Figure 6. Discard column 3.

Next, find the smallest element between row 2 and row 3 which is 1.20 at position d_{52}; then, insert city 5 before city 2 in *List-cities* {5-2-4-3} and discard its corresponding column (see Figure 7).

$$\begin{pmatrix} \infty & 1.33 & 1.60 & 1.40 & 2.00 \\ 1.33 & \infty & 1.00 & 0.80 & 1.20 \\ 1.60 & 1.00 & \infty & 0.87 & 2.20 \\ 1.40 & 0.80 & 0.87 & \infty & 1.73 \\ 2.00 & 1.20 & 2.20 & 1.73 & \infty \end{pmatrix}$$

Figure 7. Discard column 5.

Find the smallest element between row 3 and row 5 which is 1.60 at position d_{13}, insert city 1 after city 3 in *List-cities* {5-2-4-3-1} and discard its corresponding column (see Figure 8).

$$\begin{pmatrix} \infty & 1.33 & 1.60 & 1.40 & 2.00 \\ 1.33 & \infty & 1.00 & 0.80 & 1.20 \\ 1.60 & 1.00 & \infty & 0.87 & 2.20 \\ 1.40 & 0.80 & 0.87 & \infty & 1.73 \\ 2.00 & 1.20 & 2.20 & 1.73 & \infty \end{pmatrix}$$

Figure 8. Discard column 1.

Obviously, all columns are discarded and a tour is generated. The final step is to generate a cycle starting and ending by city 1. So, translate city by city, from the left to the right until city 1 will become at the first position: {1-5-2-4-3}. Finally, add city one at the last position in order to generate a cycle: {1-5-2-4-3-1}. Thus, the optimal solution generated using DM-TSP1 is: $x_{24}=1;\ x_{43}=1;\ x_{52}=1;\ x_{31}=1;\ x_{15}=1$. With a total crisp cost $z=0.80+0.87+1.20+1.60+2.00=6.47$. Consequently, the minimal pentagonal neutrosophic cost is: $z^N =< 40,50,56,62,72;15,23,30,38,44;4,10,17,23,29 >$. The graphical representation of the optimal solution obtained by DM-TSP1 is depicted in Figure 9.

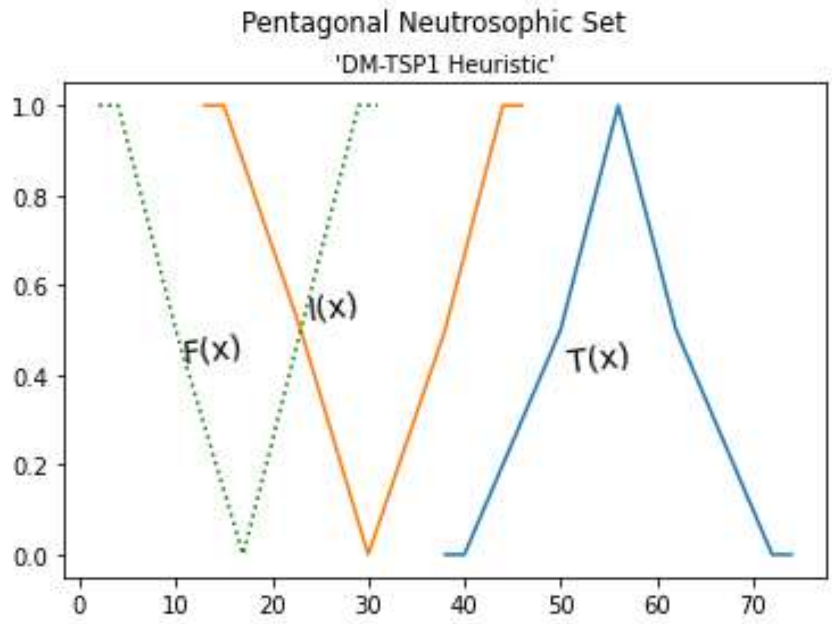

Figure 9. Graphical representation of the optimal neutrosophic solution.

Thus, the decision maker can deduce the minimal pentagonal neutrosophic cost with its truth, indeterminacy and falsity degrees. The truth membership function for the generated solution is denoted by Equation 5.

$$T_N(x)=\begin{cases}0, \ x\leq 40\\ \dfrac{x-40}{50-40}, 40\leq x\leq 50\\ \dfrac{x-50}{56-50}, 50\leq x\leq 56\\ 1, x=56\\ \dfrac{62-x}{62-56}, 56\leq x\leq 62\\ \dfrac{72-x}{72-62}, 62\leq x\leq 72\\ 0, \ x\geq 72\end{cases} \tag{5}$$

Similarly, the indeterminacy membership function is presented by Equation 6.

$$I_N(x)=\begin{cases}1, \ x\leq 15\\ \dfrac{23-x}{23-15}, 15\leq x\leq 23\\ \dfrac{30-x}{30-23}, 23\leq x\leq 30\\ 0, x=30\\ \dfrac{x-30}{38-30}, 30\leq x\leq 38\\ \dfrac{x-38}{44-38}, 38\leq x\leq 44\\ 1, \ x\geq 44\end{cases} \tag{6}$$

Also, the falsity membership function is presented by Equation 7.

$$F_N(x)=\begin{cases} 1, \ x \le 4 \\ \dfrac{10-x}{10-4}, 4 \le x \le 10 \\ \dfrac{17-x}{17-10}, 10 \le x \le 17 \\ r, x=17 \\ \dfrac{x-17}{23-17}, 17 \le x \le 23 \\ \dfrac{x-23}{29-23}, 23 \le x \le 29 \\ 1, \ x \ge 29 \end{cases} \tag{7}$$

3.2. Illustration example 2

Let consider a second numerical example for a travel salesman man who needs to generate a Hamiltonian cycle between 6 cities namely A, B, C, D, E and F (see Table 2).

Table 2. The pentagonal neutrosophic time between 6 cities

	A	**B**	**C**	**D**	**E**	**F**
A	∞	<10,11,12,13,14; 4,5,6,7,9; 2,5,6,8,12>	<19,21,24,27,30; 8,9,14,17,18; 6,7,8,9,10>	<4,5,6,7,8; 0,1,2,3,4; 0,1,1,2,3>	<15,22,23,26,29; 5,7,9,11,15; 5,6,8,9,15>	<7,11,12,16,17; 2,4,5,6,7; 1,5,8,9,11>
B	<10,11,12,13,14; 4,5,6,7,9; 2,5,6,8,12>	∞	<9,10,15,18,19; 5,6,7,8,9; 1,2,6,8,12>	<6,13,14,15,16; 4,5,8,9,11; 0,3,5,6,7>	<1,6,11,16,21; 3,4,5,6,7; 1,2,3,4,5>	<9,16,17,18,25; 3,4,6,11,12; 2,3,4,5,6>
C	<19,21,24,27,30; 8,9,14,17,18; 6,7,8,9,10>	<9,10,15,18,19; 5,6,7,8,9; 1,2,6,8,12>	∞	<9,10,15,18,19; 5,6,7,8,9; 1,2,6,8,12>	<5,7,14,15,16; 5,6,7,8,9; 0,1,2,3,4>	<13,16,17,21,27; 7,9,10,11,12; 2,5,6,7,8>
D	<4,5,6,7,8; 0,1,2,3,4; 0,1,1,2,3>	<6,13,14,15,16; 4,5,8,9,11; 0,3,5,6,7>	<9,10,15,18,19; 5,6,7,8,9; 1,2,6,8,12>	∞	<11,13,15,24,25; 3,4,6,13,14; 2,5,8,11,14>	<14,17,21,22,23; 3,4,6,11,21; 1,3,7,9,16>
E	<15,22,23,26,29; 5,7,9,11,15; 5,6,8,9,15>	<1,6,11,16,21; 3,4,5,6,7; 1,2,3,4,5>	<5,7,14,15,16; 5,6,7,8,9; 0,1,2,3,4>	<11,13,15,24,25; 3,4,6,13,14; 2,5,8,11,14>	∞	<8,9,11,17,19; 2,3,4,10,11; 1,3,4,5,7>
F	<7,11,12,16,17; 2,4,5,6,7; 1,5,8,9,11>	<9,16,17,18,25; 3,4,6,11,12; 2,3,4,5,6>	<13,16,17,21,27; 7,9,10,11,12; 2,5,6,7,8>	<14,17,21,22,23; 3,4,6,11,21; 1,3,7,9,16>	<8,9,11,17,19; 2,3,4,10,11; 1,3,4,5,7>	∞

At first convert the pentagonal neutrosophic number to crisp number using Equation (3). The generated crisp matrix is presented in Figure 10.

$$\begin{pmatrix} \infty & 0.40 & 1.67 & 1.53 & 2.33 & 1.00 \\ 0.40 & \infty & 1.13 & 1.07 & 1.67 & 2.60 \\ 1.67 & 1.13 & \infty & 2.00 & 1.47 & 1.80 \\ 1.53 & 1.07 & 2.00 & \infty & 1.20 & 1.73 \\ 2.33 & 1.67 & 1.47 & 1.20 & \infty & 1.60 \\ 1.00 & 2.60 & 1.80 & 1.73 & 1.60 & \infty \end{pmatrix}$$

Figure 10. The crisp time matrix.

Next, compute the range function (max-min) for each row (see figure 11).

$$\begin{pmatrix} \infty & 0.40 & 1.67 & 1.53 & 2.33 & 1.00 \\ 0.40 & \infty & 1.13 & 1.07 & 1.67 & 2.60 \\ 1.67 & \boxed{1.13} & \infty & 2.00 & 1.47 & 1.80 \\ 1.53 & 1.07 & 2.00 & \infty & 1.20 & 1.73 \\ 2.33 & 1.67 & 1.47 & 1.20 & \infty & 1.60 \\ 1.00 & 2.60 & 1.80 & 1.73 & 1.60 & \infty \end{pmatrix} \begin{matrix} 1.93 \\ 2.20 \\ \boxed{0.87} \\ 0.93 \\ 1.13 \\ 1.60 \end{matrix}$$

Figure 11. Compute the range of each row.

The minimal range is 0.87, so we look for the minimal element in the third row: it is 1.13 at position d_{32}. Thus, city 3 and city 2 are inserted in *List-cities* {3-2} and their corresponding columns are discarded (see Figure 12).

$$\begin{pmatrix} \infty & 0.40 & 1.67 & 1.53 & 2.33 & 1.00 \\ \boxed{0.40} & \infty & 1.13 & 1.07 & 1.67 & 2.60 \\ 1.67 & \boxed{1.13} & \infty & 2.00 & \boxed{1.47} & 1.80 \\ 1.53 & 1.07 & 2.00 & \infty & 1.20 & 1.73 \\ 2.33 & 1.67 & 1.47 & 1.20 & \infty & 1.60 \\ 1.00 & 2.60 & 1.80 & 1.73 & 1.60 & \infty \end{pmatrix}$$

Figure 12. Discard columns 2 and 3.

Besides, find the smallest element between row 3 and row 2 which is 0.40 at position d_{21}. Then, insert city 1 at the last position (after city 2) in *List-cities* {3-2-1} and discarded its corresponding column (see Figure 13).

$$\begin{pmatrix} \infty & 0.40 & 1.67 & 1.53 & 2.33 & \boxed{1.00} \\ 0.40 & \infty & 1.13 & 1.07 & 1.67 & 2.60 \\ 1.67 & \boxed{1.13} & \infty & 2.00 & \boxed{1.47} & 1.80 \\ 1.53 & 1.07 & 2.00 & \infty & 1.20 & 1.73 \\ 2.33 & 1.67 & 1.47 & 1.20 & \infty & 1.60 \\ 1.00 & 2.60 & 1.80 & 1.73 & 1.60 & \infty \end{pmatrix}$$

Figure 13. Discard column 1.

Next, find the smallest element between row 3 and row 1 which is 1.00 at position d_{16}; then, insert city 6 after city 1 in *List-cities* {3-2-1-6} and discard its corresponding column (see Figure 4).

$$\begin{pmatrix} \infty & 0.40 & 1.67 & 1.53 & 2.33 & 1.00 \\ 0.40 & \infty & 1.13 & 1.07 & 1.67 & 2.60 \\ 1.67 & 1.13 & \infty & 2.00 & 1.47 & 1.80 \\ 1.53 & 1.07 & 2.00 & \infty & 1.20 & 1.73 \\ 2.33 & 1.67 & 1.47 & 1.20 & \infty & 1.60 \\ 1.00 & 2.60 & 1.80 & 1.73 & 1.60 & \infty \end{pmatrix}$$

Figure 14. Discard column 6.

Succeeding, find the smallest element between rows 3 and 6 which is 1.47 at position d_{35}; then, insert city 5 before city 3 in *List-cities* {5-3-2-1-6} and discard its corresponding column (see Figure 15).

$$\begin{pmatrix} \infty & 0.40 & 1.67 & 1.53 & 2.33 & 1.00 \\ 0.40 & \infty & 1.13 & 1.07 & 1.67 & 2.60 \\ 1.67 & 1.13 & \infty & 2.00 & 1.47 & 1.80 \\ 1.53 & 1.07 & 2.00 & \infty & 1.20 & 1.73 \\ 2.33 & 1.67 & 1.47 & 1.20 & \infty & 1.60 \\ 1.00 & 2.60 & 1.80 & 1.73 & 1.60 & \infty \end{pmatrix}$$

Figure 15. Discard column 5.

Subsequent, select the smallest element between rows 5 and row 6 which is 1.20 at position d_{54}, insert city 4 before city 5 in *List-cities* {4-5-3-2-1-6} and discard its corresponding column (see Figure 16).

$$\begin{pmatrix} \infty & 0.40 & 1.67 & 1.53 & 2.33 & 1.00 \\ 0.40 & \infty & 1.13 & 1.07 & 1.67 & 2.60 \\ 1.67 & 1.13 & \infty & 2.00 & 1.47 & 1.80 \\ 1.53 & 1.07 & 2.00 & \infty & 1.20 & 1.73 \\ 2.33 & 1.67 & 1.47 & 1.20 & \infty & 1.60 \\ 1.00 & 2.60 & 1.80 & 1.73 & 1.60 & \infty \end{pmatrix}$$

Figure 16. Discard column 4.

The final step is to generate a cycle starting and ending by city 1. So, translate city by city, from the left to the right until city 1 will become at the first position: {1-6-4-5-3-2}. Finally, add city one at the last position in order to generate a cycle: {1-6-4-5-3-2-1}.

Thus, the optimal solution generated using DM-TSP1 is: $x_{32}=1;\ x_{21}=1;\ x_{16}=1;\ x_{53}=1;\ x_{45}=1; x_{64}=1$. With a total crisp cost $z=1.13+0.40+1.00+1.47+1.20+1.73=6.93$. Consequently, the minimal pentagonal neutrosophic cost is: $z^N =< 56,69,89,108,114;22,29,37,53,69;7,21,37,48,69 >$. The graphical representation of the optimal solution obtained by DM-TSP1 is depicted in Figure 17.

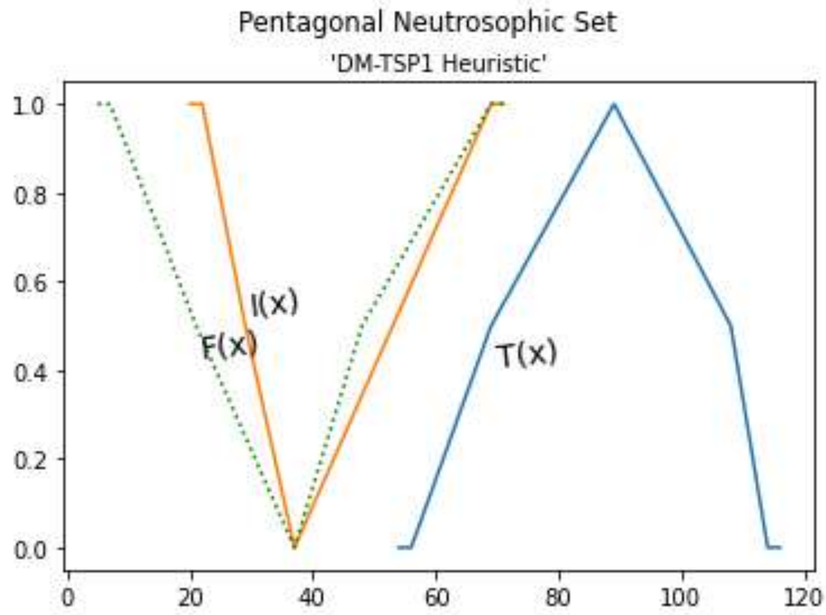

Figure 17. Graphical representation of the optimal neutrosophic solution.

All examples presented in this section are presented as type-1 neutrosophic number. For next research, the proposed constructive heuristic DM-TSP1 can be as well employed to optimize TSP under type-2 neutrosophic number which is an advancement of neutrosophic number presented in [37].

5. Conclusions

The neutrosophic concept is a new philosophy and representing the Travelling Salesman Problem (TSP) under pentagonal neutrosophic environment has not been earlier considered by any other author in the literature. Consequently, in this paper we describe the first resolution of the TSP with pentagonal neutrosophic number using the novel heuristic Dhouib-Matrix-TSP1 (DM-TSP1). Viewing that there is no instance for this problem; so, we generate novel instances. The method DM-TSP1 has been demonstrated by a suitable numerical example. Furthermore, DM-TSP1 easily generates the optimal Hamiltonian cycle after only n iterations, where n is the number of cities. As future work, the DM-TSP1 will be improvised for optimizing the TSP under type-2 neutrosophic domain.

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On Continuity in Minimal Structure Neutrosophic Topological Space

Gour Pal[1], **Binod Chandra Tripathy**[2,*], **and Runu Dhar**[3]

[1]Department of Mathematics, Dasaratha Deb Memorial College, Khowai, 799201, Tripura, India.

[1]E-mail: gourpal74@gmail.com

[2]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.

[2]E-mail: binodtripathy@tripurauniv.in, tripathybc@yahoo.com, tripathybc@rediffmail.com

[3]Department of Applied Mathematics, Maharaja Bir Bikram University, Agartala, 799004, Tripura, India.

[3]E-mail: runu.dhar@gmail.com.

***Correspondence:** tripathybc@yahoo.com; Tel.: (+91 9864087231)

Abstract:

In this paper, we introduce the notion of continuity via neutrosophic minimal structure space. Besides, we introduce the notion of product minimal space in neutrosophic topological space. Further, we investigate some basic properties of N_m–continuity in neutrosophic minimal structure space, such as composition of N_m–continuous functions, product of N_m–continuous functions in product neutrosophic minimal structure space.

Keywords: *Neutrosophic Set*; *Neutrosophic Topology*; *Minimal Structure*; *Neutrosophic Pre-Open Set*; *Neutrosophic Semi-Open Set*; m_N-*Continuity*; *Neutrosophic Product Space*.

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1. Introduction:

The existing theory of Cantor's crisp set theory was not sufficient to handle most of the problems in the real life situation. Uncertainty plays an important role in our everyday life problems. Then, L.A. Zadeh introduced notion of fuzzy set in the year 1965 to overcome the uncertainty situation on considering the membership of truthiness. This is considered as an important generalization of the two valued logic. Still the introduction of fuzzy sets was not sufficient to control the uncertainty. K. Atanaosv in the year 1986 considered non-membership value together with the membership value. He introduced the notion of intuitionistic set. Smarandache [24] realised that the existing tools are not sufficient to find solutions to all types of problems on uncertainty. He then considered the elements with truth membership, false membership and indeterministic membership values, and introduced the notion of neutrosophic set. The concept of neutrosophic set has been applied in many branches of science and technology. Das et al. [5] have studied algebraic operations neutrosophic fuzzy matrices. Das and Tripathy [6] have investigated different properties of neutrosophic multiset topological space. Das et al. [4] have applied the concept of neutrosophic sets for the solution of decision making problems.

The notion of neutrosophic topological space was introduced by Salama and Alblowi [21]. Salama and Alblowi [22] further studied the notion of generalized neutrosophic set and generalized neutrosophic topological space. Later on, Iswaraya and Bageerathi [10], Arokiarani et al. [2], Parimala et al. [17], Parimala et al. [18], Rao and Srinivasa [19], Salama et al. [23], Das and Tripathy [9] and others introduced different notions of open sets in neutrosophic topological space. Recently, Tripathy and Das [27] introduced the notion of b-locally open sets in neutrosophic topological space. The notion of b-locally open sets in bitopological space was introduced and investigated by Tripathy and Sarma [32]. In 2013, Tripathy and Sarma [33] studied the notion of weakly b-continuous functions in bitopological space. In 2020, Das and Tripathy [7] introduced the concept of pairwise neutrosophic b-open set via neutrosophic bitopological space. Later on, Tripathy and Das [26] defined pairwise neutrosophic b-continuous functions via neutrosophic bitopological space.

The notion of minimal structure in topological space was introduced by Maki et al. [12]. Thereafter, it was investigated by many others from different aspects. The notion of minimal structure in a fuzzy topological space was introduce by Alimohammady and Roohi [1], and further investigated by Tripathy and Debnath [28] and others.

Continuity on topological spaces is a very fundamental concept. It plays an important role and has successfully been applied in different areas of research in science and technology. Different types of continuity on topological spaces and fuzzy topological spaces has been investigated by Ray and Tripathy [20], Tripathy and Ray [29-31], Tripathy and Sarma [33] and others.

In this article we introduce the notion of continuity in minimal structure spaces in neutrosophic topological space and investigate its different properties.

The rest of the paper is divided into following sections:

Section 2 is on the preliminaries and definitions. All the existing definitions have been procured in this section those are very useful for the preparation of the main results of this article. Section 3 introduces mappings between neutrosophic sets, and some basic results have been proved. Section 4 is on ccontinuity in neutrosophic minimal structure spaces. Finally, in section 5, we conclude the work done in this article.

2. Materials and Methods (proposed work with more details):

In this section we procure some basic definitions and notations those will be used throughout this article.

Definition 2.1.[24] Let X be a universal set. A neutrosophic set A in X is a set contains triplet having truthness, falseness and indeterminacy membership values that can be characterized independently, denoted by T_A, F_A, I_A in [0,1]. The neutrosophic set A is denoted as follows:

$A = \{(x, T_A(x), I_A(x), F_A(x)): x\in X$, and $T_A(x), I_A(x), F_A(x) \in[0, 1]\}$. Since, no restriction on the values of $T_A(x)$, $I_A(x)$ and $F_A(x)$ is imposed, so we have $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Example 2.1. Let $X=\{n_1, n_2\}$ be a non-empty fixed set. Clearly, $W=\{(n_1,0.4,0.7,0.8), (n_2,0.2,0.7,0.8)\}$ is a neutrosophic set over X.

Definition 2.2.[24] The null and full neutrosophic set over a nonempty set X are denoted by 0_N and 1_N, given by

(*i*) $0_N=\{(x, 0, 1, 1) : x\in X\}$;

(*ii*) $1_N=\{(x, 1, 0, 0) : x\in X\}$.

There are also other representations of 0_N and 1_N. One may refer to the references cited in the article. Clearly, $0_N\subseteq 1_N$. We have, for any neutrosophic set A, $0_N\subseteq A\subseteq 1_N$.

Definition 2.3.[24] Let $A = \{(x, T_A(x), I_A(x), F_A(x)): x\in X\}$ be a neutrosophic set over X, then the complement of A is defined by $A^c=\{(x, 1-T_A(x), 1-I_A(x), 1-F_A(x)) : x\in X\}$.

Example 2.2. Let $X=\{n_1, n_2\}$ be a non-empty set. Let $W=\{(n_1,0.5,0.7,0.5), (n_2,0.5,0.8,0.7)\}$ be a neutrosophic set over X. Then, the complement of W is $W^c=\{(n_1,0.5,0.3,0.5), (n_2,0.5,0.2,0.3)\}$.

Definition 2.4.[24] A neutrosophic set $A = \{(x, T_A(x), I_A(x), F_A(x)): x\in X\}$ is contained in the other neutrosophic set $B = \{(x, T_B(x), I_B(x), F_B(x)): x\in X\}$ (i.e., $A\subseteq B$) if and only if $T_A(x)\leq T_B(x)$, $I_A(x)\geq I_B(x)$, $F_A(x)\geq F_B(x)$, for each $x\in X$.

Example 2.3. Let $X=\{n_1, n_2\}$ be a non-empty set. Let $W=\{(n_1,0.5,0.7,0.5), (n_2,0.5,0.8,0.7)\}$ and $M=\{(n_1,0.7,0.5,0.2), (n_2,0.9,0.3,0.5)\}$ be two *neutrosophic sets* over X. Then, $W\subseteq M$.

Definition 2.5.[24] If $A = \{(x, T_A(x), I_A(x), F_A(x)): x\in X\}$ and $B = \{(x, T_B(x), I_B(x), F_B(x)): x\in X\}$ are any two neutrosophic sets over X, then $A\cup B$ and $A\cap B$ is defined by

$$A\cup B =\{(x,T_A(x)\vee T_B(x), I_A(x)\wedge I_B(x), F_A(x)\wedge F_B(x)): x\in X\},$$

$$\text{and } A\cap B =\{(x, T_A(x)\wedge T_B(x), I_A(x)\vee I_B(x), F_A(x)\vee F_B(x)): x\in X\}.$$

Example 2.4. Let $X=\{n_1, n_2\}$ be a non-empty set. Let $W=\{(n_1,0.5,0.7,0.5), (n_2,0.5,0.8,0.7)\}$ and $M=\{(n_1,0.7,0.5,0.2), (n_2,0.9,0.3,0.5)\}$ be two neutrosophic sets over X. Then, $W\cup M=\{(n_1,0.7,0.5,0.2), (n_2,0.9,0.3,0.5)\}$ and $W\cap M=\{(n_1,0.5,0.7,0.5), (n_2,0.5,0.8,0.7)\}$.

The notion of neutrosophic topological space is defined as follows:

Definition 2.6.[21] Let X be a non-empty set and τ be the collection of neutrosophic subsets of X then τ is said to be a neutrosophic topology (in short *NT*) on X if the following properties holds:

(*i*) $0_N, 1_N\in\tau$,

(*ii*) $U_1, U_2\in\tau \Rightarrow U_1\cap U_2\in\tau$,

(*iii*) $\cup_{i\in\Delta}u_i\in\tau$, for every $\{u_i: i \in\Delta\}\subseteq \tau$.

Then, (X, τ) is called a neutrosophic topological space (in short NTS) over X. The members of τ are called neutrosophic open sets (in short NOS). A neutrosophic set D is called neutrosophic closed set (in short NCS) if and only if D^c is a neutrosophic open set.

Example 2.5. Let W, E and Z be three neutrosophic sets over a fixed set $X=\{p, q, r\}$ such that:

$W=\{(p,0.7,0.7,0.5), (q,0.5,0.5,0.1), (r,0.9,0.6,0.7): p, q, r\in X\}$;

E={(p,0.6,0.8,0.9), (q,0.5,0.8,0.3), (r,0.4,0.7,0.8): p, q, $r\in X$};

Z={(p,0.5,0.8,1.0), (q,0.4,0.9,0.4), (r,0.3,0.7,1.0): p, q, $r\in X$}.

Then, the collection τ={0_N, 1_N, W, E, Z} forms a neutrosophic topology on X. Here, 0_N, 1_N, W, E, Z are NOSs in (X, τ), and their complements 1_N, 0_N, W^c={(p,0.3,0.3,0.5), (q,0.5,0.5,0.9), (r,0.1,0.4,0.3): p, q, $r\in X$}, E^c={(p,0.4,0.2,0.1), (q,0.5,0.2,0.7), (r,0.6,0.3,0.2): p, q, $r\in X$} and Z={(p,0.5,0.2,0.0), (q,0.6,0.1,0.6), (r,0.7,0.3,0.0): p, q, $r\in X$} are NCSs in (X, τ).

The notion of neutrosophic interior and neutrosophic closure of a neutrosophic set is defined as follows:

Definition 2.7.[21] Let (X, τ) be a *NTS* and U be a *NS* in X. Then the neutrosophic interior (in short N_{int}) and neutrosophic closure (in short N_{cl}) of U are defined by

$$N_{int}(U) = \cup\{E : E \text{ is a } NOS \text{ in } X \text{ and } E\subseteq U\},$$

$$\text{and } N_{cl}(U) = \cap\{F : F \text{ is a } NCS \text{ in } X \text{ and } U\subseteq F\}.$$

Example 2.6. Let us consider a neutrosophic topological space as shown in **Example 2.5.** Let U={(p,0.5,0.7,0.5), (q,0.5,0.8,0.7), (r,0.3,0.7,1.0)} be a *neutrosophic set* over X. Then, $N_{int}(U)=0_N$ and $N_{cl}(U)$={(p,0.5,0.2,0.0), (q,0.6,0.1,0.6), (r,0.7,0.3,0.0)}.

Remark 2.1.[21] Clearly $N_{int}(U)$ is the largest neutrosophic open set over X which is contained in U and $N_{cl}(U)$ is the smallest neutrosophic closed set over X which contains U.

Definition 2.8.[2] Let (X, τ) be a neutrosophic topological space and G be a neutrosophic set over X. Then G is called,

(i) Neutrosophic semi-open (in short *NSO*) set if and only if $G\subseteq N_{cl}(N_{int}(G))$;

(ii) Neutrosophic pre-open (in short *NPO*) set if and only if $G\subseteq N_{int}(N_{cl}(G))$.

The collection of all NSO sets and NPO sets in (X, τ) are denoted by NSO(τ) and NPO(τ).

Example 2.7. Let X={a, b} be a non-empty set. Clearly, (X, τ) is a neutrosophic topological space, where τ={0_N, 1_N, {(a,0.3,0.4,0.3), (b,0.4,0.3,0.4): a, $b\in X$}, {(a,0.4,0.4,0.1), (b,0.5,0.1,0.3): a, $b\in X$}}. Then, the neutrosophic set Q={(a,0.6,0.4,0.1), (b,0.9,0.1,0.2): a, $b\in X$} is a NSO set and P={(a,0.3,0.9,0.2), (b,0.3,0.4,0.3): a, $b\in X$} is a NPO set in (X, τ).

Definition 2.9.[2] A neutrosophic set G is called a neutrosophic b-open set in a NTS (X, τ) if and only if $G\subseteq N_{int}(N_{cl}(G))\cup N_{cl}(N_{int}(G))$. A neutrosophic set H is said to be neutrosophic b-closed set if its complement H^c is a neutrosophic b-open. The collection of all neutrosophic b-open sets in (X, τ) is denoted by N-b-O(τ).

Example 2.8. Let (X, τ) be a neutrosophic topological space as shown in **Example 2.7**. Then, the neutrosophic set P={(a,0.3,0.9, 0.2), (b,0.3,0.4,0.3): a, $b\in X$} is a neutrosophic b-open set in (X, τ).

Definition 2.10.[27] Let (X, τ) be a neutrosophic topological space. A neutrosophic set G is said to be a neutrosophic locally open (in short *NLO*) set if G= $H\cup K$, where H is a neutrosophic open set and K is a neutrosophic closed set in X.

Example 2.9. Let (X, τ) be a neutrosophic topological space as shown in **Example 2.5**. Then, the *neutrosophic set* R={(p,0.7,0.2,0.0), (q,0.6,0.1,0.1), (r,0.9,0.3,0.0): p, q, $r \in X$} is a neutrosophic locally open set in (X, τ).

3. Mappings Between Neutrosophic Sets:

In this section, we prove some results on mappings between neurosophic subsets.

Proposition 3.1. Let f:$X \rightarrow Y$ be a mapping, and $\{U_i : i \in \Delta\}$ be a family of neurosophic subsets of Y, then we have

(i) $f^{1}(\cup_{i\in\Delta} U_i) = \cup_{i\in\Delta} f^{1}(U_i)$.

(ii) $f^{1}(\cap_{i\in\Delta} U_i) = \cap_{i\in\Delta} f^{1}(U_i)$.

Proof. The proofs are so easy, so omitted.

Theorem 3.2. If f_i: $X_i \rightarrow Y_i$ and U_i be neutrosophic sets of Y for i = 1, 2, then

$(f_1 x f_2)^{-1}(U_1 x U_2) = f_1^{-1}(U_1)\ x\ f_2^{-1}(U_2)$.

Proof. Let f_i: $X_i \rightarrow Y_i$ be mappings for i = 1, 2. Let $U_1 = \{(T_1, F_1, I_1)(x_1) : x_1 \in X_1\}$ and U_2 $\{(T_2, F_2, I_2)(x_2) : x_2 \in X_2\}$ be neutrosophic subsets in Y_1 and Y_2 respectively. Then we have for (x_1, x_2) in $X_1 x X_2$, we have

$(f_1 x f_2)^{-1}(T_1 x T_2)\ (x_1, x_2) = (T_1 x T_2)\ (f_1(x_1), f_2(x_2))$

$= \min\ \{T_1 f_1(x_1),\ T_2\, f_2(x_2)\}$

$= \min\ \{f_1^{-1}(T_1(x_1)),\ f_2^{-1}(T_2(x_2))\}$

$= (f_1^{-1}(T_1), f_2^{-1}(T_2))\ (x_1, x_2)$.

Following the above argument, we can show that

$$(f_1 x f_2)^{-1}(F_1 x F_2)\ (x_1, x_2) = (f_1^{-1}(F_1), f_2^{-1}(F_2))\ (x_1, x_2)$$

and $(f_1 x f_2)^{-1}(I_1 x I_2)\ (x_1, x_2) = (f_1^{-1}(I_1), f_2^{-1}(I_2))\ (x_1, x_2)$.

4. Continuity in Minimal Structure Neutrosophic Topological Space:

In this section we introduce the notion of continuous maps between minimal structures in neutrosophic topological spaces. We procure the following definitions on neutrosophic minimal structre spaces from the article by Pal et al. [15].

Definition 4.1. A family M of neutrosophic subsets of X if M $\subset P(X)$, where $P(X)$ denotes the power set of X is said to be a neutrosophic minimal structure on X if 0_N and 1_N belong to M. Then, the pair (X, M) is called a nutrosophic minimal space.

Example 4.1. Let W, E and Z be three neutrosophic sets over a fixed set X={p, q, r} such that:

W={(p,0.7,0.7,0.5), (q,0.5,0.5,0.1), (r,0.9,0.6,0.7): p, q, $r \in X$};

E={(p,0.6,0.8,0.9), (q,0.5,0.8,0.3), (r,0.4,0.7,0.8): p, q, $r \in X$};

Z={(p,0.5,0.8,1.0), (q,0.4,0.9,0.4), (r,0.3,0.7,1.0): p, q, $r \in X$}.

Clearly, the collection M={0_N, 1_N, W, E, Z} forms a neutrosophic minimal structure on X, and the pair (X, M) is a neutrosophic minimal structure space.

Remark 4.1. Eevery NTS is a nutrosophic minimal structure space. But every nutrosophic minimal structure space may not be a NTS in general. This follows from the following example.

Example 4.2. Let W, E and Z be three neutrosophic sets over a fixed set X={p, q, r} such that:

W={(p,0.5,0.7,0.5), (q,0.5,0.9,0.1), (r,0.9,0.6,0.7): p, q, $r\in X$};

E={(p,0.6,0.6,0.9), (q,0.5,0.8,0.3), (r,0.4,0.7,0.8): p, q, $r\in X$};

Z={(p,0.5,0.5,1.0), (q,0.4,0.7,0.4), (r,1.0,0.7,1.0): p, q, $r\in X$}.

Clearly, the collection M={0_N, 1_N, W, E, Z} forms a neutrosophic minimal structure on X, and the pair (X, M) is a neutrosophic minimal structure space. But (X, M) is not a NTS.

Definition 4.2. Let (X, M) be a neutrosophic minimal structure space. If $E\in$M, then E is called a neutrosophic m-open set, and its complement is called a neutrosophic m-closed set in (X, M).

Example 4.3. Let us consider a neutrosophic minimal structure space (X, M) as shown in **Example 4.1.** Clearly, 0_N, 1_N, W, E, Z are neutrosophic m-open sets in (X, M), and their complements 1_N, 0_N, W^c={(p,0.3,0.3,0.5), (q,0.5,0.5,0.9), (r,0.1,0.4,0.3): p, q, $r\in X$}, E^c={(p,0.4,0.2,0.1), (q,0.5,0.2,0.7), (r,0.6,0.3,0.2): p, q, $r\in X$} and Z={(p,0.5,0.2,0.0), (q,0.6,0.1,0.6), (r,0.7,0.3,0.0): p, q, $r\in X$} are neutrosophic m-closed sets in (X, M).

The notion of neutrosophic minimal interior and neutrosophic minimal closure of a neutrosophic set in a neutrosophic minimal structure space is defined as follows:

Definition 4.3. Let (X, M) be a neutrosophic minimal structure space, and U be a neutrosophic set over X. Then, the neutrosophic minimal interior (in short $N_{m\text{-}int}$) and neutrosophic minimal closure (in short $N_{m\text{-}cl}$) of U are defined as follows:

$$N_{m\text{-}int}(U) = \cup\{E : E \text{ is a neutrosophic } m\text{-open set in } X \text{ and } E\subseteq U\},$$

$$\text{and } N_{m\text{-}cl}(U) = \cap\{F : F \text{ is a neutrosophic } m\text{-closed set in } X \text{ and } U\subseteq F\}.$$

Example 4.4. Let (X, M) be a neutrosophic minimal structure space as defined in **Example 4.1.** Then, the neutrosophic minimal interior and neutrosophic minimal closure of U={(p,0.2,0.4,0.6), (q,0.4,0.7, 0.9), (r,0.0,0.5,0.4)} are $N_{m\text{-}int}(U)$={(p,0,1,1), (q,0,1,1), (r,0,1,1)} and $N_{m\text{-}cl}(U)$={(p,0.3,0.3,0.5), (q,0.5,0.5,0.9), (r,0.1,0.4,0.3)} respectively.

Exzample 4.5. From the above definitions, it is clear that every neutrosophic pre-open sets, neutrosophic semi-open sets, neutrosophic b-open sets are neutrosophic m-open sets.

Example 4.6. Let W, E and Z be three neutrosophic sets over a non-empty set X={p, q} such that:

W={(p,0.7,0.7,0.5), (q,0.5,0.5,0.1) : p, $q\in X$};

E={(p,0.6,0.8,0.9), (q,0.5,0.8,0.3) : p, $q\in X$};

Z={(p,0.5,0.8,1.0), (q,0.4,0.9,0.4) : p, $q\in X$}.

Here, the family τ={0_N, 1_N, W, E, Z} forms a neutrosophic topology on X, and so (X, τ) is a neutrosophic topological space. Suppose M = $\tau\cup$NPO(τ)$\cup$NSO(τ)$\cup$N-b-O(τ), then (X, M) is a neutrosophic minimal structure. Now, from the above it is clear that, every neutrosophic pre-open sets, neutrosophic semi-open sets, neutrosophic b-open sets in (X, τ) are neutrosophic m-open sets in (X, M). Further, it is also seen that, every neutrosophic m-open set in (X, M) is also a neutrosophic pre-open set, neutrosophic semi-open set, neutrosophic b-open set in (X, τ).

Definition 4.4. The function f: (X,M_1)⟶ (Y,M_2) is said to be minimal continuous (in short m_N continuous) if $f^{-1}(U)$ is an m-open set, where U is any m-open set in M_2 .

Definition 4.5. Let (X, M) be a minimal structure on the neutrosophic set X. Let (X, τ) and (Y, σ) be neutrosophic topological spaces. Then the mapping f: (X, τ)→(Y, σ) is said to be weakly m-continuous if for each neutrosophic point x_0 and each neutrosophic open set V with $f(x_0)\in V$, there exists a neutrosophic open set U such that $x_0\in U$ and $f(U) \subset N_{cl}(V)$.

We state the following result without proof, in view of the above definition:

Proposition 4.1. Let f: (X,M_1)⟶ (Y,M_2) and g: (X,M_2)⟶ (Y,M_3) be N_m–continuous functions. Then the composition function gof : (X,M_1)⟶ (Y,M_3) is N_m–continuous.

Theorem 4.1. Let (Y,M_2) be a minimal space and f: X ⟶(Y,M_2) be a function. Then there is a weaker minimal structure M_1 on X for which f is N_m– continuous.

Proof: Let X and Y be non-empty sets and M_2 be a minimal structure on Y. Let f: X ⟶(Y,M_2) be a function. Let $M_1 \subseteq$ P (X) be defined by $M_1 =\{f^{-1}(V) : V\in M_2\}$. Hence ($X$, M_1) is a minimal structure on X. From the definition of N_m–continuity and construction of M_1, it follows that f: (X,M_1)⟶ (Y,M2) is N_m–continuous. Further by the definition of weaker minimal structure and construction of M_1, it follows that M_1 is a weaker minimal structure on X.

We state the following result without proof.

Proposition 4.2. Let (X,M) be a minimal space and $Y\subseteq X$, then (Y,M⋂Y) is a minimal structure on Y. Further for (X, M⋂Y) there is a weaker minimal structure space.

Theorem 4.2. Let minimal (X,M_1) be a minimal structure and $Y\subseteq X$. Then there is a weakest minimal structure on Y say M_2 such the map i_f : (Y,M_2) ⟶(X,M_1⋂Y) is N_m–continuous.

Proof: In view of the above Theorem 4.2 on considering the identity map we can have the map i_f to be N_m– continuous.

Remark 4.2. The above result is true for the inclusion map i : (Y,M_2) ⟶(X,M_1). In this case M_2 is called as the induced minimal structure on Y.

Theorem 4.3. Let $Y\subseteq X$ and f:(X,M_1)⟶ (Z,N_1) be N_m–continuous. Then $f|_Y$:(Y,M2) ⟶(Z,N_1⋂$f(y)$) is m-continuous, where Y is endowed with M_2, induced minimal structure.

Proof: By the above remark and theorem 4.1, we have $f|_Y$ = foi (or foi_f) and hence $f|_Y$ is N_m–continuous.

Theorem 4.4. Let {(X_i, M_i) : $i\in\Delta$} be a family of minimal spaces, , where Δ being the index set and {f_i :X⟶(X_i, M_i) : $i\in\Delta$ } be a family of N_m–continuous functions. Then there is a weakest minimal structure M on X such that f_i's are N_m –continuous.

Proof: Let $\{(X_i, M_i) : i\in\Delta\}$, where Δ is the index set be a family of minimal spaces and $\{f_i : X\longrightarrow(X_i, M_i) : i\in\Delta\}$ be a family functions. Let $E_i = f^{-1}(M_i) = \{f^{-1}(V) : V\in M_i\}$ for $i\in\Delta$. Consider $M = \cup_{i\in\Delta}E_i$. Then (X, M) is a minimal structure on X by definition. Further from the construction of M, it is clear that $f_i :(X, M) \longrightarrow(X_i, M_i)$ are N_m–continuous. Since we have considered the union while considering the minimal structure M on X, so it will include all other minimal structures on X, so it is the weakest minimal structure on X.

Theorem 4.5. Let $\{f_i: X\longrightarrow(X_i, M_i) : i\in\Delta\}$ be a family of N_m–continuous functions, where, (X_i, M_i) are minimal spaces. Let the minimal structure M in X be generated by $\{f_i : i\in\Delta\}$. Then the function $f : (Y,N)\longrightarrow (X,M)$ is N_m–continuous if and only if $f_i o f$ is N_m–continuous function for all $i\in\Delta$.

Proof: Let $\{f_i: X\longrightarrow(X_i, M_i) : i\in\Delta\}$ be a family of N_m–continuous functions and
$f : (Y,N)\longrightarrow (X,M)$ be N_m–continuous then by proposition 4.1, $f_i o f$ is N_m– continuous.
Next, let $f_i o f$ be N_m–continuous functions for each $i\in\Delta$, but f is not N_m–continuous. Thus we have $B\in M$ such that $f^{-1}(B) \notin N$. Then we have the following possibilities:

(i) There exist $i_0\in\Delta$ and $B_{i_0} \in M_{i0}$ such that $B = f_{i_0}^{-1}(B_{i_0})$.

(ii) For every $i\in\Delta$ and every $B_i\in M_i$, $B\neq f_{i_0}^{-1}(B_{i_0})$.

Consider case (i), we have $B = f_{i_0}^{-1}(B_{i_0})$, implies $f^{-1}(f_{i_0}^{-1}(B_{i_0})) = (f_{i_0} \circ f)^{-1}(B_{i_0})$. Thus, for $B_{i_0} \in M_{i0}$, we have $(f_{i_0} \circ f)^{-1}(B_{i_0}) \notin N$, which shows that $f_{i_0} \circ f$ is not N_m–continuous. Hence we arrive at a contradiction. Thus our supposition is wrong.

Next, consider the case (ii), we have $f^{-1}(0_{N,X})=0_{N,Y}$ and $f^{-1}(1_{N,X}) = 1_{N,Y}$, which leads to $B(0_{N,Y}, 1_{N,Y})$. Hence M \{ B} is a minimal structure on X. Thus for each $i\in\Delta$, f_i:(X, M \{B})$\longrightarrow$(X_i, M_i), we have $f_i o f$: (Y, N)$\longrightarrow$(X_i, M_i) is m_N – continuous for each $i\in\Delta$,. This leads to a contradiction to the choice of the minimal structure M on X. Thus, f is N_m–continuous, whenever $f_i o f$ is N_m–continuous for each $i\in\Delta$.

Remark 4.3. Let $\{(X_i, M_i), i\in\Delta\}$ be a family of minimal structures, then the product space is defined by $\Pi_{i\in\Delta} X_i$. It can be easily verify that $(\Pi_{i\in\Delta} X_i, \Pi_{i\in\Delta} M_i)$ is a minimal structure on $\Pi_{i\in\Delta} X_i$. Further $M = \Pi_{i\in\Delta} M_i$ is the weakest minimal structure on $\Pi_{i\in\Delta} X_i$.

One can easily verify that for each $j\in\Delta$, the canonical projection $\pi_j : \Pi_{i\in\Delta} X_i\longrightarrow X_j$ is N_m–continuous.

In view of the above theorem and remark, we formulate the following results.

Proposition 4.3. Let $\{(X_i, M_i), i\in\Delta\}$ be a family of minimal spaces and $X = \Pi_{i\in\Delta} X_i$ exists.

Proposition 4.4. Let $\{(X_i, M_i), i\in\Delta\}$ be a family of minimal spaces and $X = \Pi_{i\in\Delta} X_i$. Let the minimal structure on X be generated by $\Pi_{i\in\Delta} X_i$. Let $f : (Y,N)\longrightarrow (X,M)$ be a function. Then f is N_m-continuous if and only if $\pi_i o f$ is N_m–continuous for all $i\in\Delta$.

The following result is a consequence of above results.

Corollary 4.1. Let f : (X,M)⟶ (Y,N) and g: (X,M)⟶(Z, Q) be N_m–continuous functions. Then the function f xg :(X,M) ⟶(YxZ, N x Q) defined by (fxg)(x)=(f(x), g(x)) is N_m-continuous.

5. Conclusion:

In this article, we have introduced the notion of continuity and product minimal space in neutrosophic minimal structure spaces. Besides, we have investigated some basic properties of N_m–continuity in neutrosophic minimal structure spaces, such as composition of N_m–continuous functions, product of N_m–continuous functions in product neutrosophic topological space etc. It is hoped that, these kind of notions can also be investigated in the field of Neutrosophic Multiset Topological Space [6], Neutrosophic Bitopological Space [7], Pentapartitioned Neutrosophic Topological Space [8], Neutrosophic Complex Topological Space [11], Generalized Neutrosophic Topological Space [22], etc.

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Geometric Programming in Imprecise Domain with Application

Pintu Das[1*]

[1]Department of Mathematics, Tamralipta Mahavidyalaya, Tamluk, Purba Medinipur, 721636, West Bengal, India

* Correspondence: mepintudas@yahoo.com

Abstract: The paper aims to obtain a computational algorithm to solve a geometric Programming Problem by weighted sum method with equal priority in imprecise condition i.e. in Fuzzy, Intuitionistic Fuzzy and Neutrosophic field. A contrasting study of optimal solution among three has been prescribed to show the efficiency of this method. Numerical example and an application Gravel Box Design Problem is presented to compare different designs. Proposed method is determined by maximizing the truth and indeterminacy membership degree and minimizing the negative membership degree.

Keywords: Geometric Programming, Fuzzy, Intuitionistic Fuzzy, Neutrosophic sets, Gravel Box Design Problem.

1. Introduction

Geometric programming is an advanced method to solve a nonlinear programming problem. It has certain benefits over the other optimization methods. The concept of fuzzy sets (FS) was launched by Zadeh in 1965 **[1]**. Since the fuzzy sets and fuzzy logic have been applied in many real applications to maintain uncertainty. The conventional fuzzy sets uses single real value $\mu_A(x) \in [0, 1]$ to represents the truth membership function of a fuzzy set. In some applications we should consider not only the truth membership supported by the evident but also the falsity membership against by the evident. That is out of the scope of fuzzy sets and interval valued fuzzy sets. However in reality, it may not always be true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the membership degree because there may be some negative degree. In 1986, Atanassov **[3]**, **[5]** introduced the intuitionistic fuzzy sets (IFS) which is a modification of fuzzy sets. The intuitionistic fuzzy sets consider both truth membership and falsity membership. Intuitionistic fuzzy sets can only handle incomplete information not the indeterminate information and inconsistent information. In IFS, sum of membership-degree and non-membership degree of a vague parameter is less than unity. Therefore a certain amount of incomplete information or indeterminacy arises in an intuitionistic fuzzy set. It cannot handle all types of uncertainties successfully in different real physical problems. Hence further modification of fuzzy set as well as intuitionistic fuzzy sets are need. In neutrosophic sets (NS) indeterminacy is clarified explicitly and truth membership, indeterminacy membership and falsity membership are not dependent. Neutrosophy was launched by Florentin Smarandache in 1995 **[4]** which is actually generalization of different types of FS and IFS. The term "neutrosophy" means advance information of neutral thought. This neutral concepts make the differece between NS and other sets like FS, IFS.

Fuzzy representation is analyzed by a single variable: degree of truth μ, while the degree of falsity ν has a defect value calculated by negative formula: $\nu = 1- \mu$, and the degree of neutrality has a defect value that is $\sigma =0$.

Intuitionistic fuzzy representation is described by two explicit variables: degree of truth μ and degree of falsity ν, while the degree of neutrality has a defect value that is $\sigma = 0$. Atanassov considered the incomplete variant taking into account that $\mu + \nu \leq 1$.

Neutrosophic representation of information is described by three parameters: degree of truth μ, degree of falsity ν, and degree of neutrality σ.

Intuitionistic fuzzy set is a device in formating real life problem like sale analysis, new product marketing, financial services, negotiation process, portfolio optimization, psychological investigation etc. Since there is a fair chance of the existence of a non-null hesitation part at each moment of evaluation of an unknown object (Szmidt and Kacprzyk, 1997, 2001). Atanassov (1999, 2012) carried out rigorous research based on the theory and applications of intuitionistic fuzzy sets. Geometric programming has been applied to simple riser problems by R.C. Creese [13] using Chvorinov's rule. In the last 20 yrs fuzzy geometric programming has received rapid development in the theory and application. In 2002, B.Y. Cao [11] published the first monograph of fuzzy geometric programming as applied optimization series (vol 76), fuzzy geometric programming by Kluwer academy publishing (the present spinger), the book gives a detailed exposition to theory and application of fuzzy geometric programming. In 1990 R. k. verma [1**4**] has studied fuzzy programming technique to solve geometric programming problems. Recently a paper multi-objective geometric programming problem based on intuitionistic fuzzy geometric programming technique is published by Pintu Das et al. [**15**]. Multi-objective non-linear programming problem based on Neutrosophic Optimization Technique and its application in Riser Design Problem is published by Pintu Das et al. **[16]**. In our uncertain life a decision-maker has to allow to handle indeterminacy or neutral thoughts in decision-making process. Neutrosophic optimization technique is limited in application to design optimization. The motivation of the present study is to explain computational procedure for solving Geometric Programming Problem in imprecise environment (i.e. Fuzzy, Intuitionistic Fuzzy, Neutrosophic) and as an application "Gravel Box Design" problem is represented. A contrasting study of optimal solution among three has been prescribed to show the efficiency of this method. Numerical example and an application Gravel Box Design Problem is presented to compare different designs. Proposed method is determined by maximizing the truth and indeterminacy membership degree and minimizing the negative membership degree.

2. Geometric Programming

Geometric programming (GP) is an advanced method to solve the special class of non-linear programming problems subject to linear or non-linear restriction. The original mathematical development of this method used the arithmetic–geometric mean inequality relationship between sums and products of real numbers. In 1967 Duffin, Peterson and Zener made a beginning stone to solve vast range of engineering problems by basic theories of geometric programming in the book "**Geometric Programming" [12]**. Beightler and Phillips gave a full account of whole modern theory of geometric programming and numerous examples of successful applications of geometric programming to real-world problems in their book "**Applied Geometric Programming" [6]**. The study of GP by Duffin et al. (1967) deals with the problem associating only a positive coefficient for the component cost terms. However, many real world problems comprise of positive as well as negative coefficients for the cost terms. GP method has some advantages. The advantage is that it is sometimes simple to solve the dual problem than primal.

3. Posynomial Geometric Programming Problem

Primal Problem

A single objective posynomial geometric programming problem can be written as

Minimize $f_0(x)$ (1)

subject to

$f_j(x) \leq 1$ (j=1,2,……..,m)

$x_i > 0$ (i=1,2,……...,n)

Where $f_j(x) = \sum_{k=1}^{N_j} c_{jk} \prod_{i=1}^{n} x_i^{a_{jki}}$

Where c_{jk} (> 0) and a_{jki} i =1, 2,……,n; k=1,2,………N_j;

j = 0, 1, 2, ……,m ; are real.

$x \equiv (x_1, x_2, \dots \dots \dots . x_n)^T$.

Dual Problem

The dual programming of (1) is as follows

Maximize d $(w) = \prod_{j=0}^{m} \prod_{k=1}^{N_j} (\frac{c_{jk} w_{j0}}{w_{jk}})^{w_{jk}}$ (2)

subject to

$\sum_{k=1}^{N_j} w_{0k} = 1$ (Normality condition)

$\sum_{j=0}^{m} \sum_{k=1}^{N_j} a_{jki} w_{jk} = 0$ (Orthogonality condition)

$w_{j0} = \sum_{k=1}^{N_j} w_{jk} \geq 0,\ w_{jk} \geq 0$

i = 1,2,…………n; k= 1,2,……………N_j, $w_{00} = 1$

4. Signomial Geometric Programming Problem

Primal Problem

A single objective signomial geometric programming problem can be formulated as

Min $f_0(x)$ (3)

Subject to

$f_j(x) \leq \delta_j$ (j=1,2,……..,m)

$x_i > 0$ (i=1,2,……...,n)

Where $f_j(x) = \sum_{k=1}^{N_j} \delta_{jk} c_{jk} \prod_{i=1}^{n} x_i^{a_{jki}}$ (j = 0,1,2,……….,m)

$\delta_j = \pm 1$ (j = 2,3, ………..m.), $\delta_{jk} = \pm 1$ (j=0,1,2,……….,m); k=1,2,………,N_j

$x \equiv (x_1, x_2, \ldots \ldots \ldots, x_n)^T$.

Dual Problem

The dual problem of (3) is as follows

Maximize d $(w)=\delta_0 [\prod_{j=0}^{m} \prod_{k=1}^{N_j} (\frac{c_{jk} w_{jo}}{w_{jk}})^{\delta_{jk} w_{jk}}]^{\delta_0}$ (4)

subject to

$\sum_{k=1}^{N_0} \delta_{0k} w_{0k} = \delta_0$ (Normality condition)

$\sum_{j=0}^{m} \sum_{k=1}^{N_j} \delta_{jk} a_{jki} w_{jk} = 0$ (Orthogonality condition)

i = 1,2,……..n.

Where $\delta_j = \pm 1$ (j = 2,3, ………..m.), $\delta_{jk} = \pm 1$ (j=1,2,……….m);

K= 1,2,………..N_j and $w_{00} = 1$

$\delta_0 = \pm 1$

$w_{j0} = \delta_j \sum_{k=1}^{N_j} \delta_{jk} w_{jk} \geq 0,\ \delta_{jk} \geq 0$

j = 1,2,…………m; k= 1,2…………N_j .

5. Fuzzy Geometric Programming (FGP)

A geometric programming problem with fuzzy objective can be written as

$$\widetilde{Minimize} f_0(x) \quad (5)$$

Subject to $f_j(x) \lesssim b_j$ j=1,2,…………..,m

$x \geq 0$

Here the symbol "$\widetilde{Minimize}$" denotes a flexible version of "Minimize". Similarly the symbol "$\lesssim$" denotes a fuzzy version of "$\leq$". These fuzzy requirements may be determined by taking membership functions $\mu_j(f_j(x))$ (j= 0,1,2,……..m) from the decision maker for all functions $f_j(x)$ (j=0,1,2,……..m) by taking account of the rate of increased membership functions. It is, in general strictly monotone decreasing linear or non-linear functions with respect to $f_j(x)$ (j = 0, 1, 2,……..,m). Here for simplicity, linear membership functions are considered. The linear membership functions can be presented by

$$\mu_j(f_j(x)) = \begin{cases} 1 & if\ f_j(x) \leq f_j^0 \\ \frac{f_j' - f_j(x)}{f_j' - f_j^0} & if\ f_j^0 \leq f_j(x) \leq f_j' \\ 0 & if\ f_j(x) \geq f_j' \end{cases}$$

for j= 0,1,2,3,…………m.

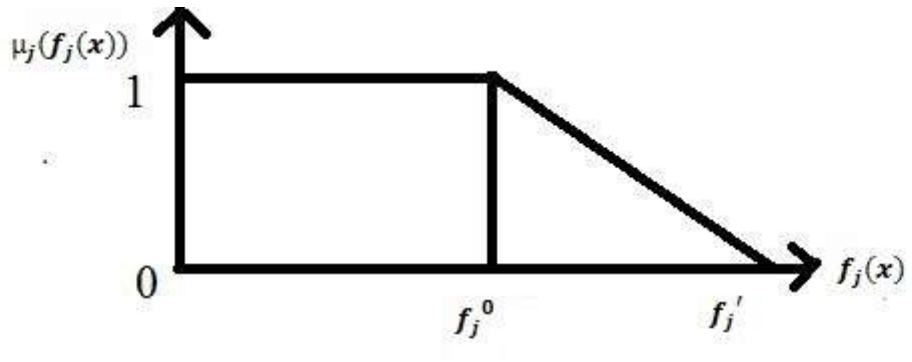

Figure-1: Membership function of a minimization-type objective function

The problem (5) reduces to FGP when f_0(t) and $f_j(x)$ are signomial and posynomial functions.

Based on fuzzy decision making of bellman and zadeh (1972), we may write

i) $\mu_D(x^*) = \max(\min \mu_j(f_j(x)))$ (Max-min operator) (6)

subject to $\mu_j(f_j(x)) = \frac{f_j{}' - f_j(x)}{f_j{}' - f_j{}^0}$

(j= 0,1,2,3,…………,m.)

$x > 0$

ii) $\mu_D(x^*) = \max(\sum_{j=0}^{m} \lambda_j \mu_j(f_j(x)))$ (Max-additive operator) (7)

subject to $\mu_j(f_j(x)) = \frac{f_j{}' - f_j(x)}{f_j{}' - f_j{}^0}$

(j= 0,1,2,3,…………m.)

$x > 0$

iii) $\mu_D(x^*) = \max\left(\prod_{j=0}^{m}\left(\mu_j\left(f_j(x)\right)\right)^{\lambda_j}\right)$ (Max - product operator) (8)

subject to $\mu_j(f_j(x)) = \frac{f_j{}' - f_j(x)}{f_j{}' - f_j{}^0}$

(j= 0, 1, 2, 3,…………,m.)

$x > 0$

Here for λ_j (j=0,1,2,…………m) are numerical weights determined by a decision making unit . For normalized weights $\sum_{j=0}^{m} \lambda_j = 1$

For equal priority of objective and constraint goals, λ_j =1 and $\lambda_j \in [0, 1]$. For equal priority of objective and constraint goals, λ_j = 1 (j=0,1,2,…………,m).

6. Numerical Example

Let us take a fuzzy posynomial geometric programming problem as

$$\widetilde{Minimize}\ f_0(x_1, x_2) = 2x_1^{-2}x_2^{-3} \quad (9)$$

Here objective goal is 57.87 with tolerance 2.91

$f_1(x_1, x_2) = x_1^{-1}x_2^{-2} \leq 6.75$ (with tolerance 0.19)

$f_2(x_1, x_2) = x_1 + x_2 \leq 1$

$x_1, x_2 > 0.$

Here, linear membership functions for the fuzzy objective and constraint goals are

$$\mu_1(f_1(x_1, x_2)) = \begin{cases} 1 & if\ x_1^{-1}x_2^{-2} \leq 6.75 \\ \frac{6.94 - x_1^{-1}x_2^{-2}}{0.19} & if\ 6.75 \leq x_1^{-1}x_2^{-2} \leq 6.94 \\ 0 & if x_1^{-1}x_2^{-2} \geq 6.94 \end{cases}$$

$$\mu_0 (f_0(x_1,x_2)) = \begin{cases} 1 & if\ 2x_1^{-2}x_2^{-3} \leq 57.87 \\ \frac{60.78-2x_1^{-2}x_2^{-3}}{2.91} & if\ 57.87 \leq 2x_1^{-2}x_2^{-3} \leq 60.78 \\ 0 & if\ 2x_1^{-2}x_2^{-3} \geq 60.78 \end{cases}$$

Based on the max-additive operator (7), FGP (9) reduces to

Maximize $V_A(x_1,x_2) = \frac{6.94-x_1^{-1}x_2^{-2}}{0.19} + \frac{60.78-2x_1^{-2}x_2^{-3}}{2.91}$

subject to $x_1 + x_2 \leq 1$

$x_1,x_2 > 0$

Neglecting the constant term in the above model we have the following crisp geometric programming

Minimize V $(x_1,x_2) = 5.263x_1^{-1}x_2^{-2} + 0.687x_1^{-2}x_2^{-3}$

subject to $x_1 + x_2 \leq 1$

$x_1,x_2 > 0$

Here D.D = 4−(2+1) = 1

The DP of this GP is

Max d (w) = $\left(\frac{5.263}{w_{01}}\right)^{w_{01}} \left(\frac{0.687}{w_{02}}\right)^{w_{02}} \left(\frac{1}{w_{11}}\right)^{w_{11}} \left(\frac{1}{w_{12}}\right)^{w_{12}} \times$

$(w_{11} + w_{12})^{(w_{11}+w_{12})}$

Such that $w_{01} + w_{02} = 1$,

$-W_{01} - 2w_{02} + w_{11} = 0$,

$-2w_{01} - 3w_{02} + w_{12} = 0$,

So $w_{02} = 1 - w_{01}, w_{11} = 2 - w_{01}, w_{12} = 3 - w_{01}$

Max d (w_{01}) $= \left(\frac{5.263}{w_{01}}\right)^{w_{01}} \left(\frac{0.687}{1-w_{01}}\right)^{1-w_{01}} \left(\frac{1}{2-w_{01}}\right)^{2-w_{01}} \left(\frac{1}{3-w_{01}}\right)^{3-w_{01}} \times$

$(5 - 2w_{01})^{(5-2w_{01})}$

Subject to $0 < w_{01} < 1$

For optimality, $\frac{d(\mathrm{d}(w_{01}))}{dw_{01}} = 0$

$5.263(1- w_{01})(2 - w_{01})(3-w_{01}) = 0.687w_{01}(5 - 2w_{01})^2$

$w_{01}{}^*$=0.7035507, $w_{02}{}^*$= 0.2964493, $w_{11}{}^*$= 1.296449, $w_{12}{}^*$= 2.296449.

$x_1{}^*$= 0.360836, $x_2{}^*$= 0.6391634

$f_0{}^*(x_1{}^*, x_2{}^*)$ = 58.82652, $f_1{}^*(x_1{}^*, x_2{}^*)$ = 6.783684.

7. Intuitionistic Fuzzy Geometric Programming

Let us consider the intuitionistic fuzzy geometric programming problem as

$\widetilde{Min}^i f_0(x)$ (10)

Subject to $f_j(x) \lesssim$i b_j j=1,2,…………..,m

$x \geq 0$

Here the symbol "≲i" denotes the intuitionistic fuzzy type of "≤".

Now for Intuitionistic fuzzy geometric programming linear membership and non-membership functions can be prescribed as follows.

$$\mu_j\ (f_j(x)) = \begin{cases} 1 & if\ f_j(x) \le f_j^{0} \\ \frac{f_j{}' - f_j(x)}{f_j{}' - f_j{}^{0}} & if\ f_j^{0} \le f_j(x) \le f_j{}' \\ 0 & if\ f_j(x) \ge f_j{}' \end{cases}$$

for j= 0,1,2,3,…………,m.

$$\nu_j\ (f_j\ (x)) = \begin{cases} 0 & if\ f_j(x) \le f_j{}' - f_j{}'' \\ \frac{f_j(x) - (f_j{}' - f_j{}'')}{f_j{}''} & if\ f_j{}' - f_j{}'' \le f_j(x) \le f_j{}' \\ 1 & if\ f_j(x) \ge f_j{}' \end{cases}$$

for j= 0,1,2,3,…………,m.

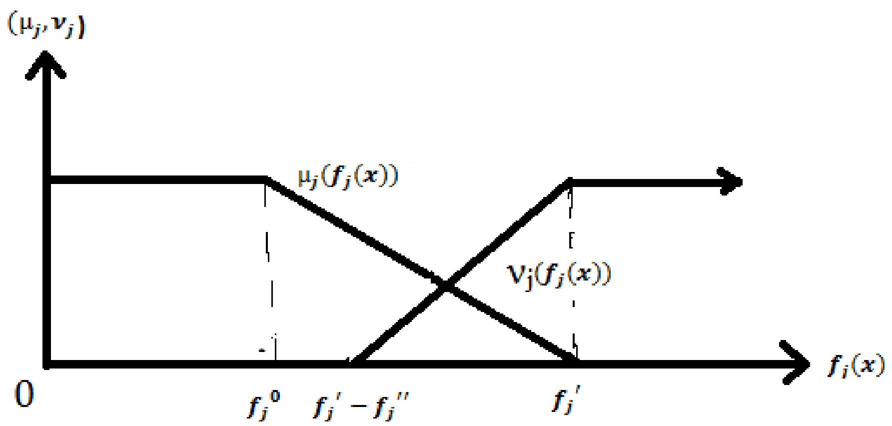

Figure-2: Membership and non-membership functions of a minimization-type objective function.

Now an intuitionistic fuzzy geometric programming problem (10) with membership and non-membership function can be written as

Maximize $\mu_j\ (f_j\ (x))$ (11)

Minimize $\nu_j\ (f_j\ (x))$

for j = 0,1,2,………..,m.

Considering equal priority on all membership and non-membership functions of (11) and using weighted sum method the above optimization problem reduces to

Maximize $V_A\ = \sum_{j=0}^{m}\{\mu_j(f_j\ (x)) - \nu_j\ (f_j\ (x))\}$

subject to $x\ \geq 0$

The above problem is equivalent to

Min $V_{A1} = \sum_{j=0}^{m}\left\{\left(\frac{1}{f_j{}' - f_j{}^0} + \frac{1}{f_j{}''}\right) f_j\ (x) - \left(\frac{f_j{}' - g_j{}''}{f''} + \frac{f_j{}'}{f_j{}' - f_j{}^0}\right)\right\}$

subject to $x\ \geq 0$ (12)

Where $f_j\ (x) = \sum_{k=1}^{N_j} c_{jk} \prod_{i=1}^{n} x_i{}^{a_{jki}}$

Where c_{jk} (> 0) and a_{jki} (i=1,2,…….n; k=1,2,………,N_j ; j=0,1,2,…….,m ;) are real.

$x \equiv (x_1, x_2, \dots \dots \dots . . x_n)^T.$

The posynomial geometric programming problem (12) can be solved by usual geometric programming technique.

8. Numerical Example

Let us consider an intuitionistic geometric programming problem with intuitionistic fuzzy goal as

$\widetilde{Min}^i\ f_0\ (x_1, x_2) = \ 2x_1{}^{-2}x_2{}^{-3}$

Here objective goal is 57.87 with tolerance 2.91

$f_1\ (x_1, x_2) = x_1{}^{-1}x_2{}^{-2} \leq\ 6.75$ (with tolerance 0.19)

$f_2\ (x_1, x_2) = x_1 + x_2 \leq\ 1$

$x_1, x_2 > 0$

Here, linear membership and non-membership functions for the fuzzy objective and constraint goals are

$$\mu_0(f_0(x_1,x_2)) = \begin{cases} 1 & if\ 2x_1^{-2}x_2^{-3} \le 57.87 \\ \frac{60.78-2x_1^{-2}x_2^{-3}}{2.91} & if\ 57.87 \le 2x_1^{-2}x_2^{-3} \le 60.78 \\ 0 & if\ 2x_1^{-2}x_2^{-3} \ge 60.78 \end{cases}$$

$$\mu_1(f_1(x_1,x_2)) = \begin{cases} 1 & if\ x_1^{-1}x_2^{-2} \le 6.75 \\ \frac{6.94-x_1^{-1}x_2^{-2}}{0.19} & if\ 6.75 \le x_1^{-1}x_2^{-2} \le 6.94 \\ 0 & if\ x_1^{-1}x_2^{-2} \ge 6.94 \end{cases}$$

$$\nu_0(f_0(x_1,x_2)) = \begin{cases} 0 & if\ 2x_1^{-2}x_2^{-3} \le 59.03 \\ \frac{2x_1^{-2}x_2^{-3}-59.03}{1.75} & if\ 59.03 \le 2x_1^{-2}x_2^{-3} \le 60.78 \\ 1 & if\ 2x_1^{-2}x_2^{-3} \ge 60.78 \end{cases}$$

$$\nu_1(f_1(x_1,x_2)) = \begin{cases} 0 & if\ x_1^{-1}x_2^{-2} \le 6.83 \\ \frac{x_1^{-1}x_2^{-2}-6.83}{0.11} & if\ 6.83 \le x_1^{-1}x_2^{-2} \le 6.94 \\ 1 & if\ x_1^{-1}x_2^{-2} \ge 6.94 \end{cases}$$

Minimize $\left(\frac{1}{0.19}+\frac{1}{0.11}\right) x_1^{-1}x_2^{-2} + \left(\frac{1}{2.91}+\frac{1}{1.75}\right) 2x_1^{-2}x_2^{-3}$

subject to $x_1 + x_2 \le 1$

$x_1, x_2 > 0$

Minimize V $(x_1,x_2) = 14.354x_1^{-1}x_2^{-2} + 1.828x_1^{-2}x_2^{-3}$

Subject to $x_1 + x_2 \le 1$

$x_1, x_2 > 0$

Here DD = 4−(2+1) = 1

The DP of this GP is

Max d (w) $=\left(\frac{14.354}{w_{01}}\right)^{w_{01}}\left(\frac{1.828}{w_{02}}\right)^{w_{02}}\left(\frac{1}{w_{11}}\right)^{w_{11}}\left(\frac{1}{w_{12}}\right)^{w_{12}}\times$

$(w_{11}+w_{12})^{(w_{11}+w_{12})}$

such that $w_{01}+w_{02}=1$,

$-w_{01}-2w_{02}+w_{11}=0$,

$-2w_{01}-3w_{02}+w_{12}=0$,

So $w_{02}=1-w_{01}, w_{11}=2-w_{01}, w_{12}=3-w_{01}$

Max d (w_{01}) $=\left(\frac{14.354}{w_{01}}\right)^{w_{01}}\left(\frac{1.828}{1-w_{01}}\right)^{1-w_{01}}\left(\frac{1}{2-w_{01}}\right)^{2-w_{01}}\left(\frac{1}{3-w_{01}}\right)^{3-w_{01}}\times$

$(5-2w_{01})^{(5-2w_{01})}$

Subject to $0 < w_{01} < 1$

For optimality, $\frac{d(\mathrm{d}(w01))}{dw_{01}}=0$

$14.354(1-w_{01})(2-w_{01})(3-w_{01}) = 1.828w_{01}(5-2w_{01})^2$

$w_{01}^{*}= 0.6454384$, $w_{02}^{*}= 0.3545616$, $w_{11}^{*}= 1.3545616$, $w_{12}^{*}= 2.3545616$

$x_1^{*}= 0.365197$, $x_2^{*} = 0.63348027$

$f_0^{*}(x_1^{*},x_2^{*}) = 58.62182$, $f_1^{*}(x_1^{*},x_2^{*}) = 6.795091$

9. Neutrosophic Geometric Programming

Let us consider a neutrosophic geometric programming problem with neutrosophic objective goal as

$$\widetilde{Min}^n f_0(x) \qquad (13)$$

Subject to $f_j(x) \lesssim n\ b_j$ j=1,2,…………..,m

$x \geq 0$

Here the symbol "≲n" denotes the Neutrosophic variant of "≤". Now for Neutrosophic geometric programming linear Truth membership (simply membership), Falsity membership (simply non-membership) and Indeterminacy membership functions can be presented as follows.

$$\mu_j\ (f_j\ (x)) = \begin{cases} 1 & if\ f_j(x) \leq f_j^{0} \\ \frac{f_j' - f_j(t)}{f_j' - f_j^{0}} & if\ f_j^{0} \leq f_j(x) \leq f_j' \\ 0 & if\ f_j(x) \geq f_j' \end{cases}$$

for j= 0,1,2,3,…………,m.

$$\nu_j\ (f_j\ (x)) = \begin{cases} 0 & if\ f_j(x) \leq f_j' - f_j'' \\ \frac{f_j(x) - (f_j' - f_j'')}{f_j''} & if\ f_j' - f_j'' \leq f_j(x) \leq f_j' \\ 1 & if\ f_j(x) \geq f_j' \end{cases}$$

for j= 0,1,2,3,…………m.

$$\sigma_j\ (f_j\ (x)) = \begin{cases} 1 & if\ f_j(x) \leq f_j^{0} \\ \frac{(f_j' - f_j''') - f_j(x)}{f_j' - f_j''' - f_j^{0}} & if\ f_j^{0} \leq f_j(x) \leq f_j' - f_j''' \\ 0 & if\ f_j(x) \geq f_j' - f_j''' \end{cases}$$

for j= 0,1,2,3,…………,m.

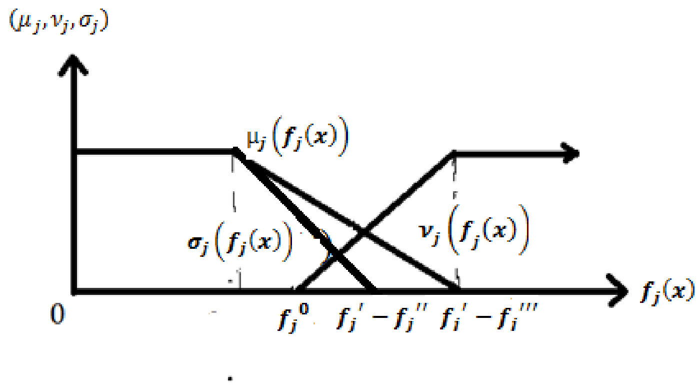

Figure-3: Truth membership, Falsity membership and Indeterminacy membership functions of a minimization-type objective function.

Now Neutrosophic geometric programming problem (13) with Truth membership, Falsity membership and Indeterminacy membership functions can be written as

Maximize $\mu_j\ (f_j\ (x))$ (14)

Minimize $\nu_j\ (f_j\ (x))$

Maximize $\sigma_j\ (f_j\ (x))$

subject to $x \geq 0$

for j = 0,1,2,………..,m.

Using weighted-sum method and giving equal priority on all Truth membership, Falsity membership and Indeterminacy membership functions the above problem (14) becomes

Maximize $V_A = \sum_{j=0}^{m}\{\mu_j(f_j\ (x)) - \nu_j(f_j\ (x)) + \sigma_j\ (f_j\ (x))\}$

subject to $x \geq 0$

The above problem is similar to

$$\text{Min } V_{A1} = \sum_{j=0}^{m}\left\{\left(\frac{1}{f_j{}'-f_j{}^{o}}+\frac{1}{f_j{}''}+\frac{1}{f_j{}'-f_j{}'''-f_j{}^{o}}\right)f_j\,(x)-\left(\frac{f_j{}'-f_j{}''}{f_j{}''}+\frac{f_j{}'}{f_j{}'-f_j{}^{o}}+\frac{f_j{}'-f_j{}'''}{f_j{}'-f_j{}'''-f_j{}^{o}}\right)\right\}$$

Subject to $x \geq 0$ (15)

Where $f_j(x) = \sum_{k=1}^{N_j} c_{jk} \prod_{i=1}^{n} x_i{}^{a_{jki}}$

Where c_{jk} (> 0) and a_{jki} (i=1,2,…….,n; k=1,2,………,N_j ; j=0,1,2,…….,m ;) are real.

$x \equiv (x_1, x_2, \dots \dots \dots, x_n)^T$.

By usual geometric programming technique the posynomial geometric programming problem (15) can be solved

10. Numerical Example

Let us take a neutrosophic geometric programming problem with neutrosophic objective goal as

$$\widetilde{Min}^n\ f_0\ (x_1, x_2) \;=\; 2x_1{}^{-2}x_2{}^{-3}$$

Here objective goal is 57.87 with tolerance 2.91

$f_1\ (x_1, x_2) \;= x_1{}^{-1}x_2{}^{-2} \leq$ 6.75 (with tolerance 0.19)

$f_2\ (x_1, x_2) \;= x_1 + x_2 \leq 1$

$x_1, x_2 > 0.$

Here, linear Truth membership, Falsity membership and Indeterminacy membership functions for the fuzzy objective and constraint goals are

$$\mu_0\,(f_0(x_1,x_2)) = \begin{cases} 1 & if\ 2x_1{}^{-2}x_2{}^{-3} \leq\ 57.87 \\ \frac{60.78-2x_1{}^{-2}x_2{}^{-3}}{2.91} & if\ \ 57.87 \leq\ 2x_1{}^{-2}x_2{}^{-3} \leq\ 60.78 \\ 0 & if\ 2x_1{}^{-2}x_2{}^{-3} \geq 60.78 \end{cases}$$

$$\mu_1(f_1(x_1,x_2)) = \begin{cases} 1 & if\ x_1^{-1}x_2^{-2} \le 6.75 \\ \frac{6.94 - x_1^{-1}x_2^{-2}}{0.19} & if\ 6.75 \le x_1^{-1}x_2^{-2} \le 6.94 \\ 0 & if x_1^{-1}x_2^{-2} \ge 6.94 \end{cases}$$

$$\nu_0(f_0(x_1,x_2)) = \begin{cases} 0 & if\ 2x_1^{-2}x_2^{-3} \le 59.03 \\ \frac{2x_1^{-2}x_2^{-3} - 59.03}{1.75} & if\ 59.03 \le 2x_1^{-2}x_2^{-3} \le 60.78 \\ 1 & if\ 2x_1^{-2}x_2^{-3} \ge 60.78 \end{cases}$$

$$\nu_1(f_1(x_1,x_2)) = \begin{cases} 0 & if\ x_1^{-1}x_2^{-2} \le 6.83 \\ \frac{x_1^{-1}x_2^{-2} - 6.83}{0.11} & if\ 6.83 \le x_1^{-1}x_2^{-2} \le 6.94 \\ 1 & if x_1^{-1}x_2^{-2} \ge 6.94 \end{cases}$$

$$\sigma_0(f_0(x_1,x_2)) = \begin{cases} 1 & if\ 2x_1^{-2}x_2^{-3} \le 57.87 \\ \frac{59.50 - 2x_1^{-2}x_2^{-3}}{1.63} & if\ 57.87 \le 2x_1^{-2}x_2^{-3} \le 59.50 \\ 0 & if\ 2x_1^{-2}x_2^{-3} \ge 59.50 \end{cases}$$

$$\sigma_1(f_1(x_1,x_2)) = \begin{cases} 1 & if\ x_1^{-1}x_2^{-2} \le 6.75 \\ \frac{6.88 - x_1^{-1}x_2^{-2}}{0.13} & if\ 6.75 \le x_1^{-1}x_2^{-2} \le 6.88 \\ 0 & if x_1^{-1}x_2^{-2} \ge 6.88 \end{cases}$$

using truth, indeterminacy, falsity membership functions above problem can be formulated as

Minimize V $(x_1,x_2) = 22.046x_1^{-1}x_2^{-2} + 3.057132x_1^{-2}x_2^{-3}$

subject to $x_1 + x_2 \le 1$

$x_1,x_2 > 0$

Here DD = 4−(2+1) = 1

The DP of this GP is

Max d (w) $=\left(\frac{22.046}{w_{01}}\right)^{w_{01}}\left(\frac{3.057132}{w_{02}}\right)^{w_{02}}\left(\frac{1}{w_{11}}\right)^{w_{11}}\left(\frac{1}{w_{12}}\right)^{w_{12}}\times$

$(w_{11}+w_{12})^{(w_{11}+w_{12})}$

such that $w_{01}+w_{02}=1$,

$-w_{01}-2w_{02}+w_{11}=0$,

$-2w_{01}-3w_{02}+w_{12}=0$,

So $w_{02}=1-w_{01}, w_{11}=2-w_{01}, w_{12}=3-w_{01}$

Max d (w_{01}) $=\left(\frac{22.046}{w_{01}}\right)^{w_{01}}\left(\frac{3.057132}{1-w_{01}}\right)^{1-w_{01}}\left(\frac{1}{2-w_{01}}\right)^{2-w_{01}}\left(\frac{1}{3-w_{01}}\right)^{3-w_{01}}\times(5-2w_{01})^{(5-2w_{01})}$

Subject to $0<w_{01}<1$

For optimality, $\frac{d(\mathrm{d}(w_{01}))}{dw_{01}}=0$

$22.046(1-w_{01})(2-w_{01})(3-w_{01})=3.057132w_{01}(5-2w_{01})^2$

$w_{01}{}^*=0.6260958,\ w_{02}{}^*=0.3739042,\ w_{11}{}^*=1.3739042,\ w_{12}{}^*=2.3739042$

$x_1{}^*=0.366588,\ x_2{}^*=0.633411$

$f_0{}^*(x_1{}^*,x_2{}^*)=58.56211,\ f_1{}^*(x_1{}^*,x_2{}^*)=6.799086$

11. Application of Neutrosophic Geometric Programming in Gravel Box Design Problem

Gravel Box Problem: A sum of 800 cubic-meters of gravel is to be carried across a river on a barrage. A box (with an open top) is to be made for this occasion. After the whole gravel has been carried, the box is to be rejected. The transport cost per round trip of barrage of box is Rs 1 and the cost of substances of the ends of the box are Rs20/m^2 and the cost of substances of other two sides and bottom are Rs 10/m^2 and Rs 80/m^2. Find the size of the box that is to be made for this occasion and the total optimal cost.

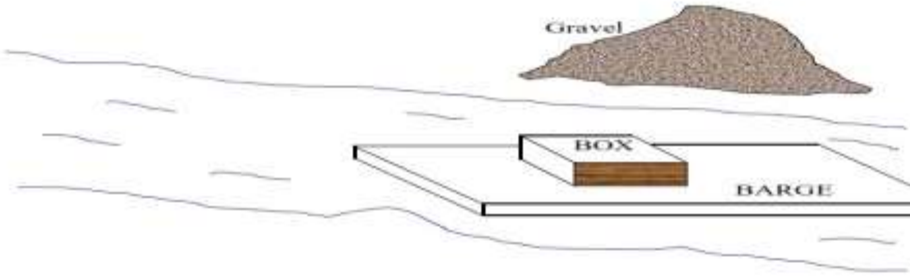

Figure -4: Gravel box design

Let length $=x_1$ m, breadth = x_2 m, height = x_3 m. The area of the end of the gravel box $=x_2x_3$ m^2. Area of the sides $=x_1x_3$ m^2. Area of the bottom $=x_1x_2$ m^2 .The volume of the gravel box$=x_1x_2x_3$ m^3. Transport cost: Rs $\frac{80}{x_1x_2x_3}$. Material cost: $40x_2x_3$.

So the geometric programming problem is

Min $f_0(x_1,x_2,x_3) = \frac{80}{x_1x_2x_3} + 40x_2x_3$

such that $f_1(x_1,x_2,x_3) = x_1x_2 + 2x_1x_3 \leq 4$.

$x_1,x_2,x_3 > 0$.

Here objective goal is 90 (with truth-flexibility 8, falsity-flexibility 5, and indeterminacy-flexibility 5)

and constrained goal

$f_1\ (x_1, x_2, x_3) \leq\ 4$ (with truth- flexibility 0.9, falsity- flexibility 0.5, indeterminacy-flexibility 0.6)

$x_1{}^*$= 2.4775, $x_2{}^*$= 1.1271, $x_3{}^*$= 0.5635

$f_0{}^*(x_1{}^*, x_2{}^*, x_3{}^*)$ = 76.237, $f_1{}^*(x_1{}^*, x_2{}^*, x_3{}^*)$ = 4.5856.

12. Conclusion:

In respect of contrasting the Neutrosophic geometric programming method with Fuzzy, Intuitionistic fuzzy geometric programming method, we also got the solution of the given numerical problem by Fuzzy and Intuitionistic fuzzy optimization method. The aims of the present study is to give the constructive algorithm for geometric programming method in imprecise conditions for obtaining optimal solutions to a single-objective non-linear programming problem.

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Single Valued Pentaparitioned Neutrosophic Off-Set / Over-Set / Under-Set

Bimal Shil[1], Rakhal Das[2], and Suman Das[3,*]

[1]Department of Statistics, Tripura University, Agartala, 799022, Tripura, India.

E-mail: [1]bimalshil738@gmail.com

[2]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.

E-mail: [2]rakhal.mathematics@tripurauniv.in, rakhaldas95@gmail.com

[3]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.

E-mail: [3]sumandas18842@gmail.com, dr.suman1995@yahoo.com

***Correspondence:** sumandas18842@gmail.com

Abstract: The main focus of this paper is to introduce the notion of single valued pentapartitioned neutrosophic off set / over set / under set. Besides, we establish several operations on single valued pentapartitioned neutrosophic off sets / over sets / under sets. Besides, we furnish some suitable examples to validate the results established in this article. Further, we establish some interesting results on single valued pentapartitioned neutrosophic off set / over set / under set.

Keywords: Neutrosophic Set; SV-PN-Set; SV-PN-off-set; SV-PN-over-set; SV-PN-under-set.

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1. Introduction: In 1965, Zadeh [33] grounded the concept of fuzzy set, where every element has membership values between 0 and 1. Afterwards, Atanassov [1] introduced the notion of intuitionistic fuzzy set as an extension of fuzzy set. In 1998, Smarandache [27] presented the concept of neutrosophic set (in short N-S) by extending the idea of fuzzy set and intuitionistic fuzzy set to deal with the uncertainty events having indeterminacy. In every N-S, each member has three independent components namely truth, indeterminacy and false membership values. Later on, Wang et al. [32] grounded the notion of single valued neutrosophic set (in short SV-N-S), which is basically a subclass of N-S. One can use SV-N-S to represent indeterminate and incomplete information which makes trouble to take decision (or in selection) in the real world. Thereafter, many researchers of different countries used the notion of SV-N-S in their model (or algorithm) in the different branches of real world such as medical diagnosis, educational problem, social problems, decision-making problems, conflict resolution, image processing, etc. In 2013, Smarandache [28] introduced the idea of n-valued refined neutrosophic logic, and applied this notion in physics. In 2016, Smarandache [29] grounded a new concept of neutrosophic over-set, neutrosophic under-set, neutrosophic off-set, and studied their various properties.

In the year 2020, Mallick and Pramanik [25] grounded the idea of single valued pentapartitioned neutrosophic set by splitting the indeterminacy into three independent components namely contradiction, ignorance and unknown-membership, and studied several properties of them. In 2021, Das and Tripathy [17] grounded the notion of pentapartitioned neutrosophic topological space and formulated several results on it. Afterwards, Das et al. [12] established an MADM strategy based on tangent similarity measure. Later on, Majumder et al. [24] presented a cosine similarity measure based MADM strategy under the single valued pentapartitioned neutrosophic set environment. Recently, Das et al. [13] established a MADM strategy using grey relational analysis method under the single valued pentapartitioned neutrosophic set environment.

In this article, we introduce the notion of single valued pentapartitioned neutrosophic off set / over set / under set. Besides, we establish several operations on single valued pentapartitioned neutrosophic off sets / over sets / under sets. Besides, we furnish some suitable examples to validate the results established in this article. Further, we establish some interesting results on single valued pentapartitioned neutrosophic off set / over set / under set.

Research gap: No investigation on single valued pentapartitioned neutrosophic over-set / under-set / off-set has been reported in the recent literature.

Motivation: To fill the research gap, we introduce and study the notion of single valued pentapartitioned neutrosophic over-set/under-set/off-set.

The remaining part of this article has been divided into following three sections:

In section 2, we recall some basic definitions and properties related to N-Ss, single valued neutrosophic over-sets / under-sets / off-sets and single valued pentapartitioned neutrosophic sets. In section 3, we introduce the notion of single valued pentapartitioned neutrosophic over-set / under-set / off-set, and study some of their basic properties. In this section, we also formulated several interesting results on single valued pentapartitioned neutrosophic over-set / under-set / off-set. In section 4, we conclude the word done in this article.

2. Some Relevant Results:

In this section, we give some relevant definitions and results for our study of the main results of this paper.

The notion of N-S was defined by Smarandache [27] in the following way:

Assume that L be a non-empty set. Then D, an N-S over L is defined by:

$D=\{(\alpha,T_D(\alpha),I_D(\alpha),F_D(\alpha)):\alpha\in L\}$, where T_D, I_D, F_D are the truth, indeterminacy and false membership functions from the whole set L to [0, 1] respectively. So, $0 \le T_D(\alpha) + I_D(\alpha) + F_D(\alpha) \le 3$, for each $\alpha\in L$.

The notions of neutrosophic over-set, neutrosophic under-set, and neutrosophic off-set was also grounded by Smarandache [29] in the year 2016, and defined as follows:

Let L be a universal set. Then, a single valued neutrosophic over set D over L is defined by:

$D=\{(\alpha,T_D(\alpha),I_D(\alpha),F_D(\alpha)):\alpha\in L\}$, such that at least one member in D has at least one of the neutrosophic component that is greater than 1. Here, T_D, I_D, $F_D : L\to[0, \check{N}]$ are the truth, indeterminacy, and false membership functions respectively such that $0 < 1 < \check{N}$, and $\check{N}$ is the over-limit of D.

For example, D={(a,0.2,0.3,1.5), (b,0.9,1.3,0.2), (c,0.2,0.1,0.6)} is an neutrosophic over set defined over L. But K={(a,0.3,0.5,0.9), (b,0.8,0.4,0.9), (c,0.2,0.5,0.5)} is not an neutrosophic over set defined over L.

Let L be a universal set. Then, a single valued neutrosophic under set Y over L is defined by:

$Y=\{(\alpha,T_Y(\alpha),I_Y(\alpha),F_Y(\alpha)):\alpha\in L\}$, such that at least one member in Y has at least one of the neutrosophic component that is smaller than 0. That is, the truth, indeterminacy, and false membership functions T_Y, I_Y, F_Y are defined from L to $[\check{N}, 1]$ such that $\check{N} < 0 < 1$, and $\check{N}$ is said to be the under-limit of Y.
For example, Y={(a,0.2,-0.3,0.9), (b,-0.5,0.2,-0.2), (c,0.2,-0.1,0.6)} is an neutrosophic under-set defined over L. But Z={(a,0.3,0.5,0.9), (b,0.8,0.4,0.9), (c,0.2,0.5,0.5)} is not an neutrosophic under-set defined over L.

A single valued neutrosophic off-set K over a fixed set L is defined by:
$K=\{(\alpha,T_K(\alpha),I_K(\alpha),F_K(\alpha)):\alpha\in L\}$, such that some members of K has at least one of the neutrosophic component that is smaller than 0 and at least one of the neutrosophic component that is greater than 1. That is, the truth, indeterminacy, and false membership functions T_K, I_K, F_K are defined from L to $[\check{N}, \tilde{N}]$ such that $\check{N} < 0 < 1 < \tilde{N}$. Here, $\check{N}$ and $\tilde{N}$ are said to be the under-limit and over-limit of K respectively.

For example, K={(a,0.2,-0.3,1.6), (b,-0.5,0.2,0.2), (c,1.3,0.1,0.6)} is an neutrosophic off set defined over L. But L={(a,0.3,1.5,0.9), (b,0.8,0.4,0.9), (c,0.2,0.5,-0.5)} is not an neutrosophic off set defined over L.

Recently, Mallick and Pramanik [14] grounded the idea of pentapartitioned neutrosophic set (in short PNS) by extending the notions of N-S.

Suppose that L be a fixed set. Then D, a PNS over L is defined as follows:
$D=\{(\alpha,T_D(\alpha),C_D(\alpha),R_D(\alpha),U_D(\alpha),F_D(\alpha)):\alpha\in L\}$, where T_D, C_D, R_D, U_D, F_D: $L\rightarrow[0, 1]$ are the truth, contradiction, ignorance, unknown and false membership functions respectively. So,

$$0\leq T_D(\alpha)+C_D(\alpha)+R_D(\alpha)+U_D(\alpha)+F_D(\alpha)\leq 5.$$

Let $X=\{(\alpha,T_X(\alpha),C_X(\alpha),R_X(\alpha),U_X(\alpha),F_X(\alpha)):\alpha\in L\}$ and $Y=\{(\alpha,T_Y(\alpha),C_Y(\alpha),R_Y(\alpha), U_Y(\alpha),F_Y(\alpha)):\alpha\in L\}$ be two PNSs over L. Then,

(i) $X\subseteq Y \Leftrightarrow T_X(\alpha)\leq T_Y(\alpha)$, $C_X(\alpha)\leq C_Y(\alpha)$, $R_X(\alpha)\geq R_Y(\alpha)$, $U_X(\alpha)\geq U_Y(\alpha)$, $F_X(\alpha)\geq F_Y(\alpha)$, for all $\alpha\in L$.

(ii) $X\cup Y=\{(\alpha, \max\{T_X(\alpha), T_Y(\alpha)\}, \max\{C_X(\alpha), C_Y(\alpha)\}, \min\{R_X(\alpha), R_X(\alpha)\}, \min\{U_X(\alpha), U_X(\alpha)\}, \min\{F_X(\alpha), F_X(\alpha)\}):\alpha\in L\}$.

(iii) $X^c=\{(\alpha,T_X(\alpha),C_X(\alpha),1-R_X(\alpha),U_X(\alpha),F_X(\alpha)):\alpha\in L\}$.

(iv) $X\cap Y=\{(\alpha, \min\{T_X(\alpha), C_Y(\alpha)\}, \min\{C_X(\alpha), C_Y(\alpha)\}, \max\{R_X(\alpha), R_X(\alpha)\}, \max\{U_X(\alpha), U_X(\alpha)\}, \max\{F_X(\alpha), F_X(\alpha)\}):\alpha\in L\}$.

3. Pentapartitioned Neutrosophic Off-set / Over-set / Under-set:

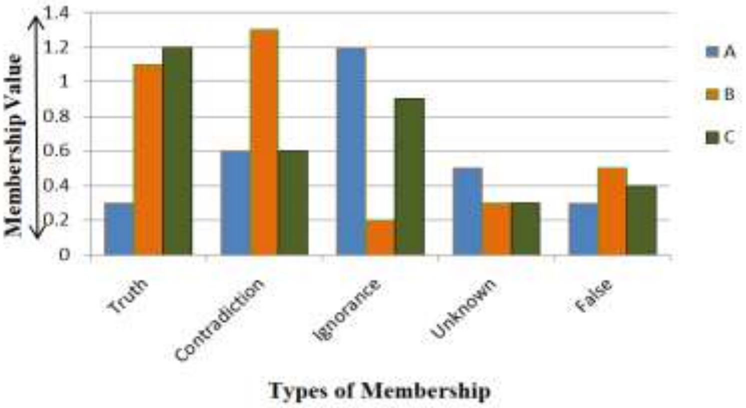
In this section, we introduce the notions of pentapartitioned neutrosophic off-set (in short PN-off-S) / pentapartitioned neutrosophic under-set (in short PN-under-S) / pentapartitioned neutrosophic over-set (in short PN-over-S). Then, we formulate and study some interesting results on them.

Definition 3.1. Let L be a universal set. Then D, a PN-over-S over L is defined by:
$D=\{(\alpha,T_D(\alpha),C_D(\alpha),G_D(\alpha),U_D(\alpha),F_D(\alpha)):\alpha\in L\}$, such that at least one member in D has at least one of the pentapartitioned neutrosophic component that is greater than 1 and no member has pentapartitioned neutrosophic components that are less than zero. Here, T_D, C_D, G_D, U_D, F_D : $L\rightarrow[0,$

Ň] are the truth, contradiction, ignorance, unknown and false membership functions respectively such that 1 < Ň, and Ň is said to be the over-limit of D.

Example 3.1. Assume that L={a, b, c} be a fixed set. Then, D={(a,0.3,0.6,1.2,0.5,0.3), (b,1.1,1.3,0.2,0.3, 0.5), (c,1.2,0.6,0.9,0.3,0.4)} is a PN-over-S defined over L. But K={(a,0.1,0.4,0.4,0.6,0.8), (b,0.9,0.5,0.8,0.6, 0.8), (c,0.1,0.6,0.7,0.8,0.9)} is not a PN-over-S defined over L.

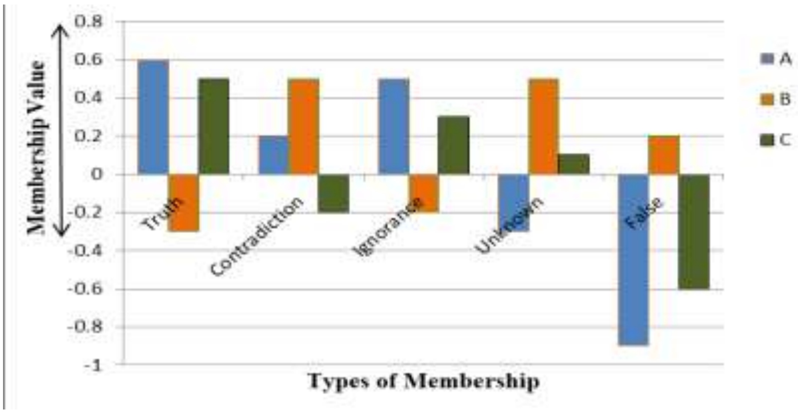
The pictorial representation of **Example 3.1** is given as follows:



Definition 3.2. Suppose that L be a fixed universal set. Then Y, a PN-under-S over L is defined by: Y={(α,T_Y(α),C_Y(α),G_Y(α),U_Y(α),F_Y(α)):α∈L}, such that at least one member in Y has at least one of the neutrosophic component that is smaller than 0 and no member has pentapartitioned neutrosophic components that are greater than one. That is, the truth, contradiction, ignorance, unknown, and false membership functions T_Y, C_Y, G_Y,U_Y, F_Y are defined from L to [Ň,1] such that Ň<0, and Ň is said to be the under-limit of Y.

Example 3.2. Assume that L={a, b, c} be a fixed set. Then, Y={(a,0.6,0.2,0.5,-0.3,-0.9), (b,-0.3,0.5,-0.2, 0.5,0.2), (c,0.5,-0.2,0.3,0.1,-0.6)} is a PN-under-S over L. But Z={(a,0.3,0.2,0.8,0.5,0.9), (b,0.9,0.8,0.5,0.4, 0.9), (c,0.2,0.5,0.3,0.5,0.5)} is not a PN-under-S over L.

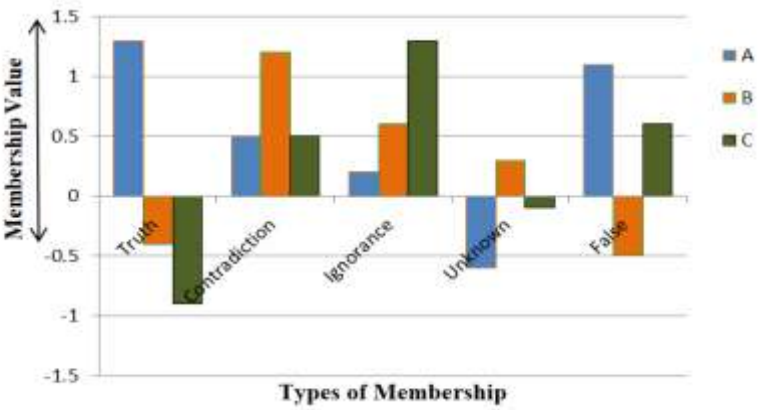
The pictorial representation of **Example 3.2** is given as follows:



Definition 3.3. Assume that L be a fixed non-empty set. Then K, a PN-off-S over L is defined by: K={(α,T_K(α),C_K(α),G_K(α),U_K(α),F_K(α)):α∈L}, such that some members of K has at least one of the pentapartitioned neutrosophic component that is smaller than 0 and at least one of the pentapartitioned neutrosophic component that is greater than 1. That is, the truth, contradiction, ignorance, unknown, and false membership functions T_K, C_K, G_K, U_K, F_K are defined from L to [$\check{N}$, $\tilde{N}$] such that $\check{N} < 0 < 1 < \tilde{N}$. Here, $\check{N}$ and $\tilde{N}$ are called the under-limit and over-limit of K respectively.

Example 3.3. Assume that L={a, b, c} be a fixed set. Then, K={(a,1.3,0.5,0.2,-0.6,1.1), (b,-0.4,1.2,0.6, 0.3,-0.5), (c,-0.9,0.5,1.3,-0.1,0.6)} is a PN-off-S defined over L. But L={(a,0.3,0.3,0.4,0.4, 0.9), (b,0.9,0.4,0.1,0.2,0.3), (c,0.6,0.4,0.4,0.3,0.3)} is not a PN-off-S defined over L.

The pictorial representation of **Example 3.3** is given as follows:



Definition 3.4. Assume that L be a fixed set. Then, null PN-over-S (0_P) and the absolute PN-over-S (1_P) over L is defined by:

(i) 0_P = {(α,0,$\tilde{N}$,$\tilde{N}$,$\tilde{N}$,0): α∈L};

(ii) 1_P = {(α,$\tilde{N}$,0,0,0,$\tilde{N}$): α∈L}.

Definition 3.5. Assume that L be a fixed set. Then, null PN-under-S (0_P) and the absolute PN-under-S (1_P) over L is defined by:

(i) 0_P = {(α,$\check{N}$,1,1,1,$\check{N}$): α∈L};

(ii) 1_P = {(α,1,$\check{N}$,$\check{N}$,$\check{N}$,1): α∈L}.

Definition 3.6. Assume that L be a fixed set. Then, null PN-off-S (0_P) and the absolute PN-off-S (1_P) over L is defined by:

(i) 0_P = {(α,$\check{N}$,$\tilde{N}$,$\tilde{N}$,$\tilde{N}$,$\check{N}$): α∈L};

(ii) 1_P = {(α,$\tilde{N}$,$\check{N}$,$\check{N}$,$\check{N}$,$\tilde{N}$): α∈L}.

Definition 3.7. Assume that K={(α,T_K(α),C_K(α),R_K(α),U_K(α),F_K(α)):α∈L} and Y={(α,T_Y(α),C_Y(α),R_Y(α), U_Y(α),F_Y(α)):α∈L} be two PN-over-Ss / PN-under-Ss / PN-off-Ss. Then, the intersection and union of K and Y is defined by

(i) K∩Y={(α, min{T_K(α),T_Y(α)}, max{C_K(α), C_Y(α)}, max{R_K(α), R_Y(α)}, max{U_K(α), U_Y(α)}, min{F_K(α), F_Y(α)): α∈L};

(ii) $K \cup Y=\{(\alpha, \max\{T_K(\alpha), T_Y(\alpha)\}, \min\{C_K(\alpha), C_Y(\alpha)\}, \min\{R_K(\alpha), R_Y(\alpha)\}, \min\{U_K(\alpha), U_Y(\alpha)\}, \max\{F_K(\alpha), F_Y(\alpha)): \alpha \in L\}$.

Example 3.4. Assume that, L={c, d} be a fixed set. Suppose that K={(c,0.9,0.8,1.3,0.4,1.5), (d,0.2,1.3,1.7, 0.2,0.9)} and Y={(c,0.6,0.3,1.6,1.2,0.8), (d,0.8,0.3,0.8,1.5,0.7)} be two PN-over-Ss over L. Then,

(i) $K \cap Y$={(c,0.6,0.8,1.6,1.2,0.8), (d,0.2,1.3,1.7,1.5,0.7)};

(ii) $K \cup Y$={(c,0.9,0.3,1.3,0.4,1.5), (d,0.8,0.3,0.8,0. 2,0.9)}.

Example 3.5. Assume that, L={c, d} be a fixed set. Suppose that K={(c,0.6,0.6,-0.2,0.5,-0.9), (d,0.5,-0.5, 0.4,0.6,-0.1)} and Y={(c,-0.4,0.7,0.5,0.4,-0.8), (d,0.8,-0.5,-0.3,0.5,0.8)} be two PN-under-Ss over L. Then,

(i) $K \cap Y$={(c,-0.4,0.7,0.5,0.5,-0.9), (d,0.5,-0.5,0.4,0.6,-0.1)};

(ii) $K \cup Y$={(c,0.6,0.6,-0.2,0.4,-0.8), (d,0.8,-0.5,-0.3,0.5,0.8)}.

Example 3.6. Assume that, L={c, d} be a universe of discourse. Let K={(c,0.8,-0.6,0.7,0.6,1.1), (d,1.5,-0.2, 0.9,0.7,0.4)} and Y={(c,1.8,0.9,-0.9,0.7,1.2), (d,0.1,0.7,1.5,-0.6,0.9)} be two PN-off-Ss. Then,

(i) $K \cap Y$={(c,0.8,0.9,0.7,0.7,1.1), (d,0.1,0.7,1.5,0.7,0.4)};

(ii) $K \cup Y$={(c,1.8,-0.6,-0.9,0.6,1.2), (d,1.5,-0.2,0.9,-0.6,0.9)}.

Definition 3.8. Let K and Y be two PN-off-Ss / PN-under-Ss / PN-over-Ss over L. Then,

(i) $K \subseteq Y$ if and only if $T_K(\alpha) \leq T_Y(\alpha)$, $C_K(\alpha) \geq C_Y(\alpha)$, $R_K(\alpha) \geq R_Y(\alpha)$, $U_K(\alpha) \geq U_Y(\alpha)$, $F_K(\alpha) \leq F_Y(\alpha), \forall \alpha \in L$;

(ii) $K^c=\{(\alpha,1-T_K(\alpha),1-C_K(\alpha),1-R_K(\alpha),1-U_K(\alpha),1-F_K(\alpha)):\alpha \in L\}$.

Example 3.7. Suppose that L={c, d} be a non-empty set. Assume that K={(c,1.5,1.9,0.9,0.9,0.5), (d,1.7, 1.3,1.3,0.4,0.3)} and Y={(c,1.6,1.2,0.8,0.3,0.6), (d,1.8,0.8,1.2,0.3,0.9)} be two PN-over-Ss. Then, $K \subseteq Y$, and K^c={(c,-0.5,-0.9,0.1,0.1,0.5), (d,-0.7,-0.3,-0.3,0.6,0.7)} and Y^c={(c,-0.6,-0.2,0.2,0.7,0.4), (d,-0.8,0.2,-0.2, 0.7,0.1)}.

Example 3.8. Suppose that L={c, d} be a non-empty set. Assume that K={(c,0.8,0.7,-0.8,0.3,0.9) (d,-0.9, 0.8,-0.2,0.3,-0.1)} and Y={(c,0.9,-0.5,-0.9,0.1,0.9), (d,0.7,0.5,-0.3,-0.5,0.1)} be two PN-under-Ss over L. Then, $K \subseteq Y$, and K^c={(c,0.2,0.3,1.8,0.7,0.1), (d,1.9,0.2,1.2,0.7,1.1)} and Y^c={(c,0.1,1.5,1.9,0.9,0.1), (d,0.3, 0.5,1.3,1.5,0.9)}.

Example 3.9. Suppose that L={c, d} be a non-empty set. Assume that K={(c,0.8,-0.7,1.5,0.6,1.5), (d,1.7, 0.5,-0.1,0.3,0.5)} and Y={(c,0.8,-0.8,0.9,0.5,1.7), (d,1.8,0.2,-0.5,0.2,0.8)} be two PN-off-Ss over L. Then, $K \subseteq Y$, and K^c={(c,0.2,1.7,-0.5,0.4,-0.5), (d,-0.7,0.5,1.1,0.7,0.5)} and Y^c={(c,0.2,1.8,0.1,0.5,-0.7), (d,-0.8,0.8, 1.5,0.8,0.2)}.

Proposition 3.1. Assume that K and Y be two PN-off-Ss / PN-under-Ss / PN-over-Ss over L. Then,

(i) $K \cup Y = Y \cup K$;

(ii) $K \cap Y = Y \cap K$.

Proof. It is known that, $K \cup Y = \{(\alpha, \max(T_K(\alpha), T_Y(\alpha)), \min(C_K(\alpha), C_Y(\alpha)), \min(R_K(\alpha), R_Y(\alpha)), \min(U_K(\alpha), U_Y(\alpha)),\max(F_K(\alpha), F_Y(\alpha)): \alpha \in L\} = \{(\alpha, \max(T_Y(\alpha), T_K(\alpha)), \min(C_Y(\alpha), C_K(\alpha)), \min(R_Y(\alpha), R_K(\alpha)), \min(U_Y(\alpha), U_K(\alpha)), \max(F_Y(\alpha), F_K(\alpha)): \alpha \in L\} = Y \cup K$.

Therefore, $K \cup Y = Y \cup K$.

Similarly, it can be established that $K \cap Y = Y \cap K$.

Proposition 3.2. Let K_1, K_2 and K_3 be three PN-off-Ss / PN-under-Ss / PN-over-Ss over L. Then, $K_1 \cup (K_2 \cup K_3) = (K_1 \cup K_2) \cup K_3$ and $K_1 \cap (K_2 \cap K_3) = (K_1 \cap K_2) \cap K_3$.

Proof. Suppose that, $\alpha_i \in K_1 \cup (K_2 \cup K_3)$. Therefore,

$\alpha_i \in K_1 \cup \{(\alpha_i, \max(T_{K_2}(\alpha_i), T_{K_3}(\alpha_i)), \min(C_{K_2}(\alpha_i), C_{K_3}(\alpha_i)), \min(R_{K_2}(\alpha_i), R_{K_3}(\alpha_i)), \min(U_{K_2}(\alpha_i), U_{K_3}(\alpha_i)), \max(F_{K_2}(\alpha_i), F_{K_3}(\alpha_i)): \alpha_i \in L\}$

$\Rightarrow \alpha_i \in \{(\alpha_i, \max(T_{K_1}(\alpha_i), T_{K_2}(\alpha_i), T_{K_3}(\alpha_i)), \min(C_{K_1}(\alpha_i), C_{K_2}(\alpha_i), C_{K_3}(\alpha_i)), \min(R_{K_1}(\alpha_i), R_{K_2}(\alpha_i), R_{K_3}(\alpha_i)), \min(U_{K_1}(\alpha_i), U_{K_2}(\alpha_i), U_{K_3}(\alpha_i)), \max(F_{K_1}(\alpha_i), F_{K_2}(\alpha_i), F_{K_3}(\alpha_i)): \alpha_i \in L\}$

$\Rightarrow \alpha_i \in \{(\alpha_i, \max(T_{K_1}(\alpha_i), T_{K_2}(\alpha_i)), \min(C_{K_1}(\alpha_i), C_{K_2}(\alpha_i)), \min(R_{K_1}(\alpha_i), R_{K_2}(\alpha_i)), \min(U_{K_1}(\alpha_i), U_{K_2}(\alpha_i)), \max(F_{K_1}(\alpha_i), F_{K_2}(\alpha_i)): \alpha_i \in L\} \cup K_3$

$\Rightarrow \alpha_i \in (K_1 \cup K_2) \cup K_3$

$\Rightarrow K_1 \cup (K_2 \cup K_3) \subset (K_1 \cup K_2) \cup K_3$ (1)

Assume that, $\beta_i \in (K_1 \cup K_2) \cup K_3$. Therefore,

$\beta_i \in \{(\beta_i, \max(T_{K_1}(\beta_i), T_{K_2}(\beta_i)), \min(C_{K_1}(\beta_i), C_{K_2}(\beta_i)), \min(R_{K_1}(\beta_i), R_{K_2}(\beta_i)), \min(U_{K_1}(\beta_i), U_{K_2}(\beta_i)), \max(F_{K_1}(\beta_i), F_{K_2}(\beta_i)): \beta_i \in L\} \cup K_3$

$\Rightarrow \beta_i \in \{(\beta_i, \max(T_{K_1}(\beta_i), T_{K_2}(\beta_i), T_{K_3}(\beta_i)), \min(C_{K_1}(\beta_i), C_{K_2}(\beta_i), C_{K_3}(\beta_i)), \min(R_{K_1}(\beta_i), R_{K_2}(\beta_i), R_{K_3}(\beta_i)), \min(U_{K_1}(\beta_i), U_{K_2}(\beta_i), U_{K_3}(\beta_i)), \max(F_{K_1}(\beta_i), F_{K_2}(\beta_i), F_{K_3}(\beta_i)) : \beta_i \in L\}$

$\Rightarrow \beta_i \in K_1 \cup \{(\beta_i, \max(T_{K_2}(\beta_i), T_{K_3}(\beta_i)), \min(C_{K_2}(\beta_i), C_{K_3}(\beta_i)), \min(R_{K_2}(\beta_i), R_{K_3}(\beta_i)), \min(U_{K_2}(\beta_i), U_{K_3}(\beta_i)), \max(F_{K_2}(\beta_i), F_{K_3}(\beta_i)) : \beta_i \in L\}$

$\Rightarrow \beta_i \in K_1 \cup (K_2 \cup K_3)$

$\Rightarrow (K_1 \cup K_2) \cup K_3 \subset K_1 \cup (K_2 \cup K_3)$ (2)

From eqs (1) and (2), we have, $K_1 \cup (K_2 \cup K_3) = (K_1 \cup K_2) \cup K_3$.

Similarly, it can be established that, $K_1 \cap (K_2 \cap K_3) = (K_1 \cap K_2) \cap K_3$.

Proposition 3.3. Let K_1, K_2 and K_3 be three PN-off-Ss / PN-under-Ss / PN-over-Ss over L. Then, $K_1 \cup (K_2 \cap K_3) = (K_1 \cup K_2) \cap (K_1 \cup K_3)$ and $K_1 \cap (K_2 \cup K_3) = (K_1 \cap K_2) \cup (K_1 \cap K_3)$.

Proof. Suppose that $\alpha_i \in K_1 \cup (K_2 \cap K_3)$. Therefore,

$\alpha_i \in K_1 \cup \{(\alpha_i, \min(T_{K_2}(\alpha_i), T_{K_3}(\alpha_i)), \max(C_{K_2}(\alpha_i), C_{K_3}(\alpha_i)), \max(R_{K_2}(\alpha_i), R_{K_3}(\alpha_i)), \max(U_{K_2}(\alpha_i), U_{K_3}(\alpha_i)), \min(F_{K_2}(\alpha_i), F_{K_3}(\alpha_i)): \alpha_i \in L\}$

$\Rightarrow \alpha_i \in \{(\alpha_i, \max(T_{K_1}(\alpha_i), \min(T_{K_2}(\alpha_i), T_{K_3}(\alpha_i))), \min(C_{K_1}(\alpha_i), \max(C_{K_2}(\alpha_i), C_{K_3}(\alpha_i))), \min(R_{K_1}(\alpha_i), \max(R_{K_2}(\alpha_i), R_{K_3}(\alpha_i))), \min(U_{K_1}(\alpha_i), \max(U_{K_2}(\alpha_i), U_{K_3}(\alpha_i))), \max(F_{K_1}(\alpha_i), \min(F_{K_2}(\alpha_i), F_{K_3}(\alpha_i))): \alpha_i \in L\}$

$\Rightarrow \alpha_i \in \{(\alpha_i, \max(T_{K_1}(\alpha_i), T_{K_2}(\alpha_i)), \min(C_{K_1}(\alpha_i), C_{K_2}(\alpha_i)), \min(R_{K_1}(\alpha_i), R_{K_2}(\alpha_i)), \min(U_{K_1}(\alpha_i), U_{K_2}(\alpha_i)), \max(F_{K_1}(\alpha_i), F_{K_2}(\alpha_i)): \alpha_i \in L\} \cap \{(\alpha_i, \max(T_{K_1}(\alpha_i), T_{K_3}(\alpha_i)), \min(C_{K_1}(\alpha_i), C_{K_3}(\alpha_i)), \min(R_{K_1}(\alpha_i), R_{K_3}(\alpha_i)), \min(U_{K_1}(\alpha_i), U_{K_3}(\alpha_i)), \max(F_{K_1}(\alpha_i), F_{K_3}(\alpha_i)): \alpha_i \in L\}$

$\Rightarrow \alpha_i \in (K_1 \cup K_2) \cap (K_1 \cup K_3)$

$\Rightarrow K_1 \cup (K_2 \cap K_3) \subset (K_1 \cup K_2) \cap (K_1 \cup K_3)$ (1)

Assume that, $\beta_i \in (K_1 \cup K_2) \cap (K_1 \cup K_3)$. Therefore,

$\beta_i \in \{(\beta_i, \max(T_{K_1}(\beta_i), T_{K_2}(\beta_i)), \min(C_{K_1}(\beta_i), C_{K_2}(\beta_i)), \min(R_{K_1}(\beta_i), R_{K_2}(\beta_i)), \min(U_{K_1}(\beta_i), U_{K_2}(\beta_i)), \max(F_{K_1}(\beta_i), F_{K_2}(\beta_i)): \beta_i \in L\} \cap \{(\beta_i, \max(T_{K_1}(\beta_i), T_{K_3}(\beta_i)), \min(C_{K_1}(\beta_i), C_{K_3}(\beta_i)), \min(R_{K_1}(\beta_i), R_{K_3}(\beta_i)), \min(U_{K_1}(\beta_i), U_{K_3}(\beta_i)), \max(F_{K_1}(\beta_i), F_{K_3}(\beta_i)): \beta_i \in L\}$

$\Rightarrow \beta_i \in \{(\beta_i, \max(T_{K_1}(\beta_i), \min(T_{K_2}(\beta_i), T_{K_3}(\beta_i))), \min(C_{K_1}(\beta_i), \max(C_{K_2}(\beta_i), C_{K_3}(\beta_i))), \min(R_{K_1}(\beta_i), \max(R_{K_2}(\beta_i), R_{K_3}(\beta_i))), \min(U_{K_1}(\beta_i), \max(U_{K_2}(\beta_i), U_{K_3}(\beta_i))), \max(F_{K_1}(\beta_i), \min(F_{K_2}(\beta_i), F_{K_3}(\beta_i))): \beta_i \in L\}$

$\Rightarrow \beta_i \in K_1 \cup (K_2 \cap K_3)$

$\Rightarrow(K_1\cup K_2)\cap(K_1\cup K_3) \subset K_1\cup(K_2\cap K_3)$ (2)

From eqs. (1) and (2), we have $K_1\cup(K_2\cap K_3) = (K_1\cup K_2)\cap(K_1\cup K_3)$.

Similarly, it can be established that, $K_1\cap(K_2\cup K_3) = (K_1\cap K_2)\cup(K_1\cap K_3)$.

Proposition 3.4. Let K_1 be a PN-off-Ss / PN-under-Ss / PN-over-Ss over L. Then, $K_1\cap K_1^c = 0_{PN}$.

Proof: Suppose that, $\alpha_i\in K_1\cap K_1^c$. This implies,

$\alpha_i\in\{(\alpha_i, T_{K_1}(\alpha_i), C_{K_1}(\alpha_i), R_{K_1}(\alpha_i), U_{K_1}(\alpha_i), F_{K_1}(\alpha_i)): \alpha_i\in L\} \cap \{(\alpha_i, 1\text{-}T_{K_1}(\alpha_i), 1\text{-}C_{K_1}(\alpha_i), 1\text{-}R_{K_1}(\alpha_i), 1\text{-}U_{K_1}(\alpha_i), 1\text{-}F_{K_1}(\alpha_i)): \alpha_i\in L\}$

$\Rightarrow \alpha_i\in\{(\alpha_i, \min(T_{K_1}(\alpha_i), 1\text{-}T_{K_1}(\alpha_i)), \max(C_{K_1}(\alpha_i), 1\text{-}C_{K_1}(\alpha_i)), \max(R_{K_1}(\alpha_i), 1\text{-}R_{K_1}(\alpha_i)), \max(U_{K_1}(\alpha_i), 1\text{-}U_{K_1}(\alpha_i)), \min(F_{K_1}(\alpha_i), 1\text{-}F_{K_1}(\alpha_i))): \alpha_i\in L\}$

$\Rightarrow \alpha_i\in 0_{PN}$

Therefore, $K_1\cap K_1^c\subset 0_{PN}$ (3)

Again, consider $\beta_i\in 0_{PN}$

$\Rightarrow\beta_i\in \{\min(T_{K_1}(\beta_i),(1\text{-}T_{K_1}(\beta_i)), \max(C_{K_1}(\beta_i),U_{K_1}(\beta_i)), \max(R_{K_1}(\beta_i),(1\text{-}R_{K_1}(\beta_i)), \max(U_{K_1}(\beta_i),C_{K_1}(\beta_i)), \min(F_{K_1}(\beta_i),(1\text{-}F_{K_1}(\beta_i))\}$

$\Rightarrow\beta_i\in \{T_{K_1}(\beta_i),C_{K_1}(\beta_i),R_{K_1}(\beta_i),U_{K_1}(\beta_i),F_{K_1}(\beta_i)\}\cap\{(1\text{-}T_{K_1}(\beta_i)),U_{K_1}(\beta_i),(1\text{-}R_{K_1}(\beta_i)),C_{K_1}(\beta_i),(1\text{-}F_{K_1}(\beta_i))\}$

$\Rightarrow\beta_i\in K_1\cap K_1^c$.

Therefore, $0_{PN} \in K_1\cap K_1^c$ (4)

From the equation (3) and (4) we can conclude that,

$K_1\cap K_1^c = 0_{PN}$

Proposition 3.5. Let K_1 and K_2 be two PN-off-Ss / PN-under-Ss / PN-over-Ss over L. Then,

(i) $(K_1\cup K_2)^c=K_1^C\cap K_2^C$

(ii) $(K_1\cap K_2)^c=K_1^C\cup K_2^C$

Proof: Suppose that, $\alpha_i\in((K_1\cup K_2)^c$

$\Rightarrow\alpha_i\in \{\max(T_{K_1}(\alpha_i),T_{K_2}(\alpha_i)), \min(C_{K_1}(\alpha_i),C_{K_2}(\alpha_i)), \min(R_{K_1}(\alpha_i),R_{K_2}(\alpha_i)), \min(U_{K_1}(\alpha_i),U_{K_2}(\alpha_i)), \max(F_{K_1}(\alpha_i),F_{K_2}(\alpha_i))\}^c$

$\Rightarrow\alpha_i\in \{\min((1\text{-}T_{K_1}(\alpha_i)),(1\text{-}T_{K_2}(\alpha_i))), \max(U_{K_1}(\alpha_i),U_{K_2}(\alpha_i)), \max((1\text{-}R_{K_1}(\alpha_i)),(1\text{-}R_{K_2}(\alpha_i))), \max(C_{K_1}(\alpha_i), C_{K_2}(\alpha_i)), \min((1\text{-}F_{K_1}(\alpha_i)),(1\text{-}F_{K_2}(\alpha_i)))\}$

$\Rightarrow\alpha_i\in \{(1\text{-}T_{K_1}(\alpha_i)), C_{K_1}(\alpha_i),(1\text{-}R_{K_1}(\alpha_i)), U_{K_1}(\alpha_i),(1\text{-}F_{K_1}(\alpha_i))\}\cap\{(1\text{-}T_{K_2}(\alpha_i)), C_{K_2}(\alpha_i),(1\text{-}R_{K_2}(\alpha_i)), U_{K_2}(\alpha_i), (1\text{-}F_{K_1}(\alpha_i))\}$.

$\Rightarrow\alpha_i\in K_1^C\cap K_2^C$

$\Rightarrow(K_1\cup K_2)^c\subset K_1^C\cap K_2^C$ (5)

Assume that, $\beta_i\in K_1^C\cap K_2^C$

$\Rightarrow\beta_i\in \{(1\text{-}T_{K_1}(\beta_i)), C_{K_1}(\beta_i),(1\text{-}R_{K_1}(\beta_i)), U_{K_1}(\beta_i),(1\text{-}F_{K_1}(\beta_i))\}\cap\{(1\text{-}T_{K_2}(\beta_i)), C_{K_2}(\beta_i),(1\text{-}R_{K_2}(\beta_i)), U_{K_2}(\beta_i), (1\text{-}F_{K_1}(\beta_i))\}$

$\Rightarrow\beta_i\in \{\min((1\text{-}T_{K_1}(\beta_i)),(1\text{-}T_{K_2}(\beta_i))), \max(U_{K_1}(\beta_i),U_{K_2}(\beta_i)), \max((1\text{-}R_{K_1}(\beta_i)),(1\text{-}R_{K_2}(\beta_i))), \max(C_{K_1}(\beta_i), C_{K_2}(\beta_i)), \min((1\text{-}F_{K_1}(\beta_i)),(1\text{-}F_{K_2}(\beta_i)))\}$

$\Rightarrow\beta_i\in \{\max(T_{K_1}(\beta_i),T_{K_2}(\beta_i)), \min(C_{K_1}(\beta_i),C_{K_2}(\beta_i)), \min(R_{K_1}(\beta_i),R_{K_2}(\beta_i)), \min(U_{K_1}(\beta_i),U_{K_2}(\beta_i)), \max(F_{K_1}(\beta_i),F_{K_2}(\beta_i))\}^c$

$\Rightarrow\beta_i\in (K_1\cup K_2)^c$

Therefore, $(K_1 \cap K_2)^c \subset K_1^C \cup K_2^C$. (6)

From the equation (5) and (6) we can conclude that,

$(K_1 \cup K_2)^c = K_1^C \cap K_2^C$.

Assume that, $\alpha_i \in (K_1 \cap K_2)^c$

$\Rightarrow \alpha_i \in \{\min(T_{K_1}(\alpha_i), T_{K_2}(\alpha_i)), \max(C_{K_1}(\alpha_i), C_{K_2}(\alpha_i)), \max(R_{K_1}(\alpha_i), R_{K_2}(\alpha_i)), \max(U_{K_1}(\alpha_i), U_{K_2}(\alpha_i)), \min(F_{K_1}(\alpha_i), F_{K_2}(\alpha_i))\}^c$

$\Rightarrow \alpha_i \in \{\max((1-T_{K_1}(\alpha_i)),(1-T_{K_2}(\alpha_i))), \min(U_{K_1}(\alpha_i), U_{K_2}(\alpha_i)), \min((1-R_{K_1}(\alpha_i)),(1-R_{K_2}(\alpha_i))), \max(C_{K_1}(\alpha_i), C_{K_2}(\alpha_i)), \max((1-F_{K_1}(\alpha_i)),(1-F_{K_2}(\alpha_i)))\}$

$\Rightarrow \alpha_i \in \{(1-T_{K_1}(\alpha_i)), U_{K_1}(\alpha_i), (1-R_{K_1}(\alpha_i)), C_{K_1}(\alpha_i), (1-F_{K_1}(\alpha_i))\} \cup \{(1-T_{K_2}(\alpha_i)), U_{K_2}(\alpha_i), (1-R_{K_2}(\alpha_i)), C_{K_2}(\alpha_i), (1-F_{K_1}(\alpha_i))\}$

$\Rightarrow \alpha_i \in (K_1^C \cup K_2^C)$

Therefore, $(K_1 \cap K_2)^c \subset K_1^C \cup K_2^C$ (7)

Assume that, $\beta_i \in (K_1^C \cup K_2^C)$

$\Rightarrow \beta_i \in \{(1-T_{K_1}(\beta_i)), U_{K_1}(\beta_i), (1-R_{K_1}(\beta_i)), C_{K_1}(\beta_i), (1-F_{K_1}(\beta_i))\} \cup \{(1-T_{K_2}(\beta_i)), U_{K_2}(\beta_i), (1-R_{K_2}(\beta_i)), C_{K_2}(\beta_i), (1-F_{K_1}(\beta_i))\}$

$\Rightarrow \beta_i \in \{\max((1-T_{K_1}(\beta_i)),(1-T_{K_2}(\beta_i))), \min(U_{K_1}(\beta_i), U_{K_2}(\beta_i)), \min((1-R_{K_1}(\beta_i)),(1-R_{K_2}(\beta_i))), \max(C_{K_1}(\beta_i), C_{K_2}(\beta_i)), \max((1-F_{K_1}(\beta_i)),(1-F_{K_2}(\beta_i)))\}$

$\Rightarrow \beta_i \in \{\min(T_{K_1}(\beta_i), T_{K_2}(\beta_i)), \max(C_{K_1}(\beta_i), C_{K_2}(\beta_i)), \max(R_{K_1}(\beta_i), R_{K_2}(\beta_i)), \max(U_{K_1}(\beta_i), U_{K_2}(\beta_i)), \min(F_{K_1}(\beta_i), F_{K_2}(\beta_i))\}^c$

$\Rightarrow \beta_i \in (K_1 \cap K_2)^c$

Therefore, $(K_1^C \cup K_2^C) \subset (K_1 \cap K_2)^c$ (8)

From eq. (7) and eq. (8), we can conclude that $(K_1 \cap K_2)^c = K_1^C \cup K_2^C$.

6. Conclusions: In this article, we have introduced the notion of single valued pentapartitioned neutrosophic over-set / under-set / off-set. Besides, we have studied several properties on them. In the future, we hoped that the notion of some algebraic structures like Groups, Field, etc. can be easily applied to the proposed sets. Furthermore, the notion of proposed sets can also be applied to real life decision making problems [5, 12, 13, 19, 22, 24, etc.].

Conflict of Interest: The authors declare that they have no conflict of interest.

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Single-Valued Pentapartitioned Neutrosophic Exponential Similarity Measure under SVPNS Environment and Its Application in the Selection of Bacteria

Srila Dey[1], Rama Debbarma[2], Binod Chandra Tripathy[3], Suman Das[4,*] and Priyanka Majumder[5]

[1,2]Civil Engineering Department, National Institute of Technology Agartala, 799046, Tripura, India.

[3,4]Department of Mathematics, Tripura University, Agartala, 799022, Tripura, India.

[5]Department of Basic Science and Humanities (Mathematics), Techno College of Engineering Agartala, Tripura, India.

E-mail: [1]srilaagt10@gmail.com, [2]ramadebbarma@gmail.com, [3]tripathybc@yahoo.com, tripathybc@gmail.com, [4]sumandas18842@gmail.com, dr.suman1995@yahoo.com, and [5]majumderpriyanka94@yahoo.com

***Correspondence:** sumandas18842@gmail.com

Abstract: The purpose of this paper is to introduce a novel similarity measure, the single-valued pentapartitioned neutrosophic exponential similarity measure (SVPNESM), and the single-valued pentapartitioned neutrosophic weighted exponential similarity measure (SVPNWESM) under the single-valued pentapartitioned neutrosophic set (SVPNS) environment for selecting bacteria on concrete mortar to improve compressive strength and to reduce water absorption, porosity and chloride permeability. In order to improve the properties of concrete, bacteria must fulfill requirements such as increased compressive strength, decreased water absorption capacities, reduced porosity, decreased chloride permeability etc. A novel approach for selecting suitable bacteria in concrete mortar is presented in this study based on such requirements. In this study, suitable bacteria is selected from four bacteria for concrete mortar based on 4 criteria with fixed bacteria concentrations of 10^5. Based on this study, Bacillus subtitles is selected among four alternatives as suitable. Furthermore, the proposed MADM method is shown to be well suited to this problem after it has been compared with two existing methods.

Keywords: *SVPNS*; *SVPNESM*; *SVPNWESM*; *MADM*.

1. Introduction:

The concept of fuzzy set (FS) was first grounded by Zadeh [48] in the year 1965 to deal with different real world problems having uncertainty. In a FS, each element has a membership value lies in the interval [0, 1]. Afterwards, Atanassov [3] felt that the non-membership of a mathematical expression has also plays a vital role in solving the problems having uncertainty, and established the concept of intuitionistic fuzzy set (IFS) by generalizing the notion of FS. In every IFS, each element has both membership and non-membership values lies in the interval [0, 1]. Till now, many researchers around the globe applied the concept of FS, IFS and their extensions in the area of theoretical research and practical research. Many times, uncertainty events will also have some indeterminacy part, which can't be expressed by using the idea of crisp set, FS and IFS. Keep in mind, Smarandache [42] grounded the idea of neutrosophic set (NS) by generalizing the concept of FS and IFS to deal with the uncertainty events having indeterminacy. In an NS, each element has truth, indeterminacy and false membership values respectively lies in the interval [0, 1]. In 2010, Wang et al. [44] introduced the idea of single-valued neutrosophic set (SVNS) by extending the notion of NS. The notion of SVNS is more effective in dealing with the uncertainty events having indeterminate information. Till now, many mathematicians around the globe used the notion of SVNS and their extensions in theoretical [5-6, 10-16, 43] as well as in the several branches of this real world such as weaver selection [18], location selection [33-34], medical diagnosis [19, 35-36], fault diagnosis [45-46], and other decision making problems [4, 21-22, 24, 28-31, 47].

The concept of single-valued pentapartitioned neutrosophic set (SVPNS) was grounded by Mallick and Pramanik [27] by dividing the indeterminacy membership function into three independent membership function namely contradiction membership function, ignorance membership function and unknown membership function. Later on, Das et al. [7] grounded the notion of single-valued pentapartitioned neutrosophic Q-ideals of single-valued pentapartitioned neutrosophic Q-algebra. In 2021, Das et al. [9] established the single-valued pentapartitioned neutrosophic tangent similarity measure of similarities between the SVPNSs under SVPNS environment, and proposed a MADM technique under the SVPNS environment. In 2021, Das et al. [8] proposed a MADM technique based on grey relational analysis under the SVPNS environment. Later on, Das and Tripathy [17] extended the notion of topology on SVPNSs, and grounded the concept of pentapartitioned neutrosophoic topological space. Thereafter, Majumder et al. [26] established an MADM strategy based on cosine similarity measure under the SVPNS environment for the selection of most significant risk factor of COVID-19 in economy. Recently, Radha and Mary [37] introduced the idea of pentapartitioned neutrosophic pythagorean soft set as an extension of quadripartitioned neutrosophic pythagorean soft set.

The rest of this article has been designed as follows:

Section-2 presents several basic definitions and operations on SVPNSs those are very useful for developing the main results of this paper. Section 3 represents the concept of SVPNESM and SVPNWESM of similarities between two SVPNSs . A MADM strategy using SVPNWESM under the SVPNS environment is discussed in section-4. In section-5 the proposed MADM strategy is applied to a real world problem. Finally, in section 6, a comparative study has been conducted to validate the results obtained from the proposed method. In section-7, wrap up the work presented in this article.

List of abbreviations are shown in below:

Short Terms	
Single-Valued Neutrosophic Set	SVNS
Multi-Attribute Decision Making	MADM
Single-Valued Pentapartitioned Neutrosophic Set	SVPNS
Single-Valued Pentapartitioned Neutrosophic Exponential Similarity Measure	SVPNESM
Single-Valued Pentapartitioned Neutrosophic Weighted Exponential Similarity Measure	SVPNWCSM
Decision Matrix	DM
Positive Ideal Alternative	PIA

2. Some Relevant Definitions:

In this section some basic definitions and results are described .

Assume that *V* be a universe of discourse. Then *A*, a SVPNS [27] over *V* is defined by:

$$A=\left\{\left(t,\partial_A(t),\wp_A(t),\Im_A(t),\square_A(t),\ell_A(t)\right):t\in V\right\}.$$

Here, ∂_A, $\wp_A$, $\Im_A$, $\square_A$ and ℓ_A are the truth, contradiction, ignorance, unknown and false membership functions from *V* to the unit interval [0, 1] respectively i.e., $\partial_A(t)$, $\wp_A(t)$, $\Im_A(t)$, $\square_A(t)$ and $\ell_A(t)\in[0,1]$, for each $t\in V$. So, $0\le\partial_A(t)+\wp_A(t)+\Im_A(t)+\square_A(t)+\ell_A(t)\le 1$, for each $t\in V$.

The absolute SVPNS (1_{PN}) [27] and the null SVPNS (0_{PN}) over a fixed set *V* are defined as follows:

(*i*) $1_{PN}=\{(t,1,1,0,0,0):t\in V\}$,

(*ii*) $0_{PN}=\{(t,0,0,1,1,1):t\in V\}$.

Let $A=\left\{\left(t,\partial_A(t),\wp_A(t),\Im_A(t),\square_A(t),\ell_A(t)\right):t\in V\right\}$ and $B=\left\{\left(t,\partial_B(t),\wp_B(t),\Im_B(t),\square_B(t),\ell_B(t)\right):t\in V\right\}$

be any two [27] SVPNSs over *V*. Then,

(*i*) $A\subseteq B$ if and only if $\partial_A(t)\le\partial_B(t)$, $\wp_A(t)\le\wp_B(t)$, $\Im_A(t)\ge\Im_B(t)$, $\square_A(t)\ge\square_B(t)$, $\ell_A(t)\ge\ell_B(t)$, for all $t\in V$.

(*ii*) $A^c=\left\{\left(t,\ell_A(t),\square_A(t),1-\Im_A(t),\wp_A(t),\partial_A(t)\right):t\in V\right\}$;

$$(iii)\ A \cup B = \left\{\left(t, \max\left\{\partial_A(t), \partial_B(t)\right\}, \max\left\{\wp_A(t), \wp_B(t)\right\}, \min\left\{\Im_A(t), \Im_B(t)\right\}, \min\left\{\square_A(t), \square_B(t)\right\}, \min\left\{\ell_A(t), \ell_B(t)\right\}\right) : t \in V\right\}$$

$$(iv)\ A \cap B = \left\{\left(t, \min\left\{\partial_A(t), \partial_B(t)\right\}, \min\left\{\wp_A(t), \wp_B(t)\right\}, \max\left\{\Im_A(t), \Im_B(t)\right\}, \max\left\{\square_A(t), \square_B(t)\right\}, \max\left\{\ell_A(t), \ell_B(t)\right\}\right) : t \in V\right\}$$

Example 2.1. Suppose that A = {(p, 0.6, 0.1, 0.3, 0.4, 0.5), (q, 0.9, 0.1, 0.2, 0.2, 0.1)} and B = {(p, 0.9, 0.2, 0.2, 0.1, 0.4), (q, 1.0, 0.3, 0.1, 0.2, 0.1)} be two SVPNSs over a universe of discourse V = {p, q}. Then,

(i) $A \subseteq B$;

(ii) A^c = {(p, 0.5, 0.4, 0.7, 0.1, 0.6), (q, 0.1, 0.2, 0.8, 0.1, 0.9)} and B^c = {(p, 0.4, 0.1, 0.8, 0.2, 0.9), (q, 0.1, 0.2, 0.9, 0.3, 1.0)};

(iii) $A \cup B$ = {(p, 0.9, 0.2, 0.2, 0.1, 0.4), (q, 1.0, 0.3, 0.1, 0.2, 0.1)};

(iv) $A \cap B$ = {(p, 0.6, 0.1, 0.3, 0.4, 0.5), (q, 0.9, 0.1, 0.2, 0.2, 0.1)}.

3. Single-Valued Pentapartitioned Neutrosophic Exponential Similarity Measure:

The notion of SVPNESM is discussed in the current section. This notion depends on similarities between two SVPNSs. In this section, some basic results on SVPNESM and SVPNWESM are discussed.

Definition 3.1. Suppose that A and B be two SVPNSs over a fixed set V such as $A = \left\{\left(t, \partial_A(t), \wp_A(t), \Im_A(t), \square_A(t), \ell_A(t)\right) : t \in V\right\}$ an $B = \left\{\left(t, \partial_B(t), \wp_B(t), \Im_B(t), \square_B(t), \ell_B(t)\right) : t \in V\right\}$. Then, the SVPNESM of similarities between A and B is denoted by $P_{SVPNESM}(A,B)$ and is defined by:

$$P_{SVPNESM}(A,B) = \frac{1}{n}\sum_{t \in V} e^{-\left[\left|\partial_A(t)-\partial_B(t)\right| + \left|\wp_A(t)-\wp_B(t)\right| + \left|\Im_A(t)-\Im_B(t)\right| + \left|\square_A(t)-\square_B(t)\right| + \left|\ell_A(t)-\ell_B(t)\right|\right]^2} \quad \text{..................(1)}$$

Theorem 3.1. Let $P_{SVPNESM}(A,B)$ be the SVPNESM between the SVPNSs A and B. Then, the following holds:

1) $0 \le P_{SVPNESM}(A,B) \le 1$;

2) $P_{SVPNESM}(A,B) = P_{SVPNESM}(B,A)$;

3) $P_{SVPNESM}(A,B) = 1 \Leftrightarrow A = B$.

Proof.

1) Since $\left|\partial_A(t)-\partial_B(t)\right| \ge 0$, $\left|\wp_A(t)-\wp_B(t)\right| \ge 0$, $\left|\Im_A(t)-\Im_B(t)\right| \ge 0$, $\left|\square_A(t)-\square_B(t)\right| \ge 0$ and $\left|\ell_A(t)-\ell_B(t)\right| \ge 0$ for all $t \in V$ then from (1) $P_{SVPNESM}(A,B) \ge 0$

Also, since exponential function is monotonically decreasing for all values in the set $\square^{+} \cup \{0\}$, so

from the equation (1) it is clear that $P_{SVPNESM}(A,B) \le 1$.

Hence, $0 \le P_{SVPNESM}(A,B) \le 1$

2) From the equation (1),

$$P_{SVPNESM}(A,B) = \frac{1}{n}\sum_{t\in V} e^{-\left[\left|\partial_A(t)-\partial_B(t)\right|+\left|\wp_A(t)-\wp_B(t)\right|+\left|\Im_A(t)-\Im_B(t)\right|+\left|\square_A(t)-\square_B(t)\right|+\left|\ell_A(t)-\ell_B(t)\right|\right]^2}$$
$$= \frac{1}{n}\sum_{t\in V} e^{-\left[\left|\partial_B(t)-\partial_A(t)\right|+\left|\wp_B(t)-\wp_A(t)\right|+\left|\Im_B(t)-\Im_A(t)\right|+\left|\square_B(t)-\square_A(t)\right|+\left|\ell_B(t)-\ell_A(t)\right|\right]^2}$$
$$= P_{SVPNESM}(B,A)$$

Hence $P_{SVPNESM}(A,B) = P_{SVPNESM}(B,A)$

3) Let us assume that *A* and *B* be two SVPNSs over *V* such that *A*=*B*. This implies,

$\partial_A(t)-\partial_B(t)=0$, $\wp_A(t)-\wp_B(t)=0$, $\Im_A(t)-\Im_B(t)=0$, $\square_A(t)-\square_B(t)=0$ and $\ell_A(t)-\ell_B(t)=0$,

for all $t \in V$. Therefore, $\left|\partial_A(t)-\partial_B(t)\right|=0$, $\left|\wp_A(t)-\wp_B(t)\right|=0$, $\left|\Im_A(t)-\Im_B(t)\right|=0$,

$\left|\square_A(t)-\square_B(t)\right|=0$ and $\left|\ell_A(t)-\ell_B(t)\right|=0$ for all $t \in V$. Hence, from (1),

$$P_{SVPNESM}(A,B) = \frac{1}{n}\sum_{t\in V} e^{0} = \frac{1}{n}\sum_{t\in V} 1 = \frac{n}{n} = 1.$$

Conversely, let $P_{SVPNESM}(A,B)=1$. This implies, $\left|\partial_A(t)-\partial_B(t)\right|=0$, $\left|\wp_A(t)-\wp_B(t)\right|=0$,

$\left|\Im_A(t)-\Im_B(t)\right|=0$, $\left|\square_A(t)-\square_B(t)\right|=0$ and $\left|\ell_A(t)-\ell_B(t)\right|=0$ for all $t \in V$. Therefore,

$\partial_A(t)=\partial_B(t)$, $\wp_A(t)=\wp_B(t)$, $\Im_A(t)=\Im_B(t)$, $\square_A(t)=\square_B(t)$ and $\ell_A(t)=\ell_B(t)$, for all $t \in V$.

Hence, *A* =*B*.

Theorem 3.2. If *A*, *B* and *C* be three SVPNSs over *U* such that *A* ⊆ *B* ⊆*C*, then $P_{SVPNESM}(A,B) \ge P_{SVPNESM}(A,C)$ $P_{SVPNESM}(B,C) \ge P_{SVPNESM}(A,C)$.

Proof. Assume that *A*, *B* and *C* be three SVPNSs over a fixed set *V* such that *A* ⊆ *B* ⊆ *C*. Therefore, $\partial_A(t) \le \partial_B(t)$, $\wp_A(t) \le \wp_B(t)$, $\Im_A(t) \ge \Im_B(t)$, $\square_A(t) \ge \square_B(t)$, $\ell_A(t) \ge \ell_B(t)$, $\partial_A(t) \le \partial_B(t)$,

$\wp_A(t) \le \wp_C(t)$, $\Im_A(t) \ge \Im_C(t)$, $\square_A(t) \ge \square_C(t)$, $\ell_A(t) \ge \ell_C(t)$, for all $t \in V$.

We have,

$\left|\partial_A(t)-\partial_B(t)\right| \le \left|\partial_A(t)-\partial_C(t)\right|$, $\left|\wp_A(t)-\wp_B(t)\right| \le \left|\wp_A(t)-\wp_C(t)\right|$, $\left|\square_A(t)-\square_B(t)\right| \le \left|\square_A(t)-\square_C(t)\right|$,

$\left|\ell_A(t)-\ell_B(t)\right| \le \left|\ell_A(t)-\ell_C(t)\right|$

Therefore,

$$\begin{aligned} P_{SVPNESM}(A,B) &= \frac{1}{n}\sum_{t\in V} e^{-\left[\left|\partial_A(t)-\partial_B(t)\right|+\left|\wp_A(t)-\wp_B(t)\right|+\left|\Im_A(t)-\Im_B(t)\right|+\left|\square_A(t)-\square_B(t)\right|+\left|\ell_A(t)-\ell_B(t)\right|\right]^2} \\ &\ge \frac{1}{n}\sum_{t\in V} e^{-\left[\left|\partial_A(t)-\partial_C(t)\right|+\left|\wp_A(t)-\wp_C(t)\right|+\left|\Im_A(t)-\Im_C(t)\right|+\left|\square_A(t)-\square_C(t)\right|+\left|\ell_A(t)-\ell_C(t)\right|\right]^2} \\ &= P_{SVPNESM}(A,C) \end{aligned}$$

Hence, $P_{SVPNESM}(A,B) \ge P_{SVPNESM}(A,C)$

Further,

$\left|\partial_B(t)-\partial_C(t)\right| \le \left|\partial_A(t)-\partial_C(t)\right|$, $\left|\wp_B(t)-\wp_C(t)\right| \le \left|\wp_A(t)-\wp_C(t)\right|$, $\left|\square_B(t)-\square_C(t)\right| \le \left|\square_A(t)-\square_C(t)\right|$,

$\left|\ell_B(t)-\ell_C(t)\right| \le \left|\ell_A(t)-\ell_C(t)\right|$

Therefore,

$$\begin{aligned} P_{SVPNESM}(B,C) &= \frac{1}{n}\sum_{t\in V} e^{-\left[\left|\partial_B(t)-\partial_C(t)\right|+\left|\wp_B(t)-\wp_C(t)\right|+\left|\Im_B(t)-\Im_C(t)\right|+\left|\square_B(t)-\square_C(t)\right|+\left|\ell_B(t)-\ell_C(t)\right|\right]^2} \\ &\ge \frac{1}{n}\sum_{t\in V} e^{-\left[\left|\partial_A(t)-\partial_C(t)\right|+\left|\wp_A(t)-\wp_C(t)\right|+\left|\Im_A(t)-\Im_C(t)\right|+\left|\square_A(t)-\square_C(t)\right|+\left|\ell_A(t)-\ell_C(t)\right|\right]^2} \\ &= P_{SVPNESM}(A,C) \end{aligned}$$

Hence, $P_{SVPNESM}(B,C) \ge P_{SVPNESM}(A,C)$.

Definition 3.2. Let us consider two SVPNSs A and B over a fixed set V such as $A=\left\{\left(t,\partial_A(t),\wp_A(t),\Im_A(t),\square_A(t),\ell_A(t)\right): t\in V\right\}$ and $B=\left\{\left(t,\partial_B(t),\wp_B(t),\Im_B(t),\square_B(t),\ell_B(t)\right): t\in V\right\}$.

Then, the single valued pentapartitioned neutrosophic weighted exponential similarity measure (SVPNWESM) of the similarities between two SVPNSs A and B is defined as follows:

$$P_{SVPNWESM}(A,B) = \frac{1}{n}\sum_{t\in V} w_t \times e^{-\left[\left|\partial_A(t)-\partial_B(t)\right|+\left|\wp_A(t)-\wp_B(t)\right|+\left|\Im_A(t)-\Im_B(t)\right|+\left|\square_A(t)-\square_B(t)\right|+\left|\ell_A(t)-\ell_B(t)\right|\right]^2} \quad \text{..................(2)}$$

where, $\sum_{t\in V} w_t = 1$

In view of the above theorems, the two following two propositions can be formulated.

Proposition 3.1. If $P_{SVPNWESM}(A,B)$ be the single valued pentapartitioned neutrosophic weighted sine similarity measure of similarities between the SVPNSs A and B. Then,

1) $0 \le P_{SVPNWESM}(A, B) \le 1$;

2) $P_{SVPNWESM}(A, B) = P_{SVPNWESM}(B, A)$;

3) $P_{SVPNWESM}(A, B) = 1 \, iff \; A = B$.

Proposition 3.2. If *A*, *B* and *C* be three SVPNSs over *U* such that *A* ⊆ *B* ⊆*C*, then $P_{SVPNWESM}(A, B) \ge P_{SVPNWESM}(A, C)$, $P_{SVPNWESM}(B, C) \ge P_{SVPNWESM}(A, C)$.

4. MADM Strategy Using SVPNWESM under SVPNS Environment:

In this section, an attempt is made to propose a MADM model under the SVPNS environment using the SVPNWESM.

In a MADM problem, let us consider two sets $E = \{\theta_1, \theta_2, \theta_3, ..., \theta_n\}$ and $F = \{\mu_1, \mu_2, \mu_3, ..., \mu_m\}$ of all possible alternatives and attributes respectively. Then, a decision maker can give their evaluation information for each alternative $\theta_i\,(i = 1, 2, ..., m)$ with respect to the each attribute $\mu_j\,(j = 1, 2, ..., k)$ by a SVPNS. By using the decision maker's whole evaluation information, a decision matrix (DM) can be formed.

The steps of the proposed MADM strategy are discussed below. Figure 1 represents the flow chart of the proposed MADM strategy.

Step-1. Formation of DM by using SVPNS.

Suppose, the decision maker gives their evaluation information by using the SVPNS $K_{\theta_i} = \{(\mu_j, \partial_{ij}(\theta_i, \mu_j), \wp_{ij}(\theta_i, \mu_j), \Im_{ij}(\theta_i, \mu_j), \square_{ij}(\theta_i, \mu_j), \ell_{ij}(\theta_i, \mu_j))\}$ for each alternative $\theta_i\,(i = 1, 2, ..., m)$ with respect to the attributes $\mu_j\,(j = 1, 2, ..., k)$, where

$(\partial_{ij}(\theta_i, \mu_j), \wp_{ij}(\theta_i, \mu_j), \Im_{ij}(\theta_i, \mu_j), \square_{ij}(\theta_i, \mu_j), \ell_{ij}(\theta_i, \mu_j)) = (\theta_i, \mu_j)$ (say) $(i = 1, 2, ..., m$ and $j = 1, 2, ..., k)$

indicates the evaluation information of alternatives $\theta_i\,(i = 1, 2, ..., m)$ with respect to the attribute $\mu_j\,(j = 1, 2, ..., k)$.

The decision matrix (DM) can be expressed as follows:

$$\begin{array}{c} \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{array} \begin{array}{c} \begin{array}{cccc} \mu_1 & \mu_2 & \cdots & \mu_k \end{array} \\ \begin{pmatrix} (\theta_1, \mu_1) & (\theta_1, \mu_2) & \cdots & (\theta_1, \mu_k) \\ (\theta_2, \mu_1) & (\theta_2, \mu_2) & \cdots & (\theta_2, \mu_k) \\ \vdots & \vdots & \vdots & \vdots \\ (\theta_m, \mu_1) & (\theta_m, \mu_2) & \cdots & (\theta_m, \mu_k) \end{pmatrix} \end{array}$$

Step-2. Determination of the Weights for Each Attribute.

In every MADM strategy, the determination of weights for every attributes is an important task. If the information of attributes' weight is completely unknown, then the decision maker can use the compromise function to calculate the weights for each attribute.

The compromise function of λ_j for each θ_j is defined as follows:

$$\lambda_j = \sum_{i=1}^{m}\left[3+\partial_{ij}\left(\theta_i,\mu_j\right)+\wp_{ij}\left(\theta_i,\mu_j\right)-\Im_{ij}\left(\theta_i,\mu_j\right)-\square_{ij}\left(\theta_i,\mu_j\right)+\ell_{ij}\left(\theta_i,\mu_j\right)\right]\Big/5 \quad\text{.................(3)}$$

. Then, the weight of the *j*-th attribute is defined by $w_j = \dfrac{\lambda_j}{\sum_{j=1}^{k}\lambda_j}$ ....(4)

Here, $\sum_{j=1}^{k} w_j = 1$

Step-3. Selection of the Positive Ideal Alternative (PIA).

In this step, the decision maker can form the PIA by using the maximum operator for all the attributes.

The positive ideal alternative (PIA) I^+ is defined as follows:

$$\mathrm{I}^+ = \left(\nu_1^+,\nu_2^+,\nu_3^+,...,\nu_k^+\right)\text{.................(5)}$$

Where, $\nu_j^+ = \Big(\max\left\{\partial_{ij}\left(\theta_i,\mu_j\right): i=1,2,...,m\right\}, \max\left\{\wp_{ij}\left(\theta_i,\mu_j\right): i=1,2,...,m\right\}, \min\left\{\Im_{ij}\left(\theta_i,\mu_j\right): i=1,2,...,m\right\},$

$\min\left\{\square_{ij}\left(\theta_i,\mu_j\right): i=1,2,...,m\right\}, \min\left\{\ell_{ij}\left(\theta_i,\mu_j\right): i=1,2,...,m\right\}\Big)$.................(6), $j=1,2,...,k$

Step-4. Determination of the SVPNWESM between the PIA and $K_{\theta_i}\left(i=1,2,...,m\right)$.

In this step, the SVPNWESM between the decision elements from the decision matrix and the PIA is calculated by using eq. (2).

Step-5. Ranking Order of the Alternatives.

Finally, the ranking order of alternatives is determined based on the ascending order of SVPNWESM between the PIA and the decision elements from the decision matrix. The alternative associated with the highest SVPNWESM value is the most suitable alternatives.

The flowchart of the proposed MADM-strategy is given as follows:

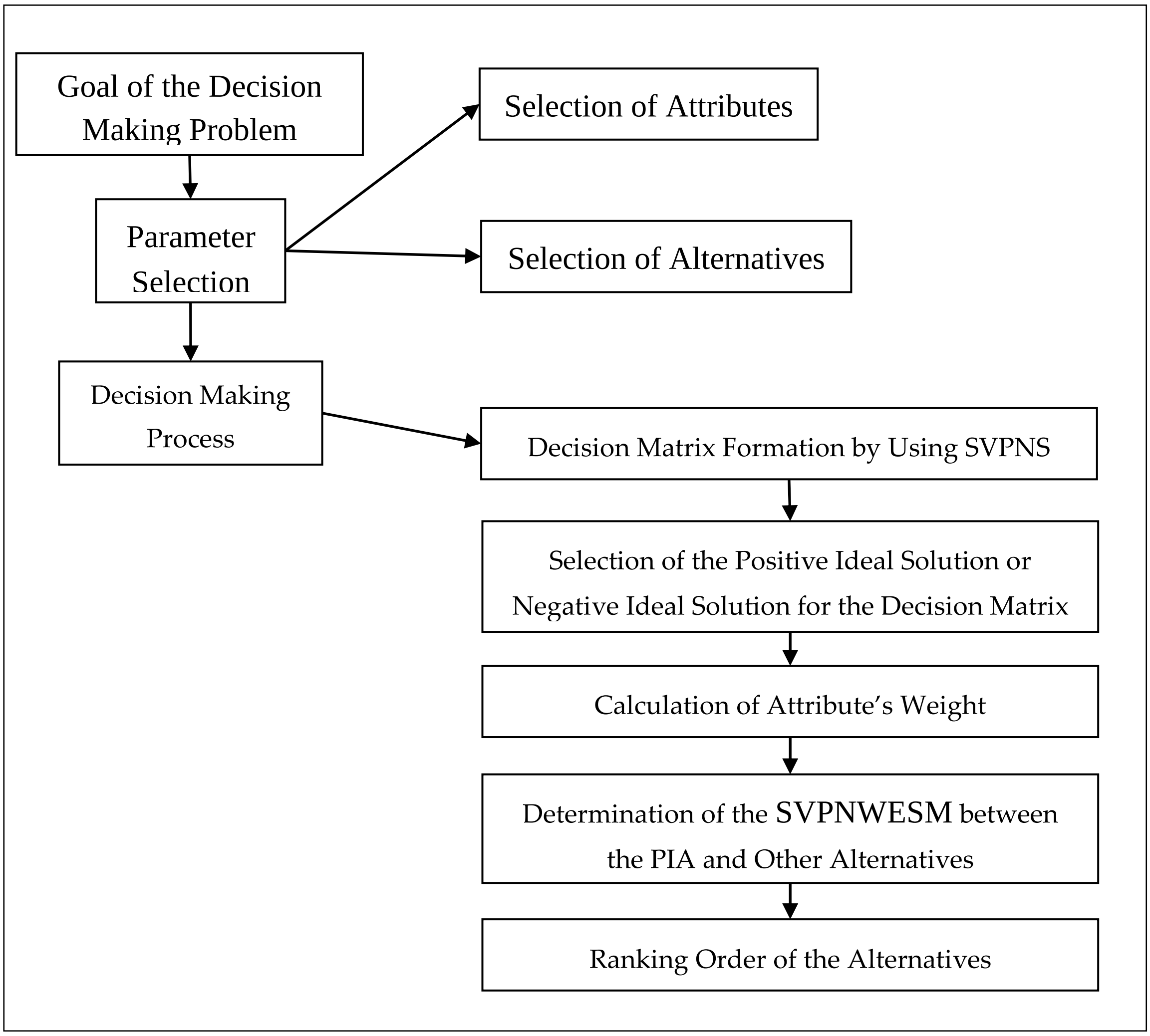

Figure- 1: Proposed MADM-Strategy

5. Application of the Proposed MADM Strategy for Selecting Suitable Bacteria in Concrete Mortar under the SVPNS Environment:

The calcite producing bacterium has been used in this research work to study its effect on strength and permeation properties of concrete. The calcite produced by the bacteria in the concrete pores, densities the matrix which results not only in improvement of compressive strength but also reduces the pore size, thereby, improving the permeation properties. Further, the rate of calcite precipitation is dependent upon the type of bacteria and the concentration of the bacteria.

Bio mineralization process depends on the types of bacteria. Selection of bacteria is a key factor in the bio mineralization process. Bacteria must fulfill some of the requirements for improving the properties of concrete. It must be able to adjust to alkaline atmosphere in concrete for the production of calcium carbonate, it should produce copious amount of calcium carbonate without being affected by calcium ion concentration, it must be able to withstand high pressure and should be oxygen brilliant to consume much oxygen and minimize corrosion of steel.

The selection of the bacteria is depend on the survive capability of bacteria in the alkaline environment. Shewanella species bacterium able to survive up to 6 to 7 days inside the concrete, due to calcite precipitation and clogging of pores inside the concrete matrix. This life span and pathogenic property are the disadvantages of using as self-healing agent for a longer period. Generally, researchers used some alkali-resistant, calcite precipitating, ureolytic bacteria of the Bacillus genus like Bacillus subtillis, Bacillus sphaericus, Bacillus cereus, Bacillus magaterium etc. [1-2, 19-20, 23, 25, 32, 38-40].

From literature review it is concluded that these bacteria could survive up to hundreds of years without nutrients and can able to withstand environmental chemicals, high mechanical stresses as well as ultraviolet radiations [41]. Generally, in case of ureolytic process, urea generates a huge amount of CO_2 and urea produces ammonia, which has a foul smell. So that to reveal from this situation researchers to investigate the calcite precipitating, alkali-resistant non-ureolytic bacteria. Afterwards the study showed aerobic alkaliphilic spore forming bacteria in concrete lead to the precipitation of $CaCO_3$. Table 1 represents the list of bacteria used to concrete mortar base on compressive strength, water absorption capacities, porosity, chloride permeability as output.

Table 1: List of Bacteria and Their Effect on Concrete Mortar

Bacteria	Concentration (μ_1)	Material (μ_2)	Compressive strength(28 days) (μ_3)	Water absorption reduction (28 days) (μ_4)	Porosity reduction (28 days) (μ_5)	Chloride permeability reduction (28 days) (μ_6)
Bacillus sphaericus (θ_1)[20]	10^5	Mortar	18.30%	89.00%	45.00%	10.00% - 40.00%
Bacillus cohnii (Nonureolytic) (θ_2) [40]	10^5	Mortar	26.23%			
Bacillus subtilis (θ_3) [32]	10^5	Mortar	27.00% (54 Map)	23.00%		
S. pasteurii (θ_4) [38]	10^5	Mortar	22.00% (28 Map)	13.00%		

The decision hierarchy of the current MADM problem is given below:

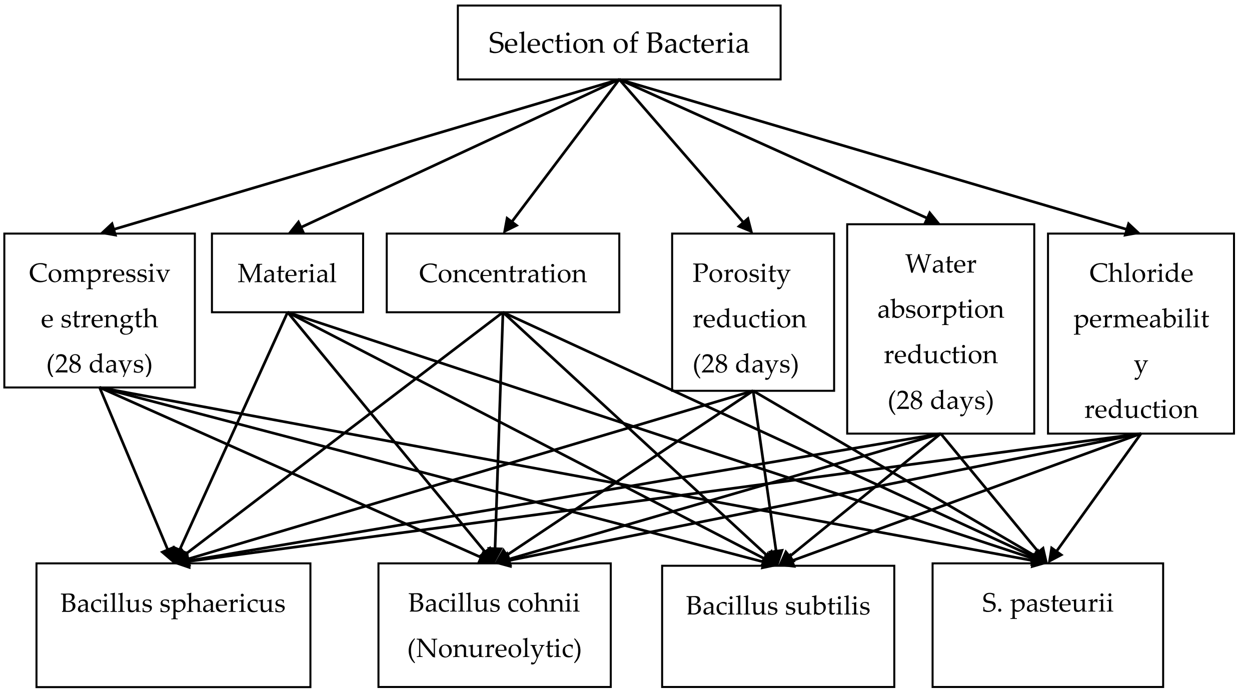

Figure- 2: Decision Hierarchy of the Current MADM-Problem

Figure-2 represents decision hierarchy of the Current MADM-Problem and steps involve in the current MADM problem is presented as follows:

By using the evaluation information for all alternatives given by the decision makers, prepare the decision matrix in Table-2.

Table-2: Decision Matrix

	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
θ_1	(0.8,0.2,0.3,0.1,0.2)	(0.9,0.1,0.1,0.0,0.1)	(0.6,0.2,0.4,0.2,0.3)	(0.8,0.1,0.2,0.0,0.2)	(0.8,0.1,0.2,0.2,0.2)	(0.9,0.1,0.1,0.1,0.1)
θ_2	(0.9,0.1,0.2,0.0,0.1)	(0.7,0.2,0.3,0.2,0.2)	(0.7,0.1,0.1,0.1,0.3)	(0.9,0.0,0.1,0.2,0.1)	(0.8,0.2,0.1,0.1,0.2)	(0.9,0.0,0.1,0.0,0.1)
θ_3	(0.9,0.1,0.2,0.0,0.1)	(0.7,0.0,0.3,0.2,0.2)	(0.9,0.0,0.1,0.0,0.1)	(0.9,0.0,0.1,0.1,0.1)	(0.9,0.0,0.1,0.0,0.1)	(0.9,0.0,0.1,0.1,0.1)
θ_4	(0.8,0.1,0.1,0.1,0.1)	(0.8,0.0,0.2,0.1,0.1)	(0.8,0.1,0.2,0.0,0.1)	(0.8,0.2,0.2,0.0,0.2)	(0.7,0.2,0.2,0.1,0.3)	(0.8,0.1,0.2,0.1,0.2)

Now, by using the eq. (5) & eq. (6), the PIA (I^+) is formed for the decision matrix is shown in Table-3:

Table-3: Positive Ideal Alternative

	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6

θ_1	(0.8,0.2,0.3,0.1,0.2)	(0.9,0.1,0.1,0.0,0.1)	(0.6,0.2,0.4,0.2,0.3)	(0.8,0.1,0.2,0.0,0.2)	(0.8,0.1,0.2,0.2,0.2)	(0.9,0.1,0.1,0.1,0.1)
θ_2	(0.9,0.1,0.2,0.0,0.1)	(0.7,0.2,0.3,0.2,0.2)	(0.7,0.1,0.1,0.1,0.3)	(0.9,0.0,0.1,0.2,0.1)	(0.8,0.2,0.1,0.1,0.2)	(0.9,0.0,0.1,0.0,0.1)
θ_3	(0.9,0.1,0.2,0.0,0.1)	(0.7,0.0,0.3,0.2,0.2)	(0.9,0.0,0.1,0.0,0.1)	(0.9,0.0,0.1,0.1,0.1)	(0.9,0.0,0.1,0.0,0.1)	(0.9,0.0,0.1,0.1,0.1)
θ_4	(0.8,0.1,0.1,0.1,0.1)	(0.8,0.0,0.2,0.1,0.1)	(0.8,0.1,0.2,0.0,0.1)	(0.8,0.2,0.2,0.0,0.2)	(0.7,0.2,0.2,0.1,0.3)	(0.8,0.1,0.2,0.1,0.2s)
v_j^+	(0.9,0.2,0.1,0.0,0.1)	(0.9,0.2,0.1,0.0,0.1)	(0.9,0.2,0.1,0.0,0.1)	(0.9,0.2,0.1,0.0,0.1)	(0.9,0.2,0.1,0,0.1)	(0.9,0.1,0.1,0,0.1)

Weights of the attributes are obtained by using the eq. (3) & eq. (4). The weights of the attribute are $w_1 = 0.1710526$, $w_2 = 0.1602871$, $w_3 = 0.1602871$, $w_4 = 0.1698565$, $w_5 = 0.1662679$, $w_6 = 0.1722488$.

By using the eq. (2), obtained SVPNWESM of similarities between the PIA and the decision elements from the decision matrix as follows:

$SVPNWESM\left(\theta_1, I^+\right) =$ 0.1928416,

$SVPNWESM\left(\theta_2, I^+\right) =$ 0.2077046,

$SVPNWESM\left(\theta_3, I^+\right) =$ 0.2141489,

$SVPNWESM\left(\theta_4, I^+\right) =$ 0.2044531.

The ascending order of the SVPNWESM of similarities between the PIS and the decision elements from the decision matrix is as follows:

$$SVPNWESM\left(\theta_3, I^+\right) < SVPNWESM\left(\theta_2, I^+\right) < SVPNWESM\left(\theta_4, I^+\right) < SVPNWESM\left(\theta_1, I^+\right)$$

6. Comparative Study:

To verify the proposed result based on the SVPNWESM, an investigation has been conducted for the purpose of comparison with the existing MADM techniques [9, 26]. From the comparative Table-4, it is observed that the existing methods support the same performance as per the proposed method for best attribute. According to the Table-4 it is clear that the weighted values of all attribute are much closed for two existing methods. In case of proposed technique the weighted values of all attribute is not closed compare to existing tool, it helps to take better decision for considering attributes. So the proposed method is more effective compare to considering MADM methods.

Table-4: Comparative Study

Methods	Θ_1	Θ_2	Θ_3	Θ_4	Ranking Order
MADM Strategy Based on Tangent Similarity Measure under SVPNS Environment [9]	0.9802728	0.9834154	0.9855075	0.9810444	$\Theta_1 < \Theta_4 < \Theta_2 < \Theta_3$
MADM Strategy Based on Cosine Similarity Measure under SVPNS Environment [26]	0.834740	0.8349405	0.8357332	0.8348665	$\Theta_1 < \Theta_2 < \Theta_4 < \Theta_3$
Proposed MADM Strategy	0.1928416	0.2077046	0.2141489	0.2044531	$\Theta_1 < \Theta_4 < \Theta_2 < \Theta_3$

7. Conclusions: In this article, a novel MADM is proposed for selecting suitable bacteria in concrete mortar based on compressive strength, water absorption capacities, porosity, chloride permeability etc. The ranking order $\Theta_1 < \Theta_4 < \Theta_2 < \Theta_3$ is derived by the proposed method. It is obvious from the ranking order generated by the new method that alternative Θ_3 is the best among all alternatives. A comparison of the results obtained by the new MADM method is performed using different existing methods. Based on all methods, alternative Θ_3 i.e., Bacillus subtilis is the best alternative, and therefore, it is concluded that the proposed method is well suited for solving such a problem.

The main limitation of this paper is that it compares alternatives based on a fixed concentration of bacteria. In future work, the effect of different concentrations of bacteria will be tested after selecting the most suitable bacteria from among the four alternatives discussed in this paper.

Further, it is hoped that, the proposed MADM technique can also be used in solving other decision-making problems such as weaver selection [18], location selection [33-34], medical diagnosis [19, 35-36], fault diagnosis [45-46], etc.

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Bipolar Neutrosophic Frank Aggregation Operator and its application in Multi Criteria Decision Making Problem

M. Lathamaheswari[1*], S. Sudha[1a,1b], Said Broumi[2] and Florentin Smarandache[3]

[1*]Department of Mathematcs, Hindustan Institute of Technology & Science, Chennai-603 103, India. lathamax@gmail.com

[1a]Department of Mathematcs, Hindustan Institute of Technology & Science, Chennai-603 103, India..sudha.aarpitha@gmail.com

[1b]Department of Mathematcs, Mohamed Sathak AJ College of Engineering, Chennai-603 103, India.sudha.aarpitha@gmail.com

[2]Laboratory of Information processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco. broumisaid78@gmail.com

[3]Department of Mathematics, University of New Mexico, 705 Gurley Avenue, Gallup, NM 87301, USA

fsmarandache@gmail.com

Abstract: Aggregation operators can be used to combine and synthesise a finite number of numerical values into a single numerical value. Many areas, including decision-making, expert systems, risk analysis, and image processing, rely heavily on aggregating functions. In real-world situations, the neutrosophic set can manage the uncertainties associated with information from any decision-making challenge, whereas the fuzzy set and intuitionistic set cannot. The term "bipolarity" refers to the propensity of the human mind to weigh pros and drawbacks when thinking through decisions. Triangular norms are aggregation operators in a variety of fields, including fuzzy set theory, probability and statistics, and decision sciences. Thus, the individual assessments in this paper's study of and approach to multi-criteria decision-making (MCDM) problems that use bipolar neutrosophic numbers as the individual evaluations. Frank operational laws of bipolar neutrosophic numbers, bipolar neutrosophic Frank weighted geometric aggregation (BNFWGA) and the bipolar neutrosophic frank ordered weighted geometric aggregation (BNFOWGA) operators have been developed with its desirable properties. Additionally, the suggested aggregation operators have been used in the selection of bridges. The outcomes demonstrate the applicability and validity of the suggested approach. Comparative analysis has been performed using the current approach.

Abbreviation:

NS	Neutrosophic Set
IFS	Intuitionistic fuzzy set
IVIFS	Interval-valued intuitionistic fuzzy set
INS	Interval neutrosophic set
SVNS	Single valued neutrosophic set
MCDM	Multi criteria decision making
BNS	Bipolar neutrosophic set
MAGDM	Multi attribute group decision making
BNNs	Bipolar neutrosophic numbers
BNFWGA	Bipolar neutrosophic frank aggregation weighted geometric aggregation
BNFOWGA	Bipolar neutrosophic ordered weighted geometric aggregation

Keywords: Bipolar neutrosophic set, Frank Triangular Norms, Operational Laws, Aggregation Operators, Decision Making

1. Introduction

A newly established model typically fixes the flaws of prior models in fuzzy theory. Because ambiguity and uncertainty present challenges in many real-world situations, routine mathematics is not always available. Many methods, including statistical hypothesis, probability, and fuzzy set hypothesis, have been presented as alternatives to traditional models and to guard against weaknesses in order to handle such difficulties. The majority of these mathematical alternatives, regrettably, have drawbacks and shortcomings of their own. Most words are in fact ambiguous and cannot be quantified, for instance, authentic and best-known. The authors of [1] started thinking about the chance based on the participation function that awards a membership grade in [0, 1] to handle such muddled and ambiguous information. Fuzzy sets are unable to handle the difficult issue since they only have one membership degree. Presented the intuitionistic fuzzy set (IFS) concept in [2]. IFS is used to provide an extremely flexible description of uncertain information. IFS offers degrees that are both membership- and non-membership-based. Introduced the idea of an intuitive fuzzy set with interval values in [3]. [4] developed the idea of a neutrosophic set (NS). NS includes membership, non-membership, and indeterminacy membership functions to define incompletes, inconsistent, and uncertain information. In order to adapt NS to real-world decision-making scenarios, [5, 6] introduced the interval Neutrosophic set (INS) and single valued Neutrosophic set (SVNS) concepts. Bipolar fuzzy sets are a generalization of fuzzy sets that were created by [7, 8]. The bipolar fuzzy relations study, in which each tuple is connected to a pair of satisfaction degrees, was first presented in [9]. Two applications for bipolar fuzzy sets in groups called the bipolar fuzzy groups and the norm have been introduced in [10].

A popular area of research in fuzzy theory of decision analysis is the study of fuzzy multi-attribute group decision-making. A sequence of judgments made in a fuzzy environment, which is usually ambiguous or uncertain, give decision information in the process of choosing the best possible options in terms of several criteria. Gaining a comprehensive understanding of the data is crucial for information fusion, particularly when making difficult decisions. Aggregation functions are one of the most efficient and simple methods for obtaining the aggregated result, although there are other methods as well. An n-tuple of data can be condensed into a single output using an aggregate function, which uses non-decreasing functions and keeps the output in the same set as the input. [11] In situations where all attribute values were defined as intuitionistic fuzzy numbers or interval-valued intuitionistic fuzzy numbers, aggregation functions were employed to handle dynamic multi-attribute decision-making in [11].

On the premise that the decision makers' criteria or preferences are unrelated and that the aggregating operators—defined by the independence axiom—are linear operators based on additive measures, multiple aggregation processes have been implemented in [12,13]. According to [14], real decision-making issues show the occurrence of unique dependencies or interactions between criteria. Decision-makers are typically invited from the same or related fields for a choice dilemma. They have a comparable social status, a similar level of knowledge, and similar tastes. Their arbitrary preferences can be demonstrated to exhibit nonlinearity. As a result, both the mutually preferred independence of these criteria and the independence of decision-makers are compromised. Advanced neutrosophic planar graph concepts and their applications were introduced in [15]. In [16], it was suggested to utilize a neutrosophic graph to predict linkages in social networks. A novel method of link prediction using the rsm index was developed by the authors of [17]. Radio fuzzy graphs and used radio k-colouring graphs to assign frequencies in radio stations introduced in [18]. [19] investigated the edge colouring of fuzzy graphs; chromatic index and the strong chromatic index have been proposed with its related properties in [19]. The colouring of directed fuzzy graphs based on the influence of relationship was proposed in [20]. Bipolar Neutrosophic TOPSIS was introduced in [21] as a method for resolving Multi Attribute Decision-Making (MADM) issues in a bipolar Neutrosophic fuzzy environment. In [22], methods based on Frank Choquet Bonferroni Mean Operators were developed to address MADM difficulties in a bipolar Neutrosophic fuzzy environment. [23, 24] discussed a few aggregation operators on different models.

The lattice of closed interval-valued fuzzy sets has been extended using Frank t-norms-based extension operations, which were proposed in [25]. These operations have been given the necessary and sufficient conditions to form a complete algebraic structure. [26] developed an analytical hierarchy process method for multi-attribute decision-making issues based on a logarithmic regression function and presented the idea of a triangular interval type-2 fuzzy set. [27] used arithmetic procedures like union and intersection between interval fuzzy linguistic numbers and Multi Attribute Group Decision Making problems to create a probabilistic linguistic framework. [28] a new signed distances-based linear assignment technique for MAGDM issues with fuzzy set information was created, and it was then applied to locate a landfill. [29] suggested a MAGDM and the idea of a trapezoidal interval type-2 fuzzy soft set. According to the aforementioned data, the research contribution based on bipolarity is relatively minimal, which is why we applied the concept in our research. MATLAB can be used to lessen the time complexity. The rest of the essay is organized as follows. Section 2 presents the fundamental antecedents. Section 3 established the Frank Aggregation operators' operating laws for bipolar neutrosophic numbers (BNNs). We go into great detail on BNFWGA and BNFOWGA, as well as their attributes, in Section 4. On the basis of bipolar neutrosophic Frank aggregation operations, we propose a number of comprehensive MCDM techniques in Section 5. The presented principles are applied to extend comprehensive techniques in Section 6. The proposed aggregation operators are used in section 7 to resolve the decision-making problem for choosing the best bridge. The comparative analysis and current methods have been discussed in Section 8. Section 9 provides the conclusion of the current study along with future directions.

2. Basic Concepts

For a better understanding, basic definitions pertaining to the current work are provided in this section.

Definition 2.1: Bipolar Neutrosophic Set (BNS) [30]

Let U be a fixed set. Then BNS can be defined as follows.

$$N(u) =< \chi_N^+(u), \xi_N^+(u), \Lambda_N^+(u), \chi_N^-(u), \xi_N^-(u), \Lambda_N^-(u) > / u \in U\},$$

Where $\chi_N^+(u), \xi_N^+(u), \Lambda_N^+(u) : U \to [0,1]$ and $\chi_N^-(u), \xi_N^-(u), \Lambda_N^-(u) : U \to [-1,0]$. The positive membership degrees $\chi_N^+(u), \xi_N^+(u), \Lambda_N^+(u)$ are the truth membership, indeterminacy membership degree and falsity membership degree of an element $u \in U$ corresponding to BNSN and the negative membership degrees $\chi_N^-(u), \xi_N^-(u), \Lambda_N^-(u)$ denote the truth membership degree, indeterminacy membership degree and falsity membership degree of an element $u \in U$ to some implicit counter property corresponding to a BNSN.

In particular, if 'U' has only one element, then

$N(u) =< \chi_N^+(u), \xi_N^+(u), \Lambda_N^+(u), \chi_N^-(u), \xi_N^-(u), \Lambda_N^-(u) >\}$, is called bipolar neutrosophic numbers (BNN).

Definition 2.2 : Algebraic operations of BNNs [30]

Let $N_1 =< \chi_1^+, \xi_1^+, \Lambda_1^+, \chi_1^-, \xi_1^-, \Lambda_1^- >$ and $N_2 =< \chi_2^+, \xi_2^+, \Lambda_2^+, \chi_2^-, \xi_2^-, \Lambda_2^- >$ be two BNNs. Then algebraic operations are defined as follows:

$$(1) N_1 \oplus N_2 =< \chi_1^+ + \chi_2^+ - \chi_1^+\chi_2^+, \xi_1^+\xi_2^+, \Lambda_1^+\Lambda_2^-, -\chi_1^+\chi_2^+, -(-\xi_1^- - \xi_2^- - \xi_1^-\xi_2^-), -(-\Lambda_1^- - \Lambda_2^- - \Lambda_1^-\Lambda_2^-) >$$

$$(2) N_1 \otimes N_2 =< \chi_1^+\chi_2^+, \xi_1^+ + \xi_2^+ - \xi_1^+\xi_2^+, \Lambda_1^+ + \Lambda_2^+ - \Lambda_1^+\Lambda_2^+, -(-\chi_1^- - \chi_2^- - \chi_1^-\chi_2^-), -\xi_1^-\xi_2^-, -\Lambda_1^-\Lambda_2^- >$$

$$(3) l.N_1 =< 1-(1-\chi_1^+)^l, \left(\xi_1^+\right)^l, \left(\Lambda_1^+\right)^l, \left(-\chi_1^-\right)^l, -\left(1-\left(1-\left(-\xi_1^-\right)^l\right)\right), -\left(1-\left(1-\left(-\Lambda_1^-\right)^l\right)\right) > (l>0),$$

$$(4) \mathrm{N}_1^l =< \left(\chi_1^+\right)^l, 1-\left(1-\xi_1^+\right)^l, 1-\left(1-\Lambda_1^+\right)^l, -\left(1-\left(1-\left(-\chi_1^-\right)^l\right)\right), -\left(-\xi_1^-\right)^l, -\left(-\Lambda_1^-\right)^l > (l>0)$$

Definition 2.3: Score and accuracy function of BNNs [26]

Let $N_1 =< \chi_1^+, \xi_1^+, \Lambda_1^+, \chi_1^-, \xi_1^-, \Lambda_1^- >$ be BNN. Then the score function s(aleph1),accuracy function $a\left(N_1\right)$ and certainty function

c(aleph1) are defined as:

$$s\left(N_1\right) = \frac{\chi_1^+ + 1 - \xi_1^+ + 1 - \Lambda_1^+ + 1 + \chi_1^- - \xi_1^- - \Lambda_1^-}{6} \quad \text{……………….(1)}$$

$$a\left(N_1\right) = \left(\chi_1^+ - \Lambda_1^+\right) + \left(\chi_1^- - \Lambda_1^-\right) \quad \text{……………….(2)}$$

$$c\left(N_1\right) = \chi_1^+ - \Lambda_1^- \quad \text{…………………….(3)}$$

Definition 2.4: Properties on Bipolar Neutrosophic Sets [30]

Let $N_1 =< \chi_1^+, \xi_1^+, \Lambda_1^+, \chi_1^-, \xi_1^-, \Lambda_1^- >$

And $N_2 =< \chi_2^+, \xi_2^+, \Lambda_2^+, \chi_2^-, \xi_2^-, \Lambda_2^- >$

Be two BNNs therefore

(1) If $s\left(N_1\right) > s\left(N_2\right), then N_1 > N_2$.

(2) If $s(N_1)=s(N_2)\&a(N_1)=a(N_2), then N_1 > N_2$.

(3) If $s(N_1)=s(N_2)\&a(N_1)=a(N_2)\&c(N_1)=c(N_2), then N_1 > N_2$

(4) If $s(N_1)=s(N_2)\&a(N_1)=a(N_2)\&c(N_1)=c(N_2), then N_1 \square N_2$

Definition 2.5: Frank Triangular Norms [27]

The sum and product of Frank triangular norms are defined as follows.

$$\mathrm{N}_1 \oplus_F \mathrm{N}_2 = 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\mathrm{N}_1}-1\right)\left(\lambda^{1-\mathrm{N}_2}-1\right)}{\lambda-1}\right)\lambda>1\forall\left(\mathrm{N}_1,\mathrm{N}_2\right)\in[0,1]^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\mathrm{N}_1 \otimes_F \mathrm{N}_2 = \log_\lambda\left(1+\frac{\left(\lambda^{\mathrm{N}_1}-1\right)\left(\lambda^{\mathrm{N}_2}-1\right)}{\lambda-1}\right)\lambda>1\forall\left(\mathrm{N}_1,\mathrm{N}_2\right)\in[0,1]^2 \quad \ldots\ldots\ldots\ldots(5)$$

3. Operational Laws of Frank Triangular Norms for Bipolar Neutrosophic Numbers:

In this section, Operational laws are proposed using Frank triangular norms for BNNs.

Definition 3.1: Let $\mathrm{N}_1=\left\langle \chi_1^+,\xi_1^+,\Lambda_1^+,\chi_1^-,\xi_1^-,\Lambda_1^-\right\rangle$ and $\mathrm{N}_2=\left\langle \chi_2^+,\xi_2^+,\Lambda_2^+,\chi_2^-,\xi_2^-,\Lambda_2^-\right\rangle$ be two BNNs and $\lambda \ge 1$. Then the operational laws are as follows.

(i). Addition:

$$N_1 \oplus_F N_2 = \begin{cases} <1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\chi_1^+}-1\right)\left(\lambda^{1-\chi_2^+}-1\right)}{\lambda-1}\right)-\log_\lambda\left(1+\frac{\left(\lambda^{\chi_1^+}-1\right)\left(\lambda^{\chi_2^+}-1\right)}{\lambda-1}\right), \log_\lambda\left(1+\frac{\left(\lambda^{\xi_1^+}-1\right)\left(\lambda^{\xi_2^+}-1\right)}{\lambda-1}\right), \\ \log_\lambda\left(1+\frac{\left(\lambda^{\Lambda_1^+}-1\right)\left(\lambda^{\Lambda_1^-}-1\right)}{\lambda-1}\right), -\log_\lambda\left(1+\frac{\left(\lambda^{\chi_1^+}-1\right)\left(\lambda^{\chi_2^+}-1\right)}{\lambda-1}\right), \\ \log_\lambda\left(1+\frac{\left(\lambda^{1+\xi_1^-}-1\right)\left(\lambda^{1+\xi_2^-}-1\right)}{\lambda-1}\right)-1+\log_\lambda\left(1+\frac{\left(\lambda^{-\xi_1^-}-1\right)\left(\lambda^{-\xi_2^-}-1\right)}{\lambda-1}\right), \\ \log_\lambda\left(1+\frac{\left(\lambda^{1+\Lambda_1^-}-1\right)\left(\lambda^{1+\Lambda_2^-}-1\right)}{\lambda-1}\right)-1+\log_\lambda\left(1+\frac{\left(\lambda^{-\Lambda_1^-}-1\right)\left(\lambda^{-\Lambda_2^-}-1\right)}{\lambda-1}\right)> \end{cases}$$

(ii). Multiplication:

$$N_1 \otimes_F N_2 = \begin{cases} <\log_\lambda\left(1+\frac{\left(\lambda^{\chi_1^+}-1\right)\left(\lambda^{\chi_2^+}-1\right)}{\lambda-1}\right), \\ 1-\log_\lambda\left(\frac{\lambda+\lambda^{2-\xi_1^+-\xi_2^+}-\lambda^{1-\xi_1^+}-\lambda^{1-\xi_2^+}}{\lambda-1}\right)-\log_\lambda\left(\frac{\lambda+\lambda^{\xi_1^++\xi_2^+}-\lambda^{\xi_1^+}-\lambda^{\xi_2^+}}{\lambda-1}\right), \\ 1-\log_\lambda\left(\frac{\lambda+\lambda^{2-\Lambda_1^+-\Lambda_2^+}-\lambda^{1-\Lambda_1^+}-\lambda^{1-\Lambda_2^+}}{\lambda-1}\right)-\log_\lambda\left(\frac{\lambda+\lambda^{\Lambda_1^++\Lambda_2^+}-\lambda^{\Lambda_1^+}-\lambda^{\Lambda_2^+}}{\lambda-1}\right). \\ \log_\lambda\left(1-\frac{\left(\lambda^{1+\chi_1^-}-1\right)\left(\lambda^{1+\chi_2^-}-1\right)}{\left(\lambda^{-\chi_1^-}-1\right)\left(\lambda^{-\chi_2^-}-1\right)}\right)-1, -\log_\lambda\left(1+\frac{\left(\lambda^{\xi_1^-}-1\right)\left(\lambda^{\xi_2^-}-1\right)}{\lambda-1}\right), \\ -\log_\lambda\left(1+\frac{\left(\lambda^{\Lambda_1^-}-1\right)\left(\lambda^{\Lambda_2^-}-1\right)}{\lambda-1}\right)> \end{cases}$$

(iii). Multiplication by an ordinary number:

$$l._F N_1 = \begin{cases} <1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\chi_1^+}-1\right)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{\xi_1^+}-1\right)^k}{(\lambda-1)^{k-1}}\right) \\ \log_\lambda\left(1+\frac{\left(\lambda^{\Omega_1^+}-1\right)^k}{(\lambda-1)^{k-1}}\right), -\log_\lambda\left(1+\frac{\left(\lambda^{\Lambda_1^-}-1\right)^k}{(\lambda-1)^{k-1}}\right), \\ \log_\lambda\left(1+\frac{\left(\lambda^{1-\xi_1^-}-1\right)^k}{(\lambda-1)^{k-1}}\right)-1, \log_\lambda\left(1+\frac{\left(\lambda^{1-\Lambda_1^-}-1\right)^k}{(\lambda-1)^{k-1}}\right)-1> \end{cases}$$

(iv) Power Operation:

$$N_1^{\wedge_F^k} = \begin{cases} < \log_\lambda\left(1+\frac{\left(\lambda^{\chi_1^+}-1\right)^k}{(\lambda-1)^{k-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\xi_1^+}-1\right)^k}{(\lambda-1)^{k-1}}\right) \\ 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\Lambda_1^+}-1\right)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{1-\chi_1^-}-1\right)^k}{(\lambda-1)^{k-1}}\right)-1, \\ -\log_\lambda\left(1+\frac{\left(\lambda^{\xi_1^-}-1\right)^k}{(\lambda-1)^{k-1}}\right), -\log_\lambda\left(1+\frac{\left(\lambda^{\Lambda_1^-}-1\right)^k}{(\lambda-1)^{k-1}}\right) > \end{cases}$$

Theorem 3.2:

Let $N_1 = \left\langle \chi_1^+, \xi_1^+, \Lambda_1^+, \chi_1^-, \xi_1^-, \Lambda_1^- \right\rangle$ and $N_2 = \left\langle \chi_2^+, \xi_2^+, \Lambda_2^+, \chi_2^-, \xi_2^-, \Lambda_2^- \right\rangle$ be two BNNs and

$l, l_1, l_2 > 0$. Then the following properties can be proven easily

(a) $N_1 \oplus_F N_2 = N_2 \oplus_F N_1$

(b) $N_1 \otimes_F N_2 = N_2 \otimes_F N_1$

(c) $l \cdot_F \left(N_1 \oplus_F N_2\right) = l \cdot_F N_2 \oplus_F l \cdot_F N_1$

(d) $\left(N_1 \oplus_F N_2\right)^{\wedge_F^l} = N_1^{\wedge_F^l} \oplus_F N_2^{\wedge_F^l}$

(e) $\left(l_1 + l_2\right) \cdot_F N_1 = l_1 \cdot_F N_2 \oplus_F l_2 \cdot_F N_1$

(f) $N_1^{\wedge_F^{(l_1+l_2)}} = N_2^{\wedge_F^{l_1}} \otimes N_2^{\wedge_F^{l_2}}$

4. Bipolar Neutrosophic Frank Weighted Geometric Aggregation Operator

In this section, we proposed bipolar neutrosophic Frank weighted geometric aggregation (BNFWGA) and the bipolar neutrosophic Frank ordered weighted geometric aggregation (BNFOWGA) operators and discussed different properties.

Definition 4.1:

Let $N_j = \left\langle \chi_j^+, \xi_j^+, \Lambda_j^+, \chi_j^-, \xi_j^-, \Lambda_j^- \right\rangle \left(j = 1, 2, 3......n\right)$ be a family of BNN's.

A mapping BNFWGA: $U^{'} = U$ is called BNFWGA operator, if it satisfies

$$BNFWGA\left(N_1, N_2, N_3.........N_n\right) = \overset{n}{\underset{j=1}{\otimes}} N_n^{\wedge_F^{\omega_n}}$$

$$= N_1^{\wedge_F^{\omega_1}} \otimes_F N_2^{\wedge_F^{\omega_2}} \otimes_F ....\otimes_F N_n^{\wedge_F^{\omega_n}}$$

and $\omega = \left(\omega_1, \omega_2, \omega_3..........\omega_n\right)^T$ be the weight vector of $\mathrm{N}_j, \omega_j \in [0,1] \,\&\, \sum_{j=1} \omega_j = 1.$

Theorem 4.1 :

Let $\mathrm{N}_j = \left\langle \chi_j^+, \xi_j^+, \Lambda_j^+, \chi_j^-, \xi_j^-, \Lambda_j^- \right\rangle \left(j = 1, 2, 3......n\right)$ be a family of BNN's and $\omega = \left(\omega_1, \omega_2, \omega_3..........\omega_n\right)^T$ be the weight vector of $\mathrm{N}_j, \omega_j \in [0,1] \,\&\, \sum_{j=1} \omega_j = 1.$ Then, the value aggregated using BNFWGA operator is still a BNN i.e).

$$BNFWGA\left(N_1, N_2, N_3.........N_n\right) = N_1^{\wedge_F^{\omega_j}} \otimes_F N_2^{\wedge_F^{\omega_j}} \otimes_F ....\otimes_F N_n^{\wedge_F^{\omega_j}}$$

$$= \begin{cases} < \log_\lambda \left(1 + \prod_{j=1}^{n} \left(\lambda^{\chi_j^+} - 1\right)^{\omega_j}\right), 1 - \log\left(1 + \prod_{j=1}^{n} \left(\lambda^{1-\xi_j^+} - 1\right)^{\omega_j}\right) \\ 1 - \log \prod_{j=1}^{n} \left(\lambda^{1-\Lambda_j^+} - 1\right)^{\omega_j}, \log_\lambda \left(1 + \prod_{j=1}^{n} \left(\lambda^{1-(-\chi_j^-)} - 1\right)^{\omega_j}\right) - 1, \\ -\log_\lambda \left(1 + \prod_{j=1}^{n} \left(\lambda^{-\xi_j^-} - 1\right)^{\omega_j}\right), -\log_\lambda \left(1 + \prod_{j=1}^{n} \left(\lambda^{-\Lambda_j^-} - 1\right)^{\omega_j}\right), > (\omega > 0) \end{cases} \quad ....(6)$$

Proof:

By mathematical induction we prove the result.

Case (1): When n=2

Based on the Definition 4.1, the following result can be obtained

$$BNFWGA\left(N_1, N_2\right) = N_1^{\wedge_F^{\omega_1}} \otimes_F N_2^{\wedge_F^{\omega_2}}$$

$$N_1^{\wedge_F^{\omega}} = \begin{cases} < \log_\lambda \left(1 + \left(\lambda^{\chi_1^+} - 1\right)^{\omega}\right), 1 - \log_\lambda \left(1 + \left(\lambda^{1-\xi_1^+} - 1\right)^{\omega}\right), \\ 1 - \log_\lambda \left(1 + \left(\lambda^{1-\Lambda_1^+} - 1\right)^{\omega}\right), \log_\lambda \left(1 + \left(\lambda^{1-(-\chi_1^-)} - 1\right)^{\omega}\right) - 1, \\ -\log_\lambda \left(1 + \left(\lambda^{-\xi_1^-} - 1\right)^{\omega}\right), -\log_\lambda \left(1 + \left(\lambda^{-\Lambda_1^-} - 1\right)^{\omega}\right) > \end{cases}$$

$$N_2^{\wedge_F^{\omega_1}} = \begin{cases} < \log_\lambda\left(1+\left(\lambda^{\chi_2^+}-1\right)^{\omega_2}\right), 1-\log_\lambda\left(1+\left(\lambda^{1-\xi_2^+}-1\right)^{\omega_2}\right), \\ 1-\log_\lambda\left(1+\left(\lambda^{1-\Lambda_2^+}-1\right)^{\omega_2}\right), \log_\lambda\left(1+\left(\lambda^{1-(-\chi_2^-)}-1\right)^{\omega_2}\right)-1, \\ -\log_\lambda\left(1+\left(\lambda^{-\xi_2^-}-1\right)^{\omega_2}\right), -\log_\lambda\left(1+\left(\lambda^{-\Lambda_2^-}-1\right)^{\omega_2}\right) > \end{cases}$$

$$N_1^{\wedge_F^{\omega_1}} \otimes_F N_2^{\wedge_F^{\omega_2}} == \begin{cases} < \log_\lambda\left(1+\prod_{j=1}^{2}\left(\lambda^{\chi_j^+}-1\right)^{\omega_j}\right), 1-\log\prod_{j=1}^{2}\left(\lambda^{1-\xi_j^+}-1\right)^{\omega_j}, \\ 1-\log\prod_{j=1}^{2}\left(\lambda^{1-\Lambda_j^+}-1\right)^{\omega_j}, \log_\lambda\left(1+\prod_{j=1}^{2}\left(\lambda^{1-(-\chi_j^-)}-1\right)^{\omega_j}\right)-1, \\ -\log_\lambda\left(1+\prod_{j=1}^{2}\left(\lambda^{-\xi_j^-}-1\right)^{\omega_j}\right), -\log_\lambda\left(1+\prod_{j=1}^{2}\left(\lambda^{-\Lambda_j^-}-1\right)^{\omega_j}\right), > (\omega > 0) \end{cases}$$

Case (ii): When n=s

Using equation (6), the following result can be obtained.

$$BNFWGA\left(N_1, N_2.....N_s\right) = \otimes_{j=1}^{s} N_j^{\wedge^{\omega j}}$$

$$= \begin{cases} < \log_\lambda\left(1+\prod_{j=1}^{s}\left(\lambda^{\chi_j^+}-1\right)^{\omega_j}\right), 1-\log\left(1+\prod_{j=1}^{s}\left(\lambda^{1-\xi_j^+}-1\right)^{\omega_j}\right), \\ 1-\log\prod_{j=1}^{s}\left(\lambda^{1-\Lambda_j^+}-1\right)^{\omega_j}, \log_\lambda\left(1+\prod_{j=1}^{s}\left(\lambda^{1-(-\chi_j^-)}-1\right)^{\omega_j}\right)-1, \\ -\log_\lambda\left(1+\prod_{j=1}^{s}\left(\lambda^{-\xi_j^-}-1\right)^{\omega_j}\right), -\log_\lambda\left(1+\prod_{j=1}^{s}\left(\lambda^{-\Lambda_j^-}-1\right)^{\omega_j}\right), > (\omega > 0) \end{cases}$$

Case (iii) When n=s+1 then following result can be obtained:

$$BNFWGA\left(N_1, N_2.....N_s, N_{s+1}\right) = \otimes_{j=1}^{s} N_j^{\wedge \omega_j} \otimes N_j^{\wedge \omega_{s+1}}$$

$$= \begin{cases} < \log_\lambda\left(1+\prod_{j=1}^{s}\left(\lambda^{\chi_j^+}-1\right)^{\omega_j}\right), 1-\log\left(1+\prod_{j=1}^{s}\left(\lambda^{1-\xi_j^+}-1\right)^{\omega_j}\right), \\ 1-\log\left(1+\prod_{j=1}^{s}\left(\lambda^{1-\Lambda_j^+}-1\right)^{\omega_j}\right), \log_\lambda\left(1+\prod_{j=1}^{s}\left(\lambda^{1-(-\chi_j^-)}-1\right)^{\omega_j}\right)-1, \\ -\log_\lambda\left(1+\prod_{j=1}^{s}\left(\lambda^{-\xi_j^-}-1\right)^{\omega_j}\right), -\log_\lambda\left(1+\prod_{j=1}^{s}\left(\lambda^{-\Lambda_j^-}-1\right)^{\omega_j}\right), > (\omega > 0) \end{cases}$$

$$\otimes_F \begin{cases} < \log_\lambda\left(1+\prod_{j=s+1}\left(\lambda^{\chi_j^+}-1\right)^{\omega_j}\right), 1-\log\left(1+\prod_{j=s+1}\left(\lambda^{1-\xi_j^+}-1\right)^{\omega_j}\right), \\ 1-\log\left(1+\prod_{j=s+1}\left(\lambda^{1-\Lambda_j^+}-1\right)^{\omega_j}\right), \log_\lambda\left(1+\prod_{j=s+1}\left(\lambda^{1-(-\chi_j^-)}-1\right)^{\omega_j}\right)-1, \\ -\log_\lambda\left(1+\prod_{j=s+1}\left(\lambda^{-\xi_j^-}-1\right)^{\omega_j}\right), -\log_\lambda\left(1+\prod_{j=s+1}\left(\lambda^{-\Lambda_j^-}-1\right)^{\omega_j}\right), > (\omega > 0) \end{cases}$$

Thus the following result can be obtained.

$$BNFWGA\left(N_1, N_2, N_3.........N_n\right) = N_1^{\wedge_F \omega_j} \otimes_F N_2^{\wedge_F \omega_j} \otimes_F ....\otimes_F N_n^{\wedge_F \omega_j}$$

$$= \begin{cases} < \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\chi_j^+}-1\right)^{\omega_j}\right), 1-\log\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\xi_j^+}-1\right)^{\omega_j}\right), \\ 1-\log\prod_{j=1}^{n}\left(\lambda^{1-\Lambda_j^+}-1\right)^{\omega_j}, \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-(-\chi_j^-)}-1\right)^{\omega_j}\right)-1, \\ -\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{-\xi_j^-}-1\right)^{\omega_j}\right), -\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{-\Lambda_j^-}-1\right)^{\omega_j}\right), > (\omega > 0) \end{cases}$$

Therefore, the theorem is true for n=s+1.

Hence the theorem.

Example 4.1: Let

$$N_1 =< 0.5, 0.7, 0.3, -0.6, -0.2, -0.6 >$$
$$N_2 =< 0.2, 0.5, 0.5, -0.8, -0.4, -0.3 >$$
$$N_3 =< 0.3, 0.6, 0.4, -0.7 - 0.3, -0.1 >$$

be three BNNs and let weight vector of BNNs $N_j \ \left(j = 1, 2, 3\right)$ be

$\omega = \left(\frac{1}{8}, \frac{3}{8}, \frac{1}{2}\right)^T, \omega_1 = \frac{1}{8}, \omega_2 = \frac{3}{8}, \omega_3 = \frac{1}{2}$ are the weight of $N_j \ \left(j = 1, 2, 3\right)$ such that $\prod_{j=1} \omega_j = 1$

Then by above theorem

$$BNFWGA\left(N_1, N_2, N_3\right) = \begin{cases} \log_3\left(1+\left(3^{0.5}-1\right)^{\frac{1}{8}}+\left(3^{0.2}-1\right)^{\frac{3}{8}}+\left(3^{0.3}-1\right)^{\frac{1}{2}}\right), \\ 1-\log_3\left(1+\left(3^{1-0.7}-1\right)^{\frac{1}{8}}+\left(3^{1-0.5}-1\right)^{\frac{3}{8}}+\left(3^{1-0.6}-1\right)^{\frac{1}{2}}\right), \\ 1-\log_3\left(1+\left(3^{1-0.3}-1\right)^{\frac{1}{8}}+\left(3^{1-0.5}-1\right)^{\frac{3}{8}}+\left(3^{1-0.4}-1\right)^{\frac{1}{2}}\right), \\ \log_3\left(1+\left(3^{1-0.6}-1\right)^{\frac{1}{8}}+\left(3^{1-0.8}-1\right)^{\frac{3}{8}}+\left(3^{1-0.7}-1\right)^{\frac{1}{2}}\right)-1, \\ \log_3\left(1+\left(3^{0.2}-1\right)^{\frac{1}{8}}+\left(3^{0.4}-1\right)^{\frac{3}{8}}+\left(3^{0.3}-1\right)^{\frac{1}{2}}\right), \\ l\log_3\left(1+\left(3^{0.6}-1\right)^{\frac{1}{8}}+\left(3^{0.3}-1\right)^{\frac{3}{8}}+\left(3^{0.1}-1\right)^{\frac{1}{2}}\right) \end{cases}$$

$$BNFWGA\left(N_1, N_2, N_3\right) = \left(1.0, 0.0391, 0.2474, -0.3072, -1.0, -1.0\right)$$

The BNFWGA operator has the following properties:

(1) Idomopotency:

Let all the BNN's be $N_j =< \chi_j^+, \xi_j^+, \Lambda_j^+, \chi_j^-, \xi_j^-, \Lambda_j^- >= N\left(j = 1, 2, 3....n\right)$ where $\omega = \left(\omega_1, \omega_2, \omega_3..........\omega_n\right)^T$ be the weight vector of $N_j.\omega\ _j \in [0,1]$ and $\sum_{j=1}^{3} \omega_j = 1$ then

$$BNFWGA\left(N_1, N_2, N_3.........N_j\right) = N$$

(2) Monotonicity: Let $N_j\ \left(j = 1, 2, 3....n\right)$ and $N_j^{'}\ \left(j = 1, 2, 3....n\right)$ be two families of BNNs, where $\omega = \left(\omega_1, \omega_2, \omega_3..........\omega_n\right)^T$ be the weight vector of N_j and $N_j^{'}, \omega_j \in [0,1]$ and $\prod_{j=1} \omega_j = 1$. For all 'j' if $N_j \geq N_j^{'}$ then

$$BNFWGA\left(N_1, N_2, N_3.........N_j\right) \geq BNFWGA\left(N_1^{'}, N_2^{'}, N_3^{'}.........N_j^{'}\right)$$

(3) Boundedness: Let $N_j =< \chi_j^+, \xi_j^+, \Lambda_j^+, \chi_j^-, \xi_j^-, \Lambda_j^- >= N\left(j=1,2,3....n\right)$ be a family of BNNs. Where $\omega = \left(\omega_1, \omega_2, \omega_3.........\omega_n\right)^T$ be the weight vector of N_j , $\omega_j \in [0,1]$ and $\sum_{j=1}^{3} \omega_j = 1$, Therefore we have

$$BNFWGA\left(N^-, N^-, ...N^-\right) \leq BNFWGA\left(N_1, N_2, N_3.........N_j\right) \leq BNFWGA\left(N^+, N^+, ...N^+\right)$$

where $N^- =< \chi_{N^-}^+, \xi_{N^-}^+, \Lambda_{N^-}^+, \chi_{N^-}^-, \xi_{N^-}^-, \Lambda_{N^-}^- >$

$$= \begin{cases} \min\left(\chi_1^+, \chi_2^+, \chi_3^+, ...\chi_n^+\right), \max\left(\xi_1^+, \xi_2^+, \xi_3^+, ...\xi_n^+\right), \\ \max\left(\Lambda_1^+, \Lambda_2^+, \Lambda_3^+, ...\Lambda_n^+\right), \max\left(\chi_1^-, \chi_2^-, \chi_3^-, ...\chi_n^-\right) \\ \min\left(\xi_1^-, \xi_2^-, \xi_3^-, ...\xi_n^-\right), \min\left(\Lambda_1^-, \Lambda_2^-, \Lambda_3^-, ...\Lambda_n^-\right) > \end{cases}$$

And $N^+ =< \chi_{N^+}^+, \xi_{N^+}^+, \Lambda_{N^+}^+, \chi_{N^+}^-, \xi_{N^+}^-, \Lambda_{N^+}^- >$

$$= \begin{cases} \max\left(\chi_1^+, \chi_2^+, \chi_3^+, ...\chi_n^+\right), \min\left(\xi_1^+, \xi_2^+, \xi_3^+, ...\xi_n^+\right), \\ \min\left(\Lambda_1^+, \Lambda_2^+, \Lambda_3^+, ...\Lambda_n^+\right), \min\left(\chi_1^-, \chi_2^-, \chi_3^-, ...\chi_n^-\right) \\ \max\left(\xi_1^-, \xi_2^-, \xi_3^-, ...\xi_n^-\right), \max\left(\Lambda_1^-, \Lambda_2^-, \Lambda_3^-, ...\Lambda_n^-\right) > \end{cases}$$

Proof: Since $N_j =< \chi_j^+, \xi_j^+, \Lambda_j^+, \chi_j^-, \xi_j^-, \Lambda_j^- >= N\left(j=1,2,3....n\right)$ then, the following result can be obtained by using Equation (6). The following result can be obtained

$$BNFWGA\left(N_1, N_2, N_3.........N_n\right) = \begin{cases} < \log_\lambda\left(1 + \prod_{j=1}^{n}\left(\lambda^{\chi_j^+} - 1\right)^{\omega_j}\right), 1 - \log\prod_{j=1}^{n}\left(\lambda^{1-\xi_j^+} - 1\right)^{\omega_j}, \\ 1 - \log\prod_{j=1}^{n}\left(\lambda^{1-\Lambda_j^+} - 1\right)^{\omega_j}, \log_\lambda\left(1 + \prod_{j=1}^{n}\left(\lambda^{1-(-\chi_j^-)} - 1\right)^{\omega_j}\right) - 1, \\ -\log_\lambda\left(1 + \prod_{j=1}^{n}\left(\lambda^{-\xi_j^-} - 1\right)^{\omega_j}\right), -\log_\lambda\left(1 + \prod_{j=1}^{n}\left(\lambda^{-\Lambda_j^-} - 1\right)^{\omega_j}\right), > (\omega > 0) \end{cases}$$

$$= \begin{cases} < \log_\lambda\left(1 + \left(\lambda^{\chi^+} - 1\right)\right), 1 - \log_\lambda\left(1 + \left(\lambda^{1-\xi^+} - 1\right)\right), \\ 1 - \log_\lambda\left(1 + \left(\lambda^{1-\Lambda^+} - 1\right)\right), \log_\lambda\left(1 + \left(\lambda^{1-(-\chi^-)} - 1\right)\right) - 1, \\ -\log_\lambda\left(1 + \left(\lambda^{-\xi^-} - 1\right)\right), -\log_\lambda\left(1 + \left(\lambda^{-\Lambda^-} - 1\right)\right) > \end{cases}$$

$=< \chi^+, \xi^+, \Lambda^+, \chi^-, \xi^-, \Lambda^- >= N$ holds

(1) The property is obvious based on the equation (6).

(2) Let $N^{-} = < \chi^{+}_{N^{-}}, \xi^{+}_{N^{-}}, \Lambda^{+}_{N^{-}}, \chi^{-}_{N^{-}}, \xi^{-}_{N^{-}}, \Lambda^{-}_{N^{-}} > and\, N^{+} = < \chi^{+}_{N^{+}}, \xi^{+}_{N^{+}}, \Lambda^{+}_{N^{+}}, \chi^{-}_{N^{+}}, \xi^{-}_{N^{+}}, \Lambda^{-}_{N^{+}} >$

There are following inequalities:

$$\begin{cases} < \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{\chi^{+}_{N^{-}}}-1\right)\right) \leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{\chi^{+}_{j}}-1\right)\right) \leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{\chi^{+}_{N^{+}}}-1\right)\right), \\ 1-\log\left(1+\left(\prod_{j=1}^{n}\left(\lambda^{1-\xi^{+}_{N^{-}}}-1\right)\right)\right) \leq 1-\log\left(1+\left(\prod_{j=1}^{n}\left(\lambda^{1-\xi^{+}_{N^{-}}}-1\right)\right)\right) \leq 1-\left(1+\left(\prod_{j=1}^{n}\left(\lambda^{1-\xi^{+}_{N^{-}}}-1\right)\right)\right) \\ 1-\log\prod_{j=1}^{n}\left(\lambda^{1-\Lambda^{+}_{N^{-}}}-1\right) \leq 1-\log\prod_{j=1}^{n}\left(\lambda^{1-\Lambda^{+}_{j}}-1\right) \leq 1-\log\prod_{j=1}^{n}\left(\lambda^{1-\Lambda^{+}_{N^{+}}}-1\right), \\ \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{1-(-\chi^{-}_{N^{-}})}-1\right)\right)-1 \leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{1-(-\chi^{-}_{N^{-}})}-1\right)\right)-1 \leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{1-(-\chi^{-}_{N^{-}})}-1\right)\right)-1 \\ -\log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{-\xi^{-}_{N^{-}}}-1\right)\right) \leq -\log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{-\xi^{-}_{j}}-1\right)\right) \leq -\log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{-\xi^{-}_{N^{+}}}-1\right)\right), \\ -\log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{-\Lambda^{-}_{N^{-}}}-1\right)\right), \leq -\log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{-\Lambda^{-}_{j}}-1\right)\right) - \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{-\Lambda^{-}_{N^{+}}}-1\right)\right) > \end{cases}$$

Hence

$$BNFWGA\left(N^{-}, N^{-}, ...N^{-}\right) \leq BNFWGA\left(N_1, N_2, N_3.........N_j\right) \leq BNFWGA\left(N^{+}, N^{+}, ...N^{+}\right)$$

holds.

5. Bipolar Neutrosophic Frank Ordered Weighted Geometric Aggregation (BNFOWGA) Operator

This section proposes the BNFOWGA operator and describes its properties in detail.

Definition 5.1:

Let $\mathrm{N}_j = \left\langle \chi^{+}_{j}, \xi^{+}_{j}, \Lambda^{+}_{j}, \chi^{-}_{j}, \xi^{-}_{j}, \Lambda^{-}_{j} \right\rangle \left(j = 1, 2, 3......n \right)$ be a family of BNNs.

A mapping BNFOWGA: $U^{'} = U$ is called BNDWGA operator, if it satisfies,

$$BNFWGA\left(N_1, N_2, N_3.........N_n\right) = \overset{n}{\underset{k=1}{\otimes}} N_{\rho(k)}^{\wedge_F^{\omega_n}}$$

$$= N_{\rho(1)}^{\wedge_F^{\omega_1}} \otimes_F N_{\rho(2)}^{\wedge_F^{\omega_2}} \otimes_F ....\otimes_F N_{\rho(n)}^{\wedge_F^{\omega_n}}$$

Where ρ is permutation that orders the elements

$N_{\rho(1)} \geq N_{\rho(2)} \geq N_{\rho(3)} \geq ....N_{\rho(n)} \geq$

and $\omega = (\omega_1, \omega_2, \omega_3..........\omega_n)^T$ be the weight vector of $\mathrm{N}_j, \omega_j \in [0,1] \& \sum_{j=1} \omega_j = 1.$

Theorem 5.1:

Let $\mathrm{N}_j = \langle \chi_j^+, \xi_j^+, \Lambda_j^+, \chi_j^-, \xi_j^-, \Lambda_j^- \rangle (j = 1, 2, 3......n)$ be a family of BNN's and $\omega = (\omega_1, \omega_2, \omega_3..........\omega_n)^T$ be the weight vector of $\mathrm{N}_j, \omega_j \in [0,1] \& \sum_{j=1} \omega_j = 1.$ Then, the value aggregated by using bipolar neutrosophic Frank ordered weighted geometric average operator is still a BNN

$$BNFWGA(N_1, N_2, N_3.........N_n) = \overset{n}{\underset{k=1}{\otimes}} N_{\rho(k)}^{\wedge_F^{\omega_n}}$$

$$= N_{\rho(1)}^{\wedge_F^{\omega_1}} \otimes_F N_{\rho(2)}^{\wedge_F^{\omega_2}} \otimes_F ....\otimes_F N_{\rho(n)}^{\wedge_F^{\omega_n}}$$

$$= \begin{cases} < \log_\lambda \left(1 + \prod_{k=1}^{n} \left(\lambda^{\chi_j^+} - 1\right)^{\omega_k}\right), 1 - \left(\log\left(1 + \prod_{k=1}^{n} \left(\lambda^{1-\xi_j^+} - 1\right)^{\omega_k}\right)\right) \\ 1 - \log \prod_{k=1}^{n} \left(\lambda^{1-\Lambda_j^+} - 1\right)^{\omega_k}, \log_\lambda \left(1 + \prod_{k=1}^{n} \left(\lambda^{1-(-\chi_j^-)} - 1\right)^{\omega_k}\right) - 1, \\ -\log_\lambda \left(1 + \prod_{k=1}^{n} \left(\lambda^{-\xi_j^-} - 1\right)^{\omega_k}\right), -\log_\lambda \left(1 + \prod_{k=1}^{n} \left(\lambda^{-\Lambda_j^-} - 1\right)^{\omega_k}\right), > (\omega > 0) \end{cases} \quad .... (1)$$

Proof:

If n=2, using Frank operations for bipolar neutrosophic numbers, the following result can be obtained $BNFWGA(N_1, N_2) = N_{\rho(1)}^{\wedge_F^{\omega_1}} \otimes_F N_{\rho(2)}^{\wedge_F^{\omega_2}}$

$$N_1^{\wedge_F^{\omega_1}} = \begin{cases} < \log_\lambda \left(1 + \left(\lambda^{\chi_1^+} - 1\right)^{\omega_1}\right), 1 - \log_\lambda \left(1 + \left(\lambda^{1-\xi_1^+} - 1\right)^{\omega_1}\right), \\ 1 - \log_\lambda \left(1 + \left(\lambda^{1-\Lambda_1^+} - 1\right)^{\omega_1}\right), \log_\lambda \left(1 + \left(\lambda^{1-(-\chi_1^-)} - 1\right)^{\omega_1}\right) - 1, \\ -\log_\lambda \left(1 + \left(\lambda^{-\xi_1^-} - 1\right)^{\omega_1}\right), -\log_\lambda \left(1 + \left(\lambda^{-\Lambda_1^-} - 1\right)^{\omega_1}\right) > \end{cases}$$

$$N_2^{\hat{\omega}_1 F} = \begin{cases} < \log_\lambda\left(1+\left(\lambda^{\chi_2^+}-1\right)^{\omega_2}\right), 1-\log_\lambda\left(1+\left(\lambda^{1-\xi_2^+}-1\right)^{\omega_2}\right), \\ 1-\log_\lambda\left(1+\left(\lambda^{1-\Lambda_2^+}-1\right)^{\omega_2}\right), \log_\lambda\left(1+\left(\lambda^{1-(-\chi_2^-)}-1\right)^{\omega_2}\right)-1, \\ -\log_\lambda\left(1+\left(\lambda^{-\xi_2^-}-1\right)^{\omega_2}\right), -\log_\lambda\left(1+\left(\lambda^{-\Lambda_2^-}-1\right)^{\omega_2}\right) > \end{cases}$$

$$N_1^{\hat{\omega}_1 F} \otimes_F N_2^{\hat{\omega}_2 F} == \begin{cases} < \log_\lambda\left(1+\prod_{j=1}^{2}\left(\lambda^{\chi_j^+}-1\right)^{\omega_j}\right), 1-\log\left(1+\prod_{j=1}^{2}\left(\lambda^{1-\xi_j^+}-1\right)^{\omega_j}\right), \\ 1-\log\prod_{j=1}^{2}\left(\lambda^{1-\Lambda_j^+}-1\right)^{\omega_j}, \log_\lambda\left(1+\prod_{j=1}^{2}\left(\lambda^{1-(-\chi_j^-)}-1\right)^{\omega_j}\right)-1, \\ -\log_\lambda\left(1+\prod_{j=1}^{2}\left(\lambda^{-\xi_j^-}-1\right)^{\omega_j}\right), -\log_\lambda\left(1+\prod_{j=1}^{2}\left(\lambda^{-\Lambda_j^-}-1\right)^{\omega_j}\right), > (\omega > 0) \end{cases}$$

If k=s then,

$$BNFWGA\left(N_1, N_2.....N_s\right) = \otimes_{k=1}^{s} N_{\rho(k)}^{\hat{\omega}_k}$$

$$= \begin{cases} < \log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{\chi_k^+}-1\right)^{\omega_k}\right), 1-\log\left(1+\prod_{j=1}^{s}\left(\lambda^{1-\xi_k^+}-1\right)^{\omega_k}\right), \\ 1-\log\prod_{j=1}^{s}\left(\lambda^{1-\Lambda_k^+}-1\right)^{\omega_k}, \log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{1-(-\chi_k^-)}-1\right)^{\omega_k}\right)-1, \\ -\log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{-\xi_k^-}-1\right)^{\omega_k}\right), -\log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{-\Lambda_k^-}-1\right)^{\omega_k}\right), > (\omega > 0) \end{cases}$$

If k=s+1 then there is following result:

$$BNFOWGA\left(N_1, N_2.....N_s, N_{s+1}\right) = \otimes_{k=1}^{s} N_{\rho(s)}^{\hat{\omega}_s} \otimes N_{\rho(s+1)}^{\hat{\omega}_{s+1}}$$

$$= \begin{cases} < \log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{\chi_k^+}-1\right)^{\omega_k}\right), 1-\log\left(1+\prod_{k=1}^{n}\left(\lambda^{1-\xi_k^+}-1\right)^{\omega_k}\right) \\ 1-\log\left(1+\prod_{k=1}^{s}\left(\lambda^{1-\Lambda_k^+}-1\right)^{\omega_k}\right), \log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{1-(-\chi_k^-)}-1\right)^{\omega_k}\right)-1, \\ -\log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{-\xi_k^-}-1\right)^{\omega_k}\right), -\log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{-\Lambda_k^-}-1\right)^{\omega_k}\right), > (\omega > 0), \end{cases}$$

Hence the theorem true for r=s+1. Thus the result

$$\begin{cases} <\log_{\lambda}\left(1+\prod_{k=1}^{n}\left(\lambda^{\chi_k^+}-1\right)^{\omega_k}\right), 1-\log\left(1+\prod_{k=1}^{n}\left(\lambda^{1-\xi_k^+}-1\right)^{\omega_k}\right) \\ 1-\log\prod_{k=1}^{n}\left(\lambda^{1-\Lambda_k^+}-1\right)^{\omega_k}, \log_{\lambda}\left(1+\prod_{k=1}^{n}\left(\lambda^{1-(-\chi_k^-)}-1\right)^{\omega_k}\right)-1, \\ -\log_{\lambda}\left(1+\prod_{k=1}^{n}\left(\lambda^{-\xi_k^-}-1\right)^{\omega_k}\right), -\log_{\lambda}\left(1+\prod_{k=1}^{n}\left(\lambda^{-\Lambda_k^-}-1\right)^{\omega_k}\right), > (\omega>0) \end{cases}$$

holds for all n.

