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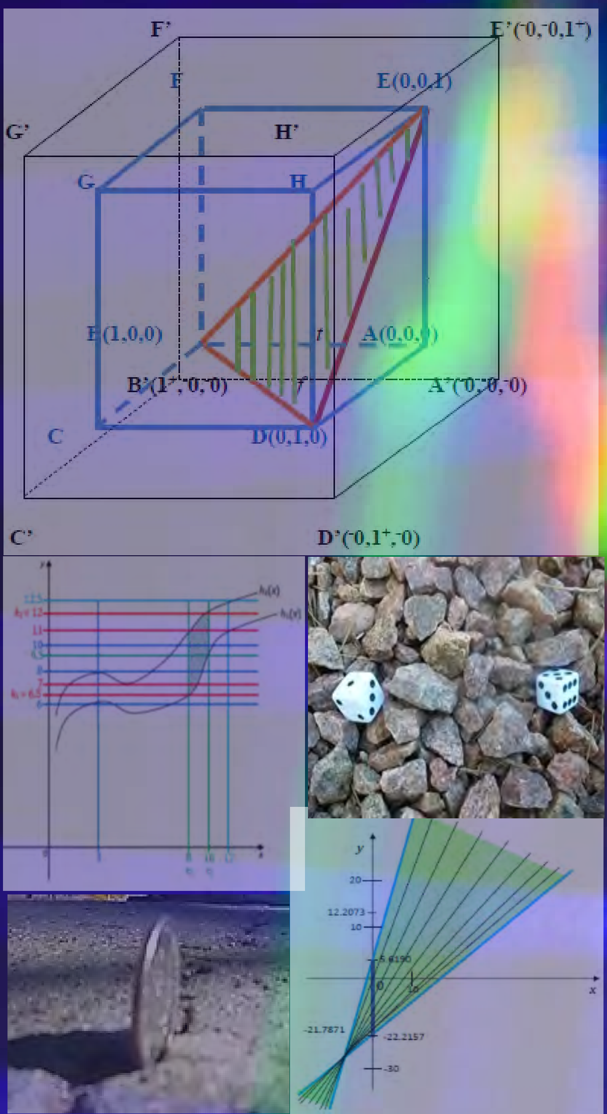

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Contents

MAGDM Model Using the Aczel-Alsina Aggregation Operators of Neutrosophic Entropy Elements in the Case of Neutrosophic Multi-Valued Sets

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Abstract: To overcome the limitations of both the conversion method based on the standard deviation and the decision flexibility in existing neutrosophic multi-valued decision-making models, this study aims to propose various new techniques including a conversion method, Aczel-Alsina aggregation operations, and a multi-attribute group decision making (MAGDM) model in the case of neutrosophic multi-valued sets (MVNSs). First, we propose a conversion method to convert neutrosophic multi-valued elements (MVNEs) into neutrosophic entropy elements (NEEs) based on the mean and normalized Shannon/probability entropy of truth, falsity, and indeterminacy sequences. Second, the score and accuracy functions of NEEs are defined for the ranking of NEEs. Third, the Aczel-Alsina t-norm and t-conorm operations of NEEs and the NEE Aczel-Alsina weighted arithmetic averaging (NEEAAWAA) and NEE Aczel-Alsina weighted geometric averaging (NEEAAWGA) operators are presented to reach the advantage of flexible operations by an adjustable parameter. Fourth, we propose a MAGDM model in light of the NEEAAWAA and NEEAAWGA operators and the score and accuracy functions in the case of NMVSs to solve flexible MAGDM problems with an adjustable parameter subject to decision makers' preference. Finally, an illustrative example is given to verify the impact of different parameter values on the decision results of the proposed MAGDM model. Compared with existing techniques, the new techniques not only overcome the defects of existing techniques but also be broader and more versatile than existing techniques when dealing with MAGDM problems in the case of NMVSs.

Keywords: neutrosophic multi-valued set; neutrosophic entropy element; Aczel-Alsina aggregation operator; group decision making

1. Introduction

In indeterminate and inconsistent situations, multi-valued neutrosophic sets (MVNSs) or neutrosophic hesitant fuzzy sets (NHFSs) can be depicted by the multi-valued sequences of the truth, falsity, and indeterminacy membership degrees, which were the extension of neutrosophic sets [1]. Then, relation operations, aggregation algorithms, and measure methods of MVNSs/NHFSs are critical research topics and play important roles in the fuzzy decision-making issues. Therefore, MVNSs/NHFSs have been used in medical diagnosis, decision making, engineering experiments, measurements, etc. Under the environment of NHFSs, some aggregation operators of single and interval valued NHFSs were presented and utilized in multi-attribute decision making (MADM) problems [2-4]. Then, MADM models based on the extended grey relation analysis [5] and the TOPSIS method [6] were introduced in the setting of NHFSs. Under the environment of MVNSs, some aggregation operators of MVNSs were proposed for multi-valued neutrosophic MADM problems [7, 8]. The Dice similarity measure of single-valued neutrosophic multisets (SVNMs) was introduced and used for medical diagnosis [9]. Furthermore, the correlation coefficient of dynamic SVNMs was presented for MADM problems [10]. The TODIM methods were introduced for MADM problems with MVNSs [11, 12]. However, there are the operational difficulty and complexity between different sequence lengths/cardinalities in multi-valued/hesitant sequences. To solve these issues, Fan et al. [13] introduced a conversion method from SVNMs to single-valued neutrosophic sets (SVNSs) by the average aggregation values of truth, indeterminacy, and falsity sequences, and then proposed the cosine similarity measure of SVNSs for MADM problems in the case of SVNMs. But this conversion method in [13] may result in some loss/distortion of information. To solve this problem, Ye et al. [14] further proposed a reasonable conversion method of neutrosophic multi-valued sets (NMVSs) (including MVNSs, NHFSs, and SVNMs) in light of the average values and consistency degrees (complement of standard deviation) of truth, indeterminacy, and falsity sequences to realize the reasonable information expression and operations of consistency neutrosophic sets/elements (CNS/CNEs), and then developed a multi-attribute group decision making (MAGDM) method using correlation coefficients of CNSs in the case of NMVSs. Then, the conversion method based on the average value and standard deviation [14] is only suitable for normal distribution, which indicates its limitation. Moreover, the existing MAGDM method based on two correlation coefficients of CNSs [14] lacks decision flexibility in the case of NMVSs. Therefore, it is difficult to satisfy the preference of decision makers and/or application needs. Under a probabilistic MVNS environment, Liu and Cheng [15] proposed a three-phase MAGDM method based on the multi-attributive border approximation area comparison (MABAC) method. Since the probability method needs a large number of evaluation values to reasonably give their probabilistic values in MAGDM problems, it is difficult to apply it in actual MAGDM problems. According to the theory of probability and statistics, it is seen that the probability value yielded from a few of the evaluation values (small-scale sample data) is unreasonable and may cause the probability distortion. Moreover, the three-phase MAGDM method also lacks its flexible decision-making feature in the setting of probabilistic MVNSs.

Recently, many researchers have proposed various Aczel-Alsina aggregation operators and their decision-making approaches in various fuzzy circumstances because the operations based on the Aczel-Alsina t-norm and t-conorm [16, 17] reflect the advantage of changeability by an adjustable parameter. For example, Fu et al. [18] proposed the Aczel-Alsina aggregation operators of entropy fuzzy elements and their MAGDM model for renal cancer surgery options in the case of fuzzy multi-sets. Yong et al. [19] introduced the Aczel-Alsina aggregation operators of simplified neutrosophic elements and their MADM approach. Senapati [20] proposed the Aczel-Alsina average aggregation operators of fuzzy picture elements and their MADM approach. Hussain et al. [21] presented the Aczel-Alsina aggregation operators of T-spherical fuzzy elements and their decision-making problems. Then, Senapati et al. [22-24] developed the Aczel-Alsina aggregation operators of (interval-valued) intuitionistic fuzzy elements and their MADM approach. Senapati et al. [25] introduced hesitant fuzzy aggregation operators and applied them to the assessment of cyclone disasters. However, these Aczel-Alsina aggregation operators cannot deal with the aggregation operations and MAGDM issues of NMVSs.

To solve the aforementioned limitations/deflects of the existing methods in the case of NMVSs, the purposes of this research are: (1) to propose a conversion method from a neutrosophic multi-valued element (NMVE) to a neutrosophic entropy element (NEE) in light of the average values and Shannon/probability entropy of truth, falsity, and indeterminacy sequences, (2) to define score and accuracy functions of NEE and ranking laws of NEEs, (3) to propose the Aczel-Alsina t-norm and t-conorm operations of NEEs and the NEE Aczel-Alsina weighted arithmetic averaging (NEEAAWAA) and NEE Aczel-Alsina weighted geometric averaging (NEEAAWGA) operators, and

(4) to develop a MAGDM method by the proposed NEEAAWAA and NEEAAWGA operators and score and accuracy functions to be effectively used for flexible decision-making issues with the information of NMVSs.

In order to verify the impact of different parameter values on the decision results of the proposed MAGDM model, an illustrative example indicates the efficiency and rationality of the proposed MAGDM model. Then, comparative analysis shows that our new techniques not only overcome the defects of the existing techniques, but also are broader and more versatile than the existing techniques when dealing with MAGDM problems in the setting of NMVSs.

However, the conversion method, the NEEAAWAA and NEEAAWGA operators, and the MAGDM model proposed in this research show new contributions and outstanding advantages of these new techniques.

The remainder of this paper contains the following sections. Section 2 proposes a conversion method from NMVE to NEE in terms of the mean and Shannon entropy of the truth, indeterminacy and falsity sequences in NMVEs, and then defines score and accuracy functions of NEE, ranking laws of NEEs, and the Aczel-Alsina t-norm and t-conorm operations of NEEs. Section 3 presents the NEEAAWAA and NEEAAWGA operators and their properties. In Section 4, a MAGDM model is established by the NEEAAWAA and NEEAAWGA operators and the score and accuracy functions of NEEs in the NMVS setting. Section 5 introduces an illustrative example and comparison with existing techniques to show the efficiency and rationality of the new techniques. The last section contains conclusions and further work.

2. NEEs Based on the Mean and Normalized Shannon Entropy in the Case of NMVSs

In the setting of NMVSs, this section first presents a NEE concept by a conversion method based on the Shannon entropy and average values of truth, falsity and indeterminacy sequences, and then defines the score and accuracy functions and ranking laws of NEEs and the Aczel-Alsina t-norm and t-conorm operations of NEEs.

Definition 1 [14]. Set $Y = \{y_k | k = 1, 2, \ldots, m\}$ as a finite universe set. A NMVS M on Y is defined as

$$M = \left\{ \left\langle y_k, M_T(y_k), M_I(y_k), M_F(y_k) \right\rangle | \, y_k \in Y \right\},$$

where $M_T(y_k)$, $M_I(y_k)$ and $M_F(y_k)$ are the truth, indeterminacy, and falsity sequences with the same and/or different fuzzy values, which are denoted by $M_T(y_k) = (\alpha_T^1(y_k), \alpha_T^2(y_k), \ldots, \alpha_T^{r_k}(y_k))$, $M_I(y_k) = (\alpha_I^1(y_k), \alpha_I^2(y_k), \ldots, \alpha_I^{r_k}(y_k))$ and $M_F(y_k) = (\alpha_F^1(y_k), \alpha_F^2(y_k), \ldots, \alpha_F^{r_k}(y_k))$ for $y_k \in Y$, along with the length of their sequence r_k and $0 \le \sup M_T(y_k) + \sup M_I(y_k) + \sup M_F(y_k) \le 3$ ($k = 1, 2, \ldots, m$).

For convenience, the kth element $\left\langle y_k, M_T(y_k), M_I(y_k), M_F(y_k) \right\rangle$ in M is denoted as the NMVE $Ms_k = \left\langle M_{Tk}, M_{Ik}, M_{Fk} \right\rangle = \left\langle (\alpha_{Tk}^1, \alpha_{Tk}^2, \ldots, \alpha_{Tk}^{r_k}), (\alpha_{Ik}^1, \alpha_{Ik}^2, \ldots, \alpha_{Ik}^{r_k}), (\alpha_{Fk}^1, \alpha_{Fk}^2, \ldots, \alpha_{Fk}^{r_k}) \right\rangle$ in decreasing sequences.

First, the concept of the Shannon/probability entropy [26] is introduced below.

Set $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ as a probability distribution on a set of random variables. Then, the Shannon entropy of the probability distribution α is expressed as

$$P(\alpha) = -\sum_{j=1}^{n} \alpha_j \ln(\alpha_j). \tag{1}$$

where $\alpha_j \in [0, 1]$ and $\sum_{j=1}^{n} \alpha_j = 1$.

If all values of α_j ($j = 1, 2, \ldots, n$) are the same, then the entropy $P(\alpha)$ reaches the maximum value, which means perfect consistency of α_j. Generally, there is an approximately linear relationship between entropy and standard deviation: the larger the standard deviation, the smaller the entropy.

In the following, we present the definition of NEE by a conversion method in light of the normalized Shannon entropy and average values of truth, falsity, and indeterminacy sequences in NMVE.

Definition 2. Set $Ms_k = \langle M_{Tk}, M_{Ik}, M_{Fk} \rangle = \langle (\alpha_{Tk}^1, \alpha_{Tk}^2, ..., \alpha_{Tk}^{r_k}), (\alpha_{Ik}^1, \alpha_{Ik}^2, ..., \alpha_{Ik}^{r_k}), (\alpha_{Fk}^1, \alpha_{Fk}^2, ..., \alpha_{Fk}^{r_k}) \rangle$ as the *k*th NMVE. Then, its NEE is represented as follows:

$$N_{Ek} = \langle (\alpha_{Tk}, e_{Tk}), (\alpha_{Ik}, e_{Ik}), (\alpha_{Fk}, e_{Fk}) \rangle,$$

where α_{Tk}, α_{Ik}, $\alpha_{Fk} \in [0, 1]$ are the average values of the truth, indeterminacy, and falsity sequences and e_{Tk}, e_{Ik}, $e_{Fk} \in [0, 1]$ are the normalized entropy values of the truth, indeterminacy, and falsity sequences, which are yielded by the following formulae:

(1) $$\alpha_{Tk} = \frac{1}{r_k}\sum_{j=1}^{r_k}\alpha_{Tk}^j \text{ and } e_{Tk} = -\frac{1}{\ln r_k}\sum_{j=1}^{r_k}\left(\frac{\alpha_{Tk}^j}{\sum_{j=1}^{r_k}\alpha_{Tk}^j}\ln\frac{\alpha_{Tk}^j}{\sum_{j=1}^{r_k}\alpha_{Tk}^j}\right);$$

(2) $$\alpha_{Ik} = \frac{1}{r_k}\sum_{j=1}^{r_k}\alpha_{Ik}^j \text{ and } e_{Ik} = -\frac{1}{\ln r_k}\sum_{j=1}^{r_k}\left(\frac{\alpha_{Ik}^j}{\sum_{j=1}^{r_k}\alpha_{Ik}^j}\ln\frac{\alpha_{Ik}^j}{\sum_{j=1}^{r_k}\alpha_{Ik}^j}\right);$$

(3) $$\alpha_{Fk} = \frac{1}{r_k}\sum_{j=1}^{r_k}\alpha_{Fk}^j \text{ and } e_{Fk} = -\frac{1}{\ln r_k}\sum_{j=1}^{r_k}\left(\frac{\alpha_{Fk}^j}{\sum_{j=1}^{r_k}\alpha_{Fk}^j}\ln\frac{\alpha_{Fk}^j}{\sum_{j=1}^{r_k}\alpha_{Fk}^j}\right).$$

Remark 1. Since the entropy of r_k components cannot exceed $\ln r_k$ ($r_k > 1$), the defined normalized Shannon entropy measures satisfy e_{Tk}, e_{Ik}, $e_{Fk} \in [0, 1]$, and also there exist the following results:

$$\sum_{j=1}^{r_k}\frac{\alpha_{Tk}^j}{\sum_{j=1}^{r_k}\alpha_{Tk}^j} = 1,\ \sum_{j=1}^{r_k}\frac{\alpha_{Ik}^j}{\sum_{j=1}^{r_k}\alpha_{Ik}^j} = 1,\ \sum_{j=1}^{r_k}\frac{\alpha_{Fk}^j}{\sum_{j=1}^{r_k}\alpha_{Fk}^j} = 1,$$

which can satisfy the Shannon entropy conditions. When all components in a multi-valued sequence are the same value, the normalized Shannon entropy is equal to one (the maximum value).

Example 1. Let Ms = <(0.8, 0.7, 0.5), (0.3, 0.2, 0.1), (0.2, 0.2, 0.2)> be NMVE. Using the formulae (1)-(3) in Definition 2, we obtain the following NEE:

N_E = <(0.6667, 0.9835), (0.2, 0.9206), (0.2, 1)>.

Then, we can give the definition of some relations of NEEs below.

Definition 3. Set N_{E1} = <(α_{T1}, e_{T1}), (α_{I1}, e_{I1}), (α_{F1}, e_{F1})> and N_{E2} = <(α_{T2}, e_{T2}), (α_{I2}, e_{I2}), (α_{F2}, e_{F2})> as two NEEs. Then, their relations are defined as follows:

(1) $N_{E1} \supseteq N_{E2} \Leftrightarrow \alpha_{T1} \geq \alpha_{T2}$, $e_{T1} \geq e_{T2}$, $\alpha_{I2} \geq \alpha_{I1}$, $e_{I2} \geq e_{I1}$, $\alpha_{F2} \geq \alpha_{F1}$, and $e_{F2} \geq e_{F1}$;

(2) $N_{E1} = N_{E2} \Leftrightarrow N_{E1} \supseteq N_{E2}$ and $N_{E2} \supseteq N_{E1}$;

(3) $N_{E1} \cup N_{E2} = \langle (\alpha_{T1} \vee \alpha_{T2}, e_{T1} \vee e_{T2}), (\alpha_{I1} \wedge \alpha_{I2}, e_{I1} \wedge e_{I2}), (\alpha_{F1} \wedge \alpha_{F2}, e_{F1} \wedge e_{F2}) \rangle$;

(4) $N_{E1} \cap N_{E2} = \langle (\alpha_{T1} \wedge \alpha_{T2}, e_{T1} \wedge e_{T2}), (\alpha_{I1} \vee \alpha_{I2}, e_{I1} \vee e_{I2}), (\alpha_{F1} \vee \alpha_{F2}, e_{F1} \vee e_{F2}) \rangle$;

(5) $(N_{E1})^c = \langle (\alpha_{F1}, e_{F1}), (1-\alpha_{I1}, 1-e_{I1}), (\alpha_{T1}, e_{T1}) \rangle$ (Complement of N_{E1}).

To sort NEEs, we define the score and accuracy functions and ranking laws of NEEs below.

Definition 4. Let N_{Ek} = <(α_{Tk}, e_{Tk}), (α_{Ik}, e_{Ik}), (α_{Fk}, e_{Fk})> for k = 1, 2 be two NEEs. Then, the score and accuracy functions of NEEs are defined as follows:

$$R(N_{Ek}) = (2 + \alpha_{Tk} \times e_{Tk} - \alpha_{Ik} \times e_{Ik} - \alpha_{Fk} \times e_{Fk}) / 3 \ \text{ for } \ R(N_{Ek}) \in [0,1], \tag{2}$$

$$Q(N_{Ek}) = \alpha_{Tk} \times e_{Tk} - \alpha_{Fk} \times e_{Fk} \ \text{ for } \ Q(N_{Ek}) \in [-1,1]. \tag{3}$$

Thus, the two NEEs N_{E1} and N_{E2} are ranked by the following laws:

(1) If $R(N_{E1}) > R(N_{E2})$, then $N_{E1} > N_{E2}$;

(2) If $R(N_{E1}) = R(N_{E2})$ and $Q(N_{E1}) > Q(N_{E2})$, then $N_{E1} > N_{E2}$;

(3) If $R(N_{E1}) = R(N_{E2})$ and $Q(N_{E1}) = Q(N_{E2})$, then $N_{E1} = N_{E2}$

Example 2. There are two NEEs N_{E1} = <(0.6333, 0.6376), (0.1333, 0.6534), (0.3, 0.6783)> and N_{E2} = <(0.4667, 0.6464), (0.2, 0.6338), (0.2333, 0.7346)>. By Eq. (2), the score values and ranking of the two NEEs are given as follows:

$R(N_{E1})$ = (2+0.6333×0.6376 – 0.1333×0.6534 – 0.3×0.6783)/3 = 0.7044,

$R(N_{E2})$ = (2+0.4667×0.6464 – 0.2×0.6338 – 0.2333×0.7346)/3 = 0.6678.

Since R(N_{E1}) > R(N_{E2}), the ranking of both is $N_{E1} > N_{E2}$.

Regarding the t-norm and t-conorm operations, Aczel and Alsina [16] and Alsina et al. [17] defined the Aczel-Alsina t-norms $G_\rho(c,d)$: $[0, 1]^2 \rightarrow [0,1]$ and the Aczel-Alsina t-conorms $H_\rho(c,d)$: $[0, 1]^2 \rightarrow [0,1]$ for all $c, d \in [0, 1]$ and $\rho \geq 0$ as follows:

(a) The Aczel-Alsina t-norms are defined as

$$G_\rho(c,d) = \begin{cases} G_D(c,d), & if\ \rho = 0 \\ \min(c,d), & if\ \rho = \infty \\ e^{-((-\ln c)^\rho + (-\ln d)^\rho)^{1/\rho}}, & \text{otherwise} \end{cases}.$$

(b) The Aczel-Alsina t-conorms are defined as

$$H_\rho(c,d) = \begin{cases} H_D(c,d), & if\ \rho = 0 \\ \max(c,d), & if\ \rho = \infty \\ 1 - e^{-((-\ln(1-c))^\rho + (-\ln(1-d))^\rho)^{1/\rho}}, & \text{otherwise} \end{cases},$$

where $G_D(c, d)$ and $H_D(c, d)$ are the drastic t-norm and the drastic t-conorm, respectively, which are denoted as

$$G_D(c,d) = \begin{cases} c, & \text{if } d = 1 \\ d, & \text{if } c = 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad H_D(c,d) = \begin{cases} c, & \text{if } d = 1 \\ d, & \text{if } c = 1 \\ 1, & \text{otherwise} \end{cases}.$$

Since the operations based on the Aczel-Alsina t-norm and t-conorm [16, 17] reflect the advantage of changeability by an adjustable parameter ρ, we can give the definition of the Aczel-Alsina t-norm and t-conorm operations of NEEs.