Example 5.1: Let $N_1 =< 0.5, 0.7, 0.3, -0.6, -0.2, -0.6 >$, $N_2 =< 0.2, 0.5, 0.5, -0.8, -0.4, -0.3 >$, $N_3 =< 0.3, 0.6, 0.4, -0.7 - 0.3, -0.1 >$ be three BNNs and let weight vector of BNNs N_k $(k=1,2,3)$ be $\omega=\left(\frac{1}{8},\frac{3}{8},\frac{1}{2}\right)^T, \omega_1=\frac{1}{8}, \omega_2=\frac{3}{8}, \omega_3=\frac{1}{2}$ are the weight of N_k $(k=1,2,3)$ such that $\prod_{k=1}\omega_k=1$. Then by above theorem

$$BNFOWGA(N_1,N_2,N_3)=\begin{cases} \log_3\left(1+\left(3^{0.3}-1\right)^{\frac{1}{8}}+\left(3^{0.2}-1\right)^{\frac{3}{8}}+\left(3^{0.5}-1\right)^{\frac{1}{2}}\right), \\ 1-\log_3\left(1+\left(3^{1-0.6}-1\right)^{\frac{1}{8}}+\left(3^{1-0.5}-1\right)^{\frac{3}{8}}+\left(3^{1-0.7}-1\right)^{\frac{1}{2}}\right), \\ 1-\log_3\left(1+\left(3^{1-0.4}-1\right)^{\frac{1}{8}}+\left(3^{1-0.5}-1\right)^{\frac{3}{8}}+\left(3^{1-0.3}-1\right)^{\frac{1}{2}}\right), \\ \log_3\left(1+\left(3^{1-0.7}-1\right)^{\frac{1}{8}}+\left(3^{1-0.8}-1\right)^{\frac{3}{8}}+\left(3^{1-0.6}-1\right)^{\frac{1}{2}}\right)-1, \\ \log_3\left(1+\left(3^{0.3}-1\right)^{\frac{1}{8}}+\left(3^{0.4}-1\right)^{\frac{3}{8}}+\left(3^{0.2}-1\right)^{\frac{1}{2}}\right), \\ l\,og_3\left(1+\left(3^{0.1}-1\right)^{\frac{1}{8}}+\left(3^{0.3}-1\right)^{\frac{3}{8}}+\left(3^{0.6}-1\right)^{\frac{1}{2}}\right) \end{cases}$$

$$BNFOWGA(N_1,N_2,N_3)=(1.0, 0.125, 0.2520, -0.0652, -1.0, -1.0).$$

The BNFOWGA operator has the following properties:

(4) Idomopotency:

Let all the BNN's be $N_n =< \chi_k^+, \xi_k^+, \Lambda_k^+, \chi_k^-, \xi_k^-, \Lambda_k^- >= N(k=1,2,3....n)$ where $\omega=(\omega_1,\omega_2,\omega_3.........\omega_n)^T$ be the weight vector of $N_k.\omega_k\in[0,1]$ and $\prod_{k=1}\omega_k=1$ then

$BNFWGA\left(N_1, N_2, N_3.........N_k\right) = N$.

(5) Monotonicity: Let N_k $\left(k = 1,2,3....n\right)$ and $N_j^{'}$ $\left(j = 1,2,3....n\right)$ be two families of BNNs, where $\omega = \left(\omega_1, \omega_2, \omega_3..........\omega_n\right)^T$ be the weight vector of N_k and $N_k^{'}$, $\omega_k \in \left[0,1\right]$ and $\prod_{k=1} \omega_k = 1$. For all 'j' if $N_j \geq N_j^{'}$ then

$$BNFWGA\left(N_1, N_2, N_3.........N_k\right) \geq BNFWGA\left(N_1^{'}, N_2^{'}, N_3^{'}.........N_k^{'}\right).$$

(6) Boundedness: Let $N_k = <\chi_k^+, \xi_k^+, \Lambda_k^+, \chi_k^-, \xi_k^-, \Lambda_j^- >= N\left(k = 1,2,3....n\right)$ be a family of BNNs. Where $\omega = \left(\omega_1, \omega_2, \omega_3..........\omega_n\right)^T$ be the weight vector of N_k, $\omega_k \in \left[0,1\right]$ and $\sum_{k=1}^{3} \omega_k = 1$, Therefore we have

$$BNFOWGA\left(N^-, N^-, ...N^-\right) \leq BNFOWGA\left(N_1, N_2, N_3.........N_j\right) \leq BNFOWGA\left(N^+, N^+, ...N^+\right)$$

where $N^- = <\chi_{N^-}^+, \xi_{N^-}^+, \Lambda_{N^-}^+, \chi_{N^-}^-, \xi_{N^-}^-, \Lambda_{N^-}^- >$

$$= \begin{cases} \min\left(\chi_1^+, \chi_2^+, \chi_3^+, ...\chi_n^+\right), \max\left(\xi_1^+, \xi_2^+, \xi_3^+, ...\xi_n^+\right), \\ \max\left(\Lambda_1^+, \Lambda_2^+, \Lambda_3^+, ...\Lambda_n^+\right), \max\left(\chi_1^-, \chi_2^-, \chi_3^-, ...\chi_n^-\right) \\ \min\left(\xi_1^-, \xi_2^-, \xi_3^-, ...\xi_n^-\right), \min\left(\Lambda_1^-, \Lambda_2^-, \Lambda_3^-, ...\Lambda_n^-\right) > \end{cases} \text{ And}$$

$N^+ = <\chi_{N^+}^+, \xi_{N^+}^+, \Lambda_{N^+}^+, \chi_{N^+}^-, \xi_{N^+}^-, \Lambda_{N^+}^- >$

$$= \begin{cases} \max\left(\chi_1^+, \chi_2^+, \chi_3^+, ...\chi_n^+\right), \min\left(\xi_1^+, \xi_2^+, \xi_3^+, ...\xi_n^+\right), \\ \min\left(\Lambda_1^+, \Lambda_2^+, \Lambda_3^+, ...\Lambda_n^+\right), \min\left(\chi_1^-, \chi_2^-, \chi_3^-, ...\chi_n^-\right) \\ \max\left(\xi_1^-, \xi_2^-, \xi_3^-, ...\xi_n^-\right), \max\left(\Lambda_1^-, \Lambda_2^-, \Lambda_3^-, ...\Lambda_n^-\right) > \end{cases}.$$

Proof:

(3) Since $N_j = <\chi_j^+, \xi_j^+, \Lambda_j^+, \chi_j^-, \xi_j^-, \Lambda_j^- >= N\left(j = 1,2,3....n\right)$ then, the following result can be obtained by using Equation (1)The following result can be obtained

$$BNFOWGA\left(N_1, N_2, N_3.........N_n\right) = \begin{cases} < \log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{\chi_k^+}-1\right)^{\omega_k}\right), 1-\log\prod_{k=1}^{n}\left(\lambda^{1-\xi_k^+}-1\right)^{\omega_k}, \\ 1-\log\prod_{k=1}^{n}\left(\lambda^{1-\Lambda_k^+}-1\right)^{\omega_k}, \log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{1-(-\chi_k^-)}-1\right)^{\omega_k}\right)-1, \\ -\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{-\xi_k^-}-1\right)^{\omega_k}\right), -\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{-\Lambda_k^-}-1\right)^{\omega_k}\right), > (\omega > 0) \end{cases}$$

$$= \begin{cases} < \log_\lambda\left(1+\left(\lambda^{\chi^+}-1\right)\right), 1-\log_\lambda\left(1+\left(\lambda^{1-\xi^+}-1\right)\right), \\ 1-\log_\lambda\left(1+\left(\lambda^{1-\Lambda^+}-1\right)\right), \log_\lambda\left(1+\left(\lambda^{1-(-\chi^-)}-1\right)\right)-1, \\ -\log_\lambda\left(1+\left(\lambda^{-\xi^-}-1\right)\right), -\log_\lambda\left(1+\left(\lambda^{-\Lambda^-}-1\right)\right) > \end{cases}$$

$$=< \chi^+, \xi^+, \Lambda^+, \chi^-, \xi^-, \Lambda^- >= N \text{ holds}$$

(4) The property is obvious based on the equation (6).

(5) Let $N^- =< \chi^+_{N^-}, \xi^+_{N^-}, \Lambda^+_{N^-}, \chi^-_{N^-}, \xi^-_{N^-}, \Lambda^-_{N^-} > and\, N^+ =< \chi^+_{N^+}, \xi^+_{N^+}, \Lambda^+_{N^+}, \chi^-_{N^+}, \xi^-_{N^+}, \Lambda^-_{N^+} >$.

There are the following inequalities:

$$\begin{cases} < \log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{\chi^+_{N^-}}-1\right)\right) \le \log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{\chi_k^+}-1\right)\right) \le \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\chi^+_{N^+}}-1\right)\right), \\ 1-\log\prod_{k=1}^{n}\left(\lambda^{1-\xi^+_{N^-}}-1\right) \le 1-\log\prod_{k=1}^{n}\left(\lambda^{1-\xi_k^+}-1\right) \le, 1-\log\prod_{k=1}^{n}\left(\lambda^{1-\xi^+_{N^+}}-1\right), \\ 1-\log\prod_{k=1}^{n}\left(\lambda^{1-\Lambda^+_{N^-}}-1\right) \le 1-\log\prod_{k=1}^{n}\left(\lambda^{1-\Lambda_k^+}-1\right) \le 1-\log\prod_{j=1}^{n}\left(\lambda^{1-\Lambda^+_{N^+}}-1\right), \\ \log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{1-(-\chi^-_{N^-})}-1\right)\right)-1 \le \log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{1-(-\chi^-_{N^-})}-1\right)\right)-1 \le \log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{1-(-\chi^-_{N^-})}-1\right)\right)-1 \\ -\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{-\xi^-_{N^-}}-1\right)\right) \le -\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{-\xi_k^-}-1\right)\right) \le -\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{-\xi^-_{N^+}}-1\right)\right), \\ -\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{-\Lambda^-_{N^-}}-1\right)\right), \le -\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{-\Lambda_k^-}-1\right)\right) -\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{-\Lambda^-_{N^+}}-1\right)\right) > \end{cases}$$

.

Hence

$$BNFOWGA\left(N^-, N^-, ...N^-\right) \le BNFOWGA\left(N_1, N_2, N_3.........N_j\right) \le BNFOWGA\left(N^+, N^+, ...N^+\right)$$

holds for all 'n'.

6. Model for MCDM Using Bipolar Neutrosophic Information

Using the BNFWGA and BNFOWGA operators that are suggested, two comprehensive MCDM approaches are expanded in this section.

For MCDM model with bipolar neutrosophic fuzzy information, Let $A=\left(A_1, A_2, A_3 \ldots\ldots A_n\right)$ be the set of alternatives and $C=\left(C_1, C_2, C_3 \ldots\ldots C_n\right)$ be a set of attributes.

Let $\omega=\left(\omega_1, \omega_2, \omega_3 \ldots\ldots\ldots \omega_n\right)^T$ be the weight to attribute C_K.

Suppose that $N=\left(N_{jk}\right)_{s\times r}=\left(\chi_{jk}^{+}, \xi_{jk}^{+}, \Lambda_{jk}^{+}, \chi_{jk}^{-}, \xi_{jk}^{-}, \Lambda_{jk}^{-}\right)_{s\times r}\left(j=1,2,3\ldots.s\right)\left(k=1,2,\ldots.r\right)$ is BNN decision matrix, where $\chi_{jk}^{+}, \xi_{jk}^{+}, \Lambda_{jk}^{+}$ indicates the truth membership degree, indeterminacy membership degree and falsity membership degree of alternative A_j under attribute C_K with respect to positive preferences and $\chi_{jk}^{-}, \xi_{jk}^{-}, \Lambda_{jk}^{-}$ indicates the truth membership degree, indeterminacy membership degree and falsity membership degree of alternative A_j under attribute C_K with respect to negative preferences. We have conditions $\chi_{jk}^{+}, \xi_{jk}^{+}, \Lambda_{jk}^{+}, \chi_{jk}^{-}, \xi_{jk}^{-}, \&\Lambda_{jk}^{-} \in [0,1]$ such that $0 \le \chi_{jk}^{+}, \xi_{jk}^{+}, \Lambda_{jk}^{+}, \chi_{jk}^{-}, \xi_{jk}^{-}, \Lambda_{jk}^{-} \le 6$ for $\left(j=1,2,3\ldots.s\right)\left(k=1,2,\ldots.r\right)$.

6.1 Proposed Algorithm using BNFOWGA operator to solve MCDM problem

Step 1 Collect information on the bipolar neutrosophic evaluation

Step 2 Calculate score and the accuracy values of collected information.

The score values $s\left(N_{jk}\right)$ and accuracy values $a\left(N_{jk}\right)$ of alternatives A_j can be calculated by using Equations (1) and (2).

Step 3 The compression method in Definition 4.1 to reorder information on evaluation under each attribute. The comparison method is used to reorder $\left(N_{jk}\right)$.

Step 4 Derive the collective BNN $N_j\left(j=1,2,\ldots.s\right)$ for the alternative $A_j\left(j=1,2,\ldots.s\right)$

Method (1) .

Utilize BNFWGA operator to calculate the collective BNN for each alternative, then $N_j = BNFWGA\left(N_{j1}, N_{j2}, N_{j3}.........N_{jn}\right) = \overset{n}{\underset{k=1}{\otimes}} N_{jk}^{\wedge_F^{\omega_k}}$.

$$N_j = BNFWGA\left(N_1, N_2.....N_{jn}\right) = \otimes_{k=1}^{n} N_{\rho(jk)}^{\wedge^{\omega_k}}$$

$$= \begin{cases} < \log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{\chi_{jk}^{+}}-1\right)^{\omega_k}\right), \\ 1-\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{1-\xi_{jk}^{+}}-1\right)^{\omega_k}\right), \\ 1-\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{1-\Lambda_{jk}^{+}}-1\right)^{\omega_k}\right), \\ \log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{1-(-\chi_{jk}^{-})}-1\right)^{\omega_k}\right)-1, \\ -\log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{-\xi_{jk}^{-}}-1\right)^{\omega_k}\right), \\ -\log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{-\Lambda_{jk}^{-}}-1\right)^{\omega_k}\right), > (\omega > 0) \end{cases}$$

Where $\omega = \left(\omega_1, \omega_2, \omega_3..........\omega_n\right)^T$ is the weight vector such that $\omega_k \in [0,1]$ and $\sum_{k=1}^{n} \omega_k = 1$

Method (2).

Utilize BNFOWGA operator to calculate the collective BNN for each alternative, then

$$N_j = BNFOWGA\left(N_1, N_2.....N_{jl}\right) = \otimes_{k=1}^{n} N_{\rho(jl)}^{\wedge^{\omega_k}}$$

$$= \begin{cases} < \log_\lambda\left(1+\prod_{l=1}^{n}\left(\lambda^{\chi_{jl}^{+}}-1\right)^{\omega_l}\right), 1-\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{1-\xi_{jl}^{+}}-1\right)^{\omega_l}\right), \\ 1-\log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{1-\Lambda_{jl}^{+}}-1\right)^{\omega_l}\right), \log_\lambda\left(1+\prod_{k=1}^{n}\left(\lambda^{1-(-\chi_{jl}^{-})}-1\right)^{\omega_l}\right)-1, \\ -\log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{-\xi_{jl}^{-}}-1\right)^{\omega_l}\right), -\log_\lambda\left(1+\prod_{k=1}^{s}\left(\lambda^{-\Lambda_{jl}^{-}}-1\right)^{\omega_l}\right), > (\omega > 0) \end{cases}$$

Where ρ is permutation that orders the elements: $N_{\rho(j1)} \geq N_{\rho(j2)} \geq N_{\rho(j3)} \geq ....N_{\rho(jn)}$

Where $\omega = \left(\omega_1, \omega_2, \omega_3..........\omega_n\right)^T$ is the weight vector such that $\omega_l \in [0,1]$ and $\sum_{l=1}^{n} \omega_l = 1$.

Step 5: Calculate the score values $s\left(N_j\right)\left(j = 1, 2, 3........s\right)$ of BNNs $\left(N_j\right)\left(j = 1, 2, 3........s\right)$ to rank all the alternatives $A_j\left(j = 1, 2, 3........s\right)$ and then select favorable one. If score values of BNNs

N_j & N_k are equal, then we calculate accuracy values $a(N_j)$ & $a(N_k)$ of BNNs N_j & N_k respectively and then rank the alternatives A_j & A_l as accuracy values $a(N_j)$ & $a(N_k)$

Step 6 Rank all the alternatives $A_j\ (j = 1, 2, 3........s)$ and select favorable one.

Step 7 End

7. Bridge Management to avoid traffic congestion using proposed Algorithm

The best solution to a bridge management problem from Amin Amini & Navid Nikraz [32] is discovered in this part in order to prevent traffic jams. To determine the optimum path that avoids traffic jams. Additionally, parametric analysis and comparison analysis are performed to confirm the adaptability and efficiency of the suggested algorithm to address the problem of decision-making.
People who reside in and travel through affected neighbourhoods, as well as on state routes, are greatly affected by traffic jams. The behaviour of road users, the safety of the road and bridge infrastructure conditions and characteristics, and vehicles interact continuously to form the traffic process. People that are delayed are late for key daily tasks including work, school, appointments, and other things. When clients and consumers have trouble contacting them, business suffers. When ambulances, rescue teams, and fire vehicles are unable to drive on their usual routes, routine incidents can quickly become life-threatening. Therefore, our programme was created to determine the optimum path while taking into consideration three factors: connectivity in a single lane, avoiding traffic incidents, and saving time for human resources. Consider three bridges A_1, A_2, A_3.

Based on the recommendations of the experts group in terms of three criteria, the roads departments chose to construct the bridge in order to reduce traffic namely multiple road connection in single lane (C_1), avoid road accidents (C_2), time saver for human resources (C_3).

Using the proposed algorithm bridge selection has been done as follows:
Step1: Collect information on bipolar neutrosophic evaluation
The information collected from expert discussion on evaluation is given in Table1

Table 1: Bipolar-neutrosophic evaluation information under

	C_1	C_2	C_3
A_1	(0.5,0.7,0.2,-0.7,-0.3,-0.6)	(0.4,0.4,0.5,-0.7,-0.8,-0.4)	(0.7,0.7,0.5,-0.8,-0.7-0.6)

A_2	(0.9,0.7,0.5,-0.7,-0.7,-0.1)	(0.7,0.6,0.8,-0.7,-0.5,-0.1)	(0.9,0.4,0.6,-0.1,-0.7,-0.5)
A_3	(0.3,0.4,0.2,-0.6,-0.3,-0.7)	(0.2,0.2,0.2,-0.4,-0.7,-0.4)	(0.9,0.5,0.5,-0.6,-0.5,-0.2)

Step 2: Calculate Score and accuracy values of collected information.

For each alternative A_j the attribute C_k ,the score values $s\left(N_{jk}\right)$ and the accuracy values $a\left(N_{jk}\right)$ can be calculated based on Equation (1) and Equation (2). The Score values $s\left(N_{jk}\right)$ and the accuracy values $a\left(N_{jk}\right)$ are shown in Tables 2 and 3 respectively.

Table 2: Score values $s\left(N_{jk}\right)$

	C_1	C_2	C_3
A_1	0.4067	0.5000	0.5000
A_2	0.4667	0.3667	0.6667
A_3	0.5167	0.5833	0.5000

Table 2: Accuracy Values $a\left(N_{jk}\right)$

	C_1	C_2	C_3
A_1	0.2000	-0.4000	0
A_2	-0.2000	-0.7000	0.7000
A_3	0.2000	0	0

Step 3: Reorder information on evaluation under each attribute

The comparison method in Definition 4.1 is used to reorder N_{jk}

Table 4 reordering bipolar neutrosophic evaluation information by using comparison method based on Definition 4.1

	C_1	C_2	C_3
A_1	(0.7,0.7,0.5,-0.8,-0.7-0.6)	(0.4,0.4,0.5,-0.7,-0.8,-0.4)	(0.5,0.7,0.2,-0.7,-0.3,-0.6)
A_2	(0.9,0.4,0.6,-0.1,-0.7,-0.5)	0.5,0.2,0.7,-0.5,-0.1,-0.9)	(0.9,0.7,0.5,-0.7,-0.7,-0.1)
A_3	(0.2,0.2,0.2,-0.4,-0.7,-0.4)	(0.3,0.4,0.2,-0.6,-0.3,-0.7)	(0.9,0.5,0.5,-0.6,-0.5,-0.2)

Table 5: Score values $s\left(N_{jk}\right)$

	C_1	C_2	C_3
A_1	0.5000	0.5000	0.4067
A_2	0.6667	0.5167	0.4667
A_3	0.5833	0.5167	0.5000

Table 6: Accuracy Values $a\left(N_{jk}\right)$

	C_1	C_2	C_3
A_1	0	-0.4000	0.2000
A_2	0.7000	0.2000	-0.2000
A_3	0	0.2000	0

Step 4: Derive the collective BNN $N_j\left(j=1,2,3...s\right)$ for the alternative $A_j\left(j=1,2,3...s\right)$ Method 1 BNFWGA operator using Eqn(8) and supporting $\lambda=7$ to calculate the collective BNN for each alternative, then

$$N_1=\langle 0.1301, 0.6857, 0.6345, -0.7517, -0.2494, -0.4480\rangle$$
$$N_2=\langle 0.5735, 0.6795, 0.7665, -0.6913, -0.1301, -0.1038\rangle$$
$$N_3=\langle 0.1301, 0.6857, 0.6345, -0.7517, -0.2494, -0.4480\rangle$$

The value of BNN by power operation when $\lambda=7$

$$\langle 0.7168, 0.2758, 0.2858, -0.2968, -0.6981, -0.6484\rangle$$

Method (2) Utilize BNFOWGA operator using Eq (9) and supporting $\lambda = 7$ to calculate the collective BNN for each alternative, then

$$N_1 = \langle 0.1301, 0.6857, 0.6345, -0.78414, -0.2512, -0.4480 \rangle$$
$$N_2 = \langle 0.5493, 0.6357, 0.7496, -0.6569, -0.1193, -0.1193 \rangle$$
$$N_3 = \langle 0.2160, 0.4527, 0.4264, -0.5661, -0.2506, -0.2335 \rangle$$

The value is $\langle 0.7100, 0.2758, 0.2668, -0.2739, -0.6343, -0.6529 \rangle$

Step 5: Calculate the score values $s(N_j)$ $A_j\,(j = 1,2,3)$ of BNNs N_j $(j = 1,2,3)$ for each alternatives $A_j\,(j = 1,2,3)$ $A_k\,(k = 1,2,3)$. The score values $s(N_{jk})$ are calculated using Equation (2)

Method 1 The following score values are obtained by using the BNFWGA operation

$s(N_1) = 0.2875 : s(N_2) = 0.2783 : s(N_3) = 0.3751$

Method 2 The following score values are obtained by using the BNFOWGA operation

$s(N_1) = 0.2875 : s(N_2) = 0.2910 : s(N_3) = 0.3758$

Step 6: Rank all the alternatives $A_j\,(j = 1,2,3)$ and select favorable one.

The alternative can be ranked in descending order based on the comparison method, and favorable alternative can be selected.

Method 1 The following ranking order based on score values is obtained by using BNFWGA operator:

$A_3 > A_2 > A_1$ Thus A_3 is favorable.

Method 2 The following ranking order based on the score values is obtained by using BNFOWGA operator: $A_3 > A_2 > A_1$ Thus A_3 is favorable.

Step 7: End

8. Comparative Analysis

To demonstrate the soundness of the suggested work, this section contrasts and compares it to the current approaches.

Table: 7 Ranking Orders obtained by Different Methods.

Methods	Rankings

A_ω [22]	$A_3 > A_2 > A_1$
G_ω [22]	$A_3 > A_2 > A_1$
BNDGA Operator $\lambda = 7$ [33]	$A_3 > A_2 > A_1$
BNDOGA Operator $\lambda = 7$ [33]	$A_3 > A_2 > A_1$
FBNCGBM operator (s,t=1)[21]	$A_3 > A_2 > A_1$
FBNCOGBM operator (s,t=1)[21]	$A_3 > A_2 > A_1$
FATTT2FFAAOWA $\lambda = 7$ []	$A_2 > A_1 > A_3$
FATTT2FFAAFWA $\lambda = 7$ [33]	$A_2 > A_3 > A_1$
BNFWGA Operator $\lambda = 7$	$A_3 > A_2 > A_1$
BNFOWGA Operator $\lambda = 7$	$A_3 > A_2 > A_1$

Table: 8 Characteristic Comparison of different methods.

Methods	Flexible measure easier
A_ω [18]	No
G_ω [18]	No
BNDGA Operator $\lambda = 7$ [29]	No
BNDOGA Operator $\lambda = 7$ [29]	No
FBNCGBM operator (s,t=1)[22]	No
FBNCOGBM operator (s,t=1)[22]	No
FATTT2FFAAOWA $\lambda = 7$ [31]	Yes
FATTT2FFAAFWA $\lambda = 7$ [31]	Yes
BNFWGA Operator $\lambda = 7$	Yes
BNFOWGA Operator $\lambda = 7$	No

In G_{ω} , BNDGA and BNDOGA methods, the proposed method based on proposed BNFWGA operator does not consider the interaction and interrelation among attributes. In contrast to A_{ω}, G_{ω}, FATTT2FFAAOWA $(\lambda = 7)$ and FATTT2FFAAFWA $(\lambda = 7)$ methods, the proposed BNFWGA operator choose the appropriate parameters according to the preferences of Decision Makings. In similar to FBNCGBM operator (s,t=1) and FBNCOGBM operator (s,t=1), BNFWGA Operator for $\lambda = 7$, BNFOWGA Operator for $\lambda = 7$, the proposed BNFWGA operators can select the appropriate parameters according to preferences of the decision making.

In G_{ω} , BNDGA and BNDOGA methods, the proposed method based on proposed BNFOWGA operator does not consider the interaction and interrelation among attributes. In contrast to A_{ω}, G_{ω}, FATTT2FFAAOWA $(\lambda = 7)$ and FATTT2FFAAFWA $(\lambda = 7)$ methods, the proposed BNFOWGA operator selects the relevant settings in accordance with Decision Makings' preferences. In similar to FBNCGBM operator (s,t=1) and FBNCOGBM operator (s,t=1), BNFWGA Operator $\lambda = 7$, BNFOWGA Operator $\lambda = 7$ the proposed BNFWGA operators can select the appropriate parameters according to preferences of the decision makings. Therefore, based on the proposed BNFWGA operator, the proposed technique may be used for decision-making. As a result, the suggested operators are more dependable and adaptable. These suggested strategies can be used in practice MCDM situations for decision-making based on the suggested operators and their requirements.

9. Conclusion

The tendency of the human mind to think about both positive and negative impacts when making decisions is referred to as bipolarity. A generalization of fuzzy, intuitionistic, and neutrosophic sets, the bipolar neutrosophic set enables it to handle ambiguous information in the decision-making process with higher adaptability. As a result, the operational laws of the proposed aggregation operation for both BNFWGA and BNFOWGA have been investigated and given in this study utilising Frank triangular norms in a bipolar neutrosophic environment. The use of the BNFWGA and BNFOWGA operators' proposed method to the MCDM bridge selection problem demonstrated its viability and cogency, and the suggested principles were used to choose the best bridge for reducing traffic congestion. Additionally, a comparison of the new method with the old method has been conducted. In the future, aggregation operators may be created employing a variety of triangular norms in different neutrosophic contexts.

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Neutrosophic Statistical Process Monitoring

Muhammad Aslam
Department of Statistics, Faculty of Science, King Abdulaziz University
Jeddah 21551, Saudi Arabia
E-mail: aslam_ravian@hotmail.com

Abstract: Woodall et al. [1] raised some issues regarding neutrosophic methodology. In this paper, we will address their comments and questions related to the use of neutrosophic statistics in process monitoring. We provided the responses to some important questions/comments related to neutrosophic statistical process monitoring.

Keywords: classical statistics; neutrosophic statistics; control chart; sampling plans; average run length

1. Introduction

Neutrosophic statistics as an extension of classical statistics was introduced by Smarandache [2]. Neutrosophic statistics is applied when the data is in the interval and has imprecise observations. Smarandache [37] provided a detailed discussion on the application of neutrosophic statistics. Smarandache [37] proved that neutrosophic statistics is more efficient than classical statistics and interval statistics. Smarandache [37] provided responses to questions of Woodall et al. [1] related to neutrosophic methodology. More applications of neutrosophic statistics can be seen in [3-5].

In this paper, we will address some important questions raised by Woodall et al. [1]. We provided the responses to some important questions/comments related to neutrosophic statistical process monitoring (NSPM)

2 Some Comments/Questions

In this section, we selected some important questions/comments from Woodall et al. [1] and provided the responses.

2.1 Neutrosophic Sample Size

The control charts using the fuzzy-based approach have been used for decades. These control charts are designed when uncertainty is found in sample size, data, and control chart parameters. Jean [6] presented a detailed discussion on the determination of the sample size for a control chart. Gülbay and Kahraman [7] proposed the control chart for imprecise data. Engin et al. [8] argued that the determination of sample size is a problem in attribute control charts. Moheb Alizadeh et al. [9] proposed control when observations in each sample are a canonical fuzzy number. Turanoğlu et al. [10] presented the sampling plan when the sample size is fuzzy. Yimnak and Intaramo [11] designed a standard deviation control chart when uncertainty (fuzziness) is found in sample size. Haridy et al. [12] contradicted the common belief that the sample size for X-bar and R chart or X-bar and standard deviation should be between [4, 6]. Hesamian et al. [13] proposed the exponentially weighted moving average (EWMA) control chart when the random variable is fuzzy. Zhou et al. [14] proposed control chart by considering the fuzzy sample number. Yalçın and Kaya [15] presented the analysis using the process capability index using a neutrosophic sample [90, 100]. More information on such as the control chart can be seen in [16] and [17]. Control charts using neutrosophic statistics are the extension of control charts using a fuzzy-based approach. Under the neutrosophic framework, the neutrosophic sample size is fixed before the sample is collected. Like the fuzzy theory, neutrosophic theory is workable under uncertainty where the decision-makers are uncertain about the sample size before the sample is collected. In addition, neutrosophic statistics reduce to classical statistics when the decision-makers are curtained about the sample before the sample is collected. Chen et al. [3] and Chen et al. [4] showed the efficiency of neutrosophic statistics over classical statistics.

2.2 Extreme Amount of Imprecision in Attribute Control Charts

In the attribute control chart, during the simulation, it was observed that there is a high jump in average run length while changing the other parameters slightly. Therefore, in attribute control charts, it is expected a high indeterminacy to meet the given conditions.

2.3 Neutrosophic Sample Size in Acceptance sampling Plan

Aslam [18] and Aslam [19] determined the neutrosophic parameters of sampling plans. The real data used in Aslam [18] and Aslam [19] is assumed to have neutrosophic numbers. Therefore, fixed sample size is selected from the indeterminate interval of the sample size. The neutrosophic sample size in the interval can be applied using the same lines when two data sets having neutrosophic numbers are available. The neutrosophic theory is flexible and can be modified according to the situation and underlying studies.

2.4 Neutrosophic Control Limits Multiplier

The development of control limits using the fuzzy-based approach can be seen in [20-22]. The smoothing constant (lambda) is expressed in intervals having the range between 0 and 1. Hunter [23] suggested that the smoothing parameter should be selected from 0.20 to 0.3. On the other hand, Montgomery [24] recommended selecting the values of the smoothing parameter from 0.05 to 0.25. The determination of the smoothing parameter is an important issue in designing the control charts; see [25-26]. Therefore, the decision-makers are not always sure about the value of the smoothing constant to be selected in designing the control chart. In addition, as neutrosophic statistics was applied for the interval data, therefore, it would be justifiable to express the parameters in intervals rather than the exact value. The control charts using neutrosophic statistics are designed when uncertainty is found in the smoothing parameter or moving average span. Note that all neutrosophic parameters of the control chart are fixed in advance. Therefore, it is important to study the behavior of control charts when uncertainty is found in observations or any parameters of the control charts.

2.5 Approximation in EWMA Control Charts

The use of approximations in the evaluation of the average run length of EWMA charts was provided, for example, by [27-29]. More information can be seen in Ziegel [30]. Using the approximation of Aslam et al. [31], it is found that the values of average run length (ARL) are close to Lucas and Saccucci [32] for the larger values of smoothing constant (lambda). It is not recommended to apply the approximation using the ARL=1/p for EWMA control charts. This type of approximation should be used for Shewhart control charts only.

2.6 Efficiency of Control Charts using Repetitive Sampling

Aslam et al. [33] found that the average sample number for the control chart using repetitive sampling is almost similar to the sample size. It means that the number of repetitions is quite small (almost close to one). But, from the comparative study of the control chart using repetitive sampling and the control chart using single sampling, the average run length for the control chart using repetitive sampling is smaller than the average run length obtained from the control chart using the single sampling when the average sample number is equal to the competitor chart using single. By increasing the sample size to get smaller values of average run length cannot be encouraged.

2.6 Equivalence of Multiple Dependent State Sampling and Runs-Rule

Recently, Woodall et al. [34] showed that multiple dependent state sampling (MDSS) is equivalent to a run-rule chart. Aslam et al. [35] proposed control chart using MDSS when the process is in-control and by following the assumption of MDSS. Designing the MDSS by following [34] has less significant chance that (i+1) samples are in intermediate region and will not be efficient as the approach adopted by Woodall et al. [34]. As suggested by Riaz et al. [36] "In real applications, having a wider indecisive zone may not be very practical, and hence we have chosen the indecisive regions of practical worth".

3 Concluding Remarks

In this paper, we addressed some questions raised by Woodall et al. [1] on the application of control charts using neutrosophic statistics. From the study, it is concluded that like the fuzzy-based approach, the control charts using neutrosophic statistics can be designed and applied in an uncertain environment.

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PCTLHS-Matrix, Time-based Level Cuts, Operators, and unified time-layer health state Model.

Shazia Rana [1], and Muhammad Saeed [1]

[1] University University of Management and Technology, Johar Town Campus, Lahore, 54000, Pakistan;
[1]shaziaranams@gmail.com; [2] muhammad.saeed@umt.edu.pk

[2] University of New Mexico, Mathematics, Physics, and Natural Science Division, 705 Gurley Ave., Gallup, NM 87301, USA;
smarand@unm.edu

Abstract: This article aims to introduce a unique hypersoft time-based matrix model that organizes and classifies higher-dimensional information scattered in numerous forms and vague appearances varying on specific time levels. Classical matrices as rank-2 tensors single-handedly relate equations and variables across rows and columns are a limited approach to organizing higher-dimensional information. This Plithogenic Crisp Time Leveled Hypersoft Matrix (PCTLHS-Matrix) model is designed to sort the higher dimensional information flowing in parallel time layers as a combined view of events. This matrix has several parallel layers of time. The time-based level cuts as time layers are introduced to present an explicit view of information on certain required time levels as a separate reality. The sub-layers are formulated as sub-level cuts that represent a partial view of the event or reality. Further subdividing these sub-levels creates sub-sub-level cuts, which are the smallest focused partial view of the event, serving the purpose of zooming. These Level cuts are utilized to construct local aggregation operators for PCTLHS-Matrix. And the concept of timelessness is introduced by unifying the time levels of the universe. This means all attributes that exist in various time levels are merged to exist in a unified time called the unified time layer. In this way, the attributes are focused and the layers of time are merged as if there is no time. The particular types of time layers are unified by local operators to introduce the concept of timelessness that is obtained by unifying time levels. Finally, for a precise description of the model, a numerical example is constructed by assuming a classification of various health states with COVID-19 patients in a hospital.
Intuitionistic Fuzzy / Neutrosophic / and other fuzzy-extension IndetermSoft Set & IndetermHyperSoft Set are presented together with their applications.

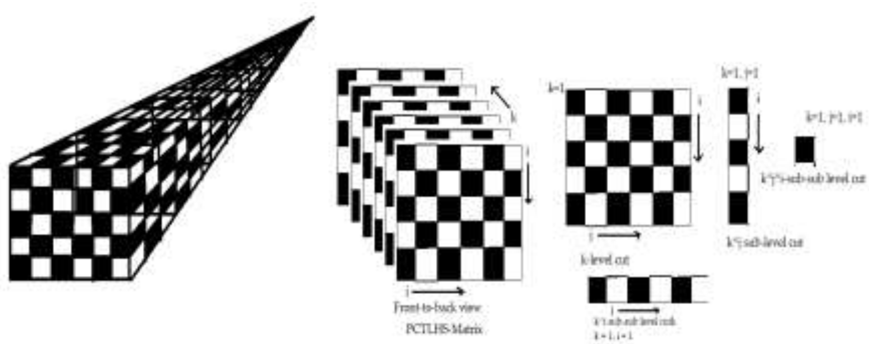
Keywords: : PCTLHS-Matrix; Time-Layers, Level-Cuts; Sub-Level Cuts; Sub-Sub- Level Cuts; Combined Event-View, Separate event-view; Partial event view; Aggregation-Operators.

1. Introduction

The discipline of modeling and decision-making in an uncertain and vague environment consisting of higher-dimensional information is an incredible task. To enhance the field modeling and decision-making in an uncertain and vague universe, the field of fuzzy theory was developed by Zadeh [1] in 1965. Later, in 1986, K. Atanassov [2] further expanded this state of vagueness by introducing intuition or hesitation into decision-making structures called intuitionistic fuzzy set theory (IFS). The three states of the human mind were represented by the level of membership, the level of non-membership, and the level of hesitation in IFS theory. In addition, K. Atanassov [3] introduced an interval-valued fuzzy set (IVFS) in 1999, which is another form of IFS (memberships and nonmemberships packed in unit intervals). Later, F. Smarandache [4-6] introduced neutrosophy by extending hesitation as an independent indeterminate neutral factor. Molodtsove formulated a soft set in 1999 [7-10], he expanded the theory of wage by considering multiple attributes parameterized by subjects. In 2018 [11] Smarandache introduced the Hypersoft set and the Plithogenic Hypersoft set earlier, giving the plithogenicy theory [12]. In these sets, he extended the attributes to the values of the attributes called sub-attributes and parametrized the subjects by several attributes and sub-attributes. By introducing the Hypersoft set and the Plithogenic Hypersoft set, he opened some problems in exploring these sets such as constructing aggregation operators and decision-making models.

Shazia et al [13], explored and extended these sets and addressed the problems opened by Smarandache. In addition, they introduced the plithogenic Fuzzy Whole Hypersoft Set (PFWHSS) and formulated a matrix representation form named Plithogenic Fuzzy Whole Hypersoft-Matrix. They developed some local aggregation operators for the plithogenic Fuzzy Hypersoft set (PFHSS). This matrix was developed for a specific combination of attributes and sub-attributes. The application of this matrix has been provided in the form of a decision-making model referred to as the Plithogenic Frequency Matrix Multi-Attribute Decision-Making technique. Later, Shazia et al [14] extended the Plithogenic Whole Hypersoft Matrix to a generalized form of the Matrix called the Plithogenic Subjective Hyper-Super-Soft Matrix.

It was a superior matrix to its previously developed matrix which has a greater capacity for expressing the variations of certain connected attributive levels. These attribute levels are presented as matrix layers. The application of this matrix is provided in the form of a new ranking model called the Plithogenic Subjective Local-Global Universal Ranking Model.



This model has provided a physical classification of the Universe (a combination of subjects with attributes). Later, in this research field, the hypersoft set expanded, and some MADM techniques and operators are developed. Saqlain, Saeed, et al [15] discussed the Generalization of TOPSIS for Neutrosophic Hypersoft set using Accuracy Function and its Application. Saqlain, et al [16] constructed some Aggregate Operators of Neutrosophic Hypersoft Set. Quek, et al [17] introduced Entropy Measures for Plithogenic Sets and Applications in Multi-Attribute Decision Making. Saqlain Smarandache [18] formulated octagonal neutrosophic numbers and discussed their different representations, properties, graphs, and de-eutrophication with the application of personnel selection. International Journal of Neutrosophic Science. Rahman et al, [19] developed a multi-attribute decision-support system that is based on aggregations of Interval-Valued Complex Neutrosophic Hypersoft Set. Saeed, Muhammad, and Atiqe Ur Rahman [20] constructed an optimal supplier selection model via a decision-making algorithmic technique that is based on a single-valued neutrosophic fuzzy hypersoft Set. Ihsan, Muhammad, Atiqe Ur Rahman, and Muhammad Saeed [21] discussed a single-valued neutrosophic hypersoft expert set with application in decision making. Saeed, Muhammad, et al. [22] formulated the model of the prognosis of allergy-based diseases using Pythagorean fuzzy hypersoft mapping structures and recommended medication. Rahman, Atiqe Ur, et al. [23] developed decision-making algorithmic techniques based on aggregation operations and similarity measures of possibility intuitionistic fuzzy hypersoft sets.

This current article provides a further upgraded plithogenic Model. In this model, a new time-level variation has been introduced. This model is programmed with a magnified angle of vision to cope with scattered time-dependent information of the plithogenic Universe in a crisp environment. First, a three-dimensional expanded view of the PCTLHS matrix is presented to show the Plithogenic Crisp Time Lined Hypersoft Set. This PCTLHS The matrix is a third-rank tensor representing three types of variation; it consists of several matrix layers, each layer being a second-rank tensor expressed in the Crisp environment. Furthermore, this PCTLHS matrix represents multiple parallel universes or parallel realities. By using this connected-matrix expression, one can grasp and categorize all the information at a glimpse, i.e., the information from a crowd of people assigned to a combination of attributes and observed at different time levels. Therefore, it is obvious that the matrix expression is the most appropriate expression to represent the multidimensional data compared to the classical set expression. This new model would help to enhance and broaden the field of decision-making and artificial intelligence. After a detailed description, specific types of level cuts are constructed on the variation indices. These level cuts are named K-level cuts obtained by splitting the PCTLHS-Matrix at one of the three given variation indices. Further, these K-level cuts would provide a structure for viewing each event or reality separately and serve as the projection of higher-dimensional events in the lower-dimensional universe. Additionally, these K-level cuts are further broken down into sub-level cuts by splitting the matrix layer at either of the two remaining variation indices, i.e: J, I. While these sub-level cuts offer the projection of the previous lower dimension into a further lower dimension and provide an interior view of the expanded universe, this view may be called an implicit expanded universal view.

At a later stage from these sub-level cuts, sub-sub-level cuts are constructed by dividing the sub-level cuts (row or column of a given layer of the PCTLHS matrix) at the second variation index of the matrix and then further at the third variation index of the matrix. After applying all splits at indices the outcomes would be reflected as points. It is obvious that these level cuts, sublevel cuts, and sub-sub-level cuts serve to program zoom-in functions to look at an inside view of the event. This can be considered as a contraction of the expanded higher-dimensional universe. However, the sub-sub level cuts provide a contracted picture of the smallest part of single or multiple universes. In this way, the expanded universe of matrix layers could possibly be contracted at a single point. Similarly, by reversing the process, one can extend the same point to other higher dimensions of rows, columns, matrix, matrix layers, and clusters of matrix layers. In the final phase, plithogenic aggregation operators are developed and used to elaborate the activity of these different types of level cuts based on variation indices. These operators are plithogenic disjunction operators, plithogenic conjunction operators, plithogenic average operators, and vice versa. For a more precise and lucid explanation of the model, a numerical example related to the classification of COVID-19 patients and their health states in a hospital at two different time levels.

2. Prelimneries

This section summarizes some basic definitions of Soft Sets, Hypersoft Sets, Crisp Hypersoft Sets, Plithogenic Hypersoft Sets, Plithogenic Crisp Hypersoft Sets. These definitions would help expand the theory of plithogecy.

Definition 2.1 [7] (Soft Set)

Let $\boldsymbol{U}$ be the initial Universe of discourse, and $\boldsymbol{E}$ be a set of parameters or attributes with respect to $\boldsymbol{U}$ let $\boldsymbol{P}(\boldsymbol{U})$ denote the power set of $\boldsymbol{U}$, and $\boldsymbol{A} \subseteq \boldsymbol{E}$ is a set of attributes. Then pair $(\boldsymbol{F}, \boldsymbol{A})$, where $\boldsymbol{F}: \boldsymbol{A} \longrightarrow \boldsymbol{P}(\boldsymbol{U})$ is called Soft Set over $\boldsymbol{U}$. , In other words, a soft set $(\boldsymbol{F}, \boldsymbol{A})$ over $\boldsymbol{U}$ is a parameterized family of subsets of $\boldsymbol{U}$. Fore $\boldsymbol{e} \in \boldsymbol{A}$, $\boldsymbol{F}(\boldsymbol{e})$ may be considered as a set of $\boldsymbol{e}$ elements or $\boldsymbol{e}$ approximate elements

$$(\boldsymbol{F}, \boldsymbol{A}) = \{(\boldsymbol{F}(\boldsymbol{e}) \in \boldsymbol{P}(\boldsymbol{U}): \boldsymbol{e} \in \boldsymbol{E}, \boldsymbol{F}(\boldsymbol{e}) = \boldsymbol{\varphi}\,\mathbf{if}\,\boldsymbol{e} \notin \boldsymbol{A}\} \tag{2.1}$$

Definition 2.2 [11] (Hypersoft set)

Let $\boldsymbol{U}$ be the initial Universe of discourse $\boldsymbol{P}(\boldsymbol{U})$ the power set of $\boldsymbol{U}$.

let $\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}$ for $\boldsymbol{n} \geq \mathbf{1}$ be $\boldsymbol{n}$ distinct attributes, whose corresponding attributes values are respectively the sets $\boldsymbol{A_1}, \boldsymbol{A_2}, \ldots, \boldsymbol{A_n}$ with $\boldsymbol{A_i} \cap \boldsymbol{A_j} = \boldsymbol{\varphi}$ for $\boldsymbol{i} \neq \boldsymbol{j}$ and $\boldsymbol{i}, \boldsymbol{j} \in \{\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}\}$.

Then the pair $(\boldsymbol{F}, \boldsymbol{A_1} \times \boldsymbol{A} \times \ldots \times \boldsymbol{A_n})$ where,

$$\boldsymbol{F}: \boldsymbol{A_1} \times \boldsymbol{A} \times \ldots \times \boldsymbol{A_n} \rightarrow \boldsymbol{P}(\boldsymbol{U}), \tag{2.2}$$

is called a Hypersoft set over $\boldsymbol{U}$

Definition 2.3 [11] (plithogenic Crisp Hypersoft set)

Let $\boldsymbol{U_c}$ be the initial Crisp Universe of discourse $\boldsymbol{P}(\boldsymbol{U_c})$ the power set of $\boldsymbol{U}$. Let $\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}$ for $\boldsymbol{n} \geq \mathbf{1}$ be $\boldsymbol{n}$ distinct attributes, whose corresponding attributes values are respectively the sets $\boldsymbol{A_1}, \boldsymbol{A_2}, \ldots, \boldsymbol{A_n}$ with $\boldsymbol{A_i} \cap \boldsymbol{A_j} = \boldsymbol{\varphi}$ for $\boldsymbol{i} \neq \boldsymbol{j}$ and $\boldsymbol{i}, \boldsymbol{j} \in \{\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}\}$. Then

$\{\boldsymbol{F_c}, \boldsymbol{A_1} \times \boldsymbol{A_2} \times \ldots \times \boldsymbol{A_n}\}$ is called plithogenic Crisp Hypersoft set over $\boldsymbol{U_c}$ where, $\boldsymbol{F_c}: \boldsymbol{A_1} \times \boldsymbol{A} \times \ldots \times \boldsymbol{A_n} \rightarrow \boldsymbol{P}(\boldsymbol{U_c})$, *Definition 2.4 [24][25] [26] (super-matrices)*

A square or rectangular arrangements of numbers in rows and columns are matrices we shall call them as simple matrices while a super-matrix is one whose elements are themselves matrices with elements that can be either scalars or other matrices.

$\boldsymbol{a} = \begin{bmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \end{bmatrix}$, where

$\boldsymbol{a_{11}} = \begin{bmatrix} \mathbf{2} & \mathbf{-4} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$, $\boldsymbol{a_{12}} = \begin{bmatrix} \mathbf{0} & \mathbf{40} \\ \mathbf{21} & \mathbf{-12} \end{bmatrix}$,

$\boldsymbol{a_{21}} = \begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{5} & \mathbf{7} \\ \mathbf{-2} & \mathbf{9} \end{bmatrix}$, $\boldsymbol{a_{22}} = \begin{bmatrix} \mathbf{4} & \mathbf{12} \\ \mathbf{-17} & \mathbf{6} \\ \mathbf{3} & \mathbf{7} \end{bmatrix}$ $\boldsymbol{a}$ is a super-matrix.

Note: The elements of super-matrices are called sub-matrices i.e. $\boldsymbol{a_{11}}, \boldsymbol{a_{12}}, \boldsymbol{a_{21}}, \boldsymbol{a_{22}}$ are submatrices of the super-matrix $\boldsymbol{a}$.

in this example, the order of super-matrix $\boldsymbol{a}$ is $\mathbf{2} \times \mathbf{2}$ and order of sub-matrices $\boldsymbol{a_{11}}$ is $\mathbf{2} \times \mathbf{2}$, $\boldsymbol{a_{12}}$ is $\mathbf{2} \times \mathbf{2}$ $\boldsymbol{a_{21}}$ is $\mathbf{3} \times \mathbf{2}$ and order of sub-matrix $\boldsymbol{a_{22}}$ is $\mathbf{3} \times \mathbf{2}$, we can see that the order of super-matrix doesn't tell us about the order of its sub-matrices.

Definition 2.5 [27] (Hypermatrices)

For $\boldsymbol{n_1}, \dots, \boldsymbol{n_d} \in \boldsymbol{N}$, a function $\boldsymbol{f}: (\boldsymbol{n_1}) \times \ldots \times (\boldsymbol{n_d}) \to \boldsymbol{F}$ is a hypermatrix, also called an order-d hypermatrix or d-hypermatrix. We often just write $\boldsymbol{a_{k_1 \ldots k_d}}$ to denote the value $\boldsymbol{f}(\boldsymbol{k_1} \ldots \boldsymbol{k_d})$ of $\boldsymbol{f}$ at $(\boldsymbol{k_1} \ldots \boldsymbol{k_d})$ and think of $\boldsymbol{f}$ (renamed as $\boldsymbol{A}$) as specified by a d-dimensional table of values, writing $\boldsymbol{A} = [\boldsymbol{a_{k_1 \ldots k_d}}]_{\boldsymbol{k_1 \ldots k_d}}^{\boldsymbol{n_1, \ldots, n_d}}$

A 3-hypermatrix may be conveniently written down on a (2-dimensional) piece of paper as a list of usual matrices, called slices. For example

$$\boldsymbol{A} = [\boldsymbol{a_{ijk}}] = \begin{bmatrix} \boldsymbol{a_{111}} & \boldsymbol{a_{121}} & \boldsymbol{a_{131}} & \cdot & \boldsymbol{a_{112}} & \boldsymbol{a_{122}} & \boldsymbol{a_{132}} \\ \boldsymbol{a_{211}} & \boldsymbol{a_{221}} & \boldsymbol{a_{231}} & \cdot & \boldsymbol{a_{212}} & \boldsymbol{a_{222}} & \boldsymbol{a_{232}} \\ \boldsymbol{a_{311}} & \boldsymbol{a_{321}} & \boldsymbol{a_{331}} & \cdot & \boldsymbol{a_{312}} & \boldsymbol{a_{322}} & \boldsymbol{a_{332}} \end{bmatrix}$$

3. Plithogenic Crisp Time-Leveled Hypersoft Matrix

Definition 3.1 (Plithogenic Crisp Time leveled Hypersoft Matrix):

Let $\boldsymbol{U_C}(\boldsymbol{X})$ be the Crisp universe of discourse, $\boldsymbol{P}(\boldsymbol{U_C})$ be the power set of $\boldsymbol{U_C}$, $\boldsymbol{A_j^k}$ is a combination of attributes sub-attributes for some $j = 1,2,3,\dots,N$ attributes, $k = 1,2,3,\dots,L$ time-leveled-attributes and x_i $i = 1,2,3,\dots, M$ subjects under consideration then Plithogenic Crisp Time-Leveled Hypersoft-Matrix (PCTLHS-Matrix), is a mapping C from the cross product of attributes / time-leveled-attributes on the power set of universe $P(U_C)$ represented in matrix form. This mapping C and its matrix form in the plithogenic crisp environment is described below in Eq.3.1 and Eq.3.2 respectively,

$$F: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \to P(\boldsymbol{U_C}) \tag{3.1}$$

$$F = \left[\mu_{A_j^k}(x_i)\right] \tag{3.2}$$

s.t $\mu_{A_j^k}(x_i) \in \{0.1\}$, are crisp states as memberships either "0" or "1",

$\mu_{A_j^k}(x_i)$ are crisp memberships for given x_i subjects regarding each given A_j^k attributes / time-leveled-attributes where, A_j^k is a combination of attributes / time leveled-attributes for some $j = 1,2,3,\ldots,N$ attributes, $k = 1,2,3,\ldots,L$ time levels associated to x_i $i = 1,2,3,\ldots, M$ subjects under consideration.

Or in simple words a Plithogenic Crisp Hypersoft Set, represented in the matrix form is called Plithogenic Crisp Time Lined Hypersoft Matrix, (PCTLHS-Matrix).

This matrix has three possible expanded forms or views described in Crisp environments.

3.2 PCTLHS-Matrix as a Tensor:

As we know, all matrices in the real vector space are rank-2 tensors, similar to how the PCTLHS matrix with its three variation indices is a rank 3 tensor. The PCTLHS matrix contains layers of ordinary matrices called matrix layers or Level cuts (plane slices).

$\boldsymbol{A} = [\boldsymbol{A_{ijk}}]$ is an example of PCTLHS-Matrix. Index i refers to variations in rows used to represent subjects under consideration j specifies a variation in columns used to represent attributes of subjects, and k provides variations of layers of rows and columns used to represent the attributes on specific time levels (varying matrix layers as clusters of rows and columns). Similarly $[\boldsymbol{A_{jki}}]$ is interpreted as the index j referred to variation of rows k gives a variation of columns, and i offers a variation of clusters of rows and columns.

3.3 Level Cuts, Sub-Level Cuts, and Sub-Sub-Level Cuts of PCTLHS-Matrix

We may define Level Cuts, Sub-Level Cuts, and Sub-Sub-Level Cuts by specifying the variation indices i, j, k for their positive integer values.

3.3.1 Level Cuts**:** Level Cuts are sub-matrices (first level splits) of PCTLHS-Matrix that can be further described as parallel matrix layers. The PCTLHS-Matrix is generated by uniting these matrix layers. These level cuts of PCTLHS-Matrix are obtained by assigning a specific integer value to one of the three variation indexes at a time.

The level cuts are of three types according to three types of views of the PCTLHS-Matrix i.e i-Level Cuts, j-Level Cuts, k-Level Cuts the detailed mathematical description of k-Level Cuts is described in section 4.

3.3.2 Sub-Level Cuts: Sub-Level Cuts are Level Cuts of Level Cuts (second splits applied over first splits) of PCTLHS-Matrix that are columns or rows of the Sub-Matrix or a Layer-Matrix. The Sub-Level Cuts are obtained by assigning a specific integer value to one of the two variation indices of a parallel layer (Sub-Matrix) of PCHS-Matrix. The detailed description and construction of sub-level cuts are presented in sec 4.

3.3.3 Sub-Sub-Level Cuts: The Sub-Sub-Level Cuts are obtained by assigning a specific Integer value to a variation index of sub-Level Cut (the third level splits over second splits). The Sub-Sub-Level Cut is one Specific element (point) of the Sub-Level Cut (Column or Row). These Level Cuts, Sub-Level Cuts, and Sub-Sub-Level Cuts are images of the higher dimensional Universe in the lower dimensional Universe and can be used as tools for getting images and transformations.

The detailed organization of these level cuts, sub-level cuts, and sub-sub-level cuts are described in the next section with the specific time based view of the PCTLHS-Matrix.

The utilization of these Cuts is that one can contract the expanded dimension of PCTLHS-Matrix to a Matrix, then to a row or column matrix, and then further to a single point. However, the reverse procedure would provide an expansion of the Universe in a similar manner. In PCHs-Matrix three types of variation indices are introduced on Crisp memberships $\boldsymbol{\mu_{A_j^k}(x_i)}$. For example, in the Universe of subjects attributes and sub-attributes, one may consider the first variation on the index $\boldsymbol{i}$ that is referred to as subjects representing $\boldsymbol{M}$ rows of an $\boldsymbol{M \times N}$ Sub-Matrix of an $\boldsymbol{M \times N \times L}$ PCHS-Matrix. The second variation on $\boldsymbol{j}$ is used to specify attributes representing $\boldsymbol{N}$ columns of a Sub-Matrix of an $\boldsymbol{M \times N \times L}$ Hs-Matrix. A third variation on $\boldsymbol{k}$ is introduced to express attributive levels and is represented in the form of $\boldsymbol{L}$ layers or $\boldsymbol{L}$ level-Cuts of $\boldsymbol{M} \times \boldsymbol{N} \times \boldsymbol{L}$ PCHS-Matrix.

4. Three-dimensional view and a front-to-back view of PCTLHS-Matrix

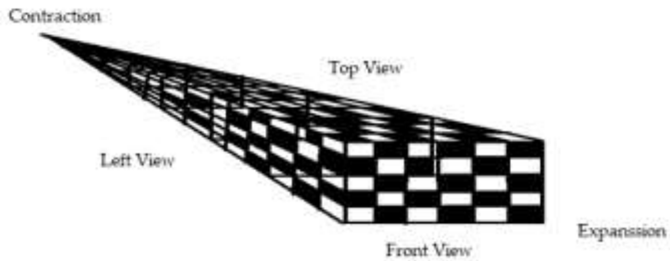


Figure 1 (Three-dimensional view of PCTLHS-Matrix)

The three-dimensional indexed-based PCTLHS-Matrix and level cuts, sub-level cuts, and Sub-Sub-Level Cuts are described below:

$$A = \begin{bmatrix} \mu_{A_1^K}(x_1) & \mu_{A_2^K}(x_1) & . & . & . & \mu_{A_N^K}(x_1) \\ \mu_{A_1^K}(x_2) & \mu_{A_2^K}(x_2) & . & . & . & \mu_{A_N^k}(x_2) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \mu_{A_N^k}(x_M) & \mu_{A_N^k}(x_M) & . & . & . & \mu_{A_N^K}(x_M) \end{bmatrix} \tag{4.1)a$$

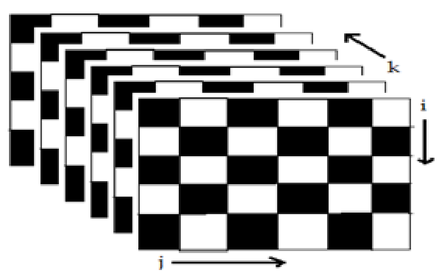


Figure 2 (front-to-back view of the matrix)

front-to-back view of PCTLHS-Matrix is described in Eq. (4.1)a figure 2

Front to back view of PCTLHS-Matrix in a more expanded form is described in Eq. (4.1)b as,

$$A=\begin{bmatrix}\begin{bmatrix}\mu_{A_1^1}(x_1) & \mu_{A_2^1}(x_1) & . & . & . & \mu_{A_N^1}(x_1)\\ \mu_{A_1^1}(x_2) & \mu_{A_2^1}(x_2) & . & . & . & \mu_{A_N^1}(x_2)\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ \mu_{A_1^1}(x_M) & \mu_{A_2^1}(x_M) & . & . & . & \mu_{A_N^1}(x_M)\end{bmatrix}\\ \begin{bmatrix}\mu_{A_1^2}(x_1) & \mu_{A_2^2}(x_1) & . & . & . & \mu_{A_N^2}(x_1)\\ \mu_{A_1^2}(x_2) & \mu_{A_2^2}(x_2) & . & . & . & \mu_{A_N^2}(x_2)\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ \mu_{A_1^2}(x_M) & \mu_{A_2^2}(x_M) & . & . & . & \mu_{A_N^2}(x_M)\end{bmatrix}\\ .\\ .\\ .\\ \begin{bmatrix}\mu_{A_1^L}(x_1) & \mu_{A_2^L}(x_1) & . & . & . & \mu_{A_N^k}(x_1)\\ \mu_{A_1^L}(x_2) & \mu_{A_2^L}(x_2) & . & . & . & \mu_{A_N^L}(x_2)\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ \mu_{A_1^L}(x_M) & \mu_{A^L}(x_M) & . & . & . & \mu_{A_N^L}(x_M)\end{bmatrix}\end{bmatrix} \quad (4.1)\text{b}$$

4.1 k-Level Cuts $A^{[k]}$ of PCTLHS-Matrix:

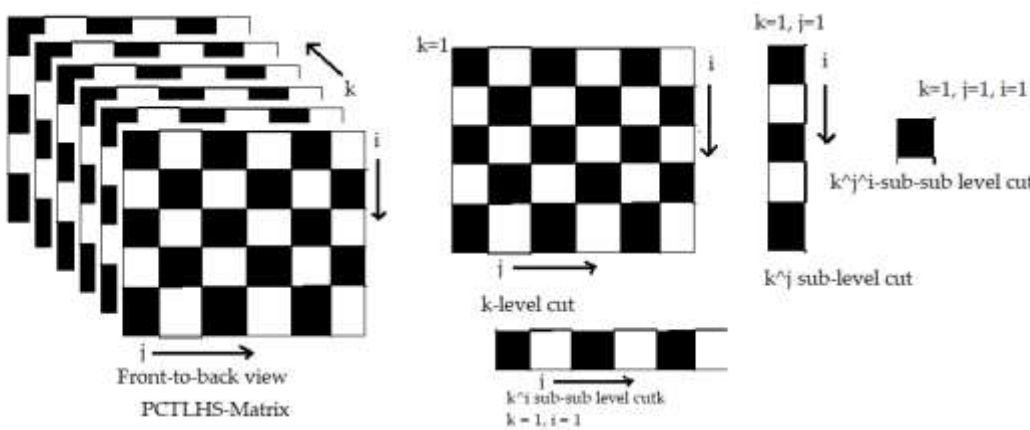


Figure 3 (level cuts sub-level cuts and sub-sub-level cuts)

k-Level Cuts of PCTLHS-Matrix are front-to-back k-splits of the matrix obtained by varying the index stepwise. These are the $\boldsymbol{L}$ number of front-to-back Matrix layers of $M \times N \times L$ PCTLHS-Matrix. Each layer is an $M \times N$ Matrix as described in Figure 3. L number of $k - level$ cuts ($A^{[k]}$) of PCTLHS-Matrix in the general contracted form are described in the matrix form by specifying $k = l$,

$$A^{[1]} = \left[\mu_{A_j^1}(x_i)\right],\ A^{[2]} = \left[\mu_{A_j^2}(x_i)\right], \ldots, A^{[M]} = \left[\mu_{A_j^L}(x_i)\right]$$

$i = 1,2,\ldots,M,\ j = 1,2,\ldots,N$ and $k = 1,2,3,\ldots,L$

k-Level Cuts of PHCTLHS-Matrix in the expanded form are described underneath in Eq. (4.2)

$$A^{[K]} = \begin{bmatrix} \mu_{A_1^k}(x_1) & \mu_{A_2^k}(x_1) & . & . & . & \mu_{A_N^k}(x_1) \\ \mu_{A_1^k}(x_2) & \mu_{A_2^k}(x_2) & . & . & . & \mu_{A_N^k}(x_2) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \mu_{A_1^k}(x_M) & \mu_{A_2^k}(x_M) & . & . & . & \mu_{A_N^k}(x_M) \end{bmatrix} \quad (4.2)$$

k-Level Cuts $A^{[k]}$ of PCTLHS-Matrix in the expanded form are given below in EQs. (4.3), (4.4), and (4.5).

$$A^{[1]} = \begin{bmatrix} \mu_{A_1^1}(x_1) & \mu_{A_2^1}(x_1) & . & . & . & \mu_{A_N^1}(x_1) \\ \mu_{A_1^1}(x_2) & \mu_{A_2^1}(x_2) & . & . & . & \mu_{A_N^1}(x_2) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \mu_{A_1^1}(x_M) & \mu_{A_2^1}(x_M) & . & . & . & \mu_{A_N^1}(x_M) \end{bmatrix} \quad (4.3)$$

$$A^{[2]} =, \begin{bmatrix} \mu_{A_1^2}(x_1) & \mu_{A_2^2}(x_1) & . & . & . & \mu_{A_N^2}(x_1) \\ \mu_{A_1^2}(x_2) & \mu_{A_2^2}(x_2) & . & . & . & \mu_{A_N^2}(x_2) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \mu_{A_1^2}(x_M) & \mu_{A_2^2}(x_M) & . & . & . & \mu_{A_N^2}(x_M) \end{bmatrix} \quad (4.4)$$

$$A^{[L]} = \begin{bmatrix} \mu_{A_1^L}(x_1) & \mu_{A_2^L}(x_1) & . & . & . & \mu_{A_N^k}(x_1) \\ \mu_{A_1^L}(x_2) & \mu_{A_2^L}(x_2) & . & . & . & \mu_{A_N^L}(x_2) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \mu_{A_1^L}(x_M) & \mu_{A^L}(x_M) & . & . & . & \mu_{A_N^L}(x_M) \end{bmatrix} \quad (4.5)$$

4.2 k_i-Sub-Level Cuts: For k-Level Cuts further offer i-splits (row-wise splits) by specifying i and varying j k_i-Sub-Level Cuts that are obtained. These are rows of $M \times N$ Matrix Layer that are obtained by specifying $k = l, i = m,$ .and varying $j = 1,2 \ldots N$

$$A^{[l_m]} = \begin{bmatrix} \mu_{A_1^l}(x_\mathrm{m}) & \mu_{A_2^l}(x_\mathrm{m}) & . & . & . & \mu_{A_N^l}(x_\mathrm{m}) \end{bmatrix} \quad (4.6)$$

4.3 k_j-Sub-Level Cuts: For k-Level Cuts further provide j-splits (column-wise splits) by specifying j and varying i k_j-Sub-Level Cuts are obtained. These are columns of $M \times N$ Matrix Layer that are obtained by specifying $k = l, j = n,$ and varying $i = 1,2 \ldots M$

$$A^{[l_n]} = \begin{bmatrix} \mu_{A_n^l}(x_1) \\ \mu_{A_n^l}(x_2) \\ . \\ . \\ . \\ \mu_{A_n^l}(x_M) \end{bmatrix} \quad (4.7)$$

4.4 k_{i_j}-Sub-Sub-Level Cuts $A^{[k_{i_j}]}$: For Specific k_i-Sub-Level Cuts further specifying j $A^{[k_{i_j}]}$ Sub-Sub-Level Cuts are obtained. These are elements of rows of $M \times N$ Matrix Layer that are obtained by specifying $k = l, i = m,$ .and varying $\boldsymbol{j} = 1,2 \ldots N$.

$$A^{[n_{m_l}]} = [\mu_{A_n^l}(x_m)] \quad (4.8)$$

It observed that k_{i_j} -Sub-Sub-Level Cuts $A^{[k_{i_j}]}$ and k_{j_i} -Sub-Sub-Level Cuts $A^{[k_{j_i}]}$ are identical i.e

$$A^{[n_{m_l}]} = [\mu_{A_n^l}(x_m)]$$

Note: It is obvious that by following the division procedure mentioned above, one would zoom into the given PCTLHS matrix. The first split would provide a zoom into the layer of the matrix, then zoom into the column or row, then next zoom into the element of the column or row. And the reverse process can serve as a zoom-out function. In this way one can approach the smallest unit of the extended universe.

5. COVID-19 Patients unified time-based health state Model

The mathematical modeling of the organization and analysis of information and observations of some patients with COVID-19 symptoms is described in the given example.

Example 5.1

Considering $U_{PC} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ representing six patients (vaccinated) who presented to the hospital with symptoms of COVID-19. Description of an investigation case of a doctor who examined three of them for four symptoms (attributes) These symptoms are observed and organized as doctor visits at two specific times, considered as symptoms observed at two distinct time levels (time leveled attributes). Patients under observation are recognized as subjects (material bodies). The observations of the visits are expressed in the Crisp environment and analyzed using Plithogenic Crisp-Time-Leveled Hyper-Soft-Matrix.

$let\ T = \{x_1, x_2, x_3\} \subset U_C$ be the set of these three patients considered by a doctor for examination.

Let the attributes be A_j^k; $j = 1,2,3,4$ with Time Leveled-attributes k = 1, 2 are described as,

A_1^k = Fever with numeric values, $k = 1\,,2$ representing first-time level and second-time level states of attributes

A_1^1 = State of fever at the first visit, A_1^2 = State of fever at the second visit

A_2^k = Dry cough, with numeric values, $k = 1{,}2$

A_2^1 = Condition of cough at the first visit, A_2^2 = Condition of cough at the second visit

A_3^k = Breathing difficulty with numeric values, $k = 1{,}2$

A_3^1 = Breathing difficulty level at first visit,

A_3^2 = Breathing difficulty level at first visit,

A_4^k = Sickness record, with numeric values $k = 1{,}2$

A_4^1 = Sickness state at first visit,) A_4^2 = Sickness state at second visit.

In the next two subsections, this information now consisted of symptoms (attributes) of patients (subjects) observed at two stages, as two levels of time are organized and presented in two ways. One as a set, i.e. PCTLHS set and the other as a connected matrix of two matrices, that is the PCTLHS matrix.

5.1 Plithogenic Crisp Time-Leveled Hypersoft set (PCTLHS-Set) representation:

Let the Function A is reflecting given attributes/Time leveled-attributes as described below,

$$\boldsymbol{A}: A_1^k \times A_2^k \times A_3^k \times A_4^k \to P(U_C)$$

$$\text{S.t}\quad \boldsymbol{A}(A_1^1, A_2^1, A_3^1, A_4^1) = \{x_1, x_2, x_3\} \tag{5.1}$$

be a Time-leveled hypersoft set. Consider $A_1^1, A_2^1, A_3^1, A_4^1$ a combination of attributes at the first visit level (α combination)

$$\boldsymbol{A}(A_1^2, A_2^2, A_3^2, A_4^2) = \{x_1, x_2, x_3\} \tag{5.2}$$

$A_1^2, A_2^2, A_3^2, A_4^2$ a combination of attributes at the second visit level (β combination)

The Individual Crisp memberships are assigned to $\boldsymbol{A}$ = $\{x_1, x_2, x_3\}$ according to the doctor's opinion and then represented in PCTLHS-Set A=$\{x_1, x_2, x_3\}$. the opinion of the physician represented in the PCTLHS-Set as Crisp memberships i.e if the given symptom is present membership is one if not present membership is zero.

$\boldsymbol{A} = \{x_1\ (\mu_{A_j^1}(x_1)), x_2\ (\mu_{A_j^1}(x_2)), x_3(\mu_{A_j^1}(x_3))\}$ as $\mu_{A_j^k}(x_i)$ for $i = 1,2,3$ and $j = 1,2, 3, 4$ in $\boldsymbol{A}$ (these plithogenic crisp memberships reflect whether the A_j^k attribute is present ($\mu_{A_j^k}(x_i) = 1$) in x_i subject or not present ($\mu_{A_j^k}(x_i) = 0$) associated to time leveled α-combination of attributes.

$$A(\alpha) = A(A_1^1, A_2^1, A_3^1, A_4^1) = \begin{Bmatrix} x_1(1,0,1,1), \\ x_2(1,1,1,1), \\ x_3(1,0,0,1) \end{Bmatrix} \tag{5.3}$$

The first visit information is organized as PCTLHS-Set would produce the first level of the PCTLHS Matrix.

Regarding the second visit (second level of time) of patients for β-combination of attributes, the information is now presented as a PCTLHS Set

$$A(\beta) = A(A_1^1, A_2^1, A_3^1, A_4^1) = \begin{Bmatrix} x_1(0,0,0,0), \\ x_2(0,1,0,0), \\ x_3(1,0,1,0) \end{Bmatrix} \tag{5.4}$$

5.2 PCTLHS -Matrix representation: Let A be the representation matrix for PCTLHS-Set. The rows of the matrix represent x_1, x_2, x_3 (physical bodies or subjects) and columns represent (the non-Physical aspect of subjects) $A_1^k, A_2^k, A_3^k, A_4^k$ Attributes.

This information is organized in the form of PCTLHS-Matrix A as,

$$F = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{bmatrix} \tag{5.5}$$

This PCTLHS-Matrix consists of two layers for the first layer, it is interpreted that patient x_1 has a fever without a dry cough at the first visit but feels suffocation and nausea. Patient x_2 suffers from a fever with a dry cough, fits of suffocation, and nausea. Patient x_3 has a fever with a dry cough, no difficulty breathing, but nausea. During the second visit, while the patient x_1 has all symptoms resolved, patient x_2 only feels difficulty in breathing, and patient x_3 suffers from fever and difficulty breathing. One can see clearly by using this connected matrix expression, we can see and classify all the information at a glance, i.e. the information from a group of patients assigned to a combination of attributes and observed at different time levels. Therefore, it is obvious that the matrix expression is the more appropriate expression to represent the multidimensional data as compared to the classical Set expression.

$A = \left[\mu_{A_j^k}(x_i)\right]$ $i = 1,2,3$ $j = 1,2,3,4$, and $k = 1,2$

This ($\boldsymbol{A}$) is a PCTLHS-Matrix of rank 3 and order $(i \times j \times k) = (3 \times 4 \times 2)$

The front-to-back view of this PCTLHS-Matrix consists of two parallel layers of ordinary 3×4 ordered matrices. These layers when separated are called k-level cuts described as underneath,

5.3 k-Level Cuts $A^{[k]}$ of PCTLHS-Matrix

$$A = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{bmatrix} \tag{5.6}$$

Eq. (5.6) represents a front-to-back view of PHS-Matrix in Crisp environment for $i = 1,2,3$ $j = 1,2,3,4$ and $k = 1,2$. This $3 \times 4 \times 2$ hyper matrix has two 3×4 Matrices as two front-to-back layers. These layers are separated to construct k-level cuts (time-wise level cuts).

The two k-Level Cuts of PCTLHS-Matrix (eq. (5.6)) are given below, which serves to focus the time levels first.

$A^{[k=1]} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ $A^{[k=2]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

5.4 k_i-Sub-Level Cuts: Sub-layers of PCTLHS-Matrix obtained by specifying $k = 1, i = 1$, and varying $j = 1,2,3,4$ are given below as k_i-Sub-Level Cuts as rows of the matrix layers.

$A^{[1_1]} = [1 \quad 0 \quad 1 \quad 1], A^{[1_2]} = [1 \quad 1 \quad 1 \quad 1], A^{[1_3]} = [1 \quad 1 \quad 0 \quad 1]$

$A^{[2_1]} = [0 \quad 0 \quad 0 \quad 0], A^{[2_2]} = [0 \quad 1 \quad 0 \quad 0], A^{[2_3]} = [1 \quad 0 \quad 1 \quad 0]$

For example the description of $A^{[2_1]} = [0 \quad 0 \quad 0 \quad 0]$ is that by the second visit to the first patient, all four symptoms are clear.

5.5 k_j-Sub-Level Cuts: Sub-layers of PCTLHS-Matrix obtained by specifying $k = 1, j = 1$, respectively, and varying $i = 1,2,3$, are given below as k_j-Sub-Level Cuts as columns of the matrix layers.

$A^{[1_1]} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, A^{[1_2]} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, A^{[1_3]} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, A^{[1_4]} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, A^{[2_1]} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, A^{[2_2]} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, A^{[2_3]} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For example the given $A^{[1_3]}$ describes that at the first time level state of the third symptom is described individually in all three patients.

5.6 k_{j_i} Sub-Sub-Level Cut: For a given k_j -Sub-Level Cuts after specifying the time and attribute respectively our final focus is the patient (subject) i.e specifying finally $i = m$ we get k_{j_i}Sub-Sub-Level Cut for a fixed k_j-Sub-Level Cuts. $A^{[l_{n_m}]}$ is obtained by specifying $k = l, j = n$ respectively, and finally $i = m$. This sub-sub level cut is the smallest unit of the matrix that is the single element as described below,

$A^{[k_{j_i}]}$: $A^{[1_{1_1}]} = [1]$ $A^{[1_{2_1}]} = [0]$ $A^{[2_{3_1}]} = [0]$

For example $A^{[2_{3_1}]} = [0]$ represents the condition of the first patient for the third symptom at the second time level.

Example 5.2

$$B = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{bmatrix} \quad (5.7)$$

Is a $3 \times 3 \times 2$ PCTLHS-Matrix with two attribute time levels.

$i = 1,2,3$ $j = 1,2,3$ and $k = 1,2$

A Front to back view of PCTLHS-Matrix with two k-level cuts, each level cut is a 3×3 Matrix is given below in Eq. (5.9)

$$B = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{bmatrix} \quad (5.9)$$

$k - level\ cuts\ of\ B$ are $B^{[k]}$

$$B^{[1]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ B^{[2]} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

6 Local Aggregation Operators of $k - Level\ Cuts$

Operations of PCTLHS-Matrix are the basic set laws of union, intersection, average, and compliment that is defined by using k-level cuts. After using these operations cumulative memberships $\Omega^{t}_{\mathbf{A}^{k}_{j}}(x_i)$ are obtained by combining corresponding memberships of k-level cuts (time-based level cuts). These local operators serve to unify the time levels of the universe. This means that all attributes that are present in different time levels are considered as present in a unified single time level being reflected from many entities of the universe. In this way, attributes are focused and time levels are merged as there is no time, therefore, these special types of k-level cuts and their local operators introduce the concept of no time that is obtained by the unification of time levels by using aggregation operators. Three local operators are formulated and described, $t = 1$ used for max -operator $t = 2$ used for min-operator, and $t = 3$ used for the $averaging$-operator. These three operators are described as under,

6.1 Union of $k - Level\ Cuts$: The union between front to back parallel layers is defined as

$$\cup_k \left[\mu_{A^k_j}(x_i)\right] = \underset{k}{Max}\left(\mu_{A^k_j}(x_i)\right) = \left[\Omega_{A_j}(x_i)\right] \tag{6.1}$$

$\left[\Omega_{A^k_j}(\mathrm{x})\right]$ is the cumulated layer of highest memberships considered as the Extreme front level layer.

Example 6.1: For Matrix **A** given in Ex-1 ∪ ($\boldsymbol{A}^{[\boldsymbol{k}]}$)) **is**

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

6.2 Intersection of $\boldsymbol{j}$ − Level Cuts: The intersection between front to back parallel layers is defined below,

$$\cap \left[\mu_{A^k_j}(x_i)\right] = \underset{k}{Min}\left(\mu_{A^k_j}(x_i)\right) = \left[\Omega_{A_j}(x_i)\right] \tag{6.2}$$

$\left[\Omega_{A^k_j}(\mathrm{x})\right]$ is the cumulated layer of lowest memberships considered as the Extreme back level layer.

Example 6.2: For Matrix **A** given in Ex-1 ∪ ($\boldsymbol{A}^{[\boldsymbol{k}]}$)) **is**

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \cap \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

This lowest back level layer is reflecting the accumulated lowest state of symptoms throughout the two time-levels.

6.3 Average of $k - Level\ Cuts$: This is the average between front to back parallel layers is defined below, $\Gamma\left[\mu_{A^k_j}(x_i)\right] = \left(\Omega_{A_j}(x_i)\right)$ such that

$$\left(\Omega_{A_j}(x_i)\right) = \begin{bmatrix} 1\ if\ \sum_{j=1}^{L} \frac{\left(\mu_{A^k_j}(x_i)\right)}{N} \geq 0.5 \\ 0\ if\ \sum_{j=1}^{L} \frac{\left(\mu_{A^k_j}(x_i)\right)}{N} < 0.5 \end{bmatrix} \tag{6.3}$$

$\left[\Omega_{A^k_j}(\mathrm{x})\right]$ is the cumulated layer of average memberships considered as the interior level layer.

Example 6.3: For Matrix **A** given in Ex-1 ∪ ($\boldsymbol{A}^{[\boldsymbol{k}]}$)) **is**

$$\begin{bmatrix}1 & 0 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 0 & 1\end{bmatrix}\Gamma\begin{bmatrix}0 & 0 & 0 & 0\\0 & 1 & 0 & 0\\1 & 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix}$$

This average middle-level layer is reflecting the accumulated average state of symptoms throughout the two distinct time levels.

6.4 Compliment of $k - Level\ Cuts$: The complement of each membership of the k-Level Cut is defined in eq (5.3)d

$$\mathrm{C}\left(A^{[k]}\right) = \left[1 - \mu_{A_j^k}(x_i)\right] \tag{6.4}$$

Example 6.4: Compliments of k-Level Cuts of A are,

$$\mathrm{C}\left(A^{[1]}\right)=\begin{bmatrix}0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix},\mathrm{C}\left(A^{[2]}\right)=\begin{bmatrix}1 & 1 & 1 & 1\\1 & 0 & 1 & 1\\0 & 1 & 0 & 1\end{bmatrix}$$

7. Conclusion & Analysis

7.1 Conclusions

1. We can portray an extensive indeterminable Plithogenic Universe by using PCTLHS-Matrix.
2. We can display Multiple-dimensional views of the Universe (subjects versus attributes and time lined-attributes) By considering all possible views of the PCTLHS-Matrix
3. We can classify and analyze the universe explicitly and implicitly through level cuts, sublevel cuts, and sub-sub-level cuts.
4. The PCTLHS -Matrix provides the broader Exterior and interior view of displaying all possible Events (realities) together.
5. The Level Cuts of PCTLHS-Matrix Portray an explicit event or reality at an instance
6. Choosing the level cut based on the Variation index (i, j, k) Provides a view of reality or events from multiple angles of vision.
7. We can analyze the Universe by choosing the best possible reality out of multiple possible realities with the help of level cuts and operators. This fact would be helpful in the development of artificial intelligence programs.
8. The disjunction operator, i.e., the $Max - operator$ Provides the optimist view of the reality.
9. The conjunction operator, i.e., the $Min - operator$ Provides the pessimist view of the reality
10. The Averaging operator Provides a neutral view of the reality.
11. The Complement operator depicts the inverted reflection of the event or reality.
12. These local aggregation operators that are designed for k-level cuts introduce the concept of timelessness by unifying the time levels of the universe. This means that all attributes that exist in different time levels are merged and considered to exist in a unified single time level. In this way, attributes are focused, and time layers are merged as if there is no time for them.

7.2 Comparisons of former fuzzy extensions and models:

This section describes a brief comparison of previous and recent fuzzy extensions.

The soft set is an improved and extended version of the fuzzy set since it handles numerous attributes at the same time regardless of the fuzzy set, which only holds one attribute at a time.

Hypersoft Set is a superior extension of Soft Set because it can adapt multidimensional information by handling various attributes and their values as sub-attributes simultaneously.

Plithogenic Hypersoft Set The Plithogenic Hypersoft Set manages multiple attributes and their values (sub-attributes) simultaneously and beyond by observing each attribute separately therefore it is a more inovative version as compared to the Hypersoft Set, Soft set, and Fuzzy Set. It manages detailed information in a single structure. The spectator can precieve the state of element x (subject) by observing each attribute separately.

Plithogenic Fuzzy Whole Hypersoft-Set/Matrix (PFWHS-Set/Matrix) is a more aplicable choice compared to the previously mentioned extensions as it manages the states of subjects (attributes/sub-attributes) at the isolated level for each attribute/sub-attribute (the case of Plithogenic Hypersoft Set) and also at the combined level for merged attributes as a whole (the case of hypersoft set). Therefore, it is an extended and a hybrid version of the hypersoft set and plithogenic hypersoft set. By using PFWHS-Set/Matrix, one can observe a more transparent inner perception (case of a single state representation) or outer view (case of a combined state representation) of the information/facts/events.

The Plithogenic Subjective Hyper-Super-Soft Matix (PSHSS-Matrix) It is a generalized and an advanced form of the PFWHS-Matrix, as it has a higher capability to manage numerous connected attributes/sub-attributes separately and as a whole by considering connected attribute / sub-attribute levels.

Plithogenic Time-Leveled Hypersoft-Matrix (PTLHS-Matrix) is a unique case of the previous mentioned form (PSHSS-Matrix). It can manage time-based connected attributes. It is more suitable than other extended fuzzy sets mentioned (Soft Set, Hypersoft, Plithogenic Hypersoft Set, PFWHS Set / Matrix PSHSS Matrix) for the subsequent valid reasons.

1. Most of the variations in this universe are time-dependent like weather graphs, stock exchange, website ratings, etc. Therefore, it is of great help if this PCTLHS-Matrix is used to manage the scattered time-varying piece of information.
2. It manages several attributes sub-attributes interiorly such that each attribute has many values varying in the flow of time called time-based attributes.
3. By using PCTLHS-Matrix one can organize and classify multidimensional information into the shape of connected matrix layers as hypermatrices.
4. The matrix expression is the most applicable expression to represent multidimensional information compared to the classic set expression.
5. The observer can see the information down to its innermost level through level cuts, sub-level cuts, and sub-level cuts of PCTLHS-Matrix.
6. PCTLHS-Matrix offers a broader view of multi-dimensional information by viewing the entire universe as a hypersoft time-leveled matrix. Therefore, the observer can see and analyze the whole universe externally at a single glance.
7. The level cuts offers the observer to focus on one required piece of information that is displayed as a single matrix layer of PTLHS-Matrix. Whereas the other information can vary in the flow of time being displayed as other matrix layers.
8. The sub-level cuts can focus on required information that is displayed as a single column or row of the given layer (sub-matrix) of PTLHS-Matrix.
9. The sub-sub-level cuts can focus on one required information that is displayed as a single element of the sub-matrix of PTLHS-Matrix.
10. The Sub-Level Cuts offer the representation of the previous lower dimension in the further lower dimension and enable us to sneak in an inside view of the expanded universe, i.e after explicitly focusing on a subject through an i-level cut (single level of the layered matrix) our next focus is on that subject's (patient's) attribute (a particular symptom) through the sub-level cut (row or column of one layer of the multi-layered matrix).

11. It also offers the unification of the information by applying the aggregation operators, in this way all the extended information of the universe that is represented as a matrix having multiple layers can be transformed into a single layer of the matrix.

Open problems:

Now, let us list some of the open problems that might be addressed in future research.

- In this article, we have portrayed the Plithognic Hypersoft Matrix in a Crisp Environment.
- The expression of this matrix in other environments like Fuzzy, intuitionistic, and neutrosophic, or any mixed or combined environment, i.e., containing several environments, would provide the variation of fuzziness levels of reflected events.
- One can extend this model in other environments like intuitionistic environment, Neutrosophic environment, or any other mixed environment according to required conditions.
- By introducing these Level Cuts, we have provided the concept of contracting the expanded dimension of PCTLHS-Matrix to a single point (serving as a zoom-in function.
- Moreover, some other kinds of the local operator can be provided for unification purposes according to the requirement of the concerned bodies.

The operations and properties of these hypersoft matrices need to be explored.

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Construction of New Similarity Measures and Entropy for Interval-Valued Neutrosophic Sets with Applications

Zarife Zararsiz[1]

[1]Department of Mathematics, Nevşehir H.B.V. University, Nevşehir, Turkey; zarifezararsiz@nevsehir.edu.tr

*Correspondence: zarifezararsiz@nevsehir.edu.tr;

Abstract. The concepts of similarity measures and entropy have practical applications in computational intelligence, machine learning, image processing, neural networks, medical diagnosis, and decision analysis. An interval-valued neutrosophic set (IVNS) is strong model for modeling and handling uncertainties by using independent intervals of truthness, indeterminacy, and untruth. We introduce new similarity measures, entropy and inclusion relation for interval-valued neutrosophic sets (IVNSs). We introduce new inclusion relation named as type-f for ordering of interval neutrosophic sets. Additionally, a robust multi-attribute decision-making (MADM) method is developed by making use of proposed measures of similarity for IVNSs. A practical application for ranking of alternatives with newly developed MADM approach is illustrated by a numerical example for the car selection. The validity and superiority of new similarity measures with existing approaches is also given with the help of a comparison analysis.

Keywords: Similarity measure; entropy;interval-valued neutrosophic set; multi-attribute decision-making.

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1. **Introduction**

Zadeh [23] advanced his significant idea of fuzzy sets in 1965 to deal with various styles of uncertainties. From that time, it has been used prevalently in so many areas. Theory of fuzzy set is a more developed version of crisp set theory. By using fuzzy numbers or linguistic numbers which have numerical representation of inaccurate information, new mathematical methods have been developed for modeling the uncertain structure of today's problems. There is not a single model in fuzzy set theory, it means that many options can be reached considering the features of the system to be modeled by using various extensions of fuzzy sets. Since the problems encountered in life and human thoughts are too complex to be limited, fuzzy numbers have been inadequate at the decision-making stage such as problems involving vague or incomplete information. In this way, a fuzzy set's extension have been proposed by Atanassov [1] as intuitionistic fuzzy sets (IFS) in 1986 that include the degrees of membership and non-membership. It means that an IFS $A = \{< a, \mu_A(a), \lambda_A(a) >: a \in X\}$ has been established by two mappings $\mu_A(a), \lambda_A(a) : X \to [0, 1]$ named as membership function and non-membership function, respectively, with the restriction $0 \leq \mu_A(a) + \lambda_A(a) \leq 1, a \in X$. Later on IFSs extended towards interval-valued intuitionistic fuzzy sets (for brief: IVFSs) by Atanassov and G.Gargov [2], Turksen [15], and Gorzalczany [4]. IVIFSs have been used by these authors in the fields of signal processing, approximate inference, and controller, etc. Smarandache [12] initiated the notion of neutrosophic sets which consider indeterminate/uncertain information in today's problems and incorporated not only membership and non-membership grades, but also indeterminacy grades assigned each component of the discourse universe with is limitation that the sum of three independent grades chosen in the interval $[0, 3]$. Later on, Wang *et al.* [17, 18] defined the notion of single valued neutrosophic set (SVNS) and interval neutrosophic set (INS). Besides, in [10] the definitions of fuzzy neutrosophic soft (FNS) σ-algebra, FNS-measure and FNS-outer measure are established considering the concepts of soft sets and neutrosophic sets. Additionally, illustrative examples are given in [10]. Saqlain *et al.* [11] suggested an algorithm involving neutrosophic soft set for decision making problems.

Ye [19] has created neutrosophic linguistic variables as well as any new assemblage operators for interval-valued neutrosophic linguistic data. A new MADM application is also suggested by [19]. Recently, Jun [20] proposed new similarity methods for neutrosophic sets of interval values by using and Hamming distances an developed an application of these measures in MADM problems. Additionally, Simsek and Kirisci [14], and Kirisci [6] defined the neutrosophic contraction mapping and established a fixed point theorem in neutrosophic metric spaces. Similarity measures have been successfully used in various fields, for instance; pattern recognition, image processing, medical diagnosis, decision-making, etc. Majumdar and Samanta [8] suggested a membership degree-based similarity measure between SVNSs. The cosine similarity measure and weighted cosine similarity measure of IVFSs with risk preference were described by Ye [22].

The remainder of this paper is structured as follows: Firstly, fundamental definitions are given about neutrosophic set theory such as interval-valued neutrosophic set, inclusion relations. After, type-f inclusion relation for INS is defined. In section 3, we propose the idea of similarity measures and entropy for interval-valued neutrosophic sets. Section 4 provides the numerical example to indicate how the calculation, correction and suitability of similarity measures were done. Finally in Section 5, a comparative study is given and some conclusions are outlined.

2. **Preliminaries**

In this section, we review some basic ideas of NSs, IVNSs, and distance measures.
Any variable of is called an interval number and is represented by u. Find the following: $\mathbb{E} = \{u = [u_\ell, u_r] : u_\ell, u_r \in \mathbb{R}, u_\ell \leq u_r\}$. For $u, v \in \mathbb{E}$, we have $u = v \Leftrightarrow u_\ell = v_\ell, u_r = v_r$. Define the fundamental functions of addition $+ : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$, multiplication of scalars $\cdot : \mathbb{R} \times \mathbb{E} \to \mathbb{E}$ and product $\cdot : \mathbb{R} \times \mathbb{E} \to \mathbb{E}$, respectively, as follows:

$$+(u, v) = u + v = [u_\ell + v_r, u_r + v_r],$$

$$\alpha u = \begin{cases} [\alpha u_\ell, \alpha u_r], & \alpha \geq 0 \\ [\alpha u_r, \alpha u_\ell], & \alpha < 0, \end{cases}$$

$$\cdot(u, v) = u \cdot v = [\min R, \max R], R = \{u_\ell v_\ell, u_\ell v_r, u_r v_\ell, u_r v_r\}.$$

Any two random elements (interval number) in E may not always be compared using the start and end points. A second way of comparing the interval numbers is given below:
Let $u = [u_\ell, u_r] \in E$. Then $B(u) = \max\{|a - a'| : a, a' \in u\} = u_r - u_\ell$ is called the length of interval number u. By using the property of $B(u)$, the ordering of two interval numbers u and v can be defined as

$$u \leq v \Leftrightarrow B(u) \leq B(v)$$

A fuzzy set F is a function $F : X \to I$ on the universe X, where $I = [0, 1]$. The set of α levels (α-cut) $[F]^\alpha$, and the support of the set F are given as follows:

$$[F]^\alpha = \{a \in X : F(x) \geq \alpha\}, \alpha \in (0, 1];$$

$$supp[F] = \{a \in X : A(a) > 0\}.$$

Definition 2.1. [13] A NS, N over universe X can be given by

$$N = \Big\{[a, (t_N(a), i_N(a), f_N(a))] : a \in X\Big\}$$

where $t_N(a), i_N(a), f_N(a)$ are standard or non-standard subsets of $]_\ell 0, 1_r[$ which represent truth-function, indeterminacy, and untruth-function of $a \in N$, respectively.

Definition 2.2. [18] A single-valued neutrosophic set (SVNS) on the universe X is defined as

$$A = \Big\{ < x, t_A(a), i_A(a), f_A(a) >: a \in X\Big\}$$

$t_A(a), i_A(a), f_A(a) \in [0, 1]$ indicate the degree of truthness, degree of indeterminacy, and degree of untruth, respectively.

Definition 2.3. [17] Given a set X with generic elements showed by a. A neutrosophic set of interval values $\tilde{N}$ (IVNS $\tilde{N}$) is described by an interval truth-membership function $t_{\widetilde{N}}(a) = [t_{\widetilde{N}\ell}, t_{\widetilde{N}r}]$, an interval indeterminacy-membership function $i_{\widetilde{N}}(a) = [i_{\widetilde{N}\ell}, i_{\widetilde{N}r}]$, and an interval untruth-membership function $f_{\widetilde{N}}(a) = [f_{\widetilde{N}\ell}, f_{\widetilde{N}r}]$ for each $x \in X$ and $t_{\widetilde{N}}(a), i_{\widetilde{N}}(a), f_{\widetilde{N}}(a) \subset [0, 1]$. An IVNS $\widetilde{N}$ can be represented as

$$\widetilde{N} = \{[a, \left(t_{\widetilde{N}}(a), i_{\widetilde{N}}(a), f_{\widetilde{N}}(a)\right)] : a \in X\}.$$

Additionally, complement of $\widetilde{N}$ will be given as

$$\widetilde{N}^c = \{[a, \left(t_{\widetilde{N}^c}(a), i_{\widetilde{N}^c}(a), f_{\widetilde{N}^c}(a)\right)] : x \in X\}$$

where $t_{\widetilde{N}^c}(a) = f_{\widetilde{N}^c}(a), i_{\widetilde{N}^c}(a) = [1 - i_{\widetilde{N}r}(a), 1 - i_{\widetilde{N}\ell}(a)]$.

Inclusion relation is a fundamental to give definitions of union and intersection operations on any sets. In literature, there are two suggestions of the inclusion relation of neutrosophic sets. First inclusion definition for neutrosophic sets is introduced by Smarandache (see [25], [26]), it's referred to it as a inclusion relationship of type-1 and represented by $\subseteq_1$; second is the type-2 inclusion relation, which is demonstrated by $\subseteq 2$. Now, we give definitions of these inclusion relations as in the following, respectively:

Definition 2.4. [12] A single valued neutrosophic set N is included in the other single valued neutrosophic set M, it means that $N \subseteq_1 M \Leftrightarrow t_N(a) \leq t_M(a), i_N(a) \geq i_M(a), f_N(a) \geq f_M(a)$ for any $a \in X$.

Definition 2.5. [18] SVNS, N is included in the other SVNS, M, it means that $N \subseteq_2 M \Leftrightarrow t_N(a) \leq t_M(a), i_N(a) \leq i_M(a), f_N(a) \geq f_M(a)$ for any $a \in X$.

Smarandache [17] proposes an original description relation for the interval neutrosophic set as follows:

Definition 2.6. An interval neutrosophic set $\widetilde{N}$ is included in the other interval neutrosophic set $\widetilde{M}$, it means that $\widetilde{N} \subseteq_1 \widetilde{M} \Leftrightarrow t_{\widetilde{N}\ell}(a) \leq t_{\widetilde{M}\ell}(a), t_{\widetilde{N}r}(a) \leq t_{\widetilde{M}r}(a), i_{\widetilde{N}\ell}(a) \geq i_{\widetilde{M}\ell}(a), i_{\widetilde{N}r}(a) \geq i_{\widetilde{M}r}(a), f_{\widetilde{N}\ell}(a) \geq f_{\widetilde{M}\ell}(a), f_{\widetilde{N}r}(a) \geq f_{\widetilde{M}r}(a)$ for any $a \in X$.

Now, we give new definition named type-f inclusion relation:

Definition 2.7. Let $u = ([u_{1\ell}, u_{1r}], [u_{2\ell}, u_{2r}], [u_{3\ell}, u_{3r}])$ and $v = ([v_{1\ell}, v_{1r}], [v_{2\ell}, v_{2r}], [v_{3\ell}, v_{3r}])$ be the interval neutrosophic values. We can say $u \leq_f v$ if and only if any conditions is satisfied given as in the following:

(1) $B^t(u) \leq B^t(v)$ and $B^f(u) \geq B^f(v)$
(2) $B^t(u) = B^t(v)$ and $B^f(u) > B^f(v)$
(3) $B^t(u) = B^t(v)$ and $B^f(u) = B^f(v)$ and $B^i(u) \geq B^i(v)$.

By this way, the inclusion relation $\widetilde{N} \subseteq_f \widetilde{M}$ between interval neutrosophic sets $\widetilde{N}$ and $\widetilde{M}$ is satisfied if and only if one of the following three conditions exist:

(1) $B^t(\widetilde{N}) \leq B^t(\widetilde{M})$ and $B^f(\widetilde{N}) \geq B^f(\widetilde{M})$
(2) $B^t(\widetilde{N}) = B^t(\widetilde{M})$ and $B^f(\widetilde{N}) > B^f(\widetilde{M})$
(3) $B^t(\widetilde{N}) = B^t(\widetilde{M})$ and $B^f(\widetilde{N}) = B^f(\widetilde{M})$ and $B^i(\widetilde{N}) \geq B^i(\widetilde{M})$.

3. **Similarity and Entropy of Interval Neutrosophic Sets**

In this section, firstly, we give definition of similarity measure between interval neutrosophic values by means of [20].

Definition 3.1. (See [20]) Letting $S : \mathfrak{D} \times \mathfrak{D} \to [0,1]$ is similarity between interval neutrosophic values u and v if S has the following properties;

(1) $0 \leq S(u,v) \leq 1$;
(2) $S(u,v) = 1 \Leftrightarrow u = v$;
(3) $S(u,v) = S(v,u)$
(4) If $u \leq v \leq z,$ then $S(u,z) \leq S(u,v), S(u,z) \leq S(v,z)$ for all $u,v,z \in \mathfrak{D}$
$= \{u : u = ([u_{1\ell}, u_{1r}], [u_{2\ell}, u_{2r}], [u_{3\ell}, u_{3r}])\}$.

Now, we introduce new similarity by considering $\subseteq_f$ as given below.

Definition 3.2. Let $u = ([u_{1\ell}, u_{1r}], [u_{2\ell}, u_{2r}], [u_{3\ell}, u_{3r}])$ and $v = ([v_{1\ell}, v_{1r}], [v_{2\ell}, v_{2r}], [v_{3\ell}, v_{3r}])$. Then the similarity measure of u and v is defined by

$$S(u,v) = 1 - \frac{\max\{|u_{2r} - v_{2r}|, |u_{2\ell} - v_{2\ell}|\}}{2} \tag{1}$$

in the case $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}]$ and $[u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}]$ and

$$S(u,v) = \frac{1}{2} - \frac{1}{4}\{\max\{|u_{1r} - v_{1r}|, |u_{1\ell} - v_{1\ell}|\} + \max\{|u_{1r} - v_{1r}|, |u_{1\ell} - v_{1\ell}|\}\} \tag{2}$$

otherwise.

Theorem 3.3. *The values $S(u,v)$ defined by (1) and (2) are similarity measure between u and v.*

Proof. Let $u = ([u_{1\ell}, u_{1r}], [u_{2\ell}, u_{2r}], [u_{3\ell}, u_{3r}]), v = ([v_{1\ell}, v_{1r}], [v_{2\ell}, v_{2r}], [v_{3\ell}, v_{3r}]) \in \mathfrak{D}$. If we choose $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}]$ and $[u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}]$; then $0.5 \leq S(u,v) \leq 1$. In otherwise, $0 \leq S(u,v) \leq 0.5$.

(1) It is clear that $0 \leq S(u,v) \leq 1$,
(2) $S(u,v) = 1 \Leftrightarrow u = v$,
(3) $S(u,v) = S(v,u)$ is clearly satisfied,
(4) Let $u,v,z \in \mathfrak{D}$ and $u \leq v \leq z$, then the following cases hold:

(a) $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}]$.

(b) $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$.

(c) $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$. From here, $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$.

(d) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}]$.

(e) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$.

(f) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$.

(g) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}], [u_{2\ell}, u_{2r}] \geq [v_{2\ell}, v_{2r}]$ and $[v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}], [u_{2\ell}, u_{2r}] \geq [v_{2\ell}, v_{2r}]$.

(h) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}], [u_{2\ell}, u_{2r}] \geq [v_{2\ell}, v_{2r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$.

(i) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}], [u_{2\ell}, u_{2r}] \geq [v_{2\ell}, v_{2r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [u_{2\ell}, u_{2r}] \geq [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$.

Finally, in all cases it is deduced that $S(u, z) \leq S(u, v), S(u, z) \leq S(v, z)$.

Consequently, it is deduced that $S(u, v)$ is similarity on u and v. □

Fuzziness is significant topic in neutrosophic sets and there exist soo many ways to measure this fuzziness. Here, firstly we give definition of entropy for interval neutrosophic value then construct original entropy for neutrosophic value u.

Definition 3.4. [7] If E has the following properties, $E : \mathfrak{D} \to [0, 1]$ is an entropy of interval neutrosophic value:

(1) $E(u) = 0 \Leftrightarrow [u_{1\ell}, u_{1r}] = [0, 0]$ or $[1, 1]$ and $[u_{3\ell}, u_{3r}] = [0, 0]$ or $[1, 1]$;

(2) $E(u) = 1 \Leftrightarrow [u_{1\ell}, u_{1r}] = [u_{2\ell}, u_{2r}] = [u_{3\ell}, u_{3r}] = [0.5, 0.5]$;

(3) $E(u) = E(u^c)$;

(4) Let $u, v \in \mathfrak{D}$ then $v^c = ([v_{3\ell}, v_{3r}], [1 - v_{2r}, 1 - v_{2\ell}], [v_{1\ell}, v_{1r}])$ and $E(u) \leq E(v)$ if $u \leq_f v$ when $v \leq_f v^c$ or $v \leq_f u$ when $v^c \leq_f v$.

Definition 3.5. Let $u = ([u_{1\ell}, u_{1r}], [u_{2\ell}, u_{2r}], [u_{3\ell}, u_{3r}])$. Then the entropy for u is defined by

$$E(u) = \begin{cases} 1 - \frac{|u_{2r}+u_{2\ell}-1|}{2}, & [u_{1\ell}, u_{1r}] = [u_{3\ell}, u_{3r}] = [\frac{1}{2}, \frac{1}{2}] \\ \frac{1}{2} - \frac{1}{2}\left\{\max\{|u_{1\ell} - u_{3\ell}|, |u_{1r} - u_{3r}|\}\right\}, & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 3.6. *$E(u)$ introduced as (3) isn entropy for u.*

Proof. If $[u_{1\ell}, u_{1r}] = [u_{3\ell}, u_{3r}] = [\frac{1}{2}, \frac{1}{2}]$, then it is easy to see that $\frac{1}{2} \leq E(u) \leq 1$. In other case, $0 \leq E(u) \leq \frac{1}{2}$.

(1) $E(u) = 0 \Leftrightarrow \frac{1}{2} - \frac{1}{2}\left\{\max\{|u_{1\ell} - u_{3\ell}|, |u_{1r} - u_{3r}|\}\right\} = 0$
$\Leftrightarrow 1 = \max\{|u_{1\ell} - u_{3\ell}|, |u_{1r} - u_{3r}|\}$
$\Leftrightarrow [u_{1\ell}, u_{1r}] = [0, 0]$ or $[1, 1], [u_{3\ell}, u_{3r}] = [1, 1]$ or $[0, 0]$.

(2) $E(u) = 1 \Leftrightarrow [u_{1\ell}, u_{1r}] = [u_{2\ell}, u_{2r}] = [u_{3\ell}, u_{3r}] = [\frac{1}{2}, \frac{1}{2}]$.

(3) $E(u) = E(u^c) =$ is clearly satisfied.

(4) Let $u, v \in \mathfrak{D}$ and $v^c = ([v_{3\ell}, v_{3r}], [1 - v_{2r}, 1 - v_{2\ell}], [v_{1\ell}, v_{1r}])$. If $u \leq_f v$ when $v \leq_f v^c$ or $v \leq_f u$ when $v^c \leq_f v$ then $E(u) \leq E(v)$.

This completes the proof. □

3.1. *Definition of Similarity and Entropy of INSs*

In [21], similarity and entropy measure definitions of interval neutrosophic values expanded to interval neutrosophic sets. Now, we introduce this definition as follows, respectively.

Definition 3.7. (See [21]) Let M, N be two interval neutrosophic sets. Then, S is called similarity measure between M and N, if the following properties are satisfied:

(1) $0 \leq S(M, N) \leq 1$;

(2) $S(M, N) = 1 \Leftrightarrow M = N$;

(3) $S(M, N) = S(N, M)$

(4) If $M \subseteq N \subseteq P$, then $S(M, P) \leq S(M, N)$, $S(M, P) \leq S(B, P)$ for all $M, N, P \in$ INSs.

Definition 3.8. (See [21]) Let M be an interval neutrosophic set, then we give the definition E as interval neutrosophic sets' entropy if E contains the following assertions:

(1) $E(M) = 0 \Leftrightarrow B^t(M) = [0, 0]$ or $[1, 1], B^f(M) = [0, 0]$ or $[1, 1]$;

(2) $E(M) = 1 \Leftrightarrow B^t(M) = B^i(M) = B^f(M) = [0.5, 0.5]$;

(3) $E(M) = E(M^c)$;

(4) Let M, N are two INSs, $E(M) \leq E(N)$ if $M \subseteq_f N$ when $N \subseteq_f N^c$, or $N \subseteq_f M$ when $N^c \subseteq_f N$.

In addition to above definitions, [21] gain the literature similarity and entropy concepts of two neutrosophic sets. It means that similarity measure of interval neutrosophic values is carried on the interval neutrosophic sets as showed in the following definition.

Definition 3.9. [21]

$$S(M,N) = \frac{1}{n}\sum_{i=1}^{n} s(M(x_i), N(x_i))$$

where $X = \{x_1, x_2, ..., x_n\}$ is a NS and $s : \mathfrak{D} \times \mathfrak{D} \to [0,1]$ is similarity of INS for $M, N \subseteq X$. And

$$E(M) = \frac{1}{n}\sum_{i=1}^{n} e(M(x_i)), e : \mathfrak{D} \to [0,1].$$

4. Multi-attribute Decision-making

Ye [20] employs a multi-attribute decision-making process for single valued neutrosophic sets. First we discuss the Hamming distance, Euclidean distance, and measure of similarities for INSs developed by Ye [20].

(1) The Hamming Distance:
$d_1(A,B) = \frac{1}{6}\sum_{i=1}^{n}(|t_{A\ell}(u_i)-t_{B\ell}(u_i)|+|t_{Ar}(u_i)-t_{Br}(u_i)|+|i_{A\ell}(u_i)-i_{B\ell}(u_i)|+|i_{Ar}(u_i)-i_{Br}(u_i)| + |f_{A\ell}(u_i) - f_{B\ell}(u_i)| + |f_{Ar}(u_i) - f_{Br}(u_i)|)$.

(2) The Euclidean Distance:
$d_2(A,B) = \frac{1}{6}\sum_{i=1}^{n} (t_{A\ell}(u_i) - t_{B\ell}(u_i))^2 + (t_{Ar}(u_i) - t_{Br}(u_i))^2 + (i_{A\ell}(u_i) - i_{B\ell}(u_i))^2 + (i_{Ar}(u_i) - i_{Br}(u_i))^2 + (f_{A\ell}(u_i) - f_{B\ell}(u_i))^2 + (f_{Ar}(u_i) - f_{Br}(u_i))^2$.

(3) Similarity Measure:
$S_1(A,B) = 1 - \frac{1}{6}\sum_{i=1}^{n}(|t_{A\ell}(u_i) - t_{B\ell}(u_i)| + |t_{Ar}(u_i) - t_{Br}(u_i)| + |i_{A\ell}(u_i) - i_{B\ell}(u_i)| + |i_{Ar}(u_i) - i_{Br}(u_i)| + |f_{A\ell}(u_i) - f_{B\ell}(u_i)| + |f_{Ar}(u_i) - f_{Br}(u_i)|)$.

(4) Similarity Measure:
$S_2(A,B) = 1 - \frac{1}{6}\sum_{i=1}^{n} (t_{A\ell}(u_i) - t_{B\ell}(u_i))^2 + (t_{Ar}(u_i) - t_{Br}(u_i))^2 + (i_{A\ell}(u_i) - i_{B\ell}(u_i))^2 + (i_{Ar}(u_i) - i_{Br}(u_i))^2 + (f_{A\ell}(u_i) - f_{B\ell}(u_i))^2 + (f_{Ar}(u_i) - f_{Br}(u_i))^2$.

Next we give a numerical example for MADM and a comparison analysis of Ye's methods, Wang's method with our proposed MADM method.

4.1. *Numerical Example*

Let $\{M_1, M_2, M_3, M_4\}$ be the set of cars (alternatives), and $\{P_1, P_2, P_3\}$ be the set criterion for the selection of a suitable car, where P_1 is fuel compatibility and performance, M_2 is resale value and affordability, M_3 is safety and ride.
Let us consider the following interval neutrosophic set

$$M = (([0.7, 0.8], [0.1, 0.2], [0.1, 0.3]), ([0.8, 0.9], [0, 0.1], [0.2, 0.4]), ([0.6, 0.8], [0.1, 0.2], [0.3, 0.6]))$$

as a model option for the selection of a best car under given criterion.

The alternatives are evaluated under the given criterion and the interval neutrosophic decision matrix is computed and it is given in Table 1, where the columns represent the criteria and the rows represent the alternatives.
First we calculate similarity measure values by using proposed similarity measures under interval neutrosophic set as given below:

$$S_z(M_1, M) = 0.408, S_z(M_2, M) = 0.425, S_z(M_1, M) = 0.366, S_z(M_4, M) = 0.4.$$

Hence,

$$S(M_2, M) \succ S(M_1, M) \succ S(M_4, M) \succ S(M_3, M)$$

That is,

$$M_2 \succ M_1 \succ M_4 \succ M_3$$

Here M_2 is the best choice for selecting a car.
Now we consider the similarity measure values by means of [20] as given below.

$$\left(\left[\max_i\left(t_{\ell M_i}(x_i)\right), \max_i\left(t_{rM_i}(x_i)\right)\right], \left[\min_i\left(i_{\ell M_i}(x_i)\right), \min_i\left(i_{rM_i}(x_i)\right)\right], \left[\min_i\left(f_{\ell M_i}(x_i)\right), \min\left(f_{rM_i}(x_i)\right)\right]\right).$$

Secondly, we use Ye's method and obtain the following results:

$$S_1(M_1, M) = 0.55, S_1(M_2, M) = 0.7, S_1(M_3, M) = 0.4, S_1(M_4, M) = 0.6$$

$$S_2(M_1, M) = 0.881, S_2(M_2, M) = 0.92, S_2(M_3, M) = 0.823, S_2(M_4, M) = 0.886.$$

Hence, it is deduced that

$$S_2(M_2, M) \succ S_2(M_4, M) \succ S_2(M_1, M) \succ S_2(M_3, M)$$

That is,

$$M_2 \succ M_4 \succ M_1 \succ M_3$$

Thus and so, M_2 is the most suitable alternative.

Lastly, we calculate similarity measure by taking into account Yang *et al.* [21] method and determine the best option also as below:

$$S(M_1, M) = 0.420, S(M_2, M) = 0.445, S(M_1, M) = 0.375, S(M_4, M) = 0.416.$$

We have following results,

$$S(M_2, M) \succ S(M_1, M) \succ S(M_4, M) \succ S(M_3, M)$$

That is,

$$M_2 \succ M_1 \succ M_4 \succ M_3$$

Finally, M_2 is the best option for car selection. As shown in Table 2, the suggested MADM method is compared to established MADM methods. It can be noted in the comparison Table 2, the selected alternative given by any one proposed method acknowledges the authenticity and the efficacy of the existing methodology.

5. Conclusion

The concept of interval neutrosophic set (INS) is a strong model for MADM. We introduced new similarity measures, entropy, and inclusion relation named as type-f for interval neutrosophic sets (INSs). Then we developed robust MADM method for car selection by using proposed similarity measures for INSs. Meanwhile, a practical application for ranking of alternatives with newly developed MADM approach is illustrated by a numerical example. We computed similarity measures by our proposed method and compared the results with existing methods of Ye [20] and Yang *et al.* [21]. The validity and superiority of new similarity measures with existing approaches is also given with the help of a comparison analysis. Finally, it is deduced that proposed similarity measure and inclusion relations are more efficient, impressive and suitable.

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Appendix

TABLE 1. The interval neutrosophic decision matrix

	C_1	C_2	C_3
M_1	([0.3,0.5],[0.2,0.4],[0.3,0.5])	([0.4,0.6],[0.1,0.2],[0.2,0.5])	([0.6,0.8],[0.2,0.4],[0.3,0.6])
M_2	([0.7,0.8],[0.1,0.2],[0.2,0.3])	([0.8,0.9],[0,0.1],[0.3,0.4])	([0.4,0.5],[0.3,0.5],[0.7,0.8])
M_3	([0.4,0.5],[0.1,0.3],[0.4,0.5])	([0.5,0.6],[0.3,0.4],[0.2,0.4])	([0.3,0.5],[0.1,0.2],[0.7,0.9])
M_4	([0.7,0.8],[0.2,0.4],[0.1,0.3])	([0.4,0.5],[0,0.2],[0.4,0.5])	([0.5,0.6],[0.1,0.2],[0.7,0.8])

TABLE 2. Comparison analysis of final ranking with existing methods.

Method	Ranking of alternatives	The optimal alternative
Proposed method	$M_2 \succ M_1 \succ M_4 \succ M_3$	M_2
Ye's method	$M_2 \succ M_4 \succ M_1 \succ M_3$	M_2
Wang's method	$M_2 \succ M_1 \succ M_4 \succ M_3$	M_2

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The neutrosophic vector spaces- another approach

A. Elrawy
Department of Mathematics, Faculty of Science, SVU, Qena 83523, Egypt.
Correspondence: amr.elsaman@sci.svu.edu.eg.

Abstract. In this paper, we introduce and define a new version of a neutrosophic vector space. Indeed, this approach is a generalization of the notion of fuzzy vector space. Also, we study the neutrosophic subspace and linear independence. Furthermore, this study builds the basis and dimension of a neutrosophic vector space. Finally, we investigate the properties of the introduced notations.

Keywords: Neutrosophic vector space, basis, dimension.

1. **Introduction**

After Zadeh [16] introduced fuzzy sets, this fundamental concept has been generalized for a variety of purposes. Atanassov [4,5] first proposed the concept of intuitionistic fuzzy sets (IFSs) in 1986. The notion of a neutrosophic set (NS) was introduced by Smarandache [14,15]. As a generalization of the fuzzy and intuitionistic sets, the theory of neutrosophic sets is expected to play an important role in modern mathematics in general. Since 2005, the concept of the neutrosophic set has gotten a lot of attention [7, 9, 11–13], and it has a lot of applications [1, 2, 6, 10].

Additionally, a number of works have been published by researchers to extend the classical and fuzzy mathematical notions to the context of neutrosophic fuzzy mathematics. The difficulty in such generalizations lies in how to choose the most rational generalization among many available approaches. The concept of a neutrosophic vector space was introduced in [3]. In this study, we provide a new definition of the neutrosophic vector space, which represents the rational generalization of the fuzzy vector space. Also, we study the purely algebraic properties of neutrosophic vector spaces. Furthermore, we establish the idea of a neutrosophic basis and show that it exists in a large class of neutrosophic vector spaces.

The work is conceived as follows. In Section 2, some basic concepts in our study are recorded. Section 3 is devoted to introducing the new definition for a neutrosophic vector space as an extension of fuzzy vector space. Also, we introduced and studied some of the concepts. The conclusion remarks are reached in section 4.

In order to clarify the picture, we present here the following standard notation. V will denote a vector space over any field $\mathbb{K}$.

2. Basic concepts

In this section, we will go over certain definitions and outcomes that will be used in the next section.

Definition 2.1. [8] Let V be a vector space over $\mathbb{K}$. A fuzzy vector space is the pair $\overline{V} = (V, \mu)$ with the property that $\forall a, b \in R$ and $u, v \in V$, we have

$$\mu(au + bv) > \mu(u) \wedge \mu(v),$$

where $\mu : V \to [0, 1]$.

Definition 2.2. [15] Let N be the universe set. A neutrosophic set $\mathcal{N}$ on N (NS $\mathcal{N}$) is defined as:

$$\mathcal{N} = \{< a, \mu(a), \gamma(a), \zeta(a) > | a \in N\}.$$

where $\mu, \gamma, \zeta : N \to [0, 1]$.

Definition 2.3. [8] A set A is said to be upper well ordered if for all non-empty subsets $B \subset A$, $\sup B \in B$

Definition 2.4. [8] A subset $A \subset [0, 1]$ is said to have an increasing monotonic limit $x \in [0, 1]$ if and only if x is a limit of a monotonically increasing sequence in A.

Proposition 2.5. *[8] A set $A \subset [0, 1]$ is without any increasing monotonic limits iff it is upper well ordered.*

Proposition 2.6. *[8] All upper well ordered subsets of $[0, 1]$ are countable.*

3. Main result

In this section, we present a new definition of a neutrosophic vector space and give examples. Also, we derive some properties concerning this definition. In addition, we define neutrosophic linear independence and investigate certain properties. Lastly, the neutrosophic basis and dimension are defined and studied.

Definition 3.1. Neutrosohic vector space is a quaternary $\overline{V} = (V, \mu, \gamma, \zeta)$ where V is a vector space over arbitrary field $\mathbb{K}$ with

$$\mu : V \to [0,1],$$
$$\gamma : V \to [0,1],$$
$$\zeta : V \to [0,1],$$

with the following properties

$$\mu(au+bv) \geq \mu(u) \wedge \mu(v),$$
$$\gamma(au+bv) \leq \gamma(u) \vee \gamma(v),$$
$$\zeta(au+bv) \leq \zeta(u) \vee \zeta(v),$$

where $u, v \in V$ and $a, b \in \mathbb{K}$.

Example 3.2. Let $\mathbb{R}^2$ be a vector space over a field $\mathbb{R}$, then $\overline{V} = (\mathbb{R}^2, \mu, \gamma, \zeta)$ is a neutrosophic vector space over a field $\mathbb{R}$, where

$$\mu(s,t) = \begin{cases} 1 & \text{if } s = t = 0 \\ \frac{1}{2} & \text{if } s = 0, t \in \mathbb{R} - \{0\} \\ \frac{1}{4} & \text{if } s \in \mathbb{R}, t = 0 \end{cases}$$

$$\gamma(s,t) = \begin{cases} \frac{1}{4} & \text{if } s = t = 0 \\ \frac{1}{2} & \text{if } s = 0, t \in \mathbb{R} - \{0\} \\ \frac{3}{4} & \text{if } s \in \mathbb{R}, t = 0 \end{cases}$$

$$\zeta(s,t) = \begin{cases} \frac{1}{3} & \text{if } s = t = 0 \\ \frac{1}{2} & \text{if } s = 0, t \in \mathbb{R} - \{0\} \\ \frac{5}{6} & \text{if } s \in \mathbb{R}, t = 0 \end{cases}$$

Proposition 3.3. *If $\overline{V} = (V, \mu, \gamma, \zeta)$ is a neutrosophic vector space over a field $\mathbb{K}$, then*

- (i) $\mu(au) = \mu(u)$, $\forall a \in \mathbb{K} - \{0\}$.
- (ii) $\gamma(au) = \gamma(u)$, $\forall a \in \mathbb{K} - \{0\}$.
- (iii) $\zeta(au) = \zeta(u)$, $\forall a \in \mathbb{K} - \{0\}$.
- (iv) *If* $u, v \in V$ *and* $\mu(u) > \mu(v)$, *then* $\mu(u+v) = \mu(v)$.
- (v) *If* $u, v \in V$ *and* $\gamma(u) < \gamma(v)$, *then* $\gamma(u+v) = \gamma(v)$.
- (vi) *If* $u, v \in V$ *and* $\zeta(u) < \zeta(v)$, *then* $\zeta(u+v) = \zeta(v)$.

Proof. We prove only (v) since the remainder are well-known.
Since $\gamma(u) < \gamma(v)$ we have $\gamma(v) \leq \gamma(u+v)$. Also, $\gamma[(u+v)-v] = \gamma(u) \leq \gamma(u+v) \vee \gamma(v)$. Since $\gamma(u) < \gamma(v)$ we have $\gamma(u+v) \leq \gamma(u)$. Consequently $\gamma(u+v) = \gamma(v)$. □

Proposition 3.4. *Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over $\mathbb{K}$ with $\mu(u) \neq \mu(v)$, $\gamma(u) \neq \gamma(v)$ and $\zeta(u) \neq \zeta(v)$, then*

$$\mu(u+v) = \mu(u) \wedge \mu(v),$$

$$\gamma(u+v) = \gamma(u) \vee \gamma(v),$$

$$\zeta(u+v) = \zeta(u) \vee \zeta(v),$$

where $u, v \in V$.

Proof. It is clear from Proposition 3.3. □

Proposition 3.5. *Let* $\overline{V} = (V, \mu, \gamma, \zeta)$ *be a neutrosophic vector space over a field* $\mathbb{K}$*, then*

(*i*) $\mu(0) = \sup_{u \in V} \mu(u) = \sup[\mu(V)]$.

(*ii*) $\gamma(0) = \inf_{u \in V} \gamma(u) = \inf[\gamma(V)]$.

(*iii*) $\zeta(0) = \inf_{u \in V} \zeta(u) = \inf[\zeta(V)]$.

Proof. We prove only (ii) since (i) and (iii) are well-known.
Let $u \in V$, then $\gamma(0) = \gamma(0u) \leq \gamma(u)$. □

Definition 3.6. Let W be a subspace of a vector space V. Then $(W, \mu_W, \gamma_W, \zeta_W)$ is called neutrosophic subspace of a neutrosophic vector (V, μ, γ, ζ) if the following conditions are satisfied:

(*i*) $\mu_W(x-y) \geq \mu_W(x) \wedge \mu_W(y)$.

(*ii*) $\mu_W(cx) = \mu_W(x)$.

(*iii*) $\gamma_W(x-y) \leq \gamma_W(x) \vee \gamma_W(y)$.

(*iv*) $\gamma_W(cx) = \gamma_W(x)$.

(*v*) $\zeta_W(x-y) \leq \zeta_W(x) \vee \zeta_W(y)$.

(*vi*) $\zeta_W(cx) = \zeta_W(x)$.

Definition 3.7. Let $\overline{V}_1 = (V, \mu_1, \gamma_1, \zeta_1)$ and $\overline{V}_2 = (V, \mu_2, \gamma_2, \zeta_2)$ be two neutrosophic vector spaces over $\mathbb{K}$, then

(*i*) The intersection of $\overline{V}_1$ and $\overline{V}_2$ define as follows: $\overline{V}_1 \cap \overline{V}_2 = (V, \mu_1 \wedge \mu_2, \gamma_1 \vee \gamma_2, \zeta_1 \vee \zeta_2)$

(*ii*) The sum of $\overline{V}_1$ and $\overline{V}_2$ define as follows: $\overline{V}_1 + \overline{V}_2 = (V, \mu_1 + \mu_2, \gamma_1 + \gamma_2, \zeta_1 + \zeta_2)$,
where $(\mu_1 + \mu_2)(a) = sup\{\mu_1(a) \wedge \mu_2(a-v)\}$, $(\gamma_1 + \gamma_2)(a) = inf\{\gamma_1(a) \vee \gamma_2(a-v)\}$, $(\zeta_1 + \zeta_2)(a) = inf\{\zeta_1(a) \vee \zeta_2(a-v)\}$ and $a = u + v$.

Proposition 3.8. *Let* $\overline{W}_i = (V, \mu_i, \gamma_i, \zeta_i)$ *be a set of family neutrosophic subspaces over a field* $\mathbb{K}$ *with* $i \in I = 1, 2, ..., n$*, then* $\cap_{i \in I} \overline{W}_i$ *is a neutrosophic vector space over a field* $\mathbb{K}$.

Proposition 3.9. *Let* $\overline{W}_i = (V, \mu_i, \gamma_i, \zeta_i)$ *be a set of family neutrosophic subspaces over a field* $\mathbb{K}$ *with* $i \in I = 1, 2, ..., n$*, then* $\sum_{i=1}^{n} \overline{W}_i$ *is a neutrosophic vector space over a field* $\mathbb{K}$.

Proof. We use contradiction to prove this result. Firstly, assume that

$$(\mu_1 + \mu_2 + ... + \mu_n)(x+y) < (\mu_1 + \mu_2 + ... + \mu_n)(x) \wedge (\mu_1 + \mu_2 + ... + \mu_n)(y).$$

Thus, there exists $u_1+u_2+...+u_{n+1}$ and $v_1+v_2+...+v_{n+1}$ such that for all $z_1+z_2+...+z_{n+1}$ we have

$$\mu_1(z_1)\wedge\mu_2(z_2)\wedge...\wedge\mu_n(x+y-z_1-z_2-...-z_n)<[\mu_1(u_1)\wedge\mu_2(u_2)\wedge...\wedge\mu_n(x-u_1-u_2-...-u_n)]$$
$$\wedge[\mu_1(v_1)\wedge\mu_2(v_2)\wedge...\wedge\mu_n(y-v_1-v_2-...-v_n)]\rightarrow(\star)$$

but

$$[\mu_1(u_1)\wedge\mu_2(u_2)\wedge...\wedge\mu_n(x-u_1-u_2-...-u_n)]\wedge[\mu_1(v_1)\wedge\mu_2(v_2)\wedge...\wedge\mu_n(y-v_1-v_2-...-v_n)]=$$
$$\mu_1(u_1)\wedge\mu_1(v_1)\wedge\mu_2(u_2)\wedge\mu_2(v_2)\wedge...\wedge\mu_n(x-u_1-u_2-...-u_n)]\wedge\mu_n(y-v_1-v_2-...-v_n)\le$$
$$\mu_1(u_1\wedge v_1)\wedge\mu_1(u_2\wedge v_2)\wedge....\wedge\mu_n(x+y-u_1-v_1-u_2-v_2...-u_n-v_n).$$

Therefore, there exists $z_i=u_i+v_i$ for which $(\star)$ is false, this we have a contradiction and therefore $(\mu_1+\mu_2+...+\mu_n)(x+y)\ge(\mu_1+\mu_2+...+\mu_n)(x)\wedge(\mu_1+\mu_2+...+\mu_n)(y)$.

Secondly, suppose that

$$(\gamma_1+\gamma_2+...+\gamma_n)(x+y)>(\gamma_1+\gamma_2+...+\gamma_n)(x)\vee(\gamma_1+\gamma_2+...+\gamma_n)(y).$$

Thus, there exists $u_1+u_2+...+u_{n+1}$ and $v_1+v_2+...+v_{n+1}$ such that for all $z_1+z_2+...+z_{n+1}$ we have

$$\gamma_1(z_1)\vee\ \gamma_2(z_2)\vee...\vee\gamma_n(x+y-z_1-z_2-...-z_n)>[\gamma_1(u_1)\vee\gamma_2(u_2)\vee...\vee\gamma_n(x-u_1-u_2-...-u_n)]$$
$$\vee[\gamma_1(v_1)\vee\gamma_2(v_2)\vee...\vee\gamma_n(y-v_1-v_2-...-v_n)]\rightarrow(\star\star)$$

but

$$[\gamma_1(u_1)\vee\gamma_2(u_2)\vee...\vee\gamma_n(x-u_1-u_2-...-u_n)]\vee[\gamma_1(v_1)\vee\gamma_2(v_2)\vee...\vee\gamma_n(y-v_1-v_2-...-v_n)]=$$
$$\gamma_1(u_1)\vee\gamma_1(v_1)\vee\gamma_2(u_2)\vee\gamma_2(v_2)\vee...\vee\gamma_n(x-u_1-u_2-...-u_n)]\vee\gamma_n(y-v_1-v_2-...-v_n)\ge$$
$$\gamma_1(u_1\vee v_1)\vee\gamma_1(u_2\vee v_2)\vee....\vee\gamma_n(x+y-u_1-v_1-u_2-v_2...-u_n-v_n).$$

Therefore, there exists $z_i=u_i+v_i$ for which $(\star\star)$ is false, this we have a contradiction and therefore $(\gamma_1+\gamma_2+...+\gamma_n)(x+y)\le(\gamma_1+\gamma_2+...+\gamma_n)(x)\vee(\gamma_1+\gamma_2+...+\gamma_n)(y)$. Similarly, we find $(\gamma_1+\gamma_2+...+\gamma_n)(x+y)\le(\gamma_1+\gamma_2+...+\gamma_n)(x)\vee(\gamma_1+\gamma_2+...+\gamma_n)(y)$. □

Now, we proceed to characterize the neutrosophic linear independence.

3.1. *Neutrosophic linear independence*

Definition 3.10. Let $\overline{V}=(V,\mu,\gamma,\zeta)$ be a neutrosophic vector space over a field $\mathbb{K}$. We say that a finite set of vectors $\{u_i\}_{i=1}^n$ is a neutrosophic linear independence in $\overline{V}$ iff $\{u_i\}_{i=1}^n$ is linear independence in V and $\forall\ \{a_i\}_{i=1}^n\subset\mathbb{K}$,

$$\mu(\sum_{i=1}^n a_iu_i)=\bigwedge_{i=1}^n\mu(a_iu_i),$$
$$\gamma(\sum_{i=1}^n a_iu_i)=\bigvee_{i=1}^n\gamma(a_iu_i),$$
$$\zeta(\sum_{i=1}^n a_iu_i)=\bigvee_{i=1}^n\zeta(a_iu_i).$$

A set of vectors is neutrosophic linearly independent in $\overline{V}$ if all finite subsets of it are neutrosophic linearly independent in $\overline{V}$.

Example 3.11. Let $\overline{V}$ be a neutrosophic vector space which define in Example 3.20. The set of vectors $\{x = (2, 0), y = (-2, 1)\}$ is linearly independent. Also, it easy to checked

$$\mu(x) = \mu(y) \textit{ and } \mu(x + y) > \mu(x),$$

$$\gamma(x) = \gamma(y) \textit{ and } \gamma(x + y) < \gamma(x),$$

$$\zeta(x) = \zeta(y) \textit{ and } \zeta(x + y) < \zeta(x).$$

This set is not neutrosophic linearly independent in $\overline{V}$.

Proposition 3.12. *If $\overline{V} = (V, \mu, \gamma, \zeta)$ is a neutrosophic vector space over a field $\mathbb{K}$, then any set of vectors $\{x_i\}_{i=1}^{n} \subset V - \{0\}$ which has distinct μ, γ, ζ-values is linearly and neutrosophic linearly independent.*

Proof. We use mathematical induction to prove this proposition. By [8], μ-values are linearly and neutrosophic linearly independent. We now show that γ-values are both linearly and neutrosophic linearly independent. In the case $n = 1$ we find the statement is true. Also, suppose that the statement is true for n. Assume that $\{x_i\}_{i=1}^{n+1}$ is a set of vectors in $V \backslash \{0\}$ with distinct γ-values. According to the inductive hypothesis we have $\{x_i\}_{i=1}^{n}$ is neutrosophic linearly independent. Suppose that $\{x_i\}_{i=1}^{n+1}$ is not linearly independent and thus $x_{n+1} = \sum_{i \in S} a_i x_i$ where $S \subset \{1, \ldots, n\}, S \neq \emptyset$ and for all $i \in S, a_i \neq 0$. Then

$$\gamma(x_{n+1}) = \bigvee_{\epsilon} \gamma(a_i x_i) = \bigvee_{\epsilon} \gamma(x_i)$$

and hence $\gamma(x_{n+1}) \in \{\gamma(x_i)\}_{i=1}^{n}$. This contradicts the fact that $\{x_i\}_{i=1}^{n+1}$ has distinct γ-values. Therefore $\{x_i\}_{i=1}^{n+1}$ is linearly independent. Finally Propositions 3.3 (ii), 3.4 and 3.5 (ii) clearly show that $\{x_i\}_{i=1}^{n+1}$ is neutrosophic linearly independent. □

We conclude this section by providing definitions of the neutrosophic base and dimension and looking at some properties.

3.2. *Neutrosophic basis and dimension*

Definition 3.13. The linearly independent basis for $\overline{V}$ is the neutrosophic basis of a neutrosophic vector space $\overline{V} = (V, \mu, \gamma, \zeta)$ over a field $\mathbb{K}$.

The following theorem illustrates how a neutrosophic basis may be used to create a large class of neutrosophic vector spaces.

Theorem 3.14. *Let V be a vector space with basis $B = \{v_\aleph\}_{\aleph \in I}$, constants $\mu_0, \gamma_0, \zeta_0 \in (0,1]$ and any sets of constants $\{\mu_\aleph\}_{\aleph \in I}, \{\gamma_\aleph\}_{\aleph \in I}, \{\zeta_\aleph\}_{\aleph \in I} \subset (0,1]$ such that $\mu_0 \geq \mu_\aleph, \gamma_0 \geq \gamma_\aleph, \zeta_0 \geq \zeta_\aleph \ \forall \ \aleph \in I$. Define*

$$\mu(u) = \bigwedge_{i=1}^{N} \mu(v_{\aleph_i}) = \bigwedge_{i=1}^{N} \mu_{\aleph_i} \text{ and } \mu(0) > \mu_0,$$

$$\gamma(u) = \bigvee_{i=1}^{N} \gamma(v_{\aleph_i}) = \bigvee_{i=1}^{N} \gamma_{\aleph_i} \text{ and } \gamma(0) > \gamma_0,$$

$$\zeta(u) = \bigvee_{i=1}^{N} \zeta(v_{\aleph_i}) = \bigvee_{i=1}^{N} \zeta_{\aleph_i} \text{ and } \zeta(0) > \zeta_0,$$

where $u \in V$ with $u = \sum_{i=1}^{N} c_i v_{\aleph_i}$ and μ, γ, ζ is well-defined. We assert that $\overline{V} = (V, \mu, \gamma, \zeta)$ neutrosophic vector space with basis B.

Proof. Let $u, w \in V - \{0\}$, then we can write $u,\ w$ in a unique formula

$$u = \sum_{i \in C \cup D_u} c_i v_{\aleph_i},$$

$$w = \sum_{i \in C \cup D_w} d_i v_{\aleph_i},$$

where $C \cap D_u = \phi$, $C \cap D_w = \phi$ and $c_i,\ d_i \in \mathbb{R} - \{0\}$. Suppose that $a,\ b \in \mathbb{R} - \{0\}$ and $au + bw \neq 0$. Let $Z = \{i \in C : ac_i + bd_i = 0\}$ and $N = C - \{Z\}$. Now, the proof boils down to showing that

$$\mu(au + bw) \geq \mu(u) \wedge \mu(w), \tag{3.21}$$

$$\gamma(au + bw) \leq \gamma(u) \vee \gamma(w), \tag{3.22}$$

$$\zeta(au + bw) \leq \zeta(u) \vee \zeta(w), \tag{3.23}$$

we prove only 3.23 and 3.23, since 3.21 see [8]. Now,

$$\begin{aligned}\gamma(au + bw) &= \gamma(\sum_{i \in C}(ac_i + bd_i)v_i + \sum_{i \in D_u} ac_i v_i + \sum_{i \in D_w} bd_i v_i) \\ &= \gamma(\sum_{i}^{N}(ac_i + bd_i)v_i + \sum_{i \in D_u} ac_i v_i + \sum_{i \in D_w} bd_i v_i)\end{aligned}$$

All of the coefficients according to the above linear combination are non-zero, and thus by definition of γ we have

$$\begin{aligned}\gamma(au+bw) &= \left(\bigvee_{i\in N}\gamma\left(v_{\aleph_i}\right)\right)\vee\left(\bigvee_{i\in D_u}\gamma\left(v_{\aleph_i}\right)\right)\vee\left(\bigvee_{i\in D_w}\gamma\left(v_{\aleph_i}\right)\right)\\ &= \left(\bigvee_{i\in N}\gamma_{\aleph_i}\right)\vee\left(\bigvee_{i\in D_u}\gamma_{\aleph_i}\right)\vee\left(\bigvee_{i\in D_w}\gamma_{\aleph_i}\right)\\ &= \bigvee_{i\in N\cup D_u\cup D_w}\gamma_{\aleph_i}\\ &\leq \bigvee_{i\in C\cup D_u\cup D_w}\gamma_{\aleph_i} = \left(\bigvee_{i\in C\cup D_u}\gamma_{\aleph_i}\right)\vee\left(\bigvee_{i\in C\cup D_w}\gamma_{\aleph_i}\right)\end{aligned}$$

Therefore if $a, b \neq 0$ and $au+bw \neq 0$ then $\gamma(au+bw) \leq \gamma(u)\vee\gamma(w)$. In case you do $au+bw = 0$, since $\gamma(0) = \mu_0 \leq \inf\gamma(B)$ we must have $\gamma(au+bw) = \gamma(0) \leq \gamma(u)\vee\gamma(w)$.

Without giving up generality, in the case where a or b is zero we may say $a = 0$, then $\gamma(0u + bw) = \gamma(bw) \leq \gamma(u)\vee\gamma(bw) \leq \gamma(u)\vee\gamma(w)$. □

Lemma 3.15. *Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over a field $\mathbb{K}$ with $\mu(V), \gamma(V), \zeta(V)$ are upper well ordered and let U be a proper subspace of V, then $\exists\, u \in V/U$ such that*

$$\mu(u+v) = \mu(u)\wedge\mu(v)$$

$$\gamma(u+v) = \gamma(u)\vee\gamma(v)$$

$$\zeta(u+v) = \zeta(u)\vee\zeta(v)$$

where $v \in V$.

Proof. We only prove $\gamma(u+v) = \gamma(u)\vee\gamma(v)$. Since $\gamma(V)$ is upper well ordered we can find $u \in V/U$ such that $\gamma(u) = inf[\gamma(V/U)]$. Now, if $\gamma(u) \neq \gamma(v)$ then $\gamma(u)\vee\gamma(v) = \gamma(u+v)$ by Proposition 3.4. If $\gamma(u) = \gamma(v)$ then $\gamma(u+v) \leq \gamma(u)\vee\gamma(v)$. Also, since $u+v \in V/U$ and $\gamma(u) = inf[\gamma(V/U)]$ we have $\gamma(u+v) \geq \gamma(u)\vee\gamma(v)$ and thus $\gamma(u)\vee\gamma(v) = \gamma(u+v)$. □

Lemma 3.16. *Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over a field $\mathbb{K}$ with $\mu(V), \gamma(V), \zeta(V)$ are upper well ordered and let U be a proper subspace of V. Assume that B is a neutrosophic basis for U, then there exists $w \in V\backslash U$ such that $B^* = B\cup\{w\}$ is a neutrosophic basis for $\overline{W} = (W = \langle B^*\rangle, \mu_W, \gamma_W, \zeta_W)$, where $\langle B^*\rangle$ is the vector space spanned by B^*.*

Proof. Suppose that $w \in V \backslash U$ such that $\mu(w) = \sup[\mu(V\backslash U)]$, $\gamma(w) = \inf[\gamma(V\backslash U)]$, and $\zeta(w) = \inf[\zeta(V\backslash U)]$, then by Lemma 3.15, we have w is neutrosophic linearly independent from B. Assume that $B^* = B \cup \{w\}$. Obvious B^* is a neutrosophic basis for $\overline{W} = (W = \langle B^* \rangle, \mu_W, \gamma_W, \zeta_W)$. □

Theorem 3.17. *Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over $\mathbb{K}$ which $\mu(V), \gamma(V), \zeta(V)$ are upper well ordered, then $\overline{V}$ has a neutrosophic basis.*

Proof. Suppose that $\overline{V} = (V, \mu, \gamma, \zeta)$ is a neutrosophic vector space over $\mathbb{K}$ which $\mu(V), \gamma(V), \zeta(V)$ are upper well ordered. Let $\vartheta = \{B \subset V \mid B$ is neutrosophic linearly independent $\}$. We find ϑ is partial order by set inclusion. Assume that χ is a totally ordered subset of ϑ and let $A = \bigcup_{B\in\chi} B$. Obviously, A is an upper bound for C. Assume $a_1, \ldots, a_n \in A$. Then there exist $B_{\alpha(1)}, \ldots, B_{\alpha(n)} \in \chi$ such that $a_i \in B_{\alpha(i)}$. Since χ is totally ordered, one of the sets, say $B_{\alpha(k)}$, is a super set of the others. Hence $a_1, \ldots, a_n \in B_{\alpha(k)}$. Since $B_{\alpha(k)}$ is neutrosophic linearly independent $a_1, \ldots, a_n$ are neutrosophic linearly independent. Thus A is upper bound of χ in ϑ. By Zorn's Lemma, there exists a maximal element B^* in ϑ. Suppose that $\langle B^* \rangle = U$ is a proper subspace of V then by Lemma 3.15, there exists $w \in V\backslash U$ such that $B^* \cup \{w\} = B^+$ is a neutrosophic basis for $\overline{W} = (W, \mu_W, \gamma_W, \zeta_W)$. This contradicts the fact that B^* is a maximal element in ϑ. Thus we must have $\langle B^* \rangle = V$ and B^* is a neutrosophic basis for V. □

Corollary 3.18. *Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over a field $\mathbb{K}$ with V finite dimensional, then $\overline{V}$ has a neutrosophic basis.*

Proof. Suppose that $\mu(V)$, $\gamma(V)$ and $\zeta(V)$ are finite and therefore upper well ordered since V is finite dimensional. Thus, $\overline{V}$ has a neutrosophic basis, according to Theorem 3.17. □

In what follows, we define the dimension of neutrosophic vector spaces.

Definition 3.19. Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over a field $\mathbb{K}$, then we define the dimension of a neutrosophic space to be

$$dim(\overline{V}) = (dim\ V, \sup_{x \text{ a base for V}} (\sum_{v\in x} \mu(v)), \inf_{x \text{ a base for V}} (\sum_{v\in x} \gamma(v)), \inf_{x \text{ a base for V}} (\sum_{v\in x} \zeta(v))).$$

There is no doubt that the dimension is a function from the class of all neutrosophic vector spaces to $[0, \infty[$. Only when $dim(\overline{V}) = e < \infty$ does a neutrosophic vector space have a finite dimension.

Example 3.20. Let $\mathbb{R}^2$ be a vector space over a field $\mathbb{R}$. It is easily checked that $\overline{V} = (\mathbb{R}^2, \mu, \gamma, \zeta)$ is a neutrosophic vector space over a field $\mathbb{R}$, where

$$\mu(s,t) = \begin{cases} 1 & \text{if } s = t = 0 \\ \frac{1}{3} & \text{if } s = t,\ s \in \mathbb{R} - \{0\} \\ \frac{1}{5} & \text{otherwise} \end{cases}$$

$$\gamma(s,t) = \begin{cases} \frac{1}{15} & \text{if } s = t = 0 \\ \frac{1}{6} & \text{if } s = t,\ s \in \mathbb{R} - \{0\} \\ \frac{1}{3} & \text{other wise} \end{cases}$$

$$\zeta(s,t) = \begin{cases} \frac{1}{9} & \text{if } s = t = 0 \\ \frac{1}{2} & \text{if } s = t,\ s \in \mathbb{R} - \{0\} \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

It is also easy to check that $dim(\mathbb{R}^2, \mu, \gamma, \zeta) = (2, \frac{8}{15}, \frac{3}{6}, \frac{5}{6})$.

4. Conclusions

Recently, it is important and applicable to study neutrosophic sets in the mathematical branch. In this paper, the author has made redefined the concept of neutrosophic vector space as an extension of the definition of fuzzy vector space. Furthermore, this definition was studied in order to define and study linear independence, basis, and dimension. The dimension of a class of neutrosophic vector space will be taken up by the author for future research. As a fuzzy vector space, we couldnt find an example of a neutrosophic vector space without a neutrosophic basis or prove that all neutrosophic vector spaces have one. It is, in my opinion, a difficult problem, and this is an open problem for the next research. However, there is a simple condition that a neutrosophic vector space must satisfy in order to have a neutrosophic basis.

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Similarity Measures on Interval-Complex Neutrosophic Soft Sets with Applications to Decision Making and Medical Diagnosis under Uncertainty

Faisal Al-Sharqi[1,2,*], Abd Ghafur Ahmad[1] and Ashraf Al-Quran[3]

[1]Department of Mathematical Sciences, Faculty of Science and Technology Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, Malaysia ; P102960@siswa.ukm.edu.my ; ghafur@ukm.edu.my

[2]Department of Mathematics, Faculty of Education for Pure Sciences, University of Anbar, Ramadi, Anbar, Iraq; faisal.ghazi@uoanbar.edu.iq

[2]Preparatory Year Deanship, King Faisal University, Hofuf 31982, Al- Ahsa, Saudi Arabia; aalquran@kfu.edu.sa

[*]Correspondence: p102960@siswa.ukm.edu.my

Abstract. The idea of an interval complex neutrosophic soft set (I-CNSS) emerges from the interval neutrosophic soft set (I-NSS) by the extension of its three membership functions (T, I, F) from real space to complex space (unit disc) to better handle uncertainties, vagueness, indeterminacy, and imprecision of information in the periodic nature. The novelty of I-CNSS lies in its more significant range of activity compared to CNS. Measures of similarity and distance are important tools that can be used to solve many problems in real life. Hence, this paper presents some interval complex neutrosophic soft similarities based on Hamming and Euclidean distances of I-CNSSs to deal with real-life problems that include uncertain information such as decision-making issues and medical diagnosis stats. Firstly, this paper reviews the definition of an interval complex neutrosophic soft set. Secondly, we defined distance Hamming measures and distance Euclidean measures on I-CNSSs . Next, the axiomatic definition of similarity measures based on Hamming and Euclidean distances of I-CNSSs is proposed. Moreover, a numerical example is given and relations between these measures are introduced and verified. Meanwhile, some applications are given to show how similarity can be used to help the user in making decisions and making medical diagnoses. Finally, a comparison of some current approaches is used to back up this study.

Keywords: similarity measure; decision making; interval complex neutrosophic soft set;distance measure.

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1. **Introduction**

The idea of a complex fuzzy set (CFS) was established by Ramot [1] as a generalization of traditional fuzzy set theory [2] from real numbers to complex plane (unit disc) to represent the uncertain information that exists in two dimensions. The enormous success of this idea has brought about the building up of many extensions of CFSs, such as complex intuitionistic fuzzy sets (CIFSs) [3], complex multi-fuzzy sets (CMFSs) [4], complex vague sets (CVSs) [5], complex hesitant fuzzy sets (CHFSs) [6] and interval-valued complex intuitionistic fuzzy sets (I-VCIFSs) [7] which have been introduced and examined in several fields in our life, such as decision-making, image restoration and medical diagnosis. Ali and Smarandache [8] introduced a new model knowing a complex neutrosophic set where the three membership functions T,I,F instead of being real-valued functions with a range of [0,1] is replaced by a complex-valued functions of the form $T = t_A(x).e^{j\mu_A(x)}$,$I = i_A(x).e^{j\beta_A(x)}$ and $F = f_A(x).e^{j\aleph_A(x)}$ with $\left(j = \sqrt{-1}\right)$ and $\mu_A(x), \beta_A(x), \aleph_A(x)$ are periodic functions.To make the CNS more flexible and adaptable to cover vague periodic information in real-life problems, Ali and Smarandache again generalize the CNS idea to interval complex neutrosophic sets (I-CNSs) [9], which are characterized by the degrees of three complex membership functions that are characterized by interval values. Researchers utilized CNS and ICNS in different application areas and introduce new contributions, such as Broumi et al. [10] presented some of the potential properties and theorems with a multi-criteria decision-making process on bipolar complex neutrosophic sets. Quek et al. [11] introduced a new approach to neutrosophic graphs named complex neutrosophic graphs. Al-Quran et al. [12] investigated the relation between CNSs depending on the cartesian product of CNSs. Dat et al. [13] provided the connotation of a linguistic ICNS number, which is categorised independently by three membership functions linguistic variables for multiple attribute group decision-making (MGDM). All the existing mentioned theories work in several fields of life without considering the parameterization factor during the analysis. Consequently, these models lack parametrization tools to handle uncertainties and ambiguous issues in parameterized form. To cope with such challenges, Molodtsov (1999) [14] benchmarked and identified the theory of soft sets (SS) to handle uncertainty and vagueness by providing a general mathematical tool that can represent the problem parameters in a more wide and more complete form. A soft set is a set-valued map that provides a rough description of the things under consideration depending on several parameters. Since its inception, soft set has been studied and extended by researchers to several different hybrid models. First of all, Maji et al.employed the soft set theory with FSs and IFSs and introduced new notions of the fuzzy soft set (FSS) [15] and intuitionistic fuzzy soft set (IFSS) [16]. Thus, the hybridization process of soft set theory with the other uncertainty concepts created many hybrid fuzzy structures, such as interval fuzzy soft set [17], interval-valued intuitionistic fuzzy soft set [18], neutrosophic soft set [19], interval neutrosophic soft set [20] Q-neutrosophic soft set [21, 22], etc. are introduced. Later, in order to incorporate the advantages of complex numbers into the concept of soft sets, fuzzy sets, and their generalizations, Thirunavukarasu et al. [23] developed the complex fuzzy soft set and tested it in decision making problems. Selvachandran and

Singh [24] extended the CFSSs theory and established a new vision which is known as interval-valued complex fuzzy soft sets (I-VCFSs). Kumar and Bajaj [25] introduced the concept of complex intuitionistic fuzzy soft sets, while Selvachandran et al. [26] proposed the concept of complex vague soft sets and defined several distance measures between these sets. Broumi et al. [27] presented the notion of CNSSs with their essential properties. Following that, these uncertainty sets have been actively used to address uncertainty in a variety of decision-making problems [28]- [35]. Recently, Al-Sharqi et al. [36] developed a hybrid model of complex neutrosophic sets with soft sets in an interval setting called the interval complex neutrosophic soft set (ICNSS). An ICNSS is defined by a complex interval-valued truth membership function which represents uncertainty with periodicity, complex interval-valued indeterminacy membership function which represents indeterminacy with periodicity, and a complex interval-valued falsity membership function which represents falsity with periodicity. This notion handles the neutrosophic environment data in a periodic manner, while the interval neutrosophic soft set provides a parameterization tool to handle the neutrosophic environment data. In addition, Al-Sharqi et al. also kept building on the idea of ICNSS by combining it with other mathematical techniques to solve problems with uncertainty more efficiently [37], [38].
Similarity measures are an important tool in fuzzy set theory and its hybrid structures. This tool indicates the degree of similarity between two objects in a fuzzy environment. Various similarity measures of fuzzy sets and their extensions have been offered and they have been successfully applied in solving real-world problems such as decision making [39], [40], medical diagnosis [41], [42], and pattern recognition [43], [44]. In neutrosophic environment, Broumi and Smarandache [45] introduced the concept of similarity of NSs. Jun Ye [46] proposed the concept of similarity measures between interval neutrosophic sets. Following that, Mukherjee and Sarkar [47] studied several similarity measures on interval neutrosophic soft sets with an application in pattern recognition. Abu Qamar and Hassan [48] applied similarity and entropy tools to Q-NSS and they examined these tools in decision-making problems and a medical diagnosis.The development of the uncertainty sets under the similarity measures environment mentioned above is not restricted to the real field but developed in the complex field. Recently, researchers [49]- [51] made noteworthy additions to the literature on similarity measures environments by using hybridized models to handle the uncertainty of periodic data, where time plays a vital role in describing it. Following this trend and to make our concept (ICNSS) more useful in modeling some problems in real life, in this article the Hamming and Euclidean distances between two interval complex neutrosophic soft sets (ICNS sets) are defined and similarity measures between two ICNS sets based on these distances are presented. Similarity measures between two ICNS sets based on a set theoretic approach are also proposed in this article. An application in decision-making and medical diagnosis methods is established based

on the proposed similarity measures. The rest of this paper is organized in the following way: In Section 2, we recall the fundamental concepts related to interval complex neutrosophic soft sets. In Section 3, we develop some similarity measures of interval complex neutrosophic soft sets based on the distance measures: Hamming distance and Euclidean distance. In Section 4, we apply these similarity measures to a decision-making problem and a medical diagnosis with interval-valued complex neutrosophic soft information.A detailed comparison between the tools used in this work and other existing tools in section 5. Finally, the conclusions are offered in Section 6.

2. Preliminaries

Now, in this current section, we recapitulate the idea of soft set(SS) [14],neutrosophic set (NS) [52, 53], interval neutrosophic set (INS) [54], complex neutrosophic set [8] and show an overview of the I-CNSS model [36].

2.1. *Neutrosophic Set (NS)*

Definition 2.1. [52, 53] A N is a neutrosophic set (NS) on universe of a non-empty universe V and defined as $N = \{\langle v, T_N(v), I_N(v), F_N(v)\rangle\}$,where $T_N(v), I_N(v), F_N(v)$ are truth ,indeterminacy and falsity memberships respectively, such that $0 \leq T_N(v)+I_N(v)+F_N(v) \leq 3$.

2.2. *Interval-Neutrosophic Set (INS)*

Definition 2.2. [54]. An INS A defined on V is given by:
$A = \left\{\left(v, \tilde{T}_A(v), \tilde{I}_A(v), \tilde{F}_A(v)\right) : v \in V\right\}$ where,
$\tilde{T}_A(v) = \left[\tilde{p}^L_A(v), \tilde{p}^U_A(v)\right] \subseteq [0, 1]$,$\tilde{I}_A(v) = \left[\tilde{q}^L_A(v), \tilde{q}^U_A(v)\right] \subseteq [0, 1]$ and $\tilde{F}_A(v) = \left[\tilde{r}^L_A(v), \tilde{r}^U_A(v)\right] \subseteq [0, 1]$ represent the interval truth membership, interval indeterminacy membership, and interval non-membership degrees such that $0^- \leq \tilde{T}_A(v) + \tilde{I}_A(v) + \tilde{F}_A(v) \leq 3^+$ for all $v \in V$.

2.3. *Complex Neutrosophic Set (CNS)*

Definition 2.3. [8] Let V be an initial universe, E be a set of parameters and $A \subset E$. Let $P(V)$ denote the complex neutrosophic power set of V. Then, a pair $\left(\ddot{S}, A\right)$ is dubbed a complex neutrosophic set (CNS) on V, where $ddotF$ defined as a mapping $\tilde{F} : A \to P(V)$ such that

$$\ddot{S}(v) = \left\{\alpha, \left(\breve{T}_{\ddot{S}(\alpha)}(v)\ , \breve{I}_{\ddot{S}(\alpha)}(v)\, , \breve{F}_{\ddot{S}(\alpha)}(v)\right) : \alpha \in A \subset E, v \in V\right\},$$

where $\breve{T}_{\ddot{S}(\alpha)}(v)$,$\breve{I}_{\ddot{S}(\alpha)}(v)$ and $\breve{F}_{\ddot{S}(\alpha)}(v)$ are complex truth-membership function, complex indeterminacy-membership function and complex false-membership function and defined as bellow:

$\breve{T}_{\ddot{S}(\alpha)}(v) = \breve{p}_{\ddot{S}(\alpha)}(v).e^{j2\pi\breve{\mu}_{\ddot{S}(\alpha)}(v)}$,

$\breve{I}_{\ddot{S}(\alpha)}(v) = \breve{q}_{\ddot{S}(\alpha)}(v).e^{j2\pi\breve{\omega}_{\ddot{S}(\alpha)}(v)}$

and

$\breve{F}_{\ddot{S}(\alpha)}(v) = \breve{r}_{\ddot{S}(\alpha)}(v).e^{j2\pi\breve{\Phi}_{\ddot{S}(\alpha)}(v)}$.

Where,

$\breve{p}_{\ddot{S}(\alpha)}(v)$,$\breve{q}_{\ddot{S}(\alpha)}(v)$ and $\breve{r}_{\ddot{S}(\alpha)}(v)$ indicate to amplitude terms and $e^{j2\pi\breve{\mu}_{\ddot{S}(\alpha)}(v)}$,$e^{j2\pi\breve{\omega}_{\ddot{S}(\alpha)}(v)}$ and $e^{j2\pi\breve{\Phi}_{\ddot{S}(\alpha)}(v)}$ indicate to phase terms.

2.4. ***Soft Set (SS)***

Definition 2.4. [14] Let V be an initial universe and E be a set of parameters.Then, a pair $\left(\tilde{F}, E\right)$ is dubbed a soft set on V, where $\tilde{F}$ defined as a mapping $\tilde{F} : E \to P(V)$ such that $P(V)$ denotes the power of parameters set in V.

2.5. ***Interval-Complex Neutrosophic Soft Set (I-CNSS)***

Faisal et al. [36] introduced the idea of I-CNSS by combined both concepts SS and INS under complex setting to address two-dimensional indeterminate and incompatible data in periodic nature.

Definition 2.5. **[36]**.An I-CNSS $\left(\hat{G}, A\right)$ defined on V is a set given by:

$\left(\hat{G}, A\right) = \left\{\left\langle \alpha, \widehat{T}_{\hat{G}(\alpha)}(v)\ , \widehat{I}_{\hat{G}(\alpha)}(v), \widehat{F}_{\hat{G}(\alpha)}(v)\right\rangle : \alpha \in A \subseteq E, v \in V\right\}$.

Where,

$\widehat{T}_{\hat{G}(\alpha)}(v) = p_{\hat{G}(\alpha)}(v).e^{j\mu_{\hat{G}(\alpha)}(v)}$, $\widehat{I}_{\hat{G}(\alpha)}(v) = q_{\hat{G}(\alpha)}(v).e^{j\omega_{\hat{G}(\alpha)}(v)}$,

$\widehat{F}_{\hat{G}(\alpha)}(v) = r_{\hat{G}(\alpha)}(v).e^{j\Phi_{\hat{G}(\alpha)}(v)}$.

And the amplitude interval terms $p_{\hat{G}(\alpha)}(v)$,$q_{\hat{G}(\alpha)}(v)$,$r_{\hat{G}(\alpha)}(v)$ can be write as

$p_{\hat{G}(\alpha)}(v) = \left[p^L_{\hat{G}(\alpha)}(v), p^U_{\hat{G}(\alpha)}(v)\right]$

$q_{\hat{G}(\alpha)}(v) = \left[q^L_{\hat{G}(\alpha)}(v), q^U_{\hat{G}(\alpha)}(v)\right]$

$r_{\hat{G}(\alpha)}(v) = \left[r^L_{\hat{G}(\alpha)}(v), r^U_{\hat{G}(\alpha)}(v)\right]$

and for the phases interval terms $\mu_{\hat{G}(\alpha)}(v)$,$\omega_{\hat{G}(\alpha)}(v)$,$\Phi_{\hat{G}(\alpha)}(v)$ can be write as

$\mu_{\hat{G}(\alpha)}(v) = \left[\mu^L_{\hat{G}(\alpha)}(v), \mu^U_{\hat{G}(\alpha)}(v)\right]$

$\omega_{\hat{G}(\alpha)}(v) = \left[\omega^L_{\hat{G}(\alpha)}(v), \omega^U_{\hat{G}(\alpha)}(v)\right]$

$\Phi_{\hat{G}(\alpha)}(v) = \left[\Phi^L_{\hat{G}(\alpha)}(v), \Phi^U_{\hat{G}(\alpha)}(v)\right]$

where $p^L_{\hat{G}(\alpha)}(v), q^L_{\hat{G}(\alpha)}(v), r^L_{\hat{G}(\alpha)}(v), p^U_{\hat{G}(\alpha)}(v), q^U_{\hat{G}(\alpha)}(v), r^U_{\hat{G}(\alpha)}(v)$, $\mu^L_{\hat{G}(\alpha)}(v), \omega^L_{\hat{G}(\alpha)}(v), \Phi^L_{\hat{G}(\alpha)}(v), \mu^U_{\hat{G}(\alpha)}(v), \omega^U_{\hat{G}(\alpha)}(v)$,and $\Phi^U_{\hat{G}(\alpha)}(v)$ represent the lower and upper bound of amplitude interval terms and phases interval terms respectively.

3. Similarity measures based on distance measures between I-CNSSs

Now, first we present several distances in the interval-complex neutrosophic soft sets (I-CNSSs) case and support it with some numerical examples. Second, based on the proposed distance measures between I-CNSSs, we give the following definition of similarity measures:

Definition 3.1. Assume that $\xi = \left(\hat{G}, A\right)$, $\bar{\xi} = \left(\hat{K}, A\right)$and $\bar{\bar{\xi}} = \left(\hat{L}, A\right)$ are an interval-complex neutrosophic soft sets (I-CNSSs) on universe of discourse V. A function $d : I - CNSS\,(V) \times I - CNSS\,(V) \to [0,1]$is called distance measure I-CNSS(V) if d fulfill the following three axioms:

(1) $d\left(\xi, \bar{\xi}\right) \geq 0$ and $d\left(\xi, \bar{\xi}\right) = 0$ if and only if $\xi = \bar{\xi}$.
(2) $d\left(\xi, \bar{\xi}\right) = d\left(\bar{\xi}, \xi\right)$.
(3) $d\left(\xi, \bar{\bar{\xi}}\right) \leq d\left(\xi, \bar{\xi}\right) + d\left(\bar{\xi}, \bar{\bar{\xi}}\right)$.(triangle inequality)

Definition 3.2. If $V = \{v_1, v_2, ..., v_n\}$is a nonempty universal set, $A = \{\alpha_1, \alpha_2, ..., \alpha_m\}$being a parameters set. Then for all $\xi = \left(\hat{G}, A\right)$, $\bar{\xi} = \left(\hat{K}, B\right)$ are a I-CNSSs(V) and d a distance measure between I-CNSSs for all $\alpha_i \in E$. Then, the diverse distances between two I-CNSSs ξ and $\bar{\xi}$ are defined as follows:

(1) The Hamming distance measure:

$$d^H_{I-CNSS}\left(\xi, \bar{\xi}\right) = \frac{1}{12}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big\{\left|p^L_{G(\alpha_i)}(v_j) - p^L_{K(\alpha_i)}(v_j)\right| + \left|p^U_{G(\alpha_i)}(v_j) - p^U_{K(\alpha_i)}(v_j)\right| + \left|q^L_{G(\alpha_i)}(v_j) - q^L_{K(\alpha_i)}(v_j)\right| + \left|q^U_{G(\alpha_i)}(v_j) - q^U_{K(\alpha_i)}(v_j)\right| + \left|r^L_{G(\alpha_i)}(v_j) - r^L_{K(\alpha_i)}(v_j)\right| + \left|r^U_{G(\alpha_i)}(v_j) - r^U_{K(\alpha_i)}(v_j)\right| + \frac{1}{2\pi}\Big(\left|\mu^L_{G(\alpha_i)}(v_j) - \mu^L_{K(\alpha_i)}(v_j)\right| + \left|\mu^U_{G(\alpha_i)}(v_j) - \mu^U_{K(\alpha_i)}(v_j)\right| + \left|\varphi^L_{G(\alpha_i)}(v_j) - \varphi^L_{K(\alpha_i)}(v_j)\right| + \left|\varphi^U_{G(\alpha_i)}(v_j) - \varphi^U_{K(\alpha_i)}(v_j)\right| + \left|\omega^L_{G(\alpha_i)}(v_j) - \omega^L_{K(\alpha_i)}(v_j)\right| + \left|\omega^U_{G(\alpha_i)}(v_j) - \omega^U_{K(\alpha_i)}(v_j)\right|\Big)\Big\}$$

(2) The normalized Hamming distance measure:

$$d^{NH}_{I-CNSS}\left(\xi, \bar{\xi}\right) = \frac{d^H_{I-CNSS}(\xi, \bar{\xi})}{mn}.$$

(3) The Euclidean distance measure:

$$d^E_{I-CNSS}\left(\xi, \bar{\xi}\right) = \Bigg\{\frac{1}{12}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\left|p^L_{G(\alpha_i)}(v_j) - p^L_{K(\alpha_i)}(v_j)\right|^2 + \left|p^U_{G(\alpha_i)}(v_j) - p^U_{K(\alpha_i)}(v_j)\right|^2 + \left|q^L_{G(\alpha_i)}(v_j) - q^L_{K(\alpha_i)}(v_j)\right|^2 +$$

$$\left|q^U_{G(\alpha_i)}(v_j) - q^U_{K(\alpha_i)}(v_j)\right|^2 + \left|r^L_{G(\alpha_i)}(v_j) - r^L_{K(\alpha_i)}(v_j)\right|^2 + \left|r^U_{G(\alpha_i)}(v_j) - r^U_{K(\alpha_i)}(v_j)\right|^2 + \frac{1}{(2\pi)^2}\left(\left|\mu^L_{G(\alpha_i)}(v_j) - \mu^L_{K(\alpha_i)}(v_j)\right|^2 + \left|\mu^U_{G(\alpha_i)}(v_j) - \mu^U_{K(\alpha_i)}(v_j)\right|^2 + \left|\varphi^L_{G(\alpha_i)}(v_j) - \varphi^L_{K(\alpha_i)}(v_j)\right|^2 + \left|\varphi^U_{G(\alpha_i)}(v_j) - \varphi^U_{K(\alpha_i)}(v_j)\right|^2 + \left|\omega^L_{G(\alpha_i)}(v_j) - \omega^L_{K(\alpha_i)}(v_j)\right|^2 + \left|\omega^U_{G(\alpha_i)}(v_j) - \omega^U_{K(\alpha_i)}(v_j)\right|^2\right)\Bigg]\Bigg\}^{\frac{1}{2}}$$

(4) The normalized Euclidean distance measure:

$$d^{NE}_{I-CNSS}\left(\xi,\bar{\xi}\right) = \frac{d^{E}_{I-CNSS}\left(\xi,\bar{\xi}\right)}{\sqrt{mn}}.$$

Based on these distance measures, the following properties clearly hold:

(a) $0 \le d^{H}_{I-CNSS}\left(\xi,\bar{\xi}\right) \le mn$.
(b) $0 \le d^{NH}_{I-CNSS}\left(\xi,\bar{\xi}\right) \le 1$.
(c) $0 \le d^{E}_{I-CNSS}\left(\xi,\bar{\xi}\right) \le \sqrt{mn}$.
(d) $0 \le d^{NE}_{I-CNSS}\left(\xi,\bar{\xi}\right) \le 1$.

Theorem 3.3. *All the distance measures on I-CNSSs which are given in Definition 3.2 are distance functions of I-CNSSs.*

Proof. It is clear that all the distance measures presented in Definition 3.2 fulfil the mentioned conditions in Definition 3.1. Thus, tracking brevity, we only go to prove condition (3) (triangle inequality) for the Hamming distance measure.

Let $\xi, \bar{\xi}$ and $\bar{\bar{\xi}}$ be three I-CNSSs and for Hamming distance measure, we have:

$$d^{H}_{I-CNSS}\left(\xi,\bar{\xi}\right) + d^{H}_{I-CNSS}\left(\bar{\xi},\bar{\bar{\xi}}\right) =$$

$$\frac{1}{12}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big\{\left|p^L_{\hat{G}(\alpha_i)}(v_j) - p^L_{\hat{K}(\alpha_i)}(v_j)\right| + \left|p^U_{\hat{G}(\alpha_i)}(v_j) - p^U_{\hat{K}(\alpha_i)}(v_j)\right| + \left|q^L_{\hat{G}(\alpha_i)}(v_j) - q^L_{\hat{K}(\alpha_i)}(v_j)\right| + \left|q^U_{\hat{G}(\alpha_i)}(v_j) - q^U_{\hat{K}(\alpha_i)}(v_j)\right| + \left|r^L_{\hat{G}(\alpha_i)}(v_j) - r^L_{\hat{K}(\alpha_i)}(v_j)\right| + \left|r^U_{\hat{G}(\alpha_i)}(v_j) - r^U_{\hat{K}(\alpha_i)}(v_j)\right| + \frac{1}{2\pi}\Big(\left|\mu^L_{\hat{G}(\alpha_i)}(v_j) - \mu^L_{\hat{K}(\alpha_i)}(v_j)\right| + \left|\mu^U_{\hat{G}(\alpha_i)}(v_j) - \mu^U_{\hat{K}(\alpha_i)}(v_j)\right| + \left|\varphi^L_{\hat{G}(\alpha_i)}(v_j) - \varphi^L_{\hat{K}(\alpha_i)}(v_j)\right| + \left|\varphi^U_{\hat{G}(\alpha_i)}(v_j) - \varphi^U_{\hat{K}(\alpha_i)}(v_j)\right| + \left|\omega^L_{\hat{G}(\alpha_i)}(v_j) - \omega^L_{\hat{K}(\alpha_i)}(v_j)\right| + \left|\omega^U_{\hat{G}(\alpha_i)}(v_j) - \omega^U_{\hat{K}(\alpha_i)}(v_j)\right|\Big)\Big\} +$$

$$\frac{1}{12}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big\{\left|p^L_{\hat{K}(\alpha_i)}(v_j) - p^L_{\hat{L}(\alpha_i)}(v_j)\right| + \left|p^U_{\hat{K}(\alpha_i)}(v_j) - p^U_{\hat{L}(\alpha_i)}(v_j)\right| + \left|q^L_{\hat{K}(\alpha_i)}(v_j) - q^L_{\hat{L}(\alpha_i)}(v_j)\right| + \left|q^U_{\hat{K}(\alpha_i)}(v_j) - q^U_{\hat{L}(\alpha_i)}(v_j)\right| + \left|r^L_{\hat{K}(\alpha_i)}(v_j) - r^L_{\hat{L}(\alpha_i)}(v_j)\right| + \left|r^U_{\hat{K}(\alpha_i)}(v_j) - r^U_{\hat{L}(\alpha_i)}(v_j)\right| + \frac{1}{2\pi}\Big(\left|\mu^L_{\hat{K}(\alpha_i)}(v_j) - \mu^L_{\hat{L}(\alpha_i)}(v_j)\right| + \left|\mu^U_{\hat{K}(\alpha_i)}(v_j) - \mu^U_{\hat{L}(\alpha_i)}(v_j)\right| + \left|\varphi^L_{\hat{K}(\alpha_i)}(v_j) - \varphi^L_{\hat{L}(\alpha_i)}(v_j)\right| + \left|\varphi^U_{\hat{K}(\alpha_i)}(v_j) - \varphi^U_{\hat{L}(\alpha_i)}(v_j)\right| + \left|\omega^L_{\hat{K}(\alpha_i)}(v_j) - \omega^L_{\hat{L}(\alpha_i)}(v_j)\right| + \left|\omega^U_{\hat{K}(\alpha_i)}(v_j) - \omega^U_{\hat{L}(\alpha_i)}(v_j)\right|\Big)\Big\}$$

$$= \frac{1}{12}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big\{\left|p^L_{\hat{G}(\alpha_i)}(v_j) - p^L_{\hat{K}(\alpha_i)}(v_j)\right| + \left|p^L_{\hat{K}(\alpha_i)}(v_j) - p^L_{\hat{L}(\alpha_i)}(v_j)\right| + \ldots + \left|r^L_{\hat{G}(\alpha_i)}(v_j) - r^L_{\hat{K}(\alpha_i)}(v_j)\right| + \left|r^U_{\hat{K}(\alpha_i)}(v_j) - r^U_{\hat{L}(\alpha_i)}(v_j)\right| + + \frac{1}{2\pi}\Big(\left|\mu^L_{\hat{G}(\alpha_i)}(v_j) - \mu^L_{\hat{K}(\alpha_i)}(v_j)\right| + \left|\mu^L_{\hat{K}(\alpha_i)}(v_j) - \mu^L_{\hat{L}(\alpha_i)}(v_j)\right| + \ldots + \left|\omega^L_{\hat{G}(\alpha_i)}(v_j) - \omega^L_{\hat{K}(\alpha_i)}(v_j)\right| + \left|\omega^U_{\hat{K}(\alpha_i)}(v_j) - \omega^U_{\hat{L}(\alpha_i)}(v_j)\right|\Big)\Big\} \ge \frac{1}{12}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big\{\left|p^L_{\hat{G}(\alpha_i)}(v_j) - p^L_{\hat{L}(\alpha_i)}(v_j)\right| +$$

$$\left|p^{U}_{\hat{G}(\alpha_i)}(v_j) - p^{U}_{\hat{L}(\alpha_i)}(v_j)\right| + \left|q^{L}_{\hat{G}(\alpha_i)}(v_j) - q^{L}_{\hat{L}(\alpha_i)}(v_j)\right| + \left|q^{U}_{\hat{G}(\alpha_i)}(v_j) - q^{U}_{\hat{L}(\alpha_i)}(v_j)\right| + \left|r^{L}_{\hat{G}(\alpha_i)}(v_j) - r^{L}_{\hat{L}(\alpha_i)}(v_j)\right| + \left|r^{U}_{\hat{G}(\alpha_i)}(v_j) - r^{U}_{\hat{L}(\alpha_i)}(v_j)\right|$$
$$+\frac{1}{2\pi}\left(\left|\mu^{L}_{\hat{G}(\alpha_i)}(v_j) - \mu^{L}_{\hat{L}(\alpha_i)}(v_j)\right| + \left|\mu^{U}_{\hat{G}(\alpha_i)}(v_j) - \mu^{U}_{\hat{L}(\alpha_i)}(v_j)\right| + \left|\varphi^{L}_{\hat{G}(\alpha_i)}(v_j) - \varphi^{L}_{\hat{L}(\alpha_i)}(v_j)\right|\right.$$
$$\left.+\left|\varphi^{U}_{\hat{G}(\alpha_i)}(v_j) - \varphi^{U}_{\hat{L}(\alpha_i)}(v_j)\right| + \left|\omega^{L}_{\hat{G}(\alpha_i)}(v_j) - \omega^{L}_{\hat{L}(\alpha_i)}(v_j)\right| + \left|\omega^{U}_{\hat{G}(\alpha_i)}(v_j) - \omega^{U}_{\hat{L}(\alpha_i)}(v_j)\right|\right)\Big\}$$
$$= d^{H}_{I-CNSS}\left(\xi, \bar{\xi}\right)$$

□

Now, we will introduce the concept of a similarity measure between I-CNSSs. We can look into ambiguous data in interval-complex neutrosophic soft sets by using these measures.

Definition 3.4. Assume that $\xi = \left(\hat{G}, A\right)$, $\tilde{\xi} = \left(\hat{K}, A\right)$and $\tilde{\tilde{\xi}} = \left(\hat{L}, A\right)$are an interval-complex neutrosophic soft sets (I-CNSSs) on universe of discourse V. A function $S : I-CNSS\left(V\right) \times I-CNSS\left(V\right) \to [0,1]$is called similarity measure between I-CNSSs if S fulfill the following axioms:

(S1) $0 \leq S\left(\xi, \tilde{\xi}\right) \leq 1$,

(S2) $S\left(\xi, \tilde{\xi}\right) = 1$if and only if $\xi = \tilde{\xi}$,

(S3) $S\left(\xi, \tilde{\xi}\right) = S\left(\tilde{\xi}, \xi\right)$,

(S4) If$\xi \subseteq \tilde{\xi} \subseteq \tilde{\tilde{\xi}}$,then $S\left(\xi, \tilde{\tilde{\xi}}\right) \leq \min\left\{S\left(\xi, \tilde{\xi}\right), S\left(\tilde{\xi}, \tilde{\tilde{\xi}}\right)\right\}$.

Distance and similarity measures are related concepts in mathematics. Thus, we can use the proposed distances in definition 3.2 to describe similarity measures between I-CNSSs. As a result, we can provide several measures of similarity between I-CNSSs, as shown below.

- $S^{H}_{I-CNSS}\left(\xi, \tilde{\xi}\right) = \frac{1}{1+d^{H}_{I-CNSS}\left(\xi,\tilde{\xi}\right)}$.
- $S^{NH}_{I-CNSS}\left(\xi, \tilde{\xi}\right) = \frac{1}{1+d^{NH}_{I-CNSS}\left(\xi,\tilde{\xi}\right)}$.
- $S^{E}_{I-CNSS}\left(\xi, \tilde{\xi}\right) = \frac{1}{1+d^{E}_{I-CNSS}\left(\xi,\tilde{\xi}\right)}$.
- $S^{NE}_{I-CNSS}\left(\xi, \tilde{\xi}\right) = \frac{1}{1+d^{NE}_{I-CNSS}\left(\xi,\tilde{\xi}\right)}$.

Example 3.5. Let $\xi = \left(\hat{G}, A\right)$and $\tilde{\xi} = \left(\hat{K}, A\right)$ be two I-CNSSs on $V = \{v_1, v_2\}$and defined as follows:

$$\xi = \left\{\left\{\alpha_1, \left(\frac{\langle [0.4,0.6].e^{j2\pi[0.5,0.6]}, [0.1,0.7].e^{j2\pi[0.1,0.3]}, [0.3,0.5].e^{j2\pi[0.8,0.9]}\rangle}{v_1}\right),\right.\right.$$
$$\left.\left.\left(\frac{[0.2,0.4].e^{j2\pi[0.3,0.6]}, [0.1,0.1].e^{j2\pi[0.7,0.9]}, [0.5,0.9].e^{j2\pi[0.2,0.5]}}{v_2}\right)\right\}\right\}$$

and,

$$\tilde{\xi} = \left\{ \left\{ \alpha_1, \left(\frac{\langle [0.2,0.7].e^{j2\pi[0.7,0.8]},[0.4,0.9].e^{j2\pi[0.3,0.5]},[0.6,0.8].e^{j2\pi[0.5,0.6]} \rangle}{v_1} \right), \left(\frac{[0.15,0.52].e^{j2\pi[0.1,0.3]},[0,0.5].e^{j2\pi[0.6,0.8]},[0.3,0.3].e^{j2\pi[0.6,0.7]}}{v_2} \right) \right\} \right\}$$

Now, by Definition 3.2, we have the following results:

$$d^{H}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \frac{1}{12}\left(|0.4-0.2| + |0.6-0.7| + |0.1-0.4| + |0.7-0.9| + |0.3-0.6| + |0.5-0.8| + \frac{1}{2\pi}\left(|0.5\pi-0.7\pi| + |0.6\pi-0.8\pi| + |0.1\pi-0.3\pi| + |0.3\pi-0.5\pi| + |0.8\pi-0.5\pi| + |0.9\pi-0.6\pi|\right)\right) + \frac{1}{12}\left(|0.2-0.15| + |0.4-0.52| + |0.1-0| + |0.1-0.5| + |0.5-0.3| + |0.9-0.3| + \frac{1}{2\pi}\left(|0.3\pi-0.1\pi| + |0.6\pi-0.3\pi| + |0.7\pi-0.6\pi| + |0.9\pi-0.8\pi| + |0.2\pi-0.6\pi| + |0.5\pi-0.7\pi|\right)\right) = 0.350$$

and

$$d^{NH}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \frac{0.350}{4} = 0.0875$$

$$d^{E}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \frac{1}{12}\left(\left(|0.4-0.2|^2 + |0.6-0.7|^2 + |0.1-0.4|^2 + |0.7-0.9|^2 + |0.3-0.6|^2 + |0.5-0.8|^2 + \frac{1}{4\pi^2}\left(|0.5\pi-0.7\pi|^2 + |0.6\pi-0.8\pi|^2 + |0.1\pi-0.3\pi|^2 + |0.3\pi-0.5\pi|^2 + |0.8\pi-0.5\pi|^2 + |0.9\pi-0.6\pi|^2\right)\right) + \frac{1}{12}\left(|0.2-0.15|^2 + |0.4-0.52|^2 + |0.1-0|^2 + |0.1-0.5|^2 + |0.5-0.3|^2 + |0.9-0.3|^2 + \frac{1}{4\pi^2}\left(|0.3\pi-0.1\pi|^2 + |0.6\pi-0.3\pi|^2 + |0.7\pi-0.6\pi|^2 + |0.9\pi-0.8\pi|^2 + |0.2\pi-0.6\pi|^2 + |0.5\pi-0.7\pi|^2\right)\right)\right)^{\frac{1}{2}} = 0.209$$

. and

$$d^{NE}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \frac{0.209}{2} = 0.1045$$

Now, by Equations in definition 3.4, respectively, we get the similarity between two I-CNSSs as following:

$$S^{H}_{I-CNSS}\left(\xi,\bar{\xi}\right) = \frac{1}{1+0.350} = 0.741,\ S^{NH}_{I-CNSS}\left(\xi,\bar{\xi}\right) = \frac{1}{1+0.0875} = 0.919.$$
$$S^{E}_{I-CNSS}\left(\xi,\bar{\xi}\right) = \frac{1}{1+0.209} = 0.827,\ S^{NE}_{I-CNSS}\left(\xi,\bar{\xi}\right) = \frac{1}{1+0.1045} = 0.905.$$

Nonetheless, in practice, a different weight may have been assigned to the different sets. i.e, $\exists w_i \geq 0,\ i = 1, 2, ...m$, and $\sum_{i=1}^{m} w_i = 1$,for each element $v_i \in V$.Therefore, in this work, we will propose the weighted Hamming and Euclidean distance measures for I-CNSSs.

- The weighted Hamming distance measure:

$$d^{wH}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \frac{1}{12}\sum_{j=1}^{n}\sum_{i=1}^{m} w_i \Big\{ \left|p^L_{G(\alpha_i)}(v_j) - p^L_{K(\alpha_i)}(v_j)\right| + \left|p^U_{G(\alpha_i)}(v_j) - p^U_{K(\alpha_i)}(v_j)\right| + \left|q^L_{G(\alpha_i)}(v_j) - q^L_{K(\alpha_i)}(v_j)\right| + \left|q^U_{G(\alpha_i)}(v_j) - q^U_{K(\alpha_i)}(v_j)\right| + \left|r^L_{G(\alpha_i)}(v_j) - r^L_{K(\alpha_i)}(v_j)\right| + \left|r^U_{G(\alpha_i)}(v_j) - r^U_{K(\alpha_i)}(v_j)\right| + \frac{1}{2\pi}\Big(\left|\mu^L_{G(\alpha_i)}(v_j) - \mu^L_{K(\alpha_i)}(v_j)\right| + \left|\mu^U_{G(\alpha_i)}(v_j) - \mu^U_{K(\alpha_i)}(v_j)\right| + \left|\varphi^L_{G(\alpha_i)}(v_j) - \varphi^L_{K(\alpha_i)}(v_j)\right| + \left|\varphi^U_{G(\alpha_i)}(v_j) - \varphi^U_{K(\alpha_i)}(v_j)\right| + \left|\omega^L_{G(\alpha_i)}(v_j) - \omega^L_{K(\alpha_i)}(v_j)\right| + \left|\omega^U_{G(\alpha_i)}(v_j) - \omega^U_{K(\alpha_i)}(v_j)\right| \Big) \Big\}$$

and

- The normalized weighted Hamming distance measure:

$$d^{nwH}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \frac{d^{wH}_{I-CNSS}\left(\xi,\bar{\xi}\right)}{mn}.$$

- The weighted Euclidean distance measure:

$$d^{wE}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \Big\{ \frac{1}{12}\sum_{j=1}^{n}\sum_{i=1}^{m} w_i \Big[\left|p^L_{G(\alpha_i)}(v_j) - p^L_{K(\alpha_i)}(v_j)\right|^2 + \left|p^U_{G(\alpha_i)}(v_j) - p^U_{K(\alpha_i)}(v_j)\right|^2 + \left|q^L_{G(\alpha_i)}(v_j) - q^L_{K(\alpha_i)}(v_j)\right|^2 + \left|q^U_{G(\alpha_i)}(v_j) - q^U_{K(\alpha_i)}(v_j)\right|^2 + \left|r^L_{G(\alpha_i)}(v_j) - r^L_{K(\alpha_i)}(v_j)\right|^2 + \left|r^U_{G(\alpha_i)}(v_j) - r^U_{K(\alpha_i)}(v_j)\right|^2 + \frac{1}{(2\pi)^2}\Big(\left|\mu^L_{G(\alpha_i)}(v_j) - \mu^L_{K(\alpha_i)}(v_j)\right|^2 + \left|\mu^U_{G(\alpha_i)}(v_j) - \mu^U_{K(\alpha_i)}(v_j)\right|^2 + \left|\varphi^L_{G(\alpha_i)}(v_j) - \varphi^L_{K(\alpha_i)}(v_j)\right|^2 + \left|\varphi^U_{G(\alpha_i)}(v_j) - \varphi^U_{K(\alpha_i)}(v_j)\right|^2 + \left|\omega^L_{G(\alpha_i)}(v_j) - \omega^L_{K(\alpha_i)}(v_j)\right|^2 + \left|\omega^U_{G(\alpha_i)}(v_j) - \omega^U_{K(\alpha_i)}(v_j)\right|^2 \Big) \Big] \Big\}^{\frac{1}{2}}$$

- The normalized weighted Euclidean distance measure:

$$d^{nwE}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \frac{d^{wE}_{I-CNSS}\left(\xi,\bar{\xi}\right)}{mn}.$$

Also, the similarity measure on weighted Hamming and Euclidean distance measures for I-CNSSs defined as following:

- $S^{wH}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \frac{1}{1+d^{wH}_{I-CNSS}\left(\xi,\bar{\xi}\right)}.$
- $S^{wnH}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \frac{1}{1+d^{nwH}_{I-CNSS}\left(\xi,\bar{\xi}\right)}.$

- $S^{wE}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \frac{1}{1+d^{wE}_{I-CNSS}\left(\xi,\tilde{\xi}\right)}$.
- $S^{wnE}_{I-CNSS}\left(\xi,\tilde{\xi}\right) = \frac{1}{1+d^{wnE}_{I-CNSS}\left(\xi,\tilde{\xi}\right)}$.

Theorem 3.6. *If S be the similarity measure between two I-CNSSs $\xi, \tilde{\xi}$. Then,*
1. $S\left(\xi,\tilde{\xi}\right) = S\left(\tilde{\xi},\xi\right)$.
2. $0 \leq S\left(\xi,\tilde{\xi}\right) \leq 1$.
3. $0 \leq S\left(\xi,\tilde{\xi}\right) = 0$ *iff* $\xi = \tilde{\xi}$.

Proof. Immediately follows from definitions 3.4. □

4. **Applications**

I-CNSS is a hybrid tool for modeling two-dimensional information of a periodic nature in our everyday lives. In this section, we introduce some practical examples of I-CNSSs to show that the proposed similarity measures play an important role in solving some real-life problems such as decision-making problems and medical diagnosis.

4.1. *Similarity Measures of I-CNSSs Applied to Medical Diagnosis*

In this current subsection, we developed an algorithm based on the Hamming similarity measure of two I-CNS sets to evaluate the possibility that a sick person having related symptoms is suffering from typhoid. This algorithm depends on data described by two I-CNSS models, and it is built with the assistance of a medicinal master person. where the first I-CNSS indicates illness, and the second I-CNSS indicates the ill person. Then we find the similarity measure of these two I-CNS sets. We also think that the person may have typhoid if the similarity measure between these two I-CNS sets is greater than or equal to 0.6, which can be fixed with the help of a medical professional.

4.1.1. *Algorithm*

Step 1: Construct a model I-CNSS over the universe V for illness, which may be developed with the aid of a medical expert person.
Step 2: Build I-CNSS over the universe V for the patient person by helping a medical expert person.
Step 3: Compute Hamming distance between the model I-CNSS for illness and the I-CNSS for the patient person.
Step 4: Compute similarity Hamming measure between the I-CNSS for illness and the I-CNSS for the patient person.

Step 5: If the similarity Hamming measure between two I-CNSSs is greater than or equal to 0.6, then the person may possibly be suffering from typhoid, while if the similarity Hamming measure between two I-CNSSs is less than 0.6, then the person may not possibly be suffering from typhoid.

Now, we give a numerical example that shows a way out of these diagnosis problems from a scientific point of view. This is done to show how the above-proposed algorithm can be used to tell a patient if they have typhoid or not.

Example 4.1. Let $V = \{v_1 = \text{typhoid}, v_2 = \text{not typhoid}\}$, and $E = \{\alpha_1 = \text{flu}, \alpha_2 = \text{headache}, \alpha_3 = \text{body pain}\}$ be set of parameters which consist of symptoms of typhoid disease.

Step 1: Construct the model I-CNSS for typhoid:

$$\xi = \left\{\left\{\alpha_1, \left(\frac{\langle[0.3,0.4].e^{j2\pi[0.4,0.5]},[0.4,0.5].e^{j2\pi[0.2,0.3]},[0.6,0.7].e^{j2\pi[0.1,0.2]}\rangle}{v_1}\right), \left(\frac{\langle[0.6,0.7].e^{j2\pi[0.2,0.3]},[0.2,0.3].e^{j2\pi[0.6,0.7]},[0.1,0.2].e^{j2\pi[0.6,0.8]}\rangle}{v_2}\right)\right\}, \left\{\alpha_2, \left(\frac{\langle[0.3,0.4].e^{j2\pi[0.7,0.8]},[0.0,0.3].e^{j2\pi[0.4,0.6]},[0.3,0.4].e^{j2\pi[0.4,0.5]}\rangle}{v_1}\right), \left(\frac{\langle[0.5,0.6].e^{j2\pi[0.2,0.3]},[0.5,0.6].e^{j2\pi[0.1,0.2]},[0.0,0.1].e^{j2\pi[0.2,0.4]}\rangle}{v_2}\right)\right\}, \left\{\alpha_3, \left(\frac{\langle[0.4,0.6].e^{j2\pi[0.5,0.6]},[0.1,0.7].e^{j2\pi[0.1,0.3]},[0.3,0.5].e^{j2\pi[0.8,0.9]}\rangle}{v_1}\right), \left(\frac{[0.6,0.7].e^{j2\pi[0.4,0.6]},[0.1,0.1].e^{j2\pi[0.7,0.9]},[0.5,0.9].e^{j2\pi[0.2,0.5]}}{v_2}\right)\right\}\right\}$$

Step 2: Create two models of I-CNSS for patients X and Y, respectively, as:

$$\tilde{\xi} = \left\{\left\{\alpha_1, \left(\frac{\langle[0.3,0.45].e^{j2\pi[0.35,0.5]},[0.3,0.5].e^{j2\pi[0.25,0.35]},[0.5,0.7].e^{j2\pi[0.1,0.2]}\rangle}{v_1}\right), \left(\frac{\langle[0.2,0.4].e^{j2\pi[0.6,0.7]},[0.05,0.3].e^{j2\pi[0.5,0.7]},[0.3,0.4].e^{j2\pi[0.6,0.65]}\rangle}{v_2}\right)\right\}, \left\{\alpha_2, \left(\frac{\langle[0.05,0.2].e^{j2\pi[0.4,0.5]},[0.1,0.2].e^{j2\pi[0.5,0.7]},[0.1,0.3].e^{j2\pi[0.3,0.3]}\rangle}{v_1}\right), \left(\frac{\langle[0.25,0.4].e^{j2\pi[0.5,0.6]},[0.6,0.6].e^{j2\pi[0.2,0.25]},[0.25,0.3].e^{j2\pi[0.4,0.8]}\rangle}{v_2}\right)\right\}, \left\{\alpha_3, \left(\frac{\langle[0.2,0.5].e^{j2\pi[0.4,0.5]},[0,0.3].e^{j2\pi[0.14,0.2]},[0.1,0.3].e^{j2\pi[0.5,0.7]}\rangle}{v_1}\right), \left(\frac{[0.3,0.5].e^{j2\pi[0.2,0.4]},[0.2,0.2].e^{j2\pi[0.3,0.5]},[0.6,0.7].e^{j2\pi[0.3,0.6]}}{v_2}\right)\right\}\right\}$$

$$\tilde{\tilde{\xi}} = \left\{ \left\{ \alpha_1, \left(\frac{\langle [0.8,0.9].e^{j2\pi[0.3,0.6]},[0.3,0.7].e^{j2\pi[0.4,0.6]},[0.2,0.3].e^{j2\pi[0.4,0.5]} \rangle}{v_1} \right), \left(\frac{\langle [0.1,0.5].e^{j2\pi[0.2,0.4]},[0,0.3].e^{j2\pi[0.3,0.3]},[0.2,0.3].e^{j2\pi[0.2,0.5]} \rangle}{v_2} \right) \right\}, \left\{ \alpha_2, \left(\frac{\langle [0.5,0.6].e^{j2\pi[0.4,0.4]},[0.1,0.2].e^{j2\pi[0.2,0.3]},[0.2,0.4].e^{j2\pi[0.1,0.4]} \rangle}{v_1} \right), \left(\frac{\langle [0.1,0.3].e^{j2\pi[0.5,0.6]},[0.4,0.4].e^{j2\pi[0.1,0.2]},[0.3,0.5].e^{j2\pi[0.2,0.5]} \rangle}{v_2} \right) \right\}, \left\{ \alpha_3, \left(\frac{\langle [0.3,0.6].e^{j2\pi[0.2,0.3]},[0.1,0.1].e^{j2\pi[0.3,0.4]},[0.4,0.5].e^{j2\pi[0.4,0.6]} \rangle}{v_1} \right), \left(\frac{[0.6,0.6].e^{j2\pi[0.3,0.2]},[0.1,0.4].e^{j2\pi[0.2,0.4]},[0.3,0.5].e^{j2\pi[0.2,0.6]}}{v_2} \right) \right\} \right\}$$

In $\bar{\xi}$ and $\tilde{\tilde{\xi}}$ above, which are based on the physician's report, the amplitude term of the lower and upper bounds of the complex interval truth membership function denotes the strength and intensity of the symptoms that the patient suffers, and the phase term of the lower and upper bounds of the complex interval truth membership function denotes the period of the symptoms. At the same time, the amplitude term of the lower and upper bounds of the complex interval indeterminacy membership function means the inability to indeterminate these symptoms during the period mentioned in the phase term. The lower and upper bounds of the complex interval falsity membership function denote the absence of these symptoms during the period mentioned in the phase term of the lower and upper bounds of the complex interval falsity membership function. Here it is necessary to point out that the values of the amplitude term closer to 1 describe heavy symptoms, and values of the amplitude term closer to zero represent moderate symptoms. The period of the phase term is defined as weeks, so for values equal to 1, the period will be maximum.

Step 3: Based on Definition 3.2, the Hamming distance between $\xi$and $\tilde{\xi}$is 0.725 while between $\xi$and $\tilde{\tilde{\xi}}$is 0.821.

Step 4: Based on Definition 3.4, the similarity Hamming measure between $\xi$and $\tilde{\xi}$is 0.579 while between $\xi$and $\tilde{\tilde{\xi}}$is 0.549.

Step 4: Here $S^H_{I-CNSS}\left(\xi,\tilde{\xi}\right) \prec 0.6$and $S^H_{I-CNSS}\left(\xi,\tilde{\tilde{\xi}}\right) \prec 0.6$ that means both patients X and Y may possibly suffer from no typhoid.

Remark 4.2. In case no such conclusion can be drawn with the given information then we need to reassess all the symptoms with the help of expert and then repeat all the steps proposed in I-CNSSM-algorithm

4.2. *Similarity Measures of I-CNSSs Applied to Multicriteria Decision Making*

In this section, we developed an algorithm based on similarity measures of interval complex neutrosophic soft sets as defined in definition 3.4 for possible application in multicriteria decision making.

4.2.1. *Algorithm*

Step 1: Construct a model I-CNSS $\xi = (K, A)$ over the universe V depends on the opinion of one of the experts and the customer satisfaction rate.
Step 2: Compute the distance measured between the optimality choice $\left([1,1]\,.e^{2\pi[1,1]}\right)$ and $\alpha_i(\text{for} i = 1,2,3)$ with the weighting vector w.
Step 3: Compute similarity measure for all the distances we got in **Step 2:**.
Step 4: Analyze the result and the decision is to select the alternative which has the highest similarity to the optimality of I-CNSS.

Example 4.3. Suppose a customer wishes to buy a new computer for his personal usage. There are four computers (four alternatives) that can be represented by $V = \{v_1, v_2, v_3, v_4\}$, a customer can pick out one of them. The customer considers three features (attributes) in his choice, namely performance, service, and price, which can be represented by $A = \{\alpha_1, \alpha_2, \alpha_3\}$ respectively with the weight vectors $w = \{0.3, 0.3, 0.4\}$. To help the customer choose the right apparatus, we will apply the proposed algorithm.

Step 1: Construct the model I-CNSS for the opinion of one of the experts and the customer satisfaction rate:

$$\xi =$$

$$\left\{\left\{\alpha_1, \left(\frac{\left\langle[0.8,0.9].e^{2\pi[0.7,0.8]},[0.2,0.3].e^{2\pi[0.1,0.2]},[0.2,0.4].e^{2\pi[0.2,0.3]}\right\rangle}{v_1}\right), \left(\frac{\left\langle[0.4,0.65].e^{j2\pi[0.55,0.7]},[0.1,0.4].e^{j2\pi[0.15,0.4]},[0.4,0.6].e^{j2\pi[0.2,0.25]}\right\rangle}{v_2}\right)\right.\right.$$

$$\left.\left(\frac{\left\langle[0.3,0.7].e^{j2\pi[0.6,0.8]},[0.4,0.5].e^{j2\pi[0.3,0.6]},[0.5,0.8].e^{j2\pi[0.6,0.64]}\right\rangle}{v_3}\right) \left(\frac{\left\langle[0.2,0.4].e^{j2\pi[0.6,0.7]},[0.05,0.3].e^{j2\pi[0.5,0.7]},[0.3,0.4].e^{j2\pi[0.6,0.65]}\right\rangle}{v_4}\right)\right\},$$

$$\left\{\alpha_2, \left(\frac{\left\langle[0.5,0.6].e^{j2\pi[0.4,0.5]},[0.1,0.2].e^{j2\pi[0.5,0.7]},[0.1,0.3].e^{j2\pi[0.3,0.3]}\right\rangle}{v_1}\right), \left(\frac{\left\langle[0.3,0.45].e^{j2\pi[0.35,0.5]},[0.3,0.5].e^{j2\pi[0.25,0.35]},[0.5,0.7].e^{j2\pi[0.1,0.2]}\right\rangle}{v_2}\right)\right.$$

$$\left.\left(\frac{\left\langle[0.2,0.5].e^{j2\pi[0.25,0.3]},[0.4,0.7].e^{j2\pi[0.45,0.5]},[0.3,0.6].e^{j2\pi[0.2,0.35]}\right\rangle}{v_3}\right) \left(\frac{\left\langle[0.25,0.4].e^{j2\pi[0.5,0.6]},[0.6,0.6].e^{j2\pi[0.2,0.25]},[0.25,0.3].e^{j2\pi[0.4,0.8]}\right\rangle}{v_4}\right)\right\},$$

$$\left\{\alpha_3, \left(\frac{\left\langle[0.2,0.5].e^{j2\pi[0.4,0.5]},[0,0.3].e^{j2\pi[0.14,0.2]},[0.1,0.3].e^{j2\pi[0.5,0.7]}\right\rangle}{v_1}\right), \left(\frac{\left\langle[0.6,0.6].e^{j2\pi[0.3,0.6]},[0.33,0.4].e^{j2\pi[0.51,0.61]},[0.4,0.8].e^{j2\pi[0.2,0.5]}\right\rangle}{v_2}\right),\right.$$

$$\left.\left.\left(\frac{\left\langle[0.3,0.45].e^{j2\pi[0.35,0.5]},[0.4,0.5].e^{j2\pi[0.18,0.5]},[0.6,0.9].e^{j2\pi[0.3,0.5]}\right\rangle}{v_3}\right), \left(\frac{[0.3,0.5].e^{j2\pi[0.2,0.4]},[0.2,0.2].e^{j2\pi[0.3,0.5]},[0.6,0.7].e^{j2\pi[0.3,0.6]}}{v_4}\right)\right\}\right\}$$

In I-CNSS model $\xi = (K, A)$the interval complex neutrosophic soft values clarify the expert opinion about the attributes of all alternatives. For instance, in the interval-complex neutrosophic soft value of the alternative v_1 under α_1 attribute $[0.8, 0.9]\,.e^{2\pi[0.7,0.8]}$, $[0.2, 0.3]\,.e^{2\pi[0.1,0.2]}$, $[0.2, 0.4]\,.e^{2\pi[0.2,0.3]}$ the truth interval membership function $[0.8, 0.9]\,.e^{2\pi[0.7,0.8]}$show that the expert here means that the performance of the computer v_1 is marked by an amplitude interval value of 0.8 to 0.9, and this percentage confirms that this computer is characterized by high performance, while the phase interval value of 0.7 to 0.8 indicate that the customer satisfied degree between 70% to 80% . While the indeterminacy interval membership function $[0.2, 0.3]\,.e^{2\pi[0.1,0.2]}$ reveal that the expert cannot determine if this computer has high performance or not by degree between 0.2 to 0.3, and the phase interval value indicates the degree of confusion of the customer for this device between 10% to 20%. Also, for the falsity interval membership function $[0.2, 0.4]\,.e^{2\pi[0.2,0.3]}$ show that the expert is unsatisfied with this device by a degree ranging from 0.2 to 0.4, and the customer is unsatisfied by a degree of 20% to 30%.

Step 2: Compute the distance measured between the optimality choice $\left([1, 1]\,.e^{2\pi[1,1]}\right)$ and α_i(for i=1,2,3) with the weighting vector $w = \{0.3, 0.3, 0.4\}$ and obtain the results as shown in Table 1.

TABLE 1. Distance Measures results

V_i	d^H	d^E	d^{nH}	d^{nE}	d^{nwH}	d^{nwE}
v_1	1.439	0.863	0.119	0.253	0.4848	0.253
v_2	1.268	0.760	0.105	0.223	0.418	0.228
v_3	1.166	0.699	0.097	0.205	0.388	0.203
v_4	1.357	0.814	0.113	0.239	0.453	0.236

Step 3: Compute similarity measure for all the distances we got in **Table 1**.

TABLE 2. Similarity Measures results

V_i	S^H	S^E	S^{nH}	S^{nE}	S^{nwH}	S^{nwE}
v_1	0.410	0.536	0.893	0.798	0.673	0.798
v_2	0.440	0.568	0.904	0.817	0.705	0.814
v_3	0.461	0.588	0.911	0.829	0.720	0.831
v_4	0.424	0.551	0.898	0.807	0.688	0.809

Table 3. Ordering of the given alternatives.

	Ordering.
S^H	$v_3 \succ v_2 \succ v_4 \succ v_1$
S^E	$v_3 \succ v_2 \succ v_4 \succ v_1$
S^{nH}	$v_3 \succ v_2 \succ v_4 \succ v_1$
S^{nE}	$v_3 \succ v_2 \succ v_4 \succ v_1$
S^{nwH}	$v_3 \succ v_2 \succ v_4 \succ v_1$
S^{nwE}	$v_3 \succ v_2 \succ v_4 \succ v_1$

Step 4: From Table 3, we analyse the similarity values in Table 2, which we got based on the distance measures given in Table 1. Clearly, the most useful alternative is v_3, which is the one with the highest similarity to the ideal choice.

5. Comparison with Existing Models/Methods in Literature

This work is based on the similarity measures of interval-complex neutrosophic soft sets (I-CNSSs) and employs these measures to solve some real-life applications such as decision making and medical diagnosis under uncertainty. In this section, we compare the I-CNSS model to other similar models in the literature based on the characteristics in Table 4.

Table 4. Characteristic comparison of the I-CNSS with other variants.

Methods	Uncertainty	Three membership function	Parameterization	Interval form	Periodicity
[52]	**T**	**T**	**F**	**F**	**F**
[55]	**T**	**F**	**T**	**T**	**F**
[20]	**T**	**T**	**T**	**T**	**F**
[50]	**T**	**F**	**T**	**F**	**T**
Our model:I-CNSS	**T**	**T**	**T**	**T**	**T**

On the other hand, compared with INSSM [20], which used the INSS setting to describe the decision-making information, our suggested I-CNSS is a new approach created to conceptualize uncertainty issues that have two dimensions. From Example 4.1, it can be noted that the concept of INSS cannot cover the factors affecting the problem (symptoms severity and period of symptoms) in two stages simultaneously. Because it is not adapted to deal with two-dimensional issues, i.e., it doesn't have enough tools to do that. But the I-CNSS model can put the phase and amplitude terms together and can be used to represent these two variables together. So, we can say that the INSSM can't directly handle the problem given above with two-dimensional information in this way. Otherwise, we can say that the INSS model is a particular case of our model I-CNSS and can be conceptualized in the form of I-CNSS. In other words, the INSS is an I-CNSS with phase terms equal to zero. For example, the INSS. On the other hand, we can say that the INSS model is a particular case of our model I-CNSS and can be conceptualized in the form of I-CNSS. In other words, the INSS is an I-CNSS with phase terms equal to zero. For example, the INSS $([0.3, 0.7], [0.1, 0.4], [0.5, 0.6])$ can be represented as $\left([0.3, 0.7].e^{j2\pi[0,0]}, [0.1, 0.4].e^{j2\pi[0,0]}, [0.5, 0.6].e^{j2\pi[0,0]}\right)$ employing I-CNSS. Furthermore, our approach I-CNSS, is appropriate for other methods such as interval intuitionistic fuzzy soft problems, since interval intuitionistic fuzzy soft sets are a particular case of INSS and I-CNSS. For example, the interval intuitionistic fuzzy soft value $([0.3, 0.3], [0.5, 0.5])$ can be written as $([0.3, 0.3], [0.5, 0.5], [0.2, 0.2])$ employing INSS and hence can be written as $\left([0.3, 0.3].e^{j2\pi[0,0]}, [0.5, 0.5].e^{j2\pi[0,0]}, [0.2, 0.2].e^{j2\pi[0,0]}\right)$ using I-CNSS, since the sum of the degrees of lower and upper interval true membership, lower and upper interval nonmembership, and lower and upper interval indeterminacy membership of an interval intuitionistic fuzzy value is equal to $([1, 1])$.Note that the lower and upper interval indeterminacy degree in an interval intuitionistic fuzzy set is provided by default. It cannot be defined alone, unlike the interval neutrosophic set, where the lower and upper interval indeterminacy membership are determined independently and quantified explicitly.As for the CVSS model [50], this model is based on a vague set model and also misses indeterminacy membership. Therefore, it is difficult for this model to deal with the data of the problem entirely like our proposed model.

5.1. ***Pros of I-CNSS model***

Based on all of the above, our proposed method has particular advantages. Firstly, the main feature of I-CNSS is the presence of a phase and its membership in the form of an interval. Researchers realized that the time period is an important factor along with the membership value so that decision-makers can make the real decision, and it is more reliable and more acceptable than the other existing theories in which there is no scope for considering time-period. So, this new concept provides more scope for the decision-makers to make real decisions with more feasibility. Secondly, a practical formula is utilised to convert the I-CNSVs (complex stats) to the INSVs (real stats), which sustains the entirety of the original data without diminishing or distorting them. Thirdly, our technique allows for decision-making using a simple computational procedure that does not require the use of directed operations on complex numbers. Finally, the I-CNSS that is used in our approach has the capability to handle the imprecise, indeterminate, inconsistent, and incomplete information that is captured by the amplitude terms and phase terms simultaneously. As a result, the proposed method is capable of dealing with more uncertain data.

- Established on all that is mentioned in this comparison, we see that the similarity measures based on I-CNSS that are given in this work are more effective in dealing with uncertainty issues than other concepts mentioned in the literature.

6. Conclusion

Al-Sharqi et al. [36] established the idea of ICNSS as a substantial and essential generalization of the soft set to deal with the uncertain, inconsistent, and incomplete information in periodic data. In this paper, we have proposed several distance measures in the case of the interval complex neutrosophic soft sets, which are very helpful in dealing with the two-dimensional data in some real-world applications. Based on these distance measures, we introduced an axiomatic definition of similarity measure to measure the degree of fuzzy information in interval complex neutrosophic soft sets. Moreover, a numerical example is given, and relations between this similarity measure and these distances are introduced and verified. In addition, a proposed algorithm based on these measures has been built and applied in some daily life applications like decision-making problems and medical diagnoses. Finally, a comparison between the existing methods and I-CNSS was given, and some features of I-CNSS were revealed. In future possible research, we can extend from soft to hypersoft set [56] (by transforming the function F into a multi-attribute function). We also want to combine these measures with other kinds of algebraic structures, such as group structures [57]- [59], ring structures [60]- [62], and topological structures [63]- [65].

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On 2-SuperHyperLeftAlmostSemihypergroups

Pairote Yiarayong[1,*]
[1]Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand; E-mail: pairote0027@hotmail.com
[*]Correspondence: pairote0027@hotmail.com

Abstract. The aim of this paper is to extend the concept of hyperideals to the SuperHyperAlgebras. In this paper, we introduce the concept of 2-SuperHyperLeftAlmostSemihypergroups which is a generalization of $\mathcal{LA}$-semihypergroups. Furthermore, we define and study 2-SuperHyper-$\mathcal{LA}$-subsemihypergroups, SuperHyper-Left(Right)HyperIdeals and SuperHyperHyperIdeals of 2-SuperHyperLeftAlmostSemihypergroups, and related properties are investigated. We give an example to show that in general these two notions are different. Finally, we show that every SuperHyperRightHyperIdeal of 2-SuperHyper-$\mathcal{LA}$-semihypergroup S with pure left identity is SuperHyperHyperIdeal.

Keywords: SuperHyperAlgebra; $\mathcal{LA}$-subsemihypergroup; 2-SuperHyperLeftAlmostSemihypergroup; SuperHyperHyperIdeals; SuperHyperLeft(Right)HyperIdeal.

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1. **Introduction**

The concept of left almost semihypergroups ($\mathcal{LA}$-semihypergroups), which is a generalization of $\mathcal{LA}$-semigroups and semihypergroups, was introduced by Hila and Dine [9] in 2011. They defined the concept of hyperideals and bi-hyperideals in $\mathcal{LA}$-semihypergroups. Until now, $\mathcal{LA}$-semihypergroups have been applied to many fields [2, 4–6, 13, 16, 18]. In 2013, Yaqoob et al. [17] have characterized intra-regular $\mathcal{LA}$-semihypergroups by using the properties of their left and right hyperideals and investigated some useful conditions for an $\mathcal{LA}$-semihypergroup to become an intra-regular $\mathcal{LA}$-semihypergroup. In 2014, Amjad et al. [1] generalized the concepts of locally associative $\mathcal{LA}$-semigroups to hypergroupoids and studied several properties. They defined the concept of locally associative $\mathcal{LA}$-semihypergroups and characterized a locally associative $\mathcal{LA}$-semihypergroup in terms of (m, n)-hyperideals. In 2016, Khan et al. [10] proved that an $\mathcal{LA}$-semigroup S is $0(0, 2)$-bisimple if and only if S is right 0-simple. In 2018, Azhar et al. [3] applied the notion of $(\in, \in \vee q_k)$-fuzzy sets to $\mathcal{LA}$-semihypergroups. They introduced the notion of $(\in, \in \vee q_k)$-fuzzy hyperideals in an ordered $\mathcal{LA}$-semihypergroup and then derived their basic properties. In 2019, Gulistan et al. [8] presented a new definition of generalized fuzzy hyperideals, generalized fuzzy bi-hyperideals and generalized fuzzy normal bi-hyperideals in an ordered $\mathcal{LA}$-semihypergroup. They characterized ordered $\mathcal{LA}$-semihypergroups by the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy normal bi-hyperideals. In 2021, Suebsung et al. [12] have introduced the notion of left almost hyperideals, right almost hyperideals, almost hyperideals and minimal almost hyperideals in $\mathcal{LA}$-semihypergroups. In 2022, Nakkhasen [11] characterized intra-regular $\mathcal{LA}$-semihyperrings by the properties of their hyperideals.

In this paper, we extend the concept of hyperideals to the SuperHyperAlgebras. In this paper, we introduce the concept of 2-SuperHyperLeftAlmostSemihypergroups which is a generalization of $\mathcal{LA}$-semihypergroups. Furthermore, we define and study 2-SuperHyper-$\mathcal{LA}$-subsemihypergroups, SuperHyperLeft(Right)HyperIdeals and SuperHyperHyperIdeals of 2-SuperHyperLeftAlmostSemihypergroups, and related properties are investigated. We give an example to show that in general these two notions are different. Finally, we show that every SuperHyperRightHyperIdeal of 2-SuperHyper-$\mathcal{LA}$-semihypergroup S with pure left identity is SuperHyperHyperIdeal.

2. Preliminaries and Basic Definitions

In this section, we give some basic definitions and properties of left almost semihypergroups and classical-type Binary SuperHyperOperations that are required in this study.

Recall that a mapping $\circ : S \times S \to \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all non empty subsets of S, is called a **hyperoperation** on S. An image of the pair (x, y) is denoted by $x \circ y$. The couple $(S, \circ)$ is called a **hypergroupoid**.

Let x be an elements of a non empty set of S and let A, B be two non empty subsets of S. Then we denote $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b, x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$.

In 2011, Hila and Dine [9] introduced the concept and notion of left almost semihypergroup as a generalization of semigroups, $\mathcal{LA}$-semigroups and semihypergroups.

Definition 2.1. [9] A hypergroupoid $(S, \circ)$ is called a **left almost semihypergroup ($\mathcal{LA}$-semihypergroup)** if $\circ$ is left invertive law, that is $(x \circ y) \circ z = (z \circ y) \circ x$ for every $x, y, z \in S$.

Clearly, every $\mathcal{LA}$-semihypergroup is $\mathcal{LA}$-semigroup. If $(S, \circ)$ is an $\mathcal{LA}$-semihypergroup, then $\bigcup\limits_{a \in x \circ y} a \circ z = \bigcup\limits_{b \in z \circ y} b \circ x$ for all $x, y, z \in S$.

The concept of classical-type binary SuperHyperOperation was introduced by Smarandache [14, 15].

Definition 2.2. [14, 15] Let $\mathcal{P}_*^n(S)$ be the n^{th}-powerset of the set S such that none of $\mathcal{P}(S), \mathcal{P}^2(S), \ldots, \mathcal{P}^n(S)$ contain the empty set. A **classical-type binary SuperHyperOperation $\bullet_n$** is defined as follows:

$$\bullet_n : S \times S \to \mathcal{P}_*^n(S)$$

where $\mathcal{P}_*^n(S)$ is the n^{th}-power set of the set S, with no empty set.

An image of the pair (x, y) is denoted by $x \bullet_n y$. The couple $(S, \bullet_n)$ is called a **2-SuperHyperGroupoid**.

The following is an example of Examples of classical-type binary SuperHyperOperation (or 2-SuperHyperGroupoid).

Example 2.3. [14] Let $S = \{a, b\}$ be a finite discrete set. Then its power set, without the empty-set $\emptyset$, is: $\mathcal{P}(S) = \{a, b, S\}$ and $\mathcal{P}^2(S) = \mathcal{P}^2(\mathcal{P}(S)) = \mathcal{P}^2(\{a, b, S\}) = \{a, b, S, \{a, S\}, \{b, S\}, \{a, b, S\}\}$. The classical-type binary SuperHyperOperation defined as follows, $\bullet_2 : S \times S \to \mathcal{P}_*^2(S)$

$\bullet_2$	a	b
a	$\{a, S\}$	$\{b, S\}$
b	a	$\{a, b, S\}$

Then $(S, \bullet_2)$ is a 2-SuperHyperGroupoid and is not a hypergroupoid.

3. 2-SuperHyperLeftAlmostSemihypergroups

In this section, we generalize this concept in left almost semihypergroup and introduce SuperHyperLeft(Right)HyperIdeals of 2-SuperHyper-$\mathcal{LA}$-semihypergroups and study their properties.

The 2-SuperHyperLeftAlmostSemihypergroups is generated with the help of left almost semihypergroups and classical-type binary SuperHyperOperations. So we can say that 2-SuperHyperLeftAlmostSemihypergroup is the generalization of previously defined concepts related to binary SuperHyperOperations. We consider the SuperHyperLeftAlmostSemihypergroup as follows.

Definition 3.1. A 2-SuperHyperGroupoid $(S, \bullet_n)$ is called a n-**SuperHyperLeftAlmostSemihypergroup (2-SuperHyper-$\mathcal{LA}$-semihypergroup)** if it satisfies the SuperHyperLeftInvertive law; $(x \bullet_n y) \bullet_n z = (z \bullet_n y) \bullet_n x$ for all $x, y, z \in S$.

The following is an example of a 2-SuperHyper-$\mathcal{LA}$-semihypergroup S.

Example 3.2. Let $S = \{a, b\}$ be a finite discrete set. The classical-type binary SuperHyperOperation defined as follows, $\bullet_2 : S \times S \to \mathcal{P}_*^2(S)$

$\bullet_2$	a	b
a	$\{a, S\}$	b
b	$\{b, S\}$	$\{a, b, S\}$

Then, as is easily seen, $(S, \bullet_2)$ is a 2-SuperHyper-$\mathcal{LA}$-semihypergroup. Since

$$\begin{aligned}(a \bullet_2 a) \bullet_2 b &= \{a, S\} \bullet_2 a \\ &= (a \bullet_2 a) \cup (S \bullet_2 a) \\ &= \{a, S\} \cup \bigcup_{x \in S} x \bullet_2 a \\ &= \{a, S\} \cup (a \bullet_2 a) \cup (b \bullet_2 a) \\ &= \{a, S\} \cup \{a, S\} \cup \{b, S\} \\ &= \{a, b, S\} \\ &\neq b \\ &= a \bullet_2 b \\ &= a \bullet_2 (a \bullet_2 b),\end{aligned}$$

we have $\bullet_2$ is not Strong SuperHyperAssociativity.

Theorem 3.3. *Every 2-SuperHyper-$\mathcal{LA}$-semihypergroup S satisfies the **SuperHyperMedial law**, that is, for all $a, b, c, d \in S$, $(a \bullet_n b) \bullet_n (c \bullet_n d) = (a \bullet_n c) \bullet_n (b \bullet_n d)$.*

Proof. Let a, b, c and d be any elements of S. Then we have

$$\begin{aligned}(a \bullet_n b) \bullet_n (c \bullet_n d) &= ((c \bullet_n d) \bullet_n b) \bullet_n a \\ &= ((b \bullet_n d) \bullet_n c) \bullet_n a \\ &= (a \bullet_n c) \bullet_n (b \bullet_n d).\end{aligned}$$

This completes the proof. □

Theorem 3.4. *If S is a 2-SuperHyper-$\mathcal{LA}$-semihypergroup, then $(a \bullet_n b)^2 = a^2 \bullet_n b^2$ for all $a, b \in S$.*

Proof. Let a and b be any elements of S. Then by Theorem 3.3,

$$\begin{aligned}(a \bullet_n b)^2 &= (a \bullet_n b) \bullet_n (a \bullet_n b) \\ &= (a \bullet_n a) \bullet_n (b \bullet_n b) \\ &= a^2 \bullet_n b^2.\end{aligned}$$

□

An element e of a 2-SuperHyper-$\mathcal{LA}$-semihypergroup S is called **left identity (resp., pure left identity)** if for all $a \in \mathcal{N}(S), a \in e \bullet_n a$ (resp., $a = e \bullet_n a$). The following is an example of a pure left identity element in 2-SuperHyper-$\mathcal{LA}$-semihypergroups.