Definition 5. Let N_{E1} = <(α_{T1}, e_{T1}), (α_{I1}, e_{I1}), (α_{F1}, e_{F1})> and N_{E2} = <(α_{T2}, e_{T2}), (α_{I2}, e_{I2}), (α_{F2}, e_{F2})> be two NEEs, $\rho \geq 1$, and $\gamma > 0$. Then, their operations are defined below:

(1) $$N_{E1}\oplus N_{E2}=\left\langle \begin{matrix} \left(1-e^{-((-\ln(1-\alpha_{T1}))^{\rho}+(-\ln(1-\alpha_{T2}))^{\rho})^{1/\rho}},1-e^{-((-\ln(1-e_{T1}))^{\rho}+(-\ln(1-e_{T2}))^{\rho})^{1/\rho}}\right), \\ \left(e^{-((-\ln\alpha_{I1})^{\rho}+(-\ln\alpha_{I2})^{\rho})^{1/\rho}},e^{-((-\ln e_{I1})^{\alpha}+(-\ln e_{I2})^{\rho})^{1/\rho}}\right), \\ \left(e^{-((-\ln\alpha_{F1})^{\rho}+(-\ln\alpha_{F2})^{\rho})^{1/\rho}},e^{-((-\ln e_{F1})^{\rho}+(-\ln e_{F2})^{\rho})^{1/\rho}}\right) \end{matrix}\right\rangle;$$

(2) $$N_{E1}\otimes N_{E2}=\left\langle \begin{matrix} \left(e^{-((-\ln\alpha_{T1})^{\rho}+(-\ln\alpha_{T2})^{\rho})^{1/\rho}},e^{-((-\ln e_{T1})^{\rho}+(-\ln e_{T2})^{\rho})^{1/\rho}}\right), \\ \left(1-e^{-((-\ln(1-\alpha_{I1}))^{\rho}+(-\ln(1-\alpha_{I2}))^{\rho})^{1/\rho}},1-e^{-((-\ln(1-e_{I1}))^{\rho}+(-\ln(1-e_{I2}))^{\rho})^{1/\rho}}\right), \\ \left(1-e^{-((-\ln(1-\alpha_{F1}))^{\rho}+(-\ln(1-\alpha_{F2}))^{\rho})^{1/\rho}},1-e^{-((-\ln(1-e_{F1}))^{\rho}+(-\ln(1-e_{F2}))^{\rho})^{1/\rho}}\right) \end{matrix}\right\rangle;$$

(3) $$\gamma N_{E1}=\left\langle \begin{matrix} \left(1-e^{-(\gamma(-\ln(1-\alpha_{T1}))^{\rho})^{1/\rho}},1-e^{-(\gamma(-\ln(1-e_{T1}))^{\rho})^{1/\rho}}\right), \\ \left(e^{-(\gamma(-\ln(\alpha_{I1}))^{\rho})^{1/\rho}},e^{-(\gamma(-\ln e_{I1})^{\rho})^{1/\rho}}\right),\left(e^{-(\gamma(-\ln\alpha_{F1})^{\rho})^{1/\rho}},e^{-(\gamma(-\ln e_{F1})^{\rho})^{1/\rho}}\right) \end{matrix}\right\rangle;$$

(4) $$(N_{E1})^{\gamma}=\left\langle \begin{matrix} \left(e^{-(\gamma(-\ln\alpha_{T1})^{\rho})^{1/\rho}},e^{-(\gamma(-\ln e_{T1})^{\rho})^{1/\rho}}\right), \\ \left(1-e^{-(\gamma(-\ln(1-\alpha_{I1}))^{\rho})^{1/\rho}},1-e^{-(\gamma(-\ln(1-e_{I1}))^{\rho})^{1/\rho}}\right), \\ \left(1-e^{-(\gamma(-\ln(1-\alpha_{F1}))^{\rho})^{1/\rho}},1-e^{-(\gamma(-\ln(1-e_{F1}))^{\rho})^{1/\rho}}\right) \end{matrix}\right\rangle.$$

Example 3. Let N_{E1} = <(0.6333, 0.6376), (0.1333, 0.6534), (0.3, 0.6783> and N_{E2} = <(0.4667, 0.6464), (0.2, 0.6338), (0.2333, 0.7346)> be two NEEs, ρ = 3, and γ = 0.6. Using the operations (1)-(4) in Definition 5, we obtain the following operational results:

(1) $$N_{E1}\oplus N_{E2}=\left\langle \begin{matrix} \left(1-e^{-((-\ln(1-0.6333))^{3}+(-\ln(1-0.4667))^{3})^{1/3}},1-e^{-((-\ln(1-0.6376))^{3}+(-\ln(1-0.6464))^{3})^{1/3}}\right), \\ \left(e^{-((-\ln 0.1333)^{3}+(-\ln 0.2)^{3})^{1/3}},e^{-((-\ln 0.6534)^{3}+(-\ln 0.6338)^{3})^{1/3}}\right), \\ \left(e^{-((-\ln 0.3)^{3}+(-\ln 0.2333)^{3})^{1/3}},e^{-((-\ln 0.6783)^{3}+(-\ln 0.7346)^{3})^{1/3}}\right) \end{matrix}\right\rangle;$$

$$=\langle(0.6603,\ 0.7260),\ (0.0991,\ 0.5735),\ (0.1845,\ 0.6411)\rangle$$

(2) $$N_{E1}\otimes N_{E2}=\left\langle \begin{matrix} \left(e^{-((-\ln 0.6333)^{3}+(-\ln 0.4667)^{3})^{1/3}},e^{-((-\ln 0.6376)^{3}+(-\ln 0.6464)^{3})^{1/3}}\right), \\ \left(1-e^{-((-\ln(1-0.1333))^{3}+(-\ln(1-0.2))^{3})^{1/3}},1-e^{-((-\ln(1-0.6534))^{3}+(-\ln(1-0.6338))^{3})^{1/3}}\right), \\ \left(1-e^{-((-\ln(1-0.3))^{3}+(-\ln(1-0.2333))^{3})^{1/3}},1-e^{-((-\ln(1-0.6783))^{3}+(-\ln(1-0.7346))^{3})^{1/3}}\right) \end{matrix}\right\rangle;$$

$$=\langle(0.4434,\ 0.5721),\ (0.2143,\ 0.7278),\ (0.3299,\ 0.7898)\rangle$$

(3) $$0.6N_{E1}=\left\langle \begin{matrix} \left(1-e^{-(0.6(-\ln(1-0.6333))^{3})^{1/3}},1-e^{-(0.6(-\ln(1-0.6376))^{3})^{1/3}}\right), \\ \left(e^{-(0.6(-\ln 0.1333)^{3})^{1/3}},e^{-(0.6(-\ln 0.6534)^{3})^{1/3}}\right),\left(e^{-(0.6(-\ln 0.3)^{3})^{1/3}},e^{-(0.6(-\ln 0.6783)^{3})^{1/3}}\right) \end{matrix}\right\rangle;$$

$$=\langle(0.5709,\ 0.5752),\ (0.1827,\ 0.6984),\ (0.3622,\ 0.7208)\rangle$$

(4) $$(N_{E1})^{0.6}=\left\langle \begin{matrix} \left(e^{-(0.6(-\ln 0.6333)^3)^{1/3}}, e^{-(0.6(-\ln 0.6376)^3)^{1/3}}\right), \\ \left(1-e^{-(0.6(-\ln(1-0.1333))^3)^{1/3}}, 1-e^{-(0.6(-\ln(1-0.6534))^3)^{1/3}}\right), \\ \left(1-e^{-(0.6(-\ln(1-0.3))^3)^{1/3}}, 1-e^{-(0.6(-\ln(1-0.6783))^3)^{1/3}}\right) \end{matrix} \right\rangle.$$

$$=\langle(0.6803, 0.6841), (0.1137, 0.5909), (0.2598, 0.6158)\rangle$$

3. Aczel-Alsina Aggregation Operators of NEEs

3.1 NEEAAWAA Operator

This part proposes the NEEAAWAA operator according to the operations in Definition 5.

Definition 6. Set N_{Ek} = <(α_{Tk}, e_{Tk}), (α_{Ik}, e_{Ik}), (α_{Fk}, e_{Fk})> (k = 1, 2, …, m) as a group of NEEs with the weight vector of N_{Ek} $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ for $\gamma_k \in [0, 1]$ and $\sum_{k=1}^{m}\gamma_k = 1$. Then, the definition of a NEEAAWAA operator is given by the following form:

$$NEEAAWAA(N_{E1}, N_{E2}, ..., N_{Em}) = \bigoplus_{k=1}^{m}\gamma_k N_{Ek}. \quad (4)$$

Thus, the NEEAAWAA operator has the following theorem.

Theorem 1. Set N_{Ek} = <(α_{Tk}, e_{Tk}), (α_{Ik}, e_{Ik}), (α_{Fk}, e_{Fk})> (k = 1, 2, …, m) as a group of NEEs with the weight vector of N_{Ek} $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ for $\gamma_k \in [0, 1]$ and $\sum_{k=1}^{m}\gamma_k = 1$. Then, the collected value of the NEEAAWAA operator is till NEE, which is given by the formula:

$$NEEAAWAA(N_{E1}, N_{E2}, ..., N_{Em}) = \bigoplus_{k=1}^{m}\gamma_k N_{Ek} = \left\langle \begin{matrix} \left(1-e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(1-\alpha_{Tk}))^{\rho}\right)^{1/\rho}}, 1-e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(1-e_{Tk}))^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln\alpha_{Ik})^{\rho}\right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln e_{Ik})^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln\alpha_{Fk})^{\rho}\right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln e_{Fk})^{\rho}\right)^{1/\rho}}\right) \end{matrix} \right\rangle. \quad (5)$$

Proof. Theorem 1 is proved by mathematical induction below.

(1) Let k = 2. According to Definition 5 and Eq. (4), the operational results are given as

$$NEEAAWAA(N_{E1}, N_{E2}) = \gamma_1 N_{E1} \oplus \gamma_2 N_{E2}$$
$$= \left\langle \begin{array}{l} \left(1-e^{-\left(\gamma_1(-\ln(1-\alpha_{T1}))^{\rho}\right)^{1/\rho}}, 1-e^{-\left(\gamma_1(-\ln(1-e_{T1}))^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\gamma_1(-\ln\alpha_{I1})^{\rho}\right)^{1/\rho}}, e^{-\left(\gamma_1(-\ln e_{I1})^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\gamma_1(-\ln\alpha_{F1})^{\rho}\right)^{1/\rho}}, e^{-\left(\gamma_1(-\ln e_{F1})^{\rho}\right)^{1/\rho}}\right) \end{array} \right\rangle \oplus \left\langle \begin{array}{l} \left(1-e^{-\left(\gamma_2(-\ln(1-\alpha_{T2}))^{\rho}\right)^{1/\rho}}, 1-e^{-\left(\gamma_2(-\ln(1-e_{T2}))^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\gamma_2(-\ln\alpha_{I2})^{\rho}\right)^{1/\rho}}, e^{-\left(\gamma_2(-\ln e_{I2})^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\gamma_2(-\ln\alpha_{F2})^{\rho}\right)^{1/\rho}}, e^{-\left(\gamma_2(-\ln e_{F2})^{\rho}\right)^{1/\rho}}\right) \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l} \left(1-e^{-\left(\sum_{k=1}^{2}\gamma_k(-\ln(1-\alpha_{Tk}))^{\rho}\right)^{1/\rho}}, 1-e^{-\left(\sum_{k=1}^{2}\gamma_k(-\ln(1-e_{Tk}))^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{2}\gamma_k(-\ln\alpha_{Ik})^{\rho}\right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{2}\gamma_k(-\ln e_{Ik})^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{2}\gamma_k(-\ln\alpha_{Fk})^{\rho}\right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{2}\gamma_k(-\ln e_{Fk})^{\rho}\right)^{1/\rho}}\right) \end{array} \right\rangle. \quad (6)$$

(2) Assume Eq. (5) for $k = s$ exists. Then, there exists the following result:

$$NEEAAWAA(N_{E1}, Nz_{E2}, ..., N_{Es}) = \mathop{\oplus}_{k=1}^{s} \gamma_k N_{Ek} = \left\langle \begin{array}{l} \left(1-e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln(1-\alpha_{Tk}))^{\rho}\right)^{1/\rho}}, 1-e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln(1-e_{Tk}))^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln\alpha_{Ik})^{\rho}\right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln e_{Ik})^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln\alpha_{Fk})^{\rho}\right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln e_{Fk})^{\rho}\right)^{1/\rho}}\right) \end{array} \right\rangle. \quad (7)$$

(3) Let $k = s+1$. By Eqs. (6) and (7), there is the following result:

$$NEEAAWAA(N_1, N_2, ..., N_s, N_{s+1}) = \mathop{\oplus}_{k=1}^{s+1} \gamma_k N_{Ek}$$

$$= \left\langle \begin{matrix} \left(1-e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln(1-\alpha_{Tk}))^{\rho}\right)^{1/\rho}}, 1-e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln(1-e_{Tk}))^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln\alpha_{Ik})^{\rho}\right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln e_{Ik})^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln\alpha_{Fk})^{\rho}\right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln e_{Fk})^{\rho}\right)^{1/\rho}}\right) \end{matrix} \right\rangle \oplus \left\langle \begin{matrix} \left(1-e^{-\left(\gamma_{m+1}(-\ln(1-\alpha_{Tm+1}))^{\rho}\right)^{1/\rho}}, 1-e^{-\left(\gamma_{m+1}(-\ln(1-e_{Tm+1}))^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\gamma_{m+1}(-\ln\alpha_{Im+1})^{\rho}\right)^{1/\rho}}, e^{-\left(\gamma_{m+1}(-\ln e_{Im+1})^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\gamma_{m+1}(-\ln\alpha_{Fm+1})^{\rho}\right)^{1/\rho}}, e^{-\left(\gamma_{m+1}(-\ln e_{Fm+1})^{\rho}\right)^{1/\rho}}\right) \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} \left(1-e^{-\left(\sum_{k=1}^{s+1}\gamma_k(-\ln(1-\alpha_{Tk}))^{\rho}\right)^{1/\rho}}, 1-e^{-\left(\sum_{k=1}^{s+1}\gamma_k(-\ln(1-e_{Tk}))^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{s+1}\gamma_k(-\ln\alpha_{Ik})^{\alpha}\right)^{1/\alpha}}, e^{-\left(\sum_{k=1}^{s+1}\gamma_k(-\ln e_{Ik})^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{s+1}\gamma_k(-\ln\alpha_{Fk})^{\rho}\right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{s+1}\gamma_k(-\ln e_{Fk})^{\rho}\right)^{1/\rho}}\right) \end{matrix} \right\rangle.$$

Based on the above (1)-(3), Eq. (5) can hold for any k. □

Moreover, the NEEAAWAA operator of Eq. (5) implies the following properties.

Theorem 2. The NEEAAWAA operator contains the properties (P1)-(P4):

(P1) *Idempotency:* Set N_{Ek} = <(α_{Tk}, e_{Tk}), (α_{Ik}, e_{Ik}), (α_{Fk}, e_{Fk})> (k = 1, 2, …, m) as a group of NEEs. If $N_{Ek} = N_E$ (k = 1, 2, …, m), there is NEEAAWAA(N_{E1}, N_{E2}, …, N_{Em}) = N_E.

(P2) *Commutativity:* Assume that a group of NEEs $(N_{E1}^{'}, N_{E2}^{'}, ..., N_{Em}^{'})$ is any permutation of (N_{E1}, N_{E2}, …, N_{Em}). Then, $NEEAAWAA(N_{E1}^{'}, N_{E2}^{'}, ..., N_{Em}^{'}) = NEEAAWAA\left(N_{E1}, N_{E2}, \ldots, N_{Em}\right)$ can exist.

(P3) *Boundedness:* If the maximum and minimum NEEs are specified as follows:

$$N_{E\max} = \left\langle \left(\max_k(\alpha_{Tk}), \max_k(e_{Tk})\right), \left(\min_k(\alpha_{Ik}), \min_k(e_{Ik})\right), \left(\min_k(\alpha_{Fk}), \min_k(e_{Fk})\right) \right\rangle,$$

$$N_{E\min} = \left\langle \left(\min_k(\alpha_{Tk}), \min_k(e_{Tk})\right), \left(\max_k(\alpha_{Ik}), \max_k(e_{Ik})\right), \left(\max_k(\alpha_{Fk}), \max_k(e_{Fk})\right) \right\rangle,$$

then $N_{E\min}$ ≤ NEEAAWAA(N_{E1}, N_{E2}, …, N_{Em}) ≤ $N_{E\max}$ can exist.

(P4) *Monotonicity:* If $N_{Ek} \le N_{Ek}^{*}$ (k = 1, 2, …, m), there is NEEAAWAA(N_{E1}, N_{E2}, …, N_{Em}) ≤ NEEAAWAA($N_{E1}^{*}, N_{E2}^{*}, ..., N_{Em}^{*}$).

Proof. (P1) If N_{Ek} = <(α_{Tk}, e_{Tk}), (α_{Ik}, e_{Ik}), (α_{Fk}, e_{Fk})> = N_E (k = 1, 2, …, m), by Eq. (4) we yield the result:

$$NEEAAWAA(N_{E1},N_{E2},...,N_{Em})=\overset{m}{\underset{k=1}{\oplus}}\gamma_k N_{Ek}$$

$$=\left\langle \begin{array}{l} \left(1-e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(1-\alpha_{Tk}))^{\rho}\right)^{1/\rho}},1-e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(1-e_{Tk}))^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln\alpha_{Ik})^{\rho}\right)^{1/\rho}},e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln e_{Ik})^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln\alpha_{Fk})^{\rho}\right)^{1/\rho}},e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln e_{Fk})^{\rho}\right)^{1/\rho}}\right) \end{array}\right\rangle = \left\langle \begin{array}{l} \left(1-e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(1-\alpha_{T}))^{\rho}\right)^{1/\rho}},1-e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(1-e_{T}))^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln\alpha_{I})^{\rho}\right)^{1/\rho}},e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln e_{I})^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln\alpha_{F})^{\rho}\right)^{1/\rho}},e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln e_{F})^{\rho}\right)^{1/\rho}}\right) \end{array}\right\rangle$$

$$=\left\langle \begin{array}{l} \left(1-e^{-\left((-\ln(1-\alpha_T))^{\rho}\right)^{1/\rho}},1-e^{-\left((-\ln(1-e_T))^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left((-\ln\alpha_I)^{\rho}\right)^{1/\rho}},e^{-\left((-\ln e_I)^{\rho}\right)^{1/\rho}}\right), \\ \left(e^{-\left((-\ln\alpha_F)^{\rho}\right)^{1/\rho}},e^{-\left((-\ln e_F)^{\rho}\right)^{1/\rho}}\right) \end{array}\right\rangle = \left\langle \begin{array}{l} \left(1-e^{\ln(1-\alpha_T)},1-e^{\ln(1-e_T)}\right), \\ \left(e^{\ln\alpha_I},e^{\ln e_I}\right), \\ \left(e^{\ln\alpha_F},e^{\ln e_F}\right) \end{array}\right\rangle = \langle(\alpha_T,e_T),(\alpha_I,e_I),(\alpha_F,e_F)\rangle = N_E.$$

(P2) The property (P2) is straightforward.

(P3) Since the inequalities $\min_k(\alpha_{Tk})\le\alpha_{Tk}\le\max_k(\alpha_{Tk})$, $\min_k(e_{Tk})\le e_{Tk}\le\max_k(e_{Tk})$, $\min_k(\alpha_{Ik})\le\alpha_{Ik}\le\max_k(\alpha_{Ik})$, $\min_k(e_{Ik})\le e_{Ik}\le\max_k(e_{Ik})$, $\min_k(\alpha_{Fk})\le\alpha_{Fk}\le\max_k(\alpha_{Fk})$, and $\min_k(e_{Fk})\le e_{Fk}\le\max_k(e_{Fk})$ exist based on the maximum and minimum NEEs, there are the following inequalities:

$$1-e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(1-\min_k(\alpha_{Tk})))^{\rho}\right)^{1/\rho}}\le 1-e^{-\left(\sum_{k=1}^{s}\gamma_k(-\ln(1-\alpha_{Tk}))^{\rho}\right)^{1/\rho}}\le 1-e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(1-\max_k(\alpha_{Tk})))^{\rho}\right)^{1/\rho}},$$

$$1-e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(1-\min_k(e_{Tk})))^{\alpha}\right)^{1/\alpha}}\le 1-e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(1-e_{Tk}))^{\rho}\right)^{1/\rho}}\le 1-e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(1-\max_k(e_{Tk})))^{\rho}\right)^{1/\rho}},$$

$$e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(\max_k(\alpha_{Ik})))^{\rho}\right)^{1/\rho}}\le e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln\alpha_{Ik})^{\rho}\right)^{1/\rho}}\le e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(\min_k(\alpha_{Ik})))^{\rho}\right)^{1/\rho}},$$

$$e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(\max_k(e_{Ik})))^{\rho}\right)^{1/\rho}}\le e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln e_{Ik})^{\rho}\right)^{1/\rho}}\le e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(\min_k(e_{Ik})))^{\rho}\right)^{1/\rho}},$$

$$e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(\max_k(\alpha_{Fk})))^{\rho}\right)^{1/\rho}}\le e^{-\left(\sum_{k=1}^{m}\gamma_k(-ln\alpha_{Fk})^{\rho}\right)^{1/\rho}}\le e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(\min_k(\alpha_{Fk})))^{\rho}\right)^{1/\rho}},$$

$$e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(\max_k(e_{Fk})))^{\rho}\right)^{1/\rho}}\le e^{-\left(\sum_{k=1}^{m}\gamma_k(-ln e_{Fk})^{\rho}\right)^{1/\rho}}\le e^{-\left(\sum_{k=1}^{m}\gamma_k(-\ln(\min_k(e_{Fk})))^{\rho}\right)^{1/\rho}}.$$

Regarding the property (P1) and the score value of Eq. (2), we can obtain $N_{E\min}\le\overset{m}{\underset{k=1}{\oplus}}\gamma_k N_{Ek}\le N_{E\max}$, then there is $N_{E\min}\le$ NEEAAWAA(N_{E1}, N_{E2}, …, N_{Em}) $\le N_{E\max}$.

(P4) When $N_{Ek}\le N_{Ek}^{*}$ (k = 1, 2, …, m), there exists $\overset{m}{\underset{k=1}{\oplus}}\gamma_k N_{Ek}\le\overset{m}{\underset{k=1}{\oplus}}\gamma_k N_{Ek}^{*}$. Thus, NEEAAWAA($N_{E1}$, N_{E2}, …, N_{Em}) $\le$ NEEAAWAA($N_{E1}^{*},N_{E2}^{*},...,N_{Em}^{*}$) can exist. □

Especially when ρ = 1, the NEEAAWAA operator of Eq. (5) is reduced to the NEE weighted arithmetic averaging (NEEWAA) operator:

$$NEEWAA(N_{E1}, Nz_{E2}, ..., N_{Em}) = \overset{m}{\underset{k=1}{\oplus}} \gamma_k N_{Ek} = \left\langle \begin{gathered} \left(1 - \prod_{k=1}^{m} (1-\alpha_{Tk})^{\gamma_k}, 1 - \prod_{k=1}^{m} (1-e_{Tk})^{\gamma_k} \right), \\ \left(\prod_{k=1}^{m} (\alpha_{Ik})^{\gamma_k}, \prod_{k=1}^{m} (e_{Ik})^{\gamma_k} \right), \left(\prod_{k=1}^{m} (\alpha_{Fk})^{\gamma_k}, \prod_{k=1}^{m} (e_{Fk})^{\gamma_k} \right) \end{gathered} \right\rangle. \quad (8)$$

3.2 NEEAAWGA Operator

This part presents the NEEAAWGA operator according to the operations in Definition 5.

Definition 7. Set N_{Ek} = <(α_{Tk}, e_{Tk}), (α_{Ik}, e_{Ik}), (α_{Fk}, e_{Fk})> (k = 1, 2, …, m) as a group of NEEs with the weight vector of N_{Ek} $\boldsymbol{\gamma}$ = (γ_1, γ_2, …, γ_m) for $\gamma_k \in$ [0, 1] and $\sum_{k=1}^{m} \gamma_k = 1$. Thus, a NEEAAWGA operator is defined as

$$NEEAAWGA(N_{E1}, N_{E2}, ..., N_{Em}) = \overset{m}{\underset{k=1}{\otimes}} (N_{Ek})^{\gamma_k}. \quad (9)$$

Then, the NEEAAWGA operator shows the following theorem.

Theorem 3. Set N_{Ek} = <(α_{Tk}, e_{Tk}), (α_{Ik}, e_{Ik}), (α_{Fk}, e_{Fk})> (k = 1, 2, …, m) as a group of NEEs with the weight vector of N_{Ek} $\boldsymbol{\gamma}$ = (γ_1, γ_2, …, γ_m) for $\gamma_k \in$ [0, 1] and $\sum_{k=1}^{m} \gamma_k = 1$. Then, the collected value of the NEEAAWGA operator is also NEE, which is obtained by the following formula:

$$NEEAAWGA(N_{E1}, N_{E2}, ..., N_{Em}) = \overset{m}{\underset{k=1}{\otimes}} (N_{Ek})^{\gamma_k} = \left\langle \begin{gathered} \left(e^{-\left(\sum_{k=1}^{m} \beta_k (-\ln \alpha_{Tk})^{\rho} \right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln e_{Tk})^{\rho} \right)^{1/\rho}} \right), \\ \left(1 - e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln(1-\alpha_{Ik}))^{\rho} \right)^{1/\rho}}, 1 - e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln(1-e_{Ik}))^{\rho} \right)^{1/\rho}} \right), \\ \left(1 - e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln(1-\alpha_{Fk}))^{\rho} \right)^{1/\rho}}, 1 - e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln(1-e_{Fk}))^{\rho} \right)^{1/\rho}} \right) \end{gathered} \right\rangle. \quad (10)$$

By the similar proof way of Theorem 1, we can easily verify Theorem 3, which is omitted.

Similarly, the NEEAAWGA operator also contains some properties.

Theorem 4. The NEEAAWGA operator includes these properties (P1)-(P4):

(P1) *Idempotency:* Set N_{Ek} = <(α_{Tk}, e_{Tk}), (α_{Ik}, e_{Ik}), (α_{Fk}, e_{Fk})> (k = 1, 2, …, m) as a group of NEEs. When N_{Ek} = N_E (k = 1, 2, …, m), NEEAAWGA(N_{E1}, N_{E2}, …, N_{Em}) = N_E exists.

(P2) *Commutativity:* Assume that a group of NEEs $(N'_{E1}, N'_{E2}, ..., N'_{Em})$ is any permutation of (N_{E1}, N_{E2}, …, N_{Em}). Then, $NEEAAWGA(N'_{E1}, N'_{E2}, ..., N'_{Em}) = NEEAAWGA\left(N_{E1}, N_{E2}, \ldots, N_{Em}\right)$ can exist.

(P3) *Boundedness:* If the maximum and minimum NEEs are specified below:

$$N_{E\max} = \left\langle \left(\max_k(\alpha_{Tk}), \max_k(e_{Tk}) \right), \left(\min_k(\alpha_{Ik}), \min_k(e_{Ik}) \right), \left(\min_k(\alpha_{Fk}), \min_k(e_{Fk}) \right) \right\rangle,$$

$$N_{E\min} = \left\langle \left(\min_k(\alpha_{Tk}), \min_k(e_{Tk}) \right), \left(\max_k(\alpha_{Ik}), \max_k(e_{Ik}) \right), \left(\max_k(\alpha_{Fk}), \max_k(e_{Fk}) \right) \right\rangle,$$

then $N_{E\min}$ ≤ NEEAAWGA(N_{E1}, N_{E2}, …, N_{Em}) ≤ $N_{E\max}$ can hold.

(P4) *Monotonicity:* Set $N_{Ek} \le N^*_{Ek}$ (k = 1, 2, …, m). Then, NEEAAWGA(N_{E1}, N_{E2}, …, N_{Em}) ≤ NEEAAWGA($N^*_{E1}, N^*_{E2}, ..., N^*_{Em}$) exists.

By the similar proof method of Theorem 2, we can easily verify Theorem 4, which is not repeated here.

Especially when $\rho = 1$, the NEEAAWGA operator is reduced to the NEE weighted geometric averaging (NEEWGA) operator:

$$NEEWGA(N_{E1}, Nz_{E2}, ..., N_{Em}) = \overset{m}{\underset{k=1}{\otimes}} (N_{Ek})^{\gamma_k}$$
$$= \left\langle \begin{matrix} \left(\prod_{k=1}^{m} (\alpha_{Tk})^{\gamma_k}, \prod_{k=1}^{m} (e_{Tk})^{\gamma_k} \right), \left(1 - \prod_{k=1}^{m} (1-\alpha_{Ik})^{\gamma_k}, 1 - \prod_{k=1}^{m} (1-e_{Ik})^{\gamma_k} \right), \\ \left(1 - \prod_{k=1}^{m} (1-\alpha_{Fk})^{\gamma_k}, 1 - \prod_{k=1}^{m} (1-e_{Fk})^{\gamma_k} \right) \end{matrix} \right\rangle. \quad (11)$$

4. MAGDM Model Based on the NEEAAWAA and NEEAAWGA Operators and the Score and Accuracy Functions

In this section, a MAGDM model is established by the proposed NEEAAWAA and NEEAAWGA operators and score and accuracy functions to solve MAGDM problems in the NMVS setting.

Regarding a MAGDM problem, a set of s alternatives $L = \{L_1, L_2, …, L_s\}$ is preliminarily provided and satisfactorily evaluated by a set of m attributes $B = \{b_1, b_2, …, b_m\}$. Then, the importance of various attributes b_k ($k = 1, 2, …, m$) is assigned by a weight vector $\gamma = (\gamma_1, \gamma_2, …, \gamma_m)$ with $\gamma_k \in [0, 1]$ and $\sum_{k=1}^{m} \gamma_k = 1$. The satisfactory evaluation values of each alternative L_i ($i = 1, 2, …, s$) over each attribute b_k ($k = 1, 2, …, m$) are assigned by a group of experts/decision makers, then the evaluated truth, indeterminacy, and falsity sequences are denoted as the NMVE $Ms_{ik} = \langle M_{Tik}, M_{Iik}, M_{Fik} \rangle = \langle (\alpha_{Tik}^1, \alpha_{Tik}^2, ..., \alpha_{Tik}^{r_k}), (\alpha_{Iik}^1, \alpha_{Iik}^2, ..., \alpha_{Iik}^{r_k}), (\alpha_{Fik}^1, \alpha_{Fik}^2, ..., \alpha_{Fik}^{r_k}) \rangle$ for $0 \le \sup M_{Tik} + \sup M_{Iik} + \sup M_{Fik} \le 3$ and α_{Tik}^j, $\alpha_{Iik}^j, \alpha_{Fik}^j \in [0, 1]$ ($j = 1, 2, …, r_k$; $i = 1, 2, …, s$; $k = 1, 2, …, m$). Then, the evaluated NMVEs are represented as the decision matrix of NMVEs $M_D = (Ms_{ik})_{s\times m}$.

Regarding this MAGDM problem, we give the decision steps below.

Step 1: By the formulae (1)-(3) in Definition 2, all NMVEs in the decision matrix M_D are conversed into the NEEs $N_{Eik} = <(\alpha_{Tik}, e_{Tik}), (\alpha_{Iik}, e_{Iik}), (\alpha_{Fik}, e_{Fik})>$ for $\alpha_{Tik}, \alpha_{Iik}, \alpha_{Fik} \in [0, 1]$ and $e_{Tik}, e_{Iik}, e_{Fik} \in [0, 1]$ ($i = 1, 2, …, s$; $k = 1, 2, …, m$), which are constructed as the decision matrix of NEEs $N_D = (N_{Eik})_{s\times m}$.

Step 2: Using one of Eq. (5) and Eq. (10), the aggregated NEE N_{Ei} ($i = 1, 2, …, s$) for L_i is given by one of two formulae:

$$N_{Ei} = NEEAAWAA(N_{Ei1}, N_{Ei2}, ..., N_{Eim}) = \overset{m}{\underset{k=1}{\oplus}} \gamma_k N_{Eik} = \left\langle \begin{matrix} \left(1 - e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln(1-\alpha_{Tik}))^{\rho} \right)^{1/\rho}}, 1 - e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln(1-e_{Tik}))^{\rho} \right)^{1/\rho}} \right), \\ \left(e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln \alpha_{Iik})^{\rho} \right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln e_{Iik})^{\rho} \right)^{1/\rho}} \right), \\ \left(e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln \alpha_{Fik})^{\rho} \right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln e_{Fik})^{\rho} \right)^{1/\rho}} \right) \end{matrix} \right\rangle, \quad (12)$$

$$N_{Ei} = NEEAAWGA(N_{Ei1}, N_{Ei2}, ..., N_{Eim}) = \underset{k=1}{\overset{m}{\otimes}} (N_{Eik})^{\gamma_k} = \left\langle \begin{array}{l} \left(e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln \alpha_{Tik})^{\rho}\right)^{1/\rho}}, e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln e_{Tik})^{\rho}\right)^{1/\rho}} \right), \\ \left(1-e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln(1-\alpha_{Iik}))^{\rho}\right)^{1/\rho}}, 1-e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln(1-e_{Iik}))^{\rho}\right)^{1/\rho}} \right), \\ \left(1-e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln(1-\alpha_{Fik}))^{\rho}\right)^{1/\rho}}, 1-e^{-\left(\sum_{k=1}^{m} \gamma_k (-\ln(1-e_{Fik}))^{\rho}\right)^{1/\rho}} \right) \end{array} \right\rangle . \quad (13)$$

Step 3: The score values of $R(N_{Ei})$ (the accuracy values of $Q(N_{Ei})$ subject to necessary) (i = 1, 2, …, s) are obtained by Eq. (2) (Eq. (3)).

Step 4: All alternatives are sorted based on the ranking laws and the best one is chosen.

Step 5: End.

5. Illustrative Example and Comparison with Existing Techniques

5.1 Example on the Performance Assessment of Service Robots

Service robotics contain many application fields, such as industrial service robots, home service robots, and medical service robots. They are improving our daily lives in various ways. Then, most of them have unique designs and different degrees of automation (from full teleoperation to fully autonomous operation) to affect the quality of our work and lives. However, the performance evaluation of the service robots is an important issue for users. To indicate the applicability of the developed MAGDM model under the environment of NMVSs, this subsection applies the developed MAGDM model to the performance assessment of service robots.

Suppose that there are four kinds of service robots/alternatives, which are denoted as their set $L = \{L_1, L_2, L_3, L_4\}$. Then, they must satisfy the requirements of the four performance indices/attributes: mobility (b_1), dexterity (b_2), working ability (b_3), and communication and control capability (b_4). The weight vector of the four attributes is given as $\boldsymbol{\gamma}$ = (0.25, 0.24, 0.26, 0.25) by experts/decision makers. The assessment of four types of service robots over the four attributes is performed by three experts/decision makers, where their evaluation values are assigned by the NMVEs $Ms_{ik} = \langle M_{Tik}, M_{Iik}, M_{Fik} \rangle = \langle (\alpha_{Tik}^1, \alpha_{Tik}^2, ..., \alpha_{Tik}^{r_k}), (\alpha_{Iik}^1, \alpha_{Iik}^2, ..., \alpha_{Iik}^{r_k}), (\alpha_{Fik}^1, \alpha_{Fik}^2, ..., \alpha_{Fik}^{r_k}) \rangle$ (consisting of the truth, indeterminacy, and falsity sequences) for $0 \le \sup M_{Tik} + \sup M_{Iik} + \sup M_{Fik} \le 3$ and α_{Tik}^j, α_{Iik}^j, $\alpha_{Fik}^j \in [0, 1]$ (j = 1, 2, 3; i, k = 1, 2, 3, 4; r_k = 3). Thus, all assessed NMVEs can be expressed as the following decision matrix of NMVEs $M_D = (Ms_{ik})_{4\times4}$:

$$M_D = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} \left\langle \begin{array}{l} (0.8, 0.7, 0.7), \\ (0.3, 0.2, 0.1), \\ (0.2, 0.2, 0.1) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.8, 0.7, 0.6), \\ (0.3, 0.1, 0.1), \\ (0.4, 0.3, 0.2) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.7, 0.7, 0.6), \\ (0.3, 0.3, 0.3), \\ (0.3, 0.2, 0.2) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.7, 0.7, 0.7), \\ (0.3, 0.1, 0.1), \\ (0.3, 0.3, 0.3) \end{array} \right\rangle \\ \left\langle \begin{array}{l} (0.7, 0.7, 0.6), \\ (0.2, 0.2, 0.1), \\ (0.2, 0.2, 0.1) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.7, 0.7, 0.7), \\ (0.3, 0.3, 0.2), \\ (0.3, 0.1, 0.1) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.8, 0.8, 0.7), \\ (0.1, 0.1, 0.1), \\ (0.3, 0.2, 0.2) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.8, 0.8, 0.8), \\ (0.2, 0.1, 0.1), \\ (0.4, 0.3, 0.3) \end{array} \right\rangle \\ \left\langle \begin{array}{l} (0.8, 0.7, 0.6), \\ (0.3, 0.3, 0.2), \\ (0.3, 0.2, 0.2) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.7, 0.7, 0.6), \\ (0.2, 0.1, 0.1), \\ (0.3, 0.3, 0.1) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.8, 0.7, 0.7), \\ (0.1, 0.1, 0.1), \\ (0.2, 0.1, 0.1) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.8, 0.8, 0.7), \\ (0.2, 0.1, 0.1), \\ (0.4, 0.4, 0.3) \end{array} \right\rangle \\ \left\langle \begin{array}{l} (0.8, 0.8, 0.6), \\ (0.2, 0.2, 0.1), \\ (0.3, 0.2, 0.2) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.9, 0.8, 0.8), \\ (0.1, 0.1, 0.1), \\ (0.3, 0.2, 0.1) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.8, 0.7, 0.7), \\ (0.4, 0.4, 0.2), \\ (0.5, 0.3, 0.3) \end{array} \right\rangle & \left\langle \begin{array}{l} (0.7, 0.7, 0.7), \\ (0.2, 0.1, 0.1), \\ (0.2, 0.2, 0.1) \end{array} \right\rangle \end{bmatrix}.$$

In the MAGDM example, the proposed MAGDM model is given by the following decision process.

First, using the formulae (1)-(3) in Definition 2 for the decision matrix $M_D = (M_{Sik})_{4\times4}$, we obtain the NEE decision matrix N_D:

$$N_D = \begin{bmatrix} \langle(0.7333, 0.9981), (0.2000, 0.9206), (0.1667, 0.9602)\rangle & \langle(0.7000, 0.9938), (0.1667, 0.8650), (0.3000, 0.9656)\rangle \\ \langle(0.6667, 0.9977), (0.1667, 0.9602), (0.1667, 0.9602)\rangle & \langle(0.7000, 1.0000), (0.2667, 0.9851), (0.1667, 0.8650)\rangle \\ \langle(0.7000, 0.9938), (0.2667, 0.9851), (0.2333, 0.9821)\rangle & \langle(0.6667, 0.9977), (0.1333, 0.9464), (0.2333, 0.9141)\rangle \\ \langle(0.7333, 0.9922), (0.1667, 0.9602), (0.2333, 0.9414)\rangle & \langle(0.8333, 0.9986), (0.1000, 1.0000), (0.2000, 0.9206)\rangle \end{bmatrix}.$$

$$\begin{matrix} \langle(0.6667, 0.9977), (0.3000, 1.0000), (0.2333, 0.9821)\rangle & \langle(0.7000, 1.0000), (0.1667, 0.8650), (0.3000, 1.0000)\rangle \\ \langle(0.7667, 0.9983), (0.1000, 1.0000), (0.2333, 0.9821)\rangle & \langle(0.8000, 1.0000), (0.1333, 0.9464), (0.3333, 0.9461)\rangle \\ \langle(0.7333, 0.9981), (0.1000, 1.0000), (0.1333, 0.9464)\rangle & \langle(0.7667, 0.9983), (0.1333, 0.9464), (0.3667, 0.9922)\rangle \\ \langle(0.7333, 0.9981), (0.3333, 0.9602), (0.3667, 0.9713)\rangle & \langle(0.7000, 1.0000), (0.1333, 0.9464), (0.1667, 0.9602)\rangle \end{matrix}$$

Then using one of Eqs. (12) and (13), the aggregated NEEs N_{Ei} (i = 1, 2, …, s) are calculated corresponding to various values of ρ, and then score values of N_{Ei} (i = 1, 2, …, s) for L_i and ranking orders of the four alternatives are given by Eq. (2) and the ranking laws, which are shown in Tables 1 and 2.

Table 1. Decision results based on Eq. (12) and Eq. (2)

ρ	Score value	Ranking	The best one
1	0.7594, 0.7953, 0.7863, 0.7909	$L_2 > L_4 > L_3 > L_1$	L_2
3	0.7673, 0.8067, 0.7981, 0.8038	$L_2 > L_4 > L_3 > L_1$	L_2
5	0.7730, 0.8148, 0.8066, 0.8133	$L_2 > L_4 > L_3 > L_1$	L_2
7	0.7777, 0.8209, 0.8130, 0.8206	$L_2 > L_4 > L_3 > L_1$	L_2
9	0.7815, 0.8255, 0.8178, 0.8264	$L_4 > L_2 > L_3 > L_1$	L_4
11	0.7846, 0.8290, 0.8217, 0.8309	$L_4 > L_2 > L_3 > L_1$	L_4

Table 2. Decision results based on Eq. (13) and Eq. (2)

ρ	Score value	Ranking	The best one
1	0.7448, 0.7820, 0.7717, 0.7728	$L_2 > L_4 > L_3 > L_1$	L_2
3	0.7340, 0.7617, 0.7496, 0.7446	$L_2 > L_3 > L_4 > L_1$	L_2
5	0.7250, 0.7455, 0.7326, 0.7247	$L_2 > L_3 > L_4 > L_1$	L_2
7	0.7183, 0.7341, 0.7212, 0.7123	$L_2 > L_3 > L_1 > L_4$	L_2
9	0.7135, 0.7262, 0.7134, 0.7043	$L_2 > L_1 > L_3 > L_4$	L_2
11	0.7100, 0.7207, 0.7079, 0.6988	$L_2 > L_1 > L_3 > L_4$	L_2

According to the decision results in Tables 1 and 2, the ranking orders produced by Eq. (12) and Eq. (13) show their difference, then the best alternative L_2 is the same by taking ρ = 1, 3. Moreover, in the proposed MAGDM model, using different values of ρ and different aggregation operators can affect the ranking orders of alternatives and show its decision flexibility, then the change of the parameter ρ is sensitive to the ranking impact of alternatives. However, the best alternative of the example is L_2 or L_4 depending on a preference selection of decision makers.

5.2 Comparison with existing techniques in the setting of NMVSs

In this part, we compare our new techniques with existing techniques [14] in the setting of NMVSs.

On the one hand, the characteristic comparison between our new techniques and the existing techniques is indicated in Table 3.

Table 3. Characteristic comparison between our new techniques and the existing techniques

Method	Evaluation information	Conversion form	Decision-making model with an adjustable parameter	Using condition
Existing techniques [14]	NMVS/NMVE	CNE based on the mean and consistency degree (complement of standard deviation)	No	Normal distribution
Our new techniques	NMVS/NMVE	NEE based on the mean and Shannon entropy	Yes	No limitation

Regarding the comparative results of Table 3, our new techniques are often broader and more versatile than the existing techniques when dealing with MAGDM problems in the setting of NMVSs.

On the other hand, we can apply the existing MAGDM model using two consistency neutrosophic correlation coefficients [14] to the above example. By the existing MAGDM model using two consistency neutrosophic correlation coefficients, we give all decision results, which are shown in Table 4.

Table 4. Decision results of the existing MAGDM model using two correlation coefficients

Existing decision-making model	Ranking	The best one
Correlation coefficient 1 [14]	$L_2 > L_3 > L_4 > L_1$	L_2
Correlation coefficient 2 [14]	$L_1 > L_3 > L_4 > L_2$	L_1

Although there is the same ranking order between the existing MAGDM model using the correlation coefficient 1 [14] and our proposed MAGDM model using the NEEAAWGA operator for ρ = 3, 5, the existing MAGDM model lacks its decision flexibility. Furthermore, in the existing MAGDM model, the conversion technique based on the mean and standard deviation only is suitable for the normal distribution of multi-valued sequences in NMVEs. Therefore, our proposed model can not only overcome the limitation and insufficiency of the existing model [14], but also show its outstanding advantage of diversified decision results to satisfy the preference order of decision makers in actual applications. However, our new conversion method and decision-making model are superior to the existing ones in the setting of NMVSs.

6. Conclusions

To overcome the shortcomings of existing MAGDM method under the environment of NMVSs, this study proposed a NEE concept based on the normalized Shannon extropy and average values of the truth, falsity, and indeterminacy sequences in NMVSs to overcome the limitation of the existing conversion method based on the mean and standard deviation of the truth, falsity, and indeterminacy sequences. Then, the proposed ranking laws based on the score and accuracy functions of NEEs and the proposed Aczel-Alsina t-norm and t-conorm operations and NEEAAWAA and NEEAAWGA operators provided important mathematical tools for solving flexible decision-making issues in the case of NMVSs. The developed MAGDM model can effectively carry out flexible decision-making issues with the information of NMVSs, where various parameter values can affect ranking orders of alternatives to satisfy decision makers' preference requirements. Finally, an illustrative example was given to verify the efficiency and rationality of the developed MAGDM model. Compared with the existing techniques, our proposed techniques are broader and more versatile than the existing techniques when dealing with MAGDM problems in the case of NMVSs. However, in this study, the proposed information expression, operations, and aggregation operators of NEEs and the established MAGDM method show the highlighting advantages of these new techniques.

Regarding these new techniques, we have many future researches to be performed in various areas, such as image processing, medical diagnosis, and information fusion. Meanwhile, the proposed Aczel-Alsina t-norm and t-conorm operations and aggregation operators of NEEs are also extended to cubic neutrosophic sets, refined neutrosophic sets, consistency neutrosophic sets, neutrosophic rough sets, etc. Then, they can be applied in engineering management, slope risk/stability evaluation, as well as clustering analysis, information retrieval, data mining, and so on in the case of NMVSs.

Data Availability: All data generated or analyzed during this study are included in this article.

Conflict of Interest: The authors declare no conflict of interest.

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Neutrosophic Pseudo-t-Norm and Its Derived Neutrosophic Residual Implication

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Abstract: First of all, on the basis of complete lattice, the concept of neutrosophic pseudo-t-norm (NPT) is given. Definitions and examples of representable neutrosophic pseudo-t-norms (RNPTs) are given, while unrepresentable neutrosophic pseudo-t-norms (UNPTs) is also given. Secondly, De Morgan neutrosophic triples (DMNTs) consists of three operators: NPTs, neutrosophic negators (NNs) and neutrosophic pseudo-s-norms (NPSs), where NPTs and NPSs are dual about NNs. Again, we study the neutrosophic residual implications (NRIs) of NPTs, as well as their underlying properties. Finally, we give a method to get NPTs from neutrosophic implications (NIs) and construct non-commutative residuated lattices (NCRLs) based on NRIs and NPTs.

Keywords: Neutrosophic set; Neutrosophic pseudo-t-norm; Neutrosophic residual implication; Non-commutative residuated lattices

1. Introduction

From the perspective of philosophy, Smarandache introduced neutrosophic sets (NSs). NSs is a expansion of fuzzy sets (FSs), and has universality [1]. Although NSs has expanded the expression of uncertain information, there are many inconveniences in practical application. From a scientific standpoint, so as to solve more practical problems, single valued neutrosophic sets (SVNSs) was put forward by Wang [2]. Some multi-attribute decision problems are solved by applying SVNSs. "SVNSs" is simply denoted as "NSs" in this article.

The t-norms, s-norms, negators, pseudo-t-norms, pseudo-s-norms and implications operators are fundamental tools in FS theory. Pseudo-t-norm and pseudo-s-norm was proposed in [3], followed by their residual implication were put forward by Wang in [4]. Pseudo-t-norm has many applications, such as resolution of finite fuzzy relation equations, linear optimization problems of mixed fuzzy relation inequalities and so on [5-10].

NSs has a lot of important neutrosophic logical operators, such as: NPTs, NPSs, NNs, NIs, NRIs and so on. In past few years, Smarandache [11] introduced n-conorm and n-norm in neutrosophic logic. Zhang et al. [12] introduced a new type of relation of inclusion for NSs. A new kind of residuated lattice obtained through neutrosophic t-norms and its derived NRIs was introduced by Hu and Zhang [13]. On the basis of neutrosophic t-norms, fuzzy reasoning triple I method was studied by Luo et al. [14]. Therefore, it is very meaningful to study the NRIs of NPTs.

The basic framework of this paper: Section 2 presents the basics knowledge that will be useful for writing this paper. We defined NPTs, NPSs, NNs and so on in Section 3. Moreover, we also provide some useful typical examples and theorems. In Section 4, the definitions of NRIs generated from NPTs are obtained, and their basic properties are discussed in depth. In addition, this paper provides a new method to generate NPTs from NIs, and at the same time prove that system (D^*; $\wedge_1$, $\vee_1$, $\otimes$, $\rightarrow$, $\rightsquigarrow$, 0_{D^*}, 1_{D^*}) is a NCRL. Section 5 concludes the whole content of this paper.

2. Preliminaries

Definition 2.1 ([3]) A mapping PT: $[0, 1]^2 \rightarrow [0, 1]$ be a pseudo-t-norm iff, $\forall m, n, r \in [0, 1]$:
(PT1) $PT(m, PT(n, r)) = PT(n, PT(m, r))$;
(PT2) if $m \leq n$, then $PT(m, r) \leq PT(n, r)$, $PT(r, m) \leq PT(r, n)$;
(PT3) $PT(1, m) = m$, $PT(m, 1) = m$.

Definition 2.2 ([3]) A mapping PS: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ be a pseudo-s-norm iff, $\forall m, n, r \in [0, 1]$:
(PS1) $PS(m, PS(n, r)) = PS(n, PS(m, r))$;
(PS2) if $m \leq n$, then $PS(m, r) \leq PS(n, r)$, $PS(r, m) \leq PS(r, n)$;
(PS3) $PS(0, m) = m$, $PS(m, 0) = m$.

Definition 2.3 ([15]) An intuitionistic fuzzy set (IFS) W in nonempty set M is depicted through two mappings: $\mu_W(m)$ and $\nu_W(m)$: $M \rightarrow [0, 1]$. W is expressed as, when $\forall m \in M$,

$$W = \{(m, \mu_W(m), \nu_W(m)) \mid m \in M\},$$

satisfy $0 \leq \mu_W(m) + \nu_W(m) \leq 1$, where $\mu_W(m)$ is affiliation function, $\nu_W(m)$ is non-affiliation function.

Definition 2.4 ([2]) Let the set M be nonempty. A SVNS W in M is depicted through $T_W(m)$, $I_W(m)$, and $F_W(m)$, all of which are functions defined on [0, 1]. Then, W is expressed as, when $\forall m \in M$,

$$W = \{\langle m, T_W(m), I_W(m), F_W(m) \rangle \mid m \in M\},$$

satisfy $0 \leq T_W(m) + I_W(m) + F_W(m) \leq 3$, where $T_W(m)$ is the function of truth-affiliation, $I_W(m)$ is the function of indeterminacy- affiliation, and $F_W(m)$ is the function of falsity- affiliation.

Proposition 2.5 ([13]) The first type of inclusion relationship is discussed in this article.

Definition 2.6 ([1,17,18]) Let the set M be nonempty. Give two NSs W, N in M, where $W = \{\langle m, T_W(m), I_W(m), F_W(m) \rangle \mid m \in M\}$, $N = \{\langle m, T_N(m), I_N(m), F_N(m) \rangle \mid m \in M\}$. The algebraic operations of the first type of inclusion relation was given as shown below, $\forall m \in M$,
(1) $W \subseteq_1 N \Leftrightarrow T_W(m) \leq T_N(m)$, $I_W(m) \geq I_N(m)$, $F_W(m) \geq F_N(m)$;
(2) $W \cup_1 N = \{\langle m, \max(T_W(m), T_N(m)), \min(I_W(m), I_N(m)), \min(F_W(m), F_N(m)) \rangle \mid m \in M\}$;
(3) $W \cap_1 N = \{\langle m, \min(T_W(m), T_N(m)), \max(I_W(m), I_N(m)), \max(F_W(m), F_N(m)) \rangle \mid m \in M\}$;
(4) $W^c = \{\langle m, I_W(m), 1 - F_W(m), T_W(m) \rangle \mid m \in M\}$.

Proposition 2.7 ([13]) We consider that set D^* defined by,

$$D^* = \{m = (m_1, m_2, m_3) \mid m_1, m_2, m_3 \in [0,1]\}.$$

$\forall m, n \in D^*$, the order relation we define $\leq_1$ on D^* is shown below:

$$m \leq_1 n \Leftrightarrow m_1 \leq n_1,\ m_2 \geq n_2,\ m_3 \geq n_3.$$

Proposition 2.8 ([13]) (D^*; $\leq_1$) is a partially ordered set.

Proposition 2.9 ([13]) $\forall m, n\in D^*$, $m \wedge_1 n$ is called maximum lower bound of m, n, and expressed as $\inf(m, n)$; $m \vee_1 n$ is called minimum upper bound of m, n, and expressed as $\sup(m, n)$. In other word, $(D^*; \leq_1)$ is a lattice.

The content of definition of operators $\wedge_1$ and $\vee_1$ refers to proposition 2 in [13].

Proposition 2.10 ([13]) $(D^*; \leq_1)$ is a complete lattice.

The maximun of D^* is indicated as $1_{D^*} = (1,0,0)$, the minimun of D^* is indicated as $0_{D^*} = (0,1,1)$.

Definition 2.11 ([16]) A pseudo-t-norm PT: $L\times L\to L$ on $(L; \leq_L)$ be undecreasing and associative mapping that meets $PT(1_L, m) = m = PT(m, 1_L)$, which $\forall m\in L$. A pseudo-s-norm PS: $L^2\to L$ on $(L; \leq_L)$ be associative and undecreasing mapping that meets $PS(0_L, m) = m = PS(m, 0_L)$, $\forall m \in L$.

Definition 2.12 ([13]) For every $m\in D^*$, we define a complement of m as follows:

$$m^c = (m_3, 1\text{-}m_2, m_1).$$

Proposition 2.13 ([13]) The system $(D^*; \wedge_1, \vee_1, {}^c, 0_{D^*}, 1_{D^*})$ is a De Morgan algebra.

3. NPTs On $(D^*; \leq_1)$

Definition 3.1 A binary function $\boldsymbol{PT}$: $D^*\times D^*\to D^*$ is called NPT, $\forall m, n, u, v, r\in D^*$, if $\boldsymbol{PT}$ satisfies:
(NPT1) $\boldsymbol{PT}(m, \boldsymbol{PT}(n, r)) = \boldsymbol{PT}(n, \boldsymbol{PT}(m, r))$;
(NPT2) $\boldsymbol{PT}(m, n) \leq_1 \boldsymbol{PT}(u, v)$ and $\boldsymbol{PT}(n, m) \leq_1 \boldsymbol{PT}(v, u)$, where $m \leq_1 u$, $n \leq_1 v$;
(NPT3) $\boldsymbol{PT}(1_{D^*}, m) = m$, $\boldsymbol{PT}(m, 1_{D^*}) = m$.