Example 3.5. 1. Let $S = \{a, b\}$ be a finite discrete set. The classical-type binary SuperHyperOperation defined as follows, $\bullet_2 : S \times S \to \mathcal{P}^2_*(S)$

$\bullet_2$	a	b
a	a	$\{a, b, S\}$
b	$\{b, S\}$	S

Then, as is easily seen, $(S, \bullet_2)$ is a 2-SuperHyper-$\mathcal{LA}$-semihypergroup with left identity a.

2. Let $S = \{a, b\}$ be a finite discrete set. The classical-type binary SuperHyperOperation defined as follows, $\bullet_2 : S \times S \to \mathcal{P}^2_*(S)$

$\bullet_2$	a	b
a	a	b
b	b	S

Then, as is easily seen, $(S, \bullet_2)$ is a 2-SuperHyper-$\mathcal{LA}$-semihypergroup with pure left identity a.

Theorem 3.6. *A 2-SuperHyper-$\mathcal{LA}$-semihypergroup S with pure left identity e satisfies the* ***SuperHyperParamedial law****, that is, for all $a, b, c, d \in S$, $(a \bullet_n b) \bullet_n (c \bullet_n d) = (d \bullet_n c) \bullet_n (b \bullet_n a)$.*

Proof. Let a, b, c and d be any elements of S. Then we have

$$\begin{aligned}(a \bullet_n b) \bullet_n (c \bullet_n d) &= [(e \bullet_n a) \bullet_n b] \bullet_n (c \bullet_n d)\\ &= [(b \bullet_n a) \bullet_n e] \bullet_n (c \bullet_n d)\\ &= [(c \bullet_n d) \bullet_n e] \bullet_n (b \bullet_n a)\\ &= [(e \bullet_n d) \bullet_n c] \bullet_n (b \bullet_n a)\\ &= (d \bullet_n c) \bullet_n (b \bullet_n a).\end{aligned}$$

This completes the proof. □

The following may be noted from the above definitions.

Lemma 3.7. *If S is a 2-SuperHyper-$\mathcal{LA}$-semihypergroup with pure left identity, then $a \bullet_n (b \bullet_n c) = b \bullet_n (a \bullet_n c)$ holds for all $a, b, c \in S$.*

Proof. Let a, b and c be any elements of S. Then by Theorem 3.3,

$$\begin{aligned}a \bullet (b \bullet_n c) &= (e \bullet_n a) \bullet (b \bullet_n c)\\ &= (e \bullet_n b) \bullet (a \bullet_n c)\\ &= b \bullet_n (a \bullet_n c).\end{aligned}$$

This completes the proof. □

Now, we give the concept of 2-SuperHyperLeftAlmostSemihypergroups (2-SuperHyper-$\mathcal{LA}$-subsemihypergroup) of 2-SuperHyper-$\mathcal{LA}$-semihypergroups.

Definition 3.8. A nonempty subset A of a 2-SuperHyper-$\mathcal{LA}$-semihypergroup S is called **2-SuperHyperLeftAlmostSemihypergroup (2-SuperHyper-$\mathcal{LA}$-subsemihypergroup)** if $A \bullet_n A \subseteq A$.

The following may be noted from the above definitions.

Proposition 3.9. *Let A and B be two 2-SuperHyper-$\mathcal{LA}$-subsemihypergroups of a 2-SuperHyper-$\mathcal{LA}$-semihypergroup S. If $A \cap B \neq \emptyset$, then $A \cap B$ is a 2-SuperHyper-$\mathcal{LA}$-subsemihypergroup of S.*

Proof. Let A and B be two 2-SuperHyper-$\mathcal{LA}$-subsemihypergroups of S such that $A \cap B \neq \emptyset$. Then have that

$$\begin{aligned}(A \cap B) \bullet_2 (A \cap B) &= [A \bullet_n (A \cap B)] \cap [B \bullet_n (A \cap B)] \\ &= (A \bullet_n A) \cap (A \bullet_n B) \cap (B \bullet_n A) \cap (B \bullet_n B) \\ &\subseteq (A \bullet_n A) \cap (B \bullet_n B) \\ &\subseteq A \cap B,\end{aligned}$$

and so $A \cap B$ is a 2-SuperHyper-$\mathcal{LA}$-subsemihypergroup of S. □

Now we mention some special class of 2-SuperHyper-$\mathcal{LA}$-subsemihypergroups in a 2-SuperHyper-$\mathcal{LA}$-semihypergroup.

Definition 3.10. A nonempty subset L of a 2-SuperHyper-$\mathcal{LA}$-semihypergroup S is called **SuperHyperLeft(Right)HyperIdeal** if

$$S \bullet_n L \subseteq L \, (R \bullet_n S \subseteq R).$$

A nonempty subset I of S is called a **SuperHyperHyperIdeal** of S if it is both a SuperHyperLeft and a SuperHyperRightHyperIdeal of S.

Proposition 3.11. *Let $\mathcal{N}(S)$ be a 2-SuperHyper-$\mathcal{LA}$-semihypergroup with pure left identity. Then the following properties hold.*

(1) *If L is a SuperHyperLeftHyperIdeal of S, then $S \bullet_n L = L$.*
(2) *If $\mathcal{N}(R)$ is a SuperHyperRightHyperIdeal of S, then $R \bullet_n S = R$.*
(3) *$S \bullet_n S = S$.*

Proof. 1. Since L is a SuperHyperLeftHyperIdeal of S, we have $S \bullet_n L \subseteq L$. On the other hand, let a be an element of S such that $a \in L$. Then we have $a = e \bullet_n a \in S \bullet_n L$ and hence $S \bullet_n L = L$.

2. Since R is a SuperHyperRightHyperIdeal of S, we have $R \bullet_n S \subseteq R$. On the other hand, let a be an element of S such that $a \in R$. Then we have

$$\begin{aligned} a &= e \bullet_n a \\ &= (e \bullet_n e) \bullet_n a \\ &= (a \bullet_n e) \bullet_n e \\ &\subseteq (R \bullet_n S) \bullet_n S \\ &\subseteq R \bullet_n S. \end{aligned}$$

Therefore we obtain that $R \subseteq R \bullet_n S$ and hence $R \bullet_n S = R$.

3. The proof is similar to the proof of (2). □

By applying the above definition, we state the following result.

Theorem 3.12. *Let S be a 2-SuperHyper-$\mathcal{LA}$-semihypergroup with pure left identity. Then the following properties hold.*

(1) *If x is an element of S, then $x \bullet_n S$ is a SuperHyperLeftHyperIdeal of S.*

(2) *If x is an element of S, then $S \bullet_n x$ is a SuperHyperLeftHyperideal of S.*

(3) *If x is an element of S, then $S \bullet_n x \cup x \bullet_n S$ is a SuperHyperRightHyperIdeal of S.*

Proof. 1. Let x be an element of S. By Lemma 3.7 and Proposition 3.11 (3), we have

$$\begin{aligned} S \bullet_n [x \bullet_n S] &= x \bullet_n [S \bullet_n S] \\ &= x \bullet_n S. \end{aligned}$$

Therefore we obtain that $x \bullet_n S$ is a SuperHyperLeftHyperIdeal of S.

2. Let x be an element of S. By Theorem 3.6 and Proposition 3.11 (3), we have

$$\begin{aligned} S \bullet_n (S \bullet_n x) &= (S \bullet_n S) \bullet_n (S \bullet_n x) \\ &= (x \bullet_n S) \bullet_n (S \bullet_n S) \\ &= [(S \bullet_n S) \bullet_n S] \bullet_n x \\ &= S \bullet_n x. \end{aligned}$$

Therefore we obtain that $S \bullet_n x$ is a SuperHyperLeftHyperIdeal of S.

3. Let x be an element of S. By Theorem 3.6, Lemma 3.7 and Proposition 3.11 (3), we have

$$\begin{aligned} (S \bullet_n x \cup x \bullet_n S) \bullet_n S &= [(S \bullet_n x) \bullet_n S] \cup [(x \bullet_n S) \bullet_n S] \\ &= [(S \bullet_n x) \bullet_n (S \bullet_n S)] \cup [(S \bullet_n S) \bullet_n x] \\ &= [(S \bullet_n S) \bullet_n (x \bullet_n S)] \cup (S \bullet_n x) \\ &= [x \bullet_n ((S \bullet_n S) \bullet_n S)] \cup (S \bullet_n x) \\ &= S \bullet_n x \cup x \bullet_n S. \end{aligned}$$

Therefore we obtain that $S \bullet_n x \cup x \bullet_n S$ is a SuperHyperRightHyperIdeal of S. □

For that, we need the following theorem.

Theorem 3.13. *Let S be a 2-SuperHyper-$\mathcal{LA}$-semihypergroup with pure left identity. Then the following properties hold.*

(1) *If x is an element of S, then $x^2 \bullet_n S$ is a SuperHyperHyperIdeal of S.*
(2) *If x is an element of S, then $S \bullet_n x^2$ is a SuperHyperHyperIdeal of S.*
(3) *If x is an element of S, then $S \bullet_n x \cup x \bullet_n S$ is a SuperHyperHyperIdeal of S.*

Proof. 1. Let x be an element of $\mathcal{N}(S)$. By Theorem 3.12 (1), we have that $x^2 \bullet_n S$ is a SuperHyperLeftHyperIdeal of $\mathcal{N}(S)$. Since

$$\begin{aligned}\left(x^2 \bullet_n S\right) \bullet_n S &= (S \bullet_n S) \bullet_n x^2 \\ &= x^2 \bullet_n (S \bullet_n S) \\ &= x^2 \bullet_n S,\end{aligned}$$

we have $x^2 \bullet_n S$ is a SuperHyperRightHyperIdeal of S and so $x^2 \bullet_n S$ is a SuperHyperHyperIdeal of S.

2. The proof is similar to the proof of (1).

3. Let x be an element of S. By Theorem 3.12 (3), we have that $S \bullet_n x \cup x \bullet_n S$ is a SuperHyperRightHyperIdeal of $\mathcal{N}(S)$. By Theorem 3.6, Lemma 3.7 and Proposition 3.11 (3), we have

$$\begin{aligned}S \bullet_n (S \bullet_n x \cup x \bullet_n S) &= [S \bullet_n (S \bullet_n x)] \cup [S \bullet_n (x \bullet_n S)] \\ &= [(S \bullet_n S) \bullet_n (S \bullet_n x)] \cup [x \bullet_n (S \bullet_n S)] \\ &= [(x \bullet_n S) \bullet_n (S \bullet_n S)] \cup (x \bullet_n S) \\ &= [((S \bullet_n S) \bullet_n S) \bullet_n x] \cup (x \bullet_n S) \\ &= S \bullet_n x \cup x \bullet_n S.\end{aligned}$$

Therefore we obtain that $S \bullet_n x \cup x \bullet_n S$ is a SuperHyperLeftHyperIdeal of S and hence $S \bullet_n x \cup x \bullet_n S$ is a SuperHyperHyperIdeal of S. □

Theorem 3.14. *Every SuperHyperRightHyperIdeal of 2-SuperHyper-$\mathcal{LA}$-semihypergroup S with pure left identity is SuperHyperHyperIdeal.*

Proof. Let R be a SuperHyperRightHyperIdeal of S. By Theorem 3.6, Lemma 3.7 and Proposition 3.11 (3), we have

$$\begin{aligned}S \bullet_n R &= (S \bullet_n S) \bullet_n R \\ &= (R \bullet_n S) \bullet_n S \\ &\subseteq R \bullet_n S \\ &\subseteq R.\end{aligned}$$

Therefore we obtain that R is a SuperHyperLeftHyperIdeal of S and hence R is a SuperHyperHyperIdeal of S. □

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Covering properties in neutrosophic topological spaces

Sudeep Dey[1,2], Gautam Chandra Ray[2,*]

[1]Department of Mathematics, Science College, Kokrajhar, Assam, India; sudeep.dey.1976@gmail.com

[2]Department of Mathematics, Central Institute of Technology, Kokrajhar, Assam, India; gautomofcit@gmail.com

[*]Correspondence: gautomofcit@gmail.com

Abstract. Single-valued neutrosophic set is being extensively used in solving real-life problems. Recently neutrosophic topological space was developed based on redefined single-valued neutrosophic set operations. The purpose of this article is to investigate some covering properties of these neutrosophic topological spaces.

Keywords: Neutrosophic Lindelöf ; Neutrosophic continuous function ; Neutrosophic compact ; Neutrosophic countably compact ; Single-valued Neutrosophic set.

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1. **Introduction**

In the year 1965, Zadeh [34] introduced the concept of a fuzzy set. But after some decades a new branch of philosophy, acknowledged as Neutrosophy, was developed and studied by Florentin Smarandache [22–24]. Smarandache [24] proved that the neutrosophic set was a generalization of the intuitionistic fuzzy set which was developed by K.Atanassov [1] in 1986 as an extension of a fuzzy set. Like an intuitionistic fuzzy set, an element in a neutrosophic set has the degree of membership and the degree of non-membership but it has another grade of membership known as the degree of indeterminacy and one very important point about the neutrosophic set is that all three neutrosophic components are independent of one another.

After Smarandache had introduced the concept of neutrosophy, it was studied by many researchers [7, 11, 29, 32]. In the year 2002, Smarandache [23] introduced the notion of neutrosophic topology on the non-standard interval. Lupiáñez [16–18] studied and investigated many properties of neutrosophic topological spaces. In the year 2012, Salama & Alblowi [25] introduced neutrosophic topological space as a generalization of intuitionistic fuzzy topological space developed by D.Coker [9] in 1997. Salma et.al. [26–28] studied generalized neutrosophic topological space, neutrosophic filters, and neutrosophic continuous functions. In the year 2016, Karatas and Kuru [15] redefined the single-valued neutrosophic set operations and introduced neutrosophic topology and then investigated some important properties of neutrosophic topological spaces. Later, various aspects of neutrosophic topology were developed by many researchers [2, 8, 12, 30, 31].

Neutrosophy, due to the fact of its flexibility and effectiveness, is attracting researchers throughout the world and is very useful not only in the development of science and technology but also in various other fields. For instance, Abdel-Basset et.al. [3–5] studied the applications of neutrosophic theory in a number of scientific fields. In 2014, Pramanik and Roy [19] studied the conflict between India and Pakistan over Jammu-Kashmir through neutrosophic game theory. Works on medical diagnosis [13, 33], decision-making problems [4, 5], image processing [14], etc. were also done in a neutrosophic environment. Recently some studies on COVID-19 [6, 10] had been done with the help of neutrosophic theory.

There are still many concepts to be developed in connection with neutrosophic topological spaces. Very recently Ray and Dey [20] introduced the idea of neutrosophic points on single-valued neutrosophic sets and studied various properties. The authors [21] also studied the relation of quasi-coincidence for neutrosophic sets. In this article, we investigate some covering properties of neutrosophic topological spaces.

2. Preliminaries

In this section we confer some basic concepts which will be helpful in the later sections.

2.1. Definition: [22]

Let X be the universe of discourse. A neutrosophic set A over X is defined as $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x)\rangle : x \in X\}$, where the functions $\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A$ are real standard or non-standard subsets of $]^-0, 1^+[$, i.e., $\mathcal{T}_A : X \to\,]^-0, 1^+[,\ \mathcal{I}_A : X \to\,]^-0, 1^+[,\ \mathcal{F}_A : X \to\,]^-0, 1^+[$ and $^-0 \leq \mathcal{T}_A(x) + \mathcal{I}_A(x) + \mathcal{F}_A(x) \leq 3^+$.

The neutrosophic set A is characterized by the truth-membership function $\mathcal{T}_A$, indeterminacy-membership function $\mathcal{I}_A$, falsehood-membership function $\mathcal{F}_A$.

2.2. Definition: [32]

Let X be the universe of discourse. A single-valued neutrosophic set A over X is defined as $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x)\rangle : x \in X\}$, where $\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A$ are functions from X to $[0, 1]$ and $0 \leq \mathcal{T}_A(x) + \mathcal{I}_A(x) + \mathcal{F}_A(x) \leq 3$.

The set of all single valued neutrosophic sets over X is denoted by $\mathcal{N}(X)$.

Throughout this article, a neutrosophic set (NS, for short) will mean a single-valued neutrosophic set.

2.3. **Definition: [15]**

Let $A, B \in \mathcal{N}(X)$. Then

(i) (Inclusion): If $\mathcal{T}_A(x) \leq \mathcal{T}_B(x), \mathcal{I}_A(x) \geq \mathcal{I}_B(x), \mathcal{F}_A(x) \geq \mathcal{F}_B(x)$ for all $x \in X$ then A is said to be a neutrosophic subset of B and which is denoted by $A \subseteq B$.

(ii) (Equality): If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(iii) (Intersection): The intersection of A and B, denoted by $A \cap B$, is defined as $A \cap B = \{\langle x, \mathcal{T}_A(x) \wedge \mathcal{T}_B(x), \mathcal{I}_A(x) \vee \mathcal{I}_B(x), \mathcal{F}_A(x) \vee \mathcal{F}_B(x)\rangle : x \in X\}$.

(iv) (Union): The union of A and B, denoted by $A \cup B$, is defined as $A \cup B = \{\langle x, \mathcal{T}_A(x) \vee \mathcal{T}_B(x), \mathcal{I}_A(x) \wedge \mathcal{I}_B(x), \mathcal{F}_A(x) \wedge \mathcal{F}_B(x)\rangle : x \in X\}$.

(v) (Complement): The complement of the NS A, denoted by A^c, is defined as $A^c = \{\langle x, \mathcal{F}_A(x), 1 - \mathcal{I}_A(x), \mathcal{T}_A(x)\rangle : x \in X\}$

(vi) (Universal Set): If $\mathcal{T}_A(x) = 1, \mathcal{I}_A(x) = 0, \mathcal{F}_A(x) = 0$ for all $x \in X$ then A is said to be neutrosophic universal set and which is denoted by $\tilde{X}$.

(vii) (Empty Set): If $\mathcal{T}_A(x) = 0, \mathcal{I}_A(x) = 1, \mathcal{F}_A(x) = 1$ for all $x \in X$ then A is said to be neutrosophic empty set and which is denoted by $\tilde{\emptyset}$.

2.4. **Definition: [25]**

Let $\{A_i : i \in \triangle\} \subseteq \mathcal{N}(X)$, where $\triangle$ is an index set. Then

(i) $\cup_{i \in \triangle} A_i = \{\langle x, \vee_{i \in \triangle} \mathcal{T}_{A_i}(x), \wedge_{i \in \triangle} \mathcal{I}_{A_i}(x), \wedge_{i \in \triangle} \mathcal{F}_{A_i}(x)\rangle : x \in X\}$.

(ii) $\cap_{i \in \triangle} A_i = \{\langle x, \wedge_{i \in \triangle} \mathcal{T}_{A_i}(x), \vee_{i \in \triangle} \mathcal{I}_{A_i}(x), \vee_{i \in \triangle} \mathcal{F}_{A_i}(x)\rangle : x \in X\}$.

2.5. **Definition: [20]**

Let $\mathcal{N}(X)$ be the set of all neutrosophic sets over X. An NS $P = \{\langle x, \mathcal{T}_P(x), \mathcal{I}_P(x), \mathcal{F}_P(x)\rangle : x \in X\}$ is called a neutrosophic point (NP, for short) iff for any element $y \in X$, $\mathcal{T}_P(y) = \alpha, \mathcal{I}_P(y) = \beta, \mathcal{F}_P(y) = \gamma$ for $y = x$ and $\mathcal{T}_P(y) = 0, \mathcal{I}_P(y) = 1, \mathcal{F}_P(y) = 1$ for $y \neq x$, where $0 < \alpha \leq 1, 0 \leq \beta < 1, 0 \leq \gamma < 1$. A neutrosophic point $P = \{\langle x, \mathcal{T}_P(x), \mathcal{I}_P(x), \mathcal{F}_P(x)\rangle : x \in X\}$ will be denoted by $P^x_{\alpha,\beta,\gamma}$ or $P < x, \alpha, \beta, \gamma >$ or simply by $x_{\alpha,\beta,\gamma}$. For the NP $x_{\alpha,\beta,\gamma}$, x will be called its support. The complement of the NP $P^x_{\alpha,\beta,\gamma}$ will be denoted by $(P^x_{\alpha,\beta,\gamma})^c$ or by $x^c_{\alpha,\beta,\gamma}$.

2.6. **Theorem: [28]**

Let $f : X \to Y$ be a function. Also let $A, A_i \in \mathcal{N}(X), i \in I$ and $B, B_j \in \mathcal{N}(Y), j \in J$. Then the following hold.

(i) $A_1 \subseteq A_2 \Leftrightarrow f(A_1) \subseteq f(A_2)$, $B_1 \subseteq B_2 \Leftrightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

(ii) $A \subseteq f^{-1}(f(A))$ and if f is injective then $A = f^{-1}(f(A))$.

(iii) $f^{-1}(f(B)) \subseteq B$ and if f is surjective then $f^{-1}(f(B)) = B$.

(iv) $f^{-1}(\cup B_j) = \cup f^{-1}(B_j)$ and $f^{-1}(\cap B_j) = \cap f^{-1}(B_j)$.

(v) $f(\cup A_i) = \cup f(A_i)$, $f(\cap A_i) \subseteq \cap f(A_i)$ and if f is injective then $f(\cap A_i) = \cap f(A_i)$.

(vi) $f^{-1}(\tilde{\emptyset}_Y) = \tilde{\emptyset}_X$, $f^{-1}(\tilde{Y}) = \tilde{X}$.

(vii) $f(\tilde{\emptyset}_X) = \tilde{\emptyset}_Y$, $f(\tilde{X}) = \tilde{Y}$ if f is surjective.

2.7. **Definition: [28]**

Let X and Y be two non-empty sets and $f : X \to Y$ be a function. Also let $A \in \mathcal{N}(X)$ and $B \in \mathcal{N}(Y)$. Then

(1) Image of A under f is defined by
$f(A) = \{\langle y, f(\mathcal{T}_A)(y), f(\mathcal{I}_A)(y), (1 - f(1 - \mathcal{F}_A))(y)\rangle : y \in Y\}$, where

$$f(\mathcal{T}_A)(y) = \begin{cases} \sup\{\mathcal{T}_A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

$$f(\mathcal{I}_A)(y) = \begin{cases} \inf\{\mathcal{I}_A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

$$(1 - f(1 - \mathcal{F}_A))(y) = \begin{cases} \inf\{\mathcal{F}_A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

(2) Pre-image of B under f is defined by
$f^{-1}(B) = \{\langle x, f^{-1}(\mathcal{T}_B)(x), f^{-1}(\mathcal{I}_B)(x), f^{-1}(\mathcal{F}_B)(x)\rangle : x \in X\}$

2.8. **Definition: [15]**

Let $\tau \subseteq \mathcal{N}(X)$. Then τ is called a neutrosophic topology on X if

(i) $\tilde{\emptyset}$ and $\tilde{X}$ belong to τ.

(ii) An arbitrary union of neutrosophic sets in τ is in τ.

(iii) The intersection of any two neutrosophic sets in τ is in τ.

If τ is a neutrosophic topology on X then the pair (X, τ) is called a neutrosophic topological space (NTS, for short) over X. The members of τ are called neutrosophic open sets in X. If for a neutrosophic set A, $A^c \in \tau$ then A is said to be a neutrosophic closed set in X.

2.9. **Definition: [30]**

Let (X, τ) and (Y, σ) be two neutrosophic topological spaces and $f : X \to Y$ be a function. Then

(i) f is called a neutrosophic open function if $f(G) \in \sigma$ for all $G \in \tau$

(ii) f is called a neutrosophic continuous function if $f^{-1}(G) \in \tau$ for all $G \in \sigma$.

3. **Main Results**

3.1. **Definition :**

Let (X, τ) be a neutrosophic topological space. A subcollection $\mathcal{B}$ of τ is called a neutrosophic base (or simply, base) for τ iff for each $A \in \tau$, there exists a subcollection $\{A_i : i \in \Delta\} \subseteq \mathcal{B}$ such that $A = \cup\{A_i : i \in \triangle\}$, where $\triangle$ is an index set.

A subcollection $\mathcal{B}_*$ of τ is called a neutrosophic subbase (or simply, subbase) for τ iff the finite intersection of members of $\mathcal{B}_*$ forms a neutrosophic base for τ.

3.2. **Definition :**

An NTS (X, τ) is said to satisfy the second axiom of countability or is said to be neutrosophic C_{II} (or simply, C_{II}) space iff τ has a countable neutrosophic base, i.e., an NTS (X, τ) is said to be C_{II} space iff there exists a countable subcollection $\mathcal{B}$ of τ such that every member of τ can be expressed as the union of some members of $\mathcal{B}$.

3.3. **Definition :**

Let (X, τ) be an NTS. A collection $\{G_\lambda : \lambda \in \Delta\}$ of neutrosophic closed sets of X is said to have the finite intersection property (FIP, in short) iff every finite subcollection $\{G_{\lambda_k} : k = 1, 2, \cdots, n\}$ of $\{G_\lambda : \lambda \in \Delta\}$ satisfies the condition $\bigcap_{k=1}^{n} G_{\lambda_k} \neq \tilde{\emptyset}$, where $\triangle$ is an index set.

3.4. **Definition :**

Let (X, τ) be an NTS and $A \in \mathcal{N}(X)$. A collection $C = \{G_\lambda : \lambda \in \triangle\}$ of neutrosophic open sets of X is called a neutrosophic open cover (NOC, in short) of A if $A \subseteq \cup_{\lambda \in \triangle} G_\lambda$. We then say C covers A. In particular, C is said to be an NOC of X iff $\tilde{X} = \cup_{\lambda \in \triangle} G_\lambda$.

Let C be an NOC of the NS A and $C' \subseteq C$. Then C' is called a neutrosophic open subcover (NOSC, in short) of C if C' covers A.

An NOC of A is said to be countable (resp. finite) if it consists of a countable (resp. finite) number of neutrosophic open sets.

3.5. **Definition :**

An NS A in an NTS (X, τ) is said to be neutrosophic compact set iff every NOC of A has a finite NOSC. In particular, the space X is said to be neutrosophic compact space iff every NOC of X has a finite NOSC.

3.6. **Definition :**

An NTS (X, τ) is said to be neutrosophic countably compact space iff every countable NOC of X has a finite NOSC.

3.7. **Definition :**

An NTS (X, τ) is said to be neutrosophic Lindelöf iff every NOC of X has a countable NOSC.

3.8. **Example :**

Let $X = \{1, 2\}$, $A = \{\langle 1, 1, 0, 0\rangle, \langle 2, 0, 1, 1\rangle\}$, $B = \{\langle 1, 0, 1, 1\rangle, \langle 2, 1, 0, 0\rangle\}$ and $\tau = \{\tilde{X}, \tilde{\emptyset}, A, B\}$. Clearly (X, τ) is an NTS. It is clear that (X, τ) is neutrosophic compact, neutrosophic countably compact as well as neutrosophic Lindelöf.

3.9. **Example :**

Let $X = \{a, b\}$ and $G_n = \{\langle a, \frac{n}{n+1}, \frac{1}{n+2}, \frac{1}{n+3}\rangle, \langle b, \frac{n+1}{n+2}, \frac{1}{n+3}, \frac{1}{n+4}\rangle\}, n \in \mathsf{N} = \{1, 2, 3, \cdots\}$ and $\tau = \{\tilde{X}, \tilde{\emptyset}\} \cup \{G_n : n \in \mathsf{N}\}$. Clearly (X, τ) is an NTS. Also it is easy to see that $\cup_{n\in\mathsf{N}} G_n = \tilde{X}$. Therefore $\{G_n : n \in \mathsf{N}\}$ is an NOC of X. Now $G_1 = \{\langle a, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\rangle, \langle b, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}\rangle\}$, $G_2 = \{\langle a, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}\rangle, \langle b, \frac{3}{4}, \frac{1}{5}, \frac{1}{6}\rangle\}$, $G_3 = \{\langle a, \frac{3}{4}, \frac{1}{5}, \frac{1}{6}\rangle, \langle b, \frac{4}{5}, \frac{1}{6}, \frac{1}{7}\rangle\}$ and so on. Clearly $G_1 \cup G_2 = G_2$, $G_1 \cup G_3 = G_3$ and $G_1 \cup G_2 \cup G_3 = G_3$. So, for any finite subcollection $\{G_{n_k} : n_k \in M$, M is a finite subset of $\mathsf{N}\}$ of $\{G_n : n \in \mathsf{N}\}$, we have $\bigcup_{n_k} G_{n_k} = G_m \neq \tilde{X}$, where $m = \max\{n_k : n_k \in M\}$. Therefore (X, τ) is not a neutrosophic compact space.

3.10. **Theorem :**

Finite union of neutrosophic compact sets is neutrosophic compact.

Proof: Very obvious.

3.11. **Theorem :**

Let (X, τ) be an NTS. An NS $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x)\rangle : x \in X\}$ in X is neutrosophic compact iff for every collection $C = \{G_\lambda : \lambda \in \triangle\}$ of neutrosophic open sets of X satisfying $\mathcal{T}_A(x) \leq \bigvee_{\lambda\in\triangle} \mathcal{T}_{G_\lambda}(x)$, $1 - \mathcal{I}_A(x) \leq \bigvee_{\lambda\in\triangle}(1 - \mathcal{I}_{G_\lambda}(x))$ and $1 - \mathcal{F}_A(x) \leq \bigvee_{\lambda\in\triangle}(1 - \mathcal{F}_{G_\lambda}(x))$, there exists a finite subcollection $\{G_{\lambda_k} : k = 1, 2, 3, ..., n\}$ such that $\mathcal{T}_A(x) \leq \bigvee_{k=1}^{n} \mathcal{T}_{G_{\lambda_k}}(x)$, $1 - \mathcal{I}_A(x) \leq \bigvee_{k=1}^{n}(1 - \mathcal{I}_{G_{\lambda_k}}(x))$ and $1 - \mathcal{F}_A(x) \leq \bigvee_{k=1}^{n}(1 - \mathcal{F}_{G_{\lambda_k}}(x))$.

Proof: Necessary Part : Let $C = \{G_\lambda : \lambda \in\triangle\}$ of neutrosophic open sets of X satisfying $\mathcal{T}_A(x) \leq \bigvee_{\lambda\in\triangle} \mathcal{T}_{G_\lambda}(x)$, $1-\mathcal{I}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{I}_{G_\lambda}(x))$ and $1-\mathcal{F}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{F}_{G_\lambda}(x))$. Now $1-\mathcal{I}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{I}_{G_\lambda}(x)) \Rightarrow 1-\mathcal{I}_A(x) \leq 1-\mathcal{I}_{G_\beta}(x)$ for some $\beta \in\triangle \Rightarrow \mathcal{I}_A(x) \geq \mathcal{I}_{G_\beta}(x) \Rightarrow \mathcal{I}_A(x) \geq \bigwedge_{\lambda\in\triangle} \mathcal{I}_{G_\lambda}(x)$. Similarly $1 - \mathcal{F}_A(x) \leq \bigvee_{\lambda\in\triangle}(1 - \mathcal{F}_{G_\lambda}(x)) \Rightarrow \mathcal{F}_A(x) \geq \bigwedge_{\lambda\in\triangle} \mathcal{F}_{G_\lambda}(x)$. Therefore $A \subseteq \cup_{\lambda\in\triangle} G_\lambda$, i.e., C is an NOC of A. Since A is compact, so C has a finite NOSC $\{G_{\lambda_k} : k = 1, 2, 3, \cdots, n\}$. Therefore $A \subseteq \cup_{k=1}^{n} G_{\lambda_k}$. Then $\mathcal{T}_A(x) \leq \bigvee_{k=1}^{n} \mathcal{T}_{G_{\lambda_k}}(x)$, $\mathcal{I}_A(x) \geq \bigwedge_{k=1}^{n} \mathcal{I}_{G_{\lambda_k}}(x)$ and $\mathcal{F}_A(x) \geq \bigwedge_{k=1}^{n} \mathcal{F}_{G_{\lambda_k}}(x)$. Now $\mathcal{I}_A(x) \geq \bigwedge_{k=1}^{n} \mathcal{I}_{G_{\lambda_k}}(x) \Rightarrow \mathcal{I}_A(x) \geq \mathcal{I}_{G_{\lambda_m}}(x)$ for some $m, 1 \leq m \leq n \Rightarrow 1 - \mathcal{I}_A(x) \leq 1 - \mathcal{I}_{G_{\lambda_m}}(x), 1 \leq m \leq n \Rightarrow 1 - \mathcal{I}_A(x) \leq$

$\bigvee_{k=1}^{n}(1-\mathcal{I}_{G_{\lambda_k}}(x))$. Similarly we can show that $\mathcal{F}_A(x) \geq \bigwedge_{k=1}^{n}\mathcal{F}_{G_{\lambda_k}}(x) \Rightarrow 1-\mathcal{F}_A(x) \leq \bigvee_{k=1}^{n}(1-\mathcal{F}_{G_{\lambda_k}}(x))$. Thus $\mathcal{T}_A(x) \leq \bigvee_{k=1}^{n}\mathcal{T}_{G_{\lambda_k}}(x)$, $1-\mathcal{I}_A(x) \leq \bigvee_{k=1}^{n}(1-\mathcal{I}_{G_{\lambda_k}}(x))$ and $1-\mathcal{F}_A(x) \leq \bigvee_{k=1}^{n}(1-\mathcal{F}_{G_{\lambda_k}}(x))$.

Sufficient Part : Let $C = \{G_\lambda : \lambda \in\triangle\}$ be an NOC of A. Then $A \subseteq \cup_{\lambda\in\triangle}G_\lambda$, i.e., $\mathcal{T}_A(x) \leq \bigvee_{\lambda\in\triangle}\mathcal{T}_{G_\lambda}(x)$, $\mathcal{I}_A(x) \geq \bigwedge_{\lambda\in\triangle}\mathcal{I}_{G_\lambda}(x)$ and $\mathcal{F}_A(x) \geq \bigwedge_{\lambda\in\triangle}\mathcal{F}_{G_\lambda}(x)$. Now $\mathcal{I}_A(x) \geq \bigwedge_{\lambda\in\triangle}\mathcal{I}_{G_\lambda}(x) \Rightarrow \mathcal{I}_A(x) \geq \mathcal{I}_{G_\alpha}(x)$ for some $\alpha \Rightarrow 1-\mathcal{I}_A(x) \leq 1-\mathcal{I}_{G_\alpha}(x) \Rightarrow 1-\mathcal{I}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{I}_{G_\lambda}(x))$. Similarly $\mathcal{F}_A(x) \geq \bigwedge_{\lambda\in\triangle}\mathcal{F}_{G_\lambda}(x) \Rightarrow 1-\mathcal{F}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{F}_{G_\lambda}(x))$. Thus the collection C satisfies the condition $\mathcal{T}_A(x) \leq \bigvee_{\lambda\in\triangle}\mathcal{T}_{G_\lambda}(x)$, $1-\mathcal{I}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{I}_{G_\lambda}(x))$ and $1-\mathcal{F}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{F}_{G_\lambda}(x))$. By the hypothesis, there exists a finite subcollection $\{G_{\lambda_k} : k = 1, 2, 3, ..., n\}$ such that $\mathcal{T}_A(x) \leq \bigvee_{k=1}^{n}\mathcal{T}_{G_{\lambda_k}}(x)$, $1-\mathcal{I}_A(x) \leq \bigvee_{k=1}^{n}(1-\mathcal{I}_{G_{\lambda_k}}(x))$ and $1-\mathcal{F}_A(x) \leq \bigvee_{k=1}^{n}(1-\mathcal{F}_{G_{\lambda_k}}(x))$. Now $1-\mathcal{I}_A(x) \leq \bigvee_{k=1}^{n}(1-\mathcal{I}_{G_{\lambda_k}}(x)) \Rightarrow 1-\mathcal{I}_A(x) \leq 1-\mathcal{I}_{G_{\lambda_m}}(x)$ for some $m \Rightarrow \mathcal{I}_A(x) \geq \mathcal{I}_{G_{\lambda_m}}(x) \Rightarrow \mathcal{I}_A(x) \geq \bigwedge_{k=1}^{n}\mathcal{I}_{G_{\lambda_k}}(x)$. Similarly we shall have $\mathcal{F}_A(x) \geq \bigwedge_{k=1}^{n}\mathcal{F}_{G_{\lambda_k}}(x)$. Therefore $A \subseteq \cup_{k=1}^{n}G_{\lambda_k}$, i.e., the NOC C of A has a finite NOSC $\{G_{\lambda_k} : k = 1, 2, 3, \cdots, n\}$. Thus A is neutrosophic compact.

Hence proved.

3.12. **Theorem :**

Let (X, τ) be an NTS. Then X is neutrosophic compact iff for every collection $C = \{G_\lambda : \lambda \in\triangle\}$ of neutrosophic open sets of X satisfying $\bigvee_{\lambda\in\triangle}\mathcal{T}_{G_\lambda}(x) = 1$, $\bigvee_{\lambda\in\triangle}(1-\mathcal{I}_{G_\lambda}(x)) = 1$ and $\bigvee_{\lambda\in\triangle}(1-\mathcal{F}_{G_\lambda}(x)) = 1$, there exists a finite subcollection $\{G_{\lambda_k} : k = 1, 2, 3, ..., n\}$ such that $\bigvee_{k=1}^{n}\mathcal{T}_{G_{\lambda_k}}(x) = 1$, $\bigvee_{k=1}^{n}(1-\mathcal{I}_{G_{\lambda_k}}(x)) = 1$ and $\bigvee_{k=1}^{n}(1-\mathcal{F}_{G_{\lambda_k}}(x)) = 1$.

Proof: Immediate from 3.11.

3.13. **Theorem :**

Let β be a neutrosophic base for an NTS (X, τ). Then X is neutrosophic compact iff every NOC of X by the members of β has a finite NOSC.

Proof: Necessary Part : Obvious.

Sufficient Part : Let $\beta = \{B_\alpha : \alpha \in\triangle\}$ be the neutrosophic base. Also let $\mathcal{C} = \{G_\lambda : \lambda \in\triangle\}$ be an NOC of X. Then each member G_λ of $\mathcal{C}$ is the union of some members of β and the totality of such members of β is evidently an NOC of X. By the hypothesis, this collection of members of β has a finite NOSC $\mathcal{D} = \{B_{\alpha_j} : j = 1, 2, 3, \cdots, n\}$, say. Clearly for each B_{α_j} in $\mathcal{D}$, we can find a G_{λ_j} in $\mathcal{C}$ such that $B_{\alpha_j} \subseteq G_{\lambda_j}$. Therefore the finite subcollection $\{G_{\lambda_j} : j = 1, 2, 3, \cdots, n\}$ of $\mathcal{C}$ is an NOC of X. Therefore X is neutrosophic compact.

3.14. **Theorem :**

If the NTS (X, τ) is C_{II} then neutrosophic compactness and neutrosophic countably compactness are equivalent.

Proof: First we show that if (X,τ) is neutrosophic compact then it is neutrosophic countably compact. Let $\mathcal{A}=\{A_i : i \in\triangle\}$ be a countable NOC of X. Since X is compact, so $\mathcal{A}$ has a finite NOSC. Therefore X is neutrosophic countably compact. Next we show that if (X,τ) is neutrosophic countably compact then it is neutrosophic compact. Let $\mathcal{A}=\{A_i : i \in\triangle\}$ be any NOC of X. Since X is C_{II}, so there exists a countable base $\mathcal{B}=\{B_n : n=1,2,3,\cdots\}$ for τ. Then each $A_i\in\mathcal{A}$ can be expressed as the union of some members of $\mathcal{B}$. Let $A_i=\bigcup_{k=1}^{i_0} B_{n_k}$, where $B_{n_k}\in\mathcal{B}$ and i_0 may be infinity. Clearly $\mathcal{B}_0=\{B_{n_k}\}$ is an NOC of X. Also $\mathcal{B}_0$ is countable as $\mathcal{B}_0\subseteq\mathcal{B}$. Since X is countably compact, so $\mathcal{B}_0$ has a finite NOSC $\mathcal{B}_1$, say. Since by construction, each member of $\mathcal{B}_1$ is contained in one member A_i, so these A_i's form a finite subfamily of $\mathcal{A}$ and certainly a cover of X. Thus the NOC $\mathcal{A}$ of X has a finite NOSC. Therefore X is neutrosophic compact. Hence Proved.

3.15. **Theorem :**

If the NTS (X,τ) is C_{II} then it is neutrosophic Lindelöf.

Proof: Let $\mathcal{A}=\{A_i : i \in\triangle\}$ be an NOC of X. Since X is C_{II}, so there exists a countable base $\mathcal{B}=\{B_n : n=1,2,3,\cdots\}$ for τ. Then each $A_i\in\mathcal{A}$ can be expressed as the union of some members of $\mathcal{B}$. Let $A_i=\bigcup_{k=1}^{i_0} B_{n_k}$, where $B_{n_k}\in\mathcal{B}$ and i_0 may be infinity. Let $\mathcal{B}_0=\{B_{n_k}\}$. Then $\mathcal{B}_0$ is an NOC of X. Also $\mathcal{B}_0$ is countable as $\mathcal{B}_0\subseteq\mathcal{B}$. By construction, each member of $\mathcal{B}_0$ is contained in one A_i . So, these A_i's form a countable NOSC of $\mathcal{A}$. Thus the NOC $\mathcal{A}$ of X has a countable NOSC. Therefore X is neutrosophic Lindelöf. Hence proved.

3.16. **Theorem :**

An NTS (X,τ) is neutrosophic compact iff every collection of neutrosophic closed sets with the FIP has a non-empty intersection.

Proof: Necessary part : Let $\mathcal{A}=\{N_i : i \in\triangle\}$ be an arbitrary collection of neutrosophic closed sets with the FIP. We show that $\cap_{i\in\triangle}N_i\neq\tilde{\emptyset}$. On the contrary, suppose that $\cap_{i\in\triangle}N_i=\tilde{\emptyset}$. Then $(\cap_{i\in\triangle}N_i)^c=(\tilde{\emptyset})^c\Rightarrow\cup_{i\in\triangle}N_i^c=\tilde{X}$. Therefore $\mathcal{B}=\{N_i^c : N_i\in\mathcal{A}\}$ is an NOC of X and so $\mathcal{B}$ has a finite NOSC $\{N_{i_1}^c,N_{i_2}^c,...,N_{i_k}^c\}$, say. Then $\cup_{j=1}^k N_{i_j}^c=\tilde{X}\Rightarrow\cap_{j=1}^k N_{i_j}=\tilde{\emptyset}$, which is a contradiction as $\mathcal{A}$ has FIP. Therefore $\cap_{i\in\triangle}N_i\neq\tilde{\emptyset}$.

Sufficient part : Let $\mathcal{C}=\{G_i : i \in\triangle\}$ be an NOC of X. Suppose that $\mathcal{C}$ has no finite NOSC. Then for every finite subcollection $\{G_{i_1},G_{i_2},...,G_{i_k}\}$ of $\mathcal{C}$, we have $\cup_{j=1}^k G_{i_j}\neq\tilde{X}\Rightarrow\cap_{j=1}^k G_{i_j}^c\neq\tilde{\emptyset}$. Therefore $\{G_i^c : G_i\in\mathcal{C}\}$ is a collection of neutrosophic closed sets having the FIP. By the assumption, $\cap_{i\in\triangle}G_i^c\neq\tilde{\emptyset}\Rightarrow\cup_{i\in\triangle}G_i\neq\tilde{X}$. This implies that $\mathcal{C}$ is not an NOC of X, which is a contradiction. Therefore $\mathcal{C}$ must have a finite NOSC. Therefore X is neutrosophic compact.

Hence proved.

3.17. **Theorem :**

Let (X,τ_1) and (Y,τ_2) be two NTSs and let $f : X \to Y$ be a neutrosophic continuous function. If A is neutrosophic compact in (X,τ_1) then $f(A)$ is neutrosophic compact in (Y,τ_2).

Proof: Let $\mathcal{B} = \{G_\lambda : \lambda \in \Delta\}$ be an NOC of $f(A)$, where $G_\lambda = \{\langle y, \mathcal{T}_{G_\lambda}(y), \mathcal{I}_{G_\lambda}(y), \mathcal{F}_{G_\lambda}(y)\rangle : y \in Y\}$. Then $f(A) \subseteq \cup_{\lambda\in\Delta} G_\lambda \Rightarrow f^{-1}(f(A)) \subseteq f^{-1}(\cup_{\lambda\in\Delta} G_\lambda) \Rightarrow f^{-1}(f(A)) \subseteq \cup_{\lambda\in\Delta} f^{-1}(G_\lambda) \Rightarrow A \subseteq \cup_{\lambda\in\Delta} f^{-1}(G_\lambda)[\because A \subseteq f^{-1}(f(A))]$. Since G_λ is open in Y, so $f^{-1}(G_\lambda)$ is open in X as f is continuous. Therefore $C = \{f^{-1}(G_\lambda) : \lambda \in \Delta\}$ is an NOC of A. Since A is compact, so C has a finite NOSC $\{f^{-1}G_{\lambda_1}, f^{-1}G_{\lambda_2}, \cdots, f^{-1}G_{\lambda_n}\}$. Therefore $A \subseteq \cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) \Rightarrow f(A) \subseteq f(\cup_{i=1}^{n} f^{-1}(G_{\lambda_i})) \Rightarrow f(A) \subseteq \cup_{i=1}^{n} f(f^{-1}(G_{\lambda_i})) \Rightarrow f(A) \subseteq \cup_{i=1}^{n} G_{\lambda_i}$. Thus the NOC $\mathcal{B}$ of $f(A)$ has a finite NOSC. Therefore $f(A)$ is neutrosophic compact. Hence proved.

3.18. **Theorem :**

Let (X,τ_1) and (Y,τ_2) be two NTSs and let $f : X \to Y$ is a neutrosophic continuous onto function. If (X,τ_1) is neutrosophic compact then (Y,τ_2) is neutrosophic compact.

Proof: Since f is onto, so $f(\tilde{X}) = \tilde{Y}$. Let $\mathcal{B} = \{G_\lambda : \lambda \in \Delta\}$ be an NOC of Y, where $G_\lambda = \{\langle y, \mathcal{T}_{G_\lambda}(y), \mathcal{I}_{G_\lambda}(y), \mathcal{F}_{G_\lambda}(y)\rangle : y \in Y\}$. Then $\cup_{\lambda\in\Delta} G_\lambda = \tilde{Y} \Rightarrow f^{-1}(\cup_{\lambda\in\Delta} G_\lambda) = f^{-1}(\tilde{Y}) \Rightarrow \cup_{\lambda\in\Delta} f^{-1}(G_\lambda) = \tilde{X}$. Since G_λ is open in Y, so $f^{-1}(G_\lambda)$ is open in X as f is continuous. Therefore $C = \{f^{-1}(G_\lambda) : \lambda \in \Delta\}$ is an NOC of X. Since X is compact, so C has a finite NOSC $\{f^{-1}G_{\lambda_1}, f^{-1}G_{\lambda_2}, \cdots, f^{-1}G_{\lambda_n}\}$. Therefore $\cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) = \tilde{X} \Rightarrow f(\cup_{i=1}^{n} f^{-1}(G_{\lambda_i})) = f(\tilde{X}) \Rightarrow \cup_{i=1}^{n} f(f^{-1}(G_{\lambda_i})) = \tilde{Y} \Rightarrow \cup_{i=1}^{n} G_{\lambda_i} = \tilde{Y}$.. Thus the NOC $\mathcal{B}$ of Y has a finite NOSC. Therefore Y is neutrosophic compact. Hence proved.

3.19. **Theorem :**

Let (X,τ_1) and (Y,τ_2) be two NTSs and let $f : X \to Y$ is a neutrosophic continuous onto function. If X is neutrosophic countably compact then Y is also neutrosophic countably compact.

Proof: Since f is onto, so $f(\tilde{X}) = \tilde{Y}$. Let $\mathcal{A} = \{G_\lambda : \lambda \in \Delta\}$ be a countable NOC of Y, where $G_\lambda = \{\langle y, \mathcal{T}_{G_\lambda}(y), \mathcal{I}_{G_\lambda}(y), \mathcal{F}_{G_\lambda}(y)\rangle : y \in Y\}$. Then $\cup_{\lambda\in\Delta} G_\lambda = \tilde{Y} \Rightarrow f^{-1}(\cup_{\lambda\in\Delta} G_\lambda) = f^{-1}(\tilde{Y}) \Rightarrow \cup_{\lambda\in\Delta} f^{-1}(G_\lambda) = \tilde{X}$. Since G_λ is open in Y, so $f^{-1}(G_\lambda)$ is open in X as f is continuous. Therefore $C = \{f^{-1}(G_\lambda) : \lambda \in \Delta\}$ is an NOC of X. Obviously C is countable as A is countable. Again since X is neutrosophic countably compact, so C has a finite NOSC $\{f^{-1}G_{\lambda_1}, f^{-1}G_{\lambda_2}, \cdots, f^{-1}G_{\lambda_n}\}$. Therefore $\cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) = \tilde{X} \Rightarrow f(\cup_{i=1}^{n} f^{-1}(G_{\lambda_i})) = f(\tilde{X}) \Rightarrow \cup_{i=1}^{n} f(f^{-1}(G_{\lambda_i})) = \tilde{Y} \Rightarrow \cup_{i=1}^{n} G_{\lambda_i} = \tilde{Y}$. Thus $\mathcal{A}$ has a finite NOSC. Hence Y is neutrosophic countably compact.

3.20. **Theorem :**

Let (X, τ_1) and (Y, τ_2) be two NTSs and let $f : X \to Y$ is a neutrosophic continuous onto function. If X is neutrosophic Lindelöf then Y is also neutrosophic Lindelöf.

Proof: Since f is onto, so $f(\tilde{X}) = \tilde{Y}$. Let $\mathcal{A} = \{A_i : i \in\triangle\}$, be an NOC of Y. Then $\tilde{Y} = \bigcup_{i\in\triangle} A_i \Rightarrow f^{-1}(\tilde{Y}) = f^{-1}(\bigcup_{i\in\triangle} A_i) \Rightarrow \tilde{X} = \bigcup_{i\in\triangle} f^{-1}(A_i) \Rightarrow \{f^{-1}(A_i) : i \in\triangle\}$ is an NOC of X. Since X is neutrosophic Lindelöf, so $\{f^{-1}(A_i) : i \in\triangle\}$ has a countable NOSC $\mathcal{B} = \{f^{-1}(A_{i_k}) : k = 1, 2, 3, \ldots\}$. Therefore $\tilde{X} = \bigcup_{k=1}^{i_0} f^{-1}(A_{i_k})$, where i_0 may be infinity. This gives $f(\tilde{X}) = f[\bigcup_{k=1}^{i_0} f^{-1}(A_{i_k})] \Rightarrow \tilde{Y} = \bigcup_{k=1}^{i_0}[f(f^{-1}(A_{i_k})] \Rightarrow \tilde{Y} = \bigcup_{k=1}^{i_0} A_{i_k} \Rightarrow \{A_{i_k} : k = 1, 2, 3, \cdots\}$ an NOC of Y. Since $\mathcal{B}$ is countable, so $\{A_{i_k} : k = 1, 2, 3, \cdots\}$ is also countable. Therefore the NOC $\mathcal{A}$ of Y has a countable NOSC $\{A_{i_k} : k = 1, 2, 3, \cdots\}$, i.e., Y is neutrosophic Lindelöf. Hence proved.

3.21. **Theorem : (Alexander subbase lemma)**

Let β be a subbase of an NTS (X, τ). Then X is neutrosophic compact iff for every collection of neutrosophic closed sets chosen from β^c having the FIP, there is a non-empty intersection.

Proof: Necessary part : Immediate.

Sufficient Part : On the contrary, let us suppose that X is not compact. Then there exists a collection $\mathcal{C} = \{G_i : i \in I\}$, where $G_i = \{\langle x, \mathcal{T}_{G_i}(x), \mathcal{I}_{G_i}(x), \mathcal{F}_{G_i}(x)\rangle : x \in X\}$, of neutrosophic closed sets of X having the FIP such that $\cap_{i\in\triangle} G_i = \tilde{\emptyset}$. The collection of all such collections $\mathcal{C}$ can be arranged in an order by using the classical inclusion($\subseteq$) and the collection will certainly have an upper bound. Therefore by Zorn's lemma, there will be a maximal collection of all the collections $\mathcal{C}$. Let $\mathcal{M} = \{M_j : j \in J\}$ be the maximal collection, where $M_j = \{\langle x, \mathcal{T}_{M_j}(x), \mathcal{I}_{M_j}(x), \mathcal{F}_{M_j}(x)\rangle : x \in X\}$. This collection $\mathcal{M}$ has the following properties : (i) $\tilde{\emptyset} \notin \mathcal{M}$ (ii) $P \in \mathcal{M}, P \subseteq Q \Rightarrow Q \in \mathcal{M}$ (iii) $P, Q \in \mathcal{M} \Rightarrow P \cap Q \in \mathcal{M}$ (iv) $\cap(\mathcal{M} \cap \beta^c) = \tilde{\emptyset}$. Clearly the property (iv) delivers a contradiction to the hypothesis. Therefore X is compact.

Hence proved.

3.22. **Definition :**

An NTS (X, τ) is said to be neutrosophic locally compact iff for every NP $x_{\alpha,\beta,\gamma}$ in X, there exists neutrosophic τ-open set G such that $x_{\alpha,\beta,\gamma} \in G$ and G is neutrosophic compact in X.

3.23. **Theorem :**

Every neutrosophic compact space is neutrosophic locally compact space.

Proof: Let (X, τ) be a neutrosophic compact space and let $x_{\alpha,\beta,\gamma}$ be an NP in X. Since X is neutrosophic compact and since $\tilde{X}$ is a neutrosophic open set containing $x_{\alpha,\beta,\gamma}$, so, X is a neutrosophic locally compact space.

3.24. **Remark :**

Every neutrosophic locally compact space need not be neutrosophic compact space. We establish it by the following example.

Let $X = \mathsf{N} = \{1, 2, 3, \cdots\}$. For $n \in \mathsf{N}$, we define $G_n = \{\langle x, \mathcal{T}_{G_n}(x), \mathcal{I}_{G_n}(x), \mathcal{F}_{G_n}(x) : x \in X\}$, where $\mathcal{T}_{G_n}(x) = 1, \mathcal{I}_{G_n}(x) = 0, \mathcal{F}_{G_n}(x) = 0$ if $x \leq n$ and $\mathcal{T}_{G_n}(x) = 0, \mathcal{I}_{G_n}(x) = 1, \mathcal{F}_{G_n}(x) = 1$ if $x > n$. Let τ be the set consisting of $\tilde{\emptyset}$, $\tilde{X}$ and the neutrosophic sets G_n, $n \in \mathsf{N}$. Obviously (X, τ) is an NTS and it is also clear that (X, τ) is a neutrosophic locally compact space but not a neutrosophic compact space.

3.25. **Theorem :**

Let f be a neutrosophic continuous function from a neutrosophic locally compact space (X, τ) onto an NTS (Y, σ). If f is neutrosophic open function then (Y, σ) is also neutrosophic locally compact space.

Proof: Let $y_{p,q,r}$ be any NP in Y. Also let $x_{\alpha,\beta,\gamma}$ be an NP in X such that $x_{\alpha,\beta,\gamma} \in f^{-1}(y_{p,q,r})$. Then $f(x_{\alpha,\beta,\gamma}) = y_{p,q,r}$. Since $x_{\alpha,\beta,\gamma} \in X$, and X neutrosophic locally compact, so there exists a τ-open set G such that $x_{\alpha,\beta,\gamma} \in G$ and G is neutrosophic compact in X. Now $x_{\alpha,\beta,\gamma} \in G \Rightarrow f(x_{\alpha,\beta,\gamma}) \in f(G) \Rightarrow y_{p,q,r} \in f(G)$. Since f is neutrosophic continuous and G is neutrosophic compact in X, so $f(G)$ is neutrosophic compact in Y. Again since f is a neutrosophic open function, so is $f(G)$ is a σ-open set. Thus for any any NP $y_{p,q,r}$ in Y, there exists a σ-open set $f(G)$ such that $y_{p,q,r} \in f(G)$ and $f(G)$ is neutrosophic compact in Y. Therefore (Y, σ) is neutrosophic locally compact space.

4. Conclusions :

In this article, we have defined neutrosophic compactness, neutrosophic countably compactness, neutrosophic Lindelöfness and investigated various covering properties. Especially we have shown that if a neutrosophic topological space is neutrosophic C_{II} then neutrosophic compactness and neutrosophic countably compactness are equivalent. We have proved that the neutrosophic compactness is preserved under neutrosophic continuous function. We have also stated and proved the neutrosophic version of "Alexander subbase lemma". Lastly, we have defined neutrosophic locally compact space and put forward two propositions with proofs. Hope that the findings in this article will assist the research fraternity to move forward for the development of different aspects of neutrosophic topology.

5. Conflict of Interest

We certify that there is no actual or potential conflict of interest in relation to this article.

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MCGDM based on TOPSIS and VIKOR using Pythagorean neutrosophic soft with aggregation operators

M.Palanikumar[1,*] and K. Arulmozhi[2]

[1] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai-602105, India; palanimaths86@gmail.com

[2] Department of Mathematics, Bharath Institute of Higher and Research, Chennai-600073, India; arulmozhiems@gmail.com

*Correspondence: palanimaths86@gmail.com;

Abstract. Pythagorean neutrosophic soft set (PNSS set) is a new approach towards decision making under uncertainty. The PNSS set has much stronger abilities than the neutrosophic soft set and the Pythagorean fuzzy soft set. In this paper, we discuss aggregated operations for aggregating the PNSS decision matrix. The TOPSIS and VIKOR methods are strong approaches for multi criteria group decision making (MCGDM), which is various extensions of neutrosophic soft sets. In this approach, we propose a score function based on aggregating TOPSIS and VIKOR methods to the PNSS-positive ideal solution and the PNSS-negative ideal solution. Also, the TOPSIS and VIKOR methods provide the weights of decision-making. Afterward, a revised closeness is introduced to identify the optimal alternative.

Keywords: Pythagorean neutrosophic soft set, MCGDM, TOPSIS, VIKOR, aggregation operator.

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1. **Introduction**

The classic article of 1965, Zadeh proposed fuzzy set theory [39]. According to this definition a fuzzy set is a function described by a membership value . It takes degrees in real unit interval. But, later it has been seen that this definition is inadequate by considering not only the degree of membership but also the degree of non-membership. Neutrosophic set is a generalization of the fuzzy set and intuitionistic fuzzy set, where the truth-membership, indeterminacy-membership, and falsity-membership are represented independently. Atanassov [3] described a set that is called an intuitionistic fuzzy set to handle mentioned ambiguity. Since this set has some problems in applications, Smarandache [31] introduced neutrosophy to deal with the problems that involves indeterminate and inconsistent information. Yager [38] as being introduced by the concept of Pythagorean fuzzy sets. It has been extended from intuitionistic fuzzy sets and is distinguished by the requirement that the square sum of their degrees of membership and non-membership not exceed unity. A neutrosophic set is used to tackle uncertainty using the truth, indeterminacy, and falsity membership grades by Smarandache [30]. The theory of soft sets was proposed by [15]. Maji et al. proposed the concepts of the fuzzy soft set [13] and the intuitionistic fuzzy soft set [14]. These two theories are applied to solve various decision making problems. In recent years, Peng et al. [29]have extended the fuzzy soft set to the Pythagorean fuzzy soft set. Smarandache et al. [5, 10] discussed the concept of Pythagorean neutrosophic set approach. A decision-making (DM) problem is the process of finding the best optional alternatives. In almost all such problems, the multiplicity of criteria for judging the alternatives is pervasive. That is, for many such problems, the decision maker wants to solve a multiple criteria decision making (MCDM) problem. A survey of the MCDM methods has been presented by Hwang and Yoon [7]. A MCDM problem can be expressed in matrix format as:

$$\mathscr{D}_{n\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_m \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \end{array}$$

where $A_1, A_2, ..., A_n$ are possible alternatives among which decision makers must choose, $\mathcal{C}_1, \mathcal{C}_2, ..., C_m$ are criteria with which alternative performance is measured, a_{ij} is the rating of alternative A_i in relation to criterion C_j.

Many researchers have studied the TOPSIS and VIKOR methods for decision making problems, including Adeel et al. [1], Akram and Arshad [2], Boran et al. [4], Eraslan and Karaaslan [6], Peng and Dai [28], Xu and Zhang [36] and Zhang and Xu [40]. In 2021, Zulqarnain discussed the TOPSIS technique as it applies to interval valued intuitionistic fuzzy soft sets (IVIFSS) information, where the mechanisms are assumed in terms of IVIFSNs. To measure the degree of dependency of IVIFSS's, [41] discussed a new correlation coefficient for IVIFSS's and examined some properties of the developed correlation coefficient. To achieve the goal accurately, the TOPSIS technique may be extended to solve MADM problems. The basic idea of TOPSIS is rather straightforward. It simultaneously considers the distances to both positive ideal solutions (PIS) and negative ideal solutions (NIS), and a preference order is ranked according to their relative closeness and a combination of these two distance measures. The VIKOR method focuses on ranking and selecting from a set of alternatives, and determining compromise solutions for a problem with conflicting criteria, which can help the decision makers reach a final decision [16,17]. Opricovic and Tzeng [18] suggested using fuzzy logic for the VIKOR method. Tzeng et al. [33] used and compared the VIKOR and TOPSIS methods in solving a public transportation problem. Newly, Pythagorean fuzzy logical with real life applications discussed many authors [8,9,32,34,35,37]. Recently, Palanikumar et al. discussed various field of applications including algebraic structures [11,12,19–27].

2. Preliminaries

Definition 2.1. [5] Let $\mathbb{U}$ be a non-empty set of the universe. A neutrosophic set A in $\mathbb{U}$ is an object having the following form : $A = \{u, \sigma_A^{\mathcal{T}}(u), \sigma_A^{\mathcal{I}}(u), \sigma_A^{\mathcal{F}}(u) | u \in \mathbb{U}\}$, where $\sigma_A^{\mathcal{T}}(u)$, $\sigma_A^{\mathcal{I}}(u)$ $\sigma_A^{\mathcal{F}}(u)$ represents the degree of truth membership, degree of indeterminacy membership and degree of falsity membership of A respectively. The mapping $\sigma_A^{\mathcal{T}}, \sigma_A^{\mathcal{I}}, \sigma_A^{\mathcal{F}} : \mathbb{U} \to [0,1]$ and $0 \leq \sigma_A^{\mathcal{T}}(u) + \sigma_A^{\mathcal{I}}(u) + \sigma_A^{\mathcal{F}}(u) \leq 3$.

Definition 2.2. [10] Let $\mathbb{U}$ be a non-empty set of the universe, Pythagorean neutrosophic set (PNSS) A in $\mathbb{U}$ is an object having the following form : $A = \{u, \sigma_A^{\mathcal{T}}(u), \sigma_A^{\mathcal{I}}(u), \sigma_A^{\mathcal{F}}(u) | u \in \mathbb{U}\}$, where $\sigma_A^{\mathcal{T}}(u)$, $\sigma_A^{\mathcal{I}}(u)$ $\sigma_A^{\mathcal{F}}(u)$ represents the degree of truth membership, degree of indeterminacy membership and degree of falsity membership of A respectively. The mapping $\sigma_A^{\mathcal{T}}, \sigma_A^{\mathcal{I}}, \sigma_A^{\mathcal{F}} :$ $\mathbb{U} \to [0,1]$ and $0 \leq (\sigma_A^{\mathcal{T}}(u))^2 + (\sigma_A^{\mathcal{I}}(u))^2 + (\sigma_A^{\mathcal{F}}(u))^2 \leq 2$. Since $A = (\sigma_A^{\mathcal{T}}, \sigma_A^{\mathcal{I}}, \sigma_A^{\mathcal{F}})$ is called a Pythagorean neutrosophic number(PNSN).

Definition 2.3. The score function for any PNSN $A = (\sigma_A^{\mathcal{T}}, \sigma_A^{\mathcal{I}}, \sigma_A^{\mathcal{F}})$ is defined as $S(A) = \sigma_A^{2T} - \sigma_A^{2I} - \sigma_A^{2F}$, where $-1 \leq S(A) \leq 1$.

3. MCGDM based on PNSS sets

Definition 3.1. Let $\mathbb{U}$ be a non-empty set of the universe and E be a set of parameter. The pair (Δ, A) or Δ_A is called a Pythagorean neutrosophic soft set (PNSS set) on $\mathbb{U}$ if $A \sqsubseteq E$ and $\Delta : A \to PNS^{\mathbb{U}}$, where $PNS^{\mathbb{U}}$ is represent the aggregate of all Pythagorean neutrosophic subsets of $\mathbb{U}$. (ie) $\Delta_A = \left\{ \left(e, \left\{ \frac{u}{\left(\sigma_{\Delta_A}^{\mathcal{T}}(u), \sigma_{\Delta_A}^{\mathcal{I}}(u), \sigma_{\Delta_A}^{\mathcal{F}}(u)\right)} \right\} \right) : e \in A, u \in \mathbb{U} \right\}$.

Remark 3.2. If we write $a_{ij} = \sigma_{\Delta_A}^{\mathcal{T}}(e_j)(u_i)$, $b_{ij} = \sigma_{\Delta_A}^{\mathcal{I}}(e_j)(u_i)$ and $c_{ij} = \sigma_{\Delta_A}^{\mathcal{F}}(e_j)(u_i)$, where $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ then the PNSS set Δ_A may be represented in matrix form as

$$\Delta_A = [(a_{ij}, b_{ij}, c_{ij})]_{m\times n} = \begin{bmatrix} (a_{11}, b_{11}, c_{11}) & (a_{12}, b_{12}, c_{12}) & \dots & (a_{1n}, b_{1n}, c_{1n}) \\ (a_{21}, b_{21}, c_{21}) & (a_{22}, b_{22}, c_{22}) & \dots & (a_{2n}, b_{2n}, c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (a_{m1}, b_{m1}, c_{m1}) & (a_{m2}, b_{m2}, c_{m2}) & \dots & (a_{mn}, b_{mn}, c_{mn}) \end{bmatrix}$$

This matrix is called Pythagorean neutrosophic soft matrix (PNSSM).

Definition 3.3. The cardinal set of the PNSS set Δ_A over $\mathbb{U}$ is a PNSS set over E and is defined as $c\Delta_A = \left\{ \frac{e}{\left(\sigma^{\mathcal{T}}_{c\delta_A}(e), \sigma^{\mathcal{I}}_{c\zeta_A}(e), \sigma^{\mathcal{F}}_{c\varphi_A}(e)\right)} : e \in E \right\}$, where $\sigma^{\mathcal{T}}_{c\delta_A}$, $\sigma^{\mathcal{I}}_{c\zeta_A}$ and $\sigma^{\mathcal{F}}_{c\varphi_A}$ are mapping from E to unit interval respectively, where $\sigma^{\mathcal{T}}_{c\delta_A}(e) = \frac{|\delta_A(e)|}{|\mathbb{U}|}$, $\sigma^{\mathcal{I}}_{c\zeta_A}(e) = \frac{|\zeta_A(e)|}{|\mathbb{U}|}$ and $\sigma^{\mathcal{F}}_{c\varphi_A}(e) = \frac{|\varphi_A(e)|}{|\mathbb{U}|}$ where $|\delta_A(e)|$, $|\zeta_A(e)|$ and $|\varphi_A(e)|$ denote the scalar cardinalities of the PNSS sets $\delta_A(e)$, $\zeta_A(e)$ and $\varphi_A(e)$respectively, and $|\mathbb{U}|$ represents cardinality of the universe $\mathbb{U}$. The collection of all cardinal sets of PNSS sets of $\mathbb{U}$ is represented as $cPNS^{\mathbb{U}}$. If $A \subseteq E = \{e_i : i = 1, 2, ..., n\}$, then $c\Delta_A \in cPNS^{\mathbb{U}}$ may be represented in matrix form as $[(a_{1j}, b_{1j}, c_{1j})]_{1\times n} = [(a_{11}, b_{11}, c_{11}), (a_{12}, b_{12}, c_{12}), ..., (a_{1n}, b_{1n}, c_{1n})]$, where $(a_{1j}, b_{1j}, c_{1j}) = \mu_{c\Delta_A}(e_j)$, for $j = 1, 2, ..., n$. This matrix is termed as cardinal matrix of $c\Delta_A$ of E.

Definition 3.4. Let $\Delta_A \in PNS^{\mathbb{U}}$ and $c\Delta_A \in cPNS^{\mathbb{U}}$. The PNSS set aggregation operator $PNSS_{agg} : cPNS^{\mathbb{U}} \times PNS^{\mathbb{U}} \to PNSS(\mathbb{U}, E)$ is defined as
$PNSS_{agg}(c\Delta_A, \Delta_A) = \left\{ \frac{u}{\mu_{\Delta^*_A}(u)} : u \in \mathbb{U} \right\} = \left\{ \frac{u}{\left(\sigma^{\mathcal{T}}_{\delta^*_A}(u), \sigma^{\mathcal{I}}_{\zeta^*_A}(u), \sigma^{\mathcal{F}}_{\varphi^*_A}(u)\right)} : u \in \mathbb{U} \right\}$. This collection is called aggregate Pythagorean neutrosophic set of PNSS set Δ_A. The degree of truth membership $\sigma^{\mathcal{T}}_{\delta^*_A}(u) : \mathbb{U} \to [0, 1]$ by $\sigma^{\mathcal{T}}_{\delta^*_A}(u) = \frac{1}{|E|} \sum_{e\in E} \left(\sigma^{\mathcal{T}}_{c\delta_A}(e), \sigma^{\mathcal{T}}_{\delta_A}(e)\right)(u)$, degree of indeterminacy membership $\sigma^{\mathcal{I}}_{\zeta^*_A}(u) : \mathbb{U} \to [0, 1]$ by $\sigma^{\mathcal{I}}_{\zeta^*_A}(u) = \frac{1}{|E|} \sum_{e\in E} \left(\sigma^{\mathcal{I}}_{c\zeta_A}(e), \sigma^{\mathcal{I}}_{\zeta_A}(e)\right)(u)$ and degree of falsity membership $\sigma^{\mathcal{F}}_{\varphi^*_A}(u) : \mathbb{U} \to [0, 1]$ by $\sigma^{\mathcal{F}}_{\varphi^*_A}(u) = \frac{1}{|E|} \sum_{e\in E} \left(\sigma^{\mathcal{F}}_{c\varphi_A}(e), \sigma^{\mathcal{F}}_{\varphi_A}(e)\right)(u)$. The set $PNSS_{agg}(c\Delta_A, \Delta_A)$ is expressed in matrix form as

$$[(a_{i1}, b_{i1}, c_{i1})]_{m\times 1} = \begin{bmatrix} (a_{11}, b_{11}, c_{11}) \\ (a_{21}, b_{21}, c_{21}) \\ \vdots \\ (a_{m1}, b_{m1}, c_{m1}) \end{bmatrix}$$

where $[(a_{i1}, b_{i1}, c_{i1})] = \mu_{\Delta^*_A}(u_i)$, for $i = 1, 2, ..., m$. This matrix is called PNSS aggregate matrix of $PNSS_{agg}(c\Delta_A, \Delta_A)$ over $\mathbb{U}$.

Definition 3.5. Let $A = (\sigma^{\mathcal{T}}_{ij}, \sigma^{\mathcal{I}}_{ij}, \sigma^{\mathcal{F}}_{ij}) \in PNSSM_{m\times n}$, then the choice matrix of PNSSM A is given by $\mathscr{C}(A) = \left[\left(\frac{\sum_{j=1}^{n}(\sigma^{\mathcal{T}}_{ij})^2}{n}, \frac{\sum_{j=1}^{n}(\sigma^{\mathcal{I}}_{ij})^2}{n}, \frac{\sum_{j=1}^{n}(\sigma^{\mathcal{F}}_{ij})^2}{n}\right)\right]_{m\times 1}$ $\forall i$ when weights are equal.

Definition 3.6. Let $A = (\sigma^{\mathcal{T}}_{ij}, \sigma^{\mathcal{I}}_{ij}, \sigma^{\mathcal{F}}_{ij}) \in PNSSM_{m\times n}$, then the weighted choice matrix of PNSSM A is given by $\mathscr{C}_w(A) = \left[\left(\frac{\sum_{j=1}^{n} w_j(\sigma^{\mathcal{T}}_{ij})^2}{\sum w_j}, \frac{\sum_{j=1}^{n} w_j(\sigma^{\mathcal{I}}_{ij})^2}{\sum w_j}, \frac{\sum_{j=1}^{n} w_j(\sigma^{\mathcal{F}}_{ij})^2}{\sum w_j}\right)\right]_{m\times 1}$ $\forall i$ where $w_j > 0$ are weights (means weights are unequal).

Theorem 3.7. *Let Δ_A be a PNSS set. Suppose that $M_{\Delta_A}, M_{c\Delta_A}, M^*_{\Delta_A}$ are matrices of $\Delta_A, c\Delta_A, \Delta^*_A$ respectively, then $M_{\Delta_A} \times M^{\mathbb{T}}_{c\Delta_A} = M^*_{\Delta_A} \times |E|$, where $M^{\mathbb{T}}_{c\Delta_A}$ is the transpose of $M_{c\Delta_A}$.*

Proof. The proof follows Definition 3.3 and Definition 3.4.

We can make a MCGDM based on PNSS sets by the following algorithms:

Algorithm-I

Step 1: Construct PNSS set Δ_A over the universal $\mathbb{U}$.
Step 2: Compute the cardinalities and find the cardinal set $c\Delta_A$ of Δ_A .
Step 3: Find aggregate PNSS set Δ_A^* of Δ_A.
Step 4: Compute the value of score function by $S(u) = \sigma_u^{2\mathcal{T}} - \sigma_u^{2\mathcal{I}} - \sigma_u^{2\mathcal{F}}, \forall\, u \in \mathbb{U}$.
Step 5: Compute $S(u)$ is maximum is the best alternative.

Example 3.8. Suppose that an automobile company produces ten different types of cars $\mathbb{U} = \{\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_{10}\}$ and lets a set of parameters $E = \{e_1, e_2, ..., e_5\}$ represent fuel economy, acceleration, top speed, ride comfort, and good power steering, respectively. Suppose that a customer has to decide which car purchase ? Following the discussion, each car is evaluated using a subset of parameters $A = \{e_1, e_2, e_4\} \subseteq E$. We apply the above algorithm as follows.

Step-1: We Construct PNSS set Δ_A of $\mathbb{U}$ is defined as below:

$$\Delta_A = \left\{ \left(e_1, \left\{\frac{\mathcal{C}_1}{(0.55,0.75,0.6)}, \frac{\mathcal{C}_4}{(0.8,0.7,0.65)}, \frac{\mathcal{C}_7}{(0.7,0.75,0.55)}, \frac{\mathcal{C}_9}{(0.9,0.5,0.8)}, \frac{\mathcal{C}_{10}}{(0.65,0.6,0.6)}\right\}\right), \right.$$
$$\left(e_2, \left\{\frac{\mathcal{C}_2}{(0.6,0.75,0.5)}, \frac{\mathcal{C}_3}{(0.65,0.55,0.8)}, \frac{\mathcal{C}_5}{(0.55,0.65,0.6)}, \frac{\mathcal{C}_8}{(0.65,0.7,0.7)}, \frac{\mathcal{C}_{10}}{(0.5,0.8,0.55)}\right\}\right),$$
$$\left.\left(e_4, \left\{\frac{\mathcal{C}_3}{(0.75,0.7,0.7)}, \frac{\mathcal{C}_4}{(0.5,0.6,0.75)}, \frac{\mathcal{C}_6}{(0.6,0.65,0.8)}, \frac{\mathcal{C}_8}{(0.7,0.75,0.7)}, \frac{\mathcal{C}_9}{(0.9,0.55,0.7)}\right\}\right) \right\}.$$

Step-2: The cardinal set of Δ_A as $c\Delta_A = \left\{\frac{e_1}{(0.36,0.33,0.32)}, \frac{e_2}{(0.295,0.345,0.315)}, \frac{e_4}{(0.345,0.325,0.365)}\right\}$.

Step-3: The aggregate PNSS set Δ_A^* of Δ_A is $M_{\Delta_A^*} = \frac{M_{\Delta_A} \times M_{c\Delta_A}^{\mathbb{T}}}{|E|}$

$$= \frac{1}{5}\left\{ \begin{bmatrix} 0.55 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0.65 & 0 & 0.75 & 0 \\ 0.8 & 0 & 0 & 0.5 & 0 \\ 0 & 0.55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.65 & 0 & 0.7 & 0 \\ 0.9 & 0 & 0 & 0.9 & 0 \\ 0.65 & 0.5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.36 \\ 0.295 \\ 0 \\ 0.345 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.75 & 0 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 & 0 \\ 0 & 0.55 & 0 & 0.7 & 0 \\ 0.7 & 0 & 0 & 0.6 & 0 \\ 0 & 0.65 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.65 & 0 \\ 0.75 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.75 & 0 \\ 0.5 & 0 & 0 & 0.55 & 0 \\ 0.6 & 0.8 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.33 \\ 0.345 \\ 0 \\ 0.325 \\ 0 \end{bmatrix}, \right.$$

$$\left.\begin{bmatrix} 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0.7 & 0 \\ 0.65 & 0 & 0 & 0.75 & 0 \\ 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \\ 0.55 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.7 & 0 \\ 0.8 & 0 & 0 & 0.7 & 0 \\ 0.6 & 0.55 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.32 \\ 0.315 \\ 0 \\ 0.365 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0.0396 \\ 0.0354 \\ 0.0901 \\ 0.0921 \\ 0.03245 \\ 0.0414 \\ 0.0504 \\ 0.08665 \\ 0.1269 \\ 0.0763 \end{bmatrix}, \begin{bmatrix} 0.0495 \\ 0.05175 \\ 0.08345 \\ 0.0852 \\ 0.04485 \\ 0.04225 \\ 0.0495 \\ 0.09705 \\ 0.06875 \\ 0.0948 \end{bmatrix}, \begin{bmatrix} 0.0384 \\ 0.0315 \\ 0.1015 \\ 0.09635 \\ 0.0378 \\ 0.0584 \\ 0.0352 \\ 0.0952 \\ 0.1023 \\ 0.07305 \end{bmatrix} \right\}.$$

Hence, $\Delta^*_A = \Big\{ \frac{\mathcal{C}_1}{(0.0396, 0.0495, 0.0384)}, \frac{\mathcal{C}_2}{(0.0354, 0.05175, 0.0315)}, \frac{\mathcal{C}_3}{(0.0901, 0.08345, 0.1015)}, \frac{\mathcal{C}_4}{(0.0921, 0.0852, 0.09635)}, \frac{\mathcal{C}_5}{(0.03245, 0.04485, 0.0378)}, \frac{\mathcal{C}_6}{(0.0414, 0.04225, 0.0584)}, \frac{\mathcal{C}_7}{(0.0504, 0.0495, 0.0352)}, \frac{\mathcal{C}_8}{(0.08665, 0.09705, 0.0952)}, \frac{\mathcal{C}_9}{(0.1269, 0.06875, 0.1023)}, \frac{\mathcal{C}_{10}}{(0.0763, 0.0948, 0.07305)} \Big\}$.

Step-4: The values of the score function $S(\mathcal{C}_i)$ for each element of $\mathbb{U}$ are tabulated as follows.

Car	$S(\mathcal{C}_i)$
$\mathcal{C}_1$	-0.00236
$\mathcal{C}_2$	-0.00242
$\mathcal{C}_3$	-0.00915
$\mathcal{C}_4$	-0.00806
$\mathcal{C}_5$	-0.00239
$\mathcal{C}_6$	-0.00348
$\mathcal{C}_7$	-0.00115
$\mathcal{C}_8$	-0.01097
$\mathcal{C}_9$	0.00091
$\mathcal{C}_{10}$	-0.0085

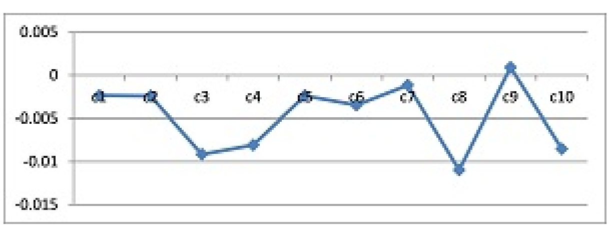

Figure 1 Graphical representation using MCGDM based on PNSS.

Step 5: Since $\max_i S(\mathcal{C}_i) = 0.00091$ which corresponds to $\mathcal{C}_9$. Therefore in this case the most suitable car $\mathcal{C}_9$ for the customer would be purchased.

Algorithm-II

Step-1: Construct Pythagorean neutrosophic soft matrix (PNSS matrix) on the basis of the parameters.

Step-2: Case-I (Equal weights) Compute the choice matrix for the positive membership, neutral membership and negative membership of PNSS matrix.

Case-II (Unequal weights) Compute the choice matrix for the positive membership, neutral membership and negative membership of PNSS matrix.

Step-3: Choose alternative with maximum score value.

Case-I: By Example 3.8,

$$\mathscr{C}(A) = \left\{ \begin{bmatrix} 0.0605 \\ 0.072 \\ 0.197 \\ 0.178 \\ 0.0605 \\ 0.072 \\ 0.098 \\ 0.1825 \\ 0.324 \\ 0.1345 \end{bmatrix}, \begin{bmatrix} 0.1125 \\ 0.1125 \\ 0.1585 \\ 0.17 \\ 0.0845 \\ 0.0845 \\ 0.1125 \\ 0.2105 \\ 0.1105 \\ 0.2 \end{bmatrix}, \begin{bmatrix} 0.072 \\ 0.05 \\ 0.226 \\ 0.197 \\ 0.072 \\ 0.128 \\ 0.0605 \\ 0.196 \\ 0.226 \\ 0.1325 \end{bmatrix} \right\}$$

Score value =

Car	$S(\mathcal{C}_i)$
$\mathcal{C}_1$	-0.01418
$\mathcal{C}_2$	-0.00997
$\mathcal{C}_3$	-0.03739
$\mathcal{C}_4$	-0.03603
$\mathcal{C}_5$	-0.00866
$\mathcal{C}_6$	-0.01834
$\mathcal{C}_7$	-0.00671
c_8	-0.04942
$\mathcal{C}_9$	0.04169
$\mathcal{C}_{10}$	-0.03947

Case-II: Weights $(w_j) = \{0.16, 0.19, 0.25, 0.22, 0.18\}$.

By Example 3.8,

$$\mathscr{C}_w(A) = \left\{ \begin{bmatrix} 0.0484 \\ 0.0684 \\ 0.204025 \\ 0.1574 \\ 0.057475 \\ 0.0792 \\ 0.0784 \\ 0.188075 \\ 0.3078 \\ 0.1151 \end{bmatrix}, \begin{bmatrix} 0.09 \\ 0.106875 \\ 0.165275 \\ 0.1576 \\ 0.080275 \\ 0.09295 \\ 0.09 \\ 0.21685 \\ 0.10655 \\ 0.1792 \end{bmatrix}, \begin{bmatrix} 0.0576 \\ 0.0475 \\ 0.2294 \\ 0.19135 \\ 0.0684 \\ 0.1408 \\ 0.0484 \\ 0.2009 \\ 0.2102 \\ 0.115075 \end{bmatrix} \right\}$$

Score value =

Car	$S(\mathcal{C}_i)$
$\mathcal{C}_1$	-0.00908
$\mathcal{C}_2$	-0.009
$\mathcal{C}_3$	-0.03831
$\mathcal{C}_4$	-0.03668
$\mathcal{C}_5$	-0.00782
$\mathcal{C}_6$	-0.02219
$\mathcal{C}_7$	-0.0043
$\mathcal{C}_8$	-0.05201
$\mathcal{C}_9$	0.0392
$\mathcal{C}_{10}$	-0.03211

Algorithm-III

Step-1: Obtain the aggregated Pythagorean neutrosophic weighted averaging (PNSWA) numbers $\mathscr{C}(A) = \left(\sum_{j=1}^{n} w_j\sigma_{ij}^{\mathcal{T}}, \sum_{j=1}^{n} w_j\sigma_{ij}^{\mathcal{I}}, \sum_{j=1}^{n} w_j\sigma_{ij}^{\mathcal{F}}\right)$.

Step-2: Compute the score function of $S(\mathcal{C}_i)$.

Step-3: Select the optimal alternative by $\max_i S(\mathcal{C}_i)$ value.

Weights $(w_j) = \{0.16, 0.19, 0.25, 0.22, 0.18\}$.

By Example 3.8,

$$\mathscr{C}(A) = \left\{ \begin{bmatrix} 0.088 \\ 0.114 \\ 0.2885 \\ 0.238 \\ 0.1045 \\ 0.132 \\ 0.112 \\ 0.2775 \\ 0.342 \\ 0.199 \end{bmatrix}, \begin{bmatrix} 0.12 \\ 0.1425 \\ 0.2585 \\ 0.244 \\ 0.1235 \\ 0.143 \\ 0.12 \\ 0.298 \\ 0.201 \\ 0.248 \end{bmatrix}, \begin{bmatrix} 0.096 \\ 0.095 \\ 0.306 \\ 0.269 \\ 0.114 \\ 0.176 \\ 0.088 \\ 0.287 \\ 0.282 \\ 0.2005 \end{bmatrix} \right\}$$

Score value =

Car	$S(\mathcal{C}_i)$
$\mathcal{C}_1$	-0.01587
$\mathcal{C}_2$	-0.01634
$\mathcal{C}_3$	-0.07723
$\mathcal{C}_4$	-0.07525
$\mathcal{C}_5$	-0.01733
$\mathcal{C}_6$	-0.034
$\mathcal{C}_7$	-0.0096
$\mathcal{C}_8$	-0.09417
$\mathcal{C}_9$	-0.00296
$\mathcal{C}_{10}$	-0.0621

3.1. *Analysis for PNSS-Methods:*

Analysis of final ranking as follows:

Methods	*Ranking of alternatives*	*Optimal alternatives*
Algorithm – I	$\mathcal{C}_8 \le \mathcal{C}_3 \le \mathcal{C}_4 \le \mathcal{C}_6 \le \mathcal{C}_2 \le \mathcal{C}_5 \le \mathcal{C}_1 \le \mathcal{C}_7 \le \mathcal{C}_{10} \le \mathcal{C}_9$	$\mathcal{C}_9$
Algorithm – II Case – (i)	$\mathcal{C}_8 \le \mathcal{C}_{10} \le \mathcal{C}_3 \le \mathcal{C}_4 \le \mathcal{C}_6 \le \mathcal{C}_1 \le \mathcal{C}_2 \le \mathcal{C}_5 \le \mathcal{C}_7 \le \mathcal{C}_9$	$\mathcal{C}_9$
Algorithm – II Case – (ii)	$\mathcal{C}_8 \le \mathcal{C}_3 \le \mathcal{C}_4 \le \mathcal{C}_{10} \le \mathcal{C}_6 \le \mathcal{C}_1 \le \mathcal{C}_2 \le \mathcal{C}_5 \le \mathcal{C}_7 \le \mathcal{C}_9$	$\mathcal{C}_9$
Algorithm – III	$\mathcal{C}_8 \le \mathcal{C}_3 \le \mathcal{C}_4 \le \mathcal{C}_{10} \le \mathcal{C}_6 \le \mathcal{C}_5 \le \mathcal{C}_2 \le \mathcal{C}_1 \le \mathcal{C}_7 \le \mathcal{C}_9$	$\mathcal{C}_9$

Therefore most suitable car $\mathcal{C}_9$ for the customer would be purchased.

4. MCGDM based on PNSS-TOPSIS aggregating operator

Algorithm-IV (PNSS-TOPSIS)

Step-1: Assume that $\mathscr{D} = \{\mathscr{D}_i : i \in \mathbb{N}\}$ is a finite set of decision makers/experts, $\mathscr{C} = \{z_i : i \in \mathbb{N}\}$ is the finite collection of alternatives and $D = \{e_i : i \in \mathbb{N}\}$ is a finite family of parameters/criterion.

Step-2: By selecting the linguistic terms and constructing weighted parameter matrix $\mathscr{P}$ can be written as

$$\mathscr{P} = [w_{ij}]_{n\times m} = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1m} \\ w_{21} & w_{22} & \dots & w_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & \dots & w_{im} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \dots & w_{nm} \end{bmatrix}$$

Where w_{ij} is the weight assigned by the expert $\mathscr{D}_i$ to the alternative $\mathscr{P}_j$ by considering linguistic variables.

Step-3: Construct weighted normalized decision matrix using the following

$$\widehat{\mathscr{N}} = [\widehat{n}_{ij}]_{n\times m} = \begin{bmatrix} \widehat{n}_{11} & \widehat{n}_{12} & \dots & \widehat{n}_{1m} \\ \widehat{n}_{21} & \widehat{n}_{22} & \dots & \widehat{n}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{n}_{i1} & \widehat{n}_{i2} & \dots & \widehat{n}_{im} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{n}_{n1} & \widehat{n}_{n2} & \dots & \widehat{n}_{nm} \end{bmatrix}$$

where $\widehat{n}_{ij} = \frac{w_{ij}}{\sqrt{\sum_{i=1}^{n} w_{ij}^2}}$ is the normalized criteria rating and obtaining the weighted vector $\mathscr{W} = (m_1, m_2, ..., m_m)$, where $m_i = \frac{w_i}{\sqrt{\sum_{l=1}^{n} w_{li}}}$ is the relative weight of the j^{th} criterion and $w_j = \frac{\sum_{i=1}^{n} \widehat{n}_{ij}}{n}$.

Step-4: Construct PNSS decision matrix can be calculate as follows

$$\mathscr{D}_i = [x^i_{jk}]_{l\times m} = \begin{bmatrix} x^i_{11} & x^i_{12} & \dots & x^i_{1m} \\ x^i_{21} & x^i_{22} & \dots & x^i_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x^i_{j1} & x^i_{j2} & \dots & x^i_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ x^i_{l1} & x^i_{l2} & \dots & x^i_{lm} \end{bmatrix}$$

Where x^i_{jk} is a PNSS element for i^{th} decision maker so that $\mathscr{D}_i$ for each i. Then obtain the aggregating matrix $\mathscr{A} = \frac{\mathscr{D}_1+\mathscr{D}_2+...+\mathscr{D}_n}{n} = [y_{jk}]_{l\times m}$.

Step-5: Find the weighted PNSS decision matrix by

$$\mathscr{Y} = [z_{jk}]_{l\times m} = \begin{bmatrix} z_{11} & z_{12} & \dots & z_{1m} \\ z_{21} & z_{22} & \dots & z_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{j1} & z_{j2} & \dots & z_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ z_{l1} & z_{l2} & \dots & z_{lm} \end{bmatrix}$$

Where $z_{jk} = m_k \times y_{jk}$.

Step-6: Calculate PNSSV-PIS and PNSSV-NIS. Now,
PNSSV-PIS $= [z_1^+, z_2^+, ..., z_l^+] = \{(\vee_k z_{jk}, \wedge_k z_{jk}, \wedge_k z_{jk}) : k = 1, 2, ..., m\}$ and PNSSV-PIS $= [z_1^-, z_2^-, ..., z_l^-] = \{(\wedge_k z_{jk}, \vee_k z_{jk}, \vee_k z_{jk}) : k = 1, 2, ..., m\}$, where $\vee$ stands for PNSS union and $\wedge$ represents PNSS intersection.

Step-7: Compute PNSS-Euclidean distances of each alternative from PNSSV-PIS and PNSSV-NIS. Now, $(d_j^+)^2 = \sum_{k=1}^m \left\{(\sigma_{jk}^{T+} - \sigma_j^{T+})^2 + (\sigma_{jk}^{I+} - \sigma_j^{I+})^2 + (\sigma_{jk}^{F+} - \sigma_j^{F+})^2\right\}$ and $(d_j^-)^2 = \sum_{k=1}^m \left\{(\sigma_{jk}^{T-} - \sigma_j^{T-})^2 + (\sigma_{jk}^{I-} - \sigma_j^{I-})^2 + (\sigma_{jk}^{F-} - \sigma_j^{F-})^2\right\}$, where $j = 1, 2, ..., n$.

Step-8: Calculate the relative closeness of each alternative to the ideal solution by $C^*(z_j) = \frac{d_j^-}{d_j^+ + d_j^-} \in [0, 1]$.

Step-9: The rank of alternatives in decreasing or increasing order of their relative closeness coefficients. The bigger $C^*(z_j)$, the more desirable alternative z_j.

Step-10: The best alternative is the one with the highest relative closeness to the ideal solution.

Example 4.1. Assume that a firm plans to invest some money in stock exchange by purchasing some shares of best five companies. In order to minimize the risk factor, they decide to invest their money 30%, 25%, 20%, 15% and 10% in accordance with the top ranked five companies.

Step-1: Assume that $\mathscr{D} = \{\mathscr{D}_i : i = 1, 2, 3, 4, 5\}$ is a finite set of decision makers/experts, $\mathscr{C} = \{z_i : i = 1, 2, ..., 10\}$ is the collection of companies/alternatives and $D = \{e_i : i = 1, 2, ..., 5\}$ is a finite family of parameters/criterion, where e_1 = Momentum, e_2 = Value, e_3 = Growth, e_4 = Volatility, e_5 = Quality.

Step-2: Forms a Linguistic terms for judging alternatives as given below:

Linguistic terms	*Fuzzy weights*
$Very\ Good\ Testing(VGT)$	0.95
$Good\ Testing(GT)$	0.80
$Average\ Testing(AT)$	0.65
$Poor\ Testing(PT)$	0.50
$Very\ Poor\ Testing(VPT)$	0.35

Construct weighted parameter matrix

$$\mathscr{P} = [w_{ij}]_{5\times5} = \begin{bmatrix} GC & VGC & PC & VPC & AC \\ AC & GC & VPC & PC & GC \\ PC & AC & VGC & VGC & VPC \\ VGC & PC & AC & GC & PC \\ AC & VPC & VGC & GC & VPC \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.95 & 0.5 & 0.35 & 0.65 \\ 0.65 & 0.8 & 0.35 & 0.5 & 0.8 \\ 0.5 & 0.65 & 0.95 & 0.95 & 0.35 \\ 0.95 & 0.5 & 0.65 & 0.8 & 0.5 \\ 0.65 & 0.35 & 0.95 & 0.8 & 0.35 \end{bmatrix}$$

Where w_{ij} is the weight provided by the specialist $\mathscr{D}_i$ to each parameter $\mathscr{P}_j$.

Step-3: The normalized weighted decision matrix is

$$\begin{aligned} \widehat{\mathscr{N}} &= [\widehat{n}_{ij}]_{5\times 5} \\ &= \begin{bmatrix} 0.4926 & 0.6214 & 0.3101 & 0.219 & 0.5208 \\ 0.4002 & 0.5233 & 0.2171 & 0.3128 & 0.641 \\ 0.3079 & 0.4251 & 0.5892 & 0.5943 & 0.2804 \\ 0.585 & 0.327 & 0.4031 & 0.5005 & 0.4006 \\ 0.4002 & 0.2289 & 0.5892 & 0.5005 & 0.2804 \end{bmatrix}. \end{aligned}$$

And weighted vector is $\mathscr{W} = (0.1231, 0.1308, 0.124, 0.1251, 0.1603)$.

Step-4: The aggregated decision matrix $\mathscr{A}$ can be written as

$$\begin{aligned} \mathscr{A} &= \frac{\mathscr{D}_1 + \mathscr{D}_2 + \mathscr{D}_3 + \mathscr{D}_4 + \mathscr{D}_5}{5} \\ &= \begin{bmatrix} (0.78, 0.48, 0.7) & (0.7, 0.45, 0.6) & (0.68, 0.6, 0.65) & (0.65, 0.75, 0.9) & (0.78, 0.57, 0.6) \\ (0.8, 0.7, 0.9) & (0.65, 0.75, 0.85) & (0.64, 0.66, 0.64) & (0.69, 0.8, 0.67) & (0.68, 0.81, 0.7) \\ (0.75, 0.65, 0.75) & (0.72, 0.68, 0.42) & (0.72, 0.87, 0.45) & (0.74, 0.7, 0.59) & (0.62, 0.56, 0.85) \\ (0.8, 0.95, 0.62) & (0.9, 0.8, 0.65) & (0.85, 0.8, 0.41) & (0.81, 0.8, 0.56) & (0.9, 0.69, 0.75) \\ (0.8, 0.55, 0.95) & (0.55, 0.65, 0.9) & (0.62, 0.61, 0.68) & (0.69, 0.54, 0.67) & (0.68, 0.62, 0.7) \\ (0.84, 0.83, 0.62) & (0.9, 0.8, 0.45) & (0.9, 0.43, 0.73) & (0.83, 0.49, 0.8) & (0.9, 0.68, 0.45) \\ (0.79, 0.65, 0.75) & (0.75, 0.55, 0.65) & (0.78, 0.65, 0.55) & (0.65, 0.75, 0.9) & (0.8, 0.57, 0.6) \\ (0.75, 0.7, 0.68) & (0.76, 0.7, 0.42) & (0.8, 0.43, 0.43) & (0.47, 0.8, 0.85) & (0.83, 0.5, 0.55) \\ (0.85, 0.61, 0.74) & (0.66, 0.58, 0.65) & (0.7, 0.62, 0.78) & (0.4, 0.9, 0.64) & (0.58, 0.77, 0.6) \\ (0.9, 0.55, 0.65) & (0.63, 0.62, 0.8) & (0.69, 0.72, 0.55) & (0.83, 0.6, 0.49) & (0.62, 0.49, 0.78) \end{bmatrix} \\ &= [y_{jk}]_{10\times 5} \end{aligned}$$

Step-5: The weighted PNSS decision matrix $\mathscr{Y}$ can be written as $\mathscr{Y} = m_k \times y_{jk} =$

$$\begin{bmatrix} (0.0961, 0.0591, 0.0862) & (0.0916, 0.0589, 0.0785) & (0.0843, 0.0744, 0.0806) & (0.0813, 0.0938, 0.1126) & (0.125, 0.0913, 0.0962) \\ (0.0985, 0.0862, 0.1108) & (0.085, 0.0981, 0.1112) & (0.0794, 0.0819, 0.0794) & (0.0863, 0.1001, 0.0838) & (0.109, 0.1298, 0.1122) \\ (0.0924, 0.08, 0.0924) & (0.0942, 0.089, 0.0549) & (0.0893, 0.1079, 0.0558) & (0.0926, 0.0876, 0.0738) & (0.0994, 0.0897, 0.1362) \\ (0.0985, 0.117, 0.0764) & (0.1177, 0.1047, 0.085) & (0.1054, 0.0992, 0.0509) & (0.1013, 0.1001, 0.0701) & (0.1442, 0.1106, 0.1202) \\ (0.0985, 0.0677, 0.117) & (0.0719, 0.085, 0.1177) & (0.0769, 0.0757, 0.0843) & (0.0863, 0.0676, 0.0838) & (0.109, 0.0994, 0.1122) \\ (0.1034, 0.1022, 0.0764) & (0.1177, 0.1047, 0.0589) & (0.1116, 0.0533, 0.0905) & (0.1039, 0.0613, 0.1001) & (0.1442, 0.109, 0.0721) \\ (0.0973, 0.08, 0.0924) & (0.0981, 0.0719, 0.085) & (0.0967, 0.0806, 0.0682) & (0.0813, 0.0938, 0.1126) & (0.1282, 0.0913, 0.0962) \\ (0.0924, 0.0862, 0.0837) & (0.0994, 0.0916, 0.0549) & (0.0992, 0.0533, 0.0533) & (0.0588, 0.1001, 0.1064) & (0.133, 0.0801, 0.0881) \\ (0.1047, 0.0751, 0.0911) & (0.0863, 0.0759, 0.085) & (0.0868, 0.0769, 0.0967) & (0.05, 0.1126, 0.0801) & (0.0929, 0.1234, 0.0962) \\ (0.1108, 0.0677, 0.08) & (0.0824, 0.0811, 0.1047) & (0.0856, 0.0893, 0.0682) & (0.1039, 0.0751, 0.0613) & (0.0994, 0.0785, 0.125) \end{bmatrix}$$

$= [z_{jk}]_{10\times 5}$.

Step-6: We find PNSSV-PIS and PNSSV-NIS can be written as

z^+	$PNSSV-PIS$
z_1^+	$(0.125, 0.0589, 0.0785)$
z_2^+	$(0.109, 0.0819, 0.0794)$
z_3^+	$(0.0994, 0.08, 0.0549)$
z_4^+	$(0.1442, 0.0992, 0.0509)$
z_5^+	$(0.109, 0.0676, 0.0838)$
z_6^+	$(0.1442, 0.0533, 0.0589)$
z_7^+	$(0.1282, 0.0719, 0.0682)$
z_8^+	$(0.133, 0.0533, 0.0533)$
z_9^+	$(0.1047, 0.0751, 0.0801)$
z_{10}^+	$(0.1108, 0.0677, 0.0613)$

z^-	$PNSSV-NIS$
z_1^-	$(0.0813, 0.0938, 0.1126)$
z_2^-	$(0.0794, 0.1298, 0.1122)$
z_3^-	$(0.0893, 0.1079, 0.1362)$
z_4^-	$(0.0985, 0.117, 0.1202)$
z_5^-	$(0.0719, 0.0994, 0.1177)$
z_6^-	$(0.1034, 0.109, 0.1001)$
z_7^-	$(0.0813, 0.0938, 0.1126)$
z_8^-	$(0.0588, 0.1001, 0.1064)$
z_9^-	$(0.05, 0.1234, 0.0967)$
z_{10}^-	$(0.0824, 0.0893, 0.125)$

Step-7: We found PNSS euclidean distances of each alternative from PNSSV-PIS and PNSSV-NIS.

$Alternative\,(z_i)$	d_i^+	d_i^-
z_1	0.0979	0.0906
z_2	0.0899	0.0964
z_3	0.0979	0.1446
z_4	0.1166	0.1174
z_5	0.0863	0.0859
z_6	0.1282	0.1025
z_7	0.0983	0.0851
z_8	0.1408	0.1381
z_9	0.0906	0.1217
z_{10}	0.0937	0.1101

Step-8: We calculate closeness coefficients of each alternative from PNSSV-PIS and PNSSV-NIS.

$Alternative\,(z_i)$	C_i^*
z_1	0.4807
z_2	0.5174
z_3	0.5962
z_4	0.5017
z_5	0.4988
z_6	0.4444
z_7	0.4639
z_8	0.4951
z_9	0.5734
z_{10}	0.5404

Step-9: The order of the alternatives for C_i^* is $z_3 \geq z_9 \geq z_{10} \geq z_2 \geq z_4 \geq z_5 \geq z_8 \geq z_1 \geq z_7 \geq z_6$.

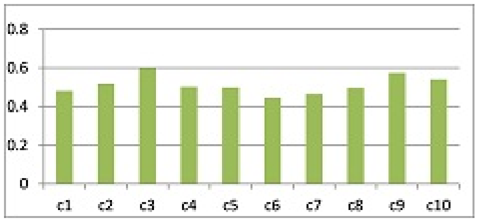

Figure 2 Graphical representation using MCGDM based on TOPSIS.

Step-10: The above ranking, it conclude that the firm should z_3 invest 30%, z_9 invest 25%, z_{10} invest 20%, z_2 invest 15% and z_4 invest 10%.

5. MCGDM based on PNSS-VIKOR aggregating operator

Algorithm-V (PNSS-VIKOR)

Step-1: Assume that $\mathscr{D} = \{\mathscr{D}_i : i \in \mathbb{N}\}$ is a finite set of decision makers/experts, $\mathscr{C} = \{z_i : i \in \mathbb{N}\}$ is the finite collection of alternatives and $D = \{e_i : i \in \mathbb{N}\}$ is a finite family of parameters/criterion.

Step-2: By selecting the linguistic terms and constructing weighted parameter matrix $\mathscr{P}$ can be written as

$$\mathscr{P} = [w_{ij}]_{n\times m} = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1m} \\ w_{21} & w_{22} & \dots & w_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & \dots & w_{im} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \dots & w_{nm} \end{bmatrix}$$

Where w_{ij} is the weight assigned by the expert $\mathscr{D}_i$ to the alternative $\mathscr{P}_j$ by considering linguistic variables.

Step-3: Construct weighted normalized decision matrix using the following

$$\widehat{\mathscr{N}} = [\widehat{n}_{ij}]_{n\times m} = \begin{bmatrix} \widehat{n}_{11} & \widehat{n}_{12} & \dots & \widehat{n}_{1m} \\ \widehat{n}_{21} & \widehat{n}_{22} & \dots & \widehat{n}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{n}_{i1} & \widehat{n}_{i2} & \dots & \widehat{n}_{im} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{n}_{n1} & \widehat{n}_{n2} & \dots & \widehat{n}_{nm} \end{bmatrix}$$

where $\widehat{n}_{ij} = \frac{w_{ij}}{\sqrt{\sum_{i=1}^{n} w_{ij}^2}}$ is the normalized criteria rating and obtaining the weighted vector $\mathscr{W} = (m_1, m_2, ..., m_m)$, where $m_i = \frac{w_i}{\sqrt{\sum_{l=1}^{n} w_{li}}}$ is the relative weight of the j^{th} criterion and $w_j = \frac{\sum_{i=1}^{n} \widehat{n}_{ij}}{n}$.

Step-4: Construct PNSS decision matrix can be calculated as

$$\mathscr{D}_i = [x^i_{jk}]_{l\times m} = \begin{bmatrix} x^i_{11} & x^i_{12} & \dots & x^i_{1m} \\ x^i_{21} & x^i_{22} & \dots & x^i_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x^i_{j1} & x^i_{j2} & \dots & x^i_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ x^i_{l1} & x^i_{l2} & \dots & x^i_{lm} \end{bmatrix}$$

Where x^i_{jk} is a PNSS element for i^{th} decision maker so that $\mathscr{D}_i$ for each i. Then obtain the aggregating matrix $\mathscr{A} = \frac{\mathscr{D}_1+\mathscr{D}_2+...+\mathscr{D}_n}{n} = [y_{jk}]_{l\times m}$.

Step-5: Construct the weighted PNSS decision matrix by

$$\mathscr{Y} = [z_{jk}]_{l\times m} = \begin{bmatrix} z_{11} & z_{12} & \dots & z_{1m} \\ z_{21} & z_{22} & \dots & z_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{j1} & z_{j2} & \dots & z_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ z_{l1} & z_{l2} & \dots & z_{lm} \end{bmatrix}$$

Where $z_{jk} = m_k \times y_{jk}$.

Step-6: Calculate the values of PNSSV-PIS and PNSSV-NIS. Now, PNSSV-PIS $= [z^+_1, z^+_2, ..., z^+_l] = \{(\vee_k z_{jk}, \wedge_k z_{jk}, \wedge_k z_{jk}) : j = 1,2,...,l\}$ and PNSSV-PIS $= [z^-_1, z^-_2, ..., z^-_l] = \{(\wedge_k z_{jk}, \vee_k z_{jk}, \vee_k z_{jk}) : j = 1,2,...,l\}$, where $\vee$ stands for PNSS union and $\wedge$ represents PNSS intersection.

Step-7: Find the values of utility $\mathscr{S}_i$, individual regret $\mathscr{R}_i$ and compromise $\mathscr{Q}_i$, where $\mathscr{S}_i = \sum_{j=1}^m m_j \left(\frac{d(z_{ij}, z^+_j)}{d(z^+_j, z^-_j)}\right)$, $\mathscr{R}_i = \max_{j=1}^m m_j \left(\frac{d(z_{ij}, z^+_j)}{d(z^+_j, z^-_j)}\right)$ and $\mathscr{Q}_i = \kappa\left(\frac{\mathscr{S}_i - \mathscr{S}^-}{\mathscr{S}^+ - \mathscr{S}^-}\right) + (1-\kappa)\left(\frac{\mathscr{R}_i - \mathscr{R}^-}{\mathscr{R}^+ - \mathscr{R}^-}\right)$. Where $\mathscr{S}^+ = \max_i \mathscr{S}_i$, $\mathscr{S}^- = \min_i \mathscr{S}_i$, $\mathscr{R}^+ = \max_i \mathscr{R}_i$ and $\mathscr{R}^- = \min_i \mathscr{R}_i$. The real number κ is called a coefficient of decision mechanism. The role of κ is that if compromise solution is to be selected by majority if $\kappa > 0.5$; for consensus if $\kappa = 0.5$ and $\kappa < 0.5$ represents veto. Let m_j represents the weight of the j^{th} criteria.

Step-8: The rank of choices and derive compromise solution. Arrange $\mathscr{S}_i$, $\mathscr{R}_i$ and $\mathscr{Q}_i$ in increasing order to make these three ranking lists. The alternative z_α will be declared compromise solution if it ranks the best in $\mathscr{Q}_i$ (having least value) and satisfies the following two requirements simultaneously:

$[C-1]$ acceptable: If z_α and z_β represent top alternatives in $\mathscr{Q}_i$, then $\mathscr{Q}(z_\beta) - \mathscr{Q}(z_\alpha) \geq \frac{1}{n-1}$, where n is the number of parameters.

$[C-2]$ acceptable: The alternative z_α should be best ranked by $\mathscr{S}_i$ and /or $\mathscr{R}_i$.

If above two conditions are not met simultaneously, then there exist multiple compromise solutions:

(i) If only condition $C-1$ is satisfied, then both alternatives z_α and z_β are called the compromise solutions:

(ii) If condition $C-1$ is not satisfied, then the alternatives z_α, z_β,..., z_ζ are called the compromise solutions, where z_ζ is founded by $\mathscr{Q}(z_\zeta) - \mathscr{Q}(z_\alpha) \geq \frac{1}{n-1}$.

Example 5.1. We resolve Example 4.1 using VIKOR method. The first five steps are the same as in Example 4.1. So we start with step 6.

Step-6: We compute PNSSV-PIS and PNSSV-NIS are listed as follows.

z^+	$PNSSV-PIS$
z_1^+	$(0.1108, 0.0591, 0.0764)$
z_2^+	$(0.1177, 0.0589, 0.0549)$
z_3^+	$(0.1116, 0.0533, 0.0509)$
z_4^+	$(0.1039, 0.0613, 0.0613)$
z_5^+	$(0.1442, 0.0785, 0.0721)$

z^-	$PNSSV-NIS$
z_1^-	$(0.0924, 0.117, 0.117)$
z_2^-	$(0.0719, 0.1047, 0.1177)$
z_3^-	$(0.0769, 0.1079, 0.0967)$
z_4^-	$(0.05, 0.1126, 0.1126)$
z_5^-	$(0.0929, 0.1298, 0.1362)$

Step-7: Taking $\kappa = 0.5$, we found that the values of utility $\mathscr{S}_i$, individual regret $\mathscr{R}_i$ and compromise $\mathscr{Q}_i$ for each alternative z_i.

$Alternative\,(z)$	$\mathscr{S}_i$	$\mathscr{R}_i$	$\mathscr{Q}_i$
z_1	0.2972	0.0897	0.2208
z_2	0.457	0.1225	0.9271
z_3	0.3763	0.1309	0.7881
z_4	0.4024	0.0997	0.5843
z_5	0.4104	0.1189	0.7732
z_6	0.3065	0.0737	0.1049
z_7	0.3031	0.0897	0.2364
z_8	0.2666	0.1033	0.2591
z_9	0.4212	0.1196	0.8079
z_{10}	0.3184	0.1148	0.4958

Step-8: The rank of alternatives for $\mathscr{Q}_i$: $z_6 \leq z_1 \leq z_7 \leq z_8 \leq z_{10} \leq z_4 \leq z_5 \leq z_3 \leq z_9 \leq z_2$. Now, $\mathscr{Q}(z_1) - \mathscr{Q}(z_6) = 0.1159 \ngeq \frac{1}{4}$. Thus, the condition C-1 is not satisfied. Further $\mathscr{Q}(z_{10}) - \mathscr{Q}(z_6) = 0.3909 \geq \frac{1}{4}$. Therefore, we decide $z_6, z_1, z_7, z_8, z_{10}$ are multiple compromise solutions. Hence the firm should invest 30% on z_6, 25% on z_1, 20% on z_7, 15% on z_8 and 10% on z_{10}.

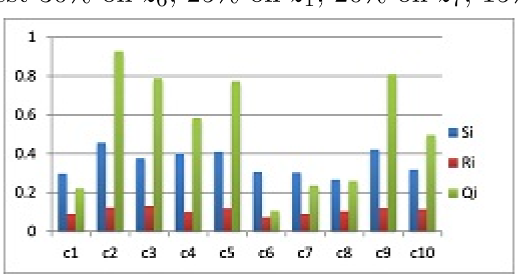

Figure 3 Graphical representation using MCGDM based on VIKOR.

6. Analysis and discussion

We used the above example to analyse the two methods in the literature. The ranking results of all ten alternatives were obtained using these two approaches. These two methods assume a scale component for each criterion. The normalisation approach is different in these two methods. The TOPSIS method utilises a vector normalisation approach and the VIKOR method utilises a linear normalisation approach. The TOPSIS method uses "n"- dimensional Euclidean distance that by itself could constitute some balance between total and individual contentment, but the VIKOR method uses a different way by which weight "κ" is introduced. The major difference between the two methods is in the aggregation function. We can find the ranking of values using an aggregating function. The best ranked alternative by VIKOR is closest to the ideal solution. However, the best ranked alternative by TOPSIS is the one using the ranking index, which does not mean the closest to the ideal solution. Hence, the advantage of the VIKOR method gives a compromise solution.

7. Conclusion:

In this communication, we studied various properties of PNSSS and PNSSM that occur in investment decision making. We proposed the first four algorithms, followed by MCGDM under PNSS. The last two algorithms are based on PNSS linguistic TOPSIS and VIKOR approaches using aggregation operators. Again, we interact with the PNSS aggregation operator and score function values based on some technique. Also, we made use of various sorts of statistical charts to imagine the rankings of different alternatives under consideration. We have analyzed an application of the new approach in a DM problem regarding the selection of particulars where we can see the different conclusions obtained by using different types of aggregation operators.

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University of New Mexico

Clustering Algorithm Based on Data indeterminacy in Neutrosophic Set

Dan Zhang [1], Yingcang Ma [1,*], Florentin Smarandache [2], Xuezhen Dai [3] and Yaqin Qiao[3]
[1] School of Science, Xi'an Polytechnic University, Xi'an, China;
[2] Mathematics and Science Division, Gallup Campus, University of New Mexico, Gallup, NM, USA;
[3] The Public Sector, Xi'an Traffic Engineering Institute, Xi'an, China;

* Correspondence: mayingcang@xpu.edu.cn;

Abstract: Clustering research is an important field in machine learning, pattern recognition and other fields. The neutrosophic set characterizes the data through true membership functions, indeterminate membership functions and false membership functions. Data clustering using neutrosophic set has become one of the current research hotspots. In this paper, first, a new definition of data uncertainty in a neutrosophic set is proposed in this paper based on the density of data. Next, a clustering model based on the uncertainty value of neutrosophic set data is proposed by considering the main cluster (true membership) and the noise cluster (false membership) in the data set. The model takes into account the distance of the data points to the cluster centers and the indeterminacy value of each data point, and then minimizes the proposed cost function by the method of Lagrangian multipliers. The true membership value and false membership value of each data point can be obtained. Finally, the effectiveness of the method is demonstrated by experiments on the various datasets. Experimental results show that the cost function has more accurate membership degree when dealing with boundary points and outliers, and outperforms existing clustering methods on datasets.

Keywords: neutrosophic set; data indeterminacy; clustering algorithm

1. Introduction

Clustering is to divide data into disjoint groups, each of which satisfies two rules: Objects are similar (or related) to each other within the same group (minimizing intra-cluster distance), and at the same time different (or unrelated) to other groups (maximizing inter-cluster distance). Data clustering is an important field in machine learning and has a wide range of applications in computer vision, image processing, medicine, geology, and pattern recognition [1-6].

In k-type clustering, the clustering method represented by k-means [7] is hard clustering, and *k*-means makes each data point belong to exactly one cluster. It divides the data into *k* clusters by minimizing the intra-cluster squared distance and the main disadvantage is that it cannot ensure a global minimum variance. *K*-medoid is a variant of *k*-means that computes the median of each cluster for its cluster center. One of the strongest assumptions in median-based clustering models is that objects must belong to one (and only one) cluster. However, Krishnapuram proposed the fuzzy *k*-center clustering algorithm (FKM) [8]. The essential difference between FKM and *k*-means is that FKM allows each data point to have membership in all clusters, rather than a single cluster with different memberships. Kannan [9] proposed a robust kernel-based FKM by combining normed kernel function and center initialization algorithm. Reference[10] introduced adaptive spatial information theory fuzzy clustering into traditional FKM to improve robustness.

Different from the hard clustering, the fuzzy clustering allows each object to be assigned to all clusters with different degrees of membership. FCM [11] is the most typical fuzzy clustering algorithm. But FCM has four major problems: 1) It just minimizes the variance within the class and does not consider the variance between clusters like the *k*-means algorithm does. 2) The result of clustering depends largely on the initialization. 3) It is sensitive to noise, and the membership degree of noise points may be high. 4) It is also sensitive to the type of distance metric and cannot distinguish between equally likely and equally less likely data points. Krishnapuram and Keller proposed a new possibility *c*-means (PCM) [12]. However, it is sensitive to cluster center initialization, requires additional parameters to be tuned, and may generate overlapping clusters. Reference [13] proposed a robust sparse fuzzy *k*-means algorithm (RSFKM), which introduced a robust function to deal with outliers and noise points to enhance the robustness and sparsity of the FCM algorithm. Reference [14] proposed a variant of fuzzy clustering and hard clustering called relational fuzzy c-means. In recent years, many clustering methods have been developed based on different theories [15-17].

The neutrosophic theory [18] was first proposed by Smarandache in 1995. Picture fuzzy set is a standardized form of neutrosophic set. Thong [19] proposed an picture fuzzy clustering algorithm(FC-PFS). This algorithm needs to calculate three matrices of the same scale, and the clustering effect is not good for high-dimensional data. Li [20] proposed a single-valued neutrosophic clustering algorithm based on Tsallis entropy maximization in the framework of picture fuzzy set clustering and single-valued neutropenic set. The algorithm showed good results in image segmentation. Another the algorithms are based on the original neutrosophic set framework. For example, Guo [21] proposed the neutrosophic c-means clustering algorithm (NCM) based on the neutrosophic set and FCM, which can effectively distinguish the sample points, boundary points and outliers in the cluster. The true membership is not affected by noise, which effectively solves the problem that the FCM algorithm cannot detect abnormal data points. Rashno [22] proposed a neutrosophic clustering algorithm based on data indeterminacy, which can effectively separate boundary points and noise points. Ye [23] proposed a single-valued neutrosophic minimum spanning tree clustering algorithm (SVNMST) by defining a generalized single-valued neutrosophic set distance measure, which showed great superiority in the clustering of single-valued neutrosophic observation data. Kandasamy [24] proposed a dual-valued neutrosophic minimum spanning tree clustering algorithm (DVNMST) to cluster data represented by dual-valued neutrosophic information. All previous methods deal with boundaries and outliers directly in the cost function. This paper mainly deals with boundary points and outliers by proposing an indeterminate set (I) in the NS set, and expressing this set as a new clustering cost function. The rest of the paper is organized as follows. Section II reviews the FKM algorithm and the NS set. Section III presents the proposed method(INCA). Section IV presents the experimental results of the method on scatter and real datasets. Finally, Section V concludes the paper.

2. Related Algorithms

2.1 Definition of NS

X is a set of objects, x is an element in X, and the neutrosophic set A on X can be expressed as

$$A=\{[x,(T_A(x),I_A(x),F_A(x))] \mid x\in X\}, \tag{1}$$

where $T_{\dot{A}}(x)$ is the true value of the object, $I_{\dot{A}}(x)$ is the indeterminate value, $F_{\dot{A}}(x)$ is the false value. They belongs to the standard and non-standard subsets in $]0^-,1^+[$, namely

$T_A(x), I_A(x), F_A(x): X \rightarrow]0^-,1^+[$. The sum of $T_A(x), I_A(x), F_A(x)$ has no limit, so there is $0^- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$.

2.2 FKM

The FKM algorithm is a clustering algorithm based on the median of objects, and its objective function is as follows

$$\min Z_{FKM} = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}\left(e_{ij}\right)^{h}, \tag{2}$$

$$\begin{aligned} s.t.\ & \sum_{j=1}^{n} e_{ij} = 1, \forall i \in \{1,\cdots,n\}; \\ & e_{ij} \le e_{jj}; \quad \sum_{j=1}^{n} e_{jj} = k; \\ & e_{ij} \in [0,1], \forall i, j \in \{1,\cdots,n\}, i \ne j; \\ & e_{jj} \in \{0,1\}, \forall j \in \{1,\cdots,n\}. \end{aligned} \tag{3}$$

where e_{ij} is the representation of the data object o_i to the cluster center o_j (if o_j is not the cluster center, then e_{ij}), and h is the fuzzy factor. The fuzzy factor is a hyperparameter that represents the expected degree of overlap between the clusters to be found. When $h \to 1+$, data objects are often assigned to a cluster, the clustering is very clear. When $h \to \infty$, objects tend to be evenly distributed in each cluster. The final membership value for each non-cluster center and each cluster center is $1/k$.

Given a set of known cluster centers (selected from sample points), the membership of each object to the selected cluster center can be found by computing the following expression:

$$e_{ij} = 1 \Big/ \sum_{t|e_{tt}=1}\left(\frac{d_{ij}}{d_{it}}\right)^{1/(h-1)}, \tag{4}$$

2.3 NCM

The neutrosophic *c*-means (NCM) [10] defines the true membership, false membership and indeterminate membership of the data. NCM can handle boundary points and outliers contained in the dataset itself . Solve the following convex optimization problem:

$$J(T,I,F) = \sum_{i=1}^{N}\sum_{j=1}^{C}\left(\varpi_1 T_{ij}\right)^m \left\|x_i - c_j\right\|^2 + \sum_{i=1}^{N}\left(\varpi_2 I_i\right)^m \left\|x_i - \overline{c}_{i\max}\right\|^2 + \sum_{i=1}^{N}\delta^2\left(\varpi_3 F_i\right)^m, \tag{5}$$

where m is a constant. T_{ij}, I_i, F_i are the membership value belongs to the determinate clusters, boundary regions and noise datasets. Define $0 < T_{ij}, I_i, F_i < 1$, satisfying the following constraints:

$$\sum_{j=1}^{C} T_{ij} + I_i + F_i = 1, \tag{6}$$

For each data point *i*, the cluster centeris $c_{i\max}$ calculated using T_{ij} with the largest and second largest value:

$$c_{i\max}=\frac{c_{pi}+c_{qi}}{2}, \tag{7}$$

$$\begin{cases} p_i=\arg\max\limits_{j=1,2,\ldots C}(T_{ij}) \\ q_j=\arg\max\limits_{j\neq pi\cap j=1,2,\ldots C}(T_{ij}) \end{cases} \tag{8}$$

3. INCA

3.1 Characterization of indeterminacy

A new clustering method is proposed in this paper, which can cluster data containing outliers and boundary points. The basic idea is to combine the FCM algorithm with the neutrosophic set. First, we define the indeterminate for each data point through Euclidean distance, and use the uncertainty in the neutrosophic set to describe it.

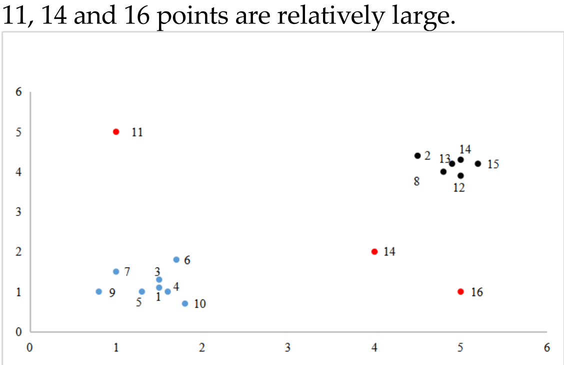
$$\rho_i=\sum_j \chi\left(d_{ij}-d_c\right), \quad \chi(x)=\begin{cases}1, & x<0\\ 0, & x>0\end{cases}, \tag{9}$$

$$\delta_i=\begin{cases}\min_{j\in I_s^i}\{d_{ij}\}, & I_s^i\neq\varnothing\\ \max_j\{d_{ij}\}, & I_s^i=\varnothing\end{cases} \tag{10}$$

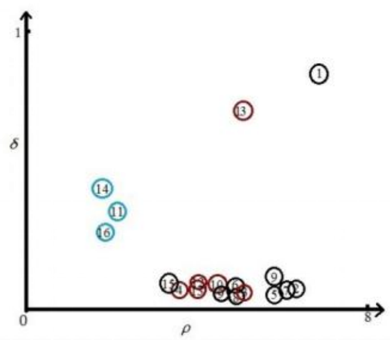
$$I_s^i=\{k:\rho_k>\rho_i\} \tag{11}$$

$$I_i=\frac{1}{\rho_i\delta_i} \tag{12}$$

where ρ_i is the local density of the i-th data sample and δ_i is the distance attribute of the *i*-th data sample. If a point is denser than its neighbors and has a relatively large distance from the more dense point, the point is considered to be within the main cluster and should have less uncertainty. Instead, the point has a larger indeterminate value. This idea makes the uncertainty close to 1 for noise points and close to 0 for the points within the main cluster. Lower uncertainty is assigned to the points in dense regions and not vice versa. As shown in Figure 1, points 1 and 18 in the left figure are the cluster centers of the two clusters. It can be seen from the right figure that the values of δ_i and ρ_i of the two points are large, so the indeterminate value is small. The indeterminate values of 11, 14 and 16 points are relatively large.



(**a**)



(**b**)

Figure 1. The distribution of data points, ρ_i and δ_i (**a**) data points; (**b**) ρ_i and δ_i

3.2 Model

In INCA, the determinate and indeterminate membership of the main cluster and noise points is considered. Set A is the union of determinate clusters and indeterminate clusters, $A = C_i \bigcup R; i = 1, 2, \cdots, k$; where C_i and R represent determinate clusters and indeterminate clusters I, and $\bigcup$ is the union operator. In clustering applications, C_i and R represent the membership degree of the true set and the false set. Therefore, C_i and R are the union of true and false set in the NS set. We hope that a smaller distance $\left\| x_i - c_j \right\|^2$ corresponds to a larger true membership T_{ij} and a smaller false membership F_i. It indicates that the data points x_i are easily divided into the corresponding clusters c_j. A larger distance $\left\| x_i - c_j \right\|^2$ corresponds to a smaller true membership T_{ij} and a larger false membership F_i. It indicates that the data points are not easily divided into the corresponding clusters c_j. The objective function of the proposed algorithm is:

$$L(T,F) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left\| x_i - x_j \right\|_2^2 \left(\omega_1 I_i T_{ij}\right)^m + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\omega_2 \left(1 - I_i\right) F_i\right)^m e^{-\left\| x_i - x_j \right\|^2}, \tag{13}$$

where T_{ij} and F_i are the membership of the data i to the main cluster j and the membership of the noise cluster. For each data point, the following conditions are simultaneously met:

$$s.t. \ \sum_{j=1}^{n} T_{ij} + F_i = 1, \quad \forall i \in \{1, \cdots, n\}, \tag{14}$$

The decision variable $T_{ij}\left(i, j \in \{1, 2, \cdots n\}\right)$ is the membership degree that assigns the data object i to the cluster center *j* (if the data point *j* is not a cluster center, $T_{ij} = 0$). To comply with the constraints of NS theory, constraints (14) are defined. As can be seen from the above model, there are two conditions for data point *i* to have the highest membership degree to the cluster *j*: a) the distance of data point *i* to cluster center *j* is less than the distance to other cluster centers. b) The data point *i* should have a small indeterminacy. Similarly, there are two conditions for data point *i* to have the highest membership to a noisy cluster: a) it has the largest sum distance from all main clusters. b) The data point i should have a large indeterminacy.

3.3 Model solution

The Lagrangian function of the model is:

$$\begin{aligned} L(T,F) = & \sum_{i=1}^{n}\sum_{j=1}^{n}\left\| x_i - x_j \right\|_2^2 \left(\omega_1 I_i T_{ij}\right)^m + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\omega_2 \left(1 - I_i\right) F_i\right)^m e^{-\left\| x_i - x_j \right\|^2} - \\ & \sum_{i=1}^{n}\lambda_i \left(\sum_{j=1}^{n} T_{ij} + F_i - 1\right) \end{aligned}, \tag{15}$$

To minimize the Lagrange objective function, we use the following operations:

$$\frac{\partial L}{\partial T_{ij'}} = m\left(\omega_1 I_i\right)^m T_{ij'}^{\ m-1} \left\| x_i - c_{j'} \right\|_2^2 - \lambda_i, \tag{16}$$

$$\frac{\partial L}{\partial F_i} = m\left(\omega_2\left(1-I_i\right)^m\right)F_i^{m-1}\sum_{j'=1}^{c} e^{-\left\|x_i-c_{j'}\right\|_2^2} - \lambda_i , \tag{17}$$

The norm is specified as the Euclidean norm. Let $\frac{\partial L}{\partial T_{ij'}} = 0$ and $\frac{\partial L}{\partial F_i} = 0$, then:

$$T_{ij'} = \left(\frac{\lambda_i}{m}\right)^{\frac{1}{m-1}}\left(\omega_1 I_i\right)^{-\frac{m}{m-1}}\left\|x_i - c_{j'}\right\|_2^{-2/m-1} , \tag{18}$$

$$F_i = \left(\frac{\lambda_i}{m}\right)^{\frac{1}{m-1}}\left(\omega_2\left(1-I_i\right)\right)^{-\frac{m}{m-1}}\left(\sum_{j'=1}^{c} e^{-\left\|x_i-c_{j'}\right\|_2^2}\right)^{-\frac{1}{m-1}} , \tag{19}$$

Let $\left(\frac{\lambda_i}{m}\right)^{\frac{1}{m-1}} = Ktemp$,

$$Ktemp = \left(\left(\omega_1 I_i\right)^{-\frac{m}{m-1}}\sum_{j'=1}^{c}\left\|x_i - c_{j'}\right\|_2^{-2/m-1} + \left(\omega_2\left(1-I_i\right)\right)^{-\frac{m}{m-1}}\left(\sum_{j'=1}^{c} e^{-\left\|x_i-c_{j'}\right\|_2^2}\right)^{-\frac{1}{m-1}}\right)^{-1} , \tag{20}$$

Therefore:

$$T_{ij'} = Ktemp\left(\omega_1 I_i\right)^{-\frac{m}{m-1}}\left\|x_i - c_{j'}\right\|_2^{-2/m-1} , \tag{21}$$

$$F_i = Ktemp\left(\omega_2\left(1-I_i\right)\right)^{-\frac{m}{m-1}}\left(\sum_{j'=1}^{c} e^{-\left\|x_i-c_{j'}\right\|_2^2}\right)^{-\frac{1}{m-1}} , \tag{22}$$

The above equations allow the formulation of INCA algorithm. It can be summarized in the following steps:

INCA algorithm:
input：X、n、k、D
output：T, F;
1：randomly select k centers;
2：Calculate T using Equation (21);
3：Calculate F using Equation(22);
4：Calculate the value of the objective function Z_1;
5：Select k centers by exhaustive method;
6：Calculate T_2;
7：Calculate F_2;
8：Calculate the value of the objective function Z_2;
9：Compare the values of Z_1 and Z_2 , if $Z_2 < Z_1$, go back to step 5. If $Z_2 > Z_1$, assign the center of Z_1 to Z_2 , T_2 to T, F_2 to F and the end.

The time complexity of INCA is divided into two parts. The first part is the calculation of the memberships T and F. It is related to the sample dimension, the number of samples and the number of categories, and needs to traverse all the data points in the data. If the dimension of the given dataset is m, the number of sample points is n, and the number of clusters is c, the algorithm complexity is $O(n^2mc+n^2m)$. The second part is the exhaustive optimization process, which needs to iteratively calculate the memberships T and F, so the complexity of this part of the algorithm is $O[n!(n^2mc+n^2m)]$. The overall algorithm complexity of this paper is $O[n!(n^2mc+n^2m)]$. We can see that the computational complexity is very high when m and n are large.

4. Results

4.1. Datasets

The performance of INCA is evaluated on artificial datasets and real datasets. The proposed method is compared with INCM [22], FC-PFS [19], RFKM [13], NCM [21], and FKM [8] methods. In the experiment of the exhaustive clustering center, we only extract the same proportion of sample points from each class, and appropriately reduce the running time of the algorithm.

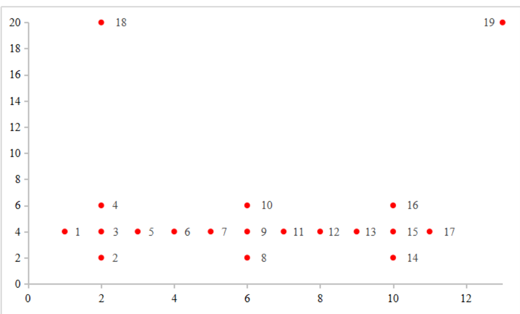
The parameter dc of the uncertainty calculation part is set by the method in the article [25]. In the cost function of INCA, the parameters are configured as $m=1.3, \omega_1=1, \omega_2=2$.

In this section, three types of datasets are used to evaluate the performance of INCA. The first is the diamond dataset, including the X19 and X24 scatter datasets proposed by Guo [25], and a scatter dataset we designed. In these datasets, border points between the main clusters and outliers far from the main clusters are considered. It is easy to see how the clustering method is affected by the main points in each dataset. The second is the UCI dataset, which includes higher-dimensional and larger-scale datasets. There are mainly dermatology, pima, TOX-171, votel, ecoli, iris, ionosphere and vote.

4.2. Results

4.2.1. Artificial datasets

The X19 dataset has three clusters in Figure 2, points 1-5, 7-11 and 13-17 are points in the main cluster, points 6 and 12 are boundary points, points 18 and 19 are noise points. Figure 3 shows the clustering results of INCA. The memberships calculated by INCA and the FKM are counted in Table 1. Although INCA and FKM assign the same cluster label to all points, INCA assigns the points(e.g. 5, 7, 11, 12) with higher indeterminate membership in their corresponding clusters. Data point 5 has the same distance between the main and border clusters, but it belongs to the main cluster. FKM cannot distinguish point 5 as a boundary or a main cluster. INCA solves this problem, and the membership of point 5 assigned to the main cluster is 0.67, while the FKM is 0.36. Figure 4 visually depicts the membership of INCA and the FKM algorithm.



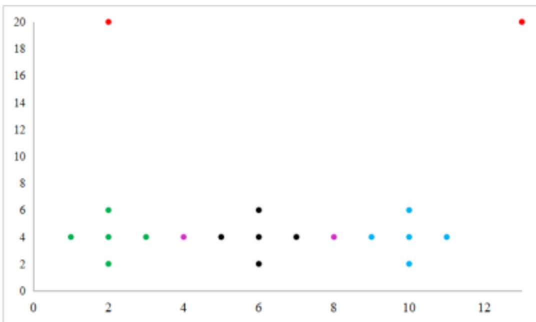
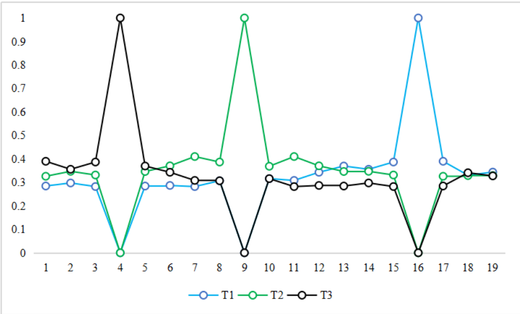
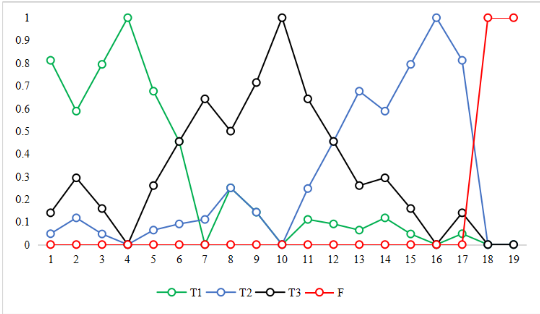


Figure 2. X19



(a)

Figure 3. Clustering results of INCA on X19



(b)

Figure 4. Membership calculated by FKM and INCA on X19

Table 1. Clustering results of X19

	FKM			INCA				
	U_1	U_2	U_3	T_1	T_2	T_3	F	
1	0.2844	0.3259	0.3897	0.8122	0.0478	0.1400	0	
2	0.2974	0.3469	0.3556	0.5882	0.1176	0.2941	0	
3	0.2821	0.3313	0.3865	0.7944	0.0467	0.1589	0	
4	0	0	1	1	0	0	0	
5	0.2843	0.3462	0.3695	0.6762	0.0638	0.2601	0	
6	0.2868	0.3704	0.3429	0.4545	0.0910	0.4545	0	boundary
7	0.2819	0.4099	0.3082	0.2470	0.1107	0.6422	0	
8	0.3068	0.3865	0.3068	0.25	0.25	0.5	0	
9	0	1	0	0.1429	0.1429	0.7143	0	
10	0.3158	0.3684	0.3158	0	0	0	1	
11	0.3082	0.4099	0.2819	0.1107	0.2470	0.6422	0	
12	0.3429	0.3704	0.2868	0.0910	0.4545	0.4545	0	boundary
13	0.3695	0.3462	0.2843	0.0638	0.6762	0.2601	0	
14	0.3556	0.3469	0.2974	0.1176	0.5882	0.2941	0	
15	0.3865	0.3313	0.2821	0.0467	0.7944	0.1589	0	
16	1	0	0	0	1	0	0	
17	0.3897	0.3259	0.2844	0.0478	0.8122	0.1400	0	
18	0.3304	0.3287	0.3409	0	0	0	1	
19	0.3437	0.3289	0.3274	0	0	0	1	

We also conduct more experiments, using the four-class X24 shown in Fig. 5 to compare INCA and FKM. Data points 6, 12 and 18 are boundaries and 24 is an outlier. Fig. 6 presents the clustering results of INCA. Table 2 lists the results of INCA and FKM. The first five data points belong to a main cluster because their T_4 values are higher for the other clusters (T_2, T_3 and T_4). It can also be inferred that similar observations data points 6, 12 and 18 are ambiguous because there are two highest T values. The last data point 24 was inferred as an outlier. Fig. 7 visually depicts the degree of membership.

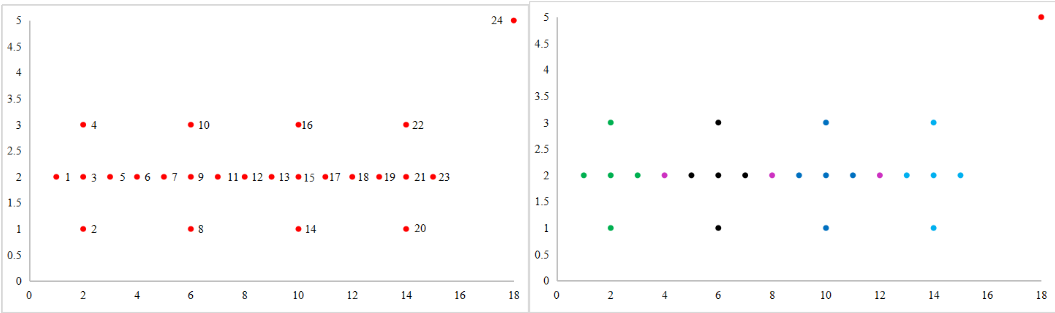
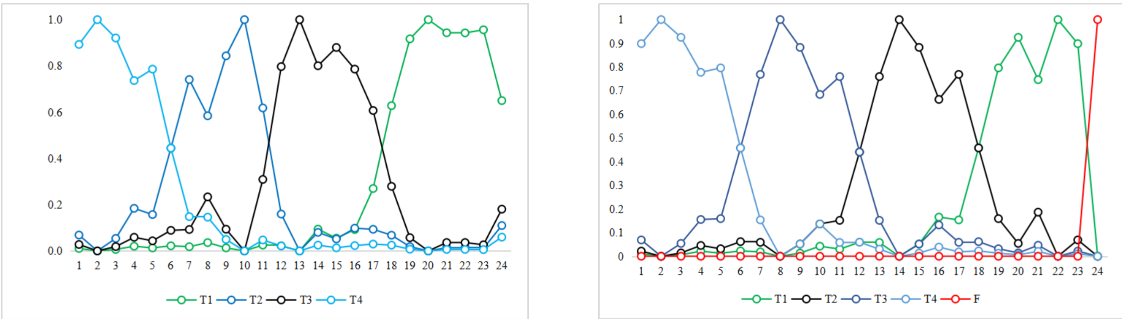


Figure 5. X24Figure 6. Clustering results of INCA on X24



(**a**) (**b**)

Figure 7. Membership calculated by FKM and INCA on X24

Table 2. Clustering results of X24

	FKM				INCA					
	U_1	U_2	U_3	U_4	T_1	T_2	T_3	T_4	F	
1	0.0106	0.0687	0.0279	0.8929	0.0106	0.0219	0.0691	0.8984	0	
2	0	0	0	1	0	0	0	1	0	
3	0.0064	0.0542	0.0188	0.9207	0.0064	0.0142	0.0544	0.9250	0	
4	0.0203	0.1842	0.0589	0.7366	0.0216	0.0457	0.1554	0.7772	0	
5	0.0130	0.1572	0.0437	0.7861	0.0130	0.0318	0.1592	0.7959	0	
6	0.0222	0.4444	0.0889	0.4444	0.0227	0.0619	0.4577	0.4577	0	boundary
7	0.0183	0.7409	0.0926	0.1482	0.0187	0.0591	0.7685	0.1537	0	
8	0.0360	0.5843	0.2337	0.1461	0	0	1	0	0	
9	0.0132	0.8435	0.0937	0.0496	0.0136	0.0519	0.8826	0.0519	0	
10	0	1	0	0	0.0427	0.1368	0.6838	0.1368	0	
11	0.0252	0.6181	0.3091	0.0476	0.0304	0.1519	0.7593	0.0584	0	
12	0.0221	0.1594	0.7969	0.0215	0.0595	0.4405	0.4405	0.0595	0	boundary
13	0	0	1	0	0.0584	0.7593	0.1519	0.0304	0	
14	0.0942	0.0801	0.8007	0.0250	0	1	0	0	0	
15	0.0550	0.0517	0.8797	0.0135	0.0519	0.8826	0.0519	0.0136	0	
16	0.0925	0.0983	0.7861	0.0231	0.1657	0.6628	0.1325	0.0340	0	
17	0.2698	0.0934	0.6071	0.0296	0.1537	0.7685	0.0591	0.0187	0	
18	0.6281	0.0679	0.2791	0.0249	0.4577	0.4577	0.0619	0.0227	0	boundary
19	0.9168	0.0183	0.0573	0.0075	0.7959	0.1592	0.0318	0.0130	0	

20	1	0	0	0	0.9250	0.0544	0.0142	0.0064	0
21	0.9433	0.0139	0.0363	0.0065	0.7461	0.1865	0.0467	0.0207	0
22	0.9426	0.0147	0.0363	0.0064	1	0	0	0	0
23	0.9562	0.0117	0.0266	0.0056	0.8984	0.0691	0.0219	0.0106	0
24	0.6499	0.1098	0.1805	0.0597	0	0	0	0	1

In this paper, a dataset is constructed as shown in Fig. 8. The dataset contains 83 data points, including 2 outliers and 3 boundary points. INCA can accurately distinguish main cluster points, boundary points and outlier points, as shown in Fig. 9. Data points 41 and 42 are outliers (blue circles in Figure 8), data points 61, 69 and 70 are boundary points (magenta circles in Figure 8), and the rest belong to the main cluster. Figure 10 visually depicts membership.

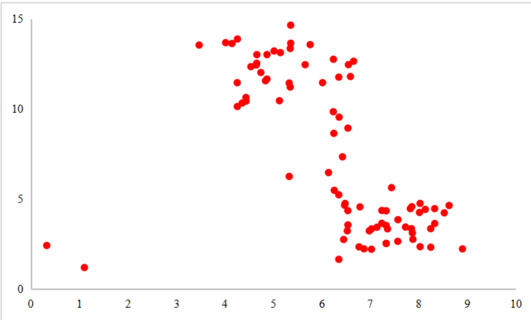


Figure 8. dataset 1

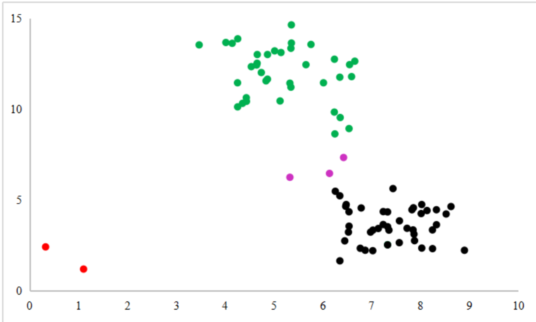


Figure 9. Clustering results of INCA on dataset 1

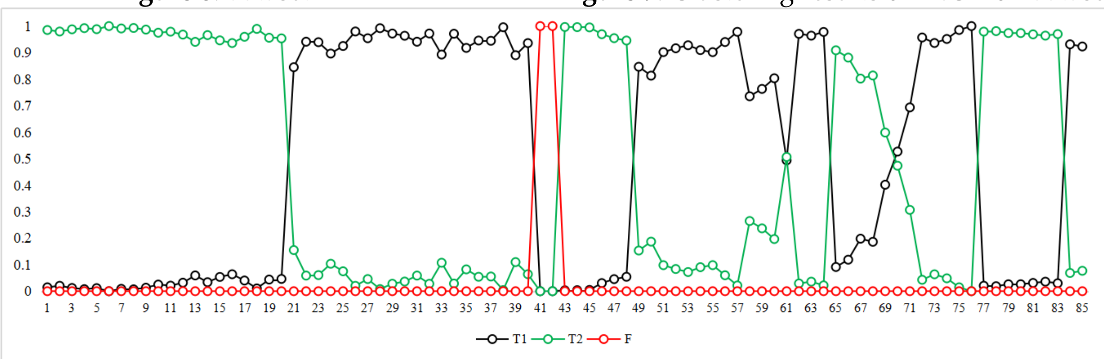


Figure 10. Membership calculated by FKM and INCA on dataset 1

4.2.2. Real dataset

To further evaluate the proposed clustering method, the UCI dataset is considered a standard dataset in the field of machine learning. In this study, the "dermatology", "pima", "TOX-171", "vowel", "ecoli", "iris" and "vote" datasets were selected among other UCI datasets. Table 3 summarizes the number of features, classes, and samples in each data. These datasets are used for traditional clustering methods such as FKM, RSFKM, FC-PFS, NCM and INCM.

Table 3. Datasets

Datasets	No. of instance	No. of feature	No. of class
dermatology	366	34	6
pima	768	8	2
TOX-171	171	5748	4

vowel	528	10	11
ecoli	336	343	8
iris	150	4	3
vote	435	16	2

Table 4 summarizes the accuracy of the proposed method and the FKM, RSFKM, FC-PFS, INCM and NCM methods. The accuracy rates of the proposed method on the "dermatology", "pima", "TOX-171", "vowel", "ecoli", "iris" and "vote" datasets were 82.24%, 74.35%, 51.46%, 40.15%, 76.79%, 98.00% and 84.60%. The accuracy of INCA is higher or second than other comparison algorithms. Table 5 summarizes the mutual information of INCA and FKM, RSFKM, FC-PFS, INCM and NCM methods. The mutual information of INCA is higher or second than other comparison algorithms.

Table 4. ACC of the different datasets

	dermatology	pima	TOX-171	vowel	ecoli	iris	vote
INCM	0.5314	0.6510	0.3918	0.2708	0.6875	0.9466	0.8000
FC-PFS	0.5027	0.6589	0.3977	0.2321	0.6250	0.8933	0.8138
RSFKM	0.8689	0.6602	0.2632	0.2746	0.6518	0.9267	0.8253
NCM	0.5000	0.6302	0.2865	0.2348	0.6280	0.9000	0.8138
FKM	0.6995	0.6563	0.4912	0.3655	0.6280	0.8933	0.8230
INCA	0.8224	0.7435	0.5146	0.4015	0.7679	0.9800	0.8460

Table 5. NMI of the different datasets

	dermatology	pima	TOX-171	vowel	ecoli	iris	vote
INCM	0.0117	0.0022	0.0685	0.2341	0.4867	0.8081	0.2918
FC-PFS	0.3193	0.0317	0.0722	0.2063	0.2614	0.7501	0.3333
RSFKM	0.8477	0.0267	0.0000	0.3027	0.3247	0.7933	0.3644
NCM	0.1998	0.0521	0.0231	0.2168	0.2711	0.7540	0.3297
FKM	0.6070	0.0294	0.2178	0.3915	0.4625	0.7515	0.3359
INCA	0.7240	0.0092	0.2248	0.3933	0.5895	0.9187	0.3636

Figure 11 shows the average accuracy of different algorithms. It can be seen that the average accuracy of INCA is higher than that of other comparison algorithms.

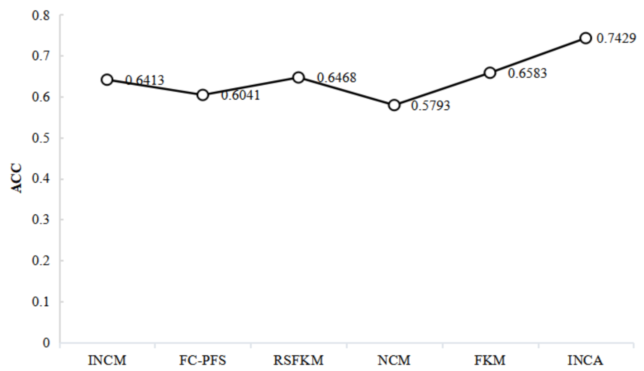


Figure 11. Average accuracy of different algorithms

4.3. Parameter analysis

In this section, the influence of parameters on the clustering results is analyzed. For this task, the Iris was selected for parameter evaluation. In each step, one parameter is changed and the others are fixed. Table 6 reports the results of the clustering methods for different parameter values. In each column, one parameter is considered to have 7 different quantities, while the other parameters are considered to be fixed and the quantities are in the fourth row. Each row in the table is a combination of parameters, and the fourth row is the best combination we chose in our experiments. The reasons for this choice will be discussed in detail in the following chapters.

Based on (13) each data point depends on two factors, namely the distance from the data to the cluster center and the uncertainty of the data, both of which influence each other. The parameter m determines the weighting effect of these factors. If m increases, $\omega_1 I_i T_{ij}$ and $\omega_2 (1-I_i) F_i$ are used more for membership assignments for main clusters and boundary points, respectively, and vice versa. By reducing m, the distance to the cluster center is a more important factor for membership assignment, which is almost the same as FKM. This parameter is 2 in this paper.

The parameters ω_1 and ω_2 are based on equation (18), on the one hand an increase in ω_1 leads to a decrease in T_{ij} and an increase in F_i, which means that the cost function pays more attention to the *F* set (boundary points) and reduces the accuracy. On the other hand, a smaller number of ω_1 has positive and negative effects on the main and border clusters, respectively. $\omega_1 = 1$ is configured, which is the best balance between the main cluster and the border cluster. The parameter ω_2 has the same effect in equation (19). Figure 12 shows the effect of different parameter combinations on the clustering results.

Table 6. Parameter sensitivity analysis

m	ω_1	ω_2
m=1.3	$\omega_1 = 0.3$	$\omega_2 = 0.5$
ACC=0.9667	ACC=0.9533	ACC=0.9667
m=1.5	$\omega_1 = 0.6$	$\omega_2 = 1.1$
ACC=0.9533	ACC=0.9267	ACC=0.9400
m=1.8	$\omega_1 = 0.7$	$\omega_2 = 1.5$
ACC=0.8800	ACC=0.9533	ACC=0.9400
m=2	$\omega_1 = 1$	$\omega_2 = 2$
ACC=0.9800	ACC=0.9667	ACC=0.9800
m=2.5	$\omega_1 = 1.5$	$\omega_2 = 3$
ACC=0.9300	ACC=0.9200	ACC=0.9567
m=3	$\omega_1 = 2$	$\omega_2 = 4$
ACC=0.9600	ACC=0.9600	ACC=0.9600
m=4	$\omega_1 = 3$	$\omega_2 = 5$
ACC=0.9400	ACC=0.9600	ACC=0.9400

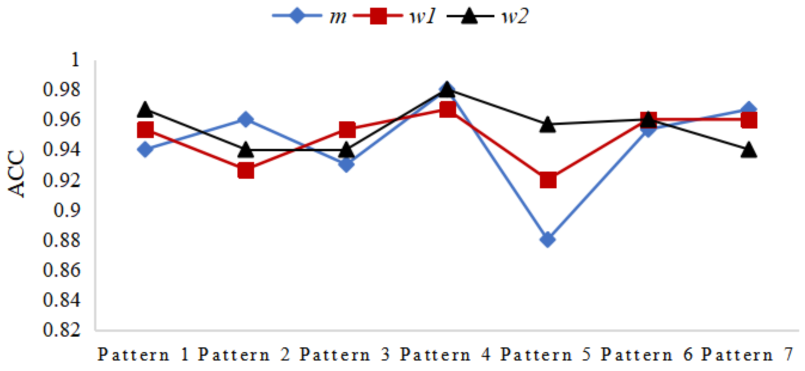


Figure 12. Parameter sensitivity analysis

In this section, the advantages and disadvantages of the proposed method are discussed. Border points and outliers are not considered in methods such as FKM. For example X19: 5, 7, 11, 13 and X24: 5, 7, 11, 13, 17, 19 are not assigned to the main cluster with a high degree of certainty. The reason is that such points are located at the same distance from the center of the main cluster and the center of the boundary cluster.For boundary points, such as X19: 6, 12 and X24: 6, 12, 18, the distances between the two main clusters are equal, but they are forcibly divided into one of the main clusters, which does not meet the actual situation and requirements.

From the above experiments, it can be seen that INCA is robust and the main cluster centers are not forced away from the boundary points. The experimental results show that INCA is more suitable for partitioning data, especially fuzzy and unclear data. Traditional methods only describe the degree of each cluster. For some samples in the boundary between different clusters, it is difficult to determine which group it belongs to. The method proposed in this paper aims to deal with these shortcomings of traditional partitioning methods.

5. Conclusions

The cost function in the neutrosophic set is proposed. Two types of clusters are considered in the proposed cost function, including main clusters and noise clusters. Experiments on different datasets show that INCA can not only deal with outliers and boundary points, but also outperform the comparative methods in both scatter data clustering and real datasets with these shortcomings of traditional partitioning methods.

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Simplified intuitionistic neutrosophic hypersoft TOPSIS method based on correlation coefficient

V.Chinnadurai [1*], A.Bobin [2] and D.Cokilavany [3]

[1] Department of Mathematics, Annamalai University, Tamilnadu, India.

[2] Department of Mathematics, IFET College of Engineering (Autonomous), Tamilnadu, India.; bobinalbert@gmail.com

[3] Department of Mathematics, Anna University, Tamilnadu, India.; cokila@annauniv.edu

*Corresponding author: chinnaduraiau@gmail.com; Tel.: (+91 9443238743)

Abstract. In psychology and sociology fields, the information is either dependent or independent and sometimes requires combinations of both. The existing structures like the fuzzy set, intuitionistic fuzzy set, and Pythagorean set have limitations when the information requires combinations of dependent and independent components. To overcome these limitations, we introduce the concept of a simplified intuitionistic neutrosophic hypersoft set. We present some properties of the correlation coefficient, weighted correlation coefficient, and aggregation operators on a simplified intuitionistic hypersoft set. Finally, we develop an algorithm and illustrate with a case study for identifying the leader; who can bring changes to society in the socio-political context.

Keywords: neutrosophic set; intuitionistic set; soft set; hypersoft set.

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1. **Introduction**

Zadeh [32] defined the concept of fuzzy set (FS). The membership value of each element in FS is specified by a real number from the closed interval of [0,1]. Atanassov [5] proposed the notion of an intuitionistic fuzzy set (IFS), an extension of FS. In IFS, the elements possess both membership and non-membership values such that their sum does not exceed unity. Smarandache [22] presented the concept of neutrosophic set (NS), characterized by the values of truth, indeterminacy, and falsity grades for each element of the set. Later, Wang et al. [27] proposed the notion of single-valued NS (SVNS) with a restricted condition for the membership values to overcome the constraints faced in NS. Molodtsov [15] introduced the concept of a soft set to deal with uncertainties. Smarandache [24] presented the concept of hypersoft set (HSS) to overcome the restriction faced in the soft set. Smarandache [23] proposed the concept of degree of dependence and the degree of independence between the components of the FS and NS. Also, for the first time, Smarandache [25] presented the concept of neutrosociology. Chinnadurai and Bobin [8], [9] introduced the concepts of the simplified intuitionistic neutrosophic soft set (SINSS) and interval-valued intuitionistic neutrosophic soft set (IVINSS) and studied some of their properties. In SINSS and IVINSS, the membership grades of truth and falsity are dependent on each other such that their sum cannot exceed one and the membership grade of indeterminacy is independent with a value less than or equal to one. Hence, in SINSS and IVINSS the sum of the membership grades cannot exceed two.

Khan et al. [14] introduced a programming language to solve multi-objective multi-product production planning problems. Smarandache [26] extended for the second time the nonstandard analysis by adding the left monad closed to the right, and right monad closed to the left. New theorems, better notations for monads and binads, and examples of nonstandard neutrosophic operations were discussed. Akram et al. [3] introduced the notion of hesitant fuzzy N-soft sets and used it in decision-making problems. Abdel-Basset et al. [1] presented the concept of type -2 neutrosophic numbers and presented a real case study using the technique of order of preference by similarity to ideal solution (TOPSIS). Abdel-Basset et al. [2] combined the neutrosophic analytical network process (ANP) method and the ViseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method for solving supplier chain management problems. Arora and Harish [4] studied the properties of aggregation operators on IFS. Ayele et al. [6] proposed a method for traffic signal control using an interval-valued neutrosophic soft set. Ejegwa et al. [10] used intuitionistic fuzzy correlation measure and programming language in the medical diagnosis field. Harish and Rishu [12] proposed TOPSIS method based on correlation measures on IFS to solve multi-criteria decision-making (MCDM) problems. Jana and Pal [13] presented the concept of aggregation operators on SVNS for solving MCDM problems. Naeem and Riaz [17] introduced Pythagorean fuzzy soft sets and established some of their algebraic properties. Naeem et al. [18] compared TOPSIS, VIKOR, and generalized aggregation operators models and showed that all the three techniques rendered the same optimal choice. Riaz et al. [20] presented an investment strategic decision making problem to illustrate the application of the Pythagorean m -Polar Fuzzy Weighted Aggregation operators and demonstrated its effectiveness. Naeem et al. [19] discussed an application of Pythagorean m-polar fuzzy sets in the decision-making problem for selecting an appropriate mode of advertisement by using the TOPSIS method. Zulqarnain et al. [33] introduced the concept of intuitionistic fuzzy HSS and used the TOPSIS method based on correlation coefficient (CC). Zulqarnain et al. [34] studied the fundamental operations of interval-valued neutrosophic HSS. Muhammad et al. [16] defined aggregation operators on neutrosophic HSS and studied some properties. Saqlain et al. [21] presented the concepts of single neutrosophic HSS and multi-valued neutrosophic HSS. They used tangent similarity measures to solve MCDM problems.

The main aim of the present study is to rank the alternatives of simplified intuitionistic neutrosophic hypersoft sets (SINHSS) by using aggregation operators and also by using the TOPSIS method based on CC. To the best of our knowledge, research on SINHSS is confined to its theory and related development and applications. Therefore, we examine and provide a suitable solution to the decision-makers in ranking the alternatives. We present a MCDM approach based on TOPSIS, and the effectiveness of this method is demonstrated through the selection of a leader who influences society in a socio-political context. To prove the efficacy of the proposed method, a comparative analysis between the proposed and existing method is illustrated with examples. Thus, the SINHSS is a robust tool to predict uncertainties when the membership grades of truth and falsity are dependent on each other.

The manuscript consists of the following sections. Section 2 briefs on existing definitions. Section 3, 4 and 5 introduces the concept of SINHSS and discusses some properties of CC and weighted CC of SINHSS. Section 6 deals with the simplified intuitionistic neutrosophic hypersoft weighted average operator(SINHSWAO) and simplified intuitionistic neutrosophic hypersoft weighted geometric operator (SINHSWGO). Section 7 highlights the combination of CC with the TOPSIS method. Section 8 shows the significance of the proposed method with comparative analysis. Section 9 ends with a conclusion.

2. **Preliminaries**

We present some of the basic definitions required for this study. Let us consider the following notations throughout this study unless otherwise specified. Let $\mathcal{V}$ be the universe and $v \in \mathcal{V}$, $P(\mathcal{V})$ be the power set of $\mathcal{V}$, $\mathbb{N}$ represents natural numbers, and $\mathcal{S}^U$ represent the collection of simplified intuitionistic neutrosophic sets (SINS) over $\mathcal{V}$.

Definition 2.1. [8] A SINS in $\mathcal{V}$ is of the form $\Omega = \{\langle v, \mathcal{T}_\Omega(v), \mathcal{I}_\Omega(v), \mathcal{F}_\Omega(v)\rangle\}$, where $\mathcal{T}_\Omega(v), \mathcal{I}_\Omega(v), \mathcal{F}_\Omega(v) : \mathcal{V} \rightarrow [0,1]$, are the membership values of truth, indeterminacy and falsity of the element $v \in \mathcal{V}$ respectively, such that $0 \leq \mathcal{T}_\Omega(v) + \mathcal{F}_\Omega(v) \leq 1$ and $0 \leq \mathcal{T}_\Omega(v) + \mathcal{I}_\Omega(v) + \mathcal{F}_\Omega(v) \leq 2$.

Definition 2.2. [24] Let $\Delta_1, \Delta_2, ..., \Delta_k$, be distinct attribute sets, whose corresponding sub-attributes are $\Delta_1 = \{\lambda_{11}, \lambda_{12}, ..., \lambda_{1f}\}, \Delta_2 = \{\lambda_{21}, \lambda_{22}, ..., \lambda_{2g}\}, ..., \Delta_k = \{\lambda_{k1}, \lambda_{k2}, ..., \lambda_{kh}\}$, where $1 \leq f \leq p$, $1 \leq g \leq q$, $1 \leq h \leq r$ and $p,\ q,\ r \in \mathbb{N}$, such that $\Delta_i \cap \Delta_j = \emptyset$, for each $i, j \in \{1, 2, ..., k\}$ and $i \neq j$. Then the Cartesian product of the distinct attribute sets $\Delta_1 \times \Delta_2 \times ... \times \Delta_k = \tilde{\Delta} = \{\lambda_{1f} \times \lambda_{2g} \times ... \times \lambda_{kh}\}$, represent a collection of multi- attributes. A pair $(\Omega, \tilde{\Delta})$ is called a hypersoft set (HSS) over $\mathcal{V}$, where $\Omega : \tilde{\Delta} \rightarrow P(\mathcal{V})$. HSS can be represented as $(\Omega, \tilde{\Delta}) = \left\{(\tilde{\lambda}, \Omega(\tilde{\lambda})) | \tilde{\lambda} \in \tilde{\Delta}, \Omega(\tilde{\lambda}) \in P(\mathcal{V})\right\}$.

3. **Simplified intuitionistic neutrosophic hypersoft set**

We present the notion of simplified intuitionistic neutrosophic hypersoft set (SINHSS). Also, we discuss some basic properties of correlation coefficient (CC) and weighted CC (WCC) on SINHSS.

Definition 3.1. A pair $(\Omega, \tilde{\Delta})$ is called a SINHSS over $\mathcal{V}$, where $\Omega : \tilde{\Delta} \to \mathcal{S}^U$. SINHSS can be represented as $(\Omega, \tilde{\Delta}) = \left\{(\tilde{\lambda}, \Omega(\tilde{\lambda}))|\tilde{\lambda} \in \tilde{\Delta}, \Omega(\tilde{\lambda}) \in \mathcal{S}^U \in [0,1]\right\}$, where $\Omega(\tilde{\lambda}) = \left\{\left\langle v, \mathcal{T}_{\Omega(\tilde{\lambda})}(v), \mathcal{I}_{\Omega(\tilde{\lambda})}(v), \mathcal{F}_{\Omega(\tilde{\lambda})}(v)\right\rangle |v \in \mathcal{V}\right\}$, $\mathcal{T}_{\Omega(\tilde{\lambda})}(v)$, $\mathcal{I}_{\Omega(\tilde{\lambda})}(v)$ and $\mathcal{F}_{\Omega(\tilde{\lambda})}(v)$ represent the membership values of truth, indeterminacy and falsity, such that $0 \leq \mathcal{T}_{\Omega(\tilde{\lambda})}(v) + \mathcal{F}_{\Omega(\tilde{\lambda})}(v) \leq 1$ and $0 \leq \mathcal{T}_{\Omega(\tilde{\lambda})}(v) + \mathcal{I}_{\Omega(\tilde{\lambda})}(v) + \mathcal{F}_{\Omega(\tilde{\lambda})}(v) \leq 2$.

Example 3.2. Let $\mathcal{V} = \{v_1, v_2, v_3\}$ be a set of sociologists responsible to evaluate a leader, the role of the leader is to bring socio-political changes to society. Let Δ_1, Δ_2 and Δ_3 be distinct attribute sets whose corresponding sub-attributes are represented as $\Delta_1 =$ leader attributes $= \{\lambda_{11} = \text{personality variables}, \lambda_{12} = \text{cognitive ability and skills}, \lambda_{13} = \text{sense making}\}$, $\Delta_2 = \text{leader behavior} = \{\lambda_{21} = \text{setting sub culture}, \lambda_{22} = \text{conflict management}\}$, $\Delta_3 = \text{group behaviors} = \{\lambda_{31} = \text{living the sub culture}\}$. Then $\tilde{\Delta} = \Delta_1 \times \Delta_2 \times \Delta_3$ be distinct attribute sets, such as

$$\tilde{\Delta} = \Delta_1 \times \Delta_2 \times \Delta_3 = \{\lambda_{11}, \lambda_{12}, \lambda_{13}\} \times \{\lambda_{21}, \lambda_{22}\} \times \{\lambda_{31}\}.$$

$$= \left\{(\lambda_{11}, \lambda_{21}, \lambda_{31}), (\lambda_{11}, \lambda_{22}, \lambda_{31}), (\lambda_{12}, \lambda_{21}, \lambda_{31}), (\lambda_{12}, \lambda_{22}, \lambda_{31}), (\lambda_{13}, \lambda_{21}, \lambda_{31}), (\lambda_{13}, \lambda_{22}, \lambda_{31})\right\}.$$

$$= \left\{\tilde{\lambda_1}, \tilde{\lambda_2}, \tilde{\lambda_3}, \tilde{\lambda_4}, \tilde{\lambda_5}, \tilde{\lambda_6}\right\}.$$

A SINHSS $(\Omega, \tilde{\Delta})$ is a collection of subsets of $\mathcal{V}$, given by the sociologists for a leader based on the description in Table 1.

TABLE 1. Shows leadership skills of a leader in SINHSS $(\Omega, \tilde{\Delta})$ form.

$\mathcal{V}$	$\tilde{\lambda_1}$	$\tilde{\lambda_2}$	$\tilde{\lambda_3}$	$\tilde{\lambda_4}$	$\tilde{\lambda_5}$	$\tilde{\lambda_6}$
v_1	$\langle 0.4, 0.9, 0.5\rangle$	$\langle 0.2, 0.5, 0.7\rangle$	$\langle 0.8, 0.9, 0.1\rangle$	$\langle 0.7, 0.9, 0.2\rangle$	$\langle 0.1, 0.4, 0.3\rangle$	$\langle 0.9, 0.9, 0.1\rangle$
v_2	$\langle 0.2, 0.8, 0.5\rangle$	$\langle 0.6, 0.2, 0.4\rangle$	$\langle 0.7, 0.4, 0.2\rangle$	$\langle 0.4, 0.5, 0.4\rangle$	$\langle 0.3, 0.4, 0.2\rangle$	$\langle 0.5, 0.2, 0.1\rangle$
v_3	$\langle 0.4, 0.4, 0.4\rangle$	$\langle 0.3, 0.3, 0.3\rangle$	$\langle 0.2, 0.1, 0.7\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.4, 1.0, 0.6\rangle$	$\langle 0.4, 0.8, 0.4\rangle$

4. **Correlation coefficient for SINHSS**

Let $(\Omega_1, \tilde{\Delta}_1) = \left\{(v_i, \mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i), \mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i), \mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i))|v_i \in \mathcal{V}\right\}$ and $(\Omega_2, \tilde{\Delta}_2) = \left\{(v_i, \mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i), \mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i), \mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))|v_i \in \mathcal{V}\right\}$ be two SINHSS over $\mathcal{V}$.

Definition 4.1. Let $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ be two SINHSS. Then the simplified intuitionistic neutrosophic informational energies of $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ are represented as

$$\Phi(\Omega_1, \tilde{\Delta}_1) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\right] \tag{1}$$

$$\Phi(\Omega_2, \tilde{\Delta}_2) = \sum_{k=1}^{m}\sum_{i=1}^{n}\left[(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\right]. \tag{2}$$

Definition 4.2. Let $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ be two SINHSS. Then the correlation measure between $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ is defined as

$$\mathcal{C}_{\mathcal{M}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))\Big]. \tag{3}$$

Proposition 4.3. *Let $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ be two SINHSS. Then,*
(i) $\mathcal{C}_{\mathcal{M}}((\Omega_1, \tilde{\Delta}_1), (\Omega_1, \tilde{\Delta}_1)) = \Phi(\Omega_1, \tilde{\Delta}_1)$
(ii) $\mathcal{C}_{\mathcal{M}}((\Omega_2, \tilde{\Delta}_2), (\Omega_2, \tilde{\Delta}_2)) = \Phi(\Omega_2, \tilde{\Delta}_2)$.

Proof. Straight forward □

Definition 4.4. Let $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ be two SINHSS. Then, the CC between $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ is given as

$$\mathcal{C}_C((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = \frac{\mathcal{C}_{\mathcal{M}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))}{\sqrt{\Phi(\Omega_1, \tilde{\Delta}_1)}\sqrt{\Phi(\Omega_2, \tilde{\Delta}_2)}} \tag{4}$$

Example 4.5. Let the values of $(\Omega_1, \tilde{\Delta}_1)$ be as in Table 1 and the values of $(\Omega_2, \tilde{\Delta}_2)$ be as in Table 2.

TABLE 2. Shows leadership skills of a leader in SINHSS $(\Omega_2, \tilde{\Delta}_2)$ form.

$\mathcal{V}$	$\tilde{\lambda_1}$	$\tilde{\lambda_2}$	$\tilde{\lambda_3}$	$\tilde{\lambda_4}$	$\tilde{\lambda_5}$	$\tilde{\lambda_6}$
v_1	$\langle 0.2, 0.8, 0.5\rangle$	$\langle 0.6, 0.2, 0.4\rangle$	$\langle 0.7, 0.4, 0.2\rangle$	$\langle 0.4, 0.5, 0.4\rangle$	$\langle 0.3, 0.4, 0.2\rangle$	$\langle 0.5, 0.2, 0.1\rangle$
v_2	$\langle 0.4, 0.4, 0.4\rangle$	$\langle 0.3, 0.3, 0.3\rangle$	$\langle 0.2, 0.1, 0.7\rangle$	$\langle 0.5, 0.5, 0.5\rangle$	$\langle 0.4, 1.0, 0.6\rangle$	$\langle 0.4, 0.8, 0.4\rangle$
v_3	$\langle 0.4, 0.9, 0.5\rangle$	$\langle 0.2, 0.5, 0.7\rangle$	$\langle 0.8, 0.9, 0.1\rangle$	$\langle 0.7, 0.9, 0.2\rangle$	$\langle 0.1, 0.4, 0.3\rangle$	$\langle 0.9, 0.9, 0.1\rangle$

Then, $\mathcal{C}_C((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = 0.7738$.

Proposition 4.6. *Let $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ be two SINHSS. Then, the following CC properties hold:*
(i) $0 \leq \mathcal{C}_C((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \leq 1$;

(ii) $\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = \mathcal{C}_C((\Omega_2,\tilde{\Delta}_2),(\Omega_1,\tilde{\Delta}_1))$;
(iii) If $(\Omega_1,\tilde{\Delta}_1) = (\Omega_2,\tilde{\Delta}_2)$, *then* $\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = 1$.

Proof. (i) Obviously, $\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \geq 0$. Now, we present the proof of $\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \leq 1$.

$$\mathcal{C}_{\mathcal{M}}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))$$
$$= \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))\Big].$$
$$= \sum_{k=1}^{m}\Bigg[\Big((\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_1))\Big)$$
$$+\Big((\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_2))\Big) + ...$$
$$+\Big((\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_n))\Big)\Bigg].$$

By applying Cauchy-Schwarz inequality, we get

$$\mathcal{C}_M((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))^2$$
$$\leq \sum_{k=1}^{m}\Bigg[\Big\{(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 +$$
$$... + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\}\Big\{(\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\}\Bigg] \times$$
$$\sum_{k=1}^{m}\Bigg[\Big\{(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 +$$
$$... + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\}\Big\{(\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\}\Bigg].$$

$$\mathcal{C}_M((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))^2$$
$$\leq \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\Big] \times \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 +$$
$$(\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big].$$

$\Rightarrow \mathcal{C}_M((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))^2 \leq \Phi(\Omega_1,\tilde{\Delta}_1) \times \Phi(\Omega_2,\tilde{\Delta}_2)$.
$\Rightarrow \mathcal{C}_M((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \leq \sqrt{\Phi(\Omega_1,\tilde{\Delta}_1)} \times \sqrt{\Phi(\Omega_2,\tilde{\Delta}_2)}$.
$\Rightarrow \frac{\mathcal{C}_M((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))}{\sqrt{\Phi(\Omega_1,\tilde{\Delta}_1)}\times\sqrt{\Phi(\Omega_2,\tilde{\Delta}_2)}} \leq 1$.
By using Definition 4.4, we get $\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \leq 1$.
Hence, $0 \leq \mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \leq 1$. □

Proof. (ii) Straight forward. □

Proof. (iii) $\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = \frac{\mathcal{C}_M((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))}{\sqrt{\Phi(\Omega_1,\tilde{\Delta}_1)}\times\sqrt{\Phi(\Omega_2,\tilde{\Delta}_2)}}$.
Since, $(\Omega_1,\tilde{\Delta}_1) = (\Omega_2,\tilde{\Delta}_2)$.
$\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))$

$$= \frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left[(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\right]}{\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left[(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\right]} \times \sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\left[(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\right]}}.$$

$\Rightarrow \mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = 1.$ □

Definition 4.7. Let $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ be two SINHSS. Then, the CC between $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2\tilde{\Delta}_2)$ is defined as

$$\tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = \frac{\mathcal{C}_{\mathcal{M}}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))}{\max\left\{\Phi(\Omega_1,\tilde{\Delta}_1),\Phi(\Omega_2,\tilde{\Delta}_2)\right\}}. \tag{5}$$

$\tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))$

$$= \frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\left[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))\right]}{\max\left\{\sum_{k=1}^{m}\sum_{i=1}^{n}\left[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\right], \sum_{k=1}^{m}\sum_{i=1}^{n}\left[(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\right]\right\}}.$$

Proposition 4.8. *Let $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ be two SINHSS. Then, the following CC properties hold:*
(i) $0 \le \tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \le 1$;
(ii) $\tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = \tilde{\mathcal{C}_C}((\Omega_2,\tilde{\Delta}_2),(\Omega_1,\tilde{\Delta}_1))$;
(iii) If $(\Omega_1,\tilde{\Delta}_1) = (\Omega_2,\tilde{\Delta}_2)$*, then* $\tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = 1$.

Proof. (i) Obviously, $\tilde{\mathcal{C}_C}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \geq 0$. Now, we present the proof of $\tilde{\mathcal{C}_C}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \leq 1$. $\mathcal{C}_{\mathcal{M}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))$

$$= \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))\Big].$$

$$= \sum_{k=1}^{m}\Bigg[\Big((\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_1))\Big)$$
$$+\Big((\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_2))\Big) + ...$$
$$+\Big((\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_n))\Big)\Bigg].$$

By applying Cauchy-Schwarz inequality, we get
$\mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))$

$$\leq \Bigg\{ \sum_{k=1}^{m}\Bigg[\Big\{(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 +$$
$$... + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\}\Big\{(\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\}\Bigg] \times$$
$$\sum_{k=1}^{m}\Bigg[\Big\{(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 +$$
$$... + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\}\Big\{(\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\}\Bigg]\Bigg\}^{\frac{1}{2}}.$$

$\mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))$

$$\leq \Bigg\{ \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\Big] \times \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]\Bigg\}^{\frac{1}{2}}.$$

$$\leq \Bigg\{\Bigg(\max\Bigg\{ \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\Big] \times \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]\Bigg\}\Bigg)^2\Bigg\}^{\frac{1}{2}}.$$

$$= \max\Bigg\{ \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\Big] \times \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]\Bigg\}.$$

$\Rightarrow \mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \leq \max\left\{\Phi(\Omega_1, \tilde{\Delta}_1) \times \Phi(\Omega_2, \tilde{\Delta}_2)\right\}$.

$\Rightarrow \frac{\mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))}{\max\left\{\Phi(\Omega_1, \tilde{\Delta}_1) \times \Phi(\Omega_2, \tilde{\Delta}_2)\right\}} \leq 1$.

By using Definition 4.7, we get $\tilde{\mathcal{C}_C}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \leq 1$.

Hence, $0 \leq \tilde{\mathcal{C}_C}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \leq 1$.

Proofs of (ii) and (iii) are same as in Proposition 4.6. □

5. Weighted correlation coefficient for SINHSS

We present the concept of weighted correlation coefficient (WCC) for SINHSS. WCC facilitates decision-makers (DMs) to provide different weights for each alternative. Consider $\mathcal{D} = \{\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_m\}$ and $\mathcal{W} = \{\mathcal{W}_1, \mathcal{W}_2, ..., \mathcal{W}_n\}$ as weight vectors for alternatives and experts, respectively, such that $\mathcal{D}_k, \mathcal{W}_i > 0$ and $\sum_{k=1}^{m} \mathcal{D}_k = 1, \sum_{i=1}^{n} \mathcal{W}_i = 1$.

Definition 5.1. Let $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ be two SINHSS. Then, the WCC between $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ is defined as

$$\mathcal{C}_{C_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = \frac{\mathcal{C}_{\mathcal{M}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))}{\sqrt{\Phi(\Omega_1, \tilde{\Delta}_1)}\sqrt{\Phi(\Omega_2, \tilde{\Delta}_2)}} \tag{6}$$

$$\mathcal{C}_{C_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))$$
$$= \frac{\sum_{k=1}^{m} \mathcal{D}_k\left(\sum_{i=1}^{n} \mathcal{W}_i\left[\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i) * \mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i) + \mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i) * \mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i) + \mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i) * \mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i)\right]\right)}{\sqrt{\sum_{k=1}^{m} \mathcal{D}_k\left(\sum_{i=1}^{n} \mathcal{W}_i\left[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\right]\right)}}$$
$$\times \sqrt{\sum_{k=1}^{m} \mathcal{D}_k\left(\sum_{i=1}^{n} \mathcal{W}_i\left[(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\right]\right)}.$$

If $\mathcal{D} = \left\{\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m},\right\}$ and $\mathcal{W} = \left\{\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n},\right\}$, then WCC given in Eq.(6) reduces to CC as in Eq.(4).

Example 5.2. Let the values of $(\Omega_1, \tilde{\Delta}_1)$ be as in Table 1 and the values of $(\Omega_2, \tilde{\Delta}_2)$ be as in Table 2.

Then, $\mathcal{C}_{C_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = 0.7903$.

Proposition 5.3. *Let $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ be two SINHSS. Then, the following WCC properties hold:*

(i) $0 \leq \mathcal{C}_{C_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \leq 1$;

(ii) $\mathcal{C}_{C_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = \mathcal{C}_{C_{\mathcal{W}}}((\Omega_2, \tilde{\Delta}_2), (\Omega_1, \tilde{\Delta}_1))$;

(iii) If $(\Omega_1, \tilde{\Delta}_1) = (\Omega_2, \tilde{\Delta}_2)$, *then* $\mathcal{C}_{C_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = 1$.

Proof. Similar to Proposition 4.6. □

Definition 5.4. Let $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ be two SINHSS. Then, the WCC between $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2\tilde{\Delta}_2)$ is defined as

$$\mathcal{C}_{\tilde{C}_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = \frac{\mathcal{C}_{\mathcal{M}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))}{\max\left\{\Phi(\Omega_1, \tilde{\Delta}_1), \Phi(\Omega_2, \tilde{\Delta}_2)\right\}}. \quad (7)$$

$$\mathcal{C}_{\tilde{C}_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = \frac{\sum_{k=1}^{m} \mathcal{D}_k\left(\sum_{i=1}^{n} \mathcal{W}_i\left[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))\right]\right)}{\max\left\{\sum_{k=1}^{m} \mathcal{D}_k\left(\sum_{i=1}^{n} \mathcal{W}_i\left[(\mathcal{T}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\right]\right), \sum_{k=1}^{m} \mathcal{D}_k\left(\sum_{i=1}^{n} \mathcal{W}_i\left[(\mathcal{T}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{I}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\mathcal{F}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\right]\right)\right\}}.$$

If $\mathcal{D} = \left\{\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m}, \right\}$ and $\mathcal{W} = \left\{\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}, \right\}$, then WCC given in Eq.(7) reduces to CC as in Eq.(5).

Proposition 5.5. *Let $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ be two SINHSS. Then, the following WCC properties hold:*
(i) $0 \le \mathcal{C}_{\tilde{C}_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \le 1$;
(ii) $\mathcal{C}_{\tilde{C}_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = \mathcal{C}_{\tilde{C}_{\mathcal{W}}}((\Omega_2, \tilde{\Delta}_2), (\Omega_1, \tilde{\Delta}_1))$;
(iii) If $(\Omega_1, \tilde{\Delta}_1) = (\Omega_2, \tilde{\Delta}_2)$, *then* $\mathcal{C}_{\tilde{C}_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = 1$.

Proof. Similiar to Proposition 4.6. □

6. Aggregation operators for SINHSS

We now present the concept of simplified intuitionistic neutrosophic hypersoft weighted average operator(SINHSWAO) and simplified intuitionistic neutrosophic hypersoft weighted geometric operator (SINHSWGO) by using operational laws. Let κ represent the collection of simplified intuitionistic neutrosophic hypersoft numbers (SINHSNs).

6.1. *Operational laws for SINHSS*

Definition 6.1. Let $\Omega_{e_{11}} = (\mathcal{T}_{11}, \mathcal{I}_{11}, \mathcal{F}_{11})$ and $\Omega_{e_{12}} = (\mathcal{T}_{12}, \mathcal{I}_{12}, \mathcal{F}_{12})$ be two SINHSS and β a positive integer. Then,
(i) $\Omega_{e_{11}} \oplus \Omega_{e_{12}} = \langle \mathcal{T}_{11} + \mathcal{T}_{12} - \mathcal{T}_{11}\mathcal{T}_{12}, \mathcal{I}_{11} + \mathcal{I}_{12} - \mathcal{I}_{11}\mathcal{I}_{12}, \mathcal{F}_{11}\mathcal{F}_{12} \rangle$;
(ii) $\Omega_{e_{11}} \otimes \Omega_{e_{12}} = \langle \mathcal{T}_{11}\mathcal{T}_{12}, \mathcal{I}_{11}\mathcal{I}_{12}, \mathcal{F}_{11} + \mathcal{F}_{12} - \mathcal{F}_{11}\mathcal{F}_{12} \rangle$;
(iii) $\beta\Omega_{e_{11}} = \left\langle \left[(1 - (1 - \mathcal{T}_{11})^{\beta}, (1 - (1 - \mathcal{I}_{11})^{\beta}, (\mathcal{F}_{11})^{\beta}\right] \right\rangle$;
(iv) $(\Omega_{e_{11}})^{\beta} = \left\langle \left[(\mathcal{T}_{11})^{\beta}, (\mathcal{I}_{11})^{\beta}, (1 - (1 - \mathcal{F}_{11})^{\beta}\right] \right\rangle$.

6.2. *Simplified intuitionistic neutrosophic hypersoft weighted average operator*

Definition 6.2. Let $\mathcal{D}_k$ and $\mathcal{W}_i$ be weight vectors for alternatives and experts, respectively, such that $\mathcal{D}_k, \mathcal{W}_i > 0$ and $\sum_{k=1}^{m} \mathcal{D}_k = 1, \sum_{i=1}^{n} \mathcal{W}_i = 1$ and $\Omega_{e_{ik}} = (\mathcal{T}_{ik}, \mathcal{I}_{ik}, \mathcal{F}_{ik})$ be a SINHSN, where $i = \{1, 2, ...n\}$, $k = \{1, 2, ...m\}$. Then, $\mathcal{A} : \kappa^n \rightarrow \kappa$, SINHSWAO is represented as

$$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{nm}}) = \bigoplus_{k=1}^{m} \mathcal{D}_k \left(\bigoplus_{i=1}^{n} \mathcal{W}_i \Omega_{e_{ik}} \right).$$

Theorem 6.3. *Let $\Omega_{e_{ik}} = (\mathcal{T}_{ik}, \mathcal{I}_{ik}, \mathcal{F}_{ik})$ be a SINHSN, where $i = \{1, 2, ...n\}$, $k = \{1, 2, ...m\}$. Then, the aggregated value of SINHSWAO is also a SINHSN, which is given by*

$$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{nm}}) = \left\langle 1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mathcal{T}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mathcal{I}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, \prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(\mathcal{F}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \right\rangle.$$

Proof. If $n = 1$, then $\mathcal{W}_1 = 1$. By using Definition 6.1, we get

$$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{1m}}) = \bigoplus_{k=1}^{m} \mathcal{D}_k \Omega_{e_{1k}}.$$
$$= \left\langle 1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{1} \left(1 - \mathcal{T}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{1} \left(1 - \mathcal{I}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, \prod_{k=1}^{m} \left(\prod_{i=1}^{1} \left(\mathcal{F}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \right\rangle.$$

If $m = 1$, then $\mathcal{D}_1 = 1$. By using Definition 6.2, we get

$$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{21}}, ..., \Omega_{e_{n1}}) = \bigoplus_{i=1}^{n} \mathcal{W}_i \Omega_{e_{i1}}.$$
$$= \left\langle 1 - \prod_{k=1}^{1} \left(\prod_{i=1}^{n} \left(1 - \mathcal{T}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{1} \left(\prod_{i=1}^{n} \left(1 - \mathcal{I}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, \prod_{k=1}^{1} \left(\prod_{i=1}^{n} \left(\mathcal{F}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \right\rangle.$$

Hence, the results hold for $n = 1$ and $m = 1$.

Now, if $m = l_1 + 1$ and $n = l_2$, then,

$$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{l_2(l_1+1)}}) = \bigoplus_{k=1}^{l_1+1} \mathcal{D}_k \left(\bigoplus_{i=1}^{l_2} \mathcal{W}_i \Omega_{e_{ik}} \right).$$
$$= \left\langle 1 - \prod_{k=1}^{l_1+1} \left(\prod_{i=1}^{l_2} \left(1 - \mathcal{T}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{l_1+1} \left(\prod_{i=1}^{l_2} \left(1 - \mathcal{I}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, \prod_{k=1}^{l_1+1} \left(\prod_{i=1}^{l_2} \left(\mathcal{F}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \right\rangle.$$

Similarly, if $m = l_1$, $n = l_2 + 1$, then,

$$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{(l_2+1)l_1}}) = \bigoplus_{k=1}^{l_1} \mathcal{D}_k \left(\bigoplus_{i=1}^{l_2+1} \mathcal{W}_i \Omega_{e_{ik}} \right).$$
$$= \left\langle 1 - \prod_{k=1}^{l_1} \left(\prod_{i=1}^{l_2+1} \left(1 - \mathcal{T}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{l_1} \left(\prod_{i=1}^{l_2+1} \left(1 - \mathcal{I}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, \prod_{k=1}^{l_1} \left(\prod_{i=1}^{l_2+1} \left(\mathcal{F}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \right\rangle.$$

Now, if $m = l_1 + 1$, $n = l_2 + 1$, then,

$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{(l_2+1)(l_1+1)}})$

$$= \bigoplus_{k=1}^{l_1+1} \mathcal{D}_k \Big(\bigoplus_{i=1}^{l_2+1} \mathcal{W}_i \Omega_{e_{ik}} \Big).$$

$$= \bigoplus_{k=1}^{l_1+1} \mathcal{D}_k \Big(\bigoplus_{i=1}^{l_2} \mathcal{W}_i \Omega_{e_{ik}} \Big) \bigoplus_{k=1}^{l_1+1} \mathcal{D}_k \Big(\mathcal{W}_{l_2+1} \Omega_{e_{(l_2+1)k}} \Big).$$

$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{(l_2+1)(l_1+1)}})$

$$= \Big\langle 1 - \prod_{k=1}^{l_1+1} \Big(\prod_{i=1}^{l_2} \Big(1 - \mathcal{T}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \oplus 1 - \prod_{k=1}^{l_1+1} \Big(\Big(1 - \mathcal{T}_{(l_2+1)k}\Big)^{\mathcal{W}_{(l_2+1)}} \Big)^{\mathcal{D}_k},$$
$$1 - \prod_{k=1}^{l_1+1} \Big(\prod_{i=1}^{l_2} \Big(1 - \mathcal{I}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \oplus 1 - \prod_{k=1}^{l_1+1} \Big(\Big(1 - \mathcal{I}_{(l_2+1)k}\Big)^{\mathcal{W}_{(l_2+1)}} \Big)^{\mathcal{D}_k},$$
$$\prod_{k=1}^{l_1+1} \Big(\prod_{i=1}^{l_2} \Big(\mathcal{F}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \oplus \prod_{k=1}^{l_1+1} \Big(\Big(\mathcal{F}_{(l_2+1)k}\Big)^{\mathcal{W}_{(l_2+1)}} \Big)^{\mathcal{D}_k} \Big\rangle.$$

$$= \Big\langle 1 - \prod_{k=1}^{l_1+1} \Big(\prod_{i=1}^{l_2+1} \Big(1 - \mathcal{T}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{l_1+1} \Big(\prod_{i=1}^{l_2+1} \Big(1 - \mathcal{I}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k}, \prod_{k=1}^{l_1+1} \Big(\prod_{i=1}^{l_2+1} \Big(\mathcal{F}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \Big\rangle.$$

Hence, the results hold for $n = l_2 + 1$ and $m = l_1 + 1$.

Therefore, by induction method, the result is true $\forall\ m, n \geq 1$.

Since

$0 \leq \mathcal{T}_{ik} + \mathcal{F}_{ik} \leq 1$ and $0 \leq \mathcal{I}_{ik} \leq 1$.

$$\Leftrightarrow 1 - \prod_{k=1}^{m} \Big(\prod_{i=1}^{n} \Big(1 - \mathcal{T}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} + \prod_{k=1}^{m} \Big(\prod_{i=1}^{n} \Big(\mathcal{F}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \leq 1$$
$$\text{and} 1 - \prod_{k=1}^{m} \Big(\prod_{i=1}^{n} \Big(1 - \mathcal{I}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \leq 1.$$
$$\Leftrightarrow 1 - \prod_{k=1}^{m} \Big(\prod_{i=1}^{n} \Big(1 - \mathcal{T}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} + \prod_{k=1}^{m} \Big(\prod_{i=1}^{n} \Big(1 - \mathcal{T}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \leq 1$$
$$\text{and } 1 - \prod_{k=1}^{m} \Big(\prod_{i=1}^{n} \Big(1 - \mathcal{I}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \leq 1.$$
$$\Leftrightarrow 1 - \prod_{k=1}^{m} \Big(\prod_{i=1}^{n} \Big(1 - \mathcal{T}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} + \prod_{k=1}^{m} \Big(\prod_{i=1}^{n} \Big(1 - \mathcal{T}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} + 1 - \prod_{k=1}^{m} \Big(\prod_{i=1}^{n} \Big(1 - \mathcal{I}_{ik}\Big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \leq 2.$$

Therefore, the aggregated value given by SINHSWAO is also a SINHSN. □

Example 6.4. Let us consider the same values mentioned in Example 3.2. Also, let $\mathcal{W}_i = \{0.50, 0.30, 0.20\}$ and $\mathcal{D}_k = \{0.14, 0.13, 0.23, 0.20, 0.18, 0.12\}$ be the weight of sociologists and attributes, respectively. Then,

$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{36}})$

$$= \left\langle 1 - \prod_{k=1}^{6}\left(\prod_{i=1}^{3}\left(1-\mathcal{T}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{6}\left(\prod_{i=1}^{3}\left(1-\mathcal{I}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}, \prod_{k=1}^{6}\left(\prod_{i=1}^{3}\left(\mathcal{F}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} \right\rangle.$$

$$= \langle 0.55, 1.00, 0.27\rangle .$$

6.3. *Simplified intuitionistic neutrosophic hypersoft weighted geometric operator*

Definition 6.5. Let $\mathcal{D}_k$ and $\mathcal{W}_i$ be weight vectors for alternatives and experts, respectively, such that $\mathcal{D}_k, \mathcal{W}_i > 0$ and $\sum_{k=1}^{m} \mathcal{D}_k = 1, \sum_{i=1}^{n} \mathcal{W}_i = 1$ and $\Omega_{e_{ik}} = (\mathcal{T}_{ik}, \mathcal{I}_{ik}, \mathcal{F}_{ik})$ be a SINHSN, where $i = \{1, 2, ...n\}$, $k = \{1, 2, ...m\}$. Then, $\mathcal{G} : \kappa^n \to \kappa$, SINHSWGO is defined as

$$\mathcal{G}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{nm}}) = \bigotimes_{k=1}^{m}\left(\bigotimes_{i=1}^{n}\left(\Omega_{e_{ik}}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}.$$

Theorem 6.6. *Let $\Omega_{e_{ik}} = (\mathcal{T}_{ik}, \mathcal{I}_{ik}, \mathcal{F}_{ik})$ be a SINHSN, where $i = \{1, 2, ...n\}$, $k = \{1, 2, ...m\}$. Then, the aggregated value of SINHSWGO is also a SINHSN, which is given by*

$$\mathcal{G}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{nm}}) = \left\langle \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{T}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}, \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{I}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{F}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} \right\rangle.$$

Proof. Similar to Theorem 6.3. □

Example 6.7. Let us consider the same values mentioned in Example 3.2 and the weight of sociologists and attributes be as in Example 6.4. Then,

$\mathcal{G}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{36}})$

$$= \left\langle \prod_{k=1}^{6}\left(\prod_{i=1}^{3}\left(\mathcal{T}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}, \prod_{k=1}^{6}\left(\prod_{i=1}^{3}\left(\mathcal{I}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{6}\left(\prod_{i=1}^{3}\left(1-\mathcal{F}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} \right\rangle.$$

$$= \langle 0.40, 0.52, 0.37\rangle .$$

7. MCDM problems based on TOPSIS and CC method

TOPSIS method helps to find the best alternative based on minimum and maximum distance from the neutrosophic positive ideal solution (NPIS) and neutrosophic negative ideal solution (NNIS). Also, when TOPSIS method is combined with CC instead of similarity measures, it provides reliable results for predicting the closeness coefficients. We present an algorithm and a case study to illustrate the SINHSS TOPSIS method based on CC.

7.1. *Algorithm to solve MCDM problems with SINHSS data based on TOPSIS and CC method*

Let $\mathcal{A} = \left\{\mathcal{A}^1, \mathcal{A}^2, ..., \mathcal{A}^x\right\}$ be a set of selected leaders aspiring to bring in socio-political changes to society and $\mathcal{V} = \{v_1, v_2, ..., v_n\}$ be a set of sociologists responsible to evaluate the leaders with weights $\mathcal{W}_i = (\mathcal{W}_1, \mathcal{W}_2, ..., \mathcal{W}_n)$, such that $\mathcal{W}_i > 0$ and $\sum_{i=1}^{n} \mathcal{W}_i = 1$. Let $\tilde{\Delta} = \left\{\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_m\right\}$ be a set of multi-valued sub-attributes with weights $\mathcal{D}_k = (\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_m)$, such that $\mathcal{D}_k > 0$ and $\sum_{k=1}^{m} \mathcal{D}_k = 1$. The evaluation of leaders $\mathcal{A}^t$, $(t = 1, 2, ..., x)$ performed by the sociologists v_i, $(i = 1, 2, ..., n)$ based on the multi-valued sub-attributes $\tilde{\lambda}_k$, $(k = 1, 2, ..., m)$ are given in SINHSS form and represented as $\Omega^t_{ik} = \left\langle \mathcal{T}^t_{ik}, \mathcal{I}^t_{ik}, \mathcal{F}^t_{ik} \right\rangle$, such that $0 \leq \mathcal{T}^t_{ik} + \mathcal{F}^t_{ik} \leq 1$ and $0 \leq \mathcal{T}^t_{ik} + \mathcal{I}^t_{ik} + \mathcal{F}^t_{ik} \leq 2 \; \forall \; i, k$.

Step 1. Construct the matrix for each multi-valued sub-attributes in SINHSS form as below:

$$[\mathcal{A}^t, \tilde{\Delta}]_{n\times m} = [\mathcal{A}^t]_{n\times m} = \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{array} \begin{array}{c} \begin{array}{cccc} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \dots & \tilde{\lambda}_m \end{array} \\ \begin{bmatrix} \langle \mathcal{T}^t_{11}, \mathcal{I}^t_{11}, \mathcal{F}^t_{11} \rangle & \langle \mathcal{T}^t_{12}, \mathcal{I}^t_{12}, \mathcal{F}^t_{12} \rangle & \dots & \langle \mathcal{T}^t_{1m}, \mathcal{I}^t_{1m}, \mathcal{F}^t_{1m} \rangle \rangle \\ \langle \mathcal{T}^t_{21}, \mathcal{I}^t_{21}, \mathcal{F}^t_{21} \rangle & \langle \mathcal{T}^t_{22}, \mathcal{I}^t_{22}, \mathcal{F}^t_{22} \rangle & \dots & \langle \mathcal{T}^t_{2m}, \mathcal{I}^t_{2m}, \mathcal{F}^t_{2m} \rangle \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathcal{T}^t_{n1}, \mathcal{I}^t_{n1}, \mathcal{F}^t_{n1} \rangle & \langle \mathcal{T}^t_{n2}, \mathcal{I}^t_{n2}, \mathcal{F}^t_{n2} \rangle & \dots & \langle \mathcal{T}^t_{nm}, \mathcal{I}^t_{nm}, \mathcal{F}^t_{nm} \rangle \rangle \end{bmatrix} \end{array}$$

Step 2. Obtain the weighted decision matrix for each multi-valued sub-attributes,

$$[\tilde{A}^t_{ik}]_{n\times m}$$
$$= \left\langle 1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mathcal{T}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mathcal{I}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, \prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(\mathcal{F}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \right\rangle$$
$$= \left\langle \tilde{\mathcal{T}}_{ik}, \tilde{\mathcal{I}}_{ik}, \tilde{\mathcal{F}}_{ik} \right\rangle.$$

Step 3. Determine the NPIS and NNIS for weighted SINHSS as below:

$$\tilde{\mathcal{A}}^+ = \left\langle \tilde{\mathcal{T}}^+, \tilde{\mathcal{I}}^+, \tilde{\mathcal{F}}^+ \right\rangle_{n\times m} = \left\langle \tilde{\mathcal{T}}^{(\vee_{ij})}, \tilde{\mathcal{I}}^{(\wedge_{ij})}, \tilde{\mathcal{F}}^{(\wedge_{ij})} \right\rangle \text{ and}$$

$$\tilde{\mathcal{A}}^- = \left\langle \tilde{\mathcal{T}}^-, \tilde{\mathcal{I}}^-, \tilde{\mathcal{F}}^- \right\rangle_{n\times m} = \left\langle \tilde{\mathcal{T}}^{(\wedge_{ij})}, \tilde{\mathcal{I}}^{(\wedge_{ij})}, \tilde{\mathcal{F}}^{(\vee_{ij})} \right\rangle,$$

where $\vee_{ij} = \arg\max_t \left\{\varphi^t_{ij}\right\}$ and $\wedge_{ij} = \arg\min_t \left\{\varphi^t_{ij}\right\}$.

Step 4. Determine the CC for each alternative from NPIS and NNIS.

$$\chi^t = \mathcal{C}_C(\tilde{A}^t, \tilde{A}^+) = \frac{\mathcal{C}_{\mathcal{M}}(\tilde{A}^t, \tilde{A}^+)}{\sqrt{\Phi(\tilde{A}^t)} * \sqrt{(\Phi \tilde{A}^+)}} \text{ and}$$

$$\lambda^t = \mathcal{C}_C(\tilde{A}^t, \tilde{A}^-) = \frac{\mathcal{C}_{\mathcal{M}}(\tilde{A}^t, \tilde{A}^-)}{\sqrt{\Phi(\tilde{A}^t)} * \sqrt{\Phi(\tilde{A}^-)}}$$

Step 5. Compute the closeness coefficient of neutrosophic ideal solution as below:

$$\epsilon^t = \frac{1 - \lambda^t}{2 - \chi^t - \lambda^t}$$

Step 6. Arrange the ϵ^t values in descending order and determine the rank of the alternatives $\mathcal{A}^t$, $(t = 1, 2, ..., x)$. The one with the maximum value is the best alternative.

7.2. *Application based on TOPSIS and CC method*

Let $\mathcal{A} = \{\mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3, \mathcal{A}^4\}$ be a set of leaders aspiring to bring in socio-political changes with their leadership skills and Δ_1 and Δ_2 be distinct attribute sets whose corresponding sub-attributes are represented as Δ_1 = leader attributes = $\{\lambda_{11} = \text{personality variables}, \lambda_{12} = \text{cognitive ability and skills}\}$, Δ_2 = leader behaviors = $\{\lambda_{21} = \text{setting sub culture}, \lambda_{22} = \text{conflict management}\}$. Then $\tilde{\Delta} = \Delta_1 \times \Delta_2$ be distinct attribute sets, such as

$$\tilde{\Delta} = \Delta_1 \times \Delta_2 = \{\lambda_{11}, \lambda_{12}\} \times \{\lambda_{21}, \lambda_{22}\}.$$

$$= \Big\{ (\lambda_{11}, \lambda_{21}), (\lambda_{11}, \lambda_{22}), (\lambda_{12}, \lambda_{21}), (\lambda_{12}, \lambda_{21}) \Big\}.$$

$$= \Big\{ \tilde{\lambda_1}, \tilde{\lambda_2}, \tilde{\lambda_3}, \tilde{\lambda_4} \Big\} \text{ with weights } \mathcal{D}_k = (0.20, 0.25, 0.30, 0.25).$$

An expert team selects a set of sociologists and provides the weightage depending on their tenure and knowledge. Let $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ be a set of sociologists responsible to evaluate the leaders with weights $\mathcal{W}_i = (0.35, 0.15, 0.30, 0.20)$. This study aims to find a leader who can bring major socio-political changes in a larger way to society.

Step 1. Construct $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{A}^3$ and $\mathcal{A}^4$ matrices for each multi-valued sub-attributes in SINHSS form.

TABLE 3. Representation of values in SINHSS form for $\mathcal{A}^1$.

$\mathcal{A}^1$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle 0.65, 0.92, 0.34 \rangle$	$\langle 0.55, 0.48, 0.25 \rangle$	$\langle 0.78, 0.88, 0.21 \rangle$	$\langle 0.23, 0.24, 0.35 \rangle$
v_2	$\langle 0.55, 0.72, 0.24 \rangle$	$\langle 0.65, 0.56, 0.25 \rangle$	$\langle 0.55, 0.77, 0.12 \rangle$	$\langle 0.43, 0.45, 0.45 \rangle$
v_3	$\langle 0.63, 0.87, 0.35 \rangle$	$\langle 0.45, 0.76, 0.35 \rangle$	$\langle 0.67, 0.55, 0.32 \rangle$	$\langle 0.41, 0.67, 0.55 \rangle$
v_4	$\langle 0.53, 0.79, 0.45 \rangle$	$\langle 0.67, 0.34, 0.31 \rangle$	$\langle 0.57, 0.66, 0.42 \rangle$	$\langle 0.32, 0.87, 0.53 \rangle$

TABLE 4. Representation of values in SINHSS form for $\mathcal{A}^2$.

$\mathcal{A}^2$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle 0.74, 0.27, 0.24 \rangle$	$\langle 0.69, 0.43, 0.25 \rangle$	$\langle 0.54, 0.22, 0.12 \rangle$	$\langle 0.32, 0.67, 0.24 \rangle$
v_2	$\langle 0.44, 0.95, 0.54 \rangle$	$\langle 0.79, 0.56, 0.15 \rangle$	$\langle 0.66, 0.33, 0.31 \rangle$	$\langle 0.42, 0.78, 0.15 \rangle$
v_3	$\langle 0.35, 0.85, 0.45 \rangle$	$\langle 0.57, 0.32, 0.25 \rangle$	$\langle 0.53, 0.44, 0.21 \rangle$	$\langle 0.52, 0.89, 0.43 \rangle$
v_4	$\langle 0.45, 0.76, 0.35 \rangle$	$\langle 0.82, 0.78, 0.16 \rangle$	$\langle 0.64, 0.55, 0.24 \rangle$	$\langle 0.34, 0.91, 0.61 \rangle$

TABLE 5. Representation of values in SINHSS form for $\mathcal{A}^3$.

$\mathcal{A}^3$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle 0.65, 0.78, 0.34\rangle$	$\langle 0.65, 0.78, 0.21\rangle$	$\langle 0.65, 0.65, 0.23\rangle$	$\langle 0.63, 0.34, 0.19\rangle$
v_2	$\langle 0.45, 0.55, 0.42\rangle$	$\langle 0.54, 0.88, 0.19\rangle$	$\langle 0.58, 0.45, 0.33\rangle$	$\langle 0.53, 0.47, 0.25\rangle$
v_3	$\langle 0.55, 0.76, 0.35\rangle$	$\langle 0.75, 0.33, 0.24\rangle$	$\langle 0.46, 0.35, 0.45\rangle$	$\langle 0.23, 0.78, 0.34\rangle$
v_4	$\langle 0.35, 0.45, 0.24\rangle$	$\langle 0.58, 0.44, 0.25\rangle$	$\langle 0.74, 0.25, 0.19\rangle$	$\langle 0.45, 0.81, 0.17\rangle$

TABLE 6. Representation of values in SINHSS form for $\mathcal{A}^4$.

$\mathcal{A}^4$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle 0.81, 0.46, 0.12\rangle$	$\langle 0.35, 0.45, 0.24\rangle$	$\langle 0.23, 0.32, 0.42\rangle$	$\langle 0.54, 0.93, 0.45\rangle$
v_2	$\langle 0.64, 0.56, 0.14\rangle$	$\langle 0.59, 0.65, 0.34\rangle$	$\langle 0.33, 0.43, 0.52\rangle$	$\langle 0.45, 0.48, 0.38\rangle$
v_3	$\langle 0.54, 0.76, 0.23\rangle$	$\langle 0.63, 0.76, 0.26\rangle$	$\langle 0.12, 0.54, 0.72\rangle$	$\langle 0.56, 0.79, 0.41\rangle$
v_4	$\langle 0.76, 0.45, 0.16\rangle$	$\langle 0.67, 0.88, 0.31\rangle$	$\langle 0.18, 0.65, 0.45\rangle$	$\langle 0.66, 0.58, 0.34\rangle$

Step 2. Obtain $\tilde{\mathcal{A}}^1$, $\tilde{\mathcal{A}}^2$, $\tilde{\mathcal{A}}^3$ and $\tilde{\mathcal{A}}^4$, the weighted matrices for each multi-valued sub-attributes.

TABLE 7. Representation of weighted values in SINHSS form for $\tilde{\mathcal{A}}^1$.

$\tilde{\mathcal{A}}^1$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle 0.0709, 0.1621, 0.9273\rangle$	$\langle 0.0675, 0.0557, 0.8858\rangle$	$\langle 0.1470, 0.1996, 0.8489\rangle$	$\langle 0.0227, 0.0238, 0.9123\rangle$
v_2	$\langle 0.0237, 0.0375, 0.9581\rangle$	$\langle 0.0387, 0.0304, 0.9494\rangle$	$\langle 0.0353, 0.0640, 0.9090\rangle$	$\langle 0.0209, 0.0222, 0.9705\rangle$
v_3	$\langle 0.0580, 0.1153, 0.9390\rangle$	$\langle 0.0439, 0.1016, 0.9243\rangle$	$\langle 0.0950, 0.0694, 0.9026\rangle$	$\langle 0.0388, 0.0798, 0.9562\rangle$
v_4	$\langle 0.0298, 0.0606, 0.9686\rangle$	$\langle 0.0540, 0.0206, 0.9432\rangle$	$\langle 0.0494, 0.0627, 0.9493\rangle$	$\langle 0.0191, 0.0970, 0.9688\rangle$

TABLE 8. Representation of weighted values in SINHSS form $\tilde{\mathcal{A}}^2$.

$\tilde{\mathcal{A}}^2$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_2$	$\tilde{\lambda}_4$
v_1	$\langle 0.0900, 0.0218, 0.9050\rangle$	$\langle 0.0975, 0.0480, 0.8858\rangle$	$\langle 0.0784, 0.0258, 0.8005\rangle$	$\langle 0.0332, 0.0925, 0.8827\rangle$
v_2	$\langle 0.0173, 0.0860, 0.9817\rangle$	$\langle 0.0569, 0.0304, 0.9314\rangle$	$\langle 0.0474, 0.0179, 0.9487\rangle$	$\langle 0.0203, 0.0552, 0.9314\rangle$
v_3	$\langle 0.0256, 0.1076, 0.9533\rangle$	$\langle 0.0614, 0.0286, 0.9013\rangle$	$\langle 0.0657, 0.0509, 0.8690\rangle$	$\langle 0.0536, 0.1526, 0.9387\rangle$
v_4	$\langle 0.0237, 0.0555, 0.9589\rangle$	$\langle 0.0822, 0.0730, 0.9125\rangle$	$\langle 0.0595, 0.0468, 0.9180\rangle$	$\langle 0.0206, 0.1135, 0.9756\rangle$

Step 3. Determine the NPIS and NNIS from the weighted matrices, $\tilde{\mathcal{A}}^1$, $\tilde{\mathcal{A}}^2$, $\tilde{\mathcal{A}}^3$ and $\tilde{\mathcal{A}}^4$.

$$\tilde{\mathcal{A}}^+ = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{cccc} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 & \tilde{\lambda}_4 \\ \end{array}$$

$$\tilde{\mathcal{A}}^+ = \begin{bmatrix} \langle 0.1098, 0.0218, 0.8621\rangle & \langle 0.0975, 0.0480, 0.8724\rangle & \langle 0.1470, 0.0258, 0.8005\rangle & \langle 0.0834, 0.0238, 0.8648\rangle \\ \langle 0.0302, 0.0237, 0.9428\rangle & \langle 0.0569, 0.0304, 0.9314\rangle & \langle 0.0474, 0.0179, 0.9090\rangle & \langle 0.0280, 0.0222, 0.9314\rangle \\ \langle 0.0580, 0.0821, 0.9156\rangle & \langle 0.0988, 0.0286, 0.8985\rangle & \langle 0.0950, 0.0381, 0.8690\rangle & \langle 0.0598, 0.0798, 0.9223\rangle \\ \langle 0.0555, 0.0237, 0.9294\rangle & \langle 0.0822, 0.0206, 0.9125\rangle & \langle 0.0777, 0.0172, 0.9052\rangle & \langle 0.0526, 0.0425, 0.9153\rangle \end{bmatrix}$$

Table 9. Representation of weighted values in SINHSS form for $\tilde{\mathcal{A}}^3$.

$\tilde{A}^3$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle 0.0709, 0.1006, 0.9273\rangle$	$\langle 0.0878, 0.1241, 0.8724\rangle$	$\langle 0.1044, 0.1044, 0.8571\rangle$	$\langle 0.0834, 0.0358, 0.8648\rangle$
v_2	$\langle 0.0178, 0.0237, 0.9744\rangle$	$\langle 0.0287, 0.0765, 0.9397\rangle$	$\langle 0.0383, 0.0266, 0.9514\rangle$	$\langle 0.0280, 0.0236, 0.9494\rangle$
v_3	$\langle 0.0468, 0.0821, 0.9390\rangle$	$\langle 0.0988, 0.0296, 0.8985\rangle$	$\langle 0.0540, 0.0381, 0.9307\rangle$	$\langle 0.0195, 0.1074, 0.9223\rangle$
v_4	$\langle 0.0171, 0.0237, 0.9446\rangle$	$\langle 0.0425, 0.0286, 0.9331\rangle$	$\langle 0.0777, 0.0172, 0.9052\rangle$	$\langle 0.0295, 0.0797, 0.9153\rangle$

Table 10. Representation of weighted values in SINHSS form for $\tilde{\mathcal{A}}^4$.

$\tilde{A}^4$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle 0.1098, 0.0423, 0.8621\rangle$	$\langle 0.0370, 0.0510, 0.8827\rangle$	$\langle 0.0271, 0.0397, 0.9130\rangle$	$\langle 0.0657, 0.2076, 0.9326\rangle$
v_2	$\langle 0.0302, 0.0244, 0.9428\rangle$	$\langle 0.0329, 0.0387, 0.9604\rangle$	$\langle 0.0179, 0.0250, 0.9711\rangle$	$\langle 0.0222, 0.0243, 0.9644\rangle$
v_3	$\langle 0.0456, 0.0821, 0.9156\rangle$	$\langle 0.0719, 0.1016, 0.9040\rangle$	$\langle 0.0115, 0.0676, 0.9709\rangle$	$\langle 0.0598, 0.1105, 0.9354\rangle$
v_4	$\langle 0.0555, 0.0237, 0.9294\rangle$	$\langle 0.0540, 0.1006, 0.9432\rangle$	$\langle 0.0119, 0.0611, 0.9533\rangle$	$\langle 0.0526, 0.0425, 0.9475\rangle$

$$\tilde{\mathcal{A}}^- = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{cccc} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 & \tilde{\lambda}_4 \\ \end{array}$$

$$\begin{bmatrix} \langle 0.0709, 0.0218, 0.9273\rangle & \langle 0.0370, 0.0480, 0.8858\rangle & \langle 0.0271, 0.0258, 0.9130\rangle & \langle 0.0227, 0.0238, 0.9326\rangle \\ \langle 0.0173, 0.0218, 0.9817\rangle & \langle 0.0287, 0.0304, 0.9604\rangle & \langle 0.0179, 0.0179, 0.9711\rangle & \langle 0.0203, 0.0222, 0.9705\rangle \\ \langle 0.0256, 0.0237, 0.9533\rangle & \langle 0.0439, 0.0286, 0.9243\rangle & \langle 0.0115, 0.0381, 0.9709\rangle & \langle 0.0195, 0.0798, 0.9562\rangle \\ \langle 0.0171, 0.0237, 0.9686\rangle & \langle 0.0425, 0.0206, 0.9432\rangle & \langle 0.0119, 0.0172, 0.9533\rangle & \langle 0.0191, 0.0425, 0.9756\rangle \end{bmatrix}$$

Step 4. Determine the CC for the alternatives by using the values of NPIS and NNIS.

$$\chi^1 = 0.9968, \chi^2 = 0.9981, \chi^3 = 0.9983 \text{ and } \chi^4 = 0.9962.$$

$$\lambda^1 = 0.9961, \lambda^2 = 0.9975, \lambda^3 = 0.9979 \text{ and } \lambda^4 = 0.9975.$$

Step 5. Compute the closeness coefficient of neutrosophic ideal solution as below.

$$\epsilon^1 = 0.5493, \epsilon^2 = 0.5682, \epsilon^3 = 0.5526 \text{ and } \epsilon^4 = 0.3968.$$

Step 6. Arrange the values in descending order.

$$\epsilon^2 > \epsilon^3 > \epsilon^1 > \epsilon^4.$$

$$\Rightarrow \mathcal{A}^2 > \mathcal{A}^3 > \mathcal{A}^1 > \mathcal{A}^4.$$

Hence, $\mathcal{A}^2$ is the best leader among the group and can play a significant role in bringing socio-political changes to society.

8. Comparative Analysis

We compare existing TOPSIS methods with the proposed method. Also, we provide examples to show the advantage of the TOPSIS method based on CC instead of distance or similarity measures.

Example 8.1. Consider the SINHSS values mentioned in Table 10. By applying the existing neutrosophic simplified TOPSIS method discussed in Elhassouny and Smarandache [11],

$$\mathcal{A}^1 = \mathcal{A}^2 = \mathcal{A}^3 = \mathcal{A}^4 = 0.47.$$

Hence, it is not possible to identify the best alternative.
By applying the proposed method,

$$\mathcal{A}^4 = 0.58, \mathcal{A}^3 = 0.56, \mathcal{A}^2 = 0.51 \text{ and } \mathcal{A}^1 = 0.48$$

Hence, the best alternative is $\mathcal{A}^4$.

TABLE 11. Representation of values in SINHSS form for $\mathcal{A}^i$.

$\mathcal{A}^i$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$	$\tilde{\lambda}_5$	$\tilde{\lambda}_6$
$\mathcal{A}^1$	$\langle 0.55, 0.89, 0.34\rangle$	$\langle 0.46, 0.87, 0.25\rangle$	$\langle 0.62, 0.54, 0.11\rangle$	$\langle 0.23, 0.91, 0.35\rangle$	$\langle 0.55, 0.77, 0.24\rangle$	$\langle 0.63, 0.44, 0.21\rangle$
$\mathcal{A}^2$	$\langle 0.65, 0.87, 0.24\rangle$	$\langle 0.72, 0.56, 0.12\rangle$	$\langle 0.45, 0.56, 0.12\rangle$	$\langle 0.25, 0.93, 0.45\rangle$	$\langle 0.44, 0.95, 0.54\rangle$	$\langle 0.78, 0.57, 0.15\rangle$
$\mathcal{A}^3$	$\langle 0.76, 0.85, 0.14\rangle$	$\langle 0.57, 0.76, 0.24\rangle$	$\langle 0.67, 0.55, 0.32\rangle$	$\langle 0.58, 0.67, 0.55\rangle$	$\langle 0.42, 0.75, 0.45\rangle$	$\langle 0.57, 0.54, 0.25\rangle$
$\mathcal{A}^4$	$\langle 0.53, 0.65, 0.45\rangle$	$\langle 0.71, 0.69, 0.11\rangle$	$\langle 0.57, 0.66, 0.33\rangle$	$\langle 0.42, 0.87, 0.54\rangle$	$\langle 0.55, 0.89, 0.14\rangle$	$\langle 0.69, 0.56, 0.16\rangle$

Example 8.2. Consider the SINHSS values mentioned in Table 11. By applying the existing TOPSIS method discussed in Biswas et al. [7],

$$\mathcal{A}^1 = \mathcal{A}^2 = \mathcal{A}^3 = \mathcal{A}^4 = 0.48.$$

Hence, it is not possible to identify the best alternative.
By applying the proposed method,

$$\mathcal{A}^3 = 0.57, \mathcal{A}^2 = 0.54, \mathcal{A}^4 = 0.51 \text{ and } \mathcal{A}^1 = 0.06$$

Hence, the best alternative is $\mathcal{A}^3$.

TABLE 12. Representation of values in SINHSS form for $\mathcal{A}^i$.

$\mathcal{A}^i$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$	$\tilde{\lambda}_5$	$\tilde{\lambda}_6$
$\mathcal{A}^1$	$\langle 0.45, 0.79, 0.45\rangle$	$\langle 0.34, 0.34, 0.65\rangle$	$\langle 0.47, 0.54, 0.52\rangle$	$\langle 0.52, 0.23, 0.42\rangle$	$\langle 0.41, 0.77, 0.49\rangle$	$\langle 0.35, 0.44, 0.61\rangle$
$\mathcal{A}^2$	$\langle 0.65, 0.85, 0.24\rangle$	$\langle 0.66, 0.88, 0.12\rangle$	$\langle 0.45, 0.56, 0.12\rangle$	$\langle 0.47, 0.93, 0.24\rangle$	$\langle 0.44, 0.95, 0.54\rangle$	$\langle 0.78, 0.57, 0.15\rangle$
$\mathcal{A}^3$	$\langle 0.76, 0.85, 0.14\rangle$	$\langle 0.57, 0.76, 0.24\rangle$	$\langle 0.67, 0.55, 0.32\rangle$	$\langle 0.71, 0.88, 0.23\rangle$	$\langle 0.52, 0.75, 0.45\rangle$	$\langle 0.59, 0.54, 0.21\rangle$
$\mathcal{A}^4$	$\langle 0.53, 0.65, 0.45\rangle$	$\langle 0.71, 0.69, 0.11\rangle$	$\langle 0.57, 0.66, 0.33\rangle$	$\langle 0.42, 0.83, 0.54\rangle$	$\langle 0.42, 0.79, 0.14\rangle$	$\langle 0.49, 0.56, 0.34\rangle$

Example 8.3. Consider the SINHSS values mentioned in Table 13. By combining the existing neutrosophic simplified TOPSIS method discussed in Elhassouny and Smarandache [11], with the similarity measures given in Table 12, it is not possible to identify the best alternative. However, by using the proposed method, the best alternative is identified for all the cases, as shown in Table 14.