Definition 3.2 A binary function $\boldsymbol{PS}$: $D^*\times D^*\to D^*$ is called NPS, $\forall m, n, u, v, r\in D^*$, if $\boldsymbol{PS}$ satisfies:
(NPS1) $\boldsymbol{PS}(m, \boldsymbol{PS}(n, r)) = \boldsymbol{PS}(n, \boldsymbol{PS}(m, r))$;
(NPS2) $\boldsymbol{PS}(m, n) \leq_1 \boldsymbol{PS}(u, v)$ and $\boldsymbol{PS}(n, m) \leq_1 \boldsymbol{PS}(v, u)$, where $m \leq_1 u$, $n \leq_1 v$;
(NPS3) $\boldsymbol{PS}(0_{D^*}, m) = m$, $\boldsymbol{PS}(m, 0_{D^*}) = m$.

Example 3.3 ([3,19]) Table 1 below gives part pseudo-t-norms, and its derived residual implications.

Table 1. Eaxmple of part pseudo-t-norms

Pseudo-t-norms	Residual implications
$PT_1(m,n)=\begin{cases}0 & \text{if } m\in[0,a_1], n\in[0,b_1],\\ \min(m,n) & \text{otherwise,}\end{cases}$ where $0<a_1<b_1<1$.	$I_{1L}(m,n)=\begin{cases}\max(a_1,n) & \text{if } m\leq b_1, m>n,\\ n & \text{if } m>b_1, m>n,\\ 1 & \text{if } m\leq n.\end{cases}$ $I_{1R}(m,n)=\begin{cases}b_1 & \text{if } m\leq a_1, m>n,\\ n & \text{if } m>a_1, m>n,\\ 1 & \text{if } m\leq n.\end{cases}$
$PT_2(m,n)=\begin{cases}\min(m,n) & \text{if } \sin(\frac{\pi}{2}m)+n>1,\\ 0 & \text{if } \sin(\frac{\pi}{2}m)+n\leq 1.\end{cases}$	$I_{2L}(m,n)=\begin{cases}1 & \text{if } m\leq n,\\ \max\{n,\frac{2}{\pi}\arcsin(1-m)\} & \text{if } m>n.\end{cases}$ $I_{2R}(m,n)=\begin{cases}1 & \text{if } m\leq n,\\ \max\{n,1-\sin(\frac{\pi}{2}m)\} & \text{if } m>n.\end{cases}$

$PT_3(m,n)=\begin{cases}\min(m,n) & \text{if } m^2+n^3>1,\\ 0 & \text{if } m^2+n^3\le 1.\end{cases}$	$I_{3L}(m,n)=\begin{cases}1 & \text{if } m\le n,\\ \max\{n,\sqrt{1-m^3}\} & \text{if } m>n.\end{cases}$ $I_{3R}(m,n)=\begin{cases}1 & \text{if } m\le n,\\ \max\{n,\sqrt[3]{1-m^2}\} & \text{if } m>n.\end{cases}$
$PT_4(m,n)=\begin{cases}\min(m,n) & \text{if } m+\sqrt{n}>1,\\ 0 & \text{if } m+\sqrt{n}\le 1.\end{cases}$	$I_{4L}(m,n)=\begin{cases}1 & \text{if } m\le n,\\ \max\{n,1-\sqrt{m}\} & \text{if } m>n.\end{cases}$ $I_{4R}(m,n)=\begin{cases}1 & \text{if } m\le n,\\ \max\{n,(1-m)^2\} & \text{if } m>n.\end{cases}$

Example 3.4 Table 2 below gives part pseudo-s-norms, and its derived residual co-implications.

Table 2. Example of part pseudo-s-norms

Pseudo-s-norms	**Residual co-implications**
$PS_1(m,n)=\begin{cases}1 & \text{if } m\ge a_1, n\ge b_1,\\ \max(m,n) & \text{otherwise,}\end{cases}$ where $0<a_1<b_1<1$.	$J_{1L}(m,n)=\begin{cases}a_1 & \text{if } m>b_1, m<n,\\ n & \text{if } m\le b_1, m<n,\\ 0 & \text{if } m\ge n.\end{cases}$ $J_{1R}(m,n)=\begin{cases}\min(a_1,n) & \text{if } m>a_1, m<n,\\ n & \text{if } m\le a_1, m<n,\\ 0 & \text{if } m\ge n.\end{cases}$
$PS_2(m,n)=\begin{cases}\max(m,n) & \text{if } \sin(\frac{\pi}{2}m)+n<1,\\ 1 & \text{if } \sin(\frac{\pi}{2}m)+n\ge 1.\end{cases}$	$J_{2L}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,\frac{2}{\pi}\arcsin(1-m)\} & \text{if } m<n.\end{cases}$ $J_{2R}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,1-\sin(\frac{\pi}{2}m)\} & \text{if } m<n.\end{cases}$
$PS_3(m,n)=\begin{cases}\max(m,n) & \text{if } m^2+n^3<1,\\ 1 & \text{if } m^2+n^3\ge 1.\end{cases}$	$J_{3L}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,\sqrt{1-m^3}\} & \text{if } m<n.\end{cases}$ $J_{3R}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,\sqrt[3]{1-m^2}\} & \text{if } m<n.\end{cases}$
$PS_4(m,n)=\begin{cases}\max(m,n) & \text{if } m+\sqrt{n}<1,\\ 1 & \text{if } m+\sqrt{n}\ge 1.\end{cases}$	$J_{4L}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,1-\sqrt{m}\} & \text{if } m<n.\end{cases}$ $J_{4R}(m,n)=\begin{cases}0 & \text{if } m\ge n,\\ \min\{n,(1-m)^2\} & \text{if } m<n.\end{cases}$

Example 3.5 Suppose that PT_i (i=1,2,3,4) are pseudo-t-norms as shown in **Example 3.3** and PS_i (i=1,2,3,4) are pseudo-s-norms as shown in **Example 3.4**. Then, the binary function $\boldsymbol{PT}_i$ (i=1,2,3,4,5,6) defined on D^* are NPTs as follows:

(1) $\boldsymbol{PT}_1(m,n)=(PT_1(m_1,n_1),PS_1(m_2,n_2),PS_1(m_3,n_3))$;

(2) $\boldsymbol{PT}_2(m,n) = (PT_2(m_1,n_1), PS_2(m_2,n_2), PS_2(m_3,n_3))$;

(3) $\boldsymbol{PT}_3(m,n) = (PT_3(m_1,n_1), PS_3(m_2,n_2), PS_3(m_3,n_3))$;

(4) $\boldsymbol{PT}_4(m,n) = (PT_4(m_1,n_1), PS_4(m_2,n_2), PS_4(m_3,n_3))$;

(5) $\boldsymbol{PT}_5(m,n) = (PT_1(m_1,n_1), PS_2(m_2,n_2), PS_3(m_3,n_3))$;

(6) $\boldsymbol{PT}_6(m,n) = (PT_1(m_1,n_1), PS_3(m_2,n_2), PS_3(m_3,n_3))$.

Example 3.6 Suppose that PT_i (i=1,2,3,4) are pseudo-t-norms as shown in **Example 3.3** and PS_i (i=1,2,3,4) are pseudo-s-norms as shown in **Example 3.4**. Then, the binary function $\boldsymbol{PS}_i$ (i=1,2,3,4,5,6) defined on D^* are NPSs as follows:

(1) $\boldsymbol{PS}_1(m,n) = (PS_1(m_1,n_1), PT_1(m_2,n_2), PT_1(m_3,n_3))$;

(2) $\boldsymbol{PS}_2(m,n) = (PS_2(m_1,n_1), PT_2(m_2,n_2), PT_2(m_3,n_3))$;

(3) $\boldsymbol{PS}_3(m,n) = (PS_3(m_1,n_1), PT_3(m_2,n_2), PT_3(m_3,n_3))$;

(4) $\boldsymbol{PS}_4(m,n) = (PS_4(m_1,n_1), PT_4(m_2,n_2), PT_4(m_3,n_3))$;

(5) $\boldsymbol{PS}_5(m,n) = (PS_1(m_1,n_1), PT_2(m_2,n_2), PT_3(m_3,n_3))$;

(6) $\boldsymbol{PS}_6(m,n) = (PS_1(m_1,n_1), PT_3(m_2,n_2), PT_3(m_3,n_3))$.

Theorem 3.7 Give a binary function $\boldsymbol{PT}$: $D^*\times D^*\to D^*$, two pseudo-s-norms PS_i (i=1,2) and a pseudo-t-norm PT. Then, $\forall m, n\in D^*$,

$$\boldsymbol{PT}(m,n) = (PT(m_1,n_1), PS_1(m_2,n_2), PS_2(m_3,n_3))$$

is a NPT.

Proof. $\forall m, u, n, v, r\in D^*$, have following:

(NPT1) According to item (PT1) of **Definition 2.1** and item (PS1) of **Definition 2.2**, it is obvious that $\boldsymbol{PT}(m, \boldsymbol{PT}(n, r)) = \boldsymbol{PT}(m, (PT(n_1, r_1), PS_1(n_2, r_2), PS_2(n_3, r_3))) = (PT(m_1, PT(n_1, r_1)), PS_1(m_2, PS_1(n_2, r_2)), PS_2(m_3, PS_2(n_3, r_3))) = (PT(n_1, PT(m_1, r_1)), PS_1(n_2, PS_1(m_2, r_2)), PS_2(n_3, PS_2(m_3, r_3))) = \boldsymbol{PT}(n, \boldsymbol{PT}(m, r))$;

(NPT2) If $m \le_1 u$, $n \le_1 v$, then $PT(m_1, n_1) \le PT(u_1, v_1)$, $PS_1(m_2, n_2) \ge PS_1(u_2, v_2)$, $PS_2(m_3, n_3) \ge PS_2(u_3, v_3)$. Therefore, $\boldsymbol{PT}(m, n) \le_1 \boldsymbol{PT}(u, v)$. Likewise, we can also get $\boldsymbol{PT}(n, m) \le_1 \boldsymbol{PT}(v, u)$.

(NPT3) $\boldsymbol{PT}(m, 1_{D^*}) = (PT(m_1, 1), PS_1(m_2, 0), PS_2(m_3, 0)) = (m_1, m_2, m_3) = m$. Similarly, $\boldsymbol{PT}(1_{D^*}, m) = m$.

Thus, $\boldsymbol{PT}(m, n)$ is a NPT.

Theorem 3.8 Give a binary function $\boldsymbol{PS}$: $D^*\times D^*\to D^*$, two pseudo-t-norms PT_i (i=1,2) and a pseudo-s-norm PS. Then,

$$\boldsymbol{PS}(m,n) = (PS(m_1,n_1), PT_1(m_2,n_2), PT_2(m_3,n_3))$$

is a NPS, for arbitrary $m, n\in D^*$.

Theorem 3.7 provieds a idea for constructing NPT on D^* with pseudo-s-norm and pseudo-t-norm. However, the reverse is not able to find two pseudo-s-norms PS_i (i=1,2) and a pseudo-t-norm PT to make $\boldsymbol{PT} = (PT, PS_1, PS_2)$.

In order to make a clear distinction between the two types of NPTs, so put forward a concept of RNPT.

Definition 3.9 If $\forall m, n\in D^*$, there exists two pseudo-s-norms PS_i (i=1,2) and a pseudo-t-norm PT such that $\boldsymbol{PT}$ holds with respect to the following equation:

$$\boldsymbol{PT}(m,n) = (PT(m_1,n_1), PS_1(m_2,n_2), PS_2(m_3,n_3)).$$

Then $\boldsymbol{PT}$ is said to be representable.

Definition 3.10 If $\forall m, n \in D^*$, there exists pseudo-s-norm PS and pseudo-t-norm PT such that $\boldsymbol{PT}$ holds with respect to the following equation:

$$\boldsymbol{PT}(m,n) = (PT(m_1,n_1), PS(m_2,n_2), PS(m_3,n_3)).$$

Then $\boldsymbol{PT}$ is said to be standard representable.

These NPTs given in **Example 3.5** are representable.

Definition 3.11 If $\forall m, n \in D^*$, there exists two pseudo-t-norms PT_i (i=1,2) and a pseudo-s-norm PS such that $\boldsymbol{PS}$ holds with respect to the following equation:

$$\boldsymbol{PS}(m,n) = (PS(m_1,n_1), PT_1(m_2,n_2), PT_2(m_3,n_3)).$$

Then $\boldsymbol{PS}$ is said to be representable.

Definition 3.12 If $\forall m, n \in D^*$, there exists pseudo-s-norm PS and pseudo-t-norm PT such that $\boldsymbol{PS}$ holds with respect to the following equation:

$$\boldsymbol{PS}(m,n) = (PS(m_1,n_1), PT(m_2,n_2), PT(m_3,n_3)).$$

Then $\boldsymbol{PS}$ is said to be standard representable.

These NPSs given in **Example 3.6** are representable.

Propositions 3.13 and **3.14** below demonstrate a approach to construct new RNPTs (RNPSs) with intuitionistic fuzzy t-norms (IFTs) and intuitionistic fuzzy s-norms (IFSs).

Proposition 3.13 $\forall x = (x_1, x_3) \in L$, $y = (y_1, y_3) \in L$, $T(x, y) = (t(x_1, y_1), s_2(x_3, y_3))$ is a representable IFT, which t and s_2 are t-norm and s-norm, respectively. If $\forall m, n \in D^*$, there is a pseudo-s-norm ps_1 that makes $0 \le t(m_1, n_1)+ps_1(m_2, n_2)+s_2(m_3, n_3) \le 3$ true, then $\boldsymbol{PT}(m, n) = (t(m_1, n_1), ps_1(m_2, n_2), s_2(m_3, n_3))$ is a RNPT.

Proposition 3.14 $\forall x = (x_1, x_3) \in L$, $y = (y_1, y_3) \in L$, $S(x, y) = (s(x_1, y_1), t_2(x_3, y_3))$ is a representable IFS, where s and t_2 are s-norm and t-norm, respectively. If $\forall m, n \in D^*$, there is a pseudo-t-norm pt_1 that makes $0 \le s(m_1, n_1)+pt_1(m_2, n_2)+t_2(m_3, n_3) \le 3$ true, then $\boldsymbol{PS}(m, n) = (s(m_1, n_1), pt_1(m_2, n_2), t_2(m_3, n_3))$ is a RNPS.

Example 3.15 ([20]) Table 3 below gives part t-norms, and its derived residual implications.

Table 3. Example of the part t-norms

t-norms	**Residual implications**
$T_M(m,n) = \min(m,n)$	$I_{GD}(m,n) = \begin{cases} 1 & \text{if } m \le n, \\ n & \text{if } m > n. \end{cases}$
$T_P(m,n) = m \cdot n$	$I_{GG}(m,n) = \begin{cases} 1 & \text{if } m \le n, \\ \dfrac{n}{m} & \text{if } m > n. \end{cases}$
$T_{LK}(m,n) = \max(m+n-1, 0)$	$I_{LK}(m,n) = \min(1, 1-m+n)$

Example 3.16 ([20]) Table 4 below gives part s-norms, and its derived residual co-implications.

Table 4. Example of the part s-norms

s-norms	**Residual co-implications**

$S_M(m,n)=\max(m,n)$	$J_{GD}(m,n)=\begin{cases}0 & \text{if } m\geq n,\\ n & \text{if } m<n.\end{cases}$
$S_P(m,n)=m+n-m\cdot n$	$J_{GG}(m,n)=\begin{cases}0 & \text{if } m\geq n,\\ \dfrac{n-m}{1-m} & \text{if } m<n.\end{cases}$
$S_{LK}(m,n)=\min(m+n,1)$	$J_{LK}(m,n)=\max(0,n-m)$

Example 3.17 Let PS_i (i=1,2,4) are pseudo-s-norms as shown in **Example 3.4**, T_M, T_P, T_{LK} are t-norms as shown in **Example 3.15**, and S_M, S_P, S_{LK} are s-norms as shown in **Example 3.16**. Then, the binary function $\boldsymbol{PT}_i$ (i=7,8,9) constructed by IFTs defined on D^* are RNPTs as follows:

(1) $\boldsymbol{PT}_7(m,n) = (T_M(m_1,n_1), PS_4(m_2,n_2), S_{LK}(m_3,n_3))$;

(2) $\boldsymbol{PT}_8(m,n) = (T_P(m_1,n_1), PS_2(m_2,n_2), S_M(m_3,n_3))$;

(3) $\boldsymbol{PT}_9(m,n) = (T_{LK}(m_1,n_1), PS_1(m_2,n_2), S_P(m_3,n_3))$.

Example 3.18 Let PT_i (i=1,2,4) are pseudo-t-norms as shown in **Example 3.3**, T_M, T_P, T_{LK} are t-norms as shown in **Example 3.15**, and S_M, S_P, S_{LK} are s-norms as shown in **Example 3.16**. Then, the binary function $\boldsymbol{PS}_i$ (i=7,8,9) constructed by IFSs defined on D^* are RNPSs as follows:

(1) $\boldsymbol{PS}_7(m,n) = (S_M(m_1,n_1), PT_4(m_2,n_2), T_{LK}(m_3,n_3))$;

(2) $\boldsymbol{PS}_8(m,n) = (S_P(m_1,n_1), PT_2(m_2,n_2), T_M(m_3,n_3))$;

(3) $\boldsymbol{PS}_9(m,n) = (S_{LK}(m_1,n_1), PT_1(m_2,n_2), T_P(m_3,n_3))$.

Definition 3.19 ([13]) A mapping $\boldsymbol{N}$: $D^*\to D^*$ be known as NN if satisfies, $\forall m, n\in D^*$:

(NN1) $m \leq_1 n$ iff $\boldsymbol{N}(m)\geq_1 \boldsymbol{N}(n)$;

(NN2) $\boldsymbol{N}(1_{D^*}) = 0_{D^*}$;

(NN3) $\boldsymbol{N}(0_{D^*}) = 1_{D^*}$.

If $\boldsymbol{N}(\boldsymbol{N}(m)) = m$ holds with $\forall m\in D^*$, then $\boldsymbol{N}$ is said to be involutive NN.

The function $\boldsymbol{N}_S$: $D^*\to D^*$ defined by, $\forall(m_1, m_2, m_3)\in D^*$,

$$\boldsymbol{N}_S(m_1, m_2, m_3) = (m_3, 1-m_2, m_1)$$

is a involutive NN, which is also called standard NN. Meanwhile, $\boldsymbol{N}(m) = (m_2, 1-m_3, m_1)$, $\boldsymbol{N}(m) = (m_2, m_1, m_1)$, $\boldsymbol{N}(m) = (m_2, 1-m_2, m_1)$ are NNs.

Definition 3.20 Assume that $\boldsymbol{PT}$ is a NPT, $\boldsymbol{N}$ is a NN and $\boldsymbol{PS}$ is a NPS. $\forall m, n\in D^*$, if the triple ($\boldsymbol{PT}$, $\boldsymbol{N}$, $\boldsymbol{PS}$) satisfied the following conditions:

$$\boldsymbol{N}(\boldsymbol{PS}(m, n)) = \boldsymbol{PT}(\boldsymbol{N}(m), \boldsymbol{N}(n)).$$
$$\boldsymbol{N}(\boldsymbol{PT}(m, n)) = \boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(n));$$

Then, we call the triple ($\boldsymbol{PT}$, $\boldsymbol{N}$, $\boldsymbol{PS}$) is a DMNT.

Theorem 3.21 Suppose $\boldsymbol{N}$ is involutive. If exists a NPS $\boldsymbol{PS}$, then such that $\boldsymbol{PT}$ be defined as

$$\boldsymbol{PT}(m, n) = \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(n)))$$

is NPT. Besides, ($\boldsymbol{PT}$, $\boldsymbol{N}$, $\boldsymbol{PS}$) is a DMNT.

Proof. According to known condition, there are as follows, $\forall m, u, n, v, r\in D^*$:

(NPT1)According to item (NPS1) of **Definition 3.2**, naturally there is $\boldsymbol{PT}(m, \boldsymbol{PT}(n, r)) = \boldsymbol{PT}(m, \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{N}(r)))) = \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(\boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{N}(r)))))) = \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{N}(r)))) = \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(r)))) = \boldsymbol{PT}(n, \boldsymbol{PT}(m, r))$.

(NPT2) If $m \leq_1 u$, $n \leq_1 v$, so $\boldsymbol{N}(m) \geq_1 \boldsymbol{N}(u)$, $\boldsymbol{N}(n) \geq_1 \boldsymbol{N}(v)$. From (NPS2) of **Definition 3.2** and (NN1) of **Definition 3.19**, we get $\boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(n)) \geq_1 \boldsymbol{PS}(\boldsymbol{N}(u), \boldsymbol{N}(v))$ and $\boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{N}(m)) \geq_1 \boldsymbol{PS}(\boldsymbol{N}(v), \boldsymbol{N}(u))$. Thus, $\boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(m), \boldsymbol{N}(n))) \leq_1 \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(u), \boldsymbol{N}(v)))$ and $\boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(n), \boldsymbol{N}(m))) \leq_1 \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(v), \boldsymbol{N}(u)))$, that is, $\boldsymbol{PT}(m, n) \leq_1 \boldsymbol{PT}(u, v)$ and $\boldsymbol{PT}(n, m) \leq_1 \boldsymbol{PT}(v, u)$.

(NPT3) $\boldsymbol{PT}(1_{D^*}, m) = \boldsymbol{N}(\boldsymbol{PS}(\boldsymbol{N}(1_{D^*}), \boldsymbol{N}(m))) = \boldsymbol{N}(\boldsymbol{PS}(0_{D^*}, \boldsymbol{N}(m))) = \boldsymbol{N}(\boldsymbol{N}(m)) = m$. Similarly, $\boldsymbol{PT}(m, 1_{D^*}) = m$.

Therefore, the statement that $\boldsymbol{PT}$ is NPT is proved.

Besides, ($\boldsymbol{PT}$, $\boldsymbol{N}$, $\boldsymbol{PS}$) is a DMNT.

Theorem 3.22 Assume $\boldsymbol{N}$ is involutive. If exists a NPT $\boldsymbol{PT}$, then such that $\boldsymbol{PS}$ be defined as

$$\boldsymbol{PS}(m, n) = \boldsymbol{N}(\boldsymbol{PT}(\boldsymbol{N}(m), \boldsymbol{N}(n)))$$

being NPS. Moreover, ($\boldsymbol{PT}$, $\boldsymbol{N}$, $\boldsymbol{PS}$) is a DMNT.

Example 3.23 A few NPTs and NPSs are dual about $\boldsymbol{N}_S$.

(1) $\boldsymbol{PT}_1(m, n) = (PT_1(m_1, n_1), PS_1(m_2, n_2), PS_1(m_3, n_3))$, $\boldsymbol{PS}_1(m, n) = (PS_1(m_1, n_1), PT_1(m_2, n_2), PT_1(m_3, n_3))$.

Indeed, $\boldsymbol{PT}_1(\boldsymbol{N}_S(m), \boldsymbol{N}_S(n)) = \boldsymbol{PT}_1((m_3, 1\text{-}m_2, m_1), (n_3, 1\text{-}n_2, n_1)) = (PT_1(m_3, n_3), PS_1(1\text{-}m_2, 1\text{-}n_2), PS_1(m_1, n_1))$, then $\boldsymbol{N}_S(\boldsymbol{PT}_1(\boldsymbol{N}_S(m), \boldsymbol{N}_S(n))) = (PS_1(m_1, n_1), 1\text{-}PS_1(1\text{-}m_2, 1\text{-}n_2), PT_1(m_3, n_3)) = (PS_1(m_1, n_1), PT_1(m_2, n_2), PT_1(m_3, n_3)) = \boldsymbol{PS}_1(m, n)$.

(2) $\boldsymbol{PT}_3(m, n) = (PT_3(m_1, n_1), PS_3(m_2, n_2), PS_3(m_3, n_3))$, $\boldsymbol{PS}_3(m, n) = (PS_3(m_1, n_1), PT_3(m_2, n_2), PT_3(m_3, n_3))$.

The theorem about UNPT is given next:

Theorem 3.24 Let $\boldsymbol{PT}$: $D^*\times D^*\rightarrow D^*$ being a function. $\forall m, n\in D^*$,

$$\boldsymbol{PT}(m,n)=\begin{cases} m & n=1_{D^*}, \\ n & m=1_{D^*}, \\ (\min(2m_1,n_1),\max(1-2m_1,1-n_1),\max(m_3,n_3)) & \text{otherwise.} \end{cases}$$

is a UNPT.

Proof. First, we show that $\boldsymbol{PT}$ is a NPT, $\forall m, u, n, v, r\in D^*$.

(NPT1) If $m = 1_{D^*}$ or $n = 1_{D^*}$, then $\boldsymbol{PT}$ satisfies the associative law. If $m \neq 1_{D^*}$, $n \neq 1_{D^*}$, $\boldsymbol{PT}(m, \boldsymbol{PT}(n, r)) = (\min(2m_1, \min(2n_1, r_1)), \max(1\text{-}2m_1, 1\text{-}\min(2n_1, r_1)), \max(m_3, \max(n_3, r_3))) = (\min(2m_1, 2n_1, r_1), \max(1\text{-}2m_1, 1\text{-}2n_1, 1\text{-}r_1,), \max(m_3, n_3, r_3)) = (\min(2n_1, \min(2m_1, r_1)), \max(1\text{-}2n_1, 1\text{-}\min(2m_1, r_1)), \max(n_3, \max(m_3, r_3))) = \boldsymbol{PT}(n, \boldsymbol{PT}(m, r))$.