Table 13. Framework of existing similarity measures.

Existing similarity measures
$\mathcal{S}_J(\psi_1,\psi_2)$ [28]$= \frac{1}{n}\sum_{i=1}^{n}\frac{\tilde{\mathcal{J}}}{(\mathcal{T}^2_{\psi_1}(u_i)+\mathcal{I}^2_{\psi_1}(u_i)+\mathcal{F}^2_{\psi_1}(u_i))+(\mathcal{T}^2_{\psi_2}(u_i)+\mathcal{I}^2_{\psi_2}(u_i)+\mathcal{F}^2_{\psi_2}(u_i))-\tilde{\mathcal{J}}}$,
where $\tilde{\mathcal{J}} = \mathcal{T}_{\psi_1}(u_i)\mathcal{T}_{\psi_2}(u_i)+\mathcal{I}_{\psi_1}(u_i)\mathcal{I}_{\psi_2}(u_i)+\mathcal{F}_{\psi_1}(u_i)\mathcal{F}_{\psi_2}(u_i)$.
$\mathcal{S}_D(\psi_1,\psi_2)$ [28]$= \frac{1}{n}\sum_{i=1}^{n}\frac{2(\mathcal{T}_{\psi_1}(u_i)\mathcal{T}_{\psi_2}(u_i)+\mathcal{I}_{\psi_1}(u_i)\mathcal{I}_{\psi_2}(u_i)+\mathcal{F}_{\psi_1}(u_i)\mathcal{F}_{\psi_2}(u_i))}{(\mathcal{T}^2_{\psi_1}(u_i)+\mathcal{I}^2_{\psi_1}(u_i)+\mathcal{F}^2_{\psi_1}(u_i))+(\mathcal{T}^2_{\psi_2}(u_i)+\mathcal{I}^2_{\psi_2}(u_i)+\mathcal{F}^2_{\psi_2}(u_i))}$.
$\mathcal{S}_1(\psi_1,\psi_2)$ [29]$= \frac{1}{n}\sum_{i=1}^{n}\cos\left[\frac{\pi}{2}\max(\lvert\mathcal{T}_{\psi_1}(u_i)-\mathcal{T}_{\psi_2}(u_i)\rvert,\lvert\mathcal{I}_{\psi_1}(u_i)-\mathcal{I}_{\psi_2}(u_i)\rvert,\lvert\mathcal{F}_{\psi_1}(u_i)-\mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.
$\mathcal{S}_2(\psi_1,\psi_2)$ [29]$= \frac{1}{n}\sum_{i=1}^{n}\cos\left[\frac{\pi}{6}(\lvert\mathcal{T}_{\psi_1}(u_i)-\mathcal{T}_{\psi_2}(u_i)\rvert+\lvert\mathcal{I}_{\psi_1}(u_i)-\mathcal{I}_{\psi_2}(u_i)\rvert+\lvert\mathcal{F}_{\psi_1}(u_i)-\mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.
$\mathcal{S}_3(\psi_1,\psi_2)$ [30]$= \frac{1}{n}\sum_{i=1}^{n}\tan\left[\frac{\pi}{4}\max(\lvert\mathcal{T}_{\psi_1}(u_i)-\mathcal{T}_{\psi_2}(u_i)\rvert,\lvert\mathcal{I}_{\psi_1}(u_i)-\mathcal{I}_{\psi_2}(u_i)\rvert,\lvert\mathcal{F}_{\psi_1}(u_i)-\mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.
$\mathcal{S}_4(\psi_1,\psi_2)$ [30]$= \frac{1}{n}\sum_{i=1}^{n}\tan\left[\frac{\pi}{12}(\lvert\mathcal{T}_{\psi_1}(u_i)-\mathcal{T}_{\psi_2}(u_i)\rvert+\lvert\mathcal{I}_{\psi_1}(u_i)-\mathcal{I}_{\psi_2}(u_i)\rvert+\lvert\mathcal{F}_{\psi_1}(u_i)-\mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.
$\mathcal{S}_5(\psi_1,\psi_2)$ [31]$= \frac{1}{n}\sum_{i=1}^{n}\cot\left[\frac{\pi}{4}+\frac{\pi}{4}\max(\lvert\mathcal{T}_{\psi_1}(u_i)-\mathcal{T}_{\psi_2}(u_i)\rvert,\lvert\mathcal{I}_{\psi_1}(u_i)-\mathcal{I}_{\psi_2}(u_i)\rvert,\lvert\mathcal{F}_{\psi_1}(u_i)-\mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.
$\mathcal{S}_6(\psi_1,\psi_2)$ [31]$= \frac{1}{n}\sum_{i=1}^{n}\cot\left[\frac{\pi}{4}+\frac{\pi}{6}(\lvert\mathcal{T}_{\psi_1}(u_i)-\mathcal{T}_{\psi_2}(u_i)\rvert+\lvert\mathcal{I}_{\psi_1}(u_i)-\mathcal{I}_{\psi_2}(u_i)\rvert+\lvert\mathcal{F}_{\psi_1}(u_i)-\mathcal{F}_{\psi_2}(u_i)\rvert)\right]$.

Table 14. Representation of values in SINHSS form for $\mathcal{A}^i$.

$\mathcal{A}^i$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$	$\tilde{\lambda}_5$	$\tilde{\lambda}_6$
$\mathcal{A}^1$	$\langle 0.55, 0.89, 0.34\rangle$	$\langle 0.46, 0.87, 0.25\rangle$	$\langle 0.62, 0.54, 0.11\rangle$	$\langle 0.23, 0.91, 0.35\rangle$	$\langle 0.55, 0.77, 0.24\rangle$	$\langle 0.63, 0.44, 0.21\rangle$
$\mathcal{A}^2$	$\langle 0.65, 0.87, 0.24\rangle$	$\langle 0.72, 0.56, 0.12\rangle$	$\langle 0.45, 0.56, 0.12\rangle$	$\langle 0.25, 0.93, 0.45\rangle$	$\langle 0.44, 0.95, 0.54\rangle$	$\langle 0.78, 0.57, 0.15\rangle$
$\mathcal{A}^3$	$\langle 0.76, 0.85, 0.14\rangle$	$\langle 0.57, 0.76, 0.24\rangle$	$\langle 0.67, 0.55, 0.32\rangle$	$\langle 0.58, 0.67, 0.55\rangle$	$\langle 0.42, 0.75, 0.45\rangle$	$\langle 0.57, 0.38, 0.25\rangle$
$\mathcal{A}^4$	$\langle 0.53, 0.65, 0.45\rangle$	$\langle 0.71, 0.69, 0.11\rangle$	$\langle 0.57, 0.66, 0.33\rangle$	$\langle 0.42, 0.87, 0.54\rangle$	$\langle 0.55, 0.89, 0.14\rangle$	$\langle 0.69, 0.46, 0.16\rangle$

Table 15. Comparison of existing similarity measures with proposed method.

Unable to rank using existing similarity measures	Able to rank using Proposed method
$\mathcal{S}_J(\psi_1,\psi_2)$ [28]$\Rightarrow \mathcal{A}^1=\mathcal{A}^2=\mathcal{A}^3=\mathcal{A}^4=0.50$	$\mathcal{A}^4=0.58,>\mathcal{A}^3=0.56,>\mathcal{A}^2=0.51,>\mathcal{A}^1=0.50.$
$\mathcal{S}_D(\psi_1,\psi_2)$ [28]$\Rightarrow \mathcal{A}^1=\mathcal{A}^2=\mathcal{A}^3=\mathcal{A}^4=0.50$	$\mathcal{A}^4=0.58,>\mathcal{A}^3=0.56,>\mathcal{A}^2=0.51,>\mathcal{A}^1=0.50.$
$\mathcal{S}_1(\psi_1,\psi_2)$ [29]$\Rightarrow \mathcal{A}^1=\mathcal{A}^2=\mathcal{A}^3=\mathcal{A}^4=0.50$	$\mathcal{A}^4=0.58,>\mathcal{A}^3=0.56,>\mathcal{A}^2=0.51,>\mathcal{A}^1=0.50.$
$\mathcal{S}_2(\psi_1,\psi_2)$ [29]$\Rightarrow \mathcal{A}^1=\mathcal{A}^2=\mathcal{A}^3=\mathcal{A}^4=0.50$	$\mathcal{A}^4=0.58,>\mathcal{A}^3=0.56,>\mathcal{A}^2=0.51,>\mathcal{A}^1=0.50.$
$\mathcal{S}_3(\psi_1,\psi_2)$ [30]$\Rightarrow \mathcal{A}^1=\mathcal{A}^2=\mathcal{A}^3=\mathcal{A}^4=0.50$	$\mathcal{A}^4=0.58,>\mathcal{A}^3=0.56,>\mathcal{A}^2=0.51,>\mathcal{A}^1=0.50.$
$\mathcal{S}_4(\psi_1,\psi_2)$ [30]$\Rightarrow \mathcal{A}^1=\mathcal{A}^2=\mathcal{A}^3=\mathcal{A}^4=0.50$	$\mathcal{A}^4=0.58,>\mathcal{A}^3=0.56,>\mathcal{A}^2=0.51,>\mathcal{A}^1=0.50.$
$\mathcal{S}_5(\psi_1,\psi_2)$ [31]$\Rightarrow \mathcal{A}^1=\mathcal{A}^2=\mathcal{A}^3=\mathcal{A}^4=0.50$	$\mathcal{A}^4=0.58,>\mathcal{A}^3=0.56,>\mathcal{A}^2=0.51,>\mathcal{A}^1=0.50.$
$\mathcal{S}_6(\psi_1,\psi_2)$ [31]$\Rightarrow \mathcal{A}^1=\mathcal{A}^2=\mathcal{A}^3=\mathcal{A}^4=0.50$	$\mathcal{A}^4=0.58,>\mathcal{A}^3=0.56,>\mathcal{A}^2=0.51,>\mathcal{A}^1=0.50.$

9. Conclusions

Existing theories fail to handle the information when each component is interrelated. To overcome this limitation, we establish the properties of a simplified intuitionistic neutrosophic hypersoft set. We propose an application based on the TOPSIS method to identify a leader in a socio-political context. We apply CC instead of the usual distance or similarity measures in the TOPSIS method to understand the closeness coefficients in a better way. We have presented a comparative study between the proposed method and the existing TOPSIS method to prove the reliability of the proposed model. The proposed concept may be extended to algebraic structures, $\mathcal{N}$ soft set, and other hybrid structures. Apart from the theoretical dimension, the discussed concepts may be implemented to real-world challenges in fields such as psychology, economics, pattern recognition, artificial intelligence, and many more.

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Interval-valued intuitionistic neutrosophic hypersoft TOPSIS method based on correlation coefficient

A.Bobin [1*] and V.Chinnadurai [2]
Department of Mathematics, IFET College of Engineering (Autonomous), Tamilnadu, India.
[2] Department of Mathematics, Annamalai University, Tamilnadu, India.; chinnaduraiau@gmail.com
*Correspondence: bobinalbert@gmail.com

Abstract. In multi-criteria decision-making problems, we may have to deal with numbers that are in interval forms, like those of membership, non-membership grades and indeterminacy grades representing unique attributes of elements. When decision-makers come across such an environment, the decisions are harder to make and the most significant factor is that we need to combine these interval numbers to generate a single real number, which can be done using aggregation operators or score functions. To overcome this hindrance, we introduce the notion of interval-valued intuitionistic neutrosophic hypersoft set. This eventually helps the decision-maker to collect the data with no misconceptions. The primary aim of this study is to establish some operational laws for interval-valued intuitionistic neutrosophic hypersoft set. Also, we present the fundamental properties of two aggregation operators, interval-valued intuitionistic neutrosophic weighted average and interval-valued intuitionistic neutrosophic weighted geometric operators. Also, we propose an algorithm for the technique of order of preference by similarity to ideal solution (TOPSIS) method based on correlation coefficients to choose a suitable employee among the alternative using Leipzig leadership model in an organization for an upcoming new project. Finally, we present a comparative study with existing similarity measures to show the effectiveness of the proposed method.

Keywords: interval-valued neutrosophic set; intuitionistic set; hypersoft set.

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1. **Introduction**

Zadeh introduced the concept of fuzzy set (FS) [32] and FS has been widely used in various fields. The idea of intuitionistic FS (IFS) was presented by Atanassov [3], an extension of FS. Smarandache [24] developed the notion of the neutrosophic set (NS) characterized by the values of truth, indeterminacy, and falsity grades for each element of the set. Later, Wang et al. [27], [28] proposed the concepts of single-valued NS (SVNS) and interval-valued NS with a restricted condition for the membership values to overcome the constraints faced in NS. Chinnadurai et al. [9] discussed a solution to find out unique ranking among the alternatives. Chinnadurai and Bobin [10] proposed a concept to identify the profit and gains in decision-making problems by using Prospect theory. Chinnadurai and Bobin [11] introduced the concept of single valued neutrosophic N soft set. Also, Chinnadurai and Bobin [12] established the properties of interval-valued neutrosophic N soft set. Molodtsov [16] introduced the idea of the soft set (SS) to deal with uncertainties. Smarandache [25] presented the notion of the hypersoft set (HSS) to overcome the restriction faced in SS. Saeed [22] briefed the fundamental concepts of HSS. Ihsan et al. [21] used a hypersoft expert set for the recruitment process in MCDM problems.

Selvachandran et al. [23] presented a modified TOPSIS deviation method of SVNS. Wang and Chen [29] proposed a TOPSIS method, in which they got the optimal weights of attributes using linear programming of interval-valued IFS. Wang and Wan [30] investigated group decision making with interval-valued IFS. Nabeeh et al. [18] contributed to the personnel selection process among different alternatives by combining the analytical hierarchy method with the TOPSIS. Abdel-Basset et al. [1] combined type 2 NS and TOPSIS for supplier selection. Abdel-Basset et al. [2] proposed the use of bipolar neutrosophic numbers in the TOPSIS method for selecting smart medical devices. Endalkachew Teshome Ayele et al. [4] presented a method for traffic signal control using an interval-valued neutrosophic soft set. Christianto and Smarandache [13] proposed the idea of a third-way leadership model, a blend of hard-style and soft-style leadership. Harish et al. [14] used a combination of TOPSIS and Choquet integral method in hesitant FS to solve multi-criteria decision-making (MCDM) problems. Rana Muhammad Zulqarnain et al. [54] introduced the concept of intuitionistic fuzzy HSS and used the TOPSIS method based on correlation coefficient (CC). Rana Muhammad Zulqarnain et al. [55] studied the fundamental operations of interval-valued neutrosophic HSS. Saqlain Muhammad et al. [17] defined aggregation operators on neutrosophic HSS and studied some properties.

Rahman et al. [19] extended the concept of HSS to complex FS, complex IFS, and complex NS. Zulqarnain et al. [45] developed the TOPSIS method in a fuzzy environment and used it in the medical staff recruitment process. Zulqarnain et al. [46] established the concept of generalized TOPSIS method to solve MCDM problems. Zulqarnain and Dayan [43] presented a method for choosing the best criteria by using the fuzzy TOPSIS method. Zulqarnain et al. [44] proposed an idea for predicting diabetes using TOPSIS analysis. Zulqarnain et al. [34] used the TOPSIS method based on correlation coefficient and aggregation operators under intuitionistic fuzzy hypersoft set (IFHSS) environment. Zulqarnain et al. [39] used aggregation operators in the IFHSS environment to solve MCDM problems. Zulqarnain [48] developed a new TOPSIS method based on the correlation coefficient of interval-valued intuitionistic fuzzy soft sets in MCDM problems. Zulqarnain [53] established aggregation operators under Pythagorean fuzzy soft environment to solve MCDM problems. Zulqarnain et al. [50] developed aggregation operators of Pythagorean fuzzy soft sets for selecting green supplier chain management. Zulqarnain [49] discussed an application towards green supply chain management by using Pythagorean fuzzy soft set. Zulqarnain et al. [37] developed the concept of Pythagorean fuzzy hypersoft set (PFHSS). Zulqarnain et al. [47] discussed an idea of solving MCDM problems by using the generalized neutrosophic TOPSIS method. Zulqarnain et al. [38] used the concept of PFHSS in selecting the antivirus mask during the pandemic. Zulqarnain et al. [41] presented an application for solving MCDM problems using neutrosophic hypersoft matrices.

Zulqarnain et al. [42] discussed MCDM problems using the aggregation operators in the PFHSS environment. Zulqarnain [51] discussed an integrated TOPSIS model in a neutrosophic environment. Zulqarnain [52] proposed algorithms for a generalized multi-polar neutrosophic soft set to solve medical diagnoses. Zulqarnain et al. [33] proposed the generalized aggregate operators on neutrosophic HSS (NHSS) such as extended union, extended intersection, OR-operation, AND operation, etc., and established their properties. Samad et al. [40] extended the TOPSIS method based on correlation coefficient under NHSS environment in selecting an effective hand sanitizer during the pandemic. Rahman et al. [20] developed the concept of neutrosophic parametrized hypersoft set theory to solve MCDM problems. Zulqarnain et al. [35] discussed the concepts of the decision-making approach based on correlation coefficient under interval-valued neutrosophic hypersoft set (IVNHSS). Zulqarnain et al. [36] presented the fundamental operations on IVHSS and established their properties. Smarandache [26] proposed the notion of dependence and independence between the components of the FS and NS. Chinnadurai and Bobin [7], [8] defined the concepts of simplified intuitionistic neutrosophic SS (SINSS) and interval-valued intuitionistic neutrosophic SS (IVINSS) and studied some of their properties. In SINSS and IVINSS, the membership grades of truth and falsity depend on each other such that their sum cannot exceed one and the membership grade of indeterminacy is independent with a value less than or equal to one. Hence, in SINSS and IVINSS, the sum of the membership grades cannot exceed two.

All the above mentioned fuzzy hybrid sets cannot accommodate the membership grades of truth and falsity, which depend on each other such that their sum cannot exceed one and the membership grade of indeterminacy is independent with a value less than or equal to one. Therefore, to solve this problem, in this article, we present some aggregation operators for IVINHSS. We develop an algorithm to solve the decision-making problem based on the established operators. We have presented a numerical example to ensure the practicality of the developed algorithm. The main aim of the present study is to rank the alternatives based on interval-valued intuitionistic neutrosophic hypersoft set (IVINHSS) data using aggregation operators and also making use of the TOPSIS method based on CC. To the best of our knowledge, research on IVINHSS is confined to its theory and related development and applications. Therefore, the new method proposed in this paper can examine and provide a suitable solution to the decision-makers in ranking the alternatives. We present an MCDM approach based on TOPSIS, and the effectiveness of this method is showed through the selection of a suitable employee who can lead the project successfully. To prove the efficacy of the proposed method, a comparative analysis between the proposed and existing similarity measures (SMs) is given. Thus, the IVINHSS is a robust tool to predict uncertainties when the grades are in interval form for all truth, falsity, and indeterminacy grades for all the attributes.

The manuscript comprises the following sections. Section 2 briefs on existing definitions. Section 3 introduces the concept of IVINHSS and discusses some properties of CC and weighted CC of IVINHSS. Section 4 deals with the interval-valued intuitionistic neutrosophic hypersoft weighted average operator (IVINHSWAO) and interval-valued intuitionistic neutrosophic hypersoft weighted geometric operator (IVINHSWGO). Section 5 highlights the combination of CC with the TOPSIS method. Section 6 shows the significance of the proposed method with comparative analysis. Section 7 ends with a conclusion.

2. **Preliminaries**

We present some of the basic definitions required for this study. Let us consider the following notations throughout this study unless otherwise specified. Let $\mathcal{V}$ be the universe and $v_i \in \mathcal{V}$, $P(\mathcal{V})$ be the power set of $\mathcal{V}$, $\mathbb{N}$ represents natural numbers, $C[0,1]$ denotes the set of all closed sub intervals of [0,1] and $\mathcal{N}^U$ represent the collection of interval-valued intuitionistic NS (IVINS) over $\mathcal{V}$.

Definition 2.1. [32] A fuzzy set (FS) is a set of the form $\mathcal{F} = \{(v, \mathcal{T}_{\mathcal{F}}(v)) : v \in \mathcal{V}\}$, where $\mathcal{T}_{\mathcal{F}} : \mathcal{V} \to [0,1]$ defines the degree of membership of the element $v \in \mathcal{V}$.

Definition 2.2. [3] An intuitionistic FS (IFS) is an object of the form $\mathcal{C} = \{(v, \mathcal{T}_{\mathcal{C}}(v), \mathcal{F}_{\mathcal{C}}(v)) : v \in \mathcal{V}\}$, where $\mathcal{T}_{\mathcal{C}} : \mathcal{V} \to [0,1]$ and $\mathcal{F}_{\mathcal{C}} : \mathcal{V} \to [0,1]$ define the degree of membership and degree of non-membership of the element $v \in \mathcal{V}$, respectively and for every $v \in \mathcal{V}$, $0 \leq \mathcal{T}_{\mathcal{C}}(v)+\mathcal{F}_{\mathcal{C}}(v) \leq 1$, where $\pi_{\mathcal{C}}(v) = 1 - \mathcal{T}_{\mathcal{C}}(v) - \mathcal{F}_{\mathcal{C}}(v)$ represents the degree of hesitancy.

Definition 2.3. [27] A single valued neutrosophic set (SVNS) is an object of the form $\mathfrak{N} = \{\langle v, \mathcal{T}_{\mathfrak{N}}(v), \mathcal{I}_{\mathfrak{N}}(v), \mathcal{F}_{\mathfrak{N}}(v)\rangle : v \in \mathcal{V}\}$, where $\mathcal{T}_{\mathfrak{N}} : \mathcal{V} \to [0,1]$, $\mathcal{I}_{\mathfrak{N}} : \mathcal{V} \to [0,1]$ and $\mathcal{F}_{\mathfrak{N}} : \mathcal{V} \to [0,1]$ represent the degree of truth membership, degree of indeterminacy membership and degree of falsity membership of the element $v \in \mathcal{V}$, respectively and for every $v \in \mathcal{V}$,

$0 \leq \mathcal{T}_{\mathfrak{N}}(v) + \mathcal{I}_{\mathfrak{N}}(v) + \mathcal{F}_{\mathfrak{N}}(v) \leq 3$. $\mathfrak{N}^U$ denote the set of all single valued neutrosophic subsets of $\mathcal{V}$.

Definition 2.4. [28] An interval valued neutrosophic set (IVNS) is a set of the form $\mathcal{R} = \{\langle v, [\underline{\mathcal{T}_{\mathcal{R}}}(v), \overline{\mathcal{T}_{\mathcal{R}}}(v)], [\underline{\mathcal{I}_{\mathcal{R}}}(v), \overline{\mathcal{I}_{\mathcal{R}}}(v)], [\underline{\mathcal{F}_{\mathcal{R}}}(v), \overline{\mathcal{F}_{\mathcal{R}}}(v)]\rangle : v \in \mathcal{V}\}$. IVNS can be represented as $\mathcal{R} = \{\langle v, \tilde{\mathcal{T}_{\mathcal{R}}}(v), \tilde{\mathcal{I}_{\mathcal{R}}}(v), \tilde{\mathcal{F}_{\mathcal{R}}}(v)\rangle : v \in \mathcal{V}\}$, where $\tilde{\mathcal{T}_{\mathcal{R}}} : \mathcal{V} \to C[0,1]$, $\tilde{\mathcal{I}_{\mathcal{R}}} : \mathcal{V} \to C[0,1]$ and $\tilde{\mathcal{F}_{\mathcal{R}}} : \mathcal{V} \to C[0,1]$ represent the degree of truth membership, degree of indeterminacy membership and degree of falsity membership in closed sub-intervals of the element $v \in \mathcal{V}$, respectively and for every $v \in \mathcal{V}$, $0 \leq \overline{T_{\mathcal{R}}}(v) + \overline{I_{\mathcal{R}}}(v) + \overline{F_{\mathcal{R}}}(v) \leq 3$. $\mathcal{R}^U$ denote the set of all interval valued neutrosophic subsets of $\mathcal{V}$.

Definition 2.5. [8] An IVINS in $\mathcal{V}$ is an object of the form $\Omega = \{\langle v, \alpha_\Omega(v), \beta_\Omega(v), \gamma_\Omega(v)\rangle\}$, where $\alpha_\Omega(v) : \mathcal{V} \to C[0,1]$, $\beta_\Omega(v) : \mathcal{V} \to C[0,1]$ and $\gamma_\Omega(v) : \mathcal{V} \to C[0,1]$. $\alpha_\Omega(v)$, $\beta_\Omega(v)$ and $\gamma_\Omega(v)$ are closed sub intervals of [0,1], representing the membership grades of truth, indeterminacy and falsity of the element $v \in \mathcal{V}$. The lower and upper ends of $\alpha_\Omega(v)$, $\beta_\Omega(v)$ and $\gamma_\Omega(v)$ are denoted, respectively by $\underline{\alpha}_\Omega(v)$, $\overline{\alpha}_\Omega(v)$, $\underline{\beta}_\Omega(v)$, $\overline{\beta}_\Omega(v)$, and $\underline{\gamma}_\Omega(v)$, $\overline{\gamma}_\Omega(v)$, where $0 \leq \overline{\alpha}_\Omega(v) + \overline{\gamma}_\Omega(v) \leq 1$ and $\underline{\alpha}_\Omega(v), \underline{\beta}_\Omega(v), \underline{\gamma}_\Omega(v) \geq 0$, $0 \leq \overline{\alpha}_\Omega(v) + \overline{\beta}_\Omega(v) + \overline{\gamma}_\Omega(v) \leq 2$, $\forall\ v \in \mathcal{V}$.

Definition 2.6. [16] A pair $(\Omega, \mathcal{E})$ is called a soft set (SS) over $\mathcal{V}$, if $\Omega : \mathcal{E} \to \mathcal{P}(\mathcal{V})$. Then for any $p \in \mathcal{E}$, $\Omega(p) = 1$ is equivalent to $v \in \Omega(p)$ and $\Omega(p) = 0$ is equivalent to $v \notin \Omega(p)$. Thus a SS is not a set, but a parameterized family of subsets of $\mathcal{V}$.

Definition 2.7. [25] Let $\Delta_1, \Delta_2, ..., \Delta_k$, be distinct attribute sets, whose corresponding sub-attributes are $\Delta_1 = \{\lambda_{11}, \lambda_{12}, ..., \lambda_{1f}\}, \Delta_2 = \{\lambda_{21}, \lambda_{22}, ..., \lambda_{2g}\}, ..., \Delta_k = \{\lambda_{k1}, \lambda_{k2}, ..., \lambda_{kh}\}$, where $1 \leq f \leq p$, $1 \leq g \leq q$, $1 \leq h \leq r$ and p, q, $r \in \mathbb{N}$, such that $\Delta_i \cap \Delta_j = \emptyset$, for each $i, j \in \{1, 2, ..., k\}$ and $i \neq j$. Then the Cartesian product of the distinct attribute sets $\Delta_1 \times \Delta_2 \times ... \times \Delta_k = \tilde{\Delta} = \{\lambda_{1f} \times \lambda_{2g} \times ... \times \lambda_{kh}\}$, represent a collection of multi- attributes. A pair $(\Omega, \tilde{\Delta})$ is called a hypersoft set (HSS) over $\mathcal{V}$, where $\Omega : \tilde{\Delta} \to P(\mathcal{V})$. HSS can be represented as $(\Omega, \tilde{\Delta}) = \left\{(\tilde{\lambda}, \Omega(\tilde{\lambda})) | \tilde{\lambda} \in \tilde{\Delta}, \Omega(\tilde{\lambda}) \in P(\mathcal{V})\right\}$.

3. Interval-valued intuitionistic neutrosophic hypersoft set

We present the notion of interval-valued intuitionistic neutrosophic hypersoft set (IVINHSS). Also, we discuss some basic properties of correlation coefficient (CC) and weighted CC (WCC) on IVINHSS.

Definition 3.1. A pair $(\Omega, \tilde{\Delta})$ is called an IVINHSS over $\mathcal{V}$, where $\Omega : \tilde{\Delta} \to \mathcal{N}^U$. IVINHSS can be represented as $(\Omega, \tilde{\Delta}) = \left\{(\tilde{\lambda}, \Omega(\tilde{\lambda})) | \tilde{\lambda} \in \tilde{\Delta}, \Omega(\tilde{\lambda}) \in \mathcal{N}^U \in C[0,1]\right\}$, where $\Omega(\tilde{\lambda}) = \left\{\left\langle v, \alpha_{\Omega(\tilde{\lambda})}(v), \beta_{\Omega(\tilde{\lambda})}(v), \gamma_{\Omega(\tilde{\lambda})}(v)\right\rangle | v \in \mathcal{V}\right\}$, where $\alpha_{\Omega(\tilde{\lambda})}(v) : \mathcal{V} \to C[0,1]$, $\beta_{\Omega(\tilde{\lambda})}(v) : \mathcal{V} \to C[0,1]$ and $\gamma_{\Omega(\tilde{\lambda})}(v) : \mathcal{V} \to C[0,1]$. $\alpha_{\Omega(\tilde{\lambda})}(v)$, $\beta_{\Omega(\tilde{\lambda})}(v)$ and $\gamma_{\Omega(\tilde{\lambda})}(v)$ are closed sub intervals of

[0,1], representing the membership grades of truth, indeterminacy and falsity. The lower and upper ends of $\alpha_{\Omega(\tilde{\lambda})}(v)$, $\beta_{\Omega(\tilde{\lambda})}(v)$ and $\gamma_{\Omega(\tilde{\lambda})}(v)$ are denoted, respectively by $\underline{\alpha}_{\Omega(\tilde{\lambda})}(v)$, $\overline{\alpha}_{\Omega(\tilde{\lambda})}(v)$, $\underline{\beta}_{\Omega(\tilde{\lambda})}(v)$, $\overline{\beta}_{\Omega(\tilde{\lambda})}(v)$, and $\underline{\gamma}_{\Omega(\tilde{\lambda})}(v)$, $\overline{\gamma}_{\Omega(\tilde{\lambda})}(v)$, where $0 \leq \overline{\alpha}_{\Omega(\tilde{\lambda})}(v) + \overline{\gamma}_{\Omega(\tilde{\lambda})}(v) \leq 1$ and $\underline{\alpha}_{\Omega(\tilde{\lambda})}(v), \underline{\beta}_{\Omega(\tilde{\lambda})}(v), \underline{\gamma}_{\Omega(\tilde{\lambda})}(v) \geq 0$, $0 \leq \overline{\alpha}_{\Omega(\tilde{\lambda})}(v) + \overline{\beta}_{\Omega(\tilde{\lambda})}(v) + \overline{\gamma}_{\Omega(\tilde{\lambda})}(v) \leq 2$.

Example 3.2. Let $\mathcal{V} = \{v_1, v_2, v_3\}$ be a set of managers who evaluate an employee based on the Leipzig leadership model for an upcoming project. Let Δ_1, Δ_2, Δ_3 and Δ_4 be distinct attribute sets whose corresponding sub-attributes are represented as Δ_1 = purpose = $\{\lambda_{11} = \text{achieve goals}\}$, Δ_2 = entrepreneurial spirit = $\{\lambda_{21} = \text{quick decision}, \lambda_{22} = \text{logical decision}\}$,
Δ_3 = responsibility = $\{\lambda_{31} = \text{inspire and motivate}, \lambda_{32} = \text{time management}\}$ and Δ_4 = effectiveness = $\{\lambda_{41} = \text{successful accomplishment}\}$. Then $\tilde{\Delta} = \Delta_1 \times \Delta_2 \times \Delta_3 \times \Delta_4$ is the distinct attribute set given by

$$\begin{aligned}\tilde{\Delta} &= \Delta_1 \times \Delta_2 \times \Delta_3 \times \Delta_4 = \{\lambda_{11}\} \times \{\lambda_{21}, \lambda_{22}\} \times \{\lambda_{31}, \lambda_{32}\} \times \{\lambda_{41}\}.\\ &= \Big\{(\lambda_{11}, \lambda_{21}, \lambda_{31}, \lambda_{41}), (\lambda_{11}, \lambda_{21}, \lambda_{32}, \lambda_{41}), (\lambda_{11}, \lambda_{22}, \lambda_{31}, \lambda_{41}), (\lambda_{11}, \lambda_{22}, \lambda_{32}, \lambda_{41})\Big\}.\\ &= \Big\{\tilde{\lambda_1}, \tilde{\lambda_2}, \tilde{\lambda_3}, \tilde{\lambda_4}\Big\}.\end{aligned}$$

An IVINHSS $(\Omega, \tilde{\Delta})$ is a collection of subsets of $\mathcal{V}$, given by the managers for each employee based on the description in Table 1.

TABLE 1. Shows leadership skills of an employee in IVINHSS $(\Omega, \tilde{\Delta})$ form.

$\mathcal{V}$	$\tilde{\lambda_1}$	$\tilde{\lambda_2}$	$\tilde{\lambda_3}$	$\tilde{\lambda_4}$
v_1	$\langle[0.3, 0.4], [0.7, 0.8], [0.2, 0.3]\rangle$	$\langle[0.2, 0.4], [0.5, 0.6], [0.5, 0.6]\rangle$	$\langle[0.6, 0.7], [0.2, 0.1], [0.1, 0.2]\rangle$	$\langle[0.3, 0.4], [0.4, 0.5], [0.2, 0.3]\rangle$
v_2	$\langle[0.2, 0.4], [0.8, 0.9], [0.1, 0.3]\rangle$	$\langle[0.6, 0.7], [0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.4, 0.5], [0.4, 0.6], [0.1, 0.2]\rangle$	$\langle[0.2, 0.5], [0.5, 0.6], [0.2, 0.4]\rangle$
v_3	$\langle[0.1, 0.2], [0.5, 0.7], [0.2, 0.3]\rangle$	$\langle[0.3, 0.4], [0.6, 0.7], [0.2, 0.4]\rangle$	$\langle[0.2, 0.3], [0.1, 0.3], [0.6, 0.7]\rangle$	$\langle[0.2, 0.3], [0.6, 0.8], [0.4, 0.6]\rangle$

3.1. *Correlation coefficient for IVINHSS*

Let the two IVINHSS over $\mathcal{V}$ be as given below.

$$(\Omega_1, \tilde{\Delta}_1) = \big\{(v_i, [\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i), \overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i)], [\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i), \overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i)], [\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i), \overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i)]\big\},$$

$$(\Omega_2, \tilde{\Delta}_2) = \big\{(v_i, [\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i), \overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i)], [\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i), \overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i)], [\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i), \overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i)]\big\}.$$

Definition 3.3. Let $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ be two IVINHSS. Then the interval-valued intuitionistic neutrosophic informational energies of $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ are represented as

$$\begin{aligned}\Phi(\Omega_1, \tilde{\Delta}_1) = \sum_{k=1}^{m}\sum_{i=1}^{n}\Bigg[&(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\\ &+ (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\Bigg], \quad (1)\end{aligned}$$

$$\Phi(\Omega_2,\tilde{\Delta}_2)=\sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]. \quad (2)$$

Definition 3.4. Let $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ be two IVINHSS. Then the correlation measure between $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ is defined as

$$\mathcal{C}_{\mathcal{M}}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))=\sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))\Big]. \quad (3)$$

Proposition 3.5. *Let $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ be two IVINHSS. Then,*
(i) $\mathcal{C}_{\mathcal{M}}((\Omega_1,\tilde{\Delta}_1),(\Omega_1,\tilde{\Delta}_1))=\Phi(\Omega_1,\tilde{\Delta}_1)$
(ii) $\mathcal{C}_{\mathcal{M}}((\Omega_2,\tilde{\Delta}_2),(\Omega_2,\tilde{\Delta}_2))=\Phi(\Omega_2,\tilde{\Delta}_2)$.

Proof. Straight forward □

Definition 3.6. Let $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ be two IVINHSS. Then, the CC between $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ is given by

$$\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))=\frac{\mathcal{C}_{\mathcal{M}}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))}{\sqrt{\Phi(\Omega_1,\tilde{\Delta}_1)}\sqrt{\Phi(\Omega_2,\tilde{\Delta}_2)}} \quad (4)$$

Proposition 3.7. *Let $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ be two IVINHSS. Then, the following CC properties hold:*
(i) $0\le\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))\le 1$;
(ii) $\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))=\mathcal{C}_C((\Omega_2,\tilde{\Delta}_2),(\Omega_1,\tilde{\Delta}_1))$;
(iii) If $(\Omega_1,\tilde{\Delta}_1)=(\Omega_2,\tilde{\Delta}_2)$, then $\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))=1$.

Proof. (i)Obviously, $\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))\ge 0$. Now, we present the proof of $\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))\le 1$.

$$\mathcal{C}_{\mathcal{M}}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))=\sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))\Big].$$

$$= \sum_{k=1}^{m}\Big[\Big((\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_1))$$
$$+ (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_1))\Big)$$
$$+\Big((\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) +$$
$$(\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_2))\Big) + ...$$
$$+\Big((\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) +$$
$$(\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_n))\Big)\Big].$$

By applying Cauchy-Schwarz inequality, we get

$$\mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))^2$$
$$\leq \sum_{k=1}^{m}\Big[\Big\{(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2$$
$$+ ... + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\} +$$
$$\Big\{(\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2$$
$$+ ... + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\}\Big] \times$$
$$\sum_{k=1}^{m}\Big[\Big\{(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 +$$
$$... + (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2$$
$$+ (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 +$$
$$... + (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\}\Big].$$

$$\mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))^2$$
$$\leq \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2$$
$$+ (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\Big] \times \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2$$
$$+ (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big].$$

$$\Rightarrow \mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))^2 \leq \Phi(\Omega_1, \tilde{\Delta}_1) \times \Phi(\Omega_2, \tilde{\Delta}_2).$$
$$\Rightarrow \mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \leq \sqrt{\Phi(\Omega_1, \tilde{\Delta}_1)} \times \sqrt{\Phi(\Omega_2, \tilde{\Delta}_2)}.$$
$$\Rightarrow \frac{\mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))}{\sqrt{\Phi(\Omega_1, \tilde{\Delta}_1)} \times \sqrt{\Phi(\Omega_2, \tilde{\Delta}_2)}} \leq 1.$$

By using Definition 3.5, we get $\mathcal{C}_C((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \leq 1$.

Hence, $0 \leq \mathcal{C}_C((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \leq 1$. □

Proof. (ii) Straight forward. □

Proof. (iii) $\mathcal{C}_C((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = \frac{\mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))}{\sqrt{\Phi(\Omega_1, \tilde{\Delta}_1)} \times \sqrt{\Phi(\Omega_2, \tilde{\Delta}_2)}}$.

Since, $(\Omega_1, \tilde{\Delta}_1) = (\Omega_2, \tilde{\Delta}_2)$.

$\mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))$

$$= \frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]}{\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]}\times\sqrt{\sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]}}.$$

$\Rightarrow \mathcal{C}_C((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = 1.$ □

Definition 3.8. Let $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ be two IVINHSS. Then, the CC between $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2\tilde{\Delta}_2)$ is defined as

$$\tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = \frac{\mathcal{C}_{\mathcal{M}}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))}{\max\Big\{\Phi(\Omega_1,\tilde{\Delta}_1),\Phi(\Omega_2,\tilde{\Delta}_2)\Big\}}. \quad (5)$$

$\tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))$

$$= \frac{\sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))+(\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))*(\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))\Big]}{\max\Big\{\sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2+(\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2+(\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2+(\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2+(\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2+(\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\Big], \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2+(\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]\Big\}}.$$

Proposition 3.9. *Let $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ be two IVINHSS. Then, the following CC properties hold:*

(i) $0 \leq \tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \leq 1$;

(ii) $\tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = \tilde{\mathcal{C}_C}((\Omega_2,\tilde{\Delta}_2),(\Omega_1,\tilde{\Delta}_1))$;

(iii) If $(\Omega_1,\tilde{\Delta}_1) = (\Omega_2,\tilde{\Delta}_2)$, *then* $\tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = 1$.

Proof. (i) Obviously, $\tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \geq 0$. Now, we present the proof of $\tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \leq 1$.

$\mathcal{C}_{\mathcal{M}}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))$

$$= \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))$$
$$+ (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))\Big].$$
$$= \sum_{k=1}^{m}\Big[\Big((\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_1))$$
$$+(\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_1)) + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_1)) * (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_1))\Big)$$
$$+\Big((\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_2))$$
$$+(\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_2)) + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_2)) * (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_2))\Big) + ...$$
$$+\Big((\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_n))$$
$$+(\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_n)) + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_n)) * (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_n))\Big)\Big].$$

By applying Cauchy-Schwarz inequality, we get

$$\mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))$$
$$\leq \Bigg\{ \sum_{k=1}^{m}\Big[\Big\{(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 +$$
$$(\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\}$$
$$+ \Big\{(\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 +$$
$$(\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_1))^2 + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_2))^2 + ... + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_n))^2\Big\}\Big] \times$$
$$\sum_{k=1}^{m}\Big[\Big\{(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 +$$
$$... + (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\}$$
$$+ \Big\{(\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 +$$
$$... + (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\} + \Big\{(\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_1))^2 + (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_2))^2 + ... + (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_n))^2\Big\}\Big]\Bigg\}^{\frac{1}{2}}.$$

$$\mathcal{C}_M((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))$$
$$\leq \Bigg\{ \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2$$
$$+ (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\Big] \times \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2$$
$$+ (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]\Bigg\}^{\frac{1}{2}}.$$
$$\leq \Bigg\{\Bigg(\max\Bigg\{ \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 +$$
$$(\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\Big] \times \sum_{k=1}^{m}\sum_{i=1}^{n}\Big[(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 +$$
$$(\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]\Bigg\}\Bigg)^2\Bigg\}^{\frac{1}{2}}.$$

$$= \max\Big\{ \sum_{k=1}^{m}\sum_{i=1}^{n} \Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\Big] \times \sum_{k=1}^{m}\sum_{i=1}^{n} \Big[(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]\Big\}.$$

$$\Rightarrow \mathcal{C}_M((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \leq \max\Big\{\Phi(\Omega_1,\tilde{\Delta}_1) \times \Phi(\Omega_2,\tilde{\Delta}_2)\Big\}.$$

$$\Rightarrow \frac{\mathcal{C}_M((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))}{\max\Big\{\Phi(\Omega_1,\tilde{\Delta}_1)\times\Phi(\Omega_2,\tilde{\Delta}_2)\Big\}} \leq 1.$$

By using Definition 3.8, we get $\tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \leq 1$.
Hence, $0 \leq \tilde{\mathcal{C}_C}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \leq 1$.
Proofs of (ii) and (iii) are same as in Proposition 3.6. □

3.2. *Weighted correlation coefficient for IVINHSS*

We present the concept of weighted correlation coefficient (WCC) for IVINHSS. WCC facilitates decision-makers (DMs) to provide different weights for each alternative. Consider $\mathcal{D} = \{\mathcal{D}_1,\mathcal{D}_2,...,\mathcal{D}_m\}$ and $\mathcal{W} = \{\mathcal{W}_1,\mathcal{W}_2,...,\mathcal{W}_n\}$ as weight vectors for alternatives and experts, respectively, such that $\mathcal{D}_k,\mathcal{W}_i > 0$ and $\sum_{k=1}^{m}\mathcal{D}_k = 1, \sum_{i=1}^{n}\mathcal{W}_i = 1$.

Definition 3.10. Let $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ be two IVINHSS. Then, the WCC between $(\Omega_1,\tilde{\Delta}_1)$ and $(\Omega_2,\tilde{\Delta}_2)$ is defined as

$$\mathcal{C}_{C_{\mathcal{W}}}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = \frac{\mathcal{C}_{\mathcal{M}}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2))}{\sqrt{\Phi(\Omega_1,\tilde{\Delta}_1)}\sqrt{\Phi(\Omega_2,\tilde{\Delta}_2)}} \tag{6}$$

$$\mathcal{C}_{C_{\mathcal{W}}}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) = \frac{\sum_{k=1}^{m}\mathcal{D}_k\Big(\sum_{i=1}^{n}\mathcal{W}_i\Big[\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i)*\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i) + \underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i)*\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i) + \underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i)*\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i) + \overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i)*\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i) + \overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i)*\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i) + \overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i)*\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i)\Big]\Big)}{\sqrt{\sum_{k=1}^{m}\mathcal{D}_k\Big(\sum_{i=1}^{n}\mathcal{W}_i\Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2\Big]\Big)} \times \sqrt{\sum_{k=1}^{m}\mathcal{D}_k\Big(\sum_{i=1}^{n}\mathcal{W}_i\Big[(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2\Big]\Big)}}.$$

If $\mathcal{D} = \big\{\frac{1}{m},\frac{1}{m},...,\frac{1}{m},\big\}$ and $\mathcal{W} = \big\{\frac{1}{n},\frac{1}{n},...,\frac{1}{n},\big\}$, then WCC given in Eq.(6) reduces to CC as in Eq.(4).

Proposition 3.11. *Let* $(\Omega_1,\tilde{\Delta}_1)$ *and* $(\Omega_2,\tilde{\Delta}_2)$ *be two IVINHSS. Then, the following WCC properties hold:*
(i) $0 \leq \mathcal{C}_{C_{\mathcal{W}}}((\Omega_1,\tilde{\Delta}_1),(\Omega_2,\tilde{\Delta}_2)) \leq 1$*;*

(ii) $\mathcal{C}_{C_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = \mathcal{C}_{C_{\mathcal{W}}}((\Omega_2, \tilde{\Delta}_2), (\Omega_1, \tilde{\Delta}_1))$;
(iii) If $(\Omega_1, \tilde{\Delta}_1) = (\Omega_2, \tilde{\Delta}_2)$, *then* $\mathcal{C}_{C_{\mathcal{W}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = 1$.

Proof. Similar to Proposition 3.6. □

Definition 3.12. Let $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2, \tilde{\Delta}_2)$ be two IVINHSS. Then, the WCC between $(\Omega_1, \tilde{\Delta}_1)$ and $(\Omega_2\tilde{\Delta}_2)$ is defined as

$$\mathcal{C}_{\tilde{C_{\mathcal{W}}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = \frac{\mathcal{C}_{\mathcal{M}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))}{\max\left\{\Phi(\Omega_1, \tilde{\Delta}_1), \Phi(\Omega_2, \tilde{\Delta}_2)\right\}}. \tag{7}$$

$$\mathcal{C}_{\tilde{C_{\mathcal{W}}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2))$$

$$= \frac{\begin{gathered}\sum_{k=1}^{m} \mathcal{D}_k \Bigg(\sum_{i=1}^{n} \mathcal{W}_i \Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) \\ + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i)) * (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i)) \Big] \Bigg)\end{gathered}}{\begin{gathered}\max\Bigg\{ \sum_{k=1}^{m} \mathcal{D}_k \Bigg(\sum_{i=1}^{n} \mathcal{W}_i \Big[(\underline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\alpha}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\beta}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\gamma}_{\Omega_1(\tilde{\lambda}_k)}(v_i))^2 \Big] \Bigg), \\ \sum_{k=1}^{m} \mathcal{D}_k \Bigg(\sum_{i=1}^{n} \mathcal{W}_i \Big[(\underline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\underline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\alpha}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\beta}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 + (\overline{\gamma}_{\Omega_2(\tilde{\lambda}_k)}(v_i))^2 \Big] \Bigg) \Bigg\}\end{gathered}}.$$

If $\mathcal{D} = \left\{\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m}, \right\}$ and $\mathcal{W} = \left\{\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}, \right\}$, then WCC given in Eq.(7) reduces to CC as in Eq.(5).

Proposition 3.13. *Let* $(\Omega_1, \tilde{\Delta}_1)$ *and* $(\Omega_2, \tilde{\Delta}_2)$ *be two IVINHSS. Then, the following WCC properties hold:*
(i) $0 \leq \mathcal{C}_{\tilde{C_{\mathcal{W}}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) \leq 1$;
(ii) $\mathcal{C}_{\tilde{C_{\mathcal{W}}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = \mathcal{C}_{\tilde{C_{\mathcal{W}}}}((\Omega_2, \tilde{\Delta}_2), (\Omega_1, \tilde{\Delta}_1))$;
(iii) If $(\Omega_1, \tilde{\Delta}_1) = (\Omega_2, \tilde{\Delta}_2)$, *then* $\mathcal{C}_{\tilde{C_{\mathcal{W}}}}((\Omega_1, \tilde{\Delta}_1), (\Omega_2, \tilde{\Delta}_2)) = 1$.

Proof. Similiar to Proposition 3.6. □

4. **Aggregation operators for IVINHSS**

We now present the concept of interval-valued intuitionistic neutrosophic hypersoft weighted average operator(IVINHSWAO) and interval-valued intuitionistic neutrosophic hypersoft weighted geometric operator (IVINHSWGO) by using operational laws. Let κ represent the collection of interval-valued intuitionistic neutrosophic hypersoft numbers (IVINHSNs).

4.1. *Operational laws for IVINHSS*

Definition 4.1.
Let $\Omega_{e_{11}} = \langle [\underline{\alpha}_{11}, \overline{\alpha}_{11}], [\underline{\beta}_{11}, \overline{\beta}_{11}], [\underline{\gamma}_{11}, \overline{\gamma}_{11}] \rangle$ and $\Omega_{e_{12}} = \langle [\underline{\alpha}_{12}, \overline{\alpha}_{12}], [\underline{\beta}_{12}, \overline{\beta}_{12}], [\underline{\gamma}_{12}, \overline{\gamma}_{12}] \rangle$ be two IVINHSS and δ a positive integer. Then,

(i) $\Omega_{e_{11}} \oplus \Omega_{e_{12}} = \langle [\underline{\alpha}_{11} + \underline{\alpha}_{12} - \underline{\alpha}_{11}\underline{\alpha}_{12}, \overline{\alpha}_{11} + \overline{\alpha}_{12} - \overline{\alpha}_{11}\overline{\alpha}_{12}], [\underline{\beta}_{11} + \underline{\beta}_{12} - \underline{\beta}_{11}\underline{\beta}_{12}, \overline{\beta}_{11} + \overline{\beta}_{12} - \overline{\beta}_{11}\overline{\beta}_{12}], [\underline{\gamma}_{11}\underline{\gamma}_{12}, \overline{\gamma}_{11}\overline{\gamma}_{12}] \rangle$;

(ii) $\Omega_{e_{11}} \otimes \Omega_{e_{12}} = \langle [\underline{\alpha}_{11}\underline{\alpha}_{12}, \overline{\alpha}_{11}\overline{\alpha}_{12}], [\underline{\beta}_{11}\underline{\beta}_{12}, \overline{\beta}_{11}\overline{\beta}_{12}], [\underline{\gamma}_{11} + \underline{\gamma}_{12} - \underline{\gamma}_{11}\underline{\gamma}_{12}, \overline{\gamma}_{11} + \overline{\gamma}_{12} - \overline{\gamma}_{11}\overline{\gamma}_{12}] \rangle$;

(iii) $\delta\Omega_{e_{11}} = \langle [(1-(1-\underline{\alpha}_{11})^{\delta}, (1-(1-\overline{\alpha}_{11})^{\delta}], [(1-(1-\underline{\beta}_{11})^{\delta}, (1-(1-\overline{\beta}_{11})^{\delta}], [(\underline{\gamma}_{11})^{\delta}, (\overline{\gamma}_{11})^{\delta}] \rangle$;

(iv) $(\Omega_{e_{11}})^{\delta} = \langle [(\underline{\alpha}_{11})^{\delta}, (\overline{\alpha}_{11})^{\delta}], [(\underline{\beta}_{11})^{\delta}, (\overline{\beta}_{11})^{\delta}], [(1-(1-\underline{\gamma}_{11})^{\delta}, (1-(1-\overline{\gamma}_{11})^{\delta}] \rangle$.

4.2. *Interval-valued intuitionistic neutrosophic hypersoft weighted average operator*

Definition 4.2. Let $\mathcal{D}_k$ and $\mathcal{W}_i$ be weight vectors for alternatives and experts, respectively, such that $\mathcal{D}_k, \mathcal{W}_i > 0$ and $\sum_{k=1}^{m} \mathcal{D}_k = 1, \sum_{i=1}^{n} \mathcal{W}_i = 1$ and $\Omega_{e_{ik}} = \langle [\underline{\alpha}_{ik}, \overline{\alpha}_{ik}], [\underline{\beta}_{ik}, \overline{\beta}_{ik}], [\underline{\gamma}_{ik}, \overline{\gamma}_{ik}] \rangle$ be an IVINHSN, where $i = \{1, 2, ...n\}$, $k = \{1, 2, ...m\}$. Then, $\mathcal{A} : \kappa^n \to \kappa$, IVINHSWAO is represented as

$$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{nm}}) = \bigoplus_{k=1}^{m} \mathcal{D}_k \left(\bigoplus_{i=1}^{n} \mathcal{W}_i \Omega_{e_{ik}} \right).$$

Theorem 4.3. *Let $\Omega_{e_{ik}} = \langle [\underline{\alpha}_{ik}, \overline{\alpha}_{ik}], [\underline{\beta}_{ik}, \overline{\beta}_{ik}], [\underline{\gamma}_{ik}, \overline{\gamma}_{ik}] \rangle$ be an IVINHSN, where $i = \{1, 2, ...n\}$, $k = \{1, 2, ...m\}$. Then, the aggregated value of IVINHSWAO is also an IVINHSN, which is given by*

$$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{nm}}) = \Bigg\langle \Bigg[1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \underline{\alpha}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \overline{\alpha}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \Bigg], \Bigg[1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \underline{\beta}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \overline{\beta}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \Bigg], \Bigg[\prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(\underline{\gamma}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, \prod_{k=1}^{m} \left(\prod_{i=1}^{n} \left(\overline{\gamma}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, \Bigg] \Bigg\rangle.$$

Proof. If $n = 1$, then $\mathcal{W}_1 = 1$. By using Definition 4.1, we get

$$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{1m}}) = \bigoplus_{k=1}^{m} \mathcal{D}_k \Omega_{e_{1k}}.$$

$$= \Bigg\langle \Bigg[1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{1} \left(1 - \underline{\alpha}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{1} \left(1 - \overline{\alpha}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \Bigg], \Bigg[1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{1} \left(1 - \underline{\beta}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{m} \left(\prod_{i=1}^{1} \left(1 - \overline{\beta}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \Bigg], \Bigg[\prod_{k=1}^{m} \left(\prod_{i=1}^{1} \left(\underline{\gamma}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, \prod_{k=1}^{m} \left(\prod_{i=1}^{1} \left(\overline{\gamma}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \Bigg] \Bigg\rangle.$$

If $m = 1$, then $\mathcal{D}_1 = 1$. By using Definition 4.2, we get

$$\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{21}}, ..., \Omega_{e_{n1}}) = \bigoplus_{i=1}^{n} \mathcal{W}_i \Omega_{e_{i1}}.$$

$$= \Bigg\langle \Bigg[1 - \prod_{k=1}^{1} \left(\prod_{i=1}^{n} \left(1 - \underline{\alpha}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{1} \left(\prod_{i=1}^{n} \left(1 - \overline{\alpha}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \Bigg], \Bigg[1 - \prod_{k=1}^{1} \left(\prod_{i=1}^{n} \left(1 - \underline{\beta}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{1} \left(\prod_{i=1}^{n} \left(1 - \overline{\beta}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \Bigg], \Bigg[\prod_{k=1}^{1} \left(\prod_{i=1}^{n} \left(\underline{\gamma}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k}, \prod_{k=1}^{1} \left(\prod_{i=1}^{n} \left(\overline{\gamma}_{ik}\right)^{\mathcal{W}_i} \right)^{\mathcal{D}_k} \Bigg] \Bigg\rangle.$$

Hence, the results hold for $n=1$ and $m=1$.

Now, if $m=l_1+1$ and $n=l_2$, then,

$$\mathcal{A}(\Omega_{e_{11}},\Omega_{e_{12}},...,\Omega_{e_{l_2(l_1+1)}})=\bigoplus_{k=1}^{l_1+1}\mathcal{D}_k\left(\bigoplus_{i=1}^{l_2}\mathcal{W}_i\Omega_{e_{ik}}\right).$$

$$=\Bigg\langle\Bigg[1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(1-\underline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(1-\overline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\Bigg],\Bigg[1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(1-\underline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},$$
$$1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(1-\overline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\Bigg],\Bigg[\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(\underline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(\overline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\Bigg]\Bigg\rangle.$$

Similarly, if $m=l_1$, $n=l_2+1$, then,

$$\mathcal{A}(\Omega_{e_{11}},\Omega_{e_{12}},...,\Omega_{e_{(l_2+1)l_1}})=\bigoplus_{k=1}^{l_1}\mathcal{D}_k\left(\bigoplus_{i=1}^{l_2+1}\mathcal{W}_i\Omega_{e_{ik}}\right).$$

$$=\Bigg\langle\Bigg[1-\prod_{k=1}^{l_1}\left(\prod_{i=1}^{l_2+1}\left(1-\underline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},1-\prod_{k=1}^{l_1}\left(\prod_{i=1}^{l_2+1}\left(1-\overline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\Bigg],\Bigg[1-\prod_{k=1}^{l_1}\left(\prod_{i=1}^{l_2+1}\left(1-\underline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},$$
$$1-\prod_{k=1}^{l_1}\left(\prod_{i=1}^{l_2+1}\left(1-\overline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\Bigg],\Bigg[\prod_{k=1}^{l_1}\left(\prod_{i=1}^{l_2+1}\left(\underline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},\prod_{k=1}^{l_1}\left(\prod_{i=1}^{l_2+1}\left(\overline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\Bigg]\Bigg\rangle.$$

Now, if $m=l_1+1$, $n=l_2+1$, then,

$$\mathcal{A}(\Omega_{e_{11}},\Omega_{e_{12}},...,\Omega_{e_{(l_2+1)(l_1+1)}})$$
$$=\bigoplus_{k=1}^{l_1+1}\mathcal{D}_k\left(\bigoplus_{i=1}^{l_2+1}\mathcal{W}_i\Omega_{e_{ik}}\right).$$
$$=\bigoplus_{k=1}^{l_1+1}\mathcal{D}_k\left(\bigoplus_{i=1}^{l_2}\mathcal{W}_i\Omega_{e_{ik}}\right)\bigoplus_{k=1}^{l_1+1}\mathcal{D}_k\left(\mathcal{W}_{l_2+1}\Omega_{e_{(l_2+1)k}}\right).$$
$$\mathcal{A}(\Omega_{e_{11}},\Omega_{e_{12}},...,\Omega_{e_{(l_2+1)(l_1+1)}})$$
$$=\Bigg\langle\Bigg[1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(1-\underline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(1-\overline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\Bigg]\oplus\Bigg[1-\prod_{k=1}^{l_1+1}\left(\left(1-\underline{\alpha}_{(l_2+1)k}\right)^{\mathcal{W}_{(l_2+1)}}\right)^{\mathcal{D}_k},$$
$$1-\prod_{k=1}^{l_1+1}\left(\left(1-\overline{\alpha}_{(l_2+1)k}\right)^{\mathcal{W}_{(l_2+1)}}\right)^{\mathcal{D}_k}\Bigg],\Bigg[1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(1-\underline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(1-\overline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\Bigg]$$
$$\oplus\Bigg[1-\prod_{k=1}^{l_1+1}\left(\left(1-\underline{\beta}_{(l_2+1)k}\right)^{\mathcal{W}_{(l_2+1)}}\right)^{\mathcal{D}_k},1-\prod_{k=1}^{l_1+1}\left(\left(1-\overline{\beta}_{(l_2+1)k}\right)^{\mathcal{W}_{(l_2+1)}}\right)^{\mathcal{D}_k}\Bigg],\Bigg[\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(\underline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},$$
$$\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2}\left(\overline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},\Bigg]\oplus\Bigg[\prod_{k=1}^{l_1+1}\left(\left(\underline{\gamma}_{(l_2+1)k}\right)^{\mathcal{W}_{(l_2+1)}}\right)^{\mathcal{D}_k},\prod_{k=1}^{l_1+1}\left(\left(\overline{\gamma}_{(l_2+1)k}\right)^{\mathcal{W}_{(l_2+1)}}\right)^{\mathcal{D}_k}\Bigg\rangle.$$
$$=\Bigg\langle\Bigg[1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2+1}\left(1-\underline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2+1}\left(1-\overline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\Bigg],\Bigg[1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2+1}\left(1-\underline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},$$
$$1-\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2+1}\left(1-\overline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\Bigg],\Bigg[\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2+1}\left(\underline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},\prod_{k=1}^{l_1+1}\left(\prod_{i=1}^{l_2+1}\left(\overline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\Bigg]\Bigg\rangle.$$

Hence, the results hold for $n=l_2+1$ and $m=l_1+1$.

Therefore, by induction method, the result is true $\forall\ m,n\geq 1$.

Since

$0 \leq \overline{\alpha}_{ik} + \overline{\gamma}_{ik} \leq 1$ and $0 \leq \overline{\beta}_{ik} \leq 1$.

$\Leftrightarrow 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\underline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} + 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\overline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} + \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(\underline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} +$
$\prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(\overline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} \leq 1$ and $1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\underline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} + 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\overline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} \leq 1.$

$\Leftrightarrow 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\underline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} + 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\overline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} + \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\underline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} +$
$\prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\overline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} \leq 1$ and $1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\underline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} + 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\overline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} \leq 1.$

$\Leftrightarrow 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\underline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} + 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\overline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} + \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\underline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} +$
$\prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\overline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} + 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\underline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} + 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\overline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k} \leq 2.$

Therefore, the aggregated value given by IVINHSWAO is also an IVINHSN. □

Example 4.4. Let us consider the same values mentioned in Example 3.2. Also, let $\mathcal{W}_i = \{0.25, 0.35, 0.40\}$ and $\mathcal{D}_k = \{0.30, 0.20, 0.40, 0.10\}$ be the weight of managers and attributes, respectively. Then,

$$\begin{aligned}\mathcal{A}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{34}}) &= \Bigg\langle \left[1 - \prod_{k=1}^{4}\left(\prod_{i=1}^{3}\left(1-\underline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}, \left[1 - \prod_{k=1}^{4}\left(\prod_{i=1}^{3}\left(1-\overline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\right], \left[1 - \prod_{k=1}^{4}\left(\prod_{i=1}^{3}\left(1-\underline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},\right.\right. \\ &\left[1 - \prod_{k=1}^{4}\left(\prod_{i=1}^{3}\left(1-\overline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\right], \left[\prod_{k=1}^{4}\left(\prod_{i=1}^{3}\left(\underline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}, \left[\prod_{k=1}^{4}\left(\prod_{i=1}^{3}\left(\overline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\right]\right\rangle. \\ &= \langle [0.32, 0.45], [0.49, 0.64], [0.20, 0.34] \rangle.\end{aligned}$$

4.3. *Interval-valued intuitionistic neutrosophic hypersoft weighted geometric operator*

Definition 4.5. Let $\mathcal{D}_k$ and $\mathcal{W}_i$ be weight vectors for alternatives and experts, respectively, such that $\mathcal{D}_k, \mathcal{W}_i > 0$ and $\sum_{k=1}^{m} \mathcal{D}_k = 1, \sum_{i=1}^{n} \mathcal{W}_i = 1$ and $\Omega_{e_{ik}} = (\alpha_{ik}, \beta_{ik}, \gamma_{ik})$ be an IVINHSN, where $i = \{1, 2, ...n\}$, $k = \{1, 2, ...m\}$. Then, $\mathcal{G} : \kappa^n \rightarrow \kappa$, IVINHSWGO is defined as

$$\mathcal{G}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{nm}}) = \bigotimes_{k=1}^{m}\left(\bigotimes_{i=1}^{n}\left(\Omega_{e_{ik}}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}.$$

Theorem 4.6. *Let* $\Omega_{e_{ik}} = \left\langle [\underline{\alpha}_{ik}, \overline{\alpha}_{ik}], [\underline{\beta}_{ik}, \overline{\beta}_{ik}], [\underline{\gamma}_{ik}, \overline{\gamma}_{ik}] \right\rangle$ *be an IVINHSN, where* $i = \{1, 2, ...n\}$, $k = \{1, 2, ...m\}$. *Then, the aggregated value of IVINHSWGO is also an IVINHSN, which is given by*

$$\begin{aligned}\mathcal{G}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{nm}}) &= \Bigg\langle \left[\prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(\underline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}, \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(\overline{\alpha}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},\right], \left[\prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(\underline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}, \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(\overline{\beta}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k},\right], \\ &\left[1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\underline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{m}\left(\prod_{i=1}^{n}\left(1-\overline{\gamma}_{ik}\right)^{\mathcal{W}_i}\right)^{\mathcal{D}_k}\right]\Bigg\rangle.\end{aligned}$$

Proof. Similar to Theorem 4.3. □

Example 4.7. Let us consider the same values mentioned in Example 3.2 and the weight of managers and attributes be as in Example 4.4. Then,

$$\mathcal{G}(\Omega_{e_{11}}, \Omega_{e_{12}}, ..., \Omega_{e_{34}})$$
$$= \Big\langle \Big[\prod_{k=1}^{4} \Big(\prod_{i=1}^{3} \big(\underline{\alpha}_{ik} \big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k}, \prod_{k=1}^{4} \Big(\prod_{i=1}^{3} \big(\overline{\alpha}_{ik} \big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \Big], \Big[\prod_{k=1}^{4} \Big(\prod_{i=1}^{3} \big(\underline{\beta}_{ik} \big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k}, \prod_{k=1}^{4} \Big(\prod_{i=1}^{3} \big(\overline{\beta}_{ik} \big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \Big],$$
$$\Big[1 - \prod_{k=1}^{4} \Big(\prod_{i=1}^{3} \big(1 - \underline{\gamma}_{ik} \big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{4} \Big(\prod_{i=1}^{3} \big(1 - \overline{\gamma}_{ik} \big)^{\mathcal{W}_i} \Big)^{\mathcal{D}_k} \Big] \Big\rangle.$$
$$= \langle [0.26, 0.40], [0.37, 0.50], [0.28, 0.41] \rangle .$$

5. MCDM problems based on TOPSIS and CC method

TOPSIS method helps to find the best alternative based on minimum and maximum distance from the interval-valued intuitionistic neutrosophic positive ideal solution (IVINPIS) and interval-valued intuitionistic neutrosophic negative ideal solution (IVINNIS). Also, when TOPSIS method is combined with CC instead of similarity measures, it provides reliable results for predicting the closeness coefficients. We present an algorithm and a case study to illustrate the IVINHSS TOPSIS method based on CC.

5.1. *Algorithm to solve MCDM problems with IVINHSS data based on TOPSIS and CC method*

Let $\mathcal{A} = \{\mathcal{A}^1, \mathcal{A}^2, ..., \mathcal{A}^x\}$ be a set of selected employees and $\mathcal{V} = \{v_1, v_2, ..., v_n\}$ be a set of managers responsible to evaluate the employees with weights $\mathcal{W}_i = (\mathcal{W}_1, \mathcal{W}_2, ..., \mathcal{W}_n)$, such that $\mathcal{W}_i > 0$ and $\sum_{i=1}^{n} \mathcal{W}_i = 1$. Let $\tilde{\Delta} = \{\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_m\}$ be a set of multi-valued sub-attributes with weights $\mathcal{D}_k = (\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_m)$, such that $\mathcal{D}_k > 0$ and $\sum_{k=1}^{m} \mathcal{D}_k = 1$. The evaluation of employees $\mathcal{A}^t$, $(t = 1, 2, ..., x)$ performed by the managers v_i, $(i = 1, 2, ..., n)$ based on the multi-valued sub-attributes $\tilde{\lambda}_k$, $(k = 1, 2, ..., m)$ are given in IVINHSS form and represented as $\Omega^t_{ik} = \langle [\underline{\alpha}_{ik}, \overline{\alpha}_{ik}], [\underline{\beta}_{ik}, \overline{\beta}_{ik}], [\underline{\gamma}_{ik}, \overline{\gamma}_{ik}] \rangle$, such that $0 \le \overline{\alpha}^t_{ik} + \overline{\gamma}^t_{ik} \le 1$ and $0 \le \overline{\alpha}^t_{ik} + \overline{\beta}^t_{ik} + \overline{\gamma}^t_{ik} \le 2\ \forall$ i, k. The managing experts aid to accommodate the multi-sub attributes values in IVINHSS form.

Step 1. Construct the matrix for each multi-valued sub-attributes in IVINHSS form as below:

$[\mathcal{A}^t, \tilde{\Delta}]_{n\times m} = [\mathcal{A}^t]_{n\times m}$

$$= \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{array} \begin{bmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \cdots & \tilde{\lambda}_m \\ \left\langle[\underline{\alpha}^t_{11},\overline{\alpha}^t_{11}],[\underline{\beta}^t_{11},\overline{\beta}^t_{11}],[\underline{\gamma}^t_{11},\overline{\gamma}^t_{11}]\right\rangle & \left\langle[\underline{\alpha}^t_{12},\overline{\alpha}^t_{12}],[\underline{\beta}^t_{12},\overline{\beta}^t_{12}],[\underline{\gamma}^t_{12},\overline{\gamma}^t_{12}]\right\rangle & \cdots & \left\langle[\underline{\alpha}^t_{1m},\overline{\alpha}^t_{1m}],[\underline{\beta}^t_{1m},\overline{\beta}^t_{1m}],[\underline{\gamma}^t_{1m},\overline{\gamma}^t_{1m}]\right\rangle \\ \left\langle[\underline{\alpha}^t_{21},\overline{\alpha}^t_{21}],[\underline{\beta}^t_{21},\overline{\beta}^t_{21}],[\underline{\gamma}^t_{21},\overline{\gamma}^t_{21}]\right\rangle & \left\langle[\underline{\alpha}^t_{22},\overline{\alpha}^t_{22}],[\underline{\beta}^t_{22},\overline{\beta}^t_{22}],[\underline{\gamma}^t_{22},\overline{\gamma}^t_{22}]\right\rangle & \cdots & \left\langle[\underline{\alpha}^t_{2m},\overline{\alpha}^t_{2m}],[\underline{\beta}^t_{2m},\overline{\beta}^t_{2m}],[\underline{\gamma}^t_{2m},\overline{\gamma}^t_{2m}]\right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle[\underline{\alpha}^t_{n1},\overline{\alpha}^t_{n1}],[\underline{\beta}^t_{n1},\overline{\beta}^t_{n1}],[\underline{\gamma}^t_{n1},\overline{\gamma}^t_{n1}]\right\rangle & \left\langle[\underline{\alpha}^t_{n2},\overline{\alpha}^t_{n2}],[\underline{\beta}^t_{n2},\overline{\beta}^t_{n2}],[\underline{\gamma}^t_{n2},\overline{\gamma}^t_{n2}]\right\rangle & \cdots & \left\langle[\underline{\alpha}^t_{nm},\overline{\alpha}^t_{nm}],[\underline{\beta}^t_{nm},\overline{\beta}^t_{nm}],[\underline{\gamma}^t_{nm},\overline{\gamma}^t_{nm}]\right\rangle \end{bmatrix}$$

Step 2. Obtain the weighted decision matrix for each multi-valued sub-attributes,

$[\tilde{A}^t_{ik}]_{n\times m}$

$$= \Big\langle \Big[1 - \prod_{k=1}^{m}\Big(\prod_{i=1}^{n}\Big(1-\underline{\alpha}_{ik}\Big)^{\mathcal{W}_i}\Big)^{\mathcal{D}_k}, 1 - \prod_{k=1}^{m}\Big(\prod_{i=1}^{n}\Big(1-\overline{\alpha}_{ik}\Big)^{\mathcal{W}_i}\Big)^{\mathcal{D}_k}\Big], \Big[1 - \prod_{k=1}^{m}\Big(\prod_{i=1}^{n}\Big(1-\underline{\beta}_{ik}\Big)^{\mathcal{W}_i}\Big)^{\mathcal{D}_k},$$
$$1 - \prod_{k=1}^{m}\Big(\prod_{i=1}^{n}\Big(1-\overline{\beta}_{ik}\Big)^{\mathcal{W}_i}\Big)^{\mathcal{D}_k}\Big], \Big[\prod_{k=1}^{m}\Big(\prod_{i=1}^{n}\Big(\underline{\gamma}_{ik}\Big)^{\mathcal{W}_i}\Big)^{\mathcal{D}_k}, \prod_{k=1}^{m}\Big(\prod_{i=1}^{n}\Big(\overline{\gamma}_{ik}\Big)^{\mathcal{W}_i}\Big)^{\mathcal{D}_k}\Big]\Big\rangle$$
$$= \Big\langle \Big[\underline{\tilde{\alpha}}_{ik}, \tilde{\overline{\alpha}}_{ik}\Big], \Big[\underline{\tilde{\beta}}_{ik}, \tilde{\overline{\beta}}_{ik}\Big], \Big[\underline{\tilde{\gamma}}_{ik}, \tilde{\overline{\gamma}}_{ik}\Big]\Big\rangle.$$

Step 3. Determine the IVINPIS and IVINNIS for weighted IVINHSS as below:

$$\tilde{\mathcal{A}}^{+} = \left\langle \left[\underline{\tilde{\alpha}}^{+}, \tilde{\overline{\alpha}}^{+}\right], \left[\underline{\tilde{\beta}}^{+}, \tilde{\overline{\beta}}^{+}\right], \left[\underline{\tilde{\gamma}}^{+}, \tilde{\overline{\gamma}}^{+}\right]\right\rangle_{n\times m} = \left\langle \left[\underline{\tilde{\alpha}}^{(\vee_{ij})}, \tilde{\overline{\alpha}}^{(\vee_{ij})}\right], \left[\underline{\tilde{\beta}}^{(\wedge_{ij})}, \tilde{\overline{\beta}}^{(\wedge_{ij})}\right], \left[\underline{\tilde{\gamma}}^{(\wedge_{ij})}, \tilde{\overline{\gamma}}^{(\wedge_{ij})}\right]\right\rangle,$$

$$\tilde{\mathcal{A}}^{-} = \left\langle \left[\underline{\tilde{\alpha}}^{-}, \tilde{\overline{\alpha}}^{-}\right], \left[\underline{\tilde{\beta}}^{-}, \tilde{\overline{\beta}}^{-}\right], \left[\underline{\tilde{\gamma}}^{-}, \tilde{\overline{\gamma}}^{-}\right]\right\rangle_{n\times m} = \left\langle \left[\underline{\tilde{\alpha}}^{(\wedge_{ij})}, \tilde{\overline{\alpha}}^{(\wedge_{ij})}\right], \left[\underline{\tilde{\beta}}^{(\wedge_{ij})}, \tilde{\overline{\beta}}^{(\wedge_{ij})}\right], \left[\underline{\tilde{\gamma}}^{(\vee_{ij})}, \tilde{\overline{\gamma}}^{(\vee_{ij})}\right]\right\rangle,$$

where $\vee_{ij} = \arg\max_t \left\{\varphi^t_{ij}\right\}$ and $\wedge_{ij} = \arg\min_t \left\{\varphi^t_{ij}\right\}$.

Step 4. Determine the CC for each alternative from IVINPIS and IVINNIS.

$$\chi^t = \mathcal{C}_C(\tilde{A}^t, \tilde{A}^{+}) = \frac{\mathcal{C}_{\mathcal{M}}(\tilde{A}^t, \tilde{A}^{+})}{\sqrt{\Phi(\tilde{A}^t)} * \sqrt{(\Phi\tilde{A}^{+})}} \text{ and}$$

$$\lambda^t = \mathcal{C}_C(\tilde{A}^t, \tilde{A}^{-}) = \frac{\mathcal{C}_{\mathcal{M}}(\tilde{A}^t, \tilde{A}^{-})}{\sqrt{\Phi(\tilde{A}^t)} * \sqrt{\Phi(\tilde{A}^{-})}}$$

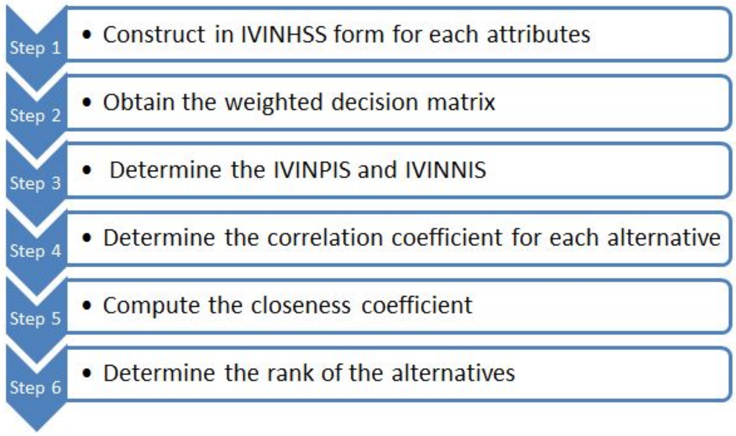
Step 5. Compute the closeness coefficient of neutrosophic ideal solution as below:

$$\epsilon^t = \frac{1-\lambda^t}{2-\chi^t-\lambda^t}$$

Step 6. Arrange the ϵ^t values in descending order and determine the rank of the alternatives $\mathcal{A}^t$, $(t = 1, 2, ..., x)$. The one with the maximum value is the suitable employee to lead the new project.

The graphical representation of the proposed method is given in Figure 1:

FIGURE 1. Flowchart of the proposed method



5.2. *Application based on TOPSIS and CC method*

Let $\mathcal{A} = \left\{\mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3, \mathcal{A}^4\right\}$ be a set of employees and let $\mathcal{V} = \{v_1, v_2, v_3\}$ be a set of managers who evaluate the employees based on the Leipzig leadership model for an upcoming project with weights $\mathcal{W}_i = (0.35, 0.15, 0.30, 0.20)$. Let Δ_1, Δ_2, Δ_3 and Δ_4 be distinct attribute sets whose corresponding sub-attributes are represented as $\Delta_1 = \text{purpose} = \left\{\lambda_{11} = \text{achieve goals}\right\}$, $\Delta_2 = \text{entrepreneurial spirit} = \left\{\lambda_{21} = \text{quick decision}, \lambda_{22} = \text{logical decision}\right\}$, $\Delta_3 = \text{responsibility} = \left\{\lambda_{31} = \text{inspire and motivate}, \lambda_{32} = \text{time management}\right\}$ and $\Delta_4 = \text{effectiveness} = \left\{\lambda_{41} = \text{successful accomplishment}\right\}$. Then $\tilde{\Delta} = \Delta_1 \times \Delta_2 \times \Delta_3 \times \Delta_4$ is the distinct attribute set given by

$$\tilde{\Delta} = \Delta_1 \times \Delta_2 \times \Delta_3 \times \Delta_4 = \{\lambda_{11}\} \times \{\lambda_{21}, \lambda_{22}\} \times \{\lambda_{31}, \lambda_{32}\} \times \{\lambda_{41}\}.$$

$$= \left\{(\lambda_{11}, \lambda_{21}, \lambda_{31}, \lambda_{41}), (\lambda_{11}, \lambda_{21}, \lambda_{32}, \lambda_{41}), (\lambda_{11}, \lambda_{22}, \lambda_{31}, \lambda_{41}), (\lambda_{11}, \lambda_{22}, \lambda_{32}, \lambda_{41})\right\}.$$

$$= \left\{\tilde{\lambda_1}, \tilde{\lambda_2}, \tilde{\lambda_3}, \tilde{\lambda_4}\right\} \text{ with weights } \mathcal{D}_k = (0.20, 0.25, 0.30, 0.25).$$

This study aims to find an employee who can successfully lead the project.

Step 1. Construct $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{A}^3$ and $\mathcal{A}^4$ matrices for each multi-valued sub-attributes in IVINHSS form.

TABLE 2. Representation of values in IVINHSS form for $\mathcal{A}^1$.

$\mathcal{A}^1$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$
v_1	$\langle[0.43, 0.55], [0.91, 0.95], [0.31, 0.36]\rangle$	$\langle[0.43, 0.52], [0.58, 0.81], [0.12, 0.21]\rangle$	$\langle[0.67, 0.71], [0.77, 0.81], [0.19, 0.29]\rangle$
v_2	$\langle[0.32, 0.45], [0.71, 0.78], [0.22, 0.29]\rangle$	$\langle[0.54, 0.63], [0.34, 0.44], [0.15, 0.24]\rangle$	$\langle[0.45, 0.48], [0.62, 0.72], [0.25, 0.35]\rangle$
v_3	$\langle[0.29, 0.53], [0.81, 0.89], [0.31, 0.41]\rangle$	$\langle[0.37, 0.41], [0.66, 0.71], [0.29, 0.35]\rangle$	$\langle[0.49, 0.51], [0.49, 0.59], [0.39, 0.42]\rangle$
v_4	$\langle[0.34, 0.43], [0.61, 0.82], [0.42, 0.53]\rangle$	$\langle[0.48, 0.59], [0.31, 0.42], [0.21, 0.41]\rangle$	$\langle[0.42, 0.47], [0.57, 0.61], [0.39, 0.45]\rangle$

$\mathcal{A}^1$	$\tilde{\lambda}_4$
v_1	$\langle[0.15, 0.19], [0.49, 0.51], [0.32, 0.34]\rangle$
v_2	$\langle[0.24, 0.29], [0.65, 0.72], [0.51, 0.55]\rangle$
v_3	$\langle[0.33, 0.39], [0.94, 0.98], [0.44, 0.45]\rangle$
v_4	$\langle[0.48, 0.49], [0.78, 0.84], [0.26, 0.34]\rangle$

TABLE 3. Representation of values in IVINHSS form for $\mathcal{A}^2$.

$\mathcal{A}^2$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$
v_1	$\langle[0.61, 0.65], [0.25, 0.35], [0.22, 0.31]\rangle$	$\langle[0.44, 0.59], [0.59, 0.71], [0.11, 0.12]\rangle$	$\langle[0.44, 0.51], [0.42, 0.45], [0.21, 0.25]\rangle$
v_2	$\langle[0.39, 0.41], [0.91, 0.99], [0.41, 0.59]\rangle$	$\langle[0.59, 0.64], [0.66, 0.76], [0.21, 0.31]\rangle$	$\langle[0.54, 0.62], [0.31, 0.36], [0.32, 0.38]\rangle$
v_3	$\langle[0.32, 0.42], [0.82, 0.88], [0.41, 0.49]\rangle$	$\langle[0.48, 0.54], [0.21, 0.37], [0.29, 0.32]\rangle$	$\langle[0.49, 0.54], [0.49, 0.59], [0.25, 0.29]\rangle$
v_4	$\langle[0.34, 0.44], [0.66, 0.77], [0.33, 0.38]\rangle$	$\langle[0.69, 0.74], [0.68, 0.79], [0.19, 0.21]\rangle$	$\langle[0.58, 0.66], [0.69, 0.71], [0.33, 0.34]\rangle$

$\mathcal{A}^2$	$\tilde{\lambda}_4$
v_1	$\langle[0.21, 0.28], [0.57, 0.59], [0.41, 0.43]\rangle$
v_2	$\langle[0.28, 0.31], [0.67, 0.68], [0.57, 0.61]\rangle$
v_3	$\langle[0.41, 0.46], [0.77, 0.81], [0.23, 0.29]\rangle$
v_4	$\langle[0.21, 0.29], [0.69, 0.71], [0.44, 0.49]\rangle$

TABLE 4. Representation of values in IVINHSS form for $\mathcal{A}^3$.

$\mathcal{A}^3$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$
v_1	$\langle[0.55, 0.56], [0.68, 0.78], [0.32, 0.37]\rangle$	$\langle[0.48, 0.55], [0.68, 0.87], [0.11, 0.28]\rangle$	$\langle[0.51, 0.54], [0.55, 0.62], [0.30, 0.32]\rangle$
v_2	$\langle[0.42, 0.46], [0.45, 0.55], [0.41, 0.48]\rangle$	$\langle[0.39, 0.45], [0.81, 0.91], [0.29, 0.31]\rangle$	$\langle[0.47, 0.49], [0.35, 0.42], [0.21, 0.42]\rangle$
v_3	$\langle[0.53, 0.55], [0.66, 0.76], [0.24, 0.42]\rangle$	$\langle[0.51, 0.65], [0.38, 0.42], [0.24, 0.29]\rangle$	$\langle[0.32, 0.34], [0.31, 0.41], [0.35, 0.41]\rangle$
v_4	$\langle[0.31, 0.43], [0.35, 0.45], [0.14, 0.29]\rangle$	$\langle[0.35, 0.48], [0.31, 0.49], [0.31, 0.38]\rangle$	$\langle[0.63, 0.64], [0.22, 0.32], [0.15, 0.21]\rangle$

$\mathcal{A}^3$	$\tilde{\lambda}_4$
v_1	$\langle[0.51, 0.53][0.41, 0.44][0.21, 0.24]\rangle$
v_2	$\langle[0.42, 0.43][0.45, 0.49][0.32, 0.34]\rangle$
v_3	$\langle[0.05, 0.12][0.65, 0.69][0.45, 0.49]\rangle$
v_4	$\langle[0.21, 0.26][0.72, 0.79][0.22, 0.23]\rangle$

TABLE 5. Representation of values in IVINHSS form for $\mathcal{A}^4$.

$\mathcal{A}^4$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$
v_1	$\langle[0.61, 0.71], [0.36, 0.55][0.09, 0.21]\rangle$	$\langle[0.31, 0.39], [0.67, 0.77], [0.29, 0.39]\rangle$	$\langle[0.27, 0.34], [0.17, 0.27], [0.32, 0.35]\rangle$
v_2	$\langle[0.44, 0.54], [0.46, 0.66][0.12, 0.25]\rangle$	$\langle[0.41, 0.57], [0.87, 0.92], [0.39, 0.41]\rangle$	$\langle[0.39, 0.41], [0.39, 0.41], [0.41, 0.49]\rangle$
v_3	$\langle[0.34, 0.44], [0.66, 0.77][0.33, 0.39]\rangle$	$\langle[0.53, 0.64], [0.64, 0.77], [0.21, 0.28]\rangle$	$\langle[0.14, 0.15], [0.49, 0.59], [0.62, 0.68]\rangle$
v_4	$\langle[0.52, 0.66], [0.35, 0.49][0.14, 0.25]\rangle$	$\langle[0.47, 0.56], [0.41, 0.45], [0.27, 0.34]\rangle$	$\langle[0.25, 0.29], [0.46, 0.66], [0.31, 0.34]\rangle$

$\mathcal{A}^4$	$\tilde{\lambda}_4$
v_1	$\langle[0.37, 0.39], [0.81, 0.91], [0.49, 0.51]\rangle$
v_2	$\langle[0.41, 0.42], [0.38, 0.42], [0.29, 0.31]\rangle$
v_3	$\langle[0.52, 0.59], [0.65, 0.69], [0.23, 0.29]\rangle$
v_4	$\langle[0.31, 0.36], [0.42, 0.51], [0.61, 0.62]\rangle$

Step 2. Obtain $\tilde{A}^1$, $\tilde{A}^2$, $\tilde{A}^3$ and $\tilde{A}^4$, the weighted matrices for each multi-valued sub-attributes.

TABLE 6. Representation of weighted values in IVINHSS form for $\tilde{A}^1$.

$\tilde{A}^1$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$
v_1	$\langle[0.0386, 0.0062], [0.1552, 0.0229], [0.9213, 0.9922]\rangle$	$\langle[0.0480, 0.0071], [0.0731, 0.0625], [0.8307, 0.0529]\rangle$
v_2	$\langle[0.0116, 0.0014], [0.0365, 0.0036], [0.9556, 0.9972]\rangle$	$\langle[0.0287, 0.0029], [0.0155, 0.0053], [0.9314, 0.0635]\rangle$
v_3	$\langle[0.0204, 0.0031], [0.0949, 0.0090], [0.9322, 0.9964]\rangle$	$\langle[0.0341, 0.0027], [0.0778, 0.0290], [0.9114, 0.0956]\rangle$
v_4	$\langle[0.0165, 0.0019], [0.0370, 0.0057], [0.9659, 0.9980]\rangle$	$\langle[0.0322, 0.0037], [0.0184, 0.0051], [0.9250, 0.1197]\rangle$

$\tilde{A}^1$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle[0.1099, 0.0143], [0.1430, 0.0745], [0.8400, 0.0904]\rangle$	$\langle[0.0142, 0.0021], [0.0573, 0.0273][0.9052, 0.0913]\rangle$
v_2	$\langle[0.0266, 0.0023], [0.0427, 0.0139], [0.9396, 0.1162]\rangle$	$\langle[0.0103, 0.0010], [0.0387, 0.0116][0.9751, 0.1737]\rangle$
v_3	$\langle[0.0589, 0.0044], [0.0589, 0.0251], [0.9188, 0.1414]\rangle$	$\langle[0.0296, 0.0026], [0.1903, 0.0887][0.9403, 0.1301]\rangle$
v_4	$\langle[0.0322, 0.0032], [0.0494, 0.0104], [0.9451, 0.1591]\rangle$	$\langle[0.0322, 0.0028], [0.0730, 0.0169][0.9349, 0.0955]\rangle$

TABLE 7. Representation of weighted values in IVINHSS form for $\tilde{A}^2$.

$\tilde{A}^2$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$
v_1	$\langle[0.0638, 0.0134], [0.0200, 0.0055], [0.8995, 0.9852]\rangle$	$\langle[0.0495, 0.0142], [0.0751, 0.0062], [0.8244, 0.0284]\rangle$
v_2	$\langle[0.0148, 0.0016], [0.0697, 0.0136], [0.9737, 0.9985]\rangle$	$\langle[0.0329, 0.0038], [0.0397, 0.0246], [0.9432, 0.0864]\rangle$
v_3	$\langle[0.0229, 0.0025], [0.0978, 0.0097], [0.9480, 0.9968]\rangle$	$\langle[0.0479, 0.0045], [0.0176, 0.0113], [0.9114, 0.0874]\rangle$
v_4	$\langle[0.0165, 0.0020], [0.0423, 0.0049], [0.9567, 0.9969]\rangle$	$\langle[0.0569, 0.0056], [0.0554, 0.0164], [0.9204, 0.0549]\rangle$

$\tilde{A}^2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle[0.0591, 0.0136], [0.0556, 0.0036], [0.8489, 0.0747]\rangle$	$\langle[0.0205, 0.0053], [0.0712, 0.0045], [0.9250, 0.1188]\rangle$
v_2	$\langle[0.0344, 0.0043], [0.0166, 0.0093], [0.9501, 0.1304]\rangle$	$\langle[0.0123, 0.0014], [0.0408, 0.0197], [0.9792, 0.2049]\rangle$
v_3	$\langle[0.0589, 0.0054], [0.0589, 0.0259], [0.8828, 0.0929]\rangle$	$\langle[0.0388, 0.0036], [0.1044, 0.0398], [0.8957, 0.0780]\rangle$
v_4	$\langle[0.0508, 0.0054], [0.0679, 0.0156], [0.9357, 0.1125]\rangle$	$\langle[0.0118, 0.0015], [0.0569, 0.0131], [0.9598, 0.1488]\rangle$

TABLE 8. Representation of weighted values in IVINHSS form for $\tilde{A}^3$.

$\tilde{A}^3$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$
v_1	$\langle[0.0544, 0.0089], [0.0767, 0.0164], [0.9234, 0.9893]\rangle$	$\langle[0.0557, 0.0109], [0.0949, 0.0384], [0.8244, 0.0731]\rangle$
v_2	$\langle[0.0163, 0.0021], [0.0178, 0.0026], [0.9737, 0.9977]\rangle$	$\langle[0.0184, 0.0025], [0.0604, 0.0107], [0.9547, 0.0864]\rangle$
v_3	$\langle[0.0443, 0.0071], [0.0627, 0.0126], [0.9180, 0.9924]\rangle$	$\langle[0.0521, 0.0116], [0.0353, 0.0086], [0.8985, 0.0756]\rangle$
v_4	$\langle[0.0148, 0.0017], [0.0171, 0.0018], [0.9244, 0.9964]\rangle$	$\langle[0.0214, 0.0025], [0.0184, 0.0029], [0.9432, 0.1046]\rangle$

$\tilde{A}^3$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle[0.0722, 0.0126], [0.0805, 0.0221], [0.8813, 0.1014]\rangle$	$\langle[0.0606, 0.0103], [0.0452, 0.0111], [0.8724, 0.0614]\rangle$
v_2	$\langle[0.0282, 0.0033], [0.0192, 0.0030], [0.9322, 0.1472]\rangle$	$\langle[0.0203, 0.0023], [0.0222, 0.0030], [0.9582, 0.0962]\rangle$
v_3	$\langle[0.0342, 0.0056], [0.0329, 0.0099], [0.9099, 0.1353]\rangle$	$\langle[0.0039, 0.0015], [0.0758, 0.0182], [0.9419, 0.1432]\rangle$
v_4	$\langle[0.0580, 0.0046], [0.0148, 0.0020], [0.8925, 0.0633]\rangle$	$\langle[0.0118, 0.0012], [0.0617, 0.0067], [0.9271, 0.0587]\rangle$

TABLE 9. Representation of weighted values in IVINHSS form for $\tilde{\mathcal{A}}^4$.

$\tilde{\mathcal{A}}^4$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$
v_1	$\langle[0.0638, 0.0157], [0.0308, 0.0102], [0.8449, 0.9803]\rangle$	$\langle[0.0320, 0.0079], [0.0925, 0.0113], [0.8974, 0.0992]\rangle$
v_2	$\langle[0.0173, 0.0027], [0.0184, 0.0038], [0.9384, 0.9953]\rangle$	$\langle[0.0196, 0.0037], [0.0737, 0.0116], [0.9654, 0.1165]\rangle$
v_3	$\langle[0.0247, 0.0029], [0.0627, 0.0073], [0.9357, 0.9954]\rangle$	$\langle[0.0551, 0.0063], [0.0738, 0.0228], [0.8896, 0.0740]\rangle$
v_4	$\langle[0.0290, 0.0063], [0.0171, 0.0039], [0.9244, 0.9920]\rangle$	$\langle[0.0313, 0.0060], [0.0261, 0.0026], [0.9367, 0.0916]\rangle$

$\tilde{\mathcal{A}}^4$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle[0.0326, 0.0080], [0.0194, 0.0030], [0.8873, 0.1035]\rangle$	$\langle[0.0397, 0.0079], [0.1353, 0.0184], [0.9395, 0.1399]\rangle$
v_2	$\langle[0.0220, 0.0028], [0.0220, 0.0030], [0.9607, 0.1727]\rangle$	$\langle[0.0196, 0.0024], [0.0178, 0.0026], [0.9547, 0.0834]\rangle$
v_3	$\langle[0.0135, 0.0013], [0.0589, 0.0167], [0.9579, 0.2738]\rangle$	$\langle[0.0536, 0.0055], [0.0758, 0.0182], [0.8957, 0.0770]\rangle$
v_4	$\langle[0.0172, 0.0030], [0.0363, 0.0056], [0.9322, 0.1089]\rangle$	$\langle[0.0184, 0.0033], [0.0269, 0.0031], [0.9756, 0.2004]\rangle$

Step 3. Determine the IVINPIS and IVINNIS from the weighted matrices, $\tilde{\mathcal{A}}^1$, $\tilde{\mathcal{A}}^2$, $\tilde{\mathcal{A}}^3$ and $\tilde{\mathcal{A}}^4$.

TABLE 10. Representation of IVINPIS ($\tilde{\mathcal{A}}^+$) from the weighted matrices.

$\tilde{\mathcal{A}}^+$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$
v_1	$\langle[0.0638, 0.0157], [0.0200, 0.0055], [0.8449, 0.9803]\rangle$	$\langle[0.0557, 0.0142], [0.0731, 0.0062], [0.8244, 0.0284]\rangle$
v_2	$\langle[0.0173, 0.0027], [0.0178, 0.0026], [0.9384, 0.9953]\rangle$	$\langle[0.0329, 0.0038], [0.0155, 0.0053], [0.9314, 0.0635]\rangle$
v_3	$\langle[0.0443, 0.0071], [0.0627, 0.0073], [0.9180, 0.9924]\rangle$	$\langle[0.0551, 0.0116], [0.0176, 0.0086], [0.8896, 0.0740]\rangle$
v_4	$\langle[0.0290, 0.0063], [0.0171, 0.0018], [0.9244, 0.9920]\rangle$	$\langle[0.0569, 0.0060], [0.0184, 0.0026], [0.9204, 0.0549]\rangle$

$\tilde{\mathcal{A}}^+$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle[0.1099, 0.0143], [0.0194, 0.0030], [0.8400, 0.0747]\rangle$	$\langle[0.0606, 0.0103], [0.0452, 0.0045], [0.8724, 0.0614]\rangle$
v_2	$\langle[0.0344, 0.0043], [0.0166, 0.0030], [0.9322, 0.1162]\rangle$	$\langle[0.0203, 0.0024], [0.0178, 0.0026], [0.9547, 0.0834]\rangle$
v_3	$\langle[0.0589, 0.0056], [0.0329, 0.0099], [0.8828, 0.0929]\rangle$	$\langle[0.0536, 0.0055], [0.0758, 0.0182], [0.8957, 0.0770]\rangle$
v_4	$\langle[0.0580, 0.0054], [0.0148, 0.0020], [0.8925, 0.0633]\rangle$	$\langle[0.0322, 0.0033], [0.0269, 0.0031], [0.9271, 0.0587]\rangle$

TABLE 11. Representation of IVINPIS ($\tilde{\mathcal{A}}^-$) from the weighted matrices.

$\tilde{\mathcal{A}}^-$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$
v_1	$\langle[0.0386, 0.0062], [0.0200, 0.0055], [0.9234, 0.9922]\rangle$	$\langle[0.0320, 0.0071], [0.0731, 0.0062], [0.8974, 0.0992]\rangle$
v_2	$\langle[0.0116, 0.0014], [0.0178, 0.0026], [0.9737, 0.9985]\rangle$	$\langle[0.0184, 0.0025], [0.0155, 0.0053], [0.9654, 0.1165]\rangle$
v_3	$\langle[0.0204, 0.0025], [0.0178, 0.0026], [0.9480, 0.9968]\rangle$	$\langle[0.0341, 0.0027], [0.0176, 0.0086], [0.9114, 0.0956]\rangle$
v_4	$\langle[0.0148, 0.0017], [0.0171, 0.0018], [0.9659, 0.9980]\rangle$	$\langle[0.0214, 0.0025], [0.0176, 0.0026], [0.9432, 0.1197]\rangle$

$\tilde{\mathcal{A}}^-$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
v_1	$\langle[0.0326, 0.0080], [0.0194, 0.003], [0.8873, 0.1035]\rangle$	$\langle[0.0142, 0.0021], [0.0452, 0.0045], [0.93950.1399]\rangle$
v_2	$\langle[0.0220, 0.0023], [0.0166, 0.003], [0.9607, 0.1727]\rangle$	$\langle[0.0103, 0.0010], [0.0178, 0.0026], [0.97920.2049]\rangle$
v_3	$\langle[0.0135, 0.0013], [0.0166, 0.003], [0.9579, 0.2738]\rangle$	$\langle[0.0039, 0.0015], [0.0178, 0.0026], [0.94190.1432]\rangle$
v_4	$\langle[0.0172, 0.0030], [0.0148, 0.002], [0.9451, 0.1591]\rangle$	$\langle[0.0118, 0.0012], [0.0269, 0.0031], [0.97560.2004]\rangle$

Step 4. Determine the CC for the alternatives by using the values of IVINPIS and IVINNIS.

$$\chi^1 = 0.9968, \chi^2 = 0.9984, \chi^3 = 0.9988 \text{ and } \chi^4 = 0.9968.$$

$$\lambda^1 = 0.9957, \lambda^2 = 0.9972, \lambda^3 = 0.9971 \text{ and } \lambda^4 = 0.9984.$$

Step 5. Compute the closeness coefficient of neutrosophic ideal solution as below.

$$\epsilon^1 = 0.5733, \epsilon^2 = 0.6364, \epsilon^3 = 0.7073 \text{ and } \epsilon^4 = 0.3333.$$

Step 6. Arrange the values in descending order.

$$\epsilon^3 > \epsilon^2 > \epsilon^1 > \epsilon^4.$$

$$\Rightarrow \mathcal{A}^3 > \mathcal{A}^2 > \mathcal{A}^1 > \mathcal{A}^4.$$

Hence, $\mathcal{A}^3$ is the best among the group who can lead the project successfully.

6. **Comparative Analysis**

We combine the proposed interval-valued intuitionistic neutrosophic TOPSIS method with existing SMs to show the reliability, validity and effectiveness of the proposed TOPSIS method based on CC.

Example 6.1. Consider the same IVINHSS values and weights mentioned in Section 5.2. We now combine the proposed TOPSIS method, with the SMs given below to rank the alternatives.

(i) $\mathcal{S}_Y(\Omega_1, \Omega_2)$ [5]

$$= 1 - \frac{1}{n}\sum_{i=1}^{n} w_j \Big[|\underline{\alpha}_{\Omega_1(q_i)}(v_j) - \underline{\alpha}_{\Omega_2(q_i)}(v_j)| + |\overline{\alpha}_{\Omega_1(q_i)}(v_j) - \overline{\alpha}_{\Omega_2(q_i)}(v_j)| + |\underline{\beta}_{\Omega_1(q_i)}(v_j) - \underline{\beta}_{\Omega_2(q_i)}(v_j)|$$
$$+ |\overline{\beta}_{\Omega_1(q_i)}(v_j) - \overline{\beta}_{\Omega_2(q_i)}(v_j)| + |\underline{\gamma}_{\Omega_1(q_i)}(v_j) - \underline{\gamma}_{\Omega_2(q_i)}(v_j)| + |\overline{\gamma}_{\Omega_1(q_i)}(v_j) - \overline{\gamma}_{\Omega_2(q_i)}(v_j)|\Big].$$

(ii) $\mathcal{S}_T(\Omega_1, \Omega_2)$ [6]

$$= \frac{\sum\limits_{i=1}^{n}\begin{pmatrix} min(\underline{\alpha}_{\Omega_1(q_i)}(v_j), \underline{\alpha}_{\Omega_2(q_i)}(v_j)) + min(\overline{\alpha}_{\Omega_1(q_i)}(v_j), \overline{\alpha}_{\Omega_2(q_i)}(v_j)) + min(\underline{\beta}_{\Omega_1(q_i)}(v_j), \underline{\beta}_{\Omega_2(q_i)}(v_j)) \\ + min(\overline{\beta}_{\Omega_1(q_i)}(v_j), \overline{\beta}_{\Omega_2(q_i)}(v_j)) + min(\underline{\gamma}_{\Omega_1(q_i)}(v_j), \underline{\gamma}_{\Omega_2(q_i)}(v_j)) + min(\overline{\gamma}_{\Omega_1(q_i)}(v_j), \overline{\gamma}_{\Omega_2(q_i)}(v_j))\end{pmatrix}}{\sum\limits_{i=1}^{n}\begin{pmatrix} max(\underline{\alpha}_{\Omega_1(q_i)}(v_j), \underline{\alpha}_{\Omega_2(q_i)}(v_j)) + max(\overline{\alpha}_{\Omega_1(q_i)}(v_j), \overline{\alpha}_{\Omega_2(q_i)}(v_j)) + max(\underline{\beta}_{\Omega_1(q_i)}(v_j), \underline{\beta}_{\Omega_2(q_i)}(v_j)) \\ + max(\overline{\beta}_{\Omega_1(q_i)}(v_j), \overline{\beta}_{\Omega_2(q_i)}(v_j)) + max(\underline{\gamma}_{\Omega_1(q_i)}(v_j), \underline{\gamma}_{\Omega_2(q_i)}(v_j)) + max(\overline{\gamma}_{\Omega_1(q_i)}(v_j), \overline{\gamma}_{\Omega_2(q_i)}(v_j))\end{pmatrix}},$$

(iii) $\mathcal{S}_H(\Omega_1, \Omega_2)$ [6]

$$= \frac{1}{6}\sum_{i=1}^{n} w_j \Big[|\underline{\alpha}_{\Omega_1(q_i)}(v_j) - \underline{\alpha}_{\Omega_2(q_i)}(v_j)| + |\overline{\alpha}_{\Omega_1(q_i)}(v_j) - \overline{\alpha}_{\Omega_2(q_i)}(v_j)| + |\underline{\beta}_{\Omega_1(q_i)}(v_j) - \underline{\beta}_{\Omega_2(q_i)}(v_j)|$$
$$+ |\overline{\beta}_{\Omega_1(q_i)}(v_j) - \overline{\beta}_{\Omega_2(q_i)}(v_j)| + |\underline{\gamma}_{\Omega_1(q_i)}(v_j) - \underline{\gamma}_{\Omega_2(q_i)}(v_j)| + |\overline{\gamma}_{\Omega_1(q_i)}(v_j) - \overline{\gamma}_{\Omega_2(q_i)}(v_j)|\Big].$$

(iv) $\mathcal{S}_E(\Omega_1, \Omega_2)$ [6]

$$= \Bigg(\sum_{i=1}^{n} w_j \Big[|\underline{\alpha}_{\Omega_1(q_i)}(v_j) - \underline{\alpha}_{\Omega_2(q_i)}(v_j)|^2 + |\overline{\alpha}_{\Omega_1(q_i)}(v_j) - \overline{\alpha}_{\Omega_2(q_i)}(v_j)|^2 + |\underline{\beta}_{\Omega_1(q_i)}(v_j) - \underline{\beta}_{\Omega_2(q_i)}(v_j)|^2$$
$$+ |\overline{\beta}_{\Omega_1(q_i)}(v_j) - \overline{\beta}_{\Omega_2(q_i)}(v_j)|^2 + |\underline{\gamma}_{\Omega_1(q_i)}(v_j) - \underline{\gamma}_{\Omega_2(q_i)}(v_j)|^2 + |\overline{\gamma}_{\Omega_1(q_i)}(v_j) - \overline{\gamma}_{\Omega_2(q_i)}(v_j)|^2\Big]\Bigg)^{\frac{1}{2}}.$$

(v) $\mathcal{S}_{C_1}(\Omega_1, \Omega_2)$ [31]

$$= \frac{1}{n}\sum_{i=1}^{n} \text{Cos}\Big[\frac{\pi}{4}\Big(|\underline{\alpha}_{\Omega_1(q_i)}(v_j) - \underline{\alpha}_{\Omega_2(q_i)}(v_j)| \vee |\underline{\beta}_{\Omega_1(q_i)}(v_j) - \underline{\beta}_{\Omega_2(q_i)}(v_j)| \vee |\underline{\gamma}_{\Omega_1(q_i)}(v_j) - \underline{\gamma}_{\Omega_2(q_i)}(v_j)|$$

$$+ |\overline{\alpha}_{\Omega_1(q_i)}(v_j) - \overline{\alpha}_{\Omega_2(q_i)}(v_j)| \vee |\overline{\beta}_{\Omega_1(q_i)}(v_j) - \overline{\beta}_{\Omega_2(q_i)}(v_j)| \vee |\overline{\gamma}_{\Omega_1(q_i)}(v_j) - \overline{\gamma}_{\Omega_2(q_i)}(v_j)|\Big)\Big].$$

(vi) $\mathcal{S}_{C_2}(\Omega_1, \Omega_2)$ [31]

$$= \frac{1}{n}\sum_{i=1}^{n} \text{Cos}\Big[\frac{\pi}{12}\Big(|\underline{\alpha}_{\Omega_1(q_i)}(v_j) - \underline{\alpha}_{\Omega_2(q_i)}(v_j)| + |\overline{\alpha}_{\Omega_1(q_i)}(v_j) - \overline{\alpha}_{\Omega_2(q_i)}(v_j)| + |\underline{\beta}_{\Omega_1(q_i)}(v_j) - \underline{\beta}_{\Omega_2(q_i)}(v_j)|$$

$$+ |\overline{\beta}_{\Omega_1(q_i)}(v_j) - \overline{\beta}_{\Omega_2(q_i)}(v_j)| + |\underline{\gamma}_{\Omega_1(q_i)}(v_j) - \underline{\gamma}_{\Omega_2(q_i)}(v_j)| + |\overline{\gamma}_{\Omega_1(q_i)}(v_j) - \overline{\gamma}_{\Omega_2(q_i)}(v_j)|\Big)\Big].$$

Table 12. Comparison of existing similarity measures with proposed method.

Determination of rank using existing similarity measures
$\mathcal{S}_Y(\psi_1, \psi_2)$ [5]$\Rightarrow \mathcal{A}^1 = \mathcal{A}^4 = 0.50$ and $\mathcal{A}^2 = \mathcal{A}^3 = 0.49$
$\mathcal{S}_T(\psi_1, \psi_2)$ [6]$\Rightarrow \mathcal{A}^1 = \mathcal{A}^4 = 0.50$ and $\mathcal{A}^2 = \mathcal{A}^3 = 0.49$
$\mathcal{S}_{C_1}(\psi_1, \psi_2)$ [31]$\Rightarrow \mathcal{A}^1 = \mathcal{A}^2 = \mathcal{A}^3 = \mathcal{A}^4 = 0.50$
$\mathcal{S}_{C_2}(\psi_1, \psi_2)$ [31]$\Rightarrow \mathcal{A}^1 = \mathcal{A}^2 = \mathcal{A}^3 = \mathcal{A}^4 = 0.50$

Analysis : From Table 12, it is evident that, when SMs of $\mathcal{S}_Y(\psi_1, \psi_2)$ [5], $\mathcal{S}_T(\psi_1, \psi_2)$ [6], $\mathcal{S}_{C_1}(\psi_1, \psi_2)$ [31] and $\mathcal{S}_{C_2}(\psi_1, \psi_2)$ [31] are used in the proposed TOPSIS method instead of CC, it is not possible to identify the best alternative. However, the best alternative is identified in the proposed method when CC is used. Hence, it is evident that the proposed TOPSIS method based on CC is more reliable and effective than SMs.

7. Conclusions

In this work, we have introduced the notion of IVINHSS and established some of its properties. The aim of this research is to introduce new operational laws for IVINHSS. Also, we have presented the aggregation operators for IVINHSS by using the operational laws and established some of their properties. We have proposed aggregation operators and an application based on the TOPSIS method to identify a suitable employee, who can handle the project successfully using the Leipzig leadership model. To study the closeness coefficients, we have applied CC instead of SMs in the proposed TOPSIS method. We have presented a comparative study between the proposed method and the existing SMS to prove the reliability of the proposed model. In the future, we can extend this structure to several aggregate operators, combine IVINHSS with N soft set and in various decision-making problems.

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A New Perspective of Neutrosophic Hyperconnected Spaces

Sasikala D[1] and Deepa M [2]
[1]PSGR Krishnammal College For Women, Coimbatore; dsasikala@psgrkcw.ac.in
[2]PSGR Krishnammal College For Women, Coimbatore; mdeepa@psgrkcw.ac.in

Abstract. The focus of this article is to introduce a new class of sets namely neutrosophic semi j-open and neutrosophic semi j-closed sets in neutrosophic topological space. Using this, we present the new spaces neutrosophic hyperconnected and neutrosophic semi j-hyperconnected. Also we explore the characteristics of neutrosophic semi j-open sets, neutrosophic semi j-closed set, neutrosophic hyperconnectedness. Finally we examine the properties of neutrosophic semi j-hyperconnectedness with some existing sets.

Keywords: Neutrosophic semi j-open, Neutrosophic semi j-closed, Neutrosophic hyperconnected, Neutrosophic semi j-hyperconnectedness.

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1. **Introduction**

In 1965, L.Zadeh introduced the concept of fuzzy sets and fuzzy logic. It is an important concept in handling uncertainity in real life where each element has a membership functions [22]. In 1986, Attanassov proposed the concept of intuitionistic fuzzy sets, which is a generalization of fuzzy sets [10]. Intuitionstic fuzzy sets are characterized by the membership function and non-membership function with each element, whereas in real life we need to handle the incompleteness and indeterminancy. In this context, Smarandache applied neutrosophic set theory to solve real world practical problems. Smarandache's neutrosophic set theory focused on medical, engineering fields, social science etc. [5, 7], Neutrosophic sets are characterized by membership, indeterminancy and non-membership functions [8, 21].

In 2012, Salama and Alblowi defined neutrosophic topological space by using neutrosophic sets [14]. Further researchers have carried out to investigate the various properties of neutrosophic sets in different fields [2–4, 6].In 1970, Steen and Seebach [20] introduced the notion of hyperconnectedness in topological spaces. Several researchers examined the properties of hyperconnectedness in general topology [1, 11, 12, 15, 17, 18]. Jayasree chakraborty, Baby bhattacharya and Arnab paul defined fuzzy hyperconnectedness in fuzzy topological space [9].

Recently Sasikala.D and Deepa.M introduced semi j-hyperconnected space using semi j-open sets [16]. This paper communicates the role of hyperconnectedness in the field of neutrosophic topological spaces. We ideate a new class of sets called neutrosophic semi j-open set and neutrosophic semi j-closed set exercised with theorems and appropriate examples. Also we proposed the novel space namely neutrosophic semi j-hyperconnected space by neutrosophic semi j-open sets and analyses the essential characteristics of this space.

Throughout this paper neutrosophic topological space $[\mathcal{X}, \tau]$ is simply denoted by $\mathcal{X}$.

2. Preliminaries

Definition 2.1. [13] Let $\mathcal{X}$ be a non empty set. A neutrosophic set P is an object having the form $P = \{< x, \lambda_P(x), \mu_P(x), \nu_P(x) >: x \in \mathcal{X}\}$ where $\lambda_P(x), \mu_P(x)$ and $\nu_P(x)$ represents the degree of membership function, the degree of indeterminancy and the degree of non membership function respectively of each element $x \in \mathcal{X}$ to the set P. It is simply denoted by $P =< \lambda_P(x), \mu_P(x), \nu_P(x) >$.

Definition 2.2. [14] Let $P =< \lambda_P(x), \mu_P(x), \nu_P(x) >$ be a neutrosophic set on $\mathcal{X}$, then the complement of the set P can be defined by the following three kinds as
$(i) C[P] = \{x, 1 - \lambda_P(x), 1 - \mu_P(x), 1 - \nu_P(x) >: x \in \mathcal{X}\}$.
$(ii) C[P] = \{x, \nu_P(x), \mu_P(x), \lambda_P(x) >: x \in \mathcal{X}\}$.
$(iii) C[P] = \{x, \nu_P(x), 1 - \mu_P(x), \lambda_P(x) >: x \in \mathcal{X}\}$.

Proposition 2.3. *[14] For any neutrosophic set P, the following conditions hold:*
(i) $0_N \subseteq P, 0_N \subseteq 0_N$.
(ii) $P \subseteq 1_N, 1_N \subseteq 1_N$.

Definition 2.4. [14] Let τ be a collection of all neutrosophic subsets on $\mathcal{X}$. Then τ is called a neutrosophic topology on $\mathcal{X}$ if the following conditions hold.
(i) $0_N, 1_N \in \tau$.
(ii) union of any numbers of neutrosophic sets in τ also belongs to τ.
(iii) intersection of any two of neutrosophic sets in τ also belongs to τ.

Then the pair $(\mathcal{X}, \tau)$ is called neutrosophic topological space. A neutrosophic set Q is neutrosophic closed if and only if complement of Q is neutrosophic open.

Definition 2.5. [13] Let $\mathcal{X}$ be neutrosophic topological space and $P =< \lambda_P(x), \mu_P(x), \nu_P(x) >$ be a neutrosophic set in $\mathcal{X}$. Then the neutrosophic closure and neutrosophic interior of P are defined by
$Ncl[P] = \cap\{M : M$ is a neutrosophic closed set in $\mathcal{X}$ and $P \subseteq M\}$.
$Nint[P] = \cup\{N : N$ is a neutrosophic open set in $\mathcal{X}$ and $N \subseteq P\}$.

It follows that $Ncl[P]$ is neutrosophic closed set and $Nint[P]$ is a neutrosophic open set in

$\mathcal{X}$.

(i) P is neutrosophic open set if and only if $P = Nint[P]$.

(ii) P is neutrosophic closed set if and only if $P = Ncl[P]$.

Proposition 2.6. *[13] For any neutrosophic set P in $\mathcal{X}$ we have*

(i) $Ncl(C[P]) = C(Nint[P])$.

(ii) $Nint(C[P]) = C(Ncl[P])$.

Definition 2.7. [21] A neutrosophic set P in a neutrosophic topological space $\mathcal{X}$ is called

(i) Neutrosophic semiopen set if $P \subseteq Ncl[Nint[P]]$.

(ii) Neutrosophic preopen set if $P \subseteq Nint[Ncl[P]]$.

(iii) Neutrosophic regular open set if $P = Nint[Ncl[P]]$.

(iv) Neutrosophic j-open set if $P \subseteq Nint[Npcl[P]]$.

Definition 2.8. [4] A neutrosophic subset P in a neutrosophic topological space $\mathcal{X}$ is called

(i) neutrosophic dense if $Ncl[P] = 1_N$.

(ii) neutrosophic nowhere dense if $Nint[Ncl[P]] = 0_N$.

Proposition 2.9. *[13] Let $\mathcal{X}$ be a neutrosophic topological space and P, Q be two neutrosophic subsets in $\mathcal{X}$. Then the following conditions hold:*

(i) $Nint[P] \subseteq P$.

(ii) $P \subseteq Ncl[P]$.

(iii) $P \subseteq Q \implies Nint[P] \subseteq Nint[Q]$.

(iv) $P \subseteq Q \implies Ncl[P] \subseteq Ncl[Q]$.

(v) $Nint[Nint[P]] = Nint[P]$.

(vi) $Ncl[Ncl[P]] = Ncl[P]$.

(vii) $Nint[P \cap Q] = Nint[P] \cap Nint[Q]$.

(viii) $Ncl[P \cup Q] = Ncl[P] \cup Ncl[Q]$.

(ix) $Nint[0_N] = 0_N$.

(x) $Nint[1_N] = 1_N$.

(xi) $Ncl[0_N] = 0_N$.

(xii) $Ncl[1_N] = 1_N$.

(xiii) $P \subseteq Q \implies C[Q] \subseteq C[P]$.

(xiv) $Ncl[P \cap Q] \subseteq Ncl[P] \cap Ncl[Q]$.

(xv) $Nint[P \cup Q] \supseteq Nint[P] \cup Nint[Q]$.

Definition 2.10. [19] A topological space $\mathcal{X}$ is said to be hyperconnected if every non empty open subset of $\mathcal{X}$ is dense in $\mathcal{X}$.

3. Neutrosophic semi j-open sets

In this part, we define a new set namely neutrosophic semi j-open set in neutrosophic topological spaces. Also some of its basic properties are discussed.

Definition 3.1. Let P be a neutrosophic subset of a neutrosophic topological space $\mathcal{X}$. Then P is said to be neutrosophic semi j-open set of $\mathcal{X}$ if and only if $P \subseteq Ncl[Nint[Npcl[P]]]$.

Example 3.2. Let $\mathcal{X} = \{s, t, r\}$ and the neutrosophic subsets P, Q, R and S in $\mathcal{X}$ as follows,
$P = \{< s, 0.4, 0.3, 0.8 >, < t, 0.5, 0.2, 0.6 >, < r, 0.4, 0.2, 0.6 >; s, t, r \in \mathcal{X}\}$,
$Q = \{< s, 0.3, 0.4, 0.5 >, < t, 0.6, 0.4, 0.6 >, < r, 0.3, 0.4, 0.6 >; s, t, r \in \mathcal{X}\}$,
$R = \{< s, 0.4, 0.4, 0.5 >, < t, 0.6, 0.4, 0.6 >, < r, 0.4, 0.4, 0.6 >; s, t, r \in \mathcal{X}\}$,
$S = \{< s, 0.3, 0.3, 0.8 >, < t, 0.5, 0.2, 0.6 >, < r, 0.3, 0.2, 0.6 >; s, t, r \in \mathcal{X}\}$.
Then $\tau = \{0_N, P, Q, R, S, 1_N\}$ is a neutrosophic topological space $\mathcal{X}$.
Let $E = \{< s, 0.4, 0.4, 0.5 >, < t, 0.5, 0.4, 0.7 >, < r, 0.4, 0.4, 0.7 >; s, t, r \in \mathcal{X}\}$ be a neutrosophic subset in $\mathcal{X}$, then $Ncl[Nint[Npcl[E]]] = \{< s, 0.5, 0.6, 0.5 >, < t, 0.6, 0.6, 0.6 >, < r, 0.6, 0.6, 0.4 >; s, t, r \in \mathcal{X}\}$ Therefore $E \subseteq Ncl[Nint[Npcl[E]]]$. Hence E is a neutrosophic semi j-open set.

Theorem 3.3. *Let $\{P_\alpha : \alpha \in \Delta\}$ be a collection of neutrosophic semi j-open sets in neutrosophic topological space $\mathcal{X}$. Then $\bigcup_{\alpha \in \Delta} P_\alpha$ is also neutrosophic semi j-open in $\mathcal{X}$.*

Proof. Since P_α is neutrosophic semi j-open set in $\mathcal{X}$. Then $P_\alpha \subseteq Ncl[Nint[Npcl[P_\alpha]]]$. $\bigcup_{\alpha \in \Delta} P_\alpha \subseteq \bigcup_{\alpha \in \Delta} Ncl[Nint[Npcl[P_\alpha]]] \subseteq Ncl[Nint[Npcl[\bigcup_{\alpha \in \Delta} P_\alpha]]]$. Hence $\bigcup_{\alpha \in \Delta} P_\alpha$ is also neutrosophic semi j-open set in $\mathcal{X}$. □

Remark 3.4. The intersection of any two neutrosophic semi j-open sets of neutrosophic topological space $\mathcal{X}$ need not be a neutrosophic semi j-open set as verified by the following example.

Example 3.5. Let $\mathcal{X} = \{s, t\}$ and the neutrosophic subsets P, Q, R and S in $\mathcal{X}$ as follows,
$P = \{< s, 0.2, 0.1, 0.8 >, < t, 0.3, 0.1, 0.4 >; s, t \in \mathcal{X}\}$,
$Q = \{< s, 0.1, 0.2, 0.5 >, < t, 0.4, 0.3, 0.4 >; s, t \in \mathcal{X}\}$,
$R = \{< s, 0.2, 0.2, 0.5 >, < t, 0.4, 0.3, 0.4 >; s, t \in \mathcal{X}\}$,
$S = \{< s, 0.1, 0.1, 0.8 >, < t, 0.3, 0.1, 0.4 >; s, t \in \mathcal{X}\}$.
Then $\tau = \{0_N, P, Q, R, S, 1_N\}$ is a neutrosophic topological space $\mathcal{X}$.
Let $E = \{< s, 0.9, 0.1, 0.6 >, < t, 0.3, 0.2, 0.6 >; s, t \in \mathcal{X}\}$ and
$F = \{< s, 0.6, 0.7, 0.3 >, < t, 0.5, 0.7, 0.4 >; s, t \in \mathcal{X}\}$ be the neutrosophic subsets in $\mathcal{X}$. Then $Ncl[Nint[Npcl[E]]] = 1_N$ and $Ncl[Nint[Npcl[F]]] = 1_N$. This implies $E \subseteq Ncl[Nint[Npcl[E]]]$ and $F \subseteq Ncl[Nint[Npcl[F]]]$. Here $E \cap F = \{< s, 0.6, 0.1, 0.6 >, <$

$t, 0.3, 0.2, 0.6 >; s, t \in \mathcal{X}\}$.Therefore E and F are neutrosophic semi j-open sets and $E \cap F$ is not neutrosophic semi j-open set in $\mathcal{X}$.

Theorem 3.6. *In a neutrosophic topological space $\mathcal{X}$, let P be a neutrosophic semi j-open set and $P \subseteq Q \subseteq Ncl[P]$. Then Q is also a neutrosophic semi j-open set in $\mathcal{X}$.*

Proof. Since P is neutrosophic semi j-open in $\mathcal{X}$. Then $P \subseteq Ncl[Nint[Npcl[P]]]$. $Ncl[P] \subseteq Ncl[Ncl[Nint[[Npcl[P]]]]$. Using proposition 2.9, $Ncl[P] \subseteq Ncl[Nint[Npcl[P]]]$. By hypothesis $P \subseteq Q \subseteq Ncl[P]$, then $Q \subseteq Ncl[Nint[Npcl[P]]]$. We have $P \subseteq Q$, therefore $Ncl[Nint[Npcl[P]]] \subseteq Ncl[Nint[Npcl[Q]]]$, which implies $Q \subseteq Ncl[Nint[Npcl[Q]]]$. Hence Q is a neutrosophic semi j-open set in $\mathcal{X}$. □

Theorem 3.7. *In a neutrosophic topological space $\mathcal{X}$, every neutrosophic j-open set is neutrosophic semi j-open.*

Proof. Let P be a neutrosophic j-open set in $\mathcal{X}$. Then $P \subseteq Nint[Npcl[P]]$, $Ncl[P] \subseteq Ncl[Nint[Npcl[P]]]$. We know that $P \subseteq Ncl[P]$. Therefore $P \subseteq Ncl[Nint[Npcl[P]]]$. Hence P is a neutrosophic semi j-open set in $\mathcal{X}$. □

Remark 3.8. Converse of the above theorem need not be true as shown in the following example.

Example 3.9. Let $\mathcal{X} = \{s, t\}$ and the neutrosophic subsets P and Q in $\mathcal{X}$ as follows,
$P = \{< s, 0.2, 0.2, 0.5 >, < t, 0.4, 0.3, 0.4 >; s, t \in \mathcal{X}\}$,
$Q = \{< s, 0.1, 0.1, 0.8 >, < t, 0.3, 0.1, 0.4 >; s, t \in \mathcal{X}\}$.
Then $\tau = \{0_N, P, Q, 1_N\}$ is a neutrosophic topological space on $\mathcal{X}$.
Let $R = \{< s, 0.2, 0.1, 0.6 >, < t, 0.4, 0.4, 0.5 >; s, t \in \mathcal{X}\}$. $Nint[Npcl[R]] = P$, this implies $R \nsubseteq P$. and $Ncl[Nint[Npcl[R]]] = P^C$. Therefore $R \subseteq P^C$. Hence R is neutrosophic semi j-open but not neutrosophic j-open.

Theorem 3.10. *Every neutrosophic open sets in $\mathcal{X}$ is neutrosophic semi j-open.*

Proof. Let P be a neutrosophic open sets in $\mathcal{X}$. Then $P = Nint[P]$. We know that $Nint[P] \subseteq P \subseteq Ncl[P]$ and $Npcl[P] \subseteq Ncl[P]$. This implies $P \subseteq Npcl[P] \subseteq Ncl[P]$. Using proposition 2.9,
$\implies Nint[P] \subseteq Nint[Npcl[P]] \subseteq Nint[Ncl[P]]$
$\implies Ncl[Nint[P]] \subseteq Ncl[Nint[Npcl[P]]] \subseteq Ncl[Nint[Npcl[P]]]$
$\implies Ncl[P] \subseteq Ncl[Nint[Npcl[P]]]$
$\implies P \subseteq Ncl[Nint[Npcl[P]]]$.
Hence P is neutrospohic semi j-open. □

Remark 3.11. Converse of the above theorem need not be true as shown in the following example.

Example 3.12. Consider $\mathcal{X} = \{s\}$ and the neutrosophic subsets P and Q as follows
$P = \{< s, 0.4, 0.5, 0.3 >; s \in \mathcal{X}\}$,
$Q = \{< s, 0.1, 0.5, 0.5 >; s \in \mathcal{X}\}$.
Then $\tau = \{0_N, P, Q, 1_N\}$ is a neutrosophic topological space $\mathcal{X}$.
Here $R = \{< s, 0.3, 0.6, 0.5 >; s \in \mathcal{X}\}$ is neutrosophic semi j-open but not neutrosophic open.

4. **Neutrosophic semi j-closed sets**

Definition 4.1. A neutrosophic subset S of a neutrosophic topological space $\mathcal{X}$ is said to be neutrosophic semi j-closed set if and only if $Nint[Ncl[Npint[S]]] \subseteq S$.

Example 4.2. Let $\mathcal{X} = \{s_1, s_2, s_3\}$ and the neutrosophic subsets Q_1, Q_2 and Q_3 as follows
$Q_1 = \{< s_1, 0.6, 0.5, 0.6 >, < s_2, 0.7, 0.4, 0.4 >, < s_3, 0.6, 0.4, 0.4 >; s_1, s_2, s_3 \in \mathcal{X}\}$,
$Q_2 = \{< s_1, 0.7, 0.6, 0.3 >, < s_2, 0.8, 0.6, 0.4 >, < s_3, 0.6, 0.6, 0.4 >; s_1, s_2, s_3 \in \mathcal{X}\}$,
$Q_3 = \{< s_1, 0.6, 0.5, 0.4 >, < s_2, 0.7, 0.5, 0.4 >, < s_3, 0.6, 0.5, 0.4 >; s_1, s_2, s_3 \in \mathcal{X}\}$.
Then $\tau = \{0_N, Q_1, Q_2, Q_3, 1_N\}$ is a neutrosophic topological space $\mathcal{X}$. Put $F = \{< s_1, 0.5, 0.4, 0.5 >, < s_2, 0.6, 0.5, 0.3 >, < s_3, 0.4, 0.4, 0.3 >; s_1, s_2, s_3 \in \mathcal{X}\}$. Then $Nint[Ncl[Npint[F]]] \subseteq F$. Therefore F is a neutrosophic semi j-closed set in $\mathcal{X}$.

Theorem 4.3. *Take S be a neutrosophic subset of $\mathcal{X}$, then S is neutrosophic semi j-closed if and only if $C(S)$ is neutrosophic semi j-open.*

Proof. Assume S is neutrosophic semi j-closed set in $\mathcal{X}$. Then $Nint[Ncl[Npint[S]]] \subseteq S$, taking compliments on both sides, we obtain $C[S] \subseteq C[Nint[Ncl[Npint[S]]]] = Ncl[Nint[Npcl[C[S]]]]$ using proposition 2.6. Hence $C[S]$ is neutrosophic semi j-open set in $\mathcal{X}$. Conversely assume $C[S]$ is a neutrosophic semi j-open set in $\mathcal{X}$. Then $C[S] \subseteq Ncl[Nint[Npcl[C[S]]]]$. We obtain $C[Ncl[Nint[Npcl[C[S]]]]] \subseteq C[C[S]]$ by taking compliments on both sides. This implies $Nint[Ncl[Npint[S]]] \subseteq S$. Hence S is a neutrosophic semi j-closed set in $\mathcal{X}$. □

Theorem 4.4. *Let $\{S_\alpha : \alpha \in \Delta\}$ be a family of neutrosophic semi j-closed set in $\mathcal{X}$. Then arbitrary intersection of neutrosophic semi j-closed sets is also neutrosophic semi j-closed.*

Proof. Let $\{S_\alpha : \alpha \in \Delta\}$ be a family of neutrosophic semi j-closed sets in $\mathcal{X}$ and $P_\alpha = \{S_\alpha\}^c$. Then $\{S_\alpha : \alpha \in \Delta\}$ is a family of neutrosophic semi j-open sets in $\mathcal{X}$. Using theorem 3.3 $\bigcup_{\alpha\in\Delta} P_\alpha$ is neutrosophic semi j-open. Then $\{\bigcup_{\alpha\in\Delta} P_\alpha\}^C$ is neutrosophic semi j-closed which implies $\bigcap_{\alpha\in\Delta} {P_\alpha}^c$ is neutrosophic semi j-closed. Hence $\bigcap_{\alpha\in\Delta} S_\alpha$ is neutrosophic semi j-closed. □

Theorem 4.5. *In a neutrosophic topological space $\mathcal{X}$, every neutrosophic j-closed set is also neutrosophic semi j-closed.*

Proof. Let S be a neutrosophic j-closed set in $\mathcal{X}$. Then $Ncl[Npint[S]] \subseteq S$. $Nint[Ncl[Npint[S]]] \subseteq Nint[S]$. We know that $Nint[S] \subseteq S$, therefore $Nint[Ncl[Npint[S]]] \subseteq S$. Hence S is neutrosophic semi j-closed. □

Remark 4.6. Converse of the above theorem need not be true, as verified by the following example.

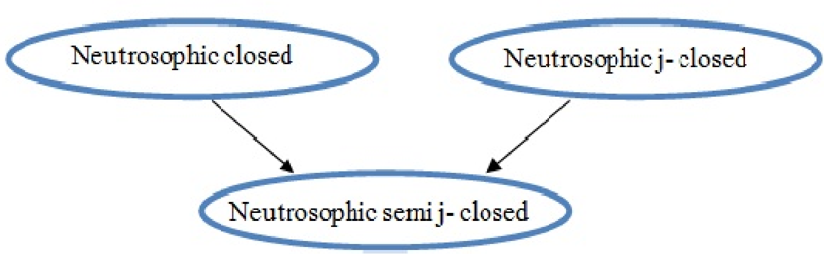
Example 4.7. Let $\mathcal{X} = \{t_1, t_2, t_3\}$ and the neutrosophic subsets Q_1, Q_2 as follows,
$Q_1 = \{< t_1, 0.2, 0.5, 0.4 >, < t_2, 0.2, 0.4, 0.5 >, < t_3, 0.1, 0.0, 0.5 >; t_1, t_2, t_3 \in \mathcal{X}\}$,
$Q_2 = \{< t_1, 0.3, 0.4, 0.5 >, < t_2, 0.4, 0.3, 0.2 >, < t_3, 0.2, 0.3, 0.4 >; t_1, t_2, t_3 \in \mathcal{X}\}$.
Put $\tau = \{0_N, Q_1, Q_2, Q_1 \cup Q_2, 1_N\}$. Let $G = \{< t_1, 0.4, 0.5, 0.4 >, < t_2, 0.3, 0.4, 0.2 >, < t_3, 0.3, 0.4, 0.5 >; t_1, t_2, t_3 \in \mathcal{X}\}$. Then G is a neutrosophic semi j-closed but not neutrosophic j-closed. Since $Nint[Ncl[Npint[G]]] = Q_1 \subseteq G$, but $Ncl[Npint[G]] = [Q_1 \cup Q_2]^c \nsubseteq G$.

Theorem 4.8. *In a neutrosophic topological space $\mathcal{X}$, every neutrosophic closed set is neutrosophic semi j-closed.*

Proof. Let S be neutrosophic closed set in $\mathcal{X}$. Then $S = Ncl[S]$. We know that $Nint[S] \subseteq Npint[S]$. $Ncl[Nint[S]] \subseteq Ncl[Npint[S]] \subseteq Ncl[S]$. It follows that $Nint[Ncl[Nint[S]]] \subseteq Nint[Ncl[Npint[S]]]$. Hence S is a semi j-closed in $\mathcal{X}$. □

Remark 4.9. The converse of the above theorem may not be true, as shown by the following example. In example 4.7, $Ncl[G] \neq G$. This implies G is neutrosophic semi j-closed set but not neutrosophic closed set.

From the above results, we have the following indications: But the converse of the above



indications need not be true as shown by 4.7 and 4.9.

5. Neutrosophic hyperconnected space

Definition 5.1. A neutrosophic topological space $\mathcal{X}$, is said to be neutrosophic hyperconnected if for every non empty neutrosophic open subsets of $\mathcal{X}$ is neutrosophic dense in $\mathcal{X}$.

Example 5.2. Consider $\mathcal{X} = \{s_1, s_2\}$ with $\tau = \{0_N, 1_N, P_1, P_2, P_3, P_4\}$ where
$P_1 = \{< s_1, 0.2, 0.4, 0.3 >, < s_2, 0.5, 0.1, 0.4 >, s_1, s_2 \in \mathcal{X}\}$,
$P_2 = \{< s_1, 0.1, 0.5, 0.6 >, < s_2, 0.4, 0.2, 0 >, s_1, s_2 \in \mathcal{X}\}$,
$P_3 = \{< s_1, 0.2, 0.5, 0.3 >, < s_2, 0.5, 0.2, 0 >, s_1, s_2 \in \mathcal{X}\}$,
$P_4 = \{< s_1, 0.1, 0.4, 0.6 >, < s_2, 0.4, 0.1, 0.4 >, s_1, s_2 \in \mathcal{X}\}$.
Here every non empty neutrosophic open sets $P_1, P_2, P_3, P_4, 1_N$ are neutrosophic dense in $\mathcal{X}$. ie.,
$Ncl[P_1] = 1_N$,
$Ncl[P_2] = 1_N$,
$Ncl(P_3) = 1_N$,
$Ncl(P_4) = 1_N$,
$Ncl(1_N) = 1_N$.
Therefore $\mathcal{X}$ is neutrosophic hyperconnected space.

Definition 5.3. A neutrosophic topological space $\mathcal{X}$ is called as neutrosophic extremely disconnected if the neutrosophic closure of each neutrosophic open set is neutrosophic open in $\mathcal{X}$.

Theorem 5.4. *In a neutrosophic topological space $\mathcal{X}$, every neutrosophic hyperconnected space is neutrosophic extremely disconnected.*

Proof. Let us take $\mathcal{X}$ be neutrosophic hyperconnected. Then for any neutrosophic open set P, $Ncl[P] = 1_N$. This implies that $Ncl[P]$ is neutrosophic open. Therefore $\mathcal{X}$ is neutrosophic extremely disconnected. □

Remark 5.5. The following example shows that the converse of the above theorem need not be true.

Example 5.6. Let $\mathcal{X} = \{s\}$ with $\tau = \{0_N, P_1, P_2, P_3, P_4, 1_N\}$ where
$P_1 = \{< s, 0.5, 0.3, 0.2 >; s \in \mathcal{X}\}$,
$P_2 = \{< s, 0.2, 0.3, 0.5 >; s \in \mathcal{X}\}$,
$P_3 = \{< s, 0.3, 0.3, 0.5 >; s \in \mathcal{X}\}$,
$P_4 = \{< s, 0.5, 0.3, 0.5 >; s \in \mathcal{X}\}$.
Here $Ncl[P_1] = \{< s, 0.5, 0.3, 0.2 >; s \in \mathcal{X}\}$,
$Ncl[P_2] = \{< s, 0.2, 0.3, 0.5 >; s \in \mathcal{X}\}$,

$Ncl(P_3) = \{< s, 0.5, 0.3, 0.5 >; s \in \mathcal{X}\}$,

$Ncl(P_4) = \{< s, 0.5, 0.3, 0.5 >; s \in \mathcal{X}\}$.

This example shows that $[\mathcal{X}, \tau]$ is neutrosophic extremely disconnected. Since $Ncl[P_1]$, $Ncl[P_2]$, $Ncl(P_3)$ and $Ncl(P_4)$ are neutrosophic open but not neutrosophic dense . Therefore, $\mathcal{X}$ is not neutrosophic hyperconnected.

Theorem 5.7. *In a neutrosophic topological space $\mathcal{X}$, the following properties are equivalent.*
(a) $\mathcal{X}$ is neutrosophic hyperconnected.
(b) In $\mathcal{X}$, the only neutrosophic regular open sets are 0_N and 1_N.

Proof. (a) $\implies$ (b)
Let $\mathcal{X}$ be a neutrosophic hyperconnected space. If P is a non-empty neutrosophic regular open set, then by the definition $P = Nint[Ncl[P]]$. This implies that $[Nint[Ncl[P]]]^c = [1_N - Nint[Ncl[P]]] = Ncl[1_N - Ncl[P]] = Ncl(P^c) = P^c \neq 1_N$. Since $P \neq 0_N$. This is a contradiction to the assumption. Hence the only neutrosophic regular open sets in $\mathcal{X}$ are 0_N and 1_N.
(b) $\implies$ (a)
Assume that 0_N and 1_N are the only neutrosophic regular open subsets in $\mathcal{X}$. Suppose that $\mathcal{X}$ is not neutrosophic hyperconnected. Then there exist a non empty neutrosophic open subset P of $\mathcal{X}$ such that $Ncl[P] \neq 1_N$. This implies $Ncl[Nint[P]] \neq 1_N$. Therefore, we have $Ncl[Nint[P]] = 0_N$. This gives $Ncl[P] = 0_N$. Since $P \neq 0_N$. It contradicts our assumption that $\mathcal{X}$ is not neutrosophic hyperconnected . Hence $\mathcal{X}$ is neutrosophic hyperconnected space.
□

Theorem 5.8. *A neutrosophic topological space $\mathcal{X}$ is neutrosophic hyperconnected if and only if for every neutrosophic subset of $\mathcal{X}$ is either neutrosophic dense or neutrosophic nowhere dense.*

Proof. Suppose $\mathcal{X}$ be a neutrosophic hyperconnected space and let P be any neutrosophic subset of $\mathcal{X}$ such that $P \subseteq 1_N$. Assume P is not neutrosophic nowhere dense. Then $Ncl[1_N - Ncl[P]] = 1_N - [Nint[Ncl[P]]] \neq 1_N$. Since $Nint[Ncl[P]] \neq 0_N$. This implies that $Ncl[Nint[Ncl[P]]] = 1_N$. Since $Ncl[Nint[Ncl[P]]] = 1_N \subseteq Ncl[P]$. Therefore, $Ncl[P] = 1_N$. Hence P is neutrosophic dense set.

For the converse part, let P_1 be any non empty neutrosophic open set in $\mathcal{X}$, then $P_1 \subset Nint[Ncl[P_1]]$. This implies that P_1 is not neutrosophic nowhere dense set. By the hypothesis, P_1 is neutrosophic dense set. □

Proposition 5.9. *If $\mathcal{X}$ be a neutrosophic hyperconnected space, then the intersection of any two neutrosophic semi open sets is also neutrosophic semi open.*

Proof. Let P_1 and P_2 be the two non empty neutrosophic semi open sets in a neutrosophic hyperconnected space $\mathcal{X}$. Then, we have $P_1 \subseteq Ncl[Nint[P_1]]$ and $P_2 \subseteq Ncl[Nint[P_2]]$. It follows that, $Ncl[P_1] = Ncl[Nint[P_1]] = 1_N$ and $Ncl[P_2] = Ncl[Nint[P_2]]] = 1_N$, where P_1 and P_2 are two non empty neutrosophic semi open sets in a neutrosophic hyperconnected space. Also we have $P_1 \wedge P_2 \neq 0_N$. Therefore, $Ncl[Nint[P_1 \wedge P_2]] = Ncl[Nint[P_1]] \wedge Ncl[Nint[P_2]] = 1_N$. This implies $P_1 \wedge P_2 \subseteq Ncl[Nint[P_1]] \wedge Ncl[Nint[P_2]] = Ncl[Nint[P_1 \wedge P_2]]$. Hence $P_1 \wedge P_2$ is neutrosophic semi open set. □

6. Neutrosophic semi j-hyperconnected spaces

Definition 6.1. A neutrosophic subset P of $\mathcal{X}$ is said to be neutrosophic semi j-interior of P if the union of all neutrosophic semi j-open sets of $\mathcal{X}$ contained in P. It is denoted by $Nint_{sj}[P]$.

A neutrosophic subset Q of $\mathcal{X}$ is said to be neutrosophic semi j-closure of Q if the intersection of all neutrosophic semi j-closed sets of $\mathcal{X}$ containing Q. It is denoted by $Ncl_{sj}[Q]$.

Example 6.2. Consider $\mathcal{X} = \{s_1, s_2, s_3\}$ and the neutrosophic subsets S_1, S_2, S_3 in $\mathcal{X}$ as follows,

$S_1 = \{< s_1, 0.3, 0.4, 0.3 >, < s_2, 0.6, 0.2, 0.4 >, < s_3, 0.5, 0.2, 0.3 >; s_1, s_2, s_3 \in \mathcal{X}\}$,

$S_2 = \{< s_1, 0.2, 0.6, 0.5 >, < s_2, 0.4, 0.2, 0.3 >, < s_3, 0.2, 0.3, 0.1 >; s_1, s_2, s_3 \in \mathcal{X}\}$,

$S_3 = \{< s_1, 0.3, 0.6, 0.3 >, < s_2, 0.6, 0.2, 0.3 >, < s_3, 0.5, 0.3, 0.1 >; s_1, s_2, s_3 \in \mathcal{X}\}$.

Take $\tau = \{0_N, S_1, S_2, S_3, 1_N\}$. For this 0_N, 1_N, S_1, S_2, $S_1 \cup S_2$, $S_1 \cup S_3$, $S_2 \cup S_3$ are the neutrosophic semi j-open sets and 0_N, 1_N, S_1^c, S_2^c, $(S_1 \cup S_2)^c$, $(S_1 \cup S_3)^c$, $(S_2 \cup S_3)^c$ are the neutrosophic semi j-closed sets. Put $T = \{< s_1, 0.5, 0.6, 0.2 >, < s_2, 0.7, 0.3, 0.3 >, < s_3, 0.6, 0.4, 0.2; s_1, s_2, s_3 \mathcal{X} >\}$ is a neutrosophic subset in $\mathcal{X}$. Then we have $Nint_{sj}(T) = S_1$ and $Ncl_{sj}(T) = 1_N$.

Definition 6.3. A neutrosophic topological space $\mathcal{X}$ is said to be neutrosophic semi j-hyperconnected space if for each nonempty neutrosophic semi j-open subset A of $\mathcal{X}$ is neutrosophic semi j-dense in $\mathcal{X}$. ie., $Ncl_{sj}(A) = 1_N$ for every A in $\mathcal{X}$.

Example 6.4. Let $\mathcal{X} = \{s_1, s_2, s_3\}$ and the neutrosophic subsets P_1, P_2, P_3 in $\mathcal{X}$ as follows,

$P_1 = \{< s_1, 0.1, 0.3, 0.2 >, < s_2, 0.4, 0.1, 0.3 >, < s_3, 0.3, 0.1, 0.2 >; s_1, s_2, s_3 \in \mathcal{X}\}$,

$P_2 = \{< s_1, 0.1, 0.4, 0.5 >, < s_2, 0.3, 0.1, 0.0 >, < s_3, 0.2, 0.0, 0.1 >; s_1, s_2, s_3 \in \mathcal{X}\}$,

$P_3 = \{< s_1, 0.2, 0.4, 0.2 >, < s_2, 0.4, 0.1, 0.0 >, < s_3, 0.3, 0.1, 0.0 >; s_1, s_2, s_3 \in \mathcal{X}\}$.

Put $\tau = \{0_N, P_1, 1_N\}$. Then the collection of neutrosophic semi j-open sets are 0_N, 1_N, $P_1 \cup P_2$ and $P_1 \cup P_3$ i.e.,$P_1 \subseteq Ncl[Nint[Npcl[P_1]]]$, $P_1 \cup P_2 \subseteq Ncl[Nint[Npcl[P_1 \cup P_2]]]$ and

$P_1 \cup P_3 \subseteq Ncl[Nint[Npcl[P_1 \cup P_2]]]$. Here every non empty neutrosophic semi j-open sets are neutrosophic semi j-dense in $\mathcal{X}$. i.e., $Ncl_{sj}[P_1] = 1_N$, $Ncl_{sj}[P_1 \cup P_2] = 1_N$, $Ncl_{sj}[P_1 \cup P_3] = 1_N$ and $Ncl_{sj}[1_N]$. Therefore a neutrosophic topological space $\tau = \{0_N, P_1, 1_N\}$ is neutrosophic semi j-hyperconnected space.

Theorem 6.5. *In a neutrosophic topological space, every neutrosophic hyperconnected space is neutrosophic semi j-hyperconnected.*

Proof. Let $\mathcal{X}$ be a neutrosophic hyperconnected space and P be a neutrosophic open subset of $\mathcal{X}$. Then $Ncl[P] = 1_N$. This implies that $Nint[Ncl[P]] = 1_N$. Therefore P is neutrosophic preopen. P^C is neutrosophic preclosed. Since every neutrosophic open set is neutrosophic preopen and its complement is neutrosophic preclosed. It follows that $Npcl[P] = Ncl[P] = 1_N$ which implies $Ncl[Nint[Npcl[P]]] = 1_N$. Therefore P is neutrosophic semi j-open $\Longrightarrow$ $Ncl_{sj}[P] = 1_N$ for any neutrosophic open set in $\mathcal{X}$. Hence $\mathcal{X}$ is neutrosophic semi j-hyperconnected space. □

Definition 6.6. Let $\mathcal{X}$ be a neutrosophic topological space and P be a neutrosophic semi j-open set of $\mathcal{X}$. Then

(a) P is said to be neutrosophic semi j-regular open set if and only if $P = Nint_{sj}[Ncl_{sj}[P]]$.

(b) P is said to be neutrosophic semi j-regular closed set if and only if $Ncl_{sj}[Nint_{sj}[P]] = P$.

Theorem 6.7. *Let $\mathcal{X}$ be a neutrosophic topological space, then each of the following statements are equivalent.*

(a) *$\mathcal{X}$ is neutrosophic semi j-hyperconnected.*

(b) *$\mathcal{X}$ has no two proper neutrosophic semi j-regular open or proper semi j-regular closed subset.*

(c) *Let P and Q be the proper disjoint neutrospohic semi j-open subsets in $\mathcal{X}$, then $\mathcal{X}$ does not have P and Q such that $Ncl_{sj}[P] \cup Q = P \cup Ncl_{sj}[Q] = 1_N$.*

(d) *$\mathcal{X}$ has no proper semi j-closed subset S and T such that $\mathcal{X} = S \cup T$ and $Nint_{sj} \cap T = S \cap Nint_{sj}(T) = 0_N$.*

Proof. (a) $\Longrightarrow$ (b) Let $0_N \neq P$ be neutrosophic semi j-regular open subset in $\mathcal{X}$. Then $P = Nint_{sj}[Ncl_{sj}[P]]$. Since $\mathcal{X}$ is a neutrosophic semi j-hyperconnected space. then $Ncl_{sj}[P] = 1_N$. This implies $P = 1_N$. Clearly P is not a proper neutrosophic semi j-regular open subset of $\mathcal{X}$. Similarly $\mathcal{X}$ cannot have a proper neutrosophic semi j-regular closed subset.

(b) $\Longrightarrow$ (c) Suppose P and Q are the neutrosophic subsets in $\mathcal{X}$ and $P \cap Q = 0_N$ such that

$Ncl_{sj}[P] \cup Q = P \cup Ncl_{sj}[Q] = 1_N$. This implies $0_N \neq Ncl_{sj}[P]$ is the neutrosophic semi j-regular closed set in $\mathcal{X}$. Since $P \cap Q = 0_N$ and $Ncl_{sj}[P] \cap Q = 0_N \implies Ncl_{sj}[P] \neq 1_N$ which implies $\mathcal{X}$ has a proper neutrosophic semi j-regular closed subset P. This contradicts (b).

(c) $\implies$ (d) Suppose, there exist two proper neutrosophic semi j-closed subset $0_N \neq S$ and $0_N \neq T$ in $\mathcal{X}$ such that $\mathcal{X} = S \cup T$, $Nint_{sj}(S) \cap T = S \cap Nint_{sj}(T) = 0_N$. Then $P = 1_N - S$, $Q = 1_N - T$ are the two non-empty neutrosophic semi j-open sets. Then $Ncl_{sj}[P] \cup Q = Ncl_{sj}(1_N - S) \cup Q = [1_N - Nint_{sj}(S)] \cup Q = 1_N. \implies Ncl_{sj}[P] \cup Q = P \cup Ncl_{sj}[Q] = 1_N$ which contradicts (c).

(d) $\implies$ (a) Suppose there exist a proper neutrosophic semi j-open set $0_N \neq P$ of $\mathcal{X}$ such that $Ncl_{sj}[P] \neq 1_N$. Then $Nint_{sj}[Ncl_{sj}[P]] \neq 1_N$. Take $S = Ncl_{sj}[P]$ and $T = 1_N - Nint_{sj}[Ncl_{sj}[P]]$. This implies $S \cup T = Ncl_{sj}[P] \cup [1_N - Nint_{sj}[Ncl_{sj}[P]]] = Ncl_{sj}[P] \cup Ncl_{sj}[1_N - Ncl_{sj}[P]] \implies Ncl_{sj}[P] \cup Ncl_{sj}[C(S)] \implies S \cup C(S) = 1_N$. Since S is neutrosophic semi j-closed set. Then $Nint_{sj}[Ncl_{sj}[P]] \cap [1_N - Nint_{sj}[Ncl_{sj}[P]]] = 0_N$. $\implies Ncl_{sj}[P] \cap Nint_{sj}[1_N - Nint_{sj}[Ncl_{sj}[P]]] = S \cap Nint_{sj} Ncl_{sj}[1_N - Ncl[P]] = S \cap Nint_{sj} Ncl_{sj}\ [C(S)] = S \cap C(S) = 0_N$. Since $C(S)$ is neutrosophic semi j-open. Thus $\mathcal{X}$ has two proper neutrosophic semi j-closed sets S and T such that $\mathcal{X} = S \cup T$ and $Nint_{sj} S \cap T = S \cap Nint_{sj} C[T] = 0_N$. This is a contradiction to (d). □

Theorem 6.8. *In a neutrosophic semi j-hyperconnected space $\mathcal{X}$. Let $0_N \neq P$ and $0_N \neq Q$ be the two neutrosophic semi j-open subsets in $\mathcal{X}$, then $P \cap Q$ is also non-empty.*

Proof. Suppose $P \cap Q = 0_N$, for any $0_N \neq P$ and $0_N \neq Q$ neutrosophic semi j-open sets in $\mathcal{X}$. Then $Ncl_{sj}[P] \cap Q = 0_N$. This implies P is not neutrosophic semi j-dense. We have P is neutrosophic semi j-open then $P \subseteq Ncl[Nint[Npcl[P]]]$ and P is not neutrosophic semi j-dense which is a contradiction to our assumption that $P \cap Q = 0_N$. Hence $P \cap Q \neq 0_N$. □

Theorem 6.9. *In a neutrosophic semi j-hyperconnected space, intersection of any two neutrosophic semi j-open sets are neutrosophic semi j-open.*

Proof. Let $0_N \neq P$, $0_N \neq Q$ be the two neutrosophic semi j-open sets in a neutrosophic semi j-hyperconnected space $\mathcal{X}$. Then $P \subseteq Ncl[Nint[Npcl[P]]]$ and $Q \subseteq Ncl[Nint[Npcl[P]]]$. We have $Ncl_{sj}[P] = 1_N$ and $Ncl_{sj} = 1_N$. This implies $Ncl[Nint[Npcl[P]]] = Ncl[Nint[Npcl[Q]]] = 1_N$, also we have $P \cap Q \neq 0_N$ using proposition 2.9. It follows that $P \cap Q \subseteq Ncl[Nint[Npcl[P]]] \cap Ncl[Nint[Npcl[Q]]] = Ncl[Nint[Npcl[P \cap Q]]] = 1_N$. Therefore $P \cap Q \subseteq Ncl[Nint[Npcl[P \cap Q]]] = 1_N$. Hence $P \cap Q$ is also neutrosophic semi j-open. □

7. Conclusion

The characteristics of neutrosophic semi j-open sets, neutrosophic semi j-closed sets, neutrosophic hyperconnectedness and neutrosophic semi j-hyperconnectedness are discussed in this paper. Nowadays neutrosophic sets have began to play a vital role by helping in the analysis of real life situations. In future, neutrosophic hyperconnected spaces will assist in determining solutions in each situations where indeterminancy occurs as the main crisis.

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Validating the Interval Valued Neutrosophic Soft Set Traffic Signal Control Model Using Delay Simulation

Endalkachew Teshome Ayele[1], Natesan Thillaigovindan [2],Said Broumi[3]

[1]Department of Mathematics,Arbaminch University,Arbaminch,Ethiopia; endalkachewteshome83@yahoo.com

[2]Department of Mathematics,Arbaminch University,Arbaminch,Ethiopia; thillagovindan.natesan@gmail.com

[3]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University of Hassan II, Casablanca, Morocco; broumisaid78@gmail.com

Abstract. Currently most signalized intersections in almost all developing countries use fixed time traffic controllers or pre-timed traffic lights. But as a real life situation, in addition to uncertainty and impreciseness there is also indeterminacy in traffic signal control constraints due to various factors like unawareness of the problem, inaccurate and imperfect data and poor forecasting in addition to uncertainty in the constraints. To overcome these interval valued neutrosophic soft set traffic signal control model at four way isolated signalized intersections has been developed. The main aim of this research is to validate the IVNSS traffic signal control model and compare it with fixed time traffic signal control model using MATLAB simulation tool. Vehicle delay at the junction is used as a measure of effectiveness. The simulation is conducted for seven consecutive days from Monday up to Sunday for eight hours to reflect the different traffic flow conditions. The simulated delay model results are analysed under 5 different scenarios. And results showed that in case of heavy traffic conditions vehicle delay under IVNSS traffic signal control model is minimized by 36 percent and under light traffic conditions the average vehicle delay is minimized by 73 percent when compared to fixed time traffic signal control model.

Keywords: Signal control- Delay- Simulation

1. INTRODUCTION

One of the major problems of both developed and developing countries is traffic congestion in urban road transportation systems. Urban traffic congestions lead to a lot of time consumption and exhaust emissions. So alleviating congestion will have a good impact on both economy and environment. The signal control at urban intersections is an effective and most important way to reduce the traffic jams and congestions. Traffic signals are signalized devices positioned at road intersections, pedestrian crossings and other such locations to control competing flow of traffic [1]. The purpose of traffic light signal control is to make the current intersection system more effective and efficient in improving traffic safety and minimizing congestion, maximizing the capacity of flow and minimizing the delay without building new roadways which is often impossible due to scarce or non-availability of land resource. The conflicts arising from movements of traffic in different directions are addressed by time sharing principle. The advantages of traffic signal include an orderly movement of traffic, an increased capacity of the intersection and require only simple geometric design. However the disadvantages of the signalized intersection are large stopped delays [2]. Traffic signal control is a measure that is commonly used at road intersections to minimize vehicular delays. In early days as well as at present, traffic is controlled by hand signs by traffic police officers or by signals and markings called the traditional traffic control systems. Researches have established that unless otherwise implemented properly the traditional traffic control system can contribute more to the congestion at intersections [3]. Currently most signalized intersections in almost all developing countries use fixed time traffic controllers or pre-timed traffic lights.The traffic lights change phase at a constant cycle time in fixed time traffic light controller, without taking into account the peak period or highly varying traffic intensity with respect to time. Pre-timed traffic light also causes traffic congestion as it is incapable of detecting traffic intensity at the junction and to allow the vehicles waiting in the lanes to cross the junction as per the urgency necessitated by the traffic conditions prevailing at that time. The present day traffic signal controller models suffer from indeterminacy due to various factors like unawareness of the problem, inaccurate and imperfect data and poor forecasting in addition to uncertainty in the constraints. To overcome these Endalkachew et al.[4] developed an interval valued neutrosophic soft set traffic signal control model for four way isolated signalized intersections. The main aim of this research is to validate the IVNSS traffic signal control model and compare it with fixed time traffic signal control model using simulation study. A MATLAB simulation model for the proposed IVNSS traffic control system is developed and the efficiency of the model is tested subject to random variation, the basic methods of generating random variables and simulating probabilistic systems are presented. The MATLAB simulation tool is utilized to compare the developed IVNSS traffic signal control model with fixed time control model for an isolated four way intersection at St. Stifanos traffic junction, Addis Ababa, Ethiopia using Webster delay model.

2. REVIEW OF LITERATURE

Literally simulation is an imitation of certain real events or a system. This technique involves building a mathematical model that sufficiently represents a given system and using a computer to imitate (simulate) the operations of the system. Basically, it is used to analyse the behaviour of the system or to estimate its performance under various circumstances in order to find ways to improve the functioning of the system. There are several criteria named MOEs (Measures of Effectiveness); delay, level of service (LoS), average queue length, max queue length, number of stops and vehicle through put that can be used to compare the proposed IVNSS traffic signal control model with the widely used pre-timed control. But in this research we use vehicle delay at the junction as a measure of effectiveness. Vehicle delay is the most important parameter used by transportation professionals in evaluating the performance of a signalized intersection. However delay is a parameter that is not easily determined due to the non-deterministic nature of the arrival and departure processes at the intersection [10]. But lot of research has been done in this field to define delay by a number of simulated delay models. Broumi Said et al.[23] reviewed some available mathematical techniques for traffic flow using rough set, fuzzy rough set, and its extension with the neutrosophic set to solve the traffic problem and found that the rough set theory can be useful for dealing the uncertain, incomplete, and indeterminate data set. Hence, the hybridization of the neutrosophic set and rough can be considered one of the efficient tools for intelligent traffic control and its approximation via automatic red, green and yellow lights. Recommended that the proposed study will be helpful for several researchers working on traffic flow, traffic accident diagnosis, and its hybridization as future research. Arshad Jamal et al. [11] developed meta-heuristic-based methods for intelligent traffic control at isolated signalized intersections, in the city of Dhahran, Saudi Arabia to optimize delay. Genetic algorithm (GA) and differential evolution (DE) were employed to enhance the intersection's level of service (LOS) by optimizing the signal timings plan. The study results indicated that both GA and DE produced a systematic signal timings plan and significantly reduced travel time delay ranging from 15 to 35 percent compared to existing conditions. Although DE converges much faster to the objective function, GA outperforms DE in terms of solution quality i.e., minimum vehicle delay. To validate the performance of proposed methods, cycle length-delay curves from GA and DE were compared with optimization outputs from TRANSYT 7F, a state-of-the-art traffic signal simulation. Nilesh Bhosale et al. [12] compared analysis of the previously developed methodology and results of delay caused due to pre-timed two way signal coordination with least time pollution and environmental pollutions. They developed suitable methodology and simulation techniques for coordination to reduce the time pollution as well as improve the traffic efficiency and concluded that coordination of signal plays a vital role to abate congestion, reduces travel time as well as waiting time at signalized intersections. The phase difference plan method is best suited for signal coordination as these results in minimal delay in overall average travel time. Zhenyu Mei et al.[13] presented the findings of a simulation study evaluating the potential benefits of implementing transit signal priority (TSP) combined with arterial signal coordination for an isolated intersection. Simulation analysis reveals the effect of TSP strategies with flow variation on the optimal cycle, and also identifies a reasonable method for selecting the gap time and initial green time of the priority phase. The volume influences both the gap time and initial green time of the TSP phase. Moreover, the efficiency of red truncation is slightly better than that of the green time extension strategy. Theresa Thuniga et al.[17] provided an open-source implementation of a decentralized, adaptive signal control algorithm in the agent-based transport simulation MATSim, which is applicable for large-scale real-world scenarios. The algorithm is extended in this paper to cope with realistic situations like different lanes per signal, small periods of overload, phase combination of non-conflicting traffic, and minimum green times. Impacts and limitations of the adaptive signal control are analysed for a real-world scenario and compared to a fixed-time and traffic-actuated signal control. Another finding is that the adaptive signal control behaves like a fixed-time control in overload situations and, therefore, ensures system with stability. Nada B et al.[14] presented a new method of developing an optimal real-time traffic signal controller using the fuzzy logic method (FLM), taking into consideration all various incoming traffic flows. The developed FLM was designed for an isolated intersection with four legs, split phasing, and three different movements (through, right, and left). Calibration and validation tests were conducted to ensure accuracy and efficiency of the developed model. Results show that using the developed FLM for controlling traffic signals with optimized conditions is promising as it provides optimal solution for all different traffic flow combinations, during all model development stages, including the simulation, calibration and the validation process. Ardavan Shojaeyan [15] carried out the design of efficient phase optimization technique using developed phase plan. CG Road was identified as a troubled corridor during reconnaissance survey and as such, selected for study. Data on geometric features were collected by Field survey using Odometer as well as with Google Earth Software. Peak and off peak hour traffic volume data were collected using ultra high resolution full HD camera. Furthermore signal cycle timing, space mean speed, discharge head way were simultaneously collected by trained enumerator's at all three intersections. Data extraction was carried out on projector screen using updated VLC media player. The geometric and traffic data collected were analyzed with Microsoft Excel. Three different Phase Optimization Technique(POT) is tested on real traffic signal data of corridor in forward and backward direction using Time Space Diagram. With change of phase plan and phase sequence POT 1 is successful in minimizing combined delay of corridor up to 28.05 percent to 76.04 percent for all 4 forward movements for analyzed two cycles. Further improvement in POT 2 is achieved by introducing 10 second offset at intersection B which reduces combined delay up to 32.52 percent to 98.6 percent in all 4 forward movements. Tracking average travel time, demand supply and prevailing signal cycle time POT 3 is applied with equal signal cycle length of 104 second at all 3 intersections. D.Nagarajan [16] analysed traffic flow control under neutrosophic environment using MATLAB. Triangular and Trapezoidal Fuzzy numbers were used. Traffic flow management has been analyzed with respect to various ranges of indeterminacy under neutrosophic environment using MATLAB program. They also compared the traffic control management for crisp sets, fuzzy and neutrosophic sets. Chandan. Ka [18] proposed a connected vehicle signal control (CVSC) strategy for an isolated intersection, which utilizes detailed information, including speeds and positions of GPS equipped vehicles on each approach at every second. The proposed strategy first aims at dispersing any queue that is built up during the red interval, and then starts minimizing the difference between cumulative arrival flow and cumulative departure flow on all approaches of the intersection. Various traffic scenarios with 100 percent GPS market penetration rate were tested in the VISSIM 8 microscopic simulation tool. Results have established that the proposed CVSC strategy showed outstanding performance in reducing travel time delays and average number of stops per vehicle when compared to the EPICS adaptive control. D.Nagarajan, et al.

[19] studied a triangular interval type-2 Schweizer and Sklar weighted arithmetic (TIT2SSWA) operator and a triangular interval type-2 Schweizer and Sklar weighted geometric (TIT2SSWG) operator based on Schweizer and Sklar triangular norms. Moreover, they proposed an improved score function for interval neutrosophic numbers (INNs) to control traffic flow that has been analyzed by identifying the junction where the traffic intensity is more. D. Nagarajan et al. [20] analysed traffic flow management with respect to various ranges of indeterminacy under neutrosophic environment using Gauss Jordan method with the support of MATLAB program. As seen from the above, traffic signal control models have been developed by a number of researchers under neutrosophic environment but no one has studied its efficiency and compared it with other existing models.

3. PRELIMINARY CONCEPTS

In this section we present the necessary preliminary ideas and some basic results needed for the present research work. We start from the definition of a neutrosophic set.

3.1. *Single valued neutrosophic set [21]*

Let X be the universal set.A neutrosophic set A in X is characterized by a truth membership function μ_A, an indeterminacy membership function υ_A and a falsity membership function ω_A,where $\mu_A, \upsilon_A, \omega_A : X \to [0,1]$ are functions and $\forall x \in X$,$x \equiv x(\mu_x, \upsilon_x, \omega_x) \in A$ is a single valued neutrosophic element of A.
A single valued neutrosophic set A(SVNS in short) over a finite universe $X = \{x_1, x_2, ..., x_n\}$ can be represented as $A = \sum_{i=1}^{n} < \mu_A(x_i), \upsilon_A(x_i), \omega_A(x_i) > /x_i$.
The three membership functions form the fundamental concepts of neutrosophic set and they are independently and explicitly quantified subject to the following conditions.

$$0 \leq \mu_A(x), \upsilon_A(x), \omega_A(x) \leq 1 \text{ and}$$
$$0 \leq \mu_A(x) + \upsilon_A(x) + \omega_A(x) \leq 3 \ \forall x \in X.$$

3.2. *Union and Intersection of SVNS* [21]

Let A and B be two SVNS defined on a common universe X .Then the union of A and B ,written as $A \cup B = C$ is defined by

$$\mu_C(x) = max(\mu_A(x), \mu_B(x))$$
$$\upsilon_C(x) = max(\upsilon_A(x), \upsilon_B(x))\text{and}$$
$$\omega_C(x) = min(\omega_A(x), \omega_B(x))\forall x \in X.$$

The intersection of A and B ,denoted by $A \cap B = C$ is defined by

$$\mu_C(x) = min(\mu_A(x), \mu_B(x)),$$
$$\upsilon_C(x) = min(\upsilon_A(x), \upsilon_B(x)) \text{ and}$$
$$\omega_C(x) = max(\omega_A(x), \omega_B(x))\forall x \in X.$$

3.3. *Interval Valued Neutrosophic Set* [22]

For an arbitrary sub interval set A of $[0,1]$ we define $\underline{A}$ = inf of A and $\overline{A}$ = sup of A.
Let X be the universal set.An interval valued neutrosophic set A in X is characterized by a truth membership function μ_A , an indeterminacy membership function υ_A and a falsity membership function ω_A for each element $x \in X$ where

$$\mu_A(x) = [\underline{\mu}_A(x), \overline{\mu}_A(x)],\ \upsilon_A(x) = [\underline{\upsilon}_A(x), \overline{\upsilon}_A(x)],\ \omega_A(x) = [\underline{\omega}_A(x), \overline{\omega}_A(x)] \text{ are closed sub-intervals of } [0,1].$$

Thus $A = < \mu_A(x), \upsilon_A(x), \omega_A(x) > /x; x \in X$.

3.4. *Union and Intersection of IVNS* [22]

let A and B be two IVNS defined over a common universe X.The union of A and B denoted by$A\tilde{\cup}B$ is defined as

$$\begin{gathered} A\tilde{\cup}B = \{< [max(\underline{\mu}_A(x), \underline{\mu}_B(x)), max(\overline{\mu}_A(x), \overline{\mu}_B(x))], \\ [max(\underline{\upsilon}_A(x), \underline{\upsilon}_B(x)), max(\overline{\upsilon}_A(x), \overline{\upsilon}_B(x))], \\ [min(\underline{\omega}_A(x), \underline{\omega}_B(x), min(\overline{\omega}_A(x), \overline{\omega}_B(x))] > /x; \forall x \in X\} \end{gathered}$$

Similarly the intersection of A and B denoted by $A\tilde{\cap}B$ is defined by

$$\begin{gathered} A\tilde{\cap}B = \{< [min(\underline{\mu}_A(x), \underline{\mu}_B(x)), min(\overline{\mu}_A(x), \overline{\mu}_B(x))], \\ [min(\underline{\upsilon}_A(x), \underline{\upsilon}_B(x)), min(\overline{\upsilon}_A(x), \overline{\upsilon}_B(x))], \\ [max(\underline{\omega}_A(x), \underline{\omega}_B(x), max(\overline{\omega}_A(x), \overline{\omega}_B(x))] > /x; \forall x \in X\} \end{gathered}$$

Traffic flow is usually interrupted by traffic signals and stop signs. These controls have different impacts on overall flow. The operational state of traffic at an interrupted traffic-flow facility is defined by the following measures [5].Classified vehicle count or traffic volume, directional movement of vehicles, queues (saturation flow rate), signal timing and phasing data and delay.

3.5. *Traffic flow at signal junction*

3.5.1. *Traffic Volume Count*

Volume is the total number of vehicles that pass over a given point or section of a lane or road way during a given time interval; it can be expressed in terms of annual, daily, hourly, or sub hourly periods. Traffic volume count for Directional movement of each vehicle (Through, Right turn and Left turn movements) is conducted to determine the number, movements and classifications of roadway movements at a given location. These data can help identify critical flow time periods. The length of the sampling period depends on the type of count being taken and the intended use of the data recorded. For example, an intersection count may be conducted during the peak flow period. Manual count with 15-minute intervals could be used to obtain the traffic volume data.

Basically, the traffic volume and saturation flow data are collected through traffic sensors installed at the junction or video graphic record or through manual count.

3.5.2. *Capacity*

Capacity is an adjustment of the saturation flow rate that takes the real signal timing into account, since most signals are not allowed to permit continuous movement of one phase for an hour. If the approach has 30 minutes of green per hour, it can be deduce that the actual capacity of the approach is about half of the saturation flow rate. The capacity, therefore, is the maximum hourly flow of vehicles that can be discharged through the intersection from the lane group in question under the prevailing traffic, roadway, and signalization conditions. The formula for calculating capacity (c) is given below.

$c = (g/C) \times s$

where:

c=capacity(vehicle per hour)

g = Effective green time for the phase in question (seconds)

C = Cycle length (seconds)

s = Saturation flow rate (vehicle per hour per green)

3.6. *Delay at signalized intersection*

To give a clear description and to understand traffic flow conditions at an individual intersection the following performance measures are being applied: delay, level of service (LoS), average queue length, max queue length, number of stops and vehicle throughput. The reasons for determining these parameters are as follows. Delay and LoS play a primary role in determining individual intersection performance. LoS can be used to understand the quality of traffic conditions on a particular intersection and delay exposes the difference between free-flow and congested traffic conditions. Frequent stops due to congestion are a typical characteristic of urban traffic. One of the reasons for this is the presence of signalized intersections. Therefore, the information of queue length and number of stops must be included as performance measure also. The vehicle throughput can provide useful information about the maximum number of vehicles which can be discharged during the time. In addition, for an in-depth analysis of the arterial section in the analyzed urban traffic network, travel time; delay and number of stops are also useful.

Vehicle delay is the most important parameter used by transportation professionals in evaluating the performance of a signalized intersection. This is perhaps because it directly relates to the time loss that a vehicle experiences while crossing an intersection .However delay is a parameter that is not easily determined due to the non deterministic nature of the arrival and departure processes at the intersection. But lot of research has been done in this field to define delay by a number of analytical delay models. But the most popular and commonly used delay model is the Webster delay model.

3.7. *Delay Components*

In analytic models for predicting delay, there are three distinct components of delay, namely, uniform delay, random delay, and overflow delay.

3.7.1. *Uniform delay*

Uniform delay is the delay based on an assumption of uniform arrivals and stable flow with no individual cycle failures. No signal cycle fails here, i.e., no vehicles are forced to wait for more than one green phase to be discharged. This type of delay is known as Uniform delay where uniform vehicle arrival is assumed.

3.7.2. *Random Delay*

Random delay is the additional delay, above and beyond uniform delay, because flow is randomly distributed rather than uniform at isolated intersections. This case represents a situation in which the overall period of analysis is stable (i.e., total demand does not exceed total capacity). Individual cycle failures within the period, however, have occurred. For these periods, there is a second component of delay in addition to uniform delay.This type of delay is referred to as Random delay.

3.7.3. *Overflow Delay*

Overflow delay is the additional delay that occurs when the capacity of an individual phase or series of phases is less than the demand or arrival flow rate. Actual vehicle arrivals vary in a random manner [72] and this randomness causes overflows in some signal cycles. If this persists for a long time period then the over-saturated conditions lead to continuous overflow delay. The effect of the overflow depends on the degree of saturation over a given time period. This is the case at which demand exceeds capacity (v/c $>$ 1.0).This type of delay is referred to as Overflow delay

3.8. *Webster's Delay Models*

3.8.1. *Uniform Delay Model*

Model is explained based on the assumptions of stable flow and a simple uniform arrival function. Thus, Webster's model [70] for uniform delay (UD) is given as

$$UD = \frac{C(1-\frac{g}{C})^2}{2(1-\frac{v}{s})}$$

Another form of the equation uses the capacity, c, rather than the saturation flow rate, s.
$s = \frac{c}{\frac{g}{C}}$ so, the relation for uniform delay changes to,

$$UD = \frac{C(1-\frac{g}{C})^2}{2(1-\frac{g}{C}\frac{v}{c})}$$

$$UD = \frac{C(1-\frac{g}{C})^2}{2(1-\frac{g}{C}X)}$$

where, UD is the uniform delay (sec/vehicle) C is the cycle length (sec), c is the capacity, v is the vehicle arrival rate (vehicle per hour), s is the saturation flow rate or departing rate of vehicles (vehicle per hour green), X is the v/c ratio or degree of saturation (ratio of the demand flow rate to saturation flow rate), and g/C is the effective green ratio for the approach.

3.8.2. *Random Delay Model*

The uniform delay model assumes that arrivals are uniform and that no signal phases fail (i.e., that arrival flow is less than capacity during every signal cycle of the analysis period). At isolated intersections, vehicle arrivals are more likely to be random. A number of stochastic models have been developed for this case, including those by Newell, Miller and Webster. These models generally assume that arrivals are Poisson distributed, with an underlying average rate of v vehicles per unit time. The models account for random arrivals and the fact that some individual cycles within a demand period with v/c $<$ 1.0 could fail due to this randomness. This additional delay is often referred to as Random delay. The most frequently used model for random delay is Webster's formulation:

$$RD = \frac{X^2}{2v(1-X)}$$

Where, RD is the average random delay second per vehicle, and X is the degree of saturation (v/c ratio).

Webster found that the above delay formula overestimate delay and hence he proposed that total delay is the sum of uniform delay and random delay multiplied by a constant for agreement with field observed values. Accordingly, the total delay is given as:

D=0.9(UD+RD)

3.8.3. *Overflow delay model*

Model is explained based on the assumption that arrival function is uniform. In this model a new term called over saturation is used to describe the extended time periods during which arrival vehicles exceeds the capacity of the intersection approach to the discharged vehicles. In such cases queue grows and there will be overflow delay in addition to the uniform delay. This is the case where v/c $>$1.0.

During the period of over-saturation delay consists of both uniform delay and overflow delay. As the maximum value of X is 1.0 for uniform delay, it can be simplified as [72],

$$UD = \frac{C(1-\frac{g}{C})^2}{2(1-\frac{g}{C}X)} = \frac{C}{2}(1-\frac{g}{C})$$

Average delay is obtained by dividing the aggregate delay by the number of vehicles discharged within the time T which is cT. T is the analysis period in seconds.

$$OD = \frac{T}{2}(\frac{v}{c}-1)$$

4. DELAY SIMULATION

4.1. *Data Requirement for the Simulation Model*

Even though there are many intersections facing with traffic congestion in Addis Ababa, due to time and budget constraints and accessibility of relevant traffic data, it is difficult to cover all such intersections in the city. So that St.Stifanos traffic junction is selected for this study.This intersection is located in front of St. Stifanos church in Kirkos sub-city at Meskel intersection and considered as the most congested traffic junction by the road users. St.Stifanos traffic junction is one of the largest intersections in Addis Ababa with its heavy traffic congestion especially in the morning, mid-day and evening peak hours due to poor signal controlling system. The geometry of the intersection is presented in Figure 1 below and the aerial shoot of Meskel square intersection is shown in Figure 2.

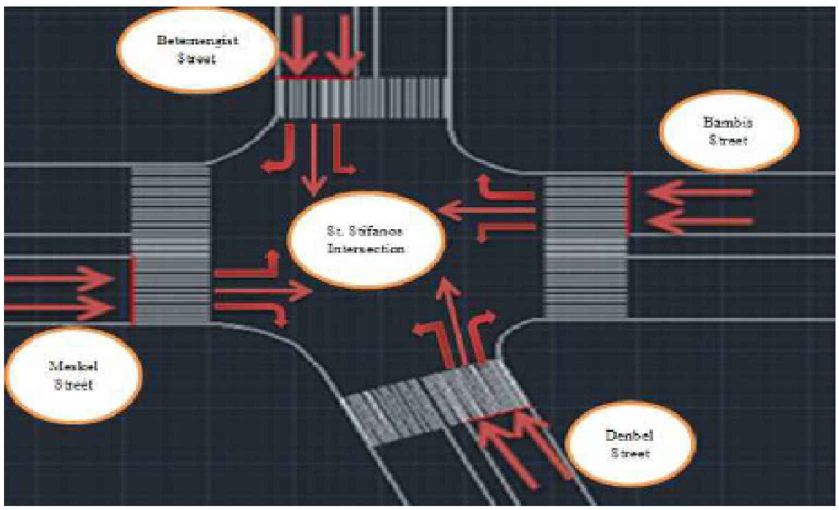

FIGURE 1. Geometric representation of the study area

4.2. *Steps for simulation*

Step 1: Obtain the input parameters (average arrival rate and average saturation flow rate (queue) for each approach per day. And use this data to simulate the vehicle arrivals and saturation flow rates using Poisson distribution.

Step 2: As categorizing arrival rates and saturation flow rates (queues) in to different interval valued neutrosophic soft sets follow uniform distribution, simulate the corresponding interval valued neutrosophic soft sets using uniform distribution using the flow rates and saturation flow rates (queues) obtained in step one. The interval valued neutrosophic data are simulated for 5 different scenarios and the average value of all the simulated values is taken as input parameter for determining the green time and cycle length for each approach. The proposed random number generation plan is simulated by using the MATLAB simulation tool. This gives the IVNSS 'A' for vehicle arrivals and IVNSS 'B' for saturation flow rate (queue).

FIGURE 2. Aerial shoot of Meskel square intersection

Step 3: Combine the two interval valued neutrosophic soft sets A and B to get a resultant interval valued neutrosophic soft set say 'C'.

Step 4: By defining an AVG-threshold value, reduce the parameters.

Step 5:

- Determine the row index for each signal group
- Row index represents the weight of the signal group.
- The first row index represents the weight of the signal group SG1.
- The second row index represents the weight of the signal group SG2.
- The third row index represents the weight of the signal group SG3 and
- The fourth row index represents the weight of the signal group SG4,
- Row index assigns value 1 if the row satisfies the given threshold value and 0 otherwise.

Step 6: Obtain the total weight values which is the sum of the weights of the signal groups with respect to the parameter e_ij which is the choice value for the signal groups.

Step 7: Select (choose) the indices (corresponding signal group) with maximum weight.

Step 8: Determine the total cycle time and the green time using the weight value obtained in Step 6.

Step 9: A MATLAB simulation is carried out for the proposed IVNSS traffic signal system to validate the model and test the efficiency of the model with respect to vehicle delay in which the variables involved are subject to random variation, we present the basic methods of generating random variables and simulating probabilistic systems.

4.3. *Assumptions*

The simulation is based on the following assumptions:

• The input parameters used for the simulation (average arrival rates and average saturation flow rates are based on the data obtained from [8].

• Maximum green time is 95 seconds and minimum green time is 27 seconds at St.Stifanos isolated traffic junction.

• The traffic movement is right, left and through.

• The yellow (amber) signal for all phases at each intersection is included in green signal and its duration is 3 seconds.

• No right turn on red and

• No pedestrian demand

• The intersection has four phases.

4.4. *Delay Simulation Analysis*

The IVNSS traffic signal control model validation is carried out comparing the results with fixed time traffic signal using Webster delay formulas. Generally for the simulation purpose the average traffic volume count (arrival rate) and average saturation flow rate (queue) are obtained by generating randomly using Poisson distribution. But in this research, the average traffic volume count (arrival rate) and average saturation flow rate (queue) are obtained directly from [8] which are used as input parameters for the simulation and the data was obtained from St. Stifanos intersection with Bambis in the East, Betemengist in the North, Dembel in the South and Meskel Square in the West as shown in Figure 1. According to the data obtained from [8], the traffic volume count was made for 8 hours per day for one week starting from the morning 7:30 AM to the evening 7:30 PM at 15 minutes interval as shown in appendix (A). This is done 3 hours in the morning (7:30-10:30) AM, 2 hours in the midday (12:00- 2:00) PM and 3 hours in the evening (4:30-7:30) PM. The traffic flow count is categorized in to two groups, the first count was made from Monday to Friday at which there is a heavy traffic flow condition and there is traffic saturations and the second count was made on Saturday and Sunday at which there is light traffic conditions or no traffic saturations. The average traffic flow rate, saturation flow rate (queue) and average vehicle delay for fixed time signal control per day for each approach for a week is shown in the table 1 and table 2 below [8].

Table 1. The average vehicle delay under fixed time per day for all intersection legs.(Monday-Friday)

Approach Leg	PH	Volume	FR	G/C	GT	CT	SFR	Capacity	V/c	Delay
Bambis	Morning	115	460	0.28	75	271	1377	381	1.2	188
Bambis	Mid-Day	85	340	0.26	55	214	1023	263	1.3	214
Bambis	Evening	114	456	0.29	76	263	1371	396	1.15	172
Dembel	Morning	173	692	0.36	108	301	2080	746	0.93	176.8
Dembel	Mid-Day	131	524	0.35	87	251	1577	546.6	0.96	148.8
Dembel	Evening	171	684	0.36	108	297	2058	748.4	0.91	173.7
Betemengist	Morning	88	352	0.2	55	265	1051	218.1	1.6	376
Betemengist	Mid-Day	66	264	0.16	38	231	794	130.6	2.01	552
Betemengist	Evening	85	340	0.2	52	257	1015	205.4	1.66	400
Meskel Sq.	Morning	111	444	0.27	72	262	1328	365	1.22	195
Meskel Sq	Mid-Day	82	328	0.23	51	219	981	228.5	1.44	278
Meskel Sq	Evening	121	484	0.29	81	277	1456	425.8	1.14	161

Table 2. The average vehicle delay under fixed time control per day for all intersection legs. (Saturday and Sunday)

Approach Leg	PH	Volume	FR	GT	CT	g/C	SFR	Capacity	V/c	Delay
Bambis	Morning	56	224	35	166	0.21	669	141	1.6	336
Bambis	Mid-Day	41	164	28	144	0.19	490	95.3	1.7	373
Bambis	Evening	52	208	34	160	0.21	621	132	1.6	333
Dembel	Morning	59	236	38	161	0.24	705	166.4	1.4	241
Dembel	Mid-Day	45	185	31	140	0.22	536	118.7	1.6	325
Dembel	Evening	60	240	39	159	0.25	715	175.4	1.4	240
Betemengist	Morning	33	132	23	157	0.15	395	57.9	2.3	652
Betemengist	Mid-Day	27	108	19	153	0.12	325	40.4	2.7	832
Betemengist	Evening	31	124	22	158	0.14	377	52.5	2.4	698
Meskel Sq.	Morning	53	212	34	164	0.2	631	130.8	1.6	336
Meskel Sq	Mid-Day	34	136	24	145	0.17	411	68	2	510
Meskel Sq	Evening	48	152	33	159	0.2	582	120.8	1.25	177

The St. Stifanos traffic signal junction is classified into four different signal groups in order to fit the developed IVNSS traffic signal control design based on the average traffic flow data obtained from St.Stifanos junction for different peak hour flow rates and peak hour saturation flow rates (queue). An IVNSS traffic signal control model is developed and simulated to estimate the signal phase, cycle length

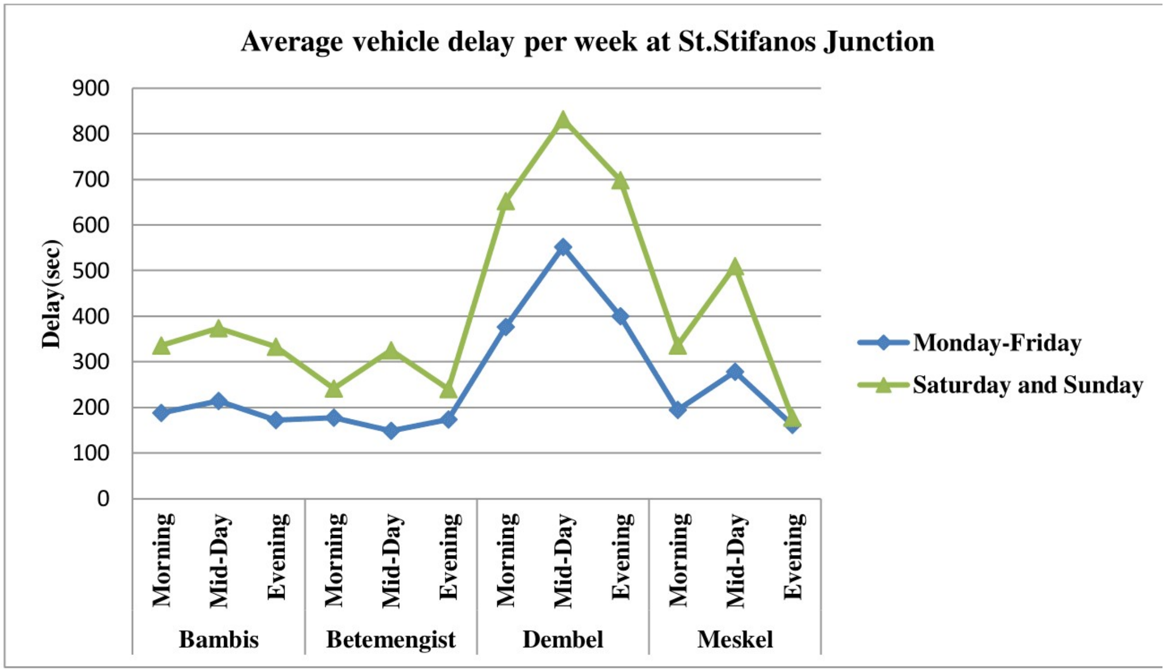

Figure 3. Weekly vehicle delay time at St. Stifanos intersection per approach under FTC

and the green time duration of the isolated intersection making use of the arrival rate and saturation flow rate (queue) at the downstreams of the intersection. The model developed is made to run for 5 different simulation scenarios, the average interval valued neutrosophic data (matrix) is extracted. The arrival rates and saturation flow rates (queue) are first simulated into interval valued neutrosophic data. The outputs of these simulation runs are then used to extract two outputs; namely, the effective green time of each signal group and the optimal cycle length. The values of cycle length and green time for different scenarios are tabulated in tables 3, 4, 5 and 6 below considering the different traffic flow conditions.

The average effective green times of each signal group and the average cycle length for the different scenarios are then used to estimate the average vehicle delay of each approach and the average vehicle delay at the junction. The results are shown in tables 7 and 8.

Table 3. Different cycle time scenarios at St.Stifanos intersection (Monday-Friday)

Phases/SG	PH	Scenario1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Average
1{maximum (EL1, WL1)}	Morning	217.4	378.3	231	231	308.7	273.3
1{maximum (EL1, WL1)}	Mid-Day	261.2	219.7	276.3	265	139	232.2
1{maximum (EL1, WL1)}	Evening	149.4	238.6	231	452	208.3	255.9
2{maximum (EL2, WL2)}	Morning	217.4	378.3	231	231	308.7	273.3
2{maximum (EL2, WL2)}	Mid-Day	261.2	219.7	276.3	265	139	232.2
2{maximum (EL2, WL2)}	Evening	149.4	238.6	231	452	208.3	255.9
3{maximum (NL1, SL1)}	Morning	217.4	378.3	231	231	308.7	273.3
3{maximum (NL1, SL1)}	Mid-Day	261.2	219.7	276.3	265	139	232.2
3{maximum (NL1, SL1)}	Evening	149.4	238.6	231	452	208.3	255.9
4{maximum (NL2, SL2)}	Morning	217.4	378.3	231	231	308.7	273.3
4{maximum (NL2, SL2)}	Mid-Day	261.2	219.7	276.3	265	139	232.2
4{maximum (NL2, SL1)}	Evening	149.4	238.6	231	452	208.3	255.9

Table 4. Different cycle time scenarios at St.Stifanos intersection (Saturday and Sunday)

Phases/SG	PH	Scenario1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Average
1	Morning	355.7	231	435	256.5	202.7	296.2
1	Mid-Day	185.7	269.9	339.8	486	187.7	293.8
1	Evening	181.1	684.3	185.7	179	333	312.6
2	Morning	355.7	231	435	256.5	202.7	296.2
2	Mid-Day	185.7	269.9	339.8	486	187.7	293.8
2	Evening	181.1	684.3	185.7	179	333	312.6
3	Morning	355.7	231	435	256.5	202.7	296.2
3	Mid-Day	185.7	269.9	339.8	486	187.7	293.8
3	Evening	181.1	684.3	185.7	179	333	312.6
4	Morning	355.7	231	435	256.5	202.7	296.2
4	Mid-Day	185.7	269.9	339.8	486	187.7	293.8
4	Evening	181.1	684.3	185.7	179	333	312.6

Table 5. Different green time scenarios at St.Stifanos intersection (Monday-Friday)

Phases/SG	PH	Scenario1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Average
1	Morning	21	122.7	81.5	44.7	74.8	68.9
1	Mid-Day	50.8	67	67.2	54.4	19.1	51.7
1	Evening	90	29.8	39.8	87.5	23.1	54
2	Morning	91.2	71.6	74.7	29.8	102.9	74
2	Mid-Day	36.3	55	97.1	88.3	28.8	61.1
2	Evening	30	44.7	39.8	145.8	63.7	64.8
3	Morning	63	122.7	6.8	96.9	37.4	65.4
3	Mid-Day	72.6	85.4	29.9	27.2	4.8	44
3	Evening	15	96.9	103.6	102.1	92.6	82
4	Morning	42.1	61.4	67.9	59.6	93.5	65
4	Mid-Day	101.6	12	82.2	95.1	86.3	75.4
4	Evening	15	67.1	47.8	116.6	28.9	55

TABLE 6. Different green time scenarios at St.Stifanos intersection (Saturday and Sunday)

Phases/SG	PH	Scenario1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Average
1	Morning	122	74.5	105	33	47.7	76.4
1	Mid-Day	92.8	108	121.4	132.5	31.3	97.2
1	Evening	90.6	203.5	83.2	26.9	70.6	95
2	Morning	112	67	135	66.2	17.9	111.8
2	Mid-Day	23.2	63	72.8	88.4	37.5	57
2	Evening	23.8	166.5	44.8	58.2	80.7	74.8
3	Morning	61	7.5	75	58	477	135.7
3	Mid-Day	34.8	54	85	147.3	25	131.9
3	Evening	19	166.5	6.4	9	121	64.4
4	Morning	61	82	120	99.3	89.4	90.3
4	Mid-Day	34.8	45	60.7	117.8	93.9	70.4
4	Evening	47.7	148	51.2	85	60.5	78.5

TABLE 7. The average signal timings and delay results for IVNSS traffic signal control model (Monday-Friday)

Phase/SG	PH	Volume	Flow rate	SFR	Capacity	V/c	CT	GT	G/C	Delay
1	Morning	112	448	2080	524	0.85	273.3	68.9	0.25	146.8
1	Mid-Day	85	340	1577	351	0.97	232.2	51.7	0.22	126.6
1	Evening	111	444	2058	434.3	1.00	255.9	54	0.21	140.3
2	Morning	61	244	2080	563.2	0.43	273.3	74	0.27	129.1
2	Mid-Day	46	184	1577	415	0.44	232.2	61.1	0.26	110.1
2	Evening	60	240	2058	521	0.46	255.9	64.8	0.25	122
3	Morning	109	436	1377	329.5	1.3	273.3	65.4	0.24	239
3	Mid-Day	81	324	1023	194	1.67	232.2	44	0.19	396
3	Evening	108	432	1456	466.6	0.93	255.9	82	0.32	146.9
4	Morning	17	68	1377	327.5	0.21	273.3	65	0.24	122
4	Mid-Day	12	48	1023	332	0.14	232.2	75.4	0.32	98
4	Evening	18	72	1456	313	0.23	255.9	55	0.22	115.5

5. Results and Discussion

From tables 1 and 7, one can see that from Monday-Friday, the average vehicle delay at the junction in the morning under fixed traffic control is ((188+177+276+195))/4= 209 whereas the average vehicle delay at the junction in the morning under IVNSS model is (147+129+239+122)/4=159.

From tables 1 and 7, one can see that from Monday-Friday, the average vehicle delay at the junction in the mid -day under fixed traffic control is ((214+149+552+278))/4= 298 whereas the average vehicle delay at the junction in the mid-day under IVNSS model is (127+110.1+396.4+98)/4=183.

From tables 1 and 7, one can see that from Monday-Friday, the average vehicle delay at the junction in the evening under fixed traffic control is ((172+174+400+161))/4= 227 whereas the average vehicle delay at the junction in the evening IVNSS model is (140.3+122+146.9+115.5)/4=131.

From tables 2 and 8, one can see that in Saturday and Sunday, the average vehicle delay at the junction

TABLE 8. The average signal timings and delay results for IVNSS traffic signal control model (Saturday and Sunday)

Phase/SG	PH	Volume	Flow rate	SFR	Capacity	V/c	Cycle time	Green time	G/C	Delay
1	Morning	38	152	705	181.8	0.84	296.2	76.4	0.26	158.6
1	Mid-Day	29	116	536	177	0.65	293.8	97.2	0.33	150.3
1	Evening	39	156	715	217.3	0.72	312.6	95	0.3	163.3
2	Morning	21	84	705	266	0.32	296.2	111.8	0.38	129.8
2	Mid-Day	16	64	536	104	0.62	293.8	57	0.19	144.5
2	Evening	21	84	715	171	0.49	312.6	74.8	0.24	150.3
3	Morning	53	212	669	306.5	0.69	296.2	135.7	0.46	154.2
3	Mid-Day	39	156	4907	220	0.70	293.8	131.9	0.45	154.9
3	Evening	49	196	621	128	1.53	312.6	64.4	0.2	364
4	Morning	8	32	669	204	0.16	296.2	90.3	0.3	127
4	Mid-Day	5	20	490	117.5	0.17	293.8	70.4	0.24	130
4	Evening	7	28	621	156	0.18	312.6	78.5	0.25	138

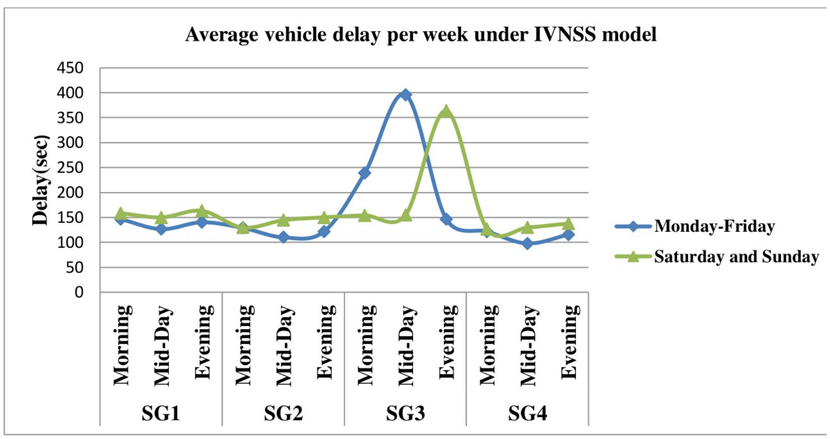

FIGURE 4. Weekly vehicle delay time at St. Stifanos intersection per approach under IVNSS model

in the morning under fixed traffic control is ((336+241+652+336))/4= 391 whereas the average vehicle delay at the junction in the morning under IVNSS model is (158.6+129.8+154.2+127)/4=142.
From tables 2 and 8, one can see that that in Saturday and Sunday, the average vehicle delay at the junction in the mid -day under fixed traffic control is ((373+325+832+510))/4= 510 whereas the average vehicle delay at the junction in the mid-day under IVNSS model is ((150.3+144.5+154.9+130))/4= 145.
From tables 2 and 8, one can see that that in Saturday and Sunday, the average vehicle delay at the junction in the evening under fixed traffic control is ((333+240+698+177))/4= 362 whereas the average vehicle delay at the junction in the evening under IVNSS model is (163.3+150.3+364+138)/4=204.
Comparison of the summarized average vehicle delay estimations for different flow rates and saturation flow rates is given in Table 9. As can be seen from the above discussion, from Monday-Friday the average vehicle delay at St.Estifanos traffic intersection per day is 244.6 sec/vehicle under FTC and 157.6 sec/vehicle under IVNSS traffic control model where as in Saturday and Sunday the average

Table 9. Summery of the average vehicle delay at St.Estifanos traffic junction per day.

Junction	Day	PH	Delay(FTC)	Delay(IVNSS)
St.Estifanos	Monday-Friday	Morning	209	159
St.Estifanos	Monday-Friday	Mid-day	298	183
St.Estifanos	Monday-Friday	Evening	227	131
St.Estifanos	Saturday and Sunday	Morning	391	142
St.Estifanos	Saturday and Sunday	Mid-day	510	145
St.Estifanos	Saturday and Sunday	Evening	362	204

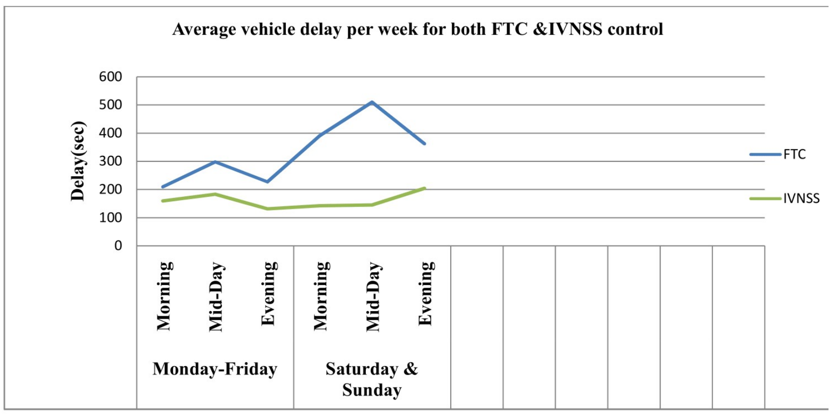

Figure 5. Weekly vehicle delay time at St. Stifanos intersection per approach under both FTC and IVNSS model

vehicle delay at St.Estifanos traffic intersection is 421 sec/vehicle under FTC and 163.6 sec/vehicle under IVNSS traffic signal control model. From Monday up to Friday under IVNSS traffic signal control model the average vehicle delay at the junction is reduced by 36 percent and on Saturday and Sunday the average vehicle delay is reduced by 73 percent under IVNSS control model.

6. CONCLUSION

A comparative study of the IVNSS traffic signal control with the existing fixed time traffic control shows that IVNSS traffic signal control model gives better performance in terms of delay both from Monday up to Friday at which heavy traffic (traffic saturation) is experienced and in Saturday and Sunday (holidays).in which it is expected that there is no heavy traffic conditions. Thus IVNSS model performs better than FTC, especially for both high and low traffic volumes and for both unsaturated and saturated traffic conditions. Under fixed time signal control even if the flow of traffic on Saturday and Sunday is low ,the average delay per vehicle is very high this is due to the geometric design of the junction because left-turning movements at signalized intersections are not only difficult to accommodate but also often cause accidents. Such problems can be reduced by adopting an exclusive left-turn signal phase [9].

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An Application of Pentagonal Neutrosophic Linear Programming for Stock Portfolio Optimization

Hamiden Abd El- Wahed Khalifa[1,2], Muhammad Saeed[3], Atiqe Ur Rahman[3,*] and Salwa El-Morsy[2,4]

[1]*Department of Operations Research, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt, E-Mail: hamiden@cu.edu.eg*

[2]*Department of Mathematics, College of Science and Arts, Al- Badaya, Qassim University, Saudi Arabia, E-Mail: ha.ahmed@qu.edu.sa, s.elmorsy@qu.edu.sa*

[3]*Department of Mathematics, University of Management and Technology, Lahore, 54000, Pakistan, E-Mail: muhammad.saeed@umt.edu.pk, aurkhb@gmail.com*

[4]*Basic Science Department, Nile Higher Institute for Engineering and Technology, Mansoura, Egypt.*

* Correspondence: aurkhb@gmail.com;

Abstract: The Linear programming problems (LPP) have been widely applied to many real-world problems. In this study, a formulation of stock portfolio problem is proposed. The problem is formulated by involving neutrosophic pentagonal fuzzy numbers (NPFN) in the rate of risked return, expected return rate and portfolio risk amount. Based on score function, the problem is transformed to its corresponding crisp form. A solution algorithm is investigated to provide the decision of the portfolio investment joined with investors in savings and securities. The main features of this study are: the investor can choose freely the risk coefficients to maximize the expected returns; also, the investors may determine their strategies under consideration of their own conditions. The optimal return rate is obtained by using TORA software. An example is introduced to indicate the efficiency and reliability of the technique.

Keywords: Portfolio; Investment: Stock Portfolio Investment; Pentagonal Fuzzy Numbers; Score Function, TORA Software; Neutrosophic Pentagonal Fuzzy Return Rate.

1. Introduction

Portfolio optimization is one of the essential problems in asset management of financial, its main goal is to minimize the risk of an investment by dividing it into many assets expected to fluctuate independently (Elton et al., 2009). A portfolio is a set of financial assets like cash equivalents, stocks, commodities, currencies and bonds. Portfolio can also include non-publicly tradable securities as, arts, private investment and real estate. Portfolio are directly held by investors and/ or managed by money managers and financial professionals [1]. Skrinjaric and Sego [2] applied Grey Relational Analysis (GRA) method to study the performance for a sample of stocks under various factors.

Fuzzy set theory initiated by Zadeh [3] has gained a great attention of researchers to solve real-life issues, like the supervision of economic threat. It permits us to illustrate and control vagueness in decision-support system. The imprecise facts of assets reports and the vagueness associated with the behavior of monetary markets can also be considered by means of fuzzy quantities or constraints. Fuzzy numerical data may be described using the phenomena of fuzzy subsets of R, are fuzzy numbers. Dubois and Prade [4] used a fuzzification principle to extended algebraic operations on real numbers to fuzzy numbers (FN).

Portfolio selection (PS) is the problem where investor selects the optimal portfolio from a set of possible portfolios. Also, it focuses on the optimal investment of one's wealth for maximizing profitable return and minimizing risk control [5]. According to lack of clarity of the real-world applications, the exact return of each security cannot be predetermined. The theory of optimal portfolios has been developed by Markowitz [6],where he has firstly introduced the mean-variance models. The PS problem is typically a LPP when all return of securities is constants. Numerous studies for PS have been done in the last few decades such as [7 –15]. Many researchers studied stock price assessment, in [16] Lindberg introduced new parameterization of the drift rates to modify the n stock Black-choles model, and solved Markowitz' continuous time PS in this framework.

Neutrosophic set (NS) theory was introduced by Smarandache [17] it is a generalization of fuzzy set; each element of NS has a truth, indeterminacy and falsity membership function. So, NS can describe inaccurate and maladjusted information effectively. Neutrosophic linear programming (NLP) problem is a LP problem that contains at least one neutrosophic coefficient or parameter. The NLP problem is more efficient than regular LP problems due to imperfect data. Many researchers studied NLP problems; Hussein et al. [18] transformed the NLP problem into its corresponding crisp model. Abdel-Basset et al. [19] proposed a novel method for solving a fully NLP problem. Ahmed [20], developed a new method for solving LR- type NLP problems. Ahmad et al. [21] developed a method for solving bipolar single-valued NLP problem. In [22], Bera studies the applications of NLP in real life. Das and Dash [23], introduced a modified Solution for NLP Problems with Mixed Constraints. Thamaraiselvi and Santhi [24] presented a new method for optimizing a real-life transportation problem in neutrosophic environment.

The rest of the paper is outlined as follows:

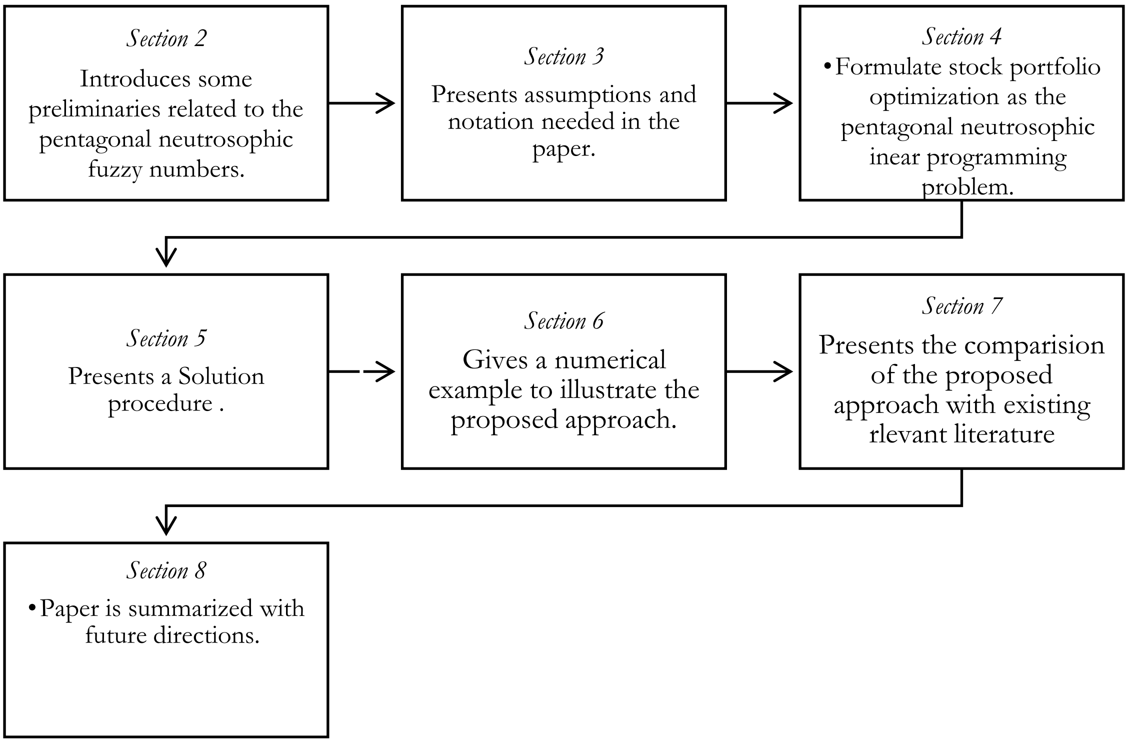

Fig.1. Rest of the paper

2. Preliminaries

In this section, some essential definitions and terminologies are recalled from fuzzy-like literature for proper understanding of the proposed work.

Definition 1. [3] A fuzzy set $\tilde{\wp}$ defined on the set of real numbers $\mathcal{R}$ is said to be fuzzy numbers when its membership function $\mu_{\tilde{\wp}}(x)$: $\mathcal{R} \to [0,1]$, have the following properties:

1. $\mu_{\tilde{\wp}}(x)$ is an upper semi- continuous membership function;
2. $\tilde{\wp}$ is convex fuzzy set, i.e., $\mu_{\tilde{\wp}}(\mathcal{F}x + (1-\mathcal{F})y) \geq \min\left\{\mu_{\tilde{\wp}}(x), \mu_{\tilde{\wp}}(y)\right\}$ for all $x, y \in \mathcal{R}; 0 \leq \mathcal{F} \leq 1$;
3. $\tilde{\wp}$ is normal, i.e., $\exists x_0 \in \mathcal{R}$ such that$\mu_{\tilde{\wp}}(x_0) = 1$;
4. Supp $(\tilde{\wp}) = \left\{x \in \mathcal{R}: \mu_{\tilde{\wp}}(x) > 0\right\}$ is the support of $\tilde{\wp}$, and the closure Cl (Supp($\tilde{\wp}$)) is a compact set.

Definition 2. [25]A fuzzy number $\widetilde{A}_{\wp} = (r, s, t, u, v), r \leq s \leq t \leq u \leq v$, on $\mathcal{R}$ is said to be a pentagonal fuzzy number if its membership function is:

$$\mu_{\tilde{A}_P} = \begin{cases} 0, & x < r, \\ w_1\left(\frac{x-r}{s-r}\right), & r \le x \le s, \\ 1-(1-w_1)\left(\frac{x-s}{t-s}\right), & s \le x \le t \\ 1, & x = t, \\ 1-(1-w_2)\left(\frac{u-x}{u-t}\right), & t \le x \le u, \\ w_2\left(\frac{v-x}{v-u}\right), & u \le x \le v, \\ 0, & x > v. \end{cases} \quad (1)$$

The graphical representation of the pentagonal fuzzy number is illustrated in the following figure

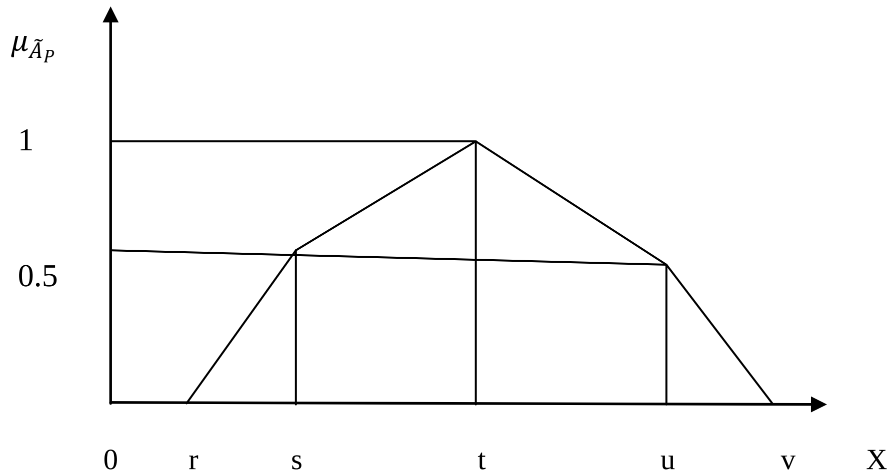

Fig.2. Graphical Representation of Pentagonal Fuzzy number [25]

Definition 3. [17] A neutrosophic set $\tilde{B}^N$ of non-empty set $\mathcal{X}$ is defined as

$\tilde{B}^N = \{\langle x;\ I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x), V_{\tilde{B}^N}(x)\rangle : x \in \mathcal{X}, I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x), V_{\tilde{B}^N}(x) \in]0_-, 1^+[\}$, where $I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x)$, and $V_{\tilde{B}^N}(x)$ are truth membership function, an indeterminacy- membership function, and a falsity-membership function and there is no restriction on the sum of $I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x)$, and $V_{\tilde{B}^N}(x)$, so

$0^- \le \text{Sup}\{I_{\tilde{B}^N}(x)\} + \text{Sup}\{J_{\tilde{B}^N}(x)\} + \text{Sup}\{V_{\tilde{B}^N}(x)\} \le 3^+$, and $]0^-, 1^+[$ is a nonstandard unit interval.

Definition 4. [17] A single- valued neutrosophic set $\tilde{B}^{SVN}$ of a non-empty set $\mathcal{X}$ is defined as

$\bar{B}^{SVN} = \{\langle x, I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x), V_{\tilde{B}^N}(x)\rangle : x \in X\}$, where $I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x)$, and $V_{\tilde{B}^N}(x) \in [0, 1]$ for each $x \in \mathcal{X}$ and $0 \le I_{\tilde{B}^N}(x) + J_{\tilde{B}^N}(x) + V_{\tilde{B}^N}(x) \le 3$.

Definition 5. [23] Let $\tau_{\tilde{p}}, \phi_{\tilde{p}}, \omega_{\tilde{p}} \in [0, 1]$ and $r, s, t, u, v \in \mathbb{R}$ such that $r \le s \le t \le u \le v$. Then a single-valued pentagonal fuzzy neutrosophic set (SVPFN), $\tilde{p}^{PN} = \langle (r, s, t, u, v);\ \tau_{\tilde{p}}, \phi_{\tilde{p}}, \omega_{\tilde{p}}\rangle$ is a special neutrosophic set on $\mathcal{R}$, whose truth-membership, hesitant- membership, and falsity-membership functions are

$$\tau_{\tilde{p}^{PN}}(x)=\begin{cases}0, & x<r;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r\le x\le s;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s\le x\le t;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t\le x\le u;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u\le x\le v;\\ 0, & x>v.\end{cases} \quad (2)$$

$$\phi_{\tilde{p}^{PN}}=\begin{cases}0, & x<r;\\ \phi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r\le x\le s;\\ \phi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s\le x\le t;\\ \phi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t\le x\le u;\\ \phi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u\le x\le v;\\ 0, & x>v.\end{cases} \quad (3)$$

$$\omega_{\tilde{p}^{PN}}=\begin{cases}0, & x<r;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r\le x\le s;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s\le x\le t;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t\le x\le u;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u\le x\le v;\\ 0, & x>v.\end{cases} \quad (4)$$

Where $\tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}$, and $\omega_{\tilde{p}^{PN}}$ denote the maximum truth, minimum-hesitant, and minimum falsity membership degrees, respectively. SVPFN $\tilde{p}^{PN}=\langle(r,s,t,u,v): \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}}\rangle$ may express in ill-defined quantity about p, which is approximately similar to $[s,u]$.

Definition 6. [25]

Let $\tilde{p}^{PN}=\langle(r,s,t,u,v);\ \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}}\rangle$ and $\tilde{q}^{PN}=\langle(r^*, s^*, t^*, u^*, v^*);\ \tau_{\tilde{q}^{PN}}, \phi_{\tilde{q}^{PN}}, \omega_{\tilde{q}^{PN}}\rangle$ be two single-valued PFNs, the arithmetic operations on $\tilde{p}^{PN}$ and $\tilde{q}^{PN}$ are:

1. $\tilde{p}^{PN}\oplus\tilde{q}^{PN}=\langle(r+r^*, s+s^*, t+t^*, u+u^*, v+v^*);\ \tau_{\tilde{p}^{PN}}\wedge\tau_{\tilde{q}^{PN}}, \phi_{\tilde{p}^{PN}}\vee\phi_{\tilde{q}^{PN}}, \omega_{p^{PN}}\vee$

   ωqPN,
2. $\tilde{p}^{PN}\ominus\tilde{q}^{PN}=\langle(r-v^*, s-u^*, t-t^*, u-s^*, v-r^*);\ \tau_{\tilde{p}^{PN}}\wedge\tau_{\tilde{q}^{PN}}, \phi_{\tilde{p}^{PN}}\vee\phi_{\tilde{q}^{PN}}, \omega_{p^{PN}}\vee\omega_{\tilde{q}^{PN}}\rangle$,

3. $\tilde{p}^{PN} \otimes \tilde{q}^{PN} = \frac{1}{5}\gamma_{q} \langle (r,s,t,u,v);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \phi_{\tilde{p}^{PN}} \vee \phi_{\tilde{q}^{PN}}, \omega_{p^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle, \gamma_{q} = \frac{1}{3}(r^* + s^* +$

t∗+ u∗+ v∗2+τqPN−ϕqPN≠0,

4. $\tilde{p}^{PN} \oslash \tilde{q}^{PN} = \frac{5}{\gamma_{q}} \langle (r,s,t,u,v);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \phi_{\tilde{p}^{PN}} \vee \phi_{\tilde{q}^{PN}}, \omega_{p^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle, \gamma_{q} \neq 0,$

5. $m\tilde{p}^{PN} = \begin{cases} \langle (mr, ms, mt, mu, mv);\ \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle, m > 0, \\ \langle (mv, mu, mt, ms, mr);\ \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle, m < 0, \end{cases}$

6. $\tilde{p}^{PN^{-1}} = \langle \left(\frac{1}{v}, \frac{1}{u}, \frac{1}{t}, \frac{1}{s}, \frac{1}{r}\right);\ \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle, \tilde{p}^{PN} \neq 0.$

Definition7. [26] Let $\tilde{p}^{PN} = \langle (r,s,t,u,v); \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle$ be a single- valued pentagonal fuzzy neutrosophic numbers, then

1. Accuracy function $AC(\tilde{p}^{PN}) = \left(\frac{1}{15}\right)(r + s + t + u + v) * [2 + \tau_{\tilde{p}^{PN}} - \phi_{\tilde{p}^{PN}}].$

2. Score function $SC(\tilde{p}^{PN}) = \left(\frac{1}{15}\right)(r + s + t + u + v) * \left[2 + \tau_{\tilde{p}^{PN}} - \phi_{\tilde{p}^{PN}} - \omega_{\tilde{p}^{PN}}\right].$

Definition 8. [27] The order relations between $\tilde{p}^{PN}$ and $\tilde{q}^{PN}$ based on $SC(\tilde{p}^{NP})$ and $AC(\tilde{q}^{NP})$ are defined as

1. If $SC(\tilde{p}^{PN}) > SC(\tilde{q}^{NP})$, then $\tilde{p} > \tilde{q}$,
2. If $SC(\tilde{p}^{PN}) < SC(\tilde{q}^{NP})$, then $\tilde{p} < \tilde{q}$,
3. If $SC(\tilde{p}^{PN}) = SC(\tilde{q}^{NP})$, then

   i. If $AC(\tilde{p}^{PN}) < AC(\tilde{q}^{NP})$, then $\tilde{p} < \tilde{q}$,

   ii. If $AC(\tilde{p}^{PN}) > AC(\tilde{q}^{NP})$, then $\tilde{p} > \tilde{q}$,

   iii. If $AC(\tilde{p}^{PN}) = AC(\tilde{q}^{NP})$, then $\tilde{p} = \tilde{q}$.

3. Assumptions and Notations

3.1 Assumptions

In reality, small changes influence in selecting portfolio, since the investment environment is quite sensitive. For facilitating problem formulation, we assumed that:

1) The securities are evaluated based on the expected return rate and the loss-risk rate;
2) Securities are imperfect and can be divided;
3) In the course of transaction, there is no need to pay for transactions;
4) Investors must obey the assumptions of avoiding risk and of non-satisfaction;

5) During the investment period, the interest rate of the bank is fixed;

6) The operation of short selling is not allowed;

7) There are n different risk securities.

3.2 Notations

r_0: Bank interest rate;

r_i: Expected return rates, $i = 1, 2, \dots, n$;

A_{ij}: Risked return rates, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$;

x_0: Proportion of total investments during the investment period

x_i: Proportion of funds invested in the secondary securities, $i = 1, 2, \dots, n$;

R: Total expected return rate;

$\mathfrak{b}$: Risk coefficient of portfolio investment;

V: Maximum value of all securities risks.