(NPT2) If $m = 1_{D^*}$ or $n = 1_{D^*}$, we can prove $\boldsymbol{PT}$ is undecreasing in each variable. If $m \neq 1_{D^*}$, $n \neq 1_{D^*}$, at the same time satisfy $m \leq_1 u$, $n \leq_1 v$, and $m_1 \leq u_1$, $n_1 \leq v_1$, $m_3 \geq u_3$, $n_3 \geq v_3$. Thus, $\min(2m_1, n_1) \leq \min(2u_1, v_1)$, $\max(1\text{-}2m_1, 1\text{-}n_1) \geq \max(1\text{-}2u_1, 1\text{-}v_1)$, $\max(m_3, n_3) \geq \max(u_3, v_3)$. That is, $\boldsymbol{PT}(m, n) \leq_1 \boldsymbol{PT}(u, v)$. Likewise, we can also have $\boldsymbol{PT}(n, m) \leq_1 \boldsymbol{PT}(v, u)$.

(NPT3) $\boldsymbol{PT}(m, 1_{D^*}) = m$, $\boldsymbol{PT}(1_{D^*}, m) = m$. Therefore, $\boldsymbol{PT}$ is a NPT.

Second, assume NPT $\boldsymbol{PT}$ is representable, $m = (m_1, m_2, m_3)\in D^*$, $n = (n_1, n_2, n_3)\in D^*$, there are a pseudo-t-norm PT and two pseudo-s-norms PS_i (i=1,2) such that $\boldsymbol{PT}(m, n) = (PT(m_1, n_1), PS_1(m_2, n_2), PS_2(m_3, n_3))$. Let $m = (0.2, 0.5, 0.4)$, $u = (0.4, 0.3, 0.2)$, $n = (0.5, 0.7, 0.6)$. From $\boldsymbol{PT}(m, n) = (0.4, 0.6, 0.6)$ and $\boldsymbol{PT}(u, n) = (0.5, 0.5, 0.6)$, we get $PS_1(m_2, n_2) = 0.6$ and $PS_1(u_2, n_2) = 0.5$, so $PS_1(m_2, n_2) \neq PS_1(u_2, n_2)$. Thus $PS_1(m, n)$ is not independent from m_1, that is to say $\boldsymbol{PT}$ is UNPT.

Moreover, $\forall m, n\in D^*$, the dual of NPT $\boldsymbol{PT}$ about standard NN N_S is NPS $\boldsymbol{PS}$, which be defined as:

$$\boldsymbol{PS}(m,n)=\begin{cases} m & n=0_{D^*}, \\ n & m=0_{D^*}, \\ (\max(m_1,n_1),\min(2m_3,n_3),\min(2m_3,n_3)) & \text{otherwise.} \end{cases}$$

Then, ***PS*** is unrepresentable.

Remark 3.25 On the one hand, suppose ***PT*** and $\boldsymbol{N}_S$ are UNPT and standard NN on *D**, respectively. The dual of ***PT*** about $\boldsymbol{N}_S$ is ***PS***. Then, we have that ***PS*** is UNPS. On the other hand, let ***N*** be involutive NN, the dual NPT about ***N*** of UNPS is unrepresentable.

4. NRI Induced by NPT on *D**

Definition 4.1 ([13]) A NI is a mapping $\boldsymbol{I}$: $D^*\times D^*\to D^*$, $\forall m, u, n, v\in D^*$, if it satisfies:
(NI1) $m \leq_1 u \Rightarrow \boldsymbol{I}(m, n) \geq_1 \boldsymbol{I}(u, n)$;
(NI2) $n \leq_1 v \Rightarrow \boldsymbol{I}(m, n) \leq_1 \boldsymbol{I}(m, v)$;
(NI3) $\boldsymbol{I}(1_{D^*}, 1_{D^*}) = \boldsymbol{I}(0_{D^*}, 0_{D^*}) = 1_{D^*}$;
(NI4) $\boldsymbol{I}(1_{D^*}, 0_{D^*}) = 0_{D^*}$.

Since NPT without commutativity, we can define left and right NRIs which satisfy the residual property induced by NPT.

Definition 4.2 Let ***PT*** be a NPT. Define two functions $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$: $D^*\times D^*\to D^*$,

$$\boldsymbol{I}^{(L)}(m, n) = \sup\{k \mid k\in D^*, \boldsymbol{PT}(k, m) \leq_1 n\};$$
$$\boldsymbol{I}^{(R)}(m, n) = \sup\{k \mid k\in D^*, \boldsymbol{PT}(m, k) \leq_1 n\}.$$

Then, $\boldsymbol{I}^{(L)}$ ($\boldsymbol{I}^{(R)}$) is called left NRI (right NRI) induced by ***PT***.

We note that the two NRIs induced by ***PT*** as $\boldsymbol{I}_{PT^{(L)}}$, $\boldsymbol{I}_{PT^{(R)}}$.

Besides, Let ***PT*** be a NPT, then $\forall m, n, k\in D^*$, ***PT*** satisfies the residual criteria iff,

$$\boldsymbol{PT}(k, m) \leq_1 n \text{ iff } k \leq_1 \boldsymbol{I}_{PT^{(L)}}(m, n);$$
$$\boldsymbol{PT}(m, k) \leq_1 n \text{ iff } k \leq_1 \boldsymbol{I}_{PT^{(R)}}(m, n).$$

Likewise, the concept of neutrosophic co-implications (NCIs) and related knowledge are also given as follows:

Definition 4.3 ([13]) A NCI is a binary function $\boldsymbol{J}$: $(D^*)^2\to D^*$, $\forall m, u, n, v\in D^*$, if it satisfies:
(NJ1) $m \leq_1 u \Rightarrow \boldsymbol{J}(m, n) \geq_1 \boldsymbol{J}(u, n)$;
(NJ2) $n \leq_1 v \Rightarrow \boldsymbol{J}(m, n) \leq_1 \boldsymbol{J}(m, v)$;
(NJ3) $\boldsymbol{J}(0_{D^*}, 0_{D^*}) = \boldsymbol{J}(1_{D^*}, 1_{D^*}) =0_{D^*}$;
(NJ4) $\boldsymbol{J}(0_{D^*}, 1_{D^*}) = 1_{D^*}$.

Analogously, we can also define left and right neutrosophic residual co-implications (NRCIs) which satisfie the residual property induced by NPS.

Definition 4.4 Suppose that ***PS*** is a NPS. Define two functions $\boldsymbol{J}^{(L)}$, $\boldsymbol{J}^{(R)}$: $D^*\times D^*\to D^*$,

$$\boldsymbol{J}^{(L)}(m, n) = \inf\{k \mid k\in D^*, \boldsymbol{PS}(k, m) \geq_1 n\};$$
$$\boldsymbol{J}^{(R)}(m, n) = \inf\{k \mid k\in D^*, \boldsymbol{PS}(m, k) \geq_1 n\}.$$

Then, $\boldsymbol{J}^{(L)}$ ($\boldsymbol{J}^{(R)}$) is called left NRCI (right NRCI) induced by ***PS***.

We remark that two NRCIs induced by ***PS*** as $\boldsymbol{J}_{PS^{(L)}}$, $\boldsymbol{J}_{PS^{(R)}}$.

Let ***PS*** be a NPS, then $\forall m, n, k\in D^*$, ***PS*** satisfies the residual criteria iff

$$\boldsymbol{PS}(k, m) \geq_1 n \text{ iff } k \geq_1 \boldsymbol{J}_{PS^{(L)}}(m, n);$$
$$\boldsymbol{PS}(m, k) \geq_1 n \text{ iff } k \geq_1 \boldsymbol{J}_{PS^{(R)}}(m, n).$$

Through learning above definitions, we give the NRIs (NRCIs) of NPTs (NPSs) discussed in **Section 3** as follows:

Example 4.5 Suppose that I_{iL} (i=1,2,3,4) and I_{iR} (i=1,2,3,4) are left and right residual implications induced by pseudo-t-norms PT_i (i=1,2,3,4) as shown in **Example 3.3**; J_{iL} (i=1,2,3,4) and J_{iR} (i=1,2,3,4) are left and right residual co-implications induced by pseudo-s-norms PS_i (i=1,2,3,4) as shown in **Example 3.4**. Then, the binary functions $\boldsymbol{I}_{PT(i)^{(L)}}$ (i=1,2,3,4,5,6) and $\boldsymbol{I}_{PT(i)^{(R)}}$ (i=1,2,3,4,5,6) induced by RNPTs $\boldsymbol{PT}_i$ (i=1,2,3,4,5,6) of **Example 3.5** defined on D^* are left and right NRIs as follows:

(1) $\boldsymbol{I}_{\boldsymbol{PT}_1}{}^{(L)}(m,n) = (I_{1L}(m_1,n_1), J_{1L}(m_2,n_2), J_{1L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_1}{}^{(R)}(m,n) = (I_{1R}(m_1,n_1), J_{1R}(m_2,n_2), J_{1R}(m_3,n_3))$;

(2) $\boldsymbol{I}_{\boldsymbol{PT}_2}{}^{(L)}(m,n) = (I_{2L}(m_1,n_1), J_{2L}(m_2,n_2), J_{2L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_2}{}^{(R)}(m,n) = (I_{2R}(m_1,n_1), J_{2R}(m_2,n_2), J_{2R}(m_3,n_3))$;

(3) $\boldsymbol{I}_{\boldsymbol{PT}_3}{}^{(L)}(m,n) = (I_{3L}(m_1,n_1), J_{3L}(m_2,n_2), J_{3L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_3}{}^{(R)}(m,n) = (I_{3R}(m_1,n_1), J_{3R}(m_2,n_2), J_{3R}(m_3,n_3))$;

(4) $\boldsymbol{I}_{\boldsymbol{PT}_4}{}^{(L)}(m,n) = (I_{4L}(m_1,n_1), J_{4L}(m_2,n_2), J_{4L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_4}{}^{(R)}(m,n) = (I_{4R}(m_1,n_1), J_{4R}(m_2,n_2), J_{4R}(m_3,n_3))$;

(5) $\boldsymbol{I}_{\boldsymbol{PT}_5}{}^{(L)}(m,n) = (I_{1L}(m_1,n_1), J_{2L}(m_2,n_2), J_{3L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_5}{}^{(R)}(m,n) = (I_{1R}(m_1,n_1), J_{2R}(m_2,n_2), J_{3R}(m_3,n_3))$;

(6) $\boldsymbol{I}_{\boldsymbol{PT}_6}{}^{(L)}(m,n) = (I_{1L}(m_1,n_1), J_{3L}(m_2,n_2), J_{3L}(m_3,n_3))$;

$\boldsymbol{I}_{\boldsymbol{PT}_6}{}^{(R)}(m,n) = (I_{1R}(m_1,n_1), J_{3R}(m_2,n_2), J_{3R}(m_3,n_3))$.

Example 4.6 Suppose that I_{iL} (i=1,2,3,4) and I_{iR} (i=1,2,3,4) are left and right residual implications induced by pseudo-t-norms PT_i (i=1,2,3,4) as shown in **Example 3.3**; J_{iL} (i=1,2,3,4) and J_{iR} (i=1,2,3,4) are left and right residual co-implications induced by pseudo-s-norms PS_i (i=1,2,3,4) as shown in **Example 3.4**. Then, the binary functions $\boldsymbol{J}_{PS(i)^{(L)}}$ (i=1,2,3,4,5,6) and $\boldsymbol{J}_{PS(i)^{(R)}}$ (i=1,2,3,4,5,6) induced by RNPSs $\boldsymbol{PS}_i$ (i=1,2,3,4,5,6) of **Example 3.6** defined on D^* are left and right NRCIs as follows:

(1) $\boldsymbol{J}_{\boldsymbol{PS}_1}{}^{(L)}(m,n) = (J_{1L}(m_1,n_1), I_{1L}(m_2,n_2), I_{1L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_1}{}^{(R)}(m,n) = (J_{1R}(m_1,n_1), I_{1R}(m_2,n_2), I_{1R}(m_3,n_3))$;

(2) $\boldsymbol{J}_{\boldsymbol{PS}_2}{}^{(L)}(m,n) = (J_{2L}(m_1,n_1), I_{2L}(m_2,n_2), I_{2L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_2}{}^{(R)}(m,n) = (J_{2R}(m_1,n_1), I_{2R}(m_2,n_2), I_{2R}(m_3,n_3))$;

(3) $\boldsymbol{J}_{\boldsymbol{PS}_3}{}^{(L)}(m,n) = (J_{3L}(m_1,n_1), I_{3L}(m_2,n_2), I_{3L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_3}{}^{(R)}(m,n) = (J_{3R}(m_1,n_1), I_{3R}(m_2,n_2), I_{3R}(m_3,n_3))$;

(4) $\boldsymbol{J}_{\boldsymbol{PS}_4}{}^{(L)}(m,n) = (J_{4L}(m_1,n_1), I_{4L}(m_2,n_2), I_{4L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_4}{}^{(R)}(m,n) = (J_{4R}(m_1,n_1), I_{4R}(m_2,n_2), I_{4R}(m_3,n_3))$;

(5) $\boldsymbol{J}_{\boldsymbol{PS}_5}{}^{(L)}(m,n) = (J_{1L}(m_1,n_1), I_{2L}m_2,n_2), I_{3L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_5}{}^{(R)}(m,n) = (J_{1R}(m_1,n_1), I_{2R}(m_2,n_2), I_{3R}(m_3,n_3))$;

(6) $\boldsymbol{J}_{\boldsymbol{PS}_6}{}^{(L)}(m,n) = (J_{1L}(m_1,n_1), I_{3L}(m_2,n_2), I_{3L}(m_3,n_3))$;

$\boldsymbol{J}_{\boldsymbol{PS}_6}{}^{(R)}(m,n) = (J_{1R}(m_1,n_1), I_{3R}(m_2,n_2), I_{3R}(m_3,n_3))$.

Because NPS ***PS*** are dual operator of NPT ***PT*** about $\boldsymbol{N}_S$, so the NRIs induced by NPT and the NRCIs induced by NPS are dual. For **Examples 3.5** and **3.6** given above, If ***PS*** and ***PT*** are dual, then the NRCIs $\boldsymbol{J}_{PS}$ derived by ***PS*** is the dual operator of the NRIs $\boldsymbol{I}_{PT}$ induced by ***PT***.

The following we will show an important theorem which proves sufficient conditions that the residual operator derived by a NPT is always a NI.

Theorem 4.7 Assume ***PT*** be a NPT on D^*. Then, $\forall m, n\in D^*$,

$$\boldsymbol{I}_{PT^{(L)}}(m, n) = \sup\{k \mid k\in D^*, \boldsymbol{PT}(k, m) \leq_1 n\};$$
$$\boldsymbol{I}_{PT^{(R)}}(m, n) = \sup\{k \mid k\in D^*, \boldsymbol{PT}(m, k) \leq_1 n\}.$$

are NIs.

Proof. First give the proof that $\boldsymbol{I}_{PT^{(L)}}$ is a NI, $\forall m, u, n, v\in D^*$:

We get $\boldsymbol{I}_{PT^{(L)}}(m, 1_{D^*}) = \sup\{k \mid k\in D^*, \boldsymbol{PT}(k, m) \leq_1 1_{D^*}\} = 1_{D^*}$ by **Definition 4.2**. Thus $\boldsymbol{I}_{PT^{(L)}}(1_{D^*}, 1_{D^*}) = 1_{D^*}$. From (NPT2) in **Definition 3.1**, we get $\boldsymbol{I}_{PT^{(L)}}(1_{D^*}, 0_{D^*}) = \sup\{k \mid k\in D^*, \boldsymbol{PT}(k, 1_{D^*}) \leq_1 0_{D^*}\} = 0_{D^*}$. $\boldsymbol{I}_{PT^{(L)}}(0_{D^*}, 0_{D^*}) = \sup\{k \mid k\in D^*, \boldsymbol{PT}(k, 0_{D^*}) \leq_1 0_{D^*}\} = 1_{D^*}$.

If $m \leq_1 u$. By (NPT2) in **Definition 3.1**, $\{k \mid k\in D^*, \boldsymbol{PT}(k, m) \leq_1 n\} \supseteq_1 \{k \mid k\in D^*, \boldsymbol{PT}(k, u) \leq_1 n\}$, then $\sup\{k \mid k\in D^*, \boldsymbol{PT}(k, m) \leq_1 n\} \geq_1 \sup\{k \mid k\in D^*, \boldsymbol{PT}(k, u) \leq_1 n\}$. Thus, $\boldsymbol{I}_{PT^{(L)}}(m, n) \geq_1 \boldsymbol{I}_{PT^{(L)}}(u, n)$.

If $n \leq_1 v$. Since the undecreasingness of ***PT***, we have $\{k \mid k\in D^*, \boldsymbol{PT}(k, m) \leq_1 n\} \subseteq_1 \{k \mid k\in D^*, \boldsymbol{PT}(k, m) \leq_1 v\}$, then $\sup\{k \mid k\in D^*, \boldsymbol{PT}(k, m) \leq_1 n\} \leq_1 \sup\{k \mid k\in D^*, \boldsymbol{PT}(k, m) \leq_1 v\}$. Thus, $\boldsymbol{I}_{PT^{(L)}}(m, n) \leq_1 \boldsymbol{I}_{PT^{(L)}}(m, v)$.

To sum up, $\boldsymbol{I}_{PT^{(L)}}$ is a NI. Likewise, $\boldsymbol{I}_{PT^{(R)}}$ is a NI can also be proved.

Some relevant properties of NRI are given below.

Theorem 4.8 Suppose that ***PT*** be a NPT on D^*, $\boldsymbol{I}_{PT^{(L)}}$, $\boldsymbol{I}_{PT^{(R)}}$ are NRIs. Then, $\forall m, n, r\in D^*$,

(1) $\boldsymbol{I}_{PT^{(L)}}(0_{D^*}, n) = 1_{D^*}$;

(2) $\boldsymbol{I}_{PT^{(L)}}(m, 1_{D^*}) = 1_{D^*}$;

(3) $\boldsymbol{I}_{PT^{(L)}}(m, m) = 1_{D^*}$;

(4) $\boldsymbol{I}_{PT^{(L)}}(1_{D^*}, n) = n$;

(5) $\boldsymbol{I}_{PT^{(L)}}(m, n) \geq_1 n$;

(6) $\boldsymbol{I}_{PT^{(L)}}(m, n) = 1_{D^*}$ iff $m \leq_1 n$;

(7) $\boldsymbol{I}_{PT^{(L)}}(\boldsymbol{PT}(m, n), \boldsymbol{PT}(m, r)) \geq_1 \boldsymbol{I}_{PT^{(L)}}(n, r)$;

(8) $m \leq_1 \boldsymbol{I}_{PT^{(L)}}(n, \boldsymbol{PT}(m, n))$.

Similarly, $\boldsymbol{I}_{PT^{(R)}}$ also satisfies the properties (1)-(7) in **Theorem 4.8**. However, it should be noted that NI induced by NPT, because pseudo-t-norm removes commutativity, leads to the difference in property (8) in the corresponding **Theorem 4.8** of $\boldsymbol{I}_{PT^{(R)}}$, as shown below:

(8) $m \leq_1 \boldsymbol{I}_{PT^{(R)}}(n, \boldsymbol{PT}(n, m))$.

Proof. The proofs of (1)-(4) are straightforward to obtain by **Definition 4.2**, so the proof is ignored.

(5) From (NI1) in **Definition 4.1,** we get that $\boldsymbol{I}_{PT^{(L)}}(m, n) \geq_1 \boldsymbol{I}_{PT^{(L)}}(1_{D^*}, n) = n$.

(6) ($\Rightarrow$) if $\boldsymbol{I}_{PT^{(L)}}(m, n) = 1_{D^*}$, then $\boldsymbol{PT}(1_{D^*}, m) \leq_1 n$. Thus, $m \leq_1 n$. ($\Leftarrow$) since $m \leq_1 n$, $\boldsymbol{PT}(1_{D^*}, m) \leq_1 n$. Thus, $\boldsymbol{I}_{PT^{(L)}}(m, n) \geq_1 1_{D^*}$, that is $\boldsymbol{I}_{PT^{(L)}}(m, n) = 1_{D^*}$.

(7) $\boldsymbol{I}_{PT^{(L)}}(\boldsymbol{PT}(m, n), \boldsymbol{PT}(m, r)) = \sup\{k \mid k\in D^*, \boldsymbol{PT}(k, \boldsymbol{PT}(m, n)) \leq_1 \boldsymbol{PT}(m, r)\} = \sup\{k \mid k\in D^*, \boldsymbol{PT}(m, \boldsymbol{PT}(k, n)) \leq_1 \boldsymbol{PT}(m, r)\} \geq_1 \sup\{k \mid k\in D^*, \boldsymbol{PT}(k, n) \leq_1 r\} = \boldsymbol{I}_{PT^{(L)}}(n, r)$.

(8) Since $\boldsymbol{PT}(m, n) \leq_1 \boldsymbol{PT}(m, n)$, so we get $m \leq_1 \boldsymbol{I}_{PT^{(L)}}(n, \boldsymbol{PT}(m, n))$.

The proof which $\boldsymbol{I}_{PT^{(R)}}$ satisfies the properties (1)-(8) is similar to the proof of $\boldsymbol{I}_{PT^{(L)}}$.

In the same way, we give two theorems about NPS on D^*.

Theorem 4.9 Let $\boldsymbol{PS}$ be a NPS on D^*. Then, $\forall m, n \in D^*$,

$$\boldsymbol{J}_{PS^{(L)}}(m, n) = \inf\{k \mid k \in D^*, \boldsymbol{PS}(k, m) \geq_1 n\};$$
$$\boldsymbol{J}_{PS^{(R)}}(m, n) = \inf\{k \mid k \in D^*, \boldsymbol{PS}(m, k) \geq_1 n\}.$$

are NCIs.

Proof. According to the **Definition 4.4**, we can use the proof of **Theorem 4.7** method to prove it.

Theorem 4.10 Let $\boldsymbol{PS}$ is a NPS on D^*, $\boldsymbol{J}_{PS^{(L)}}$, $\boldsymbol{J}_{PS^{(R)}}$ are NRCIs. Then, $\forall m, n, r \in D^*$,
(1) $\boldsymbol{J}_{PS^{(L)}}(1_{D^*}, n) = 0_{D^*}$;
(2) $\boldsymbol{J}_{PS^{(L)}}(m, 0_{D^*}) = 0_{D^*}$;
(3) $\boldsymbol{J}_{PS^{(L)}}(m, m) = 0_{D^*}$;
(4) $\boldsymbol{J}_{PS^{(L)}}(0_{D^*}, n) = n$;
(5) $\boldsymbol{J}_{PS^{(L)}}(m, n) \leq_1 n$;
(6) $\boldsymbol{J}_{PS^{(L)}}(m, n) = 0_{D^*}$ iff $m \geq_1 n$;
(7) $\boldsymbol{J}_{PS^{(L)}}(\boldsymbol{PS}(m, n), \boldsymbol{PS}(m, r)) \leq_1 \boldsymbol{J}_{PS^{(L)}}(n, r)$;
(8) $m \geq_1 \boldsymbol{J}_{PS^{(L)}}(n, \boldsymbol{PS}(m, n))$.

Similarly, $\boldsymbol{J}_{PS^{(R)}}$ also satisfies the properties (1)-(7) in **Theorem 4.10**. However, it should be noted that NCI induced by NPS, because pseudo-s-norm removes commutativity, leads to the difference in property (8) in the corresponding Theorem 4.10 of $\boldsymbol{J}_{PS^{(R)}}$, as shown below:

(8) $m \geq_1 \boldsymbol{J}_{PS^{(R)}}(n, \boldsymbol{PS}(n, m))$.

Definition 4.11 Let $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$: $D^*\times D^* \rightarrow D^*$ are NIs. $\forall m, n \in D^*$, the induced operators $\boldsymbol{PT}_{I^{(L)}}$, $\boldsymbol{PT}_{I^{(R)}}$ by $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$ are defined as follows:

$$\boldsymbol{PT}_{I^{(L)}}(m, n) = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\};$$
$$\boldsymbol{PT}_{I^{(R)}}(m, n) = \inf\{k \mid k \in D^*, n \leq_1 \boldsymbol{I}^{(R)}(m, k)\}.$$

Theorem 4.12 Let $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$ are NIs on D^*. $\forall m, n, r \in D^*$, if $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$ satisfies below conditions:

(a) $r \leq_1 \boldsymbol{I}^{(L)}(n, m)$ iff $n \leq_1 \boldsymbol{I}^{(R)}(r, m)$;
(b) $\boldsymbol{I}^{(L)}(m, \boldsymbol{I}^{(L)}(n, r)) = \boldsymbol{I}^{(L)}(n, \boldsymbol{I}^{(L)}(m, r))$; $\boldsymbol{I}^{(R)}(m, \boldsymbol{I}^{(R)}(n, r)) = \boldsymbol{I}^{(R)}(n, \boldsymbol{I}^{(R)}(m, r))$;
(c) $\boldsymbol{I}^{(L)}(m, n) = 1_{D^*}$ iff $m \leq_1 n$; $\boldsymbol{I}^{(R)}(m, n) = 1_{D^*}$ iff $m \leq_1 n$;
(d) $\boldsymbol{I}^{(L)}(1_{D^*}, m) = m$; $\boldsymbol{I}^{(R)}(1_{D^*}, m) = m$.

Then, the induced operators $\boldsymbol{PT}_{I^{(L)}}$, $\boldsymbol{PT}_{I^{(R)}}$ by $\boldsymbol{I}^{(L)}$, $\boldsymbol{I}^{(R)}$ in Definition 4.11 are NPTs.

Proof. $\forall m, u, n, v \in D^*$, there are below:

(NPT1) From (a) and (b), $\boldsymbol{PT}_{I^{(L)}}(m, \boldsymbol{PT}_{I^{(L)}}(n, r)) = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(\boldsymbol{PT}_{I^{(L)}}(n, r), k)\} = \inf\{k \mid k \in D^*, \boldsymbol{PT}_{I^{(L)}}(n, r) \leq_1 \boldsymbol{I}^{(R)}(m, k)\} = \inf\{k \mid k \in D^*, r \leq_1 \boldsymbol{I}^{(R)}(n, \boldsymbol{I}^{(R)}(m, k))\} = \inf\{k \mid k \in D^*, r \leq_1 \boldsymbol{I}^{(R)}(m, \boldsymbol{I}^{(R)}(n, k))\} = \inf\{k \mid k \in D^*, \boldsymbol{PT}_{I^{(L)}}(m, r) \leq_1 \boldsymbol{I}^{(R)}(n, k)\} = \inf\{k \mid k \in D^*, n \leq_1 \boldsymbol{I}^{(L)}(\boldsymbol{PT}_{I^{(L)}}(m, r), k)\} = \boldsymbol{PT}_{I^{(L)}}(n, \boldsymbol{PT}_{I^{(L)}}(m, r))$.