4. Formulation of the Problem

Consider the stock problem introduced by Yin [28]. The expected rate of return of a combination of investments, takes the form:

$$R = \sum_{i=0}^{n} r_i x_i$$

Investors aim to maximize investments interest and minimize risk in their risk securities. The risk coefficient of portfolio $\mathfrak{b}$indicates the market risk. In case of $\mathfrak{b} > 1$, risk of stock portfolio is more than the average value of the market risk; in the case of $\mathfrak{b} < 1$, the risk of stock portfolio is less than the average value of market risk; when $\mathfrak{b} = 1$, the average market risk and stock portfolio risk are equal. The maximum value of all securities risks, denoted

$$V = \max(A_1 x_1, A_2 x_2, \dots, A_n x_n)$$

Now we can formulate the following linear programming model:

$$\max R = \sum_{i=0}^{n} r_i x_i$$

$$s.t.\begin{cases} Ax \leq \mathfrak{b} \\ \sum_{i=0}^{n} x_i = 1 \\ x_i \geq 0, \quad i = 1,2,\dots,n \end{cases} \quad (5)$$

The above model is the classical linear programming problem. For more generalization and flexibility, it is more reasonable to describe r_i, $\mathfrak{b}_i$ and A_i as pentagonal fuzzy neutrosophic numbers. So, we set up the following model:

$$\max \tilde{R}^{NP} = r_0 x_0 + \sum_{i=1}^{n} \tilde{r}_i^{NP} x_i$$

$$s.t.\begin{cases} \tilde{A}^{NP} x \leq \tilde{\mathfrak{b}}^{NP} \\ \sum_{j=0}^{n} x_j = 1 \\ x_j \geq 0, \quad j = 1,2,\dots,n;\ i = 1,2,\dots,m. \end{cases} \quad (6)$$

5. **Solution Procedure of Pentagonal Fuzzy Neutrosophic LPP**

In this section we will illustrate the solution procedure of the pentagonal fuzzy neutrosophic linear programming problem. The model associated with pentagonal fuzzy neutrosophic numbers in expected return rates, risk loss rates and risk coefficients

5.1 Formulation of pentagonal fuzzy neutrosophic LPP

Assume that $\tilde{A}^{NP} = \left(\tilde{A}_{ij}^{NP}\right)_{m\times n}$, $\tilde{b}^{NP} = \left(\tilde{b}_1^{NP}, \tilde{b}_2^{NP}, \dots, \tilde{b}_m^{NP}\right)^{\boldsymbol{T}}$, $\tilde{r}^{NP} = (\tilde{r}_1^{NP}, \tilde{r}_2^{NP}, \dots, \tilde{r}_n^{NP})$ and $X = (x_1, x_2, \dots., x_n)^T$. The following pentagonal neutrosophic linear programming model has been set up:

$$\max \tilde{R}^{NP} = r_0 x_0 + \sum_{j=1}^{n} \tilde{r}_j^{NP} x_j$$

$$s.t.\begin{cases} \sum_{j=1}^{n} \tilde{A}_{ij}^{NP} x_j \leq \tilde{b}_i^{NP} \\ \sum_{j=0}^{n} x_j = 1 \\ x_j \geq 0, \quad j = 1,2,\dots,n;\ i = 1,2,\dots,m. \end{cases} \quad (7)$$

Based on the score function defined in section 2, the pentagonal neutrosophic linear programming model transformed to regular linear programming model which is quite easy and solvable.

$$\max R = r_0 x_0 + \sum_{i=1}^{n} \mathrm{SC}(\tilde{r}_i^{NP}) x_i$$

$$s.t.\begin{cases}\sum_{j=1}^{n} \mathrm{SC}\,(\tilde{A}_{\mathrm{ij}}^{NP})x_{\mathrm{j}} \leq \mathrm{SC}(\tilde{\mathrm{b}}_{\mathrm{i}}^{NP}) \\ \sum_{j=0}^{n} x_{\mathrm{j}} = 1 \\ x_{\mathrm{j}} \geq 0, \quad \mathrm{j} = 1,2,\dots,n;\ \mathrm{i} = 1,2,\dots,m. \end{cases} \tag{8}$$

6. Numerical Example

In this section, a numerical example is studied to demonstrate the proposed approach. Consider the choice of an investor in five available stocks; the first one is a portfolio of bank savings with annual rate of interest $r_0 = 0.07$. The data of the other four stocks are given in the following table 1, table 2 and table 3.

Table 1.Expected return rate %

Stocks	$\tilde{r}^{\mathrm{PN}}$
S_1 CNPC (601857)	⟨11.5, 12.0, 12.2, 12.5, 12.9; 1.0, 0.0,0.0 ⟩
S_2 CNPC (600028)	⟨15.8, 16.0, 16.2, 16.5, 16.8; 1.0, 0.0,0.0 ⟩
S_3 Wanke (000002)	⟨13.7, 14.0, 14.3, 14.5, 14.9; 1.0, 0.0,0.0 ⟩
S_4 Poly (600048)	⟨13.0, 13.5, 14.0, 14.5, 15.0; 1.0, 0.0,0.0 ⟩

Table 2. Risk Loss Rate %

$\tilde{A}_{\mathrm{ij}}^{NP}$	Risk loss rate
$\tilde{A}_{11}^{NP}$	⟨3.8, 4.0, 5.2, 5.6, 5.9; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{12}^{NP}$	⟨9.0, 10.0, 12.5, 14.0, 16.9; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{13}^{NP}$	⟨3.2, 4.0, 4.8, 5.5, 6.0; 1.0, 0.0,0.0⟩
$\tilde{A}_{14}^{NP}$	⟨8.7, 9.0, 11.9, 14.0, 16.3; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{21}^{NP}$	⟨0.9, 1.0, 1.1, 1.2, 1.3; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{22}^{NP}$	⟨1.39, 1.7, 2.15, 3.0, 3.32; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{23}^{NP}$	⟨1.2, 3.0, 3.2, 4.0, 4.8; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{24}^{NP}$	⟨1.59, 1.8, 2.27, 3.0, 3.3; 1.0, 0.0,0.0 ⟩

Table 3. Risk coefficient%

$\tilde{\mathrm{b}}_{\mathrm{i}}^{NP}$	Risk coefficient rate
$\tilde{\mathrm{b}}_{1}^{NP}$	⟨1.2, 1.5, 2.0, 2.2, 2.4; 1.0, 0.0,0.0 ⟩
b_{2}^{NP}	⟨0.6, 0.9, 2.0, 2.6, 3.0; 1.0, 0.0,0.0 ⟩

The given problem can be formulated in the following model:

$$\max \tilde{R}^{NP} = r_0 x_0 + \sum_{j=1}^{n} \mathrm{SC}(\tilde{r}_i^{NP}) x_j$$

$$s.t. \begin{cases} \mathrm{SC}(\tilde{A}_{11}^{NP})x_1 + \mathrm{SC}(\tilde{A}_{12}^{NP})x_2 + \mathrm{SC}(\tilde{A}_{13}^{NP})x_3 + \mathrm{SC}(\tilde{A}_{14}^{NP})x_4 \le \mathrm{SC}(\tilde{b}_1^{NP}) \\ \mathrm{SC}(\tilde{A}_{21}^{NP})x_1 + \mathrm{SC}(\tilde{A}_{22}^{NP})x_2 + \mathrm{SC}(\tilde{A}_{23}^{NP})x_3 + \mathrm{SC}(\tilde{A}_{24}^{NP})x_4 \le \mathrm{SC}(\tilde{b}_2^{NP}) \\ x_0 + x_1 + x_2 + x_3 + x_4 = 1 \\ x_j \ge 0, \quad 0 \le j \le 4 \end{cases} \quad (9)$$

According to properties and arithmetic operations on pentagonal fuzzy neutrosophic numbers, we obtain the following mathematical model:

$$\max R = 0.07\ x_0 + 0.09165\ x_1 + 0.12195 x_2 + 0.1071 x_3 + 0.105 x_4$$

$$s.t. \begin{cases} 3.675\ x_1 + 9.36\ x_2 + 3.525\ x_3 + 8.985\ x_4 \le 1.395, \\ 0.825\ x_1 + 1.734\ x_2 + 2.43\ x_3 + 1.794\ x_4 \le 1.365, \\ x_0 + x_1 + x_2 + x_3 + x_4 = 1, \\ x_j \ge 0,\ 0 \le j \le 4. \end{cases} \quad (10)$$

The optimal solution is:

$x_0 = 0.6042553,\ x_1 = 0.0, x_2 = 0.0, x_3 = 0.3957447, x_4 = 0.0,$

The optimal value $R = 0.084682$

The obtained results indicate that the optimal investment under the offered information occurred when 60.0426% of all capital is saved to the bank with interest rate 7%and 39.57% of the total capital is invested into security of S_3. This strategy leads to the maximum expected return 8.4682% on the premise of risk coefficients $\tilde{b}_1^{NP}$ and $\tilde{b}_2^{NP}$.

7. Comparative Study

This section, introduces a comparative study between the topics covered by our proposed approach and those studied by some other researchers in related work in solving PS problems.

Table 4. Comparisons with some researcher's contributions

Reference no.	Efficient solution	Environment	Type of number
[28]	NO	Fuzzy	Triangle interval valued
[29]	NO	Neutrosophic	Neutrosophic
[30]	NO	Fuzzy	Fuzzy-valued function
[31]	NO	realistic	Real
[32]	NO	stochastic	random variables
[33]	NO	Fuzzy	Triangle
Our investigation	YES	Neutrosophic	Neutrosophic

8. Conclusion Remarks and Future Work

A formulation of stock portfolio problem involving neutrosophic pentagonal fuzzy numbers in the rate of risked return, expected return rate and portfolio risk amount is proposed. Using score function, the problem is converted to its corresponding crisp form. A solution approach is investigated to provide the decision of the portfolio investment joined with investors in savings and securities. The main advantages of this study are: the freedom in choosing the risk coefficients to maximize the expected returns; also, the investors may select their strategies under consideration of their own conditions. The optimal return rate is obtained using TORA software. A numerical example indicates that the approach is reliable and efficient for studying pentagonal neutrosophic stock portfolio. Future work may include the further extension of this study to other fuzzy- like structure (i. e., interval- valued fuzzy set, Neutrosophic set, Pythagorean fuzzy set, Spherical fuzzy set etc. with more discussion and suggestive comments.

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Conflict of Interest

The authors do not have conflict of interest.

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University of New Mexico

Connectedness on Hypersoft Topological Spaces

Sagvan Y. Musa[1,*], Baravan A. Asaad[2,3]

[1] Department of Mathematics, Faculty of Education, University of Zakho, Zakho, Iraq; sagvan.musa@uoz.edu.krd

[2] Department of Computer Science, College of Science, Cihan University-Duhok, Duhok, Iraq

[3] Department of Mathematics, Faculty of Science, University of Zakho, Zakho, Iraq; baravan.asaad@uoz.edu.krd

*Correspondence: sagvan.musa@uoz.edu.krd

Abstract. Connectedness (resp. disconnectedness), which reflects the key characteristic of topological spaces and helps in the differentiation of two topologies, is one of the most significant and fundamental concept in topological spaces. In light of this, we introduce hypersoft connectedness (resp. hypersoft disconnectedness) in hypersoft topological spaces and investigate its properties in details. Furthermore, we present the concepts of disjoint hypersoft sets, separated hypersoft sets, and hypersoft hereditary property. Also, some examples are provided for the better understanding of these ideas.

Keywords: hypersoft connected (resp. hypersoft disconnected); hypersoft topology; hypersoft sets; disjoint hypersoft sets; separated hypersoft sets; hypersoft hereditary property.

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1. **Introduction**

Some mathematical concepts, such as theory of fuzzy sets, theory of rough sets, and theory of vague sets, can be considered as mathematical tools for dealing with uncertainties. However, each of these theories has its own difficulties. Molodtsov [1] first proposed the concept of soft sets as a general mathematical tool for dealing with uncertain objects. He successfully applied soft set theory to a variety of fields, including the smoothness of functions, game theory, operation research, Riemann integration, and elsewhere [1,2]. Applications have been made to decision-making, business competitive capacity information systems, classification of natural textures, optimization problems, data analysis, similarity measures, algebraic structures of soft sets, soft matrix theory, parameter reduction in soft set theory, classification of natural textures, and soft sets and their relation to rough and fuzzy sets. In 2003, Maji et al. [3]

presented the basic operations of soft sets. After that, the properties and applications of soft set theory have been studied increasingly [4–7]. In 2011, Shabir and Naz [8] and Çağman et al. [9] introduced and studied the notion of soft topological spaces in different ways. Then, some authors has began to study some of basic concepts and properties of soft topological spaces [10–20]. Moreover, the concept of connectedness attracted the interest of researchers [21, 22].

In 2018, Smarandache [23] proposed the notion of hypersoft set as a generalization of soft set. Then, Saeed et al. [24] put forward the basic concepts of hypersoft set theory. They defined the operators of the intersection, union, and difference between two hypersoft sets as well as a complement of a hypersoft set. In [25], Saeed et al. modified some operators in [24] and presented some new types. Abbas et al. [26], in a unique approach, presented new types of these operators as well as they introduced the concept of hypersoft points.

The concept of bipolar hypersoft sets (a hybridization of hypersoft set and bipolarity) was introduced and discussed in detail by Musa and Asaad [27]. In [28], they initiated the study of bipolar hypersoft topological spaces and studied some topological structures via bipolar hypersoft sets. Musa and Asaad [29] continued studying bipolar hypersoft topological spaces by presenting the notion of bipolar hypersoft connected (resp. bipolar hypersoft disconnected) spaces. The concepts of separated bipolar hypersoft sets and bipolar hypersoft hereditary property were also investigated by them.

Recently, Musa and Asaad [30] initiated the study of hypersoft topological spaces. They defined hypersoft topology as a collection $\mathcal{T}_{\mathcal{H}}$ of hypersoft sets over the universe $\mathcal{U}$ with a fixed set of parameters $\mathcal{E}$. Consequently, they defined basic concepts of hypersoft neighborhood, hypersoft limit point, and hypersoft subspace and investigated their several properties. Furthermore, Musa and Asaad explored and studied in detail hypersoft closure, hypersoft interior, hypersoft exterior, and hypersoft boundary, as well as the relationship between them were discussed.

In this work, we introduce a new concept in hypersoft topological spaces called hypersoft connected (resp. hypersoft disconnected) spaces. Preliminaries on basic notions related to hypersoft sets and hypersoft topological spaces are presented in Section 2. Section 3 gives the concepts of disjoint hypersoft sets, separated hypersoft sets, hypersoft connected (resp. hypersoft disconnected) spaces, and hypersoft hereditary property as well as some examples are given for the better understanding of these ideas. A summary of the recent work and an idea for additional research are provided in Section 4.

2. Preliminaries

In this section, we present the necessary concepts and results that are related to hypersoft set and hypersoft topology.

2.1. *Hypersoft Sets*

Let $\mathcal{U}$ be an initial universe, $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$, and $E_1, E_2, ..., E_n$ the pairwise of disjoint sets of parameters. Let $A_i, B_i \subseteq E_i$ for $i = 1, 2, ..., n$.

Definition 2.1. [23] A pair $(\mathbb{F}, A_1 \times A_2 \times ... \times A_n)$ is called a hypersoft set over $\mathcal{U}$, where $\mathbb{F}$ is a mapping given by $\mathbb{F} : A_1 \times A_2 \times ... \times A_n \to \mathcal{P}(\mathcal{U})$.

From now on, we write the symbol $\mathcal{E}$ for $E_1 \times E_2 \times ... \times E_n$, $\mathcal{A}$ for $A_1 \times A_2 \times ... \times A_n$, and $\mathcal{B}$ for $B_1 \times B_2 \times ... \times B_n$ where $\mathcal{A}$, $\mathcal{B} \subseteq \mathcal{E}$. Clearly, each element in $\mathcal{A}, \mathcal{B}$ and $\mathcal{E}$ is an n-tuple element.

Moreover, we represent hypersoft set $(\mathbb{F}, \mathcal{A})$ as an ordered pair,

$$(\mathbb{F}, \mathcal{A}) = \{(\alpha, \mathbb{F}(\alpha)) : \alpha \in \mathcal{A}\}.$$

Definition 2.2. [24] For two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$, we say that $(\mathbb{F}, \mathcal{A})$ is a hypersoft subset of $(\mathbb{G}, \mathcal{B})$ if

(1) $\mathcal{A} \subseteq \mathcal{B}$, and

(2) $\mathbb{F}(\alpha) \subseteq \mathbb{G}(\alpha)$ for all $\alpha \in \mathcal{A}$.

We write $(\mathbb{F}, \mathcal{A}) \,\widetilde{\sqsubseteq}\, (\mathbb{G}, \mathcal{B})$.

$(\mathbb{F}, \mathcal{A})$ is said to be a hypersoft superset of $(\mathbb{G}, \mathcal{B})$, if $(\mathbb{G}, \mathcal{B})$ is a hypersoft subset of $(\mathbb{F}, \mathcal{A})$. We denote it by $(\mathbb{F}, \mathcal{A}) \,\widetilde{\sqsupseteq}\, (\mathbb{G}, \mathcal{B})$.

Definition 2.3. [24] Two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$ are said to be hypersoft equal if $(\mathbb{F}, \mathcal{A})$ is a hypersoft subset of $(\mathbb{G}, \mathcal{B})$ and $(\mathbb{G}, \mathcal{B})$ is a hypersoft subset of $(\mathbb{F}, \mathcal{A})$.

Definition 2.4. [24] The complement of a hypersoft set $(\mathbb{F}, \mathcal{A})$ is denoted by $(\mathbb{F}, \mathcal{A})^c$ and is defined by $(\mathbb{F}, \mathcal{A})^c = (\mathbb{F}^c, \mathcal{A})$ where $\mathbb{F}^c : \mathcal{A} \to \mathcal{P}(\mathcal{U})$ is a mapping given by $\mathbb{F}^c(\alpha) = \mathcal{U} \setminus \mathbb{F}(\alpha)$ for all $\alpha \in \mathcal{A}$.

Definition 2.5. [25] A hypersoft set $(\mathbb{F}, \mathcal{A})$ over $\mathcal{U}$ is said to be a relative null hypersoft set, denoted by $(\Phi, \mathcal{A})$, if for all $\alpha \in \mathcal{A}$, $\mathbb{F}(\alpha) = \phi$.

The relative null hypersoft set with respect to the universe set of parameters $\mathcal{E}$ is called the null hypersoft set over $\mathcal{U}$ and is denoted by $(\Phi, \mathcal{E})$.

A hypersoft set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$ is said to be a non-null hypersoft set if $\mathbb{F}(\alpha) \neq \phi$ for some $\alpha \in \mathcal{E}$.

Definition 2.6. [25] A hypersoft set $(\mathbb{F}, \mathcal{A})$ over $\mathcal{U}$ is said to be a relative whole hypersoft set, denoted by $(\Psi, \mathcal{A})$, if for all $\alpha \in \mathcal{A}$, $\mathbb{F}(\alpha) = \mathcal{U}$.

The relative whole hypersoft set with respect to the universe set of parameters $\mathcal{E}$ is called the whole hypersoft set over $\mathcal{U}$ and is denoted by $(\Psi, \mathcal{E})$.

A hypersoft set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$ is said to be a non-whole hypersoft set if $\mathbb{F}(\alpha) \neq \mathcal{U}$ for some $\alpha \in \mathcal{E}$.

Definition 2.7. [25] Difference of two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$, is a hypersoft set $(\mathbb{H}, C)$, where $C = \mathcal{A} \cap \mathcal{B}$ and for all $\alpha \in C$, $\mathbb{H}(\alpha) = \mathbb{F}(\alpha) \setminus \mathbb{G}(\alpha)$. We write $(\mathbb{F}, \mathcal{A}) \setminus (\mathbb{G}, \mathcal{B}) = (\mathbb{H}, C)$.

Definition 2.8. [25] Union of two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$, is a hypersoft set $(\mathbb{H}, C)$, where $C = \mathcal{A} \cap \mathcal{B}$ and for all $\alpha \in C$, $\mathbb{H}(\alpha) = \mathbb{F}(\alpha) \cup \mathbb{G}(\alpha)$. We write $(\mathbb{F}, \mathcal{A}) \,\widetilde{\sqcup}\, (\mathbb{G}, \mathcal{B}) = (\mathbb{H}, C)$.

Definition 2.9. [25] Intersection of two hypersoft sets $(\mathbb{F}, \mathcal{A})$ and $(\mathbb{G}, \mathcal{B})$ over a common universe $\mathcal{U}$, is a hypersoft set $(\mathbb{H}, C)$, where $C = \mathcal{A} \cap \mathcal{B}$ and for all $\alpha \in C$, $\mathbb{H}(\alpha) = \mathbb{F}(\alpha) \cap \mathbb{G}(\alpha)$. We write $(\mathbb{F}, \mathcal{A}) \,\widetilde{\sqcap}\, (\mathbb{G}, \mathcal{B}) = (\mathbb{H}, C)$.

Definition 2.10. [30] Let Υ be a non-empty subset of $\mathcal{U}$. Then $(\mathcal{Y}, \mathcal{A})$ denotes the hypersoft set over $\mathcal{U}$ defined by $\mathcal{Y}(\alpha) = \Upsilon$ for all $\alpha \in \mathcal{A}$.

Definition 2.11. [30] Let $(\mathbb{F}, \mathcal{A})$ be a hypersoft set over $\mathcal{U}$ and Υ be a non-empty subset of $\mathcal{U}$. Then the sub hypersoft set of $(\mathbb{F}, \mathcal{A})$ over Υ denoted by $(\mathbb{F}_\Upsilon, \mathcal{A})$ is defined as $\mathbb{F}_\Upsilon(\alpha) = \Upsilon \cap \mathbb{F}(\alpha)$ for all $\alpha \in \mathcal{A}$.

In other words, $(\mathbb{F}_\Upsilon, \mathcal{E}) = (\mathcal{Y}, \mathcal{A}) \,\widetilde{\sqcap}\, (\mathbb{F}, \mathcal{A})$.

The following results are obvious.

Proposition 2.12. *Let $(\mathbb{F}_1, \mathcal{A})$ and $(\mathbb{F}_2, \mathcal{A})$ be two hypersoft sets over a universe $\mathcal{U}$. Then the following holds.*

(1) $(\mathbb{F}_1, \mathcal{A}) \,\widetilde{\sqcap}\, (\mathbb{F}_2, \mathcal{A}) = (\Phi, \mathcal{A})$ if and only if $(\mathbb{F}_1, \mathcal{A}) \,\widetilde{\sqsubseteq}\, (\mathbb{F}_2, \mathcal{A})^c$ and $(\mathbb{F}_2, \mathcal{A}) \,\widetilde{\sqsubseteq}\, (\mathbb{F}_1, \mathcal{A})^c$;
(2) If $(\mathbb{F}_1, \mathcal{A}) \,\widetilde{\sqsubseteq}\, (\mathbb{F}_2, \mathcal{A})$ then $(\mathbb{F}_1, \mathcal{A}) \,\widetilde{\sqcap}\, (\mathbb{F}_2, \mathcal{A}) = (\mathbb{F}_1, \mathcal{A})$;
(3) If $(\mathbb{F}_1, \mathcal{A}) \,\widetilde{\sqsubseteq}\, (\mathbb{F}_2, \mathcal{A})$ then $(\mathbb{F}_1, \mathcal{A}) \,\widetilde{\sqcup}\, (\mathbb{F}_2, \mathcal{A}) = (\mathbb{F}_2, \mathcal{A})$.

2.2. *Hypersoft Topological Spaces*

Let $\mathcal{U}$ be an initial universe and $\mathcal{E}$ be a set of parameters.

Definition 2.13. [30] Let $\mathcal{T}_\mathcal{H}$ be the collection of hypersoft sets over $\mathcal{U}$, then $\mathcal{T}_\mathcal{H}$ is said to be a hypersoft topology on $\mathcal{U}$ if

(1) $(\Phi, \mathcal{E})$, $(\Psi, \mathcal{E})$ belong to $\mathcal{T}_{\mathcal{H}}$,

(2) the intersection of any two hypersoft sets in $\mathcal{T}_{\mathcal{H}}$ belongs to $\mathcal{T}_{\mathcal{H}}$,

(3) the union of any number of hypersoft sets in $\mathcal{T}_{\mathcal{H}}$ belongs to $\mathcal{T}_{\mathcal{H}}$.

We call $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ a hypersoft topological space over $\mathcal{U}$.

Definition 2.14. [30] Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$, then the members of $\mathcal{T}_{\mathcal{H}}$ are said to be hypersoft open sets in $\mathcal{U}$.

Definition 2.15. [30] Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$. A hypersoft set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$ is said to be a hypersoft closed set in $\mathcal{U}$, if its complement $(\mathbb{F}, \mathcal{E})^c$ belongs to $\mathcal{T}_{\mathcal{H}}$.

Proposition 2.16. *[30] Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$. Then*

(1) $(\Phi, \mathcal{E})$, $(\Psi, \mathcal{E})$ are hypersoft closed set over $\mathcal{U}$,

(2) the union of any two hypersoft closed sets is a hypersoft closed set over $\mathcal{U}$,

(3) the intersection of any number of hypersoft closed sets is a hypersoft closed set over $\mathcal{U}$.

Definition 2.17. [30] Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$ and Υ be a non-empty subset of $\mathcal{U}$. Then

$$\mathcal{T}_{\mathcal{H}_\Upsilon} = \{(\mathbb{F}_\Upsilon, \mathcal{E}) \mid (\mathbb{F}, \mathcal{E}) \in \mathcal{T}_{\mathcal{H}}\}$$

is said to be the relative hypersoft topology on Υ and $(\Upsilon, \mathcal{T}_{\mathcal{H}_\Upsilon}, \mathcal{E})$ is called a hypersoft subspace of $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$.

The following results are obvious.

Proposition 2.18. *Let $(\Upsilon, \mathcal{T}_{\mathcal{H}_\Upsilon}, \mathcal{E})$ be a hypersoft subspace of hypersoft topological space $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$, then*

(1) $(\mathbb{F}, \mathcal{E})$ is hypersoft open in Υ if and only if $(\mathbb{F}, \mathcal{E}) = (\mathcal{Y}, \mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{G}, \mathcal{E})$ for some $(\mathbb{G}, \mathcal{E}) \in \mathcal{T}_{\mathcal{H}}$;

(2) $(\mathbb{F}, \mathcal{E})$ is hypersoft closed in Υ if and only if $(\mathbb{F}, \mathcal{E}) = (\mathcal{Y}, \mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{G}, \mathcal{E})$ for some hypersoft closed set $(\mathbb{G}, \mathcal{E})$ in $\mathcal{U}$.

Definition 2.19. [30] Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space and $(\mathbb{F}, \mathcal{E})$ be a hypersoft set over $\mathcal{U}$. The intersection of all hypersoft closed supersets of $(\mathbb{F}, \mathcal{E})$ is called the hypersoft closure of $(\mathbb{F}, \mathcal{E})$ and is denoted by $\overline{(\mathbb{F}, \mathcal{E})}$.

Definition 2.20. [30] Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$. Then hypersoft interior of hypersoft set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$ is denoted by $(\mathbb{F}, \mathcal{E})^o$ and is defined as the union of all hypersoft open set contained in $(\mathbb{F}, \mathcal{E})$.

Definition 2.21. [30] Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$, then hypersoft boundary of hypersoft set $(\mathbb{F}, \mathcal{E})$ over $\mathcal{U}$ is denoted by $(\mathbb{F}, \mathcal{E})^b$ and is defined as $(\mathbb{F}, \mathcal{E})^b = \overline{(\mathbb{F}, \mathcal{E})} \,\widetilde{\sqcap}\, \overline{(\mathbb{F}, \mathcal{E})^c}$.

3. Hypersoft Connected (resp. Hypersoft Disconnected) Spaces

In this section, we introduce and characterize one of the most important property of hypersoft topological spaces called hypersoft connectedness (resp. hypersoft disconnectedness).

Definition 3.1. Two hypersoft sets $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are said to be disjoint hypersoft sets if $(\mathbb{F}_1, \mathcal{E}) \,\widetilde{\sqcap}\, (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$, that is, $F_1(\alpha) \cap \mathbb{F}_2(\alpha) = \phi$ for all $\alpha \in \mathcal{E}$.

Definition 3.2. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft topological space over $\mathcal{U}$. Two non-null hypersoft sets $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are said to be separated hypersoft sets if and only if $(\mathbb{F}_1, \mathcal{E}) \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\mathbb{F}_1, \mathcal{E})} \,\widetilde{\sqcap}\, (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$.

Note that any two separated hypersoft sets are disjoint hypersoft sets.

Remark 3.3. The following example shows that two disjoint hypersoft sets are not necessarily separated hypersoft sets.

Example 3.4. Let $\mathcal{U} = \{u_1, u_2, u_3, u_4\}$, $E_1 = \{e_1, e_2\}$, $E_2 = \{e_3\}$, and $E_3 = \{e_4\}$. Let $\mathcal{T}_{\mathcal{H}}$ $= \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{F}_1, \mathcal{E}), (\mathbb{F}_2, \mathcal{E}), (\mathbb{F}_3, \mathcal{E})\}$ be a hypersoft topology defined on $\mathcal{U}$ where $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$, and $(\mathbb{F}_3, \mathcal{E})$ are hypersoft sets over $\mathcal{U}$, defined as follows

$$(\mathbb{F}_1, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_2, u_3\}), ((e_2, e_3, e_4), \{u_3, u_4\})\}.$$

$$(\mathbb{F}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \phi), ((e_2, e_3, e_4), \{u_3\})\}.$$

$$(\mathbb{F}_3, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_1, u_4\}), ((e_2, e_3, e_4), \{u_1, u_2, u_3\})\}.$$

Suppose that $(\mathbb{G}_1, \mathcal{E})$ and $(\mathbb{G}_2, \mathcal{E})$ are two hypersoft sets over $\mathcal{U}$, defined as follows

$$(\mathbb{G}_1, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_2, u_3, u_4\}), ((e_2, e_3, e_4), \phi)\}.$$

$$(\mathbb{G}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_1\}), ((e_2, e_3, e_4), \mathcal{U})\}.$$

It is easy to see that the two hypersoft sets $(\mathbb{G}_1, \mathcal{E})$ and $(\mathbb{G}_2, \mathcal{E})$ are disjoint hypersoft sets but they are not separated hypersoft sets.

Proposition 3.5. *If $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are separated hypersoft sets over $\mathcal{U}$ and $(\mathbb{G}_1, \mathcal{E}) \,\widetilde{\sqsubseteq}\, (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{G}_2, \mathcal{E}) \,\widetilde{\sqsubseteq}\, (\mathbb{F}_2, \mathcal{E})$, then $(\mathbb{G}_1, \mathcal{E})$ and $(\mathbb{G}_2, \mathcal{E})$ are also separated hypersoft sets.*

Proof. We are given that $(\mathbb{F}_1, \mathcal{E}) \,\widetilde{\sqcap}\, \overline{(\mathbb{F}_2, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\mathbb{F}_1, \mathcal{E})} \,\widetilde{\sqcap}\, (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$. Also, $(\mathbb{G}_1, \mathcal{E}) \,\widetilde{\sqsubseteq}\, (\mathbb{F}_1, \mathcal{E})$ implies $\overline{(\mathbb{G}_1, \mathcal{E})} \,\widetilde{\sqsubseteq}\, \overline{(\mathbb{F}_1, \mathcal{E})}$ and $(\mathbb{G}_2, \mathcal{E}) \,\widetilde{\sqsubseteq}\, (\mathbb{F}_2, \mathcal{E})$ implies $\overline{(\mathbb{G}_2, \mathcal{E})} \,\widetilde{\sqsubseteq}\, \overline{(\mathbb{F}_2, \mathcal{E})}$.

It follows that $(\mathbb{G}_1, \mathcal{E}) \widetilde{\sqcap} \overline{(\mathbb{G}_2, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\mathbb{G}_1, \mathcal{E})} \widetilde{\sqcap} (\mathbb{G}_2, \mathcal{E}) = (\Phi, \mathcal{E})$. Hence, $(\mathbb{G}_1, \mathcal{E})$ and $(\mathbb{G}_2, \mathcal{E})$ are separated hypersoft sets. □

Proposition 3.6. *Two hypersoft closed (resp. hypersoft open) sets* $(\mathbb{F}_1, \mathcal{E})$ *and* $(\mathbb{F}_2, \mathcal{E})$ *of a hypersoft topological space are separated hypersoft sets if and only if they are disjoint hypersoft sets.*

Proof. Since any two separated hypersoft sets are disjoint hypersoft sets, we need only show that two disjoint hypersoft closed (resp. hypersoft open) sets are separated hypersoft sets. If $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are both disjoint hypersoft sets and hypersoft closed sets, then $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$, $\overline{(\mathbb{F}_1, \mathcal{E})} = (\mathbb{F}_1, \mathcal{E})$, $\overline{(\mathbb{F}_2, \mathcal{E})} = (\mathbb{F}_2, \mathcal{E})$ so that $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} \overline{(\mathbb{F}_2, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$ showing that $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are separated hypersoft sets. If $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are both disjoint hypersoft sets and hypersoft open sets, then $(\mathbb{F}_1, \mathcal{E})^c$ and $(\mathbb{F}_2, \mathcal{E})^c$ are hypersoft closed so that $\overline{(\mathbb{F}_1, \mathcal{E})^c} = (\mathbb{F}_1, \mathcal{E})^c$, $\overline{(\mathbb{F}_2, \mathcal{E})^c} = (\mathbb{F}_2, \mathcal{E})^c$. Also, $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$ then $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})^c$ and $(\mathbb{F}_2, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})^c$. Implies that $\overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqsubseteq} \overline{(\mathbb{F}_2, \mathcal{E})^c} = (\mathbb{F}_2, \mathcal{E})^c$ and $\overline{(\mathbb{F}_2, \mathcal{E})} \widetilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})^c} = (\mathbb{F}_1, \mathcal{E})^c$. It follows that $\overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$ and $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} \overline{(\mathbb{F}_2, \mathcal{E})} = (\Phi, \mathcal{E})$. Hence, $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are separated hypersoft sets. □

Proposition 3.7. *Let* $(\mathbb{F}_1, \mathcal{E})$ *and* $(\mathbb{F}_2, \mathcal{E})$ *are separated hypersoft sets of a hypersoft topological space* $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$*. If* $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$ *is a hypersoft closed, then* $(\mathbb{F}_1, \mathcal{E})$ *and* $(\mathbb{F}_2, \mathcal{E})$ *are hypersoft closed.*

Proof. Suppose that $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$ is a hypersoft closed so that $\overline{(\mathbb{F}_1, \mathcal{E})\widetilde{\sqcup}(\mathbb{F}_2, \mathcal{E})} = (\mathbb{F}_1, \mathcal{E})\widetilde{\sqcup}(\mathbb{F}_2, \mathcal{E})$. To prove that $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are hypersoft closed, we have to prove that $\overline{(\mathbb{F}_1, \mathcal{E})} = (\mathbb{F}_1, \mathcal{E})$ and $\overline{(\mathbb{F}_2, \mathcal{E})} = (\mathbb{F}_2, \mathcal{E})$. Since we have $\overline{(\mathbb{F}_1, \mathcal{E})\widetilde{\sqcup}(\mathbb{F}_2, \mathcal{E})} = \overline{(\mathbb{F}_1, \mathcal{E})}\widetilde{\sqcup}\overline{(\mathbb{F}_2, \mathcal{E})}$, then $\overline{(\mathbb{F}_1, \mathcal{E})}\widetilde{\sqcup}\overline{(\mathbb{F}_2, \mathcal{E})} = (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$. Evidently, $\overline{(\mathbb{F}_1, \mathcal{E})} = \overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcap} (\overline{(\mathbb{F}_1, \mathcal{E})}\widetilde{\sqcup}\overline{(\mathbb{F}_2, \mathcal{E})})$ $= \overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcap} ((\mathbb{F}_1, \mathcal{E})\widetilde{\sqcup}(\mathbb{F}_2, \mathcal{E})) = (\overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_1, \mathcal{E})) \widetilde{\sqcup} (\overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})) = (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\Phi, \mathcal{E}) = (\mathbb{F}_1, \mathcal{E})$. Thus, $\overline{(\mathbb{F}_1, \mathcal{E})} = (\mathbb{F}_1, \mathcal{E})$. Similarly, we can prove that $\overline{(\mathbb{F}_2, \mathcal{E})} = (\mathbb{F}_2, \mathcal{E})$. □

Definition 3.8. A hypersoft topological space $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is said to be hypersoft disconnected if and only if $(\Psi, \mathcal{E})$ can be expressed as the union of two non-null separated hypersoft sets. Otherwise, $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is said to be hypersoft connected.

Example 3.9. Let $\mathcal{U} = \{u_1, u_2\}$, $E_1 = \{e_1, e_2\}$, $E_2 = \{e_3\}$, and $E_3 = \{e_4\}$. Let $\mathcal{T}_{\mathcal{H}} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{F}, \mathcal{E})\}$ be a hypersoft topology defined on $\mathcal{U}$ where $(\mathbb{F}, \mathcal{E}) = \{((e_1, e_3, e_4), \mathcal{U}), ((e_2, e_3, e_4), \phi)\}$. Then, it is easy to see that $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is a hypersoft connected space.

Example 3.10. Let $\mathcal{U} = \{u_1, u_2, u_3\}$, $E_1 = \{e_1, e_2\}$, $E_2 = \{e_3\}$, and $E_3 = \{e_4\}$. Let $\mathcal{T}_{\mathcal{H}} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{F}_1, \mathcal{E}), (\mathbb{F}_2, \mathcal{E})\}$ be a hypersoft topology defined on $\mathcal{U}$ where $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are hypersoft sets over $\mathcal{U}$, defined as follows

$$(\mathbb{F}_1, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_1, u_2\}), ((e_2, e_3, e_4), \{u_1\})\}.$$

$$(\mathbb{F}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_3\}), ((e_2, e_3, e_4), \{u_2, u_3\})\}.$$

Then, obviously $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is a hypersoft disconnected space.

Proposition 3.11. *A hypersoft topological space $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is hypersoft disconnected if and only if any one of the following statements holds.*

i. $(\Psi, \mathcal{E})$ *is the union of two non-null disjoint hypersoft open sets;*

ii. $(\Psi, \mathcal{E})$ *is the union of two non-null disjoint hypersoft closed sets.*

Proof. Follows from Definition 3.8 and Proposition 3.6. □

Corollary 3.12. *A hypersoft subspace $(\Upsilon, \mathcal{T}_{\mathcal{H}_\Upsilon}, \mathcal{E})$ of a hypersoft topological space $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is hypersoft disconnected if and only if $(\mathcal{Y}, \mathcal{E})$ is the union of two non-null disjoint hypersoft sets both hypersoft open (resp. hypersoft closed) sets. Thus, $(\Upsilon, \mathcal{T}_{\mathcal{H}_\Upsilon}, \mathcal{E})$ is hypersoft disconnected if and only if there exist two non-null hypersoft sets $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})$ in $\mathcal{U}$ both hypersoft open (resp. hypersoft closed) sets such that $(\mathbb{F}, \mathcal{E}) \tilde{\sqcap} (\mathcal{Y}, \mathcal{E}) \neq (\Phi, \mathcal{E})$, $(\mathbb{G}, \mathcal{E}) \tilde{\sqcap} (\mathcal{Y}, \mathcal{E}) \neq (\Phi, \mathcal{E})$, $((\mathbb{F}, \mathcal{E}) \tilde{\sqcap} (\mathcal{Y}, \mathcal{E})) \tilde{\sqcap} ((\mathbb{G}, \mathcal{E}) \tilde{\sqcap} (\mathcal{Y}, \mathcal{E})) = (\Phi, \mathcal{E})$, and $((\mathbb{F}, \mathcal{E}) \tilde{\sqcap} (\mathcal{Y}, \mathcal{E})) \tilde{\sqcup} ((\mathbb{G}, \mathcal{E}) \tilde{\sqcap} (\mathcal{Y}, \mathcal{E})) = (\mathcal{Y}, \mathcal{E})$.*

Proposition 3.13. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ be two hypersoft connected spaces on $\mathcal{U}$, then $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is a hypersoft connected space over $\mathcal{U}$.*

Proof. Suppose to the contrary that $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is not a hypersoft connected space. By Proposition 3.11, there exist two non-null disjoint hypersoft sets $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}$ such that their union is $(\Psi, \mathcal{E})$ in $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$. Since $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}$ then $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_1}$ and $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E}) \tilde{\in} \mathcal{T}_{\mathcal{H}_2}$. This implies that $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ are two non-null disjoint hypersoft sets such that their union is $(\Psi, \mathcal{E})$ in $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ and $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$ are two non-null disjoint hypersoft sets such that their union is $(\Psi, \mathcal{E})$ in $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ which is a contradiction to given hypothesis. Thus, $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is a hypersoft connected space over $\mathcal{U}$. □

Remark 3.14. The following example shows that the union of two hypersoft connected spaces over the same universe need not be a hypersoft connected.

Example 3.15. Let $\mathcal{U} = \{u_1, u_2\}$, $E_1 = \{e_1, e_2\}$, $E_2 = \{e_3\}$, and $E_3 = \{e_4\}$. Let $\mathcal{T}_{\mathcal{H}_1} = \mathcal{T}_{\mathcal{H}}$ as in Example 3.9 and $\mathcal{T}_{\mathcal{H}_2} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{F}_2, \mathcal{E})\}$ be a hypersoft topology defined on $\mathcal{U}$ where $(\mathbb{F}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \phi), ((e_2, e_3, e_4), \mathcal{U})\}$. Then, $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ are hypersoft connected spaces.

Now, $\mathcal{T}_{\mathcal{H}_1} \tilde{\sqcup} \mathcal{T}_{\mathcal{H}_2} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{F}_1, \mathcal{E}), (\mathbb{F}_2, \mathcal{E})\}$ then $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcup} \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is not a hypersoft connected space since the two hypersoft open sets $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are disjoint hypersoft sets and their union is $(\Psi, \mathcal{E})$ in $\mathcal{T}_{\mathcal{H}_1} \tilde{\sqcup} \mathcal{T}_{\mathcal{H}_2}$.

Proposition 3.16. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ be two hypersoft disconnected spaces on $\mathcal{U}$, then $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcup} \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is a hypersoft disconnected space over $\mathcal{U}$.*

Proof. This is straightforward. □

Remark 3.17. The following example shows that the intersection of two hypersoft disconnected spaces over the same universe need not be a hypersoft disconnected.

Example 3.18. Let $\mathcal{U} = \{u_1, u_2, u_3\}$, $E_1 = \{e_1, e_2\}$, $E_2 = \{e_3\}$, and $E_3 = \{e_4\}$. Let $\mathcal{T}_{\mathcal{H}_1} = \mathcal{T}_{\mathcal{H}}$ as in Example 3.10 and $\mathcal{T}_{\mathcal{H}_2} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{H}_1, \mathcal{E}), (\mathbb{H}_2, \mathcal{E})\}$ be a hypersoft topology defined on $\mathcal{U}$ where

$$(\mathbb{H}_1, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_1, u_3\}), ((e_2, e_3, e_4), \{u_2\})\}.$$

$$(\mathbb{H}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_2\}), ((e_2, e_3, e_4), \{u_1, , u_3\})\}.$$

Then, $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ and $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ are hypersoft disconnected spaces over $\mathcal{U}$. Now, $\mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E})\}$ then $(\mathcal{U}, \mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is not a hypersoft disconnected since there do not exist two non-null disjoint hypersoft open sets such that their union is $(\Psi, \mathcal{E})$ in $\mathcal{T}_{\mathcal{H}_1} \tilde{\sqcap} \mathcal{T}_{\mathcal{H}_2}$.

Proposition 3.19. *A hypersoft topological space $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is hypersoft disconnected if and only if there exists non-null, non-whole hypersoft set which is both hypersoft open and hypersoft closed.*

Proof. Let $(\mathbb{F}, \mathcal{E})$ be a non-null, non-whole hypersoft set which is both hypersoft open and hypersoft closed. We have to show that $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is hypersoft disconnected. Let $(\mathbb{G}, \mathcal{E}) = (\mathbb{F}, \mathcal{E})^c$. Then, $(\mathbb{G}, \mathcal{E})$ is non-null since $(\mathbb{F}, \mathcal{E})$ is non-whole hypersoft set. Moreover, $(\mathbb{F}, \mathcal{E}) \tilde{\sqcup} (\mathbb{G}, \mathcal{E}) = (\Psi, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E}) \tilde{\sqcap} (\mathbb{G}, \mathcal{E}) = (\Phi, \mathcal{E})$. Since $(\mathbb{F}, \mathcal{E})$ is both hypersoft open and hypersoft closed, then $(\mathbb{G}, \mathcal{E})$ is also hypersoft open and hypersoft closed. Hence, $\overline{(\mathbb{F}, \mathcal{E})} = (\mathbb{F}, \mathcal{E})$, $\overline{(\mathbb{G}, \mathcal{E})} = (\mathbb{G}, \mathcal{E})$. It follows that $(\mathbb{F}, \mathcal{E}) \tilde{\sqcap} \overline{(\mathbb{G}, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\mathbb{F}, \mathcal{E})} \tilde{\sqcap} (\mathbb{G}, \mathcal{E}) =$

$(\Phi, \mathcal{E})$. Thus, $(\Psi, \mathcal{E})$ has been expressed as the union of two separated hypersoft sets and so $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is a hypersoft disconnected.

Conversely, let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft disconnected. Then, there exists non-null hypersoft sets $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})$ such that $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} \overline{(\mathbb{G}, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\mathbb{F}, \mathcal{E})} \widetilde{\sqcap} (\mathbb{G}, \mathcal{E}) = (\Phi, \mathcal{E})$ and $(\Psi, \mathcal{E}) = (\mathbb{F}, \mathcal{E}) \widetilde{\sqcup} (\mathbb{G}, \mathcal{E})$. Since $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} \overline{(\mathbb{F}, \mathcal{E})}$, $\overline{(\mathbb{F}, \mathcal{E})} \widetilde{\sqcap} (\mathbb{G}, \mathcal{E}) = (\Phi, \mathcal{E})$ then $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{G}, \mathcal{E}) = (\Phi, \mathcal{E})$. Hence, $(\mathbb{F}, \mathcal{E}) = (\mathbb{G}, \mathcal{E})^c$. Since $(\mathbb{G}, \mathcal{E})$ is non-null and $(\mathbb{G}, \mathcal{E}) \widetilde{\sqcup} (\mathbb{G}, \mathcal{E})^c = (\Psi, \mathcal{E})$, it follows that $(\mathbb{G}, \mathcal{E}) = (\mathbb{F}, \mathcal{E})^c$ is a non-whole hypersoft set. Now, $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcup} \overline{(\mathbb{G}, \mathcal{E})} = (\Psi, \mathcal{E})$. Also, $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} \overline{(\mathbb{G}, \mathcal{E})} = (\Phi, \mathcal{E})$ then $(\mathbb{F}, \mathcal{E}) = [\overline{(\mathbb{G}, \mathcal{E})}]^c$ and similarly $(\mathbb{G}, \mathcal{E}) = [\overline{(\mathbb{F}, \mathcal{E})}]^c$. Since $\overline{(\mathbb{F}, \mathcal{E})}$ and $\overline{(\mathbb{G}, \mathcal{E})}$ are hypersoft closed sets, it follows that $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})$ are hypersoft open sets. Since $(\mathbb{F}, \mathcal{E}) = (\mathbb{G}, \mathcal{E})^c$, $(\mathbb{F}, \mathcal{E})$ is also hypersoft closed set. Thus, $(\mathbb{F}, \mathcal{E})$ is non-null, non-whole hypersoft set which is both hypersoft open and hypersoft closed. We have shown incidentally that $(\mathbb{G}, \mathcal{E})$ is also a non-null, non-whole hypersoft set which is both hypersoft open and hypersoft closed. □

Corollary 3.20. *A hypersoft topological space $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is hypersoft connected if and only if the only hypersoft sets which are both hypersoft open and hypersoft closed are $(\Phi, \mathcal{E})$ and $(\Psi, \mathcal{E})$.*

Corollary 3.21. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft topological space and let Υ be a non-empty subset of $\mathcal{U}$. Then, $(\Upsilon, \mathcal{T}_{\mathcal{H}_\Upsilon}, \mathcal{E})$ is hypersoft disconnected if and only if there exists non-null, non-whole hypersoft set, say, $(\mathbb{F}_\Upsilon, \mathcal{E})$ which is both hypersoft open and hypersoft closed. That is, if and only if there exists a hypersoft open set, say, $(\mathbb{F}, \mathcal{E})$ in $\mathcal{U}$ and a hypersoft closed set, say, $(\mathbb{G}, \mathcal{E})$ in $\mathcal{U}$ such that $(\mathbb{F}_\Upsilon, \mathcal{E}) = (\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathcal{Y}, \mathcal{E})$ and $(\mathbb{F}_\Upsilon, \mathcal{E}) = (\mathbb{G}, \mathcal{E}) \widetilde{\sqcap} (\mathcal{Y}, \mathcal{E})$.*

Proposition 3.22. *A hypersoft topological space $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is hypersoft connected if and only if every non-null, non-whole hypersoft set has a non-null hypersoft boundary.*

Proof. Let every non-null, non-whole hypersoft set has a non-null hypersoft boundary. To show that $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is hypersoft connected. Suppose, if possible $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is hypersoft disconnected. Then there exist non-null disjoint hypersoft sets $(\mathbb{F}, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E})$ both hypersoft open and hypersoft closed sets such that $(\Psi, \mathcal{E}) = (\mathbb{F}, \mathcal{E}) \widetilde{\sqcup} (\mathbb{G}, \mathcal{E})$. Therefore, $(\mathbb{F}, \mathcal{E}) = \overline{(\mathbb{F}, \mathcal{E})} = (\mathbb{F}, \mathcal{E})^o$. But, $(\mathbb{F}, \mathcal{E})^b = \overline{(\mathbb{F}, \mathcal{E})} \setminus (\mathbb{F}, \mathcal{E})^o$. Hence, $(\mathbb{F}, \mathcal{E})^b = (\mathbb{F}, \mathcal{E}) \setminus (\mathbb{F}, \mathcal{E}) = (\Phi, \mathcal{E})$, which is contrary to our hypothesis. Hence, $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ must be hypersoft connected.

Conversely, let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be hypersoft connected and suppose, if possible, there exists a non-null, non-whole hypersoft set $(\mathbb{F}, \mathcal{E})$ such that $(\mathbb{F}, \mathcal{E})^b = (\Phi, \mathcal{E})$. Now, $\overline{(\mathbb{F}, \mathcal{E})} = (\mathbb{F}, \mathcal{E})^o \widetilde{\sqcup} (\mathbb{F}, \mathcal{E})^b = (\mathbb{F}, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}, \mathcal{E})^b$. Hence, $\overline{(\mathbb{F}, \mathcal{E})} = (\mathbb{F}, \mathcal{E})^o = (\mathbb{F}, \mathcal{E})$ showing that $(\mathbb{F}, \mathcal{E})$ is both both hypersoft open and hypersoft closed set and therefore $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is hypersoft disconnected by Proposition 3.19. But this is a contradiction. Hence, every non-null, non-whole hypersoft set must have a non-null hypersoft boundary. □

Proposition 3.23. *Let $(\mathcal{U}_1, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ and $(\mathcal{U}_2, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ be two hypersoft topological spaces. If $(\mathcal{U}_1, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ is hypersoft disconnected and $\mathcal{T}_{\mathcal{H}_1} \widetilde{\sqsubseteq} \mathcal{T}_{\mathcal{H}_2}$, then $(\mathcal{U}_2, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is hypersoft disconnected.*

Proof. Since $(\mathcal{U}_1, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ is hypersoft disconnected, then there exists non-null, non-whole hypersoft set $(\mathbb{F}, \mathcal{E})$ which is both hypersoft open and hypersoft closed in $\mathcal{U}_1$. Since $\mathcal{T}_{\mathcal{H}_2}$ is finer than $\mathcal{T}_{\mathcal{H}_1}$, then $(\mathbb{F}, \mathcal{E})$ is a hypersoft open set belonging to $\mathcal{U}_2$. Again, since $(\mathbb{F}, \mathcal{E})$ is a hypersoft closed set in $\mathcal{U}_1$, then $(\mathbb{F}, \mathcal{E})^c$ is a hypersoft open set. Since $\mathcal{T}_{\mathcal{H}_2}$ is finer than $\mathcal{T}_{\mathcal{H}_1}$, then $(\mathbb{F}, \mathcal{E})^c$ is a hypersoft open set belonging to $\mathcal{U}_2$ and consequently $(\mathbb{F}, \mathcal{E})$ is a hypersoft closed set in $\mathcal{U}_2$. Thus, $(\mathbb{F}, \mathcal{E})$ is a non-null, non-whole hypersoft set which is both hypersoft open and hypersoft closed in $\mathcal{U}_2$. It follows that $(\mathcal{U}_2, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is hypersoft disconnected. □

Corollary 3.24. *Let $(\mathcal{U}_1, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ and $(\mathcal{U}_2, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ be two hypersoft topological spaces. If $(\mathcal{U}_1, \mathcal{T}_{\mathcal{H}_1}, \mathcal{E})$ is hypersoft connected and $\mathcal{T}_{\mathcal{H}_2} \widetilde{\sqsubseteq} \mathcal{T}_{\mathcal{H}_1}$, then $(\mathcal{U}_2, \mathcal{T}_{\mathcal{H}_2}, \mathcal{E})$ is hypersoft connected.*

Definition 3.25. Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft space over $\mathcal{U}$. A hypersoft set $(\mathbb{F}, \mathcal{E})$ is said to be hypersoft disconnected if and only if it is the union of two non-null separated hypersoft sets, that is, if and only if there exists two non-null hypersoft sets $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ such that $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} \overline{(\mathbb{F}_2, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E}) = (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$. A hypersoft set $(\mathbb{F}, \mathcal{E})$ is said to be hypersoft connected if it is not hypersoft disconnected.

Proposition 3.26. *Let $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ be a hypersoft topological space and let $(\mathbb{F}, \mathcal{E})$ be a hypersoft connected set such that $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$ where $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are separated hypersoft sets. Then $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$ or $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$.*

Proof. Since $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ are separated hypersoft sets, then $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} \overline{(\mathbb{F}_2, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$. Now, $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$ then $(\mathbb{F}, \mathcal{E}) = (\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} ((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E})) = ((\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_1, \mathcal{E})) \widetilde{\sqcup} ((\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}))$. We claim that at least one of the hypersoft sets $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})$ is null hypersoft set. For , if possible, suppose none of these hypersoft sets is null, that is, suppose that $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_1, \mathcal{E}) \neq (\Phi, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \neq (\Phi, \mathcal{E})$. Then, $((\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_1, \mathcal{E})) \widetilde{\sqcap} \overline{((\mathbb{F}, \mathcal{E})\widetilde{\sqcap}(\mathbb{F}_2, \mathcal{E}))} \widetilde{\sqsubseteq} ((\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_1, \mathcal{E})) \widetilde{\sqcap} (\overline{(\mathbb{F}, \mathcal{E})} \widetilde{\sqcap} \overline{(\mathbb{F}_2, \mathcal{E})}) = ((\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} \overline{(\mathbb{F}, \mathcal{E})}) \widetilde{\sqcap} ((\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} \overline{(\mathbb{F}_2, \mathcal{E})}) = (\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\Phi, \mathcal{E}) = (\Phi, \mathcal{E})$. Similarly, $\overline{((\mathbb{F}, \mathcal{E})\widetilde{\sqcap}(\mathbb{F}_1, \mathcal{E}))} \widetilde{\sqcap} ((\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})) = (\Phi, \mathcal{E})$. Hence, $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})$ are separated hypersoft sets. Thus, $(\mathbb{F}, \mathcal{E})$ has been expressed as union of two separated hypersoft sets and consequently $(\mathbb{F}, \mathcal{E})$ is hypersoft disconnected.

But this is a contradiction. Hence, at least one of the hypersoft sets $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})$ is null hypersoft set. If $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_1, \mathcal{E}) = (\Phi, \mathcal{E})$, then $(\mathbb{F}, \mathcal{E}) = (\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})$ which implies that $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$. Similarly, if $(\mathbb{F}, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$, then $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$. Hence, either $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$ or $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$. □

Proposition 3.27. *Let $(\mathbb{F}, \mathcal{E})$ be a hypersoft connected set and $(\mathbb{G}, \mathcal{E})$ be any hypersoft set such that $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} \overline{(\mathbb{F}, \mathcal{E})}$, then $(\mathbb{G}, \mathcal{E})$ is hypersoft connected. In particular, $\overline{(\mathbb{F}, \mathcal{E})}$ is hypersoft connected.*

Proof. Suppose $(\mathbb{G}, \mathcal{E})$ is hypersoft disconnected. Then, there exist non-null hypersoft sets $(\mathbb{F}_1, \mathcal{E})$ and $(\mathbb{F}_2, \mathcal{E})$ such that $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcap} \overline{(\mathbb{F}_2, \mathcal{E})} = (\Phi, \mathcal{E})$ and $\overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$ and $(\mathbb{G}, \mathcal{E}) = (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$. Since $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{G}, \mathcal{E}) = (\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E})$, it follows from Proposition 3.26 that $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$ or $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_2, \mathcal{E})$. Let $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{F}_1, \mathcal{E})$ which implies that $\overline{(\mathbb{F}, \mathcal{E})} \widetilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})}$ then $\overline{(\mathbb{F}, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) \widetilde{\sqsubseteq} \overline{(\mathbb{F}_1, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$, but $(\Phi, \mathcal{E}) \widetilde{\sqsubseteq} \overline{(\mathbb{F}, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E})$, then we have $\overline{(\mathbb{F}, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$. Also, $(\mathbb{F}_1, \mathcal{E}) \widetilde{\sqcup} (\mathbb{F}_2, \mathcal{E}) = (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} \overline{(\mathbb{F}, \mathcal{E})}$ then $(\mathbb{F}_2, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} \overline{(\mathbb{F}, \mathcal{E})}$ implies that $\overline{(\mathbb{F}, \mathcal{E})} \widetilde{\sqcap} (\mathbb{F}_2, \mathcal{E}) = (\mathbb{F}_2, \mathcal{E})$. Hence, $(\mathbb{F}_2, \mathcal{E}) = (\Phi, \mathcal{E})$ which is a contradiction since $(\mathbb{F}_2, \mathcal{E})$ is non-null. Hence, $(\mathbb{G}, \mathcal{E})$ must be hypersoft connected. Again, since $(\mathbb{F}, \mathcal{E}) \widetilde{\sqsubseteq} (\mathbb{G}, \mathcal{E}) \widetilde{\sqsubseteq} \overline{(\mathbb{F}, \mathcal{E})}$, we see that $\overline{(\mathbb{F}, \mathcal{E})}$ is hypersoft connected. □

Proposition 3.28. *Let $\{(\mathbb{F}_i, \mathcal{E}) \mid i \in I\}$ be the family of hypersoft connected sets such that $\widetilde{\sqcap}\{(\mathbb{F}_i, \mathcal{E}) \mid i \in I\} \neq (\Phi, \mathcal{E})$. Then $\widetilde{\sqcup}\{(\mathbb{F}_i, \mathcal{E}) \mid i \in I\}$ is hypersoft connected sets.*

Proof. Suppose $(\digamma, \mathcal{E}) = \widetilde{\sqcup}\{(\mathbb{F}_i, \mathcal{E}) \mid i \in I\}$ is not hypersoft connected. Then, there exist two non-null disjoint hypersoft sets $(\digamma_1, \mathcal{E})$ and $(\digamma_2, \mathcal{E})$ both hypersoft open such that $(\digamma, \mathcal{E}) = (\digamma_1, \mathcal{E}) \widetilde{\sqcup} (\digamma_2, \mathcal{E})$. For each i, $(\digamma_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_i, \mathcal{E})$ and $(\digamma_2, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_i, \mathcal{E})$ are disjoint hypersoft sets both hypersoft open in $(\mathbb{F}_i, \mathcal{E})$ such that $((\digamma_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_i, \mathcal{E})) \widetilde{\sqcup} ((\digamma_2, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_i, \mathcal{E})) = ((\digamma_1, \mathcal{E}) \widetilde{\sqcup} (\digamma_2, \mathcal{E})) \widetilde{\sqcap} (\mathbb{F}_i, \mathcal{E}) = (\mathbb{F}_i, \mathcal{E})$. Since $(\mathbb{F}_i, \mathcal{E})$ is hypersoft connected, one of the hypersoft sets $(\digamma_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_i, \mathcal{E})$ and $(\digamma_2, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_i, \mathcal{E})$ must be null hypersoft set, say, $(\digamma_1, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_i, \mathcal{E}) = (\Phi, \mathcal{E})$. Then, we have $(\digamma_2, \mathcal{E}) \widetilde{\sqcap} (\mathbb{F}_i, \mathcal{E}) = (\mathbb{F}_i, \mathcal{E})$ which implies that $(\mathbb{F}_i, \mathcal{E}) \widetilde{\sqsubseteq} (\digamma_2, \mathcal{E})$ for all $i \in I$ and hence $\widetilde{\sqcup}\{(\mathbb{F}_i, \mathcal{E}) \mid i \in I\} \widetilde{\sqsubseteq} (\digamma_2, \mathcal{E})$, that is, $(\digamma_1, \mathcal{E}) \widetilde{\sqcup} (\digamma_2, \mathcal{E}) \widetilde{\sqsubseteq} (\digamma_2, \mathcal{E})$. This gives, $(\digamma_1, \mathcal{E}) = (\Phi, \mathcal{E})$ which is a contradiction since $(\digamma_1, \mathcal{E})$ is non-null. Hence, $(\digamma, \mathcal{E})$ must be hypersoft connected. □

Definition 3.29. A property of a hypersoft topological space is said to be hypersoft hereditary if every hypersoft subspace of the space has that property.

Remark 3.30. The hypersoft disconnectedness (resp. hypersoft connectedness) is not a hypersoft hereditary.

Example 3.31. Let $\mathcal{U} = \{u_1, u_2, u_3\}$, $E_1 = \{e_1, e_2\}$, $E_2 = \{e_3\}$, and $E_3 = \{e_4\}$. Let $\mathcal{T}_{\mathcal{H}} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{F}_1, \mathcal{E}), (\mathbb{F}_2, \mathcal{E})\}$ be a hypersoft topology defined on $\mathcal{U}$, where $(\mathbb{F}_1, \mathcal{E})$, and $(\mathbb{F}_2, \mathcal{E})$, are hypersoft sets over $\mathcal{U}$, defined as follows

$$(\mathbb{F}_1, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_2\}), ((e_2, e_3, e_4), \{u_1\})\}.$$

$$(\mathbb{F}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_1, u_3\}), ((e_2, e_3, e_4), \{u_2, u_3\})\}.$$

Then, $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is a hypersoft disconnected space.

Now, let $\Upsilon = \{u_3\}$, then $\mathcal{T}_{\mathcal{H}_\Upsilon} = \{(\Phi, \mathcal{E}), (\mathcal{Y}, \mathcal{E})\}$ is a hypersoft topology defined on $\mathcal{Y}$. Since $(\Phi, \mathcal{E})$ and $(\mathcal{Y}, \mathcal{E})$ are the only hypersoft open and hypersoft closed sets then by Corollary 3.20, $(\Upsilon, \mathcal{T}_{\mathcal{H}_\Upsilon}, \mathcal{E})$ is a hypersoft connected subspace of hypersoft disconnected space.

Example 3.32. Let $\mathcal{U} = \{u_1, u_2, u_3\}$, $E_1 = \{e_1, e_2\}$, $E_2 = \{e_3\}$, and $E_3 = \{e_4\}$. Let $\mathcal{T}_{\mathcal{H}} = \{(\Phi, \mathcal{E}), (\Psi, \mathcal{E}), (\mathbb{F}_1, \mathcal{E}), (\mathbb{F}_2, \mathcal{E}), (\mathbb{F}_3, \mathcal{E})\}$ be a hypersoft topology defined on $\mathcal{U}$, where $(\mathbb{F}_1, \mathcal{E})$, $(\mathbb{F}_2, \mathcal{E})$, and $(\mathbb{F}_3, \mathcal{E})$ are hypersoft sets over $\mathcal{U}$, defined as follows

$$(\mathbb{F}_1, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_1\}), ((e_2, e_3, e_4), \{u_2\})\}.$$

$$(\mathbb{F}_2, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_2\}), ((e_2, e_3, e_4), \{u_1\})\}.$$

$$(\mathbb{F}_3, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_1, u_2\}), ((e_2, e_3, e_4), \{u_1, u_2\})\}.$$

Then, $(\mathcal{U}, \mathcal{T}_{\mathcal{H}}, \mathcal{E})$ is a hypersoft connected space.

Now, let $\Upsilon = \{u_1, u_2\}$, then $\mathcal{T}_{\mathcal{H}_\Upsilon} = \{(\Phi, \mathcal{E}), (\mathcal{Y}, \mathcal{E}), (\mathbb{F}_{1_\Upsilon}, \mathcal{E}), (\mathbb{F}_\Upsilon, \mathcal{E}), (\mathbb{F}_{3_\Upsilon}, \mathcal{E})\}$ is a hypersoft topology defined on $\mathcal{Y}$, where $(\mathbb{F}_{1_\Upsilon}, \mathcal{E})$, $(\mathbb{F}_{2_\Upsilon}, \mathcal{E})$, and $(\mathbb{F}_{3_\Upsilon}, \mathcal{E})$ are hypersoft sets over $\mathcal{Y}$, defined as follows

$$(\mathbb{F}_{1_\Upsilon}, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_1\}), ((e_2, e_3, e_4), \{u_2\})\}.$$

$$(\mathbb{F}_{2_\Upsilon}, \mathcal{E}) = \{((e_1, e_3, e_4), \{u_2\}), ((e_2, e_3, e_4), \{u_1\})\}.$$

$$(\mathbb{F}_{3_\Upsilon}, \mathcal{E}) = \{((e_1, e_3, e_4), \Upsilon), ((e_2, e_3, e_4), \Upsilon)\} = (\mathcal{Y}, \mathcal{E}).$$

It is easy to see that $(\Upsilon, \mathcal{T}_{\mathcal{H}_\Upsilon}, \mathcal{E})$ is a hypersoft disconnected subspace of hypersoft connected space.

4. Conclusions

In this paper, we have initiated the concept of hypersoft connected (resp. hypersoft disconnected) spaces. Then, some results of this concept were discussed. Furthermore, we have presented the concepts of disjoint hypersoft sets, separated hypersoft sets, and hypersoft hereditary property along with some illustrative examples. In future studies, we can define some other topological structures in the frame of hypersoft topological spaces such as hypersoft locally connected space, hypersoft component, hypersoft compact space, and hypersoft paracompact space. Moreover, we can define hypersoft separation axioms by using both ordinary points and hypersoft points.

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Classification of Ordered Semigroups Through Neutrosophic Generalized bi-ideals with Applications

Faiz Muhammad Khan [1,*], Madad Khan [2] and Ihsanullah[1]

[1]Department of Mathematics and Statistics, University of Swat

[2] Department of Mathematics,COMSATS University Islamabad, Pakistan

*Correspondence: drfaiz@uswat.edu.pk

Abstract. An icebreaking theory known as neutrosophic theory opened a new direction for researchers of philosophy, logics, set theory and probability/statistics. Neutrosophy put the point base for a entire household of new mathematical speculations that summarized classical and fuzzy correspondence theories. In this article, we introduced the conception of neutrosophic fuzzy ideal theory of ordered semigroups based on belongs to relation and quasi-coincident with relation. Particularly, neutrosophic fuzzy generalized bi-ideal (resp. bi-ideal) of type $(\in, \in \vee q)$ have been developed and detail symposium on multi-dimension of the neutrosophic said ideals in ordered semigroup has given. Further, a verity of depictions of ordered semigroups in expression of $(\in, \in \vee q)$-fuzzy generalized bi-ideals have been constructed and several related examples have been formulated. Finally, the lower parts of neutrosophic $(\in, \in \vee q)$-fuzzy generalized bi-ideals were proposed and ordered semigroups have been discussed by the properties of these newly developed neutrosophic fuzzy generalized bi-ideals.

Keywords: Ordered Semigroup; Neutrosophic set; Neutrosophic $(\in, \in \vee q)$-fuzzy bi-ideal; Neutrosophic $(\in, \in \vee q)$-fuzzy generalized bi-ideal; lower Parts of the Neutrosophic generalized bi-ideals.)

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1. **Introduction**

In the modern times, economics and technological progress play a pivotal part in the evolution of at all particular country. Caused by high-quality analysis in the new field such as computer science, control system engineering, analyses of the data , economics, error-correction coding, answerable, prediction and automated, most realms have fallen back. These new realms spend a large scrap of their budgets in these areas. From another point of view, the above-mentioned meadows face several complex issues calling for uncertainty. These completed issues cannot be solved by traditional techniques. There are definite types of speculations, such as theoretical probability, fuzzy set theory, rough set theory, and soft set theory, which can be use for the above problems. However, all these theories have their importance and inherent limitations. One of the main problems accepted by these speculations is their incompatibility with parametric implements. In order to control such labours, in 1965, Zadeh [1] introduce the ice breaking conception of fuzzy subset, which could handling imprecision and uncertainties of these king of problems. So, here we specify some terms which is used throughout my thesis, FG is used for fuzzy group, OSG is used for ordered semigroup, OSGs used for ordered semigroups, SUBG used for subgroup, for fuzzy left (resp. right) ideal used FL(resp. right)I, for fuzzy generalized bi-ideal FGB-I is used, for quasi-ideal Q-I is used, for bi-ideal B-I is used, for fuzzy subgroup FSUBG is used, for regular RG is used, for completely regular CRG is used, for intra-regular intra-RG is used, for semigroup SG is used, for prime P is used, for semiprime SP is used, for simple SMP is used, for left simple LSMP is used, for quasi-prim Q-P is used, for weakly quasi-prim Q-P is used, for fuzzy left ideal FLI is used, $(\in, \in \vee q)$-fuzzy left ideal $(\in, \in \vee q)$-FLI is used, for subsemigroup SUBSG is used, for interior ideal II is used, for fuzzy set FS is used, for $(\in, \in \vee q)$-fuzzy generalized bi-ideal $(\in, \in \vee q)$-FGB-I is used, for $(\in, \in \vee q)$-fuzzy bi-ideal $(\in, \in \vee q)$-FB-I is used, for right simple RSMP is used, for fuzzy quasi-prime ideal FQ-PI is used, for strongly regular SRG is used, for fuzzy quasi-ideal FQ-I is used, and for plural only small "s" is added at the end e.g fuzzy ideals FIs, for weakly prime fuzzy ideal WPFI is used, for completely prime fuzzy ideal CPFI is used, for completely semiprime fuzzy ideal CSPFI is used, for fuzzy point FP is used, for quaasi Q is used, for subgroup SUBG is used, for $(\in, \in \vee q)$-fuzzy left (resp. right) ideal $(\in, \in \vee q)$-FL(resp. right)I is used, for intutionistic fuzzy set IFS is used, for neutrosophic set NS is used, for neutrosophic$(\in, \in \vee q)$-fuzzy generalized bi-ideal neutrosophic $(\in, \in \vee q)$-FGB-I is used, for intutionistic set IS is used, for paraconsistent set PS is used, for strongly simple SSMP is used, for characteristic function CF is used, for lower part LP used, and other also expressed on the same way else stated. Further Zadeh [4-7] elaborated the conception of fuzzy set to a large extent. After that 1971, Rosenfeld [2] proposed the new conception of fuzzy group(FG) it opened a new direction for the scientists to assessment different conceptions and consequence from the principality of algebra in the larger flow of fuzzy surroundings. Possess the inspiration considering, Das [3] suggested the conception of level subgroup of the FG. Further, Kruoki [12-16] described the notions of fuzzy left (resp. right, bi-, quasi-, generalised bi-) ideals of SGs and thereby identified various classes (regular, intra-regular, completely regular, semiprime, left simple) of semigroups in terms of these conceptions. The renowned research group of Kehayopulu [17-21] studied fuzzy left (right, bi-, interior and quasi-) ideals in OSGs to a greater extent. Moreover, The conception of (α, β)-FSUBG by utilizing the "belongs to" relation $(\in)$ and "quasi-coincident with "relation (q) of fuzzy point(FP) with fuzzy set(FS) by studied by Bakat and Das [9, 10] and Bakat [11]. Further the conception of the sort of (α, β)-FIIs, and new conception of sort an $(\in, \in \vee q_k)$-FII of an OSG of S is determined by khan et al. [32], here k is an testimonial component of $[0, 1)$. Otherwise expressed. Further demonstrated that in a regular(resp. semisimple) of OSG, the ideas of $(\in, \in \vee q_k)$-FI and $(\in, \in \vee q_k)$-FIIs matched. Similarly, the conception of (α, β)-FL(resp. right)Is of an OSG of S and the new kind of FL(resp. right)Is of the type $(\in, \in \vee q_k)$-F (resp. right)Is, here $k \in [0,\ 1)$. Special in this paper, reported the relation between ordinary FI and $(\in, \in \vee q_k)$-FIs of an OSG was initiated by Khan et al. [33]. After this, the conception of IFSs, which is a extension of FSs and provably equivalent to interval valued FSs are initiated by Atanassov [22], in 1986.

Further, in 1998 Smarandche [23] generalized the ice breaking conception of IFS, PS, and IS to the NS, and initiated the new conception of the NS.
After this big achievement Maji [24] reported the conception of neutrosophic soft set. Moreover, using the conception neutrosophic solution to make MCDM standard decisions. In addition to studying some interesting mathematical properties of the method, the algorithm neut-MCDM is also proposed. This work also provides a concise basis for the MCDM community with the first introduction of the NS this work proposes a multi approach which was investigated by Kharal [25]. However Salama et al. [26] investigate the notion of "neutosophic crisp neighborhoods system for the neutrosophic crisp point". In addition, to introduced and investigated the notion of the local function of the neutrosophic crispness, and constructed a new type of neutrosophic crisp topological space through the ideals of the neutrosophic crispness. It involves the possible application of GIS topology rules. Further the notion of rough NS was Studied by Broumi et al. [27], in this article they developed a hybrid structure said to be " Rough Neutrosophic Sets(RNSs)" and also, investigate their possessions. Therefore, both the NS theory and rough set theory are becoming a powerful tools for managing uncertainty, incompleteness and imprecision information. Mover the operation on the interval NS were investigated by Broumi and Smarandache [28], in this paper they further defines three new operation on interval NSs whic is based on arthmetic mean, geometric mean and harmonic mean. So the interval NS is an example of NS, which can be used in actual science and engineering

2. **Some Basic Definitions and Results**

Definition 2.1. If $(S, .)$ is semigroup, then the structure $(S, ., \leq)$ is called an OSG, $(S, \leq)$ is a partially ordered set (poset) i.e $\alpha \leq \alpha$ (reflexive), $\alpha \leq \beta, \beta \leq \gamma \Rightarrow \alpha = \beta$ (anti-symmetric), $\alpha \leq \beta, \beta \leq \gamma \Rightarrow \alpha \leq \gamma$,(transitive) $\forall \alpha,\ \beta,\ \gamma\ \in S$ and $x \leq y \Rightarrow x\alpha \leq y\beta$ and $xa \leq xb$ $\forall\ a, b, x\ \in S$.

Definition 2.2. Let X be an OSG S. Then interpret the subset (X] of S as.

$(X] = \{y \in S | y \leq x$ for some $x \in X\}$.
If $X = \{x\}$, then the notion (x] is used of $(\{x\}]$. For any subset X and Y of S, $XY = \{xy | x \in X$ and $y \in Y\}$ so throughout my thesis S is an OSG unless otherwise indicated.
The definition of SUBSG and left (right) ideal are discussed as follow.

Lemma 2.3. *If S be an OSG, then the understated condition are equivalently:*
(1) S is left WRG.
(2) $\Gamma \cap \Omega \subseteq (\Gamma\Omega]$, $\forall$ *ideal* Γ *and GB-I* Ω *of S.*
(3) $\Gamma(c) \cap \Omega(c) \subseteq (\Gamma(c)\Omega(c)]$, $\forall\ c \in S$.

Lemma 2.4. *If S be an OSG, then the understated axioms are equivalently:*
(1) S is LWRG.
(2) $\Gamma \cap \Psi \subseteq (\Gamma\Psi]$, $\forall$ *ideal* Γ *and left ideal* Ψ *of S.*
(3) $\Gamma(x) \cap \Psi(x) \subseteq (\Gamma(x)\Psi(\alpha)]$, $\forall\ x \in S$.

Lemma 2.5. *If S be an OSG, then the understated axioms are equivalently:*
(1) S is RG.
(2) $\Gamma \cap \Omega \cap \Psi \subseteq (\Gamma\Omega\Psi]$, $\forall$ *right ideal* Γ*, GB-I* Ω *and left ideal* Ψ *of S.*
(3) $\Gamma(z) \cap \Omega(z) \cap \Psi(z) \subseteq (\Gamma(z)\Omega(z)\Psi(z)]$, $\forall\ z \in S$.

Lemma 2.6. *If S be an OSG, then the understated axioms are equivalently:*
(1) S is RG.
(2) $\Omega \cap \Gamma \subseteq (\Omega\Gamma\Omega]$, $\forall\ \Omega$ *GB-I and* Γ *ideal of S.*
(3) $(\Omega(k) \bigcap \Psi(k) \subseteq (\Omega(k)\Gamma(k)]$, $\forall\ k \in S$.

Lemma 2.7. *An OSG S is completely regular* $\Leftrightarrow \forall\ X \subseteq S$*, we have,* $X \subseteq (X^2SX^2]$.

Lemma 2.8. *An OSG S is L(resp. right)SMP if,* $\forall$ *(Sx]=S, (resp.(xS]=S for all* $x \in S$.

Proposition

If χ and ψ are any subsets of an OSG S, then

(1) $\chi \subseteq (\chi]$.
(2) $(\chi](\psi] \subseteq (\chi\psi]$.
(3) $((\chi]] = (\chi]$.
(4) $((\chi](\psi] = (\chi\psi]$.

Proposition

Let X and $Y \neq \phi$ subsets of S, then we have the understated condition holds:

(1) $X \subseteq Y$ iff $A_X \preceq B_Y$.

(2) $A_X \wedge A_Y = A_{X \cap Y}$.

(3) $A_X \circ A_Y = A_{(XY]}$.

3. **Neutrosophic sets (Basic Operation)**

In the past two decennaries, the utilizes of soft set theory has made one more climacteric in mathematics. In mathematics, some mathematical enigmas contain indeterminacy in different paddock, such as answerable, automaton1 theory, coding theory, economics and memorandums of understanding, while other mathematical problems cannot be solved by ordinary mathematics. Due to the influence of parameterizations, tools (such as fuzzy set theory, probability theory,etc), the newest investigation on in this managment and the new investigation on theory of soft are fruitful due to the diversified uses of soft sets in the above-mentioned fields [27, 28]. It is worth noting that Sezgum and Atagum [29] studied various new actions on theory of soft and explained soft sets the following way:

Definition 3.1. If X is non-empty set, then structure λ in X is of the structure $\lambda = \{\langle a; \lambda_T(a), \lambda_I(a),$
$\lambda_F(a)\rangle | a \in X\}$ is called NS, where $\lambda_T : X \to [0,1]$ is a truth membership function, $\lambda_I : X \to [0,1]$ is an indeterminate membership function and $\lambda_F : X \to [0,1]$ is false membership function. Generalizing the notion of an ordered FP, we introduce a new concept called neutrosophic ordered points(NOPs) as follows:

Suppose S is an OSG, t $\in$ S *and* $u, v, w \in [0,1]$. By a NOPs, we mean $t_{\tilde{p}}(x) = \langle t_u(x), t_v(x), t_w(x)\rangle$ where $\tilde{p} = (u, v, w)$ and

$$t_u(x) = \begin{cases} u, \; if \; x \in (t], \\ \quad 1, \; ifnot. \end{cases}$$

$$t_v(x) = \begin{cases} v, \; if \; x \in (t], \\ \quad 1, \; ifnot. \end{cases}$$

$$t_w(x) = \begin{cases} w, \; if \; x \in (t], \\ \quad 1, \; ifnot. \end{cases}$$

Let $\lambda = \{\langle a; \; \lambda_T(a), \; \lambda_I(a), \; \lambda_F(a)\rangle\}$ be a NS and $t_{\tilde{p}}$ be a NOP, we define

$$i \rightarrow t_{\tilde{p}} \ \in \ \lambda \ if \begin{cases} \lambda_T(t) \leq \ u, \\ \lambda_I(t) \leq \ v, \\ \lambda_F(t) \geq \ w, \end{cases}$$

$$ii \rightarrow t_{\tilde{p}} \ q \ \lambda \ if \begin{cases} \lambda_T(t) + u \ < 1, \\ \lambda_I(t) + v \ < 1, \\ \lambda_F(t) + w \ > 1, \end{cases}$$

$iii \rightarrow t_{\tilde{p}} \ \in \ \vee q \ \lambda \Rightarrow t_{\tilde{p}} \ \in \ \lambda \ or \ t_{\tilde{p}} q \lambda.$

$iv \rightarrow t_{\tilde{p}} \ \in \ \wedge q \ \lambda \Rightarrow t_{\tilde{p}} \ \in \ \lambda \ and \ t_{\tilde{p}} q \lambda.$

$v \rightarrow t_{\tilde{p}} \ \overline{\in \wedge q} \ \lambda \Rightarrow t_{\tilde{p}} \ \in \ \wedge q \ \lambda$ does not hold.

If $\lambda = \langle x; \lambda_T(x), \ \lambda_I(x), \ \lambda_F(x)\rangle$ and $\eta = \langle x; \ \eta_T(x), \ \eta_I(x), \ \eta_F(x)\rangle$ be neutrosophic sets, then $\lambda \otimes \eta, \ \lambda \tilde{\cap} \eta$ and $\lambda \tilde{\cup} \eta$ are defined as follow:

$\lambda \tilde{\cap} \eta = \{\langle x; \ ((\lambda_T \ o \ \eta_T), \ (\lambda_I \ o \ \eta_I), \ (\lambda_F \ o \ \eta_F)) \ (x)\rangle\}$
where $\lambda_T \ o \ \eta_T, \ \lambda_I \ o \ \eta_I \ and \ \lambda_F \ o \ \eta_F$ are defined as

$$(\lambda_T \ o \ \eta_T)(x) = \begin{cases} \wedge_{(y,z)\in A_x} [\lambda_T(y) \vee \eta_T(z)] \ if \ A_x \neq \phi, \\ 1 \ if \ A_x = \phi. \end{cases}$$

$$(\lambda_I \ o \ \eta_I)(x) = \begin{cases} \wedge_{(y,z)\in A_x} [\lambda_I(y) \vee \eta_I(z)] \ if \ A_x \neq \phi, \\ 1 \ if \ A_x = \phi. \end{cases}$$

$$(\lambda_F \ o \ \eta_F)(x) = \begin{cases} \vee_{(y,z)\in A_x} [\lambda_F(y) \wedge \eta_F(z)] \ if \ A_x \neq \phi, \\ 1 \ if \ A_x = \phi. \end{cases}$$

$\lambda \tilde{\cap} \eta = \{\langle x; \ ((\lambda_T \cap \eta_T), \ (\lambda_I \cap \eta_I), \ (\lambda_F \cap \eta_F))(x)\rangle\},$

where

$(\lambda_T \cap \eta_T)(x)$=max$\{\lambda_T(x), \ \eta_T(x), \ 0.5\}$

$(\lambda_I \cap \eta_I)(x)$=max$\{\lambda_I(x), \ \eta_I(x), \ 0.5\}$

$(\lambda_F \cap \eta_F)(x)$=min$\{\lambda_F(x), \eta_F(x), \ 0.5\}$

and

$\lambda \tilde{\cup} \eta = \{\langle x;\ ((\lambda_T \cup \eta),\ (\lambda_I \cup \eta_I),\ (\lambda_F \cup \eta_F))(x)\rangle\},$

where

$(\lambda_T \cup \eta_T)(x)$=min$\{\lambda_T(x),\ \eta_T(x),\ 0.5\}$

$(\lambda_I \cup \eta_I)(x)$=min$\{\lambda_I(x),\ \eta_I(x),\ 0.5\}$

$(\lambda_F \cup \eta_F)(x)$=max$\{\lambda_F(x),\ \eta_F(x),\ 0.5\}$

Note that if NOP (S) is the group of all NOPs in OSG S,

then

$t_{\tilde{p}}.s_{\tilde{q}} = (ts)\begin{pmatrix} u_1 \vee u_2 \\ v_1 \vee v_2 \\ w_1 \wedge w_2 \end{pmatrix} \in NOP\ (S)$ where$\tilde{p} = (u_1, v_1, w_1)$ and $\tilde{q} = (u_2, v_2, w_2)$.

Example

General example of neutrosophic set. The premise " Today is Sunny" or "Today will be Sunny" it does not convey a fixed rate constituent structure; this may assume to 50% true, 45% indeterminate and 40% false at time t_n where $n \geq 0$,but at the time t_{n+1} may be alter at 55% true, 46% indeterminate, and 28% false,(as stated to the new conformation source) and today at utter t_{n+60} the same premise may be 100% true, 0% indeterminate and 0% false (if today indeed sunny) this structure is dynamic; so the truth value change from time to time, another point of view, the truth value of the premise may be change from place to place
e.g;
the premise "It is sunny" in Islamabad, 100% true, 0% uncertain, and 0% false, but on the move to another site the city of Karachi the truth rate will be altered and may be 0% true, 0% indeterminate and 100% false It is also alter w.r.t viewer(subject to the parameter of the function T, I, F)
e.g;
"Simith is longer" $(.42\%, .64\%, .56\%)$ as stated to his mother, but (0.86% $0.23\%, 0.7\%)$ as stated to his personal Secretary, or $(0.48\%, 0.21\%, 0.31\%)$ as stated to his Boss.

4. **Neutrosophic generalized bi-ideals of ordered semigroups**

In this part, we initiate the conceptions of NS, neutrosophic $(\in, \in \vee q)$-SUBSG, neutrosophic $(\in, \in \vee q)$-FGB-I, neutrosophic $(\in, \in \vee q)$-FB-I, neutrosophic $(\in, \in \vee q)$-FL(resp. right)Is, neutrosophic level subset ,regular, weakly regular, related examples, theorems and propositions in detail.

For simplicity throughout the paper λ will be denoted for NS instead of$\lambda = \langle a;\ \lambda_T(a),\ \lambda_I(a),\ \lambda_F(a)\rangle$ unless otherwise stated.

Definition 4.1. A NS λ of an OSG S is called a neutrosophic $(\in, \in \vee q)$-SUBSG of S if the understated axiom is satisfied:
$(\forall t,\ s \in S)\ (\tilde{p} = (u_1,\ v_1,\ w_1),\ \tilde{q} = (u_2,\ v_2,\ w_2) \in [0,1])$

$$\left(t_{\tilde{p}} \in \lambda;\ s_{\tilde{q}} \in \lambda \Rightarrow (ts)_{\begin{pmatrix} u_1 \vee u_2 \\ v_1 \vee v_2 \\ w_1 \wedge w_2 \end{pmatrix}} \in \vee q\lambda \right).$$

Definition 4.2. A neutrosophic set λ of an OSG S is said to be a neutrosophic $(\in, \in \vee q)$-FL(resp. right)I of S if the understated axioms are contented:

$(i)(\forall t,\ s \in S\ with\ t \leq s)(\tilde{p} = (u_1,\ v_1,\ w_1) \in [0,\ 1])(s_{\tilde{p}} \in \lambda \Rightarrow t_{\tilde{p}} \in \vee q\lambda)$.

$(ii)(\forall t,\ s \in S)(\tilde{p} = (u_1,\ v_1,\ w_1) \in [0,\ 1])(s_{\tilde{p}} \in \lambda \Rightarrow (ts)_{\tilde{p}} \in \vee q\lambda(resp.(st)$
$_{\tilde{p}} \in \vee q\lambda))$.
Note that aneutrosophic set λ of S is a neutrosophic $(\in, \in \vee q)$-FI of S if it is both neutrosophic $(\in, \in \vee q)$-FLI and neutrosophic $(\in, \in \vee q)$-FRI of S.

Definition 4.3. A neutrosophic set(NS) λ of an OSG S is said to be a neutrosophic $(\in, \in \vee q)$-FGB-I of S if the understating axioms are contented:

$(i)(\forall t,\ s \in S\ with\ t \leq s)(\tilde{p} = (u_1,\ v_1,\ w_1) \in [0,\ 1])(s_{\tilde{p}} \in \lambda \Rightarrow t_{\tilde{p}} \in \vee q\lambda)$.

$$(ii)(\forall t,a,s\in S\ t\leq s)(\tilde{p}=(u_1,v_1,w_1),\tilde{q}=(u_2,v_2,w_2)\in[0,1])$$
$$\left(t_{\tilde{p}}\in\lambda; s_{\tilde{q}}\in\lambda\Rightarrow(tas)_{\begin{pmatrix}u_1\vee u_2\\ v_1\vee v_2\\ w_1\wedge w_2\end{pmatrix}}\in\vee q\lambda\right).$$

Definition 4.4. A neutrosophic set λ of an ordered semigroup S is said to be a neutrosophic $(\in,\in\vee q)$-FB-I of S if it is both a neutrosophic $(\in,\in\vee q)$-FGB-I and neutrosophic$(\in,\in\vee q)$-SUBSG of S.

Theorem 4.5. *Suppose that G is a GB-I of an OSG S and λ is a neutrosophic subset of S such that:*

$$\lambda_T(x)=\begin{cases}1\ if\ x\ \in S-G,\\ \leq 0.5\ if\ x\in G.\end{cases}$$
$$\lambda_I(x)=\begin{cases}1\ if\ x\ \in S-G,\\ \leq 0.5\ if\ x\in G.\end{cases}$$
$$\lambda_F(x)=\begin{cases}1\ if\ x\ \in S-G,\\ \geq 0.5\ if\ x\in G.\end{cases}.$$

Then,

(i) λ is a neutrosophic$(q,\in\vee q)$-FGB-I of S.

(ii) λ is a neutrosophic $(\in,\in\vee q)$-FGB-I of S.

Proof. (i) Let $t,s\in S\ and\ u_1,v_1,w_1\ \in[0,\ 1]$ with $t\leq s$ be such that $s_{\tilde{p}}q\lambda$ where $\tilde{p}=(u_1,\ v_1,\ w_1)$.

Then $s_{\tilde{p}}q\lambda$ implies that $\begin{cases}\lambda_T(s)+u_1<1,\\ \lambda_I(s)+v_1<1,\\ \lambda_F(s)+w_1>1.\end{cases}$.

Thus, $s\in G$ but G is FGB-I.

Therefore, $t\in G$ which implies that

$\lambda_T(t)\leq 0.5,\ \lambda_I(t)\leq 0.5$ and

$\lambda_F(t)\geq 0.5$. Now, if $u_1\geq 0.5, v_1\geq 0.5$ and $w_1\leq 0.5$,then $\lambda_T(t)\leq 0.5\leq u_1,\lambda_I(tas)\leq 0.5\leq v_1$ and $\lambda_F(tas)\geq 0.5\geq w_1$.Hence $t_{\tilde{p}}\in\lambda$.

If $u_1<0.5,v_1<0.5,w_1>0.5$, then $\lambda_T(t)+u_1<0.5+0.5=1$.

$\lambda_I(t)+v_1<0.5+0.5=1,\lambda_F(t)+w_1>0.5+0.5=1,$

Therefore, $t_{\tilde{p}}q\lambda$.

Thus, $t_{\tilde{p}} \in \vee q\lambda$.

Now assume $t,\ a,\ s \in S$ and $u_1,\ u_2,\ v_1,\ v_2,\ w_1,\ w_2 \in [0,\ 1]$ be such that $t_{\tilde{p}}q\lambda$ and $s_{\tilde{q}}q\lambda$ where $\tilde{p} = (u_1,\ v_1,\ w_1)$ and $\tilde{q} = (u_2, v_2, w_2)$.

$$\text{Then } t_{\tilde{p}}q\lambda \text{ implies} \begin{cases} \lambda_T(t) + u_1 < 1, \\ \lambda_I(t) + v_1 < 1, \\ \lambda_F(t) + w_1 > 1. \end{cases}$$

$$\text{And } s_{\tilde{q}}q\lambda \text{ implies} \begin{cases} \lambda_T(s) + u_1 < 1, \\ \lambda_I(s) + v_1 < 1, \\ \lambda_F(s) + w_1 > 1. \end{cases}.$$

Hence, $t, s \in G$ but G is generalized bi-ideal. Therefore, $tas \in G$ which implies that $\lambda_T(tas) \leq 0.5$, $\lambda_T(tas) \leq 0.5$ and $\lambda_F(tas) \geq 0.5$. Now, if $u_1 \geq 0.5,\ v_1 \geq 0.5$ and $w_1 \leq 0.5$, then $\lambda_T(tas) \leq 0.5 \leq u_1 \leq u_1 \vee u_1$, $\lambda_I(tas) \leq 0.5 \leq v_1 \leq v_1 \vee v_2$ and $\lambda_F(tas) \geq 0.5 \geq w_1 \geq w_1 \wedge w_2$. Hence, $(tas)_{\begin{pmatrix} u_1 \vee u_2 \\ v_1 \vee v_2 \\ w_1 \wedge w_2 \end{pmatrix}} \in \lambda$

The similar case is also hold if $u_2 \geq 0.5,\ v_2 \geq 0.5$ and $w_2 \leq 0.5$.

If $u_1 < 0.5,\ v_1 < 0.5,\ w_1 > 0.5,\ u_2 < 0.5,\ v_2 < 0.5$ and $w_2 > 0.5$, then $\lambda_T(tas) + u_1 < 0.5 + 0.5 = 1, \lambda_I(tas) + v_1 < 0.5 + 0.5 = 1, \lambda_F(tas) + w_1 > 0.5 + 0.5 = 1,\ \lambda_T(tas) + u_2 < 0.5 + 0.5 = 1,\ \lambda_I(tas) + v_2 < 0.5 + 0.5 = 1$ and $\lambda_F(tas) + w_2 > 0.5 + 0.5 = 1$.

Consequently, $\lambda_T(tas) + u_1 \vee u_2 < 0.5 + 0.5 = 1$,

$\lambda_I(tas) + v_1 \vee v_2 < 0.5 + 0.5 = 1,\ \lambda_F(tas) + w_1 \wedge w_2 > 0.5 + 0.5 = 1$.

Resultantly, (tas)$_{\begin{pmatrix} u_1 \vee u_2 \\ v_1 \vee v_2 \\ w_1 \wedge w_2 \end{pmatrix}} q\lambda$.

Thus, $(tas)_{\begin{pmatrix} u_1 \vee u_2 \\ v_1 \vee v_2 \\ w_1 \wedge w_2 \end{pmatrix}} \in \vee q\lambda$.

Consequently, λ is a neutrosophic $(q, \in \vee q)$-FGB-I of S.

(ii) Assume that $t, s \in S$ and $u,\ v,\ w \in [0, 1]$ with $t \leq s$ be such that $s_{\tilde{p}} \in \lambda$ where $\tilde{p} = (u,\ v,\ w)$.

Then $s_{\tilde{q}} \in \lambda$ implies that $\begin{cases} \lambda_T(t) \leq u, \\ \lambda_I(t) \leq v, \\ \lambda_F(t) \geq w. \end{cases}$

Thus, $s \in G$ but G is FGB-I.

Therefore, $t \in G$ which implies that $\lambda_T(t) \leq 0.5,\ \lambda_I(t) \leq 0.5$ and $\lambda_F(t) \geq 0.5$. Now, if $u \geq 0.5,\ v \geq 0.5$ and $w \leq 0.5$, then $\lambda_T(t) \leq 0.5 \leq u,\ \lambda_I(tas) \leq 0.5 \leq v$ and $\lambda_F(tas) \geq 0.5 \geq w$.

Hence $t_{\tilde{p}} \in \lambda$.

If $u < 0.5,\ v < 0.5,\ w > 0.5$, then $\lambda_T(t) + u < 0.5 + 0.5 = 1, \lambda_I(t) + v < 0.5 + 0.5 = 1,\ \lambda_F(t) + w > 0.5 + 0.5 = 1$, therefore, $t_{\tilde{p}} q \lambda$.

Thus, $t_{\tilde{p}} \in \vee q \lambda$.

Now suppose that $t,\ a,\ s \in S$ and $u_1, u_2,\ v_1,\ v_2,\ w_1, w_2 \in [0,1]$ be such that $t_{\tilde{p}} \in \lambda$ and $s_{\tilde{q}} \in \lambda$ where $\tilde{p} = (u_1, v_1, w_1)$ and $\tilde{q} = (u_2, v_2, w_2)$.

Then $t_{\tilde{p}} \in \lambda$. Implies $\begin{cases} \lambda_T(t) \leq u_1, \\ \lambda_I(t) \leq v_1, \\ \lambda_F(t) \geq w_1. \end{cases}$

And $s_{\tilde{q}} \in \lambda$ implies $\begin{cases} \lambda_T(s) \leq u_2, \\ \lambda_I(s) \leq v_2, \\ \lambda_F(s) \geq w_2. \end{cases}$.

Thus, $t,\ s \in G$ but G is FGB-I.

So, $tas \in G$ which implies that $\lambda_T(tas) \leq 0.5,\ \lambda_I(tas) \leq 0.5$ and $\lambda_F(tas) \geq 0.5$.

Now, if $u_1 \geq 0.5,\ v_1 \geq 0.5$ and $w_1 \leq 0.5$, then $\lambda_T(tas) \leq 0.5 \leq u_1 \leq u_1 \vee u_2, \lambda_I(tas) \leq 0.5 \leq v_1 \leq v_1 \vee v_2$ and $\lambda_F(tas) \geq 0.5 \geq w_1 \geq w_1 \wedge w_2$.

Hence, $(tas)_{\begin{pmatrix} u_1 \vee u_2 \\ v_1 \vee v_2 \\ w_1 \wedge w_2 \end{pmatrix}} \in \lambda$.

The similar case is also hold if $u_2 \geq 0.5,\ v_2 \geq 0.5$ and $w_2 \leq 0.5$.

If $u_1 < 0.5, v_1 < 0.5,\ w_1 > 0.5, u_2 < 0.5, v_2 < 0.5$ and $w_2 > 0.5$, then $\lambda_T(tas) + u_1 < 0.5 + 0.5 = 1$,

$\lambda_I(tas) + v_1 < 0.5 + 0.5 = 1, \lambda_T(tas) + w_1 > 0.5 + 9.5 = 1, \lambda_T(tas) + u_2 < 0.5 + 0.5 = 1, \lambda_I(tas) + v_2 < 0.5 + 0.5 = 1$ and $\lambda_F(tas) + w_2 > 0.5 + 0.5 = 1$.

Consequently, $\lambda_T(tas) + u_1 \vee u_2 < 0.5 + 0.5 = 1$,

$\lambda_I(tas) + v)_1 \vee v_2 < 0.5 + 0.5 = 1$,

$\lambda_F(tas) + w_1 \wedge w_2 > 0.5 + 0.5 = 1$.

Resultantly, $(tas)_{\begin{pmatrix} u_1 \vee u_2 \\ v_1 \vee v_2 \\ w_1 \wedge w_2 \end{pmatrix}} q\lambda.$

Thus, $(tas)_{\begin{pmatrix} u_1 \vee u_2 \\ v_1 \vee v_2 \\ w_1 \bigwedge w_2 \end{pmatrix}} \in \vee q\lambda.$

Therefore, λ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S.□

Theorem 4.6. *If λ be a neutrosophic subset of an OSG S, then show that the understated condition are equivalently:*

(I) *λ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S.*

(II) *(i)* $(\forall s, t \in S$ *such that* $s \leq t)$ $\left(\begin{cases} \lambda_T(s) \leq max\{\lambda_T(t), 0.5\}, \\ \lambda_T(s) \leq max\{\lambda_T(t), 0.5\}, \\ \lambda_T(s) \geq min\{\lambda_T(t), 0.5\}. \end{cases} \right)$

(ii) $(\forall t, a, s \in S)$ $\left(\begin{cases} \lambda_T(tas) \leq max\{\lambda_T(tas), \lambda_T(t), 0.5\}, \\ \lambda_T(tas) \leq max\{\lambda_I(tas), \lambda_T(t), 0.5\}, \\ \lambda_T(tas) \geq min\{\lambda_F(tas), \lambda_T(t), 0.5\}. \end{cases} \right).$

Proof. $(I) \Rightarrow (II)$: Let λ be a n neutrosophic $(\in, \in \vee q)$-FGB-I of S and assume on contrary bases that $\lambda_T(s) >$ max $\{\lambda_T(t),\ 0.5\}, \lambda_I(sat) >$ max $\{\lambda_I(t),\ 0.5\}$ and $\lambda_F(s) <$ min $\{\lambda_T(t),\ 0.5\}$, then $\exists\ u,\ v,\ w \in [0,\ 1] \ni \lambda_T(s) > u \geq$ max $\{\lambda_T(t),\ 0.5\},\ \lambda_I(tas) > v \geq$ max $\{\lambda_I(t),\ 0.5\}$ and $\lambda_F(s) \leq w <$ min $\{\lambda_T(t),\ 0.5\}$. It is clear that $\lambda_T(t) \leq u,\ \lambda_I(sat) \leq v$ and $\lambda_F(t) \leq w$ shows that $t_{\tilde{p}} \in \lambda$ but $s_{\tilde{q}} \overline{\in \vee q} \lambda$ which is a contradicts to the fact λ is a neutrosophic $(\in, \in \vee q)$-FGB-I.
Hence (i) hold. By similar argument we can also show that (ii) hold.
Thus, $(I) \Rightarrow (II)$.

$(II) \Rightarrow (I)$. Suppose (i) and (ii) hold, we need to manifest that λ is a neutrosophic $(\in, \in \vee q)$-FGB-I. For this let $s,\ t \in S$ such that $s \leq t,\ u,\ v,\ w \in [0,\ 1]$ and $t_{\tilde{p}} \in \lambda$ where $\tilde{p} = (u,\ v,\ w)$. Therefore,

$$t_{\tilde{p}} \in \lambda \text{ implies that } \begin{cases} \lambda_T(t) \leq u, \\ \lambda_I(t) \leq v, \\ \lambda_F(t) \geq w. \end{cases}.$$

Since by (i) $\begin{cases} \lambda_T(s) \leq \{\lambda_T(t),\ 0.5\} \leq u, \\ \lambda_I(s) \leq \{\lambda_T(t),\ 0.5\} \leq v, \\ \lambda_F(s) \geq \{\lambda_T(t), 0.5\} \geq w. \end{cases}$

which shows that $\begin{cases} \lambda_T(s) \leq u, \\ \lambda_I(s) \leq v, \\ \lambda_F(s) \geq w. \end{cases}$.

Hence $s_{\tilde{p}} \in \lambda$. If $u, v < 0.5$ and $w > 0.5$,

then $\begin{cases} \lambda_T(t) \leq u < 0.5, \\ \lambda_I(t) \leq v < 0.5, \\ \lambda_F(t) \geq w > 0.5. \end{cases}$

implies that $\lambda_T(t) < 0.5,\ \lambda_I(t) < 0.5$ and $\lambda_F(t) > 0.5$.

Consequently, $\lambda_T(s) < 0.5,\ \lambda_I(s) < 0.5$ and $\lambda_F(s) > 0.5$.

Therefore, $\lambda_T(s) + u < 0.5 + 0.5 = 1,\ \lambda_I(s) + u < 0.5 + 0.5 = 1$ and $\lambda_F(s) + u > 0.5 + 0.5 = 1$.

Hence, $s_{\tilde{p}} q\lambda$, So $s_{\tilde{p}} \in \vee q\lambda$.

Similarly, for $s,\ a,\ t \in S$ such that $t_{\tilde{p}} \in \lambda,\ s_{\tilde{q}} \in \lambda$.

$\Rightarrow\ (tas)_{\begin{pmatrix} u_1 \vee u_2 \\ v_1 \vee v_2 \\ w_1 \wedge w_2 \end{pmatrix}} \in \vee q\lambda.$

Resultantly, λ is a neutrosophic$(\in, \in \vee q)$-FGB-I of S.

Since every neutrosophiuc $(\in, \in \vee q)$-FB-I of S is a neutrosophic $(\in, \in \vee q)$-FGB-I of S but the opposite statement is generally inncorrect.□

Definition 4.7. Let λ be a neutrosophic subset(NSUBS) of an OSG S, for any $u,\ v,\ w \in [0,\ 1]$ the set

$U(\lambda, \tilde{p}) = \left\{ x \in S | \begin{array}{c} \lambda_T(x) \leq u, \\ \lambda_I(x) \leq v, \\ \lambda_F(x) \geq w. \end{array} \right\}$ is SAID TO BE a neutrosophic level subset(NLSUBS) of λ.

Theorem 4.8. *Suppose that λ is a neutrosophic subset of an OSG S. Then show that λ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S $\Leftrightarrow U(\lambda,\ \tilde{p})(\neq \phi)$ is FGB-I of S for $(u,\ v \in (0,\ 0.5],\ w \in (0,\ 0.5])$.*

Proof. Let λ is a neutrosophic $(\in, \in\ \vee q)$-FGB-I of S. Consider such that $s,\ t \in S$ and $t \in U(\lambda,\ \tilde{p})$.

$$\text{Then}\quad \begin{array}{l} \lambda_T(t) \le u, \\ \lambda_I(t) \le v, \\ \lambda_F(t) \ge w. \end{array} .$$

Since λ is a neutrosophic $(\in, \in \vee q)$-FGB-I. Therefore,

$$\text{by Theorem [4.2]},\ \begin{cases} \lambda_T(s) \le max\{\lambda_T(t),\ 0.5\}, \\ \lambda_I(s) \le max\{\lambda_I(t),\ 0.5\}, \\ \lambda_F(s) \ge min\{\lambda_F(t),\ 0.5\}. \end{cases}$$

$$\text{which implies that}\ \begin{cases} \lambda_T(s) \le max\{\lambda_T(t),\ 0.5\} = u, \\ \lambda_I(s) \le max\{\lambda_I(t),\ 0.5\} = v, \\ \lambda_F(s) \ge min\{\lambda_F(t),\ 0.5\} = w. \end{cases} \text{because } (u,\ v \in (0.5,\ 1],\ w \in (0,\ 0.5]).$$

Thus, $s \in U(\lambda,\ \tilde{p})$.

Similarly, for $s,\ a,\ t \in S$ such that $s,\ t \in U(\lambda,\ \tilde{p})$ implies $sat \in U(\lambda,\ \tilde{p})$.

Hence, $U(\lambda,\ \tilde{p})$ is a FGB-I of S.

$\Longleftarrow$, assume that $U(\lambda,\ \tilde{p})$ is FGB-I of S for $(u,\ v \in (0.5,\ 1],\ w \in (0,\ 0.5])$.

Let $s,\ t \in S \ni s \le t$.

$$\text{Suppose by contradiction}\ \begin{cases} \lambda_T(s) > max\{\lambda_T(t),\ 0.5\}, \\ \lambda_I(s) > max\{\lambda_I(t),\ 0.5\}, \\ \lambda_F(s) < min\{\lambda_F(t),\ 0.5\}. \end{cases} .$$

Then for some $u,\ v \in (0.5,\ 1],\ w \in (0,\ 0.5]$,

$$\begin{cases} \lambda_T(s) > u \ge max\{\lambda_T(t),\ 0.5\}, \\ \lambda_I(s) > v \ge max\{\lambda_I(t),\ 0.5\}, \\ \lambda_F(s) < w \le min\{\lambda_F(t),\ 0.5\}. \end{cases} .$$

Implies that $t \in U(\lambda,\ \tilde{p})$ but $s\overline{\in}U(\lambda,\ \tilde{p})$ which is a contradicts to the fact that $U(\lambda,\ \tilde{p})$ is FGB-I of S.

$$\text{Therefore,}\ \begin{cases} \lambda_T(s) \le max\{\lambda_T(s),\ \lambda_T(t),\ 0.5\}, \\ \lambda_I(s) \le max\{\lambda_I(s),\ \lambda_I(t),\ 0.5\}, \\ \lambda_F(s) \ge min\{\lambda_F(s),\ \lambda_F(t),\ 0.5\}. \end{cases} .$$

$$\text{Similarly, for } s,\ a,\ t \in S,\ \left(\begin{cases} \lambda_T(tas) \le max\{\lambda_T(t), 0.5\}, \\ \lambda_T(tas) \le max\{\lambda_T(t), 0.5\}, \\ \lambda_T(tas) \ge min\{\lambda_T(t), 0.5\}. \end{cases} \right) \text{also hold.}$$

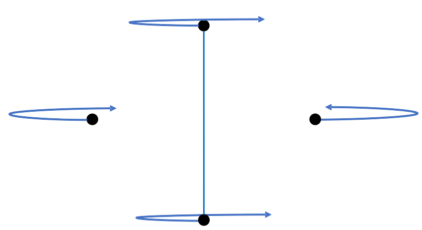
Thus, by Theorem [4.2], λ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S. □

Example

Let $S = \{a, b, c, d\}$ be an OSG with understated multiplication table and ordered relation "$\leq$" as follows: $\leq := \{(a,a), (b,b), (c,c), (d,d), (a,b)\}$

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

TABLE 1. Multiplicative table of Ordered Semigroup



S	$\lambda_T(x)$	$\lambda_I(x)$	$\lambda_F(x)$
a	0.18	0.15	0.30
b	0.20	0.19	0.28
c	0.17	0.16	0.33
d	0.20	0.20	0.32

TABLE 2. Example of Neutrosophic $(\in, \in \vee q)$-fuzzy generalized bi-ideals

Using definition (4.3) λ is neutrosophic $(\in, \in \vee q)$-FGB-I where $\tilde{p} = (0.25,\ 0.20,\ 0.22)$ and $\tilde{q} = (0.26,\ 0.30,\ 0.28)\ \in\ [0,\ 1]$

Definition 4.9. Let S be an OSG. The neutrosophic characteristic function $X_A = (X_{\lambda_T},\ X_{\lambda_I},\ X_{\lambda_F})$ of $A = \langle x, (\lambda_T,\ \lambda_I,\ \lambda_F)(x)\rangle$ is defined as

$$X_{\lambda_T}(x) = \begin{cases} 1\ if\ x \in A, \\ 0\ ifnot. \end{cases}$$

$$X_{\lambda_I}(x) = \begin{cases} 1\ if\ x \in A, \\ 0\ ifnot. \end{cases}$$

$$X_{\lambda_F}(x) = \begin{cases} 0\ if\ x \notin A, \\ 1\ ifnot. \end{cases}$$

Theorem 4.10. *A non-empty set B of an OSG S is a FGB-I of S $\Leftrightarrow$ the characteristic function X_B is a neutrosophic $(\in, \in \vee q)$-FGB-I of S.*

Proof. The proof is follows from theorem [4.3]. □

Theorem 4.11. *Suppose S is an OSG and L is a L(resp. right)I of S. If λ is defined by the neutrosophic subset of S*

$$\lambda_T(x) = \begin{cases} 1\ if\ x \in S - L, \\ \leq 0.5\ if\ x \in L. \end{cases}$$

$$\lambda_I(x) = \begin{cases} 1\ if\ x \in S - L, \\ \leq 0.5\ if\ x \in L. \end{cases}$$

$$\lambda_F(x) = \begin{cases} 1\ if\ x \in S - L, \\ \geq 0.5\ if\ x \in L. \end{cases}$$

Then

(i) λ is a neutrosophic $(q, \in \vee q)$-FL(res. right)I of S.

(ii) λ is a neutrosophic $(\in, \in \vee q)$-FL(res. right)I of S.

Proof. .□

Proved by theorem [4.1].

Theorem 4.12. *Assume that S is an OSG and I is an ideal of S. If* λ *be a neutrosophic subset of S defined as in Theorem [4.5], then* λ *is both a neutrosophic* $(q, \in \vee q)$*-FI and a neutrosophic* $(\in, \in \vee q)$*-FI of s.*

Proof. the proof follow by combining Theorem [4.5] and Theorem [4.1].□

Theorem 4.13. *If* λ *be a NSUBS of an OSG S, then show that the understating condition are equivalently:*

(I) λ *is a neutrosophic* $(\in, \in \vee q)$*-FL(resp. right)I of S.*

(II) *(i)*$(\forall s,\ t \in S\ suchthat\ s \leq t)$ $\left(\begin{cases} \lambda_T(s) \leq max\{\lambda_T(t),\ 0.5\}, \\ \lambda_I(s) \leq max\{\lambda_T(t),\ 0.5\}, \\ \lambda_F(s) \geq min\{\lambda_T(t),\ 0.5\}. \end{cases}\right)$.

(ii) $(\forall s, t \in S)$ $\left(\begin{cases} \lambda_T(st) \leq max\{\lambda_T(t),\ 0.5\},\ (resp.max\{\lambda_T(s),\ 0.5\}), \\ \lambda_I(st) \leq max\{\lambda_I(t),\ 0.5\},\ (resp.max\{\lambda_I(s), 0.5\}), \\ \lambda_F(st) \geq min\{\lambda_F(t),\ 0.5\}.\ (resp.min\{\lambda_F(s), 0.5\}) \end{cases}\right)$.

Proof. Proved by theorem [4.2]. □

Theorem 4.14. *Suppose that* λ *is a neutrosophic subset of an OSG S. Then* λ *is a neutrosophis* $(\in, \in \vee q)$*-FL(resp. right)I of S* $\Leftrightarrow$
$U(\lambda, \tilde{p})(\neq\ \phi)$ $\left(x \in S |\ \begin{matrix} \lambda_T(x) \leq u, \\ \lambda_I(x) \leq v, \\ \lambda_F(x) \geq w. \end{matrix} \right\}$ *is a L(resp. right)I of S for* $(u,\ v \in (0,\ 0.5],\ w \in (0,\ 0.5])$.

Proof. Proved by theorem [4.3]. □

Definition 4.15. If S is an OSG, then S is RG $\Leftrightarrow \forall\ x \in S\ \exists\ a \in S \ni x \leq xax$ or $A \subseteq (XSX]$ $\forall\ X \subseteq S$.

Definition 4.16. If S is an OSG, then S is left weakly RG $\Leftrightarrow x \in S\ \exists\ a, b \in S \ni x \leq axay$ or $X \subseteq ((SX)^2]\ \forall\ X \subseteq S$.

Proposition

Let λ be a neutrosophic subset of a regular OSG S. Then every neutrosophic $(\in, \in \vee q)$-FGB-I of S is a neutrosophic $(\in, \in \vee q)$-FB-I of S.

Proof. Assume that $s,\ t \in S$ and λ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S. Since S is RG so $\exists\ x \in S \ni s \leq sxs. \implies \lambda_T(s) \leq \max\{\lambda_T(sxs), 0.5\}$.

Hence $\lambda_T(st) \leq \max\{\lambda_T(sxst),\ 0.5\}$
$= max\{\lambda_T(s(xs)t),\ 0.5\}$
$\leq \max\{\lambda_T(s),\ \lambda_T(t),\ 0.5\}$,
by similar argument
$\lambda_I(st) \leq \max\{\lambda_I(s),\ \lambda_I(t),\ 0.5\}$ and $\lambda_F(st) \geq \min\{\lambda_F(s),\ \lambda_F(t),\ 0.5\}$ also hold.
Hence λ is a neutrosophic $(\in, \in \vee q)$-FB-I of S.□

Proposition

Consider that λ is a neutrosophic subset of a left weakly regular OSG S.Then every neutrosophic $(\in, \in \vee q)$-FGB-I of S is a neutrosophic $(\in, \in \vee q)$-FB-I of S.

Proof. Proved by proposition [3]. □

5. **Lower Parts of Neutrosophic** $(\in, \in \vee q)$**-generalized bi-ideals**

In this section, we will start the fundamental operations of the lower parts of the neutrosophic subset, the neutrosophic characteristic function(CF) lower parts, left (resp. right)RG, left (resp. right)SMP, the related theorems and the lammas of the lower parts.

Definition 5.1. Let λ be a neutrosophic subset of an OSG S, we stated the LP as $\lambda^{-} = \langle x, \lambda_T^{-}, \lambda_I^{-},\ \lambda_F^{-}\rangle$ of λ as follows;

$\lambda_T^{-}(x)$=max$\{\lambda_T(x),\ 0.5\}$

$\lambda_I^{-}(x)$=max$\{\lambda_I(x),\ 0.5\}$

$\lambda_F^{-}(x)$=min$\{\lambda_F(x),\ 0.5\}$

For any subset $A \neq \phi$ and neutrosophic subsets(NSUBSs) λ of OSG S, then LP of neutrosophic CF $(X_A)^{-}$ will be denoted by X_A^{-}.

Definition 5.2. Let λ and η any tow NSUBSs of an OSG S, we stated $(\lambda\tilde{\cap}\eta)^{-}$, $(\lambda\tilde{\cup}\eta)^{-}$ and $(\lambda\otimes\eta)^{-}$ as follows:

$$(\lambda\tilde{\cap}\eta)^{-}(x)=\max\{\lambda\tilde{\cap}\eta(x),\ 0.5\}$$

$$(\lambda\tilde{\cup}\eta)^{-}(x)=\max\{\lambda\tilde{\cup}\eta(x),\ 0.5\}$$

$$(\lambda\otimes\eta)^{-}(x)=\max\{\lambda\otimes\eta(x),\ 0.5\}.$$

Lemma 5.3. *Let λ and η be any tow NSUBSs of an OSG S, then $(\lambda^{-})^{-}=\lambda^{-}$ where $\lambda^{-}=\langle x,\ \lambda_T^{-},\ \lambda_I^{-},\ \lambda_F^{-}\rangle$ is the LP of λ.*

Proof. Assume that λ_0^{-} is the lower part of λ, then by definition [5.2]□

$\lambda_T^{-}\ (x)=\max\{\lambda_T(x),0.5\}$
$(\lambda_T^{-})_T^{-}(x)=\max\{\{max\{\lambda_T(x),0.5\}\},0.5\}$
$=\max\{\lambda_T(x),0.5\}=\lambda_T^{-}(x)$
Similarly $(\lambda_I^{-})_I^{-}=\lambda_I^{-}$ and $(\lambda_F^{-})_F^{-}=\lambda_F^{-}$ also hold.
Thus, $(\lambda^{-})^{-}=\lambda^{-}$.

Lemma 5.4. *Let λ and η be any tow NSUBSs of an OSG S, then*

- $(\lambda\tilde{\cap}\eta)^{-}=\lambda^{-}\tilde{\cap}\eta^{-}$.
- $(\lambda\tilde{\cup}\eta)^{-}=\lambda^{-}\tilde{\cup}\eta^{-}$
- $(\lambda\otimes\eta)^{-}=\lambda^{-}\otimes\eta^{-}$.

Proof. Proof is straightforward.□

Definition 5.5. Let S be an OSG. Then the LP of the neutrosophic CF $X_A^{-}=(X_{\lambda_T}^{-},\ X_{\lambda_I}^{-},\ X_{\lambda_F}^{-})$ of $A=\langle x,\ (\lambda_T,\lambda_I,\ \lambda_F)(x)\rangle$ is defined as

$$X^{-}{}_{\lambda_T}(x)=\begin{cases}0.5\ if\ x\in A,\\ 1\ otherwise.\end{cases}$$
$$X^{-}{}_{\lambda_I}(x)=\begin{cases}0.5\ if\ x\in A,\\ 1\ otherwise.\end{cases}$$
$$X^{-}{}_{\lambda_F}(x)=\begin{cases}1\ if\ x\notin A,\\ 0.5\ otherwise.\end{cases}$$

Theorem 5.6. *Let $A = \langle x,\ (\lambda_T,\ \lambda_I,\ \lambda_F)(x)\rangle$ and $B = \langle x,\ (\eta_T, \eta_I,\ \lambda_F)(x)\rangle$ are any tow NSUBS of an OSG S, then*

(1) $(X_A \tilde{\cap} X_B)^{-} = X^{-}_{A\cap B}$

(2) $(X_A \tilde{\cup} X_B)^{-} = X^{-}_{A\cup B}$

(3) $(X_A \otimes X_B)^{-} = X^{-}(_{AB}]$.

Proof. The proof of (1) and (2) is simple here. So for the proof of (3), Suppose that $x \in (AB]$, then $X^{-}(_{AB}](x) = 0.5$. Since $x \in (AB]$, then $x \leq ab$ for some $a \in A$ and $b \in B$ implies that $(a, b) \in A_x$.Thus $A_x \neq \phi$. Therefore,

$$(X_A \otimes X_B)^{-}(x) = \{(X_A \otimes X_B)(x), 0.5\}$$

Since $a \in A$ and $b \in B$, therefore, $X_{\lambda_T}(x) = 0.5 = X_{\eta_T}(b)$. Hence,

$$(X_{\lambda_T} \circ X_{\eta_T})(x) = \left\{ \wedge_{(a,\ b)\in A_x} [X_{\lambda_T}(a) \vee X_{\eta_T}(b)] \right\}$$

$$= \left\{ \wedge_{(a,\ b)\in A_x} [0.5, 0.5] \right\} = 0.5.$$

Similarly,
$(X_{\lambda_T} \circ X_{\eta_T})(x) = 0.5$ and
$(X_{\lambda_F} \circ X_{\eta_F})(x) = 0.5$.

Consequently, $(X_A \otimes X_B)(x) = 0.5$
$\Rightarrow\ (X_A \otimes X_B)^{-}(x) = 0.5$.
Thus $(X_A \otimes X_B)^{-}(x) = X^{-}{}_{(AB]}$.
If $x \notin (Ab]$, then $X^{-}{}_{(AB](x)} = 1$. Let $(y, z) \in A_x$, then

$$(X_{\lambda_T} \circ X_{\eta_T})(x) = \left\{ \wedge_{(y,z)\in A_x} [X_{\lambda_T}(y) \vee X_{\eta_T}(z)] \right\}$$

Since$(y,\ z) \in A_x$ then $x \leq yz$. If $y \in A$ and $z \in B$, then $yz \in AB$ implies that $x \in (AB]$ which goes to contradiction. Therefore, if $y \notin A$ and $z \in B$, then $X_{\lambda_T}(y) = 1,\ X_{\eta_T}(b) = 0.5$. Hence

$$(X_{\lambda_T} \circ X_{\eta_T})(x) = \left\{ \wedge_{(y,z)\in A_x} [1 \vee 0.5] \right\} = 1.$$

The similar case hold if $y \in A$ and $z \notin B$. By similar way,
$(X_{\lambda_I} \circ X_{\eta_I})(x) = 1$ and $(X_{\lambda_F} \circ X_{\eta_F})(x) = 1$.

Consequently, $(X_A \otimes X_B)(x) = 1$.

Hence, $(X_A \otimes X_B)^{-} = X^{-}{}_{(AB]}$.□

Lemma 5.7. *The LP $X^{-}{}_A$ of the CF X_A of A is a neutrosophic $(\in, \in \vee q)$-FGB-I of an OSG $S \Leftrightarrow$ A is a GB-I of S.*

Proof. Let A is a GB-I of S, then by theorem [4.4], $X^{-}{}_A$ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S.

$\Leftarrow$, suppose that $X^{-}{}_A$ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S.

Let $x, y \in S \ni x \leq y$ and $y \in A$, then $X^{-}{}_A(y) = 0.5$.

Since $X^{-}{}_A$ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S.

Thus $X^{-}{}_A(x) \leq max\{X^{-}{}_A(y),\ 0.5\} = 0.5$. Also $X^{-}{}_A \geq 0.5$ (always).

Therefore, $X^{-}{}_A(x) = 0.5$ shows that $x \in A$.

Similarly, for $x,\ y,\ z \in S$ and $x, z \in A$, then $X^{-}{}_A(y) = 0.5 = X^{-}{}_A(z)$.

Since $X^{-}{}_A$ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S.

So $X^{-}{}_A(xyz) \leq max\{X^{-}{}_A(x),\ X^{-}{}_A(z),\ 0.5\} = 0.5$. also $X^{-}{}_A(xyz) \geq 0.5$ (always).

Therefore, $X^{-}{}_A(xyz) = 0.5$. Shows that $xyz \in A$.

Hence, A is a GB-I of S. □

Lemma 5.8. *The LP $X^{-}{}_A$ of the CF X_A of A is a neutrosophic $(\in, \in \vee q)$-FL(resp. right)I of $S \Leftrightarrow$ A is L(resp. right)I of S.*

Proof. Follows from the lemma [5.4].□

Definition 5.9. Let S be an OSG . Then S is L(resp. right)RG if $\forall\ a \in S, \exists\ x \in S \ni a \leq xa^2$ $(resp. a \leq a^2 x)$ or $A \subseteq (SA^2]$ (resp. $A \subseteq (A^2 S]$).

Definition 5.10. S is L(resp. right)SMP $\forall$ L(resp. right)I A of S, A=S. S is SMP if it is both left and right SMP , and is left, right and RG then S is CRG.

Lemma 5.11. *An OSG S is CRG $\Leftrightarrow \forall\ A \subseteq S$, we have, $A \subseteq (A^2 S A^2]$.*

Lemma 5.12. *An OSG S is L(resp. right)SMP $\Leftrightarrow \forall$ (Sa]=S, (resp. (aS]=S $\forall\ a \in S$.*

Theorem 5.13. *If S is RG, left and right SMP, then $\lambda^{-}(a) = \lambda^{-}(b)\ \forall\ a, b \in S$ where λ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S.*

Proof. Suppose that S is RG, left and right SMP and λ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S. Let $E_s = \{s \in S | s \leq s^2\}$. Since S is RG, therefore $\forall\ a \in S, \exists\ x \in S \ni a \leq axa$ also $ax \leq axax = (ax)^2$. Thus $ax \in E_s$ implies that $E_s \neq \phi$. Now let $b,\ e \in S$, using Lemma

[3.3.7], $S = (Sb]$ and $S = (bS]$. Since $e \in S$ it implies that $e \in (Sb]$ and $e \in (bS]$, then $e \leq xb$, by for some $x,\ y \in S$.
Hence $e^2 \leq (by)(xb) = b(yx)b$. Now as λ is a neutrosophic $(\in, \in \vee q)$-FGB-I of S.

$\lambda_T(e^2) \leq$max$\{\lambda_T(b(yx)b),\ 0.5\}$

$\leq \max\{\lambda_T(b),\ \lambda_T(b), 0.5\}$

$=$max$\{\lambda_T(b),\ 0.5\}$

$\lambda_T(e^2) \vee 0.5 \leq \max\{\lambda_T(b),\ 0.5\} \vee 0.5$

$=$max$\{\lambda_T(b),\ 0.5\}$

$\bar{\lambda}_T(e^2) \leq \bar{\lambda}_T(b)$

Since $e \in E_s$, so $e \leq e^2$ that is $\lambda_T(e) \leq \max\{\lambda_T(e^2), 0.5\}$
implies that $\bar{\lambda}_T(e) \leq \bar{\lambda}_T(b)$.
By similar way $\bar{\lambda}_T(e) \leq \bar{\lambda}_I(b)$ and $\bar{\lambda}_F(e) \geq \bar{\lambda}_F(b)$. Therefore, $\bar{\lambda}$ is a constant on E_s. Now since S is RG so for $a \in S,\ ax,\ xa \in E_s$ follows that $\bar{\lambda}_T(ax) = \bar{\lambda}_T(b) = \bar{\lambda}_T(xa)$. Since $a \leq ax(axa) = (ax)a(xa)$. Therefore

$\lambda_T(a) \leq \max\ \{\lambda_T((ax)a(xa)),\ 0.5\}$

$\leq \max\ \{\lambda_T(ax),\ \lambda_T(xa),\ 0.5\}$

$= \max\ \{(\lambda_T(ax),\ 0.5), (\lambda_T(xa),\ 0.5)\}$

$\lambda_T(a) \vee 0.5 = \max\ \{(\lambda_T(ax),\ 0.5),\ (\lambda_T(xa),\ 0.5)\} \bigvee 0.5$

$\bar{\lambda}_T(a) \leq \max\ \{\bar{\lambda}_T(ax),\ \bar{\lambda}_T(xa)\} = \bar{\lambda}_T(b)$.
By similar way, $\bar{\lambda}_I(a) \leq \bar{\lambda}_I(b),\ \bar{\lambda}_F(a) \geq \bar{\lambda}_F(b)$. Since $b \in (Sa],\ (aS]$, therefore, $b \leq sa,\ at$ for some $s,\ t \in S$. Thus

$\lambda_T(b^2) \leq \max\ \{\lambda_t(a(ts)a),\ 0.5\}$

$\leq \max\{\lambda_T(a),\ \lambda_T(a),\ 0.5\}$

$= \max\ \{\lambda_T(a),\ 0.5\}$

$\lambda_T(b^2) \leq \max\{\lambda_T(a),\ 0.5\} \vee 0.5$

$=\max\{\lambda_T(a),\ 0.5\}$

$\lambda^{-}{}_T(b^2) \leq \lambda^{-}{}_T(a)$

since $b \in E_s$ so $b \leq b^2$ that is $\lambda_T(b) \leq \max\ \{\lambda_T(b^2), 0.5\}$ implies that $\lambda^{-}{}_T(b) \leq \lambda^{-}{}_T(b^2)$.
Thus $\lambda^{-}{}_T(b) \leq \lambda^{-}{}_T(b^2) \leq \lambda^{-}{}_T(a)$.
By similar way $\lambda^{-}{}_I(b) \leq \lambda^{-}{}_I(a)$ and $\lambda^{-}{}_F(b) \geq \lambda^{-}{}_F(a)$.
Thus $\lambda^{-}{}_T(b) = \lambda^{-}{}_T(a), \lambda^{-}{}_I(b) = \lambda^{-}{}_I(a)$, and $\lambda^{-}{}_F(b) = \lambda^{-}{}_F(a)$. Resultantly, $\lambda^{-}(a) = \lambda^{-}(b)$. □

Theorem 5.14. *If S is an OSG, then it is RG* $\Leftrightarrow$ $\forall$ *neutrosophic* $(\in, \in \vee q)$*-FGB-I of S,* $\lambda^{-}(a) = \lambda^{-}(a^2) \forall a \in S$.

Proof. The direct part of the theorem derived from Theorem [5.8].
$\Leftarrow$, suppose that $a \in S$, assume $B(a^2) = (a^2 \cup a^2Sa^2]$ is GB-I of S

$$X^{-}{}_{B(a^2)}(a) = \begin{cases} 0.5\ if\ a \in B(a^2) \\ 1\ otherwise \end{cases}$$

is a neutrosophic $(\in, \in \vee q)$-FGB-I S, $X^{-}{}_{B(a^2)}(a^2) = X^{-}{}_{B(a^2)}(a)$. Thus,$a \in B(a^2)$, hence $a \leq a^2$ or $a \leq a^2xa^2$. Now if $a \leq a^2$, then $a \leq a^2 = aa \leq a^2a^2 = aaa^2 \leq a^2aa^2 \in a^2Sa^2$ and $a \in (a^2Sa^2]$. If $a \leq a^2xa^2$, then $a \in (a^2Sa^2]$. $\therefore$, S is CRG.□

Lemma 5.15. *If S is an OSG, then the understating axioms are equivalently:*
(1) S is RG.
(2) $G \cap L \subseteq (GL]$ $\forall$ *GB-I G and LI L of S.*
(3) $G(k) \cap L(k) \subseteq (G(k)L(k)]$ $\forall$ $k \in S$.

Theorem 5.16. *If S be an OSG, then the understating condition are equivalently:*
(I) S is RG.
(II) $(\lambda \otimes \eta)^{-} \preceq (\lambda \tilde{\cap} \eta)^{-}$ $\forall$ *neutrosophic* $(\in, \in \vee q)$*-FGB-I of* λ *and neutrosophic* $(\in, \in \vee q)$*-FLI* η *of S.*

Proof. $(I) \Rightarrow (II)$:. Assume that S is RG, λ is a neutrosophic $(\in, \in \vee q)$-FGB-I and η is a neutrosophic $(\in, \in \vee q)$-FLI of S. Then $\exists\ x \in S \ni a \leq axa \leq (axa)(xa)$ implies that $(axa, xa) \in A_a$

which shows that $A_a \neq \phi$. Hence

$$(\lambda_T \circ \eta_T)^{-}(a) = \max\{(\lambda_T \circ \eta_T)(a),\ 0.5\}$$

$$= \max\Big\{\Big(\ \wedge_{(y,\ z)\in A_a}[\lambda_T(y) \vee \eta_T(z)].0.5\ \Big)\Big\}$$

$$\leq \max\ \{(\lambda_T(axa) \vee \eta_T(xa),\ 0.5)\}$$

Since λ is a neutrosophic $(\in, \in \bigvee q)$-FGB-I and η is a neutrosophic $(\in, \in \vee q)$-FLI of S. Then, $\lambda_T(axa) \leq \max\ \{\lambda_T(a),\ \lambda_T(a),\ 0.5\} = \max\{\lambda_T(a),\ 0.5\}$ and $\lambda_T(xa) \leq \max\ \{\lambda_T(a),\ 0.5\}$. Therefore, $(\lambda_T \circ \eta_T)^{-}(a) \leq (\lambda_T \cap \eta_T)^{-}(a)$.

Similarly, $(\lambda_I \circ \eta_I)^{-}(a) \leq (\lambda_I \cap \eta_I)^{-}(a)$ and $(\lambda_F \circ \eta_F)^{-}(a) \geq (\lambda_F \cap \eta_F)^{-}(a)$

Consequently, $(\lambda \otimes \eta)^{-} \preceq (\lambda \tilde{\cap} \eta)^{-}$

$\Leftarrow$, suppose that $(\lambda \otimes \eta)^{-} \preceq (\lambda \tilde{\cap} \eta)^{-}$. To demonstrate that S is RG, by Lemma [5.10], it is adequate to demonstrate that $G \cap L \subseteq (GL]$ for GB-I G and LI L of S. Let $x \in G \cap L$, then $x \in G$ and $x \in L$. Thus by Lemma [5.10], ${X^{-}}_G$ is a neutrosophic $(\in, \in \vee q)$-FGB-I and ${X^{-}}_L$ is a neutrosophic $(\in, \in \vee q)$-FLI of S. By supposition, $(X_G \otimes X_L)^{-}(x) \leq (X_G \tilde{\cap} X_L)^{-}(x) = \max\ \{(X_G \tilde{\cap} X_L)(x),\ 0.5\}$.

Since, $x \in G$ and $x \in L$, then $X_G(x) = 0.5$ or $X_G(x) \vee 0.5 = 0.5 = 0.5 \vee 0.5$ implies that ${X^{-}}_G(x) = 0.5$ similarly ${X^{-}}_L(x) = 0.5$ which show that ${X^{-}}_G \tilde{\cap} {X^{-}}_L = 0.5$. Follows that $(X_G \tilde{\cap} X_L)^{-}(x) = 0.5$. By Lemma [5.10], $(X_G \tilde{\cap} X_L)^{-}(x) = {X^{-}}_{(GL]} = 0.5$ therefore, $x \in (GL]$.

Hence,S is RG. □

Lemma 5.17. *If S be an OSG, then the understating axioms are equivalently:*

(1) S is RG.

(2) $G \cap T \subseteq (GT]\ \forall$ *GB-I G and ideal T of S.*

(3) $G(k) \bigcap L(k) \subseteq (G(k)L(a)]\ \forall\ k \in S$.

Theorem 5.18. *If S be an OSG, then the understating condition are equivalently:*

(I) S is RG.

(II) $(\lambda \otimes \eta \otimes \lambda)^{-} \preceq (\lambda \tilde{\cap} \eta)^{-}\ \forall$ *neutrosophic* $(\in, \in \vee q)$*-FGB-I* λ *and neutrosophic* $(\in, \in \vee q)$*-FLI* η *of S.*

Proof. The proof of the theorem can be obtained by following the same procedure as follows in the proof of Theorem [5.11].□

Lemma 5.19. *If S be an OSG, then the understating axioms are equivalently:*

(1) S is RG.

(2) $R \cap G \cap L \subseteq (RGL]$ $\forall$ *RI R, GB-I G and LI L of S.*

(3) $R(k) \cap G(k) \cap L(k) \subseteq (R(k)G(k)L(k)]$ $\forall$ $k \in S$.

Theorem 5.20. *If S be an OSG, then the understating condition are equivalent:*

(I) S is RG.

(II) $(\lambda \otimes \eta \otimes \xi)^{-} \preceq (\lambda \tilde{\cap} \eta \tilde{\cap} \xi)^{-}$ $\forall$ *neutrosophic* $(\in, \in \vee q)$*-FRI* λ*, neutrosophic* $(\in, \in \vee q)$*-FGB-I* η *and neutrosophic* $(\in, \in \vee q)$*-FLI* ξ *of S.*

Proof. Follows from the theorem [5.11]. □

Lemma 5.21. *If S be an OSG, then the understating axioms are equivalently:*

(1) S is LWRG.

(2) $T \cap L \subseteq (TL]$ $\forall$ *ideal T and LI L of S.*

(3) $T(k) \cap L(k) \subseteq (T(k)L(k)]$ $\forall$ $k \in S$.

Theorem 5.22. *If S be an OSG, then the understating condition are equivalently:*

(I) S is LWRG.

(II) $(\lambda \otimes \eta)^{-} \preceq (\lambda \tilde{\cap} \eta)^{-}$ $\forall$ *neutrosophic* $(\in, \in \vee q)$*-FI* λ *and neutrosophuic* $(\in, \in \vee q)$*-FLI* η *of S.*

Proof. Follows from the Theorem [5.11] and Lemma [5.16]. □

Lemma 5.23. *If S be an OSG, then the understating condition are equivalent:*

(1) S is LWRG.

(2) $T \cap G \subseteq (TG]$ $\forall$ *ideal T and GB-I G of S.*

(3) $T(k) \cap G(k) \subseteq (T(k)G(k)]$ $\forall$ $k \in S$.

Theorem 5.24. *If S be an OSG, then the understating condition are equivalently:*

(I) S is LWRG.

(II) $(\lambda \otimes \eta)^{-} \preceq (\lambda \tilde{\cap} \eta)^{-}$ $\forall$ *neutrosophic* $(\in, \in \vee q)$*-FI* λ *and neutrosophic* $(\in, \in \vee q)$*-FGB-I* η *of S.*

Proof. Follows from Theorem [5.11] and Lemma [5.18]. □

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Baire Spaces on Fuzzy Neutrosophic Topological Spaces

E. Poongothai[#1], E. Padmavathi[#2]
[1&2]Department of Mathematics, Shanmuga Industries of Arts and Science college, Tiruvannamalai-606603, Tamil Nadu, India.

E-mail: epoongothai5@gmail.com[1], epadmavathisu20@gmail.com[2].

Abstract:

In this paper a property which can be used to Baire spaces in fy. neutrosophic top. Spaces (simply as fy. – fuzzy, top. – topological) are introduced and studied. For this purpose, introduced fy. neutrosophic F_σ – set, fy. neutrosophic G_δ – set, fy. neutrosophic dense, fy. neutrosophic nowh. (Simply as nowh. - nowhere) dense, fy. neutrosophic one (one denotes first) category, fy. neutrosophic two (two denotes second) category and fy. neutrosophic re. (Simply as re. – residual) set are defined. Also, some characterizations about these concepts are obtained.

Keywords:

Fy. neutrosophic dense set, Fy. neutrosophic nowh. dense set, Fy. neutrosophic re. set, Fy. neutrosophic Baire spaces, Fy. neutrosophic one and two category.

AMS subject classification: 54A40, 03E72

1. Introduction:

The concept of fy. sets were introduced by L.A. Zadeh in 1965 [10]. Then the fy. set theory is extension by many researchers. The important concept of fuzzy topological space was offered by C.L. Chang [3] and from that point forward different ideas in topology have been reached out to fuzzy topological space. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a morden theory of fuzzy topology has been developed. The concept of fuzzy $\sigma-$ Baire spaces was introduced and studied by G. Thangaraj and E. Poongothai in [7]. The concept of neutrosophic sets was defined with membership, non-membership and indeterminacy degrees. In 2017, Veereswari [9] introduced fy. neutrosophic top. spaces. This concept is the solution and representation of the problems with various fields.

In this paper, we define new concepts of fy. neutrosophic $F_\sigma -$ set, fy. neutrosophic $G_\delta -$ set, fy. neutrosophic dense, fy. neutrosophic nowh. dense, fy. neutrosophic one and two category sets, fy. neutrosophic re. set, fy. neutrosophic Baire spaces, fy. neutrosophic one and two category spaces in fy. neutrosophic top. spaces, and we also discussed some new properties and examples based of this defined concept.

2. Preliminaries:

Definition 2.1 [2]:

A fy. neutrosophic set A on the universe of discourse X is defined as $A = \langle x, T_A(x), I_A(x), F_A(x)\rangle$, $x \in X$ where $T, I, F: X \to [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

With the condition $0 \leq T_{A^*}(x) + \mathrm{I}_{A^*}(x) + F_{A^*}(x) \leq 2$.

Definition 2.2 [2]:

A fy. neutrosophic set A is a subset of a fy. neutrosophic set B (i.e.,) $A \subseteq B$ for all x if $T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x), F_A(x) \geq F_B(x)$.

Definition 2.3 [2]:

Let X be a non-empty set, and $A = \langle x, T_A(x), I_A(x), F_A(x) \rangle$, $B = \langle x, T_B(x), I_B(x), F_B(x) \rangle$ be two fy. neutrosophic sets. Then

$A \cup B = \langle x, \max(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \min(F_A(x), F_B(x)) \rangle$

$A \cap B = \langle x, \min(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \max(F_A(x), F_B(x)) \rangle$

Definition 2.4 [2]:

The difference between two fy. neutrosophic sets A and B is defined as

$A \backslash B(x) = \langle x, \min(T_A(x), F_B(x)), \min(I_A(x), 1 - I_B(x)), \max(F_A(x), T_B(x)) \rangle$

Definition 2.5 [2]:

A fy. neutrosophic set A over the universe X is said to be null or empty fy. neutrosophic set if $T_A(x) = 0,\ I_A(x) = 0,\ F_A(x) = 1\ for\ all\ x \in X$. It is denoted by 0_N.

Definition 2.6 [2]:

A fy. neutrosophic set A over the universe X is said to be absolute (universe) fy. neutrosophic set if $T_A(x) = 1,\ I_A(x) = 1,\ F_A(x) = 0\ for\ all\ x \in X$. It is denoted by 1_N.

Definition 2.7 [2]:

The complement of a fy. neutrosophic set A is denoted by A^C and is defined as

$A^C = \langle x, T_{A^C}(x), I_{A^C}(x), F_{A^C}(x) \rangle$ where

$T_{A^C}(x) = F_A(x), I_{A^C}(x) = 1 - I_A(x), F_{A^C}(x) = T_A(x)$

The complement of fy. neutrosophic set A can also be defined as $A^C = 1_N - A$.

Definition 2.8 [1]:

A fy. neutrosophic topology on a non-empty set X is a τ of fy. neutrosophic sets in X

$(i)\ 0_N, 1_N \in \tau$

$(ii)\ A_1 \cap A_2 \in \tau\ \ for\ any\ A_1, A_2 \in \tau$

$(iii)\ \cup A_i \in \tau\ for\ any\ arbitrary\ family\ \{A_i : i \in J\}\ \in \tau$

Satisfying the following axioms.

In this case the pair (X, τ) is called fy. neutrosophic top. space and any Fy. neutrosophic set in τ is known as fy. neutrosophic open set in X.

Definition 2.9 [1]:

The complement A^C of a fy. neutrosophic set A in a fy. neutrosophic top. space (X, τ) is called fy. neutrosophic closed set in X.

Definition 2.10 [1]:

Let (X, τ_N) be a fy. neutrosophic top. space and $A = \langle x, T_A(x), I_A(x), F_A(x) \rangle$ be a fy. neutrosophic set in X. Then the closure and interior of A are defined by

$$int(A) = \cup \{G : G\ is\ a\ fuzzy\ neutrosophic\ open\ set\ in\ X\ and\ G \subseteq A\}$$

$$cl(A) = \cap \{G : G\ is\ a\ fuzzy\ neutrosophic\ closed\ set\ in\ X\ and\ A \subseteq G\}$$

3. On Fuzzy Neutrosophic Nowhere Dense Sets

Throughout the present paper, P denote the fy. neutrosophic top. spaces. Let A_N be a fy. neutrosophic set on P. The fy. neutrosophic interior and closure of A_N is denoted by $fn(A_N)^+$, $fn(A_N)^-$ respectively. A fy. neutrosophic set A_N is defined to be fy.

neutrosophic open set $(fnOS)$ if $A_N \leq fn\left(((A_N)^-)^+\right)^-$. The complement of a fy. neutrosophic open set is called fy. neutrosophic closed set $(fnCS)$.

Definition 3.1:

A fy. neutrosophic set A_N in a fy. neutrosophic top. space (P, τ_N) is called a fy. neutrosophic F_σ − set if $A_N = \vee_{i=1}^{\infty} A_{N_i}$, where $\overline{A_{N_i}} \in \tau_N$ for $i \in I$.

Definition 3.2:

A fy. neutrosophic set A_N in a fy. neutrosophic top. space (P, τ_N) is called a fy. neutrosophic G_δ − set in (P, τ_N) if $A_N = \wedge_{i=1}^{\infty} A_{N_i}$, where $A_{N_i} \in \tau_N$ for $i \in I$.

Definition 3.3:

A fy. neutrosophic set A_N in a fy. neutrosophic top. space (P, τ_N) is called a fy. neutrosophic semi-open if $A_N \leq fn(\,((A_N)^+)^-)$. The complement of A_N in (P, τ_N) is called a fy. neutrosophic semi-closed set in (P, τ_N).

Definition 3.4:

A fy. neutrosophic set A_N in a fy. neutrosophic top. space (P, τ_N) is called a fy. neutrosophic dense if there exist no $fnCS$ B_N in (P, τ_N) s.t $A_N \subset B_N \subset 1_X$. That is, $fn\,(A_N)^- = 1_N$.

Definition 3.5:

A fy. neutrosophic set A_N in a fy. neutrosophic top. space (P, τ_N) is called a fy. neutrosophic nowh. dense set if there exist no non-zero $fnOS$ B_N in (P, τ_N) s.t $B_N \subset fn(A_N)^-$. That is, $fn(((A_N)^-)^+) = 0_N$.

Example 3.1:

Let $P = \{a, b, c\}$ and consider the family $\tau_N = \{0_N, 1_N, A_N, B_N, C_N\}$ where

$A_N = \{\langle a, 0.3,0.3,0.5\rangle, \langle b, 0.6,0.6,0.5\rangle, \langle c, 0.6,0.6,0.5\rangle\}$

$B_N = \{\langle a, 0.6,0.6,0.6\rangle, \langle b, 0.3,0.3,0.3\rangle, \langle c, 0.6,0.6,0.6\rangle\}$

$C_N = \{\langle a, 0.3,0.3,0.4\rangle, \langle b, 0.7,0.7,0.4\rangle, \langle c, 0.3,0.3,0.4\rangle\}$

Thus (P, τ_N) is a fy. neutrosophic top. spaces.

Now $fn(((\overline{A_N})^-)^+) = 0_N, fn(((\overline{B_N})^-)^+) = 0_N$ and $fn(((\overline{C_N})^-)^+) = 0_N$. This gives that $\overline{A_N}, \overline{B_N}$ and $\overline{C_N}$ are fy. neutrosophic nowh. dense sets in (P, τ_N).

Definition 3.6:

Let (P, τ_N) be a fy. neutrosophic top. space. A fy. neutrosophic set A_N in (P, τ_N) is called fy. neutrosophic one category set if $A_N = \vee_{i=1}^{\infty} A_{N_i}$, where A_{N_i}'s are fy. neutrosophic nowh. dense sets in (P, τ_N). Any other fy. neutrosophic set in (P, τ_N) is said to be of fy. neutrosophic two category.

Definition 3.7:

A fy. neutrosophic top. space (P, τ_N) is called fy. neutrosophic one category space if the fy. neutrosophic set 1_X is a fy. neutrosophic one category set in (P, τ_N). That is $1_X = \vee_{i=1}^{\infty} A_{N_i}$, where A_{N_i}'s are fy. neutrosophic nowh. dense sets in (P, τ_N). Otherwise (P, τ_N) will be called a fy. neutrosophic two category space.

Definition 3.8:

Let A_N be a fy. neutrosophic one category set in (P, τ_N). Then $\overline{A_N}$ is called fy. neutrosophic re. set in (P, τ_N).

Proposition 3.1:

If A_N is a $fnCS$ in (P, τ_N) with $fn(A_N)^+ = 0_N$ then A_N is a Fy. neutrosophic nowh. dense set in (P, τ_N).

Proof:

Let A_N is a $fnCS$ (P, τ_N). Then $fn(A_N)^- = A_N$.

Now $fn(((A_N)^-)^+) = fn(A_N)^+ = 0_N$. and hence A_N is a fy. neutrosophic nowh. dense set in (P, τ_N).

Proposition 3.2:

If A_N is a fy. neutrosophic nowh. dense set in (P, τ_N) then $fn(A_N)^+ = 0_N$.

Proof:

Let A_N be a fy. neutrosophic nowh. dense set in (P, τ_N). Now $A_N \leq fn(A_N)^-$ gives that $fn(A_N)^+ \leq fn(((A_N)^-)^+) = 0_N$. Hence, we have $fn(A_N)^+ = 0_N$.

Remark 3.1:

The complement of a fy. neutrosophic nowh. dense set need not be a fy. neutrosophic nowh. dense set. For, consider the following example.

Example 3.2:

Let $P = \{a, b, c\}$ and consider the family $\tau_N = \{0_N, 1_N, A_N, B_N, C_N\}$ where

$A_N = \{\langle a, 0.5,0.5,0.4\rangle, \langle b, 0.5,0.3,0.5\rangle, \langle c, 0.5,0.5,0.4\rangle\}$

$B_N = \{\langle a, 0.5,0.5,0.3\rangle, \langle b, 0.4,0.2,0.5\rangle, \langle c, 0.5,0.4,0.3\rangle\}$

$C_N = \{\langle a, 0.5,0.4,0.5\rangle, \langle b, 0.5,0.3,0.2\rangle, \langle c, 0.5,0.6,0.3\rangle\}$

Now $fn(((B_N)^-)^+) = 0_N$ is a fy. neutrosophic nowh. dense sets in (P, τ_N).

But $fn(((\overline{B_N})^-)^+) \neq 0_N$. Therefore $\overline{B_N}$ is not a fy. neutrosophic nowh. dense sets in (P, τ_N).

Proposition 3.3:

If A_N is a fy. neutrosophic dense, $fnOS$ in (P, τ_N) s.t $B_N \leq \overline{A_N}$, then B_N is a fy. neutrosophic nowh. dense set in (P, τ_N).

Proof:

Let A_N be a $fnOS$ in (P, τ_N) s.t $fn(A_N)^- = 1$. Now $B_N \leq \overline{A_N}$ gives that $fn(B_N)^- \leq fn(\overline{A_N})^- = (1 - A_N)$ [$(\overline{A_N})$ is a $fnCS$ in (P, τ_N)] Then we have $fn(((B_N)^-)^+) \leq fn(\overline{A_N})^+ = \overline{(fn(A_N))^-} = 1 - 1 = 0_N$. and hence $fn(((B_N)^-)^+) = 0_N$. Therefore B_N is a fy. neutrosophic nowh. dense set in (P, τ_N).

Proposition 3.4:

If A_N is a non-zero fy. neutrosophic nowh. dense set in (P, τ_N), is a fy. neutrosophic nowh. dense set then A_N is fy. neutrosophic semi-closed set in (P, τ_N).

Proof:

Let A_N be a fy. neutrosophic nowh. dense set in (P, τ_N). Then $fn(((A_N)^-)^+) = 0_N$. and therefore $fn(((A_N)^-)^+) \leq A_N$. Hence, A_N is fy. neutrosophic semi-closed set in (P, τ_N).

Proposition 3.5:

If a $fnCS$ A_N is a fy. neutrosophic nowh. dense set in (P, τ_N) if and only if $fn(A_N)^+ = 0_N$.

Proof:

Let A_N be a $fnCS$ in (P, τ_N) with $fn(A_N)^+ = 0_N$. Then by proposition 3.1, A_N is a fy. neutrosophic nowh. dense set in (P, τ_N). Conversely, let A_N is a fy. neutrosophic nowh. dense set in (P, τ_N), then $fn(((A_N)^-)^+) = 0_N$, which gives that $fn(A_N)^+ = 0_N$, [since A_N is $fnCS$ in $fn(A_N)^- = A_N$].

Proposition 3.6:

If A_N is a fy. neutrosophic nowh. dense set in (P, τ_N), then $\overline{A_N}$ is a fy. neutrosophic dense set in (P, τ_N).

Proof:

Let A_N be a fy. neutrosophic nowh. dense set in (P, τ_N).

Then by proposition 3.2, we have, $fn(A_N)^+ = 0_N$. Now $fn(\overline{A_N})^- = \overline{(fn(\overline{A_N}))^+} = 1 - 0_N = 1_N$. Therefore $\overline{A_N}$ is a fy. neutrosophic dense set in (P, τ_N).

Proposition 3.7:

If A_N is a fy. neutrosophic nowh. dense set and $fnOS$ in (P, τ_N), then $\overline{A_N}$ is a fy. neutrosophic nowh. dense set in (P, τ_N).

Proof:

Let A_N be a $fnOS$ in (P, τ_N) s.t, $fn(A_N)^- = 1$. Now $fn(((\overline{A_N})^-)^+) = \overline{(fn((A_N)^+)^-)} = \overline{fn(A_N)^-} == 1 - 1 = 0_N$. Hence $\overline{A_N}$ is a fy. neutrosophic nowh. dense set in (P, τ_N).

Proposition 3.8:

If A_N is a fy. neutrosophic nowh. dense set in (P, τ_N), then $fn(A_N)^-$ is a fy. neutrosophic nowh. dense set in (P, τ_N).

Proof:

Let $fn(A_N)^- = B_N$, Now $fn(((B_N)^-)^+)$ $= fn((((A_N)^-)^-)^+) = fn(((A_N)^-)^+) = 0_N$.

Hence $B_N = fn(A_N)^-$ is a fy. neutrosophic nowh. dense set in (P, τ_N).

Proposition 3.9:

If A_N is a fy. neutrosophic nowh. dense set in (P, τ_N), then $\overline{fn(A_N)^-}$ is a fy. neutrosophic dense set in (P, τ_N).

Proof:

By proposition 3.8, we have $fn(A_N)^-$ is a fy. neutrosophic nowh. dense set in (P, τ_N). By proposition 3.7, we have $\overline{fn(A_N)^-}$ is a fy. neutrosophic nowh. dense set in (P, τ_N).

Proposition 3.10:

Let A_N be a fy. neutrosophic dense set in (P, τ_N).

If B_N is any fy. neutrosophic set in (P, τ_N), then B_N is a fy. neutrosophic nowh. dense set in (P, τ_N), if and only if $A_N \wedge B_N$ is a fy. neutrosophic nowh. dense set in (P, τ_N).

Proof:

Let B_N be a Fy. neutrosophic nowh. dense set in (P, τ_N).

Now,

$fn(((A_N \wedge B_N)^-)^+) = (fn(fn(A_N)^- \wedge fn\ (B_N)^-))^+ = (fn(1 \wedge fn(B_N)^-))^+$

$$= fn(((B_N)^-)^+) = 0_N.$$

Therefore $A_N \wedge B_N$ is a fy. neutrosophic nowh. dense set in (P, τ_N).

Conversely let $A_N \wedge B_N$ is a fy. neutrosophic nowh. dense set in (P, τ_N). Then $fn(((A_N \wedge B_N)^-)^+) = 0_N$ Gives that $(fn(fn(A_N)^- \wedge fn\ (B_N)^-))^+$.

Hence $(fn(1 \wedge fn(B_N)^-))^+ = 0_N$ and therefore $fn(((B_N)^-)^+) = 0_N$ which means that B_N is a fy. neutrosophic nowh. dense set in (P, τ_N).

Proposition 3.11:

Every fy. neutrosophic nowh. dense sets is a $fnCS$.

Proof:

Let A_N be any fy. neutrosophic nowh. dense set in a fy. neutrosophic top. space (P, τ_N). Therefore, we have $fn(((A_N)^-)^+) = 0_N$ and it means that there does not exist any $fnOS$ in between A_N and $(A_N)^-$. Also, let us suppose that $A_N \leq B_N$, where B_N is $fnOS$ and obviously $(A_N)^- \leq B_N$. Therefore B_N is a $fnCS$.

4. Fuzzy Neutrosophic Baire Space

Definition 4.1:

A fy. neutrosophic top. space (P, τ_N) is called fy. neutrosophic Baire space if $fn(\vee_{i=1}^{\infty}(A_{N_i}))^+ = 0_N$, where A_{N_i}'s are fy. neutrosophic nowh. dense sets in (P, τ_N).

Example 4.1:

Let $P = \{a, b, c\}$ and consider the family $\tau_N = \{0_N, 1_N, A_N, B_N, C_N, D_N\}$ where

$A_N = \{\langle a, 0.3,0.3,0.5\rangle, \langle b, 0.6,0.6,0.5\rangle, \langle c, 0.6,0.6,0.5\rangle\}$

$B_N = \{\langle a, 0.3,0.3,0.3\rangle, \langle b, 0.6,0.6,0.5\rangle, \langle c, 0.6,0.6,0.6\rangle\}$

$C_N = \{\langle a, 0.7,0.7,0.4\rangle, \langle b, 0.4,0.3,0.3\rangle, \langle c, 0.3,0.3,0.4\rangle\}$

$D_N = \{\langle a, 0.3,0.3,0.3\rangle, \langle b, 0.7,0.7,0.7\rangle, \langle c, 0.3,0.3,0.3\rangle\}$

Now $\overline{A_N}$, $\overline{B_N}$ and $\overline{C_N}$ are fy. neutrosophic nowh. dense sets in (P, τ_N). Also $fn(\overline{A_N} \vee \overline{B_N} \vee \overline{C_N})^+ = 0_N$. Hence (P, τ_N) be a fy. neutrosophic Baire Space.

Proposition 4.1:

Let (P, τ_N) be a fy. neutrosophic top. space. Then the following are equivalent.

i) (P, τ_N) is a fy. neutrosophic baire space.

ii) $fn\ (A_N)^+ = 0_N$, for every fy. neutrosophic one category set A_N in (P, τ_N).

iii) $fn\ (B_N)^+ = 1_N$, for every fy. neutrosophic re. set B_N in (P, τ_N).

Proof:

$(i) \Rightarrow (ii)$

Let A_N be a fy. neutrosophic one category set in (P, τ_N). Then $A_N = (\vee_{i=1}^{\infty} A_{N_i})$, where A_{N_i}' s are fy. neutrosophic nowh. dense sets in (P, τ_N). Now $fn\ (A_N)^+ = fn\ (\vee_{i=1}^{\infty} A_{N_i})^+ = 0_N$. Since (P, τ_N) is a fy. neutrosophic Baire space. Therefore $fn\ (A_N)^+ = 0_N$.

$(ii) \Rightarrow (iii)$

Let B_N be a fy. neutrosophic re. set in (P, τ_N). Then $\overline{B_N}$ is a fy. neutrosophic one category set in (P, τ_N). By hypothesis, $fn(\overline{B_N})^+ = 0_N$ which gives that $\overline{fn(A_N)^-} = 0_N$. Hence $fn\ (A_N)^- = 1_N$.

$(iii) \Rightarrow (i)$

Let A_N be a fy. neutrosophic one category set in (P, τ_N). Then $A_N = (\vee_{i=1}^{\infty} A_{N_i})$, where A_{N_i}'s are fy. neutrosophic nowh. dense sets in (P, τ_N). Now A_N is a fy. neutrosophic one category set gives that $\overline{A_N}$ is a fy. neutrosophic re. set in (P, τ_N). By hypothesis, we have $fn(\overline{A_N})^- = 1_N$ which gives that $\overline{fn(A_N)^+} = 1_N$. Hence $fn\,(A_N)^+ = 0_N$. That is, $fn\,(\vee_{i=1}^{\infty} A_{N_i})^+ = 0_N$, where A_{N_i}'s are fy. neutrosophic nowh. dense sets in (P, τ_N). Hence (P, τ_N) is a fy. neutrosophic Baire space.

Proposition 4.2:

If A_N be a fy. neutrosophic one category set in (P, τ_N) then $\overline{A_N} = \wedge_{i=1}^{\infty} B_{N_i}$, where $fn\,(B_{N_i})^- = 1_N$.

Proof:

Let A_N be a fy. neutrosophic one category set in (P, τ_N).

Then $A_N = (\vee_{i=1}^{\infty} A_{N_i})$, where A_{N_i}'s are fy. neutrosophic nowh. dense sets in (P, τ_N). Now $\overline{A_N} = \overline{\vee_{i=1}^{\infty} A_{N_i}} = \wedge_{i=1}^{\infty} \bar{A}_{N_i}$ Now A_{N_i} is a fy. neutrosophic nowh. dense sets in (P, τ_N). Then by proposition [3.6] we have $\overline{A_{N_i}}$. is a fy. neutrosophic dense sets in (P, τ_N). Let us put $B_{N_i} = \overline{A_{N_i}}$. Then $\overline{A_N} = \wedge_{i=1}^{\infty} B_{N_i}$, where $fn\,(B_{N_i})^- = 1_N$.

Proposition 4.3:

If $fn\,(\vee_{i=1}^{\infty} A_{N_i})^+ = 0_N$ where $fn\,(A_{N_i})^+ = 0_N$ and $A_{N_i} \in \tau_N$, then (P, τ_N) is a fy. neutrosophic Baire space.

Proof:

Now $A_{N_i} \in \tau_N$ gives that A_{N_i} is a $fnOS$ in (P, τ_N). Since $fn\,(A_{N_i})^+ = 0_N$. By proposition (3.2), A_{N_i} is a fy. neutrosophic nowh. dense sets in (P, τ_N). Therefore

$fn\ (\vee_{i=1}^{\infty} A_{N_i})^{+} = 0_N$, where $A_{N_i}{}'s$ is a fy. neutrosophic nowh. dense sets in (P, τ_N). Hence (P, τ_N) is a fy. neutrosophic Baire space.

Proposition 4.4:

If $fn\ (\vee_{i=1}^{\infty} A_{N_i})^{+}$ where $fn\ (A_{N_i})^{+} = 0_N$ and A_{N_i}'s are $fnCS$ in fy. neutrosophic top. space in (P, τ_N) then (P, τ_N) is a fy. neutrosophic Baire space.

Proof:

Let $A_{N_i}{}'s$ be $fnCS$ in (P, τ_N). Since $fn\ (A_{N_i})^{+} = 0_N$, by proposition (3.2), $A_{N_i}{}'s$ are fy. neutrosophic nowh. dense sets in (P, τ_N). Thus $fn\ (\vee_{i=1}^{\infty} A_{N_i})^{+} = 0_N$, where $A_{N_i}{}'s$ are fy. neutrosophic nowh. dense sets in (P, τ_N). Hence (P, τ_N) is a fy. neutrosophic Baire space.

Proposition 4.5

If $fn\ (\wedge_{i=1}^{n} A_{N_i},)^{-} = 1_N$, where $A_{N_i}{}'s$ are fy. neutrosophic dense and $fnOS$ in fy. neutrosophic top. space (P, τ_N) if and only if (P, τ_N) is a fy. neutrosophic Baire space.

Proof:

Let $A_{N_i}{}'s$ be fy. neutrosophic dense sets in (P, τ_N). Then $fn\ (\wedge_{i=1}^{n} A_{N_i},)^{-} = 1_N$ which gives that $1 - fn\ (\wedge_{i=1}^{n} A_{N_i},)^{-} = 0_N$. That is $fn\ ((1 - \wedge_{i=1}^{n} A_{N_i})\ )^{+} = 0_N$ gives that $fn\ (1 - \vee_{i=1}^{\infty} A_{N_i})^{+} = 0_N$. Since $A_{N_i}{}'s$ be fy. neutrosophic dense, $fn\ (A_{N_i})^{-} = 1_N$. Hence $fn\ (1 - A_{N_i})^{+} = 1 - fn\ \left(A_{N_i}\right)^{-} = 0_N$. Consequently $fn\ (1 - \vee_{i=1}^{\infty} A_{N_i})^{+} = 0_N$, where $fn\ (1 - A_{N_i})^{+} = 0_N$ and $A_{N_i}{}'s$ be $fnCS$ in (P, τ_N). By proposition 4.4, (P, τ_N) is a fy. neutrosophic Baire space.

Conversely, let A_{N_i}'s are fy. neutrosophic dense and $fnCS$ in (P, τ_N). By proposition (3.7), $1 - A_{N_i}$'s are fy. neutrosophic nowh. dense sets in (P, τ_N). Then $A_N = \vee_{i=1}^{\infty} 1 - A_{N_i}$ is a fy. neutrosophic one category set in (P, τ_N). Now $fn\,(A_N)^+ = fn\,(\vee_{i=1}^{\infty}(1 - A_{N_i}))^+ = fn\,(1 - \wedge_{i=1}^{\infty} A_{N_i})^+ = (1 - fn\,(\vee_{i=1}^{\infty} A_{N_i}))^-$.

Since (P, τ_N) is a fy. neutrosophic Baire space, by proposition 4.1, we set $fn\,(A_N)^+ = 0_N$. Then $(1 - fn\,(\wedge_{i=1}^{\infty} A_{N_i})^-) = 0_N$. This gives that $(fn\,(\wedge_{i=1}^{\infty} A_{N_i})^-) = 1_N$.

Conclusion:

In this paper, the concept of a new class of sets, spaces and called them fy. neutrosophic dense, fy. neutrosophic nowh. dense, fy. neutrosophic re. set, fy. neutrosophic one category set, fy. neutrosophic two category sets, fy. neutrosophic Baire spaces, fy. neutrosophic one category space, fy. neutrosophic two category space. Some of its characterizations of fy. neutrosophic Baire spaces are also studied. As fuzzy neutrosophic have many applications in many fields: information technology, information system, decision support system. In the future research presented some of the applications.

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Dombi Aggregation Operators of Linguistic Neutrosophic Numbers for Multiple Attribute Group Decision-Making Problems in Landslide Treatment Schemes

Yingying Zhang[1], Rui Yong*[1], Jun Ye[1], Zhen Zhong[2], Shuang Zheng[2]

[1] School of Civil and Environmental Engineering, Ningbo University, Ningbo, Zhejiang, 315211, P.R. China;

[2] Key Laboratory of Rock Mechanics and Geohazards of Zhejiang Province, Shaoxing University, Shaoxing, Zhejiang, 32200, P.R. China

* Correspondence: yongrui@nbu.edu.cn

Abstract: The landslide disaster caused huge losses to human lives and property, and the research on the selection of landslide treatment schemes has attracted global attention. Fuzzy multi-attribute decision-making methods are widely used for selecting the slope treatment schemes in engineering practice. But they do not take into account human linguistic arguments in the linguistic decision-making environment, which usually contains incomplete and uncertain information, and still lack a qualitative evaluation method. To deal with multiple attribute group decision-making (MAGDM) problems of landslide treatment schemes in the linguistic neutrosophic environment, a linguistic neutrosophic number Dombi weighted arithmetic averaging (LNNDWAA) operator and a linguistic neutrosophic number Dombi weighted geometric averaging (LNNDWGA) operator are developed to aggregate linguistic neutrosophic information. Then, a new MAGDM method using these aggregation operators is proposed in view of the Dombi operational flexibility. Finally, the proposed method is applied to select the optimal landslide treatment scheme under the linguistic neutrosophic environment. The results show that this method can effectively solve the decision-making problem of landslide treatment schemes and make the decision result more reasonable and flexible than other existing methods.

Keywords: landslide treatment scheme; multiple attribute decision; linguistic neutrosophic number; Dombi operation

1. Introduction

Landslides have caused immeasurable economic losses to human society. For example, China has an average of almost 30,000 landslides, rock falls, and debris flows every year, many of which have caused catastrophic disasters. On average, nearly 800 people are killed each year, and the direct economic loss exceeds 1 billion US dollars. In 2019, a total of 6,181 geological disasters occurred nationwide (Figure 1), where slope failure accounted for 68.27%, accounting for the vast majority [1]. Therefore, great attention has been paid to the prevention and treatment of landslides. At present, there are many prevention and control plans for slopes. To choose the most reasonable plan, some decision-making (DM) methods are needed. In recent years, research on DM methods of slope treatment schemes has received increasing attention. These DM methods include the empirical discriminant method [2], the analytic hierarchy process [3], the fuzzy multi-attribute DM method [4], the subjective and objective weighting method [5], and so on. However, most of these methods are based on the analysis or subjective judgment of expert experience, which leads to unreasonable, uneconomical, and sometimes even huge waste in the final choice of the treatment schemes. Fuzzy multi-attribute DM methods have been widely used because they can deal with fuzzy evaluation and DM problems. But they does not take into account human linguistic arguments in the DM issue of the treatment schemes, which usually contains incomplete and uncertain information.

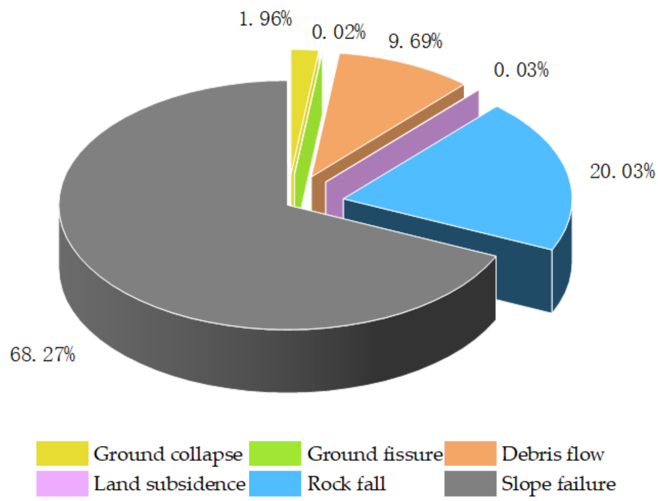

Figure 1. Types of geological disasters in 2019

Due to the indeterminacy and ambiguity of human cognition of objective things and the intricacy of multi-attribute group decision making (MAGDM) environment, linguistic variables can more effectively describe decision information than numerical values [6,7,8]. Therefore, to improve the DM effectiveness, many researchers performed extensive studies on DM challenges in linguistic environments. Zadeh [7] initially proposed the concept of linguistic variables (LVs), which can employ words or sentences to represent qualitative information. In order to solve DM problems with linguistic information, Herrera et al. [9, 10] created a technique for linguistic decision analysis. Later, Xu [11,12,13] developed linguistic aggregation operators and goal programming models to handle DM problems. To tackle incompleteness and ambiguity in DM situations more effectively, Merigó et al. [14,15,16] proposed some linguistic aggregation operators for the aggregation of LVs. Xu [17,18] proposed uncertain LVs given by interval values. Then, some scholars developed various aggregation operators of uncertain LVs for the MAGDM problems with uncertain linguistic information [19–23]. Chen et al. [24] proposed the concept of linguistic intuitionistic fuzzy numbers (LIFNs), which enables the direct description of real and false linguistic information using linguistic items. Liu et al. [25,26] put forward some LIFN aggregation operators for multi-attribute DM. However, LIFN cannot express uncertain and inconsistent linguistic information in DM problems. But the neutrosophic numbers (NNs) [27–30] and neutrosophic sets [28–30] proposed by Smarandache make up for the above shortcomings. Some scholars put forward new concepts focusing on the combination of neutrosophic set and linguistic set. Subsequently, Fang and Ye [8] introduced the linguistic neutrosophic number (LNN) which includes three independent LVs for describe true, false, and uncertain linguistic information. They also introduced the LNN weighted geometric and LNN weighted arithmetic averaging operators to handle MAGDM problems containing LNN information. However, this MAGDM method [8] can be better applicable to the expression and processing of inconsistent and uncertain linguistic information in DM problems. After that, some LNN aggregation operators were successively proposed, such as LNN normalized weighted geometric Bonferroni mean (LNNNWGBM) and LNN normalized weighted Bonferroni mean (LNNNWBM) operators [31] and linguistic neutrosophic power weighted Heronian aggregation (LNPWHA) operators [32]. These aggregation operators can effectively deal with linguistic DM problems with inconsistent and uncertain linguistic information. Furthermore, Shi and Ye developed three correlation coefficients of LNNs [33] and two cosine similarity measures of LNNs [34] for MAGDM problems with LNN information. Cui and Ye [35] defined the concept of hesitant linguistic neutrosophic number (HLNN) sets and introduced the normalized generalized distance and similarity measures of HLNNs for DM problems with HLNN information.

In 1982, Dombi [36] developed Dombi t-conorm and Dombi t-norm operations, which contain the advantage of changeability by adjusting a parameter value. Hence, Liu et al. [37] introduced the Dombi operations of intuitionistic fuzzy sets (IFSs) and proposed some Dombi aggregation operators for the MAGDM problem with intuitionistic fuzzy information. Ye and Chen [38] introduced the Dombi operations of single-valued neutrosophic numbers (SVNNs), then presented the single-valued neutrosophic Dombi weighted geometric average (SVNDWGA) operator and the single-valued neutrosophic Dombi weighted arithmetic average (SVNDWAA) operator to handle DM problems with LNNs. Ye and Lu [39] extended the Dombi operations to the environment of linguistic cubic variables (LCVs) and developed the linguistic cubic variable Dombi weighted geometric average (LCVDWGA) operator and linguistic cubic variable Dombi weighted arithmetic average (LCVDWAA) operator for MAGDM problems. However, in the available research, the Dombi operations have not yet been extended to LNNs. Therefore, the main goals of this study are (1) to propose some Dombi operations of LNNs, (2) to propose the LNN Dombi weighted geometric averaging (LNNDWGA) and LNN Dombi weighted arithmetic averaging (LNNDWAA) operators, (3) to develop a DM method based on the LNNDWAA or LNNDWGA operator for performing MAGDM problems in the LNN information environment, and (4) to validate the viability of this method through a case study.

The following sections constitute the rest of this paper. Section 2 introduces some preliminaries. In Section 3, we define the Dombi operations of LNNs and propose the LNNDWAA and LNNDWGA operators and their properties. Section 4 introduces a new MAGDM method using the LNNDWAA or LNNDWGA operator. In Section 5, the application of the proposed method is demonstrated by an application example and then a comparative analysis is given to show its superiority over existing approaches. Section 6 gives the conclusions of this article.

2. Preliminaries

2.1 Several Concepts of LNNs

Definition 1 [8]. Suppose that $Fr^{Ro}= \{Fr_0^{Ro}, Fr_1^{Ro}, \ldots, Fr_\Phi^{Ro}\}$ is a set of linguistic terms with an odd cardinality $\Phi + 1$. Then, LNN is defined as $N= \langle Fr_x^{Ro}, Fr_y^{Ro}, Fr_z^{Ro}\rangle$ for $Fr_x^{Ro}, Fr_y^{Ro}, Fr_z^{Ro} \in Fr^{Ro}$ and $x, y, z \in [0, \Phi]$, where Fr_x^{Ro}, Fr_y^{Ro} and Fr_z^{Ro} independently represent truth, uncertainty, and falsity LVs, respectively.

Definition 2 [8]. Set $N = \langle Fr_x^{Ro}, Fr_y^{Ro}, Fr_z^{Ro}\rangle$ as LNN in Fr^{Ro}. The score and accuracy functions of N are determined by the following eqations:

$$U(N) = (2\Phi + x - y - z)/(3\Phi) \text{ for } U(N) \in [0,1], \tag{1}$$

$$V(N) = (x - z)/\Phi \text{ for } V(N) \in [-1,1]. \tag{2}$$

Definition 3 [8]. Let $N_1 = \langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro}\rangle$ and $N_2 = \langle Fr_{x_2}^{Ro}, Fr_{y_2}^{Ro}, Fr_{z_2}^{Ro}\rangle$ be two LNNs in Fr^{Ro}, and they imply the following ranking relations:

(1) When $U(N_1) > U(N_2) \Rightarrow N_1 \succ N_2$;

(2) When $U(N_1) < U(N_2) \Rightarrow N_1 \prec N_2$;

(3) When $V(N_1) = V(N_2)$ and $U(N_1) = U(N_2) \Rightarrow N_1 = N_2$;

(4) When $V(N_1) < V(N_2)$ and $U(N_1) = U(N_2) \Rightarrow N_1 \prec N_2$;

(5) When $V(N_1) > V(N_2)$ and $U(N_1) = U(N_2) \Rightarrow N_1 \succ N_2$.

Definition 4 [8]. Let $N_1 = \left\langle Fr^{Ro}_{x_1}, Fr^{Ro}_{y_1}, Fr^{Ro}_{z_1} \right\rangle$ and $N_2 = \left\langle Fr^{Ro}_{x_2}, Fr^{Ro}_{y_2}, Fr^{Ro}_{z_2} \right\rangle$ be two LNNs in Fr^{Ro}, and λ is a positive real number ($\lambda > 0$). Their operational laws are introduced as follows:

(1) $N_1 \oplus N_2 = \left\langle Fr^{Ro}_{x_1}, Fr^{Ro}_{y_1}, Fr^{Ro}_{z_1} \right\rangle \oplus \left\langle Fr^{Ro}_{x_2}, Fr^{Ro}_{y_2}, Fr^{Ro}_{z_2} \right\rangle = \left(Fr^{Ro}_{x_1+x_2-\frac{x_1x_2}{\Phi}}, Fr^{Ro}_{\frac{y_1y_2}{\Phi}}, Fr^{Ro}_{\frac{z_1z_2}{\Phi}} \right)$;

(2) $N_1 \otimes N_2 = \left\langle Fr^{Ro}_{x_1}, Fr^{Ro}_{y_1}, Fr^{Ro}_{z_1} \right\rangle \otimes \left\langle Fr^{Ro}_{x_2}, Fr^{Ro}_{y_2}, Fr^{Ro}_{z_2} \right\rangle = \left(Fr^{Ro}_{\frac{x_1x_2}{\Phi}}, Fr^{Ro}_{y_1+y_2-\frac{y_1y_2}{\Phi}}, Fr^{Ro}_{z_1+z_2-\frac{z_1z_2}{\Phi}} \right)$;

(3) $\lambda N_1 = \lambda \left\langle Fr^{Ro}_{x_1}, Fr^{Ro}_{y_1}, Fr^{Ro}_{z_1} \right\rangle = \left\langle Fr^{Ro}_{\Phi-\Phi\left(1-\frac{x_1}{\Phi}\right)^{\lambda}}, Fr^{Ro}_{\Phi\left(\frac{y_1}{\Phi}\right)^{\lambda}}, Fr^{Ro}_{\Phi\left(\frac{z_1}{\Phi}\right)^{\lambda}} \right\rangle$;

(4) $N_1^{\lambda} = \left\langle Fr^{Ro}_{x_1}, Fr^{Ro}_{y_1}, Fr^{Ro}_{z_1} \right\rangle^{\lambda} = \left\langle Fr^{Ro}_{\Phi\left(\frac{x_1}{\Phi}\right)^{\lambda}}, Fr^{Ro}_{\Phi-\Phi\left(1-\frac{y_1}{\Phi}\right)^{\lambda}}, Fr^{Ro}_{\Phi-\Phi\left(1-\frac{z_1}{\Phi}\right)^{\lambda}} \right\rangle$.

2.2 Weighted Aggregation Operators of LNNs

Definition 5 [8]. Set $N_g = \left\langle Fr^{Ro}_{x_g}, Fr^{Ro}_{y_g}, Fr^{Ro}_{z_g} \right\rangle$ (g = 1, 2, …, h) as an assemblage of LNNs in Fr^{Ro}. The LNNWAA operator is defined below:

$$LNNWAA(N_1, N_2, ..., N_h) = \sum_{g=1}^{h} \gamma_g N_g , \tag{3}$$

where γ_g is the weight of N_g (g = 1, 2, …, h) for $0 \le \gamma_g \le 1$ and $\sum_{g=1}^{h} \gamma_g = 1$.

Theorem 1 [8]. Let $N_g = \left\langle Fr^{Ro}_{x_g}, Fr^{Ro}_{y_g}, Fr^{Ro}_{z_g} \right\rangle$ (g = 1, 2, …, h) as an assemblage of LNN in Fr^{Ro}, then the aggregation result is obtained based on the following aggregation equation:

$$LNNWAA(N_1, N_2, ..., N_h) = \sum_{g=1}^{h} \gamma_g N_g = \left\langle Fr^{Ro}_{\Phi-\Phi\prod_{g=1}^{h}(1-\frac{x_g}{\Phi})^{\gamma_g}}, Fr^{Ro}_{\Phi\prod_{g=1}^{h}(\frac{y_g}{\Phi})^{\gamma_g}}, Fr^{Ro}_{\Phi\prod_{g=1}^{h}(\frac{z_g}{\Phi})^{\gamma_g}} \right\rangle. \tag{4}$$

Definition 6 [8]. Suppose that $N_g = \left\langle Fr^{Ro}_{x_g}, Fr^{Ro}_{y_g}, Fr^{Ro}_{z_g} \right\rangle$ (g = 1, 2, …, h) is a group of LNNs in Fr^{Ro}, the LNNWGA operator is defined by

$$LNNWGA(N_1, N_2, ..., N_h) = \prod_{g=1}^{h} N_g^{\gamma_g} , \tag{5}$$

where γ_g is the weight of N_g (g = 1, 2, …, h) for $0 \le \gamma_g \le 1$ and $\sum_{g=1}^{h}\gamma_g = 1$.

Theorem 2 [8]. Let $N_g = \left\langle Fr_{x_g}^{Ro}, Fr_{y_g}^{Ro}, Fr_{z_g}^{Ro} \right\rangle$ (g = 1, 2, …, h) as an assemblage of linguistic neutrosophic numbers in Fr^{Ro}, then the result of aggregation is obtained based on the following aggregation equation:

$$LNNWGA(N_1, N_2, ..., N_h) = \prod_{g=1}^{h} N_g^{\gamma_g} = \left\langle Fr^{Ro}_{\Phi\prod_{g=1}^{h}(\frac{x_g}{\Phi})^{\gamma_g}}, Fr^{Ro}_{\Phi-\Phi\prod_{g=1}^{h}(1-\frac{y_g}{\Phi})^{\gamma_g}}, Fr^{Ro}_{\Phi-\Phi\prod_{g=1}^{h}(1-\frac{z_g}{\Phi})^{\gamma_g}} \right\rangle. \quad (6)$$

3. Dombi Operations of LNNs

The Dombi operations contain the advantage of flexible aggregations by modifying the value of the parameter. In 1982, Dombi [36] proposed the Dombi T-norm and T-conorm operations for the first time. Although many researchers have introduced Dombi operations in various linguistic decision-making environments and decision-making methods [37–43], the Dombi operations have not yet expanded to LNNs. Therefore, this section proposes the Dombi T-norm and T-conorm operations of LNNs, then presents the LNNDWAA and LNNDWGA operators and their properties.

3.1 Dombi Operational Laws of LNNs

Definition 7 [36]. For any two real-values *Th* and *Tj*, the Dombi T-norm and T-conorm operations between *Th* and *Tj* are defined below:

$$O_D(Th, Tj) = \frac{1}{1+\left\{\left(\frac{1-Th}{Th}\right)^{\rho} + \left(\frac{1-Tj}{Tj}\right)^{\rho}\right\}^{1/\rho}}, \quad (7)$$

$$O_D^c(Th, Tj) = 1 - \frac{1}{1+\left\{\left(\frac{Th}{1-Th}\right)^{\rho} + \left(\frac{Tj}{1-Tj}\right)^{\rho}\right\}^{1/\rho}}, \quad (8)$$

where the parameter $\rho \ge 1$ and $(Th, Tj) \in [0, 1] \times [0, 1]$.

Definition 8. Assume that $N_1 = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle$ and $N_2 = \left\langle Fr_{x_2}^{Ro}, Fr_{y_2}^{Ro}, Fr_{z_2}^{Ro} \right\rangle$ are two LNNs, $\lambda > 0$, and $\rho > 0$. Then, the Dombi T-norm and T-conorm operational laws of LNNs are expressed below:

$$N_1 \oplus N_2 = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle \oplus \left\langle Fr_{x_2}^{Ro}, Fr_{y_2}^{Ro}, Fr_{z_2}^{Ro} \right\rangle$$

$$= \left(Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\left(\frac{x_1/\Phi}{1-x_1/\Phi}\right)^{\rho}+\left(\frac{x_2/\Phi}{1-x_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}\right)}, Fr^{Ro}_{\Phi\times\frac{1}{1+\left\{\left(\frac{1-y_1/\Phi}{y_1/\Phi}\right)^{\rho}+\left(\frac{1-y_2/\Phi}{y_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi\times\frac{1}{1+\left\{\left(\frac{1-z_1/\Phi}{z_1/\Phi}\right)^{\rho}+\left(\frac{1-z_2/\Phi}{z_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}} \right), \quad (9)$$

$$= \left(Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\left(\frac{x_1}{\Phi-x_1}\right)^{\rho}+\left(\frac{x_2}{\Phi-x_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\left(\frac{\Phi-y_1}{y_1}\right)^{\rho}+\left(\frac{\Phi-y_2}{y_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\left(\frac{\Phi-z_1}{z_1}\right)^{\rho}+\left(\frac{\Phi-z_2}{z_2}\right)^{\rho}\right\}^{1/\rho}}} \right)$$

$$N_1 \otimes N_2 = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle \otimes \left\langle Fr_{x_2}^{Ro}, Fr_{y_2}^{Ro}, Fr_{z_2}^{Ro} \right\rangle$$

$$= \left(Fr^{Ro}_{\Phi\times\frac{1}{1+\left\{\left(\frac{1-x_1/\Phi}{x_1/\Phi}\right)^{\rho}+\left(\frac{1-x_2/\Phi}{x_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\left(\frac{y_1/\Phi}{1-y_1/\Phi}\right)^{\rho}+\left(\frac{y_2/\Phi}{1-y_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}\right)}, Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\left(\frac{z_1/\Phi}{1-z_1/\Phi}\right)^{\rho}+\left(\frac{z_2/\Phi}{1-z_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}\right)} \right), \quad (10)$$

$$= \left(Fr^{Ro}_{\frac{\Phi}{1+\left\{\left(\frac{\Phi-x_1}{x_1}\right)^{\rho}+\left(\frac{\Phi-x_2}{x_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\left(\frac{y_1}{\Phi-y_1}\right)^{\rho}+\left(\frac{y_2}{\Phi-y_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\left(\frac{z_1}{\Phi-z_1}\right)^{\rho}+\left(\frac{z_2}{\Phi-z_2}\right)^{\rho}\right\}^{1/\rho}}} \right)$$

$$\lambda N_1 = \lambda \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle$$

$$= \left(Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\lambda\left(\frac{x_1/\Phi}{1-x_1/\Phi}\right)^{\rho}\right\}^{1/\rho}}\right)}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{1-y_1/\Phi}{y_1/\Phi}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{1-z_1/\Phi}{z_1/\Phi}\right)^{\rho}\right\}^{1/\rho}}} \right), \quad (11)$$

$$= \left(Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\lambda\left(\frac{x_1}{\Phi-x_1}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{\Phi-y_1}{y_1}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{\Phi-z_1}{z_1}\right)^{\rho}\right\}^{1/\rho}}} \right)$$

$$N_1^{\lambda}=\left\langle Fr_{x_1}^{Ro},Fr_{y_1}^{Ro},Fr_{z_1}^{Ro}\right\rangle^{\lambda}$$

$$=\left(Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{1-x_1/\Phi}{x_1/\Phi}\right)^{\rho}\right\}^{1/\rho}}},Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\lambda\left(\frac{y_1/\Phi}{1-y_1/\Phi}\right)^{\rho}\right\}^{1/\rho}}\right)},Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\lambda\left(\frac{z_1/\Phi}{1-z_1/\Phi}\right)^{\rho}\right\}^{1/\rho}}\right)}\right). \quad (12)$$

$$=\left(Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{\Phi-x_1}{x_1}\right)^{\rho}\right\}^{1/\rho}}},Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\lambda\left(\frac{y_1}{\Phi-y_1}\right)^{\rho}\right\}^{1/\rho}}},Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\lambda\left(\frac{z_1}{\Phi-z_1}\right)^{\rho}\right\}^{1/\rho}}}\right)$$

However, the operational results of the equations (9)–(12) are also LNNs.

Example 1. Let N_1 = $<Fr_2^{Ro}, Fr_1^{Ro}, Fr_3^{Ro}>$ and N_2 = $<Fr_3^{Ro}, Fr_2^{Ro}, Fr_4^{Ro}>$ in Fr^{Ro}= $\{Fr_0^{Ro}, Fr_1^{Ro}, \ldots, Fr_6^{Ro}\}$ be two LNNs, λ = 0.5, ρ = 1. Based on the equations (9)–(12), we have the following operational results:

$$N_1\oplus N_2=\left\langle Fr_2^{Ro},Fr_1^{Ro},Fr_3^{Ro}\right\rangle\oplus\left\langle Fr_3^{Ro},Fr_2^{Ro},Fr_4^{Ro}\right\rangle$$

$$=\left(Fr^{Ro}_{6-\frac{6}{1+\left\{\left(\frac{2}{6-2}\right)^1+\left(\frac{3}{6-3}\right)^1\right\}^{1/1}}},Fr^{Ro}_{\frac{6}{1+\left\{\left(\frac{6-1}{1}\right)^1+\left(\frac{6-2}{2}\right)^1\right\}^{1/1}}},Fr^{Ro}_{\frac{6}{1+\left\{\left(\frac{6-3}{3}\right)^1+\left(\frac{6-4}{4}\right)^1\right\}^{1/1}}}\right)=\left\langle Fr_{3.6000}^{Ro},Fr_{0.7500}^{Ro},Fr_{2.4000}^{Ro}\right\rangle,$$

$$N_1\otimes N_2=\left\langle Fr_2^{Ro},Fr_1^{Ro},Fr_3^{Ro}\right\rangle\otimes\left\langle Fr_3^{Ro},Fr_2^{Ro},Fr_4^{Ro}\right\rangle$$

$$=\left(Fr^{Ro}_{\frac{6}{1+\left\{\left(\frac{6-2}{2}\right)^1+\left(\frac{6-3}{3}\right)^1\right\}^{1/1}}},Fr^{Ro}_{6-\frac{6}{1+\left\{\left(\frac{1}{6-1}\right)^1+\left(\frac{2}{6-2}\right)^1\right\}^{1/1}}},Fr^{Ro}_{6-\frac{6}{1+\left\{\left(\frac{3}{6-3}\right)^1+\left(\frac{4}{6-4}\right)^1\right\}^{1/1}}}\right)=\left\langle Fr_{1.5000}^{Ro},Fr_{2.4706}^{Ro},Fr_{4.5000}^{Ro}\right\rangle,$$

$$\lambda N_1=\lambda\left\langle Fr_2^{Ro},Fr_1^{Ro},Fr_3^{Ro}\right\rangle$$

$$=\left(Fr^{Ro}_{6-\frac{6}{1+\left\{0.5\times\left(\frac{2}{6-2}\right)^1\right\}^{1/1}}},Fr^{Ro}_{\frac{6}{1+\left\{0.5\times\left(\frac{6-1}{1}\right)^1\right\}^{1/1}}},Fr^{Ro}_{\frac{6}{1+\left\{0.5\times\left(\frac{6-3}{3}\right)^1\right\}^{1/1}}}\right)=\left\langle Fr_{1.2000}^{Ro},Fr_{1.7143}^{Ro},Fr_{4.0000}^{Ro}\right\rangle,$$

$$N_1^{\lambda}=\left\langle Fr_2^{Ro},Fr_1^{Ro},Fr_3^{Ro}\right\rangle^{\lambda}$$

$$=\left(Fr^{Ro}_{\frac{6}{1+\left\{0.5\times\left(\frac{6-2}{2}\right)^1\right\}^{1/1}}},Fr^{Ro}_{6-\frac{6}{1+\left\{0.5\times\left(\frac{1}{6-1}\right)^1\right\}^{1/1}}},Fr^{Ro}_{6-\frac{6}{1+\left\{0.5\times\left(\frac{3}{6-3}\right)^1\right\}^{1/1}}}\right)=\left\langle Fr_{3.0000}^{Ro},Fr_{0.5455}^{Ro},Fr_{2.0000}^{Ro}\right\rangle.$$

3.2 Dombi Weighted Aggregation Operators of LNNs

Definition 9. Set $N_g = \left\langle Fr_{x_g}^{Ro}, Fr_{y_g}^{Ro}, Fr_{z_g}^{Ro} \right\rangle$ (g = 1, 2, …, h) as a group of LNNs. Let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_h)$ be the weight vector of N_g such that $\gamma_g \in [0, 1]$ and $\sum_{g=1}^{h} \gamma_g = 1$. The LNNDWAA and LNNDWGA operators are proposed below:

$$LNNDWAA(N_1, N_2, ..., N_h) = \mathop{\oplus}_{g=1}^{h} \gamma_g N_g , \tag{13}$$

$$LNNDWGA(N_1, N_2, ..., N_h) = \mathop{\otimes}_{g=1}^{h} N_g^{\gamma_g} . \tag{14}$$

Theorem 3. Let $N_g = \left\langle Fr_{x_g}^{Ro}, Fr_{y_g}^{Ro}, Fr_{z_g}^{Ro} \right\rangle$ (g = 1, 2, …, h) be an assemblage of LNNs and $\gamma = (\gamma_1, \gamma_2, ..., \gamma_h)$ be the weight vector of N_g such that $\gamma_g \in [0, 1]$ and $\sum_{g=1}^{h} \gamma_g = 1$. The aggregated result of the LNNDWAA operator is still an LNN, which can be expressed by

$$LNNDWAA(N_1, N_2, ..., N_h) = \mathop{\oplus}_{g=1}^{h} \gamma_g N_g = \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h} \gamma_g \left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h} \gamma_g \left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h} \gamma_g \left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle . \tag{15}$$

Theorem 3 is proved through mathematical induction below.

Proof:

(a) Let h = 2. Based on Definition 8 we can obtain

$$LNNDWAA(N_1, N_2) = N_1 \oplus N_2$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\gamma_1 \left(\frac{x_1}{\Phi - x_1}\right)^{\rho} + \gamma_2 \left(\frac{x_2}{\Phi - x_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\gamma_1 \left(\frac{\Phi - y_1}{y_1}\right)^{\rho} + \gamma_2 \left(\frac{\Phi - y_2}{y_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\gamma_1 \left(\frac{\Phi - z_1}{z_1}\right)^{\rho} + \gamma_2 \left(\frac{\Phi - z_2}{z_2}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle .$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{2} \gamma_g \left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{2} \gamma_g \left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{2} \gamma_g \left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle$$

(b) If $h = k$, we can keep the following result from the equation (15):

$$LNNDWAA(N_1, N_2, ..., N_k) = \mathop{\oplus}_{g=1}^{k} \gamma_g N_g$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle.$$

(c) Set $h = k + 1$. Based on Definition 9 and the equation (15), there exists the following result:

$$LNNDWAA(N_1, N_2, ..., N_k, N_{k+1}) = \mathop{\oplus}_{g=1}^{k+1} \gamma_g N_g$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle \oplus \gamma_{k+1} N_{k+1}.$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle$$

In terms of the above results, the equation (15) can hold for all h.

Then, the LNNDWAA operator has some properties:

(1) Reducibility: When $\gamma = (1/h, 1/h, ..., 1/h)$, there exists

$$LNNDWAA(N_1, N_2, ..., N_h) = \mathop{\oplus}_{g=1}^{h} \gamma_g N_g$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle.$$

(2) Idempotency: Let all LNNs be $N_g = \left\langle Fr^{Ro}_{x_g}, Fr^{Ro}_{y_g}, Fr^{Ro}_{z_g} \right\rangle = N$ (g = 1, 2, ..., h). Then, LNNDWAA(N_1, N_2, ..., N_h) = N.

(3) Commutativity: Let the LNN sequence (N_1', N_2', ..., N_h') be an arbitrary arrangement of (N_1, N_2, ..., N_h). Then, there is LNNDWAA(N_1', N_2', ..., N_h') = LNNDWAA(N_1, N_2, ..., N_h).

(4) Boundedness: If the maximum and minimum LNNs are $N_{\max} = \left\langle Fr^{Ro}_{\max_g(x_g)}, Fr^{Ro}_{\min_g(y_g)}, Fr^{Ro}_{\min_g(z_g)} \right\rangle$ and $N_{\min} = \left\langle Fr^{Ro}_{\min_g(x_g)}, Fr^{Ro}_{\max_g(y_g)}, Fr^{Ro}_{\max_g(z_g)} \right\rangle$, then $N_{\min} \le$ LNNDWAA(N_1, N_2, ..., N_h) $\le N_{\max}$.

Proof:

(1) Based on the equation (15), we can see that the property (1) is valid.

(2) Since $N_g = \left\langle Fr_{x_g}^{Ro}, Fr_{y_g}^{Ro}, Fr_{z_g}^{Ro} \right\rangle = N$ (g = 1, 2, ..., h), by the equation (15) we can obtain the following result:

$$LNNDWAA(N_1, N_2, ..., N_h) = \mathop{\oplus}_{g=1}^{h} \gamma_g N_g$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\left(\frac{x}{\Phi - x}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\left(\frac{\Phi - y}{y}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\left(\frac{\Phi - z}{z}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left(\frac{x}{\Phi - x}\right)}}, Fr^{Ro}_{\frac{\Phi}{1+\left(\frac{\Phi - y}{y}\right)}}, Fr^{Ro}_{\frac{\Phi}{1+\left(\frac{\Phi - z}{z}\right)}} \right\rangle = \left\langle Fr_x^{Ro}, Fr_y^{Ro}, Fr_z^{Ro} \right\rangle = N.$$

Hence, LNNDWAA(N_1, N_2, ..., N_h) = N holds.

(3) The property (3) is obvious.

(4) Since $\min_g(x_g) \le x_g \le \max_g(x_g), \max_g(y_g) \le y_g \le \min_g(y_g), \max_g(z_g) \le z_g \le \min_g(z_g)$, there are the following inequalities:

$$\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\min_g(x_g)}{\Phi - \min_g(x_g)}\right)^{\rho}\right\}^{1/\rho}} = \min_g(x_g) \le \Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}} \le \max_g(x_g) = \Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\max_g(x_g)}{\Phi - \max_g(x_g)}\right)^{\rho}\right\}^{1/\rho}},$$

$$\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - \max_g(y_g)}{\max_g(y_g)}\right)^{\rho}\right\}^{1/\rho}} = \max_g(y_g) \le \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}} \le \min_g(y_g) = \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - \min_g(y_g)}{\min_g(y_g)}\right)^{\rho}\right\}^{1/\rho}},$$

$$\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - \max_g(z_g)}{\max_g(z_g)}\right)^{\rho}\right\}^{1/\rho}} = \max_g(z_g) \le \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}} \le \min_g(z_g) = \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - \min_g(z_g)}{\min_g(z_g)}\right)^{\rho}\right\}^{1/\rho}}.$$

Therefore, $N_{\min}$ ≤ LNNDWAA(N_1, N_2, ..., N_h) ≤ $N_{\max}$ is true.

Theorem 4. Let $N_g = \left\langle Fr^{Ro}_{x_g}, Fr^{Ro}_{y_g}, Fr^{Ro}_{z_g} \right\rangle$ (g = 1, 2, ..., h) be a group of LNNs and $\gamma = (\gamma_1, \gamma_2, ..., \gamma_h)$ be the weight vector of N_g (g = 1, 2, ..., h) for $\gamma_g \in [0, 1]$ and $\sum_{g=1}^{h} \gamma_g = 1$. The aggregated result of the LNNDWAA operator is still an LNN, which can be expressed by

$$LNNDWGA(N_1, N_2, ..., N_h) = \overset{h}{\underset{g=1}{\otimes}} N_g^{\gamma_g}$$

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle. \quad (16)$$

Theorem 4 is also proved based on mathematical induction, which is given below.

Proof:

(a) Let h = 2. Based on Definition 8 we can obtain

$$LNNDWGA(N_1, N_2) = N_1 \otimes N_2$$

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\gamma_1\left(\frac{\Phi-x_1}{x_1}\right)^{\rho}+\gamma_2\left(\frac{\Phi-x_2}{x_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\gamma_1\left(\frac{y_1}{\Phi-y_1}\right)^{\rho}+\gamma_2\left(\frac{y_2}{\Phi-y_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\gamma_1\left(\frac{z_1}{\Phi-z_1}\right)^{\rho}+\gamma_2\left(\frac{z_2}{\Phi-z_2}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle.$$

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{2}\gamma_g\left(\frac{\Phi-x_g}{x_g}\right)\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{2}\gamma_g\left(\frac{y_g}{\Phi-y_g}\right)\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{2}\gamma_g\left(\frac{z_g}{\Phi-z_g}\right)\right\}^{1/\rho}}} \right\rangle$$

(b) If $h = k$, we can get the following equation from the equation (16):

$$LNNDWGA(N_1, N_2, ..., N_k) = \overset{k}{\underset{g=1}{\otimes}} N_g^{\gamma_g}$$

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle.$$

(c) If $h = k + 1$, there exists the following result:

$$LNNDWGA(N_1, N_2, ..., N_k, N_{k+1}) = \underset{g=1}{\overset{k+1}{\otimes}} N_g^{\gamma_g}$$

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle \otimes \gamma_{k+1} N_{k+1} .$$

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle$$

In terms of the above results, the equation (16) is true for all h.

The LNNDWGA operator also contains some properties:

(1) Reducibility: When the weight vector is $\gamma = (1/h, 1/h, ..., 1/h)$, the equation (16) yields the following result:

$$LNNDWGA(N_1, N_2, ..., N_h) = \underset{g=1}{\overset{h}{\otimes}} N_g^{\gamma_g}$$

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle .$$

(2) Idempotency: Let all LNNs be $N_g = \left\langle Fr^{Ro}_{x_g}, Fr^{Ro}_{y_g}, Fr^{Ro}_{z_g} \right\rangle = N$ (g = 1, 2, ..., h). Then, LNNDWGA(N_1, N_2, ..., N_h) = N.

(3) Commutativity: Let the LNN sequence (N_1', N_2', ..., N_h') be any permutation of (N_1, N_2, ..., N_h). Then, there is LNNDWGA(N_1', N_2', ..., N_h') = LNNDWGA(N_1, N_2, ..., N_h).

(4) Boundedness: If the maximum and minimum LNNs are $N_{\max} = \left\langle Fr^{Ro}_{\max_g(x_g)}, Fr^{Ro}_{\min_g(y_g)}, Fr^{Ro}_{\min_g(z_g)} \right\rangle$ and $N_{\min} = \left\langle Fr^{Ro}_{\min_g(x_g)}, Fr^{Ro}_{\max_g(y_g)}, Fr^{Ro}_{\max_g(z_g)} \right\rangle$, then $N_{\min} \le$ LNNDWGA(N_1, N_2, ..., N_h) $\le N_{\max}$.

Since the characteristics of the LNNDWGA operator can be easily proved by the similar proof process of the characteristics of the LNNDWAA operator, it is omitted here.

4. MAGDM Method based on the LNNDWAA and LNNDWGA Operators

This section proposed a new DM method based on the LNNDWAA and LNNDWGA operators to solve MAGDM problems in the LNN environment.

In a MAGDM problem, let $P = \{P_1, P_2, ..., P_u\}$ be a set of alternatives and $\Psi = \{\Psi_1, \Psi_2, ..., \Psi_h\}$ be a set of attributes. The weight vector of the attributes Ψ_g (g = 1, 2, ..., h) is $\gamma = (\gamma_1, \gamma_2, ..., \gamma_h)$. Assume that there is a group of decision-makers $\Omega = \{\Omega_1, \Omega_2, ..., \Omega_r\}$ with their weight vector $\eta = (\eta_1, \eta_2, ..., \eta_r)$.

Each decision maker evaluates the value of each attribute Ψ_g (g = 1, 2, ..., h) for each alternative P_i from the set of linguistic terms Fr^{Ro} = {Fr_0^{Ro} = very low, Fr_1^{Ro} = low, Fr_2^{Ro} = slightly low, Fr_3^{Ro} = medium, Fr_4^{Ro} = slightly high, Fr_5^{Ro} = high, Fr_6^{Ro} = very high}. According to the linguistic terms, each decision-maker can assign the three linguistic values of indeterminacy, falsity, and truth to each attribute Ψ_g for the alternative P_v. Thus, LNN is composed of the obtained linguistic values. Hence, the LNN assessment information of the attributes Ψ_g (g = 1, 2, ..., h) for the alternatives P_v (v = 1, 2, ..., u) provided by each decision-maker Ω_s (s = 1, 2, ..., r) can establish the LNN decision matrix $M_s = \left(N_{vg}^s\right)_{u\times h}$, where $N_{vg}^s = \left\langle Fr_{x_{vg}^s}^{Ro}, Fr_{y_{vg}^s}^{Ro}, Fr_{z_{vg}^s}^{Ro}\right\rangle$ (s = 1, 2, ..., r; v = 1, 2, ..., u; g = 1, 2, ..., h) are LNNs.

Then, we present a MAGDM method using the score function (accuracy function) and the LNNDWAA and LNNDWGA operators to perform the MAGDM problem with LNN information. Here, the MAGDM method is described by the specific decision-making steps below.

Step 1: Aggregate all M_s (s = 1, 2, ..., r) by using the following LNNDWAA or LNNDWGA operator:

$$N_{vg} = LNNDWAA(N_{vg}^1, N_{vg}^2, ..., N_{vg}^r) = \mathop{\oplus}_{s=1}^{r} \eta_s N_{vg}^s$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{x_{vg}^s}{\Phi - x_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{\Phi - y_{vg}^s}{y_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{\Phi - z_{vg}^s}{z_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle \quad (17)$$

or

$$N_{vg} = LNNDWGA(N_{vg}^1, N_{vg}^2, ..., N_{vg}^r) = \mathop{\otimes}_{s=1}^{r} \left(N_{vg}^s\right)^{\eta_s}$$

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{\Phi - x_{vg}^s}{x_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{G - \frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{y_{vg}^s}{\Phi - y_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{z_{vg}^s}{\Phi - z_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle \quad (18)$$

to obtain the integrated matrix $R = \left(N_{vg}\right)_{u\times h}$, where $N_{vg} = \left\langle Fr_{x_{vg}}^{Ro}, Fr_{y_{vg}}^{Ro}, Fr_{z_{vg}}^{Ro}\right\rangle$ (v = 1, 2, ..., u; g = 1, 2, ..., h) are integrated LNNs.

Step 2: Use the following LNNDWAA or LNNDWGA operator to obtain the collective overall LNNs N_v for P_v (v = 1, 2, ..., u):

$$N_v = LNNDWAA(N_{v1}, N_{v2}, ..., N_{vh}) = \mathop{\oplus}_{g=1}^{h} \gamma_g N_{vg}$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle \quad (19)$$

$$N_v = LNNDWGA(N_{v1}, N_{v2}, ..., N_{vh}) = \underset{g=1}{\otimes} N_{vg}^{\gamma_g}$$

or

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle. \quad (20)$$

Step 3: Calculate the score values of $U(N_v)$ (the accuracy values of $V(N_v)$) (v = 1, 2, ..., u) through the equation (1) (the equation (2)).

Step 4: Rank all alternatives in decreasing order, then select the more reasonable one.

Step 5: End.

5. An Illustrative Example on Slope Treatment Scheme Selection

The application of the MAGDM method proposed in this paper is illustrated by the selection of slope treatment schemes. To avoid slope instability, a set of four slope treatment options $P = \{P_1, P_2, P_3, P_4\}$ is proposed, where P_1 is gravity retaining wall + lattice protection; P_2 is anti-slide retaining wall + anti-slide pile; P_3 is anchor retaining wall + lattice protection; and P_4 is pile-plate retaining wall. The evaluation of the schemes should meet the following attribute requirements: (1) Ψ_1 is the economic status; (2) Ψ_2 is the security situation; (3) Ψ_3 is the construction feasibility; and (4) Ψ_4 is the environment situation. The weight vector of the four attributes is assigned as γ = (0.23, 0.28, 0.26, 0.23). Assume that three experts are invited as a group of decision makers $\Omega = \{\Omega_1, \Omega_2, \Omega_3\}$, then the weight vecror η = (0.29, 0.33, 0.38) is given to indicate the importance of the various decision makers.

Decision makers need to assess the four attributes on the four alternatives from the linguistic term set $Fr^{Ro} = \{Fr_0^{Ro}$ = very low, Fr_1^{Ro} = low, Fr_2^{Ro} = slightly low, Fr_3^{Ro} = medium, Fr_4^{Ro} = slightly high, Fr_5^{Ro} = high, Fr_6^{Ro} = very high$\}$ with Φ = 6. Thus, the linguistic evaluation results of each decision-maker Ω_s (s = 1, 2, 3) can be established as the LNN decision matrices M_1, M_2, and M_3:

$$M_1 = \begin{bmatrix} \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_3^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_1^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_4^{Ro}, Fr_5^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_3^{Ro}\rangle \\ \langle Fr_5^{Ro}, Fr_2^{Ro}, Fr_1^{Ro}\rangle & \langle Fr_5^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_4^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_4^{Ro}, Fr_5^{Ro}\rangle \\ \langle Fr_4^{Ro}, Fr_4^{Ro}, Fr_5^{Ro}\rangle & \langle Fr_5^{Ro}, Fr_3^{Ro}, Fr_3^{Ro}\rangle & \langle Fr_5^{Ro}, Fr_2^{Ro}, Fr_1^{Ro}\rangle & \langle Fr_5^{Ro}, Fr_4^{Ro}, Fr_3^{Ro}\rangle \\ \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_4^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_4^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_3^{Ro}\rangle \\ \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_1^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_3^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_4^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_4^{Ro}, Fr_4^{Ro}\rangle \\ \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_3^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_4^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle \\ \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_3^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_1^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_4^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle \end{bmatrix}$$

$$M_3=\begin{bmatrix} \left\langle Fr_3^{Ro},Fr_2^{Ro},Fr_3^{Ro}\right\rangle & \left\langle Fr_4^{Ro},Fr_2^{Ro},Fr_3^{Ro}\right\rangle & \left\langle Fr_4^{Ro},Fr_3^{Ro},Fr_3^{Ro}\right\rangle & \left\langle Fr_3^{Ro},Fr_3^{Ro},Fr_1^{Ro}\right\rangle \\ \left\langle Fr_4^{Ro},Fr_2^{Ro},Fr_2^{Ro}\right\rangle & \left\langle Fr_3^{Ro},Fr_3^{Ro},Fr_2^{Ro}\right\rangle & \left\langle Fr_3^{Ro},Fr_3^{Ro},Fr_4^{Ro}\right\rangle & \left\langle Fr_4^{Ro},Fr_3^{Ro},Fr_3^{Ro}\right\rangle \\ \left\langle Fr_3^{Ro},Fr_4^{Ro},Fr_2^{Ro}\right\rangle & \left\langle Fr_4^{Ro},Fr_3^{Ro},Fr_2^{Ro}\right\rangle & \left\langle Fr_4^{Ro},Fr_2^{Ro},Fr_3^{Ro}\right\rangle & \left\langle Fr_4^{Ro},Fr_3^{Ro},Fr_2^{Ro}\right\rangle \\ \left\langle Fr_3^{Ro},Fr_2^{Ro},Fr_5^{Ro}\right\rangle & \left\langle Fr_3^{Ro},Fr_2^{Ro},Fr_3^{Ro}\right\rangle & \left\langle Fr_3^{Ro},Fr_2^{Ro},Fr_1^{Ro}\right\rangle & \left\langle Fr_3^{Ro},Fr_2^{Ro},Fr_3^{Ro}\right\rangle \end{bmatrix}$$

The decision procedures based on the LNNDWAA operator are indicated below.

Step 1: Aggregate the decision matrices M_1, M_2, and M_3 by the equation (17) for ρ = 1 and obtain the integrated matrix $R=\left(N_{vg}\right)_{4\times 4}$:

$$R=\begin{bmatrix} \left\langle Fr_{3.4249}^{Ro},Fr_{2.0000}^{Ro},Fr_{2.5751}^{Ro}\right\rangle & \left\langle Fr_{4.0000}^{Ro},Fr_{2.0000}^{Ro},Fr_{2.0033}^{Ro}\right\rangle & \left\langle Fr_{3.4790}^{Ro},Fr_{3.2345}^{Ro},Fr_{2.8599}^{Ro}\right\rangle & \left\langle Fr_{3.4249}^{Ro},Fr_{2.5751}^{Ro},Fr_{1.7045}^{Ro}\right\rangle \\ \left\langle Fr_{4.4496}^{Ro},Fr_{2.2472}^{Ro},Fr_{1.2346}^{Ro}\right\rangle & \left\langle Fr_{4.2808}^{Ro},Fr_{2.5751}^{Ro},Fr_{2.2472}^{Ro}\right\rangle & \left\langle Fr_{3.0000}^{Ro},Fr_{2.6201}^{Ro},Fr_{4.0000}^{Ro}\right\rangle & \left\langle Fr_{4.0000}^{Ro},Fr_{2.5751}^{Ro},Fr_{3.7430}^{Ro}\right\rangle \\ \left\langle Fr_{3.3799}^{Ro},Fr_{3.6036}^{Ro},Fr_{2.7933}^{Ro}\right\rangle & \left\langle Fr_{4.3051}^{Ro},Fr_{3.2698}^{Ro},Fr_{2.2140}^{Ro}\right\rangle & \left\langle Fr_{4.4496}^{Ro},Fr_{2.2472}^{Ro},Fr_{1.7192}^{Ro}\right\rangle & \left\langle Fr_{4.4496}^{Ro},Fr_{3.2345}^{Ro},Fr_{2.2140}^{Ro}\right\rangle \\ \left\langle Fr_{3.3799}^{Ro},Fr_{2.2140}^{Ro},Fr_{3.0211}^{Ro}\right\rangle & \left\langle Fr_{3.4249}^{Ro},Fr_{1.5038}^{Ro},Fr_{2.2901}^{Ro}\right\rangle & \left\langle Fr_{3.0000}^{Ro},Fr_{2.2472}^{Ro},Fr_{1.8692}^{Ro}\right\rangle & \left\langle Fr_{3.3799}^{Ro},Fr_{2.5210}^{Ro},Fr_{2.2901}^{Ro}\right\rangle \end{bmatrix}.$$

Step 2: Through the equation (19), obtain the collective overall LNNs of N_v for P_v (v = 1, 2, 3, 4) below:

$N_1 = <Fr_{3.6290}^{Ro}, Fr_{2.3546}^{Ro}, Fr_{2.1981}^{Ro}>$, $N_2 = <Fr_{4.0502}^{Ro}, Fr_{2.6660}^{Ro}, Fr_{2.2865}^{Ro}>$, $N_3 = <Fr_{4.2426}^{Ro}, Fr_{2.9738}^{Ro}, Fr_{2.1555}^{Ro}>$, and $N_4 = <Fr_{3.3043}^{Ro}, Fr_{2.0120}^{Ro}, Fr_{2.2835}^{Ro}>$.

Step 3: Calculate the score values of $U(N_v)$ (v = 1, 2, 3, 4) by the equation (1):

$U(N_1)$ = 0.6154, $U(N_2)$ = 0.6165, $U(N_3)$ = 0.6174, and $U(N_4)$ = 0.6116.

Step 4: Rank the four alternatives: $P_3 \succ P_2 \succ P_1 \succ P_4$. It can be seen that P_3 is the most reasonable option among the four ones.

Or the decision procedures based on the LNNDWGA operator are indicated below.

Step 1: Aggregate the decision matrices M_1, M_2, and M_3 by the equation (18) for ρ = 1 and obtain the integrated matrix $R=\left(N_{vg}\right)_{4\times 4}$:

$$R=\begin{bmatrix} \left\langle Fr_{3.2698}^{Ro},Fr_{2.0000}^{Ro},Fr_{2.7302}^{Ro}\right\rangle & \left\langle Fr_{4.0000}^{Ro},Fr_{2.0000}^{Ro},Fr_{3.1401}^{Ro}\right\rangle & \left\langle Fr_{3.3149}^{Ro},Fr_{3.3799}^{Ro},Fr_{3.9967}^{Ro}\right\rangle & \left\langle Fr_{3.2698}^{Ro},Fr_{2.7302}^{Ro},Fr_{2.4623}^{Ro}\right\rangle \\ \left\langle Fr_{4.2463}^{Ro},Fr_{2.3964}^{Ro},Fr_{1.4338}^{Ro}\right\rangle & \left\langle Fr_{3.7430}^{Ro},Fr_{2.7302}^{Ro},Fr_{2.3964}^{Ro}\right\rangle & \left\langle Fr_{3.0000}^{Ro},Fr_{2.7655}^{Ro},Fr_{4.0000}^{Ro}\right\rangle & \left\langle Fr_{4.0000}^{Ro},Fr_{3.7099}^{Ro},Fr_{4.2808}^{Ro}\right\rangle \\ \left\langle Fr_{3.2345}^{Ro},Fr_{3.7528}^{Ro},Fr_{3.9798}^{Ro}\right\rangle & \left\langle Fr_{3.8023}^{Ro},Fr_{3.4249}^{Ro},Fr_{2.3526}^{Ro}\right\rangle & \left\langle Fr_{4.2463}^{Ro},Fr_{2.3964}^{Ro},Fr_{2.2570}^{Ro}\right\rangle & \left\langle Fr_{4.2463}^{Ro},Fr_{3.3799}^{Ro},Fr_{2.3526}^{Ro}\right\rangle \\ \left\langle Fr_{3.2345}^{Ro},Fr_{2.3526}^{Ro},Fr_{4.2222}^{Ro}\right\rangle & \left\langle Fr_{3.2698}^{Ro},Fr_{1.7173}^{Ro},Fr_{2.4497}^{Ro}\right\rangle & \left\langle Fr_{3.0000}^{Ro},Fr_{2.3964}^{Ro},Fr_{3.4093}^{Ro}\right\rangle & \left\langle Fr_{3.2345}^{Ro},Fr_{2.6851}^{Ro},Fr_{2.4497}^{Ro}\right\rangle \end{bmatrix}.$$

Step 2: Through the equation (20), obtain the collective overall LNNs N_v for P_v (v = 1, 2, 3, 4) below:

$N_1 = <Fr_{3.4588}^{Ro}, Fr_{2.6338}^{Ro}, Fr_{3.2455}^{Ro}>$, $N_2 = <Fr_{3.6611}^{Ro}, Fr_{2.9722}^{Ro}, Fr_{3.4480}^{Ro}>$, $N_3 = <Fr_{3.8440}^{Ro}, Fr_{3.3047}^{Ro}, Fr_{2.9054}^{Ro}>$, and $N_4 = <Fr_{3.1795}^{Ro}, Fr_{2.2959}^{Ro}, Fr_{3.3218}^{Ro}>$.

Step 3: Obtain the score values of $U(N_v)$ (v = 1, 2, 3, 4) by the equation (1):

$U(N_1)$ = 0.5322, $U(N_2)$ = 0.5134, $U(N_3)$ = 0.5352, and $U(N_4)$ = 0.5312.

Step 4: Rank the four alternatives: $P_3 \succ P_1 \succ P_4 \succ P_2$. It can be seen that P_3 is the most reasonable choice among the four ones.

We can repeat the above decision process by changing the parameter ρ from 2 to 4. The sorting results obtained by using the LNNDWAA operator are shown in Figure 2. Then, the ranking orders based on the LNNDWGA operator are indicated in Figure 3.

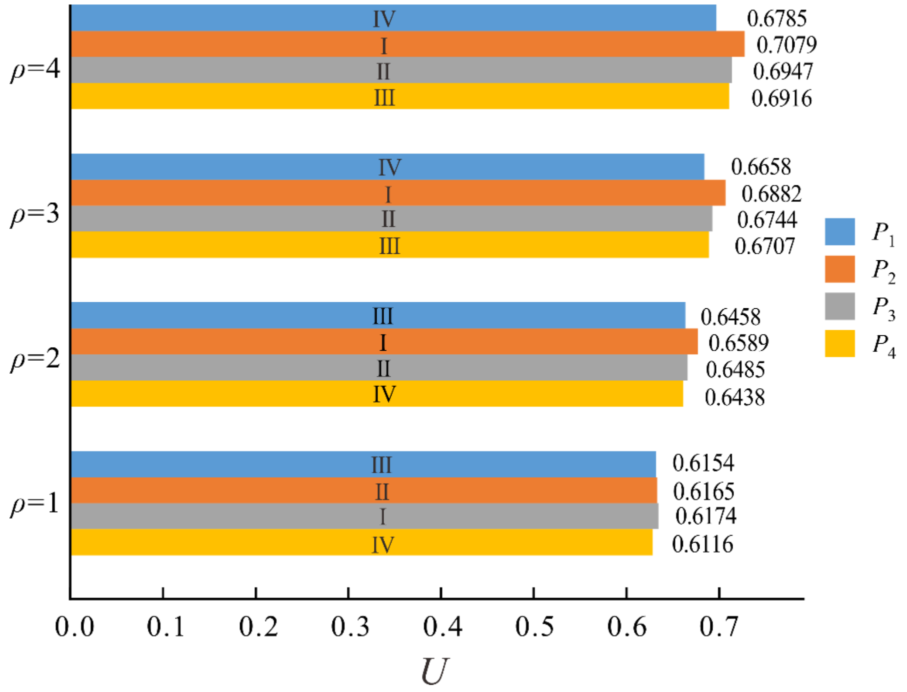

Figure 2. Ranking orders of the four alternatives based on the LNNDWAA operator (I, II, III, IV are ranking numbers)

As shown in Figure 2, the sorting results obtained based on the LNNDWAA operator change with the change of the parameter values of ρ. With an increase of ρ, the score values of the four alternatives gradually increase. However, the ranking orders tend to robustness when $\rho > 3$.