(NPT2) If $m \leq_1 u$, $n \leq_1 v$. So $\boldsymbol{I}^{(L)}(v, k) \leq_1 \boldsymbol{I}^{(L)}(n, k)$ for $\forall k \in D^*$. $\forall k_0 \in \{k \mid k \in D^*, u \leq_1 \boldsymbol{I}^{(L)}(v, k)\}$, it can be concluded that $u \leq_1 \boldsymbol{I}^{(L)}(v, k_0)$. Since $m \leq_1 u$, and $\boldsymbol{I}^{(L)}(v, k_0) \leq_1 \boldsymbol{I}^{(L)}(n, k_0)$, $m \leq_1 \boldsymbol{I}^{(L)}(n, k_0)$, namely $k_0 \in \{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\}$. Thus, $\{k \mid k \in D^*, u \leq_1 \boldsymbol{I}^{(L)}(v, k)\} \subseteq_1 \{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\}$. Hence, $\inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\} \leq_1 \inf\{k \mid k \in D^*, u \leq_1 \boldsymbol{I}^{(L)}(v, k)\}$, that is, $\boldsymbol{PT}_{I^{(L)}}(m, n) \leq_1 \boldsymbol{PT}_{I^{(L)}}(u, v)$. Likewise, we can prove that $\boldsymbol{PT}_{I^{(L)}}(n, m) \leq_1 \boldsymbol{PT}_{I^{(L)}}(v, u)$.

(NPT3) $\boldsymbol{PT}_{I^{(L)}}(1_{D^*}, m) = \inf\{k \mid k \in D^*, 1_{D^*} \leq_1 \boldsymbol{I}^{(L)}(m, k)\} = \inf\{k \mid k \in D^*, \boldsymbol{I}^{(L)}(m, k) = 1_{D^*}\} = \inf\{k \mid k \in D^*, m \leq_1 k\} = m$; $\boldsymbol{PT}_{I^{(L)}}(m, 1_{D^*}) = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(1_{D^*}, k)\} = \inf\{k \mid k \in D^*, m \leq_1 k\} = m$.

Therefore $\boldsymbol{PT}_{I^{(L)}}$ is a NPT, and in the same way, we can also show that $\boldsymbol{PT}_{I^{(R)}}$ is a NPT.

Theorem 4.13 If $\boldsymbol{PT}$ is a NPT on D^*, so there is $\boldsymbol{PT} = \boldsymbol{PT}_{I^{(L)}} = \boldsymbol{PT}_{I^{(R)}}$.

Proof. $\forall m, n \in D^*$, from **Definitions 4.2** and **4.11**, we get $\boldsymbol{PT}_{I^{(L)}}(m, n) = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\} = \inf\{k \mid k \in D^*, \boldsymbol{PT}(m, n) \leq_1 k\} = \boldsymbol{PT}(m, n)$ and $\boldsymbol{PT}_{I^{(R)}}(m, n) = \inf\{k \mid k \in D^*, n \leq_1 \boldsymbol{I}^{(R)}(m, k)\} = \inf\{k \mid k \in D^*, \boldsymbol{PT}(m, n) \leq_1 k\} = \boldsymbol{PT}(m, n)$. Thus, $\boldsymbol{PT} = \boldsymbol{PT}_{I^{(L)}} = \boldsymbol{PT}_{I^{(R)}}$.

Definition 4.14 ([21]) An algebraic system $S=(S; \wedge, \vee, \otimes, \rightarrow, \rightsquigarrow, 0, 1)$ is said to be a NCRL, $\forall m, n, r \in S$, if S satisfies:

(1) $(S; \wedge, \vee, 0, 1)$ be a bounded lattice on S, its corresponding order is $\leq$, 0 is minimal element and 1 is maximal element of S;

(2) $(S; \otimes, 1)$ be non-commutative monoid and its neutral element is 1;

(3) $m \otimes n \leq r \Leftrightarrow m \leq n \rightarrow r \Leftrightarrow n \leq m \rightsquigarrow r$.

Sections 3 and **4** focus on NPTs and their NRIs. Next, a NCRL is established, which is constructed from three neutrosophic logic operators.

Theorem 4.15 Suppose $(D^*; \wedge_1, \vee_1, {}^c, 0_{D^*}, 1_{D^*})$ is a system and $\boldsymbol{PT}$ is a NPT on D^*. $\forall m, n \in D^*$, define the following three equations:

$$m \otimes n = \boldsymbol{PT}(m, n);\ m \rightarrow n = \boldsymbol{I}_{PT^{(L)}}(m, n);\ m \rightsquigarrow n = \boldsymbol{I}_{PT^{(R)}}(m, n).$$

Then, $(D^*; \wedge_1, \vee_1, \otimes, \rightarrow, \rightsquigarrow, 0_{D^*}, 1_{D^*})$ is NCRL.

Proof. First, by **Proposition 2.9**, we get that $(D^*; \wedge_1, \vee_1, 0_{D^*}, 1_{D^*})$ be a bounded lattice on D^*.

Second, the fact that $(D^*; \otimes, 1_{D^*})$ is non-commutative monoid is proved. (1) $m \otimes 1_{D^*} = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(1_{D^*}, k)\} = \inf\{k \mid k \in D^*, m \leq_1 k\} = m$ and $1_{D^*} \otimes m = \inf\{k \mid k \in D^*, 1_{D^*} \leq_1 \boldsymbol{I}^{(L)}(m, k)\} = \inf\{k \mid k \in D^*, \boldsymbol{I}^{(L)}(m, k) = 1_{D^*}\} = \inf\{k \mid k \in D^*, m \leq_1 k\} = m$, i.e. $\forall m \in D^*$, the equation $1_{D^*} \otimes m = m \otimes 1_{D^*} = m$ is true. (2) Theorem 4.13 proves that $\boldsymbol{PT}_{I^{(L)}} = \boldsymbol{PT}$ is a NPT. Thus, $\boldsymbol{PT}$ does not satisfy the commutative law. (3) From (NPT1) of **Definition 3.1**, $\otimes$ satisfies the associative law.

Finally, $\forall m, n, k \in D^*$, we prove the below equivalence relation

$$m \otimes n \leq_1 k \Leftrightarrow m \leq n \rightarrow k \Leftrightarrow n \leq m \rightsquigarrow k$$

holds. On the one hand, by what we know about $\otimes$, there are $m \otimes n = \inf\{k \mid k \in D^*, m \leq_1 \boldsymbol{I}^{(L)}(n, k)\}$, $m \otimes n \leq_1 k$. Thus, there are $n \leq m \rightsquigarrow k$ and $m \leq n \rightarrow k$. On the other hand, by what we know about $\rightarrow$, we get $n \rightarrow k = \sup\{t \mid t \in D^*, \boldsymbol{PT}(t, n) \leq_1 k\}$. Since $m \leq n \rightarrow k$, therefore $m \otimes n \leq_1 k$. Likewise, there are $n \leq m \rightsquigarrow k \Rightarrow m \otimes n \leq_1 k$.

Thus, $(D^*; \wedge_1, \vee_1, \otimes, \rightarrow, \rightsquigarrow, 0_{D^*}, 1_{D^*})$ is NCRL.

5. Conclusions

Neutrosophic logic is an important part of NS theory. Common neutrosophic logic operators are: NPTs, NPSs NIs, NNs and so on. On the basis of complete lattice $(D^*; \leq_1)$, We define NPTs and NPSs. In addition, DMNTs are defined, describing that NPT and NPS are dual with regard to the standard NN. Then, on the basis of complete lattice $(D^*; \leq_1)$, the concepts of NRI and NRCI is given, and we present a theorem which states that residual operators derived by NPTs must be NIs, and further study their fundamental properties. Finally, we provide a method to get NPT from NI and construct NCRLs. In the future, we will investigate neutrosophic inference methods and neutrosophic pseudo overlap functions based on some new results [22-36], and further study their fundamental properties.

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Multicriteria Decision Making on 3D printers for economic manufacturing using Neutrosophic environment

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Abstract

Multicriteria group decision-making scenarios with a large number of criteria values may be challenging for experts to control. This is a result of the specialists' need to consider an excessive amount of data. They find it difficult to make the optimal decision since the possibilities overwhelm them. We propose a novel multicriteria group decision-making method that methodically eliminates the initial set of criterion values in order to address this issue. One of the most promising emerging technologies currently in development is an additive manufacturing (AM), which includes 3D printing. It has been hypothesized that 3D printing technology could eventually replace the conventional production machinery that is commonly used in the industrial sector. Making conclusions through accurate figures is difficult for decision-makers due to the complexity and ambiguity of reality. Neutrosophic ensembles are used to tackle uncertainty and indeterminacy in a practical environment. By concentrating on ranking the smaller set of criterion values, the proposed method enables the experts to carry out the group decision-making process. As a result, a relaxed decision-making environment is created, allowing the experts to handle a reasonable amount of information while still making decisions. To demonstrate the decision process of a 3D printer, we combine a single valued Neutrosophic with hybrid score and accuracy function, the single valued Neutrosophic number ranking approach, and the single valued Neutrosophic score and accuracy function. To determine the best attributes, the score function was used to rank the total values of each possibility. Concrete examples have been given to support the suggested solution to the multi-attribute decision-making problem (MADM).

Keywords: 3D printer, Neutrosophic Logic; Multicriteria decision making, knowledge based system, Decision support system.

1.Introduction:

Numerous fields are affected by Decision Making (DM) difficulties. Assessment of the proof in DM cases often depends on many factors rather than one. It's also becoming more difficult for decision-makers to evaluate all relevant aspects of a problem as the intricacy of the technical environment rises. Therefore, complex decision problems are often tackled by experts who pool their knowledge and experience. There are a variety of approaches that have been developed to handle these complex Multi Criteria Decision making (MCDM) problems. [17] The field of additive manufacturing (AM), which includes 3D printing, is often regarded as the most promising of the developing technologies currently under development. It has been suggested that 3D printing technology could eventually take the place of conventional production equipment widely used in the industrial industry. There are seven main categories of 3D printing technology. Regarding 3D printing, each method is entirely distinct from the others (i.e., operation, material usage, and no wastage) [44]. A product's final cost includes every cost incurred during production. The most crucial step is to think about the machine's characteristics and cost before production begins. Furthermore, the price of the American-made object will be determined by the total production expenses. The pricing of the product could then be reduced by selecting an AMM with desirable characteristics at an affordable price [7]. Therefore, the purpose of this study is to use one of the multi-criteria decision-making (MCDM) tools—the analytical hierarchy process—to address a decision-making issue raised by the AMM selection (AHP). The MSME begins selecting an FDM machine for its structural and doll product by requesting price quotes from several AM manufacturers. Extruder type, and machine weight should all be taken into account while selecting the optimum machinery for the project. AHP relies on criteria developed in collaboration with those making the calls. AHP requires decision-makers to answer the common Saaty's scale criterion questions to produce the pair-wise matrix [67].

AM has the ability to produce materials with the most intricate geometric features while using little bulk and waste. AM is a compelling material-saving solution with its low material costs and independent characteristics that allow for control of process parameter customization. In recent years, 3D printing, also known as additive manufacturing, has garnered interest from every primary industry. The conventional industrial production system is in crisis, and AM is the root of the problem (CM) [23]. "Compared to traditional manufacturing methods, additive manufacturing is superior at creating geometrically rigid material structures [81]. As an added benefit as seen in CM, AM excels at bypassing the need for an integrated assembly in favor of a more straightforward, layer-by-layer approach to material preparation. The high prototype production cost, the increased production rates, the high prices of the product itself, and the difficulty of performing real-time operational tests are all reasons that make AM challenging to implement" stated by [47]. Although AM prototypes are more expensive than CM ones, they provide significant benefits in terms of reduced production time [33].

Over the past few decades, the Neutrosophic has evolved alongside its ecosystem. Multiple subjects benefit from using a Neutrosophic environment, including logic, statistics, algebra, neural networks, etc. Given the inherent uncertainty in most real-world decision-making scenarios, philosophers' sets provide a promising solution. Uncertainty is inherent in situations that occur in the real world, and environmental factors often contribute significantly to it. [49]. The outcomes of neutron star environments are applied to a new facet of traffic management. When it comes to managing traffic, neutrality plays a crucial role. The data is indeterminate, and the issues of membership and non-membership are addressed.

The potential for sustainability is enhanced by the fact that 3D printing is a novel manufacturing method with far-reaching environmental impacts across the entire product life cycle. Additive manufacturing constructs items layer by layer rather than chopping away from a greater volume, drastically lowering resource requirements and production waste. Traditional manufacturing generates waste because raw materials must undergo subtractive procedures to be transformed into finished goods [46]. Due to the additive nature of 3D printing, no waste is generated. With most 3D printing technologies, the only waste is the support structures created with the product and then taken out after manufacturing.

Further, 3D printing helps cut down on defective product waste [34]. Due to this improvement in resource efficiency, less energy is needed to produce or transport materials. According to [21], 3D printing might drastically cut down on or even eliminate production waste, but the technique has to be validated before it can be widely utilized.

3D printing reduces manufacturing energy requirements by eliminating intermediaries and speeding up production. These energy-efficient production methods also reduce carbon dioxide emissions. The carbon footprint of shipping may also be reduced thanks to 3D printing [55]. The carbon footprint associated with production and transportation of these goods can be reduced. Suppose 3D printing technologies are adopted on a large scale. In that case, production speeds are increased, and additional printable materials are made accessible, [35] argues the industry's net CO2 emissions and energy usage might be reduced. Additionally, they warn that the sustainability gains from 3D printing might be nullified by a "rebound effect" if the production volume is raised because of the technology's enhanced efficiency.

A pool of molten metal receives metal powder, which is then deposited as new metal by laser melting. This process is repeated until the pool of molten metal tank is full. Fused deposition modeling [27] involves heating a filament of thermoplastic polymer and then extruding it onto a surface. Despite the fact that the materials employed in this method are more easily recyclable, this method is still preferred.

These pillars have not received the attention they deserve since it was believed that they were superfluous [24]. Although this is a difficulty, there is a solution described in [20] that involves crushing the supports and then extruding them into a filament using the new material. This is a technique that may be used to solve the problem. The acrylonitrile butadiene styrene (ABS) plastic, on the other hand, deteriorates under the effect of heat generated by the FDM printer. Research is being carried out on new polymers with the hope of improving their potential for recycling [22]. which is also an area for development [16]. The material choice might influence the sustainability of 3D printing.

Laser metal deposition and other processes involving the melting metal powders offer the environmental benefits when reusing the material but are also very energy intensive. Although FDM techniques offer a reduced energy footprint, they have the additional drawback of increased emissions [59]. "Toluene, ethylbenzene, and formaldehyde are all recognized carcinogens", and [45] discovered that "FDM with ABS plastic generated substantial amounts of these chemicals. Although [26] demonstrated that "VOCs are released during 3D printing procedures using PLA, these emissions were minimized when PLA was used instead of ABS, yielding substantially less particulate matter and no VOCs". In both the best and worst-case scenarios for printing, the amount of volatile organic compounds is far below occupational exposure levels, posing no threat to human health.

EcoPrinting, suggested by [69], is a 3D printing procedure that uses waste polymers as a source material and has a negligible environmental impact. An integrated solar battery charging system and other low-power components help the EcoPrinting system significantly reduce energy consumption [15]. To help the people of the Solomon Islands, this technique has been used to print pipe couplers and plumbing seals as part of a humanitarian assistance project [14]. ABS plastics were employed in the EcoPrinting process applied to plastic from vehicle parts and technological debris collected from a Solomon Islands landfill and local companies [43]. The Tolerances for filaments produced from recycled material are equivalent to those of filaments sold commercially [1]. By functioning without an electricity connection and utilizing collected ABS plastic waste as the printing feedstock, the EcoPrinting system has successfully demonstrated the potential improvements in 3D printing.

Unlike the CM mode, AM mode allows for a more targeted counter design. [2] studied the topic of choosing individualized procedures in healthcare. This work enhanced a strategy for selecting functions in AM's fabrication of novel materials and replacement components [3]. A practical method for reducing the likelihood of AMM failure is to systematically evaluate which AMM is the most effective. It'll boost AMM's productivity. Art critics [13] examined additive manufacturing, which decreases waste by utilizing fewer powder particles to create end products, and how the absence of process selection tools is a wasted economic opportunity.

AM stocks are beginning to register as part of the third wave of manufacturing by creating industry-spanning prototype jewels. [6] demonstrated the AM sectors manufacturing benefits in terms of lead time and time to market by comparing several fast prototype methods. Unlike traditional manufacturing methods, AM does not require the use of tools throughout the creation process, as noted by [37]. The result of this is mass manufacturing. How to choose the RP processes is in detail by [63], who use a modified matrix technique and graph theory to explain their findings. [78] detailed a comparison of genetic models for determining the optimal procedure in fast prototyping in terms of build cost, build duration, and surface roughness. According to the findings of [12], the rule-based expectation system will address the issues with AM's process selection that have been posed in the business and academic communities. Research on RP methods follows a topic-specific methodology that takes into consideration criteria such as strength of building materials, accuracy, prototypes, cost, elongation, and build time [11]. A product's roughness, precision, speed, price, and mechanical qualities were quantitatively investigated [10]. Furthermore, studies have investigated how AMMs use marginally fewer raw resources. [80] found that, when it comes to orthodontists' applications and multimodal 3D face recognition, 3D printing offers the most outstanding accuracy and lowest material waste.Studies by [79] and [77] have shown that the structure of AMMs affects their mechanical characteristics and is crucial for maintaining these qualities stable. Newer multiple-choice problems with numerous criteria and solutions may be solved with the help of the MCDM method. Using prior studies' literature, the following are the advantages and applications of MCDM.

3D printing as part of a more significant educational effort to produce pre-scanned bones for use in teaching anatomy" [3] and [4].

Researchers at "Australia's Monash University" have invented a revolutionary technique for 3D printing cadaveric orbital separation incisions in ophthalmology and optometry teaching and training [25]. 3D printing is expected to significantly improve students' educational experience in STEM fields. In reviewing the efforts of the "European Federation of Chemical Engineers" Working Party on Education's efforts over the previous decade, Gillet[56] 200 highlights three crucial aspects of chemical engineering education: curriculum creation, individual growth, and ongoing education. Middle and high school teachers were the target audience for [64] a three-day 3D printing workshop, which included online educational and visual resources. They demonstrated the potential of open-source 3D printing technology to enhance learning by encouraging active students' participation in all subjects. Loy [60] showed how 3D printing can combine eLearning and making to revolutionize how product design is taught in the classroom. In their research, [76] shows how a conceive-design-implement-operate framework may balance pedagogical theory, technology training, and classroom instruction. However, in 2013, an MIT research team led by [18] launched 4D printing, which has aided in developing intelligent materials. Fourth-dimensional printing (4D printing) is a development of 3D printing in which time is included as a fourth dimension. Depending on the stimuli (such as heat, ultraviolet light, or water), the printed form may change over time, making it a dynamic structure with malleable features and functions [74],[75]. Although this development has expanded the application of digital manufacturing's application, it still needs expertise in several fields (such as mathematics, mechatronics, mechanical engineering, and chemical engineering). New intelligent engineering materials have been exhibited and studied recently, including temperature- and pH-controlling smart valves, adaptive pipes, sensors, and soft robotics [50].

I3Mote is only one example of open-source software that has been made available to facilitate the creation of products based on integrated hardware and software. Industry 4.0, derived from the notion of a "smart factory," is an umbrella term for the IoT built on cyber-physical systems (CPS) that integrate virtual and real-world settings [43],[42],[41].

"Advanced robotics, additive manufacturing, augmented reality, simulation, horizontal and vertical integration, the industrial internet, the cloud, cybersecurity, and big data and analytics" are the nine pillars upon which Industry 4.0 rests [48]. Germany's "Industry 4.0" program, launched in 2011, aims to digitalize the manufacturing process [51]. This program is responsible for coining the phrase "Industry 4.0" and developing the architectural reference model that underlies the concept.

Regarding consolidation and integration of the high-tech industries and guaranteeing the country's technical leadership, Industry 4.0 in Germany is based on the High Tech 2020 Strategy. Singapore's "Smart Nation Program," Japan's "Industrial Value Chain Initiative," China's "Made in China 2025," and the United States' own "Smart Manufacturing" all outline similarly ambitious goals [73] and [72]. Most of Malaysia's industrial industries are either highly mechanized or involved in mass production. To raise awareness of and contribute to the development of a comprehensive national strategy for Industry 4.0, the government has held discussions with a wide range of stakeholders and implemented several public outreach initiatives [5].

Optimizing part geometries, for instance, is crucial in the design phase because it can affect the environmental impact [52]. In some other cases, though, the decision-maker doesn't consider them.

seeing widespread usage in fast prototyping [53] and [54]. Others share some features of one 3D printing technology because they share similar underlying principles. In contrast, other features are distinct because they reflect the technology's regulations and lead to distinctive differences in the characteristics of the printed parts.

Among these procedures. AM technique that uses a liquid bonding agent dropped over powder particles to hold them together. By repositioning the print head and carefully depositing the bonding agent, a BJ printed component is created [57]. When printing with expensive materials, the ability to print without anchoring the powder on a build plate is a significant time and cost saver. Manufacturing ceramic components is a typical use case for BJ. Recent research [58] demonstrates that BJ-printed components have enhanced bulk density, making them ideal for metallic foam frameworks and approaching completely dense stainless-steel parts. One of the benefits of BJ's is the high resolution that makes it possible to apply finishes with a lot of detail.

Direct Energy Deposition (DED) "processes use focused energy to melt materials directly as they are added to the workpiece in a layer-by-layer fashion. A laser, electron beam, or arc lights are often used as the focused energy source, while the raw materials take the shape of a wire or powder [61],[62] and [65]. DED printing requires a greater quantity of materials than other methods. DED-printed components can better resist fatigue" [66]. New studies reveal that the particle capture efficiency of DED varies with the working distance and may improve with an increasing material in the surface temperature [68].

Melt extrusion (ME) is like the conventional plastic extrusion method; both involve the melting of the material being shaped. A typical ME method is fused deposition modeling (FDM), where a nozzle deposits the material in a soft and semi-liquid condition onto a building platform, a little bit at a time, to create an 3D object. However, ME can create more intricate components than the extrusion method while having a slower manufacturing rate. ME often has the lowest price tag among AM technologies, which helps explain why it is the most widely used 3D printing option. The print speed, process parameters, and thermal activities of the FDM method have all been improved, and much work has been put into developing new materials [70],[71] and [40].

Additive manufacturing (AM) techniques that use a bed of powdered input material that falls under the umbrella term of powder bed fusion (PBF) [38]. By spreading a small layer of powdery material and then fusing it at particular points, the 3D object is printed. Possible sources of energy include a laser, electron beam, or infrared light. Post-processing steps, such as blowing away debris or lifting the printed product off the platform, are commonplace in print-on-demand fabrication (PBF). Their distinct microstructures cause bulk anisotropy of the PBF components; however, recent research suggests that it may be reduced by using a broad beam.

A vat is used to perform the VP technique, during which the liquid-photosensitive resin is polymerized. When the resin is exposed to a laser or an arc light source, a solid three-dimensional component could be produced as a result of a chemical reaction. There are many different kinds of VP, but some examples are stereolithography, the digital light process, and the continuous liquid interface product [39] and [36]. Stereolithography is one example. New research suggests that the "bottom-up" and "top-down" print methods may provide different results [8] and [9] for components that have certain geometries, including those with parts with a length/diameter ratio that is greater than 2. One of the first steps in developing a useful cost model is determining the extent to which the model will be used. Several broad AM processes that include supplementary AM process phases have been reported in the literature. At this point in the process, the raw ingredients are additionally assembled [19]. Powdered materials may require sieving or mixing, and "material formulation" may be required for liquid materials in order to get depositable materials. Raw materials may also need to be placed into cartridges or containers and stored in a method that prevents them from degrading for a sufficient amount of time, but this will depend on the specifics of the process and the machine's design [28],[29] and [30].

Setting up the AM machine and its control system is a prerequisite before the construction begins. There are the AM systems, the energy exposure devices, the climate control system, and the control computers to configure. Once this process is complete, the appropriate control proper control parameters may be adjusted [31]. The method for matter deposition or energy delivery varies between AM techniques [32].

A relationship function called FuzzySet(FS) was used to tackle the majority of the uncertainty problems that exist in the actual world, and it was thoroughly explained.[11] uses to expand upon the intuitionistic fuzzy set (IFS) notion discussed above. Instead, a number of approaches to tackling the uncertainty problem have been developed, such as generalised orthopairfuzzy sets [18] N-valued interval Neutrosophic [31] generalised interval-valued triangular IFS, JY. Neutrosophic multicriteria is a method of decision-making that integrates various criteria or elements, occasionally with scant or unclear data, to reach a result [82]. With the use of a mathematical model created using a double bounded rough Neutrosophic set, the expression of the students is evaluated using real-time data gathered by photographing them in relation to various subjects [83]. The suggested study mentions the principal medical areas that NIP can provide for image segmentation from DICOM pictures. It has been found to be a superior method due to the way it handles ambiguous information [84]. Understanding stress and creating strategies to lessen its effects on voice recognition and human-computer interaction systems are the goals of this research [85]. In this article, we present a method for calculating a system's expected costs under various conditions. The uncertainty in the various model parameters are managed using the trapezoidal bipolar Neutrosophic numbers [86]. In this paper, complex group decision-making scenarios are addressed using the dynamic programming method, where the preference data is represented by linguistic variables [87]. The method's advantage is that it can be used without a lower membership function for falsehood, which results in a sizable reduction in calculation time [88]. In an effort to address the traffic problem, this study made an effort to provide a general overview of each approach. Numerous academics that are now studying traffic flow, traffic accident diagnostics, and its hybridization are expected to benefit from the proposed study [89]. This work reveals that neurosophic multiple regression is the most useful model for uncertainty, as opposed to conventional regression models [90], [91], and [92]. To achieve the lowest inspection cost possible, we will compose the issue language appropriately for such a case in this study before building the appropriate mathematical model [93]. This framework takes into account the components of Industry 5.0. The most important related elements and strategies can be found by first reviewing the relevant experts and body of published research [94]. Reducing HCWT through appropriate treatment is vital for the region's economic and environmental wellbeing. In order to address single-valued neutrosophic group decision-making issues with a shortage of weight data, this research develops a novel multi-criteria decision-making technique [95].

2.Neutrosophic sets

Assume that X = x1, x2,..., xm (m ≥2) is the set of decision-makers or experts, y= y1, y2,..., yq (q≥ 2) is the collection of criteria, and A=A1,..., An (n ≥ 2) is the set of logistics centres.

The weights of the decision-makers are completely unknown in the group decision-making problem, but the weights of the criteria are only partially understood. These weights have never been assigned before.We create a method based on the hybrid score-accuracy function using linguistic variables to address the MCDM problem with uncertain weights in a single-valued Neutrosophic environment. The suggested method's steps for resolving MCGDM are listed below.

Definition 2.1(F. Smarandache 2005)

Let X be the universal set, then Neutrosophic set is defined as $S = \{(T_S(x), I_S(x), F_S(x)), x \in X\}$ where $T_S(x), I_S(x), F_S(x) \in [0,1]$ and $0 \leq T_S(x) + I_S(x) + F_S(x) \leq 3$.

Definition 2.2(H. Wang 2010)

Let X be the universal set, then SVNS is defined as $\dot{S} = \{(T_{\dot{S}}(x), I_{\dot{S}}(x), F_{\dot{S}}(x)), x \in X\}$ where $T_{\dot{S}}(x), I_{\dot{S}}(x), F_{\dot{S}}(x) \in [0,1]$ and $0 \leq T_{\dot{S}}(x) + I_{\dot{S}}(x) + F_{\dot{S}}(x) \leq 3$.

Definition 2.3 (H. Wang 2005)

Let X be the universal set, then IVNS is defined as $\dot{S} = \left\{\left(\left(T_{\dot{S}}^U(x), T_{\dot{S}}^L(x)\right),\ \left(I_{\dot{S}}^U(x), I_{\dot{S}}^L(x)\right), \left(F_{\dot{S}}^U(x), F_{\dot{S}}^L(x)\right)\right), x \in X\right\}$

where $T_{\dot{S}}(x) = \left(T_{\dot{S}}^U(x), T_{\dot{S}}^L(x)\right) \in [0,1], \left(I_{\dot{S}}^U(x), I_{\dot{S}}^L(x)\right) \in [0,1], (F_{\dot{S}}^U(x), F_{\dot{S}}^L(x)) \in [0,1]$ and $0 \leq T_{\dot{S}}^U(x) + I_{\dot{S}}^U(x) + F_{\dot{S}}^U(x) \leq 3$.

Algorithm

1.Creating the choice matrix.

2.Evaluate the hybrid score accuracy matrix.

3.Calculate the average matrix in step three.

4.Calculating the weights.

5.Calculate the average accuracy matrix.

6.find the weight model for criterion

7.compare the options

8.Finish in end step

3. Score accuracy functions using the Multi criteria decision making(MCDM) technique in a single-valued Neutrosophic environment

Based on the five factors, including precision (C1), speed (C2), price (C3), surface roughness (C4), and friendly use (C5). The weights of the decision-makers are completely unknown in the group decision-making problem, but the weights of the criteria are only partially understood. These weights have never been assigned before. We create a method based on the hybrid score-accuracy function using linguistic variables to address the MCDM problem with uncertain weights in a single-valued Neutrosophic environment. The following is a list of the stages for resolving MCGDM using the suggested method. Assume that the best 3D printing requires an optimization 3D printer. Three 3D printers are available: P1, P2, and P3. To choose the most relevant criterion based on five criteria (C1,C2,C3,C4 and C5), Four decision-makers or specialists (D1, D2, and D3) have been assembled into a committee.

Thus, linguistic factors are used by the three decision-makers. Conversion of linguistic variables and Single value is shown in Table 1.in table 2,3 and 4 is linguistic phrase presented.

Table 1 linguistic scale and corresponding single value Neutrosophic

Linguistic variable	Single value Neutrosophic
Very poor (VP)	(.01 .98 .98)
Poor (P)	(.15 .75 .85)
Good(G)	(.65 .45 . 35)
Very good (VG)	(.95 .05 .05)

Table 2 linguistic phrase for D1

	C1	*C2*	*C3*	*C4*	*C5*
P1	G	G	P	G	VG
P2	VG	VP	G	G	P
P3	G	P	G	P	G

Table 3 linguistic phrase for D2

	C1	C2	C3	C4	C5
P1	VG	G	G	G	P
P2	G	VG	P	G	G
P3	G	VG	VP	G	P

Table 4 linguistic phrase for D3

	C1	C2	C3	C4	C5
P1	G	G	VG	P	G
P2	VG	G	G	G	P
P3	G	VG	G	P	G

Decision matrix for corresponding linguistic phrase

$$Dm1 = \begin{bmatrix} (.65_{.}45_{.}35) & (.65_{.}45_{.}35) & (.15_{.}75_{.}85) & (.65_{.}45_{.}35) & (.95_{.}05_{.}05) \\ (.95_{.}05_{.}05) & (.01_{.}98_{.}98) & (.65_{.}45_{.}35) & (.65_{.}45_{.}35) & (.15_{.}75_{.}85) \\ (.65.45.35) & (.15.85.85) & (.65.45.35) & (.15.75.85) & (.65.45.35) \end{bmatrix}$$

$$Dm2 = \begin{bmatrix} (.95_{.}05_{.}05) & (.65_{.}45_{.}35) & (.65_{.}45_{.}35) & (.65_{.}45_{.}35) & (.15_{.}75_{.}85) \\ (.65_{.}45_{.}35) & (.95_{.}05_{.}05) & (.15_{.}75_{.}85) & (.65_{.}45_{.}35) & (.65_{.}45_{.}35) \\ (.65.45.35) & (.95.05.05) & (.01.98.98) & (.65.45.35) & (.15.75.85) \end{bmatrix}$$

$$Dm3 = \begin{bmatrix} (.65_{.}45_{.}35) & (.65_{.}45_{.}35) & (.95_{.}05_{.}05) & (.15_{.}75_{.}85) & (.65_{.}35_{.}45) \\ (.95_{.}05_{.}05) & (.65_{.}45_{.}35) & (.65_{.}45_{.}35) & (.65_{.}45_{.}35) & (.65_{.}45_{.}35) \\ (.65.45.45) & (.95.05.05) & (.65.45.35) & (.15.75.85) & (.15.75.85) \end{bmatrix}$$

Now, we choose the best 3D printing option using the mentioned method. We choose = 0.5 to illustrate the computation process.

Equation can be used to extract the hybrid score-accuracy matrix from the decision matrix.

The existing method Surapati Pramanik(2016)

$A_{ij}^{s} = \frac{1}{2}\alpha\left(1 + T_{ij}^{s} - F_{ij}^{s}\right) + \frac{1}{3}(1-\alpha)\left(2 + T_{ij}^{s} - I_{ij}^{s} - F_{ij}^{s}\right)$ (1)

Proposed model

$A_{ij}^{s} = \frac{1}{6}\alpha\left(T_{ij}^{s} + 2I_{ij}^{s} - 1\right) + \frac{1}{3}\left(2 + T_{ij}^{s} - I_{ij}^{s}\right)$ (2)

Using the above equation to find the hybrid score matrix for existing and proposed methods hybrid score matrix existing methods

$$Hybrid\ Score\ matrix\ 1 = \begin{bmatrix} .633 & .633 & .167 & .633 & .95 \\ .95 & .016 & .633 & .633 & .167 \\ .633 & .167 & .633 & .167 & .633 \end{bmatrix}$$

$$Hybrid\ Score\ matrix\ 2 = \begin{bmatrix} .95 & .633 & .633 & .633 & .167 \\ .633 & .95 & .167 & .633 & .633 \\ .633 & .95 & .016 & .633 & .167 \end{bmatrix}$$

$$Hybrid\ Score\ matrix\ 3 = \begin{bmatrix} .633 & .633 & .95 & .883 & .742 \\ .95 & .633 & .633 & .633 & .167 \\ .633 & .95 & .633 & .167 & .633 \end{bmatrix}$$

hybrid score matrix proposed methods

$$Hybrid\ Score\ matrix\ 1 = \begin{bmatrix} .813 & .633 & .688 & .813 & .779 \\ .971 & .609 & .279 & .424 & .938 \\ .813 & .779 & .504 & .488 & .821 \end{bmatrix}$$

$$Hybrid\ Score\ matrix\ 2 = \begin{bmatrix} .971 & .396 & .563 & .813 & .613 \\ .813 & .513 & .871 & .971 & .563 \\ .813 & .513 & .871 & .971 & .609 \end{bmatrix}$$

$$Hybrid\ Score\ matrix\ 3 = \begin{bmatrix} .813 & .613 & .688 & .813 & .513 \\ .971 & .396 & .563 & .813 & .613 \\ .813 & .513 & .513 & .971 & .396 \end{bmatrix}$$

Average matrix

$$H_{ij}^{\#} = \frac{1}{n}\sum_{x=1}^{m} H_{ij}^{x}$$

The average matrix existing methods

$$H_{ij}^{\#} = \begin{bmatrix} .738 & .633 & .583 & .717 & .635 \\ .844 & .533 & .477 & .633 & .322 \\ .633 & .688 & .428 & .322 & .478 \end{bmatrix}$$

The average matrix proposed methods

$$H_{ij}^{\#} = \begin{bmatrix} .865 & .540 & .646 & .813 & .635 \\ .918 & .506 & .571 & .736 & .704 \\ .813 & .601 & .748 & .809 & .609 \end{bmatrix}$$

3.1 CORRELATION COEFFICIENT BETWEEN $H_{ij}^{\#}$ AND H_{ij}^{x}

$$c_x = \sum_{i=1}^{m} \frac{\sum_{j=i}^{n} H_{ij}^{x} H_{ij}^{*}}{\sqrt{\sum_{j=1}^{n}\left(H_{ij}^{x}\right)^2}} \frac{1}{\sqrt{\sum_{j=1}^{n}\left(H_{ij}^{*}\right)^2}} \quad (3)$$

correlation coefficient of existing methods

$$H1*H_{ij}^{\#} = \begin{bmatrix} .468 & .401 & .097 & .454 & .588 \\ .802 & .008 & .302 & .401 & .053 \\ .401 & .011 & .271 & .053 & .302 \end{bmatrix}$$

$$H2*H_{ij}^{\#} = \begin{bmatrix} .702 & .401 & .369 & .454 & .103 \\ .534 & .506 & .079 & .401 & .204 \\ .401 & .654 & .006 & .204 & .079 \end{bmatrix}$$

$$H3*H_{ij}^{\#} = \begin{bmatrix} .467 & .401 & .554 & .633 & .459 \\ .802 & .337 & .302 & .401 & .053 \\ .401 & .654 & .271 & .053 & .302 \end{bmatrix}$$

correlation coefficient of proposed methods

$$H1*H_{ij}^{\#} = \begin{bmatrix} .703 & .330 & .444 & .660 & .494 \\ .891 & .308 & .159 & .312 & .661 \\ .660 & .469 & .377 & .394 & .499 \end{bmatrix}$$

$$H2*H_{ij}^{\#} = \begin{bmatrix} .840 & .214 & .363 & .660 & .389 \\ .745 & .259 & .497 & .714 & .396 \\ .660 & .308 & .652 & .786 & .370 \end{bmatrix}$$

$$H3*H_{ij}^{\#} = \begin{bmatrix} .703 & .331 & .444 & .660 & .325 \\ .891 & .200 & .321 & .597 & .431 \\ .660 & .308 & .652 & .786 & .240 \end{bmatrix}$$

Table 5 proposed and existing values

	Proposed values	Existing value
$\sum_{j=i}^{\rho} h_{ij}^{1} h_{ij}^{*}$	2.632679398	2.184668
$\sqrt{\sum_{j=1}^{\rho} \left(h_{ij}^{1}\right)^2}$	2.612882	2.134
$\sqrt{\sum_{j=1}^{\rho} \left(h_{ij}^{*}\right)^2}$	2.46342	1.478062

$\sum_{j=i}^{\rho} h_{ij}^{2} h_{ij}^{*}$	2.331893056	1.730445
$\sqrt{\sum_{j=1}^{\rho} \left(h_{ij}^{2}\right)^{2}}$	2.466105	1.733
$\sqrt{\sum_{j=1}^{\rho} \left(h_{ij}^{*}\right)^{2}}$	2.441935	1.315464
$\sum_{j=i}^{\rho} h_{ij}^{3} h_{ij}^{*}$	1.390534	2.777134
$\sqrt{\sum_{j=1}^{\rho} \left(h_{ij}^{3}\right)^{2}}$	2.777134	1.259
$\sqrt{\sum_{j=1}^{\rho} \left(h_{ij}^{*}\right)^{2}}$	2.647297	1.179209

$$C_1 = \frac{2.184668}{12.134\, x 1.478062} + \frac{1.730445}{1.1.733\ X\ 1.315464} + \frac{31.390534}{1.259\ X\ 1.179209} = 3.7385$$

C2=3.8986,*C3*=4.260

Table 6 proposed and existing C value

	proposed	Existing
C1	2.942516	3.7385
C2	2.908786	3.8986
C3	2.980633	4.260

Table 6 present the C value of the both methods

3.2 Decision maker's weights determination

$$\vartheta_x = \frac{C_1}{\sum_{x=1}^{n} C_x}, 0 \leq \vartheta_x \leq 1 \; for \; x = 1,2,3,\ldots,m$$

$$\vartheta_1 = 0.314, \vartheta_2 = 0.3276, \vartheta_3 = 0.3581$$

Table 7 proposed and existing Decision makers weights determination

	Proposed	Existing
ϑ_1	0.333168	0.314
ϑ_2	0.329349	0.3276
ϑ_3	0.337484	0.3581

present the weight determination

4. Calculate hybrid score accuracy matrix

In order to aggregate the hybrid score-accuracy values of the various decision makers' choices, the equation $\sum H_{ij}\,\vartheta_1$ used. and the following can be written as the overall hybrid score-accuracy matrix.

Hybrid score accuracy with $\sum H_{ij}\,\vartheta_1$ existing

$$\sum H_{ij}\,\vartheta_1 = \begin{bmatrix} .199 & .199 & .523 & .199 & .298 \\ .298 & .004 & .199 & .199 & .052 \\ .199 & .523 & .199 & .052 & .199 \end{bmatrix}$$

$$\sum H_{ij}\,\vartheta_2 = \begin{bmatrix} .311 & .207 & .207 & .207 & .054 \\ .207 & .311 & .054 & .207 & .207 \\ .207 & .311 & .005 & .207 & .054 \end{bmatrix}$$

$$\sum H_{ij}\,\vartheta_3 = \begin{bmatrix} .226 & .226 & .340 & .316 & .265 \\ .340 & .226 & .226 & .226 & .059 \\ .226 & .340 & .059 & .059 & .227 \end{bmatrix}$$

Hybrid score accuracy with $\sum H_{ij}\,\vartheta_1$ proposed

$$\sum H_{ij}\,\vartheta_1 = \begin{bmatrix} .270 & .204 & .229 & .270 & .259 \\ .323 & .203 & .093 & .141 & .312 \\ .270 & .259 & .167 & .162 & .273 \end{bmatrix}$$

$$\sum H_{ij}\,\vartheta_2 = \begin{bmatrix} .319 & .130 & .185 & .267 & .201 \\ .267 & .168 & .286 & .319 & .185 \\ .267 & .168 & .286 & .319 & .200 \end{bmatrix}$$

$$\sum H_{ij}\,\vartheta_3 = \begin{bmatrix} .274 & .206 & .232 & .274 & .172 \\ .327 & .133 & .189 & .274 & .206 \\ .274 & .172 & .293 & .327 & .133 \end{bmatrix}$$

Sum of the hybrid score accuracy matrix existing method

$$Sum\,of\;hybrid\;score\;matrix = \begin{bmatrix} .737 & .633 & .600 & .722 & .618 \\ .846 & .543 & .480 & .633 & .319 \\ .633 & .703 & .430 & .319 & .480 \end{bmatrix}$$

hybrid score accuracy in proposed methods

$$Sum\,of\;hybrid\;score\;matrix = \begin{bmatrix} .864 & .541 & .646 & .812 & .634 \\ .918 & .505 & .569 & .735 & .704 \\ .813 & .601 & .748 & .809 & .607 \end{bmatrix}$$

4.1 Weight model for criteria

Assume that the information about criteria weights is incompletely known given as follows: weight vectors,

Using the linear programming model Weighted criterion model

Assume that the following criteria weights information is incompletely known: weight matrices,

the linear programming paradigm

model $Max\ \omega = \frac{1}{n}\sum_{j=1}^{m} \omega_j H_{ij}$, we obtain the weight vector of the criteria as ω =[0.3 0.6 0.25 0.2 0.15].

We calculate the over all hybrid score-accuracy values

$\emptyset(m_i), i = 1,2,3,$in table 34 and 35 weighted criterion methods in exixting and proposed values

$$weight\ existing\ model\,matrix = \begin{bmatrix} .221 & .380 & .150 & .144 & .092 \\ .254 & .325 & .120 & .126 & .047 \\ .190 & .422 & .108 & .063 & .072 \end{bmatrix}$$

$$weight\,proposed\;\;model\,matrix = \begin{bmatrix} .259 & .324 & .161 & .162 & .095 \\ .275 & .303 & .142 & .147 & .105 \\ .243 & .360 & .187 & .161 & .091 \end{bmatrix}$$

$$\emptyset(m_1) = 0.9558, \emptyset(m_2) = 0.8772, \emptyset(m_3) = 0.8562$$

Table: 8 weight model criteria value

	Proposed	Existing
$\emptyset(m_1)$	1.003304	0.9558
$\emptyset(m_2)$	0.973961	.8772
$\emptyset(m_3)$	1.044839	0.8562

Table: 9 comparisons on Proposed and existing

Proposed model	$\emptyset(m_1) > \emptyset(m_2) > \emptyset(m_3)$
Existing model	$\emptyset(m_1) > \emptyset(m_2) > \emptyset(m_3)$

Fig 1 and fig 2 shows the pictorial representation of the values.

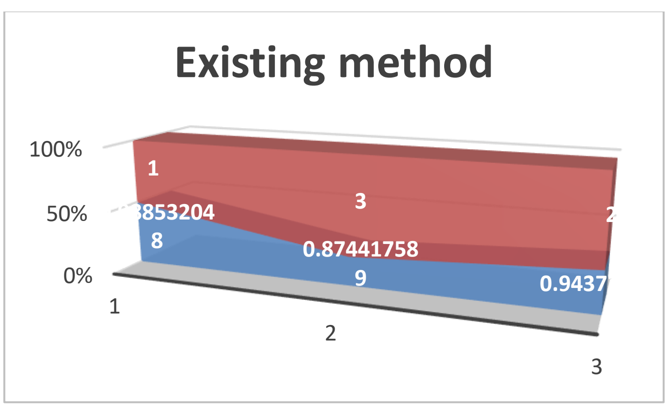

Fig 1 Existing method

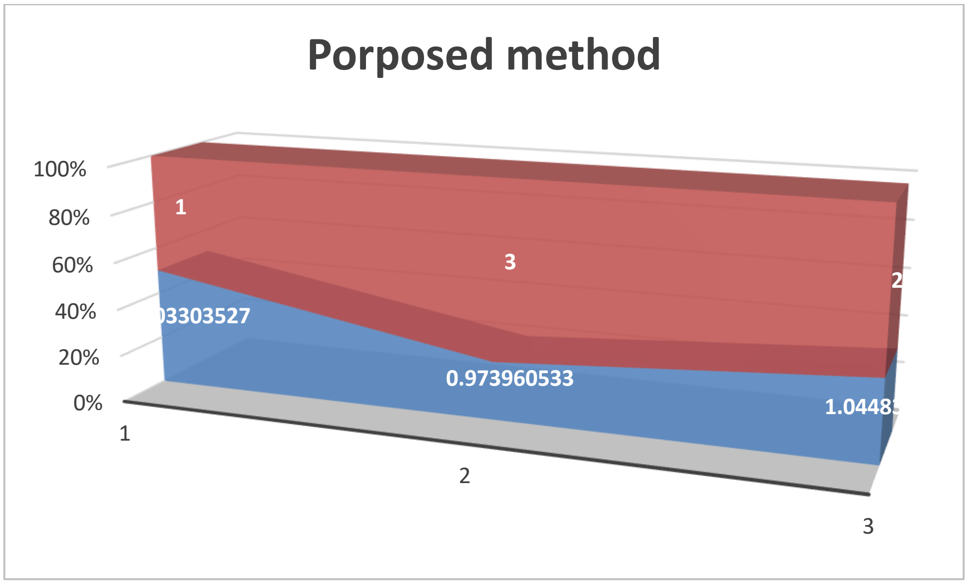

Fig 2 Proposed methods

5. Advantages and Limitation on various sets.

The table 10 below illustrates how different types of sets can manage different conditions or significant scenarios in relation to practical problems, as well as how they can't.

Table 10 limitation and drawback on various sets

Various Set Types	**Advantages**	**Limitations**
Crisp sets	can make an accurate determination without hesitating	unable to fully express the ambiguous information
Fuzzy sets	can explain the ambiguous information	Uncertain Information cannot be described with a non-membership degree.
Interval valued fuzzy sets	able to cope with interval data rather than exact data	Uncertain Information cannot be handled at the non-membership level.
Intuitionistic fuzzy sets	can simultaneously represent the uncertain information using degrees of membership (MS) and non-membership (NMS)	Cannot describe the sum of more than one MS and NMS degree.

Interval valued Intuitionistic fuzzy sets	ability to work with interval data	Cannot depict the sum of the MS and NMS degrees as greater than 1
Vague sets	can simultaneously explain ambiguous information with MS and NMS grades.	Cannot describe a degree sum of more than one in MS and NMS.
Pythagorean fuzzy sets	It provides sufficient room to discuss the total of MS and NMS degrees that is larger than 1.	cannot describe anything greater than the square of the MS and NMS degrees.
Interval valued Pythagorean fuzzy sets	dealing with interval data	undefined square sum of MS and NMS degrees higher than 1
Neutrosophic Sets	able to deal with data uncertainty and thoroughly acquire the optimum solution.	incapable of handling interval data
Interval valued Neutrosophic sets	able to handle the interval data's indeterminacy and produce the optimal solution.	unable to handle weight information that is incomplete

6. Conclusion

In this paper, the score function is used to evaluate the concept of a single valued Neutrosophic set used with the best 3D printers. Using the use of SVNS, a potential application has been addressed. This will not only be helpful on its own, but will also assist motivated researchers in resolving other uncertainty-related problems through comparative techniques. Based on actual decision-making challenges, the following paper illustrates a novel method for solving Neutrosophic fuzzy sets with the contraction value. This process has proven to be quite practical in many real-world situations where goal-oriented decision-making is required. In this paper, we model the problem of choosing the best 3D printer using the score and accuracy function, hybrid score-accuracy function of SVNNs, and linguistic variables in a single-valued Neutrosophic environment, where the weight of the decision makers is completely unknown and the weights of criteria are only partially known. From the analysis the proposed method is best for decision making. Future the work is extended to Plithogenic sets.

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Single-valued pentapartitioned neutrosophic weighted hyperbolic tangent similarity measure to determine the most significant environmental risks during the COVID-19 pandemic

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Abstract:

This paper investigates the concept of Single-Valued Pentapartitioned Neutrosophic Hyperbolic Tangent Similarity Measure (SVPNHTSM) and Single-Valued Pentapartitioned Neutrosophic Weighted Hyperbolic Tangent Similarity Measure (SVPNWHTSM) under the Single-Valued Pentapartitioned Neutrosophic Set (SVPNS) environment. SVPNHTSM and SVPNHTCSM also produce some interesting results relating to similarities between the two SVPNSs. In an SVPNS environment, SVPNHTSM is also used to bring the development of the Multi-Attribute Decision-Making (MADM) strategy. To determine the most affected factor of the environment affected by COVID-19, the novel SVPNHTSM is used. Results obtained from this study show that the incremental rate of water pollution is the major effect of COVID-19 on our environment. Finally, validation of the obtained results is done using comparative studies and sensitivity analysis.

Keywords: Fuzzy set, neutrosophic set, single-valued neutrosophic set, single-valued pentapartitioned neutrosophic set, multi-attribute decision-making, hyperbolic tangent similarity measure, COVID-19

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1. Introduction

COVID-19 is a respiratory illness caused by the novel Coronavirus SARS-CoV-2. It was first identified in December 2019 in Wuhan, China, and has since spread globally, leading to the ongoing COVID-19 pandemic [24]. The disease has affected millions of people and caused significant morbidity and mortality worldwide. Here, we present a literature review on COVID-19, focusing on the epidemiology, clinical features, and management of the disease.COVID-19 has affected people of all ages and backgrounds, but certain populations have been at higher risk of severe disease and death. Older adults, those with underlying health conditions such as diabetes, cardiovascular disease, and respiratory disease, and those with weakened immune systems are more likely to experience severe illness and complications. The virus is primarily transmitted through respiratory droplets when an infected person coughs, sneezes, or talks. It can also be transmitted by touching contaminated surfaces and then touching one's mouth, nose, or eyes. The incubation period for the virus is typically between 2 and 14 days, with an average of 5 days [44, 45].