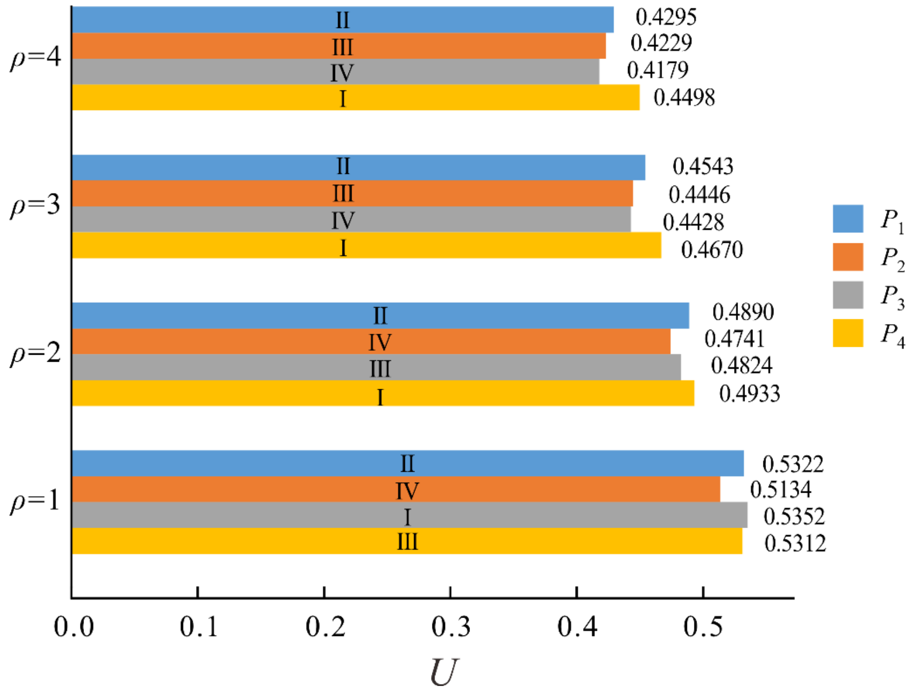

Figure 3. Ranking orders of the four alternatives based on the LNNDWGA operator

For the evaluation results using the LNNDWGA operator in Figure 3, the ranking orders also change with the change of ρ. With an increase of ρ, the score values of the four alternatives gradually decrease. The ranking orders tend to robustness when the value of the parameter ρ exceeds 3.

Furthermore, a comparison is made between the new MAGDM method and the existing relative MAGDM methods by the operators of LNNWAA and LNNWGA proposed by Fan and Ye [24]. According to the calculational steps given by Fan and Ye [24], the alternatives are evaluated as follows.

Step 1: By the LNNWAA operator of the equation (4), we can obtain the integrated matrix $R=\left(N_{vg}\right)_{u\times h}$:

$$R=\begin{bmatrix} \left\langle Fr^{Ro}_{3.3757}, Fr^{Ro}_{2.0000}, Fr^{Ro}_{2.6243}\right\rangle & \left\langle Fr^{Ro}_{4.0000}, Fr^{Ro}_{2.0000}, Fr^{Ro}_{2.3988}\right\rangle & \left\langle Fr^{Ro}_{3.4284}, Fr^{Ro}_{3.2610}, Fr^{Ro}_{3.0433}\right\rangle & \left\langle Fr^{Ro}_{3.3757}, Fr^{Ro}_{2.6243}, Fr^{Ro}_{1.9761}\right\rangle \\ \left\langle Fr^{Ro}_{4.3642}, Fr^{Ro}_{2.2863}, Fr^{Ro}_{1.3013}\right\rangle & \left\langle Fr^{Ro}_{4.0917}, Fr^{Ro}_{2.6243}, Fr^{Ro}_{2.2863}\right\rangle & \left\langle Fr^{Ro}_{3.0000}, Fr^{Ro}_{2.6672}, Fr^{Ro}_{4.0000}\right\rangle & \left\langle Fr^{Ro}_{4.0000}, Fr^{Ro}_{3.5858}, Fr^{Ro}_{3.8255}\right\rangle \\ \left\langle Fr^{Ro}_{3.3328}, Fr^{Ro}_{3.6377}, Fr^{Ro}_{2.9822}\right\rangle & \left\langle Fr^{Ro}_{4.1300}, Fr^{Ro}_{3.2988}, Fr^{Ro}_{2.2496}\right\rangle & \left\langle Fr^{Ro}_{4.3642}, Fr^{Ro}_{2.2863}, Fr^{Ro}_{1.9083}\right\rangle & \left\langle Fr^{Ro}_{4.3642}, Fr^{Ro}_{3.2610}, Fr^{Ro}_{2.2496}\right\rangle \\ \left\langle Fr^{Ro}_{3.3328}, Fr^{Ro}_{2.2496}, Fr^{Ro}_{3.3286}\right\rangle & \left\langle Fr^{Ro}_{3.3757}, Fr^{Ro}_{1.5911}, Fr^{Ro}_{2.3332}\right\rangle & \left\langle Fr^{Ro}_{3.0000}, Fr^{Ro}_{2.2863}, Fr^{Ro}_{2.3620}\right\rangle & \left\langle Fr^{Ro}_{3.3328}, Fr^{Ro}_{2.5716}, Fr^{Ro}_{2.3332}\right\rangle \end{bmatrix}.$$

Step 2: The collective overall linguistic neutrosophic numbers of N_v for P_v (v = 1, 2, 3, 4) was determined below:

$N_1 = \langle Fr^{Ro}_{3.5807}, Fr^{Ro}_{2.4175}, Fr^{Ro}_{2.4916}\rangle$, $N_2 = \langle Fr^{Ro}_{3.9258}, Fr^{Ro}_{2.7432}, Fr^{Ro}_{2.6147}\rangle$, $N_3 = \langle Fr^{Ro}_{4.0996}, Fr^{Ro}_{3.0590}, Fr^{Ro}_{2.2998}\rangle$, and $N_4 = \langle Fr^{Ro}_{3.2625}, Fr^{Ro}_{2.1144}, Fr^{Ro}_{2.5240}\rangle$.

Step 3: Calculate the score values of $U(N_v)$ (v = 1, 2, 3, 4) for the collective overall linguistic neutrosophic numbers of N_v:

$U(N_1) = 0.5929$, $U(N_2) = 0.5860$, $U(N_3) = 0.5967$, and $U(N_4) = 0.5902$.

Step 4: We can get the ranking of the four alternatives: $P_3 \succ P_1 \succ P_4 \succ P_2$. It can be seen that P_3 is the most reasonable choice among the four ones.

Or by the LNNWGA operator of the equation (5), the calculational steps are given below.

Step 1: This step is the same as Step 1 mentioned above.

Step 2: Through the equation (5), the collective overall LNNs of N_v for P_v (v = 1, 2, 3, 4) below:

$N_1 = \langle Fr^{Ro}_{3.5543}, Fr^{Ro}_{2.5138}, Fr^{Ro}_{2.5421}\rangle$, $N_2 = \langle Fr^{Ro}_{3.8110}, Fr^{Ro}_{2.8160}, Fr^{Ro}_{3.0491}\rangle$, $N_3 = \langle Fr^{Ro}_{4.0389}, Fr^{Ro}_{3.1457}, Fr^{Ro}_{2.3507}\rangle$, and $N_4 = \langle Fr^{Ro}_{3.2545}, Fr^{Ro}_{2.1658}, Fr^{Ro}_{2.5717}\rangle$.

Step 3: Calculate the score values of $U(N_v)$ (v = 1, 2, 3, 4):

$U(N_1) = 0.5832$, $U(N_2) = 0.5525$, $U(N_3) = 0.5857$, and $U(N_4) = 0.5843$.

Step 4: We can get the ranking of the four alternatives: $P_3 \succ P_4 \succ P_2 \succ P_1$. It can be seen that P_3 is the most reasonable option among the four ones.

Figure 4 shows the comparison of the decision results obtained using the LNNWGA and LNNWAA operators [24] and the proposed LNNDWAA and LNNDWGA operators in this study. The ranking orders in this MAGDM example are influenced by different aggregation operators and values of the parameter ρ. According to the results obtained using the LNNWGA and LNNWAA operators, the scheme P_3 is the most reasonable option among the four alternatives. It is the same as the result based on the proposed LNNDWAA and LNNDWGA operators when $\rho = 1$. However, the best alternative is P_2 according to the proposed LNNDWAA operator when $\rho = 2, 3, 4$. According to the result of the proposed LNNDWGA operator, when $\rho = 2, 3, 4$, the best alternative is P_4.

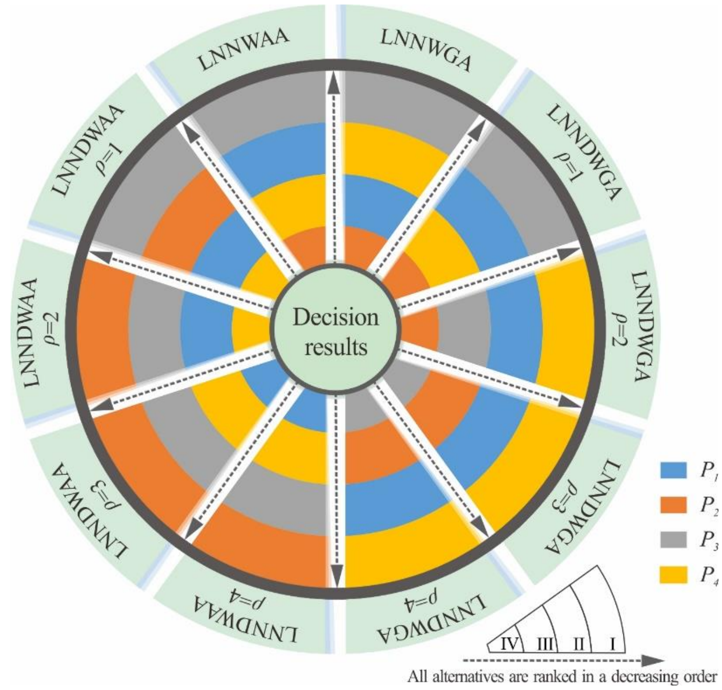

Figure 4. Comparison of the decision results based on different aggregation operators and values of ρ

6. Conclusion

In this study, the LNNDWAA and LNNDWGA operators and their properties were proposed in view of the Dombi operations in the LNN environment. A novel technique for MAGDM problems was proposed using the LNNDWAA or LNNDWGA operator. In the proposed MAGDM process, regarding the satisfactory assessment of alternatives over multiple attributes, we established a decision matrix based on the suitable evaluation results given by the decision makers. Then, we used the LNNDWAA/LNNDWGA operator to aggregate LNN information. Finally, the score values (accuracy values if necessary) was calculated and the ranking results of alternatives are given in a descending order to obtain the optimal choice. In the DM application, an illustrative example of the selection of landslide treatment schemes was presented to verify the feasibility of the proposed method. Compared with the related MAGDM methods in previous studies, this new method can influence the sorting order of alternatives by changing the parameter values of ρ. Thus, it can overcome the insufficiency of decision flexibility in the existing MAGDM method with LNNs. Therefore, we can more effectively deal with the DM problem of landslide treatment schemes by specifying various parameter values according to the preferences and demands of decision makers. It is obvious that this new method can better solve the DM problem of landslide treatment schemes and make the DM results more reasonable and flexible in the uncertainty and inconsistency of human linguistic judgments.

Data Availability: The data used to support the findings of the study are available in the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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Theory of Hypersoft Sets: Axiomatic Properties, Aggregation Operations, Relations, Functions and Matrices

Muhammad Saeed [1], Atiqe Ur Rahman[2,*], Muhammad Ahsan[3], and Florentin Smarandache[4]
[1,2,3] University of Management and Technology, Lahore, Pakistan.
muhammad.saeed@umt.edu.pk, aurkhb@gmail.com, ahsan1826@gmail.com
[4] Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA ; smarand@unm.edu
*Correspondence: aurkhb@gmail.com

Abstract. There are several decision-making based situations in which it is necessary to categorize the evaluating parameters into their respective sub-parametric values based non-overlapping sets. The existing soft set model is not compatible with such situations therefore hypersoft set ($\hat{H}s$-set) is developed which manages such situations by utilizing a novel mapping called multi-argument approximate mapping which broadens the domain of soft approximate mapping. This research presents the characterization of several essential axiomatic properties and set-based operations of $\hat{H}s$-set which will help the researchers to implement this emerging theory to other fields of study. The brief discussion on some hybridized structures of $\hat{H}s$-set with fuzzy set-like models is also provided.

Keywords: $\hat{H}s$-set; $\hat{H}s$-Relation; $\hat{H}s$-Function; $\hat{H}s$-Matrix.

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1. Introduction

There are several models in literature to deal uncertainties but fuzzy set [1] is the most significant in this regards. It has its own intricacies which limit it to tackle uncertain decision-making scenarios effectively. The justification behind these obstacles is, potentially, the deficiency of parameterization tool. A novel model is required for managing vagueness and uncertainties which should be liberated from all such obstacles. In 1999, Molodtsov [2] established a set-structure known as soft set ($\hat{s}$-set) in literature as a novel parameterized sub-class of universal set. In the year 2003, Maji et al. [3] broadened the idea and investigated several rudimental axiomatic properties and set-operations of $\hat{s}$-sets . They also validated several results. Later on Pei et al. [4] introduced an information system (Inf-sys) by using the idea of $\hat{s}$-sets. It is proved that $\hat{s}$-set can be considered as a particular class of Inf-sys. Afterwards, Ali et al. [5] identified many assertions in the research proposed by Maji et al. and introduced novel notions by using the concept of restricted and extended $\hat{s}$-set aggregation operations. In the same way, Babitha et al. [6, 7] made investigation on $\hat{s}$-set relation, $\hat{H}s$-set function by using the Cartesian product of $\hat{H}s$-sets. Sezgin et al. [8], Ge et al. [9], Fuli [10] provided few amendments in previous work by establishing few novel results. In order to utilize the concept of $\hat{s}$-sets in the development of algebraic structures, Saeed et al. [11] characterized the classical notions of elements and points under $\hat{s}$-set environment. Many researchers [12–23] discussed various gluing structures of $\hat{s}$-sets with other fuzzy set-like models to resolve several real-life decision making issues.

It is a matter of common observation that in various decision making problems, parameters have to be partitioned into their related sub-parametric valued sets whereas the previous researches on $\hat{s}$-set are not sufficient to manage such settings therefore Smarandache [24] initiated the notion of hypersoft set ($\hat{H}s$-set) as an extension of $\hat{s}$-set by introducing a novel multi-argument approximate mapping (maa-mapping). Any novel theory can not be implemented in real-world situations without the characterization of its elementary axiomatic-properties. Although Saeed et al. [25] made a good effort to investigate various basic properties of $\hat{H}s$-set but it does not cover many of the aspects of $\hat{H}s$-set theory. Therefore this paper aims to (i). generalize the research works described in [3, 5–10] for $\hat{H}s$-set environment and (ii). to modify the results discussed by Saeed et al. [25]. In the present work, all the necessary rudiments of $\hat{H}s$-set are investigated for its further developments. The Figure 1 explains the sectional-outlines of the paper.

Section 2	Recollects few essential definitions and results to assist the main results.	Section 3	Introduces few basic axiomatic properties of hypersoft sets and Illustrates set theoretic operations of hypersoft sets.
Section 4	Discusses some basic results and laws on hypersoft sets.	Section 5	Explains hypersoft relations and hypersoft functions.
Section 6	Presents the matrix representation of hypersoft sets with some operations	Section 7	Investigates few hybrids of hypersoft sets.
Section 8	Summarizes the paper with the provision of some future directions.		

FIGURE 1. Outlines of the paper

2. Preliminaries

The purpose of this section is to review some basic properties of $\hat{s}$-set for clear understanding of proposed study. The symbol $\hat{\Pi}$ will represent initial universe in the remaining parts of the article.

Definition 2.1. [2]
A $\hat{s}$-set $\mathbb{S}$ on $\hat{\Pi}$ is usually stated by a pair $(\Psi_{\mathbb{S}}, \mathfrak{G})$ in which $\Psi_{\mathbb{S}} : \mathfrak{G} \rightarrow P^{\hat{\Pi}}$ is an approximate mapping & $\mathfrak{G}$ be a sub-family of parameters. The family of $\hat{s}$-sets is symbolized as $\Sigma_{(\Psi_{\mathbb{S}},\mathfrak{G})}$.

Definition 2.2. [3]
For $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ & $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2) \in \Sigma_{(\Psi_{\mathbb{S}},\mathfrak{G})}$, if $\mathfrak{G}_1 \subseteq \mathfrak{G}_2$, & $\Psi_{\mathbb{S}_1}(\hat{\epsilon}) \subseteq \Psi_{\mathbb{S}_2}(\hat{\epsilon})$ for all $\hat{\epsilon} \in \mathfrak{G}_1$ then $\hat{s}$-set $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ is a *soft-subset* of $\hat{s}$-set $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2)$.

Definition 2.3. [3]
For $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ & $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2) \in \Sigma_{(\Psi_{\mathbb{S}},\mathfrak{G})}$, their union is a $\hat{s}$-set $(\Psi_{\mathbb{S}_3}, \mathfrak{G}_3)$ with $\mathfrak{G}_3 = \mathfrak{G}_1 \cup \mathfrak{G}_2$ & for $\hat{\epsilon} \in \mathfrak{G}_3$,

$$\Psi_{\mathbb{S}_3}(\hat{\epsilon}) = \begin{cases} \Psi_{\mathbb{S}_1}(\hat{\epsilon}) & \hat{\epsilon} \in (\mathfrak{G}_1 \setminus \mathfrak{G}_2) \\ \Psi_{\mathbb{S}_2}(\hat{\epsilon}) & \hat{\epsilon} \in (\mathfrak{G}_2 \setminus \mathfrak{G}_1) \\ \Psi_{\mathbb{S}_1}(\hat{\epsilon}) \cup \Psi_{\mathbb{S}_2}(\hat{\epsilon}) & \hat{\epsilon} \in (\mathfrak{G}_1 \cap \mathfrak{G}_2) \end{cases}$$

Definition 2.4. [3]
For $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ & $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2) \in \Sigma_{(\Psi_{\mathbb{S}},\mathfrak{G})}$, their intersection is a $\hat{s}$-set $(\Psi_{\mathbb{S}_4}, \mathfrak{G}_4)$ with $\mathfrak{G}_4 = \mathfrak{G}_1 \cap \mathfrak{G}_2$ & for $\hat{\epsilon} \in \mathfrak{G}_4$, $\Psi_{\mathbb{S}_4}(\hat{\epsilon}) = \Psi_{\mathbb{S}_1}(\hat{\epsilon}) \cap \Psi_{\mathbb{S}_2}(\hat{\epsilon})$.

One can refer [2–10] for detailed description on $\hat{s}$-sets.

3. **Hypersoft Set**

This part of the paper provides the basic axiomatic-properties of $\hat{H}s$-set along with the modification of some notions stated in [25].

Definition 3.1. [22]
Let $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, ..., \mathfrak{A}_n$ are non-overlapping sets having sub-parametric values of parameters $\ddot{a}_1, \ddot{a}_2, \ddot{a}_3, ..., \ddot{a}_n$ respectively, then a $\hat{H}s$-set on $\hat{\Pi}$, is usually stated by a pair $(\Theta, \mathfrak{A})$ in which $\Theta : \mathfrak{A} \rightarrow P^{\hat{\Pi}}$ is a maa-mapping and $\mathfrak{A} = \prod_{i=1}^{n} \mathfrak{A}_i$. The family of $\hat{H}s$-sets is symbolized by $\Sigma_{(\Theta,\mathfrak{A})}$. The model of $\hat{H}s$-set is presented in Figure 2.

Example 3.2. Mrs. Smith visits a mobile mall to purchase a mobile for her personal use. She is accompanied by her two friends who are experts in mobile purchasing. They collectively observed 8 types of mobiles which are considered as elements of universal set $\hat{\Pi} = \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}$. They have fixed some parameters for this purchase with their mutual consensus that are $\hat{e}_1$ = random only memory in giga bytes, $\hat{e}_2$ = Resolution of camera in pixels, $\hat{e}_3$ = length in inches, $\hat{e}_4$ = random access memory in giga bytes, and $\hat{e}_5$ = power of battery in mAh. These parameters have their sub-collections as:
$\mathfrak{B}_1 = \{\hat{e}_{11} = 32, \hat{e}_{12} = 64\}$, $\mathfrak{B}_2 = \{\hat{e}_{21} = 8, \hat{e}_{22} = 16\}$, $\mathfrak{B}_3 = \{\hat{e}_{31} = 6.5, \hat{e}_{32} = 6.7\}$
$\mathfrak{B}_4 = \{\hat{e}_{41} = 4, \hat{e}_{42} = 8\}$, $\mathfrak{B}_5 = \{\hat{e}_{51} = 4000\}$ then $\mathfrak{A} = \mathfrak{B}_1 \times \mathfrak{B}_2 \times \mathfrak{B}_3 \times \mathfrak{B}_4 \times \mathfrak{B}_5$

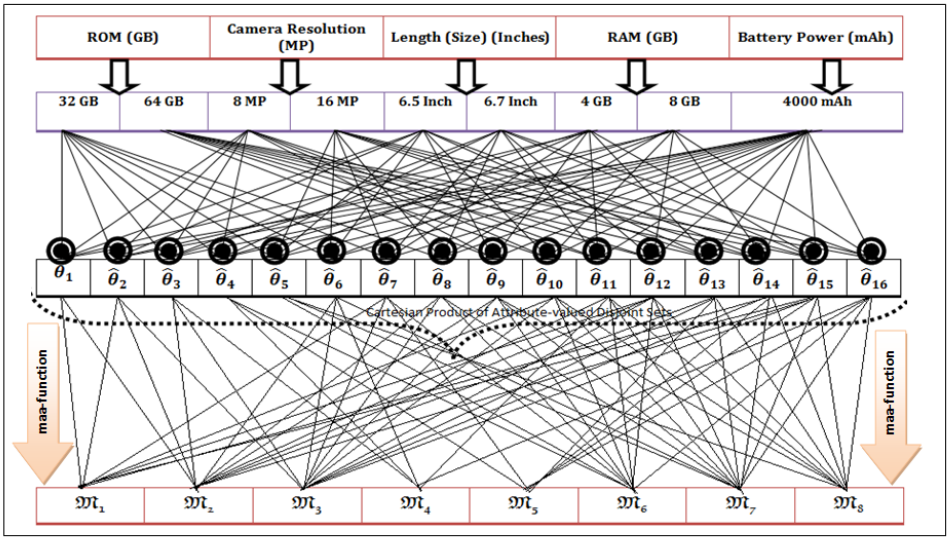

FIGURE 2. Pictorial Version of $\hat{H}s$-set

$\mathfrak{A} = \{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, ..., \hat{\theta}_{16}\}$ and every $\hat{\theta}_i$, ${}_{(1)}i^{(16)}$, is a 5-tuple member. Then the $\hat{H}s$-set $(\Theta, \mathfrak{A})$ is constructed as

$$(\Theta, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2\}\right), \left(\hat{\theta}_2, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3\}\right), \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right), \left(\hat{\theta}_4, \{\hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_6\}\right), \\ \left(\hat{\theta}_5, \{\hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_6, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_7\}\right), \left(\hat{\theta}_7, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_6\}\right), \\ \left(\hat{\theta}_8, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7\}\right), \left(\hat{\theta}_9, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{10}, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \\ \left(\hat{\theta}_{11}, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{12}, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \\ \left(\hat{\theta}_{13}, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{14}, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \\ \left(\hat{\theta}_{15}, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{16}, \{\hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right) \end{array} \right\}$$

Definition 3.3. Let $\mathcal{F}^{\hat{\Pi}}$ be a collection consisting of fuzzy subsets on $\hat{\Pi}$. Let $\ddot{a}_i, n \geq 1$, ${}_1i^n$ are parameters having their relevant sub-parametric values in the sets $\mathfrak{A}_i$ respectively, with $\mathfrak{A}_i \cap \mathfrak{A}_j = \varnothing$, for $i \neq j$, & ${}_1i^n$, ${}_1j^n$. Then a *fuzzy $\hat{H}s$-set* $(\Theta_{fhs}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as,

$$(\Theta_{fhs}, \mathfrak{A}) = \{(\hat{\theta}, \Theta_{fhs}(\hat{\theta})) : \hat{\theta} \in \mathfrak{A}, \Theta_{fhs}(\hat{\theta}) \in \mathcal{F}^{\hat{\Pi}}\}$$

where $\Theta_{fhs} : \mathfrak{A} \to \mathcal{F}^{\hat{\Pi}}$ and for all $\hat{\theta} \in \mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times ..... \times \mathfrak{A}_n$

$$\Theta_{fhs}(\hat{\theta}) = \{\mu_{\Theta_{fhs}(\hat{\theta})}(\varpi)/\varpi : \varpi \in \hat{\Pi}, \mu_{\Theta_{fhs}(\hat{\theta})}(\varpi) \in \mathbb{C}(\mathbb{I}) = [0,1]\}$$

is a fuzzy set on $\hat{\Pi}$.

One can consider this definition as modified form of fuzzy $\hat{H}s$-set stated in [22] and [24].

Example 3.4. Assuming the Example 3.2, Fuzzy $\hat{H}s$-set $(\Theta_{fhs}, \mathfrak{A})$ is constructed as

$$(\Theta_{fhs}, \mathfrak{A}) = \left\{ \begin{array}{l} (\hat{\theta}_1, \{0.1/\hat{\mathfrak{M}}_1, 0.2/\hat{\mathfrak{M}}_2\}), (\hat{\theta}_2, \{0.1/\hat{\mathfrak{M}}_1, 0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3\}), (\hat{\theta}_3, \{0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.4/\hat{\mathfrak{M}}_4\}), \\ (\hat{\theta}_4, \{0.4/\hat{\mathfrak{M}}_4, 0.5/\hat{\mathfrak{M}}_5, 0.6/\hat{\mathfrak{M}}_6\}), (\hat{\theta}_5, \{0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}), (\hat{\theta}_6, \{0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.4/\hat{\mathfrak{M}}_4, 0.7/\hat{\mathfrak{M}}_7\}), \\ (\hat{\theta}_7, \{0.1/\hat{\mathfrak{M}}_1, 0.3/\hat{\mathfrak{M}}_3, 0.5/\hat{\mathfrak{M}}_5, 0.6/\hat{\mathfrak{M}}_6\}), (\hat{\theta}_8, \{0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7\}), \\ (\hat{\theta}_9, \{0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}), (\hat{\theta}_{10}, \{0.1/\hat{\mathfrak{M}}_1, 0.3/\hat{\mathfrak{M}}_3, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}), \\ (\hat{\theta}_{11}, \{0.2/\hat{\mathfrak{M}}_2, 0.4/\hat{\mathfrak{M}}_4, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}), (\hat{\theta}_{12}, \{0.1/\hat{\mathfrak{M}}_1, 0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}), \\ (\hat{\theta}_{13}, \{0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.5/\hat{\mathfrak{M}}_5, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}), (\hat{\theta}_{14}, \{0.1/\hat{\mathfrak{M}}_1, 0.3/\hat{\mathfrak{M}}_3, 0.5/\hat{\mathfrak{M}}_5, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}), \\ (\hat{\theta}_{15}, \{0.1/\hat{\mathfrak{M}}_1, 0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.5/\hat{\mathfrak{M}}_5, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}), (\hat{\theta}_{16}, \{0.4/\hat{\mathfrak{M}}_4, 0.5/\hat{\mathfrak{M}}_5, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}) \end{array} \right\}$$

Definition 3.5. Let $(\Theta_1, \mathfrak{A}_1), (\Theta_2, \mathfrak{A}_2) \in \Sigma_{(\Theta, \mathfrak{A})}$ then $(\Theta_1, \mathfrak{A}_1)$ is said to be $\hat{H}s$- *subset* of $(\Theta_2, \mathfrak{A}_2)$ if $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ and $\forall \hat{\theta} \in \mathfrak{A}_1, \Theta_1(\hat{\theta}) \subseteq \Theta_2(\hat{\theta})$.

Example 3.6. Assuming Example 3.2, if

$(\Theta_1, \mathfrak{A}_1) = \left\{ (\hat{\theta}_1, \{\hat{\mathfrak{M}}_1\}), (\hat{\theta}_2, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2\}), (\hat{\theta}_3, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3\}) \right\}$

$(\Theta_2, \mathfrak{A}_2) = \left\{ (\hat{\theta}_1, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2\}), (\hat{\theta}_2, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3\}), (\hat{\theta}_3, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}), (\hat{\theta}_4, \{\hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_6\}) \right\}$

then $(\Theta_1, \mathfrak{A}_1) \subseteq (\Theta_2, \mathfrak{A}_2)$.

Definition 3.7. A set $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times ..... \times \mathfrak{A}_n$ in $\hat{H}s$-set $(\Theta, \mathfrak{A})$ is said to be *Not set* if it has the representation as $\ltimes\mathfrak{A} = \{\ltimes\hat{\theta}_1, \ltimes\hat{\theta}_2, \ltimes\hat{\theta}_3, \ltimes\hat{\theta}_4, ..., \ltimes\hat{\theta}_m\}$ where $m = \prod_{i=1}^{n} |\mathfrak{A}_i|$.

Example 3.8. Reconsidering $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4, , \mathfrak{A}_5$ from Example 3.2, we get

$$\ltimes\mathfrak{A} = \{\ltimes\hat{\theta}_1, \ltimes\hat{\theta}_2, \ltimes\hat{\theta}_3, \ltimes\hat{\theta}_4, ..., \ltimes\hat{\theta}_{16}\}$$

.

Definition 3.9. A $\hat{H}s$-set $(\Theta, \mathfrak{A}_1)$ is stated as a *relative null $\hat{H}s$-set* w.r.t $\mathfrak{A}_1 \subseteq \mathfrak{A}$, symbolized by $(\Theta, \mathfrak{A}_1)_\Phi$, if $\Theta(\hat{\theta}) = \varnothing, \forall \hat{\theta} \in \mathfrak{A}_1$.

Example 3.10. Assuming Example 3.2, if $(\Theta, \mathfrak{A}_1)_\Phi = \left\{ (\hat{\theta}_1, \varnothing), (\hat{\theta}_2, \varnothing), (\hat{\theta}_3, \varnothing) \right\}$ where $\mathfrak{A}_1 \subseteq \mathfrak{A}$.

Definition 3.11. A $\hat{H}s$-set $(\Theta, \mathfrak{A}_1)$ is stated as a *relative whole $\hat{H}s$-set* w.r.t $\mathfrak{A}_1 \subseteq \mathfrak{A}$, symbolized by $(\Theta, \mathfrak{A}_1)_{\hat{\Pi}}$, if $\Theta(\hat{\theta}) = \hat{\Pi}, \forall \hat{\theta} \in \mathfrak{A}_1$.

Example 3.12. Assuming Example 3.2, if $(\Theta, \mathfrak{A}_1)_{\hat{\Pi}} = \left\{ (\hat{\theta}_1, \hat{\Pi}), (\hat{\theta}_2, \hat{\Pi}), (\hat{\theta}_3, \hat{\Pi}) \right\}$ where $\mathfrak{A}_1 \subseteq \mathfrak{A}$.

Definition 3.13. A $\hat{H}s$-set $(\Theta, \mathfrak{A})$ is stated as a *absolute whole $\hat{H}s$-set* on $\hat{\Pi}$, symbolized by $(\Theta, \mathfrak{A})_{\hat{\Pi}}$, if $\Theta(\hat{\theta}) = \hat{\Pi}, \forall \hat{\theta} \in \mathfrak{A}$.

Example 3.14. Assuming Example 3.2, if

$$(\Theta, \mathfrak{A})_{\hat{\Pi}} = \left\{ \begin{array}{l} (\hat{\theta}_1, \hat{\Pi}), (\hat{\theta}_2, \hat{\Pi}), (\hat{\theta}_3, \hat{\Pi}), (\hat{\theta}_4, \hat{\Pi}), (\hat{\theta}_5, \hat{\Pi}), (\hat{\theta}_6, \hat{\Pi}), (\hat{\theta}_7, \hat{\Pi}), (\hat{\theta}_8, \hat{\Pi}), \\ (\hat{\theta}_9, \hat{\Pi}), (\hat{\theta}_{10}, \hat{\Pi}), (\hat{\theta}_{11}, \hat{\Pi}), (\hat{\theta}_{12}, \hat{\Pi}), (\hat{\theta}_{13}, \hat{\Pi}), (\hat{\theta}_{14}, \hat{\Pi}), (\hat{\theta}_{15}, \hat{\Pi}), (\hat{\theta}_{16}, \hat{\Pi}) \end{array} \right\}$$

Proposition 3.15. *Let* $(\Theta_1,\mathfrak{A}_1),(\Theta_2,\mathfrak{A}_2),(\Theta_3,\mathfrak{A}_3) \in \Sigma_{(\Theta,\mathfrak{A})}$ *with* $\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_3 \subseteq \mathfrak{A}$ *then*

(i) $(\Theta_1,\mathfrak{A}_1) \subseteq (\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}}$

(ii) $(\Theta_1,\mathfrak{A}_1)_{\Phi} \subseteq (\Theta_1,\mathfrak{A}_1)$

(iii) $(\Theta_1,\mathfrak{A}_1) \subseteq (\Theta_1,\mathfrak{A}_1)$

(iv) *If* $(\Theta_1,\mathfrak{A}_1) \subseteq (\Theta_2,\mathfrak{A}_2)$ *&* $(\Theta_2,\mathfrak{A}_2) \subseteq (\Theta_3,\mathfrak{A}_3)$ *then* $(\Theta_1,\mathfrak{A}_1) \subseteq (\Theta_3,\mathfrak{A}_3)$

(v) *If* $(\Theta_1,\mathfrak{A}_1) = (\Theta_2,\mathfrak{A}_2)$ *&* $(\Theta_2,\mathfrak{A}_2) = (\Theta_3,\mathfrak{A}_3)$ *then* $(\Theta_1,\mathfrak{A}_1) = (\Theta_3,\mathfrak{A}_3)$

Definition 3.16. The *complement* of a $\hat{H}s$-set $(\Theta,\mathfrak{A})$, symbolized by $(\Theta,\mathfrak{A})^{\ominus}$, is stated as $(\Theta,\mathfrak{A})^{\ominus} = (\Theta^{\ominus},\ltimes\mathfrak{A})$ where $\Theta^{\ominus} : \ltimes\mathfrak{A} \to P^{\hat{\Pi}}$ with $\Theta^{\ominus}(\ltimes\hat{\theta}) = \hat{\Pi} \setminus \Theta(\hat{\theta}), \forall\hat{\theta} \in \mathfrak{A}$.

Example 3.17. From Example 3.2, we get

$$(\Theta,\mathfrak{A})^{\ominus} = \left\{ \begin{array}{l} (\ltimes\hat{\theta}_1,\{\widehat{\mathfrak{M}}_3,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5,\widehat{\mathfrak{M}}_6,\widehat{\mathfrak{M}}_7,\widehat{\mathfrak{M}}_8\}),(\ltimes\hat{\theta}_2,\{\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5,\widehat{\mathfrak{M}}_6,\widehat{\mathfrak{M}}_7,\widehat{\mathfrak{M}}_8\}),(\ltimes\hat{\theta}_3,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_5,\widehat{\mathfrak{M}}_6,\widehat{\mathfrak{M}}_7,\widehat{\mathfrak{M}}_8\}), \\ (\ltimes\hat{\theta}_4,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_3,\widehat{\mathfrak{M}}_7,\widehat{\mathfrak{M}}_8\}),(\ltimes\hat{\theta}_5,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_3,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5\}),(\ltimes\hat{\theta}_6,\{\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_3,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_7\}), \\ (\ltimes\hat{\theta}_7,\{\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_7,\widehat{\mathfrak{M}}_8\}),(\ltimes\hat{\theta}_8,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5,\widehat{\mathfrak{M}}_8\}),(\ltimes\hat{\theta}_9,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5\}),(\ltimes\hat{\theta}_{10},\{\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5\}), \\ (\ltimes\hat{\theta}_{11},\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_3,\widehat{\mathfrak{M}}_5\}),(\ltimes\hat{\theta}_{12},\{\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5\}),(\ltimes\hat{\theta}_{13},\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_6\}),(\ltimes\hat{\theta}_{14},\{\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_6\}), \\ (\ltimes\hat{\theta}_{15},\{\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_6\}),(\ltimes\hat{\theta}_{16},\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_3\}) \end{array} \right\}$$

Definition 3.18. The *relative complement* of a $\hat{H}s$-set $(\Theta,\mathfrak{A})$, symbolized by $(\Theta,\mathfrak{A})^{\circledast}$, is stated as $(\Theta,\mathfrak{A})^{\circledast} = (\Theta^{\circledast},\mathfrak{A})$ where $\Theta^{\circledast} : \mathfrak{A} \to P^{\hat{\Pi}}$ with $\Theta^{\circledast}(\hat{\theta}) = \hat{\Pi} \setminus \Theta(\hat{\theta}), \forall\hat{\theta} \in \mathfrak{A}$.

Example 3.19. Reconsidering Example 3.2, we get

$$(\Theta,\mathfrak{A})^{\circledast} = \left\{ \begin{array}{l} (\hat{\theta}_1,\{\widehat{\mathfrak{M}}_3,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5,\widehat{\mathfrak{M}}_6,\widehat{\mathfrak{M}}_7,\widehat{\mathfrak{M}}_8\}),(\hat{\theta}_2,\{\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5,\widehat{\mathfrak{M}}_6,\widehat{\mathfrak{M}}_7,\widehat{\mathfrak{M}}_8\}),(\hat{\theta}_3,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_5,\widehat{\mathfrak{M}}_6,\widehat{\mathfrak{M}}_7,\widehat{\mathfrak{M}}_8\}), \\ (\hat{\theta}_4,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_3,\widehat{\mathfrak{M}}_7,\widehat{\mathfrak{M}}_8\}),(\hat{\theta}_5,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_3,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5\}),(\hat{\theta}_6,\{\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_3,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_7\}), \\ (\hat{\theta}_7,\{\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_7,\widehat{\mathfrak{M}}_8\}),(\hat{\theta}_8,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5,\widehat{\mathfrak{M}}_8\}),(\hat{\theta}_9,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5\}),(\hat{\theta}_{10},\{\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5\}), \\ (\hat{\theta}_{11},\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_3,\widehat{\mathfrak{M}}_5\}),(\hat{\theta}_{12},\{\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5\}),(\hat{\theta}_{13},\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_6\}),(\hat{\theta}_{14},\{\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_6\}), \\ (\hat{\theta}_{15},\{\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_6\}),(\hat{\theta}_{16},\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_3\}) \end{array} \right\}$$

Proposition 3.20. *Let* $(\Theta,\mathfrak{A}) \in \Sigma_{(\Theta,\mathfrak{A})}$ *then*

(i) $((\Theta,\mathfrak{A})^{\ominus})^{\ominus} = (\Theta,\mathfrak{A})$

(ii) $((\Theta,\mathfrak{A})^{\circledast})^{\circledast} = (\Theta,\mathfrak{A})$

(iii) $((\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}})^{\ominus} = (\Theta_1,\mathfrak{A}_1)_{\Phi} = ((\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}})^{\circledast}$*;* $\mathfrak{A}_1 \subseteq \mathfrak{A}$

(iv) $((\Theta_1,\mathfrak{A}_1)_{\Phi})^{\ominus} = (\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}} = ((\Theta_1,\mathfrak{A}_1)_{\Phi})^{\circledast}$*;* $\mathfrak{A}_1 \subseteq \mathfrak{A}$

Definition 3.21. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2) \in \Sigma_{(\Theta,\mathfrak{A})}$, the union-operation $(\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3 = \mathfrak{A}_1 \cup \mathfrak{A}_2$ and for $\hat{\theta} \in \mathfrak{A}_3$,

$$\Theta_3(\hat{\theta}) = \begin{cases} \Theta_1(\hat{\theta}) & \hat{\theta} \in (\mathfrak{A}_1 \setminus \mathfrak{A}_2) \\ \Theta_2(\hat{\theta}) & \hat{\theta} \in (\mathfrak{A}_2 \setminus \mathfrak{A}_1) \\ \Theta_1(\hat{\theta}) \cup \Theta_2(\hat{\theta}) & \hat{\theta} \in (\mathfrak{A}_1 \cap \mathfrak{A}_2) \end{cases}.$$

Example 3.22. Let

$$(\Theta_1,\mathfrak{A}_1) = \left\{ (\hat{\theta}_1,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_2\}),(\hat{\theta}_2,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_3\}),(\hat{\theta}_3,\{\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_3,\widehat{\mathfrak{M}}_4\}) \right\}$$
$$(\Theta_2,\mathfrak{A}_2) = \left\{ (\hat{\theta}_3,\{\widehat{\mathfrak{M}}_1,\widehat{\mathfrak{M}}_2\}),(\hat{\theta}_4,\{\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_5,\widehat{\mathfrak{M}}_6\}),(\hat{\theta}_5,\{\widehat{\mathfrak{M}}_2,\widehat{\mathfrak{M}}_4,\widehat{\mathfrak{M}}_6\}) \right\}$$

then

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\hat{\theta}_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\left(\hat{\theta}_2,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\}\right),\left(\hat{\theta}_3,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4\}\right),\\ \left(\hat{\theta}_4,\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\left(\hat{\theta}_5,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right)\end{array}\right\}$$

Definition 3.23. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, the intersection-operation $(\Theta_1,\mathfrak{A}_1)\cap(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\cap\mathfrak{A}_2$ & for $\hat{\theta}\in\mathfrak{A}_3$, $\Theta_3(\hat{\theta})=\Theta_1(\hat{\theta})\cap\Theta_2(\hat{\theta})$.

Example 3.24. Reconsidering Example 3.22, we get $(\Theta_3,\mathfrak{A}_3)=\left\{\left(\hat{\theta}_3,\{\hat{\mathfrak{M}}_2\}\right)\right\}$.

Definition 3.25. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their extended-intersection $(\Theta_1,\mathfrak{A}_1)\cap_\varepsilon(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\cup\mathfrak{A}_2$ and for $\hat{\theta}\in\mathfrak{A}_3$,

$$\Theta_3(\hat{\theta})=\begin{cases}\Theta_1(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_1\setminus\mathfrak{A}_2)\\ \Theta_2(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_2\setminus\mathfrak{A}_1)\\ \Theta_1(\hat{\theta})\cap\Theta_2(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_1\cap\mathfrak{A}_2)\end{cases}$$

Example 3.26. Taking assumptions of Example 3.22, we get

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\hat{\theta}_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\left(\hat{\theta}_2,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\}\right),\left(\hat{\theta}_3,\{\hat{\mathfrak{M}}_2\}\right),\\ \left(\hat{\theta}_4,\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\left(\hat{\theta}_5,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right)\end{array}\right\}$$

Definition 3.27. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their AND-operation $(\Theta_1,\mathfrak{A}_1)\bigwedge(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\times\mathfrak{A}_2$ and for $(\hat{\theta}_i,\hat{\theta}_j)\in\mathfrak{A}_3,\hat{\theta}_i\in\mathfrak{A}_1,\hat{\theta}_j\in\mathfrak{A}_2$,

$$\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\Theta_1(\hat{\theta}_i)\cup\Theta_2(\hat{\theta}_j).$$

Example 3.28. Taking assumptions of Example 3.22, we get

$$\mathfrak{A}_1\times\mathfrak{A}_2=\left\{\begin{array}{l}\pi_1=\left(\hat{\theta}_1,\hat{\theta}_3\right),\pi_2=\left(\hat{\theta}_1,\hat{\theta}_4\right),\pi_3=\left(\hat{\theta}_1,\hat{\theta}_5\right),\pi_4=\left(\hat{\theta}_2,\hat{\theta}_3\right),\pi_5=\left(\hat{\theta}_2,\hat{\theta}_4\right),\\ \pi_6=\left(\hat{\theta}_2,\hat{\theta}_5\right),\pi_7=\left(\hat{\theta}_3,\hat{\theta}_3\right),\pi_8=\left(\hat{\theta}_3,\hat{\theta}_4\right),\pi_9=\left(\hat{\theta}_3,\hat{\theta}_5\right)\end{array}\right\}$$

then

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\pi_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\left(\pi_2,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\\ \left(\pi_3,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right),\left(\pi_4,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\}\right),\\ \left(\pi_5,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\left(\pi_6,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right),\\ \left(\pi_7,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4\}\right),\left(\pi_8,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\\ \left(\pi_9,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right),\end{array}\right\}$$

Definition 3.29. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their OR-operation $(\Theta_1,\mathfrak{A}_1)\bigvee(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\times\mathfrak{A}_2$ and for $(\hat{\theta}_i,\hat{\theta}_j)\in\mathfrak{A}_3,\hat{\theta}_i\in\mathfrak{A}_1,\hat{\theta}_j\in\mathfrak{A}_2$,

$$\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\Theta_1(\hat{\theta}_i)\cap\Theta_2(\hat{\theta}_j).$$

Example 3.30. Taking assumptions of Examples 3.22 and 3.30, we get

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\pi_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\left(\pi_2,\{\}\right),\left(\pi_3,\{\hat{\mathfrak{M}}_2\}\right),\left(\pi_4,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\\ \left(\pi_5,\{\}\right),\left(\pi_6,\{\hat{\mathfrak{M}}_2\}\right),\left(\pi_7,\{\hat{\mathfrak{M}}_2\}\right),\left(\pi_8,\{\hat{\mathfrak{M}}_4\}\right),\left(\pi_9,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4\}\right),\end{array}\right\}$$

Definition 3.31. For $(\Theta_1, \mathfrak{A}_1) \& (\Theta_2, \mathfrak{A}_2) \in \Sigma_{(\Theta,\mathfrak{A})}$, their restricted-union $(\Theta_1, \mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2, \mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3, \mathfrak{A}_3)$ with $\mathfrak{A}_3 = \mathfrak{A}_1 \cap \mathfrak{A}_2$ and for $\hat{\theta} \in \mathfrak{A}_3$,

$$\Theta_3(\hat{\theta}) = \Theta_1(\hat{\theta}) \cup \Theta_2(\hat{\theta}).$$

Example 3.32. Taking assumptions of Example 3.22, we get

$$(\Theta_3, \mathfrak{A}_3) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$$

Definition 3.33. For $(\Theta_1, \mathfrak{A}_1) \& (\Theta_2, \mathfrak{A}_2) \in \Sigma_{(\Theta,\mathfrak{A})}$, their restricted-difference $(\Theta_1, \mathfrak{A}_1) \setminus_{\mathcal{R}} (\Theta_2, \mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3, \mathfrak{A}_3)$ with $\mathfrak{A}_3 = \mathfrak{A}_1 \cap \mathfrak{A}_2$ and for $\hat{\theta} \in \mathfrak{A}_3$,

$$\Theta_3(\hat{\theta}) = \Theta_1(\hat{\theta}) - \Theta_2(\hat{\theta}).$$

Example 3.34. Taking suppositions of Example 3.22, we get $(\Theta_3, \mathfrak{A}_3) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$.

Definition 3.35. For $(\Theta_1, \mathfrak{A}_1) \& (\Theta_2, \mathfrak{A}_2) \in \Sigma_{(\Theta,\mathfrak{A})}$, their restricted-symmetric-difference $(\Theta_1, \mathfrak{A}_1) \blacktriangle (\Theta_2, \mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3, \mathfrak{A}_3)$ stated by

$$(\Theta_3, \mathfrak{A}_3) = \left\{ ((\Theta_1, \mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2, \mathfrak{A}_2)) \setminus_{\mathcal{R}} ((\Theta_1, \mathfrak{A}_1) \cap (\Theta_2, \mathfrak{A}_2)) \right\}$$

or

$$(\Theta_3, \mathfrak{A}_3) = \left\{ ((\Theta_1, \mathfrak{A}_1) \setminus_{\mathcal{R}} (\Theta_2, \mathfrak{A}_2)) \cup_{\mathcal{R}} ((\Theta_2, \mathfrak{A}_2) \setminus_{\mathcal{R}} (\Theta_1, \mathfrak{A}_1)) \right\}$$

Example 3.36. Taking suppositions of Example 3.22, we get

$$((\Theta_1, \mathfrak{A}_1) \setminus_{\mathcal{R}} (\Theta_2, \mathfrak{A}_2)) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$$

&

$$((\Theta_2, \mathfrak{A}_2) \setminus_{\mathcal{R}} (\Theta_1, \mathfrak{A}_1)) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_1\}\right) \right\}$$

then

$$(\Theta_3, \mathfrak{A}_3) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$$

4. Axioms-based Results of $\hat{H}s$-sets

This part presents some classical axioms-based results of set theory that are also valid for $\hat{H}s$-settings.

(1) **Idempotent Laws**
   (a) $(\Theta, \mathfrak{A}) \cup (\Theta, \mathfrak{A}) = (\Theta, \mathfrak{A}) = (\Theta, \mathfrak{A}) \cup_{\mathcal{R}} (\Theta, \mathfrak{A})$
   (b) $(\Theta, \mathfrak{A}) \cap (\Theta, \mathfrak{A}) = (\Theta, \mathfrak{A}) = (\Theta, \mathfrak{A}) \cap_{\varepsilon} (\Theta, \mathfrak{A})$

(2) **Identity Laws**
   (a) $(\Theta, \mathfrak{A}) \cup (\Theta, \mathfrak{A})_{\Phi} = (\Theta, \mathfrak{A}) = (\Theta, \mathfrak{A}) \cup_{\mathcal{R}} (\Theta, \mathfrak{A})_{\Phi}$
   (b) $(\Theta, \mathfrak{A}) \cap (\Theta, \mathfrak{A})_{\hat{\Pi}} = (\Theta, \mathfrak{A}) = (\Theta, \mathfrak{A}) \cap_{\varepsilon} (\Theta, \mathfrak{A})_{\hat{\Pi}}$
   (c) $(\Theta, \mathfrak{A}) \setminus_{\mathcal{R}} (\Theta, \mathfrak{A})_{\Phi} = (\Theta, \mathfrak{A}) = (\Theta, \mathfrak{A}) \blacktriangle (\Theta, \mathfrak{A})_{\Phi}$
   (d) $(\Theta, \mathfrak{A}) \setminus_{\mathcal{R}} (\Theta, \mathfrak{A}) = (\Theta, \mathfrak{A})_{\Phi} = (\Theta, \mathfrak{A}) \blacktriangle (\Theta, \mathfrak{A})$

(3) **Domination Laws**

(a) $(\Theta,\mathfrak{A}) \cup (\Theta,\mathfrak{A})_{\hat{\Pi}} = (\Theta,\mathfrak{A})_{\hat{\Pi}} = (\Theta,\mathfrak{A}) \cup_{\mathcal{R}} (\Theta,\mathfrak{A})_{\hat{\Pi}}$

(b) $(\Theta,\mathfrak{A}) \cap (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) \cap_{\varepsilon} (\Theta,\mathfrak{A})_{\Phi}$

(4) **Property of Exclusion**

$$(\Theta,\mathfrak{A}) \cup (\Theta,\mathfrak{A})^{\circledast} = (\Theta,\mathfrak{A})_{\hat{\Pi}} = (\Theta,\mathfrak{A}) \cup_{\mathcal{R}} (\Theta,\mathfrak{A})^{\circledast}$$

(5) **Property of Contradiction**

$$(\Theta,\mathfrak{A}) \cap (\Theta,\mathfrak{A})^{\circledast} = (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) \cap_{\varepsilon} (\Theta,\mathfrak{A})^{\circledast}$$

(6) **Absorption Laws**

(a) $(\Theta_1,\mathfrak{A}_1) \cup ((\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(b) $(\Theta_1,\mathfrak{A}_1) \cap ((\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(c) $(\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} ((\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(d) $(\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} ((\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(7) **Commutative Laws**

(a) $(\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cup (\Theta_1,\mathfrak{A}_1)$

(b) $(\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cup_{\mathcal{R}} (\Theta_1,\mathfrak{A}_1)$

(c) $(\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cap (\Theta_1,\mathfrak{A}_1)$

(d) $(\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cap_{\varepsilon} (\Theta_1,\mathfrak{A}_1)$

(e) $(\Theta_1,\mathfrak{A}_1) \blacktriangle (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \blacktriangle (\Theta_1,\mathfrak{A}_1)$

(8) **Associative Laws**

(a) $(\Theta_1,\mathfrak{A}_1) \cup ((\Theta_2,\mathfrak{A}_2) \cup (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2)) \cup (\Theta_3,\mathfrak{A}_3)$

(b) $(\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} ((\Theta_2,\mathfrak{A}_2) \cup_{\mathcal{R}} (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) \cup_{\mathcal{R}} (\Theta_3,\mathfrak{A}_3)$

(c) $(\Theta_1,\mathfrak{A}_1) \cap ((\Theta_2,\mathfrak{A}_2) \cap (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2)) \cap (\Theta_3,\mathfrak{A}_3)$

(d) $(\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} ((\Theta_2,\mathfrak{A}_2) \cap_{\varepsilon} (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2)) \cap_{\varepsilon} (\Theta_3,\mathfrak{A}_3)$

(e) $(\Theta_1,\mathfrak{A}_1) \bigvee ((\Theta_2,\mathfrak{A}_2) \bigvee (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \bigvee (\Theta_2,\mathfrak{A}_2)) \bigvee (\Theta_3,\mathfrak{A}_3)$

(f) $(\Theta_1,\mathfrak{A}_1) \bigwedge ((\Theta_2,\mathfrak{A}_2) \bigwedge (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \bigwedge (\Theta_2,\mathfrak{A}_2)) \bigwedge (\Theta_3,\mathfrak{A}_3)$

(9) **De Morgans Laws**

(a) $((\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2)^{\ominus}$

(b) $((\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \cup (\Theta_2,\mathfrak{A}_2)^{\ominus}$

(c) $((\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \cap (\Theta_2,\mathfrak{A}_2)^{\circledast}$

(d) $((\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)^{\circledast}$

(e) $((\Theta_1,\mathfrak{A}_1) \bigvee (\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \bigwedge (\Theta_2,\mathfrak{A}_2)^{\ominus}$

(f) $((\Theta_1,\mathfrak{A}_1) \bigwedge (\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \bigvee (\Theta_2,\mathfrak{A}_2)^{\ominus}$

(g) $((\Theta_1,\mathfrak{A}_1) \bigvee (\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \bigwedge (\Theta_2,\mathfrak{A}_2)^{\circledast}$

(h) $((\Theta_1,\mathfrak{A}_1) \bigwedge (\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \bigvee (\Theta_2,\mathfrak{A}_2)^{\circledast}$

(10) **Distributive Laws**

(a) $(\Theta_1,\mathfrak{A}_1)\cup((\Theta_2,\mathfrak{A}_2)\cap(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cup(\Theta_2,\mathfrak{A}_2))\cap((\Theta_1,\mathfrak{A}_1)\cup(\Theta_3,\mathfrak{A}_3))$

(b) $(\Theta_1,\mathfrak{A}_1)\cap((\Theta_2,\mathfrak{A}_2)\cup(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cap(\Theta_2,\mathfrak{A}_2))\cup((\Theta_1,\mathfrak{A}_1)\cap(\Theta_3,\mathfrak{A}_3))$

(c) $(\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}((\Theta_2,\mathfrak{A}_2)\cap_{\varepsilon}(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_2,\mathfrak{A}_2))\cap_{\varepsilon}((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))$

(d) $(\Theta_1,\mathfrak{A}_1)\cap_{\varepsilon}((\Theta_2,\mathfrak{A}_2)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cap_{\varepsilon}(\Theta_2,\mathfrak{A}_2))\cup_{\mathcal{R}}((\Theta_1,\mathfrak{A}_1)\cap_{\varepsilon}(\Theta_3,\mathfrak{A}_3))$

(e) $(\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}((\Theta_2,\mathfrak{A}_2)\cap(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_2,\mathfrak{A}_2))\cap((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))$

(f) $(\Theta_1,\mathfrak{A}_1)\cap((\Theta_2,\mathfrak{A}_2)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cap(\Theta_2,\mathfrak{A}_2))\cup_{\mathcal{R}}((\Theta_1,\mathfrak{A}_1)\cap(\Theta_3,\mathfrak{A}_3))$

5. Relations-based Operations of $\hat{H}s$-sets

Here some relations-based classical notions and results are generalized for $\hat{H}s$-sets.

Definition 5.1. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their Cartesian product $(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ where $\mathfrak{A}_3=\mathfrak{A}_1\times\mathfrak{A}_2$ & $\Theta_3:\mathfrak{A}_3\to P(\hat{\Pi}\times\hat{\Pi})$ stated by $\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\Theta_1(\hat{\theta}_i)\times\Theta_2(\hat{\theta}_j)\ \forall\ (\hat{\theta}_i,\hat{\theta}_j)\in\mathfrak{A}_3$ that is $\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\{(\hat{\theta}_i,\hat{\theta}_j):\hat{\theta}_i\in\Theta_1(\hat{\theta}_i),\hat{\theta}_j\in\Theta_2(\hat{\theta}_j)\}$.

Definition 5.2. If $(\Theta_1,\mathfrak{A}_1),(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$ then a relation from $(\Theta_1,\mathfrak{A}_1)$ to $(\Theta_2,\mathfrak{A}_2)$ is stated as *$\hat{H}s$-relation* $(\hat{\Xi},\mathfrak{A}_4)$ (conveniently $\hat{\Xi}$) which is the $\hat{H}s$-subset of $(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$ where $\mathfrak{A}_4\subseteq\mathfrak{A}_1\times\mathfrak{A}_2$ & $\forall\ (\hat{\theta}_1,\hat{\theta}_2)\in\mathfrak{A}_4,\hat{\Xi}(\hat{\theta}_1,\hat{\theta}_2)=\Theta_3(\hat{\theta}_1,\hat{\theta}_2)$, where $(\Theta_3,\mathfrak{A}_3)=(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$.

Definition 5.3. Let $\hat{\Xi}$ be a $\hat{H}s$-relation from $(\Theta_1,\mathfrak{A}_1)$ to $(\Theta_2,\mathfrak{A}_2)$ such that $(\Theta_3,\mathfrak{A}_3)=(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$. Then

(i) The DoM $\hat{\Xi}$ (the domain of $\hat{\Xi}$) is a $\hat{H}s$-set $(\Theta,\mathfrak{W})\subset(\Theta_1,\mathfrak{A}_1)$ where $\mathfrak{W}=\{\hat{\theta}_i\in\mathfrak{A}_1:\Theta_3(\hat{\theta}_i,\hat{\theta}_j)\in\hat{\Xi}\ forsome\ \hat{\theta}_j\in\mathfrak{A}_2\}$ & $\Theta(\hat{\theta}_1)=\Theta_1(\hat{\theta}_1),\forall\ \hat{\theta}_1\in\mathfrak{W}$.

(ii) The RNG $\hat{\Xi}$ (the range of $\hat{\Xi}$) is a $\hat{H}s$-set $(\xi,\mathfrak{L})\subset(\Theta_2,\mathfrak{A}_2)$ where $\mathfrak{L}\subset\mathfrak{A}_2$ & $\mathfrak{L}=\{\hat{\theta}_j\in\mathfrak{A}_2:\Theta_3(\hat{\theta}_i,\hat{\theta}_j)\in\hat{\Xi}\ forsome\ \hat{\theta}_i\in\mathfrak{A}_1\}$ & $\xi(\hat{\theta}_2)=\Theta_1(\hat{\theta}_2),\forall\ \hat{\theta}_2\in\mathfrak{L}$.

(iii) The $\hat{\Xi}^{-1}$ (inverse of $\hat{\Xi}$) is a $\hat{H}s$-relation from $(\Theta_2,\mathfrak{A}_2)$ to $(\Theta_1,\mathfrak{A}_1)$ stated by $\hat{\Xi}^{-1}=\{\Theta_2(\hat{\theta}_j)\times\Theta_1(\hat{\theta}_i):\Theta_1(\hat{\theta}_i)\hat{\Xi}\Theta_2(\hat{\theta}_j)\}$.

Example 5.4. Let

$$(\Theta_1,\mathfrak{A}_1)=\left\{\ \Theta_1(\hat{\theta}_1),\Theta_1(\hat{\theta}_2),\Theta_1(\hat{\theta}_3)\ \right\},(\Theta_2,\mathfrak{A}_2)=\left\{\ \Theta_2(\hat{\theta}_4),\Theta_2(\hat{\theta}_5),\Theta_2(\hat{\theta}_6)\ \right\}$$

$$(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)=\left\{\begin{array}{c}(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_5)),(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_6)),\\(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_5)),(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_6)),\\(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_5)),(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_6))\end{array}\right\}$$

then

$$\hat{\Xi}=\left\{\ (\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_6)),(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_6)),(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_6))\ \right\}$$

(i) $DoM\,\hat{\Xi}=(\Theta,\mathfrak{W})$ where $\mathfrak{W}=\{\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3\}\subseteq\mathfrak{A}_1$ & $\Theta(\hat{\theta}_i)=\Theta_1(\hat{\theta}_i)\forall\ \hat{\theta}_i\in\mathfrak{W}$.

(ii) $RNG\,\hat{\Xi}=(\xi,\mathfrak{L})$ where $\mathfrak{L}=\{\hat{\theta}_4,\hat{\theta}_6\}\subset\mathfrak{A}_2$ & $\xi(\hat{\theta}_j)=\Theta_2(\hat{\theta}_j)\forall\ \hat{\theta}_j\in\mathfrak{L}$.

(iii)

$$\hat{\Xi}^{-1} = \left\{ \ (\Theta_2(\hat{\theta}_4) \times \Theta_1(\hat{\theta}_1)), (\Theta_2(\hat{\theta}_6) \times \Theta_1(\hat{\theta}_1)), (\Theta_2(\hat{\theta}_6) \times \Theta_1(\hat{\theta}_2)), (\Theta_2(\hat{\theta}_6) \times \Theta_1(\hat{\theta}_3)) \ \right\}.$$

Definition 5.5. Let $\hat{\Xi}$ & $\mathfrak{S}$ are two $\hat{H}s$-relations on $\hat{H}s$-set $(\Theta, \mathfrak{W})$, then we get

(i) $\hat{\Xi} \subset \mathfrak{S}$, if for all $\varpi, \varsigma \in \mathfrak{W}, \Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ then $\Theta(\varpi) \times \Theta(\varsigma) \in \mathfrak{S}$.

(ii) $\hat{\Xi}^{©} = \{\Theta(\varpi) \times \Theta(\varsigma) : \Theta(\varpi) \times \Theta(\varsigma) \notin \hat{\Xi}, \forall\ \varpi, \varsigma \in \mathfrak{W}\}$.

(iii) $\hat{\Xi} \cup \mathfrak{S} = \{\Theta(\varpi) \times \Theta(\varsigma) : \Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}\ or\ \Theta(\varpi) \times \Theta(\varsigma) \in \mathfrak{S}, \forall\ \varpi, \varsigma \in \mathfrak{W}\}$.

(iv) $\hat{\Xi} \cap \mathfrak{S} = \{\Theta(\varpi) \times \Theta(\varsigma) : \Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}\ \&\ \Theta(\varpi) \times \Theta(\varsigma) \in \mathfrak{S}, \forall\ \varpi, \varsigma \in \mathfrak{W}\}$.

Example 5.6. Let $(\Theta, \mathfrak{W}) = \left\{ \ \Theta(\hat{\theta}_1), \Theta(\hat{\theta}_2), \Theta(\hat{\theta}_3) \ \right\}$ then

$$(\Theta, \mathfrak{W}) \times (\Theta, \mathfrak{W}) = \left\{ \begin{array}{l} (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), \\ (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), \\ (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \end{array} \right\}$$

then we get

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \ \right\}$$

&

$$\mathfrak{S} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)) \ \right\}$$

now

(1) $\hat{\Xi}^{©} = \left\{ \begin{array}{l} (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), \\ (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)) \end{array} \right\}$. &
$\mathfrak{S}^{©} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \ \right\}$.

(2) $\hat{\Xi} \cup \mathfrak{S} = \left\{ \begin{array}{l} (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), \\ (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \end{array} \right\}$.

(3) $\hat{\Xi} \cap \mathfrak{S} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)) \ \right\}$.

Definition 5.7. Let $\hat{\Xi}$ be a $\hat{H}s$-relation on $(\Theta, \mathfrak{W})$, then

(i) if $\Theta(\varpi) \times \Theta(\varpi) \in \hat{\Xi} \forall \varpi \in \mathfrak{W}$, then $\hat{\Xi}$ is reflexive, e.g. $\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)) \ \right\}$.

(ii) if $\Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ then $\Theta(\varsigma) \times \Theta(\varpi) \in \hat{\Xi} \forall \varpi, \varsigma \in \mathfrak{W}$, so $\hat{\Xi}$ is symmetric, e.g.

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)) \ \right\}.$$

(iii) if $\Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ & $\Theta(\varsigma) \times \Theta(w) \in \hat{\Xi}$ then $\Theta(\varpi) \times \Theta(w) \in \hat{\Xi} \forall \varpi, \varsigma, w \in \mathfrak{W}$, so $\hat{\Xi}$ is transitive. e.g. $\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)) \ \right\}$.

(iv) if properties (i)-(iii) are satisfied then $\hat{\Xi}$ is stated as equivalence relation. E.g.

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)) \ \right\}.$$

(v) if $\Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ then $\varpi = \varsigma \forall \varpi, \varsigma \in \mathfrak{W}$, so $\hat{\Xi}$ is stated as identity. e.g.

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \ \right\}.$$

Definition 5.8. If $\hat{\Xi}$ is a $\hat{H}s$-relation from $(\Theta_1,\mathfrak{A}_1)$ to $(\Theta_2,\mathfrak{A}_2)$ & $\mathfrak{S}$ is a $\hat{H}s$-relation from $(\Theta_2,\mathfrak{A}_2)$ to $(\Theta_3,\mathfrak{A}_3)$ then composition of $\hat{\Xi}$ & $\mathfrak{S}$, symbolized by $\hat{\Xi}\circ\mathfrak{S}$, is also a $\hat{H}s$-relation $\mathfrak{T}$ from $(\Theta_1,\mathfrak{A}_1)$ to $(\Theta_3,\mathfrak{A}_3)$ stated as if $\Theta_1(\varpi)\in(\Theta_1,\mathfrak{A}_1)$ & $\Theta_3(w)\in(\Theta_3,\mathfrak{A}_3)$ then $\Theta_1(\varpi)\times\Theta_3(w)\in\hat{\Xi}\circ\mathfrak{S}$ i.e. $\Theta_1(\varpi)\times\Theta_3(w)\in\hat{\Xi}\circ\mathfrak{S}$ iff $\Theta_1(\varpi)\times\Theta_2(\varsigma)\in\hat{\Xi}$ & $\Theta_2(\varsigma)\times\Theta_3(w)\in\hat{\Xi}$.

Example 5.9. Let

$$\hat{\Xi}=\left\{\begin{array}{l}(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_1)),(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_3)),\\(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_3)),(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_3))\end{array}\right\}\ \&$$

$$\mathfrak{S}=\left\{\begin{array}{l}(\Theta_2(\hat{\theta}_1)\times\Theta_3(\hat{\theta}_1)),(\Theta_2(\hat{\theta}_1)\times\Theta_3(\hat{\theta}_2)),\\(\Theta_2(\hat{\theta}_2)\times\Theta_3(\hat{\theta}_2)),(\Theta_2(\hat{\theta}_3)\times\Theta_3(\hat{\theta}_2))\end{array}\right\}$$

then

$$\hat{\Xi}\circ\mathfrak{S}=\left\{\begin{array}{l}(\Theta_1(\hat{\theta}_1)\times\Theta_3(\hat{\theta}_1)),(\Theta_1(\hat{\theta}_1)\times\Theta_3(\hat{\theta}_2)),\\(\Theta_1(\hat{\theta}_2)\times\Theta_3(\hat{\theta}_2)),(\Theta_1(\hat{\theta}_3)\times\Theta_3(\hat{\theta}_2))\end{array}\right\}.$$

Definition 5.10. A $\hat{H}s$-relation $\mathfrak{F}$ from $(\Theta_1,\mathfrak{A}_1)$ to $(\Theta_2,\mathfrak{A}_2)$, represented by $\mathfrak{F}:(\Theta_1,\mathfrak{A}_1)\to(\Theta_2,\mathfrak{A}_2)$, is stated as $\hat{H}s$-function when (a). $DoM\,\mathfrak{F}=\mathfrak{A}_1$, (b). $DoM\,\mathfrak{F}$ has not repeated members & (c). Element-based uniqueness exists between $RNG\,\mathfrak{F}$ & $DoM\,\mathfrak{F}$ i.e. if $\Theta_1(\varpi)\,\mathfrak{F}\,\Theta_2(\varsigma)$ (or $\Theta_1(\varpi)\times\Theta_2(\varsigma)\in\mathfrak{F}$) then $\mathfrak{F}(\Theta_1(\varpi))=\Theta_2(\varsigma)$.

Example 5.11. Let $\mathfrak{A}_1=\{\varpi_1,\varpi_2,\varpi_3\}$ & $\mathfrak{A}_2=\{\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4\}$ then
$(\Theta_1,\mathfrak{A}_1)=\left\{\ \Theta_1(\varpi_1),\Theta_1(\varpi_2),\Theta_1(\varpi_3)\ \right\}$, $(\Theta_2,\mathfrak{A}_2)=\left\{\ \Theta_2(\varsigma_1),\Theta_2(\varsigma_2),\Theta_2(\varsigma_3),\Theta_2(\varsigma_4)\ \right\}$
so $\hat{H}s$-functions is

$$\mathfrak{F}=\left\{\ (\Theta_1(\varpi_1)\times\Theta_2(\varsigma_1)),(\Theta_1(\varpi_2)\times\Theta_2(\varsigma_3)),(\Theta_1(\varpi_3)\times\Theta_2(\varsigma_4))\ \right\}$$

Definition 5.12. A $\hat{H}s$-function $\mathfrak{F}:(\Theta_1,\mathfrak{A}_1)\to(\Theta_2,\mathfrak{A}_2)$ is stated as

(i) if $RNG\,\mathfrak{F}\subset\mathfrak{A}_2$, then INTO-$\hat{H}s$-function. E.g. Let $\mathfrak{A}_1=\{\varpi_1,\varpi_2,\varpi_3\}$ & $\mathfrak{A}_2=\{\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4\}$ then $\mathfrak{F}=\left\{\ (\Theta_1(\varpi_1)\times\Theta_2(\varsigma_1)),(\Theta_1(\varpi_2)\times\Theta_2(\varsigma_3)),(\Theta_1(\varpi_3)\times\Theta_2(\varsigma_4))\ \right\}$

(ii) if $RNG\,\mathfrak{F}=\mathfrak{A}_2$, then ONTO-$\hat{H}s$-function. E.g. Let $\mathfrak{A}_1=\{\varpi_1,\varpi_2,\varpi_3,\varpi_4\}$ & $\mathfrak{A}_2=\{\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4\}$ then

$$\mathfrak{F}=\left\{\ (\Theta_1(\varpi_1)\times\Theta_2(\varsigma_1)),(\Theta_1(\varpi_2)\times\Theta_2(\varsigma_3)),(\Theta_1(\varpi_3)\times\Theta_2(\varsigma_4)),(\Theta_1(\varpi_4)\times\Theta_2(\varsigma_2))\ \right\}$$

(iii) *1-1 $\hat{H}s$-function* if $\Theta_1(\varpi_1)\neq\Theta_1(\varpi_2)$ then $\mathfrak{F}(\Theta_1(\varpi_1))\neq\mathfrak{F}(\Theta_1(\varpi_2))$. E.g.

$$\mathfrak{F}=\left\{\ (\Theta_1(\varpi_1)\times\Theta_2(\varsigma_1)),(\Theta_1(\varpi_2)\times\Theta_2(\varsigma_4)),(\Theta_1(\varpi_3)\times\Theta_2(\varsigma_2)),(\Theta_1(\varpi_4)\times\Theta_2(\varsigma_3))\ \right\}$$

(iv) if it is both INTO and ONTO then bijective $\hat{H}s$-function. E.g.

$$\mathfrak{F}=\left\{\ (\Theta_1(\varpi_1)\times\Theta_2(\varsigma_1)),(\Theta_1(\varpi_2)\times\Theta_2(\varsigma_2)),(\Theta_1(\varpi_3)\times\Theta_2(\varsigma_3)),(\Theta_1(\varpi_4)\times\Theta_2(\varsigma_4))\ \right\}$$

Definition 5.13. The identity $\hat{H}s$-function on $\hat{H}s$-set $(\Theta,\mathfrak{L})$ is stated by $\mathfrak{I}:(\Theta,\mathfrak{L})\to(\Theta,\mathfrak{L})$ such that $\mathfrak{I}(\Theta(l))=\Theta(l)\ \ \forall\ \Theta(l)\in(\Theta,\mathfrak{L})$. E.g. Let $\mathfrak{L}=\{l_1,l_2,l_3,l_4\}$ then
$\mathfrak{I}=\left\{\ (\Theta(l_1)\times\Theta(l_1)),(\Theta(l_2)\times\Theta(l_2)),(\Theta(l_3)\times\Theta(l_3)),(\Theta(l_4)\times\Theta(l_4))\ \right\}$

6. Matrix-theory Based on $\hat{H}s$-sets

Here some classical matrix-based notions are generalized for $\hat{H}s$-sets.

Definition 6.1.

(i) Let $(\Theta, \mathfrak{A})$ be a $\hat{H}s$-set on $\hat{\Pi}$. A set $\mathbb{R}_{\mathfrak{A}} \subseteq \hat{\Pi} \times \mathfrak{A}$ is a relation version of $(\Theta, \mathfrak{A})$ stated as

$$\mathbb{R}_{\mathfrak{A}} = \left\{ \ (\varpi, \hat{\theta}) : \hat{\theta} \in \mathfrak{A}, \varpi \in \Theta(\hat{\theta}) \ \right\}.$$

(ii) *The characteristic function* $\mathcal{X}_{\mathbb{R}_{\mathfrak{A}}}$ is stated by $\mathcal{X}_{\mathbb{R}_{\mathfrak{A}}} : \hat{\Pi} \times \mathfrak{A} \to \{0, 1\}$, where

$$\mathcal{X}_{\mathbb{R}_{\mathfrak{A}}}(\varpi, \hat{\theta}) = \begin{cases} 1 & ; \ (\varpi, \hat{\theta}) \in \mathbb{R}_{\mathfrak{A}} \\ 0 & ; \ (\varpi, \hat{\theta}) \notin \mathbb{R}_{\mathfrak{A}} \end{cases}$$

(iii) If $|\hat{\Pi}| = m$ & $|\mathfrak{A}| = n$ then $(\ddot{s}_{ij})$ is an $m \times n$ $\hat{H}s$-matrix of $(\Theta, \mathfrak{A})$ on $\hat{\Pi}$ and stated as

$$(\ddot{s}_{ij})_{m\times n} = \begin{pmatrix} \ddot{s}_{11} & \ddot{s}_{12} & \cdots & \ddot{s}_{1n} \\ \ddot{s}_{21} & \ddot{s}_{22} & \cdots & \ddot{s}_{2n} \\ \vdots & \vdots & & \vdots \\ \ddot{s}_{m1} & \ddot{s}_{m2} & \cdots & \ddot{s}_{mn} \end{pmatrix}$$

Note: The family of all $m \times n$ $\hat{H}s$- matrices on $\hat{\Pi}$ is symbolized by $(\hat{\Pi})^{(hsm)}_{m\times n}$.

Example 6.2. Let $\hat{\Pi} = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$ & $\mathfrak{A} = \{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5\}$. Then $\Theta(\hat{\theta}_1) = \{\varpi_1, \varpi_2\}$, $\Theta(\hat{\theta}_2) = \varnothing$, $\Theta(\hat{\theta}_3) = \{\varpi_4, \varpi_5\}$, $\Theta(\hat{\theta}_4) = \{\varpi_2, \varpi_3, \varpi_4, \}$, $\Theta(\hat{\theta}_5) = \varnothing$, therefore we get $(\Theta, \mathfrak{A}) = \left\{ \ (\hat{\theta}_1, \{\varpi_1, \varpi_2\}), (\hat{\theta}_3, \{\varpi_4, \varpi_5\}), (\hat{\theta}_4, \{\varpi_2, \varpi_3, \varpi_4, \}) \ \right\}$ &
$\mathbb{R}_{\mathfrak{A}} = \left\{ \ (\varpi_1, \hat{\theta}_1), (\varpi_2, \hat{\theta}_1), (\varpi_4, \hat{\theta}_3), (\varpi_5, \hat{\theta}_3), (\varpi_2, \hat{\theta}_4), (\varpi_3, \hat{\theta}_4), (\varpi_4, \hat{\theta}_4) \ \right\}$. Hence $\hat{H}s$- matrix is given as

$$(\ddot{s}_{ij})_{5\times 5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} {}_1i^5, {}_1j^5.$$

Definition 6.3. Let $(\ddot{s}_{ij})_{m\times n} \in (\hat{\Pi})^{(hsm)}_{m\times n}$ then $(\ddot{s}_{ij})_{m\times n}$ is characterized as:

(i) The $(0)_{m\times n}$ is stated as a null $\hat{H}s$- matrix if $\ddot{s}_{ij} = 0 \ \forall \ i, j$ e.g.

$$(0)_{5\times 5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} {}_1i^5, {}_1j^5.$$

(ii) An $\mathfrak{A}_1$-*universal* $\hat{H}s$- *matrix*, symbolized by $(\ddot{s}_{ij})^{\mathfrak{A}_1}_{m\times n}$, if $\ddot{s}_{ij}=1, \forall j \in J_{\mathfrak{A}_1}=\{j:\hat{\theta}_j \in \mathfrak{A}_1\}$ & i. E.g. Let $\mathfrak{A}$ be as provided in 6.2 & $\mathfrak{A}_1=\{\hat{\theta}_2,\hat{\theta}_4,\hat{\theta}_5\}\subseteq\mathfrak{A}$ with $\Theta(\hat{\theta}_2)=\Theta(\hat{\theta}_4)=\Theta(\hat{\theta}_5)=\hat{\Pi}$ then

$$(\ddot{s}_{ij})^{\mathfrak{A}_1}_{5\times 5}=\begin{pmatrix} 0&1&0&1&1\\ 0&1&0&1&1\\ 0&1&0&1&1\\ 0&1&0&1&1\\ 0&1&0&1&1 \end{pmatrix}\ {}_1i^5,\ {}_1j^5.$$

(iii) The $(\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n}$ is stated as universal $\hat{H}s$- matrix if $\ddot{s}_{ij}=1, \forall\ i,j$. E.g. Let $\mathfrak{A}$ as stated in 6.2 with $\Theta(\hat{\theta}_1)=\Theta(\hat{\theta}_2)=\Theta(\hat{\theta}_3)=\Theta(\hat{\theta}_4)=\Theta(\hat{\theta}_5)=\hat{\Pi}$ then

$$(\ddot{s}_{ij})^{\hat{\Pi}}_{5\times 5}=\begin{pmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{pmatrix}\ {}_1i^5,\ {}_1j^5.$$

Definition 6.4. Let $\mathfrak{L}_1=(\ddot{s}_{ij})_{m\times n}, \mathfrak{L}_2=\left(\ddot{t}_{ij}\right)_{m\times n}\in(\hat{\Pi})^{(hsm)}_{m\times n}$ then

(a) $\mathfrak{L}_1$ is stated as $\hat{H}s$- *sub-matrix* of $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1\subseteq\mathfrak{L}_2$ if $\ddot{s}_{ij}\le\ddot{t}_{ij}$ e.g. $\mathfrak{L}_1=\begin{pmatrix} 0&1&0&1&1\\ 0&1&0&1&1\\ 0&1&0&1&1\\ 0&1&0&1&1\\ 0&1&0&1&1 \end{pmatrix}$ & $\mathfrak{L}_2=\begin{pmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{pmatrix}$.

(b) $\mathfrak{L}_1$ & $\mathfrak{L}_2$ are stated as comparable, symbolized by $\mathfrak{L}_1\parallel\mathfrak{L}_2$, if $\mathfrak{L}_1\subseteq\mathfrak{L}_2$ or $\mathfrak{L}_2\subseteq\mathfrak{L}_1$.

(c) $\mathfrak{L}_1$ is stated as *proper* $\hat{H}s$- *sub-matrix* of $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1\subset\mathfrak{L}_2$ if for atleast one term $\ddot{s}_{ij}\le\ddot{t}_{ij}$ e.g. $\mathfrak{L}_1=\begin{pmatrix} 1&1&0&1&1\\ 1&1&1&1&1\\ 0&1&0&1&1\\ 1&1&1&1&1\\ 0&1&0&1&1 \end{pmatrix}$ & $\mathfrak{L}_2=\begin{pmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{pmatrix}$.

(d) $\mathfrak{L}_1$ is stated as *strictly* $\hat{H}s$- *sub-matrix* of $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1\subsetneq\mathfrak{L}_2$ if for each term $\ddot{s}_{ij}\le\ddot{t}_{ij}$ e.g. $\mathfrak{L}_1=\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$ & $\mathfrak{L}_2=\begin{pmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{pmatrix}$.

(e) *union* of $\mathfrak{L}_1$ & $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1\cup\mathfrak{L}_2$, is also a $\hat{H}s$- matrix $\mathfrak{L}_3=(\delta_{ij})_{m\times n}$ if $\delta_{ij}=max\{\ddot{s}_{ij},\ddot{t}_{ij}\}\ \forall\ i,j$ e.g.

Let $\mathfrak{L}_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ then

$$\mathfrak{L}_3 = \mathfrak{L}_1 \cup \mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

(f) *intersection* of $\mathfrak{L}_1$ & $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1 \cap \mathfrak{L}_2$, is also a $\hat{H}s$- matrix $\mathfrak{L}_3 = (\delta_{ij})_{m\times n}$ if $\delta_{ij} = min\{\ddot{s}_{ij}, \ddot{t}_{ij}\}\ \forall\ i, j$ e.g.

Let $\mathfrak{L}_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ then

$$\mathfrak{L}_3 = \mathfrak{L}_1 \cap \mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

(g) The $\mathfrak{L}^{©}$ $(\mu_{ij})_{m\times n}$ (complement of $\mathfrak{L} = (\ddot{s}_{ij})_{m\times n}$), is also a $\hat{H}s$- matrix if $\mu_{ij} = 1 - \ddot{s}_{ij}\ \forall\ i, j$ e.g.

Let $\mathfrak{L} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ then $\mathfrak{L}^{©} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$.

(h) The difference $\mathfrak{L}_2 \setminus \mathfrak{L}_1$, is also a $\hat{H}s$- matrix $\mathfrak{L}_3$ such that $\mathfrak{L}_3 = \mathfrak{L}_2 \cap \mathfrak{L}_1^{©}$ e.g.

$\mathfrak{L}_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ then

$\mathfrak{L}_3 = \mathfrak{L}_2 \cap \mathfrak{L}_1^{©}$

$$\mathfrak{L}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Proposition 6.5. *For $\mathfrak{C}_1 = (\ddot{s}_{ij})_{m\times n}, \mathfrak{C}_2 = (\ddot{t}_{ij})_{m\times n}, \mathfrak{C}_3 = (\ddot{u}_{ij})_{m\times n} \in (\hat{\Pi})^{(hsm)}_{m\times n}$, the following axiomatic results are valid:*

(1) $\mathfrak{C}_1 \cup \mathfrak{C}_1 = \mathfrak{C}_1, \;\; \mathfrak{C}_1 \cap \mathfrak{C}_1 = \mathfrak{C}_1$

(2) $\mathfrak{C}_1 \cup (0)_{m\times n} = \mathfrak{C}_1, \;\; \mathfrak{C}_1 \cap (\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n} = \mathfrak{C}_1$

(3) $\mathfrak{C}_1 \cap (0)_{m\times n} = (0)_{m\times n}, \;\; \mathfrak{C}_1 \cup (\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n} = (\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n}$

(4) $((0)_{m\times n})^{©} = (\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n}, \;\; ((\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n})^{©} = (0)_{m\times n}$

(5) $\mathfrak{C}_1 \cup \mathfrak{C}_1{}^{©} = (\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n}, \;\; \mathfrak{C}_1 \cap \mathfrak{C}_1{}^{©} = (0)_{m\times n}$

(6) $(\mathfrak{C}_1 \cup \mathfrak{C}_2)^{©} = \mathfrak{C}_1{}^{©} \cap \mathfrak{C}_2{}^{©}, \;\; (\mathfrak{C}_1 \cap \mathfrak{C}_2)^{©} = \mathfrak{C}_1{}^{©} \cup \mathfrak{C}_2{}^{©}$

(7) $(\mathfrak{C}_1{}^{©})^{©} = \mathfrak{C}_1$

(8) $\mathfrak{C}_1 \cup \mathfrak{C}_2 = \mathfrak{C}_2 \cup \mathfrak{C}_1, \;\; \mathfrak{C}_1 \cap \mathfrak{C}_2 = \mathfrak{C}_2 \cap \mathfrak{C}_1$

(9) $\mathfrak{C}_1 \cup (\mathfrak{C}_2 \cup \mathfrak{C}_3) = (\mathfrak{C}_1 \cup \mathfrak{C}_2) \cup \mathfrak{C}_3, \;\; \mathfrak{C}_1 \cap (\mathfrak{C}_2 \cap \mathfrak{C}_3) = (\mathfrak{C}_1 \cap \mathfrak{C}_2) \cap \mathfrak{C}_3$

(10) $\mathfrak{C}_1 \cup (\mathfrak{C}_2 \cap \mathfrak{C}_3) = (\mathfrak{C}_1 \cup \mathfrak{C}_2) \cap (\mathfrak{C}_1 \cup \mathfrak{C}_3), \;\; \mathfrak{C}_1 \cap (\mathfrak{C}_2 \cup \mathfrak{C}_3) = (\mathfrak{C}_1 \cap \mathfrak{C}_2) \cup (\mathfrak{C}_1 \cap \mathfrak{C}_3)$

Definition 6.6. Let $\mathfrak{P} = (\ddot{c}_{ij})_{m\times n}, \;\; \mathfrak{Q} = (\ddot{d}_{ik})_{m\times n} \in (\hat{\Pi})^{(hsm)}_{m\times n}$, then

(i) *AND-product* is stated as
$\wedge : (\hat{\Pi})^{(hsm)}_{m\times n} \times (\hat{\Pi})^{(hsm)}_{m\times n} \to (\hat{\Pi})^{(hsm)}_{m\times n^2}$ with $(\ddot{c}_{ij}) \wedge (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = min\{\ddot{c}_{ij}, \ddot{d}_{ik}\}$ & $l = n(j-1)+k$.

(ii) *OR-product* is stated as
$\vee : (\hat{\Pi})^{(hsm)}_{m\times n} \times (\hat{\Pi})^{(hsm)}_{m\times n} \to (\hat{\Pi})^{(hsm)}_{m\times n^2}$ with $(\ddot{c}_{ij}) \vee (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = max\{\ddot{c}_{ij}, \ddot{d}_{ik}\}$.

(iii) *AND-NOT-product* is stated as
$\overline{\wedge} : (\hat{\Pi})^{(hsm)}_{m\times n} \times (\hat{\Pi})^{(hsm)}_{m\times n} \to (\hat{\Pi})^{(hsm)}_{m\times n^2}$ with $(\ddot{c}_{ij}) \overline{\wedge} (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = min\{\ddot{c}_{ij}, 1-\ddot{d}_{ik}\}$.

(iv) *OR-NOT-product* is stated as
$\underline{\vee} : (\hat{\Pi})^{(hsm)}_{m\times n} \times (\hat{\Pi})^{(hsm)}_{m\times n} \to (\hat{\Pi})^{(hsm)}_{m\times n^2}$ with $(\ddot{c}_{ij}) \underline{\vee} (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = max\{\ddot{c}_{ij}, 1-\ddot{d}_{ik}\}$.

Example 6.7. Let $\mathfrak{P} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ & $\mathfrak{Q} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ then

(i) $$\mathfrak{P} \wedge \mathfrak{Q} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\text{(ii)}\ \mathfrak{P} \vee \mathfrak{Q} = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$

$$\text{(iii)}\ \mathfrak{P} \barwedge \mathfrak{Q} = \begin{pmatrix} 1&1&0&1 \\ 1&1&1&1 \\ 0&1&0&1 \\ 1&1&1&1 \end{pmatrix} \wedge \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\text{(iv)}\ \mathfrak{P} \veebar \mathfrak{Q} = \begin{pmatrix} 1&1&0&1 \\ 1&1&1&1 \\ 0&1&0&1 \\ 1&1&1&1 \end{pmatrix} \vee \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$

7. Hybridized Structures of $\hat{H}s$-sets

Here the notions of some hybridized model of $\hat{H}s$-sets are presented. The set $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots\ldots \times \mathfrak{A}_m$ with $\mathfrak{A}_{\hat{\alpha}} \cap \mathfrak{A}_{\hat{\beta}} = \varnothing\ \forall\ \hat{\alpha}, \hat{\beta} = 1, 2, ..., m$ and $\mathfrak{A}_{\hat{\alpha}}$ are same as stated in Definition 3.1. The Figure 3 presents the notations and their full names that are used in this section.

Abbreviations	Used for	Abbreviations	Used for
$iv\hat{f}Hs$-set	interval-valued fuzzy hypersoft set	ivf-set	interval-valued fuzzy set
$\mathcal{F}^{ivf}(\hat{\Pi})$	collection of interval-valued fuzzy subsets over $\hat{\Pi}$	$fphs$-set	fuzzy parameterized hypersoft set
maa-function	multi-argument approximate function	iv-$fphs$-set	interval-valued fuzzy parameterized hypersoft set
$ifphs$-set	intuitionistic fuzzy parameterized hypersoft set	$nphs$-set	neutrosophic parameterized hypersoft set
if-set	intuitionistic fuzzy set	n-set	neutrosophic set

FIGURE 3. Notations

Definition 7.1. An $iv\hat{f}Hs$*-set* $(\Gamma, \mathfrak{A})$ on $\hat{\Pi}$ is stated by

$$(\Gamma, \mathfrak{A}) = \left\{ \ (\hat{\theta}, \Gamma(\hat{\theta})); \hat{\theta} \in \mathfrak{A}, \Gamma(\hat{\theta}) \in \mathcal{F}^{ivf}(\hat{\Pi}) \ \right\}$$

where $\Gamma : \mathfrak{A} \rightarrow \mathcal{F}^{ivf}(\hat{\Pi})$ & $\Gamma(\hat{\theta}) = \left\{ \ddot{\psi}_{\Gamma(\hat{\theta})}(\varpi)/\varpi : \varpi \in \hat{\Pi}, \ddot{\psi}_{\Gamma(\hat{\theta})}(\varpi) \in \mathbb{C}(\mathbb{I}) \right\}$ is an ivf-set on $\hat{\Pi}$.

Example 7.2. Let $\hat{\Pi} = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6, \varpi_7, \varpi_8\}$ & $\mathfrak{A} = \{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5, \hat{\theta}_6, \hat{\theta}_7, \hat{\theta}_8\}$, $iv\hat{f}Hs$-set $(\Gamma, \mathfrak{A})$ is constructed as

$$(\Gamma, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{[0.1,0.2]/\varpi_1, [0.2,0.3]/\varpi_2, [0.4,0.5]/\varpi_4, [0.5,0.6]/\varpi_5\}\right), \\ \left(\hat{\theta}_2, \{[0.1,0.3]/\varpi_1, [0.2,0.4]/\varpi_2, [0.3,0.4]/\varpi_3, [0.6,0.8]/\varpi_6\}\right), \\ \left(\hat{\theta}_3, \{[0.2,0.3]/\varpi_2, [0.3,0.4]/\varpi_3, [0.4,0.5]/\varpi_4, , [0.5,0.7]/\varpi_5\}\right), \\ \left(\hat{\theta}_4, \{[0.4,0.5]/\varpi_4, [0.5,0.6]/\varpi_5, [0.6,0.7]/\varpi_6, [0.7,0.8]/\varpi_7\}\right), \\ \left(\hat{\theta}_5, \{[0.3,0.6]/\varpi_3, [0.6,0.7]/\varpi_6, [0.7,0.8]/\varpi_7, [0.8,0.9]/\varpi_8\}\right), \\ \left(\hat{\theta}_6, \{[0.2,0.4]/\varpi_2, [0.3,0.5]/\varpi_3, [0.4,0.6]/\varpi_4, [0.7,0.8]/\varpi_7\}\right), \\ \left(\hat{\theta}_7, \{[0.1,0.4]/\varpi_1, [0.3,0.4]/\varpi_3, [0.5,0.7]/\varpi_5, [0.6,0.8]/\varpi_6\}\right), \\ \left(\hat{\theta}_8, \{[0.2,0.5]/\varpi_2, [0.3,0.6]/\varpi_3, [0.6,0.8]/\varpi_6, [0.7,0.8]/\varpi_7\}\right) \end{array} \right\}$$

Definition 7.3. A $fphs$-set $(\mathcal{D}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{D}, \mathfrak{A}) = \left\{ \left(\varphi_{\mathcal{F}}(\hat{\theta})/\hat{\theta}, \Theta_{\mathcal{F}}(\hat{\theta})\right), \hat{\theta} \in \mathfrak{A},\ \Theta_{\mathcal{F}}(\hat{\theta}) \in P^{\hat{\Pi}},\ \varphi_{\mathcal{F}}(\hat{\theta}) \in \mathbb{C}(\mathbb{I}) \right\}$$

where $\mathcal{F}$ is a fuzzy set with $\varphi_{\mathcal{F}} : \mathfrak{A} \rightarrow \mathbb{C}(\mathbb{I})$ as membership function of $fphs$-set & $\Theta_{\mathcal{F}} : \mathfrak{A} \rightarrow P^{\hat{\Pi}}$ is maa-function of $fphs$-set.

Example 7.4. From Example 7.2, we get

$$(\mathcal{D}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(0.1/\hat{\theta}_1, \{\varpi_1, \varpi_2\}\right), \left(0.2/\hat{\theta}_2, \{\varpi_1, \varpi_2, \varpi_3\}\right), \left(0.3/\hat{\theta}_3, \{\varpi_2, \varpi_3, \varpi_4\}\right), \\ \left(0.4/\hat{\theta}_4, \{\varpi_4, \varpi_5, \varpi_6\}\right), \left(0.5/\hat{\theta}_5, \{\varpi_6, \varpi_7, \varpi_8\}\right), \left(0.6/\hat{\theta}_6, \{\varpi_2, \varpi_3, \varpi_4, \varpi_7\}\right), \\ \left(0.7/\hat{\theta}_7, \{\varpi_1, \varpi_3, \varpi_5, \varpi_6\}\right), \left(0.8/\hat{\theta}_8, \{\varpi_2, \varpi_3, \varpi_6, \varpi_7\}\right) \end{array} \right\}$$

Definition 7.5. An iv-$fphs$-set $(\mathcal{E}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{E}, \mathfrak{A}) = \left\{ \left(\Psi_{\mathcal{F}^{iv}}(\hat{\theta})/\hat{\theta}, \psi_{\mathcal{F}^{iv}}(\hat{\theta})\right), \hat{\theta} \in \mathfrak{A},\ \psi_{\mathcal{F}_{iv}}(\hat{\theta}) \in P^{\hat{\Pi}},\ \Psi_{\mathcal{F}^{iv}}(\hat{\theta}) \in \mathbb{C}(\mathbb{I}) \right\}$$

where $\mathcal{F}^{iv}$ is an ivf-set with $\Psi_{\mathcal{F}^{iv}} : \mathfrak{A} \rightarrow \mathbb{C}(\mathbb{I})$ as membership function of $fphs$-set and $\psi_{\mathcal{F}_{iv}} : \mathfrak{A} \rightarrow P^{\hat{\Pi}}$ is maa-function of iv-$fphs$-set.

Example 7.6. From Example 7.2, we get

$$(\mathcal{E}, \mathfrak{A}) = \left\{ \begin{array}{l} \left([0.1,0.2]/\hat{\theta}_1, \{\varpi_1, \varpi_2\}\right), \left([0.2,0.3]/\hat{\theta}_2, \{\varpi_1, \varpi_2, \varpi_3\}\right), \\ \left([0.3,0.4]/\hat{\theta}_3, \{\varpi_2, \varpi_3, \varpi_4\}\right), \left([0.4,0.5]/\hat{\theta}_4, \{\varpi_4, \varpi_5, \varpi_6\}\right), \\ \left([0.5,0.6]/\hat{\theta}_5, \{\varpi_6, \varpi_7, \varpi_8\}\right), \left([0.6,0.7]/\hat{\theta}_6, \{\varpi_2, \varpi_3, \varpi_4, \varpi_7\}\right), \\ \left([0.7,0.8]/\hat{\theta}_7, \{\varpi_1, \varpi_3, \varpi_5, \varpi_6\}\right), \left([0.8,0.9]/\hat{\theta}_8, \{\varpi_2, \varpi_3, \varpi_6, \varpi_7\}\right) \end{array} \right\}$$

Definition 7.7. An $ifphs$-set $(\mathcal{H}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{H}, \mathfrak{A}) = \left\{ \left(< \varsigma_1(\hat{\theta}), \varsigma_2(\hat{\theta}) > /\hat{\theta}, \psi^{\mathcal{IF}}(\hat{\theta})\right); \hat{\theta} \in \mathfrak{A},\ \psi^{\mathcal{IF}}(\hat{\theta}) \in P^{\hat{\Pi}},\ \varsigma_1(\hat{\theta}), \varsigma_2(\hat{\theta}) \in \mathbb{C}(\mathbb{I}) \right\}$$

where $\mathcal{IF}$ is an if-set with $\varsigma_1(\hat{\theta}), \varsigma_2(\hat{\theta}) : \mathfrak{A} \rightarrow \mathbb{C}(\mathbb{I})$ as membership and non-membership functions of $ifphs$-set and $\psi^{\mathcal{IF}} : \mathfrak{A} \rightarrow P^{\hat{\Pi}}$ is maa-function of $ifphs$-set.

Example 7.8. From Example 7.2, we get

$$(\mathcal{H},\mathfrak{A}) = \left\{ \begin{array}{l} \left(<0.1,0.2>/\hat{\theta}_1,\{\varpi_1,\varpi_2\}\right),\left(<0.2,0.3>/\hat{\theta}_2,\{\varpi_1,\varpi_2,\varpi_3\}\right), \\ \left(<0.3,0.4>/\hat{\theta}_3,\{\varpi_2,\varpi_3,\varpi_4\}\right),\left(<0.4,0.5>/\hat{\theta}_4,\{\varpi_4,\varpi_5,\varpi_6\}\right), \\ \left(<0.5,0.6>/\hat{\theta}_5,\{\varpi_6,\varpi_7,\varpi_8\}\right),\left(<0.6,0.7>/\hat{\theta}_6,\{\varpi_2,\varpi_3,\varpi_4,\varpi_7\}\right), \\ \left(<0.7,0.8>/\hat{\theta}_7,\{\varpi_1,\varpi_3,\varpi_5,\varpi_6\}\right),\left(<0.8,0.9>/\hat{\theta}_8,\{\varpi_2,\varpi_3,\varpi_6,\varpi_7\}\right) \end{array} \right\}$$

Definition 7.9. A *nphs*-set $(\mathcal{N},\mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{N},\mathfrak{A}) = \left\{ \begin{array}{l} \left(<\lambda_1(\hat{\theta}),\lambda_2(\hat{\theta}),\lambda_3(\hat{\theta})>/\hat{\theta},\psi^{\mathcal{N}}(\hat{\theta})\right);\hat{\theta}\in\mathfrak{A},\ \psi^{\mathcal{N}}(\hat{\theta})\in P^{\hat{\Pi}}, \\ \lambda_1(\hat{\theta})\in\mathbb{C}(\mathbb{I}),\lambda_2(\hat{\theta})\in\mathbb{C}(\mathbb{I}),\lambda_3(\hat{\theta})\in\mathbb{C}(\mathbb{I}) \end{array} \right\}$$

where $\mathcal{N}$ is a neutrosophic set with $\lambda_1(\hat{\theta}),\lambda_2(\hat{\theta}),\lambda_3(\hat{\theta}):\mathfrak{A}\to\mathbb{C}(\mathbb{I})$ as membership, indeterminate and falsity of *nphs*-set and $\psi^{\mathcal{N}}:\mathfrak{A}\to P^{\hat{\Pi}}$ is maa-function of *nphs*-set.

Example 7.10. From Example 7.2, we get

$$(\mathcal{N},\mathfrak{A}) = \left\{ \begin{array}{l} \left(<0.1,0.2,0.2>/\hat{\theta}_1,\{\varpi_1,\varpi_2\}\right),\left(<0.2,0.3,0.3>/\hat{\theta}_2,\{\varpi_1,\varpi_2,\varpi_3\}\right), \\ \left(<0.3,0.4,0.4>/\hat{\theta}_3,\{\varpi_2,\varpi_3,\varpi_4\}\right),\left(<0.4,0.5,0.5>/\hat{\theta}_4,\{\varpi_4,\varpi_5,\varpi_6\}\right), \\ \left(<0.5,0.6,0.6>/\hat{\theta}_5,\{\varpi_6,\varpi_7,\varpi_8\}\right),\left(<0.6,0.7,0.7>/\hat{\theta}_6,\{\varpi_2,\varpi_3,\varpi_4,\varpi_7\}\right), \\ \left(<0.7,0.5,0.8>/\hat{\theta}_7,\{\varpi_1,\varpi_3,\varpi_5,\varpi_6\}\right),\left(<0.8,0.4,0.9>/\hat{\theta}_8,\{\varpi_2,\varpi_3,\varpi_6,\varpi_7\}\right) \end{array} \right\}$$

Definition 7.11. A $\hat{H}s$-set $(\ddot{\mathfrak{B}},\mathfrak{A})$ is known as *bijective* $\hat{H}s$*-set* (*bhs*-set) on $\hat{\Pi}$ if

(i) $\bigcup_{j\in\mathfrak{A}}\ddot{\mathfrak{B}}(\hat{\theta})=\hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}},\hat{\theta}_{\hat{\beta}}\in\mathfrak{A},\hat{\alpha}\neq\hat{\beta},\ddot{\mathfrak{B}}(\hat{\theta}_{\hat{\alpha}})\cap\ddot{\mathfrak{B}}(\hat{\theta}_{\hat{\beta}})=\varnothing$

Example 7.12. Reconsidering Example 7.2, we get

$$(\ddot{\mathfrak{B}},\mathfrak{A}) = \left\{ \left(\hat{\theta}_1,\{\varpi_1\}\right),\left(\hat{\theta}_2,\{\varpi_2\}\right),\left(\hat{\theta}_3,\{\varpi_3\}\right),\left(\hat{\theta}_4,\{\varpi_4\}\right),\left(\hat{\theta}_5,\{\varpi_5\}\right),\left(\hat{\theta}_6,\{\varpi_6\}\right),\left(\hat{\theta}_7,\{\varpi_7\}\right),\left(\hat{\theta}_8,\{\varpi_8\}\right) \right\}$$

Definition 7.13. A $f\hat{h}s$-set $(\ddot{\mathfrak{B}}^f,\mathfrak{A})$ is stated as *bijective* $f\hat{h}s$*-set* on $\hat{\Pi}$ if

(i) $\bigcup_{\hat{\theta}\in\mathfrak{A}}\ddot{\mathfrak{B}}^f(\hat{\theta})=\hat{\Pi}$ with $\sum_{\varpi\in\hat{\Pi}}\ddot{\psi}_f(\varpi)\in\mathbb{C}(\mathbb{I})$ where $\ddot{\psi}_f(\varpi)$ is a f-membership for each $\varpi\in\hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}},\hat{\theta}_{\hat{\beta}}\in\mathfrak{A},\hat{\alpha}\neq\hat{\beta},\ddot{\mathfrak{B}}^f(\hat{\theta}_{\hat{\alpha}})\cap\ddot{\mathfrak{B}}^f(\hat{\theta}_{\hat{\beta}})=\varnothing$

Example 7.14. Reconsidering Example 7.2, we get

$$(\ddot{\mathfrak{B}}^f,\mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1,\{0.1/\varpi_1\}\right),\left(\hat{\theta}_2,\{0.2/\varpi_2\}\right), \\ \left(\hat{\theta}_3,\{0.13/\varpi_3\}\right),\left(\hat{\theta}_4,\{0.14/\varpi_4\}\right), \\ \left(\hat{\theta}_5,\{0.05/\varpi_5\}\right),\left(\hat{\theta}_6,\{0.06/\varpi_6\}\right), \\ \left(\hat{\theta}_7,\{0.07/\varpi_7\}\right),\left(\hat{\theta}_8,\{0.08/\varpi_8\}\right) \end{array} \right\}$$

Definition 7.15. An $iv\hat{f}hs$-set $(\ddot{\mathfrak{B}}^{ivf},\mathfrak{A})$ is stated as *bijective* $iv\hat{f}hs$*-set* on $\hat{\Pi}$ if

(i) $\bigcup_{\hat{\theta}\in\mathfrak{A}}\ddot{\mathfrak{B}}^{ivf}(\hat{\theta})=\hat{\Pi}$ with $\sum_{\varpi\in\hat{\Pi}}Sup(\ddot{\psi}_f(\varpi))\in\mathbb{C}(\mathbb{I})$ where $\ddot{\psi}_f(\varpi)$ is an ivf-membership for each $\varpi\in\hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}}, \hat{\theta}_{\hat{\beta}} \in \mathfrak{A}, \hat{\alpha} \neq \hat{\beta}, \ddot{\mathfrak{B}}^{ivf}(\hat{\theta}_{\hat{\alpha}}) \cap \ddot{\mathfrak{B}}^{ivf}(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.16. Reconsidering Example 7.2, we get

$$(\ddot{\mathfrak{B}}^{ivf}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{[0.01, 0.1]/\varpi_1\}\right), \left(\hat{\theta}_2, \{[0.02, 0.2]/\varpi_2\}\right), \\ \left(\hat{\theta}_3, \{[0.03, 0.13]/\varpi_3\}\right), \left(\hat{\theta}_4, \{[0.04, 0.14]/\varpi_4\}\right), \\ \left(\hat{\theta}_5, \{[0.03, 0.05]/\varpi_5\}\right), \left(\hat{\theta}_6, \{[0.02, 0.06]/\varpi_6\}\right), \\ \left(\hat{\theta}_7, \{[0.03, 0.07]/\varpi_7\}\right), \left(\hat{\theta}_8, \{[0.04, 0.08]/\varpi_8\}\right) \end{array} \right\}$$

Definition 7.17. An $if\hat{h}s$-set $(\ddot{\mathfrak{B}}^{if}, \mathfrak{A})$ is known as *bijective* $if\hat{h}s$*-set* on $\hat{\Pi}$ if

(i) $\bigcup_{\hat{\theta}\in\mathfrak{A}} \ddot{\mathfrak{B}}^{if}(\hat{\theta}) = \hat{\Pi}$ with $\sum_{\varpi\in\hat{\Pi}} T_{if}(\varpi)$ & $\sum_{\varpi\in\hat{\Pi}} F_{if}(\varpi) \in \mathbb{C}(\mathbb{I})$ where $T_{if}(\varpi)$ & $F_{if}(\varpi)$ are membership and non-membership grades for each $\varpi \in \hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}}, \hat{\theta}_{\hat{\beta}} \in \mathfrak{A}, \hat{\alpha} \neq \hat{\beta}, \ddot{\mathfrak{B}}^{if}(\hat{\theta}_{\hat{\alpha}}) \cap \ddot{\mathfrak{B}}^{if}(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.18. Reassuming Example 7.2, we get

$$(\ddot{\mathfrak{B}}^{if}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{< 0.01, 0.1 > /\varpi_1\}\right), \left(\hat{\theta}_2, \{< 0.02, 0.2 > /\varpi_2\}\right), \\ \left(\hat{\theta}_3, \{< 0.03, 0.13 > /\varpi_3\}\right), \left(\hat{\theta}_4, \{< 0.04, 0.14 > /\varpi_4\}\right), \\ \left(\hat{\theta}_5, \{< 0.03, 0.05 > /\varpi_5\}\right), \left(\hat{\theta}_6, \{< 0.02, 0.06 > /\varpi_6\}\right), \\ \left(\hat{\theta}_7, \{< 0.03, 0.07 > /\varpi_7\}\right), \left(\hat{\theta}_8, \{< 0.04, 0.08 > /\varpi_8\}\right) \end{array} \right\}$$

Definition 7.19. A $n\hat{h}s$-set $(\ddot{\mathfrak{B}}^{\mathcal{N}}, \mathfrak{A})$ is known as *bijective* $n\hat{h}s$*-set* on $\hat{\Pi}$ if

(i) $\bigcup_{\hat{\theta}\in\mathfrak{A}} \ddot{\mathfrak{B}}^{\mathcal{N}}(\hat{\theta}) = \hat{\Pi}$ with $\sum_{\varpi\in\hat{\Pi}} T_{\mathcal{N}}(\varpi), \sum_{\varpi\in\hat{\Pi}} I_{\mathcal{N}}(\varpi)$ & $\sum_{\varpi\in\hat{\Pi}} F_{\mathcal{N}}(\varpi) \in \mathbb{C}(\mathbb{I})$ where $T_{\mathcal{N}}(\varpi), I_{\mathcal{N}}(\varpi)$ & $F_{\mathcal{N}}(\varpi)$ are membership, indeterminacy and non-membership grades for each $\varpi \in \hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}}, \hat{\theta}_{\hat{\beta}} \in \mathfrak{A}, \hat{\alpha} \neq \hat{\beta}, \ddot{\mathfrak{B}}^{\mathcal{N}}(\hat{\theta}_{\hat{\alpha}}) \cap \ddot{\mathfrak{B}}^{\mathcal{N}}(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.20. Reassuming Example 7.2, we get

$$(\ddot{\mathfrak{B}}^{\mathcal{N}}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{< 0.01, 0.02, 0.1 > /\varpi_1\}\right), \left(\hat{\theta}_2, \{< 0.02, 0.03, 0.2 > /\varpi_2\}\right), \\ \left(\hat{\theta}_3, \{< 0.03, 0.04, 0.13 > /\varpi_3\}\right), \left(\hat{\theta}_4, \{< 0.04, 0.05, 0.14 > /\varpi_4\}\right), \\ \left(\hat{\theta}_5, \{< 0.03, 0.04, 0.05 > /\varpi_5\}\right), \left(\hat{\theta}_6, \{< 0.02, 0.05, 0.06 > /\varpi_6\}\right), \\ \left(\hat{\theta}_7, \{< 0.03, 0.04, 0.07 > /\varpi_7\}\right), \left(\hat{\theta}_8, \{< 0.04, 0.05, 0.08 > /\varpi_8\}\right) \end{array} \right\}$$

8. Conclusions

In this research work, several important rudiments (i.e. axioms-based properties, set-based aggregations etc.,) of $\hat{H}s$-set are investigated and explained with the support of real-scenarios based examples. In order to attract the intellectual attention of researchers, definitions of some glued models of $\hat{H}s$-set are also presented which will motivate them to extend the theory to other branches of mathematical-cum-computational sciences. Some future directions and scope of $\hat{H}s$-sets are presented in Figure 4.

Conflicts of Interest: The authors declare no conflict of interest.

Discipline	Scope
Fuzzy sets and systems	Development of intutionistic neutrosophic hypersoft set , spherical hypersoft set, picture fuzzy hypersoft set, geometric hypersoft set etc.
Graph Theory	Development of fuzzy hypersoft graph, intutionistic fuzzy hypersoft graph, neutrosophic fuzzy hypersoft graph, intutionistic neutrosophic hypersoft graph, possibility fuzzy hypersoft graph, possibility intutionistic fuzzy hypersoft graph, possibility neutrosophic hypersoft graph etc.
Algebra	Development of hypersoft groups, hypersoft rings, hypersoft vector spaces and their related structures.
Functional Analysis	Characterization of hypersoft metric spaces, hypersoft inner product spaces, normed hypersoft spaces, hypersoft measure theory, hypersoft Hilbert spaces etc.
Topology	Characterization of topological spaces, separation axioms, connected spaces and their relevant spaces.
Mathematical Analysis	Development of hypersoft fixed point theory, hypersoft real analysis, hypersoft modular inequalities, hypersoft complex analysis etc.

Figure 4. Future Directions and Scope of $\hat{H}s$-sets

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Neutrosophic DEMATEL approach for financial ratio performance evaluation of the NASDAQ Exchange

C.Veeramani[1]*, R.Venugopal[2] and S.A.Edalatpanah[3]

[1]Department of Applied Science(Mathematics), PSG College of Technology, Coimbatore- 641 004, India; veerasworld@yahoo.com

[2]Department of Applied Science(Mathematics), PSG College of Technology, Coimbatore - 641 004, India; venusm80@gmail.com

[3]Department of Mathematics, University of Guilan & Ayandegan Institute of Higher Education, Iran; saedalatpanah@gmail.com

*Correspondence: veerasworld@yahoo.com

Abstract. The relative performance analysis and ranking of the financial ratios are highly important for optimal portfolio selection in the stock market. However, the relative performance evaluation of the financial ratios is highly complex and nonlinear. Thus, the main goal of this study is to measure the relative importance of the financial ratios of two groups as Accounting based financial measures (AFM) and Economic value-based financial measures (EFM) through Decision Making Trial and Evaluation Laboratory (DEMATEL) method under the neutrosophic environment. In this regard, one-year data (June 2018-May 2019) has been collected from 8 industries in the IT sector. The AFM and EFM values have been evaluated for each firm through the balance sheet. The obtained values have been given to the two experts: an experienced investor in the NASDAQ exchange and a Professor in Finance. They have given their opinion in terms of linguistic terms. Then, the AFM and EFM have ranked based on the neutrosophic DEMATEL approach. Finally, the neutrosophic DEMATEL approach has compared with the fuzzy DEMATEL and classical DEMATEL approach. The empirical results assist the investor and traders in selecting among the selected stock.

Keywords: Neutrosophic number; Neutrosophic DEMATEL; Accounting based financial measures; Economic value-based financial measures

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1. **Introduction**

The performance assessment of the companies is usually carried out in the context of financial analysis. From a financial point of view, the notion of performance is defined as terms such as profit, profitability, production and economic growth, and so on. The use of financial ratios in the performance review process may be useful for all businesses and relevant industries. Financial ratios obtained from the data in the balance sheets and income statements are considered to be important metrics for assessing the output and financial assets of businesses. A large number of studies have been underway for many years (Chen and Shimerda [10], Halkos and Tzeremes [14], etc.) that prove that financial ratios are crucial indicators of the financial performance of firms. They allow users to review and analyze relevant data in order to provide useful information for decision-making. Singh and Schmidgall [24] have shown that the value of the financial ratios also illustrates the strengths and weaknesses sides of the company in terms of flexibility, productivity, and profitability. The financial ratios also measure the various funding aspects of the stock and influence the movement of the stock price [25]. As the financial performance measures demonstrate the productivity of the company and competitiveness, they should be carefully identified in the assessment process [10].

One of the widely accepted techniques in group decision making is the Multi-criteria decision-making (MCDM) technique. Traditional MCDM methods consist of a group of DMs providing a qualitative and quantitative assessment of the performance for every alternative with respect to the criteria and the relative significance of the criteria with regard to the entire judgments. Analytic Hierarchy Process (AHP), Simple Additive Weighting (SAW), Analytic Network Process (ANP), Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), etc., which are the existing MCDM methods in the literature. Decision-Making Trial and Evaluate on Laboratory (DEMATEL) is one of the popular MCDM approaches for the search for interaction effects between parameters and dimensions in decision-making problems. The DEMATEL method was initially developed to describe the causal relationship among the subcomponents via a causal diagram. It was shown to be a powerful tool for solving complex problems and has several benefits for describing the interrelated relationship between the criteria. Most of the existing research work has successfully applied to the financial stock market environment. Lee et al. [17] combined DEMATEL and ANP to analyze the interdependence between key factors of stock investment decision making. The DEMATEL method is used to analyze the causal relationship between the item groups instead of the ANP approach. Golcuk and Baykasoglu [12] have suggested for ranking the alternatives based on integrating the ANP and DEMATEL method. Recently, Venugopal et al. [30] developed a Fuzzy DEMATEL Approach for Financial Ratio Performance Evaluation of NASDAQ Exchange.

However, the decision-makers (DMs) provide linguistic evaluation for several alternatives and criteria. Fuzzy MCDM methods have been effectively handled these types of circumstances. Serkan and Turkay [29] have introduced a novel DEMATEL method to the Priority investment project by calculating their possibility of decreasing foreign trade deficit and creating new investment together. Mills et al. [19] proposed a hybrid MCDM approach by comprising an integrated ANP and DEMATEL for optimal portfolio selection. These results indicated that return, financial ratios, dividends, and risk are causal criteria group, which are the most influential determinants for obtaining high benefits of stock portfolio selection in the Shanghai Stock Exchange. Rezaeian and Akbari [22] developed a new approach, which combines ANP and DEMATEL for stock portfolio selection in the fuzzy environment. The fuzzy DEMATEL method is used for different applications and has been used to change the ANP by analyzing the causal relationship between the item classes. This approach is called DEMATEL-ANP, as suggested by Golcuk and Baykasoglu [12]. Wu et al. [31] used the fuzzy and gray Delphi approach to determine a set of reliable attributes. Varma and Kumar [29], Tabrizi et al. [28] have evaluated the different criteria that apply to companies and that can assist in the creation of portfolio construction and causal relations between the criteria defined. Perin [21], Aydn and Kahraman [7] has developed a fuzzy DEMATEL system for dealing with interactions between evaluation parameters and proposed a fuzzy ANP method to calculate the relative importance of each criterion, which was evaluating and the quality of service achievement of airlines in Turkey is ranked.

Recently, [[13], [26], [32], [4], [15]] many authors have used the idea of neutrosophic set in MCDM methods. The concept of the neutrosophic set was introduced by Smarandache [23], which is distinguished by the role of truth-membership function, indeterminacy-membership function, and falsity-membership function. Therefore the neutrosophic set theory can be used to rationalize the confusion associated with ambiguity in an analogous way to human thought. This handles vague data as distributions of possibilities in terms of membership functions. Using the concept of triangular neutrosophic additive reciprocal preference relations Basset et.al, [6] developed a novel method for the group decision-making problem. Bhattacharya [8], [9], [16] discussed the concept of rule-based neutrosophic reasoning applied to the options Market. Basset et al. [6], [3] have presented a navel hybrid multiple criteria group decision-making framework for the project selection under the neutrosophic environment. Altuntas and Dereli [1] studied a novel approach based on a process called DEMATEL and patent quote analysis to prioritize investment project portfolios. The suggested strategy represents the viewpoint of the Government and takes into account foreign trade deficits and attract new investments for prioritization. The objective of this paper is to measure the relative importance of the financial ratios of two groups such as AFM and FM by using the DEMATEL approach under the neutrosophic environment.

1.1. *Motivation and Contributions*

Stock markets are unpredictable frameworks impacted by many interrelated financial, political, and internal factors and described by implicit non-linearities. Understanding whenever and how to invest in stock markets and to make the decisions are very difficult for investors. In this regard, investors need knowledge about the stocks and an intensive analysis associated with the markets along with an excellent experience. At present, there are numerous marketplaces, different variables, indicators, etc. that must be become analyzed before taking the financial decisions in the short interval of the time. The performance evaluation of companies is one of the most important measures that is considered by investors. Thus, the performance analysis is required in optimal stock selection to make use of mathematical and statistical tools to assist investors to decide at the optimum moment. However, there are many Accounting based financial measures(AFM) and Economic value-based financial measures (EFM) available in the stock market. Hence, the ranking of the AFM and EFM is important and essential in the stock market selection, which is motivated to research this field. The main purpose of the analysis is to evaluate which accounting earnings performance measures and value-based performance measures are best expressed in adjustments in the market value of the product. In general, most of the performance measures are not deterministic and can not be accurately predicted. Fuzzy set theory is vividly used to predict the performance values of securities in an uncertain environment. However, the fuzzy set focuses only on the degree of truth-membership and it does not take into account the non-membership and indeterminacy. Atanassov [5] developed intuitionist fuzzy set theory, which takes into account both degrees of truth and degree of falsity but does not find indeterminacy. So, it fails to deals with indeterminacy existing in the real world. To overcome these drawbacks of the fuzzy set, we are used the neutrosophic set in an uncertain environment. The neutrosophic set is an extent or generalization of the intuitionistic fuzzy set. It represents real-world problems effectively and efficiently by considering all aspects of decision situations (Abdel-Basset et al. [2]).

The neutrosophic DEMATEL model is used to deal with interdependencies between criteria and then to draw up a casual diagram between criteria for the assessment of financial performance ratios. This study intends to establish an investment decision model to provide investors with the MCDM model consisting of neutrosophic DEMATEL. The empirical results assist the investor and traders to select stock. To the best of our knowledge, there is no work studied yet for financial ratio performance selection by using the neutrosophic DEMATEL approach. The contributions of the paper as follows:

- The financial data of 8 companies, which are listed in the NASDAQ Exchange for a years time period between June 2018 - May 2019 have collected.

- AFM and EFM values are calculated from the balance sheet for each firm, which is given to the two experts.
- Opinion has been collected from two experts: an investor in the NASDAQ exchange, and a Professor in Finance.
- The relative performance ranking of the financial ratios is evaluated through the Neutrosophic DEMATEL framework.
- The Neutrosophic DEMATEL method is compared with the fuzzy DEMATEL and classical DEMATEL approach.

2. **Neutrosophic sets**

In this section, we discuss the definitions of neutrosophic sets, single-valued neutrosophic sets, triangular neutrosophic numbers, and operations on triangular neutrosophic numbers.

Definition 2.1. [23]

Let E be be an universe of discourse and $\xi \in E$. A neutrosophic set X in E is characterized by a truth truth-membership function $T_X(\xi)$ an indeterminacy-membership function $I_X(\xi)$ and a falsity membership function $F_X(\xi)$. $T_X(\xi)$, $I_X(\xi)$ and $F_X(\xi)$ are real standard or real nonstandard subsets of $]-0,\, 1+[$. That is $T_X(\xi) : E \rightarrow]-0,\, 1+[I_X(\xi) : E \rightarrow]-0,\, 1+[$ and $F_X(\xi) : E \rightarrow]-0,\, 1+[$. There is no restriction on the sum of $T_X(\xi), I_X(\xi)$ and $F_X(\xi)$, so $0 \leq supT_X(\xi) + supI_X(\xi) + supF_X(\xi) \leq 3$.

Definition 2.2. [23]

Let E be be a space of points. A single valued neutrosophic set X over E is an object taking the form $\{\langle \xi, T_X(\xi), I_X(\xi), F_X(\xi), \rangle : \xi \in E\}$, where $T_X(\xi) : E \rightarrow [0,1], I_X(\xi) : E \rightarrow [0,1]$ and $F_X(\xi) : E \rightarrow [0,1]$ with $0 \leq T_X(\xi)+I_X(\xi)+F_X(\xi) \leq 3$ for all $\xi \in E$. The intervals $T_X(\xi), I_X(\xi)$ and $F_X(\xi)$ represent the truth membership degree, the indeterminacy-membership degree and the falsity membership degree of x to, respectively.

Definition 2.3. [23]

Suppose $\alpha_l, \theta_l, \beta_l \in [0,1]$ and $l^{(1)}, l^{(2)}, l^{(3)} \in R$ where $l^{(1)} \leq l^{(2)} \leq l^{(3)}$. Then single value triangular neutrosophic number $\tilde{l} = \langle (l^{(1)}, l^{(2)}, l^{(3)}); \alpha_l, \theta_l, \beta_l \rangle$ is a special neutrosophic set on the real line set R, whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as:

$$T_l(\xi) = \begin{cases} \alpha_l \left(\frac{\xi - l^{(1)}}{l^{(2)} - l^{(1)}} \right), & l^{(1)} \leq \xi \leq l^{(2)} \\ \alpha_l, & \xi = l^{(2)} \\ \alpha_l \left(\frac{l^{(3)} - \xi}{l^{(3)} - l^{(2)}} \right), & l^{(2)} \leq \xi \leq l^{(3)} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

$$I_l(\xi) = \begin{cases} \frac{(l^{(2)}-\xi+\theta_l(\xi-l^{(1)})}{l^{(2)}-l^{(1)}}, & l^{(1)} \le \xi \le l^{(2)} \\ \theta_l, & \xi = l^{(2)} \\ \frac{(\xi-l^{(2)}+\theta_l(l^{(3)}-\xi))}{l^{(3)}-l^{(2)}}, & l^{(2)} \le \xi \le l^{(3)} \\ 1, & \text{otherwise} \end{cases} \tag{2}$$

$$F_l(\xi) = \begin{cases} \frac{(l^{(2)}-\xi+\beta_l(\xi-l^{(1)}))}{l^{(2)}-l^{(1)}}, & l^{(1)} \le \xi \le l^{(2)} \\ \beta_l & \xi = l^{(2)} \\ \frac{(\xi-l^{(2)}+\beta_l(l^{(3)}-\xi)}{l^{(3)}-l^{(2)}}, & l^{(2)} \le \xi \le l^{(3)} \end{cases} \tag{3}$$

Definition 2.4. [23]

Suppose that $l = \langle (l^{(1)}, l^{(2)}, l^{(3)}); \alpha_l, \theta_l, \beta_l \rangle$ and $m = \langle (m^{(1)}, m^{(2)}, m^{(3)}); \alpha_m, \theta_m, \beta_m \rangle$ are two single valued triangular neutrosophic numbers and $\gamma \neq 0$ be any real number. Then the arithmetic operations are defined as follows:

(i) $l + m = \langle (l^{(1)} + m^{(1)}, l^{(2)} + m^{(2)}, l^{(3)} + m^{(3)}); \alpha_l \wedge \alpha_m, \theta_l \vee \theta_m, \beta_l \vee \beta_m \rangle$

(ii) $l - m = \langle (l^{(1)} - m^{(3)}, l^{(2)} - m^{(2)}, l^{(3)} - m^{(1)}); \alpha_l \wedge \alpha_m, \theta_l \vee \theta_l, \beta_l \vee \beta_m \rangle$

(iii) $l^{-1} = \langle (\frac{1}{l^{(3)}}, \frac{1}{l^{(2)}}, \frac{1}{l^{(1)}}); \alpha_l, \theta_l, \beta_l \rangle, \quad \text{where} \quad l \neq 0$

(iv) $\gamma l = \begin{cases} \langle (\gamma l^{(1)}, \gamma l^{(2)}, \gamma l^{(3)}); \alpha_l, \theta_l, \beta_l \rangle, & \text{if } (\gamma > 0) \\ \langle (\gamma l^{(3)}, \gamma l^{(2)}, \gamma l^{(1)}); \alpha_l, \theta_l, \beta_l \rangle, & \text{if } (\gamma < 0) \end{cases}$

(v) $\frac{l}{m} = \begin{cases} \langle (\frac{l^{(1)}}{m^{(3)}}, \frac{l^{(2)}}{m(2)}, \frac{l^{(3)}}{m^{(1)}}); \alpha_l \wedge \alpha_m, \theta_l \vee \theta_m, \beta_l \vee \beta_m \rangle, & \text{if } (l^{(3)} > 0, m^{(3)} > 0) \\ \langle (\frac{l^{(3)}}{m^{(3)}}, \frac{l^{(2)}}{m^{(2)}}, \frac{l^{(1)}}{m^{(1)}}); \alpha_l \wedge \alpha_m, \theta_l \vee \theta_m, \beta_l \vee \beta_m \rangle, & \text{if } (l^{(3)} < 0, m^{(3)} > 0) \\ \langle (\frac{l^{(3)}}{m^{(3)}}, \frac{l^{(2)}}{m^{(2)}}, \frac{l^{(1)}}{m^{(1)}}); \alpha_l \wedge \alpha_m, \theta_l \vee \theta_m, \beta_l \vee \beta_m \rangle, & \text{if } (l^{(3)} < 0, m^{(3)} < 0) \end{cases}$

(vi) $l/m =$

$$\begin{cases} \langle (l^{(1)}m^{(1)}, l^{(2)}m^{(2)}, l^{(3)}m^{(3)}); \alpha_l \wedge \alpha_m, \theta_l \vee \theta_m, \beta_l \vee \beta_m \rangle, & \text{if } (l^{(3)} > 0, m^{(3)} > 0) \\ \langle (l^{(1)}m^{(3)}, l^{(2)}m^{(2)}, l^{(3)}m^{(1)}); \alpha_l \wedge \alpha_m, \theta_l \vee \theta_m, \beta_l \vee \beta_m \rangle, & \text{if } (l^{(3)} < 0, m^{(3)} > 0) \\ \langle (l^{(3)}m^{(3)}, l^{(2)}m^{(2)}, l^{(1)}m^{(1)}); \alpha_l \wedge \alpha_m, \theta_l \vee \theta_m, \beta_l \vee \beta_m \rangle, & \text{if } (l^{(3)} < 0, m^{(3)} < 0) \end{cases}$$

3. Neutrosophic DEMATEL Approach

Smarandache [23] proposed the neutrosophic set theory. Neutrosophy handles vagueness and uncertainty, and attend the indeterminacy of values. Neutrosophy has some of the advantages:

(i) Neutrosophy provides the ability to present unknown information in our model using the indeterminacy degree, so the experts can present opinions about the unsure preferences.

(ii) Neutrosophy depicts the disagreement between decision-makers and experts.

(iii) Neutrosophy heeds all aspects of decision making situations by considering truthiness, indeterminacy, and falsity altogether.

Fontela and Gabus [11] have suggested that DEMATEL is used to be an important tool for defining the cause-and-effect chain components of a vast system. It deals with the evaluation of collaborative interaction between factors and the identification of critical relationships via a graphic conceptual framework. The neutrosophy DEMATEL model is used to deal with internal dependencies among criteria and then to construct a casual graph between the criteria for the financial performance ratio assessment. The neutrosophy DEMATEL method is briefly discussed as follows and the flow chart of the proposed framework is shown in Fig. 1.

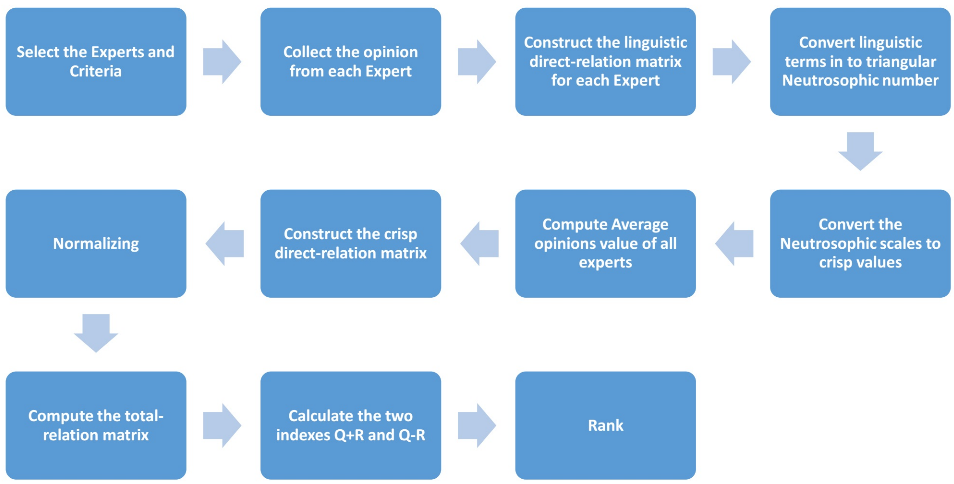

FIGURE 1. The framework of proposed neutrosophy DEMATEL method

Step 1 : Identify the experts who have well experience in the investment field.

Step 2 : Select the most important criteria which will influence the given problem.

Step 3 : Construct the linguistic direct-relation Matrix. This shows the degree of effect that each criterion has on other criteria. In this regard, collect the opinion from each expert and make the pairwise comparisons matrix for each expert, whose elements are linguistic terms such as Equally important, Slightly important, Strongly important, very strongly important, Absolutely important, etc., which is represented by the following matrix. This matrix is called linguistic the direct-relation matrix, which is a $n \times n$ matrix whose elements t_{ij} indicates the degree of effect between criteria i and criteria j, where t_{ij} takes any one the linguistic terms like equally important, slightly important, strongly important, very strongly important, absolutely important.

Step 4 : Convert the linguistic terms of direct-relation into the triangular neutrosophic scale, which is shown in table 2.

The triangular neutrosophic scale is in the form of $t_{ij} = \langle (t_{ij}^{(1)}, t_{ij}^{(2)}, t_{ij}^{(3)}; \alpha_{ij}, \theta_{ij}, \beta_{ij}) \rangle$ such that $t_{ij}^{(1)}, t_{ij}^{(2)}, t_{ij}^{(3)}$ are the lower, median and upper bound of neutrosophic number of i^{th} over j^{th} criteria, $\alpha_{ij}, \theta_{ij}, \beta_{ij}$ are the truth-membership, indeterminacy and falsity membership functions of i^{th} over j^{th} criteria.

Step 5 : Convert the neutrosophic scales to crisp values by using the following equations [27]:

	C_1	C_2	$\cdots$	C_n
C_1	t_{11}	t_{12}	$\cdots$	t_{1n}
C_2	t_{21}	t_{22}	$\cdots$	t_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
C_n	t_{n1}	t_{n2}	$\cdots$	t_{nn}

TABLE 1. Linguistic direct relation matrix

	C_1	C_2	$\cdots$	C_n
C_1	$\langle(t_{11}^{(1)}, t_{11}^{(2)}, t_{11}^{(3)}); \alpha_{11}, \theta_{11}, \beta_{11}\rangle$	$\langle(t_{12}^{(1)}, t_{12}^{(2)}, t_{12}^{(3)}); \alpha_{12}, \theta_{12}, \beta_{12}\rangle$	$\cdots$	$\langle(t_{1n}^{(1)}, t_{1n}^{(2)}, t_{1n}^{(3)}); \alpha_{1n}, \theta_{1n}, \beta_{1n}\rangle$
C_2	$\langle(t_{21}^{(1)}, t_{21}^{(2)}, t_{21}^{(3)}); \alpha_{21}, \theta_{21}, \beta_{21}\rangle$	$\langle(t_{22}^{(1)}, t_{22}^{(2)}, t_{22}^{(3)}); \alpha_{22}, \theta_{22}, \beta_{22}\rangle$	$\cdots$	$\langle(t_{2n}^{(1)}, t_{2n}^{(2)}, t_{2n}^{(3)}); \alpha_{2n}, \theta_{2n}, \beta_{2n}\rangle$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
C_n	$\langle(t_{n1}^{(1)}, t_{n1}^{(2)}, t_{n1}^{(3)}); \alpha_{n1}, \theta_{n1}, \beta_{n1}\rangle$	$\langle(t_{n2}^{(1)}, t_{n2}^{(2)}, t_{n2}^{(3)}); \alpha_{n2}, \theta_{n2}, \beta_{n2}\rangle$	$\cdots$	$\langle(t_{nn}^{(1)}, t_{nn}^{(2)}, t_{nn}^{(3)}); \alpha_{nn}, \theta_{nn}, \beta_{nn}\rangle$

TABLE 2. Neutrosophic Direct relation matrix

$$r(t_{ij}) = \left|(t_{ij}^{(1)} \times t_{ij}^{(2)} \times t_{ij}^{(3)})\frac{\alpha_{ij} + \theta_{ij} + \beta_{ij}}{9}\right| \quad (4)$$

Step 6 : Combine the opinions of all experts in one integration matrix and measure the average opinions of the experts by dividing the opinion of all experts for each criterion by the number of experts (n) considered in the question. Each expert average value is determined by dividing each value by the number of experts (n) as shown in the equation (5), and then add all the expert's average values.

$$s_{ij} = \frac{\sum_{k=1}^{m} r^k}{n} \quad (5)$$

where s_{ij} represents the average opinions value of i^{th} criteria and j^{th} criteria and r^k indicates the opinions crisp value of i^{th} criteria and j^{th} criteria for the $k^{th}(k = 1, ..., m)$ decision maker.

Step 7 : Construct the crisp direct-relation matrix S. This matrix is obtained from previous step 6 i.e. the integrating of all averaged opinions of experts. The initial direct-relation matrix denoted as S, which is a $n \times n$ matrix whose elements t_{ij} indicates the degree of effect between criteria i and criteria j.

$$S = \begin{bmatrix} 1 & s_{12} & \cdots & s_{1n} \\ s_{21} & 1 & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & 1 \end{bmatrix}$$

Step 8 : Normalizing the direct relation matrix by using the following equations.

$$U = K \times S$$

$$K = Min(\frac{1}{Max \sum_{i=1}^{n} s_{ij}}, \frac{1}{Max \sum_{j=1}^{n} s_{ij}}), 1 \leq i \leq n, 1 \leq j \leq n \tag{6}$$

Step 9 : Computing the total-relation matrix P by using the following equation

$$P = U \times (I - U)^{-1} \tag{7}$$

where I is the $n \times n$ identity matrix

Step 10 : Calculate the two indexes Q+R and Q-R for each criterion and draw the causal diagram. The first step to compute the sum of row (Q) and the sum of column (R) for each criterion separately. The (Q) and (R) are two vectors and the vector is calculated by using the following equations, where $P = [z_{ij}],\ i,\ j \in\ 1, 2, ..., n$

$$Q = \sum_{j=1}^{n} z_{ij}, \quad \forall i = 1, 2, ...n \tag{8}$$

$$R = \sum_{j=1}^{n} z_{ij}, \quad \forall i = 1, 2, ...n \tag{9}$$

4. **Case Study in NASDAQ Exchange**

In the present study, the neutrosophic DEMATEL approach is used for evaluation of relative importance of the financial ratio measure under the stock market environment. The proposed method is explained with a case study example as follows:

(1) Select the expects in the stock market field: We consider eight potential protable companies such as Apple, Micro-soft, Google, Intel Corporation, Adobe Inc, NVIDIA Corporation, and Micron Technology, Inc., Cognizant Technology Solutions Corp. The data for one-year performance (June 2018-May 2019) of 8 industries in the IT sector has been gained by distributing a questionnaire among two experts: (i) investors in the NASDAQ exchange (DM1), and (ii) a professor in Finance (DM2). The decision-maker collected opinion two different group financial measures: the Accounting based financial measures (AFM) and Economic value-based financial measures (EFM).

(2) Identify the most important criteria in financial ratio measure [33]: AFM based four criteria such Return On Assets (ROA), Return On Equity (ROE), Earnings per share (EPS), price for earnings ratio (P/E) Ratio which is shown in Fig. 2 and EFM based four criteria such that Economic Value Added (EVA), Market Value Added (MVA), Cash Value Added (CVA), Cash Flow Return on Investment (CFROI) whcih is shown in Fig.3.

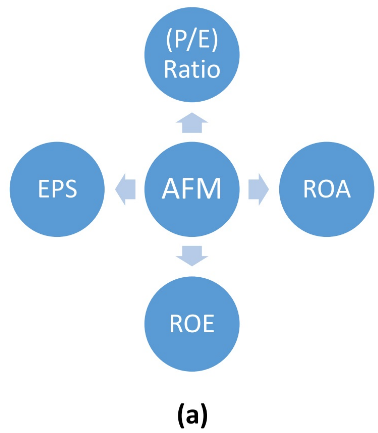

Figure 2. Accounting based financial measures

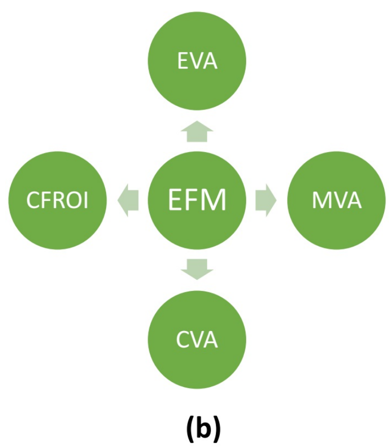

Figure 3. Economic value-based financial measures

(3) Construct the pairwise comparison matrix: In order to compare the interrelation between the four criteria such as ROA, ROE, EPS, P/E Ratio in AFM, and four criteria such as EVA, MVA, CVA, CFROI in EVA, we collect the linguistic information from the ‘two experts. Then, we design a range of values for each linguistic expression based on the (DM) expert evaluation as represented as A 5-point Likert scale (see Table 3), which is given in Tables 4, 5, 6, and 7.

Explanation	Scale	Neutrosophic Triangular Scale
Equally important	1	$\langle(1, 1, 1); 0.5, 0.5, 0.5\rangle$
Slightly important	3	$\langle(2, 3, 4); 0.30, 0.75, 0.70\rangle$
Strongly important	5	$\langle(4, 5, 6); 0.80, 0.15, 0.20\rangle$
very strongly important	7	$\langle(6, 7, 8); 0.90, 0.10, 0.10\rangle$
Absolutely important	9	$\langle(9, 9, 9); 1.00, 0.00, 0.00\rangle$
sporadic values between two close scales	2	$\langle(1, 2, 3); 0.40, 0.60, 0.65\rangle$
	4	$\langle(3, 4, 5); 0.35, 0.60, 0.40\rangle$
	6	$\langle(5, 6, 7); 0.70, 0.25, 0.30\rangle$
	8	$\langle(7, 8, 9); 0.85, 0.10, 0.15\rangle$

Table 3. The Neutrosophic Triangular scale value

	C_1(ROA	C_2(ROE)	C_3(EPS)	C_4(P/E) Ratio
C_1(ROA)	(1,1,1;0.5,0.5,0.5)	(2,3,4;0.3,0.75,0.7)	(6,7,8;0.9,0.1,0.1)	(9,9,9;1,0,0)
C_2(ROE)	(4,5,6;0.8,0.15,0.2)	(1,1,1;0.5,0.5,0.5)	(7,8,9;0.85,0.1,0.15)	(6,7,8;0.9,0.1,0.1)
C_3(EPS)	(2,3,4;0.3,0.75,0.7)	(3,4,5;0.35,0.6,0.4)	(1,1,1;0.5,0.5,0.5)	(4,5,6;0.8,0.15,0.2)
C_4(P/E) Ratio	(1,2,3;0.4,0.6,0.65)	(2,3,4;0.3,0.75,0.7)	(5,6,7;0.7,0.25,0.3)	(1,1,1;0.5,0.5,0.5)

Table 4. The pairwise Neutrosophic comparison matrix of AFM's criteria given by expert 1

	C_1(ROA)	C_2(ROE)	C_3(EPS)	C_4(P/E) Ratio
C_1(ROA)	(1,1,1;0.5,0.5,0.5)	(2,3,4;0.3,0.75,0.7)	(9,9,9;1,0,0)	(4,5,6;0.8,0.15,0.2)
C_2(ROE)	(4,5,6;0.8,0.15,0.2)	(1,1,1;0.5,0.5,0.5)	(1,2,3;0.4,0.6,0.65)	(6,7,8;0.9,0.1,0.1)
C_3(EPS)	(6,7,8;0.9,0.1,0.1)	(3,4,5;0.35,0.6,0.4)	(1,1,1;0.5,0.5,0.5)	(4,5,6;0.8,0.15,0.2)
C_4(P/E)Ratio	(1,2,3;0.4,0.6,0.65)	(2,3,4;0.3,0.75,0.7)	(7,8,9;0.85,0.1,0.15)	(1,1,1;0.5,0.5,0.5)

Table 5. The pairwise Neutrosophic comparison matrix of AFM's criteria given by expert 2

(4) Convert the neutrosophic AFM and EFM matrices into crisp matrix by using equation (4), which is shown in Table 8 and 9
(5) In order to construct the initial direction relation-matrix, measure the average opinions of the experts by using equation (5). The initial direction relation-matrix is shown in table 10.

	E_1(EVA)	E_2(MVA)	E_3(CVA)	E_4(CFROI)
E_1(EVA)	(1,1,1;0.5,0.5,0.5)	(5,6,7;0.7,0.25,0.3)	(5,6,7;0.7,0.25,0.3)	(9,9,9;1,0,0)
E_2(MVA)	(6,7,8;0.9,0.1,0.1)	(1,1,1;0.5,0.5,0.5)	(6,7,8;0.9,1,1)	(7,8,9;0.8,0.1,0.15)
E_3(CVA)	(4,5,6;0.8,0.15,0.2)	(5,6,7;0.7,0.25,0.3)	(1,1,1;0.5,0.5,0.5)	(7,8,9;0.8,0.1,0.15)
E_4(CFROI)	(1,2,3;0.4,0.6,0.65)	(9,9,9;1,0,0)	(9,9,9;1,0,0)	(1,1,1;0.5,0.5,0.5)

Table 6. The pairwise Neutrosophic comparison matrix of EFM's criteria given by expert 1

	E_1(EVA)	E_2(MVA)	E_3(CVA)	E_4(CFROI)
E_1(EVA)	(1,1,1;0.5,0.5,0.5)	(5,6,7;0.7,0.25,0.3)	(3,4,5;0.35,0.6,0.4)	(9,9,9;1,0,0)
E_2(MVA)	(2,3,4;0.3,0.75,0.7)	(1,1,1;0.5,0.5,0.5)	(6,7,8;0.9,0.1,0.1)	(5,6,7;0.7,0.2,0.35)
E_3(CVA)	(4,5,6;0.8,0.15,0.2)	(3,4,5;0.35,0.6,0.4)	(1,1,1;0.5,0.5,0.5)	(5,6,7;0.7,0.2,0.35)
E_4(CFROI)	(5,6,7;0.7,0.25,0.3)	(6,7,8;0.9,0.1,0.1)	(6,7,8;0.9,0.1,0.1)	(1,1,1;0.5,0.5,0.5)

Table 7. The pairwise Neutrosophic comparison matrix of EFM's criteria given by expert 2

	Expert-1				Expert-2			
	C_1(ROA)	C_2(ROE)	C_3(EPS)	C_4(P/E) Ratio	C_1(ROA)	C_2(ROE)	C_3(EPS)	C_4(P/E) Ratio
C_1(ROA)	1.0000	4.6667	41.0667	81.0000	1.0000	4.6667	81.0000	15.3333
C_2(ROE)	15.3333	1.0000	61.6000	41.0667	15.3333	1.0000	1.1000	41.0667
C_3(EPS)	4.6667	9.0000	1.0000	15.3333	41.0667	9.0000	1.0000	15.3333
C_4(P/E) Ratio	1.1000	4.6667	29.1667	1.0000	1.1000	4.6667	61.6000	1.0000

Table 8. The crisp values of pairwise comparison matrix for AFM

	Expert-1				Expert-2			
	E_1(EVA)	E_2(MVA)	E_3(CVA)	E_4(CFROI)	E_1(EVA)	E_2(MVA)	E_3(CVA)	E_4(CFROI)
E_1(EVA)	1.0000	29.1667	29.1667	81.0000	1.0000	29.1667	9.0000	81.0000
E_2(MVA)	108.2667	1.0000	108.2667	58.8000	4.6667	1.0000	41.0667	29.1667
E_3(CVA)	15.3333	29.1667	1.0000	58.8000	15.3333	9.0000	1.0000	29.1667
E_4(CFROI)	1.1000	0.0000	0.0000	1.0000	29.1667	41.0667	41.0667	1.0000

Table 9. The crisp values of pairwise comparison matrix for EFM

(6) Normalizing the initial direct relation matrix by using equations (6) and (7). The normalized matrix is presented in Table 11.

	AFM					EFM			
	C_1(ROA)	C_2(ROE)	C_3(EPS)	C_4(P/E) Ratio		E_1(EVA)	E_2(MVA)	E_3(CVA)	E_4(CFROI)
C_1(ROA)	1.0000	4.6667	61.0333	48.1667	E_1(EVA)	1.0000	29.1667	19.0833	81.0000
C_2(ROE)	15.3333	1.0000	31.3500	41.0667	E_2(MVA)	56.4667	1.0000	74.6667	43.9833
C_3(EPS)	22.8667	9.0000	1.0000	15.3333	E_3(CVA)	15.3333	19.0833	1.0000	43.9833
C_4(P/E) Ratio	1.1000	4.6667	45.3833	1.0000	E_4(CFROI)	15.1333	20.5333	20.5333	1.0000

Table 10. Direct-relation matrix for AFM and EFM

	AFM					EFM			
	C_1(ROA)	C_2(ROE)	C_3(EPS)	C_1(P/E) Ratio		E_1(EVA)	E_2(MVA)	E_3(CVA)	E_4(CFROI)
C_1(ROA)	0.0348	0.1625	2.1254	1.6773	E_1(EVA)	0.0227	0.6624	0.4334	1.8397
C_2(ROE)	0.5340	0.0348	1.0917	1.4301	E_2(MVA)	1.2825	0.0227	1.6958	0.9990
C_3(EPS)	0.7963	0.3134	0.0348	0.5340	E_3(CVA)	0.3483	0.4334	0.0227	0.9990
C_4(P/E) Ratio	0.0383	0.1625	1.5804	0.0348	E_4(CFROI)	0.3437	0.4664	0.4664	0.0227

Table 11. Normalized decision matrix for AFM and EFM ratio

(7) Compute the total-relation matrix by using equation (8). The total-relation matrix, is given in Table 12.

	AFM					EFM			
	C_1(ROA)	C_2(ROE)	C_3(EPS)	C_1(P/E) Ratio		E_1(EVA)	E_2(MVA)	E_3(CVA)	E_4(CFROI)
C_1(ROA)	0.0071	-0.0501	-1.6525	-0.8965	E_1(EVA)	0.0060	-0.1977	-0.2637	-0.7966
C_2(ROE)	-0.2348	0.0209	-1.0010	-0.5474	E_2(MVA)	-0.4555	0.0026	-0.7370	-0.9952
C_3(EPS)	-0.1016	-0.0350	-0.0021	-0.2244	E_3(CVA)	-0.0972	-0.0908	0.0081	-0.3740
C_4(P/E) Ratio	-0.0105	-0.0153	-0.4471	0.0091	E_4(CFROI)	-0.0723	-0.0701	-0.1170	0.0049

Table 12. Total relation matrix

(8) By using equations (9) and (10), calculate the indexes Q+R and Q-R for each criterion and rank the criteria, which is shown in Table 13. Finally, draw the causal diagram for financial measures, which is shown in Fig. 3 and 4.

5. **Result and discussion**

In this section, we analyze the results of the proposed method. Table 14, presents the ranking of AFM and EFM, which has been used for financial performance evaluation. From the result, it is observed that ROE is the highest Q+R score value (-1.8418) secured the first rank, and P/E Ratio is indicated the Q+R value is -2.1230. Hence, it secured the second rank. The EPS has indicated the Q+R value is -3.4657. It has secured the least rank. Hence, ROE has secured the first rank, which shows that ROE is the most important criterion in AFM. The company management and investor are

AFM				EFM	
Criteria	Q+R	Q-R	Criteria	Q+R	Q-R
C_1(ROA)	-2.9319	-2.2521	E_1(EVA)	-1.8711	-0.6329
C_2(ROE)	-1.8418	-1.6829	E_2(MVA)	-2.5412	-1.8291
C_3(EPS)	-3.4657	2.7395	E_3(CVA)	-1.6634	0.5557
C_4(P/E) Ratio	-2.1230	1.1955	E_4(CFROI)	-2.4153	1.9063

TABLE 13. The comparative neutrosophic DEMATEL technique $Q+R$ and $Q-R$ value

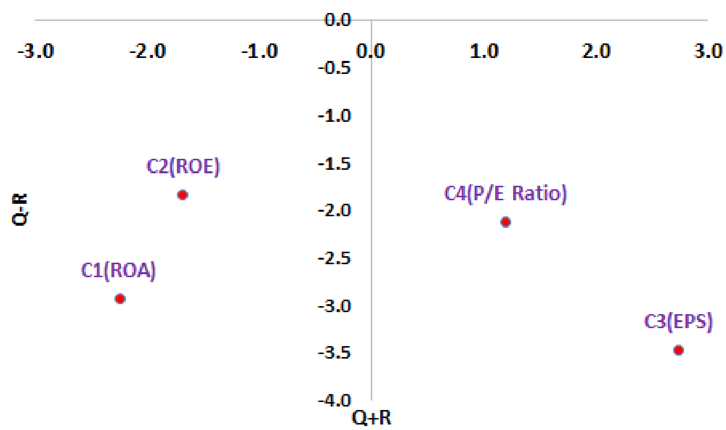

FIGURE 4. The causal diagram for Accounting based financial measures criteria

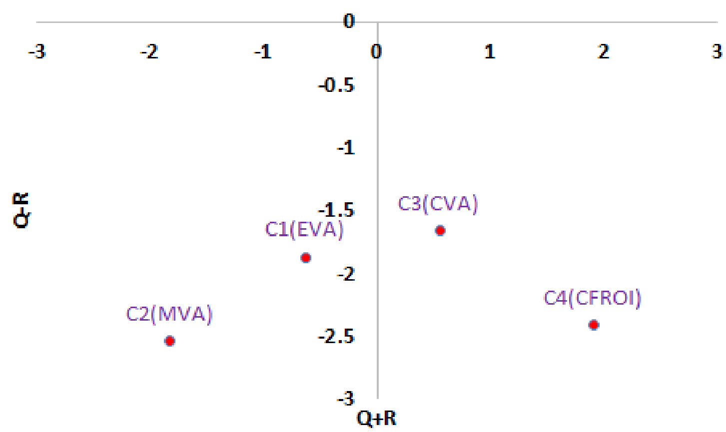

FIGURE 5. The causal diagram for Economic value-based financial measures criteria

recommended to pay more attention to ROE for achieving their best competitiveness in the organization. According to the degree of importance, Q+R the AFM criteria are ranked as follows: $C_3 > C_1 > C_4 > C_2$.

In addition, table 14 presents a summary of financial measure EFM. From the result, we conclude that the criteria CVA is the most important of the criteria since it has the highest Q+R priority value (-1.6634). The Q+R value of EVA is -1.8711, which has secured the second rank. Similarly, MVA is the least performance of the criteria. The company management and investor are recommended to pay more attention to CVA for achieving their best competitiveness in the organization. According to the degree of importance,(S+R) the criteria EFM has ranked which are $E_2 > E_4 > E_1 > E_3$. However, the causal diagram constructed with the horizontal axis is Q+R and the vertical axis is Q-R. The causal diagram of AFM's and EFM's criteria are shown in Figures 3 and 4 respectively.

6. Conclusion

Financial ratios provide useful quantitative financial information about the performance of a company. The proposed approach (Neutrosophic-DEMATEL) is used to evaluate the relative importance of financial ratios compared to two groups: Accounting based financial measures (AFM) and Economic value-based financial measures (EFM). The empirical results are recommended the following results to the investor: ROE is the most important financial measure in AFM and CVA is the most influential measure in EFM. Hence, the proposed method suggests to the investors pay more attention to ROE in AFM and CVA in EFM. Moreover, the proposed neutrosophic- DEMATEL model gives a different result for both financial measures. Because neutrosophic DEMATEL has provided us with more degrees of freedom to represent uncertainty and indeterminacy in real-world information. The discussed results will help the companies, investors, and traders before making profound decisions.

In the future, we consider the other economic value measures such as shareholder value-added, equity economic value-added, and other performance measures by using different MCDM techniques like AHP, ELECTRE, and PROMETHEE under an interval valued neutrosophic environment.

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Rough Semigroups in Connection with Single Valued Neutrosophic $(\in, \in)$-Ideals

Young Bae Jun [1], Anas Al-Masarwah [2,*] and Majdoleen Abu Qamar [3]

[1] Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea; skywine@gmail.com

[2] Department of Mathematics, Faculty of Science, Ajloun National University, P.O. Box 43, Ajloun 26810, Jordan; almasarwah85@gmail.com, anas.almasarwah@anu.edu.jo

[3] Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor DE, Malaysia; mjabuqamar@gmail.com

[*] Correspondence: almasarwah85@gmail.com

Abstract. The scheme of rough sets is an effective procedure that handle ambiguous, inexact or uncertain information configuration. Rough set theory for algebraic structures like semigroups is a formal approximation space consisting of a universal set and an equivalence relation. This article achieves a new utilization of rough sets in the theory of semigroups via single valued neutrosophic (SVN) subsemigroups/ideals. The conceptions of an SVN $(\in, \in)$-subsemigroup and an SVN $(\in, \in)$-ideal in semigroups are introduced, and its properties are investigated. Special congruence relations induced by an SVN $(\in, \in)$-ideal are introduced in semigroups. Using these notions, the lower and upper approximations, so called the $\mathcal{R}_q$-lower approximation and the $\mathcal{R}_q$-upper approximation for $q \in \{T, I, F\}$ based on an SVN $(\in, \in)$-ideal in semigroups are presented, and related characteristics are discussed. The notions of lower and upper subsemigroups/ideals, so called the $\mathcal{R}_q$-lower subsemigroup/ideal and the $\mathcal{R}_q$-upper subsemigroup/ideal for $q \in \{T, I, F\}$, are defined, and then the relationships between subsemigroups/ideals and $\mathcal{R}_q$-lower (upper) subsemigroups/ideals are considered.

Keywords: single valued neutrosophic $(\in, \in)$-subsemigroup/ideal; $\mathcal{R}_q$-lower subsemigroup/ideal; $\mathcal{R}_q$-upper subsemigroup/ideal.

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1. Introduction

Rough sets were originally suggested by Pawlak (see [1]), as an official approximation of the classical set in terms of a couple of sets that specify the upper and lower approximations of the crisp set. The approach of rough set is adequate for rule induction from sets of imperfect information. This approach helps in set apart between three patterns of missing attribute values; those are lost value, attribute-concept value and "do not care" conditions. Rough set can be seen as being used in a variety of fields (see [2–9]).

In 1965, Zadeh fetched up the idea of fuzzy set to handle imprecise information (see [10]). He used a single value to represent the degree of membership of the fuzzy set defined in a universe. There is a difficulty that not all problems with imprecise information are expressed in the class of single point membership value. To defeat such difficulties, an interval valued fuzzy set is adopted by Turksen (see [11]). As an extended notion of fuzzy sets, Atanassov attained a new scope called intuitionistic fuzziness sets (see [12]). In intuitionistic fuzzy sets, the membership (resp. nonmembership) function represents truth (resp. false) part. Smarandache used indeterminacy membership function as an independent component to introduce neutrosophic sets, which are a widen of intuitionistic fuzzy sets, by using three independent components: truth, indeterminacy and falsehood (see [13–15]). Wang et al. formed the idea of SVN sets which is an instance of neutrosophic sets which can be utilized in various disciplines of real-life issues, etc. (see [16]). It is already well known that neutrosophic sets are being applied in almost every field of study.

In this article, we state a SVN $(\in, \in)$-subsemigroup and a SVN $(\in, \in)$-ideal in semigroups, and investigate their properties. We define some special congruence relations $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ induced by a SVN $(\in, \in)$-ideal, and discuss a few properties in semigroups. Using these notions, we introduce the lower and upper approximations, so called the $\mathcal{R}_q$-lower approximation and the $\mathcal{R}_q$-upper approximation for $q \in \{T, I, F\}$, based on a SVN $(\in, \in)$-ideal in semigroups, and investigate related properties. Using the notion of $\mathcal{R}_q$-lower approximation and $\mathcal{R}_q$-upper approximation, we define lower and upper subsemigroups/ideals, so called the $\mathcal{R}_q$-lower subsemigroup/ideal and the $\mathcal{R}_q$-upper subsemigroup/ideal for $q \in \{T, I, F\}$, are defined, and then we provide the relationships between subsemigroups/ideals and $\mathcal{R}_q$-lower (upper) subsemigroups/ideals.

2. Preliminaries

This segment lists the basic well-known contents that are relevant to the current paper.

Definition 2.1. A set $S \neq \phi$ together with a binary operation "$\cdot$" such that $(w\cdot z)\cdot\hbar = w\cdot(z\cdot\hbar)$ for all $w, z, \hbar \in S$ is called a *semigroup*.

We use wz instead of $w \cdot z$ in what follows. Given two subsets G and H of a semigroup S, we define:

$$GH := \{wz | w \in G,\ z \in H\}.$$

Definition 2.2. A subset $N \neq \phi$ of a semigroup S is a *subsemigroup* of S if $NN \subseteq N$, and a *left ideal* (resp., *right ideal*) of S if $SN \subseteq N$ (resp., $NS \subseteq N$). We say that N is an *ideal* of S if it is both a left and a right ideal of S.

Definition 2.3 ([16])**.** Let $S \neq \phi$. An SVN set in S is defined as:

$$\Psi_{\text{TIF}} := \{\langle w; \Psi_T(w), \Psi_I(w), \Psi_F(w)\rangle | w \in S\} \tag{1}$$

where $\Psi_T, \Psi_I, \Psi_F : S \to [0,1]$ are functions.

For the sake of clarity, the SVN set in (1) will be symbolized by $\Psi_{\text{TIF}} := (\Psi_T,\ \Psi_I,\ \Psi_F)$.

Given an SVN set $\Psi_{\text{TIF}} := (\Psi_T, \Psi_I, \Psi_F)$ in S, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we describe:

$$\begin{aligned} T_\in(\Psi_{\text{TIF}};\alpha) &:= \{w \in S | \Psi_T(w) \geq \alpha\}, \\ I_\in(\Psi_{\text{TIF}};\beta) &:= \{w \in S | \Psi_I(w) \geq \beta\}, \\ F_\in(\Psi_{\text{TIF}};\gamma) &:= \{w \in S | \Psi_F(w) \leq \gamma\}, \end{aligned}$$

which are called SVN $\in$-subsets.

Definition 2.4 ([17])**.** An SVN set Ψ_{TIF} in a semigroup S is an SVN $(\in, \in)$-subsemigroup of S if it satisfies:

$$\begin{aligned} &w \in T_\in(\Psi_{\text{TIF}};\alpha_w),\ z \in T_\in(\Psi_{\text{TIF}};\alpha_z) \ \Rightarrow\ wz \in T_\in(\Psi_{\text{TIF}};\min\{\alpha_w,\alpha_z\}), \\ &w \in I_\in(\Psi_{\text{TIF}};\beta_w),\ z \in I_\in(\Psi_{\text{TIF}};\beta_z) \ \Rightarrow\ wz \in I_\in(\Psi_{\text{TIF}};\min\{\beta_w,\beta_z\}), \\ &w \in F_\in(\Psi_{\text{TIF}};\gamma_w),\ z \in F_\in(\Psi_{\text{TIF}};\gamma_z) \ \Rightarrow\ wz \in F_\in(\Psi_{\text{TIF}};\max\{\gamma_w,\gamma_z\}). \end{aligned} \tag{2}$$

Lemma 2.5 ([17])**.** *An* SVN *set* Ψ_{TIF} *in a semigroup* S *is an* SVN $(\in,\in)$*-subsemigroup of* S *if and only if it satisfies:*

$$(\forall w, z \in S) \left(\begin{array}{l} \Psi_T(wz) \geq \min\{\Psi_T(w), \Psi_T(z)\} \\ \Psi_I(wz) \geq \min\{\Psi_I(w), \Psi_I(z)\} \\ \Psi_F(wz) \leq \max\{\Psi_F(w), \Psi_F(z)\} \end{array} \right). \tag{3}$$

3. Rough semigroups based on single valued neutrosophic $(\in, \in)$-ideals

Here, let S be a semigroup unless otherwise stated.

Definition 3.1. An SVN set Ψ_{TIF} in S is a left SVN $(\in,\in)$-ideal of S if it is an SVN $(\in,\in)$-subsemigroup of S satisfying the following condition:

$$(\forall w, z \in S) \left(\begin{array}{l} z \in T_\in(\Psi_{\text{TIF}};\alpha) \ \Rightarrow\ wz \in T_\in(\Psi_{\text{TIF}};\alpha) \\ z \in I_\in(\Psi_{\text{TIF}};\beta) \ \Rightarrow\ wz \in I_\in(\Psi_{\text{TIF}};\beta) \\ z \in F_\in(\Psi_{\text{TIF}};\gamma) \ \Rightarrow\ wz \in F_\in(\Psi_{\text{TIF}};\gamma) \end{array} \right). \tag{4}$$

Definition 3.2. An SVN set Ψ_{TIF} in S is a right SVN $(\in, \in)$-ideal of S if it is an SVN $(\in, \in)$-subsemigroup of S satisfying the following condition:

$$(\forall w, z \in S)\begin{pmatrix} z \in T_{\in}(\Psi_{\text{TIF}};\alpha) \ \Rightarrow \ zw \in T_{\in}(\Psi_{\text{TIF}};\alpha) \\ z \in I_{\in}(\Psi_{\text{TIF}};\beta) \ \Rightarrow \ zw \in I_{\in}(\Psi_{\text{TIF}};\beta) \\ z \in F_{\in}(\Psi_{\text{TIF}};\gamma) \ \Rightarrow \ zw \in F_{\in}(\Psi_{\text{TIF}};\gamma) \end{pmatrix}. \tag{5}$$

If Ψ_{TIF} is a left and a right SVN $(\in, \in)$-ideal of S, we say that Ψ_{TIF} is an SVN $(\in, \in)$-ideal of S.

Example 3.3. Consider a semigroup $S = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ with the "$\cdot$" operation given by Table 1.

TABLE 1. Table for "$\cdot$" operation

$\cdot$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_2	ς_2	ς_4
ς_2	ς_2	ς_2	ς_2	ς_4
ς_3	ς_2	ς_2	ς_2	ς_4
ς_4	ς_4	ς_4	ς_4	ς_4

Let Ψ_{TIF} be an SVN set in S which is shown as:

$$\Psi_{\text{TIF}} = \{\langle \varsigma_1, (0.33, 0.27, 0.68)\rangle, \langle \varsigma_2, (0.55, 0.47, 0.57)\rangle,$$
$$\langle \varsigma_3, (0.11, 0.17, 0.89)\rangle, \langle \varsigma_4, (0.88, 0.77, 0.36)\rangle\}.$$

It is routine to show that Ψ_{TIF} is an SVN $(\in, \in)$-ideal of S.

Theorem 3.4. *An* SVN *set* Ψ_{TIF} *in* S *is a left (resp. right)* SVN $(\in, \in)$*-ideal of* S $\Leftrightarrow$ *it satisfies* (3) *and*

$$(\forall w, z \in S)\begin{pmatrix} \Psi_T(wz) \geq \Psi_T(z) \text{ (resp. } \Psi_T(w)) \\ \Psi_I(wz) \geq \Psi_I(z) \text{ (resp. } \Psi_I(w)) \\ \Psi_F(wz) \leq \Psi_F(z) \text{ (resp. } \Psi_F(w)) \end{pmatrix}. \tag{6}$$

Proof. Let Ψ_{TIF} be a left SVN $(\in, \in)$-ideal of S. Obviously, the condition (3) is true by Lemma 2.5. If $\exists\ w, z \in S$ such that $\Psi_T(wz) < \Psi_T(z)$, then $z \in T_{\in}(\Psi_{\text{TIF}}; \Psi_T(z))$ but $wz \notin T_{\in}(\Psi_{\text{TIF}}; \Psi_T(z))$, a contradiction. So $\Psi_T(wz) \geq \Psi_T(z)\ \forall w, z \in S$. Assume that $\Psi_I(ab) < \Psi_I(b)$ for some $a, b \in S$ and take $\beta := \frac{1}{2}(\Psi_I(ab) + \Psi_I(b))$. Then, $b \in I_{\in}(\Psi_{\text{TIF}}; \beta)$ and $ab \notin I_{\in}(\Psi_{\text{TIF}}; \beta)$, which is a contradiction. Hence, $\Psi_I(wz) \geq \Psi_I(z)$ for all $w, z \in S$. If $\Psi_F(wz) > \Psi_F(z)$ for some $w, z \in S$, then $\exists\ \gamma \in [0, 1)$ such that $\Psi_F(wz) \geq \gamma > \Psi_F(z)$. Then, $z \in F_{\in}(\Psi_{\text{TIF}}; \gamma)$ and $wz \notin F_{\in}(\Psi_{\text{TIF}}; \gamma)$, which induces a contradiction. Therefore, $\Psi_F(wz) \leq$

$\Psi_F(z)\ \forall w, z \in S$. Similarly, if Ψ_{TIF} is a right SVN $(\in, \in)$-ideal of S, then $\Psi_T(wz) \geq \Psi_T(w)$, $\Psi_I(wz) \geq \Psi_I(w)$ and $\Psi_F(wz) \leq \Psi_F(w)$ for all $w, z \in S$.

Conversely, suppose that Ψ_{TIF} satisfies $\Psi_T(wz) \geq \Psi_T(w)$, $\Psi_I(wz) \geq \Psi_I(w)$ and $\Psi_F(wz) \leq \Psi_F(w)\ \forall w, z \in S$. Let $w \in T_\in(\Psi_{\text{TIF}}; \alpha) \cap I_\in(\Psi_{\text{TIF}}; \beta) \cap F_\in(\Psi_{\text{TIF}}; \gamma)$. Then,

$$\Psi_T(wz) \geq \Psi_T(w) \geq \alpha,$$

$$\Psi_I(wz) \geq \Psi_I(w) \geq \beta$$

and

$$\Psi_F(wz) \leq \Psi_F(w) \leq \gamma,$$

which imply that $wz \in T_\in(\Psi_{\text{TIF}}; \alpha) \cap I_\in(\Psi_{\text{TIF}}; \beta) \cap F_\in(\Psi_{\text{TIF}}; \gamma)$. Hence, Ψ_{TIF} is a right SVN $(\in, \in)$-ideal of S. Similarly, if Ψ_{TIF} satisfies $\Psi_T(wz) \geq \Psi_T(z)$, $\Psi_I(wz) \geq \Psi_I(z)$ and $\Psi_F(wz) \leq \Psi_F(z)$ for all $w, z \in S$, then Ψ_{TIF} is a left SVN $(\in, \in)$-ideal of S. □

Let Δ be the diagonal relation on S and let χ_Δ be the characteristic function of Δ in $S \times S$. Given an SVNS Ψ_{TIF} in S, consider the following relations on S:

$$\begin{aligned}
\mathcal{R}_{(T,\alpha)} &:= \{(w, z) \in S \times S | \max\{\chi_\Delta(w, z), \min\{\Psi_T(w), \Psi_T(z)\}\} \geq \alpha\} \\
\mathcal{R}_{(I,\beta)} &:= \{(w, z) \in S \times S | \max\{\chi_\Delta(w, z), \min\{\Psi_I(w), \Psi_I(z)\}\} \geq \beta\} \\
\mathcal{R}_{(F,\gamma)} &:= \{(w, z) \in S \times S | \min\{f_\Delta(w, z), \max\{\Psi_F(w), \Psi_F(z)\}\} \leq \gamma\}
\end{aligned} \tag{7}$$

where $\alpha, \beta \in (0, 1]$, $\gamma \in [0, 1)$ and

$$f_\Delta : S \times S \to [0, 1],\ (w, z) \mapsto 1 - \chi_\Delta(w, z).$$

It is simple to demonstrate that $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are equivalence relations on S. Let Ψ_{TIF} be an SVN $(\in, \in)$-ideal of S. Let $a, w, z \in S$ be such that $(w, z) \in \mathcal{R}_{(T,\alpha)}$. If $aw = az$, then $\chi_\Delta(aw, az) = 1$ and so

$$\max\{\chi_\Delta(aw, az), \min\{\Psi_T(aw), \Psi_T(az)\}\} = 1 \geq \alpha.$$

Thus $(aw, az) \in \mathcal{R}_{(T,\alpha)}$. Similarly, we can verify that

$$\max\{\chi_\Delta(aw, az), \min\{\Psi_I(aw), \Psi_I(az)\}\} = 1 \geq \beta,$$

that is, $(aw, az) \in \mathcal{R}_{(I,\beta)}$. If $aw = az$, then $f_\Delta(w, z) = 1 - (w, z) = 0$ and so

$$\min\{f_\Delta(w, z), \max\{\Psi_F(w), \Psi_F(z)\}\} = 0 \leq \gamma,$$

i.e., $(aw, az) \in \mathcal{R}_{(F,\gamma)}$. Suppose that $aw \neq az$. Then, $\chi_\Delta(aw, az) = 0$ and $w \neq z$. Since Ψ_{TIF} is a left SVN $(\in, \in)$-ideal of S, it follows that

$$\begin{aligned}\max\{\chi_\Delta(aw, az), \min\{\Psi_T(aw), \Psi_T(az)\}\} &= \min\{\Psi_T(aw), \Psi_T(az)\}\\ &\geq \min\{\Psi_T(w), \Psi_T(z)\}\\ &\geq \alpha,\end{aligned}$$

$$\begin{aligned}\max\{\chi_\Delta(aw, az), \min\{\Psi_I(aw), \Psi_I(az)\}\} &= \min\{\Psi_I(aw), \Psi_I(az)\}\\ &\geq \min\{\Psi_I(w), \Psi_I(z)\}\\ &\geq \beta\end{aligned}$$

and

$$\begin{aligned}\min\{f_\Delta(ax, ay), \max\{\Psi_F(aw), \Psi_F(az)\}\} &= \max\{\Psi_F(aw), \Psi_F(az)\}\\ &\leq \max\{\Psi_F(w), \Psi_F(z)\}\\ &\leq \gamma.\end{aligned}$$

Thus $(aw, az) \in \mathcal{R}_{(T,\alpha)}$, $(aw, az) \in \mathcal{R}_{(I,\beta)}$ and $(aw, az) \in \mathcal{R}_{(F,\gamma)}$. Similarly, we can verify that $(wa, za) \in \mathcal{R}_{(T,\alpha)}$, $(wa, za) \in \mathcal{R}_{(I,\beta)}$ and $(wa, za) \in \mathcal{R}_{(F,\gamma)}$. Therefore, $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are congruence relations on S.

We summarize the result as a lemma.

Lemma 3.5. *If Ψ_{TIF} is an* SVN $(\in, \in)$*-ideal of S, then $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are congruence relations on S.*

Given $w \in S$, let $[w]_{(T,\alpha)}$ (resp., $[w]_{(I,\beta)}$ and $[w]_{(F,\gamma)}$) denote the equivalence class of x which is called T*-equivalence class* (resp. I*-equivalence class* and F*-equivalence class*) of x.

Lemma 3.6. *If Ψ_{TIF} is an* SVN $(\in, \in)$*-ideal of S, then $[w]_{(T,\alpha)}[z]_{(T,\alpha)} \subseteq [wz]_{(T,\alpha)}$, $[w]_{(I,\beta)}[z]_{(I,\beta)} \subseteq [wz]_{(I,\beta)}$ and $[w]_{(F,\gamma)}[z]_{(F,\gamma)} \subseteq [wz]_{(F,\gamma)}$ for every $\alpha, \beta, \gamma \in [0, 1]$.*

Proof. Let $a \in [w]_{(T,\alpha)}[z]_{(T,\alpha)}$. Then, $a = w'z'$ for some $w' \in [w]_{(T,\alpha)}$ and $z' \in [z]_{(T,\alpha)}$. Thus $\Psi_T(w, w') \geq \alpha$ and $\Psi_T(z, z') \geq \alpha$. Since $\mathcal{R}_{(T,\alpha)}$ is a congruence relation on S, it follows that $\Psi_T(wz, w'z') \geq \alpha$, that is, $a = w'z' \in [wz]_{(T,\alpha)}$. Hence, $[w]_{(T,\alpha)}[z]_{(T,\alpha)} \subseteq [wz]_{(T,\alpha)}$. If $b \in [w]_{(I,\beta)}[z]_{(I,\beta)}$, then $b = w'z'$ for some $w' \in [w]_{(I,\beta)}$ and $z' \in [z]_{(I,\beta)}$. Hence, $\Psi_I(w, w') \geq \beta$ and $\Psi_I(z, z') \geq \beta$ which imply that $\Psi_I(wz, w'z') \geq \beta$, that is, $b = w'z' \in [wz]_{(I,\beta)}$. This shows that $[w]_{(I,\beta)}[z]_{(I,\beta)} \subseteq [wz]_{(I,\beta)}$. Suppose that $c \in [w]_{(F,\gamma)}[z]_{(F,\gamma)}$. Then, $c = ab$ for some $a \in [w]_{(F,\gamma)}$ and $b \in [z]_{(F,\gamma)}$. Thus, $\Psi_F(a, w) \leq \gamma$ and $\Psi_F(b, z) \leq \gamma$, and so $\Psi_F(ab, wz) \leq \gamma$ since $\mathcal{R}_{(F,\gamma)}$ is a congruence relation on S. Therefore, $c = ab \in [wz]_{(F,\gamma)}$, which proves $[w]_{(F,\gamma)}[z]_{(F,\gamma)} \subseteq [wz]_{(F,\gamma)}$. □

The following example illustrates Lemma 3.6.

Example 3.7. Consider the SVN $(\in, \in)$-ideal Ψ_{TIF} of S in Example 3.3. If we take $(\alpha, \beta, \gamma) = (0.44, 0.37, 0.63)$, then

$$\mathcal{R}_{(T,\alpha)} = \{(\varsigma_1, \varsigma_1), (\varsigma_2, \varsigma_2), (\varsigma_3, \varsigma_3), (\varsigma_4, \varsigma_4), (\varsigma_2, \varsigma_4)\},$$

$$\mathcal{R}_{(I,\beta)} = \{(\varsigma_1, \varsigma_1), (\varsigma_2, \varsigma_2), (\varsigma_3, \varsigma_3), (\varsigma_4, \varsigma_4), (\varsigma_2, \varsigma_4)\}$$

and

$$\mathcal{R}_{(F,\gamma)} = \{(\varsigma_1, \varsigma_1), (\varsigma_2, \varsigma_2), (\varsigma_3, \varsigma_3), (\varsigma_4, \varsigma_4), (\varsigma_2, \varsigma_4)\}.$$

Hence, $[\varsigma_1]_{(T,\alpha)} = \{\varsigma_1\}$, $[\varsigma_2]_{(T,\alpha)} = \{\varsigma_2, \varsigma_4\}$, $[\varsigma_3]_{(T,\alpha)} = \{\varsigma_3\}$, and $[\varsigma_4]_{(T,\alpha)} = \{\varsigma_2, \varsigma_4\}$. It follows that $[\varsigma_1]_{(T,\alpha)}[\varsigma_3]_{(T,\alpha)} = \{\varsigma_2\} \subseteq \{\varsigma_2, \varsigma_4\} = [\varsigma_2]_{(T,\alpha)} = [\varsigma_1\varsigma_3]_{(T,\alpha)}$. In the same way, we can check $[w]_{(I,\beta)}[z]_{(I,\beta)} \subseteq [wz]_{(I,\beta)}$ and $[w]_{(F,\gamma)}[z]_{(F,\gamma)} \subseteq [wz]_{(F,\gamma)}$ for $w, z \in S$.

Definition 3.8. The congruence relation $\mathcal{R}_{(T,\alpha)}$ (resp., $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$) on S is said to be *complete* if $[w]_{(T,\alpha)}[z]_{(T,\alpha)} = [wz]_{(T,\alpha)}$ (resp., $[w]_{(I,\beta)}[z]_{(I,\beta)} = [wz]_{(I,\beta)}$ and $[w]_{(F,\gamma)}[z]_{(F,\gamma)} = [wz]_{(F,\gamma)}$) for all $w, z \in S$.

Example 3.9. Consider a semigroup $S = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ with the "$\cdot$" operation given by Table 2.

TABLE 2. Table for "$\cdot$" operation

$\cdot$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_2	ς_3	ς_4
ς_2	ς_2	ς_2	ς_3	ς_4
ς_3	ς_3	ς_3	ς_3	ς_4
ς_4	ς_4	ς_4	ς_4	ς_3

Let Ψ_{TIF} be an SVNS in S which is shown as:

$$\Psi_{\text{TIF}} = \{\langle \varsigma_1, (0.11, 0.27, 0.68)\rangle, \langle \varsigma_2, (0.44, 0.47, 0.57)\rangle,$$
$$\langle \varsigma_3, (0.77, 0.67, 0.29)\rangle, \langle \varsigma_4, (0.77, 0.67, 0.29)\rangle\}.$$

Then, Ψ_{TIF} is an SVN $(\in, \in)$-ideal of S. It is routine to verify that $[w]_{(T,\alpha)}[z]_{(T,\alpha)} = [wz]_{(T,\alpha)}$, $[w]_{(I,\beta)}[z]_{(I,\beta)} = [wz]_{(I,\beta)}$ and $[w]_{(F,\gamma)}[z]_{(F,\gamma)} = [wz]_{(F,\gamma)}$ for all $w, z \in S$ where $(\alpha, \beta, \gamma) = (0.77, 0.67, 0.29)$. Therefore, $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are complete congruence relations on S for $(\alpha, \beta, \gamma) = (0.77, 0.67, 0.29)$.

Definition 3.10. Let Ψ_{TIF} be an SVN $(\in, \in)$-ideal of S and let N be a nonempty subset of S. Given $q \in \{T, I, F\}$, the $\mathcal{R}_q$*-lower approximation* and $\mathcal{R}_q$*-upper approximation* of X are defined to be the sets

$$\begin{aligned}
\underline{\mathcal{R}}_T(N;\alpha) &:= \{w \in S | [w]_{(T,\alpha)} \subseteq N\}\\
\underline{\mathcal{R}}_I(N;\beta) &:= \{w \in S | [w]_{(I,\beta)} \subseteq N\}\\
\underline{\mathcal{R}}_F(N;\gamma) &:= \{w \in S | [w]_{(F,\gamma)} \subseteq N\}
\end{aligned}$$

and

$$\begin{aligned}
\overline{\mathcal{R}}_T(N;\alpha) &:= \{w \in S | [w]_{(T,\alpha)} \cap N \neq \emptyset\}\\
\overline{\mathcal{R}}_I(N;\beta) &:= \{w \in S | [w]_{(I,\beta)} \cap N \neq \emptyset\}\\
\overline{\mathcal{R}}_F(N;\gamma) &:= \{w \in S | [w]_{(F,\gamma)} \cap N \neq \emptyset\},
\end{aligned}$$

respectively.

By routine calculations, we have the next proposition.

Proposition 3.11. *Let* Ψ_{TIF} *be an* SVN $(\in, \in)$*-ideal of* S*. For any nonempty subsets* G *and* H *of* S*, the following assertions are valid.*

$$\begin{aligned}
&\underline{\mathcal{R}}_T(G;\alpha) \subseteq G \subseteq \overline{\mathcal{R}}_T(G;\alpha),\\
&\underline{\mathcal{R}}_I(G;\beta) \subseteq G \subseteq \overline{\mathcal{R}}_I(G;\beta),\\
&\underline{\mathcal{R}}_F(G;\gamma) \subseteq G \subseteq \overline{\mathcal{R}}_F(G;\gamma),
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\underline{\mathcal{R}}_T(G \cap H;\alpha) = \underline{\mathcal{R}}_T(G;\alpha) \cap \underline{\mathcal{R}}_T(H;\alpha),\\
&\underline{\mathcal{R}}_I(G \cap H;\beta) = \underline{\mathcal{R}}_I(G;\beta) \cap \underline{\mathcal{R}}_I(H;\beta),\\
&\underline{\mathcal{R}}_F(G \cap H;\gamma) = \underline{\mathcal{R}}_F(G;\gamma) \cap \underline{\mathcal{R}}_F(H;\gamma),
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\overline{\mathcal{R}}_T(G \cap H;\alpha) \subseteq \overline{\mathcal{R}}_T(G;\alpha) \cap \overline{\mathcal{R}}_T(H;\alpha),\\
&\overline{\mathcal{R}}_I(G \cap H;\beta) \subseteq \overline{\mathcal{R}}_I(G;\beta) \cap \overline{\mathcal{R}}_I(H;\beta),\\
&\overline{\mathcal{R}}_F(G \cap H;\gamma) \subseteq \overline{\mathcal{R}}_F(G;\gamma) \cap \overline{\mathcal{R}}_F(H;\gamma),
\end{aligned} \tag{10}$$

$$G \subseteq H \;\Rightarrow \begin{pmatrix} \underline{\mathcal{R}}_T(G;\alpha) \subseteq \underline{\mathcal{R}}_T(H;\alpha),\\ \underline{\mathcal{R}}_I(G;\beta) \subseteq \underline{\mathcal{R}}_I(H;\beta),\\ \underline{\mathcal{R}}_F(G;\gamma) \subseteq \underline{\mathcal{R}}_F(H;\gamma), \end{pmatrix}, \tag{11}$$

$$G \subseteq H \;\Rightarrow \begin{pmatrix} \overline{\mathcal{R}}_T(G;\alpha) \subseteq \overline{\mathcal{R}}_T(H;\alpha),\\ \overline{\mathcal{R}}_I(G;\beta) \subseteq \overline{\mathcal{R}}_I(H;\beta),\\ \overline{\mathcal{R}}_F(G;\gamma) \subseteq \overline{\mathcal{R}}_F(H;\gamma), \end{pmatrix}, \tag{12}$$

$$\begin{aligned}&\underline{\mathcal{R}}_T(G;\alpha)\cup\underline{\mathcal{R}}_T(H;\alpha)\subseteq\underline{\mathcal{R}}_T(G\cup H;\alpha),\\&\underline{\mathcal{R}}_I(G;\beta)\cup\underline{\mathcal{R}}_I(H;\beta)\subseteq\underline{\mathcal{R}}_I(G\cup H;\beta),\\&\underline{\mathcal{R}}_F(G;\gamma)\cup\underline{\mathcal{R}}_F(H;\gamma)\subseteq\underline{\mathcal{R}}_F(G\cup H;\gamma),\end{aligned}\tag{13}$$

$$\begin{aligned}&\overline{\mathcal{R}}_T(G\cup H;\alpha)=\overline{\mathcal{R}}_T(G;\alpha)\cup\overline{\mathcal{R}}_T(H;\alpha),\\&\overline{\mathcal{R}}_I(G\cup H;\beta)=\overline{\mathcal{R}}_I(G;\beta)\cup\overline{\mathcal{R}}_I(H;\beta),\\&\overline{\mathcal{R}}_F(G\cup H;\gamma)=\overline{\mathcal{R}}_F(G;\gamma)\cup\overline{\mathcal{R}}_F(H;\gamma),\end{aligned}\tag{14}$$

$$\begin{aligned}&\underline{\mathcal{R}}_T(\underline{\mathcal{R}}_T(G;\alpha);\alpha)=\underline{\mathcal{R}}_T(G;\alpha),\\&\underline{\mathcal{R}}_I(\underline{\mathcal{R}}_I(G;\beta);\beta)=\underline{\mathcal{R}}_I(G;\beta),\\&\underline{\mathcal{R}}_F(\underline{\mathcal{R}}_F(G;\gamma);\gamma)=\underline{\mathcal{R}}_F(G;\gamma),\end{aligned}\tag{15}$$

$$\begin{aligned}&\overline{\mathcal{R}}_T(\overline{\mathcal{R}}_T(G;\alpha);\alpha)=\overline{\mathcal{R}}_T(G;\alpha),\\&\overline{\mathcal{R}}_I(\overline{\mathcal{R}}_I(G;\beta);\beta)=\overline{\mathcal{R}}_I(G;\beta),\\&\overline{\mathcal{R}}_F(\overline{\mathcal{R}}_F(G;\gamma);\gamma)=\overline{\mathcal{R}}_F(G;\gamma),\end{aligned}\tag{16}$$

$$\begin{aligned}&\underline{\mathcal{R}}_T(\overline{\mathcal{R}}_T(G;\alpha);\alpha)=\underline{\mathcal{R}}_T(G;\alpha),\\&\underline{\mathcal{R}}_I(\overline{\mathcal{R}}_I(G;\beta);\beta)=\underline{\mathcal{R}}_I(G;\beta),\\&\underline{\mathcal{R}}_F(\overline{\mathcal{R}}_F(G;\gamma);\gamma)=\underline{\mathcal{R}}_F(G;\gamma),\end{aligned}\tag{17}$$

$$\begin{aligned}&\overline{\mathcal{R}}_T(\underline{\mathcal{R}}_T(G;\alpha);\alpha)=\overline{\mathcal{R}}_T(G;\alpha),\\&\overline{\mathcal{R}}_I(\underline{\mathcal{R}}_I(G;\beta);\beta)=\overline{\mathcal{R}}_I(G;\beta),\\&\overline{\mathcal{R}}_F(\underline{\mathcal{R}}_F(G;\gamma);\gamma)=\overline{\mathcal{R}}_F(G;\gamma).\end{aligned}\tag{18}$$

Proposition 3.12. *Let Ψ_{TIF} be an* SVN $(\in, \in)$*-ideal of S. For any nonempty subsets G and H of S, we have the following assertion.*

$$\begin{aligned}&\overline{\mathcal{R}}_T(G;\alpha)\overline{\mathcal{R}}_T(H;\alpha)\subseteq\overline{\mathcal{R}}_T(GH;\alpha),\\&\overline{\mathcal{R}}_I(G;\beta)\overline{\mathcal{R}}_I(H;\beta)\subseteq\overline{\mathcal{R}}_I(GH;\beta),\\&\overline{\mathcal{R}}_F(G;\gamma)\overline{\mathcal{R}}_F(H;\gamma)\subseteq\overline{\mathcal{R}}_F(GH;\gamma).\end{aligned}\tag{19}$$

Proof. Let $w\in\overline{\mathcal{R}}_T(G;\alpha)\overline{\mathcal{R}}_T(H;\alpha)$. Then, $w=ab$ for some $a\in\overline{\mathcal{R}}_T(G;\alpha)$ and $b\in\overline{\mathcal{R}}_T(H;\alpha)$. It follows that $\exists\ w_a,w_b\in S$ such that $w_a\in[a]_{(T,\alpha)}\cap G$ and $w_b\in[b]_{(T,\alpha)}\cap H$. Since $\mathcal{R}_{(T,\alpha)}$ is a congruence relations on S, we have $w_aw_b\in[ab]_{(T,\alpha)}\cap GH$, and so $w=ab\in\overline{\mathcal{R}}_T(GH;\alpha)$. Similarly, we get $\overline{\mathcal{R}}_I(G;\beta)\overline{\mathcal{R}}_I(H;\beta)\subseteq\overline{\mathcal{R}}_I(GH;\beta)$. If $w\in\overline{\mathcal{R}}_F(G;\gamma)\overline{\mathcal{R}}_F(H;\gamma)$, then $\exists\ a\in\overline{\mathcal{R}}_F(G;\gamma)$ and $b\in\overline{\mathcal{R}}_F(H;\gamma)$ such that $w=ab$. Hence, $[a]_{(F,\gamma)}\cap G\neq\emptyset$ and $[b]_{(F,\gamma)}\cap H\neq\emptyset$, which imply that $\exists\ w_a\in[a]_{(F,\gamma)}\cap G$ and $w_b\in[b]_{(F,\gamma)}\cap H$. Since $\mathcal{R}_{(F,\gamma)}$ is a congruence relations on S, it follows that $w_aw_b\in[ab]_{(F,\gamma)}\cap GH$. Therefore, $w=ab\in\overline{\mathcal{R}}_F(GH;\gamma)$, and so $\overline{\mathcal{R}}_F(G;\gamma)\overline{\mathcal{R}}_F(H;\gamma)\subseteq\overline{\mathcal{R}}_F(GH;\gamma)$. □

In Proposition 3.12, the reverse inclusion relationship does not hold as seen in the next example.

Example 3.13. Consider the SVN $(\in, \in)$-ideal $\Psi_{\rm TIF}$ of S in Example 3.3. If we take $(\alpha, \beta, \gamma) = (0.44, 0.37, 0.63)$, then $\overline{\mathcal{R}}_T(\{\varsigma_1\}; \alpha)\overline{\mathcal{R}}_T(\{\varsigma_3\}; \alpha) = \{\varsigma_1\}\{\varsigma_3\} = \{\varsigma_2\}$, $\overline{\mathcal{R}}_I(\{\varsigma_1\}; \beta)\overline{\mathcal{R}}_I(\{\varsigma_3\}; \beta) = \{\varsigma_1\}\{\varsigma_3\} = \{\varsigma_2\}$, and $\overline{\mathcal{R}}_F(\{\varsigma_1\}; \gamma)\overline{\mathcal{R}}_F(\{\varsigma_3\}; \gamma) = \{\varsigma_1\}\{\varsigma_3\} = \{\varsigma_2\}$. Also $\overline{\mathcal{R}}_T(\{\varsigma_1\}\{\varsigma_3\}; \alpha) = \{\varsigma_2, \varsigma_4\}$, $\overline{\mathcal{R}}_I(\{\varsigma_1\}\{\varsigma_3\}; \beta) = \{\varsigma_2, \varsigma_4\}$ and $\overline{\mathcal{R}}_F(\{\varsigma_1\}\{\varsigma_3\}; \gamma) = \{\varsigma_2, \varsigma_4\}$. Therefore, $\overline{\mathcal{R}}_T(\{\varsigma_1\}\{\varsigma_3\}; \alpha) \nsubseteq \overline{\mathcal{R}}_T(\{\varsigma_1\}; \alpha)\overline{\mathcal{R}}_T(\{\varsigma_3\}; \alpha)$, $\overline{\mathcal{R}}_I(\{\varsigma_1\}\{\varsigma_3\}; \beta) \nsubseteq \overline{\mathcal{R}}_I(\{\varsigma_1\}; \beta)\overline{\mathcal{R}}_I(\{\varsigma_3\}; \beta)$, and $\overline{\mathcal{R}}_F(\{\varsigma_1\}\{\varsigma_3\}; \gamma) \nsubseteq \overline{\mathcal{R}}_F(\{\varsigma_1\}; \gamma)\overline{\mathcal{R}}_F(\{\varsigma_3\}; \gamma)$.

Proposition 3.14. *If congruence relations $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ on S are complete, then*

$$\begin{aligned} &\underline{\mathcal{R}}_T(G; \alpha)\underline{\mathcal{R}}_T(H; \alpha) \subseteq \underline{\mathcal{R}}_T(GH; \alpha), \\ &\underline{\mathcal{R}}_I(G; \beta)\underline{\mathcal{R}}_I(H; \beta) \subseteq \underline{\mathcal{R}}_I(GH; \beta), \\ &\underline{\mathcal{R}}_F(G; \gamma)\underline{\mathcal{R}}_F(H; \gamma) \subseteq \underline{\mathcal{R}}_F(GH; \gamma) \end{aligned} \tag{20}$$

for all nonempty subsets G and H of S.

Proof. Let $w \in \underline{\mathcal{R}}_T(G; \alpha)\underline{\mathcal{R}}_T(H; \alpha)$. Then, $w = ab$ for some $a \in \underline{\mathcal{R}}_T(G; \alpha)$ and $b \in \underline{\mathcal{R}}_T(H; \alpha)$. Since $\mathcal{R}_{(T,\alpha)}$is a complete congruence relations on S, we get $[a]_{(T,\alpha)}[b]_{(T,\alpha)} = [ab]_{(T,\alpha)} \subseteq GH$. Hence, $w = ab \in \underline{\mathcal{R}}_T(GH; \alpha)$. Therefore, $\underline{\mathcal{R}}_T(G; \alpha)\underline{\mathcal{R}}_T(H; \alpha) \subseteq \underline{\mathcal{R}}_T(GH; \alpha)$. Similarly, we have $\underline{\mathcal{R}}_I(G; \beta)\underline{\mathcal{R}}_I(H; \beta) \subseteq \underline{\mathcal{R}}_I(GH; \beta)$. If $w \in \underline{\mathcal{R}}_F(G; \gamma)\underline{\mathcal{R}}_F(H; \gamma)$, then $\exists\, a, b \in S$ such that $w = ab$, $a \in \underline{\mathcal{R}}_F(G; \gamma)$ and $b \in \underline{\mathcal{R}}_F(H; \gamma)$. Hence, $[a]_{(F,\gamma)}[b]_{(F,\gamma)} = [ab]_{(F,\gamma)} \subseteq GH$, and so $w = ab \in \underline{\mathcal{R}}_F(GH; \alpha)$. Therefore, $\underline{\mathcal{R}}_F(G; \gamma)\underline{\mathcal{R}}_F(H; \gamma) \subseteq \underline{\mathcal{R}}_F(GH; \gamma)$. □

In Proposition 3.14, if congruence relations $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ on S are not complete, then the inclusion relationship does not hold as seen in the next example.

Example 3.15. Consider the SVN $(\in, \in)$-ideal $\Psi_{\rm TIF}$ of S in Example 3.3, and take $(\alpha, \beta, \gamma) = (0.44, 0.37, 0.63)$. Then, $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are not complete. Obviously, $\underline{\mathcal{R}}_T(G; \alpha)\underline{\mathcal{R}}_T(H; \alpha) = \{\varsigma_2\} \nsubseteq \emptyset = \underline{\mathcal{R}}_T(GH; \alpha)$, $\underline{\mathcal{R}}_I(G; \beta)\underline{\mathcal{R}}_I(H; \beta) = \{\varsigma_2\} \nsubseteq \emptyset = \underline{\mathcal{R}}_I(GH; \beta)$, and $\underline{\mathcal{R}}_F(G; \gamma)\underline{\mathcal{R}}_F(H; \gamma) = \{\varsigma_2\} \nsubseteq \emptyset = \underline{\mathcal{R}}_F(GH; \gamma)$ where $G = H = \{\varsigma_2, \varsigma_3\}$.

The results discussed above will contribute to the study of rough subsemigroups and ideals.

Definition 3.16. Let $\Psi_{\rm TIF}$ be an SVN $(\in, \in)$-ideal of S and let X be a nonempty subset of S. Given $q \in \{T, I, F\}$, if $\mathcal{R}_q$-lower approximation (resp., $\mathcal{R}_q$-upper approximation) of X is a subsemigroup of S, then we say that X is a *$\mathcal{R}_q$-lower rough subsemigroup* (resp., *$\mathcal{R}_q$-upper rough subsemigroup*) of S. If $\mathcal{R}_q$-lower approximation (resp., $\mathcal{R}_q$-upper approximation) of X is an ideal of S, then we say that X is a *$\mathcal{R}_q$-lower rough ideal* (resp., *$\mathcal{R}_q$-upper rough ideal*) of S.

Theorem 3.17. *Let* Ψ_{TIF} *be an* SVN $(\in, \in)$*-ideal of* S *and* $(\alpha, \beta, \gamma) \in (0,1] \times (0,1] \times [0,1)$. *If* G *is a subsemigroup (resp., ideal) of* S*, then it is an* $\mathcal{R}_q$*-upper rough subsemigroup (resp.,* $\mathcal{R}_q$*-upper rough ideal) of* S *for* $q \in \{T, I, F\}$.

Proof. Suppose G is a subsemigroup of S, then $GG \subseteq G$, and so

$$\overline{\mathcal{R}}_T(G;\alpha)\overline{\mathcal{R}}_T(G;\alpha) \subseteq \overline{\mathcal{R}}_T(GG;\alpha) \subseteq \overline{\mathcal{R}}_T(G;\alpha),$$
$$\overline{\mathcal{R}}_I(G;\beta)\overline{\mathcal{R}}_I(G;\beta) \subseteq \overline{\mathcal{R}}_I(GG;\beta) \subseteq \overline{\mathcal{R}}_I(G;\beta)$$

and

$$\overline{\mathcal{R}}_F(G;\gamma)\overline{\mathcal{R}}_F(G;\gamma) \subseteq \overline{\mathcal{R}}_F(GG;\gamma) \subseteq \overline{\mathcal{R}}_F(G;\gamma)$$

by (12) and Proposition 3.12. Hence, $\overline{\mathcal{R}}_T(G;\alpha)$, $\overline{\mathcal{R}}_I(G;\beta)$ and $\overline{\mathcal{R}}_F(G;\gamma)$ are subsemigroups of S, and so G is an $\mathcal{R}_q$-upper rough subsemigroup of S for $q \in \{T, I, F\}$. If G is an ideal of S, then $SGS \subseteq G$. Using (12) and Proposition 3.12, we have

$$\overline{\mathcal{R}}_T(S;\alpha)\overline{\mathcal{R}}_T(G;\alpha)\overline{\mathcal{R}}_T(S;\alpha) \subseteq \overline{\mathcal{R}}_T(SGS;\alpha) \subseteq \overline{\mathcal{R}}_T(G;\alpha),$$
$$\overline{\mathcal{R}}_I(S;\beta)\overline{\mathcal{R}}_I(G;\beta)\overline{\mathcal{R}}_I(S;\beta) \subseteq \overline{\mathcal{R}}_I(SGS;\beta) \subseteq \overline{\mathcal{R}}_I(G;\beta)$$

and

$$\overline{\mathcal{R}}_F(S;\gamma)\overline{\mathcal{R}}_F(G;\gamma)\overline{\mathcal{R}}_F(S;\gamma) \subseteq \overline{\mathcal{R}}_F(SGS;\gamma) \subseteq \overline{\mathcal{R}}_F(G;\gamma).$$

This shows that $\overline{\mathcal{R}}_T(G;\alpha)$, $\overline{\mathcal{R}}_I(G;\beta)$ and $\overline{\mathcal{R}}_F(G;\gamma)$ are ideals of S. Therefore, G is an $\mathcal{R}_q$-upper rough ideal of S for $q \in \{T, I, F\}$. □

Next example demonstrates that there is an $\mathcal{R}_q$-upper rough ideal for $q \in \{T, I, F\}$ which is not an ideal.

Example 3.18. Let $S = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ be a semigroup with the "$\cdot$" operation given by Table 3.

TABLE 3. Table for "$\cdot$" operation

$\cdot$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_2	ς_3	ς_4
ς_2	ς_2	ς_2	ς_2	ς_2
ς_3	ς_3	ς_3	ς_3	ς_3
ς_4	ς_4	ς_3	ς_2	ς_1

Let Ψ_{TIF} be an SVNS in S which is shown as :

$$\Psi_{\text{TIF}} = \{\langle\varsigma_1, (0.5, 0.6, 0.6)\rangle, \langle\varsigma_2, (0.7, 0.9, 0.2)\rangle,$$
$$\langle\varsigma_3, (0.7, 0.9, 0.2)\rangle, \langle\varsigma_4, (0.3, 0.4, 0.8)\rangle\}.$$

Clearly, Ψ_{TIF} is an SVN $(\in, \in)$-ideal of S. Consider $(\alpha, \beta, \gamma) \in (0, 1] \times (0, 1] \times [0, 1)$ such that the subsets $\{\varsigma_1\}$, $\{\varsigma_4\}$ and $\{\varsigma_2, \varsigma_3\}$ are the $\mathcal{R}_q$-congruence classes for $q \in \{(T, \alpha), (I, \beta), (F, \gamma)\}$. Then, $\overline{\mathcal{R}}_T(\{\varsigma_2\}; \alpha) = \{\varsigma_2, \varsigma_3\}$, $\overline{\mathcal{R}}_I(\{\varsigma_2\}; \beta) = \{\varsigma_2, \varsigma_3\}$ and $\overline{\mathcal{R}}_F(\{\varsigma_2\}; \gamma) = \{\varsigma_2, \varsigma_3\}$ which are ideals of S. Hence, $\{\varsigma_2\}$ is an $\mathcal{R}_q$-upper rough ideal for $q \in \{T, I, F\}$. But it is not an ideal of S since $S\{\varsigma_2\} = \{\varsigma_2, \varsigma_3\} \nsubseteq \{\varsigma_2\}$.

Theorem 3.19. *Let Ψ_{TIF} be an* SVN $(\in, \in)$*-ideal of S. in which $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are complete congruence relations on S. If G is a subsemigroup (resp., ideal) of S, then it is an $\mathcal{R}_q$-lower rough subsemigroup (resp., $\mathcal{R}_q$-lower rough ideal) of S for $q \in \{T, I, F\}$.*

Proof. If G is a subsemigroup of S, then $GG \subseteq G$ and thus

$$\underline{\mathcal{R}}_T(G; \alpha)\underline{\mathcal{R}}_T(G; \alpha) \subseteq \underline{\mathcal{R}}_T(GG; \alpha) \subseteq \underline{\mathcal{R}}_T(G; \alpha),$$
$$\underline{\mathcal{R}}_I(G; \beta)\underline{\mathcal{R}}_I(G; \beta) \subseteq \underline{\mathcal{R}}_I(GG; \beta) \subseteq \underline{\mathcal{R}}_I(G; \beta),$$
$$\underline{\mathcal{R}}_F(G; \gamma)\underline{\mathcal{R}}_F(G; \gamma) \subseteq \underline{\mathcal{R}}_F(GG; \gamma) \subseteq \underline{\mathcal{R}}_F(G; \gamma)$$

by (11) and (20). Therefore, $\underline{\mathcal{R}}_T(G; \alpha)$, $\underline{\mathcal{R}}_I(G; \alpha)$ and $\underline{\mathcal{R}}_F(G; \alpha)$ are subsemigroups of S, that is, G is an $\mathcal{R}_q$-lower rough subsemigroup of S for $q \in \{T, I, F\}$. If G is an ideal of S, then $SGS \subseteq G$. It follows from (11) and (20) that

$$\underline{\mathcal{R}}_T(S; \alpha)\underline{\mathcal{R}}_T(G; \alpha)\underline{\mathcal{R}}_T(S; \alpha) \subseteq \underline{\mathcal{R}}_T(SGS; \alpha) \subseteq \underline{\mathcal{R}}_T(G; \alpha),$$
$$\underline{\mathcal{R}}_I(S; \beta)\underline{\mathcal{R}}_I(G; \beta)\underline{\mathcal{R}}_I(S; \beta) \subseteq \underline{\mathcal{R}}_I(SGS; \beta) \subseteq \underline{\mathcal{R}}_I(G; \beta),$$
$$\underline{\mathcal{R}}_F(S; \gamma)\underline{\mathcal{R}}_F(G; \gamma)\underline{\mathcal{R}}_F(S; \gamma) \subseteq \underline{\mathcal{R}}_F(SGS; \gamma) \subseteq \underline{\mathcal{R}}_F(G; \gamma).$$

Hence, $\underline{\mathcal{R}}_T(G; \alpha)$, $\underline{\mathcal{R}}_I(G; \alpha)$ and $\underline{\mathcal{R}}_F(G; \alpha)$ are ideals of S, and therefore G is an $\mathcal{R}_q$-lower rough ideal of S for $q \in \{T, I, F\}$. □

The example below demonstrates that there is an $\mathcal{R}_q$-lower rough subsemigroup for $q \in \{T, I, F\}$ which is not a subsemigroup.

Example 3.20. Consider the SVN $(\in, \in)$-ideal Ψ_{TIF} of S in Example 3.9. Then, $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are complete congruence relations on S for $(\alpha, \beta, \gamma) = (0.77, 0.67, 0.29)$. Also, $\underline{\mathcal{R}}_T(\{\varsigma_1, \varsigma_2, \varsigma_4\}; \alpha) = \{\varsigma_1, \varsigma_2\}$, $\underline{\mathcal{R}}_I(\{\varsigma_1, \varsigma_2, \varsigma_4\}; \beta) = \{\varsigma_1, \varsigma_2\}$ and $\underline{\mathcal{R}}_F(\{\varsigma_1, \varsigma_2, \varsigma_4\}; \gamma) = \{\varsigma_1, \varsigma_2\}$ are subsemigroups of S. Hence, $\{\varsigma_1, \varsigma_2, \varsigma_4\}$ is an $\mathcal{R}_q$-lower rough subsemigroup of S for $q \in \{T, I, F\}$,. but it is not a subsemigroup of S since $\{\varsigma_1, \varsigma_2, \varsigma_4\}\{\varsigma_1, \varsigma_2, \varsigma_4\} = S \nsubseteq \{\varsigma_1, \varsigma_2, \varsigma_4\}$.

4. Conclusions

The application of the SVN set gained attention among researchers. This paper found a new link between semigroups and SVNSs by introducing an SVN $(\in,\in)$-subsemigroup and an SVN $(\in,\in)$-ideal in semigroups, and studying their properties. Special congruence relations induced by an SVN $(\in,\in)$-ideal in semigroups have been introduced. We have introduced the lower ($\mathcal{R}_q$-lower approximation) and upper approximations ($\mathcal{R}_q$-upper approximation) for $q \in \{T, I, F\}$ based on an SVN $(\in,\in)$-ideal in semigroups, and have discussed related properties. We also have defined the concepts of lower and upper subsemigroups/ideals, so called the $\mathcal{R}_q$-lower subsemigroup/ideal and the $\mathcal{R}_q$-upper subsemigroup/ideal for $q \in \{T, I, F\}$, and have considered the relationships between subsemigroups/ideals and $\mathcal{R}_q$-lower (upper) subsemigroups/ideals. In future work, various types of rough SVN ideals in semigroups will be defined and discussed. In addition, the idea in this research article can be analyzed according to the works in [18–22], which will be the way for much future work.

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EOQ model with price, marketing, service and green dependent neutrosophic demand under uncertain resource constraint: A geometric programming approach

Chaitali Kar [1,*], Tapan Kumar Roy [1] and Manoranjan Maiti [2]

[1] Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, India;

[1,*] chaitalikar12@gmail.com, [1] roy_t_k@yahoo.co.in

[2]Department of Applied Mathematics, Vidyasagar University, India ; mmaiti2005@yahoo.co.in

*Correspondence: chaitalikar12@gmail.com; Tel.: +919432583376

Abstract. In the competitive market, a customer's choice for an item depends on several factors like management's marketing strategy and service, the item's price and greenness. Demand increases with the marketing strategy, service and item's greenness, but it is inversely related to the item's price. These relations are non-linear and imprecise. Recently, neutrosophic set has been introduced to represent impreciseness more realistically. Moreover, resources (capital, storage space, etc.) are generally uncertain (random or imprecise). Considering the above business scenarios, profit maximization EOQ models with price, marketing, service, and green dependent neutrosophic demand and order quantity dependent unit production cost are developed under different uncertain resource constraints. Models' parameters are pentagonal neutrosophic (PN) numbers. The proposed models are first made deterministic and then solved using the geometric programming technique. The PN parameters are made crisp using the score function. The random, fuzzy, rough and trapezoidal neutrosophic resource constraints in different models are converted to crisp using possibility measure, chance-constrained technique, trust measure and (α, β, γ)-cut with weighted mean, respectively. These processes reduce the objective function and constraints to signomial forms, and the reduced problems are solved by geometric programming technique with the degree of difficulty 2. Numerical experiments and sensitivity analyses are performed to illustrate the models.

Keywords: Inventory; Pentagonal neutrosophic number; Possibility; Chance constrained programming; Trust measure;

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1. **Introduction**

Nowadays, integration of the effects of marketing cost, service cost, green cost, etc., into demand in an EOQ model is a realistic production and business strategy. Marketing costs are generally the total expenditure of a manufacturing company on marketing activities. This cost includes advertisement of the products, campaigning, promotional events, market research, etc. Now these activities, and hence marketing costs, directly affect the demand of items. Again, some manufacturing companies spend incentives on their sales representatives for better performance. Sometimes incentives are given to the delivery agents to have perfect timing in delivering the items. These types of expenditure are termed service cost and this cost also directly affects total demand. The demands for green goods are always very high in any market. Green costs include the extra expenditure to produce green goods. Thus the demand of an item increases with its greenness. Moreover, it is well known that an item's demand is inversely related to its selling price, i.e., demand continuously decreases with the price. In practice, the relations mentioned above are not linear and deterministic. Demand is always related to the marketing effort, service provided, greenness and price non-linearly in an imprecise sense, i.e., fuzzy. Recently, neutrosophic set more realistically represents the impreciseness. Following these real-life facts, demand is taken as a non-linear function of marketing and service costs, item's greenness and price in a neutrosophic fuzzy sense. This presentation of demand is new in the literature. Lee and Kim [1] first identified the idea of marketing planning into a classical inventory problem. They formulated the model with price and marketing cost dependent demand and solved using the geometric programming (GP) method. Later, Lee [2, 3] investigated profit maximization problems with optimal selling price and order quantity as decision variables along with some constraints. A multi-objective marketing planning EOQ problem was studied by Islam [4]. Later, marketing cost, selling price and service cost dependent demand was considered by Samadi et al. [5]. They solved the model under a fuzzy environment. Recently, Aggarwal et al. [6] developed an inventory model with price and advertising expenditure dependent demand.

In reality, an inventory model is formulated along with one or more restrictions like a limitation on storage space, order, production cost, etc. Among these restrictions, storage space constraint is very common. A manufacturing company builds or hires a warehouse to store its products at the beginning of production or business. These warehouses bear certain dimension that limits total storage space. In practical situations, this space may not be adequate all the time. Hence, space may be augmented if necessary. This augmentation is usually uncertain, i.e., the total available area may be considered as imprecise, random, rough, etc., in nature. Roy and Maiti [7] investigated a fuzzy EOQ problem under space constraints where demand depends on unit cost. Later, multi-objective inventory problems were considered for deteriorating items with space constraints under fuzzy (cf. [8]) and intuitionistic fuzzy (cf. [9]) environments. Again, Mandal and Islam [10], Panda and Maiti [11] solved an EOQ model with space constraint having fuzzy coefficients by applying the GP method. Recently, Kar et al. [12, 13] proposed neutrosophic GP technique to solve inventory problems with space constraints under neutrosophic environment. Das et al. [14] investigated a multi-item production inventory model with limited storage area under fuzzy environment. Moreover, Karimi and Sadjadi [15] developed a deteriorating multi-item EOQ model under capacity constraint and solved by a dynamic programming approach.

Chance constrained programming is introduced in an optimization problem when the chances of satisfying a certain constraint are above a certain level. In other words, when any constraint involves one or more random parameters, it is called a chance constraint. Charnes and Cooper [16] first developed a chance-constrained programming technique to solve stochastic optimization. Later, it has been extended in various directions [17–19]. An EOQ model for stochastically imperfect products was investigated and solved using chance-constrained programming by Panda et al. [20]. In the recent era, Widyan [21] developed a multi-criteria inventory model with random constraints. Furthermore, Hajiagha et al. [22] solved a multi-criteria fuzzy inventory model using chance-constrained and probabilistic programming based hybrid algorithm.

In the real world scenario, the demand for items in the manufacturing companies changes frequently. We can only note down the sales data of an item for one period and suggest an estimated demand of that item for the next period based on the previous one. Depending on this estimate, which may be fuzzy, rough, random, etc., the amount of resources, such as budget, warehouse space, etc., are determined. In such situations, the rough set theory, developed by Pawlak [23] and others [24], is used to deal with imprecise, inconsistent, incomplete information and knowledge. Later, many researchers have studied the rough set theory in various working fields [25, 26]. Also, Xu and Zhao [27] solved a fuzzy rough multi-objective decision making problem. De et al. [28] investigated an imperfect economic production model over different time horizons. Xu and Yao [29] proposed randomness and roughness simultaneously and established that some parameters follow random distribution with rough expected value. Recently, Bera and Mandal [30] and Midya and Roy [31] investigated multi-objective transportation problems under rough environment.

The theory of impreciseness has been used in various fields in the recent era. The concept of fuzzy set was first came up with Zadeh [32], and after that, many researchers [33–39] extended it and applied it in different problems. Later, Atanassov [40] successfully introduced the generalization of fuzzy set, called intuitionistic fuzzy set (IFS). In IFS, there are two-degree functions: membership and non-membership. Many researchers [41–43] have applied IFS in various fields. Nowadays, to deal with indeterminate/inconsistent information, Smarandache [44–46] developed neutrosophic set (NS). Unlike IFS, NS has three independent components: membership, indeterminacy and falsity. These independent degrees lie within $]0^-, 1^+[$. Later, Wang et al. [47] developed single valued neutrosophic set, and Peng et al. [48] proposed simplified NS. In recent era, Chakraborty et al. [49–51] applied the idea of pentagonal neutrosophic number on different problems. Moreover, Khalid et al. [52–54] and Pramanik et al. [55] introduced neutrosophic GP technique in several fields.

Generally, the GP technique is a very effective method for solving a class of non-linear optimization problems. The GP technique's most remarkable advantage is that this converts a complicated non-linear optimization problem involving highly non-linear constraints into an equivalent linear optimization problem with only linear constraints. The basic concept of GP was initially introduced by Duffin [56]. GP has contributed several applications in various areas such as inventory systems, circuit design, system design, project management, etc. Kochenberger [57] first tackled the inventory problem by GP technique. Later, several researchers [58–61] efficiently applied the GP technique for solving various non-linear problems in different fields.

In spite of the above developments in GP and its application in inventory control problems under uncertain environments, very few researchers have used GP in EOQ with neutrosophic uncertainty (cf. Kar et al. [12]). Only a few analytically expressed the limited resource amounts using neutrosophic numbers. Moreover, there are very few inventory control problems for green products using GP. We have tried to fill up the above lacunas in the present investigation.

In this paper, single item profit maximization inventory models are formulated with selling price, marketing, service and green dependent neutrosophic demand. The models are developed with different uncertain storage space constraints. Parameters of all models' objective functions are considered as PN numbers to formulate the models more realistically. Again, the resource constraint is taken in various environments as fuzzy, random, rough and trapezoidal neutrosophic (TN) numbers to derive particular models. At first, all the models are transformed into equivalent crisp forms using score function, possibility measure, chance-constrained programming, trust measure and (α, β, γ)-cut for PN, fuzzy, random, rough and TN environments respectively. These processes lead both objective function and constraint expression to signomial forms, which are solved using the GP technique. Solution procedures for all models are numerically illustrated. Sensitivity analysis is presented to observe the changes in optimum results against various parameters.

Thus, the main contributions of this investigation are

- For the first time, marketing, service and green expenditures, along with the item's selling price, is integrated into the item's demand to develop the model more realistically. Relations of these parameters with the demand are imprecise, expressed by PN numbers.
- Realistically, in this investigation, uncertain resource capacities are considered as fuzzy, random, rough and neutrosophic.
- Unit production cost is taken as a non-linear function of the order quantity.
- Models are appropriately solved by the GP technique to get the exact values/expressions of the decision variable.
- Due to the presence of neutrosophic parameters in the model, the concept of score function is introduced to convert the model into a crisp maximization problem.
- For converting particular models from fuzzy, random and rough environments to an equivalent crisp form, possibility measure, chance-constrained programming and trust measure

are respectively applied. In the case of TN resource constraint, (α, β, γ)-cuts and weighted arithmetic mean are used.

The remaining part of the present investigation is arranged as follows: Section 2 represents the formulation of the proposed models. Section 3 derives some particular cases. The solution procedures for all the models are explained in Section 4. In Section 5, numerical experiments are performed and the optimum results are described. Section 6 represents a sensitivity analysis. Conclusion and future extensions are presented in Section 7. All required preliminaries are explained in Appendix.

2. Formulation of the proposed models

The proposed models are established using the following notations and assumptions:

Notations:

Inventory related parameters:

Symbol	Explanation
D:	demand rate per unit time
C:	production cost per unit item
A:	set-up cost per period
H:	holding cost per item per unit time
T:	period of each cycle
w:	available total storage capacity area
w_0:	capacity area to store per unit quantity
I(t):	inventory level at any time t (≥ 0)
a:	selling price elasticity to demand
b:	marketing expenditure elasticity to demand
c:	service expenditure elasticity to demand
d:	green expenditure elasticity to demand
θ:	lot size elasticity to unit production cost
Decision variables:	
P:	selling price per unit quantity
Q:	number of order quantity
M:	marketing expenditure per unit item
R:	service expenditure per unit item
G:	green expenditure per unit item

Assumptions:

(a) Replenishment rate is instantaneous.

(b) Shortages are not allowed.

(c) Lead time is negligible.

(d) The inventory system allows a single item.

(e) In real life, it is always seen that when the items are ordered in a lot, the per item production cost reduces with the size of ordered units. Therefore, the unit production cost is inversely related to order quantity. So it is taken as $C = rQ^{-\theta}$, where r is the scaling factor and $0 < \theta < 1$.

(f) It is universally accepted that the demand of an item is negatively influenced by its price - either inversely or linearly. Marketing effort in the form of advertisement in electronic media, displayed hoardings on roadsides, etc., always uplifts the demand. Similarly, the service sector, in the form of timely dispatch, availability, good handling of customers, etc., plays an important role in increasing the sale/demand of an item (cf. Samadi et al. [5]). Nowadays, due to increased environmental consciousness, the demand for green products gradually increases day by day. Thus, greenness also plays a role in increased demand. Hence, demand can be expressed as $D = kP^{-a}M^bR^cG^d$, where k is the scaling factor/market size and $a > 1, 0 < b, c, d < 1$. Normally, the market size for a customized product is considered to be very high. So, the market size parameter, k, is estimated by an uncertain high number (cf. Samadi et al. [5]).

2.1. *Mathematical model*

In the present investigation, the inventory level continuously decreases to satisfy the demand (See Figure 1). If I(t) be the inventory level at time t, then the governing differential equation over the time (0,T) is given by

$$D^{'}I(t) = -D,\ 0 \leqslant t \leqslant T\ , \qquad \left(D^{'} \equiv \frac{d}{dt}\right) \tag{1}$$

where $I(0) = Q$ and $I(T) = 0$.

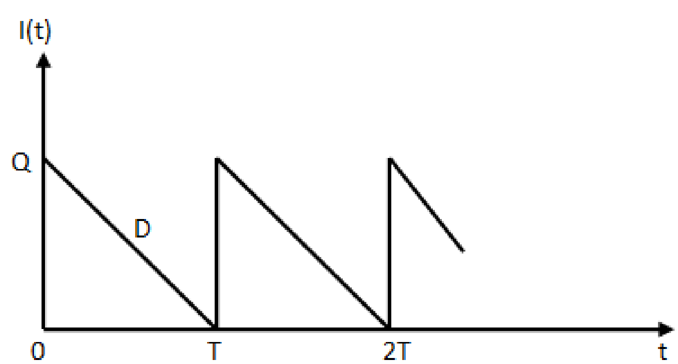


FIGURE 1. Crisp inventory model

Solving the above differential equation, we get $I(t) = Q - Dt$ and $T = \frac{Q}{D}$

Now, the average profit in the system involves the following:

Average sales revenue $= \frac{PQ}{T}$, average production cost $= \frac{CQ}{T}$, average marketing cost $= \frac{MQ}{T}$, average service cost $= \frac{RQ}{T}$, average green cost $= \frac{GQ}{T}$, average set-up cost $= \frac{A}{T}$ and average holding cost $= \int_0^T HI(t)\ dt = \frac{HQ}{2}$.

Hence total profit per unit time is

$$f = \textit{Sales revenue - Production cost - Merketing cost - Service cost - Green cost}$$
$$\textit{- Set-up cost - Holding cost}$$
$$= kP^{1-a}M^bR^cG^d - krP^{-a}M^bR^cG^dQ^{-\theta} - kP^{-a}M^{1+b}R^cG^d - kP^{-a}M^bR^{1+c}G^d$$
$$- kP^{-a}M^bR^cG^{1+d} - kAP^{-a}M^bR^cG^dQ^{-1} - 0.5HQ$$

Here, the space constraint is expressed as $w_0Q \leq w$.

2.2. *Model-1: Model in neutrosophic environment*

In reality, all data in an inventory model may not be found accurately. There may arise the case when some of the data are uncertain, incomplete and/or indeterminant. To deal with such a situation, the model's parameters are expressed in an imprecise environment considering fuzzy sets, intuitionistic sets, neutrosophic sets, rough sets, etc. The present inventory model considers all the elasticity parameters, scaling factors, holding cost, set-up cost, total available space, and per unit quantity area in the neutrosophic environment using PN numbers. Let us assume,

$$\left.\begin{array}{l}
\tilde{a}^n = \langle (a_1,a_2,a_3,a_4,a_5)\,;\mu_{\tilde{a}^n},\sigma_{\tilde{a}^n},\nu_{\tilde{a}^n}\rangle\,,\ \tilde{b}^n = \langle (b_1,b_2,b_3,b_4,b_5)\,;\mu_{\tilde{b}^n},\sigma_{\tilde{b}^n},\nu_{\tilde{b}^n}\rangle \\
\tilde{c}^n = \langle (c_1,c_2,c_3,c_4,c_5)\,;\mu_{\tilde{c}^n},\sigma_{\tilde{c}^n},\nu_{\tilde{c}^n}\rangle\,,\ \tilde{d}^n = \langle (d_1,d_2,d_3,d_4,d_5)\,;\mu_{\tilde{d}^n},\sigma_{\tilde{d}^n},\nu_{\tilde{d}^n}\rangle \\
\tilde{\theta}^n = \langle (\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)\,;\mu_{\tilde{\theta}^n},\sigma_{\tilde{\theta}^n},\nu_{\tilde{\theta}^n}\rangle\,,\ \tilde{k}^n = \langle (k_1,k_2,k_3,k_4,k_5)\,;\mu_{\tilde{k}^n},\sigma_{\tilde{k}^n},\nu_{\tilde{k}^n}\rangle \\
\tilde{r}^n = \langle (r_1,r_2,r_3,r_4,r_5)\,;\mu_{\tilde{r}^n},\sigma_{\tilde{r}^n},\nu_{\tilde{r}^n}\rangle\,,\ \tilde{H}^n = \langle (H_1,H_2,H_3,H_4,H_5)\,;\mu_{\tilde{H}^n},\sigma_{\tilde{H}^n},\nu_{\tilde{H}^n}\rangle \\
\tilde{A}^n = \langle (A_1,A_2,A_3,A_4,A_5)\,;\mu_{\tilde{A}^n},\sigma_{\tilde{A}^n},\nu_{\tilde{A}^n}\rangle\,,\ \tilde{w}^n = \langle (w_1,w_2,w_3,w_4,w_5)\,;\mu_{\tilde{w}^n},\sigma_{\tilde{w}^n},\nu_{\tilde{w}^n}\rangle \\
\tilde{w_0}^n = \langle (w_{01},w_{02},w_{03},w_{04},w_{05})\,;\mu_{\tilde{w_0}^n},\sigma_{\tilde{w_0}^n},\nu_{\tilde{w_0}^n}\rangle
\end{array}\right\} \quad (2)$$

Thus, the inventory model in neutrosophic environment is formulated as

$$\text{Max } f = \tilde{k}^nP^{1-\tilde{a}^n}M^{\tilde{b}^n}R^{\tilde{c}^n}G^{\tilde{d}^n} - \tilde{k}^n\tilde{r}^nP^{-\tilde{a}^n}M^{\tilde{b}^n}R^{\tilde{c}^n}G^{\tilde{d}^n}Q^{-\tilde{\theta}^n} - \tilde{k}^nP^{-\tilde{a}^n}M^{1+\tilde{b}^n}R^{\tilde{c}^n}G^{\tilde{d}^n}$$
$$-\tilde{k}^nP^{-\tilde{a}^n}M^{\tilde{b}^n}R^{1+\tilde{c}^n}G^{\tilde{d}^n} - \tilde{k}^nP^{-\tilde{a}^n}M^{\tilde{b}^n}R^{\tilde{c}^n}G^{1+\tilde{d}^n} - \tilde{k}^n\tilde{A}^nP^{-\tilde{a}^n}M^{\tilde{b}^n}R^{\tilde{c}^n}G^{\tilde{d}^n}Q^{-1} - 0.5\tilde{H}^nQ \quad (3)$$
$$\text{subject to} \quad \tilde{w_0}^n\,Q \leq \tilde{w}^n \quad (4)$$
$$P, M, R, G, Q > 0 \quad (5)$$

3. **Particular cases**

3.1. *Model-1.1: (Model with fuzzy space constraint)*

In this consideration, the inventory problem remains similar as formulated previously, except that the space constraint is taken under fuzzy environment. Practically, the total available space in a production source point may not be predicted precisely. Keeping this fact in mind, it is assumed that the total storage space is imprecise in nature, and it is expressed by triangular fuzzy number $\tilde{w} = (w_1, w_2, w_3)$. Thus, for Model-1.1, the profit function (Max $f_{1.1}$, say) is same as in expression (3) subject to $w_0\ Q \leq \tilde{w}$ and positivity condition (5).

3.2. *Model-1.2: (Model with random space constraint)*

In this case, total available space $\bar{w}$ is assumed to be random in nature and all the other terms in the model are left same as in Model-1. Hence, the Model-1.2 is formulated with the profit expression (Max $f_{1.2}$, say) same as in equation (3) and the constraints are given by $w_0\ Q\ \leq \bar{w}$ and positivity condition (5).

3.3. *Model-1.3: (Model with rough space constraint)*

When the available total storage space $\hat{w}$ is a rough variable, the constraint reduces to the rough environment and is expressed as $w_0 Q \leq \hat{w}$, where $\hat{w} = ([w_1, w_2][w_3, w_4])$, $0 \leq w_3 \leq w_1 \leq w_2 \leq w_4$ is a rough variable. Therefore, the Model-1.3 is described by the same profit function (say, $f_{1.3}$) as in equation (3) with the restriction $w_0\ Q\ \leq \hat{w}$ and condition (5).

3.4. *Model-1.4: (Model with neutrosophic space constraint)*

In this case, we express the total available storage area capacity using a TN number $\breve{w} = \langle (w_1, w_2, w_3, w_4); \mu_{\breve{w}}, \sigma_{\breve{w}}, \nu_{\breve{w}} \rangle$. The corresponding neutrosophic inventory model becomes maximize profit (say, $f_{1.4}$) as represented in (3) subject to the constraints $w_0\ Q\ \leq \breve{w}$ and positivity restriction (5).

3.5. *Model-1.5: (Model without space constraint)*

In this particular case, the inventory model is considered as an unconstrained profit maximization problem by omitting the space constraint from Model-1. Here, the expression for optimal profit (say, $f_{1.5}$) remains same as in equation (3) along with the positivity constraint (5) only.

4. **Solution procedure**

4.1. *Solution procedure for Model-1*

In this Section, a solution procedure is described to find a solution space for the neutrosophic inventory Model-1. Firstly, the model is converted into a crisp one from the neutrosophic one by applying the definition of the score function for PN numbers. It is then transformed into a posynomial problem. After that GP technique is used to get an ideal solution space.

Let us find the score values for the neutrosophic parameters as follows: (See Appendix)

$$\left.\begin{aligned}
S(\tilde{a}^n) &= \frac{1}{15}\left\{(a_1+a_2+a_3+a_4+a_5)\times(2+\mu_{\tilde{a}^n}-\sigma_{\tilde{a}^n}-\nu_{\tilde{a}^n})\right\}\\
S(\tilde{b}^n) &= \frac{1}{15}\left\{(b_1+b_2+b_3+b_4+b_5)\times(2+\mu_{\tilde{b}^n}-\sigma_{\tilde{b}^n}-\nu_{\tilde{b}^n})\right\}\\
S(\tilde{c}^n) &= \frac{1}{15}\left\{(c_1+c_2+c_3+c_4+c_5)\times(2+\mu_{\tilde{c}^n}-\sigma_{\tilde{c}^n}-\nu_{\tilde{c}^n})\right\}\\
S(\tilde{d}^n) &= \frac{1}{15}\left\{(d_1+d_2+d_3+d_4+d_5)\times(2+\mu_{\tilde{d}^n}-\sigma_{\tilde{d}^n}-\nu_{\tilde{d}^n})\right\}\\
S(\tilde{\theta}^n) &= \frac{1}{15}\left\{(\theta_1+\theta_2+\theta_3+\theta_4+\theta_5)\times(2+\mu_{\tilde{\theta}^n}-\sigma_{\tilde{\theta}^n}-\nu_{\tilde{\theta}^n})\right\}\\
S(\tilde{k}^n) &= \frac{1}{15}\left\{(k_1+k_2+k_3+k_4+k_5)\times(2+\mu_{\tilde{k}^n}-\sigma_{\tilde{k}^n}-\nu_{\tilde{k}^n})\right\}\\
S(\tilde{r}^n) &= \frac{1}{15}\left\{(r_1+r_2+r_3+r_4+r_5)\times(2+\mu_{\tilde{r}^n}-\sigma_{\tilde{r}^n}-\nu_{\tilde{r}^n})\right\}\\
S(\tilde{H}^n) &= \frac{1}{15}\left\{(H_1+H_2+H_3+H_4+H_5)\times(2+\mu_{\tilde{H}^n}-\sigma_{\tilde{H}^n}-\nu_{\tilde{H}^n})\right\}\\
S(\tilde{A}^n) &= \frac{1}{15}\left\{(A_1+A_2+A_3+A_4+A_5)\times(2+\mu_{\tilde{A}^n}-\sigma_{\tilde{A}^n}-\nu_{\tilde{A}^n})\right\}\\
S(\tilde{w}^n) &= \frac{1}{15}\left\{(w_1+w_2+w_3+w_4+w_5)\times(2+\mu_{\tilde{w}^n}-\sigma_{\tilde{w}^n}-\nu_{\tilde{w}^n})\right\}\\
S(\tilde{w_0}^n) &= \frac{1}{15}\left\{(w_{01}+w_{02}+w_{03}+w_{04}+w_{05})\times(2+\mu_{\tilde{w_0}^n}-\sigma_{\tilde{w_0}^n}-\nu_{\tilde{w_0}^n})\right\}
\end{aligned}\right\} \quad (6)$$

Using these values in Model-1, the converted crisp model can be formulated as follows:

$$\begin{aligned}
\text{Max } f = {} & S(\tilde{k}^n)P^{1-S(\tilde{a}^n)}M^{S(\tilde{b}^n)}R^{S(\tilde{c}^n)}G^{S(\tilde{d}^n)} - S(\tilde{k}^n)S(\tilde{r}^n)P^{-S(\tilde{a}^n)}M^{S(\tilde{b}^n)}R^{S(\tilde{c}^n)}G^{S(\tilde{d}^n)}Q^{-S(\tilde{\theta}^n)}\\
& -S(\tilde{k}^n)P^{-S(\tilde{a}^n)}M^{1+S(\tilde{b}^n)}R^{S(\tilde{c}^n)}G^{S(\tilde{d}^n)} - S(\tilde{k}^n)P^{-S(\tilde{a}^n)}M^{S(\tilde{b}^n)}R^{1+S(\tilde{c}^n)}G^{S(\tilde{d}^n)}\\
& -S(\tilde{k}^n)P^{-S(\tilde{a}^n)}M^{S(\tilde{b}^n)}R^{S(\tilde{c}^n)}G^{1+S(\tilde{d}^n)} - S(\tilde{k}^n)S(\tilde{A}^n)P^{-S(\tilde{a}^n)}M^{S(\tilde{b}^n)}R^{S(\tilde{c}^n)}G^{S(\tilde{d}^n)}Q^{-1}\\
& -0.5S(\tilde{H}^n)Q
\end{aligned} \quad (7)$$

$$\text{subject to} \quad S(\tilde{w_0}^n)\ Q \ \le S(\tilde{w}^n) \quad (8)$$

$$P, M, R, G, Q > 0$$

This transformed form of the inventory problem represents by a signomial GP problem, and its degree of difficulty is 2. Some necessary modifications are needed to convert the model into a posynomial GP problem. For the conversion, an appropriate lower bound is considered for the objective function with the aim that the maximization of that lower bound will be equivalent to the maximization of the objective function of the model. In this way, the signomial model is converted into the following equivalent form by introducing another additional auxiliary variable and a constraint:

$$\begin{aligned}
\text{Max} \quad & f_0 \\
\text{subject to } & S(\tilde{k}^n)P^{1-S(\tilde{a}^n)}M^{S(\tilde{b}^n)}R^{S(\tilde{c}^n)}G^{S(\tilde{d}^n)} - S(\tilde{k}^n)S(\tilde{r}^n)P^{-S(\tilde{a}^n)}M^{S(\tilde{b}^n)}R^{S(\tilde{c}^n)}G^{S(\tilde{d}^n)}Q^{-S(\tilde{\theta}^n)} \\
& -S(\tilde{k}^n)P^{-S(\tilde{a}^n)}M^{1+S(\tilde{b}^n)}R^{S(\tilde{c}^n)}G^{S(\tilde{d}^n)} - S(\tilde{k}^n)P^{-S(\tilde{a}^n)}M^{S(\tilde{b}^n)}R^{1+S(\tilde{c}^n)}G^{S(\tilde{d}^n)} \\
& -S(\tilde{k}^n)P^{-S(\tilde{a}^n)}M^{S(\tilde{b}^n)}R^{S(\tilde{c}^n)}G^{1+S(\tilde{d}^n)} - S(\tilde{k}^n)S(\tilde{A}^n)P^{-S(\tilde{a}^n)}M^{S(\tilde{b}^n)}R^{S(\tilde{c}^n)}G^{S(\tilde{d}^n)}Q^{-1} \\
& -0.5S(\tilde{H}^n)Q \le f_0 \qquad (9) \\
& S(\tilde{w_0}^n)\ Q\ \le S(\tilde{w}^n) \qquad (10) \\
& P, M, R, G, Q > 0 \qquad (11)
\end{aligned}$$

Again, this reduced model is equivalent to the following minimization problem:

$$\begin{aligned}
\text{Min} \quad & F = {f_0}^{-1} \\
\text{subject to } & f_0 S(\tilde{k^n})^{-1}P^{S(\tilde{a}^n)-1}M^{-S(\tilde{b}^n)}R^{-S(\tilde{c}^n)}G^{-S(\tilde{d}^n)} + S(\tilde{r}^n)P^{-1}Q^{-S(\tilde{\theta}^n)} + P^{-1}M + P^{-1}R \\
& +P^{-1}G + S(\tilde{A}^n)P^{-1}Q^{-1} + 0.5S(\tilde{H}^n)S(\tilde{k^n})^{-1}P^{S(\tilde{a}^n)-1}M^{-S(\tilde{b}^n)}R^{-S(\tilde{c}^n)}G^{-S(\tilde{d}^n)}Q \le 1 \\
& S(\tilde{w_0}^n)\ Q\ \le S(\tilde{w}^n) \\
& P, M, R, G, Q > 0 \qquad (12)
\end{aligned}$$

The derived inventory model (12) is a posynomial GP problem whose degree of difficulty is 2. To solve this problem the dual geometric programming problem is expressed as follows:

$$\begin{aligned}
\text{Max } d(\dot{w}) = & \left(\frac{1}{w_{00}}\right)^{w_{00}} \left(\frac{S(\tilde{k^n})^{-1}}{w_{01}}\right)^{w_{01}} \left(\frac{S(\tilde{r^n})}{w_{02}}\right)^{w_{02}} \left(\frac{1}{w_{03}}\right)^{w_{03}} \left(\frac{1}{w_{04}}\right)^{w_{04}} \left(\frac{1}{w_{05}}\right)^{w_{05}} \\
& \left(\frac{S(\tilde{A^n})}{w_{06}}\right)^{w_{06}} \left(\frac{0.5S(\tilde{H^n})}{w_{07}}\right)^{w_{07}} \left(\sum_{i=1}^{7} w_{0i}\right)^{\sum_{i=1}^{7} w_{0i}} \left(\frac{S(\tilde{w_0}^n)}{S(\tilde{w}^n)}\right)^{w_{11}} \qquad (13)
\end{aligned}$$

subject to the following conditions:

Normality condition: $w_{00} = 1$

Orthogonality conditions: $-w_{00} + w_{01} = 0$

$$\begin{aligned}
& (S(\tilde{a}^n) - 1)w_{01} - w_{02} - w_{03} - w_{04} - w_{05} - w_{06} + (S(\tilde{a}^n) - 1)w_{07} = 0 \\
& -S(\tilde{b}^n)w_{01} + w_{03} - S(\tilde{b}^n)w_{07} = 0 \\
& -S(\tilde{c}^n)w_{01} + w_{04} - S(\tilde{c}^n)w_{07} = 0 \\
& -S(\tilde{d}^n)w_{01} + w_{05} - S(\tilde{d}^n)w_{07} = 0 \\
& -S(\tilde{\theta}^n)w_{02} - w_{06} + w_{07} + w_{11} = 0
\end{aligned}$$

Positivity condition: $w_{00}, w_{01}, w_{02}, w_{03}, w_{04}, w_{05}, w_{06}, w_{07}, w_{11} > 0$

where $\dot{w} = (w_{00}, w_{01}, w_{02}, w_{03}, w_{04}, w_{05}, w_{06}, w_{07}, w_{11})^T$ and $w_{0i}(i = 0, 1, ..., 7), w_{11}$ are the dual variables against the primal variables P, M, R, G and Q for the problem defined by equation (12).
As the model has 2 degree of difficulty, we cannot calculate all the dual variables directly from the conditions. Therefore, solving the above constraints and expressing the dual variables $w_{0i},\ i = 2, 3, ..., 6$ in terms of w_{07} and w_{11} we get

$$\left.\begin{aligned}
w_{00} &= 1 = w_{01}\\
w_{02} &= \frac{S(\tilde{a}^n) - S(\tilde{b}^n) - S(\tilde{c}^n) - S(\tilde{d}^n) - 2}{1 - S(\tilde{\theta}^n)} w_{07} - \frac{1}{1 - S(\tilde{\theta}^n)} w_{11}\\
&\quad + \frac{S(\tilde{a}^n) - S(\tilde{b}^n) - S(\tilde{c}^n) - S(\tilde{d}^n) - 1}{1 - S(\tilde{\theta}^n)}\\
w_{03} &= S(\tilde{b}^n)(1 + w_{07})\\
w_{04} &= S(\tilde{c}^n)(1 + w_{07})\\
w_{05} &= S(\tilde{d}^n)(1 + w_{07})\\
w_{06} &= \frac{1 - S(\tilde{\theta}^n)\left(S(\tilde{a}^n) - S(\tilde{b}^n) - S(\tilde{c}^n) - S(\tilde{d}^n) - 1\right)}{1 - S(\tilde{\theta}^n)} w_{07} + \frac{1}{1 - S(\tilde{\theta}^n)} w_{11}\\
&\quad - \frac{S(\tilde{\theta}^n)\left(S(\tilde{a}^n) - S(\tilde{b}^n) - S(\tilde{c}^n) - S(\tilde{d}^n) - 1\right)}{1 - S(\tilde{\theta}^n)}
\end{aligned}\right\} \tag{14}$$

Letting $k_1 = \frac{S(\tilde{a}^n) - S(\tilde{b}^n) - S(\tilde{c}^n) - S(\tilde{d}^n) - 1}{1 - S(\tilde{\theta}^n)}$ and $k_2 = \frac{1}{1 - S(\tilde{\theta}^n)}$ we get,

$$w_{02} = (k_1 - k_2)\, w_{07} - k_2 w_{11} + k_1 \ \text{ and } \ w_{06} = \left(k_2 - S(\tilde{\theta}^n)k_1\right) w_{07} + k_2 w_{11} - S(\tilde{\theta}^n)k_1$$

Substituting these dual variables in equation (13), the dual objective function is expressed in terms of w_{07} and w_{11} as given below:

$$\begin{aligned}
d(w_{07}, w_{11}) = S(\tilde{k^n})^{-1} &\times \left(\frac{S(\tilde{r^n})}{(k_1 - k_2)\, w_{07} - k_2 w_{11} + k_1}\right)^{(k_1 - k_2) w_{07} - k_2 w_{11} + k_1}\\
&\times \left(\frac{1}{S(\tilde{b}^n)(1 + w_{07})}\right)^{S(\tilde{b}^n)(1+w_{07})} \times \left(\frac{1}{S(\tilde{c}^n)(1 + w_{07})}\right)^{S(\tilde{c}^n)(1+w_{07})}\\
&\times \left(\frac{1}{S(\tilde{d}^n)(1 + w_{07})}\right)^{S(\tilde{d}^n)(1+w_{07})} \times \left(\frac{0.5 S(\tilde{H}^n)}{S(\tilde{k^n}) w_{07}}\right)^{w_{07}}\\
&\times \left(\frac{S(\tilde{A}^n)}{(k_2 - S(\tilde{\theta}^n)k_1) w_{07} + k_2 w_{11} - S(\tilde{\theta}^n)k_1}\right)^{\left(k_2 - S(\tilde{\theta}^n)k_1\right) w_{07} + k_2 w_{11} - S(\tilde{\theta}^n)k_1}\\
&\times (S(\tilde{a}^n)(1 + w_{07}))^{S(\tilde{a}^n)(1+w_{07})} \times \left(\frac{S(\tilde{w_0}^n)}{S(\tilde{w}^n)}\right)^{w_{11}}
\end{aligned} \tag{15}$$

To evaluate the optimum dual variables w^*_{07} and w^*_{11} that optimize the dual objective $d(w_{07}, w_{11})$, we first take logarithm of equation (15) and get the following expression:

$$
\begin{aligned}
\log\ d(w_{07}, w_{11}) = \log S(\tilde{k^n})^{-1} &+ ((k_1 - k_2)w_{07} - k_2 w_{11} + k_1)\log S(\tilde{r^n}) \\
&- ((k_1 - k_2)\, w_{07} - k_2 w_{11} + k_1) \log((k_1 - k_2)\, w_{07} - k_2 w_{11} + k_1) \\
&-(S(\tilde{b}^n)(1 + w_{07}))\log(S(\tilde{b}^n)(1 + w_{07})) - (S(\tilde{c}^n)(1 + w_{07}))\log(S(\tilde{c}^n)(1 + w_{07})) \\
&-(S(\tilde{d}^n)(1 + w_{07}))\log(S(\tilde{d}^n)(1 + w_{07})) + w_{07}\log\left(\frac{0.5S(\tilde{H}^n)}{S(\tilde{k^n})}\right) - w_{07}\log w_{07} \\
&+((k_2 - S(\tilde{\theta}^n)k_1)w_{07} + k_2 w_{11} - S(\tilde{\theta}^n)k_1)\log S(\tilde{A}^n) \\
&-((k_2 - S(\tilde{\theta}^n)k_1)w_{07} + k_2 w_{11} - S(\tilde{\theta}^n)k_1)\log((k_2 - S(\tilde{\theta}^n)k_1)w_{07} + k_2 w_{11} - S(\tilde{\theta}^n)k_1) \\
&+ (S(\tilde{a}^n)(1 + w_{07}))\log(S(\tilde{a}^n)(1 + w_{07})) + w_{11}\log\left(\frac{S(\tilde{w_0}^n)}{S(\tilde{w}^n)}\right) \qquad (16)
\end{aligned}
$$

Since there are two variables w_{07} and w_{11} in the above logarithmic expression (16), to derive these optimal dual variables, we set the first order partial derivatives of log d(w_{07}, w_{11}) with respect to w_{07} and w_{11}, respectively, to zero and get the followings:

$$
\begin{aligned}
\frac{\partial \log d(w_{07}, w_{11})}{\partial w_{07}} &= (k_1 - k_2)\log S(\tilde{r^n}) - (k_1 - k_2)\log((k_1 - k_2)w_{07} - k_2 w_{11} + k_1) \\
&-S(\tilde{b}^n)\log(S(\tilde{b}^n)(1 + w_{07})) - S(\tilde{c}^n)\log(S(\tilde{c}^n)(1 + w_{07})) \\
&-S(\tilde{d}^n)\log(S(\tilde{d}^n)(1 + w_{07})) + (k_2 - S(\tilde{\theta}^n)k_1)\log S(\tilde{A}^n) \\
&-(k_2 - S(\tilde{\theta}^n)k_1)\log((k_2 - S(\tilde{\theta}^n)k_1)w_{07} + k_2 w_{11} - S(\tilde{\theta}^n)k_1) \\
&+\log\left(\frac{0.5S(\tilde{H}^n)}{S(\tilde{k^n})}\right) + \log w_{07} + S(\tilde{a^n})\log(S(\tilde{a}^n)(1 + w_{07})) \\
&= 0 \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log d(w_{07}, w_{11})}{\partial w_{11}} &= -k_2\log S(\tilde{r^n}) + k_2\log((k_1 - k_2)w_{07} - k_2 w_{11} + k_1) + k_2\log S(\tilde{A}^n) \\
&-k_2\log((k_2 - S(\tilde{\theta}^n)k_1)w_{07} + k_2 w_{11} - S(\tilde{\theta}^n)k_1) + \log\left(\frac{S(\tilde{w_0}^n)}{S(\tilde{w}^n)}\right) \\
&= 0 \qquad (18)
\end{aligned}
$$

By using any search method or any software the above equations (17) and (18) can be solved to get the optimal dual variables w^*_{07} and w^*_{11}. With the help of these optimal values, other optimal dual variables can easily be evaluated from equation (14). Consequently, the optimum dual objective function $d^*(w^*)$ can be calculated. Now, the primal-dual relations of the problem (12) are derived as:

$$
\lambda = \sum_{i=1}^{7} w^*_{0i},\ S(\tilde{k^n})^{-1} P^{*S(\tilde{a}^n)-1} M^{*-S(\tilde{b}^n)} R^{*-S(\tilde{c}^n)} G^{*-S(\tilde{d}^n)} = \frac{w^*_{01}}{\lambda},\ S(\tilde{r}^n)P^{*-1}Q^{*-S(\tilde{\theta}^n)} = \frac{w^*_{02}}{\lambda}
$$

$$
P^{*-1}M^* = \frac{w^*_{03}}{\lambda},\ P^{*-1}R^* = \frac{w^*_{04}}{\lambda},\ P^{*-1}G^* = \frac{w^*_{05}}{\lambda},\ S(\tilde{A}^n)P^{*-1}Q^{*-1} = \frac{w^*_{06}}{\lambda},
$$

$$0.5S(\tilde{H}^n)S(\tilde{k^n})^{-1}P^{*S(\tilde{a}^n)-1}M^{*-S(\tilde{b}^n)}R^{*-S(\tilde{c}^n)}G^{*-S(\tilde{d}^n)}Q^* = \frac{w_{07}^*}{\lambda},\ \left(\frac{S(\tilde{w_0}^n)}{S(\tilde{w}^n)}\right)Q^* = \frac{w_{11}^*}{w_{11}^*}$$

With the help of these equations, we obtain the following expressions for the optimal primal variables:

$$Q^* = \frac{S(\tilde{w}^n)}{S(\tilde{w_0}^n)},\ P^* = S(\tilde{A}^n)\frac{S(\tilde{w_0}^n)}{S(\tilde{w}^n)}\frac{\sum_{i=1}^7 w_{0i}^*}{w_{06}^*},\ M^* = S(\tilde{A}^n)\frac{S(\tilde{w_0}^n)}{S(\tilde{w}^n)}\frac{w_{03}^*}{w_{06}^*},$$
$$R^* = S(\tilde{A}^n)\frac{S(\tilde{w_0}^n)}{S(\tilde{w}^n)}\frac{w_{04}^*}{w_{06}^*},\ G^* = S(\tilde{A}^n)\frac{S(\tilde{w_0}^n)}{S(\tilde{w}^n)}\frac{w_{05}^*}{w_{06}^*}$$

Substituting the above obtained optimal variables in neutrosophic inventory model i.e. Model-1, the optimal profit becomes

$$f^* = \left(\frac{S(\tilde{A}^n)}{w_{06}^*}\frac{S(\tilde{w_0}^n)}{S(\tilde{w}^n)}\right)^{1-S(\tilde{a}^n)+S(\tilde{b}^n)+S(\tilde{c}^n)+S(\tilde{d}^n)} \sum_{i=1}^{7} w_{0i}^{*\ -S(\tilde{a}^n)} w_{03}^{*\ S(\tilde{b}^n)} w_{04}^{*\ S(\tilde{c}^n)} w_{05}^{*\ S(\tilde{d}^n)} \times \left[\sum_{i=1}^{7} w_{0i}^*\right.$$
$$\left. -S(\tilde{r}^n)\left(\frac{S(\tilde{A}^n)}{w_{06}^*}\frac{S(\tilde{w_0}^n)}{S(\tilde{w}^n)}\right)^{S(\tilde{\theta}^n)-1} - w_{03}^* - w_{04}^* - w_{05}^* - S(\tilde{A}^n)\right] S(\tilde{k}^n) - 0.5S(\tilde{H}^n)\frac{S(\tilde{w}^n)}{S(\tilde{w_0}^n)}$$

4.2. *Solution procedure for Model-1.1*

In this case, the constraint is formulated under fuzzy environment. To deal with such type of constraint in inventory model, we follow the possibility theory. After defuzzification the space constraint reduces to $Pos\,(w_0 Q \leq \tilde{w}) \geq \eta,\ \eta$ represents the degree of fuzziness and $\tilde{w} = (w_1, w_2, w_3)$. Following Lemma 1, the crisp formulation of the constraint becomes :

$$w_0 Q \leq \eta w_2 + (1-\eta) w_3 \tag{19}$$

Here the above obtained constraint is under crisp environment. Using the score function formula, the neutrosophic objective function is converted into a crisp one, and consequently, the equivalent crisp model becomes:

Maximize profit $f_{1.1}$ as given in equation (7) subject to conditions (19) and (5).

Now the model reduces to a signomial GP problem with degree of difficulty 2. Therefore, this problem can be solved by the GP technique. Following the similar approach as computed in section 4.1, the optimum results are evaluated as follows:

$$Q^* = \frac{\eta w_2 + (1-\eta)w_3}{w_0}, \qquad P^* = S(\tilde{A}^n)\frac{w_0}{\eta w_2 + (1-\eta)w_3}\frac{\sum_{i=1}^7 w_{0i}^*}{w_{06}^*},$$
$$M^* = S(\tilde{A}^n)\frac{w_0}{\eta w_2 + (1-\eta)w_3}\frac{w_{03}^*}{w_{06}^*}, \qquad R^* = S(\tilde{A}^n)\frac{w_0}{\eta w_2 + (1-\eta)w_3}\frac{w_{04}^*}{w_{06}^*},$$
$$G^* = S(\tilde{A}^n)\frac{w_0}{\eta w_2 + (1-\eta)w_3}\frac{w_{05}^*}{w_{06}^*}$$

and the optimum profit becomes

$$f_{1.1}^* = \left(\frac{S(\tilde{A}^n)}{w_{06}^*}\frac{w_0}{\eta w_2 + (1-\eta)w_3}\right)^{1-S(\tilde{a}^n)+S(\tilde{b}^n)+S(\tilde{c}^n)+S(\tilde{d}^n)} \sum_{i=1}^{7} w_{0i}^{*}{}^{-S(\tilde{a}^n)} w_{03}^{*}{}^{S(\tilde{b}^n)} w_{04}^{*}{}^{S(\tilde{c}^n)} w_{05}^{*}{}^{S(\tilde{d}^n)}$$
$$\times\left[\sum_{i=1}^{7} w_{0i}^* - S(\tilde{r}^n)\left(\frac{S(\tilde{A}^n)}{w_{06}^*}\frac{w_0}{\eta w_2 + (1-\eta)w_3}\right)^{S(\tilde{\theta}^n)-1} - w_{03}^* - w_{04}^* - w_{05}^* - S(\tilde{A}^n)\right] S(\tilde{k}^n)$$
$$-0.5S(\tilde{H}^n)\frac{\eta w_2 + (1-\eta)w_3}{w_0}$$

4.3. *Solution procedure for Model-1.2*

For this particular case, the space constraint is considered in a random sense. To deal with such types of constraints, we follow the chance-constrained programming approach and the corresponding constraint becomes:

$$Pr\,(w_0 Q \le \bar{w}) \ge p\ ,\ 0 < p < 1$$

where 'Pr' indicates probability and p represents the prescribed permissible probability.
Now, assume that $\bar{w}$ be normally distributed random variable with m_w and σ_w as mean and standard deviation respectively. Then, the constraint can be expressed as

$$Pr\left[\frac{w - m_w}{\sigma_w} \ge \frac{w_0 Q - m_w}{\sigma_w}\right] \ge p$$

where $\frac{w-m_w}{\sigma_w}$ is a standard normal variate.
If we consider $\phi(p)$, such that $\int_{\phi(p)}^{\infty}\phi(t)\,dt$, where $\phi(t)$ is the standard normal density function, then we get

$$\frac{w - m_w}{\sigma_w} \le \phi(p)$$

Thus, using chance-constrained programming the reduced crisp constraint can be written as

$$w_0 Q \le m_w + \sigma_w \phi(p) \tag{20}$$

Again, Definition 7 is used to transform the objective function into a crisp expression from a neutrosophic one. Thus, the corresponding crisp model can be expressed by the objective function given in equation (7) subject to the constraints (20) and (5). The above is again a signomial GP problem having degree of difficulty 2. Now we follow the GP approach as presented in section 4.1 to solve the model. Finally, the optimal solution is obtained in the following form:

$$Q^* = \frac{m_w + \sigma_w\phi(p)}{w_0},\ \ P^* = S(\tilde{A}^n)\frac{w_0}{m_w + \sigma_w\phi(p)}\frac{\sum_{i=1}^{7} w_{0i}^*}{w_{06}^*},\ \ M^* = S(\tilde{A}^n)\frac{w_0}{m_w + \sigma_w\phi(p)}\frac{w_{03}^*}{w_{06}^*},$$
$$R^* = S(\tilde{A}^n)\frac{w_0}{m_W + \sigma_W\phi(p)}\frac{w_{04}^*}{w_{06}^*},\ \ G^* = S(\tilde{A}^n)\frac{w_0}{m_w + \sigma_w\phi(p)}\frac{w_{05}^*}{w_{06}^*}$$

and corresponding optimum profit becomes

$$f^*_{1.2} = \left(\frac{S(\tilde{A}^n)}{w^*_{06}}\frac{w_0}{m_w+\sigma_w\phi(p)}\right)^{1-S(\tilde{a}^n)+S(\tilde{b}^n)+S(\tilde{c}^n)+S(\tilde{d}^n)} \sum_{i=1}^{7} w^*_{0i}{}^{-S(\tilde{a}^n)} w^*_{03}{}^{S(\tilde{b}^n)} w^*_{04}{}^{S(\tilde{c}^n)} w^*_{05}{}^{S(\tilde{d}^n)}$$
$$\times\left[\sum_{i=1}^{7} w^*_{0i} - S(\tilde{r}^n)\left(\frac{S(\tilde{A}^n)}{w^*_{06}}\frac{w_0}{m_w+\sigma_w\phi(p)}\right)^{S(\tilde{\theta}^n)-1} - w^*_{03} - w^*_{04} - w^*_{05} - S(\tilde{A}^n)\right] S(\tilde{k}^n)$$
$$-0.5S(\tilde{H}^n)\frac{m_w+\sigma_w\phi(p)}{w_0}$$

4.4. *Solution procedure for Model-1.3*

In this case, the space constraint is supposed to be in rough environment. Therefore, using Definition 12, the trust measure to the crisp conversion becomes

$$Tr\left(\hat{w} \geq w_0 Q\right) \geq \eta_1$$

where $\eta_1 \in [0,1]$ is the trust level.

Now, following Theorem 2, the rough constraint can be written in equivalent crisp form as expressed below:

$$w_0 Q \leq \begin{cases} w_4 - \frac{\eta_1(w_4-w_3)}{\zeta} & \text{if } w_2 \leq w_0 Q \leq w_4 \\ \frac{\zeta(w_2-w_1)+(1-\zeta)w_2(w_4-w_3)-\eta_1(w_4-w_3)(w_2-w_1)}{\zeta(w_2-w_1)+(1-\zeta)(w_4-w_3)} & \text{if } w_1 \leq w_0 Q \leq w_2 \\ w_4 + \frac{(1-\zeta-\eta_1)(w_4-w_3)}{\zeta} & \text{if } w_3 \leq w_0 Q \leq w_1 \\ w_3 \end{cases} \tag{21}$$

where $\zeta \in (0,1)$.

After reducing the neutrosophic objective function into its crisp form, the consequent model can be expressed by the profit function $f_{1.3}$ as given in expression (7) subject to restrictions (21) and(5). This obtained signomial GP problem bearing degree of difficulty 2 is then solved using GP technique as described in section 4.1. The optimal decision variables and profit are respectively obtained as:

$$Q^* = \frac{T}{w_0}, \quad P^* = S(\tilde{A}^n)\frac{w_0}{T}\frac{\sum_{i=1}^{7} w^*_{0i}}{w^*_{06}}, \quad M^* = S(\tilde{A}^n)\frac{w_0}{T}\frac{w^*_{03}}{w^*_{06}}, \quad R^* = S(\tilde{A}^n)\frac{w_0}{T}\frac{w^*_{04}}{w^*_{06}},$$
$$G^* = S(\tilde{A}^n)\frac{w_0}{T}\frac{w^*_{05}}{w^*_{06}} \quad \text{and}$$

$$f^*_{1.3} = \left(\frac{S(\tilde{A}^n)}{w^*_{06}}\frac{w_0}{T}\right)^{1-S(\tilde{a}^n)+S(\tilde{b}^n)+S(\tilde{c}^n)+S(\tilde{d}^n)} \sum_{i=1}^{7} w^*_{0i}{}^{-S(\tilde{a}^n)} w^*_{03}{}^{S(\tilde{b}^n)} w^*_{04}{}^{S(\tilde{c}^n)} w^*_{05}{}^{S(\tilde{d}^n)}$$
$$\times\left[\sum_{i=1}^{7} w^*_{0i} - S(\tilde{r}^n)\left(\frac{S(\tilde{A}^n)}{w^*_{06}}\frac{w_0}{T}\right)^{S(\tilde{\theta}^n)-1} - w^*_{03} - w^*_{04} - w^*_{05} - S(\tilde{A}^n)\right] S(\tilde{k}^n) - 0.5S(\tilde{H}^n)\frac{T}{w_0}$$

where

$$T = \begin{cases} w_4 - \frac{\eta_1(w_4-w_3)}{\zeta} & \text{if } w_2 \leq w_0Q \leq w_4 \\ \frac{\zeta(w_2-w_1)+(1-\zeta)w_2(w_4-w_3)-\eta_1(w_4-w_3)(w_2-w_1)}{\zeta(w_2-w_1)+(1-\zeta)(w_4-w_3)} & \text{if } w_1 \leq w_0Q \leq w_2 \\ w_4 + \frac{(1-\zeta-\eta_1)(w_4-w_3)}{\zeta} & \text{if } w_3 \leq w_0Q \leq w_1 \\ w_3 \end{cases}$$

and $\zeta \in (0, 1)$.

4.5. *Solution procedure for Model-1.4*

To convert this neutrosophic model into a crisp representation, we first need to calculate the $\alpha - cut$, $\beta - cut$ and $\gamma - cut$ of the TN number. Using Definition 10 , we derive the following $\alpha - cut$, $\beta - cut$ and $\gamma - cut$ for the neutrosophic total space

$$\breve{w}_\alpha = [L_{\breve{w}}(\alpha), R_{\breve{w}}(\alpha)] = \left[\frac{(\mu_{\breve{w}} - \alpha)w_1 + \alpha w_2}{\mu_{\breve{w}}}, \frac{(\mu_{\breve{w}} - \alpha)w_4 + \alpha w_3}{\mu_{\breve{w}}}\right]$$

$$\breve{w}_\beta = [L'_{\breve{w}}(\beta), R'_{\breve{w}}(\beta)] = \left[\frac{(1-\beta)w_2 + (\beta - \sigma_{\breve{w}})w_1}{1-\sigma_{\breve{w}}}, \frac{(1-\beta)w_3 + (\beta - \sigma_{\breve{w}})w_4}{1-\sigma_{\breve{w}}}\right]$$

$$\breve{w}_\gamma = [L''_{\breve{w}}(\gamma), R''_{\breve{w}}(\gamma)] = \left[\frac{(1-\gamma)w_2 + (\gamma - \nu_{\breve{w}})w_1}{1-\nu_{\breve{w}}}, \frac{(1-\gamma)w_3 + (\gamma - \nu_{\breve{w}})w_4}{1-\nu_{\breve{w}}}\right]$$

After that, we transform the neutrosophic parameters in the constraint into a crisp interval number by using Theorem 1. Thus, we have the crisp constraint as $w_0Q \leq [L_{\breve{w}}, R_{\breve{w}}]$,
where $L_{\breve{w}} = max\left\{L_{\breve{w}}(\alpha),\ L'_{\breve{w}}(\beta),\ L''_{\breve{w}}(\gamma)\right\},\ R_{\breve{w}} = min\left\{R_{\breve{w}}(\alpha),\ R'_{\breve{w}}(\beta),\ R''_{\breve{w}}(\gamma)\right\}$
Now applying the weighted mean approach (cf. Lemma 2), convert the interval number into parametric function and get the crisp formulation of the constraint as follows:

$$w_0Q \leq L_{\breve{w}}(1-\rho) + R_{\breve{w}}\rho \tag{22}$$

After reducing the PN objective of the model-1.4 into a crisp one using the score function, the converted equivalent crisp formulation is expressed with the objective function same as in (7) along with the constraints given in (22) and (5). Now the reduced model is again a signomial GP problem having 2 degree of difficulty. Following the solution procedure, explained in Section 4.1, we get the following optimal solutions:

$$Q^* = \frac{L_{\breve{w}}(1-\rho) + R_{\breve{w}}\rho}{w_0}, \qquad P^* = S(\tilde{A}^n)\frac{w_0}{L_{\breve{w}}(1-\rho) + R_{\breve{w}}\rho}\frac{\sum_{i=1}^{7} w_{0i}^*}{w_{06}^*},$$

$$M^* = S(\tilde{A}^n)\frac{w_0}{L_{\breve{w}}(1-\rho) + R_{\breve{w}}\rho}\frac{w_{03}^*}{w_{06}^*}, \qquad R^* = S(\tilde{A}^n)\frac{w_0}{L_{\breve{w}}(1-\rho) + R_{\breve{w}}\rho}\frac{w_{04}^*}{w_{06}^*},$$

$$G^* = S(\tilde{A}^n)\frac{w_0}{L_{\breve{w}}(1-\rho) + R_{\breve{w}}\rho}\frac{w_{05}^*}{w_{06}^*}$$

along with the optimum profit given by

$$f^*_{1.4} = \left(\frac{S(\tilde{A}^n)}{w^*_{06}}\frac{w_0}{L_{\breve{w}}(1-\rho)+R_{\breve{w}}\rho}\right)^{1-S(\tilde{a}^n)+S(\tilde{b}^n)+S(\tilde{c}^n)+S(\tilde{d}^n)} \sum_{i=1}^{7} w^*_{0i}{}^{-S(\tilde{a}^n)} \; w^*_{03}{}^{S(\tilde{b}^n)} w^*_{04}{}^{S(\tilde{c}^n)} w^*_{05}{}^{S(\tilde{d}^n)}$$
$$\times\left[\sum_{i=1}^{7} w^*_{0i} - S(\tilde{r}^n)\left(\frac{S(\tilde{A}^n)}{w^*_{06}}\frac{w_0}{L_{\breve{w}}(1-\rho)+R_{\breve{w}}\rho}\right)^{S(\tilde{\theta}^n)-1} - w^*_{03} - w^*_{04} - w^*_{05} - S(\tilde{A}^n)\right] S(\tilde{k}^n)$$
$$-0.5S(\tilde{H}^n)\frac{L_{\breve{w}}(1-\rho)+R_{\breve{w}}\rho}{w_0}$$

4.6. *Solution procedure for Model-1.5*

The objective function of this particular model is transformed into crisp environment by using the definition of score function, and obtained crisp expression is given in the equation (7). Since there is only the positivity restriction in this model, the problem is a signomial GP problem with 1 degree of difficulty. Thus, the model can be solved by the GP technique, and the optimum solutions are as follows:

$$Q^* = \left(\frac{S(\tilde{A}^n)}{S(\tilde{r}^n)}\frac{w^*_{02}}{w^*_{06}}\right)^{\frac{1}{1-S(\tilde{\theta}^n)}}, \; P^* = \left(\frac{S(\tilde{r}^n)}{w^*_{02}}\right)^{\frac{1}{1-S(\tilde{\theta}^n)}}\left(\frac{w^*_{06}}{S(\tilde{A}^n)}\right)^{\frac{S(\tilde{\theta}^n)}{1-S(\tilde{\theta}^n)}},$$
$$M^* = w^*_{03}\left(\frac{S(\tilde{r}^n)}{w^*_{02}}\right)^{\frac{1}{1-S(\tilde{\theta}^n)}}\left(\frac{w^*_{06}}{S(\tilde{A}^n)}\right)^{\frac{S(\tilde{\theta}^n)}{1-S(\tilde{\theta}^n)}}, \; R^* = w^*_{04}\left(\frac{S(\tilde{r}^n)}{w^*_{02}}\right)^{\frac{1}{1-S(\tilde{\theta}^n)}}\left(\frac{w^*_{06}}{S(\tilde{A}^n)}\right)^{\frac{S(\tilde{\theta}^n)}{1-S(\tilde{\theta}^n)}},$$
$$G^* = w^*_{05}\left(\frac{S(\tilde{r}^n)}{w^*_{02}}\right)^{\frac{1}{1-S(\tilde{\theta}^n)}}\left(\frac{w^*_{06}}{S(\tilde{A}^n)}\right)^{\frac{S(\tilde{\theta}^n)}{1-S(\tilde{\theta}^n)}}.$$

Now using these optimum decision variables, optimal profit can be calculated easily.

5. Numerical experiments

Input data for Model-1:

The proposed models (Model-1,-1.1,-1.2,-1.3,-1.4 and -1.5) are illustrated with a theoretical example in this section. For this, the following inputs in appropriate units are considered:

$$\tilde{a}^n = \langle(2,3,7,8,10);0.6,0.5,0.6\rangle, \; \tilde{b}^n = \langle(0.2,0.32,0.45,0.52,0.61);0.9,0.1,0.3\rangle,$$
$$\tilde{c}^n = \langle(0.09,0.18,0.34,0.43,0.52);0.9,0.3,0.1\rangle, \tilde{d}^n = \langle(0.09,0.12,0.19,0.28,0.37);0.8,0.5,0.3\rangle,$$
$$\tilde{\theta}^n = \langle(0.01,0.02,0.03,0.04,0.05);0.8,0.4,0.4\rangle, \tilde{r}^n = \langle(2,5.5,8,10,12);0.6,0.2,0.4\rangle,$$
$$\tilde{k}^n = \langle(200000,400000,600000,750000,1050000);0.9,0.3,0.1\rangle,$$
$$\tilde{H}^n = \langle(0.9,1.2,2.5,3.8,4.1);0.9,0.8,0.6\rangle, \tilde{A}^n = \langle(36,58,86,110.5,122);0.7,0.4,0.3\rangle,$$
$$\tilde{w_0}^n = \langle(1,6,10,15,18);0.5,0.5,0.2\rangle, \tilde{w}^n = \langle(280,450,695,860,1390);0.7,0.4,0.3\rangle.$$

Input data for particular models:

For the particular models (Model-1.1,-1.2,1.3 and -1.4) the required inputs are presented in Table 1. All the other parameters for these models are similar as in Model-1.

TABLE 1. Other input data for particular models

Environments	Related data
Fuzzy	$\tilde{w} = (410, 490, 570)$ $\eta = 0.5$
Random	$(m_w, \sigma_w) = (480, 20)$ $p = 0.96$
Rough	$\hat{w} = ([460, 510][420, 580]),\ \zeta = 0.5,\ \eta_1 = 0.6$
TN	$\check{w} = \langle(420, 465, 500, 550); 0.7, 0.3, 0.3\rangle,\ \alpha = 0.1,\ \beta = 0.7,\ \gamma = 0.9$

Optimum results:

All models are formulated with the help of above considered parameters and optimized using the proposed solution approach. The optimum values of selling price P^*, number of order quantity Q^*, marketing expenditure M^*, service expenditure R^*, green expenditure G^* and profit f^* are evaluated for the inventory models with the constraint in different environments, and the results are indicated in Table 2.

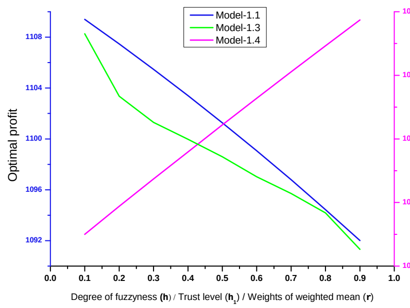
TABLE 2. Optimal solutions for all models

Models	Environments		P^*	M^*	R^*	G^*	Q^*	f^*
	Objective	constraint						
1	PN	PN	12.600	1.470	1.092	0.588	81.67	1089.50
1.1	PN	Fuzzy	12.466	1.454	1.080	0.581	88.33	1101.27
1.2	PN	Random	12.790	1.492	1.108	0.597	74.17	1073.64
1.3	PN	Rough	12.610	1.471	1.093	0.588	81.43	1089.04
1.4	PN	TN	12.710	1.483	1.102	0.593	77.21	1080.44
1.5	PN	No constraint	11.463	1.337	0.993	0.535	197.87	1160.63

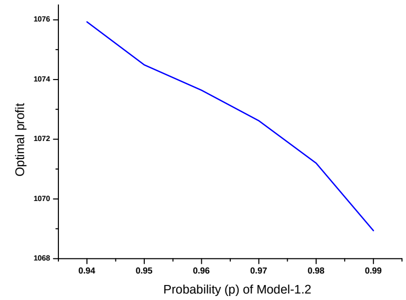
From Table 2, it is observed that the optimal profit for the model with PN numbers (Model-1) is 1089.50 \$. Again, the optimal selling price per unit item is 12.6 \$ for this model, and the total order quantity is 81.67 units. Moreover, the marketing, service and green expenditure per item are 1.47 \$, 1.092 \$ and 0.588 \$, respectively. Similarly, we note that the optimal profit for the particular cases, i.e., models with fuzzy constraint (Model-1.1), random constraint (Model-1.2), rough constraint (Model-1.3) and TN constraint (Model-1.4) are 1101.27 \$, 1073.64 \$, 1089.04 \$ and 1080.44 \$ respectively. The optimal values of these particular models' decision variables are presented in Table 2. Again, as per expectation, the unconstrained model (Model-1.5) gives the maximum profit among all models. Strictly speaking, as the environments under which the models are formulated are different, their optimum results can not be compared.

6. Sensitivity analysis

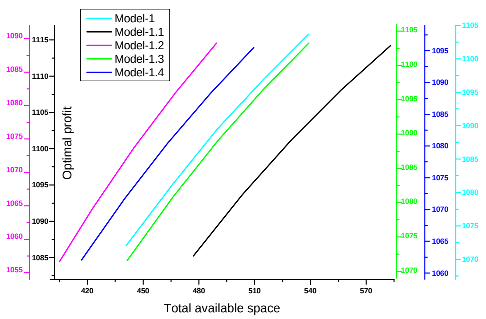
In this Section, we perform some sensitivity analysis to observe the changes in optimal profit with respect to the changes in parameters for the particular models. From Figure 2a, it is noted that whenever the degree of fuzziness (η) or the trust level (η_1) increases from 0.1 to 0.9, the optimal profit decreases continuously. But, with the increases of weights (ρ) of the weighted mean used in Model-1.4, optimal profit increases (cf. Figure 2a). Again, the decrease of optimal profit with respect to the considered probability (p) of Model-1.2 is plotted in Figure 2b. The changes in optimal profit of all the models along with the changes in total available space and set-up cost are figured in Figures 2c and 2d respectively.



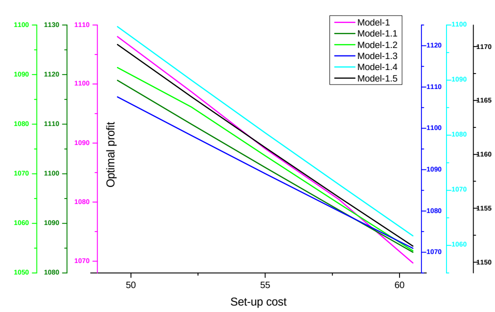
(A)



(B)



(C)



(D)

Figure 2. (A) Optimal profit vs. degree of fuzzyness (η)/ trust measure (η_1)/ weight of weighted mean (ρ); (B) Optimal profit vs. permissible probability (p) for Model-1.2; (C) Optimal profit of all models vs. total available space; (D) Optimal profit of all models vs. set-up cost

7. Conclusions

The main goal of the present study is to develop an economic order quantity model with space constraint having all its parameters as PN numbers. In reality, marketing expenditure and service quality play a significant role in the demand of any manufacturing company. Again, the demand for green items is always very high in the market. Therefore, the demand in this model is considered as a function of the selling price, marketing, service and green expenditure to be more realistic. Moreover, order quantity dependent unit production cost is assumed here. Again, the space constraint is considered in several environments like fuzzy, random, rough, and TN. Possibility measure and chance-constrained programming are used to deal with the fuzzy and random constraint goals, respectively. For all the models, a solution procedure is suggested. Finally, the GP technique is applied to solve the converted crisp models. Moreover, some numerical experiments and sensitivity analyses are done to illustrate the models.

In the future, the model can be developed more realistically by assuming ramp type demand, power demand, probabilistic demand, etc. Here, the model is considered for a single item, and hence it can be extended to a multi-item model. Also, shortages can be allowed in the problem. Furthermore, the items may be damageable, and preservation technology may be introduced. Moreover, different environments can be considered, such as intuitionistic, fuzzy-random, fuzzy-rough, type-2 fuzzy, etc. Again, the researchers may suggest different solution procedures and compare the optimum results with the present one.

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Appendix

Definition 1 (Fuzzy set). *[32] A fuzzy set $\tilde{B}$ in X (space of points/objects) is an object having the form $\tilde{B} = \{(x, \mu_{\tilde{B}}(x)) : x \in X\}$ where $\mu_{\tilde{B}} : X \to [0, 1]$ is the membership function of the fuzzy set $\tilde{B}$.*

Definition 2 (Triangular fuzzy number). *[20] A triangular fuzzy number $\tilde{B} = (b_1, b_2, b_3)$ is a fuzzy number whose membership function $\mu_{\tilde{B}}(x)$ is defined as*

$$\mu_{\tilde{B}}(x) = \begin{cases} \frac{x-b_1}{b_2-b_1} & b_1 \leq x < b_2 \\ \frac{b_3-x}{b_3-b_2} & b_2 \leq x \leq b_3 \\ 0 & otherwise \end{cases}$$

where $[b_1, b_3]$ is the supporting interval and the point $(b_2, 1)$ is the peak.

Definition 3 (Possibility measure). *[20] Let $\tilde{m}$ and $\tilde{n}$ be two fuzzy numbers and the corresponding memberships functions be $\mu_{\tilde{m}}(x)$ and $\mu_{\tilde{n}}(x)$ respectively. Then the possibility measure of $\tilde{m}$ and $\tilde{n}$ is defined as:* $Pos(\tilde{m} * \tilde{n}) = Sup\,\{min(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y)),\ x, y \in \mathbb{R},\ x * y\}$

Here, the abbreviation 'Pos' stands for possibility measure and '' represents any one of the relations $<, >, =, \leq, \geq$. Analogously if $\tilde{n}$ be a crisp number,say n, then we have*

$$Pos(\tilde{m} * n) = Sup\,\{min(\mu_{\tilde{m}}(x),\ x \in \mathbb{R},\ x * n\}$$

Lemma 1. *Let $\tilde{m} = (m_1, m_2, m_3)$ be any triangular fuzzy number and $0 < m_1$ and n be any crisp number. Then,*

$$Pos(\tilde{m} > n) \geq \alpha \ iff \ \frac{m_3 - n}{m_3 - m_2} \geq \alpha$$

Definition 4 (Neutrosophic set)**.** *[45] Let the set X be a space of points (objects) and $x \in X$. A neutrosophic set (NS) $\tilde{B}^n \subset X$ is represented by three independent functions: membership function $\mu_{\tilde{B}^n}(x)$, hesitation function $\sigma_{\tilde{B}^n}(x)$ and non-membership function $\nu_{\tilde{B}^n}(x)$ and expressed as*

$$\tilde{B}^n = \{(x, \mu_{\tilde{B}^n}(x), \sigma_{\tilde{B}^n}(x), \nu_{\tilde{B}^n}(x)) : x \in X\}.$$

where $\mu_{\tilde{B}^n}(x),\ \sigma_{\tilde{B}^n}(x),\ \nu_{\tilde{B}^n}(x) : X \to]0^-, 1^+[\ \forall\ x \in X$ are real standard or non-standard subset of $]0^-, 1^+[$. The sum of these three independent functions is related as follows:

$$0^- \leq Sup\ \mu_{\tilde{B}^n}(x) + Sup\ \sigma_{\tilde{B}^n}(x) + Sup\ \nu_{\tilde{B}^n}(x) \leq 3^+\ \forall\ x \in X.$$

Definition 5 (Single valued neutrosophic set)**.** *[45] Let the set X be a space of points (objects). A single valued neutrosophic set (SVNS) $\tilde{B} \subset X$ is expressed as*

$$\tilde{B}^n = \{(x, \mu_{\tilde{B}^n}(x), \sigma_{\tilde{B}^n}(x), \nu_{\tilde{B}^n}(x)) : x \in X\}$$

where $\mu_{\tilde{B}^n}, \sigma_{\tilde{B}^n}, \nu_{\tilde{B}^n} : X \to [0, 1]$ satisfy the condition $0 \leq \mu_{\tilde{B}^n}(x) + \sigma_{\tilde{B}^n}(x) + \nu_{\tilde{B}^n}(x) \leq 3\ \forall\ x \in X$. $\mu_{\tilde{B}^n}(x)$, $\sigma_{\tilde{B}^n}(x)$ and $\nu_{\tilde{B}^n}(x)$ denote the membership, hesitation and non-membership function respectively.

Definition 6 (Single valued pentagonal neutrosophic number)**.** *[51] A single valued pentagonal neutrosophic number (SVPN-number) $\tilde{B}^n$ having the form*

$$\tilde{B}^n = \langle [(b_1^{'}, b_2^{'}, b_3^{'}, b_4^{'}, b_5^{'}); w], [(b_1^{''}, b_2^{''}, b_3^{''}, b_4^{''}, b_5^{''}); u], [(b_1^{'''}, b_2^{'''}, b_3^{'''}, b_4^{'''}, b_5^{'''}); y] \rangle$$

where $w, u, y \in [0, 1]$. Here, membership function $\mu_{\tilde{B}^n}(x) : \mathbb{R} \to [0, w]$, hesitation function $\sigma_{\tilde{B}^n}(x) : \mathbb{R} \to [u, 1]$ and non-membership function $\nu_{\tilde{B}^n}(x) : \mathbb{R} \to [y, 1]$ are defined as follows:

$$\mu_{\tilde{B}^n}(x) = \begin{cases} \mu_{\widetilde{B^n l1}}(x) & b_1^{'} \leq x < b_2^{'} \\ \mu_{\widetilde{B^n l2}}(x) & b_2^{'} \leq x < b_3^{'} \\ w & x = b_3^{'} \\ \mu_{\widetilde{B^n r2}}(x) & b_3^{'} \leq x < b_4^{'} \\ \mu_{\widetilde{B^n r1}}(x) & b_4^{'} \leq x < b_5^{'} \\ 0 & otherwise \end{cases}, \quad \sigma_{\tilde{B}^n}(x) = \begin{cases} \sigma_{\widetilde{B^n l1}}(x) & b_1^{''} \leq x < b_2^{''} \\ \sigma_{\widetilde{B^n l2}}(x) & b_2^{''} \leq x < b_3^{''} \\ u & x = b_3^{''} \\ \sigma_{\widetilde{B^n r2}}(x) & b_3^{''} \leq x < b_4^{''} \\ \sigma_{\widetilde{B^n r1}}(x) & b_4^{''} \leq x < b_5^{''} \\ 0 & otherwise \end{cases}$$

and

$$\nu_{\tilde{B}^n}(x) = \begin{cases} \nu_{\widetilde{B^n l1}}(x) & b_1''' \leq x < b_2''' \\ \nu_{\widetilde{B^n l2}}(x) & b_2''' \leq x < b_3''' \\ y & x = b_3''' \\ \nu_{\widetilde{B^n r2}}(x) & b_3''' \leq x < b_4''' \\ \nu_{\widetilde{B^n r1}}(x) & b_4''' \leq x < b_5''' \\ 0 & otherwise \end{cases}$$

respectively.

Definition 7 (Score function of PN number). *[51] The score function is significantly used in the conversion of PN number into a crisp real number. The value of the score function utterly depends on the value of the membership, hesitation and non-membership degree of the PN number. Let* $\tilde{B}^n = \langle (b_1, b_2, b_3, b_4, b_5); \mu_{\tilde{B}^n}, \sigma_{\tilde{B}^n}, \nu_{\tilde{B}^n} \rangle$ *be any SVPN-number. Then the value of the score function is evaluated as*

$$S(\tilde{B}^n) = \frac{1}{15}\left\{(b_1 + b_2 + b_3 + b_4 + b_5) \times (2 + \mu_{\tilde{B}^n} - \sigma_{\tilde{B}^n} - \nu_{\tilde{B}^n})\right\}$$

Definition 8 (Single valued trapezoidal neutrosophic number). *[62] A single valued trapezoidal neutrosophic number (SVTN-number)* $\check{B}$ *is a special neutrosophic set on R (set of real number) having the form*

$$\check{B} = \langle (b_1, b_2, b_3, b_4); w_{\check{B}}, u_{\check{B}}, y_{\check{B}} \rangle$$

where $w_{\check{B}}, u_{\check{B}}, y_{\check{B}} \in [0, 1]$ *be any real numbers and* $b_1, b_2, b_3, b_4 \in \mathbb{R}$, $b_1 \leq b_2 \leq b_3 \leq b_4$ *are the values of the trapezoidal number. Here, membership function* $\mu_{\check{B}}(x)$*, hesitation function* $\sigma_{\check{B}}(x)$ *and non-membership function* $\nu_{\check{B}}(x)$ *are defined as follows:*

$$\mu_{\check{B}}(x) = \begin{cases} \frac{x-b_1}{b_2-b_1} w_{\check{B}} & if\ b_1 \leq x < b_2 \\ w_{\check{B}} & if\ b_2 \leq x \leq b_3 \\ \frac{b_4-x}{b_4-b_3} w_{\check{B}} & if\ b_3 < x \leq b_4 \\ 0 & otherwise \end{cases}, \quad \sigma_{\check{B}}(x) = \begin{cases} \frac{b_2-x+u_{\check{B}}(x-b_1)}{b_2-b_1} & if\ b_1 \leq x < b_2 \\ u_{\check{B}} & if\ b_2 \leq x \leq b_3 \\ \frac{x-b_3+u_{\check{B}}(b_4-x)}{b_4-b_3} & if\ b_3 < x \leq b_4 \\ 0 & otherwise \end{cases}$$

and

$$\nu_{\check{B}}(x) = \begin{cases} \frac{b_2-x+y_{\check{B}}(x-b_1)}{b_2-b_1} & if\ b_1 \leq x < b_2 \\ y_{\check{B}} & if\ b_2 \leq x \leq b_3 \\ \frac{x-b_3+y_{\check{B}}(b_4-x)}{b_4-b_3} & if\ b_3 < x \leq b_4 \\ 0 & otherwise \end{cases}$$

respectively.

Definition 9 ($\langle\alpha, \beta, \gamma\rangle$-cut set of SVTN-number). *[62] Let* $\check{B} = \langle (b_1, b_2, b_3, b_4); w_{\check{B}}, u_{\check{B}}, y_{\check{B}} \rangle$ *be a SVTN-number. Then,* $\langle\alpha, \beta, \gamma\rangle$*-cut set of the SVTN-number* $\check{B}$ *is denoted by* $\check{B}_{\langle\alpha,\beta,\gamma\rangle}$ *and defined as:*

$$\check{B}_{\langle\alpha,\beta,\gamma\rangle} = \{x|\mu_{\check{B}}(x) \geq \alpha, \sigma_{\check{B}}(x) \leq \beta, \nu_{\check{B}}(x) \leq \gamma, x \in R\}$$

which satisfies following the conditions:

$$0 \leq \alpha \leq w_{\check{B}},\ u_{\check{B}} \leq \beta \leq 1,\ y_{\check{B}} \leq \gamma \leq 1 \text{ and } 0 \leq \alpha + \beta + \gamma \leq 3$$

where $\mu_{\check{B}}$, $\sigma_{\check{B}}$ and $\nu_{\check{B}}$ represent membership, hesitation and non-membership function respectively.

Definition 10 (α-cut, β-cut, γ-cut of SVTN-number)**.** *[62] Let $\check{B} = \langle(b_1, b_2, b_3, b_4); w_{\check{B}}, u_{\check{B}}, y_{\check{B}}\rangle$ be an arbitrary SVTN-number. Then*

1. α-cut of $\check{B}$ is defined by

$$\check{B}_\alpha = [L_{\check{B}}(\alpha), R_{\check{B}}(\alpha)] = \left[\tfrac{(w_{\check{B}}-\alpha)b_1+\alpha b_2}{w_{\check{B}}}, \tfrac{(w_{\check{B}}-\alpha)b_4+\alpha b_3}{w_{\check{B}}}\right]$$

where $\alpha \in [0, w_{\check{B}}]$.

2. β-cut of $\check{B}$ is defined by

$$\check{B}_\beta = [L'_{\check{B}}(\beta), R'_{\check{B}}(\beta)] = \left[\tfrac{(1-\beta)b_2+(\beta-u_{\check{B}})b_1}{1-u_{\check{B}}}, \tfrac{(1-\beta)b_3+(\beta-u_{\check{B}})b_4}{1-u_{\check{B}}}\right]$$

where $\beta \in [u_{\check{B}}, 1]$.

3. γ-cut of $\check{B}$ is defined by

$$\check{B}_\gamma = [L''_{\check{B}}(\gamma), R''_{\check{B}}(\gamma)] = \left[\tfrac{(1-\gamma)b_2+(\gamma-y_{\check{B}})b_1}{1-y_{\check{B}}}, \tfrac{(1-\gamma)b_3+(\gamma-y_{\check{B}})b_4}{1-y_{\check{B}}}\right]$$

where $\gamma \in [y_{\check{B}}, 1]$.

Theorem 1. *Let $\check{B} = \langle(b_1, b_2, b_3, b_4); w_{\check{B}}, u_{\check{B}}, y_{\check{B}}\rangle$ be an arbitrary SVTN-number. Then, $\check{B}_{\langle\alpha,\beta,\gamma\rangle} = \check{B}_\alpha \cap \check{B}_\beta \cap \check{B}_\gamma$ is hold for any $0 < \alpha < w_{\check{B}},\ u_{\check{B}} < \beta < 1$ and $y_{\check{B}} < \gamma < 1$ where $0 \leq \alpha + \beta + \gamma \leq 3$.*

Proof: See [62]

Lemma 2. *Let $A = [a, b], a, b > 0$ be a closed interval with weights $w_1(> 0), w_2(> 0)$. Then the interval can be represented by a function using the weighted arithmetic mean*

$$WAM_A(\rho) = \frac{w_1 a + w_2 b}{w_1 + w_2} = a(1-\rho) + b\rho$$

where $\rho = \frac{w_2}{w_1+w_2}$, $\rho \in [0, 1]$.

Definition 11 (Rough variable)**.** *[28] Let $(\Lambda, \delta, \mathcal{A}, \pi)$ be a rough space. A rough variable ξ is a measurable function from the rough space $(\Lambda, \delta, \mathcal{A}, \pi)$ to $\mathbb{R}$ (the set of real numbers). That is, for every Borel set B of $\mathbb{R}$, we have $\{\eta \in \Lambda : \xi(\eta) \in B\} \in \mathcal{A}$*

The upper and lower approximations of the rough variable ξ are denoted and defined by $\bar{\xi} = \{\xi(\eta) : \eta \in \Lambda\}$ and $\underline{\xi} = \{\xi(\eta) : \eta \in \delta\}$ respectively.

Definition 12 (Trust measure)**.** *[28] Let $(\Lambda, \delta, \mathcal{A}, \pi)$ be a rough space. The trust measure of event A is denoted by $Tr\{A\}$ and defined by $Tr\{A\} = \frac{1}{2}(\underline{Tr}\{A\} + \overline{Tr}\{A\})$, where the lower and upper trust measure of event A are defined by $\underline{Tr}\{A\} = \frac{\pi\{A\cap\delta\}}{\pi\{\delta\}}$ and $\overline{Tr}\{A\} = \frac{\pi\{A\}}{\pi\{\Lambda\}}$ respectively.*

For a real life problem when sufficient information is given about the measurement of π, it may be treated as Lebesgue measure. More generally, the trust measure can be considered in the form as $Tr\{A\} = (1-\eta)\underline{Tr}\{A\} + \eta\overline{Tr}\{A\}, \; 0 < \eta < 1$.

Let $\hat{\xi} = ([m,n][p,q]), \; p \le m \le n \le q$ *be a rough variable. Lebesgue measure is considered for the trust measure of the rough event associated with* $\hat{\xi} \ge r$. *Then the trust measure of the rough event* $\hat{\xi} \ge r$ *is defined by the following function curve*

$$Tr\left\{\hat{\xi} \ge r\right\} = \begin{cases} 0 & if \; q \le r \\ \frac{\eta(q-r)}{q-p} & if \; n \le r \le q \\ \frac{\eta(q-r)}{q-p} + \frac{(1-\eta)(n-r)}{n-m} & if \; m \le r \le n \\ \frac{\eta(q-r)}{q-p} + (1-\eta) & if \; p \le r \le m \\ 1 & if \; r \le p \end{cases}$$

Theorem 2. *[28] Let* $\hat{\xi} = ([m,n][p,q]), \; p \le m \le n \le q$ *be a rough variable and* $\hat{\xi} \ge r$ *be a rough event. Then* $Tr\left\{\hat{\xi} \ge r\right\} \ge \alpha$ *iff*

$$r \le \begin{cases} q - \frac{\alpha(q-p)}{\eta} & if \; n \le r \le q \\ \frac{\eta(n-m)+(1-\eta)n(q-p)-\alpha(q-p)(n-m)}{\eta(n-m)+(1-\eta)n(q-p)} & if \; m \le r \le n \\ q + \frac{(1-\eta-\alpha)(q-p)}{\eta} & if \; p \le r \le m \\ p \end{cases}$$

for any predetermined level $\alpha \in [0,1]$

Proof. For any $\alpha \in [0,1]$, we have

$$\begin{aligned} & Tr\left\{\hat{\xi} \ge r\right\} \ge \alpha \\ \Leftrightarrow \quad & \alpha \le Tr\left\{\hat{\xi} \ge r\right\} \\ \Leftrightarrow \quad & \alpha \le \begin{cases} 0 & if \; q \le r \\ \frac{\eta(q-r)}{q-p} & if \; n \le r \le q \\ \frac{\eta(q-r)}{q-p} + \frac{(1-\eta)(n-r)}{n-m} & if \; m \le r \le n \\ \frac{\eta(q-r)}{q-p} + (1-\eta) & if \; p \le r \le m \\ 1 & if \; r \le p \end{cases} \end{aligned}$$

[With the help of definition of trust measure of an event]

$$\Leftrightarrow r \le \begin{cases} q - \frac{\alpha(q-p)}{\eta} & if \; n \le r \le q \\ \frac{\eta(n-m)+(1-\eta)n(q-p)-\alpha(q-p)(n-m)}{\eta(n-m)+(1-\eta)n(q-p)} & if \; m \le r \le n \\ q + \frac{(1-\eta-\alpha)(q-p)}{\eta} & if \; p \le r \le m \\ p \end{cases}$$

Hence the proof is done.

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Similarity Measures for Interval-Valued Neutrosophic Hypersoft Set With Their Application to Solve Decision Making Problem

Rana Muhammad Zulqarnain[1], Aiyared Iampan[2], Hamiden Abd El-Wahed Khalifa[3,4], Imran Siddique[5*]

1 College of Mathematics and Computer Science, Zhejiang Normal University, Jinhua, Zhejiang 321004, China. E-mail: ranazulqarnain7777@gmail.com
2 Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand. E-mail: aiyared.ia@up.ac.th
3 Department of Mathematics, College of Science and Arts, Al-Badaya, Qassim University E-mail: Ha.Ahmed@qu.edu.sa
4 Operations Research Department, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt, E-mail: hamiden@cu.edu.eg
5 Department of Mathematics, University of Management and Technology Lahore, Pakistan. E-mail: imransiddique@umt.edu.pk

Correspondence: E-mail: imransiddique@umt.edu.pk

Abstract:

A similarity measure is used to tackle many issues that include indistinct and blurred information, excluding is not able to deal with the general fuzziness and obscurity of the problems that have various information. In this paper, we study some basic concepts which are helpful for us to build the structure of the article, such as soft set, neutrosophic soft set, hypersoft set, neutrosophic hypersoft set (NHSS), and interval-valued NHSS, etc. The main objective of the present research is to develop a cosine similarity measure and set-theoretic similarity measure for an IVNHSS with their necessary properties. A decision-making approach has been established by using cosine and set-theoretic similarity measures. Furthermore, we used to develop a technique to solve multi-criteria decision-making problems. Finally, the advantages, effectiveness, flexibility, and comparative analysis of the algorithms are given with prevailing methods.

Keywords: Neutrosophic soft-set; hypersoft set; neutrosophic hypersoft set; interval-valued neutrosophic hypersoft set; similarity measures

1. Introduction

Decision-making is an interesting concern to pick the perfect alternate for any specific persistence. Firstly, it is pretended that details about alternatives are accumulated in crisp numbers, but in real-life situations, collective farm information always conquers wrong and inaccurate information. Fuzzy sets are like sets having an element of membership (Mem) degree. In classical set theory, the Mem degree of the elements in a set is examined in binary form to see that the element is not entirely concomitant. In contrast, the fuzzy set theory enables advanced Mem categorization of the components in the set. The Mem function portrays it, and also the multipurpose unit interval of the Mem function is [0, 1]. In some circumstances, decision-makers consider objects' Mem and nonmember-ship (Nmem) values. Zadeh's FS cannot handle imprecise and vague information in such cases. Atanassov [2] developed the notion of intuitionistic fuzzy sets (IFS) to deal above mentioned difficulties. The IFS accommodates the imprecise and inaccurate information using Mem and Nmem values.

Atanassov IFS was unable to solve those problems in which decision-makers considered the membership degree (MD) and nonmembership degrees (NMD) such as MD = 0.5 and NDM = 0.8, then $0.5 + 0.8 \nleq 1$. Yager [3, 4] extended the notion of IFS to Pythagorean fuzzy sets (PFSs) to overcome above-discussed complications by modifying $MD + NMD \leq 1$ to $MD^2 + NMD^2 \leq 1$. After developing PFSs, Zhang and Xu [5] proposed operational laws for PFSs and established a DM approach to resolving the MCDM problem. Wei and Lu [6] planned some power aggregation operators (AOs) and proposed a DM technique to solve multi-attribute decision-making (MADM) issues under the Pythagorean fuzzy environment. Wang and Li [7] presented power Bonferroni mean operators for PFSs with their basic properties using interaction. Gao et al. [8] presented several aggregation operators by considering the interaction and proposed a DM approach to solving MADM difficulties utilizing the developed operators. Wei [9] developed the interaction operational laws for Pythagorean fuzzy numbers (PFNs) by considering interaction and established interaction aggregation operators by using the developed interaction operations. Zhang [10] developed the accuracy function and presented a DM approach to solving multiple criteria group decision-making (MCGDM) problems using PFNs. Wang et al. [11] extended the PFS and introduced an interactive Hamacher operation with some novel AOs. They also established a DM method to solve MADM problems using their proposed operators. Wang and Li [12] developed some interval-valued PFSs and utilized their operators to resolve multi-attribute group decision-making (MAGDM) issues. Peng and Yuan [13] established novel operators such as Pythagorean fuzzy point operators and developed a DM technique using their proposed operators. Peng and Yang [14] introduced fundamental operations and their necessary possessions under PFSs and planned DM methodology. Garg [15] developed the logarithmic operational laws for PFSs and proposed some AOs. Arora and Garg [16] presented the operational laws for linguistic IFS and developed prioritized AOs. Ma and Xu [17] presented some innovative AOs for PFSs and proposed the score and accuracy functions for PFNs.

Above mentioned theories and their DM methodologies have been used in several fields of life. But, these theories cannot deal with the parametrization of the alternatives. Molodtsov [18] developed soft sets (SS) to overcome the complications, as mentioned earlier. Molodtsov's SS competently deals with imprecise, vague, and unclear objects considering their parametrization. Maji et al. [19] prolonged the notion of SS and introduced some necessary operators with their properties. Maji et al. [20] established a DM technique using their developed operations for SS. They also merged two well-known theories, such as FS and SS, and established the concept of fuzzy soft sets (FSS) [21]. They also proposed an intuitionistic fuzzy soft set (IFSS) [22] and discussed their basic operations. Garg and Arora [23] extended the notion of IFSS and presented a generalized form of IFSS with AOs. They also planned a DM technique to resolve undefined and inaccurate information under IFSS information. Garg and Arora [24] presented the correlation and weighted correlation coefficients for IFSS and developed the TOPSIS approach utilizing established correlation procedures. Zulqarnain et al. [25] introduced some AOs and correlation coefficients for interval-valued IFSS. They also extended the TOPSIS technique using their developed correlation measures to solve the MADM problem. Peng et al. [26] proposed the Pythagorean fuzzy soft sets (PFSSs) and presented fundamental operations of PFSSs with their desirable properties by merging PFS and SS. Athira et al. [27] proposed the entropy measure for PFSSs. They also presented some distance measures for PFSSs and utilized their developed distance measures to solve DM [28] issues. Zulqarnain et al. [29] introduced Einstein operational laws for Pythagorean fuzzy soft numbers (PFSNs) and developed Einstein AOs utilizing defined operational laws for PFSNs. They also planned a DM approach to solve MADM problems with the help of presented operators. Riaz et al. [30] prolonged the idea of PFSSs and developed the m polar PFSSs. They also established the TOPSIS method under the considered hybrid structure and proposed a DM methodology to solve the MCGDM problem. Riaz et al. [31] developed the similarity measures for PFSS with their fundamental properties. Zulqarnain et al. [32, 33] protracted the Einstein ordered AOs for PFSSs and utilized their developed approach to solving the MAGDM problem.

All the above studies only deal the inadequate information because of membership and non-membership values. However, these theories cannot handle the overall incompatible and imprecise data. To address such inconsistent and vague records, the idea of the neutrosophic set (NS) was developed by Smarandache [34]. Maji [35] offered the perception of a neutrosophic soft set (NSS) with necessary operations. The idea of the possibility of NSS was developed by Karaaslan [36] and introduced a possibility of neutrosophic soft DM method based on And-product. Broumi [37] developed the generalized NSS with some operations and properties and used the projected concept for DM. Deli and Subas [38] developed the single-valued Neutrosophic numbers (SVNNs) to solve MCDM problems. They also established the cut sets for SVNNs. Wang et al. [39] proposed the correlation coefficient (CC) for SVNSs. Ye [40] introduced the simplified NSs with operational laws and AOs. Also, he presented an MCDM technique utilizing his planned AOs. Masooma et al. [41] progressed a new concept by combining the multipolar fuzzy set and NS, known as the multipolar NS. They also established various characterization and operations with examples. Zulqarnain et al. [42] offered the generalized neutrosophic TOPSIS and used their presented technique for supplier selection.

All the above studies have some limitations. When any attribute from a set contains further sub-attributes, then the above-presented theories fail to solve such problems. To overcome the limitations above, Smarandache [43] protracted the idea of SS to hypersoft sets (HSS) by substituting the one-parameter function f to a multi-parameter (sub-attribute) function. Samarandache claimed that the established HSS competently deals with uncertain objects compared to SS. Several researchers explored the HSS and presented a lot of extensions for HSS [44, 45]. Zulqarnain et al. [46] presented the IFHSS, the generalized version of IFSS. They established the TOPSIS method utilizing the developed correlation coefficient. Zulqarnain et al. [47] proposed the Pythagorean fuzzy hypersoft sets with AOs and correlation coefficients. They also established the TOPSIS technique using their developed correlation coefficient and utilized the presented approach to select appropriate anti-virus face masks. Zulqarnain et al. [48, 49] presented some fundamental operations with their properties for interval-valued NHSS. Also, they proposed the CC and WCC for interval-valued NHSSS and established a decision-making approach utilizing their developed CC. Several researchers extended the notion of HSS and introduced different operators for hybrid structures of fuzzy hypersoft sets [50-55]. Broumi et al. [56] extended the mathematical algebra of neutrosophic graphs to the interval-valued neutrosophic graph. Broumi et al. [57] discussed several operations for interval-valued neutrosophic graphs and discussed their desirable properties. Singh [58] introduced a fuzzy three-way context using the properties of the interval-valued neutrosophic set, and its graphical properties deal with partial ignorance. Zulqarnain et al. [59, 60] proposed some fundamental operations for generalized multipolar neutrosophic soft sets and multipolar interval-valued neutrosophic soft sets. However, all existing studies only deal with the scenario by using MD and NMD of sub-attributes of the considered attributes. If any decision-maker considers the MD = 0.7 and NDM = 0.6, then $0.7 + 0.6 \leq 1$ of any sub-attribute of the alternatives. We can observe that the theories mentioned above cannot handle it. Zulqarnain et al. [47] proposed the more generalized notion of PFHSS comparative to IFHSS. The PFHSS accommodates more uncertainty compared to IFHSS by updating the condition $\boldsymbol{MD} + \boldsymbol{NMD} \leq \mathbf{1}$ to $\left(\mathcal{T}_{\mathcal{F}(\breve{a})}(\boldsymbol{\delta})\right)^{2} + \left(\mathcal{J}_{\mathcal{F}(\breve{a})}(\boldsymbol{\delta})\right)^{2} \leq 1$. All existing hybrid structures of FS cannot handle the indeterminacy of sub-attributes of considered n-tuple attributes. On the other hand, developed aggregation operators can accommodate the sub-attributes of considered attributes using truthiness, indeterminacy, and falsity objects of sub-attributes with the following condition $0 \leq \boldsymbol{\sigma}_{\mathcal{F}(\breve{\alpha}_k)}(\boldsymbol{\delta}) + \boldsymbol{\tau}_{\mathcal{F}(\breve{\alpha}_k)}(\boldsymbol{\delta}) + \boldsymbol{\gamma}_{\mathcal{F}(\breve{\alpha}_k)}(\boldsymbol{\delta}) \leq 3$. It may be seen that the best selection of the suggested approach is to resemble the verbalized own method, and that ensures the liableness along with the effectiveness of the recommended approach.

The following research is organized: In section 2, we recollected some basic definitions used in the subsequent sequel, such as NS, SS, NSS, HSS, NHSS, and IVNHSS. Section 3 proposes the similarity measures such as cosine and set-theoretic for NHSS with its properties. We established a decision-making technique to solve decision-making complications utilizing our developed similarity measures. In section 4, we use the proposed similarity measures for decision-making. A brief comparative analysis has been conducted between proposed techniques with existing methodologies in section 5. Finally, the conclusion and future directions are presented in section 6.

2. Preliminaries

In this section,"we recollect some basic definitions which are helpful to build the structure of the following manuscript such as soft set, hypersoft set, and neutrosophic hypersoft set.

Definition 2.1 [4]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $A \subseteq \mathcal{E}$. A pair $(\mathcal{F}, A)$ is called a soft set over $\mathcal{U}$, and its mapping is given as

$$\mathcal{F}: A \to \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F}, A) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \; if \; e \notin A\}$$

Definition 2.2 [38]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}$, $(n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathcal{A}} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of multi-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{A})$ is said to be HSS over $\mathcal{U}$, and its mapping is defined as

$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{A} \to \mathcal{P}(\mathcal{U})$.

It is also defined as

$(\mathcal{F}, \ddot{A}) = \{\breve{a}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}): \breve{a} \in \ddot{A}, \mathcal{F}_{\ddot{\mathcal{A}}}(\breve{a}) \in \mathcal{P}(\mathcal{U})\}$

Definition 2.3 [38]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}$, $(n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{A} = \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$ and $NS^{\mathcal{U}}$ be a collection of all neutrosophic subsets over $\mathcal{U}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{A})$ is said to be NHSS over $\mathcal{U}$, and its mapping is defined as

$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{A} \to NS^{\mathcal{U}}$.

It is also defined as

$(\mathcal{F}, \ddot{A}) = \{(\breve{a}, \mathcal{F}_{\ddot{A}}(\breve{a})): \breve{a} \in \ddot{A}, \mathcal{F}_{\ddot{A}}(\breve{a}) \in NS^{\mathcal{U}}\}$, where $\mathcal{F}_{\ddot{A}}(\breve{a}) = \{\langle \delta, \sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta), \gamma_{\mathcal{F}(\breve{a})}(\delta) \rangle : \delta \in \mathcal{U}\}$, where $\sigma_{\mathcal{F}(\breve{a})}(\delta)$, $\tau_{\mathcal{F}(\breve{a})}(\delta)$, and $\gamma_{\mathcal{F}(\breve{a})}(\delta)$ represent the truth, indeterminacy, and falsity grades of the attributes such as $\sigma_{\mathcal{F}(\breve{a})}(\delta), \tau_{\mathcal{F}(\breve{a})}(\delta), \gamma_{\mathcal{F}(\breve{a})}(\delta) \in [0, 1]$, and $0 \leq \sigma_{\mathcal{F}(\breve{a})}(\delta) + \tau_{\mathcal{F}(\breve{a})}(\delta) + \gamma_{\mathcal{F}(\breve{a})}(\delta) \leq 3$.

Example 2.4

Consider the universe of discourse $\mathcal{U} = \{\delta_1, \delta_2\}$ and $\mathfrak{L} = \{\ell_1 = Teaching\ methdology, \ell_2 = Subjects, \ell_3 = Classes\}$ be a collection of attributes with following their corresponding attribute values are given as teaching methodology $= L_1 = \{a_{11} = project\ base, a_{12} = class\ discussion\}$, Subjects $= L_2 = \{a_{21} = Mathematics, a_{22} = Computer\ Science, a_{23} = Statistics\}$, and Classes $= L_3 = \{a_{31} = Masters, a_{32} = Doctorol\}$. Let $\ddot{A} = L_1 \times L_2 \times L_3$ be a set of attributes

$\ddot{A} = L_1 \times L_2 \times L_3 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}, a_{23}\} \times \{a_{31}, a_{32}\}$

$= \begin{Bmatrix} (a_{11}, a_{21}, a_{31}), (a_{11}, a_{21}, a_{32}), (a_{11}, a_{22}, a_{31}), (a_{11}, a_{22}, a_{32}), (a_{11}, a_{23}, a_{31}), (a_{11}, a_{23}, a_{32}), \\ (a_{12}, a_{21}, a_{31}), (a_{12}, a_{21}, a_{32}), (a_{12}, a_{22}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{12}, a_{23}, a_{31}), (a_{12}, a_{23}, a_{32}), \end{Bmatrix}$

$\ddot{A} = \{\breve{a}_1, \breve{a}_2, \breve{a}_3, \breve{a}_4, \breve{a}_5, \breve{a}_6, \breve{a}_7, \breve{a}_8, \breve{a}_9, \breve{a}_{10}, \breve{a}_{11}, \breve{a}_{12}\}$

Then the NHSS over $\mathcal{U}$ is given as follows

$(\boldsymbol{\mathcal{F}}, \text{Ä}) =$

$$\left\{\begin{array}{c} \left(\breve{\boldsymbol{a}}_{1},(\boldsymbol{\delta}_{1},(.\mathbf{6},.\mathbf{3},.\mathbf{8})),(\boldsymbol{\delta}_{2},(.\mathbf{9},.\mathbf{3},.\mathbf{5}))\right),\left(\breve{\boldsymbol{a}}_{2},(\boldsymbol{\delta}_{1},(.\mathbf{5},.\mathbf{2},.\mathbf{7})),(\boldsymbol{\delta}_{2},(.\mathbf{7},.\mathbf{1},.\mathbf{5}))\right),\left(\breve{\boldsymbol{a}}_{3},(\boldsymbol{\delta}_{1},(.\mathbf{5},.\mathbf{2},.\mathbf{8})),(\boldsymbol{\delta}_{2},(.\mathbf{4},.\mathbf{3},.\mathbf{4}))\right), \\ \left(\breve{\boldsymbol{a}}_{4},(\boldsymbol{\delta}_{1},(.\mathbf{2},.\mathbf{5},.\mathbf{6})),(\boldsymbol{\delta}_{2},(.\mathbf{5},.\mathbf{1},.\mathbf{6}))\right),\left(\breve{\boldsymbol{a}}_{5},(\boldsymbol{\delta}_{1},(.\mathbf{8},.\mathbf{4},.\mathbf{3})),(\boldsymbol{\delta}_{2},(.\mathbf{2},.\mathbf{3},.\mathbf{5}))\right),\left(\breve{\boldsymbol{a}}_{6},(\boldsymbol{\delta}_{1},(.\mathbf{9},.\mathbf{6},.\mathbf{4})),(\boldsymbol{\delta}_{2},(.\mathbf{7},.\mathbf{6},.\mathbf{8}))\right), \\ \left(\breve{\boldsymbol{a}}_{7},(\boldsymbol{\delta}_{1},(.\mathbf{6}..\mathbf{5},.\mathbf{3})),(\boldsymbol{\delta}_{2},(.\mathbf{4},.\mathbf{2},.\mathbf{8}))\right),\left(\breve{\boldsymbol{a}}_{8},(\boldsymbol{\delta}_{1},(.\mathbf{8},.\mathbf{2},.\mathbf{5})),(\boldsymbol{\delta}_{2},(.\mathbf{6},.\mathbf{8},.\mathbf{4}))\right),\left(\breve{\boldsymbol{a}}_{9},(\boldsymbol{\delta}_{1},(.\mathbf{7},.\mathbf{4},.\mathbf{9})),(\boldsymbol{\delta}_{2},(.\mathbf{7}..\mathbf{3},.\mathbf{5}))\right), \\ \left(\breve{\boldsymbol{a}}_{10},(\boldsymbol{\delta}_{1},(.\mathbf{8},.\mathbf{4},.\mathbf{6})),(\boldsymbol{\delta}_{2},(.\mathbf{7},.\mathbf{2},.\mathbf{9}))\right),\left(\breve{\boldsymbol{a}}_{11},(\boldsymbol{\delta}_{1},(.\mathbf{8},.\mathbf{4},.\mathbf{5})),(\boldsymbol{\delta}_{2},(.\mathbf{4},.\mathbf{2},.\mathbf{5}))\right),\left(\breve{\boldsymbol{a}}_{5},(\boldsymbol{\delta}_{1},(.\mathbf{7},.\mathbf{5},.\mathbf{8})),(\boldsymbol{\delta}_{2},(.\mathbf{7},.\mathbf{5},.\mathbf{9}))\right) \end{array}\right\}$$

Definition 2.5 [42]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k = \{k_1, k_2, k_3, \ldots, k_n\}$, $(n \geq 1)$ be a set of attributes and set K_i a set of corresponding sub-attributes of k_i respectively with $K_i \cap K_j = \varphi$ for $n \geq 1$ for each $i, j \in \{1,2,3 \ldots n\}$ and $i \neq j$. Assume $K_1 \times K_2 \times K_3 \times \ldots \times K_n =$ Ä $= \{a_{1h} \times a_{2k} \times \cdots \times a_{nl}\}$ be a collection of sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$ and $IVNS^{\mathcal{U}}$ be a collection of all interval-valued neutrosophic subsets over $\mathcal{U}$. Then the pair ($\mathcal{F}$, $K_1 \times K_2 \times K_3 \times \ldots \times K_n =$ Ä) is said to be IVNHSS over $\mathcal{U}$, and its mapping is defined as

$\mathcal{F}$: $K_1 \times K_2 \times K_3 \times \ldots \times K_n =$ Ä $\rightarrow IVNS^{\mathcal{U}}$.

It is also defined as

($\mathcal{F}$, Ä) $= \{(\breve{a}_k, \mathcal{F}_{\text{Ä}}(\breve{a}_k)) : \breve{a}_k \in$ Ä, $\mathcal{F}_{\text{Ä}}(\breve{a}_k) \in NS^{\mathcal{U}}\}$, where $\mathcal{F}_{\text{Ä}}(\breve{a}) = \left\{\langle \delta, \sigma_{\mathcal{F}(\breve{a}_k)}(\delta), \tau_{\mathcal{F}(\breve{a}_k)}(\delta), \gamma_{\mathcal{F}(\breve{a}_k)}(\delta)\rangle : \delta \in \mathcal{U}\right\}$, where $\sigma_{\mathcal{F}(\breve{a}_k)}(\delta)$, $\tau_{\mathcal{F}(\breve{a}_k)}(\delta)$, and $\gamma_{\mathcal{F}(\breve{a}_k)}(\delta)$ represent the interval truth, indeterminacy, and falsity grades of the attributes such as $\sigma_{\mathcal{F}(\breve{a}_k)}(\delta) = \left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \sigma^{u}_{\mathcal{F}(\breve{a}_k)}(\delta)\right]$, $\tau_{\mathcal{F}(\breve{a}_k)}(\delta) = \left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \tau^{u}_{\mathcal{F}(\breve{a}_k)}(\delta)\right]$, $\gamma_{\mathcal{F}(\breve{a}_k)}(\delta) = \left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \gamma^{u}_{\mathcal{F}(\breve{a}_k)}(\delta)\right]$, where $\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \sigma^{u}_{\mathcal{F}(\breve{a}_k)}(\delta)$, $\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \tau^{u}_{\mathcal{F}(\breve{a}_k)}(\delta)$, $\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta), \gamma^{u}_{\mathcal{F}(\breve{a}_k)}(\delta) \subseteq [0,1]$, and $0 \leq \sigma^{u}_{\mathcal{F}(\breve{a}_k)}(\delta) + \tau^{u}_{\mathcal{F}(\breve{a}_k)}(\delta) + \gamma^{u}_{\mathcal{F}(\breve{a}_k)}(\delta) \leq 3$.

Simply an interval-valued neutrosophic hypersoft number (IVNHSN) can be expressed as $\boldsymbol{\mathcal{F}} = \left\{\left[\boldsymbol{\sigma}^{\boldsymbol{\ell}}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}), \boldsymbol{\sigma}^{u}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta})\right], \left[\boldsymbol{\tau}^{\boldsymbol{\ell}}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}), \boldsymbol{\tau}^{u}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta})\right], \left[\boldsymbol{\gamma}^{\boldsymbol{\ell}}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}), \boldsymbol{\gamma}^{u}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta})\right]\right\}$, where $0 \leq \boldsymbol{\sigma}^{u}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}) + \boldsymbol{\tau}^{u}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}) + \boldsymbol{\gamma}^{u}_{\mathcal{F}(\breve{a}_k)}(\boldsymbol{\delta}) \leq 3$.

3. Similarity Measures for Interval-Valued Neutrosophic Hypersoft Set

Many mathematicians developed various methodologies to solve MCDM problems in the past few years, such as aggregation operators for different hybrid structures, CC, similarity measures, and decision-making applications. The following section develops the cosine and set-theoretic similarity measure for IVNHSS.

Definition 3.1

Let ($\mathcal{F}$, Ä) $= \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \sigma^{u}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \tau^{u}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i), \gamma^{u}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and ($\mathcal{G}$, $\ddot{\mathbb{A}}$) $= \left\{\left(\delta_i, \left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \sigma^{u}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \tau^{u}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right], \left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i), \gamma^{u}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs defined over a universe of discourse $\mathcal{U}$. Then, the then cosine similarity measure of ($\mathcal{F}$, Ä) and ($\mathcal{G}$, $\ddot{\mathbb{A}}$) can be described as follows:

$$\mathcal{S}^1_{IVNHSS}((\mathcal{F},\ddot{A}),\ (\mathcal{G},\ddot{\mathbb{A}})) = \frac{1}{mn}\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\left(\begin{array}{c}\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)+\sigma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\sigma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)+\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)+\tau^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)+\\ \left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)+\gamma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)\end{array}\right)}{\left(\begin{array}{c}\sqrt{\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)}\\ \sqrt{\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)}\end{array}\right)}$$

Proposition 3.2

Let $(\mathcal{F},\ddot{A})$, $(\mathcal{G},\ddot{\mathbb{A}})$, and $(\mathcal{H},\breve{C}) \in$ NHSS, then the following properties hold

1. $0 \le \mathcal{S}^1_{IVNHSS}((\mathcal{F},\ddot{A}),\ (\mathcal{G},\ddot{\mathbb{A}})) \le 1$
2. $\mathcal{S}^1_{IVNHSS}((\mathcal{F},\ddot{A}),\ (\mathcal{G},\ddot{\mathbb{A}})) = \mathcal{S}^1_{IVNHSS}((\mathcal{G},\ddot{\mathbb{A}}),(\mathcal{F},\ddot{A}))$
3. If $(\mathcal{F},\ddot{A}) \subseteq (\mathcal{G},\ddot{\mathbb{A}}) \subseteq (\mathcal{H},\breve{C})$, then $\mathcal{S}^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{H},\breve{C})) \le \mathcal{S}^1_{IVNHSS}((\mathcal{F},\ddot{A}),\ (\mathcal{G},\ddot{\mathbb{A}}))$ and $\mathcal{S}^1_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{H},\breve{C})) \le \mathcal{S}^1_{IVNHSS}((\mathcal{G},\ddot{\mathbb{A}}),(\mathcal{H},\breve{C}))$.

Proof: Using the above definition, the proof of these properties can be done easily.

Definition 3.3

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i,\left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\sigma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right],\left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\tau^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right],\left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\gamma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i,\left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\sigma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right],\left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\tau^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right],\left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\gamma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs defined over a universe of discourse $\mathcal{U}$. Then, the then cosine similarity measure of $(\mathcal{F},\ddot{A})$ and $(\mathcal{G},\ddot{\mathbb{A}})$ can be described as follows:

$$\boldsymbol{\mathcal{S}^1_{IVNHSS}}\left((\boldsymbol{\mathcal{F}},\ddot{A}),\ (\boldsymbol{\mathcal{G}},\ddot{\mathbb{A}})\right) = \frac{1}{mn}\sum_{k=1}^{m}\sum_{i=1}^{n}\frac{\left(\begin{array}{c}\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)+\sigma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\sigma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)+\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)+\tau^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\tau^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)+\\ \left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)+\gamma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)+\gamma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)\end{array}\right)}{max\left(\begin{array}{c}\left(\left(\left(\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right)^2\right)\right),\\ \left(\left(\left(\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\sigma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\tau^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2+\left(\gamma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right)^2\right)\right)\end{array}\right)}$$

Proposition 3.4

Let $(\mathcal{F},\ddot{A}) = \left\{\left(\delta_i,\left[\sigma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\sigma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right],\left[\tau^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\tau^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right],\left[\gamma^{\ell}_{\mathcal{F}(\breve{a}_k)}(\delta_i),\gamma^{\mho}_{\mathcal{F}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ and $(\mathcal{G},\ddot{\mathbb{A}}) = \left\{\left(\delta_i,\left[\sigma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\sigma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right],\left[\tau^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\tau^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right],\left[\gamma^{\ell}_{\mathcal{G}(\breve{a}_k)}(\delta_i),\gamma^{\mho}_{\mathcal{G}(\breve{a}_k)}(\delta_i)\right]\right) \mid \delta_i \in \mathcal{U}\right\}$ be two IVNHSSs. Then, the following properties hold.

1. $0 \le \mathcal{S}^2_{IVNHSS}((\mathcal{F},\ddot{A}),\ (\mathcal{G},\ddot{\mathbb{A}})) \le 1$
2. $\mathcal{S}^2_{IVNHSS}((\mathcal{F},\ddot{A}),\ (\mathcal{G},\ddot{\mathbb{A}})) = \mathcal{S}^2_{IVNHSS}((\mathcal{G},\ddot{\mathbb{A}}),(\mathcal{F},\ddot{A}))$
3. If $(\mathcal{F},\ddot{A}) \subseteq (\mathcal{G},\ddot{\mathbb{A}}) \subseteq (\mathcal{H},\breve{C})$, then $\mathcal{S}^2_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{H},\breve{C})) \le \mathcal{S}^2_{IVNHSS}((\mathcal{F},\ddot{A}),\ (\mathcal{G},\ddot{\mathbb{A}}))$ and $\mathcal{S}^2_{IVNHSS}((\mathcal{F},\ddot{A}),(\mathcal{H},\breve{C})) \le \mathcal{S}^2_{IVNHSS}((\mathcal{G},\ddot{\mathbb{A}}),(\mathcal{H},\breve{C}))$.

Proof: Using the above definition, the proof of these properties can be done easily.

4. Application of Correlation Coefficient for Decision Making Under IVNHSS Environment

This section utilized the developed approaches based on cosine and set-theoretic similarity measures for decision making.

4.1 Algorithm for Similarity Measures of IVNHSS

Step 1. Pick out the set containing parameters.

Step 2. Construct the IVNHSS according to experts.

Step 3. Compute the cosine similarity measure by using definition 3.1.

Step 4. Compute the set-theoretic similarity measure for NHSS by utilizing definition 3.3.

Step 5. An alternative with a maximum value with cosine similarity measure has the maximum rank according to considered numerical illustration.

Step 6. An alternative with a maximum value with a set-theoretic similarity measure has the maximum rank according to considered numerical illustration.

Step 7. Analyze the ranking."

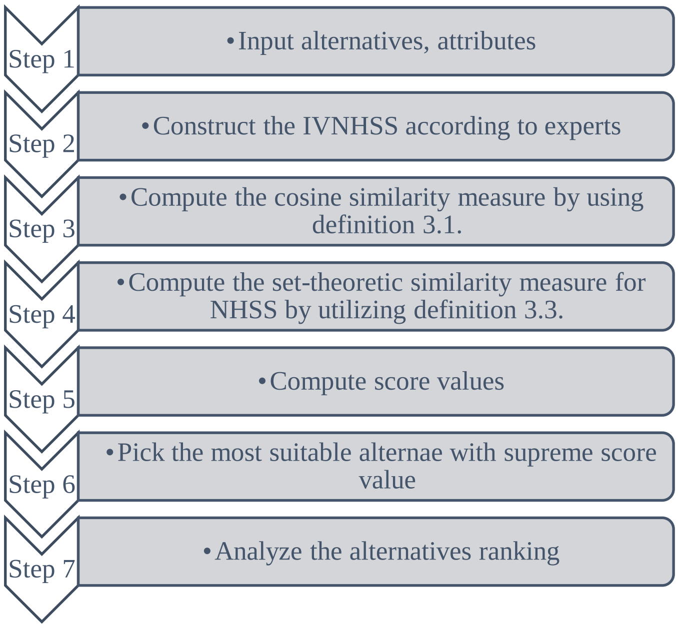

Figure 1: Flow chart of the presented similarity measures

4.2. Problem Formulation and Application of IVNHSS For Decision Making

A construction company calls for the appointment of a civil engineer to supervise the workers. Several engineers apply for the civil engineer post, simply four engineers call for an interview based on experience for undervaluation such as $S = \{S_1,\ S_2,\ S_3,\ S_4\}$ be a set of selected engineers call for the interview. The managing director hires a committee of four experts $\mathcal{U} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}\}$ for the selection of civil engineer. The group of experts chooses the set of attributes for the selection of an appropriate civil engineer such as $\mathfrak{L} = \{\ell_1 = Experiance, \ell_2 = Dealing\ skills, \ell_3 = Qualification\}$ be a set of parameters for the selection of MS. Experience $= \ell_1 = \{a_{11} = more\ than\ 20, a_{12} = \ less\ than\ 20\}$, Dealing skills $= \ell_2 = \{a_{21} = public\ dealing, a_{22} = \ Staff\ dealing\}$, and Qualification $= \ell_3 = \{a_{31} = Doctoral\ degree\ in\ medical\ education, a_{32} = \ Masters\ degree\ in\ medical\ education\}$. Let $\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3$ be a set of sub-attributes. The experts evaluate the applicants under defined parameters and forward the evaluation performa to the company's managing director. Finally, the director scrutinizes the best applicant based on the expert's evaluation report. Let $\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3$ be a set of sub-attributes

$\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}\} \times \{a_{31}, a_{32}\}$

$= \begin{Bmatrix} (a_{11}, a_{21}, a_{31}), (a_{11}, a_{21}, a_{32}), (a_{11}, a_{22}, a_{31}), (a_{11}, a_{22}, a_{32}), \\ (a_{12}, a_{21}, a_{31}), (a_{12}, a_{21}, a_{32}), (a_{12}, a_{22}, a_{31}), (a_{12}, a_{22}, a_{32}) \end{Bmatrix}$, $\mathfrak{L}' = \{\breve{a}_1, \breve{a}_2, \breve{a}_3, \breve{a}_4, \breve{a}_5, \breve{a}_6, \breve{a}_7, \breve{a}_8\}$ be a set of all multi sub-attributes. Each DM will evaluate the ratings of each alternative in the form of IVNHSNs under the considered multi sub-attributes. The developed method to find the best alternative is as follows.

4.2.1. Application of IVNHSS For Decision Making

Assume $S = \{S_1,\ S_2,\ S_3,\ S_4\}$ be a set of alternatives which are shortlisted for interview and $\mathfrak{L} = \{\ell_1 = Experiance, \ell_2 = Dealing\ skills, \ell_3 = Qualification\}$ be a set of parameters for the selection of MS. Experience = ℓ_1 = $\{a_{11} = more\ than\ 20, a_{12} = \ less\ than\ 20\}$, Dealing skills = ℓ_2 = $\{a_{21} = public\ dealing, a_{22} = \ Staff\ dealing\}$, and Qualification = ℓ_3 = $\{a_{31} = Doctoral\ degree\ in\ medical\ education, a_{32} = \ Masters\ degree\ in\ medical\ education\}$. Let $\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3$ be a set of sub-attributes. The health ministry defines a criterion for selecting MS for all alternatives in terms of IVNHSNs given in Table 1.

Table 1. Decision Matrix of Concerning Department

$\wp$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	([.3,.5], [.2,.4], [.2,.6])	([.2,.3], [.5,.7], [.1,.3])	([.5,.6], [.1,.3], [.4,.6])	([.2,.4], [.3,.5], [.3,.6])	([.2,.3], [.2,.4], [.4,.5])	([.4,.6], [.1,.3], [.2,.4])	([.6,.7], [.2,.3], [.3,.4])	([.4,.5], [.5,.8], [.1,.2])
δ_2	([.5,.6], [.1,.3], [.4,.6])	([.5,.7], [.1,.2], [.4,.6])	([.2,.4], [.3,.4], [.2,.5])	([.6,.8], [.1,.2], [.3,.5])	([.4,.6], [.4,.5], [.3,.5])	([.3,.5], [.4,.5], [.1,.3])	([.1,.2], [.5,.8], [.2,.4])	([.5,.7], [.1,.2], [.5,.6])
δ_3	([.2,.4], [.5,.6], [.4,.6])	([.2,.4], [.3,.4], [.2,.5])	([.4 .6], [.2,.3], [.1,.4])	([.2,.5], [.2,.3], [.1,.6])	([.3,.4], [.2,.5], [.5,.7])	([.3,.5], [.4,.5], [.1,.3])	([.2,.4], [.7,.8], [.1,.2])	([.1,.2], [.7,.8], [.2,.3])
δ_4	([.2,.3], [.5,.7], [.1,.3])	([.3,.4], [.2,.5], [.5,.7])	([.2,.4], [.3,.5], [.3,.6])	([.5,.7], [.1,.2], [.4,.6])	([.4,.6], [.1,.3], [.2,.4])	([.1,.2], [.5,.8], [.2,.4])	([.2,.4], [.3,.4], [.2,.5])	([.5,.6], [.1,.3], [.4,.6])

Table 2. Decision Matrix for Alternative $\aleph^{(1)}$

$\aleph^{(1)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	([.2,.4], [.4,.5], [.3,.4])	([.3,.4], [.4,.5], [.2,.5])	([.3,.6], [.2,.3], [.1,.2])	([.2,.4], [.4,.6], [.1,.2])	([.1,.3], [.6,.7], [.2,.3])	([.4,.5], [.2,.5], [.2,.3])	([.6,.7], [.1,.2], [.2,.3])	([.4,.6], [.2,.3], [.4,.5])
δ_2	([.3,.4], [.2,.5], [.5,.7])	([.4,.7], [.1,.2], [.1,.2])	([.4,.5], [.2,.5], [.1,.2])	([.5,.7], [.1,.2], [.2,.4])	([.6,.8], [.1,.2], [.1,.5])	([.2,.4], [.7,.8], [.1,.2])	([.2,.4], [.3,.5], [.3,.6])	([.3,.4], [.4,.5], [.2,.4])
δ_3	([.5,.6], [.2,.3], [.4,.5])	([.5,.7], [.1,.2], [.2,.4])	([.7,.8], [.1,.2], [.2,.4])	([.1,.3], [.1,.5], [.2,.5])	([.1,.4], [.2,.4], [.1,.2])	([.2,.5], [.2,.4], [.3,.5])	([.3,.5], [.2,.4], [.4,.6])	([.5,.7], [.1,.2], [.5,.6])
δ_4	([.3,.5], [.3,.4], [.6,.7])	([.2,.4], [.3,.4], [.2,.5])	([.2,.4], [.7,.8], [.1,.2])	([.4,.7], [.1,.2], [.1,.2])	([.5,.6], [.2,.3], [.4,.5])	([.2,.4], [.3,.5], [.3,.6])	([.4,.6], [.2,.3], [.4,.5])	([.1,.3], [.1,.5], [.2,.5])

Table 3. Decision Matrix for Alternative $\aleph^{(2)}$

$\aleph^{(2)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	([.2,.4], [.4,.6], [.4,.5])	([.2,.3], [.4,.6], [.3,.5])	([.1,.2], [.6,.8], [.2,.5])	([.4,.5], [.2,.5], [.1,.2])	([.2,.3], [.4,.6], [.3,.5])	([.1,.2], [.6,.8], [.2,.5])	([.7,.8], [.1,.2], [.2,.3])	([.1,.3], [.6,.7], [.2,.5])
δ_2	([.4,.5], [.2,.5], [.1,.2])	([.5,.7], [.1,.2], [.2,.4])	([.1,.3], [.6,.7], [.2,.6])	([.1,.4], [.2,.5], [.4,.6])	([.1,.4], [.2,.4], [.1,.2])	([.1,.2], [.2,.5], [.4,.6])	([.1,.4], [.2,.5], [.4,.6])	([.1,.4], [.2,.5], [.4,.6])
δ_3	([.3,.4], [.2,.6], [.4,.6])	([.2,.4], [.3,.4], [.2,.5])	([.4,.5], [.2,.5], [.1,.2])	([.1,.2], [.2,.5], [.4,.6])	([.3,.5], [.3,.5], [.6,.7])	([.3,.5], [.3,.5], [.6,.7])	([.1,.2], [.2,.5], [.4,.6])	([.5,.7], [.1,.2], [.2,.4])
δ_4	([.2,.4], [.4,.5], [.6,.8])	([.3,.5], [.3,.5], [.6,.7])	([.1,.2], [.2,.5], [.4,.6])	([.1,.4], [.2,.4], [.1,.2])	([.4,.5], [.2,.5], [.1,.2])	([.1,.2], [.2,.5], [.4,.6])	([.4,.5], [.2,.5], [.1,.2])	([.1,.2], [.2,.5], [.4,.6])

Table 4. Decision Matrix for Alternative $\aleph^{(3)}$

$\aleph^{(3)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	([.6,.7], [.1,.2], [.3,.5])	([.6,.8], [.1,.2], [.2,.3])	([.6,.7], [.3,.5], [.1,.2])	([.7,.8], [.1,.2], [.2,.5])	([.6,.7], [.1,.2], [.1,.2])	([.5,.8], [.1,.2], [.2,.4])	([.1,.3], [.6,.7], [.2,.5])	([.7,.8], [.1,.2], [.2,.3])
δ_2	([.5,.7], [.3,.4], [.2,.3])	([.5,.7], [.2,.5], [.2,.3])	([.5,.6], [.3,.4], [.1,.2])	([.7,.8], [.3,.5], [.1,.3])	([.1,.2], [.2,.5], [.4,.6])	([.1,.4], [.2,.5], [.4,.6])	([.4,.6], [.2,.3], [.1,.2]])	([.4,.6], [.2,.3], [.1,.2])
δ_3	([.2,.4], [.3,.4], [.2,.5])	([.4,.7], [.2,.3], [.3,.7])	([.4,.6], [.2,.3], [.1,.2])	([.3,.5], [.3,.5], [.6,.7])	([.6,.8], [.1,.2], [.1,.2])	([.7,.8], [.1,.2], [.2,.4])	([.1,.2], [.2,.5], [.4,.6])	([.6,.8], [.1,.2], [.1,.3])
δ_4	([.6,.8], [.3,.4], [.1,.2])	([.5,.7], [.1,.2], [.4,.5])	([.1,.2], [.2,.5], [.4,.6])	([.5,.6], [.3,.4], [.1,.2])	([.2,.4], [.3,.4], [.2,.5])	([.1,.3], [.6,.7], [.2,.5])	([.7,.8], [.1,.2], [.2,.5])	([.4,.6], [.2,.3], [.1,.2]])

Table 5. Decision Matrix for Alternative $\aleph^{(4)}$

$\aleph^{(4)}$	$\breve{a}_1$	$\breve{a}_2$	$\breve{a}_3$	$\breve{a}_4$	$\breve{a}_5$	$\breve{a}_6$	$\breve{a}_7$	$\breve{a}_8$
δ_1	([.3,.5], [.2,.4], [.1,.2])	([.3,.6], [.1,.2], [.4,.7])	([.4,.7], [.3,.4], [.2,.3])	([.7,.8], [.2,.4], [.3,.5])	([.5,.7], [.3,.4], [.2,.4])	([.4,.6], [.2,.5], [.3,.4])	([.2,.3], [.5,.7], [.2,.4])	([.5,.7], [.2,.4], [.3,.5])
δ_2	([.4,.5], [.5,.7], [.2,.4])	([.4,.7], [.3,.5], [.2,.4])	([.5,.8], [.3,.4], [.2,.3])	([.2,.4], [.2,.3], [.4,.5])	([.3,.5], [.2,.3], [.3,.5])	([.2,.4], [.2,.3], [.3,.6])	([.5,.8], [.3,.6], [.2,.3])	([.4,.6], [.2,.3], [.1,.2]])
δ_3	([.2,.4], [.3,.4], [.2,.5])	([.4,.6], [.2,.3], [.3,.5])	([.3,.5], [.3,.5], [.1,.2])	([.3,.5], [.4,.6], [.6,.7])	([.5,.7], [.1,.2], [.4,.5])	([.4,.6], [.3,.5], [.1,.2])	([.6,.7], [.1,.2], [.3,.5])	([.2,.5], [.2,.3], [.4,.6])
δ_4	([.1,.2], [.2,.5], [.4,.6])	([.5,.7], [.2,.4], [.1,.3])	([.3,.5], [.2,.5], [.1,.3])	([.4,.6], [.2,.5], [.3,.4])	([.5,.8], [.3,.4], [.2,.3])	([.4,.6], [.2,.3], [.1,.2]])	([.4,.7], [.3,.5], [.2,.4])	([.2,.4], [.3,.4], [.2,.5])

Step 3. Compute the cosine similarity measure by using definition 3.1.
By using Tables 1-5, compute the cosine similarity measure between $\mathcal{S}^1_{IVNHSS}(S,S_1)$, $\mathcal{S}^1_{IVNHSS}(S,S_2)$, $\mathcal{S}^1_{IVNHSS}(S,S_3)$, and $\mathcal{S}^1_{IVNHSS}(S,S_t)$ by using equation 3.1, such as
$\mathcal{S}^1_{IVNHSS}(S,S_1) = 0.07007$.
Similarly, we can find the cosine similarity measure between $\mathcal{S}^1_{IVNHSS}(S,S_2)$, $\mathcal{S}^1_{IVNHSS}(S,S_3)$, and $\mathcal{S}^1_{IVNHSS}(S,S_4)$ given as
$\mathcal{S}^1_{IVNHSS}(S,S_2) = 0.06771$, $\mathcal{S}^1_{IVNHSS}(S,S_3) = 0.06943$, and $\mathcal{S}^1_{IVNHSS}(S,S_4) = 0.06874$. This shows that $\mathcal{S}^1_{IVNHSS}(S,S_1) > \mathcal{S}^1_{IVNHSS}(S,S_3) > \mathcal{S}^1_{IVNHSS}(S,S_4) > \mathcal{S}^1_{IVNHSS}(S,S_2)$. It can be seen from this ranking alternative S_1 is most relevant and similar to S. Therefore S_1 is the best alternative for the civil engineer, the ranking of other alternatives given as $S_1 > S_3 > S_4 > S_2$.
Now we compute the set-theoretic similarity measure by using Definition 3.3 between $\mathcal{S}^2_{IVNHSS}(S,S_1)$, $\mathcal{S}^2_{IVNHSS}(S,S_2)$, $\mathcal{S}^2_{IVNHSS}(S,S_3)$, and $\mathcal{S}^2_{IVNHSS}(S,S_4)$ given as From Tables 1-5, we can find the set-theoretic similarity measure for each alternative by using definition 3.3 given as $\mathcal{S}^2_{IVNHSS}(S,S_1) = 0.06986$, $\mathcal{S}^2_{IVNHSS}(S,S_2) = 0.06379$, $\mathcal{S}^2_{IVNHSS}(S,S_3) = 0.06157$, and $\mathcal{S}^2_{IVNHSS}(S,S_4) = 0.06176$. $\mathcal{S}^1_{IVNHSS}(S,S_1) > \mathcal{S}^1_{IVNHSS}(S,S_2) > \mathcal{S}^1_{IVNHSS}(S,S_4) > \mathcal{S}^1_{IVNHSS}(S,S_3)$. It can be seen from this ranking alternative S_1 is most relevant and similar to S. Therefore S_1 is the best alternative for the civil engineer, the ranking of other alternatives given as $S_1 > S_2 > S_4 > S_3$.

5. Discussion and Comparative Analysis

In the subsequent section, we will talk over the usefulness, easiness, manageability, and assistance of the planned method. We also performed a short evaluation of the undermentioned: the planned technique and some prevailing methodologies.

5.1. Superiority of the Proposed Approach

Through this study and comparison, it could be determined that the consequences acquired by the suggested approach have been more common than either available method. Overall, the DM procedure associated with the prevailing DM methods accommodates extra information to address hesitation. Also, FS's various hybrid structures are becoming a particular feature of NHSS, along with some appropriate circumstances added. The general info associated with the object could be stated precisely and analytically, see Table 6. Therefore, it is a suitable technique to syndicate inaccurate and ambiguous information in the DM process. Hence, the suggested approach is practical, modest, and ahead of fuzzy sets' distinctive hybrid structures.

Table 6. Comparison between NHSS and some existing techniques

	Set	Truthiness	Indeterminacy	Falsity	Parametrization	Attributes	Sub-attributes
Zadeh [1]	FS	✓	×	×	×	✓	×
Atanassov [2]	IFS	✓	×	✓	×	✓	×
Smarandache [34]	NS	✓	✓	✓	×	✓	×
Maji et al. [21]	FSS	✓	×	×	✓	✓	×
Maji et al. [22]	IFSS	✓	×	✓	✓	✓	×
Peng et al. [26]	PFSS	✓	×	✓	✓	✓	×
Maji [35]	NSS	✓	✓	✓	✓	✓	×
Zulqarnain et al. [46]	IFHSS	✓	×	✓	✓	✓	✓
Zulqarnain et al. [47]	PFHSS	✓	×	✓	✓	✓	✓
Proposed approach	NHSS	✓	✓	✓	✓	✓	✓

It turns out that this is a contemporary issue. Why do we have to embody novel algorithms based on the proposed novel structure? Many indications compared with other existing methods; the recommended method may be an exception. We remember the following fact: the mixed structure limits IFS, picture fuzzy sets, FS, fuzzy hesitation sets, NS, and other fuzzy sets and cannot provide complete information about the situation. But our m-polar model GmPNSS can deal with truthiness, indeterminacy, and falsity, so it is most suitable for MCDM. Due to the exaggerated multipolar neutrosophy, these three degrees are independent and provide a lot of information about alternative norms. Other similarity measures of available hybrid structures are converted into exceptional cases of GmPNSS. A comparative analysis of some already existing techniques is listed above in Table 6. Therefore, this model has more versatility and can efficiently resolve complications than intuitionistic, neutrosophic, hesitant, image, and ambiguity substitution. The similarity measure established for GmPNSS becomes better than the existing similarity measure for MCDM.

5.1 Comparative Analysis

In the following section, we recommend another algorithmic rule under NHSS by utilizing the progressed cosine similarity measure and set-theoretic similarity measure. Subsequently, we use the suggested algorithm to a realistic problem, namely the appropriate civil engineer in a company. The overall outcomes prove that the algorithmic rule is valuable and practical. It can be observed that S_1 is the most acceptable alternative for the civil engineer position. The recommended approach may be compared to other available methods. From the research findings, it has been concluded that the outcomes acquired by the planned approach exceed the consequences of the prevailing ideas. Therefore, compared to existing techniques, the established similarity measures handled the uncertain and ambiguous information competently. However, under current DM strategies, the core advantage of the planned method is that it can accommodate extra info in data comparative to existing techniques. It is also a beneficial tool to solve inaccurate and imprecise information in DM procedures. The benefit of the planned approach and related measures over current methods is evading conclusions grounded on adverse reasons.

5.2. Discussion

Zadeh's [1] FS handled the inaccurate and imprecise information using MD of sub-attributes of considered attributes for each alternative. But the FS has no evidence around the NMD of the considered parameters. Atanassov's [2] IFS accommodates the unclear and undefined objects using MD and NMD. However, IFS cannot handle the circumstances when MD + NMD ≥ 1, conversely, is presented notion competently deals with such difficulties. Meanwhile, these theories have no information about the indeterminacy of the attributes. To overcome such problems, Smarandache [34] proposed the idea of NS. Maji et al. [21] presented the notion of FSS to deal with the parametrization of the objects, which contains uncertainty by considering the MD of the attributes. But, the presented FSS provides no information about the NMD of the thing. To overcome the presented drawback, Maji et al. [22] offered the concept of IFSS. The proposed notion handles the uncertain object more accurately by using the MD and NMD of the attributes with their parametrization, and the sum of MD and NMD does not exceed 1. To handle this scenario, Peng et al. [26] proposed the notion of PFSS by modifying the condition $MD + NMD \leq 1$ to $MD^2 + NMD^2 \leq 1$ with their parametrization. The PFSS is unable to deal with the indeterminacy of the attributes. Maji [35] introduced the concept of NSS, in which decision-makers competently solve the DM problems comparative to the above-studied theories using truthiness, falsity, and indeterminacy of the object. But all the studies mentioned above have no information about the sub-attributes of the considered attributes. So the theories discussed earlier cannot handle the scenario when characteristics have their corresponding sub-attributes. Utilizing the MD and NMD, Zulqarnain et al. [46] extended the IFSS to IFHSS and proposed the CC and WCC for IFHSS in which $MD + NMD \leq 1$ for each sub-attribute. But IFHSS cannot provide any information on the NMem values of the sub-attribute of the considered attribute. Zulqarnain et al. [47] proposed the more generalized notion of

PFHSS comparative to IFHSS. The PFHSS accommodates more uncertainty compared to IFHSS by updating the condition $MD + NMD \leq 1$ to $\left(\sigma_{\mathcal{F}(\breve{a})}(\delta)\right)^2 + \left(\tau_{\mathcal{F}(\breve{a})}(\delta)\right)^2 \leq 1$. All existing hybrid structures of FS cannot handle the indeterminacy of sub-attributes of considered n-tuple attributes. On the other hand, developed aggregation operators can accommodate the sub-attributes of considered attributes using truthness, indeterminacy, and falsity objects of sub-attributes with the following condition $0 \leq \sigma_{\mathcal{F}(\breve{a})}(\delta),\ \tau_{\mathcal{F}(\breve{a})}(\delta),\ \gamma_{\mathcal{F}(\breve{a})}(\delta) \leq 3$. It may be seen that the best selection of the suggested approach is to resemble the verbalized own method, and that ensures the liableness along with the effectiveness of the recommended approach.

6. Conclusion

The interval-valued neutrosophic hypersoft set is a novel concept that is an extension of the interval-valued neutrosophic soft set. This paper studies some basic concepts such as soft set, NSS, HSS, IFHSS, PFHSS, and NHSS. We developed the idea of cosine similarity measure and set-theoretic similarity measure for IVNHSS and described their desirable properties. Furthermore, a decision-making approach has been developed for IVNHSS based on the proposed technique. To verify the effectiveness of our developed techniques, we presented an illustration to solve MCDM problems. We introduced a comprehensive comparative analysis of proposed techniques with existing methods. In the future, the concept of IVNHSS will be extended to m polar interval-valued NHSS. It will solve real-life problems such as medical diagnoses, decision-making, etc. Future research will concentrate on presenting numerous other operators under the mPIVNHSS environment to solve decision-making issues. Many other structures such as topological, algebraic, ordered structures, etc., can be developed and discussed under-considered environment.

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Separation axioms in neutrosophic supra topological space and neutrosophic supra bi-topological space

Riad K. Al-Hamido

Department of Mathematics, College of Science, AlFurat University, Deir-ez-Zor, Syria.
riad-hamido1983@hotmail.com ; Tel.: (+963988619783)

Abstract

In this paper, as a new separation axioms in neutrosophic supra topological space, N_S-T_i- space (i=0,1,2) is built in this space. Moreover, SN.T_1T_2-open(closed) sets are defined in neutrosophic supra bi-topological spaces. Also SN_{Bi}-T_i-space (i=0,1,2) is built on this new neutrosophic sets. And their basic properties are presented. The relations between these new neutrosophic separation axioms is studied. Finally, many examples are presented.

Keywords: Neutrosophic supra topological spaces, Neutrosophic supra bi-topological spaces, SN.T_1T_2-open set, SN.T_1T_2-closed set, neutrosophic separation axioms, N_S-T_i- space, N_{SBi}-T_i- space (i=0,1,2).

1. Introduction

The idea of neutrosophic was invented and presented by F. Smarandache [1,2]. This science has many applications in all science, including topology, where neutrosophic topological space was defined by A. Salama and et al. in [3]. Also, the neutrosophic bi-topological space was defined by R. K. Al-Hamido [4] as an extension of neutrosophic topological spaces in 2019. The concept of neutrosophic supra bi-topological space has been studied in [5]. Also, the neutrosophic Tri-topological space was defined by R.K.Al-Hamido [6] as an extension of neutrosophic bi-topological spaces in 2018. Also, in 2018, R. K. Al-Hamido, extended neutrosophic bi-topological spaces to Neutrosophic Crisp Bi-Topological Spaces[7].

Later [8] studied the separations axioms but in neutrosophic crisp topological space via neutrosophic crisp points which is defined in this paper. Moreover, these new definitions of neutrosophic crisp points open the door to defined new types of separations axioms in neutrosophic crisp topological space. such as neutrosophic crisp semi separation axioms in[9] and separation axioms via neutrosophic crisp pre-open sets [10] in neutrosophic crisp topological spaces.

Based on neutrosophic crisp bi-topological space [7], separations axioms in neutrosophic crisp bi-topological space were grounded by R. K. Al-Hamido et al. in [11].

Khattak et al. [12] worked on soft b-separation axioms in NSTS. Suresh and Palaniammal [13] presented NS(WG) separation axioms in NTS.

Gunnuz Aras et al. [14] studied the separation axioms but in neutrosophic soft topological spaces(NSTS).

Mehmood et al. [15] worked on generalized neutrosophic separation axioms in NSTS.

Recently, Neutrosophic crisp set theory has been employed to model uncertainty in several areas of application such as image processing [16],[17], and in geographic information systems[18] and possible applications to database[19]. Also, neutrosophic sets [20] may have applications in the medical field [21-22].

Recently, in 2021, A.Acikgoza et al. studied separations axioms in neutrosophic topological space[23] for the first time.

Finally, R.Narmada et al. studied separation axioms in an ordered neutrosophic bitopological space in [24]. For more detail about neutrosophic topology see [25-32].

In this paper, we will defined new patterns from neutrosophic sets in neutrosophic supra bi-topological spaces, moreover we will defined separations axioms in neutrosophic supra topological space and in neutrosophic supra bi-topological space depending on SN.T_1T_2-open(closed) sets are defined in neutrosophic supra bi-topological spaces.

We will study the relationships among these new types of separations axioms, and we will also examine the relationship between separations axioms in neutrosophic supra topological space and neutrosophic supra bi-topological space.

2. Preliminaries

This section will discuss some basic definitions and properties of neutrosophic supra topology, which are helpful in sequel.

Definition 2.1.[8]

let X be a non-empty set, D be a neutrosophic set in X, then:

D is said to be neutrosophic quasi-coincident (neutrosophic q-coincident, for short) with *L*, denoted by *DqL* if and only if $D \not\subset L^c$. If *D* is not neutrosophic quasi-coincident with *L*, we denote by $D \tilde{q} L$.

Definition 2.2: [25]

A neutrosophic supra topology (NST) on a non-empty set X is a family Γ of neutrosophic subsets in X satisfying the following axioms.

1. 1_N and 0_N belong to Γ.

2. Γ is closed under arbitrary union.

The pair (X, Γ) is called neutrosophic supra topological space (NSTS) in X. Moreover, members of Γ are known as neutrosophic supra open sets (NSOS).

The set of all neutrosophic supra open (closed) set is denoted NSOS(X) (NSCS(X)).

Definition 2.3. [5]

Let T_1,T_2 be two neutrosophic supra topology on a nonempty set X then (X,T_1,T_2) be a neutrosophic supra Bi-topological space (SBi-NTS for short).

3. Separation axioms in neutrosophic supra topological space

In this part, we have defined a new separation axioms in neutrosophic supra topological space, namely N_S-T_i-space (i=0,1,2), for first time.

Definition 3.1.

A neutrosophic set F in NSTS (X,T) is called N_S-T_0-space if for any pair of neutrosophic points (NP) $x \neq y \in X$, there exists an $U \in$ NOS(X) such that ($x \in U$ and $y \notin U$) or there exists $V \in$ NOS(X); ($y \in V$ and $x \notin V$).

Example 3.2.

Let X={n,m}, T= { { $n_{s,s,1-s}$, $m_{e,e,1-e}$ } : $s \in [0,1]$, $e \in [0,1]$ }

Then (X,T) is NSTS, (X,T) is N_S-T_0-space.

Definition 3.3.

A neutrosophic set F in NSTS (X,T) is called N_S-T_1-space if for any pair of neutrosophic points (NP) $x \neq y \in X$, there exists U,V$\in$ NOS(X); ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$).

Example 3.4.

Let X={f,g},T= { { $f_{s,s,1-s}$, $g_{e,e,1-e}$ }: $s \in [0,1]$, $e \in [0,1)$ }.

Then (X,T) is NSTS, (X,T) is N_S-T_0-space. But, (X,T) is not N_S-T_1-space, because, $f_{1,1,0}$ and $g_{1,1,0}$ are neutrosophic points in (X, τ) ; $f_{1,1,0} \neq g_{1,1,0}$ and the only neutrosophic supra open set that contains $g_{1,1,0}$ is 1_N.

Definition 3.5.

A neutrosophic set F in NSTS (X,T) is called N_S-T_2-space if for any pair of neutrosophic points (NP) $x \neq y \in X$, there exists U,V$\in$ NOS(X); ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$) ; $U \tilde{q} V$.

Theorem 3.6.

Let (X,T) be a NSTS, then:

If (X,T) is N_S-T_2-space then (X,T) is N_S-T_1-space.

Proof:

Let (X,T) is N_S-T_2-space, Then for any pair of neutrosophic points (NP) $x \neq y \in X$ there exists U,V$\in$ NOS(X); ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$) ; $U \tilde{q} V$. so there exists U,V$\in$NOS(X); ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$).

Therefore (X,T) is N_S-T_1-space.

Remark 3.7.

The converse of the theorem 3.6 is not true; see the following example:

Example 3.8.

In example 3.4, (X,T) is NSTS, X is N_S-T_0-space but not N_S-T_1-space, because, $f_{1,1,0}$ and $g_{1,1,0}$ are neutrosophic points in (X, T) ; $f_{1,1,0} \neq g_{1,1,0}$ and the only neutrosophic supra open set that contains $g_{1,1,0}$ is 1_N. Therefore, X is N_S-T_0-space but not N_S-T_2-space.

Theorem 3.9.

Let (X,T) be a NSTS, then:

If (X,T) is N_S-T_1-space, then (X,T) is N_S-T_0-space.

Proof:

Let (X,T) is N_S-T_1-space, Then for any pair of neutrosophic points (NP) $x \neq y \in X$, there exists U,V$\in$ NOS(X); ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$), so there exists U,V$\in$NOS(X); ($x \in U$ and $y \notin U$) or ($y \in V$ and $x \notin V$).

Therefore (X,T) is N_S-T_0-space.

Remark 3.10.

The converse of the theorem 3.9 is not true, see the following example:

Example 3.11.
In example 3.5, (X,T) is NSTS, X is N_S-T_0-space but not N_S-T_1-space, because, $f_{1,1,0}$ and $g_{1,1,0}$ are neutrosophic points in (X, τ) ; $f_{1,1,0} \neq g_{1,1,0}$ and the only neutrosophic supra open set that contains $g_{1,1,0}$ is 1_N.

Remark 3.12.
Let (X,T) be a NSTS, then:
(X,T) is N_S-T_2-space $\Rightarrow$ (X,T) is N_S-T_1-space $\Rightarrow$ (X,T) is N_S-T_0-space.

Proof:
Proof following from theorem 3.9 and theorem 3.6.

4. Separation axioms in neutrosophic supra bi-topological space

In this part, we have defined for the first time a new separation axioms in neutrosophic supra topological space, which named N_{SBi}-T_i-space (i=0,1,2).

Definition 4.1.
A neutrosophic set A in SBi-NTS (X,T_1,T_2) is called " $N_S.T_1T_2$-open set " if it is a neutrosophic open set in (X,T_1) or in (X,T_2).

- A neutrosophic set B in SBi-NTS (X,T_1,T_2) is called " $N_S.T_1T_2$-closed set " iff its complement is " $N_S.T_1T_2$-open set ".
- The set of all " $N_S.T_1T_2$-open (closed) sets " is denoted to be" $N_S.T_1T_2$-NOS ($N_S.T_1T_2$-NCS)".

Definition 4.2.
A SBi-NTS (X,T_1,T_2) is called N_{SBi}-T_0-space if :$\forall x \neq y \in X$, $\exists$ U$\in$ $N_S.T_1T_2$-NOS; ($x \in U$ and $y \notin U$) or $\exists$ V$\in$ $N_S.T_1T_2$-NOS; ($y \in V$ and $x \notin V$).

Example 4.3.
Let X={n,m}, A_1={< n, 0.4, 0.4, 0.4>, < m, 0.5, 0.5, 0.5>}, A_2={< n, 0.3, 0.3, 0.3>, < m, 0.6, 0.6, 0.6>},
T_1={ 0_N, A_1, A_2, $A_1 \vee A_2$, 1_N}, T_2={ { $n_{s,s,1-s}$, $m_{e,e,1-e}$ } : s$\in$[0,1] , e$\in$[0,1] }
Then (X,T_1,T_2) is SBi-NTS, (X,T_1,T_2) is N_{SBi}-T_0-space.

Definition 4.4.
A SBi-NTS (X,T_1,T_2) is called N_{SBi}-T_1-space if :$\forall x \neq y \in X$, $\exists$ U,V$\in$ $N_S.T_1T_2$-NOS; ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$).

Example 4.5.
Let X={f,g}, A_1={< f, 0.4, 0.4, 0.4>, < g, 0.5, 0.5, 0.5>}, A_2={< f, 0.3, 0.3, 0.3>, < g, 0.6, 0.6, 0.6>},
T_1={ 0_N, A_1, A_2, $A_1 \vee A_2$, 1_N}, T_2={ {$f_{s,s,1-s}$, $g_{e,e,1-e}$ } : s$\in$[0,1] , e$\in$[0,1) }
Then (X,T_1,T_2) is SBi-NTS, (X,T_1,T_2) is N_{SBi}-T_0-space. But (X,T_1,T_2) is not N_{SBi}-T_1-space.

Definition 4.6.
A SBi-NTS (X,T_1,T_2) is called N_{SBi}-T_2-space if :$\forall x \neq y \in X$, $\exists$ U,V$\in$ $N_S.T_1T_2$-NOS; ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$) ; $U \tilde{q} V$.

Theorem 4.7.
Let (X,T_1,T_2) be a SBi-NTS, then:
(X,T_1,T_2) is N_{SBi}-T_0-space $\Leftrightarrow$ (X,T_1) is N_S-T_0-space or (X,T_2) is N_S-T_0-space.

Proof :
$\Rightarrow$:
Let (X,T_1,T_2) is N_{SBi}-T_0-space then, $\forall x \neq y \in X$, $\exists$ U,V$\in$ $N_S.T_1T_2$-NOS; ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$) so there existe U,V$\in$ T_1-NOS; ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$) or there existe U,V$\in$ T_2-NOS; ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$) therefore (X,T_1) is N_S-T_0-space or (X,T_2) is N_S-T_0-space.
$\Leftarrow$:

Let (X,T_1) be a N_S-T_0-space or (X,T_2) be a N_S-T_0-space. Then, for every $x \neq y \in X$, there existe $U,V \in T_1$-NOS; ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$)or $U,V \in T_2$-NOS; ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$), so there exist $U,V \in N_S.T_1T_2$-NOS; ($x \in U$ and $y \notin U$) or ($y \in V$ and $x \notin V$). Therefore (X,T_1,T_2) is N_{SBi}-T_0-space.

Theorem 4.8.

Let (X,T_1,T_2) be a SBi-NTS, then:

(X,T_1,T_2) is N_{SBi}-T_i-space $\Leftrightarrow$ (X,T_1) is N_S-T_i-space or (X,T_2) is N_S-T_i-space (i=1,2).

Proof:

In the same way of proof theorem 4.7.

Theorem 4.9.

Let (X,T_1,T_2) be a SBi-NTS, then:

(X,T_1,T_2) is N_{SBi}-T_2-space $\Rightarrow$ (X,T_1,T_2) is N_{SBi}-T_1-space

Proof:

Let (X,T_1,T_2) is N_{SBi}-T_2-space, Then $\forall x \neq y \in X$, $\exists$ $U,V \in N_S.T_1T_2$-NOS; ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$) ; U$\tilde{q}$V. so there existe $U,V \in N_S.T_1T_2$-NOS; ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$).

Therefore (X,T_1,T_2) is N_{SBi}-T_1-space.

Remark 4.10.

The converse of the theorem 4.9 is not true, see the following example:

Example 4.11.

In example 4.5, (X,T_1,T_2) is SBi-NTS, X is N_{SBi}-T_0-space but not N_{SBi}-T_1-space.

Therefore X is N_{SBi}-T_0-space but not N_{SBi}-T_2-space.

Theorem 4.12.

Let (X,T_1,T_2) be a SBi-NTS, then:

(X,T_1,T_2) is N_{SBi}-T_1-space $\Rightarrow$ (X,T_1,T_2) is N_{SBi}-T_0-space.

Proof:

Let (X,T_1,T_2) is N_{SBi}-T_1-space, Then $\forall x \neq y \in X$, $\exists$ $U,V \in N_S.T_1T_2$-NOS; ($x \in U$ and $y \notin U$) and ($y \in V$ and $x \notin V$), so there existe $U,V \in N_S.T_1T_2$-NOS; ($x \in U$ and $y \notin U$) or ($y \in V$ and $x \notin V$).

Therefore (X,T_1,T_2) is N_{SBi}-T_0-space.

Remark 4.13.

The converse of the theorem 4.12 is not true, see the following example:

Example 4.14.

In example 4.5, (X,T_1,T_2) is SBi-NTS, X is N_{SBi}-T_0-space but not N_{SBi}-T_1-space.

Remark 4.15.

Let (X,T_1,T_2) be a SBi-NTS, then:

(X,T_1,T_2) is N_{SBi}-T_2-space $\Rightarrow$ (X,T_1,T_2) is N_{SBi}-T_1-space $\Rightarrow$ (X,T_1,T_2) is N_{SBi}-T_0-space.

Proof:

Proof following from theorem 4.9 and theorem 4.12.

Theorem 4.16.

Let (X,T_1,T_2) be a SBi-NTS, then:

If (X,T_1) is N_S-T_i-space and (X,T_2) is N_S-T_i-space, then(X,T_1,T_2) is N_{SBi}-T_i-space(i=0,1,2).

Proof :

From the theorem 4.7 and theorem 4.8.

Remark 4.17:

If (X,T_1,T_2) is N_{SBi}-T_i-space(i=0,1,2) then may be(X,T_1) or (X,T_2) is not N_S-T_i-space, so the converse of the Theorem 4.16 is not true.

5. Conclusion

In this paper, we have defined for first time a new separation axioms in neutrosophic supra topological space and neutrosophic supra bi-topological space which namely NS-Ti-space and NSBi-Ti-space (i=0,1,2).

In the future, using these notions, various classes of separation axioms in neutrosophic supra topological space and neutrosophic supra bi-topological space as NS-Ti-space and NSBi-Ti-space (i=3,4,5) , and many researchers can be studied.

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Ranking of Interval Valued Neutrosophic Numbers by Qualitative and Quantitative Criteria

D. Jeni Seles Martina[1] and G. Deepa[2,*]

[1] Research Scholar, Department of Mathematics, VIT Vellore, Tamilnadu, India; djeniseles11@gmail.com

[2,*] Assistant Professor, Department of Mathematics, VIT Vellore, Tamilnadu, India; deepa.g@vit.ac.in

*Correspondence: deepa.g@vit.ac.in

Abstract. In Operations Research, making decisions based on multiple criteria is crucial. Neutrosophic numbers produce a more efficient conclusion when dealing with Fuzzy Multi Criteria Decision Making (MCDM) problems. In this paper, we determine qualitative and quantitative criteria for selecting the best building construction project. The major goal of this work is to show how to use interval valued neutrosophic sets in solving MCDM issues using the Max-Product formula. To calculate the weighted average for interval valued neutrosophic numbers, we offer a new technique in max product. Three approaches are used to rank the interval valued neutrosophic numbers, and their application is demonstrated numerically.

Keywords: Fuzzy sets; Multiple Criteria Decision Making; Neutrosophic Fuzzy sets; Interval Valued Neutrosophic Set; Interval Valued Neutrosophic Numbers.

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1. **Introduction**

L.A. Zadeh [1] introduced fuzzy sets, fuzzy membership functions, and fuzzy logic in 1965. K. Atanassov [2] introduced the Intuitionistic Fuzzy set in 1986. It's a fuzzy set generalization with a membership grade, non-membership grade, and degree of indeterminacy. MCDM is a very significant and rapidly increasing subject in operations research. Indeterminacy should be incorporated into the model formulation of difficulties because MCDM problems are well addressed in fuzzy and intuitionistic fuzzy. In the decision-making process, indeterminacy is very significant. As a result of the growth of the MCDM field in a fuzzy environment, the Neutrosophic Fuzzy MCDM was proposed, and it was used in SAW, AHP, GP, TOPSIS, and other applications. Neutrosophic set, the generalization of fuzzy set and intuitionstic fuzzy sets. In Multiple Criteria Decision Making, neutrosophic numbers are ranked to rate tough problems. Using Neutrosophic Sets in MCDM and ranking methodologies will provide the best possible solution to challenging situations. Smarandache [3] introduced Neutrosophic set in 1998. The membership functions of Neutrosophic sets are Truth, Indeterminacy, and False. Smarandache and Wang proposed interval valued neutrosophic sets in 2005, and they introduced single valued neutrosophic sets in 2010. It independently expresses truth, indeterminacy, and false membership degree.

The paper Interval Neutrosophic Sets was published by Haibin Wang, et al. in 2004 [4]. They introduce and verify the convexity of interval valued neutrosophic sets, as well as many features, operations, and relations of interval neutrosophic sets. Athar Kharal published a paper A Neutrosophic Multi-Criteria Decision Making Method [14] in 2014. This study presents a method of MCDM based on Neutrosophic sets. It is the first time that neutrosophic sets have been introduced to the MCDM community. In 2014, Based on Bhattacharya's distance, Broumi S and Smarandache F [7] define a novel cosine similarity between two Interval valued neutrosophic sets. They used the cosine similarity measure in pattern recognitionin this research. Jun Ye published a paper Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making [10] in 2014. The Hamming and Euclidean distances between interval neutrosophic sets (INSs) are described in this study, and similarity measures between INSs are provided based on the relationship between distances and similarity measures. The article, Interval neutrosophic sets and their application in MCDM problems was published by Zhang et al in 2014 [8]. They established Interval neutrosophic numbers operators and presented a comparativeapproach between INN and aggregation operators for INSs in this work. Saha and Broumi [15] presented New Operators on Interval Valued Neutrosophic Sets in 2019. They defined some new IVNS operators and examined their properties in this study. Theoperators are highly useful when dealing with two interval-valued neutrosophic sets. In the decision-making process, the similarity measure is essential in determining the degree of similarity between the ideal and each alternative. In 2019, Wang, et al. [13] proposed a multi-criteria decision-making system based on improved cosine similarity measures with interval neutrosophic sets. The purpose of this study is to develop an MCDM technique for INSs based on a similarity measure.

In 2017, Deli and Subas [17] published the paper The concept of a single valued neutrosophic number (SVNN) is important for quantifying an unknown quantity, and the ranking of SVNNis atough problem in multi-attribute decision making problems. The goal of this work is to offer a methodology for using SVNNs to solve multi-attribute decision-making problems. They created a ranking approach based on the concept of values and uncertainties, which they used to multi attribute decision making issues where the ratings of alternatives on criteria are expressed as SVTN-numbers. Ranking methods of Single Valued Neutrosophic number and Its Applications to Multiple Criteria Decision Making [12] was published by D. Stanujkic, et al. in 2019. They demonstrate the utility of single-valued Neutrosophic sets in solving MCDM problems in this work. The proposed Ranking method's approach and numerical example were presented. Although single valued neutrosophic sets apply ranking methods, interval valued neutrosophic sets and numbers are also highly effective in ranking the alternatives.Ranking of Pentagonal Neutrosophic Numbers and its Applications to Solve Assignment Problem was published in 2020 by Radhika and Arun [18]. They suggest a new method for ranking neutrosophic numbers based on their magnitude in this work. Theyoffer a method for solving neutrosophic assignment issues with pentagonal neutrosophic numbers. The article, Ranking of single-valued neutrosophic numbers through the index of optimism and its reasonable properties was published by R. Chutia and F. Smarandache in 2021 [19]. The significance and vagueness of a single-valued neutrosophic number are used to construct a novel way of ranking neutrosophic numbers in this study. The method is unique inthe reasonable features of a ranking system.

There are many ranking methods that are applied in MCDM problems using the various types of neutrosophic numbers. The motive of our paper is to use Interval Valued Nutrosophic Numbers to build ranking techniques in MCDM. It gives better results when similarity measures, score function, and hamming distance are used to rank the interval valued neutrosophic numbers. The paper contains preliminaries and Basic elements of Interval Valued Neutrosophic sets, and ranking of IVNNs in section 2. The MCDM method based on Interval Valued Neutrosophic Numbers is provided in section 3. This proposed ranking approach is given numerical illustration in section 4. Finally, there is a ranking and a conclusion.

2. **Preliminaries**

Definition 2.1. Neutrosophic Set (NS) [3]

Let U be the universal set and every element x ∈ U has degree of True, Indeterminacy, False membership in S. Then the Neutrosophic set can be written as

$$S = \{\langle x, T_S(x), I_S(x), F_S(x)\rangle : x \in U\}$$

where, $0 \le T_S(x) + I_S(x) + F_S(x) \le 3$
and Truth Membership function $T_S : U \to [0, 1]$
Indeterminacy Membership function $I_S : U \to [0, 1]$
False Membership function $F_S : U \to [0, 1]$

Definition 2.2. Interval Valued Neutrosophic Set (IVNS) [8]

Let U be a nonempty set with generic elements in U denoted by x. The Interval Valued Neutrosophic set S in U is as follows

$$S = \{x : \langle x, T_S(x), I_S(x), F_S(x)\rangle \, ; x \in U\}$$

where, Interval Truth Membership Function $T_S(x) = [T_S^L, T_S^U]$
Interval Indeterminacy Membership Function $I_S(x) = [I_S^L, I_S^U]$
Interval False Membership Function $F_S(x) = [F_S^L, F_S^U]$
and for each point x ∈ U. $T_S(x), I_S(x), F_S(x) \in [0,1]$

Definition 2.3. Interval Valued Neutrosophic Number (IVNN)

For an IVNS S in U the triple interval $\langle [t_s^L, t_s^U], [i_s^L, i_s^U], [f_s^L, f_s^U] \rangle$ is called the Interval Valued Neutrosophic Number.

Operations on IVNN

Let $s_1 = \langle [t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U] \rangle$ and $s_2 = \langle [t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U] \rangle$ be two IVNN and $\lambda > 0$, then the basic operations are defined as follows [5],

$$i) s_1 + s_2 = \langle [t_1^L + t_2^L - t_1^L t_2^L, t_1^U + t_2^U - t_1^U t_2^U], [i_1^L i_2^L, i_1^U i_2^U], [f_1^L f_2^L, f_1^U f_2^U] \rangle$$

$$ii) s_1.s_2 = \langle [t_1^L t_2^L, t_1^U t_2^U], [i_1^L + i_2^L - i_1^L i_2^L, i_1^U + i_2^U - i_1^U i_2^U], [f_1^L + f_2^L - f_1^L f_2^L, f_1^U + f_2^U - f_1^U f_2^U] \rangle$$

$$iii) \lambda s_1 = \left\langle [1 - (1 - t_1^L)^\lambda, 1 - (1 - t_1^U)^\lambda], [(i_1^L)^\lambda, (i_1^U)^\lambda], [(f_1^L)^\lambda, (f_1^U)^\lambda] \right\rangle$$

$$iv) s_1^\lambda = \left\langle [(t_1^L)^\lambda, (t_1^U)^\lambda], [1 - (1 - i_1^L)^\lambda, 1 - (1 - i_1^U)^\lambda], [1 - (1 - f_1^L)^\lambda, 1 - (1 - f_1^U)^\lambda] \right\rangle$$

Definition 2.4. Score Function of IVNN

Let $s_1 = \langle [t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U] \rangle$ be an IVNN then a Score Funtion $S_{(s_1)}$ is [6]

$$S_{(s_1)} = \frac{1}{4}[2 + t_1^L + t_1^U - 2(i_1^L + i_1^U) - (f_1^L + f_1^U)]$$

Definition 2.5. Cosine Similarity Measure

Let $s_1 = \langle [t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U] \rangle$ and $s_2 = \langle [t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U] \rangle$ be two IVNN then a Cosine Similarity Measure $C_{(s)}$ between two IVNN s_1 and s_2 is as follows [7] ,

$$C(s_1, s_2) = \frac{\frac{1}{n}\sum_{i=1}^{n} [(t_1^L + t_1^U)(t_2^L + t_2^U) + (i_1^L + i_1^U)(i_2^L + i_2^U) + (f_1^L + f_1^U)(f_2^L + f_2^U)]}{\sqrt{(t_1^L + t_1^U)^2 + (i_1^L + i_1^U)^2 + (f_1^L + f_1^U)^2}\sqrt{(t_2^L + t_2^U)^2 + (i_2^L + i_2^U)^2 + (f_2^L + f_2^U)^2}}$$

where, n = 1

Definition 2.6. Hamming Distance

Let $s_1 = \langle [t_1^L, t_1^U], [i_1^L, i_1^U], [f_1^L, f_1^U] \rangle$ and $s_2 = \langle [t_2^L, t_2^U], [i_2^L, i_2^U], [f_2^L, f_2^U] \rangle$ be two IVNN then the Hamming Distance $H_{(s)}$ between two IVNN s_1 and s_2 is as follows [10],

$$D(s_1, s_2) = \frac{1}{6}\sum_{i=1}^{n} [|t_1^L - t_2^L| + |t_1^U - t_2^U| + |i_1^L - i_2^L| + |i_1^U - i_2^U| + |f_1^L - f_2^L| + |f_1^U - f_2^U|]$$

2.1. **Ranking of Interval Valued Neutrosophic numbers**

Let s_1 and s_2 be two IVNNs, then the ranking method for comparing two IVNS is defined as follows, [12]

(i) Score Function

If $S_{(s_1)} > S_{(s_2)}$ then $s_1 > s_2$

(ii) Cosine Similarity Measure

$C_{(s_1)} > C_{(s_2)}$ then $s_1 > s_2$

(iii) Hamming Distance

$H_{(s_1)} > H_{(s_2)}$ then $s_1 < s_2$

3. **A MCDM approach based on Interval Valued Neutrosophic Numbers**

In this section, we proposed a new max-product approach for determining the weighted average for interval-valued neutrosophic numbers. This formula can be applied to any order of matrices containing interval-valued neutrosophic numbers, as well as two or more matrices of the same order. The remaining part contains the suggested method's procedure and flowchart.

Result 3.1. Let $A_{x\times y}$ and $B_{x\times y}$ be two matrix with an interval valued neutrosophic numbers. Then the Max-product for A and B is defined as follows,

$$\left\langle max\left(\prod_{m=1}^{n} m^L_{txy_1}, \prod_{m=1}^{n} m^L_{txy_2}, \dots\right), max\left(\prod_{m=1}^{n} m^U_{txy_1}, \prod_{m=1}^{n} m^U_{txy_2}, \dots\right)\right\rangle$$
$$\left\langle max\left(\prod_{m=1}^{n} m^L_{ixy_1}, \prod_{m=1}^{n} m^L_{ixy_2}, \dots\right), max\left(\prod_{m=1}^{n} m^U_{ixy_1}, \prod_{m=1}^{n} m^U_{ixy_2}, \dots\right)\right\rangle$$
$$\left\langle max\left(\prod_{m=1}^{n} m^L_{fxy_1}, \prod_{m=1}^{n} m^L_{fxy_2}, \dots\right), max\left(\prod_{m=1}^{n} m^U_{fxy_1}, \prod_{m=1}^{n} m^U_{fxy_2}, \dots\right)\right\rangle$$

Where m denotes the number of matrices.

We know, the fuzzy max-product composition,

Let A and B be x × y and y × z matrices respectively. The Fuzzy Max product composition of A and B is defined by,

$$\mu_{A\circ B} = max[\mu_A(x,y).\mu_B(y,z)]$$

From this we can extend the concept of Interval valued fuzzy number and Interval valued neutrosophic number.

Let A and B be $x \times y$ matrices with an interval valued fuzzy numbers. For A and B matrices, we should find the maximum product. The lower and upper limits are independent in this case. As a result, we calculate the max-product separately for the lower and upper limit values.

Assume that $A_{1\times 2}$ and $B_{1\times 2}$ be two matix with interval valued numbers.
we take A = $\Big((a_1^L, a_1^U) \quad (a_2^L, a_2^U)\Big)$ and B = $\Big((b_1^L, b_1^U) \quad (b_2^L, b_2^U)\Big)$

Max-product of A and B = $\Big(max[(a_1^L.b_1^L),(a_2^L.b_2^L)] \quad max[(a_1^U.b_1^U),(a_2^U.b_2^U)]\Big)$

Let A and B be two matrices with interval valued neutrosophic numbers.
we take A = $\Big[\Big\langle(a_{t1}^L, a_{t1}^U),(a_{i1}^L, a_{i1}^U),(a_{f1}^L, a_{f1}^U)\Big\rangle \quad \Big\langle(a_{t2}^L, a_{t2}^U),(a_{i2}^L, a_{i2}^U),(a_{f2}^L, a_{f2}^U)\Big\rangle\Big]$

B = $\Big[\Big\langle(b_{t1}^L, b_{t1}^U),(b_{i1}^L, b_{i1}^U),(b_{f1}^L, b_{f1}^U)\Big\rangle \quad \Big\langle(b_{t2}^L, b_{t2}^U),(b_{i2}^L, b_{i2}^U),(b_{f2}^L, b_{f2}^U)\Big\rangle\Big]$

Max-product of A and B =

$$[\langle max[(a_{t1}^L.b_{t1}^L),(a_{t2}^L.b_{t2}^L)], max[(a_{t1}^U.b_{t1}^U),(a_{t2}^U.b_{t2}^U)]\rangle$$
$$\langle max[(a_{i1}^L.b_{i1}^L),(a_{i2}^L.b_{i2}^L)], max[(a_{i1}^U.b_{i1}^U),(a_{i2}^U.b_{i2}^U)]\rangle$$
$$\langle max[(a_{f1}^L.b_{f1}^L),(a_{f2}^L.b_{f2}^L)], max[(a_{f1}^U.b_{f1}^U),(a_{f2}^U.b_{f2}^U)]\rangle]$$

This equation represents the max product value of 1×2 matrices, and we calculate the values for $x \times y$ matrices in the same way.

Max-product of m matrices =

$$\left\langle max\left(\prod_{m=1}^{n} m_{txy_1}^L, \prod_{m=1}^{n} m_{txy_2}^L, \dots\right), max\left(\prod_{m=1}^{n} m_{txy_1}^U, \prod_{m=1}^{n} m_{txy_2}^U, \dots\right)\right\rangle$$
$$\left\langle max\left(\prod_{m=1}^{n} m_{ixy_1}^L, \prod_{m=1}^{n} m_{ixy_2}^L, \dots\right), max\left(\prod_{m=1}^{n} m_{ixy_1}^U, \prod_{m=1}^{n} m_{ixy_2}^U, \dots\right)\right\rangle$$
$$\left\langle max\left(\prod_{m=1}^{n} m_{fxy_1}^L, \prod_{m=1}^{n} m_{fxy_2}^L, \dots\right), max\left(\prod_{m=1}^{n} m_{fxy_1}^U, \prod_{m=1}^{n} m_{fxy_2}^U, \dots\right)\right\rangle \quad (1)$$

Here m denotes the number of matrices. The max product of more than two matrices with x rows and y columns is represented by Equation (1). For Interval valued neutrosophic numbers, this equation is used as the weighted average max product formula.

3.1. **Procedure and Flowchart for the proposed method**

The ranking of interval-valued neutrosophic numbers is used to solve some difficult problems. Here we use score function, cosine similarity function, and hamming distance for ranking the values and we use two types of criteria which as qualitative and quantitative that are used for the more accurate outcome. The method's procedure is as follows, when we take k alternatives over m criteria by n experts.

Step 1: Define an available alternatives based on selected problem.

Step 2: Define a set of qualitative and quantitative criteria for evaluating the alternatives.

Step 3: The performance of the alternatives are evaluted by the group of experts. These performance are taken into interval valued neutrosophic numbers.

Step 4: Calculated overall ratings for qualitative and quantitative criteria separately by using weighted average max product formula given in Equation 1.

Step 5: Calculate the score function, cosine similarity function and hamming distance between qualitative and quantitative values.

Step 6: Rank the alternatives using the ranking of IVNNs and select the best one among those alternatives.

We take 4 alternatives over 3 qualitative criteria and 3 quantitative criteria by 3 experts. By these expressions, the following flowchart is the steps for solving the problem.

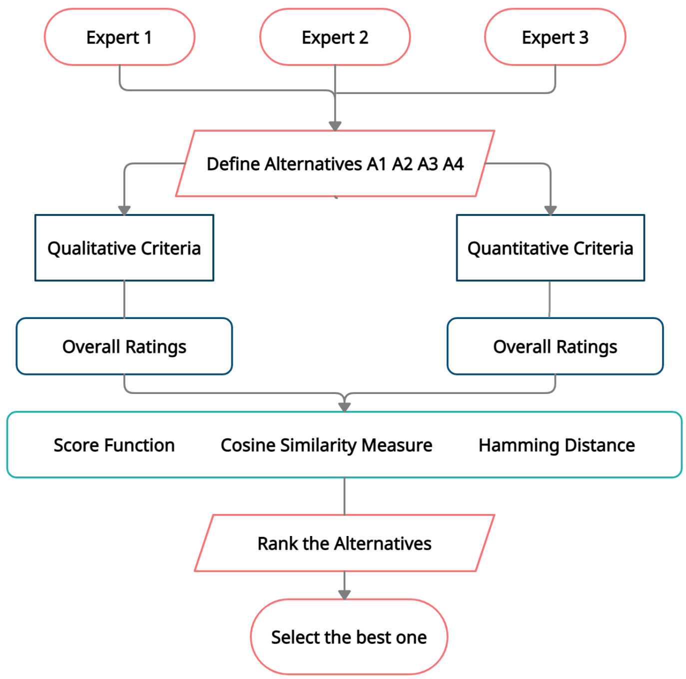


FIGURE 1. Flowchart

4. Numerical Illustration

An example of choosing the optimal construction for a building project to show how IVNNs may be used to solve MCDM challenges. Assume the manager is tasked with choosing the best tender construction for their structure. As a result, a group of three experts (E_1, E_2, E_3) was formed. On the basis of the following Qualitative and Quantitative criteria, the experts choose the best option out of four (A_1, A_2, A_3, A_4) alternatives.

Qualitative : C_1- Technical skills, C_2- Architectural Design, C_3- Reliability

Quantitative : C_4- Performance, C_5- Price, C_6- Period of work

The experts give the rating values to each alternative for the given criteria. The values are taken as Interval-valued neutrosophic numbers. When the alternatives have good criteria it has a high value in truth membership value. From this concept, experts have directly rated the alternatives in Interval values. In some critical situations, we use linguistic variables for collecting ratings from experts. Tables 1, 2 and 3 illustrate the ratings given by the three experts for qualitative criteria. Tables 4, 5, and 6 provide the ratings for qualitative criteria.

TABLE 1. Qulitative ratings by Expert 1

	C_1	C_2	C_3
A_1	[(.5,.7),(.2,.4),(.6,.7)]	[(.6,.8),(.3,.5),(.4,.6)]	[(.7,.8),(.4,.5),(.3,.5)]
A_2	[(.4,.5),(.3,.6),(.3,.4)]	[(.4,.5),(.2,.3),(.3,.4)]	[(.3,.5),(.2,.4),(.5,.6)]
A_3	[(.5,.6),(.1,.3),(.4,.5)]	[(.4,.5),(.2,.4),(.4,.5)]	[(.6,.8),(.3,.5),(.4,.5)]
A_4	[(.6,.7),(.3,.4),(.2,.3)]	[(.7,.8),(.3,.4),(.6,.7)]	[(.5,.7),(.4,.5),(.6,.7)]

TABLE 2. Qulitative ratings by Expert 2

	C_1	C_2	C_3
A_1	[(.3,.4),(.5,.6),(.6,.7)]	[(.6,.7),(.2,.3),(.4,.5)]	[(.4,.5),(.1,.3),(.5,.6)]
A_2	[(.6,.7),(.2,.3),(.3,.4)]	[(.4,.6),(.2,.4),(.3,.4)]	[(.5,.7),(.3,.5),(.3,.4)]
A_3	[(.6,.8),(.2,.4),(.4,.5)]	[(.5,.7),(.3,.4),(.4,.5)]	[(.4,.6),(.3,.4),(.4,.5)]
A_4	[(.3,.5),(.2,.4),(.6,.8)]	[(.4,.6),(.3,.5),(.5,.7)]	[(.3,.4),(.5,.6),(.5,.7)]

TABLE 3. Qulitative ratings by Expert 3

	C_1	C_2	C_3
A_1	[(.5,.6),(.2,.3),(.4,.5)]	[(.6,.8),(.1,.3),(.5,.6)]	[(.7,.8),(.3,.4),(.4,.5)]
A_2	[(.3,.4),(.3,.5),(.5,.7)]	[(.4,.6),(.3,.4),(.5,.6)]	[(.3,.5),(.1,.3),(.4,.5)]
A_3	[(.4,.5),(.2,.3),(.5,.7)]	[(.5,.6),(.2,.4),(.6,.7)]	[(.6,.7),(.3,.5),(.3,.4)]
A_4	[(.4,.6),(.2,.3),(.5,.6)]	[(.6,.7),(.2,.3),(.6,.8)]	[(.3,.4),(.5,.6),(.5,.7)]

TABLE 4. Quantitative ratings by Expert 1

	C_4	C_5	C_6
A_1	[(.3,.4),(.2,.3),(.6,.7)]	[(.4,.5),(.3,.4),(.5,.6)]	[(.3,.5),(.3,.4),(5,.6)]
A_2	[(.4,.5),(.3,.5),(.5,.6)]	[(.6,.7),(.3,.5),(.4,.5)]	[(.4,.5),(.2,.3),(.6,.7)]
A_3	[(.7,.8),(.2,.4),(.3,.5)]	[(.6,.7),(.3,.4),(.4,.6)]	[(.5,.7),(.2,.4),(.3,.4)]
A_4	[(.7,.8),(.2,.3),(.4,.5)]	[(.5,.7),(.3,.5),(.2,.4)]	[(.6,.8),(.3,.4),(.5,.6)]

Table 5. Quantitative ratings by Expert 2

	C_4	C_5	C_6
A_1	[(.2,.4),(.3,.4),(.6,.7)]	[(.3,.4),(.2,.3),(.5,.6)]	[(.4,.5),(.3,.4),(.6,.8)]
A_2	[(.4,.6),(.2,.4),(.6,.8)]	[(.5,.6),(.1,.2),(.7,.8)]	[(.6,.7),(.2,.3),(.7,.8)]
A_3	[(.7,.8),(.3,.5),(.4,.5)]	[(.6,.8),(.2,.3),(.4,.5)]	[(.8,.9),(.3,.5),(.5,.6)]
A_4	[(.5,.6),(.3,.4),(.2,.3)]	[(.4,.6),(.1,.3),(.5,.6)]	[(.5,.6),(.2,.3),(.3,.5)]

Table 6. Quantitative ratings by Expert 3

	C_4	C_5	C_6
A_1	[(.4,.5),(.2,.3),(.5,.7)]	[(.4,.5),(.2,.4),(.6,.7)]	[(.7,.8),(.3,.4),(.6,.7)]
A_2	[(.7,.9),(.4,.5),(.3,.4)]	[(.6,.7),(.3,.4),(.4,.5)]	[(.5,.6),(.4,.5),(.3,.4)]
A_3	[(.4,.6),(.2,.4),(.4,.5)]	[(.4,.5),(.2,.3),(.6,.7)]	[(.6,.7),(.3,.4),(.4,.5)]
A_4	[(.6,.8),(.4,.5),(.3,.4)]	[(.7,.8),(.3,.4),(.4,.5)]	[(.6,.7),(.4,.5),(.3,.4)]

We generate the overall rating values for qualitative and quantitative criteria using the weighted average max product formula.

$$
\begin{aligned}
A_1(T^L) &= max\{(.5 \times .3 \times .5), (.6 \times .6 \times .6), (.7 \times .4 \times .7)\} \\
&= max\{0.075, 0.216, 0.112\} = 0.216 \\
A_1(T^U) &= max\{(.7 \times .4 \times .6), (.8 \times .7 \times .8), (.8 \times .5 \times .8)\} \\
&= max\{0.168, 0.448, 0.32\} = 0.448
\end{aligned}
$$

Similarly, the interminacy and false values are calculated for A_1 and the remaining alternatives are calculated in the same way.

Overall ratings for qualitative criteria by three experts

$$
\begin{array}{ll}
A_1 & [(0.216, 0.448), (0.02, 0.072), (0.144, 0.245)] \\
A_2 & [(0.072, 0.18), (0.018, 0.09), (0.06, 0.12)] \\
A_3 & [(0.15, 0.336), (0.027, 0.1), (0.08, 0.175)] \\
A_4 & [(0.168, 0.336), (0.1, 0.18), (0.18, 0.392)]
\end{array}
$$

Overall ratings for quantitative criteria by three experts

$$
\begin{array}{ll}
A_1 & [(0.084, 0.2), (0.027, 0.064), (0.18, 0.343)] \\
A_1 & [(0.18, 0.294), (0.024, 0.1), (0.126, 0.224)] \\
A_1 & [(0.24, 0.441), (0.018, 0.08), (0.096, 0.21)] \\
A_1 & [(0.21, 0.384), (0.024, 0.06), (0.045, 0.12)]
\end{array}
$$

Score Function

$$\begin{aligned} Qualitative\ A_1 &= \frac{1}{4}[2 + 0.216 + 0.448 - 2(0.02 + 0.072) - (0.144 + 0.245)] \\ &= 0.5227 \\ Quantitative\ A_1 &= \frac{1}{4}[2 + 0.084 + 0.2 - 2(0.027 + 0.064) - (0.18 + 0.343)] \\ &= 0.3948 \\ Average\ S(A_1) &= \frac{A_1 + A_1}{2} \\ S(A_1) &= 0.4587 \end{aligned}$$

Cosine Similarity Measure

$$C(A_1) = \frac{(0.664)(0.284) + (0.092)(0.091) + (0.389)(0.523)}{\sqrt{(0.664)^2 + (0.092)^2 + (0.389)^2}\sqrt{(0.284)^2 + (0.091)^2 + (0.523)^2}}$$

$$C(A_1) = 0.8581$$

Hamming Distance

$$D(A_1) = \frac{1}{6}[|0.216 - 0.084| + |0.448 - 0.2| + |0.02 - 0.027| + |0.072 - 0.064| + |0.144 - 0.18| + |0.245 - 0.343|]$$

$D(A_1) = 0.0882$

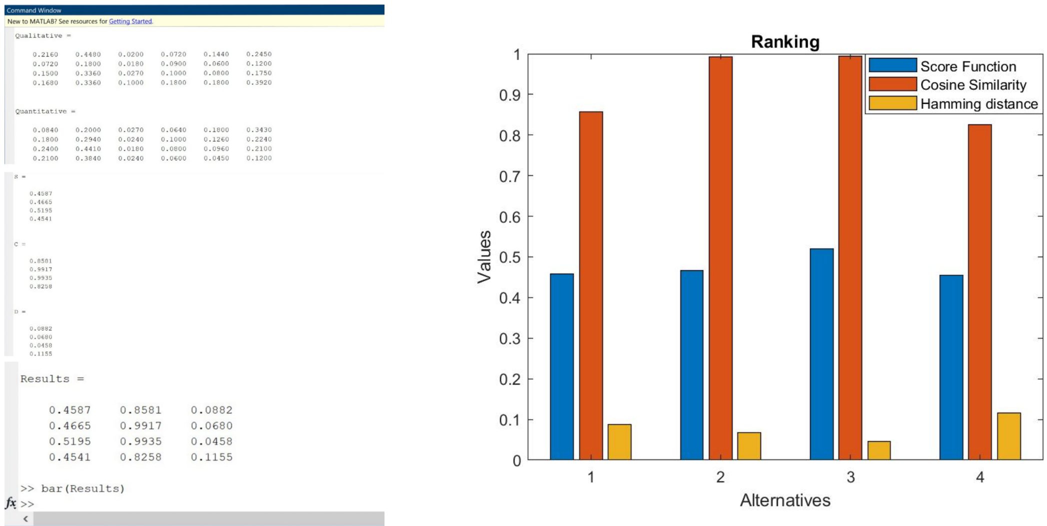
The remaining values of score function, cosine similarity measure and hamming distance are calculated for each alternatives in the same way.

Finally, the following table shows the ranking of the score function, cosine similarity function, and hamming distance.

TABLE 7. The ranking results of three approaches

	$S(A)$	Rank	$C(A)$	Rank	$D(A)$	Rank
A_1	0.4587	III	0.8581	III	0.0882	III
A_2	0.4665	II	0.9917	II	0.0680	II
A_3	0.5195	I	0.9935	I	0.0458	I
A_4	0.4541	IV	0.8258	IV	0.1155	IV

The values are calculated manually and then ran through MATLAB R2020a. We can solve a large number of matrices using Matlab code. This helps to solve the difficult situations in multiple criteria. Figure (a) shows the final output results, and Figure (b) shows the final ranking as a bar chart.



(A) Ouput Result

(B) Ranking Bar chart

Figure 2. MATLAB R2020a

The colour blue represents the score function and the colour orange represents the cosine similarity measure in this bar chart, and A_3 being the highest value in both. The hamming distance is represented by the colour yellow. When the distance between two points is small, the value receives the highest ranking. Clearly, A_3 is the highest. In comparison to the other alternatives, A_3 is the best one.

Ranking order of the alternatives : $A_3 > A_2 > A_1 > A_4$

5. Conclusions

Ranking methods always give a good result in decision-making. Particularly comparison of neutrosophic numbers uses to rank the values very easily. In that situvation, single-valued neutrosophic numbers and interval-valued neutrosophic numbers play the most part. In this paper, the basic concepts of interval valued neutrosophic sets and ranking of IVNNs are presented. We proposed a new technique using max product to calculate the weighted average for interval valued neutrosophic numbers, which achieved a very efficient result. The ranking values of three approaches produced an ideal outcome for choosing the best option which is established in Numerical example. When compared to other alternatives, A_3 is the best choice in terms of both qualitative and quantitative criteria. The output result verified through Matlab. This work can further be developed to solve more complex multi criteria decision making problems using many types of Neutrosophic numbers such as bi-polar, m-polar neutrosophic numbers.

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Semi-Separation Axioms and Semi-Regularity Axioms in Linguistic Neutrosophic Topological Spaces

N. Gayathri[1,*] and Dr.M.Helen[2]

[1]Research scholar; Nirmala college for women; Coimbatore ;Tamil Nadu; India

[2]Associate professor; Nirmala college for women;Coimbatore; Tamil Nadu; India

[*]Correspondence: gayupadmagayu@gmail.com

Abstract. Several new linguistic neutrosophic semi-separation axioms and semi-regularity axioms are presented in this article and example cases are also given for non justifiable results. Additionally, a new class of spaces called linguistic neutrosophic semi-D_0, linguistic neutrosophic semi-D_1 and linguistic neutrosophic semi-D_2 is described and the inter relationships are analyzed with appropriate illustrations.

Keywords: Linguistic Neutrosophic semi-T_i spaces(i=0,1,2); Linguistic Neutrosophic semi-R_p spaces(p=0,1); Linguistic Neutrosophic semi-D_k spaces(k=0,1,2);

1. Introduction

It is known as separation axioms in topology and related fields of mathematics that one often makes several restrictions on the kinds of topological spaces that are to be considered. Maheswari and Prasad [9] generalized T_0, T_1 and T_2 spaces to semi-T_0, semi-T_1 and semi-T_2 respectively. The separation axioms R_0 and R_1 are introduced in topological spaces by Shanin [14] in 1943. Several intriguing results have been obtained by Murdashwar and Naimpally [11] studying the properties of R_0 topological spaces. Also, they proposed a second concept, R_1, was introduced which is independent of T_0 and T_1, but stronger than T_2.

As a continuation of fuzzy sets [17] and eventually intuitionistic fuzzy sets [1], Smarandache [16] introduced the idea of neutrosophic sets. Chang [2], Coker [3] and Salama, Alblowi [14] are the topologists who have instigated the notion of fuzzy topology, intuitionistic fuzzy topology and neutrosophic topology respectively. Meanwhile, Fang [5] found linguistic neutrosophic number which has led to the concept of linguistic neutrosophic topology introduced in 2021 by Gayathri and Helen [6]. In this article, linguistic neutrosophic semi-$T_k, (k = 0, 1, 2)$ spaces and linguistic neutrosophic semi-R_p spaces($p = 0, 1,$) are discussed. Aside from that, the new spaces called linguistic neutrosophic semi-D_k spaces($k = 0, 1, 2$) are introduced and their properties analyzed and numerous relationships are discussed.

Throughout this article, (S_{LN}, τ_{LN}) denotes the linguistic neutrosophic topological space

2. Preliminaries

Definition 2.1. [16] Let S be a space of points (objects), with a generic element in x denoted by S. A neutrosophic set A in S is characterized by a truth-membership function T_A, an indeterminacy membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is $T_A : S \to]0^-, 1^+[, I_A : S \to]0^-, 1^+[, F_A : S \to]0^-, 1^+[$. There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2.2. [16] Let S be a space of points (objects), with a generic element in x denoted by S. A single valued neutrosophic set (SVNS) A in S is characterized by truth-membership function T_A, indeterminacy-membership function I_A and falsity-membership function F_A. For each point S in S, $T_A(x), I_A(x), F_A(x) \in [0, 1]$. When S is continuous, a SVNS A can be written as $A = \int \langle T(x), I(x), F(x)\rangle / x \in S$. When S is discrete, a SVNS A can be written as $A = \sum \langle T(x_i), I(x_i), F(x_i)\rangle / x_i \in S$.

Definition 2.3. [5] Let $S = \{s_\theta | \theta = 0, 1, 2, ..., \tau\}$ be a finite and totally ordered discrete term set, where τ is the even value and s_θ represents a possible value for a linguistic variable.

Definition 2.4. [5] Let $Q = \{s_0, s_1, s_2, ..., s_t\}$ be a linguistic term set (LTS) with odd cardinality $t+1$ and $\overline{Q} = \{s_h/s_0 \leq s_h \leq s_t, h \in [0, t]\}$. Then, a linguistic single valued neutrosophic set A is defined by,
$A = \{\langle x, s_\theta(x), s_\psi(x), s_\sigma(x)\rangle | x \in S\}$, where $s_\theta(x), s_\psi(x), s_\sigma(x) \in \overline{Q}$ represent the linguistic truth, linguistic indeterminacy and linguistic falsity degrees of S to A, respectively, with condition $0 \leq \theta + \psi + \sigma \leq 3t$. This triplet $(s_\theta, s_\psi, s_\sigma)$ is called a linguistic single valued neutrosophic number.

Definition 2.5. [6]Let $\alpha = (l_\theta, l_\psi, l_\sigma)$ be a LSVNN. The set of all labels is, $L = \{l_0, l_1, l_2, ..., l_t\}$.
Then the unit linguistic neutrosophic(LN in short) set (1_{LN}) is defined as $1_{LN} = (l_t, l_0, l_0)$, which is the truth membership,and the zero linguistic neutrosophic set (0_{LN}) is defined as $0_{LN} = (l_0, l_t, l_t)$, which is the falsehood membership.

Definition 2.6. [6] For a linguistic neutrosophic topology τ_{LN}, the collection of linguistic neutrosophic(LN in short) sets should obey,

(1) $0_{LN}, 1_{LN} \in \tau_{LN}$

(2) $K_1 \bigcap K_2 \in \tau_{LN}$ for any $K_1, K_2 \in \tau_{LN}$

(3) $\bigcup K_i \in \tau_{LN}, \forall \{K_i : i \in J\} \subseteq \tau_{LN}$

We call, the pair (S_{LN}, τ_{LN}), a linguistic neutrosophic topological space.

Definition 2.7. Let (S_{LN}, τ_{LN}) be a LNTS. Then, the LN semi-closure, for a LN subset E_{LN} is defined as the intersection of all LNSCSs in S_{LN}, that are contained in E_{LN}, (i.e) $LNSCl(E_{LN}) = \cap\{K_{LN} : K_{LN}$ is a LNSCS in S_{LN} and $K_{LN} \supseteq E_{LN}\}$.

Definition 2.8. A topological space (S_{LN}, τ_{LN}) is said to be

(1) semi-T_0 [9](semi-Kolomogorov) if for each pair of distinct points in X, there exists a semi-open set containing one but not the other.

(2) semi-T_1 [9](semi-Frechet) if for each pair of distinct points x and y in X, there exist semi-open sets U and V containing x and y such that $x \in U, y \notin V$ and $x \notin U, y \in V$.

(3) semi-T_2 [9](semi-Hausdorff) if every two points can be separated by disjoint semi-open sets.

(4) R_0 [11] if for each open set G, $x \in G \Rightarrow cl(\{x\}) \subseteq G$.

(5) R_1 [11] if for each $x, y \in X$ with $cl(\{x\} \neq cl(\{y\})$, there exist two disjoint open sets U and V such that $cl(\{x\} \subseteq U$ and $cl(\{y\} \in V$.

Definition 2.9. Let S_{LN} be a non-void set and $K_{LN} = \{\langle s, [T_{K_{LN}}, I_{K_{LN}}, F_{K_{LN}}]\rangle\}$ and $H_{LN} = \{\langle s, [T_{H_{LN}}, I_{H_{LN}}, F_{H_{LN}}]\rangle\}$ are LN sets in LNTS.

(I) $K_{LN} \cup H_{LN}$ can be defined as

 (a) $K_{LN} \cup H_{LN} = \{\langle s, [T_{K_{LN}} \wedge T_{H_{LN}}, I_{K_{LN}} \wedge I_{H_{LN}}, F_{K_{LN}} \vee F_{H_{LN}}]\rangle\}$

(II) $K_{LN} \cap H_{LN}$ can be defined as

 (a) $K_{LN} \cap H_{LN} = \{\langle s, [T_{K_{LN}} \wedge T_{H_{LN}}, I_{K_{LN}} \wedge I_{H_{LN}}, F_{K_{LN}} \vee F_{H_{LN}}]\rangle\}$

(III) The complement of $K_{LN} = \{\langle s, [T_{K_{LN}}, I_{K_{LN}}, F_{K_{LN}}]\rangle\}$ is defined as,

 (a) $(K_{LN})^c = \{\langle s, [F_{K_{LN}}, I_{K_{LN}}, T_{K_{LN}}]\rangle\}$

 (b) $((K_{LN})^c)^c = K_{LN}$

 (c) $(K_{LN} \cap H_{LN})^c = (K_{LN})^c \cup (H_{LN})^c$

 (d) $(K_{LN} \cup H_{LN})^c = (K_{LN})^c \cap (H_{LN})^c$

3. Linguistic Neutrosophic Separation Axioms

Definition 3.1. A LNS $P_{LN} = \{\langle s_1, T_{P_{LN}}(s_1), I_{P_{LN}}(s_1), F_{P_{LN}}(s_1)\rangle : s_1 \in S_{LN}\}$ is called a linguistic neutrosophic point(LNP in short) if and only if for any element $s_2 \in S_{LN}$,

$$\begin{cases} T_{P_{LN}}(s_1) = l_p, I_{P_{LN}}(s_1) = l_q, F_{P_{LN}}(s_1) = l_r, & \text{for} \quad s_2 = s_1, \\ T_{P_{LN}}(s_1) = 0, I_{P_{LN}}(s_1) = 0, F_{P_{LN}}(s_1) = 1, & \text{for} \quad s_2 \neq s_1. \end{cases}$$

where $0 <\text{p} \leq t, 0 \leq q < t, 0 \leq r < t.$

Definition 3.2. A LNP $P_{LN} = \{\langle s, T_{P_{LN}}(s), I_{P_{LN}}(s), F_{P_{LN}}(s)\rangle : s \in S_{LN}\}$ will be denoted by ${P_{LN}}^{s}{}_{\langle l_p,l_q,l_r\rangle}$ or $P_{LN}\langle s, l_p, l_q, l_r\rangle$ or simply by $s_{\langle l_p,l_q,l_r\rangle}$.
The complement of the LNP ${P_{LN}}^{s}{}_{\langle l_p,l_q,l_r\rangle}$ will be denoted by $({P_{LN}}^{s}{}_{\langle l_p,l_q,l_r\rangle})^c$ or ${s^c}_{\langle l_p,l_q,l_r\rangle}$.

Definition 3.3. A LNTS (S_{LN}, τ_{LN}) is LN semi-T_0 space if for a couple of distinct points in S_{LN}, there lies a LNSO set containing one point not the other.

Example 3.4. Let the universe of discourse be $U = \{a, b, c\}$. The set of all linguistic term is, $L = \{$ very salt(l_0), salt(l_1), very sour(l_2), sour(l_3), bitter(l_4), sweet(l_5), very sweet(l_6)$\}$. Let $S_{LN} = \{c\}$. Let $s_{1\langle a,l_0,l_2,l_6\rangle}, s_{2\langle a,l_1,l_0,l_6\rangle}$ be any two distinct LN points in S_{LN}. Then $A_{LN} = \langle a, (l_0, l_4, l_6)\rangle$ and $B_{LN} = \langle a, (l_2, l_0, l_6)\rangle$ the LNSOSs that contains the points s_1 and s_2 respectively such that $s_2 \notin A_{LN}$ and $s_1 \notin B_{LN}$.

Theorem 3.5. *A LNTS (S_{LN}, τ_{LN}) is semi-T_0 iff each couple of points s_1, s_2 of $S_{LN}, LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$.*

Proof:
Necessity Part: Let the space (S_{LN}, τ_{LN}) is LN semi-T_0 and $s_1 \neq s_2$ where $s_1, s_2 \in S_{LN}$. Then there lies a LNSO set V_{LN} with $s_1 \in V_{LN}$ and $s_2 \notin V_{LN}$. So, $S_{LN}\backslash V_{LN}$ is a LNSC set containing s_2 but not s_1. Also, $s_2 \in LNSCl(\{s_2\}) \subseteq S_{LN}\backslash V_{LN}$ but $s_1 \notin LNSCl(\{s_2\})$.
Sufficiency Part: Let $s_1, s_2 \in S_{LN}$ with $s_1 \neq s_2$ where $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$. Then, there lies an element $r \in S_{LN}$ with $r \in LNSCl(\{s_1\})$ and $r \notin LNSCl(\{s_2\})$. If $s_1 \in LNSCl(\{s_2\})$, then $LNSCl(\{s_1\}) \subseteq LNSCl(\{s_2\})$.(i.e) $r \in LNSCl(\{s_2\})$ which is a contradiction. Therefore, $s_1 \notin LNSCl(\{s_2\})$. Also, $s_2 \notin S_{LN}\backslash LNSCl(\{s_2\})$ where the set $S_{LN}\backslash LNSCl(\{s_2\})$ is LNSO.

Definition 3.6. Let A_{LN} be a LN subset of (S_{LN}, τ_{LN}). Then LN semi-kernel of A_{LN} is defined by, $LNSKer(A_{LN}) = \cap\{K_{LN} \subseteq S_{LN} | A_{LN} \subseteq K_{LN}$ and $K_{LN} \in LNSO(S_{LN}, \tau_{LN})\}$.

Theorem 3.7. *A LN topological space (S_{LN}, τ_{LN}) is semi-T_0 iff for any couple of points s_1, s_2 of $S_{LN}, LNSKer(\{s_1\}) \neq LNSKer(\{s_2\})$.*

Proof: Necessity Part: Suppose S_{LN} is a LN semi-T_0 space, then $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$. Ergo, $LNSKer(\{s_1\}) \neq LNSKer(\{s_2\})$.
Sufficiency Part: Let $s_1 \neq s_2$ where $s_1, s_2 \in S_{LN}$ and $LNSKer(\{s_1\}) \neq LNSKer(\{s_2\})$. Then, $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$.

Definition 3.8. A LNTS (S_{LN}, τ_{LN}) is LN semi-T_1 space if for every couple of distinct points in S_{LN}, there lies LNSO sets E_{LN} and F_{LN} containing two points respectively with $E_{LN} \cap F_{LN} = \phi$, (i.e) the intersection must be an empty set rather than zero element.

Example 3.9. Let the universe of discourse be $\mathcal{U} = \{a, b, c\}$ and the LTS be as in the example 3.4. Let $S_{LN} = \{c\}$. Let $s1_{\langle a,l_0,l_2,l_6\rangle}, s2_{\langle a,l_1,l_0,l_6\rangle}$ be any two distinct LN points in S_{LN}. Then $A_{LN} = \langle a, (l_0, l_4, l_6)\rangle$ and $B_{LN} = \langle a, (l_2, l_0, l_6)\rangle$ the LNSOSs that contains the points s_1 and s_2 respectively such that A_{LN} and B_{LN}.

Theorem 3.10. *The upcoming characterizations of a LN semi-T_1 space imply each other.*

1. *The space S_{LN} is LN semi-T_1 space.*
2. $\{s\} = LNSCl(\{s\})$ *for every* $s \in S_{LN}$.
3. *For each $s \in S_{LN}$, the intersection of all LNSO sets containing s is $\{s\}$.*

Proof: (1) $\Rightarrow$ (2): There lies a LNSO set V_{LN} in S_{LN} with $s_1 \in V_{LN}$ and $s_2 \notin V_{LN}$. If $s_1 \in LNSCl(\{s_2\})$, then s_1 is a LN semi-cluster point of $\{s_2\}$. So, U_{LN} is a LNSO set containing s_1 and $\{s_2\} \cap V_{LN} \neq \phi$, which arrives at a contradiction. Thus, $s_1 \notin LNSCl(\{s_2\})$.
(2) $\Rightarrow$ (3): Suppose $\{s_1\} = LNSCl(\{s_1\})$. Then, $\{s_1\} \subseteq LNSCl(\{s_1\})$. If $s_2 \in LNSCl(\{s_1\})$, then $s_1 \in LNSKer(\{s_2\})$. Thus, $LNSKer(\{s_1\}) \subseteq \{s_1\}$. Thus, $\{s_1\} = LNSKer(\{s_1\})$. Also, $\{s_1\} = \cap\{V_{LN} : V_{LN} \in LNSO(S_{LN}, \tau_{LN})$ and $s_1 \in U_{LN}\}$.
(3) $\Rightarrow$ (1): Let the intersections of all LNSO sets containing s is $\{s\}$. And let $s_1 \neq s_2$, where $s_1, s_2 \in S_{LN}$. By the hypothesis, $\{s_1\} = \cap\{V_{LN} : V_{LN} \in LNSO(S_{LN}, \tau_{LN})$ and $s_1 \in U_{LN}\}$. Thus, we can find a LNSO set V_{LN} containing s_1 but not s_2. Therefore, S_{LN} is LN semi-T_1 space.

Theorem 3.11. *The space (S_{LN}, τ_{LN}) is LN semi-T_1 iff the singleton sets are LNSC.*

Proof: Necessity Part: For every singleton set, $\{s\} = LNSCl(\{s\})$.
Sufficiency Part: Let $\{s\}$ is LNSC, $\{s\} = LNSCl(\{s\})$. Then the LNTS S_{LN} is LN semi-T_1.

Definition 3.12. A LNTS (S_{LN}, τ_{LN}) is LN semi-T_2 space if two distinct points s_1 and s_2 can be separated by disjoint LNSO sets U_{LN} and V_{LN} respectively.

Example 3.13. In example 3.9, A_{LN} and B_{LN} are disjoint LNSOSs.

Theorem 3.14. *The underneath characterizations of a LN semi-T_2 space imply each other.*

1. *The space S_{LN} is LN semi-T_2 space.*
2. *For every $s_2 \neq s_1$, there is a LNSO set U_{LN} containing s_1 with $s_2 \notin LNSCl(U_{LN})$.*

Proof: (1) $\Rightarrow$ (2): For each $s_2 \neq s_1$, there lie LNSO sets K_{LN} and H_{LN} with $s_1 \in K_{LN}$ and $s_2 \in H_{LN}$ with $K_{LN} \cap H_{LN} = \phi$. Also, $K_{LN} \subseteq S_{LN} \backslash H_{LN}$ and $s_2 \notin S_{LN} \backslash H_{LN}$, which shows that $s_2 \notin \cap\{S_{LN} \backslash H_{LN} : S_{LN} \backslash H_{LN}$ is LNSC and $K_{LN} \subseteq S_{LN} \backslash H_{LN}\}$.
(2) $\Rightarrow$ (1): Let $s_2 \neq s_1$, then there lies a LNSO set U_{LN} containing s_1 with $s_2 \notin LNSCl(U_{LN})$. Now, $s_1 \in U_{LN} \subseteq LNSCl(U_{LN})$ and $S_{LN} \backslash LNSCl(U_{LN})$ is LNSO which is evident that $U_{LN} \cap (S_{LN} \backslash LNSCl(U_{LN})) = \phi$.

4. **Linguistic Neutrosophic Regulation Axioms**

Definition 4.1. A LNTS (S_{LN}, τ_{LN}) is semi-R_0 if for each LNSO set K_{LN}, $s \in K_{LN} \Rightarrow LNSCl(\{s\}) \subseteq K_{LN}$.

Example 4.2. Let the universe of discourse be $U = \{x, y, z, w\}$ and let $S_{LN} = \{x, y\}$. The set of all LTS be L= {very strongly disagree(l_0), strongly disagree(l_1), disagree(l_2), mostly disagree(l_3), slightly disagree(l_4), neither disagree nor agree(l_5), slightly agree(l_6), mostly agree(l_7), agree(l_8), strongly agree(l_9), very strongly agree(l_{10})}. Let $F_{LN} = \{(x, \langle l_{10}, l_9, l_2\rangle), (y, \langle l_5, l_7, l_0\rangle)\}$ be an LNSOS. Let $s_{\{(x,\langle l_3,l_6,l_5\rangle),(y,\langle l_7,l_2,l_6\rangle)\}}$ be a LNP. Now $LNSCl(\{s\}) = (E_{LN})^c \subseteq F_{LN}$, where E_{LN} is a LNSOS.

Theorem 4.3. *A LNTS (S_{LN}, τ_{LN}) is LN semi-R_0 iff each LN subset of S_{LN} is the union of LNSC sets.*

Proof: Necessity Part: Let S_{LN} be a LN semi-R_0 space and $A_{LN} \subseteq S_{LN}$. Then for any $s \in A_{LN}, LNSCl(\{s\}) \subseteq A_{LN}$. Also, $\cup\{LNSCl(\{s\}) : s \in A_{LN}\} \subseteq A_{LN}$. Thus, $A_{LN} = LNSCl(\{s\}) = \cup\{LNSCl(\{s\}) : s \in A_{LN}\}$.
Sufficiency Part: Let $s \in A_{LN}$ where A_{LN} is LNSO. Then, there lie LNSC sets U_i with $A_{LN} = \cup\{U_i : i \in I\}$. Since $s \in A_{LN} \Rightarrow s \in U_i : i \in I$. Ergo, $s \in LNSCl(\{s\}) \subseteq U_i \subseteq A_{LN}$.

Remark 4.4. Every LN semi-T_1 space is LN semi-R_0 but not the reverse implication holds true.

Example 4.5. In example 4.2, the space is semi-R_0 but not semi-T_1 as $E_{LN} \cap F_{LN} \neq 0_{LN}$.

Theorem 4.6. *For any LNTS (S_{LN}, τ_{LN}) the upcoming statements imply each other.*

(1) (S_{LN}, τ_{LN}) *is LN semi-R_0.*
(2) *For any LNSC set V_{LN} and for $s \notin V_{LN}$, there lies a $U_{LN} \in LNSO(S_{LN}, \tau_{LN})$ with $s \notin U_{LN}$ and $V_{LN} \subseteq U_{LN}$.*
(3) $LNSCl(\{s\}) \cap V_{LN} = \phi$, *where V_{LN} is a LNSC set and $s \notin V_{LN}$.*

Proof: (1) $\Rightarrow$ (2): Let K_{LN} be a LNSC set with $s \notin K_{LN}$. By the definition, $LNSCl(\{s\}) \subseteq S_{LN}\backslash K_{LN}$ and so $K_{LN} \subseteq S_{LN}\backslash LNSCl(\{s\})$. Then, $S_{LN}\backslash LNSCl(\{s\})$ is the required LNSC set containing K_{LN} and $s \notin S_{LN}\backslash LNSCl(\{s\})$.
(2) $\Rightarrow$ (3): Let K_{LN} be a LNSC set with $s \notin K_{LN}$. By hypothesis, we can find a $U_{LN} \in LNSO(S_{LN}, \tau_{LN})$ with $s \notin U_{LN}$ and $K_{LN} \subseteq U_{LN}$. Suppose $LNSCl(\{s\}) \cap U_{LN} \neq \phi$, then there exists $r \in S_{LN}$ with $r \in U_{LN}$ and $r \in LNSCl(\{s\})$. Now, $H_{LN} \cap \{s\} \neq \phi$,(i.e)$s \in H_{LN}$. The result is that U_{LN} is a LNSO set that contains r and $s \in U_{LN}$, which arrives at a contradiction.
(3) $\Rightarrow$ (1): If H_{LN} is a LNSO set and $s \in H_{LN}$, then $S_{LN}\backslash H_{LN}$ is LNSC and $s \notin S_{LN}\backslash H_{LN}$. By the assumption, $LNSCl(\{s\}) \cap (S_{LN}\backslash H_{LN}) = \phi$.

Theorem 4.7. *In a LNTS (S_{LN}, τ_{LN}) for any two points s, r, the result $LNSCl(\{s\}) \neq LNSCl(\{r\}) \Rightarrow LNSCl(\{s\}) \cap LNSCl(\{r\}) = \phi$ holds iff the LNTS is LN semi-R_0.*

Proof: Necessity Part: Let $s, r \in S_{LN}$ with $LNSCl(\{s\}) \neq LNSCl(\{r\})$. Now, suppose we can find an element in $x \in S_{LN}$ with $x \in LNSCl(\{s\})$ and $x \in LNSCl(\{r\})$. There lies a LNSO set U_{LN} containing x with $\{r\} \cap U_{LN} = \phi$ so that $r \notin U_{LN}$. As $x \in LNSCl(\{s\})$, for each LNSO set H_{LN} containing x so that $H_{LN} \cap \{s\} = \phi$, which results in $s \in U_{LN}$. Now, $LNSCl(\{s\}) \subseteq S_{LN} \backslash LNSCl(\{t\})$.
Sufficiency Part: Let $LNSCl(\{s\}) \neq LNSCl(\{r\})$ implies $LNSCl(\{s\}) \cap LNSCl(\{t\}) = \phi$. Let H_{LN} be a LNSO set with $s \in H_{LN}$. If $t \notin H_{LN}$ and so $s \notin LNSCl(\{r\})$. By assumption, $LNSCl(\{s\}) \cap LNSCl(\{t\}) = \phi$ and $t \notin LNSCl(\{s\})$.

Theorem 4.8. *In a space (S_{LN}, τ_{LN}) for any two points s, r, the result $LNSKer(\{s\}) \neq LNSKer(\{r\})$ implies $LNSKer(\{s\}) \cap LNSKer(\{r\}) = \phi$ holds iff the LNTS is LN semi-R_0.*

Proof: Necessity Part: Let $s, t \in S_{LN}$ with $LNSKer(\{s\}) \neq LNSKer(\{r\})$ and let $x \in LNSKer(\{s\}) \cap LNSKer(\{r\})$. Then $x \in LNSKer(\{s\})$ and $x \in LNSKer(\{t\})$. Then, $s \in LNSCl(\{x\})$ and $t \in LNSCl(\{x\})$ and also $LNSCl(\{s\}) \cap LNSCl(\{x\}) \neq \phi$ and $LNSCl(\{r\}) \cap LNSKer(\{x\}) \neq \phi$. Now, $LNSCl(\{s\}) = LNSCl(\{r\})$. Then, $LNSKer(\{s\}) = LNSKer(\{r\})$ which arrives at a contradiction.
Sufficiency Part: Suppose $LNSKer(\{s\}) \neq LNSKer(\{r\})$ implies $LNSKer(\{s\}) \cap LNSKer(\{r\}) = \phi$. Let $s, t \in S_{LN}$ with $LNSCl(\{s\}) \neq LNSCl(\{r\})$ and let $x \in LNSCl(\{s\}) \cap LNSCl(\{r\})$, then $x \in LNSCl(\{s\})$ and $x \in LNSCl(\{t\})$. Then, $s \in LNSKer(\{x\})$ and $t \in LNSKer(\{x\})$. Now, $LNSKer(\{s\}) = LNSKer(\{x\})$ and $LNSKer(\{t\}) = LNSKer(\{x\})$, also $LNSKer(\{s\}) = LNSKer(\{t\})$. Then, $LNSCl(\{s\}) = LNSCl(\{r\})$ which is a contradiction.

Theorem 4.9. *For any LNTS (S_{LN}, τ_{LN}) the following imply each other.*

(1) *The space (S_{LN}, τ_{LN}) is LN semi-R_0.*
(2) *For a non-zero LN setA_{LN} and for a LNSO set K_{LN}, $A_{LN} \cap K_{LN} \neq \phi$, we can find a $U_{LN} \in LNSC(S_{LN}, \tau_{LN})$ with $A_{LN} \cap U_{LN} \neq \phi$ and $U_{LN} \subseteq K_{LN}$.*
(3) *For any $H_{LN} \in LNSO(S_{LN}, \tau_{LN}), H_{LN} = \cup\{U_{LN} : U_{LN} \in LNSC(S_{LN}, \tau_{LN})$ and $U_{LN} \subseteq H_{LN}\}$.*
(4) *For any $K_{LN} \in LNSC(S_{LN}, \tau_{LN}), K_{LN} = \cap\{H_{LN} : H_{LN} \in LNSO(S_{LN}, \tau_{LN})\}$ and $U_{LN} \subseteq H_{LN}\}$.*
(5) *For any $s \in S_{LN}, LNSCl(\{s\}) \subseteq LNSKer(\{s\})$.*
(6) *For any $s, t \in S_{LN}$, $t \in LNSCl(\{s\}) \Leftrightarrow s \in LNSCl(\{t\})$.*

Proof: (1) $\Rightarrow$ (2): Let $(\phi \neq A_{LN}) \subseteq S_{LN}$ and $K_{LN} \in LNSO(S_{LN}, \tau_{LN})$ with $A_{LN} \cap K_{LN} \neq \phi$ and let $s \in A_{LN} \cap K_{LN}$. Thus, $LNSCl(\{s\}) \cap A_{LN} \neq \phi$.

(2) $\Rightarrow$ (3): If $K_{LN} \in LNSO(S_{LN}, \tau_{LN})$ and $s \in K_{LN}$, then one can find a LNSC set U_{LN} with $\{s\} \cap U_{LN} \neq \phi$ and $U_{LN} \subseteq K_{LN}$. This implies $s \in U_{LN}$ and $U_{LN} \subseteq K_{LN}\}$ and so $K_{LN} \subseteq \cup\{U_{LN} : U_{LN} \in LNSC(S_{LN}, \tau_{LN})$ and $U_{LN} \subseteq K_{LN}\}$. Also, $\cup\{U_{LN} : U_{LN} \in LNSC(S_{LN}, \tau_{LN})$ and $U_{LN} \subseteq K_{LN}\} \subseteq K_{LN}$.

(3) $\Rightarrow$ (4): If $U_{LN} \in LNSC(S_{LN}, \tau_{LN}$, then $S_{LN} \backslash U_{LN} \in LNSO(S_{LN}, \tau_{LN})$. By hypothesis, $S_{LN} \backslash U_{LN} = \cup\{S_{LN} \backslash K_{LN} : S_{LN} \backslash K_{LN} \in LNSC(S_{LN}, \tau_{LN})$ and $S_{LN} \backslash K_{LN} \subseteq S_{LN} \backslash U_{LN}\}$. This implies that $U_{LN} = \cap\{K_{LN} : K_{LN} \in LNSO(S_{LN}, \tau_{LN})$ and $U_{LN} \subseteq K_{LN}\}$.

(4) $\Rightarrow$ (5): If $t \notin LNSKer(\{s\})$, then $s \notin LNSCl(\{t\})$. Then we can find a LNSO set V_{LN} containing s with $V_{LN} \cap \{t\} = \phi$, which implies that $LNSCl(\{t\}) \cap V_{LN} = \phi$. Then, $LNSCl(\{t\}) = \cap\{K_{LN} : K_{LN} \in LNSO(S_{LN}, \tau_{LN})$ and $LNSCl(\{t\}) \subseteq K_{LN}\}$. Since $s \in V_{LN}$, we have $s \notin LNSCl(\{t\})$ and so there exists $K_{LN} \in LNSO(S_{LN}, \tau_{LN})$ with $LNSCl(\{t\}) \subseteq K_{LN}$ and $s \notin K_{LN}$. This follows that $LNSCl(\{s\}) \cap K_{LN} = \phi$. Thus, $t \notin LNSCl(\{s\})$ and so $LNSCl(\{s\}) \subseteq LNSKer(\{s\})$.

(5) $\Rightarrow$ (6): If $t \in LNSCl(\{s\})$ then by hypothesis, $t \in LNSKer(\{s\})$ and $s \in LNSCl(\{t\})$. Similarly, if $s \in LNSCl(\{t\})$ and $s \in LNSKer(\{t\})$ then $t \in LNSKer(\{s\})$. This shows that $s \in LNSCl(\{y\}) \Leftrightarrow t \in LNSKer(\{s\})$.

(6) $\Rightarrow$ (1): Let K_{LN} be a LNSO set in (S_{LN}, τ_{LN}) and let $s \in K_{LN}$. If $t \notin K_{LN}$,then $t \in S_{LN} \backslash K_{LN}$. Since $LNSCl(\{t\})$ is the smallest LNSC set that contains t, we have $t \in LNSCl(\{t\}) \subseteq S_{LN} \backslash K_{LN}$. Then $LNSCl(\{t\}) \cap K_{LN} = \phi$, which results that $s \notin LNSCl(\{t\})$.

Theorem 4.10. *For any LNTS (S_{LN}, τ_{LN}) the following imply each other.*

(1) *(S_{LN}, τ_{LN}) is LN semi-R_0.*
(2) *If H_{LN} is LNSC, then $H_{LN} = LNSKer(H_{LN})$.*
(3) *If H_{LN} is LNSC and $s \in H_{LN}$, then $LNSKer(\{s\}) \subseteq H_{LN}$.*
(4) *If $s \in S_{LN}$, then $LNSKer(\{s\}) \subseteq LNSCl(\{s\})$.*

Proof: Proof is direct. (1) $\Rightarrow$ (2): Let H_{LN} be a LNSC and $s \notin H_{LN}$. Then $S_{LN} \backslash H_{LN}$ is a LNSO set containing s. Then by definition, $LNSCl(\{s\}) \subseteq S_{LN} \backslash H_{LN}$ and also $LNSCl(\{s\}) \cap H_{LN} = \phi$. Also, $s \notin LNSKer(H_{LN})$. This means that $LNSKer(H_{LN}) \subseteq H_{LN}$.

(2) $\Rightarrow$ (3): Proof is direct.

(3) $\Rightarrow$ (4): Let $s \in LNSCl(\{s\})$ and the set $LNSCl(\{s\})$ is LNSC. From the assumption, $LNSKer(\{s\}) \subseteq LNSCl(\{s\})$.

(4) $\Rightarrow$ (1): Let $s \in LNSCl(\{t\})$. Then $t \in LNSKer(\{s\})$ and by hypothesis $t \in LNSCl(\{s\})$. On the other hand, let $t \in LNSCl(\{s\})$. Then, $s \in LNSKer(\{t\})$ and $s \in LNSCl(\{t\})$. This reveals that $s \in LNSCl(\{t\})$ iff $t \in LNSCl(\{s\})$.

Theorem 4.11. *For any LNTS (S_{LN}, τ_{LN}) the following imply each other.*

(1) *The space (S_{LN}, τ_{LN}) is LN semi-R_0.*
(2) *For $A_{LN} \neq \phi$ and for a LNSO set K_{LN}, $A_{LN} \cap K_{LN} \neq \phi$, we can find a $U_{LN} \in LNSC(S_{LN}, \tau_{LN})$ with $A_{LN} \cap U_{LN} \neq \phi$ and $U_{LN} \subseteq K_{LN}$.*
(3) *For any $H_{LN} \in LNSO(S_{LN}, \tau_{LN}), H_{LN} = \cup\{U_{LN} : U_{LN} \in LNSC(S_{LN}, \tau_{LN})$ and $U_{LN} \subseteq H_{LN}\}$.*
(4) *For any $K_{LN} \in LNSC(S_{LN}, \tau_{LN}), K_{LN} = \cap\{H_{LN} : H_{LN} \in LNSO(S_{LN}, \tau_{LN})\}$ and $U_{LN} \subseteq H_{LN}\}$.*
(5) *For any $s \in S_{LN}, LNSCl(\{s\}) \subseteq LNSKer(\{s\})$.*
(6) *For any $s, t \in S_{LN}$, $t \in LNSCl(\{s\}) \Leftrightarrow s \in LNSCl(\{t\})$.*

Proof: (1) $\Rightarrow$ (2): Proof is direct. (2) $\Rightarrow$ (3): If $K_{LN} \in LNSO(S_{LN}, \tau_{LN})$ and $s \in K_{LN}$, then we can find a LNSC set U_{LN} in S_{LN} with $\{s\} \cap U_{LN} \neq \phi$ where $U_{LN} \subseteq K_{LN}$. This implies that $s \in U_{LN}$ and also $s \in \cup\{U_{LN} : U_{LN} \in LNSC(S_{LN}, \tau_{LN})$ and $U_{LN} \subseteq K_{LN}$ and so $K_{LN} \subseteq \cup\{U_{LN} : U_{LN} \in LNSC(S_{LN}, \tau_{LN})$. Also, $\cup\{U_{LN} : U_{LN} \in LNSC(S_{LN}, \tau_{LN})$ and $U_{LN} \subseteq K_{LN}\} \subseteq K_{LN}$.
(3) $\Rightarrow$ (4): If $U_{LN} \in LNSC(S_{LN}, \tau_{LN})$, then $S_{LN} \backslash U_{LN} \in LNSO(S_{LN}, \tau_{LN})$. By hypothesis, $S_{LN} \backslash U_{LN} = \cup\{S_{LN} \backslash K_{LN} : S_{LN} \backslash K_{LN} \in LNSC(S_{LN}, \tau_{LN})$ and $S_{LN} \backslash K_{LN} \subseteq S_{LN} \backslash U_{LN}\}$. Thus $U_{LN} = \cap\{K_{LN} : K_{LN} \in LNSO(S_{LN}, \tau_{LN})$ and $U_{LN} \subseteq K_{LN}\}$.
(4) $\Rightarrow$ (5): If $t \notin LNSKer(\{s\})$, then $s \notin LNSCl(\{t\})$. Then we can find a LNSO set V_{LN} containing s with $V_{LN} \cap \{t\} = \phi$, which implies that $LNSCl(\{t\}) \cap V_{LN} = \phi$. Then $LNSCl(\{t\}) = \cap\{K_{LN} : K_{LN} \in LNSO(S_{LN}, \tau_{LN})$ and $LNSCl(\{t\}) \subseteq K_{LN}\}$. Since $s \in V_{LN}$, $s \notin LNSCl(\{t\})$ and so there lies $K_{LN} \in LNSO(S_{LN}, \tau_{LN})$ with $LNSCl(\{t\}) \subseteq K_{LN}$ and $s \notin K_{LN}$.
Proof is direct for (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (1).

Theorem 4.12. *For any LNTS (S_{LN}, τ_{LN}) the following are equivalent.*

(1) *(S_{LN}, τ_{LN}) is LN semi-R_0.*
(2) *If H_{LN} is LNSC, then $H_{LN} = LNSKer(H_{LN})$.*
(3) *If H_{LN} is LNSC and $s \in H_{LN}$, then $LNSKer(\{s\}) \subseteq H_{LN}$.*
(4) *If $s \in S_{LN}$, then $LNSKer(\{s\}) \subseteq LNSCl(\{s\})$.*

Proof: (1) $\Rightarrow$ (2): Let H_{LN} be a LNSC and $s \notin H_{LN}$. Then $S_{LN} \backslash H_{LN}$ is a LNSO set containing s. Then, $LNSCl(\{s\}) \subseteq S_{LN} \backslash H_{LN}$ and also $LNSCl(\{s\}) \cap H_{LN} = \phi$. $s \notin LNSKer(H_{LN})$. This means that $LNSKer(H_{LN}) \subseteq H_{LN}$. Now, $H_{LN} \subseteq LNSKer(H_{LN})$.
(2) $\Rightarrow$ (3): Proof is direct.
(3) $\Rightarrow$ (4): Proof is direct.
(4) $\Rightarrow$ (1): Let $s \in LNSCl(\{t\})$. Then, $t \in LNSKer(\{s\})$ and $t \in LNSCl(\{s\})$. Let

$t \in LNSCl(\{s\})$. Then, $s \in LNSKer(\{t\})$ and $s \in LNSCl(\{t\})$. This results that $s \in LNSCl(\{t\})$ iff $t \in LNSCl(\{s\})$. Also , the LNTS is LN semi-R_0.

Definition 4.13. A LNTS (S_{LN}, τ_{LN}) is semi-R_1 if for any couple of points $s_1, s_2 \in S_{LN}$, with $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$, there lie LNSO sets K_{LN} and H_{LN} with $LNSCl(\{s_1\}) \subseteq K_{LN}$ and $LNSCl(\{s_2\}) \subseteq H_{LN}$ where $K_{LN} \cap H_{LN} = \phi$.

Example 4.14. Let the universe of discourse be $U = \{x, y, z, w\}$ and let $S_{LN} = \{x, y, z\}$. The set of all LTS be L= { very poor (l_0), poor(l_1), very weak(l_2), weak(l_3), below average (l_4), average (l_5), above average (l_6), good (l_7), very good (l_8), excellent (l_9), outstanding (l_{10})}. Let $s_{1\{}(x, \langle l_1, l_2, L_5\rangle), (y, \langle l_0, l_4, L_6\rangle), (z, \langle l_4, l_4, l_9\rangle)\}$, $s_{2\{}(x, \langle l_2, l_2, l_4\rangle), (y, \langle l_1, l_4, l_5\rangle), (z, \langle l_7, l_4, l_9\rangle)\}$ be two distinct LNPs. Now, $LNSCl(\{s_1\}) = A_{LN} \subseteq K_{LN}$ and $LNSCl(\{s_2\}) = B_{LN} \subseteq H_{LN}$, where A_{LN}, B_{LN} are LNSCSs and $K_{LN}.H_{LN}$ are LNSOSs given by,
$A_{LN} = \{(x, \langle l_1, l_3, l_4\rangle), (y, \langle l_1, l_6, l_5\rangle), (z, \langle l_5, l_8, l_9\rangle)\}$
$B_{LN} = \{(x, \langle l_4, l_5, l_3\rangle), (y, \langle l_2, l_6, l_1\rangle), (z, \langle l_8, l_5, l_7\rangle)\}$
$K_{LN} = \{(x, \langle l_5, l_6, l_1\rangle), (y, \langle l_4, l_6, l_1\rangle), (z, \langle l_9, l_8, l_5\rangle)\}$
$H_{LN} = \{(x, \langle l_5, l_6, l_1\rangle), (y, \langle l_4, l_7, l_1\rangle), (z, \langle l_9, l_6, l_8\rangle)\}$

Theorem 4.15. *Every LN semi-R_1 space is LN semi-R_0 space.*

Proof: If $S_{LN}, \tau_{LN})$ is LN semi-R_1 and K_{LN} be a LNSO set in S_{LN}. Then for any $s_1 \in S_{LN}$ and $s_2 \in S_{LN} \backslash K_{LN}$, $s_1 \neq s_2$ which implies $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$. We can find two disjoint LNSO sets K_{s_2} and H_{s_2} with $LNSCl(\{s_1\}) \subseteq K_{s_2}$ and $LNSCl(\{s_2\}) \subseteq H_{s_2}$. Let $H_{LN} = \cup\{H_{s_2}/s_2 \in S_{LN} \backslash K_{LN}\}$. If $s_1 \in LNSCl(\{s_1\}) \subseteq K_{s_2}$ and $K_{s_2} \cap H_{s_2} = \phi$ for every $s_2 \in S_{LN} \backslash K_{LN}$ and so $s_1 \notin H_{s_2}$ for each $s_2 \in S_{LN} \backslash K_{LN}$ which implies $s_1 \notin H_{LN}$. Now, $s_1 \in LNSCl(\{s_1\}) \subseteq S_{LN} \backslash H_{LN} K_{LN}$.

Remark 4.16. The reverse implication of the above theorem need not be true unless one condition is satisfied.

Theorem 4.17. *A LN semi-R_0 space is LN semi-R_1, if for each couple of points s_1 and s_2 in S_{LN} satisfying $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$ with we can find two disjoint LNSO sets K_{LN} and H_{LN} so that $s_1 \in K_{LN}$ and $s_2 \in H_{LN}$ respectively.*

Proof: Let the space S_{LN} be LN semi-R_0. Also for each couple of points s_1 and s_2 in S_{LN} satisfying $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$, there lie LNSO sets K_{LN} and H_{LN} so that $s_1 \in K_{LN}$ and $s_2 \in H_{LN}$ with $K_{LN} \cap H_{LN} = \phi$. Then, $LNSCl(\{s_1\}) \subseteq K_{LN}$ and $LNSCl(\{s_1\}) \subseteq H_{LN}$.

Theorem 4.18. *A LNTS is semi-R_1 iff for every couple of points $s_1, s_2 \in S_{LN}$ with $LNSKer(\{s_1\}) \neq LNSKer(\{s_2\})$, there lie LNSO sets K_{LN} and H_{LN} in S_{LN} with $LNSCl(\{s_1\}) \subseteq K_{LN}$ and $LNSCl(\{s_2\}) \subseteq H_{LN}$ and $K_{LN} \cap H_{LN} = \phi$.*

Proof: Let s_1 and s_2 be any two points of $S_{LN}, \tau_{LN})$ with $LNSKer(\{s_1\}) \neq LNSKer(\{s_2\})$. Now we can find disjoint LNSO subsets K_{LN} and H_{LN} with $LNSCl(\{s_1\}) \subseteq K_{LN}$ and $LNSCl(\{s_2\}) \subseteq H_{LN}$, which reveals that S_{LN} is semi-R_0.

Conversely, suppose for each couple of points $s_1, s_2 \in S_{LN}$ with $LNSKer(\{s_1\}) \neq LNSKer(\{s_2\})$, there lie disjoint LNSO subsets K_{LN} and H_{LN} with $LNSCl(\{s_1\}) \subseteq K_{LN}$ and $LNSCl(\{s_2\}) \subseteq H_{LN}$. Assume that $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$. Then, $LNSKer(\{s_1\}) \neq LNSKer(\{s_2\})$.

Theorem 4.19. *For the LNTS $S_{LN}, \tau_{LN})$, the following imply each other.*

(1) (S_{LN}, τ_{LN}) *is LN semi-T_2.*
(2) (S_{LN}, τ_{LN}) *is both LN semi-R_1 and LN semi-T_1.*
(3) (S_{LN}, τ_{LN}) *is both LN semi-R_1 and LN semi-T_0.*

Proof: (1) $\Rightarrow$ (2): Let the space be LN semi-T_2, then the LNTS is LN semi-T_1. If there exist two points $s_1, s_2 \in (S_{LN}, \tau_{LN})$ with $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$, then $s_1 \neq s_2$. Since the points are distinct, there lie LNSO subsets U_{LN} and V_{LN} with $s_1 \in U_{LN}$, $s_2 \in V_{LN}$ and $U_{LN} \cap V_{LN} = \phi$. This implies $\{s_1\} = LNSCl(\{s_1\}) \subseteq U_{LN}$ and $\{s_2\} = LNSCl(\{s_2\}) \subseteq V_{LN}$.

(2) $\Rightarrow$ (3):Proof is direct.

(3) $\Rightarrow$ (1): Suppose the space is both LN semi-R_1 and LN semi-T_0. Then for any two distinct points $s_1, s_2 \in (S_{LN}, \tau_{LN})$, we have $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$. Then we can find LNSO subsets U_{LN} and V_{LN} with $LNSCl(\{s_1\}) \subseteq U_{LN}$ and $LNSCl(\{s_2\}) \subseteq V_{LN}$.

Theorem 4.20. *A LNTS (S_{LN}, τ_{LN}) is semi-R_1 iff for each points $s_1, s_2 \in (S_{LN}, \tau_{LN})$ with $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$, there lie LNSC subsets U_{LN} and V_{LN} with $s_1 \in U_{LN}, s_2 \notin U_{LN}, s_2 \in V_{LN}, s_1 \notin V_{LN}$ and $S_{LN} = U_{LN} \cup V_{LN}$.*

Proof: Let the space (S_{LN}, τ_{LN}) be LN semi-R_1 with $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$. Then we can find LNSO subsets K_1 and K_2 with $LNSCl(\{s_1\}) \subseteq K_1$ and $LNSCl(\{s_2\}) \subseteq K_2$. Then $U_{LN} = S_{LN} \backslash K_2$ and $V_{LN} = S_{LN} \backslash K_1$ which are LNSC subsets with $s_1 \in U_{LN}, s_2 \notin U_{LN}, s_2 \in V_{LN}, s_1 \notin V_{LN}$ and $S_{LN} = U_{LN} \cup V_{LN}$.

Let $LNSCl(\{s_1\}) \neq LNSCl(\{s_2\})$ for any two disjoint points $s_1, s_2 \in S_{LN}$. Now by the hypothesis, there exists LNSC subsets U_{LN} and V_{LN} with $s_1 \in U_{LN}, s_2 \notin U_{LN}, s_2 \in V_{LN}, s_1 \notin V_{LN}$ and $S_{LN} = U_{LN} \cup V_{LN}$. Then $K_1 = S_{LN} \backslash V_{LN}$ and $K_2 = S_{LN} \backslash U_{LN}$, which are LNSO subsets with $s_1 \in KO1$ and $s_2 \in K_2$ and $K_1 \cap K_2 = \phi$. Ergo, the space is LN semi-T_2 and also LN semi-R_1.

5. Linguistic Neutrosophic Semi-D$_0$, Semi-D$_1$, Semi-D$_2$ Spaces

Definition 5.1. Let $A_{LN} = \langle s, (T_{A_{LN}}, I_{A_{LN}}, F_{A_{LN}})\rangle$ and $B_{LN} = \langle s, (T_{B_{LN}}, I_{B_{LN}}, F_{B_{LN}})\rangle$ be LNS's, then $A_{LN}\backslash B_{LN}$ is defined by,
$A_{LN}\backslash B_{LN} = \langle s, min(T_{A_{LN}}, F_{B_{LN}}), min(I_{A_{LN}}.I_{B_{LN}}), max(F_{A_{LN}}, T_{B_{LN}})\rangle$.

Definition 5.2. Let A_{LN} be a LN subset of (S_{LN}, τ_{LN}) is a LN semi-difference set(LNSDS in short) if there exist LNSOSs U_{LN} and V_{LN} such that $U_{LN} \subset S_{LN}$ and $A_{LN} = U_{LN}\backslash V_{LN}$. The collection of all LNSDS's is denoted by $LNSD(S_{LN}, \tau_{LN})$.

Remark 5.3. Every LNSOS A_{LN} that is different from S_{LN} is a LNSDS if $U_{LN} = A_{LN}$ and $V_{LN} = \phi$. The reverse implication need not be true, which is given by a counter example.

Example 5.4. Let the universe of discourse $\mathcal{U}$ and LTS be as in example 4.2. The LN sets $E_{LN} = \langle (x, l_4, l_6, l_4), (y, l_8, l_3, l_4)\rangle$ and $F_{LN} = \langle (x, l_{10}, l_9, l_2), (y, l_5, l_7, l_0)\rangle$ are LNSOSs. Now, the LNDS K_{LN} is $E_{LN}\backslash F_{LN} = \langle (x, l_2, l_6, l_0), (y, l_0, l_3, l_5)\rangle$ which is not a LNSOS because $LNInt(LNCl(K_{LN})) = 1_{LN}$.

Definition 5.5. A LNTS (S_{LN}, π_{LN}) is

(1) LN semi-D_0 if for two distinct points $s_1, s_2 \in S_{LN}$ there lies LNSD set containing one of the point but not the other.

(2) LN semi-D_1 if for two distinct points $s_1, s_2 \in S_{LN}$ there lie LNSD sets U_{LN} and V_{LN} with $s_1 \in U_{LN}, s_2 \notin U_{LN}$ and $s_2 \in V_{LN}, s_1 \notin V_{LN}$.

(3) LN semi-D_2 if for two distinct points $s_1, s_2 \in S_{LN}$ there lie LNSD sets U_{LN} and V_{LN} with $s_1 \in U_{LN}, s_2 \in V_{LN}$ and $U_{LN} \cap V_{LN} = \phi$.

Theorem 5.6. *A LNTS (S_{LN}, τ_{LN}) is LN semi-T_0 iff it is LN semi-D_0.*

Proof: Let the space be LN semi-T_0 and let $S_1 \neq s_2$ for $s_1, s_2 \in S_{LN}$. Then by the definition of semi-T_0, there exists a LNSO set U_{LN} containing one of the points but not the other, (i.e) $s_1 \in U_{LN}$ but $s_2 \notin U_{LN}$. Then $U_{LN} \neq S_{LN}$. By remark above, U_{LN} is a LNSD set containing s_1 but not s_2. Hence, (S_{LN}, τ_{LN}) is LN semi-D_0.
Conversely, let the space be LN semi-D_0. Then by the definition of semi-D_0, there exists a LNSD set A_{LN} containing one of the point but not the other, (i.e)$s_1 \in A_{LN}, s_2 \notin A_{LN}$. Thus, there exists LNSO sets $U_{LN}$and V_{LN} with $U_{LN} \neq S_{LN}$ and $A_{LN} = U_{LN}\backslash V_{LN}$. As $s_1 \in A_{LN}, s_1 \in U_{LN}$ but $s_1 \notin V_{LN}$. For $s_2 \notin A_{LN}$, we have two cases.

(i) $s_2 \notin U_{LN}$ but $s_1 \in U_{LN}$.

(ii) $s_2 \in U_{LN}$ and $s_2 \in V_{LN}$. But $s_1 \notin V_{LN}$.

In both cases, the LNTS is LN semi-T_0.

Theorem 5.7. *A LNTS (S_{LN}, τ_{LN}) is LN semi-D_1 iff it is LN semi-D_2.*

Proof: Let the space be LN semi-D_1, then for each pair of distinct points $s_1, s_2 \in S_{LN}$, we can find LNSD sets K_{LN} and H_{LN} with $s_1 \in K_{LN}, s_2 \notin K_{LN}$ and $s_2 \in H_{LN}, s_2 \notin K_{LN}$. Thus, there lie LNSO sets $A_1, A_2, B_1, B_2 \in S_{LN}$ with $A_1 \neq S_{LN}, A_2 \neq S_{LN}, K_{LN} = A_1 B_1$ and $H_{LN} = A_2 \backslash B_2$. For $s_1 \notin H_{LN}$, the two cases are given below.

Case 1: $s_1 \notin B_2$. As $s_2 \notin K_{LN}$, either $s_2 \notin A_1$ or ($s_2 \in A_1$ and $s_2 \in B_1$). If $s_2 \notin A_1$, from $s_2 \in H_{LN} = A_2 \backslash B_2$, (i.e) $s_2 \in A_2 \backslash (B_2 \cup B_1)$ and $s_1 \notin A_2$. Now, we have $s_1 \in A_1 \backslash (B_1 \cup B_2)$ and $(A_2 \backslash (B_2 \cup B_1)) \cap (A_1 \backslash (B_1 \cup B_2)) = \phi$. If $s_2 \in A_1$ and $s_2 \in B_1$, then we have $s_1 \in K_{LN} = A_1 \backslash B_1$ and $\overline{(A_1 \backslash B_1)} \cap B_1 = \phi$.

Case 2: $s_1 \in A_2$ and $s_2 \in B_2$ Now, $s_2 \in H_{LN} = A_2 \backslash B_2, s_1 \in B_2$ and $(A_2 \backslash B_2) \cap B_2 = \phi$. Since $s_1 \notin B_1$ and $s_2 \notin B_2$, B_1 and B_2 are LNSO sets different from (S_{LN}, τ_{LN}). Then by remark(5.2), B_1 and B_2 are LNSD sets. Since $B_2 \cup A_1$ and $B_1 \cup A_2$ are LNSO sets, we have $A_2 \backslash (B_2 \cup A_1)$ and $A_1 \backslash (B_1 \cup A_2)$ are LNSD sets.

Conversely, suppose the space is LN semi-D_2 and let $s_1, s_2 \in S_{LN}$ with $s_1 \neq s_2$. There lie distinct LNSD sets K_{LN} and H_{LN} with $s_1 \in K_{LN}$ and $s_2 \in H_{LN}$, (i.e) $s_1 \in K_{LN}, s_2 \notin K_{LN}$ and $s_2 \in H_{LN}, s_1 \notin H_{LN}$.

6. Conclusion

Separation axioms and axioms of regularity are outlined in a new space referred to as linguistic neutrosophic topological spaces. A variety of concepts and ideas are explored with suitable examples. In addition, semi-different axioms are introduced and discussed through the use of linguistic neutrosophic semi-difference sets, and numerous intriguing results are obtained.

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Cauchy Single-Valued Neutrosophic Numbers and Their Applications in MAGDM

Faruk Karaaslan[1,*] and Fatih Karamaz[2]

[1] Çankırı Karatekin University, Faculty of Sciences, Department of Mathematics, 18100 Çankırı, TURKEY; karaaslan.faruk@gmail.com

[2]Çankırı Karatekin University, Faculty of Sciences, Department of Mathematics, 18100 Çankırı, TURKEY; karamaz@karamaz.com

*Correspondence: fkaraaslan@karatekin.edu.tr

Abstract. The neutrosophic sets and numbers have an important role in modeling the problems. Recently, studies on neutrosophic numbers and single-valued neutrosophic numbers which is a subclass of neutrosophic numbers have increased, rapidly. Cauchy distribution is an important concept in the statistic. In this paper, the notion of Cauchy single-valued neutrosophic numbers (CSVNNs) and $\alpha-$cuts are introduced based on the Cauchy distribution formula. Summation, multiplication, and division operations between two CSVNNs are defined and given related examples. Also, the score functions of CSVNNs, arithmetic and geometric aggregation operators of them are described. Based on the defined new concepts, a multi-attribute group decision-making method is developed. Finally, to illustrate how the proposed method works, an application of the proposed method in the selection of a project to be supported and funded is developed. In this method, for each of the criteria, different score functions are determined by using the aggregation operators and score functions of the CSVNNs. Then, evaluations of the decision-makers are transformed into new values under the derived score functions for the decision. In applications, in general, decision-makers assign to criteria some values between 0 and 1 directly. In the method proposed in this paper, weights of the criteria are considered as the different functions. Therefore this method presents a more general perspective on decision-making problems.

Keywords: Single-valued neutrosophic set; Cauchy Single-valued neutrosophic number; decision-making; aggregation operators.

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1. Introduction

The fuzzy set theory is a notable theory put forward by Zadeh [38] as a useful tool for decision-making problems and as a generalization of classical sets. After introducing fuzzy sets, many researchers needed to study many generalizations of fuzzy sets in order to model the problems they encountered. The best known of these are intuitionistic fuzzy set defined by Atanassov [1], Pythagorean Fuzzy set introduced by Yager [35], Picture fuzzy set introduced by Cuong [11, 12], q-rung orthopair fuzzy set defined by Yager [36] set, spherical fuzzy sets proposed by Gundogdu and Kahraman [15], T-spherical fuzzy sets introduced by [20] and neutrosophic set (NS) by Smarandache [32]. An NS is described by three mappings defined from a non-empty set to a real standard or non-standard subset of $]^{-}0, 1^{+}[$. These functions are called truth, indeterminacy, and falsity functions and are represented by notations T, I, and F, respectively. The basis of the NS is based on neutrosophy which is a branch of philosophy. Neutrosophic sets have a very important role in modeling and solving decision-making problems. However, real standard or nonstandard subsets of $]^{-}0, 1^{+}[$ are not useful in modeling real-life problems. Therefore, Wang et al. [34] revealed the notion of a single-valued neutrosophic (SVN) set (SVNS) identified by three functions which are defined from a nonempty set into the unit interval $[0, 1]$. Many researcher studied on SVN number (SVNN) [4, 9, 13, 14, 16] and applications in decision-making (DM) based on similarity measures, distance measures, entropy and aggregation operators [2, 3, 5–8, 10, 17, 18, 22–30, 37, 39, 46–48]. In addition, after the definition of hypersoft sets [33] as a generalization of soft sets [21], Martin and Smarandache [19] combined the hypersoft sets of neutrosophic sets and introduced the concept of neutrosophic hypersoft set as a generalization of hypersoft sets. Recently, studies related to the neutrosophic hypersoft sets and hypersoft sets have been rapidly increasing. Some of them are aggregation operators [41], interval-valued neutrosophic hypersoft set [42], corrlation coefficient of interval-valued neutrosophic hypersoft set [43] Pythagorean fuzzy hypersoft set [44], correlation coefficient of neutrosophic hypersoft set [31], neutrosophic hypersoft matrices [45].

In 2018, Karaaslan [16] defined the Gaussian SVNNs and developed a multi-attribute decision-making method under the Gaussian SVN environment. He also presented an application of the proposed method in order to illustrate the progress of the developed method. The following points motivate us to present this paper:

- Single-valued trapezoidal neutrosophic number (SVTrNN), single-valued triangular neutrosophic numbers (SVTNN) and GSVNNs are important tools to model decision-making problems involving indeterminate, and inconsistent data. SVTrNN and SVTNN are expressed by partial functions involving straight line. However, sometimes indeterminate and inconsistent data may not be expressed linearly. Therefore, In order to represent nonlinear states, we introduce a new concept of neutrophic numbers based on the Cauchy distribution.
- In MADMPs, weights of the attributes are determined as real values between 0 and 1 such that their summation is equal to 1. For weights of the attributes, different functions are not considered. In this paper, we consider different score functions for each

attribute according to the common opinions of the decision-makers. Thus, developing a more flexible decision-making approach is aimed.

Following are the contributions of this article:

- The concept of CSVNNs is defined. Also, $\alpha-$cut and arithmetic operations of CSVNNs are introduced, and some results are obtained related to $\alpha-$cut of CSVNNs.
- The score functions of CSVNNs and their aggregation operators are defined.
- Based on novel definitions and operations introduced in this paper, a multi-attribute group decision-making method is proposed and given an illustrative example in order to explain the process of the proposed method.

This paper is organized as follows:In section 2, some basic concepts are recalled. In section 3, The concept of CSVNNs, α-cuts of CSVNNs, arithmetic operations between two CSVNNs, score function of CSVNN, and arithmetic and geometric aggregarin operators of them are defined and given examples of them. In section 4, a MAGDM method is developed and presented an illustrative example to show the working of the proposed method.

2. **Preliminaries**

In this section, some basic definitions related to neutrosophic sets are recalled.

Definition 2.1. [32] Let $\mathbb{X} \neq \emptyset$. Then, a neutrosophic set $\tilde{\mathfrak{A}}$ on $\mathbb{X}$ is a set of quadruplets, defined by

$$\tilde{\mathfrak{A}} = \left\{\langle \theta, \tilde{\mathfrak{A}}_t(\theta), \tilde{\mathfrak{A}}_i(\theta), \tilde{\mathfrak{A}}_f(\theta))\rangle : \theta \in \mathbb{X}\right\}.$$

Here $\tilde{\mathfrak{A}}_t, \tilde{\mathfrak{A}}_i, \tilde{\mathfrak{A}}_f : \mathbb{X} \to]^-0, 1^+[$ called truth, indeterminacy and falsity membership functions (MF) of the neutrosophic set $\tilde{\mathfrak{A}}$, respectively and $^-0 \leq \tilde{\mathfrak{A}}_t(\theta) + \tilde{\mathfrak{A}}_i(\theta) + \tilde{\mathfrak{A}}_f(\theta) \leq 3^+$.

Definition 2.2. [34] Let $\mathbb{X} \neq \emptyset$. Then, a single-valued neutrosophic set $(SVNS)$ $\hat{\mathfrak{A}} = \left\{\langle \theta, \hat{\mathfrak{A}}_t(\theta), \hat{\mathfrak{A}}_i(\theta), \hat{\mathfrak{A}}_f(\theta)\rangle : \theta \in \mathbb{X}\right\}$ is defined as follows:

If $\mathbb{X}$ is continuous, an $SVNS$ $\hat{\mathfrak{A}}$ can be expressed by

$$\hat{\mathfrak{A}} = \int_{\mathbb{X}} \left\langle \hat{\mathfrak{A}}_t(\theta), \hat{\mathfrak{A}}_i(\theta), \hat{\mathfrak{A}}_f(\theta) \right\rangle / \theta, \ \text{ for all } \theta \ \in \mathbb{X}.$$

If $\mathbb{X}$ is crisp set, an $SVNS$ $\hat{\mathfrak{A}}$ can be expressed by

$$\hat{\mathfrak{A}} = \sum_{\theta} \left\langle \hat{\mathfrak{A}}_t(\theta), \hat{\mathfrak{A}}_i(\theta), \hat{\mathfrak{A}}_f(\theta) \right\rangle / \theta, \ \text{ for all } \theta \ \in \mathbb{X}.$$

Note that $0 \leq \hat{\mathfrak{A}}_t(\theta) + \hat{\mathfrak{A}}_i(\theta) + \hat{\mathfrak{A}}_f(\theta) \leq 3$ for all $\theta \in \mathbb{X}$. For convenience, an SVNN is denoted by $\hat{\mathfrak{A}} = \langle \hat{\mathfrak{A}}_t, \hat{\mathfrak{A}}_i, \hat{\mathfrak{A}}_f \rangle$.

3. Cauchy Single-valued Neutrosophic Number

In this part, we define the Cauchy fuzzy number (CFN) and its α -cuts. Then we introduce the concept of Cauchy single-valued number number by similar way.

3.1. *Cauchy Fuzzy Number*

Definition 3.1. [40] A fuzzy number is said to be Cauchy fuzzy number $\mathbb{A} = CFN(\mathfrak{p}, \mathfrak{q})$ whose membership function of is given by

$$\mu_{\mathbb{A}}(\theta) = \frac{1}{1 + \left(\frac{\theta - \mathfrak{p}}{\mathfrak{q}}\right)^2}.$$

Definition 3.2. Let us consider membership function of $\mathbb{A} = CFN(\mathfrak{p}, \mathfrak{q})$ as follows:

$$\mu_{\mathbb{A}}(\theta) = \frac{1}{1 + \left(\frac{\theta - \mathfrak{p}}{\mathfrak{q}}\right)^2}.$$

The α-cut set of $CFN(\mathfrak{p}, \mathfrak{q})$ is defined as follows:

$$\mathbb{A}_\alpha = \left[\mathfrak{p} - \mathfrak{q}\sqrt{\frac{1-\alpha}{\alpha}}, \mathfrak{p} + \mathfrak{q}\sqrt{\frac{1-\alpha}{\alpha}}\right]$$

3.2. *Cauchy Single-valued Neutrosophic Number*

Definition 3.3. A Cauchy single-valued neutrosophic number (CSVNN) is defined by truth, indeterminacy and falsity MFs as follows:

$$\wp(\theta_t) = \frac{1}{1 + \left(\frac{\theta_t - \mathfrak{p}_t}{\mathfrak{q}_t}\right)^2},$$

$$\wp(\theta_i) = 1 - \frac{1}{1 + \left(\frac{\theta_i - \mathfrak{p}_i}{\mathfrak{q}_i}\right)^2} = \frac{\left(\frac{\theta_i - \mathfrak{p}_i}{\mathfrak{q}_i}\right)^2}{1 + \left(\frac{\theta_i - \mathfrak{p}_i}{\mathfrak{q}_i}\right)^2},$$

$$\wp(\theta_f) = 1 - \frac{1}{1 + \left(\frac{\theta_f - \mathfrak{p}_f}{\mathfrak{q}_f}\right)^2} = \frac{\left(\frac{\theta_f - \mathfrak{p}_f}{\mathfrak{q}_f}\right)^2}{1 + \left(\frac{\theta_f - \mathfrak{p}_f}{\mathfrak{q}_f}\right)^2},$$

respectively.

A CSVNN is denoted by $\tilde{\mathbb{A}} = CSVNN\big((\mathfrak{p}_t, \mathfrak{q}_t), (\mathfrak{p}_i, \mathfrak{q}_i), (\mathfrak{p}_f, \mathfrak{q}_f)\big)$. Set of all CSVNNs over $\mathbb{X}$ is denoted by $\mathcal{CSVNN}(\mathbb{X})$.

Example 3.4. Let $\tilde{\mathbb{A}} = CSVNN\big((0.8, 0.1), (0.7, 0.3), (0.5, 0.2)\big)$ be CSVNN. Truth, indeterminacy, and falsity MFs of CSVNN are shown in Fig 1.

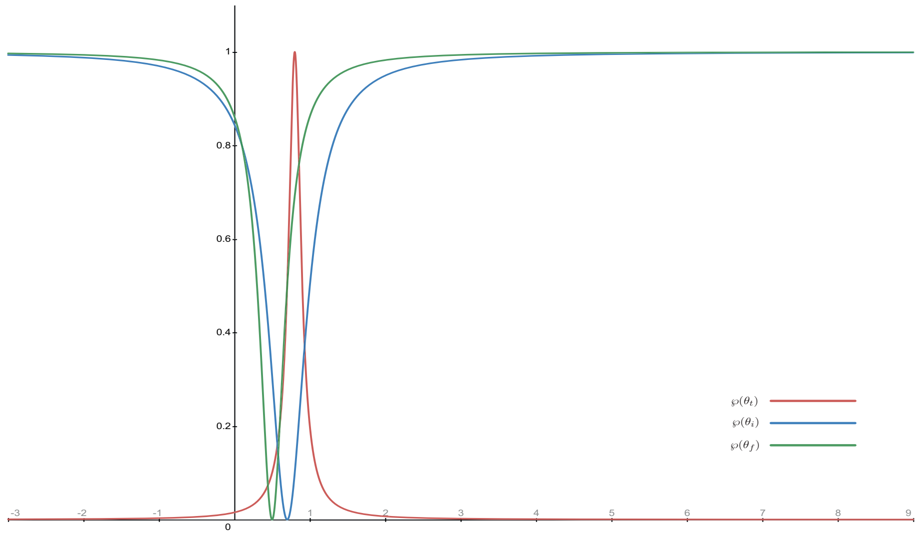

Figure 1. CSVNN $\tilde{\mathbb{A}}$

Definition 3.5. Let truth, indeterminacy and falsity MFs of CSVNN $\tilde{\mathbb{A}}$ be given as follows:

$$\begin{aligned} \wp(\theta_t) &= \frac{1}{1+\left(\dfrac{\theta_t-\mathfrak{p}_t}{\mathfrak{q}_t}\right)^2} \\ \wp(\theta_i) &= 1-\frac{1}{1+\left(\dfrac{\theta_i-\mathfrak{p}_i}{\mathfrak{q}_i}\right)^2} \\ \wp(\theta_f) &= 1-\frac{1}{1+\left(\dfrac{\theta_f-\mathfrak{p}_f}{\mathfrak{q}_f}\right)^2} \end{aligned}$$

respectively.

Then, α-cuts of above functions can be expressed as follows:

$$\begin{aligned} \tilde{\mathbb{A}}_{t_\alpha} &= \left[\mathfrak{p}_t-\mathfrak{q}_t\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_t+\mathfrak{q}_t\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}\right], \\ \tilde{\mathbb{A}}_{i_\alpha} &= \left[\mathfrak{p}_i-\mathfrak{q}_i\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_i+\mathfrak{q}_i\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right], \\ \tilde{\mathbb{A}}_{f_\alpha} &= \left[\mathfrak{p}_f-\mathfrak{q}_f\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_f+\mathfrak{q}_f\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right], \end{aligned}$$

respectively.

3.3. *Arithmetic operations of CSVNNs*

Let $\tilde{\mathbb{A}} = CSVNN\big((\mathfrak{p}_{\tilde{\mathbb{A}}_t}, \mathfrak{q}_{\tilde{\mathbb{A}}_t}), (\mathfrak{p}_{\tilde{\mathbb{A}}_i}, \mathfrak{q}_{\tilde{\mathbb{A}}_i}), (\mathfrak{p}_{\tilde{\mathbb{A}}_f}, \mathfrak{q}_{\tilde{\mathbb{A}}_f})\big)$ and $\tilde{\mathbb{B}} = CSVNN\big((\mathfrak{p}_{\tilde{\mathbb{B}}_t}, \mathfrak{q}_{\tilde{\mathbb{B}}_t}), (\mathfrak{p}_{\tilde{\mathbb{B}}_i}, \mathfrak{q}_{\tilde{\mathbb{B}}_i}), (\mathfrak{p}_{\tilde{\mathbb{B}}_f}, \mathfrak{q}_{\tilde{\mathbb{B}}_f})\big)$ be two CSVNNs. Then, α-cuts ($\alpha \in (0,1]$) of them are as follows:

$$\begin{aligned}
\tilde{\mathbb{A}}_{t_\alpha} &= \left[\mathfrak{p}_{\tilde{\mathbb{A}}_t} - \mathfrak{q}_{\tilde{\mathbb{A}}_t}\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{A}}_t} + \mathfrak{q}_{\tilde{\mathbb{A}}_t}\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}\right], \\
\tilde{\mathbb{A}}_{i_\alpha} &= \left[\mathfrak{p}_{\tilde{\mathbb{A}}_i} - \mathfrak{q}_{\tilde{\mathbb{A}}_i}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{A}}_i} + \mathfrak{q}_{\tilde{\mathbb{A}}_i}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right], \\
\tilde{\mathbb{A}}_{f_\alpha} &= \left[\mathfrak{p}_{\tilde{\mathbb{A}}_f} - \mathfrak{q}_{\tilde{\mathbb{A}}_f}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{A}}_f} + \mathfrak{q}_{\tilde{\mathbb{A}}_f}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right]
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbb{B}}_{t_\alpha} &= \left[\mathfrak{p}_{\tilde{\mathbb{B}}_t} - \mathfrak{q}_{\tilde{\mathbb{B}}_t}\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{B}}_t} + \mathfrak{q}_{\tilde{\mathbb{B}}_t}\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}\right], \\
\tilde{\mathbb{B}}_{i_\alpha} &= \left[\mathfrak{p}_{\tilde{\mathbb{B}}_i} - \mathfrak{q}_{\tilde{\mathbb{B}}_i}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{B}}_i} + \mathfrak{q}_{\tilde{\mathbb{B}}_i}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right], \\
\tilde{\mathbb{B}}_{f_\alpha} &= \left[\mathfrak{p}_{\tilde{\mathbb{B}}_f} - \mathfrak{q}_{\tilde{\mathbb{B}}_f}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{B}}_f} + \mathfrak{q}_{\tilde{\mathbb{B}}_f}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right]
\end{aligned}$$

respectively.

By using α-cuts of CSVNNs $\tilde{\mathbb{A}}$ and $\tilde{\mathbb{B}}$, arithmetic operations between CSVNN $\tilde{\mathbb{A}}$ and CSVNN $\tilde{\mathbb{B}}$ are defined as follows:

(1) **Addition:** By using interval arithmetic, we have

$$\begin{aligned}
\tilde{\mathbb{A}}_{t_\alpha} + \tilde{\mathbb{B}}_{t_\alpha} &= \left[(\mathfrak{p}_{\tilde{\mathbb{A}}_t} + \mathfrak{p}_{\tilde{\mathbb{B}}_t}) - (\mathfrak{q}_{\tilde{\mathbb{A}}_t} + \mathfrak{q}_{\tilde{\mathbb{B}}_t})\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}, (\mathfrak{p}_{\tilde{\mathbb{A}}_t} + \mathfrak{p}_{\tilde{\mathbb{B}}_t}) + (\mathfrak{q}_{\tilde{\mathbb{A}}_t} + \mathfrak{q}_{\tilde{\mathbb{B}}_t})\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}\right], \\
\tilde{\mathbb{A}}_{i_\alpha} + \tilde{\mathbb{B}}_{i_\alpha} &= \left[(\mathfrak{p}_{\tilde{\mathbb{A}}_i} + \mathfrak{p}_{\tilde{\mathbb{B}}_i}) - (\mathfrak{q}_{\tilde{\mathbb{A}}_i} + \mathfrak{q}_{\tilde{\mathbb{B}}_i})\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, (\mathfrak{p}_{\tilde{\mathbb{A}}_i} + \mathfrak{p}_{\tilde{\mathbb{B}}_i}) + (\mathfrak{q}_{\tilde{\mathbb{A}}_i} + \mathfrak{q}_{\tilde{\mathbb{B}}_i})\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right], \\
\tilde{\mathbb{A}}_{f_\alpha} + \tilde{\mathbb{B}}_{f_\alpha} &= \left[(\mathfrak{p}_{\tilde{\mathbb{A}}_f} + \mathfrak{p}_{\tilde{\mathbb{B}}_f}) - (\mathfrak{q}_{\tilde{\mathbb{A}}_f} + \mathfrak{q}_{\tilde{\mathbb{B}}_f})\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, (\mathfrak{p}_{\tilde{\mathbb{A}}_f} + \mathfrak{p}_{\tilde{\mathbb{B}}_f}) + (\mathfrak{q}_{\tilde{\mathbb{A}}_f} + \mathfrak{q}_{\tilde{\mathbb{B}}_f})\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right].
\end{aligned}$$

Truth, indeterminacy, and falsity MFs of $\tilde{\mathbb{A}}+\tilde{\mathbb{B}}$ can be expressed as follows:

$$\wp_{(\tilde{\mathbb{A}}+\tilde{\mathbb{B}})}(\theta_t) = \frac{1}{1+\left(\frac{\theta_t-(\mathfrak{p}_{\tilde{\mathbb{A}}t}+\mathfrak{p}_{\tilde{\mathbb{B}}t})}{(\mathfrak{q}_{\tilde{\mathbb{A}}t}+\mathfrak{q}_{\tilde{\mathbb{B}}t})}\right)^2}$$

$$\wp_{(\tilde{\mathbb{A}}+\tilde{\mathbb{B}})}(\theta_i) = 1-\frac{1}{1+\left(\frac{\theta_i-(\mathfrak{p}_{\tilde{\mathbb{A}}i}+\mathfrak{p}_{\tilde{\mathbb{B}}i})}{(\mathfrak{q}_{\tilde{\mathbb{A}}i}+\mathfrak{q}_{\tilde{\mathbb{B}}i})}\right)^2}$$

$$\wp_{(\tilde{\mathbb{A}}+\tilde{\mathbb{B}})}(\theta_f) = 1-\frac{1}{1+\left(\frac{\theta_f-(\mathfrak{p}_{\tilde{\mathbb{A}}f}+\mathfrak{p}_{\tilde{\mathbb{B}}f})}{(\mathfrak{q}_{\tilde{\mathbb{A}}f}+\mathfrak{q}_{\tilde{\mathbb{B}}f})}\right)^2}$$

(2) **Substraction:** By using interval arithmetic, we have

$$\tilde{\mathbb{A}}_{t_\alpha}-\tilde{\mathbb{B}}_{t_\alpha} = \left[(\mathfrak{p}_{\tilde{\mathbb{A}}_t}-\mathfrak{p}_{\tilde{\mathbb{B}}_t})-(\mathfrak{q}_{\tilde{\mathbb{A}}_t}-\mathfrak{q}_{\tilde{\mathbb{B}}_t})\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}, (\mathfrak{p}_{\tilde{\mathbb{A}}_t}-\mathfrak{p}_{\tilde{\mathbb{B}}_t})+(\mathfrak{q}_{\tilde{\mathbb{A}}_t}-\mathfrak{q}_{\tilde{\mathbb{B}}_t})\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}\right],$$

$$\tilde{\mathbb{A}}_{i_\alpha}-\tilde{\mathbb{B}}_{i_\alpha} = \left[(\mathfrak{p}_{\tilde{\mathbb{A}}_i}-\mathfrak{p}_{\tilde{\mathbb{B}}_i})-(\mathfrak{q}_{\tilde{\mathbb{A}}_i}-\mathfrak{q}_{\tilde{\mathbb{B}}_i})\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, (\mathfrak{p}_{\tilde{\mathbb{A}}_i}-\mathfrak{p}_{\tilde{\mathbb{B}}_i})+(\mathfrak{q}_{\tilde{\mathbb{A}}_i}-\mathfrak{q}_{\tilde{\mathbb{B}}_i})\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right],$$

$$\tilde{\mathbb{A}}_{f_\alpha}-\tilde{\mathbb{B}}_{f_\alpha} = \left[(\mathfrak{p}_{\tilde{\mathbb{A}}_f}-\mathfrak{p}_{\tilde{\mathbb{B}}_f})-(\mathfrak{q}_{\tilde{\mathbb{A}}_f}-\mathfrak{q}_{\tilde{\mathbb{B}}_f})\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, (\mathfrak{p}_{\tilde{\mathbb{A}}_f}-\mathfrak{p}_{\tilde{\mathbb{B}}_f})+(\mathfrak{q}_{\tilde{\mathbb{A}}_f}-\mathfrak{q}_{\tilde{\mathbb{B}}_f})\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right].$$

Truth, indeterminacy, and falsity MFs of CSVNN $\tilde{\mathbb{A}}-\tilde{\mathbb{B}}$ can be expressed as follows:

$$\wp_{(\tilde{\mathbb{A}}-\tilde{\mathbb{B}})}(\theta_t) = \frac{1}{1+\left(\frac{\theta_t-(\mathfrak{p}_{\tilde{\mathbb{A}_t}}-\mathfrak{p}_{\tilde{\mathbb{B}_t}})}{(\mathfrak{q}_{\tilde{\mathbb{A}_t}}-\mathfrak{q}_{\tilde{\mathbb{B}_t}})}\right)^2},$$

$$\wp_{(\tilde{\mathbb{A}}-\tilde{\mathbb{B}})}(\theta_i) = 1-\frac{1}{1+\left(\frac{\theta_i-(\mathfrak{p}_{\tilde{\mathbb{A}_i}}-\mathfrak{p}_{\mathbb{B}_i})}{(\mathfrak{q}_{\tilde{\mathbb{A}_i}}-\mathfrak{q}_{\tilde{\mathbb{B}_i}})}\right)^2},$$

$$\wp_{(\tilde{\mathbb{A}}-\tilde{\mathbb{B}})}(\theta_f) = 1-\frac{1}{1+\left(\frac{\theta_f-(\mathfrak{p}_{\tilde{\mathbb{A}_f}}-\mathfrak{p}_{\tilde{\mathbb{B}_f}})}{(\mathfrak{q}_{\tilde{\mathbb{A}_f}}-\mathfrak{q}_{\tilde{\mathbb{B}_f}})}\right)^2},$$

respectively.

(3) **Multiplication:** Let

$$\tilde{\mathbb{A}}_{t_\alpha} = \left[\tilde{\mathbb{A}}^L_{t_\alpha}, \tilde{\mathbb{A}}^U_{t_\alpha}\right] = \left[\mathfrak{p}_{\tilde{\mathbb{A}}_t}-\mathfrak{q}_{\tilde{\mathbb{A}}_t}\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{A}}_t}+\mathfrak{q}_{\tilde{\mathbb{A}}_t}\left(\tfrac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}\right],$$
$$\tilde{\mathbb{A}}_{i_\alpha} = \left[\tilde{\mathbb{A}}^L_{i_\alpha}, \tilde{\mathbb{A}}^U_{i_\alpha}\right] = \left[\mathfrak{p}_{\tilde{\mathbb{A}}_i}-\mathfrak{q}_{\tilde{\mathbb{A}}_i}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{A}}_i}+\mathfrak{q}_{\tilde{\mathbb{A}}_i}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right],$$
$$\tilde{\mathbb{A}}_{f_\alpha} = \left[\tilde{\mathbb{A}}^L_{f_\alpha}, \tilde{\mathbb{A}}^U_{f_\alpha}\right] = \left[\mathfrak{p}_{\tilde{\mathbb{A}}_f}-\mathfrak{q}_{\tilde{\mathbb{A}}_f}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{A}}_f}+\mathfrak{q}_{\tilde{\mathbb{A}}_f}\left(\tfrac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right],$$

and

$$\begin{aligned}\tilde{\mathbb{B}}_{t_\alpha} &= \left[\tilde{\mathbb{B}}^L_{t_\alpha}, \tilde{\mathbb{B}}^U_{t_\alpha}\right] = \left[\mathfrak{p}_{\tilde{\mathbb{B}}_t} - \mathfrak{q}_{\tilde{\mathbb{B}}_t}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{B}}_t} + \mathfrak{q}_{\tilde{\mathbb{B}}_t}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}\right],\\ \tilde{\mathbb{B}}_{i_\alpha} &= \left[\tilde{\mathbb{B}}^L_{i_\alpha}, \tilde{\mathbb{B}}^U_{i_\alpha}\right] = \left[\mathfrak{p}_{\tilde{\mathbb{B}}_i} - \mathfrak{q}_{\tilde{\mathbb{B}}_i}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{B}}_i} + \mathfrak{q}_{\tilde{\mathbb{B}}_i}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right],\\ \tilde{\mathbb{B}}_{f_\alpha} &= \left[\tilde{\mathbb{B}}^L_{f_\alpha}, \tilde{\mathbb{B}}^U_{f_\alpha}\right] = \left[\mathfrak{p}_{\tilde{\mathbb{B}}_f} - \mathfrak{q}_{\tilde{\mathbb{B}}_f}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}, \mathfrak{p}_{\tilde{\mathbb{B}}_f} + \mathfrak{q}_{\tilde{\mathbb{B}}_f}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right],\end{aligned}$$

Then,

$$\begin{aligned}\tilde{\mathbb{A}}_{t_\alpha}\tilde{\mathbb{B}}_{t_\alpha} &= \left[\min\{\tilde{\mathbb{A}}^L_{t_\alpha}\tilde{\mathbb{B}}^L_{t_\alpha}, \tilde{\mathbb{A}}^L_{t_\alpha}\tilde{\mathbb{B}}^U_{t_\alpha}, \tilde{\mathbb{A}}^U_{t_\alpha}\tilde{\mathbb{B}}^L_{t_\alpha}, \tilde{\mathbb{A}}^U_{t_\alpha}\tilde{\mathbb{A}}^U_{t_\alpha}\}, \max\{\tilde{\mathbb{A}}^L_{t_\alpha}\tilde{\mathbb{B}}^L_{t_\alpha}, \tilde{\mathbb{A}}^L_{t_\alpha}\tilde{\mathbb{B}}^U_{t_\alpha}, \tilde{\mathbb{A}}^U_{t_\alpha}\tilde{\mathbb{B}}^L_{t_\alpha}, \tilde{\mathbb{A}}^U_{t_\alpha}\tilde{\mathbb{A}}^U_{t_\alpha}\}\right],\\ \tilde{\mathbb{A}}_{i_\alpha}\tilde{\mathbb{B}}_{i_\alpha} &= \left[\min\{\tilde{\mathbb{A}}^L_{i_\alpha}\tilde{\mathbb{B}}^L_{i_\alpha}, \tilde{\mathbb{A}}^L_{i_\alpha}\tilde{\mathbb{B}}^U_{i_\alpha}, \tilde{\mathbb{A}}^U_{i_\alpha}\tilde{\mathbb{B}}^L_{i_\alpha}, \tilde{\mathbb{A}}^U_{i_\alpha}\tilde{\mathbb{A}}^U_{i_\alpha}\}, \max\{\tilde{\mathbb{A}}^L_{i_\alpha}\tilde{\mathbb{B}}^L_{i_\alpha}, \tilde{\mathbb{A}}^L_{i_\alpha}\tilde{\mathbb{B}}^U_{i_\alpha}, \tilde{\mathbb{A}}^U_{i_\alpha}\tilde{\mathbb{B}}^L_{i_\alpha}, \tilde{\mathbb{A}}^U_{i_\alpha}\tilde{\mathbb{A}}^U_{i_\alpha}\}\right],\\ \tilde{\mathbb{A}}_{f_\alpha}\tilde{\mathbb{B}}_{f_\alpha} &= \left[\min\{\tilde{\mathbb{A}}^L_{f_\alpha}\tilde{\mathbb{B}}^L_{f_\alpha}, \tilde{\mathbb{A}}^L_{f_\alpha}\tilde{\mathbb{B}}^U_{f_\alpha}, \tilde{\mathbb{A}}^U_{f_\alpha}\tilde{\mathbb{B}}^L_{f_\alpha}, \tilde{\mathbb{A}}^U_{f_\alpha}\tilde{\mathbb{A}}^U_{f_\alpha}\}, \max\{\tilde{\mathbb{A}}^L_{f_\alpha}\tilde{\mathbb{B}}^L_{f_\alpha}, \tilde{\mathbb{A}}^L_{f_\alpha}\tilde{\mathbb{B}}^U_{f_\alpha}, \tilde{\mathbb{A}}^U_{f_\alpha}\tilde{\mathbb{B}}^L_{f_\alpha}, \tilde{\mathbb{A}}^U_{f_\alpha}\tilde{\mathbb{A}}^U_{f_\alpha}\}\right].\end{aligned}$$

If $\tilde{\mathbb{A}}_{t_\alpha}, \tilde{\mathbb{A}}_{i_\alpha}, \tilde{\mathbb{A}}_{f_\alpha}, \tilde{\mathbb{B}}_{t_\alpha}, \tilde{\mathbb{B}}_{i_\alpha}$ and $\tilde{\mathbb{B}}_{f_\alpha} \subseteq \mathbb{R}^+$, then we can directly write it as follows:

$$\begin{aligned}\tilde{\mathbb{A}}_{t_\alpha}\tilde{\mathbb{B}}_{t_\alpha} &= \left[\left(\mathfrak{p}_{\tilde{\mathbb{A}}_t} - \mathfrak{q}_{\tilde{\mathbb{A}}_t}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}\right).\left(\mathfrak{p}_{\tilde{\mathbb{B}}_t} - \mathfrak{q}_{\tilde{\mathbb{B}}_t}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}\right), \left(\mathfrak{p}_{\tilde{\mathbb{A}}_t} + \mathfrak{q}_{\tilde{\mathbb{A}}_t}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}\right).\left(\mathfrak{p}_{\tilde{\mathbb{B}}_t} + \mathfrak{q}_{\tilde{\mathbb{B}}_t}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{2}}\right)\right],\\ \tilde{\mathbb{A}}_{i_\alpha}\tilde{\mathbb{B}}_{i_\alpha} &= \left[\left(\mathfrak{p}_{\tilde{\mathbb{A}}_i} - \mathfrak{q}_{\tilde{\mathbb{A}}_i}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right).\left(\mathfrak{p}_{\tilde{\mathbb{B}}_i} - \mathfrak{q}_{\tilde{\mathbb{B}}_i}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right), \left(\mathfrak{p}_{\tilde{\mathbb{A}}_i} + \mathfrak{q}_{\tilde{\mathbb{A}}_i})\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right)\left(\mathfrak{p}_{\tilde{\mathbb{B}}_i} + \mathfrak{q}_{\tilde{\mathbb{B}}_i}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right)\right],\\ \tilde{\mathbb{A}}_{f_\alpha}\tilde{\mathbb{B}}_{f_\alpha} &= \left[\left(\mathfrak{p}_{\tilde{\mathbb{A}}_f} - \mathfrak{q}_{\tilde{\mathbb{A}}_i}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right).\left(\mathfrak{p}_{\tilde{\mathbb{B}}_f} - \mathfrak{q}_{\tilde{\mathbb{B}}_f}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right), \left(\mathfrak{p}_{\tilde{\mathbb{A}}_f} + \mathfrak{q}_{\tilde{\mathbb{A}}_f})\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right)\left(\mathfrak{p}_{\tilde{\mathbb{B}}_f} + \mathfrak{q}_{\tilde{\mathbb{B}}_f}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}}\right)\right],\end{aligned}$$

(4) **Division:**

$$\begin{aligned}\frac{\tilde{\mathbb{A}}_{t\alpha}}{\tilde{\mathbb{B}}_{t\alpha}} &= \left[\frac{(\mathfrak{p}_{\tilde{\mathbb{A}_t}} - \mathfrak{q}_{\tilde{\mathbb{A}_t}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{2}})}{(\mathfrak{p}_{\tilde{\mathbb{B}_t}} - \mathfrak{q}_{\tilde{\mathbb{B}_t}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{2}})}, \frac{(\mathfrak{p}_{\tilde{\mathbb{A}_t}} + \mathfrak{q}_{\tilde{\mathbb{A}_t}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{2}})}{((\mathfrak{p}_{\tilde{\mathbb{B}_t}} + \mathfrak{q}_{\tilde{\mathbb{B}_t}})\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{2}})}\right],\ 0 \notin \tilde{\mathbb{B}}_{t\alpha}\\ \frac{\tilde{\mathbb{A}}_{i\alpha}}{\tilde{\mathbb{B}}_{i\alpha}} &= \left[\frac{(\mathfrak{p}_{\tilde{\mathbb{A}_i}} - \mathfrak{q}_{\tilde{\mathbb{A}_i}}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}})}{(\mathfrak{p}_{\tilde{\mathbb{B}_i}} - \mathfrak{q}_{\tilde{\mathbb{B}_i}}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}})}, \frac{(\mathfrak{p}_{\tilde{\mathbb{A}_i}} + \mathfrak{q}_{\tilde{\mathbb{A}_i}}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}})}{((\mathfrak{p}_{\tilde{\mathbb{B}_i}} + \mathfrak{q}_{\tilde{\mathbb{B}_i}})\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}})}\right],\ 0 \notin \tilde{\mathbb{B}}_{i\alpha}\\ \frac{\tilde{\mathbb{A}}_{f\alpha}}{\tilde{\mathbb{B}}_{f\alpha}} &= \left[\frac{(\mathfrak{p}_{\tilde{\mathbb{A}_f}} - \mathfrak{q}_{\tilde{\mathbb{A}_f}}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}})}{(\mathfrak{p}_{\tilde{\mathbb{B}_f}} - \mathfrak{q}_{\tilde{\mathbb{B}_f}}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}})}, \frac{(\mathfrak{p}_{\tilde{\mathbb{A}_f}} + \mathfrak{q}_{\tilde{\mathbb{A}_f}}\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}})}{((\mathfrak{p}_{\tilde{\mathbb{B}_f}} + \mathfrak{q}_{\tilde{\mathbb{B}_f}})\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2}})}\right],\ 0 \notin \tilde{\mathbb{B}}_{f\alpha}\end{aligned}$$

Example 3.6. Let us consider CSVNNs $\tilde{\mathbb{A}} = CSVNN((0.42, 0.68), (0.54, 0.32), (0.69, 0.21))$ and $\tilde{\mathbb{B}} = CSVNN((0.73, 0.32), (0.64, 0.23), (0.57, 0.41))$. The graphics of $\tilde{\mathbb{A}}$ and $\tilde{\mathbb{B}}$ are depicted in Figures (2) and (3).

Truth, indeterminacy, and falsity MFs of CSVNNs $\tilde{\mathbb{A}} + \tilde{\mathbb{B}}$ and $\tilde{\mathbb{A}} - \tilde{\mathbb{B}}$ are obtained as follows:

$$\wp_{(\tilde{\mathbb{A}}+\tilde{\mathbb{B}})}(\theta_t) = \frac{1}{1 + \left(\frac{\theta_t - 1.15}{1.0}\right)^2},$$

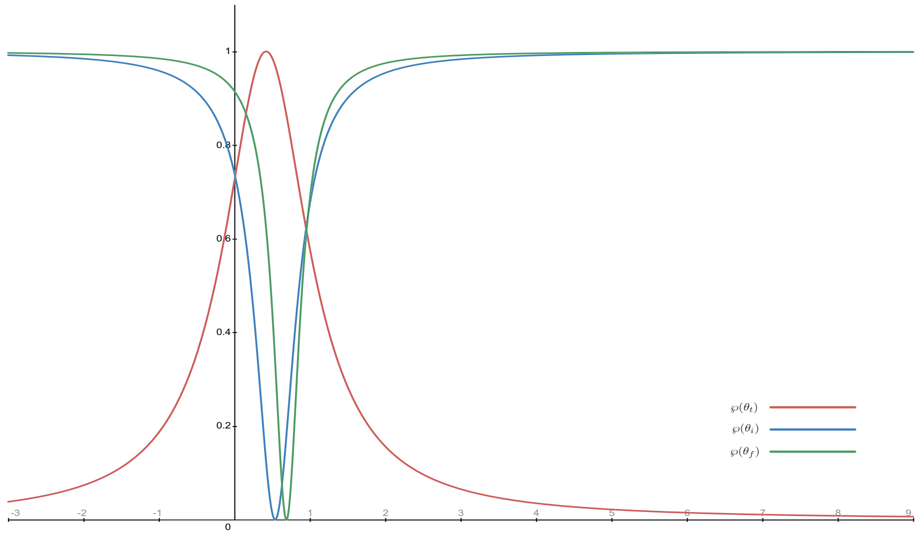

FIGURE 2. CSVNN $\tilde{\mathbb{A}}$

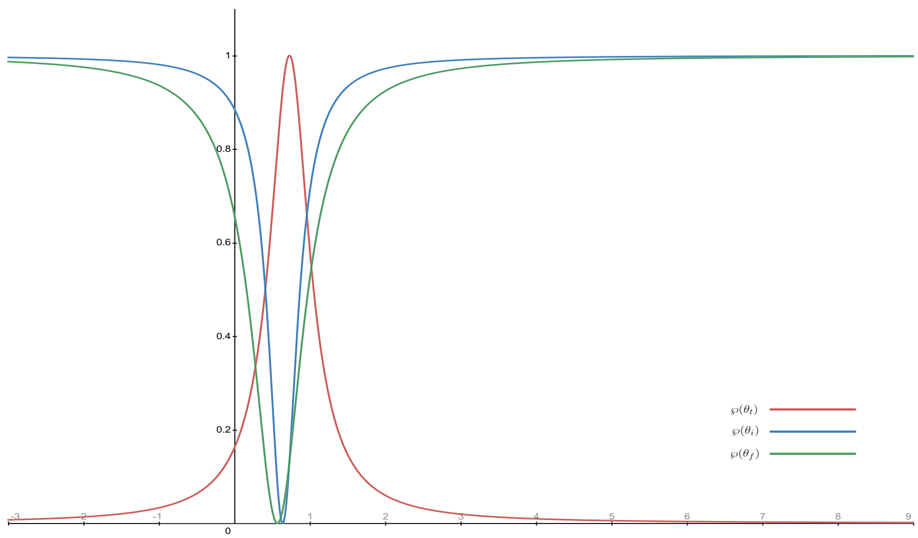

FIGURE 3. CSVNN $\tilde{\mathbb{B}}$

$$\wp_{(\tilde{\mathbb{A}}+\tilde{\mathbb{B}})}(\theta_i) = 1 - \frac{1}{1+\left(\frac{\theta_i - 1.18}{0.55}\right)^2},$$

$$\wp_{(\tilde{\mathbb{A}}+\tilde{\mathbb{B}})}(\theta_f) = 1 - \frac{1}{1+\left(\frac{\theta_f - 1.26}{0.62}\right)^2},$$

and

$$\wp_{(\tilde{\mathbb{A}}-\tilde{\mathbb{B}})}(\theta_t) = \frac{1}{1+\left(\frac{\theta_t - (-0.31)}{0.36}\right)^2},$$

$$\wp_{(\tilde{\mathbb{A}}-\tilde{\mathbb{B}})}(\theta_i) = 1 - \frac{1}{1+\left(\frac{\theta_i-(-0.1)}{0.09}\right)^2},$$

$$\wp_{(\tilde{\mathbb{A}}-\tilde{\mathbb{B}})}(\theta_f) = 1 - \frac{1}{1+\left(\frac{\theta_f-0.12}{(-0.2)}\right)^2}.$$

Figures (4) and (5) show the graphical representations of CSVNNs $\tilde{\mathbb{A}}+\tilde{\mathbb{B}}$ and $\tilde{\mathbb{A}}-\tilde{\mathbb{B}}$.

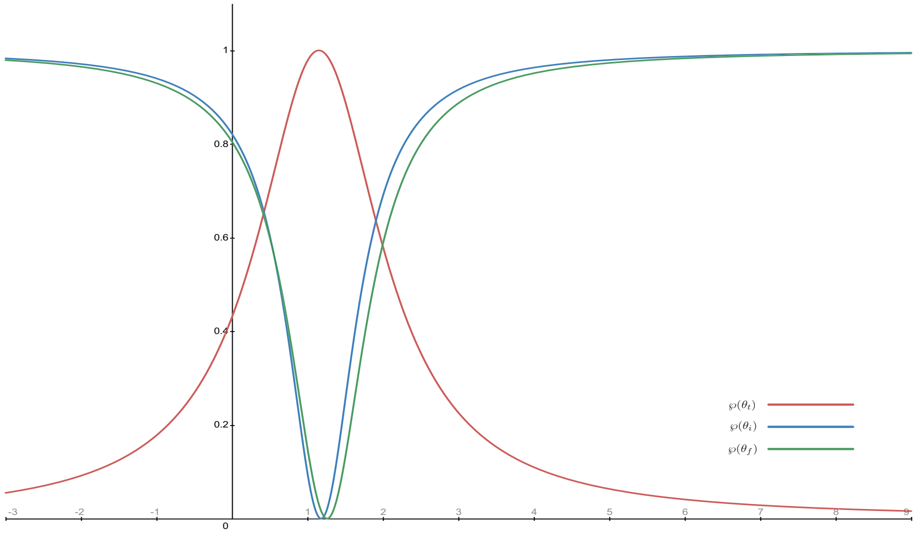

FIGURE 4. CSVNN $\tilde{\mathbb{A}}+\tilde{\mathbb{B}}$

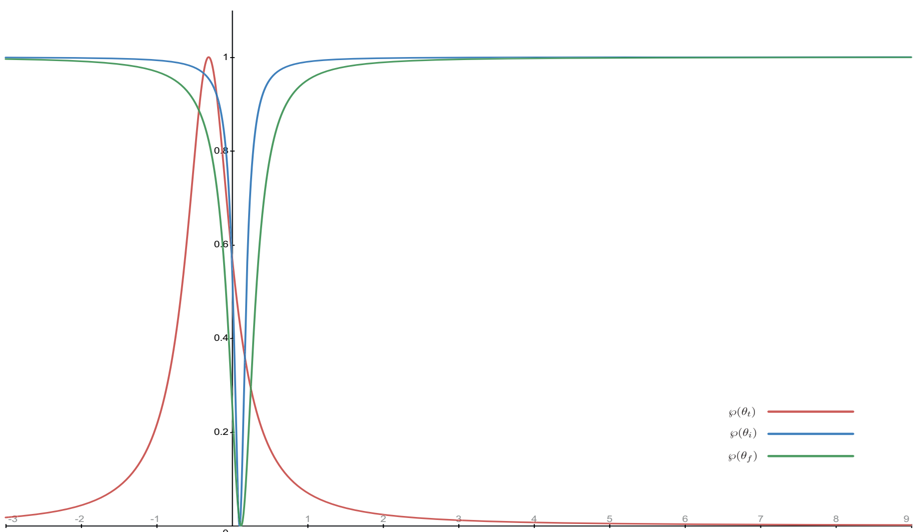

FIGURE 5. CSVNN $\tilde{\mathbb{A}}-\tilde{\mathbb{B}}$

Definition 3.7. Let $\tilde{\mathbb{A}} = ((\mathfrak{p}_t,\mathfrak{q}_t),(\mathfrak{p}_i,\mathfrak{q}_i),(\mathfrak{p}_f,\mathfrak{q}_f))$ be CSVNN. Then, score function of CSVNN $\tilde{\mathbb{A}}$, denoted by $S(\tilde{\mathbb{A}})$, is defined as follows:

$$S(\tilde{\mathbb{A}}) = \frac{1}{3}\Big(\frac{\mathfrak{q}_t^2}{\mathfrak{q}_t^2 + (\theta_t - \mathfrak{p}_t)^2} + \frac{\mathfrak{q}_i^2}{\mathfrak{q}_i^2 + (\theta_i - \mathfrak{p}_i)^2} + \frac{\mathfrak{q}_f^2}{\mathfrak{q}_f^2 + (\theta_f - \mathfrak{p}_f)^2}\Big)$$

Note that score functions are functions depending on neutrosophic variables $\langle\theta_t, \theta_i, \theta_f\rangle$. Furthermore, score functions of CSVNNs can be changed according to CSVNN.

Example 3.8. Let us consider CSVNNs $\tilde{\mathbb{A}} = ((0.5, 0.3), (0.7, 0.2), (0.3, 0.5))$ and $\tilde{\mathbb{B}} = ((0.6, 0.4), (0.2, 0.6), (0.5, 0.7))$. Then, score functions of CSVNNs $\tilde{\mathbb{A}}$ and $\tilde{\mathbb{B}}$ are obtained as follows:

$$S(\tilde{\mathbb{A}}) = \frac{1}{3}\Big(\frac{0.09}{0.09 + (\theta_t - 0.5)^2} + \frac{0.04}{0.04 + (\theta_i - 0.7)^2} + \frac{0.25}{0.25 + (\theta_t - 0.3)^2}\Big)$$

and

$$S(\tilde{\mathbb{B}}) = \frac{1}{3}\Big(\frac{0.16}{0.16 + (\theta_t - 0.6)^2} + \frac{0.36}{0.36 + (\theta_i - 0.4)^2} + \frac{0.49}{0.49 + (\theta_t - 0.5)^2}\Big).$$

If we consider SVN value $\langle 0.5, 0.6, 0.7\rangle$, then score values of this SVN according to score functions of $\tilde{\mathbb{A}}$ and $\tilde{\mathbb{B}}$ are obtained as follows:

$$S(\tilde{\mathbb{A}})(\langle 0.5, 0.6, 0.7\rangle) = 0.803 \; and \; S(\tilde{\mathbb{B}})(\langle 0.5, 0.6, 0.7\rangle) = 0.923$$

Definition 3.9. Let $\mathbb{X}$ be a nonempty set and $\sum^n = \Big\{\Psi_k = \Big((\mathfrak{p}_{t_k}, \mathfrak{q}_{t_k}), (\mathfrak{p}_{i_k}, \mathfrak{q}_{i_k}), (\mathfrak{p}_{f_k}, \mathfrak{q}_{f_k})\Big) : (k = 1, 2, ..., n)\Big\}$ be a set of CSVNNs of which weight vector $\varsigma = (\varsigma_1, \varsigma_2, ..., \varsigma_n)^T$ such that $\varsigma_k > 0$, $\sum_{k=1}^{u} \varsigma_k = 1$. Then, CSVN weighted arithmetic aggregation (CSVNWAA) operator is defined by a mapping $CSVNWAA : \sum^n \to \mathcal{CSVNN}(\mathbb{X})$, where

$$CSVNWAA(\Psi_1, \Psi_2, ..., \Psi_n) = \bigoplus_{k=1}^{n} \varsigma_k \Psi_k.$$

Theorem 3.10. *Let $\mathbb{X}$ be a nonempty set and $\sum^n = \Big\{\Psi_k = \Big((\mathfrak{p}_{t_k}, \mathfrak{q}_{t_k}), (\mathfrak{p}_{i_k}, \mathfrak{q}_{i_k}), (\mathfrak{p}_{f_k}, \mathfrak{q}_{f_k})\Big) : (k = 1, 2, ..., n)\Big\}$ be a set of CSNNs of which weight vector $\varsigma = (\varsigma_1, \varsigma_2, ..., \varsigma_n)^T$ such that $\varsigma_k > 0$, $\sum_{k=1}^{u} \varsigma_k = 1$.*
Then, aggregated value of set using CSNNWAA is a CSVNN defined as follows:

$$\begin{aligned} CSNNWAA(\Psi_1, \Psi_2, ..., \Psi_n) &= \bigoplus_{k=1}^{n} (\varsigma_k \Psi_k) \\ &= \Big((1 - \prod_{k=1}^{n}(1 - \tilde{\mathfrak{p}}_{t_k})^{\varsigma_k}, 1 - \prod_{k=1}^{n}(1 - \tilde{\mathfrak{q}}_{t_k})^{\varsigma_k}), \\ &\quad (\prod_{k=1}^{n}(\tilde{\mathfrak{p}}_{i_k})^{\varsigma_k}, \prod_{k=1}^{n}(\tilde{\mathfrak{q}}_{i_k})^{\varsigma_k}), (\prod_{k=1}^{n}(\tilde{\mathfrak{p}}_{f_k})^{\varsigma_k}, \prod_{k=1}^{n}(\tilde{\mathfrak{p}}_{f_k})^{\varsigma_k})\Big). \end{aligned} \quad (1)$$

Here there are two cases for $\mathfrak{p}_{\Delta_k}$ and $\mathfrak{q}_{\Delta_k}$ $(\Delta \in \{t,i,f\})$ and $(k = 1,2,...,n)$. If one of $\mathfrak{p}_{\Delta_k}$ and $\mathfrak{q}_{\Delta_k}$ is greater then 1, then the following formula are used $\tilde{\mathfrak{p}}_{\Delta_k} = \frac{\mathfrak{p}_{\Delta_k}}{\sqrt{\sum_{k=1}^{n}\mathfrak{p}_{\Delta_k}^2}}$, $\tilde{\mathfrak{q}}_{\Delta_k} = \frac{\mathfrak{p}_{\Delta_k}}{\sqrt{\sum_{k=1}^{n}\mathfrak{p}_{\Delta_k}^2}}$, If $\mathfrak{p}_{\Delta_k}$ and $\mathfrak{q}_{\Delta_k}$ $(\Delta \in \{t,i,f\})$ $(k = 1,2,...,n)$ are in interval $[0,1]$, then $\mathfrak{p}_{\Delta_k}$ and $\mathfrak{q}_{\Delta_k}$ $(\Delta \in \{t,i,f\})$ and $(k = 1,2,...,n)$ are used directly in $CSNNWAA(\Psi_1,\Psi_2,...,\Psi_n)$ and other aggregation operations defined in the next .

Proof. The proof can be easily made based on aggregation operations of the SVNNs. Therefore, it is omitted. □

Definition 3.11. Let $\mathbb{X}$ be a nonempty set and $\sum^n = \Big\{\Psi_k = \Big((\mathfrak{p}_{t_k},\mathfrak{q}_{t_k}),(\mathfrak{p}_{i_k},\mathfrak{q}_{i_k}),(\mathfrak{p}_{f_k},\mathfrak{q}_{f_k})\Big) : (k = 1,2,...,n)\Big\}$ be a set of CSVNNs of which weight vector $\varsigma = (\varsigma_1,\varsigma_2,...,\varsigma_n)^T$ such that $\varsigma_k > 0$, $\sum_{k=1}^{u}\varsigma_k = 1$. Then, CSVN weighted geometric aggregation operator (CSNNWGA) operator is defined by a mapping $CSVNWGA : \sum^n \to \mathcal{CSVNN}(\mathbb{X})$, where

$$CSVNWGA(\Psi_1,\Psi_2,...,\Psi_n) = \bigotimes_{k=1}^{n}\Psi_k^{\varsigma_k}$$

Theorem 3.12. *Let $\mathbb{X}$ be a universe and $\sum^n = \Big\{\Psi_k = \Big((\mathfrak{p}_{t_k},\mathfrak{q}_{t_k}),(\mathfrak{p}_{i_k},\mathfrak{q}_{i_k}),(\mathfrak{p}_{f_k},\mathfrak{q}_{f_k})\Big) : (k = 1,2,...,n)\Big\}$ be a collection of CSNNs of which weight vector $\varsigma = (\varsigma_1,\varsigma_2,...,\varsigma_n)^T$ such that $\varsigma_k > 0$, $\sum_{k=1}^{u}\varsigma_k = 1$.*
Then, aggregated value of set using CSVNWGA is a CSVNN defined as follows:

$$\begin{aligned} CSNNWGA(\Psi_1,\Psi_2,...,\Psi_n) &= \bigotimes_{k=1}^{n}\Psi_k^{\varsigma_k} \\ &= \Big((\prod_{k=1}^{n}(\tilde{\mathfrak{p}}_{t_k})^{\varsigma_k},\prod_{k=1}^{n}(\tilde{\mathfrak{q}}_{t_k})^{\varsigma_k}),(1-\prod_{k=1}^{n}(1-\tilde{\mathfrak{p}}_{i_k})^{\varsigma_k}, \\ &\quad 1-\prod_{k=1}^{n}(1-\tilde{\mathfrak{q}}_{i_k})^{\varsigma_k}),(1-\prod_{k=1}^{n}(1-\tilde{\mathfrak{p}}_{f_k})^{\varsigma_k},1-\prod_{k=1}^{n}(1-\tilde{\mathfrak{p}}_{f_k})^{\varsigma_k}\Big) \end{aligned} \quad (2)$$

Proof. The proof can be easily made based on aggregation operations of the SVNNs. Therefore, it is omitted. □

4. Multi-attribute group decision making method under CSVN environment

In the following table, beginning data and some notation are shown:

Notaions	Explanation
$\varepsilon = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$	Set of alternatives
$\Omega = \{\Omega_1, \Omega_2, ..., \Omega_k\}$	Set of attributes
$\delta = \{\delta_1, \delta_2, ..., \delta_m\}$	Set of decision makers
τ_{ij}	Evaluation of the criteria Ω_j made by decision maker δ_i
$\beta = (\beta_1, \beta_2, ..., \beta_m)$	weight vector of decision-makers

TABLE 1. Notation table for MAGDM method

4.1. *Decision making method*

Steps of the proposed method are explained as follows:

Step 1: Constructing decision-criteria (DA) matrix. In this step, each of decision makers $\delta_i, (i = 1, 2, ..., m)$ whose weight vector $\beta = (\beta_1, \beta_2, ..., \beta_m)$ such that $\beta_s > 0 (s = 1, 2, ..., m)$ and $\sum_{s=1}^{m} \beta_s = 1$, evaluates the attributes $\Omega_j, (j = 1, 2, ..., k)$ by CSVNNs and DA matrix is constructed as follows:

$$DA = \begin{bmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1k} \\ \tau_{21} & \tau_{22} & \cdots & \tau_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{m1} & \tau_{m2} & \cdots & \tau_{mk} \end{bmatrix}_{m \times k} .$$

Here each of $\tau_{ij} = ((\mathfrak{p}_{t_{ij}}, \mathfrak{q}_{t_{ij}}), (\mathfrak{p}_{i_{ij}}, \mathfrak{q}_{i_{ij}}), (\mathfrak{p}_{f_{ij}}, \mathfrak{q}_{f_{ij}}))$ is a CSVNN.

Step 2: Finding the aggregation of attributes. By adapting Eqs. 1 and 2 for weigh vector $\beta = (\beta_1, \beta_2, ..., \beta_m)$ of decision-makers, aggregated weight for each attribute is calculated the following formula

$$\begin{aligned} Agg_A(\Omega_j) &= CSVNWAA(\tau_{1j}, \tau_{2j}, ..., \tau_{nj}) = \bigoplus_{k=1}^{m} \beta_k \tau_{kj} \\ &= \Big((1 - \prod_{k=1}^{m} (1 - \tilde{\mathfrak{p}}_{t_{kj}})^{\beta_k}, 1 - \prod_{k=1}^{m} (1 - \tilde{\mathfrak{q}}_{t_{kj}})^{\beta_k}), \\ &\quad (\prod_{k=1}^{m} \tilde{\mathfrak{p}}_{i_{kj}}^{\beta_k}, \prod_{k=1}^{m} \tilde{\mathfrak{q}}_{i_{kj}}^{\beta_k}), (\prod_{k=1}^{m} \tilde{\mathfrak{p}}_{f_{kj}}^{\beta_k}, \prod_{k=1}^{m} \tilde{\mathfrak{q}}_{f_{kj}}^{\beta_k}) \Big) \end{aligned} \quad (3)$$

and

$$\begin{aligned} Agg_G(\Omega_j) &= CSVNWGA(\tau_{1j}, \tau_{2j}, ..., \tau_{nj}) = \bigoplus_{k=1}^{m} \tau_{kj}^{\beta_k} \\ &= \Big(\big(\prod_{k=1}^{m} \tilde{\mathfrak{p}}_{t_{kj}}^{\beta_k}, \prod_{k=1}^{m} \tilde{\mathfrak{q}}_{t_{kj}}^{\beta_k} \big), \big(\prod_{k=1}^{m} \tilde{\mathfrak{p}}_{i_{kj}}^{\beta_k}, \prod_{k=1}^{m} \tilde{\mathfrak{q}}_{i_{kj}}^{\beta_k} \big), \\ & \big(1 - \prod_{k=1}^{m} (1 - \tilde{\mathfrak{p}}_{f_{kj}})^{\beta_k}, 1 - \prod_{k=1}^{m} (1 - \tilde{\mathfrak{q}}_{f_{kj}})^{\beta_k} \big) \Big) \end{aligned} \tag{4}$$

respectively.

Step 3: Obtaining score functions of aggregated attributes. By using Definition 3.7 , for $j = 1, 2, ..., k$ $S(Agg_A(\Omega_j))$ $(S(Agg_G(\Omega_j)))$ are found.

Step 4: Construction of evaluation matrices by decision-makers δ_s $(s = 1, 2, ..., m)$. For each of decision-makers, evaluation matrices denoted EM_s $(s = 1, 2, ..., k)$ are obtained as follows:

$$EM_s = \begin{bmatrix} \kappa_{11}^s & \kappa_{12}^s & \cdots & \kappa_{1k}^s \\ \kappa_{21}^s & \kappa_{22}^s & \cdots & \kappa_{2k}^s \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{n1}^s & \kappa_{n2}^s & \cdots & \kappa_{nm}^s \end{bmatrix}.$$

Here $\kappa_{ij}^s = \langle \mathfrak{t}_{ij}^s, \mathfrak{i}_{ij}^s, \mathfrak{f}_{ij}^s \rangle$ denotes an SVNN which implies evaluation of alternative ε_i according to criteria Ω_j made by the decision maker δ_s.

Step 5: Compiling the EM_s (s=1,2,...,k). Compiling matrix (CM) is obtained as follows:

$$CM = \begin{bmatrix} \cup_{s=1}^{k} \kappa_{11}^s & \cup_{s=1}^{k} \kappa_{12}^s & \cdots & \cup_{s=1}^{k} \kappa_{1k}^s \\ \cup_{s=1}^{k} \kappa_{21}^s & \cup_{s=1}^{k} \kappa_{22}^s & \cdots & \cup_{s=1}^{k} \kappa_{2k}^s \\ \vdots & \vdots & \ddots & \vdots \\ \cup_{s=1}^{k} \kappa_{n1}^s & \cup_{s=1}^{k} \kappa_{n2}^s & \cdots & \cup_{s=1}^{k} \kappa_{nm}^s \end{bmatrix}.$$

Here $\cup_{s=1}^{k} \kappa_{11}^s = \langle \vee_{s=1}^{k} \mathfrak{t}_{ij}^s, \wedge_{s=1}^{k} \mathfrak{i}_{ij}^s, \wedge_{s=1}^{k} \mathfrak{f}_{ij}^s \rangle$ where $\vee$ and $\wedge$ denote the maximum and minimum, respectively.

Step 6: Finding score matrix $SM = [\partial_{ij}]_{nk}$: By using score functions obtained from DA matrix in Step 2 for each attribute, score values of elements in CM matrix are found.

Step 7: Evaluation of the alternatives. For $i = 1, 2, ..., n$, grade of the alternative $\mathfrak{g}_i$ are calculated by

$$\mathfrak{g}_i = \frac{1}{k} \sum_{j=1}^{k} \partial_{ij}$$

Step 8: Choosing the optimum alternative: Alternatives are ordered according to grades of them and alternative having maximum grade is selected as an optimum alternative.

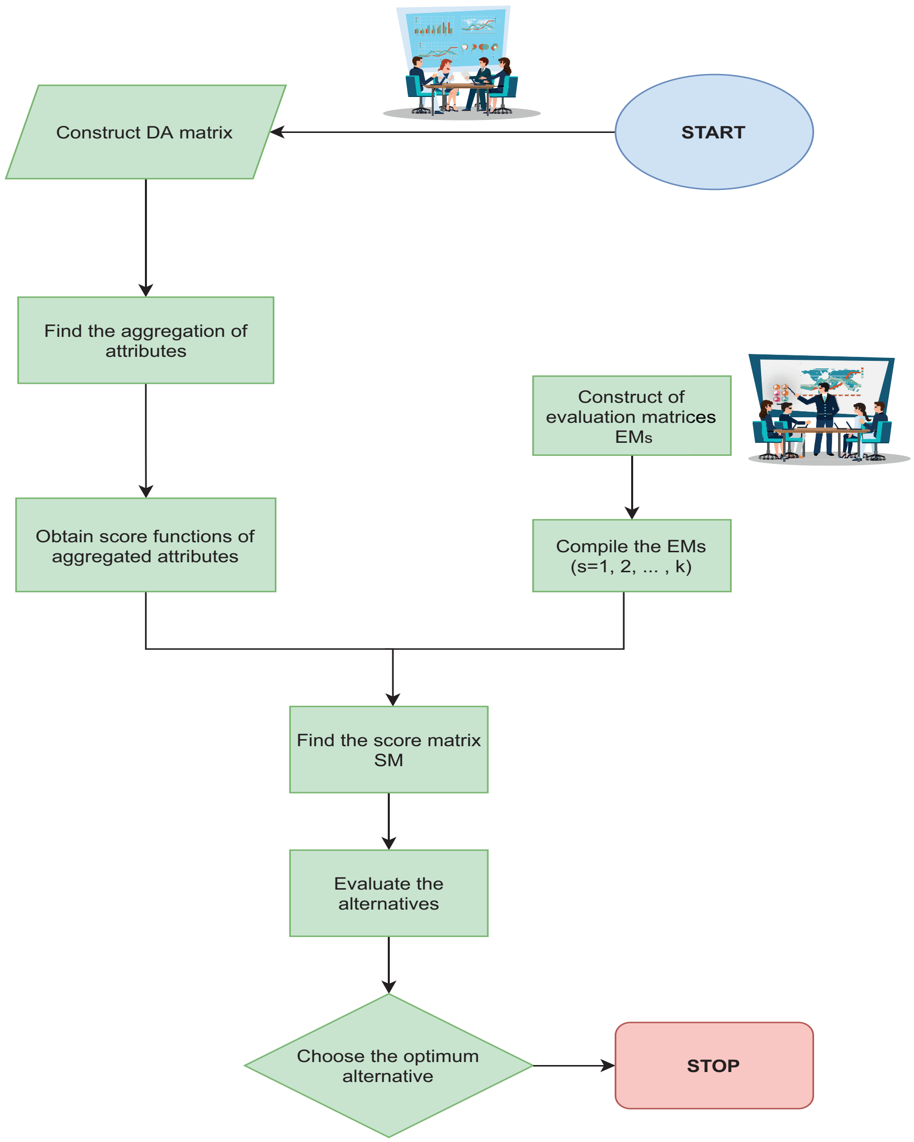

FIGURE 6. Flow chart of the proposed method

Flowchart of the algorithm is showed in Figure 6.

4.2. *Illustrative Example*

In this section, an example is provided to display the functioning of the developed decision-making method.

Example 4.1. Suppose that two projects are wanted to be selected among five projects to provide financial support. There are three experts $\delta = \{\delta_1, \delta_2, \delta_3\}$ with different academic qualifications in the panel. Their weight vector is $(0.5, 0.3, 0.2)$. These experts evaluate the project $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ and ε_5 according to four attributes Ω_1=Content, Ω_2=practicability, Ω_3=Originality and Ω_4= widespread impact.

The method proposed for choosing the best two projects that need financial support is applied as follows.

Step 1: Constructing decision-attribute (DA) matrix: Each of the experts evaluates the attribute under CSVN environment and they construct DA matrix as follows:

$$DA = \begin{bmatrix} ((0.4,0.2),(0.1,0.1),(0.6,0.2)) & ((0.7,0.1),(0.5,0.4),(0.7,0.3)) & ((0.6,0.2),(0.4,0.3),(0.5,0.1)) & ((0.8,0.1),(0.5,0.3),(0.6,0.3)) \\ ((0.7,0.1),(0.3,0.1),(0.8,0.3)) & ((0.9,0.6),(0.9,0.4),(0.6,0.4)) & ((0.9,0.8),(0.4,0.1),(0.8,0.6)) & ((0.1,0.1),(0.9,0.3),(0.9,0.2)) \\ ((0.8,0.6),(0.9,0.2),(0.7,0.3)) & ((0.9,0.2),(0.8,0.6),(0.7,0.5)) & ((0.4,0.2),(0.9,0.4),(0.7,0.4)) & ((0.7,0.6),(0.7,0.3),(0.7,0.5)) \end{bmatrix}$$

Step 2: By using Equation (4) the arithmetic aggregate of each column is obtained as follows:

$$Agg_A(\Omega_j) = \Big(((0.609,0.279),(0.216,0.115),(0.675,0.245)) \quad ((0.827,0.311),(0.655,0.434),(0.668,0.362)) \quad ((0.714,0.472),(0.470,0.229),(0.616,0.226)) \quad ((0.659,0.235),(0.638,0.300),(0.699,0.294)) \Big)$$

Step 3: By using Eq. 3.7, for $j = 1,2,...,k$ $S(Agg_A(\Omega_j))$ are found.

$$S(Agg_A(\Omega_1)) = \frac{1}{3}\Big(\frac{0.078}{0.078+(\theta_t-0.609)^2} + \frac{0.013}{0.013+(\theta_i-0.216)^2} + \frac{0.060}{0.060+(\theta_f-0.675)^2}\Big),$$

$$S(Agg_A(\Omega_2)) = \frac{1}{3}\Big(\frac{0.097}{0.097+(\theta_t-0.827)^2} + \frac{0.188}{0.188+(\theta_i-0.655)^2} + \frac{0.131}{0.131+(\theta_f-0.668)^2}\Big),$$

$$S(Agg_A(\Omega_3)) = \frac{1}{3}\Big(\frac{0.223}{0.223+(\theta_t-0.714)^2} + \frac{0.052}{0.052+(\theta_i-0.470)^2} + \frac{0.051}{0.051+(\theta_f-0.616)^2}\Big),$$

$$S(Agg_A(\Omega_4)) = \frac{1}{3}\Big(\frac{0.055}{0.055+(\theta_t-0.659)^2} + \frac{0.090}{0.090+(\theta_i-0.638)^2} + \frac{0.087}{0.087+(\theta_f-0.699)^2}\Big).$$

Also, by using $CSVNWGA$ operator score functions of the attributes are obtained as follows:

$$S(Agg_G(\Omega_1)) = \frac{1}{3}\Big(\frac{0.041}{0.041+(\theta_t-0.543)^2} + \frac{0.015}{0.015+(\theta_i-0.462)^2} + \frac{0.063}{0.063+(\theta_f-0.693)^2}\Big),$$

$$S(Agg_G(\Omega_2)) = \frac{1}{3}\Big(\frac{0.039}{0.039+(\theta_t-0.794)^2} + \frac{0.200}{0.200+(\theta_i-0.743)^2} + \frac{0.141}{0.141+(\theta_f-0.673)^2}\Big),$$

$$S(Agg_G(\Omega_3)) = \frac{1}{3}\Big(\frac{0.049}{0.049+(\theta_t-0.588)^2} + \frac{0.072}{0.072+(\theta_i-0.581)^2} + \frac{0.122}{0.122+(\theta_f-0.657)^2}\Big),$$

$$S(Agg_G(\Omega_4)) = \frac{1}{3}\Big(\frac{0.020}{0.020 + (\theta_t - 0.417)^2} + \frac{0.090}{0.090 + (\theta_i - 0.721)^2} + \frac{0.102}{0.102 + (\theta_f - 0.751)^2}\Big).$$

Step 4: For each of decision-makers, evaluation matrices are obtained as follows:

$$EM_1 = \begin{pmatrix} \langle 0.6, 0.8, 0.3\rangle & \langle 0.2, 0.5, 0.4\rangle & \langle 0.5, 0.1, 0.6\rangle & \langle 0.6, 0.9, 0.4\rangle \\ \langle 0.8, 0.3, 0.1\rangle & \langle 0.2, 0.6, 0.8\rangle & \langle 0.7, 0.2, 0.8\rangle & \langle 0.4, 0.5, 0.5\rangle \\ \langle 0.4, 0.7, 0.2\rangle & \langle 0.1, 0.1, 0.2\rangle & \langle 0.9, 0.3, 0.7\rangle & \langle 0.3, 0.4, 0.2\rangle \\ \langle 0.2, 0.3, 0.9\rangle & \langle 0.4, 0.3, 0.9\rangle & \langle 0.3, 0.4, 0.9\rangle & \langle 0.8, 0.8, 0.3\rangle \\ \langle 0.5, 0.1, 0.9\rangle & \langle 0.8, 0.4, 0.5\rangle & \langle 0.2, 0.5, 0.1\rangle & \langle 0.7, 0.6, 0.9\rangle \end{pmatrix},$$

$$EM_2 = \begin{pmatrix} \langle 0.5, 0.1, 0.2\rangle & \langle 0.8, 0.8, 0.7\rangle & \langle 0.1, 0.6, 0.4\rangle & \langle 0.7, 0.4, 0.3\rangle \\ \langle 0.3, 0.2, 0.7\rangle & \langle 0.2, 0.6, 0.3\rangle & \langle 0.7, 0.4, 0.3\rangle & \langle 0.6, 0.8, 0.6\rangle \\ \langle 0.8, 0.6, 0.9\rangle & \langle 0.4, 0.7, 0.4\rangle & \langle 0.4, 0.5, 0.4\rangle & \langle 0.7, 0.6, 0.8\rangle \\ \langle 0.4, 0.7, 0.5\rangle & \langle 0.6, 0.3, 0.2\rangle & \langle 0.3, 0.6, 0.2\rangle & \langle 0.2, 0.7, 0.5\rangle \\ \langle 0.9, 0.3, 0.3\rangle & \langle 0.7, 0.5, 0.1\rangle & \langle 0.1, 0.8, 0.6\rangle & \langle 0.5, 0.2, 0.1\rangle \end{pmatrix},$$

and

$$EM_3 = \begin{pmatrix} \langle 0.7, 0.2, 0.9\rangle & \langle 0.2, 0.6, 0.7\rangle & \langle 0.1, 0.7, 0.8\rangle & \langle 0.7, 0.6, 0.6\rangle \\ \langle 0.6, 0.4, 0.3\rangle & \langle 0.1, 0.2, 0.9\rangle & \langle 0.6, 0.6, 0.6\rangle & \langle 0.6, 0.3, 0.1\rangle \\ \langle 0.5, 0.6, 0.5\rangle & \langle 0.6, 0.3, 0.3\rangle & \langle 0.4, 0.2, 0.1\rangle & \langle 0.4, 0.2, 0.3\rangle \\ \langle 0.7, 0.8, 0.6\rangle & \langle 0.7, 0.4, 0.5\rangle & \langle 0.5, 0.6, 0.3\rangle & \langle 0.4, 0.1, 0.4\rangle \\ \langle 0.2, 0.6, 0.8\rangle & \langle 0.9, 0.6, 0.3\rangle & \langle 0.6, 0.4, 0.2\rangle & \langle 0.8, 0.8, 0.6\rangle \end{pmatrix}.$$

Here $\kappa_{ij}^s = \langle \mathfrak{t}_{ij}^s, \mathfrak{i}_{ij}^s, \mathfrak{f}_{ij}^s\rangle$ denote a SVN number which implies evaluation of alternative ε_i according to criteria Ω_j $(j = 1, 2, 3, 4)$ made by the decision maker δ_s, $(s = 1, 2, 3)$.

Step 5: Compiling matrix (CM) is obtained as follows:

$$CM = \begin{bmatrix} \langle 0.7, 0.1, 0.2\rangle & \langle 0.8, 0.5, 0.4\rangle & \langle 0.5, 0.1, 0.4\rangle & \langle 0.7, 0.4, 0.2\rangle \\ \langle 0.8, 0.2, 0.1\rangle & \langle 0.8, 0.2, 0.3\rangle & \langle 0.7, 0.2, 0.3\rangle & \langle 0.6, 0.2, 0.1\rangle \\ \langle 0.8, 0.6, 0.2\rangle & \langle 0.6, 0.1, 0.3\rangle & \langle 0.9, 0.2, 0.1\rangle & \langle 0.7, 0.2, 0.2\rangle \\ \langle 0.7, 0.3, 0.5\rangle & \langle 0.9, 0.2, 0.1\rangle & \langle 0.5, 0.4, 0.1\rangle & \langle 0.8, 0.1, 0.3\rangle \\ \langle 0.9, 0.1, 0.3\rangle & \langle 0.9, 0.4, 0.1\rangle & \langle 0.9, 0.4, 0.1\rangle & \langle 0.8, 0.2, 0.1\rangle \end{bmatrix}.$$

Here $\cup_{s=1}^k \kappa_{11}^s = \langle \vee_{s=1}^k \mathfrak{t}_{ij}^s, \wedge_{s=1}^k \mathfrak{i}_{ij}^s, \wedge_{s=1}^k \mathfrak{f}_{ij}^s\rangle$ where $\vee$ and $\wedge$ denote the maximum and minimum, respectively.

Step 6: By using score functions obtained from DA matrix for each attribute, score values of elements in CM matrix are found by using CSVNWAA and CSVNWGAO operators as follows:

$$SM_A = \begin{bmatrix} 0.537 & 0.842 & 0.543 & 0.614 \\ 0.605 & 0.653 & 0.585 & 0.485 \\ 0.324 & 0.508 & 0.481 & 0.516 \\ 0.739 & 0.571 & 0.635 & 0.442 \\ 0.424 & 0.660 & 0.647 & 0.417 \end{bmatrix}.$$

and

$$SM_G = \begin{bmatrix} 0.311 & 0.808 & 0.583 & 0.307 \\ 0.237 & 0.635 & 0.539 & 0.274 \\ 0.342 & 0.445 & 0.316 & 0.235 \\ 0.537 & 0.492 & 0.611 & 0.215 \\ 0.212 & 0.568 & 0.435 & 0.188 \end{bmatrix},$$

respectively.

Step 7: Evaluation of the alternatives: For $i = 1, 2, 3, 4, 5$, grade of the projects ε_i are calculated by

$$\varepsilon_i = \frac{1}{k}\sum_{j=1}^{k} \partial_{ij}$$

by using matrices SM_A and SM_G

	ε_1	ε_2	ε_3	ε_4	ε_5
SM_A	0.634	0.582	0.457	0.597	0.537
SM_G	0.502	0.421	0.334	0.464	0.35

Step 8: Projects are ordered according to grades of them and two projects having maximum grade are selected as projects to be provided financial support. Then, according to SM_A and SM_G ranking order is as follows:

$$\varepsilon_1 > \varepsilon_4 > \varepsilon_2 > \varepsilon_5 > \varepsilon_3.$$

It is seen that same ranking order is obtained according to both of SM_A and SM_G. Therefore, the projects ε_1 , ε_4 are the projects to be provided the financial support.

5. Conclusion

Recently, SVNN has a very important place in modeling decision making problems. Many researchers have studied on the types of SVNNs. The best known of them are SVTNN and SVTrNNs. These numbers are SVNNs containing a maximum point and a flatness, respectively. They are represented by piecewise functions using lines. However, problems in daily life may not always follow a linear course. Therefore, in this study, the concept of CSVNN was defined based on the Cauchy distribution function. CSVNNs are important in that they are non-linear and a generalization of other neutrosophic numbers. Since the score functions defined in this study are defined separately for each CSVNN, it can be considered as a generalization of type-2 fuzzy structures that include modeling the uncertainty of the membership function. In future, many mathematical structures can be defined in the environment containing CSVNNs. In addition, distance measures and similarity measures between CSVNNs may be defined and integrated TOPSIS, VIKOR and other classical DMs under CSVN environment. its more advanced potential applications under such a flexible and versatile CSVN environment

Conflicts of Interest: The authors declare that there is no conflict of interests.

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University of New Mexico

Pentagonal Neutrosophic Transportation Problems with Interval Cost

Ummey Habiba[1], and Abdul Quddoos[2,*]

[1]Integral University, Lucknow, India ; say2habiba@gmail.com

[2]Integral University, Lucknow, India ; abdulquddoos.or.amu@gmail.com

[*]Correspondence: abdulquddoos.or.amu@gmail.com

Abstract. The present paper aims to deal with a new variant of uncertain classical transportation problem namely 'Interval Pentagonal Neutrosophic Transportation Problem' (IPNTP) in which the uncertainty of source & destination parameters are described by pentagonal neutrosophic numbers while an estimated range (interval number) is used to represent uncertain cost of transportation. The objective consisting interval cost has been splitted into two equivalent crisp objectives using the concept of expected value & uncertainty of interval number. The constraints containing pentagonal neutrosophic quantities has also been converted into crisp constraints with the help of score function. The pareto optimal solution of the transformed bi-objective crisp transportation problem has been obtained using fuzzy programming approach. A numerical illustration is used to demonstrate the computational procedure of the proposed variant. Lingo 18.0 software is used to solve the problem.

Keywords: Fuzzy Set; Intuitionistic Set; Neutrosophic Set; Interval Numbers; Neutrosophic Transportation Problem; Pentagonal Neutrosophic Number

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1. Introduction

In classical transportation problem, the cost-coefficients of transporting a unit product from source i to destination j (c_{ij}), availability of the product at source i (a_i) and demand of destination j (b_j) parameters are priorly known certain quantities. However in today's uncertain world it is not at all astonishing if these quantities reflect some uncertainty. So if we logically introduce some impreciseness in these quantities, we may get a new variant of classical transportation problem. Usually the uncertainty in the problems can be dealt in three ways (i) fuzzy (ii) interval and (iii) stochastic. Fuzzy sets were introduced by Zadeh [2] in 1965 which is characterized by a membership function. The membership function assigns a grade of membership to each element of set. This grade only gives an idea about the possibility of the inclusion of respective element in the set but it does not reveal any information about non-inclusion of an element in the set. Later on Atanassov [3] proposed an extension of fuzzy set in which he also associated another grade of non-membership with each element in the fuzzy set. Such sets are known as intuitionistic fuzzy sets. Further in the year 1999, Smarandache [4] identified a situation of indeterminacy about inclusion of element in set. So he suggested the association of another grade of indeterminacy along with the grades of inclusion and non-inclusion. To overcome the limitation of intuitionistic fuzzy set, Smarandache [4] proposed the concept of neutrosophic sets. Alike fuzzy number, the concept of triangular and trapezoidal neutrosophic number and their deneutrosophication techniques have been developed by several authors (see [5]- [8]). Wang *et* al. [9] introduced the idea of single valued neutrosophic sets. In past few years, ample contribution of neutrosophic sets can be observed in multicriteria decision making ([10]- [22]), graph theory ([23]- [29]), optimization techniques ([30]- [33]) etc. The concept of neutrosophic sets has been extensively applied by several authors by logically introducing uncertainty in different ways in the parameters of classical transportation problem. For example, Thamaraiselvi and Santhi [34] presented a new approach for solving a classical transportation problem. They considered transportation problem in which the transportation cost, availability of product and demand were represented by trapezoidal neutrosophic numbers. Singh *et* al. [35] suggested a modified version of Thamaraiselvi and Santhi [34]. The uncertainty in cost, availability and demand parameters differently introduced by some authors depending various possibilities has been summarized in Table 1. In 2019, Chakaraborty *et* al. [46] discussed the advantages of using pentagonal neutrosophic numbers over triangular and trapezoidal neutrosophic numbers.

This paper considers a transportation problem with pentagonal neutrosophic availability and demand but interval cost of transportation. Since a membership grade to availability and demand of product can be assigned easily but in case of transportation cost, it varies uncontrollably within a specific range due to many reasons like; maintenance of carrier, disloyalty of drivers, fluctuation of fuel cost etc. To the best of our knowledge no such variant of the transportation problem has been considered in literature. So it is quiet advocable to represent the transportation cost in the form of interval numbers in a reasonable manner depending on past experience. The concepts of score function [47], expected value and uncertainty [1] has been applied to convert the developed uncertain transportation problem into an equivalent crisp problem. To demonstrate the procedure of transportation of IPNTP to crisp problem and its solution a numerical example is taken and solved using Lingo 18.0 software.

TABLE 1. Summary

Refrences	Interval Cost	Neutrosophic Supply & Demand
Narayanamoorthy & Anukokila [36]	✓	✗
Pramanik & Dey [37]	✗	✓
Das *et* al. [38]	✓	✗
Saini & Sangal [39]	✗	✓
Paul *et* al. [40]	✗	✓
Kumar Das [41]	✗	✓
Habiba & Quddoos, [42]	✓	✗
Habiba & Quddoos, [43]	✓	✗
Chakraborty *et* al. [44]	✗	✓
Sikkannan & Shanmugavel [45]	✗	✓
Akilbasha *et* al. [46]	✓	✗
Proposed IPNTP	✓	✓

2. Preliminaries

Definition 2.1 (Fuzzy Set [2]). A Set $\widetilde{F}$ over X represented as $\widetilde{F}=\{(x,\mu(x)) : x \in X, \mu(x) \in [0,1]\}$ is called a fuzzy set where X is the collection of points x and $\mu(x)$ is its truth membership function.

Definition 2.2 (Intuitionistic Fuzzy Set [3]). A set $\widetilde{I}$ over X represented as $\widetilde{I}=\{x;\mu(x),\delta(x);x \in X,$ where $\mu(x),\delta(x) \in [0,1]\}$ is called an intuitionistic fuzzy set where $\mu(x)$ and $\delta(x)$ are the truth and indeterminacy membership functions of x which satisfy the following relation:

$$0 \leq \mu(x) + \delta(x) \leq 1$$

Definition 2.3 (Neutrosophic Set [4]). A set $\widetilde{N}$ over X represented as $\widetilde{N}=\{x;\mu(x),\delta(x),\sigma(x);x \in X,$ and $\mu(x),\delta(x),\sigma(x) \in [0,1]\}$ is called a neutrosophic sets where $\mu(x)$, $\delta(x)$ and $\sigma(x)$ are the truth, indeterminacy and falsity membership functions of x which displays the following relation:

$$0 \leq \mu(x) + \delta(x) + \sigma(x) \leq 3$$

Definition 2.4 (Single-Valued Neutrosophic Set [9]). A Neutrosophic set $\widetilde{N}$ in definition 2.3 is called as a single- valued neutrosophic set $\widetilde{SN}$ if x is a single-valued independent variable. $\widetilde{SN}=\{x;\mu(x),\delta(x),\sigma(x);x \in X,\}$ where $\mu(x)$, $\delta(x)$ and $\sigma(x)$ are the truth, indeterminacy and falsity membership functions of x respectively.

Definition 2.5 (Single-Valued Pentagonal Neutrosophic Number [47]). A Neutrosophic Number (S), $S=\langle[(s^1,t^1,u^1,v^1,w^1);\mu],[(s^2,t^2,u^2,v^2,w^2);\delta],[(s^3,t^3,u^3,v^3,w^3);\sigma]\rangle$, where $\mu, \delta, \sigma \in [0,1]$ is called single-valued pentagonal neutrosophic number, if its truth membership function (μ), indeterminacy membership function (δ) and the falsity membership function (σ) are respective given as follows:

$$\mu(x) = \begin{cases} \mu_{l_1}(x) & s^1 \leq x \leq t^1 \\ \mu_{l_2}(x) & t^1 \leq x \leq u^1 \\ \mu & x = u^1 \\ \mu_{r_2}(x) & u^1 \leq x \leq v^1 \\ \mu_{r_1}(x) & v^1 \leq x \leq w^1 \\ 0 & otherwise \end{cases}$$

$$\delta(x) = \begin{cases} \delta_{l_1}(x) & s^2 \leq x \leq t^2 \\ \delta_{l_2}(x) & t^2 \leq x \leq u^2 \\ \delta & x = u^2 \\ \delta_{r_2}(x) & u^2 \leq x \leq v^2 \\ \delta_{r_1}(x) & v^2 \leq x \leq w^2 \\ 1 & otherwise \end{cases}$$

$$\sigma(x) = \begin{cases} \sigma_{l_1}(x) & s^3 \leq x \leq t^3 \\ \sigma_{l_2}(x) & t^3 \leq x \leq u^3 \\ \sigma & x = u^3 \\ \sigma_{r_2}(x) & u^3 \leq x \leq v^3 \\ \sigma_{r_1}(x) & v^3 \leq x \leq w^3 \\ 1 & otherwise \end{cases}$$

Definition 2.6 (**Score Function [47]**)**.** Score function for any pentagonal single typed neutrosophic number $(P_1, P_2, P_3, P_4, P_5; \mu, \delta, \sigma)$ is defined as follows:

$$\widetilde{S} = \frac{1}{15}\{(P_1 + P_2 + P_3 + P_4 + P_5) \times (2 + \mu - \delta - \sigma)\}$$

where μ, δ and σ are the truth, indeterminacy and falsity membership functions.

Definition 2.7 (**Interval Numbers [48]**)**.**

$$A = [a_L, a_R] = \{\ a :\ a_L \leq\ a \leq\ a_R, a \in \mathbb{R}\},$$

where a_L and a_R are the left-limit and right-limit of the interval A on the real line $\mathbb{R}$.

$$A = \langle a_c, a_w \rangle = \{\ a :\ a_c - a_w \leq\ a \leq\ a_c + a_w, a \in \mathbb{R}\},$$

where a_c and a_w are the mid-point and half-width (or simply be termed as 'width') of interval A on the real line $\mathbb{R}$, i.e.

$$a_c = \frac{a_R + a_L}{2}$$

$$a_w = \frac{a_R - a_L}{2}$$

Definition 2.8 (Ishibuchi and Tanaka's ranking for intervals [1]). Let $A = [a_L, a_R]{=}\langle a_c, a_w\rangle$ and $B = [b_L, b_R]{=}\langle b_c, b_w\rangle$ be two given intervals then the order relation $\leq_{CW}$ is defined as:

$$\begin{cases} A \leq_{CW} B \text{ iff } a_c \leq b_c, \text{ and } \; a_w \leq b_w \\ A <_{CW} B \text{ iff } A \leq_{CW} B, \text{ and } \; A \neq B \end{cases}$$

3. Mathematical Model of IPNTP

Consider a transportation problem in which the cost-coefficients are represented in the form of interval number & source and destination parameters are represented as pentagonal single typed neutrosophic numbers. The mathematical model of such IPNTP may be given as follows:

Problem-I:

$$\text{Minimize : } Z = \sum_{i=1}^{m}\sum_{j=1}^{n}[c_{L_{ij}}, c_{R_{ij}}]x_{ij} \tag{1}$$

Subject to;

$$\sum_{j=1}^{n} x_{ij} \leq {a_i}^S, \; i = 1, 2, \ldots, m \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} \geq {b_j}^S, \; j = 1, 2, \ldots, n \tag{3}$$

$$x_{ij} \geq 0, \; i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n \tag{4}$$

where,

$[c_{L_{ij}}, c_{R_{ij}}]$: interval cost of transporting one unit of product from source i to destination j,

$c_{L_{ij}}$: lowest possible cost of transporting one unit of product from source i to destination j,

$c_{R_{ij}}$: highest possible cost of transporting one unit of product from source i to destination j

${a_i}^S$: pentagonal single typed neutrosophic availability of source i,

${b_j}^S$: pentagonal single typed neutrosophic demand of destination j,

x_{ij}: quantity transported from source i to destination j

4. Equivalent crisp model of IPNTP

4.1. *Construction of crisp Objective Function*

Let us consider the interval objective function Z of the Problem-I which can be denoted as $Z = \langle z_c, z_w\rangle$, where $z_c = (\frac{c_R + c_L}{2})$ and $z_w = (\frac{c_R - c_L}{2})$ are the center and width of the interval Z respectively.

According to Ishibuchi and Tanaka [1] the center and width of an interval can be taken as the expected value and uncertainty of interval respectively. Since the objective function of Problem-I is the cost function which is to be minimized, so our interest is to obtain minimum cost with minimum uncertainty. Then the interval objective function (1) is transformed into a two crisp functions in terms of expected value and uncertainty by definition (2.8) as follows:

$$\text{Minimize } z_c = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{c_{ij}} x_{ij} \tag{5}$$

$$\text{Minimize } z_w = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{w_{ij}} x_{ij} \tag{6}$$

where, $c_c = \frac{c_R + c_L}{2}$ and $c_w = \frac{c_R - c_L}{2}$ are the center and width of the interval respectively.

4.2. *Construction of crisp Constraints*

Let us consider the constraint (2) of Problem-I where the right hand side ${a_i}^S$ representing the pentagonal single typed neutrosophic availability of product at source i. This pentagonal single typed neutrosophic number ${a_i}^S$ can be represented by a crisp value using the score function defined in definition (2.6). Thus corresponding crisp constraint may be written as follows:

$$\sum_{j=1}^{n} x_{ij} \leq \widetilde{S}(a_i),\ i = 1, 2, \ldots, m \tag{7}$$

Similarly, the crisp destination constraint can also be obtained as follows:

$$\sum_{i=1}^{m} x_{ij} \geq \widetilde{S}(b_j),\ j = 1, 2, \ldots, n \tag{8}$$

Using equations (5-6) and (7-8), we can write the equivalent bi-objective crisp problem of IPNTP as follows:

Problem-II:

$$\text{Minimize } z_c = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{c_{ij}} x_{ij} \tag{9}$$

$$\text{Minimize } z_w = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{w_{ij}} x_{ij} \tag{10}$$

Subject to;

$$\sum_{j=1}^{n} x_{ij} \leq \widetilde{S}(a_i),\ i = 1, 2, \ldots, m \tag{11}$$

$$\sum_{i=1}^{m} x_{ij} \geq \widetilde{S}(b_j),\ j = 1, 2, \ldots, n \tag{12}$$

$$x_{ij} \geq 0,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n \tag{13}$$

5. Fuzzy Programming Technique for solving bi-objective transportation problem (Problem-II)

First we find the best L_k and worst U_k for the $k^{th}, k = c, w$ objective function, where L_k is the aspired level and U_k is the highest acceptable level for the k^{th} objective function. Thereafter we create a fuzzy linear programming problem using membership function. The stepwise procedure of fuzzy programming technique is given as follows:

Step 1: Solve the bi-objective transportation problem (Problem-II) as a single objective problem using only one objective at a time and ignoring other.

Step 2: From each solution derived in Step1 determine the values of both objective functions.

Step 3: Find the best L_k and worst U_k for both objectives corresponding to the set of solutions. Define a fuzzy membership function $\mu_k(Z_k)$ as follows:

$$\mu_k(Z_k) = \begin{cases} 1, & \text{if } Z_k \leq L_k \\ 1 - \frac{Z_k - L_k}{U_k - L_k}, & \text{if } L_k \leq Z_k \leq U_k \\ 0, & \text{if } Z_k \geq L_k \end{cases}$$

The equivalent linear programming problem for the vector minimum problem can be written as follows:

$$\text{Maximize}: \lambda,$$

$$\text{Subject to; } \lambda \leq \frac{U_k - Z_k}{U_k - L_k}$$

$$\text{Constraints; } (11 - 13)$$

$$0 \leq \lambda \leq 1$$

The above linear programming problem may further be simplified as:

Problem-III:

$$\text{Maximize}: \lambda,$$

$$\text{Subject to; } Z_k + \lambda(U_k - L_k) \leq U_k$$

$$\text{Constraints; } (11 - 13)$$

$$0 \leq \lambda \leq 1$$

Step 4: Solve Problem-III using any method and obtain the required pareto optimal solution.

6. Numerical Illustration

A company has three factories F_1, F_2 and F_3. A homogenous product is to be transported from these factories to four destinations D_1, D_2, D_3 and D_4 in such a way that the total shipment cost becomes minimum. The availability at each factories and requirement at each destinations and unit interval cost transportation cost from each factory to each destination are given in Table 2.

TABLE 2. Interval pentagonal neutrosophic transportation table

	D_1	D_2	D_3	D_4	Availability
F_1	[5,7]	[5,9]	[3,5]	[7,8]	(22,26,28,32,35; 0.7,0.3,0.4)
F_2	[8,12]	[7,10]	[4,8]	[5,6]	(30,33,36,38,40; 0.6,0.4,0.5)
F_3	[6,7]	[1,2]	[7,9]	[5,6]	(21,28,32,37,39; 0.8,0.2,0.4)
Demand	(13,16,18,21,25; 0.5,0.5,0.6)	(17,21,24,28,30; 0.8,0.2,0.4)	(24,29,32,35,37; 0.9,0.5,0.3)	(6,10,13,15,18; 0.7,0.3,0.4)	

The mathematical model of the given problem is as follows:

$$\text{Minimize}: Z = \sum_{i=1}^{3}\sum_{j=1}^{4}[c_{L_{ij}}, c_{R_{ij}}]x_{ij}$$

Subject to;

$$\sum_{j=1}^{4} x_{1j} \le (22, 26, 28, 32, 35; 0.7, 0.3, 0.4),\ \sum_{j=1}^{4} x_{2j} \le (30, 33, 36, 38, 40; 0.6, 0.4, 0.5),$$

$$\sum_{j=1}^{4} x_{3j} \le (21, 28, 32, 37, 39; 0.8, 0.2, 0.4),\ \sum_{i=1}^{3} x_{i1} \ge (13, 16, 18, 21, 25; 0.5, 0.5, 0.6),$$

$$\sum_{i=1}^{3} x_{i2} \ge (17, 21, 24, 28, 30; 0.8, 0.2, 0.4),\ \sum_{i=1}^{3} x_{i3} \ge (24, 29, 32, 35, 37; 0.9, 0.5, 0.3),$$

$$\sum_{i=1}^{3} x_{i4} \ge (6, 10, 13, 15, 18; 0.7, 0.3, 0.4),\ x_{ij} \ge 0,\ i = 1, 2, 3,\ j = 1, 2, 3, 4$$

Using Problem-II the above problem can be written as follows:

$$\text{Minimize } z_c = \sum_{i=1}^{3}\sum_{j=1}^{4} c_{c_{ij}} x_{ij}, \qquad \text{Minimize } z_w = \sum_{i=1}^{3}\sum_{j=1}^{4} c_{w_{ij}} x_{ij}$$

where,

$$c_{c_{ij}} = \begin{bmatrix} 6 & 7 & 4 & 7.5 \\ 10 & 8.5 & 6 & 5.5 \\ 6.5 & 1.5 & 8 & 5.5 \end{bmatrix}, \qquad c_{w_{ij}} = \begin{bmatrix} 1 & 2 & 1 & 0.5 \\ 2 & 1.5 & 2 & 0.5 \\ 1 & 0.5 & 1 & 0.5 \end{bmatrix}$$

Subject to;

$$\sum_{j=1}^{4} x_{1j} \leq 19.06,\ \sum_{j=1}^{4} x_{2j} \leq 20.06,\ \sum_{j=1}^{4} x_{3j} \leq 23.01,\ \sum_{i=1}^{3} x_{i1} \geq 8.68,$$

$$\sum_{i=1}^{3} x_{i2} \geq 17.6,\ \sum_{i=1}^{3} x_{i3} \geq 21.96,\ \sum_{i=1}^{3} x_{i4} \geq 9.32,\ x_{ij} \geq 0,\ i = 1,2,3,\ j = 1,2,3,4$$

On solving the above problem using lingo 18.0 software, the pareto optimal solution of the problem is obtained as, $x_{11} = 3.27, x_{13} = 15.79, x_{23} = 6.17, x_{24} = 9.32, x_{31} = 5.41, x_{32} = 17.6$, $Z = [185.06, 280.19]$.

The above result shows that minimum total cost of transportation lies between 185.06 to 280.19. The optimal policy of transportation to be adopted by decision maker is given in Table (3).

TABLE 3. Optimal policy of transportation

Factoy		Destination	Suggested optimal policy of transportation
F_1	$\longrightarrow$	D_1	3.27 Units of product are to be transported from first factory to first destination
F_1	$\longrightarrow$	D_3	15.79 Units of product are to be transported from first factory to third destination
F_2	$\longrightarrow$	D_3	6.17 Units of product are to be transported from second factory to third destination
F_2	$\longrightarrow$	D_4	9.32 Units of product are to be transported from second factory to fourth destination
F_3	$\longrightarrow$	D_1	5.41 Units of product are to be transported from third factory to first destination
F_3	$\longrightarrow$	D_2	17.6 Units of product are to be transported from third factory to second destination

7. **Advantages and Limitations of IPNTP**

Every new study carry some limitations along with the advantages. Two major advantages and their corresponding limitations have been discussed in Table 4.

8. **Conclusion and Future Work**

In this paper, a more realistic variant of transportation problem namely IPNTP has been introduced with interval cost and pentagonal neutrosophic availability and demand parameters. Since the transportation cost greatly depends on many factors like sudden change in fuel prices, load carrying capacity of carrier, disloyalty of drivers and many more. So it becomes tedious

Table 4. Advantages and limitations

Advantages	Limitations
• Use of interval cost reduces their tedious task of assigning membership grade to every associated cost • Solution approaches work well for single objective problem as it converts into a bi-objective crisp problems	• Relatively more information needed to reduce the range of interval • Multiobjective problems increases the computational complexity of the problem because it doubles the number of objectives while converting into crisp one

for decision maker to assign grades to truth, indeterminacy and falsity membership functions for unit cost of transporting product from each source to every destination. To overcome this issue this article suggest the decision maker to represent the transportation cost in the form of interval number. But this issue is not valid in case of availability and demand because membership grade for availability and demand parameters may easily be assigned with the help of information received from sources and destinations. To the best of our knowledge no such variant of transportation problem is considered in literature previously.

An extension of the IPNTP with interval valued neutrosophic cost may be proposed in future research work.

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University of New Mexico

Neutro-Sigma Algebras and Anti-Sigma Algebras

Memet Şahin

Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey. mesahin@gantep.edu.tr

Abstract: Neutro-algebra structures play a significant role in the neutrosophic theory. Especially, with the help of neutro-algebraic structures; neutrosophic theory makes a valuable addition to the classical theory. In this article, neutro-sigma algebras and anti-sigma algebras are obtained. Furthermore, basic properties and examples for neutro-sigma algebras and anti-sigma algebras are obtained and proved. Also, classical sigma algebra, neutro-sigma algebra and anti-sigma algebra are compared to each other. Neutro-sigma algebra is shown to have a more general structure with respect to classical sigma algebra and anti-sigma algebra. Thus, (T, I, F) components that constitute the neutrosophic theory are added into classical sigma algebra (without using neutrosophic sets) and a new structure is obtained. In addition, we show that a neutro-sigma algebra can be obtained from every classical sigma algebra and a neutro-sigma algebra can be obtained from every anti-sigma algebra.

Keywords: sigma algebra, neutro-algebra, anti-algebra, neutro-sigma algebra, anti-sigma algebra.

1 Introduction

We encounter many uncertainties in every moment of our lives. Often, classical mathematical logic is insufficient to get rid of these uncertainties. The reason is that when explaining a situation or a problem, it is not always possible to say that it is correct or certain. Neutrosophic logic and the concept of the neutrosophic set are defined in 1998 by Florentin Smarandache [1]. In the concept of neutrosophic logic and neutrosophic sets, there is a degree of membership T, a degree of uncertainty I, and a degree of non-membership F. These degrees are defined independently from each other. A neutrosophic value has the form (T, I, F). In other words, a situation is handled in neutrosophy according to its trueness, its indeterminacy, and its falsity. Thus, neutrosophic sets are a more general form of fuzzy sets [2] and intuitionistic fuzzy sets [3]. Several researchers have conducted studies on neutrosophic set theory [4-7]. Recently, Şahin and Uz studied multi-criteria decision-making applications based on set-valued generalized neutrosophic quadruple sets for law [8]; Şahin et al. obtained neutrosophic triplet partial g-metric spaces [9]; Kargın et al. introduced neutrosophic triplet m–Banach spaces [10]; Zhang et al. defined singular neutrosophic extended triplet groups and generalized groups [11]; Alhasan et al. studied neutrosophic reliability [12]; Mostafa et al. obtained hybridization between deep learning algorithms and neutrosophic theory in medical image processing [13]; Şahin et al. studied Hausdorff measures on generalized set-valued neutrosophic quadruple numbers and decision-making applications for the adequacy of online education [14].

Sigma algebra theory [15] has an important place in mathematics, especially in real analysis and probability theory. Sigma algebras have widespread use, especially in measurable functions and measurement theory. Also, Borel algebras are a widely used structure built on sigma algebras.

Florentin Smarandache defined neutro-structures and anti-structures in 2019 [16] and 2020 [17]. Similar to neutrosophic logic, an algebraic structure is divided into three regions: the set of elements that satisfy the conditions of the algebraic structure, the truth region A; the set of elements that do not meet the conditions of the algebraic structure, the uncertainty region Neutro-A, and the set of elements that do not meet the conditions of the algebraic structure, the inaccuracy region Anti-A. Thus, the structure of neutrosophic logic has been transferred into the structure of classical algebras, without using neutrosophic sets and neutrosophic numbers. Therefore, neutro-algebraic structures, which are more general structures than classical algebras, can be obtained. In addition, the region of the elements that do not satisfy any of the classical algebras is also taken as anti-algebraic structures. For this reason, many researchers have conducted studies on neutro-algebraic structures and anti-algebraic structures [18 - 21]. Recently, Smarandache studied the neutroalgebra [22]; Smarandache worked on neutro-algebraic structures and anti-algebraic structures [23]; Smarandache and Hamidi, defined neutro-bck-algebra [24]; Ibrahim and Agboola introduced neutro – vector spaces [25]; Şahin et al. studied neutro–G modules and anti–G modules [26]; Şahin et al. obtained neutro-topological space and anti-topological space [27]; Mirvakili et al. studied neutro-H v-semigroups [28]; Kargın and Şahin introduced neutro-law [29]; Hamidi and Smarandache defined single-valued neutro hyper BCK-Subalgebras [30].

In the second section, we define sigma algebra [15] and give basic definitions of neutro-structures [22]. Also, we give definitions of neutro-topology and anti-topology [27]. In the third section, we define the neutro-sigma algebra and we obtain its basic properties. Also, we give similarities and differences between the classical sigma algebra and the neutro-sigma algebra. We show that neutro-sigma algebras have a more general structure with respect to classical sigma algebra and a neutro-sigma algebra can be obtained from every classic sigma algebra. In the fourth section, we define anti-sigma algebra and we give its basic features. Furthermore, we obtain similarities and differences between the classic sigma algebra and the anti-sigma algebra. Also, we show that a neutro-sigma algebra can be obtained from every anti-sigma algebra and neutro-sigma algebras have a more general structure with respect to anti-sigma algebras. In the last part, results and suggestions are given.

2 Preliminaries

Definition 1. [22] The Neutro-sophication of the Law (degree of well-defined, degree of indeterminacy, degree of outer-defined)

Let X be a non-empty set. * be binary operation. For at least a pair (x, y) ∈ (X, X), $x * y \in X$ (degree of well defined, corresponding in neutrosophy to Truth (T)) and for at least two pairs (a, b), (c, d) ∈ (X, X), [$a * b =$ indeterminate (degree of indeterminacy, corresponding in neutrosophy to Indeterminate (I)) or $c * d \notin X$ (degree of outer-defined, corresponding in neutrosophy to Falsehood (F)].

Property 2. [22] In neutro-algebra, the classical well-defined for the binary operation * is divided into three regions: degree of well-defined (T), degree of indeterminacy (I), and degree of outer-defined (F) similar to neutrosophic set and neutrosophic logic.

Definition 3. **[22] The Anti-sophication of the Law** (Total OuterFunction)

Let X be a non-empty set and let * be a binary operation. For all pairs (x, y) ∈ (X, X), $x * y \notin X$ (totally outer-defined)

Definition 4. [27] Let X be a non-empty set and $\mathcal{T}$ be a collection of subsets of X. If at least one of the following conditions {i, ii, iii} is satisfied, then $\mathcal{T}$ is called a neutro-topology on X and $(X, \mathcal{T})$ is called a neutro-topological space.

i) $[\emptyset \in \mathcal{T}, X \notin \mathcal{T}$ or $X \in \mathcal{T}, \emptyset \notin \mathcal{T}]$ or $[\emptyset, X \in_I \mathcal{T}]$

ii) For at least n elements $p_1, p_2, \ldots\ p_n \in \mathcal{T}$,

$$\bigcap_{i=1}^{n} p_i \in \mathcal{T}$$

and for at least n elements $q_1, q_2, \ldots\ q_n \in \mathcal{T}$, and for at least n elements $r_1, r_2, \ldots\ r_n \in \mathcal{T}$;

$$[(\cap_{i=1}^{n} q_i \notin \mathcal{T} \text{ or } (\cap_{i=1}^{n} r_i \in_I \mathcal{T})]$$

where n is finite.

iii) For at least n elements $p_1, p_2, \ldots\ p_n \in \mathcal{T}$,

$$\bigcup_{i \in I} p_i \in \mathcal{T}$$

and for at least n elements $q_1, q_2, \ldots\ q_n \in \mathcal{T},\ r_1, r_2, \ldots\ r_n \in \mathcal{T}$;

$$[(\cup_{i \in I} q_i \notin \mathcal{T} \text{ or } (\cup_{i \in I} r_i \notin \mathcal{T})].$$

Definition 5. [27] Let X be a non-empty set and $\mathcal{T}$ be a collection of subsets of X. If the following conditions {Ai, Aii, Aiii} are satisfied, then $\mathcal{T}$ is called an anti-topology on X and $(X, \mathcal{T})$ is called an anti-topological space.

Ai) $\emptyset, X \notin \mathcal{T}$,

Aii) For all $q_1, q_2, \dots\ q_n \in \mathcal{T}$, $(\bigcap_{i=1}^{n} q_i \notin \mathcal{T})$ where n is finite,

Aiii) For all $q_1, q_2, \dots\ q_n \in \mathcal{T}$, $(\bigcup_{i \in I} q_i \notin \mathcal{T})$.

Definition 6. [15] Let S be a non-empty set and σ be a collection of subsets of S. If σ satisfies the following conditions, then σ is called a sigma algebra:

i) $\emptyset$ and S belongs to σ,

ii) For A $\in \sigma$, $A^c \in \sigma$,

iii) Any union of elements of σ belongs to σ,

iv) Any finite intersection of elements of σ belongs to σ.

3 Neutro-Sigma Algebras

Definition 7. Let S be a non-empty set and σ be a collection of subsets of S. If at least one of the following conditions {i, ii, iii, iv} is satisfied, then σ is called a neutro-sigma algebra.

i) [$\emptyset \in \sigma, S \notin \sigma$ or $S \in \sigma, \emptyset \notin \sigma$] or [$\emptyset, S \in_I \sigma$].

ii) For at least one element

$$s_1 \in \sigma,\ {s_1}^c \in \sigma$$

and for at least two elements

$$t_1 \in \sigma,\ m_1 \in \sigma;\ [({t_1}^c \notin \sigma \text{ or } ({m_1}^c \in_I \sigma)].$$

iii) For at least n elements

$$s_1, s_2, \dots\ s_n \in \sigma,\ \bigcap_{i=1}^{n} p_i \in \sigma$$

and for at least n elements $t_1, t_2, \dots\ t_n \in \sigma$, $m_1, m_2, \dots\ m_n \in \sigma$;

$$[(\bigcap_{i=1}^{n} t_i \notin \sigma \text{ or } (\bigcap_{i=1}^{n} m_i \in_I \sigma)],$$

where n is finite.

iv) For at least n elements

$$s_1, s_2, \dots\ s_n \in \sigma,\ \bigcup_{i\in I} p_i \in \sigma$$

and for at least n elements $t_1, t_2, \dots\ t_n \in \sigma$, $m_1, m_2, \dots\ m_n \in \sigma$;

$$[(\bigcup_{i\in I} t_i \notin \sigma \text{ or } (\bigcup_{i\in I} m_i \notin \sigma)].$$

We obtain Definition 7 using Definition 1 and Property 2.

Note 8: In this chapter, the symbol "$=_I$" will be used for situations where equality is uncertain. For example, if it is not certain whether "a" and "b" are equal, then it is denoted by a $=_I$ b.

Note 9: In this chapter, the symbol "$\in_I$" will be used for situations where the inclusion is not obvious. For example, if it is not certain whether "a" is a member of the set B, then it is denoted by a $\in_I$ B.

The notation in Note 8 and Note 9 is the same as in [27].

Corollary 10:

i) By condition {ii} of Definition 7, neutro-sigma algebras differ from neutro-topology (in Definition 4). In addition, every neutro-sigma algebra is a neutro-topology. However, the reverse is not always true.

ii) From Definition 7, the neutro-sigma algebras differ from the classical sigma algebras in Definition 6. Also, the neutro-sigma algebras are given as an alternative to the classical sigma algebras. However, for a neutro-sigma algebra, instead of the ones that are not met in Definition 7, classical sigma algebra conditions are valid.

Example 11. Let $S = \{k, l, m, n\}$ be a set and $\sigma = \{\emptyset, \{k\}, \{k, l\}, \{m, n\}, \{l, m\}\}$ be a collection of subsets of S. Then,

i) It is clear that $S \notin \sigma$ and $\emptyset \in \sigma$.

ii) Let $S_1 = \{k, l\}$ and $S_2 = \{k\}$. It is clear that

$${S_1}^c \in \sigma \text{ and } {S_2}^c \notin \sigma.$$

iii) Let $S_1 = \{k\}$, $S_2 = \{k, l\}$, $T_1 = \{m, n\}$ and $T_2 = \{l, m\}$. We obtain that

$$S_1 \cap S_2 \in \sigma \text{ and } T_1 \cap T_2 \notin \sigma.$$

iv) Let $S_1 = \{k, l\}$, $S_2 = \{m, n\}$, $T_1 = \{k\}$ and $T_2 = \{l, m\}$. We obtain that

$$S_1 \cup S_2 \in \sigma \text{ and } T_1 \cup T_2 \notin \sigma.$$

Hence, σ satisfies the conditions {i, ii, iii, iv} of Definition 7. Thus, σ is a neutro-sigma algebra.

Example 12. Let S $= \{k, l, m, n\}$ be a set and $\sigma = \{\emptyset, \text{S}, \{k\}, \{l\}, \{m\}, \{n\}, \{m, n\}, \{k, l\}\}$ be a collection of subsets of S. Then,

ii) Let $S_1 = \{m, n\}$ and $S_2 = \{l\}$. It is clear that

$${S_1}^c \in \sigma \text{ and } {S_2}^c \notin \sigma.$$

iv) Let $S_1 = \{k\}$, $S_2 = \{l\}$, $T_1 = \{n\}$ and $T_2 = \{k, l\}$. We obtain that

$$S_1 \cup S_2 \in \sigma \text{ and } T_1 \cup T_2 \notin \sigma.$$

Thus, σ satisfies the conditions {ii, iv} of Definition 7. Hence σ is a neutro-sigma algebra.

We note that σ satisfies the classical sigma algebra conditions {i, iii}.

Corollary 13. In Example 11, σ is a neutro-sigma algebra, but σ is not a classical sigma algebra. In Example 12; σ is a neutro-sigma algebra but σ is not a classical sigma algebra. Thus, neutro-sigma algebras have a more general structure than classical sigma algebras.

Theorem 14. Let σ be a classical sigma algebra. Then, $\sigma - \emptyset$ is a neutro-sigma algebra.

Proof: Since σ is a classical sigma algebra, it is clear that S $\in \sigma$. Hence,

$$S^c = \emptyset \notin \sigma - \emptyset. \tag{1}$$

Also, since σ is a classical sigma algebra; we have, for all A $\in \sigma - \emptyset$

$$A^c \in \sigma - \emptyset. \tag{2}$$

Therefore, from (1) and (2); $\sigma - \emptyset$ satisfies the condition {ii} of Definition 7. Thus, $\sigma - \emptyset$ is a neutro-sigma algebra.

Also, $\sigma - \emptyset$ satisfies the condition {i} of Definition 7.

Theorem 15. Let σ be a classical sigma algebra. Then, $\sigma - \mathrm{S}$ is a neutro-sigma algebra.

Proof: Since σ is a classical sigma algebra, it is clear that $\emptyset \in \sigma$. Hence,

$$\emptyset^c = \mathrm{S} \notin \sigma - \mathrm{S}\,. \tag{3}$$

Also, since σ is a classical sigma algebra; we obtain that for all $\mathrm{A} \in \sigma - \mathrm{S}$

$$A^c \in \sigma - \mathrm{S}\,. \tag{4}$$

Therefore, from (3) and (4); $\sigma - \mathrm{S}$ satisfies the condition {ii} of Definition 7. Thus, $\sigma - \mathrm{S}$ is a neutro-sigma algebra.

Also, $\sigma - \mathrm{S}$ satisfies the condition {i} in Definition 7.

Theorem 16. Let σ be a classical sigma algebra and A be a set such that $\mathrm{A^c} \notin \sigma$. Then, $\sigma \cup \mathrm{A}$ is a neutro-sigma algebra.

Proof: It is clear that since $\mathrm{A^c} \notin \sigma$, we obtain

$$\mathrm{A^c} \notin \sigma \cup \mathrm{A} \tag{5}$$

Also, since σ is a classical sigma algebra; we obtain that for all $\mathrm{B} \in \sigma$

$$\mathrm{B^c} \in \sigma \tag{6}$$

Therefore, from (5) and (6); $\sigma \cup \mathrm{A}$ satisfies the condition {ii} of Definition 7. Thus, $\sigma \cup \mathrm{A}$ is a neutro-sigma algebra.

Theorem 17. Let σ be a classical sigma algebra and A be an element of σ. Then, $\sigma - \mathrm{A}$ is a neutro-sigma algebra.

Proof: It is clear that since $\mathrm{A} \in \sigma$, we obtain

$$(\mathrm{A^c})^c = \mathrm{A} \notin \sigma - \mathrm{A}. \tag{7}$$

Also, since σ is a classical sigma algebra; we obtain that for all $\mathrm{B} \in \sigma - \mathrm{A}$,

$$\mathrm{B^c} \in \sigma - \mathrm{A}. \tag{8}$$

Therefore, from (7) and (8); $\sigma - \mathrm{A}$ satisfies the condition {ii} of Definition 7. Thus, $\sigma - \mathrm{A}$ is a neutro-sigma algebra.

Corollary 18.

i) From Theorem 14, Theorem 15, Theorem 16, and Theorem 17, we obtain that a neutro-sigma algebra can be obtained from every classical sigma algebra.

ii) The classical sigma algebras do not satisfy the Theorem 14, Theorem 15, Theorem 16, and Theorem 17. However, neutro-sigma algebras satisfy Theorem 19 and Theorem 21.

Theorem 19. Let (σ_i) be a non-empty family of neutro-sigma algebras σ_i such that $\emptyset \in \sigma_i$, $S \notin \sigma_i$ $(i = 1, 2, \dots, n)$. Then, $\bigcup_{i=1}^{n}(\sigma_i)$ is a neutro-sigma algebra on S.

Proof: Since (σ_i) is a non-empty family of neutro-sigma algebras σ_i such that $\emptyset \in \sigma_i$, $S \notin \sigma_i$; it is clear that

$$\emptyset^c = S \notin \bigcup_{i=1}^{n}(\sigma_i) \; . \tag{9}$$

Also, since (σ_i) is a non-empty family of neutro-sigma algebras σ_i such that $\emptyset \in \sigma_i$, $S \notin \sigma_i$; we obtain that for all $B \in \bigcup_{i=1}^{n}(\sigma_i) - \emptyset$,

$$B^c \in \bigcup_{i=1}^{n}(\sigma_i) - \emptyset \; . \tag{10}$$

Therefore, from (9) and (10); $\bigcup_{i=1}^{n}(\sigma_i)$ satisfies the condition {ii} in Definition 7. Thus, $\bigcup_{i=1}^{n}(\sigma_i)$ is a neutro-sigma algebra.

Also, $\bigcup_{i=1}^{n}(\sigma_i)$ satisfies the condition {i} of Definition 7.

Example 20. Let $S = \{k, l, m, n\}$ be a set and $\sigma_1 = \{\emptyset, \{k\}, \{k, l\}, \{m, n\}, \{l, m\}\}$, $\sigma_2 = \{\emptyset, \{k\}, \{l\}, \{m\}, \{n\}, \{m, n\}, \{k, l\}\}$ be a collection of subsets of S. Then, from Example 12, σ_1 is a neutro-sigma algebra. Also, σ_2 is a neutro-sigma algebras since σ_2 satisfies conditions {i, ii, iv} of Definition 7.

Now, we show that

$$\sigma_1 \cup \sigma_2 = \{\emptyset, \{k\}, \{l\}, \{m\}, \{n\}, \{m, n\}, \{k, l\}, \{l, m\}\}$$

is a neutro-sigma algebra.

i) $\emptyset \in \sigma_1 \cup \sigma_2$ and $S \notin \sigma_1 \cup \sigma_2$.

ii) $\{m, n\}^c \in \sigma_1 \cup \sigma_2$ and $\{m\}^c \notin \sigma_1 \cup \sigma_2$.

iii) $\{k\} \cup \{l\} \in \sigma_1 \cup \sigma_2$ and $\{k, l\} \cup \{m, n\} \notin \sigma_1 \cup \sigma_2$.

iv) $\{k\} \cap \{k,l\} \in \sigma_1 \cup \sigma_2$ and $\{m,n\} \cap \{l,m\} \notin \sigma_1 \cup \sigma_2$.

Hence, $\sigma_1 \cup \sigma_2$ satisfies the conditions {i, ii, iii, iv} of Definition 7 and $\sigma_1 \cup \sigma_2$ is a neutro-sigma algebra.

Theorem 21. Let (σ_i) be a non-empty family of neutro-sigma algebras σ_i such that $\emptyset \notin \sigma_i$, $S \in \sigma_i$ $(i = 1, 2, \dots, n)$. Then, $\bigcup_{i=1}^{n}(\sigma_i)$ is a neutro-sigma algebra.

Proof: Since (σ_i) is a non-empty family of neutro-sigma algebras σ_i such that $\emptyset \notin \sigma_i$, $S \in \sigma_i$, it is clear that

$$S^c = \emptyset \notin \bigcup_{i=1}^{n}(\sigma_i) \tag{11}$$

Also, since (σ_i) is a non-empty family of neutro-sigma algebras σ_i such that $\emptyset \in \sigma_i$, $S \notin \sigma_i$; we obtain that for all $B \in \bigcup_{i=1}^{n}(\sigma_i) - S$,

$$B^c \in \bigcup_{i=1}^{n}(\sigma_i) - S . \tag{12}$$

Therefore, from (11) and (12); $\bigcup_{i=1}^{n}(\sigma_i)$ satisfies the condition {ii} of Definition 7. Thus, $\bigcup_{i=1}^{n}(\sigma_i)$ is a neutro-sigma algebra.

Also, $\bigcup_{i=1}^{n}(\sigma_i)$ satisfies the condition {i} of Definition 7.

Example 22. Let $S = \{k,l,m,n\}$ be a set and $\sigma_1 = \{S, \{k\}, \{k,l\}, \{m,n\}, \{l,m\}\}$, $\sigma_2 = \{S, \{k\}, \{l\}, \{m\}, \{n\}, \{m,n\}, \{k,l\}\}$ be a collection of subsets of S. Then, σ_1 is a neutro-sigma algebra since σ_1 satisfies conditions {i, ii, iii, iv} of Definition 7. Also, σ_2 is a neutro-sigma algebras since σ_2 satisfies the conditions {i, ii, iv} of Definition 7.

Now, we show that

$$\sigma_1 \cup \sigma_2 = \{S, \{k\}, \{l\}, \{m\}, \{n\}, \{m,n\}, \{k,l\}, \{l,m\}\}$$

is a neutro-sigma algebra.

i) $\emptyset \notin \sigma_1 \cup \sigma_2$ and $S \in \sigma_1 \cup \sigma_2$.

ii) $\{k,l\}^c \in \sigma_1 \cup \sigma_2$ and $\{l\}^c \notin \sigma_1 \cup \sigma_2$.

iii) $\{m\} \cup \{n\} \in \sigma_1 \cup \sigma_2$ and $\{k\} \cup \{l,m\} \notin \sigma_1 \cup \sigma_2$.

iv) $\{k\} \cap \{l\} \in \sigma_1 \cup \sigma_2$ and $\{m\} \cap \{l\} \notin \sigma_1 \cup \sigma_2$.

Hence, $\sigma_1 \cup \sigma_2$ satisfies the conditions {i, ii, iii, iv} of Definition 7 and $\sigma_1 \cup \sigma_2$ is a neutro-sigma algebra.

Corollary 23. The classical sigma algebras do not satisfy Theorem 19, Theorem 21, Example 20, and Example 22. However, neutro-sigma algebras satisfy Theorem 19, Theorem 21, Example 20, and Example 22.

4 Anti-Sigma Algebras

Definition 24. Let S be a non-empty set and σ be a collection of subsets of S. If the following conditions {i, ii, iii, iv} are satisfied, then σ is called an anti-sigma algebra on S.

i) $\emptyset$, S $\notin \sigma$,

ii) for all $S_i \in \sigma$, ${S_i}^c \notin \sigma$,

iii) for all $S_1, S_2, \dots\ S_n \in \sigma$, $(\bigcap_{i=1}^{n} S_i \notin \sigma)$,

iv) for all $S_1, S_2, \dots\ S_n \in \sigma$, $(\bigcup_{i\in I} S_i \notin \sigma)$.

Example 25. Let S $= \{k, l, m, n\}$ be a set and $\sigma = \{\{k\}, \{l\}, \{m\}\}$ be a collection of subsets of S. Then,

i) It is clear that $\emptyset \notin \sigma$ and S $\notin \sigma$.

ii) Let

$$S_1 = \{k\}\,, S_2 = \{l\}\,, S_3 = \{m\}.$$

Thus, we have

$$(S_1)^c \notin \sigma,\ (S_2)^c \notin \sigma,\ (S_3)^c \notin \sigma.$$

iii) Let

$$S_1 = \{k\}\,, S_2 = \{l\}\,, S_3 = \{m\}.$$

So,

$$\bigcap_{i=1}^{3} S_i \notin \sigma.$$

iv) Let

$$S_1 = \{k\}\,, S_2 = \{l\}\,, S_3 = \{m\}.$$

Then, we have

$$\bigcup_{i=1}^{3} S_i \notin \sigma.$$

Thus, σ satisfies the {i, ii, iii, iv} conditions of Definition 24. Therefore, σ is an anti-sigma algebra on S.

Corollary 26: In Example 25, σ is an anti-sigma algebra. But σ is not a neutro-sigma algebra or a classical sigma algebra. Thus, anti-sigma algebras are different from neutro-sigma algebras and classical sigma algebras.

Theorem 27. Let σ be an anti-sigma algebra on S, A and A^c be two sets such that

$$A \notin \sigma \text{ and } A^c \notin \sigma.$$

Then, $\sigma \cup A \cup A^c$ is a neutro-sigma algebra.

Proof: As $A \notin \sigma$ and $A^c \notin \sigma$, it is clear that

$$A \in \sigma \cup A \cup A^c \text{ and } A^c \in \sigma \cup A \cup A^c . \quad (13)$$

Also, since σ is an anti-sigma algebra; we obtain that for all $B \in \sigma$,

$$B^c \notin \sigma . \quad (14)$$

So, by (13) and (14); $\sigma \cup A \cup A^c$ satisfies the condition {ii} of Definition 7. Thus, $\sigma \cup A \cup A^c$ is a neutro-sigma algebra.

Theorem 28. Let σ be an anti-sigma algebra on S. Then, $\sigma \cup \emptyset \cup S$ is a neutro-sigma algebra.

Proof: Since σ is an anti-sigma algebra on S, for all $S_i \in \sigma$, we have ${S_i}^c \notin \sigma$.

$$S^c = \emptyset \in \sigma \cup \emptyset \cup S \text{ and } \emptyset^c = S \in \sigma \cup \emptyset \cup S.$$

Thus, $\sigma \cup \emptyset \cup S$ satisfies condition {ii} of Definition 7. Hence, $\sigma \cup \emptyset \cup S$ is a neutro-sigma algebra.

Also, $\sigma \cup \emptyset \cup S$ satisfies condition {iii, iv} of Definition 7.

In addition, in proof of Theorem 27; if we assume that

$$A = \emptyset \text{ and } A^c = S \text{ or } A = S \text{ and } A^c = \emptyset,$$

then $\emptyset$ and S satisfy Theorem 28.

Theorem 29. Let σ be an anti-sigma algebra on S and A be an element of σ. Then, $\sigma \cup A^c$ is a neutro-sigma algebra.

Proof: Since σ is an anti-sigma algebra on S, for all $S_i \in \sigma$, ${S_i}^c \notin \sigma$. However, it is clear that

$$A^c \in \sigma \cup \mathrm{A^c}.$$

Thus, $\sigma \cup \mathrm{A^c}$ satisfies condition {ii} of Definition 7. Hence, $\sigma \cup \mathrm{A^c}$ is a neutro-sigma algebra.

Example 30. In Example 25, $\sigma = \{\{k\},\{l\},\{m\}\}$ is an anti-sigma algebra on S. By Theorem 27, $\sigma \cup \emptyset$ is a neutro-sigma algebra.

Example 31. In Example 25, $\sigma = \{\{k\},\{l\},\{m\}\}$ is an anti-sigma algebra on S. By Theorem 28, $\sigma \cup \mathrm{S}$ is a neutro-sigma algebra.

Example 32. In Example 25, $\sigma = \{\{k\},\{l\},\{m\}\}$ is an anti-sigma algebra on S. By Theorem 29, $\sigma \cup \emptyset \cup \mathrm{S}$ is a neutro-sigma algebra.

Example 33. In Example 25, $\sigma = \{\{k\},\{l\},\{m\}\}$ is an anti-sigma algebra on S. By Theorem 30,

$$\sigma \cup \{k\}^c = \{\{k\},\{l\},\{m\},\{l,m,n\}\}$$

is a neutro-sigma algebra.

Corollary 34.

i) From Theorem 27, Theorem 28, and Theorem 29, we obtain that a neutro-sigma algebra can be obtained from every anti-sigma algebra.

ii) Neutro-topology and anti-topology do not satisfy Theorem 27, Theorem 28, and Theorem 29. Thus, neutro-sigma algebras and anti-sigma algebras have a more general structure than neutro-topology and anti-topology.

5 Conclusions

In this study, a neutro-sigma algebra is defined and relevant basic properties are given. Similarities and differences between the classical sigma algebras and neutro-sigma algebras are discussed. We show that a neutro-sigma algebra can be obtained from every classical sigma algebra. In addition, we define anti-sigma algebras and we give corresponding basic properties. We discuss similarities and differences between the classical sigma algebra and anti- sigma algebras. Also, we show that a neutro-sigma algebra can be obtained from every anti-sigma algebra. In addition, we show that neutro-sigma algebras and anti-sigma algebras have a more general structure than neutro-topology and anti-topology. Thus, we add new structures to the neutro-algebra theory.

By using the definition of neutro-sigma algebras and anti-sigma algebras, researchers can define neutro-sigma measurable functions, anti-sigma measurable functions, neutro-Borel algebras, and anti-Borel algebras.

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Model Based on Neutrosophic Ontologies for the Study of Entrepreneurship Competence

Noel Batista Hernández[1], Lisbeth Josefina Reales Chacón[2], and Lilia Esther Valencia Cruzaty [3], Gina Real Zumba[4], Wilmer Ortega Chávez[5], Jesús Tito Quispe[6]

[1] Universidad Regional Autónoma de Los Andes; Babahoyo, Ecuador. E-mail noelbatista1965@gmail.com
[2] Universidad Técnica de Ambato, Ecuador; E-mail lj.reales@uta.edu.ec
[3] Universidad Estatal Península de Santa Elena; Santa Elena, Ecuador. E-mail: lvalencia@upse.edu.ec
[4 2]Universidad Técnica de Babahoyo; Babahoyo, Ecuador. E-mail: gina_realz26@hotmail.com
[5] Universidad Nacional Intercultural de la Amazonía; Pucallpa, Perú. E-mail: wortegac@unia.edu.pe
[6] Universidad Nacional Intercultural de la Amazonía; Pucallpa, Perú. E-mail: jtitoq@unia.edu.pe

Abstract. The entrepreneurship competence is of vital importance in the teacher training of students. It is inconceivable that citizens who are trained in schools do not have the ability to be autonomous, creative and daring, to make decisions that transform the social life in which we are immersed for the better. However, we do not always find institutions with the right education to develop this competence. That is why the aim of this article is to propose a model for the representation and obtaining of knowledge about the entrepreneurship competence in any educational system. For this end, there are neutrosophic ontologies, which is a tool to easily reflect the relationships between concepts. The use of neutrosophy allows us taking into account the truthfulness, falseness and indeterminacy of the belonging of an object to a specific set, for its evaluation and classification. To the knowledge of the authors, this is the first time that neutrosophic ontology has been used to model entrepreneurship competence.

Keywords: Entrepreneurship competence, ontology, neutrosophic ontology, education.

1 Introduction

An intention of change in education demands integrating training actions from the continuity that implies the verification, distribution and use of knowledge. It is transcendent that the learner manages diverse concepts and faces dissimilar interpretations that favor their critical thinking in the field of their reality.

Training by competencies confers the approach of the educational process from the complex and organic appreciation, which compose, knowledge, abilities, skills, attitudes and values, in synergistic interaction that makes viable the autonomous performance of the individual, by equipping him (her) with tools to create, manage, interpret, understand and transform the social environment with a proactive, holistic and innovative vision. This relevant appropriation specifies the need to develop the entrepreneurship competence in education as a contribution to the integral formation of the student, from its essence of systemic realization, which nurtures and agrees with other generic, basic and specific competences.

In this sense, the programs to develop the entrepreneurship competence in educational institutions, in general, do not educate for entrepreneurship, but rather guide about it and do not focus on skills, attributes and competencies of an entrepreneur, on the contrary, it focuses teaching on the creation of new companies and business administration, aspects that are insufficient to guarantee the training of entrepreneurial students, so the advancement of attributes should be strengthened as a priority, thinking, attitude and values [1].

The theoretical validation of any strategy expresses the establishment that the purposes for which it is established meet the requirements for the intended applications. It is the process by which it is demonstrated that the procedures and actions to be developed are pertinent to consummate the indicated objectives, the suitability of the strategy to achieve the expected performance is certified by experts or other methodologies. When we study the entrepreneurship competence, we must define which projects we consider most valuable for society, because they will be the ones that we will try to get students to undertake. We are not interested in educating the competence to undertake if we do not establish personal and social objectives, specified in a set of shared values.

One way to measure this process is by representing knowledge through what is known as Knowledge Engineering, which is the theoretical basis for dealing with the knowledge acquired by experts and reflected in publications, whether monographs, books, scientific articles, among other supports.

A tool that allows us the representation of knowledge is ontology [2-4]. The term ontology in computer science refers to the formulation of an exhaustive and rigorous conceptual scheme within one or several given domains; in order to facilitate communication and the exchange of information between different systems and entities. Although it takes its name by analogy, this is the difference with the philosophical point of view of the word ontology.

A current technological common use of the concept of ontology, in this semantic sense, is found in artificial intelligence and knowledge representation. In some applications, several schemas are combined into a *de facto* complete data structure, containing all the relevant entities and their relationships within the domain.

Computer programs can thus use this view of ontology for a variety of purposes, including inductive, and a variety of problem-solving techniques.

Typically, ontologies in computers are closely related to fixed vocabularies (a foundational ontology) with whose terms everything else must be described. Because this can lead to poor representations for certain problem domains, more specialized schemas must be created to make the data useful for real-world decision-making.

Classical ontologies based on bivalent logic allow us the representation of knowledge only in a strict way, where the measurement of the belonging of an entity or object to a certain concept can be assumed without nuances, where either it is white or it is black. This does not correspond to reality, where there is uncertainty. That is why fuzzy ontologies allow us a more exact modeling of reality, where the object belongs to a fuzzy set with a certain degree of veracity, which is given by the lack of certainty that exists in the world around us, especially the world of social relationships, [3,4].

Although fuzzy ontologies better reflect reality, because they include shades of gray in the evaluations, even more precision is needed at the cost of greater indeterminacy; this can be solved with the help of neutrosophic logic. Neutrosophic logic allows the inclusion of degrees of truthfulness, falsity and indeterminacy explicitly. That is why in this paper we use neutrosophic ontologies as a way of representing the knowledge of the entrepreneurship competence. Other papers that can be found on the application of neutrosophy in pedagogy can be consulted on [5-11].

In other words, this paper aims to offer a model for representing and obtaining knowledge about the entrepreneurship competence with the help of neutrosophic ontologies. To do this, in the following section the fundamental concepts are developed, such as some details about the entrepreneurial competence and the basic notions about neutrosophic ontologies. Section 3 contains the fundamentals of the proposed model, while the last section is dedicated to giving the conclusions.

2 Preliminaries

2.1 Entrepreneurship competence

Learning development strategies are complex constructs that are oriented towards decision-making resulting from a training need, containing actions that activate knowledge in close correspondence with the search for the achievement of pre-established goals, effectively ([1,5,9,12-14]).

The definition of competence as the relevant appropriation of cross-cutting skills, knowledge, attitudes and values, which, when persistently updated, allow the individual to effectively and responsibly guide their interaction and development in dissimilar social settings.

The entrepreneurship competence has the duality of integrating terminal objectives and procedural elements, which combine substantive and adjective functions at the same time, all that it allows us from its decomposition into relevant cognitive nuclei and to group its dimensions into four cardinal groups:

Instrumental Dimension*:* (relates the procedural-adjective components) planning, organization, execution, control, management, evaluation, communication, project design, negotiation, manifest skills and abilities.

Cognitive dimension*:* (brings together the resources of appropriation and use of knowledge) learning to learn, interpreting in social reality, understanding the environment, understanding, solving problems, establishing judgments and reasoning, information management and its management for the development of comprehensive general culture.

Attitudinal dimension: (summarizes the motivational-volitional compositions of the competence) creativity, initiative, critical thinking, holistic vision, leadership, decision-making, teamwork, proactivity, risk management, motivation and audacity to achieve one's own goals or that of the group, perseverance, autonomy in action and the ability to delay the need for immediate satisfaction, development of the will to innovate.

Axiological dimension*:* (provides resources related to values and the conditioning of acting) resilience, optimism, responsibility, sustainability, altruism, preponderance of social interest, equity, respect for differences and equality, care for the environment.

Each one of these dimensions reaches its own magnitudes that distinguish them, but they cannot be isolated, since they form a unit. Its operating system is systemic, which increases its synergistic behavior to the extent that they are enhanced.

The dimensions function as a complex and coherent organization, in which each element fulfills a function, establishes an order, involves a logic of relationships, which give fullness to the whole and distinguish it, by directing and complementing the development that gives rise to competition.

Approaching the entrepreneurship competence from the pedagogical performance of teachers in any level of education means the development of constructs that promote participation, inclusion and social responsibility. It is to develop a quality dedicated to individual and group protagonism.

It is to train the student by reinforcing the attitudinal, cognitive and axiological elements that lead to the mutation of the role of passive executors to men and women with critical thinking, actors of change, producers of innovative ideas, of viable projects, with the aptitudes and attitudes to materialize it. This is graphically summarized in Figure 1.

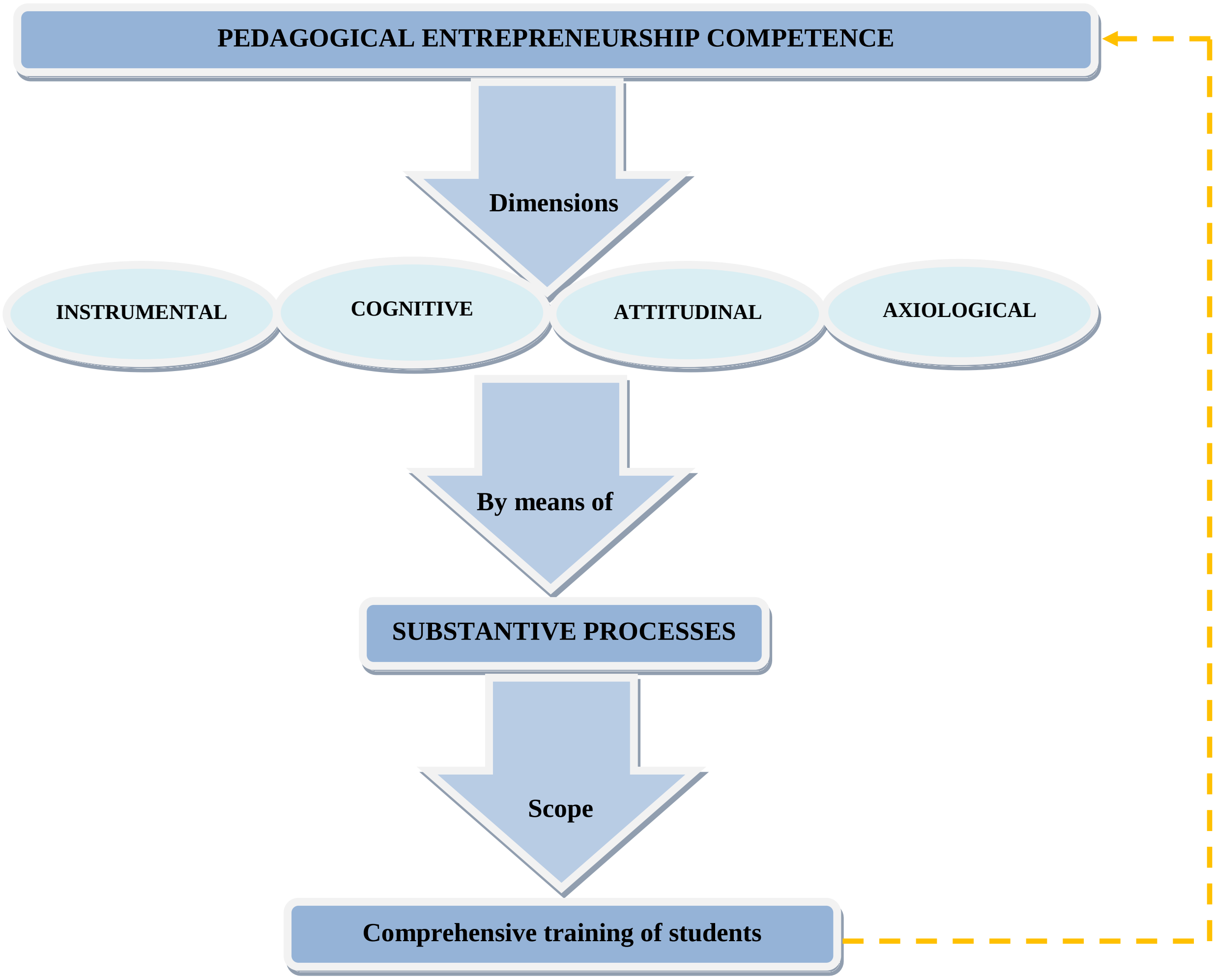

Figure 1: Graphical representation of the development of the entrepreneurship competence. Source ([1]).

2.2 Neutrosophic Ontologies

Classical ontologies are based on objects and their relationships and have the following components ([2]):

- Individuals: Instances or objects.
- Classes: Sets, collections, concepts, etc.
- Attributes: Aspects, properties, features, characteristics, etc.
- Relations: How classes and individuals relate to one another.
- Function terms: Complex structures designed from certain relations.
- Restrictions: Constrains which describe what is true to be accepted as input.
- Rules: Statements in the form of IF-THEN sentences.
- Axioms: Assertions which include rules, in a logical form that comprise the overall theory that the ontology describes in its domain of application.

Figure 2 represents an example of ontology:

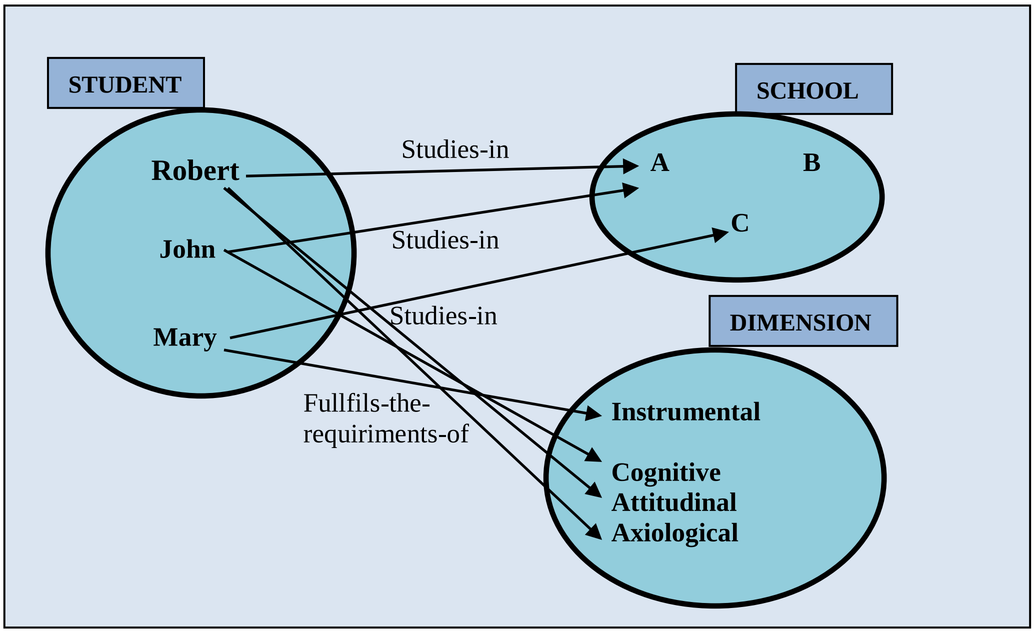

Figure 2: Graphical representation of a basic generic ontology. Source: The authors.

Neutrosophic logic is a logic in which every proposition is estimated to have the degree of truthfulness, indeterminacy, and falsity (T, I, F).

Definition 1: ([15,16]) The *Neutrosophic set* N is characterized by three membership functions, which are the truth-membership function T_A, indeterminacy-membership function I_A, and falsity-membership function F_A, where U is the Universe of Discourse and $\forall x \in U$, $T_A(x), I_A(x), F_A(x) \subseteq]^-0, 1^+[$, and $^-0 \leq \inf T_A(x) + \inf I_A(x) + \inf F_A(x) \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

See that according to Definition 1, $T_A(x), I_A(x), F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$ and hence, $T_A(x), I_A(x), F_A(x)$ can be subintervals of $[0, 1]$.

Definition 2: ([15,16]) The *Single-Valued Neutrosophic Set* (SVNS) N over U is $A = \{< x; T_A(x), I_A(x), F_A(x) >: x \in U\}$, where $T_A: U \rightarrow [0, 1]$, $I_A: U \rightarrow [0, 1]$, and $F_A: U \rightarrow [0, 1]$, $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

The *Single-Valued Neutrosophic number* (SVNN) is symbolized by $N = (t, i, f)$, such that $0 \leq t, i, f \leq 1$ and $0 \leq t + i + f \leq 3$.

Definition 3: ([17]) A neutrosophic ontology is a sextuple $NO = < I, C, T, N, X, indeterminacy >$ where I is the set of instances, C is the set of classes. T denotes the taxonomy relations among the set of concepts C. N denotes the set of non-taxonomy neutrosophic associative relationships. X is the set of axioms expressed in a proper logical language. Indeterminacy is the degree of indeterminacy existing in the overlapping region.

A Neutrosophic Ontology in the example of Figure 2 contains triple truth values for truthfulness, indeterminacy, and falseness. For example, "Studies-in" can be associated with $< 1,0,0 >$ in all the three cases because a student usually studies in only one school. On the other hand, for the relationship "Fulfils-the-requirements-of" there are different degrees of satisfaction, for example Robert could satisfy the Instrumental dimension with $< 0.03,0.2,0.9 >$, the Cognitive with $< 0.1,0.1,0.8 >$, the Attitudinal with $< 0.9,0.1,0.2 >$, and the Axiological with $< 0.85,0.15,0.2 >$.

3 The Model

In the proposed model, the neutrosophic ontology is used to represent knowledge and to evaluate each student in terms of its ability to undertake.

The set I in this case is the set of students that are going to be evaluated on their entrepreneurship competence. Set C is that of the classes involved, in this case "Has the entrepreneurship competence", which are measured by the attributes of having the "Instrumental", "Cognitive", "Attitudinal" and "Axiological" dimensions. N contains the opposite of the concepts within the classes, that is, "does not have the entrepreneurship competence".

The rules defined in this model are very simple and obvious:

1. If student S satisfies the "Instrumental" dimension, then he or she has a greater capacity for "Entrepreneurship Competence".

2. If student S satisfies the "Cognitive" dimension, then he or she has a greater capacity for “Entrepreneurship Competence”.
3. If student S satisfies the “Attitudinal” dimension, then he or she has a greater capacity for “Entrepreneurship Competence”.
4. If student S satisfies the “Axiological dimension”, then he or she has a greater capacity for “Entrepreneurship Competence”.
5. If student S satisfies a greater number of dimensions, then he or she will have a greater capacity for “Entrepreneurship Competence”.

The teachers of the students to evaluate are invited to give an evaluation out of 100 points that reflects the behavior of the students on each of the previous dimensions. This must be stored in a database. The evaluators are explained that for each evaluation they must give 3 values out of 100 points, the first value corresponds to how certainty they have that the student satisfies the indicated attribute, which is one of the 4 dimensions, the second value corresponds within the same scale to the indeterminacy that the evaluator has about the satisfaction or not of the dimension, while the third value corresponds to the certainty that the student does not satisfy the attribute. Although evaluators are asked for a greater number of elements to evaluate, this will result in greater accuracy, taking into account that teachers are not always satisfied with giving a grade in a single number that often leaves them dissatisfied.

The proposed ontology is visualized in Figure 3.

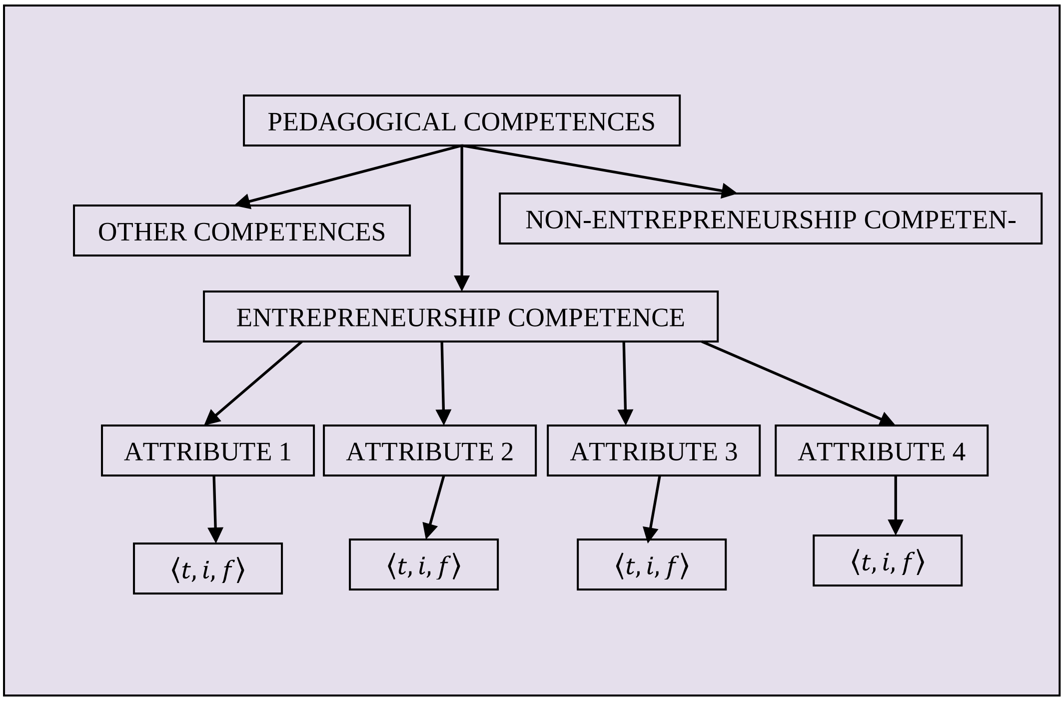

Figure 3: Neutrosophic Ontology of entrepreneurship competence. Source: The authors.

Note that the triple $\langle t, i, f\rangle$ corresponds to the neutrosophic evaluation of each student, as explained above.

Following the 5 rules, they can be summarized in that the entrepreneurship competence is measured according to the satisfaction of the 4 attributes (or dimensions). In addition to the fact that the greater the number of satisfied attributes, the greater the entrepreneurship competence, this is reflected in the following logical predicate:

$$EC \leftrightarrow Inst\ AND\ Cogn\ AND\ Attit\ AND\ Axiol \quad (1)$$

Where *EC* denotes the entrepreneurship competence, *Int*, *Cogn*, *Attit* and *Axiol*, represents each one of the dimensions and AND is the logical conjunction operator.

To perform the calculations, the grades given to the student are de-neutrosified using the following formula ([18]):

$$S(\langle t, i, f\rangle) = \frac{1}{3}\left(2 + \frac{t-i-f}{100}\right) \quad (2)$$

Then the AND is applied using the classical t-norm min(·) formula.

Let us illustrate the use of the proposed model with a generic example:

Example 1. Let us revisit the example in Figure 2 where the evaluations of the 3 students are summarized in Table 1.

Student (Instance)	Attrib1(Int)	Attrib2 (Cogn)	Attrib3 (Attit)	Attrib4 (Axiol)	CE
Robert	$< 3,20,90 >$	$< 10,10,80 >$	$< 90,10,20 >$	$< 85,15,20 >$	0.31
John	$< 60,10,30 >$	$< 70,5,15 >$	$< 63,7,24 >$	$< 80,15,20 >$	0.73333
Mary	$< 90,1,5 >$	$< 95,2,1 >$	$< 80,0,20 >$	$< 81,1,21 >$	0.86333

Table 1: Neutrosophic Ontology of the example.

We can see, according to Table 1 that Mary has a better development of the entrepreneurship competence, followed by John. On the other hand, the assessment of Robert is low.

Conclusion

The Entrepreneurship Competence is essential for the useful development of the student within the school. This competence will provide us with citizens who have initiatives that promote the progress of the society in which we live, while being critical and who have their own innovative thinking. One of the challenges that modern pedagogy has is to give it the importance that this competence deserves within the curriculum. That is why in this article we propose a model where this competence is measured in students of any level of education. The model is based on the use of neutrosophic ontologies, where ontology is combined as a technology to represent knowledge and neutrosophy that allows us the evaluation of relationships among concepts using neutrosophic numbers. To the knowledge of the authors, neutrosophic ontologies are used for the first time in the modeling of the Entrepreneurship Competence.

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Neutrosophic Mathematical Formulas of Transportation Problems

Maissam Jdid [1], **Huda E. Khalid** *[2].

[1] Faculty of Informatics Engineering, Al-Sham Private University, Damascus, Syria; m.j.foit@aspu.edu.sy

[2] Telafer University, The Administration Assistant for the President of the Telafer University, Telafer, Iraq;

https://orcid.org/0000-0002-0968-5611

*Corresponding Author: Huda E. Khalid. Email: dr.huda-ismael@uotelafer.edu.iq

Abstract.

This paper comes side by side with the complement paper (Original Methods for Obtaining the Initial Solution in Neutrosophic Transportation Problems), these two papers regarded as twins, they are both dedicated to sounding the transportation problems from the perspective of neutrosophic theory, having kinds of indeterminacy in three aspects are:

1- The entries of the payment matrix are neutrosophic values (i.e. $Nc_{ij} = [c_{ij} \pm \varepsilon]$); the indexes i&j have their usual meaning representing the transportation cost of one unit from the production center i to the consumption center j. Assume the indeterminate $\varepsilon = [\lambda_1, \lambda_2]$.

2- The available and the required quantities are both having neutrosophic values represented by $Na_i = a_i \pm \varepsilon_i$, $Nb_j = b_j \pm \delta_j$ respectively, where $\varepsilon_i = [\lambda_{i1}, \lambda_{i2}]$, $\delta_j = [\mu_{j1}, \mu_{j2}]$.

3- This kind of neutrosophic transportation problem is represented gathered from the above two cases.

Keywords: Linear Programming; Neutrosophic Transportation Problem (NTP); Neutrosophic Production Quantities; Neutrosophic Consumption Quantities.

Introduction

The operation research specifically mathematical programming is used in the daily recurrent problems that appear each time when we need to transfer materials from the production centers to the consumption centers.

After the transportation problems have been formulated, the yielded linear models will be solved by simplex method and its modifications. [1-5]. This manuscript contains a modeling study transportation problems using neutrosophic logic that first adopted by F Smarandache (1995) [6-9] is took recently a huge solicitude in effectively addressing the potential uncertainties in the real world, neutrosophic logic comes as a replacement to the fuzzy logic presented by L. Zadeh (1965) [10], intuitionistic fuzzy logic presented by K. Atanassov (1983) [11].

1. Discussion and the General Formulation of the NTP

There is no doubt of the TP importance in any Inc., because of the high costs paid by institutions and companies to secure their needs of raw materials or through the marketing of their products or even the process of transferring their administrative and functional members, so it was necessary to present a study that keeps with the frontiers of modern science in which this article studies transportation issues using the neutrosophic logic that takes into account all the changes that can occur during work, and provides companies with a safe working environment.

Assume that one material may be transferred from the production center $A_i, i = 1,2, \dots, m$ to the consumption center $B_j, j = 1,2, \dots, n$, where $a_1, a_2, \dots, a_m$ are available quantities, $b_1, b_2, \dots, b_n$ are required quantities, C_{ij} is the transfer cost of one unit from the production center i to the consumption center j, and are represent the entries of the payment matrix $C = [c_{ij}]$. To construct the mathematical model, x_{ij} denotes the transferred amount of material from the production center i to the consumption center j. The following tableau contains the basic symbols of the any transportation problems, so any later table will be read out of this table:

CC / PC	B_1	B_2	B_3	…	B_n	**AQ**
A_1	c_{11} x_{11}	c_{12} x_{12}	c_{13} x_{13}	…	c_{1n} x_{1n}	a_1
A_2	c_{21} x_{21}	c_{22} x_{22}	c_{23} x_{23}	…	c_{2n} x_{2n}	a_2
A_3	c_{31} x_{31}	c_{32} x_{32}	c_{33} x_{33}	…	c_{3n} x_{3n}	a_3
. .	…	…	…	…	…	. .
A_m	c_{m1} x_{m1}	c_{m2} x_{m2}	c_{m3} x_{m3}	…	c_{mn} x_{mn}	a_m
RQ	b_1	b_2	b_3	…	b_n	

In any TP, there are two cases:

1- Balanced model satisfying $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

2- Unbalanced model at $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$.

For the purposes of generalizing any mathematical symbol in the above-mentioned tableau from its classical model to its neutrosophic meaning, it is enough to add a prefix N to indicate that this symbol will hold some indeterminacy as follow:

a. The matrix of transferring cost of one unit from the production center i to the consumption center j in its neutrosophic value is $Nc_{ij} = [c_{ij} \pm \varepsilon]$, where $\varepsilon = [\lambda_1, \lambda_2]$ represents the indeterminate value.

b. The symbol Nx_{ij} refers to the materials' amount that transported from the production center i to the consumption center j, so the matrix of the unknowns is written as $NX = [Nx_{ij}]$.

c. The neutrosophic meaning of the available quantities and the required quantities are $Na_i = a_i \pm \varepsilon_i$, $Nb_j = b_j \pm \delta_j$ respectively, where $\varepsilon_i = [\lambda_{i1}, \lambda_{i2}]$, $\delta_j = [\mu_{j1}, \mu_{j2}]$.

d. The neutrosophic encoding of the objective function in the linear programming is $NZ = \sum_{i=1}^{m} \sum_{j=1}^{n} Nc_{ij}\, x_{ij}$, or $NZ = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, Nx_{ij}$, or $NZ = \sum_{i=1}^{m} \sum_{j=1}^{n} Nc_{ij}\, Nx_{ij}$.

So, in this article, the authors assumed the representation of neutrosophic numbers as intervals such as $\boldsymbol{\varepsilon_i = [\lambda_{i1}, \lambda_{i2}]}$, $\boldsymbol{\delta_j = [\mu_{j1}, \mu_{j2}]}$. It is important to notice that the authors did not adopt (trapezoidal numbers, pentagonal numbers, or any other neutrosophic numbers which need to specify using the membership functions, this kind of neutrosophic numbers or parameters represented by intervals have been firstly introduced by Smarandache F. in his main published books [12-14]),

2. Types of Unbalanced Neutrosophic Transportation Problems

Without loss of generality, any solver can faces unbalanced NTP (i.e. $\sum_{i=1}^{m} Na_i \neq \sum_{j=1}^{n} Nb_j$) in which two types of problems can be extracted:

1. Overproduction problems occur when $\sum_{i=1}^{m} Na_i > \sum_{j=1}^{n} Nb_j$. To treat this case, the solver should balance this problem by adding an artificial consumption center B_{n+1} has need of

$Nb_{n+1} = \sum_{i=1}^{m} Na_i - \sum_{j=1}^{n} Nb_j$, where the transformation cost of one unit from the all production centers to this artificial consumption center equal to zero (i.e. $c_{i\ n+1} = 0; i = 1,2, \dots, m$), also, in the data table, the solver should add a new column concerning the new consumption center. The conditions of this linear programming problems that should be satisfied are:

$\sum_{j=1}^{n+1} Nx_{ij} = Na_i \quad ; \quad \sum_{i=1}^{m} Nx_{ij} = Nb_j \ ; \ Nx_{ij} \geq 0 \ ; \ j = 1,2, \dots n+1 \, ; i = 1,2, \dots, m$

2. production deficient case (type of problems having production shortfall) occur when $\sum_{i=1}^{m} Na_i < \sum_{j=1}^{n} Nb_j$. The same above strategy of balancing the problem has been applied by adding an artificial production center A_{m+1} has production power of $Na_{m+1} = \sum_{j=1}^{n} Nb_j - \sum_{i=1}^{m} Na_i$, where the transformation cost of one unit from this artificial production center to all consumption centers equal to zero (i.e. $c_{m+1\ j} = 0; j = 1,2, \dots, n$), also, in the data table, the solver should add a new row concerning the new production center. The conditions of this linear programming problems that should be satisfied are:

$\sum_{j=1}^{n} Nx_{ij} = Na_i \quad ; \quad \sum_{i=1}^{m+1} Nx_{ij} = Nb_j \ ; \ Nx_{ij} \geq 0 \ ; \ j = 1,2, \dots, n \, ; i = 1,2, \dots, m+1$

3. Miscellaneous NT Problems

In the upcoming subsections (3.1,3.2,3.3), the problem text will be: A quantity of fuel is intended to be shipped from three stations A_1, A_2, A_3 to four cities B_1, B_2, B_3, B_4. The available quantities at each station, and the demand quantities in each city, with the transportation costs in each direction are demonstrated in accompanied tables;

3.1 Balanced Neutrosophic Transportation Problem (NTP)

CC / PC	B_1	B_2	B_3	B_4	AQ
A_1	$7+\varepsilon$ Nx_{11}	$4+\varepsilon$ Nx_{12}	$15+\varepsilon$ Nx_{13}	$9+\varepsilon$ Nx_{14}	$120+\varepsilon_1$
A_2	$11+\varepsilon$ Nx_{21}	$2+\varepsilon$ Nx_{22}	$7+\varepsilon$ Nx_{23}	$3+\varepsilon$ Nx_{24}	$80+\varepsilon_2$

A_3	$4+\varepsilon$ Nx_{31}	$5+\varepsilon$ Nx_{32}	$2+\varepsilon$ Nx_{33}	$8+\varepsilon$ Nx_{34}	$100+\varepsilon_3$
RQ	$85+\delta_1$	$65+\delta_2$	$90+\delta_3$	$60+\delta_4$	

By assuming $\varepsilon=[0,2], \varepsilon_1=[0,35], \varepsilon_2=[0,10], \varepsilon_3=[0,15], \delta_1=[0,7], \delta_2=[0,18], \delta_3=[0,25], \delta_4=[0,10]$, the above table can be rewritten as:

CC / PC	B_1	B_2	B_3	B_4	**AQ**
A_1	[7,9] Nx_{11}	[4,6] Nx_{12}	[15,17] Nx_{13}	[9,11] Nx_{14}	[120,155]
A_2	[11,13] Nx_{21}	[2,4] Nx_{22}	[7,9] Nx_{23}	[3,5] Nx_{24}	[80,90]
A_3	[4,6] Nx_{31}	[5,7] Nx_{32}	[2,4] Nx_{33}	[8,10] Nx_{34}	[100,115]
RQ	[85,92]	[65,83]	[90,115]	[60,70]	

Obviously, the problem is balanced cause $\sum_{i=1}^{3} Na_i = \sum_{j=1}^{4} Nb_j = [300,360]$

The model of the linear programming is

$$\text{Min } NZ = [7,9]Nx_{11} + [4,6]Nx_{12} + [15,17]Nx_{13} + [9,11]Nx_{14} + [11,13]Nx_{21} + [2,4]Nx_{22} + [7,9]Nx_{23} + [3,5]Nx_{24} + [4,6]Nx_{31} + [5,7]Nx_{32} + [2,4]Nx_{33} + [8,10]Nx_{34}$$

Subject to

$$\sum_{j=1}^{4} Nx_{ij} = a_i \quad ; \quad \sum_{i=1}^{3} Nx_{ij} = b_j \ ; \ Nx_{ij} \geq 0 \ ; \ j = 1,2,3,4 \ ; i = 1,2,3$$

3.2 Unbalanced Overproduction NTP Case Study

CC / PC	B_1	B_2	B_3	B_4	**AQ**
A_1	$7+\varepsilon$ Nx_{11}	$4+\varepsilon$ Nx_{12}	$15+\varepsilon$ Nx_{13}	$9+\varepsilon$ Nx_{14}	$120+\varepsilon_1$

A_2	$11+\varepsilon$ Nx_{21}	$2+\varepsilon$ Nx_{22}	$7+\varepsilon$ Nx_{23}	$3+\varepsilon$ Nx_{24}	$95+\varepsilon_2$
A_3	$4+\varepsilon$ Nx_{31}	$5+\varepsilon$ Nx_{32}	$2+\varepsilon$ Nx_{33}	$8+\varepsilon$ Nx_{34}	$100+\varepsilon_3$
RQ	$85+\delta_1$	$65+\delta_2$	$90+\delta_3$	$60+\delta_4$	

By assuming $\varepsilon=[0,2],\varepsilon_1=[0,35],\varepsilon_2=[0,10],\varepsilon_3=[0,15],\delta_1=[0,7],\delta_2=[0,18],\delta_3=[0,25],\delta_4=[0,10]$, the above table can be rewritten as:

PC \ CC	B_1	B_2	B_3	B_4	**AQ**
A_1	[7,9] Nx_{11}	[4,6] Nx_{12}	[15,17] Nx_{13}	[9,11] Nx_{14}	[120,155]
A_2	[11,13] Nx_{21}	[2,4] Nx_{22}	[7,9] Nx_{23}	[3,5] Nx_{24}	[95,105]
A_3	[4,6] Nx_{31}	[5,7] Nx_{32}	[2,4] Nx_{33}	[8,10] Nx_{34}	[100,115]
RQ	[85,92]	[65,83]	[90,115]	[60,70]	

It is obvious that $\sum_{i=1}^{3} Na_i=[315,375]>\sum_{j=1}^{4} Nb_j=[300,360]$; this lead to add an artificial consumption center b_5 has need of the value $b_5=\sum_{i=1}^{m} a_i-\sum_{j=1}^{n} b_j=[315,375]-[300,360]=[15,15]=15$

Hence, the value of the objective function is

$$\text{Min } NZ=[7,9]Nx_{11}+[4,6]Nx_{12}+[15,17]Nx_{13}+[9,11]Nx_{14}+0.Nx_{15}+[11,13]Nx_{21}+[2,4]Nx_{22}+[7,9]Nx_{23}+[3,5]Nx_{24}+0.Nx_{25}+[4,6]Nx_{31}+[5,7]Nx_{32}+[2,4]Nx_{33}+[8,10]Nx_{34}+0.Nx_{35}$$

Subject to

$$\sum_{j=1}^{5} Nx_{ij}=a_i \quad ; \quad \sum_{i=1}^{3} Nx_{ij}=b_j \ ; \ Nx_{ij}\geq 0 \ ; \ j=1,2,3,4,5 \ ; i=1,2,3$$

3.3 Unbalanced Production Deficient NTP Case Study

PC \ CC	B_1	B_2	B_3	B_4	**AQ**
A_1	$7+\varepsilon$ Nx_{11}	$4+\varepsilon$ Nx_{12}	$15+\varepsilon$ Nx_{13}	$9+\varepsilon$ Nx_{14}	$120+\varepsilon_1$
A_2	$11+\varepsilon$ Nx_{21}	$2+\varepsilon$ Nx_{22}	$7+\varepsilon$ Nx_{23}	$3+\varepsilon$ Nx_{24}	$80+\varepsilon_2$
A_3	$4+\varepsilon$ Nx_{31}	$5+\varepsilon$ Nx_{32}	$2+\varepsilon$ Nx_{33}	$8+\varepsilon$ Nx_{34}	$100+\varepsilon_3$
RQ	$85+\delta_1$	$100+\delta_2$	$90+\delta_3$	$60+\delta_4$	

Where, $\varepsilon=[0,2], \varepsilon_1=[0,35], \varepsilon_2=[0,10], \varepsilon_3=[0,15], \delta_1=[0,7], \delta_2=[0,18], \delta_3=[0,25], \delta_4=[0,10]$, the above table can be rewritten as:

PC \ CC	B_1	B_2	B_3	B_4	**AQ**
A_1	$[7,9]$ Nx_{11}	$[4,6]$ Nx_{12}	$[15,17]$ Nx_{13}	$[9,11]$ Nx_{14}	$[120,155]$
A_2	$[11,13]$ Nx_{21}	$[2,4]$ Nx_{22}	$[7,9]$ Nx_{23}	$[3,5]$ Nx_{24}	$[80,90]$
A_3	$[4,6]$ Nx_{31}	$[5,7]$ Nx_{32}	$[2,4]$ Nx_{33}	$[8,10]$ Nx_{34}	$[100,115]$
RQ	$[85,92]$	$[100,118]$	$[90,115]$	$[60,70]$	

It is worthy to mention that $\sum_{i=1}^{3} Na_i=[300,360] < \sum_{j=1}^{4} Nb_j=[335,395]$; this lead to add an artificial production center a_4 has production power equal to $a_4=\sum_{j=1}^{4} b_j-\sum_{i=1}^{3} a_i=[315,395]-[300,360]=[35,35]=35$

Hence, the model of the linear programming is

$$\text{Min } NZ = [7,9]Nx_{11} + [4,6]Nx_{12} + [15,17]Nx_{13} + [9,11]Nx_{14} + [11,13]Nx_{21} + [2,4]Nx_{22} + [7,9]Nx_{23} + [3,5]Nx_{24} + [4,6]Nx_{31} + [5,7]Nx_{32} + [2,4]Nx_{33} + [8,10]Nx_{34} + 0.Nx_{41} + 0.Nx_{42} + 0.Nx_{43} + 0.Nx_{44}$$

Subject to

$$\sum_{j=1}^{4} Nx_{ij} = a_i \quad ; \quad \sum_{i=1}^{4} Nx_{ij} = b_j \; ; \; Nx_{ij} \geq 0 \; ; \; j = 1,2,3,4 \, ; i = 1,2,3,4$$

4. Conclusion and Results

This manuscript presented new insights into remodeling transportation problems from classical models to the corresponding neutrosophic ones through three basic aspects containing some indeterminacies either in transformation costs, or in the available quantities existed in production centers side by side with required quantities existing in consumption centers, or in the general case study that represents the existence of indeterminacies in all previous states. The solutions to these neutrosophic transportation problems NTP give the solver a margin of freedom and reduce the loss resulting from transporting process through the availability of the indetermination notion in neutrosophic theory. This study was presented simultaneously with the paper entitled (Original Methods for Obtaining the Initial Solution in Neutrosophic Transportation Problems) regarded as complementary to this recent paper in which the authors discuss the initial solution for the NTP models and study the modifications of this initial solution aiming to get the optimal solutions.

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Practical Applications of IndetermSoft Set and IndetermHyperSoft Set and Introduction to TreeSoft Set as an extension of the MultiSoft Set

Prof. **Florentin Smarandache**[1]*, PhD, Postdoc

[1] Mathematics Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA

* Correspondence: smarand@unm.edu

Abstract: The IndetermSoft Set is as an extension of the Soft Set, because the data, or the function, or the sets involved in the definition of the soft set have indeterminacy - as in our everyday life, and we still need to deal with such situations.

And similarly, IndetermHyperSoft Set as extension of the HyperSoft Set, when there is indeterminate data, or indeterminate functions, or indeterminate sets.

Herein, 'Indeterm' stands for 'Indeterminate' (uncertain, conflicting, incomplete, not unique outcome).

We now introduce for the first time the TreeSoft Set as extension of the MultiSoft Set.

Several applications are presented for each type of soft set.

Keywords: Soft Set, IndetermSoft Set, HyperSoft Set, IndetermHyperSoft Set, MultiSoft Set, TreeSoft Set

1. Introduction

We have extended the Soft Set to HyperSoft Set [2, 3] in 2018, then both of them to IndetermSoft Set and IndetermHyperSoft Set [4, 8] respectively in 2022, and we have introduced Indeterminate Soft and HyperSoft operators.

The operations (complement, intersection, union) for IndetermSoft Set and IndetermHyperSoft Set respectively are to be done in the future research.

And in this paper a new type of soft set, called TreeSoft Set, is introduced for the first time as an extension of the MultiSoft Set.

Several applications are presented for each type of soft set.

2. Definition of Soft Set

Let U be a universe of discourse, H a non-empty subset of U, with $P(H)$ the powerset of H, and a an attribute (parameter, factor, etc.), with its set of attribute-values denoted by A. Then, the pair (F, A), with $F: A \rightarrow P(H)$, is called a (Classical) Soft Set over H.

Molodtsov [1] has defined in 1999 the Soft Set.

3. Real Example of (Classical) Soft Set

Let $H = \{h_1, h_2, h_3, h_4\}$ be a set of houses, and a an attribute, a = *color*, and its set of attribute-values A = {*white, green, red*}. The function $F : A \to P(H)$, as:

$F(white) = \{h_1, h_2, h_4\}$, $F(green) = h_3$, $F(red) = \phi$ *(no house).*

4. Definition of IndetermSoft Set

Smarandache [4, 8] introduced it in 2022.

Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ the powerset of H. Let a be an attribute, and A be a set of this attribute-values.

Then $F: A \to P(H)$ is called an IndetermSoft Set if:

i) the set A has some indeterminacy;
ii) or the set $P(H)$ has some indeterminacy;
iii) or there exist at least an attribute-value $v \in A$, such that $F(v)$ = indeterminate (unclear, incomplete, conflicting, or not unique);
iv) or any two or all three of the above situations.

The IndetermSoft Set has some degree of indeterminacy, and as such it is a particular case of the NeutroFunction [5, 6], defined in 2014 – 2015, which is a function that is only partially well-defined (inner-defined), partially indeterminate, and partially outer-defined. The NeutroFunction is a generalization of the classical function, that is totally well-defined.

IndetermSoft Set, as extension of the classical (determinate) Soft Set, deals with indeterminate data, because there are sources [4, 8] unable to provide exact or complete information on the sets A, H or $P(H)$, and on the function F.

We did not add any indeterminacy, we found the indeterminacy in our real world. Because many sources give approximate/uncertain/incomplete/conflicting information, not exact information as in the Soft Set, as such we still need to deal with such situations.

For more information on IndetermSoft Set consult [4, 8].

5. Real Example of IndetermSoft Set:

Assume a town has many houses.

1) Indeterminacy with respect to the function.

1a) You ask a source:
- What houses have the red color in the town?

The source:
- I am not sure, I think the houses h_1 or h_2.

Therefore, $F(red) = h_1$ *or* h_2
(indeterminate / uncertain answer).

1b) You ask again:
- But, what houses are yellow?

The source:
- I do not know, the only thing I know is that the house h_5 is not yellow because I have visited it.

Therefore, $F(yellow)$ = *not* h_5
(again indeterminate / uncertain answer).

1c) Another question you ask:
- Then what houses are blue?

The source:
- For sure, *either* h_8 *or* h_9

Therefore, $F(blue)$ = *either* h_8 *or* h_9
(again indeterminate / uncertain answer).

2) Indeterminacy with respect to the set H of houses.

You ask the source:

- How many houses are in the town?

The source:

- I never counted them, but I estimate their number to be *between 100-120 houses*.

3) Indeterminacy with respect to the set A of attributes.

You ask the source:

What are all colors of the houses?

The source:

I know for sure that there are houses of colors *red, yellow, and blue*, but I do not know if there are houses of other colors (?)

This is the IndetermSoft Set.

6. Definition of HyperSoft Set

Smarandache has extended in 2018 the Soft Set to the HyperSoft Set [3, 4, 8] by transforming the function F from a uni-attribute function into a multi-attribute function.

Let $\mathcal{U}$ be a universe of discourse, H a non-empty set included in U, and $P(H)$ the powerset of H. Let a_1, a_2, …, a_n, where $n \geq 1$, be n distinct attributes, whose corresponding attribute-values are respectively the sets A_1, A_2, … , A_n, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, n\}$.

Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where $A_1 \times A_2 \times \dots \times A_n$ represents the Cartesian product, with $F: A_1 \times A_2 \times \dots \times A_n \to P(H)$ is called a HyperSoft Set.

In other words, for any $(e_1, e_2, ..., e_n) \in A_1 \times A_2 \times ... \times A_n$, $F(e_1, e_2, ..., e_n) \in P(H)$

7. Real Example of HyperSoft Set

Let $H = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$ a set of houses, and two attributes a_1 and a_2, where a_1 = *color*, and its set of attribute-values A_1 = *{white, green, red}*, and a_2 = *size*, and its attribute-values

A_2 = *{small, big}*. The function $F: A_1 \times A_2 \to P(H)$, as :

F(white, small) = $\{h_1, h_2\}$, *F(green, big)* = $\{h_4, h_6, h_7\}$, *F(red, big)* = $\{h_3, h_5\}$.

8. Definition of IndetermHyperSoft Set

Smarandache [4, 8] introduced it in 2022.

Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ the powerset of H. Let a_1, a_2, …, a_n, where $n \geq 1$, be n distinct attributes, whose corresponding attribute-values are respectively the sets A_1, A_2, … , A_n, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, n\}$.

Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where $A_1 \times A_2 \times \dots \times A_n$ represents the Cartesian product, with $F: A_1 \times A_2 \times \dots \times A_n \to P(H)$ is called an IndetermHyperSoft Set if:

i) at least one of the sets $A_1, A_2, \dots, A_n$ has some indeterminacy;

ii) or the set $P(H)$ has some indeterminacy;

iii) or there exist at least one n-plet $(e_1, e_2, ..., e_n) \in A_1 \times A_2 \times ... \times A_n$ such that $F(e_1, e_2, ..., e_n)$ = indeterminate (unclear, uncertain, conflicting, or not unique);

iv) or any two or all three of the above situations.

The IndetermHyperSoft Set has some degree of indeterminacy, and it is as extension of the (determinate) HyperSoft Set.

Similarly, we did not add any indeterminacy, we found the indeterminacy in our real world. Because many sources give approximate/uncertain/incomplete/conflicting information, not exact information as in the Soft Set and in the HyperSoft Set, as such we still need to deal with such situations.

9. Real Example of IndetermSoft Set

Assume a town has many houses.

1) *Indeterminacy with respect to the function*.
1a) You ask a source:
- What houses are of red color and big size in the town?
The source:
- I am not sure, I think the houses h_1 *or* h_2.
Therefore, *F(red, big) =* h_1 *or* h_2
(indeterminate / uncertain answer).
1b) You ask again:
- But, what houses are *yellow and small*?
The source:
- I do not know, the only thing I know is that the house h_5 is neither yellow nor small because I have visited it.
Therefore, *F(yellow, small) = not* h_5
(again indeterminate / uncertain answer).
1c) Another question you ask:
- Then what houses are *blue and big*?
The source:
- For sure, *either* h_8 *or* h_9
Therefore, *F(blue, big) = either* h_8 *or* h_9
(again indeterminate / uncertain answer).

2) *Indeterminacy with respect to the set H of houses*.
You ask the source:
- How many houses are in the town?
The source:
- I never counted them, but I estimate their number to be *between 100-120 houses*.

3) *Indeterminacy with respect to the set A of attributes*.
You ask the source:
What are all colors and sizes of the houses?
The source:
I know for sure that there are houses of colors of *red, yellow,* and *blue*, but I do not know if there are houses of other colors (?)
About the size, I saw many houses that are *small*, but I do not remember to have seing be *big* houses.

This is the IndetermHyperSoft Set.

10. Definition of MultiSoft Set [7]

Let *U* be a universe of discourse, and *H* a non-empty subset of *U*.

And *P(H)* is the power set of *H*. Let $A_1, A_2, \dots, A_n$ be $n \geq 2$ sets of attributes (parameters) whose intersection $A_1 \cap A_2 \cap \ldots \cap A_n = \phi$.

Let $A = A_1 \cup A_2 \cup \ldots \cup A_n$ and $P(A)$ be the power set of *A*.

Then $F : P(A) \rightarrow P(H)$ is a MultiSoft Set over *H*.

For $\varepsilon \in P(A)$ one considers that $F(\varepsilon)$ is the set of ε - approximate sets of the multisoft set *(F, P(A))*.

11. Extension of the MultiSoft Set to a HyperSoft Set

One introduces the empty-element ϕ to each set of attribute-values, and let denote

$A_1^{'} = A_1 \cup \{\phi\}, A_2^{'} = A_2 \cup \{\phi\}, ..., A_n^{'} = A_n \cup \{\phi\}$.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) \in A_1^{'} \times A_2^{'} \times ... \times A_n^{'}$,

then $\varepsilon_1 \in A_1^{'} = A_1 \cup \{\phi\}$ means that either $\varepsilon_1 \in A_1$ or $\varepsilon_1 = \phi$ (discarded) ;

similarly for all $\varepsilon_i \in A_i^{'} = A_i \cup \{\phi\}$, $1 \le i \le n$.

Thus, $F : A_1^{'} \times A_2^{'} \times ... \times A_n^{'} \to P(H)$ is a hypersoft set.

12. Real Example of MultiSoft Set

We retake the previous example and adjust it to a MultiSoft Set.

Let $H = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$ a set of houses, and two attributes a_1 and a_2, where a_1 = *color*, and its set of attribute-values A_1 = *{white, green, red}*, and a_2 = *size*, and its attribute-values

A_2 = *{small, big}*. Let $A = A_1 \cup A_2$ = {*white, green, red ; small, big*}, and *P(A)* be the power set of *A*. Then $F : P(A) \to P(H)$ is defined as follows:

F(white) = $\{h_1\}$, *F(green, big)* = $\{h_4, h_6\}$, *F(big)* = $\{h_3, h_5\}$.

13. Real MultiSoft Set extended to a HyperSoft Set

Let's enlarge A1 and A2:

$A_1^{'}$ = {white, green, red, ϕ}, and $A_2^{'}$ = {small, big, ϕ}

Then $F^{'} : A_1^{'} \times A_2^{'} \to P(H)$

F'(white, ϕ) $\equiv$ F(white) ={h1} (since the attribute-value ϕ was discarded).

F'(green, big) $\equiv$ F(green, big) = {h4, h6}.

F'(ϕ, big) $\equiv$ F(big) = {h3, h5} (since the attribute-value ϕ was discarded).

14. Generalization of MultiSoft Set to the TreeSoft Set

Let *U* be a universe of discourse, and *H* a non-empty subset of *U*, with *P(H)* the powerset of *H*.

Let *A* be a set of attributes (parameters, factors, etc.),

$A = \{A_1, A_2, ..., A_n\}$, for integer $n \ge 1$,

where $A_1, A_2, ..., A_n$ are <u>*attributes of first level*</u> (since they have one-digit indexes).

Each attribute $A_i, 1 \le i \le n$, is formed by sub-attributes:

$$A_1 = \{A_{1,1}, A_{1,2}, ...\}$$
$$A_2 = \{A_{2,1}, A_{2,2}, ...\}$$
$$.$$
$$.$$
$$.$$
$$A_n = \{A_{n,1}, A_{n,2}, ...\}$$

where $A_{i,j}$ are <u>*sub-attributes (or attributes of second level)*</u> (since they have two-digit indexes).

Again, each sub-attribute $A_{i,j}$ is formed by <u>*sub-sub-attributes (or attributes of third level)*</u>:

$$A_{i,j,k}$$

And so on, as much refinement as needed into each application, up to *sub-sub-…-sub-attributes (or attributes of m-level* (or having m digits into the indexes):

$$A_{i_1,i_2,\ldots,i_m}$$

Therefore, a graph-tree is formed, that we denote as *Tree(A)*, whose root is A (considered of *level zero*), then nodes of *level 1, level 2,* up to *level m.*

We call *leaves* of the graph-tree, all terminal nodes (nodes that have no descendants).

Then the TreeSoft Set is:

$$F : P(Tree(A)) \to P(H) .$$

Tree(A) is the set of all nodes and leaves (from *level 1* to *level m*) of the graph-tree, and $P(Tree(A))$ is the powerset of the *Tree(A).*

All node sets of the *TreeSoft Set of level m* are:

$$Tree(A) = \{A_{i_1} \mid i_1 = 1,2,\ldots\} \cup \{A_{i_1,i_2} \mid i_1,i_2 = 1,2,\ldots\} \cup \{A_{i_1,i_2,i_3} \mid i_1,i_2,i_3 = 1,2,\ldots\} \cup \ldots \cup \{A_{i_1,i_2,\ldots,i_m} \mid i_1,i_2,\ldots,i_m = 1,2,\ldots\}$$

The first set is formed by the *nodes of level 1*, second set by the *nodes of level 2*, third set by the *nodes of level 3*, and so on, the last set is formed by the *nodes of level m*.

If the graph-tree has only two levels ($m = 2$), then the TreeSoft Set is reduced to a MultiSoft Set.

15. Example of TreeSoft Set of Level 3

Node of *level 0* (the graph-tree root): A.

Nodes of *level 1*: A_1, A_2.

Nodes of *level 2*: A_{11}, A_{12}; A_{21}, A_{22}.

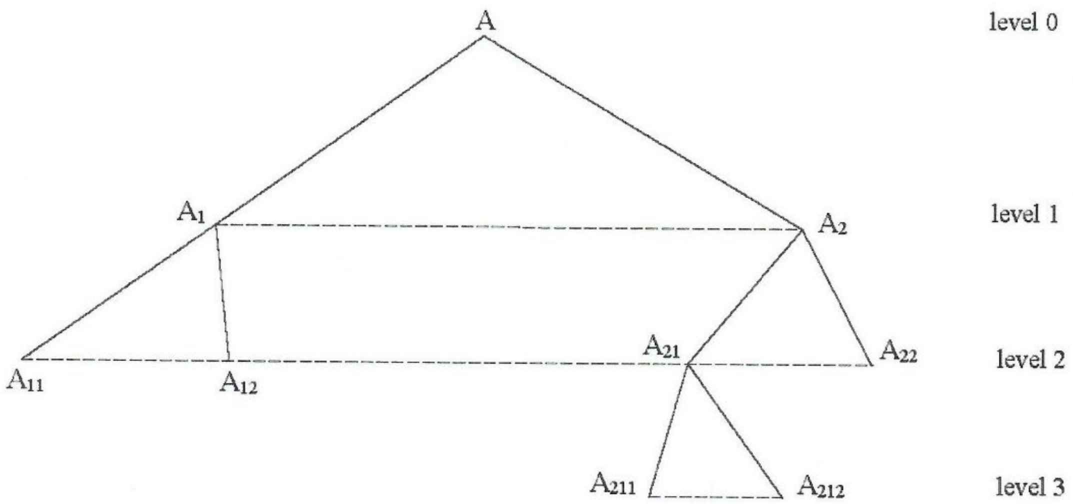
Nodes of *level 3*: A_{211}, A_{212}.

Whence *Tree(A)* = {A_1, A_2; A_{11}, A_{12}; A_{21}, A_{22}; A_{211}, A_{212}}.

The *leaves* are: A_{11}, A_{12}; A_{211}, A_{212}; A_{22}. As we see, the leaves may have various levels, in this case: *2*, or *3*.

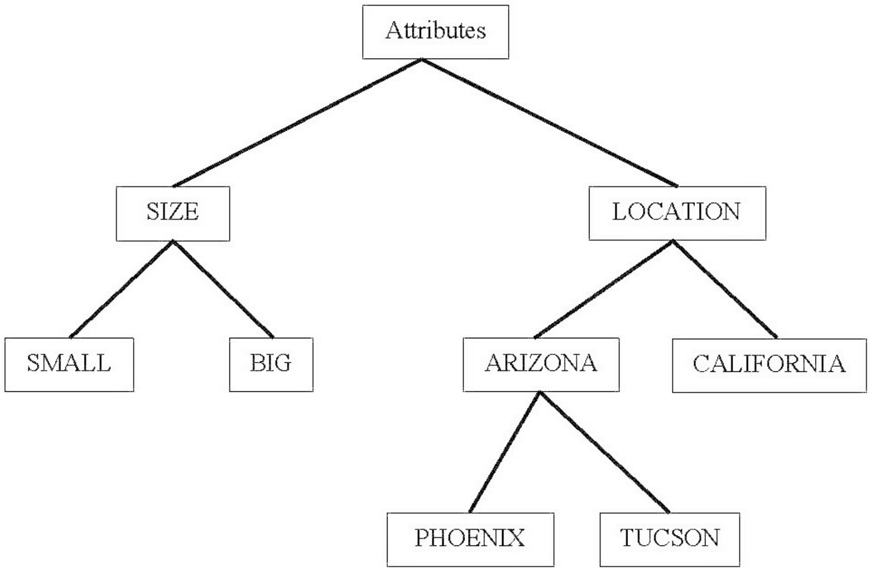
$P(Tree(A))$ is the powerset of $Tree(A)$.

F: $P(Tree(A)) \to P(H)$ is a TreeSoft Set of Level 3.



Graph 1: TreeSoft Set of Level 3

16. Practical Example of TreeSoft Set of Level 3



Graph 2: Practical TreeSoft Set of Level 3

et's consider H = $\{h_1, h_2, \ldots, h_{10}\}$ be a set of houses, and *P(H)* the powerset of *H*.
And the set of attributes: *A* = $\{A_1, A_2\}$,
where A_1 = *size*, A_2 = *location.*
Then A_1 = $\{A_{11}, A_{12}\}$ = *{small, big}.*
A_2 = $\{A_{21}, A_{22}\}$ = *{Arizona, California},* American states.
Further on, $A_{21} = \{A_{211}, A_{212}\}$ = *{Phoenix, Tucson},* Arizonian cities.
Let's assume that the function F gets the following values:
F(big, Arizona, Phoenix) = $\{h_9, h_{10}\}$
F(big, Arizona, Tucson) = $\{h_1, h_2, h_3, h_4\}$
F(big, Arizona) = all big houses from both cities, Phoenix and Tucson,
= *F(big, Arizona, Phoenix)* ∪ *F(big, Arizona, Tucson)*= $\{h_1, h_2, h_3, h_4, h_9, h_{10}\}$.

17. Properties of the TreeSoft Set

17.1. Theorem 1

F(node) includes all node's descendants, and sub-descendants, then sub-sub-descendants, and so on up to the corresponding leaves.

From previous Example 15, one has:
$F(A_{21}) = F(A_{211}) \cup F(A_{212})$,
and consequently
$F(A_{12}, A_{21}) = F(A_{12}, A_{211}) \cup F(A_{12}, A_{212})$.

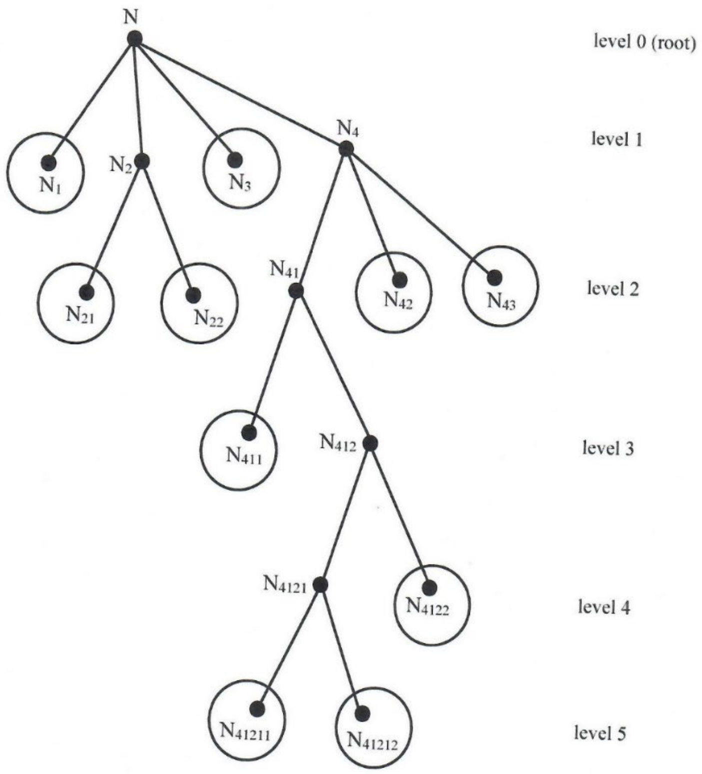
17.2. Theorem 2

Let $N \in Tree(A)$ be a node.
N generates a *SubTree(N)* whose root is *N* itself.
Then $F(N) = \cup_{\varphi(i)} F(N_{\varphi(i)})$
where $N_{\varphi(i)}$ are all leaves of the SubTree (*N*).
From previous Example 15:

$F(A_2) = F(A_{21}) \cup F(A_{22}) = (F(A_{211}) \cup F(A_{212})) \cup F(A_{22}) = F(A_{211}) \cup F(A_{212}) \cup F(A_{22})$

where A_{211}, A_{212}, A_{22} are all leaves of the SubTree whose root is A_2 {i.e. *SubTree(A_2)*}.

The *proof* of Theorem 2 is obvious, no matter what graph-tree one has, and it is similar to the below Example:



Graph 3: Tree(N)

The circled nodes are the leaves.

$F(N) = F(N_1) \cup F(N_2) \cup F(N_3) \cup F(N_4)$

$= F(N_1) \cup [F(N_{21}) \cup F(N_{22}) \cup F(N_3) \cup F(N_{41}) \cup F(N_{42}) \cup F(N_{43})]$

$= F(N_1) \cup F(N_{21}) \cup F(N_{22}) \cup F(N_3) \cup [F(N_{411}) \cup F(N_{412})] \cup F(N_{42}) \cup F(N_{43})$

$= F(N_1) \cup F(N_{21}) \cup F(N_{22}) \cup F(N_3) \cup F(N_{411}) \cup [F(N_{4121}) \cup F(N_{4122})] \cup F(N_{42}) \cup F(N_{43})$

$= F(N_1) \cup F(N_{21}) \cup F(N_{22}) \cup F(N_3) \cup F(N_{411}) \cup F(N_{41211}) \cup F(N_{41212}) \cup F(N_{42}) \cup F(N_{43})$

which is the union of the soft-values *F(.)* of all leaves of the *SubTree(N)*.

Actually Theorems 1 and 2 are equivalent.

17.3. Theorem 3

$F\left(N_{i_1}, N_{i_2}, \ldots, N_{i_p}\right) = F(N_{i_1}) \cap F(N_{i_2}) \cap \ldots \cap F\left(N_{i_p}\right)$,

where $N_{i_1}, N_{i_2}, \ldots, N_{i_p}$ are nodes of various levels into the TreeSoft Set of N.

The *proof* results from the fact that $F(N_{i_1})$ represents the subset H_1 of elements in H that have the attribute-value N_{i_1}, and $F(N_{i_2})$ represents the subset H_2 of elements in H that have the attribute-value N_{i_2}, and so on $F(N_{i_p})$ represents the subset H_p of elements in H that have the attribute-value N_{i_p}, therefore to get the elements that have all these attribute-values one needs to intersect these subsets $H_1 \cap H_2 \cap \ldots \cap H_p$.

18. Future Research

To define the operations (complement, intersection, union) for IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set respectively and to use them in real applications.

19. Conclusion

We introduced the TreeSoft Set as an extension of the MultiSoft Set. We presented simple practical applications of IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set respectively for better understanding.

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Single Valued Neutrosophic Kruskal-Wallis and Mann Whitney Tests

Mahmoud Miari [1], Mohamad Taher Anan [2] and Mohamed Bisher Zeina [3]*

[1] Department of Mathematical Statistics, Faculty of Science, University of Aleppo, Aleppo, Syria; mahmoudmiari1994@gmail.com
[2] Department of Mathematical Statistics, Faculty of Science, University of Aleppo, Aleppo, Syria; mtanan200988@gmail.com
[3] Department of Mathematical Statistics, Faculty of Science, University of Aleppo, Aleppo, Syria; bisher.zeina@gmail.com

* Correspondence: bisher.zeina@gmail.com

Abstract: In this paper, Kruskal-Wallis test is extended to deal with neutrosophic data in single valued form using score, accuracy and certainty functions to calculate ranks of SVNNs, also Mann-Whitney test is extended to deal with same data type which makes it possible to do a post-hoc test after rejecting null hypothesis using Neutrosophic Statistics Kruskal-Wallis test. Numerical examples were successfully solved showing the power of this new idea to deal with SVNNs and make statistical decisions on them.

Keywords: Kruskal-Wallis; Test Statistic; Chi Square Distribution; Hypothesis Testing; Significance Level; Single Valued Neutrosophic Number.

1. Introduction

F. Smarandache presented neutrosophic logic as an extension to fuzzy logic [1] and intuitionistic fuzzy logic [2] to deal with indeterminacy, ambiguity, uncertainty, contradiction, unsureness, nihilness, vagueness and emptiness [3], this new extension make decisions more flexible and reliable [4] [5] and has been applied in many scientific fields including abstract algebra, mathematical modelling, probability theory, statistics, operations research, artificial intelligence, machine learning, etc. [6] [5] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18]. He also introduced the Neutrosophic Statistics as an extension of the Interval Statistics, since the neutroosphic statistics may deal with all types of indeterminacies (with respect to the data, inferential procedures, probability distributions, graphical representations, etc.), it allows the reduction of indeterminacy, and it uses the neutrosophic probability that is more general than imprecise and classical probabilities, and has more detailed corresponding probability density functions - while Interval Statistics only deals with indeterminacy that can be represented by intervals. [27].

In statistics, M. Aslam presented many neutrosophic statistical tests to deal with indeterminacy in data considering that observations are classical neutrosophic numbers of the form $N = D + I$ where D is the determinant part of the number and I is its indeterminant part [19] [20] [21] [22].

Comparing population means is one of the most important statistical tests to test whether several drawn samples are from one population (then we say that means are equal) or from different populations (here we say that means are not equal). This procedure is done using hypothesis testing with respect to a test statistic having a previously known probability distribution comparing its value with acceptance region and rejection region.

The problem arises when dealing with neutrosophic number or judges, e.g., if a doctor says that a patient is 70% infected with COVID-19 with 20% indeterminacy because of similar flu syndromes and with 50% chance to be wrong diagnosis, here we cannot deal with this data type using classical statistical tests neither with previously studied neutrosophic statistical tests.

A mathematical solve for this problem in lattice theory and abstract algebra was presented in [23] where ranking of observations was done and presented in [24] to compare between judges. also, previous work was generalized in [25] [26].

In this paper we are going to solve this problem from statistical point of view where we are dealing with samples data derived from different populations to make generalize decisions made based on samples to population extending Kruskal-Wallis test to deal with (T, I, F) data sets which is the well-known single valued neutrosophic numbers and make it possible to compare several samples and take decision if those samples are drawn from same population or from different populations, then we will extend Mann-Whitney test to make a multiple comparison between each two groups.

2. Preliminaries

We recall here some basic definitions of single valued neutrosophic sets and single valued neutrosophic numbers and some operations on them.

2.1 Single Valued Neutrosophic Sets:

Suppose that Ω is the universe and let A be a subset of Ω then A is said to be Single Valued Neutrosophic Set (SVNS) with truth, indeterminacy and falsity memberships and denoted as follows:

$$A = \{(x | T_A(x), I_A(x), F_A(x))\}$$

Where:

$$T_A: \Omega \to [0,1]$$
$$I_A: \Omega \to [0,1]$$
$$F_A: \Omega \to [0,1]$$

And:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$$

2.2 Single Valued Neutrosophic Numbers:

Single Valued Neutrosophic Number (SVNN) takes the form (T, I, F) where T reflects truth, I reflects indeterminacy and F reflects falsity where $0 \le T, I, F \le 1$ and $0 \le T + I + F \le 3$.

2.3 Operations on Single Valued Neutrosophic Numbers:

Suppose that $A = (t_1, i_1, f_1), B = (t_2, i_2, f_2)$ are two SVNNs then operations on A, B are defined as follows:

$$A \oplus B = (t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2)$$
$$A \otimes B = (t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2)$$
$$A \ominus B = \left(\frac{t_1 - t_2}{1 - t_2}, \frac{i_1}{i_2}, \frac{f_1}{f_2}\right); t_2 \ne 1; i_2 \ne 0; f_2 \ne 0$$
$$\frac{A}{B} = \left(\frac{t_1}{t_2}, \frac{i_1 - i_2}{1 - i_2}, \frac{f_1 - f_2}{1 - f_2}\right); t_2 \ne 0; i_2 \ne 1; f_2 \ne 1$$
$$\lambda A = (1 - (1 - t_1)^\lambda, {i_1}^\lambda, {f_1}^\lambda); \lambda > 0$$
$$A^\lambda = ({t_1}^\lambda, 1 - (1 - i_1)^\lambda, 1 - (1 - f_1)^\lambda); \lambda > 0$$

2.4 Ranking of Single Valued Neutrosophic Numbers

Let $A(T, I, F)$ be a SVNN, the score function $s(A)$, accuracy function $a(A)$ and certainty function $c(A)$ are defined as follows:

$$s(A) = \frac{2 + T - I - F}{3}$$
$$a(A) = T - F$$
$$c(A) = T$$

We can rank A, B using the following algorithm:

1) If $s(A) > s(B)$ $then$ $A > B$.
2) If $s(A) = s(B)$ and $a(A) > a(B)$ $then$ $A > B$.
3) If $s(A) = s(B)$ and $a(A) = a(B)$ and $c(A) > c(B)$ $then$ $A > B$.
4) If $s(A) = s(B)$ and $a(A) = a(B)$ and $c(A) = c(B)$ $then$ $A = B$

3. Classical Kruskal-Wallis and Mann Whitney Tests

Kruskal-Wallis Test (H Test) one of the nonparametric tests that based on ranks used to compare the means of c independent random samples of sizes $n_1, \dots, n_c$ drawn from c univariate populations with unknown cumulative distribution functions $F_1, \dots, F_C$.
The technique of (H Test) performed by ranking all observation and defined as follows:

Formally, letting the distribution function of X over the group i be of the form $F_i(x) = F(y - \theta_i)$, we'd like to test

$$H_0: \theta_1 = \theta_2 = \cdots = \theta_C \quad \text{against} \quad H_1: \theta_i \neq \theta_j \; for \; some \; i, j$$

The test is based on $\chi^2(c-1)$ distribution using test statistic:

$$H = \frac{12}{N(N+1)} \sum_{i=1}^{C} \frac{R_i^2}{n_i} - 3(N+1)$$

Where:

c number of samples
n_i number of observations in the i[th] group
$N = \sum n_i$ number of observations in all samples
R_i sum of ranks for the i[th] group
Notice that H test tells us whether the samples are drawn from same population (when accepting H_0) or those sample are drawn from different populations.
If we reject H_0 then we must determine the true differences location, i.e. we must do a post hoc test, and one of the famous used tests is Mann Whitney test that tests the following hypothesis:

$$H_0: \theta_i = \theta_j$$
$$H_1: \theta_i \neq \theta_j$$

Using test statistic:

$$Z = \frac{U - \overline{U}}{std_U}$$

Where:

$$\overline{U} = \frac{n_i n_j}{2}$$

$$std_U = \sqrt{\frac{n_i n_j (n_i + n_j + 1)}{12}}$$

$$U = \min\left(n_i n_j + \frac{n_i(n_j+1)}{2} - R_i, n_i n_j + \frac{n_j(n_j+1)}{2} - R_j\right)$$

4. Single Valued Neutrosophic Kruskal Wallis and Mann Whitney Tests

Suppose that we have c random samples as follows:

Table 1. Neutrosophic Observations.

Sample 1	Sample 2	…	Sample c
S_{11}	S_{21}		S_{c1}
S_{12}	S_{22}	$\ddots$	S_{c2}
$\vdots$	$\vdots$		$\vdots$

S_{1n_1}	S_{2n_2}		S_{cn_c}

Where $S_{11}, S_{12}, \dots, S_{cn_c}$ are SVNNs, e.g., judgments, sentiments, point of views, considerations, … etc. and we would like to check whether these judgments are consistent. Kruskal Wallis test can answer our question but the problem that arises is how to calculate the ranks of these judges since it is base on neutrosophic numbers. We will present the following algorithm to solve this problem:

1. Merge all the observation from different samples and deal with it as one sample.
2. Calculate score, accuracy and certainty of each observation.
3. Compare and rank these observations based on its score, accuracy and certainty.
4. Give the ranked observations ranks from 1 to N and if we have two equal observation the we average its ranks.
5. Compute Kruskal Wallis test statistic using the formula:
$$H_N = \frac{12}{N(N+1)} \sum_{i=1}^{C} \frac{(R_i^2)_N}{n_i} - 3(N+1)$$
where $(R_i^2)_N$ is sum of i[th] sample neutrosophic rank, hence H_N is neutrosophic test statistic.
6. Compare the test statistic with $\chi^2_{1-\alpha}(c-1)$ critical values, if $H_N < \chi^2_{1-\alpha}(c-1)$ then samples are drawn from same population, i.e., judgments are consistent and here test is done. elsewhere judgments are inconsistent and we must go to step 7.
7. Compute Mann Whitney test statistic pairwise based on ranked data using steps 1-4 using the formula:
$$Z_N = \frac{U_N - \overline{U}_N}{std_{U_N}}$$
Where:
$$\overline{U_N} = \frac{n_i n_j}{2}$$
$$std_{U_N} = \sqrt{\frac{n_i n_j (n_i + n_j + 1)}{12}}$$
$$U_N = \min\left(n_i n_j + \frac{n_i(n_j+1)}{2} - (R_i)_N, n_i n_j + \frac{n_j(n_j+1)}{2} - (R_j)_N\right)$$
8. if $|Z_N| < Z_{1-\frac{\alpha}{2}}$ then two compared samples are drawn from same population and otherwise samples are drawn from different populations.

Example 4.1

We would like to compare judgments of 3 independent doctors on infecting with COVID-19 for 10 sick people, each doctor is confident T% and unsure I% and may be giving wrong judgment F%.

Table 2. Neutrosophic judgments of infecting with COVID-19.

A			B			C		
T	**I**	**F**	**T**	**I**	**F**	**T**	**I**	**F**
0.207	0.922	0.550	0.905	0.808	0.657	0.949	0.034	0.000
0.879	0.968	0.419	0.555	0.238	0.571	0.057	0.842	0.398
0.200	0.825	0.208	0.726	0.552	0.689	0.845	0.042	0.662
0.824	0.378	0.011	0.230	0.046	0.825	0.858	0.622	0.833
0.859	0.988	0.654	0.779	0.470	0.897	0.853	0.055	0.383
0.874	0.347	0.499	0.599	0.293	0.607	0.416	0.092	0.972
0.842	0.772	0.402	0.007	0.013	0.371	0.407	0.330	0.140
0.855	0.999	0.378	0.688	0.027	0.571	0.978	0.257	0.495
0.368	0.458	0.078	0.940	0.628	0.441	0.048	0.109	0.983

0.698	0.220	0.712	0.614	0.003	0.628	0.110	0.509	0.063

First, we calculate score, accuracy and certainty of the previous data as follows:

Table 3. Score, accuracy and certainty of judgments.

S(A)	S(B)	S(C)	A(A)	A(B)	A(C)	C(A)	C(B)	C(C)
0.245	0.480	0.972	-0.343	0.248	0.949	0.207	0.905	0.949
0.497	0.582	0.272	0.460	-0.016	-0.341	0.879	0.555	0.057
0.389	0.495	0.714	-0.008	0.037	0.183	0.200	0.726	0.845
0.812	0.453	0.468	0.813	-0.595	0.025	0.824	0.230	0.858
0.406	0.471	0.805	0.205	-0.118	0.470	0.859	0.779	0.853
0.676	0.566	0.451	0.375	-0.008	-0.556	0.874	0.599	0.416
0.556	0.541	0.646	0.440	-0.364	0.267	0.842	0.007	0.407
0.493	0.697	0.742	0.477	0.117	0.483	0.855	0.688	0.978
0.611	0.624	0.319	0.290	0.499	-0.935	0.368	0.940	0.048
0.589	0.661	0.513	-0.014	-0.014	0.047	0.698	0.614	0.110

Then we rank our neutrosophic numbers based on its score, accuracy and certainty as follows:

Table 4. Ranks of judgments.

Doctor	Score	Accuracy	Certainty	Rank
A	0.139	-0.598	0.074	1
A	0.33	-0.148	0.754	5
A	0.383	-0.535	0.31	9
A	0.426	0.003	0.638	14
A	0.44	-0.047	0.803	15
A	0.507	0.379	0.733	21
A	0.56	0.06	0.746	23
A	0.568	-0.115	0.723	24
A	0.665	0.206	0.442	28
A	0.822	0.584	0.642	30
B	0.206	-0.449	0.023	3
B	0.288	-0.434	0.541	4
B	0.352	-0.085	0.569	6
B	0.37	0.03	0.906	8
B	0.385	-0.658	0.23	11
B	0.406	0.194	0.342	12
B	0.424	-0.382	0.545	13
B	0.559	-0.2	0.614	22
B	0.594	-0.058	0.343	25
B	0.624	0.125	0.826	26
C	0.181	-0.921	0.022	2
C	0.362	-0.353	0.231	7
C	0.383	-0.037	0.472	10
C	0.468	0.363	0.446	16
C	0.474	-0.217	0.393	17

C	0.489	-0.158	0.737	18
C	0.499	-0.238	0.755	19
C	0.504	0.017	0.854	20
C	0.66	0.107	0.653	27
C	0.705	0.399	0.709	29

Now we rearrange samples and calculate sum of each sample neutrosophic ranks and we get:

$$(R_A)_N = 170, (R_B)_N = 130, (R_C)_N = 165$$

And test statistic is:

$$H_N = \frac{12}{30(30+1)}\left(\frac{170^2 + 130^2 + 165^2}{10}\right) - 3(30+1) = 1.2258$$

Comparing with critical value say at 0.05 significance level we find that $H_N = 1.2258 < \chi^2(2) = 5.9915$ so we accept the null hypothesis and we say that all judgments are consistent.

Example 4.2

3 samples of students were drawn to test whether there is a significant difference between nervous before exam where 3 sets of students were following three strategies of learning, data is shown in Table 5:

Table 5. Nervous Before Exam.

A			B			C		
T	I	F	T	I	F	T	I	F
0.399	0.056	0.457	0.127	0.4545	0.3855	0.152	0.622	0.292
0.4155	0.0705	0.373	0.0025	0.0735	0.083	0.498	0.143	0.748
0.037	0.5	0.206	0.0095	0.171	0.4055	0.357	0.831	0.625
0.4635	0.137	0.3055	0.442	0.2785	0.4225	0.464	0.761	0.551
0.0755	0.029	0.171	0.003	0.4755	0.3055			
0.3335	0.2995	0.207	0.0615	0.072	0.184			

First, we calculate score, accuracy and certainty of the previous data as follows:

Table 6. Score, accuracy and certainty of nervous.

S(A)	S(B)	S(C)	A(A)	A(B)	A(C)	C(A)	C(B)	C(C)
0.629	0.429	0.413	-0.058	-0.259	-0.140	0.399	0.127	0.152
0.657	0.615	0.536	0.043	-0.081	-0.250	0.416	0.003	0.498
0.444	0.478	0.300	-0.169	-0.396	-0.268	0.037	0.010	0.357
0.674	0.580	0.384	0.158	0.020	-0.087	0.464	0.442	0.464
0.625	0.407		-0.096	-0.303		0.076	0.003	
0.609	0.602		0.127	-0.123		0.334	0.062	

Then we rank our neutrosophic numbers based on its score, accuracy and certainty as follows:

Table 7. Ranks of nervous.

Learning Strategy	Score	Accuracy	Certainty	Rank
A	0.674	0.158	0.4635	16
A	0.657	0.0425	0.4155	15
A	0.629	-0.058	0.399	14
A	0.625	-0.0955	0.0755	13
A	0.609	0.1265	0.3335	11

A	0.444	-0.169	0.037	6
B	0.615	-0.0805	0.0025	12
B	0.602	-0.1225	0.0615	10
B	0.580	0.0195	0.442	9
B	0.478	-0.396	0.0095	7
B	0.429	-0.2585	0.127	5
B	0.407	-0.3025	0.003	3
C	0.536	-0.25	0.498	8
C	0.413	-0.14	0.152	4
C	0.384	-0.087	0.464	2
C	0.300	-0.268	0.357	1

Now we rearrange samples and calculate sum of each sample neutrosophic ranks and we get:

$$(R_A)_N = 75, (R_B)_N = 46, (R_C)_N = 15$$

And test statistic is:

$$H_N = \frac{12}{16(16+1)}\left(\frac{75^2}{6} + \frac{46^2}{6} + \frac{15^2}{4}\right) - 3(16+1) = 8.4007$$

Comparing with critical value, say at 0.05 significance level, we find that $H_N = 8.4007 > \chi^2(2) = 5.9915$ so we reject the null hypothesis and we say that level of nervous are not equal, so we must perform Neutrosophic Mann Whitney Test and we have three cases:

Case 1 between A, B:

$$U_N = \min\left(n_1 n_2 + \frac{n_1(n_1+1)}{2} - (R_A)_N, n_1 n_2 + \frac{n_2(n_2+1)}{2} - (R_B)_N\right) = \min(5,31) = 5$$

$$\overline{U_N} = \frac{n_1 n_2}{2} = 18$$

$$std_{U_N} = \sqrt{\frac{n_1 n_2(n_1 + n_2 + 1)}{12}} = 6.244998$$

$$Z_N = \frac{U_N - \overline{U}_N}{std_{U_N}} = -2.08167$$

So $|Z_N| > Z_{0.975} = 1.96$ and hence we reject the null hypothesis and take alternative hypothesis and methods A, B making different nervous level, since $\overline{R_A} = \frac{75}{6} = 12.5 > \overline{R_B} = 7.667$ then nervous level of group A is higher than nervous level of group B.

Case 2 between B, C:

Following same steps, we see that $|Z_N| = |-1.7056|$ <1.96 so there is no difference in nervous level between group B and C.

Case 3 between A, C:

Following same steps, we see that $|Z_N| = |-2.3452|$ >1.96 so there is a significant difference in nervous level between group A and C and nervous level of group A is higher than nervous level of group C because $\overline{R_A} = \frac{75}{6} = 12.5 > \overline{R_C} = \frac{15}{4} = 3.75$.

5. Conclusions

In this paper we have solved the problem of making statistical tests on single valued neutrosophic number-based problems which wasn't solved before. An algorithm to perform Kruskal-Wallis test and Mann Whitney test when dealing with SVNNs is presented and numerical examples were solved successfully in two fields of real-life problems, medical field and educational field. In future we are looking forward to extend other statistical tests which are important in decision making problems.

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