The clinical features of COVID-19 vary widely, with some people experiencing mild symptoms or no symptoms at all, while others develop severe respiratory illness and other complications. The most common symptoms of COVID-19 include fever, cough, and shortness of breath. Other symptoms can include fatigue, muscle or body aches, headache, loss of taste or smell, sore throat, congestion, and diarrhea. Severe cases can lead to pneumonia, acute respiratory distress syndrome (ARDS), and multiple organ failure [44]. Management of COVID-19 depends on the severity of the illness and the individual patient's risk factors. Mild cases may not require hospitalization and can be managed with supportive care, such as rest, hydration, and fever-relieving medications. More severe cases may require hospitalization, oxygen therapy, and other supportive treatments. In some cases, antiviral medications such as Remdesivir and monoclonal antibodies may be used to reduce the severity of the illness. Vaccination is also a valuable tool in the management of COVID-19, as it can prevent infection and reduce the severity of illness in those who do become infected [16]. COVID-19 continues to be a significant public health threat, with ongoing research aimed at improving our understanding of the disease and developing more effective treatments and preventive measures. Early detection, isolation, and contact tracing remain significant strategies for controlling the spread of the virus, along with vaccination and public health measures such as social distancing, mask-wearing, and hand hygiene [2].

Multi-criteria decision analysis (MCDA) refers to a decision-making technique that involves evaluating and comparing alternatives based on multiple criteria or factors. This approach has been widely used in the context of COVID-19 to help decision-makers make informed choices regarding various aspects of the pandemic. Here are some examples of how MCDA has been applied in relation to COVID-19:One of the most pressing issues related to COVID-19 is the prioritization of vaccines, given the limited supply. MCDA can help decision-makers weigh various factors, such as the risk of severe illness or death, the risk of transmission, and the potential impact on essential workers or vulnerable populations, to determine which groups should receive priority access to the vaccine. For example, the World Health Organization (WHO) used an MCDA approach to develop its framework for vaccine allocation and prioritization, which took into account criteria such as the epidemiology of the disease, the impact on health systems, and ethical and social considerations [45].

Another significant decision related to COVID-19 is selecting which containment measures to implement in order to slow the spread of the virus. MCDA can be used to evaluate the effectiveness of different interventions, such as social distancing, mask mandates, or travel restrictions, based on various criteria, such as their impact on public health, the economy, and social well-being.

MCDA can also be used to assess the risk levels associated with different activities or settings, such as schools, workplaces, or public gatherings. By considering factors such as the number of people involved, the duration of the activity, the degree of ventilation, and the prevalence of the virus in the community, decision-makers can determine which activities pose the greatest risk and which measures should be implemented to mitigate that risk. For example, researchers in the United Kingdom used MCDA to evaluate the risk of COVID-19 transmission in different sports and physical activities [6]. Overall, MCDA is an effective tool for decision-makers who must weigh multiple criteria and factors when making decisions related to COVID-19. By considering a range of factors and evaluating alternative options, decision-makers can make more informed choices and prioritize interventions that are most likely to have a positive impact on public health and well-being.

Alamoodi et al. [1] presented a systematic review of Multi-Criteria Decision Making (MCDM) strategies employed in medical case studies of COVID-19. As research progresses, researchers are very interested in developing new MCDM/MCGDM techniques that use several types of sets and operators [14, 18-23]in various uncertain environments.

Mallick and Pramanik [26] developed the Pentapartitioned Neutrosophic Set (PNS) [26] in 2020 using the Neutrosophic Set (SVNS) [43] and multi-valued neutrosophic logic [41] to cope with uncertainty comprehensively by decomposing the indeterminacy Membership Function (MF) into three independent ingredients, namely, contradiction MF, ignorance MF, as well as unknown MF. Pramanik [36] developed interval PNS. Details studies of SVNSs and their applications and extensions can be found in the studies [3-4, 18-23, 30-35, 37, 42]. In 2021, Das et al. [7] rendered the Q-ideals of Q-algebra in PNS settings. Das and Tripathy [12] discussed topological space in PNS environments. Das et al. [9] presented probability distributions in PNS settings. So, PNSs are getting more attention in conducting research.

Das et al. [11] extended the tangent Similarity Measure (SM)[27, 28, 38] to the SVPNS environment. Das et al. [10] developed the MADM strategy under the SVPNS environment using Grey Relational Analysis (GRA). Cosine SM-based MCDM strategy [25] was presented for identifying the environmental risk factor due to COVID-19 under the SVPNS environment. Saha et al. [39] introduced the Dice SM-based MADM strategy under the SVPNS setting. Das et al. [8] developed the single-valued bipolar PNS and presented its application to the MADM problem.

Research gap:

- Single Valued Pentapartitioned Neutrosophic Hyperbolic Tangent SM (SVPNHTSM) has not been reported in the literature.
- There is no literature on an MADM strategy based on SVPNHTSM.

Motivation:

To fill the research chasm, we initiate to examine SVPNHTSM and Single Valued Pentapartitioned Neutrosophic Weighted Hyperbolic Tangent Similarity Measure (SVPNWHTSM) and present a few theorems and propositions on SVPNHTSM and SVPNWHTSM in the SVPNS environments. An innovative MADM strategy that is based on SVPNHTM under the SVPNS environment is developed in this paper.

Contributions of the paper are as follows:

1. This paper establishes the properties of the SVPNWHTSM and SVPNWHTSM.
2. This paper develops a novel MADM strategy using the proposed SVPNWHTSM to determine the most significant risk factors in the environment. Based on the proposed strategy, a range of alternatives are created and the ranking order is determined.
3. The novel MADM method's findings are compared to those of other existing strategies. The proposed strategy reveals that under the SVPNS environment, COVID-19 negatively impacts Water Pollution more than other alternatives.

The structure of the remaining paper is shown in Table 1.

Table 1. Structure of the paper

Name of the section	Content
Section 2	recalls some definitions of relevant terms.
Section 3	presents SVPNHTSM and SVPNWHTSM and some of their basic properties.
Section 4	develops SVPNHTSM based MADM strategy under SVPNS environment.
Section 5	presents of application of the developed MADM Strategy for selecting the poignant environmental risk factor at the time of the coronavirus.
Section 6	presents a comparative study.
Section 7	describes the sensitivity analysis.
Section 8	**Presents the advantage and disadvantage of the study**
Section 9	Presents the conclusions of the paper.

2. A list of relevant terms with definition

An overview of some of the results and definitions is presented here.

Let V be the sphere of discourse. An SVPNS [26] is presented as follows:

$$G' = \{(\kappa, a_{G'}(\kappa), b_{G'}(\kappa), c_{G'}(\kappa), d_{G'}(\kappa), e_{G'}(\kappa)) : \kappa \in \Omega\}.$$

Here, $a_{G'}(\kappa), b_{G'}(\kappa), c_{G'}(\kappa), d_{G'}(\kappa)$, and $e_{G'}(\kappa)$ are the truth, contradiction, ignorance, unknown and false MFs such that $a_{G'}(\kappa)$, $b_{G'}(\kappa)$, $c_{G'}(\kappa)$, $d_{G'}(\kappa)$, and $e_{G'}(\kappa) \in [0,1]$, for each $\kappa \in \Omega$. So, $0 \le a_{G'}(\kappa) + b_{G'}(\kappa) + c_{G'}(\kappa) + d_{G'}(\kappa) + e_{G'}(\kappa) \le 1$, for each $\kappa \in \Omega$.

Norms for the null SVPNS (0_{PNN}) and the absolute SVPNS (1_{PNN}) [26] for a fixed set Ω are presented as follows:

(a) $1_{PNN} = \{(\kappa, 1, 1, 0, 0, 0) : \kappa \in \Omega\}$,

(b) $0_{PNN} = \{(\kappa, 0, 0, 1, 1, 1) : \kappa \in \Omega\}$.

Let $H' = \{(\kappa, a_{H'}(\kappa), b_{H'}(\kappa), c_{H'}(\kappa), d_{H'}(\kappa), e_{H'}(\kappa)) : \kappa \in \Omega\}$ and

$G' = \{(\kappa, a_{G'}(\kappa), b_{G'}(\kappa), c_{G'}(\kappa), d_{G'}(\kappa), e_{G'}(\kappa)) : \kappa \in \Omega\}$ be any two SVPNSs [26] over *V*. Then,

(a) $H' \subseteq G'$ if and only if $a_{H'}(\kappa) \le a_{G'}(\kappa)$, $b_{H'}(\kappa) \le b_{G'}(\kappa)$, $c_{H'}(\kappa) \ge c_{G'}(\kappa)$, $d_{H'}(\kappa) \ge d_{G'}(\kappa)$,

$e_{H'}(\kappa) \ge e_{G'}(\kappa)$, for all $\kappa \in \Omega$.

(b) $G'^{c} = \{(\kappa, e_{G'}(\kappa), d_{G'}(\kappa), 1 - c_{G'}(\kappa), b_{G'}(\kappa), a_{G'}(\kappa)) : \kappa \in \Omega\}$;

(c) $H' \cup G'$

$= \{(\kappa, \max\{a_{H'}(\kappa), a_{G'}(\kappa)\}, \max\{b_{H'}(\kappa), b_{G'}(\kappa)\}, \min\{c_{H'}(\kappa), c_{G'}(\kappa)\}, \min\{d_{H'}(\kappa), d_{G'}(\kappa)\}, \min\{e_{H'}(\kappa), e_{G'}(\kappa)\}) : \kappa \in \Omega\}$

(d)
$H' \cap G'$
$= \{(\kappa, \min\{a_{H'}(\kappa), a_{G'}(\kappa)\}, \min\{b_{H'}(\kappa), b_{G'}(\kappa)\}, \max\{c_{H'}(\kappa), c_{G'}(\kappa)\}, \max\{d_{H'}(\kappa), d_{G'}(\kappa)\}, \max\{e_{H'}(\kappa), e_{G'}(\kappa)\}) : \kappa \in \Omega\}$

Consider H' = {(r′, 0.21, 0.37, 0.67, 0.14, 0.34), (s′, 0.41, 0.25, 0.48, 0.61, 0.11)} and G' = {(r′, 0.31, 0.48, 0.71, 0.24, 0.44), (s′, 0.49, 0.36, 0.50, 0.72, 0.25)} be two SVPNSs over a set of discourses V = {r′, s′}.
Then,
(*i*) $H' \subseteq G'$;
(*ii*) H'^{C} = {(*r*′, 0.79, 0.63, 0.33, 0.0.86, 0.66), (*s*′, 0.59, 0.75, 0.52, 0.39, 0.89)} and G'^{C} = {(*r*′, 0.69, 0.52, 0.29, 0.76, 0.56), (*s*′, 0.51, 0.64, 0.50, 0.28, .0.75)};
(*iii*) $H' \cup G'$ = {(*r*′, 0.31, 0.48, 0.71, 0.24, 0.44), (*s*′, 0.49, 0.36, 0.50, 0.72, 0.25)};
(*iv*) $H' \cap G'$ = {(*r*′, 0.21, 0.37, 0.67, 0.14, 0.34), (*s*′, 0.41, 0.25, 0.48, 0.61, 0.11)}.

3. SVPNHTSM and SVPNWHTSM and some of their basic properties

SVPNHTSM and SVPNWHTSM are here presented. Various interesting consequences have been drawn up under the SVPNS environment.

Definition 3.1 Suppose $H' = \{(\kappa, a_{H'}(\kappa), b_{H'}(\kappa), c_{H'}(\kappa), d_{H'}(\kappa), e_{H'}(\kappa)) : \kappa \in \Omega\}$ and

$G' = \{(\kappa, a_{G'}(\kappa), b_{G'}(\kappa), c_{G'}(\kappa), d_{G'}(\kappa), e_{G'}(\kappa)) : \kappa \in \Omega\}$ are two SVPNSs within the set *V*. Now, the SVPNHTSM between H' and G' is defined as:

$$\mathrm{P}_{\text{SVPNHTSM}}(H',G') =$$

$$\frac{1}{n}\sum_{\kappa\in\Omega}\tanh\left[\left|a_{H'}(\kappa)-a_{G'}(\kappa)\right|+\left|b_{H'}(\kappa)-b_{G'}(\kappa)\right|+\left|c_{H'}(\kappa)-c_{G'}(\kappa)\right|+\left|d_{H'}(\kappa)-d_{G'}(\kappa)\right|+\left|e_{H'}(\kappa)-e_{G'}(\kappa)\right|\right] \quad (1)$$

Theorem 3.2 The following properties hold, if $\mathrm{P}_{\text{SVPNHTSM}}(H',G')$ is the SVPNHTSM between the SVPNSs H' and G':

(a) $0 \le \mathrm{P}_{\text{SVPNHTSM}}(H',G') \le 1;$

(b) $\mathrm{P}_{\text{SVPNHTSM}}(H',G') = \mathrm{P}_{\text{SVPNHTSM}}(G',H');$

(c) $H' = G' \Leftrightarrow \mathrm{P}_{\text{SVPNHTSM}}(H',G') = 0.$

Proof:(a) Since the hyperbolic tangent function is monotonic increasing function in the number line, therefore, it also belongs to the interval $[-1,1]$. Hence, $0 \le \mathrm{P}_{\text{SVPNHTSM}}(H',G') \le 1.$

(b) $\mathrm{P}_{\text{SVPNHTSM}}(H',G')$

$$=\frac{1}{n}\sum_{\kappa\in\Omega}\tanh\begin{bmatrix}\left|a_{H'}(\kappa)-a_{G'}(\kappa)\right|+\left|b_{H'}(\kappa)-b_{G'}(\kappa)\right|+\left|c_{H'}(\kappa)-c_{G'}(\kappa)\right|+\\ \left|d_{H'}(\kappa)-d_{G'}(\kappa)\right|+\left|e_{H'}(\kappa)-e_{G'}(\kappa)\right|\end{bmatrix}$$

$$=\frac{1}{n}\sum_{\kappa\in\Omega}\tanh\begin{bmatrix}\left|a_{G'}(\kappa)-a_{H'}(\kappa)\right|+\left|b_{G'}(\kappa)-b_{H'}(\kappa)\right|+\left|c_{G'}(\kappa)-c_{H'}(\kappa)\right|+\\ \left|d_{G'}(\kappa)-d_{H'}(\kappa)\right|+\left|e_{G'}(\kappa)-e_{H'}(\kappa)\right|\end{bmatrix}=\mathrm{P}_{\text{SVPNHTSM}}(G',H')$$

Therefore, $\mathrm{P}_{\text{SVPNHTSM}}(H',G') = \mathrm{P}_{\text{SVPNHTSM}}(G',H')$

(c) Assume that H' and G' are any two SVPNSs over Ω such that $H' = G'.$
Since, $H' = G'$

$$\Rightarrow a_{H'}(\kappa)=a_{G'}(\kappa), b_{H'}(\kappa)=b_{G'}(\kappa), c_{H'}(\kappa)=c_{G'}(\kappa), d_{H'}(\kappa)=d_{G'}(\kappa), e_{H'}(\kappa)=e_{G'}(\kappa), \text{for each } \kappa\in\Omega.$$

$$\Rightarrow \left|a_{H'}(\kappa)-a_{G'}(\kappa)\right|=0, \left|b_{H'}(\kappa)-b_{G'}(\kappa)\right|=0, \left|c_{H'}(\kappa)-c_{G'}(\kappa)\right|=0, \left|d_{H'}(\kappa)-d_{G'}(\kappa)\right|=0,$$

$$\left|e_{H'}(\kappa)-e_{G'}(\kappa)\right|=0, \text{for each } \kappa\in\Omega.$$

Hence $\mathrm{P}_{\text{SVPNHTSM}}(H',G') = \frac{1}{n}\sum_{\kappa\in\Omega}\tanh(0) = 0.$

Conversely, suppose that $\mathrm{P}_{\text{SVPNHTSM}}(H',G') = 0.$

$$\Rightarrow \left|a_{H'}(\kappa)-a_{G'}(\kappa)\right|=0, \left|b_{H'}(\kappa)-b_{G'}(\kappa)\right|=0, \left|c_{H'}(\kappa)-c_{G'}(\kappa)\right|=0, \left|d_{H'}(\kappa)-d_{G'}(\kappa)\right|=0,$$

$$\left|e_{H'}(\kappa)-e_{G'}(\kappa)\right|=0, \text{ for each } \kappa\in\Omega..$$

$a_{H'}(\kappa)=a_{G'}(\kappa), b_{H'}(\kappa)=b_{G'}(\kappa), c_{H'}(\kappa)=c_{G'}(\kappa), d_{H'}(\kappa)=d_{G'}(\kappa)$ and $\chi_{H'}(\kappa)=e_{G'}(\kappa)$,

Hence $H'=G'$.

Theorem 3.3 If H', G' and Z' are any three SVPNSs over a fixed set V such as $H'\subseteq G'\subseteq Z'$, then

$$\mathrm{P}_{\mathrm{SVPNHTSM}}(H',G')\le \mathrm{P}_{\mathrm{SVPNHTSM}}(H',Z') \text{ and } \mathrm{P}_{\mathrm{SVPNHTSM}}(G',Z')\le \mathrm{P}_{\mathrm{SVPNHTSM}}(H',Z').$$

Proof. Let H', G' and Z' be any three SVPNSs over a fixed set V such as v $H'\subseteq G'\subseteq Z'$, .So,

$$a_{H'}(\kappa)\le a_{G'}(\kappa),\; b_{H'}(\kappa)\le b_{G'}(\kappa),\; c_{H'}(\kappa)\ge c_{G'}(\kappa),\; d_{H'}(\kappa)\ge d_{G'}(\kappa),\; e_{H'}(\kappa)\ge e_{G'}(\kappa),$$

$$a_{G'}(\kappa)\le a_{Z'}(\kappa),\quad b_{G'}(\kappa)\le b_{Z'}(\kappa),\quad c_{G'}(\kappa)\ge c_{Z'}(\kappa),\quad d_{G'}(\kappa)\ge d_{Z'}(\kappa),\quad e_{G'}(\kappa)\ge e_{Z'}(\kappa),$$

for each $\kappa\in\Omega$..

Therefore $|a_{H'}(\kappa)-a_{G'}(\kappa)|\le|a_{H'}(\kappa)-a_{Z'}(\kappa)|, |b_{H'}(\kappa)-b_{G'}(\kappa)|\le|b_{H'}(\kappa)-b_{Z'}(\kappa)|,$

$$|c_{H'}(\kappa)-c_{G'}(\kappa)|\le|c_{H'}(\kappa)-c_{Z'}(\kappa)|, |d_{H'}(\kappa)-d_{G'}(\kappa)|\le|d_{H'}(\kappa)-d_{Z'}(\kappa)|,$$
$$|e_{H'}(\kappa)-e_{G'}(\kappa)|\le|e_{H'}(\kappa)-e_{Z'}(\kappa)| \text{ for each } \kappa\in\Omega.$$

Therefore

$$\begin{aligned}
&\mathrm{P}_{\mathrm{SVPNHTSM}}(H',G')\\
&=\frac{1}{n}\sum\nolimits_{\kappa\in\Omega}\tanh\left[|a_{H'}(\kappa)-a_{G'}(\kappa)|+|b_{H'}(\kappa)-b_{G'}(\kappa)|+|c_{H'}(\kappa)-c_{G'}(\kappa)|+|d_{H'}(\kappa)-d_{G'}(\kappa)|+|e_{H'}(\kappa)-e_{G'}(\kappa)|\right]\\
&\le\frac{1}{n}b\sum\nolimits_{\kappa\in\Omega}\tanh\left[|a_{H'}(\kappa)-a_{Z'}(\kappa)|+|b_{H'}(\kappa)-b_{Z'}(\kappa)|+|c_{H'}(\kappa)-c_{Z'}(\kappa)|+d\,|\Phi_{H'}(\kappa)-d_{Z'}(\kappa)|+|e_{H'}(\kappa)-e_{G'}(\kappa)|\right]\\
&=\mathrm{P}_{\mathrm{SVPNHTSM}}(H',Z')
\end{aligned}$$

Thus $\mathrm{P}_{\mathrm{SVPNHTSM}}(H',G')\le\mathrm{P}_{\mathrm{SVPNHTSM}}(H',Z')$.

Moreover,

$$|a_{G'}(\kappa)-a_{Z'}(\kappa)|\le|a_{H'}(\kappa)-a_{Z'}(\kappa)|, |b_{G'}(\kappa)-b_{Z'}(\kappa)|\le|b_{H'}(\kappa)-b_{Z'}(\kappa)|,$$

$$|c_{G'}(\kappa)-c_{Z'}(\kappa)|\le|c_{H'}(\kappa)-c_{Z'}(\kappa)|, |d_{G'}(\kappa)-d_{Z'}(\kappa)|\le|d_{H'}(\kappa)-d_{Z'}(\kappa)|,$$

$$|e_{G'}(\kappa)-e_{Z'}(\kappa)|\le|e_{H'}(\kappa)-e_{Z'}(\kappa)| \text{ for all } \kappa\in\Omega..$$

Therefore,

$$\mathrm{P}_{\mathrm{SVPNHTSM}}(G', Z')$$
$$= \frac{1}{n}\sum_{\kappa\in\Omega}\tanh\begin{bmatrix}|a_{G'}(\kappa)-a_{Z'}(\kappa)|+|b_{G'}(\kappa)-b_{Z'}(\kappa)|+|c_{G'}(\kappa)-c_{Z'}(\kappa)|+|d_{G'}(\kappa)-d_{Z'}(\kappa)|\\+|e_{G'}(\kappa)-e_{Z'}(\kappa)|\end{bmatrix}$$
$$\le \frac{1}{n}\sum_{b\in U}\tanh\begin{bmatrix}|a_{H'}(\kappa)-a_{Z'}(\kappa)|+|b_{H'}(\kappa)-b_{Z'}(\kappa)|+|c_{H'}(\kappa)-c_{Z'}(\kappa)|+|d_{H'}(\kappa)-d_{Z'}(\kappa)|\\+|e_{H'}(\kappa)-e_{Z'}(\kappa)|\end{bmatrix}$$
$$= \mathrm{P}_{\mathrm{SVPNHTSM}}(H', Z')$$

Hence $\mathrm{P}_{\mathrm{SVPNHTSM}}(G', Z') \le \mathrm{P}_{\mathrm{SVPNHTSM}}(H', Z')$.

Definition 3.4

Consider two SVPNSs $H' = \{(\kappa, \xi_{H'}(\kappa), \zeta_{H'}(\kappa), \vartheta_{H'}(\kappa), \Phi_{H'}(\kappa), \chi_{H'}(\kappa)) : \kappa \in \Omega\}$ and

$\omega'' = \{(\kappa, \xi_{\omega''}(\kappa), \zeta_{\omega''}(\kappa), \vartheta_{\omega''}(\kappa), \Phi_{\omega''}(\kappa), \chi_{\omega''}(\kappa)) : \kappa \in \Omega\}$ within a universe of discourse V, the SVPNWHTSM between H' and ω'' is defined by:

$$\mathrm{P}_{\mathrm{SVPNWHTSM}}(H', \omega'') =$$

$$\frac{1}{n}\sum_{\kappa\in\Omega}\omega''_f\tanh\begin{bmatrix}|\xi_{H'}(\kappa)-\xi_{\omega''}(\kappa)|+|\zeta_{H'}(\kappa)-\zeta_{\omega''}(\kappa)|+|\vartheta_{H'}(\kappa)-\vartheta_{\omega''}(\kappa)|+|\Phi_{H'}(\kappa)-\Phi_{\omega''}(\kappa)|\\+|\chi_{H'}(\kappa)-\chi_{\omega''}(\kappa)|\end{bmatrix} \quad (2)$$

where $\sum_{\kappa\in\Omega}\omega''_e = 1.$

The following sub sequent effects are derived in view of the above theorem:

Proposition 3.5 Assume that $\mathrm{P}_{\mathrm{SVPNWHTSM}}(H', \omega'')$ is the SVPNWHTSM of similarities between the SVPNSs H' and ω''.Then,

(a) $0 \le \mathrm{P}_{\mathrm{SVPNWHTSM}}(H', \omega'') \le 1;$

(b) $\mathrm{P}_{\mathrm{SVPNWHTSM}}(H', \omega'') = \mathrm{P}_{\mathrm{SVPNWHTSM}}(\omega'', H');$

(c) $H' = \omega'' \Leftrightarrow \mathrm{P}_{\mathrm{SVPNWHTSM}}(H', \omega'') = 0.$

Proposition 3.6 If H', G' and Z' over the hippodrome of discourse V so $H' \subseteq G' \subseteq \omega''$,

$\mathrm{P}_{\mathrm{SVPNWHTSM}}(H', G') \le \mathrm{P}_{\mathrm{SVPNWHTSM}}(R', \omega'')$ and $\mathrm{P}_{\mathrm{SVPNWHTSM}}(G', \omega'') \le \mathrm{P}_{\mathrm{SVPNWHTSM}}(H', \omega'').$

4. MADM Strategy Based on SVPNHTSM in an SVPNS Environment

This section focuses on creating the MADM approach through the employment of the SVPNHTSMs in SVPNS situations. Consider an MADM problem in which $V = \{V'_1, V'_2, ..., V'_p\}$ and

$B' = \{B'_1, B'_2, ..., B'_p\}$ represent the collection of feasible alternatives and attributes. In terms of Pentapartitioned neutrosophic numbers, the Decision-Maker (DM) provides all estimation details for all alternatives. Then, construct a decision matrix by applying the decision maker's entire evaluation details. In the next section, a new MADM strategy (see figure 1) is developed.

Phase 1: The decision matrix's construction

The estimation details are combined to create the decision matrix. $P_{V_i} = \{(B'_j, \xi_{ij}(V'_i, B'_j), \zeta_{ij}(V'_i, B'_j), \vartheta_{ij}(V'_i, B'_j), \Phi_{ij}(V'_i, B'_j), \chi_{ij}(V'_i, B'_j) : B'_j \in B'\}$ of the DM for each alternative $V'_i (i = 1(1)p)$ based on the attribute $B'_j (j = 1(1)q)$, where $\left(\xi_{ij}(V'_i, B'_j), \zeta_{ij}(V'_i, B'_j), \vartheta_{ij}(V'_i, B'_j), \Phi_{ij}(V'_i, B'_j), \chi_{ij}(V'_i, B'_j)\right) = \left(V'_i, B'_j\right)$ (say) $(i = 1(1)p$ and $j = 1(1)q)$ indicates the metrics used to evaluate alternative $V'_i (i = 1(1)p)$ with respect to attribute $B'_j (j = 1(1)q)$.

Decision matrix is delineated below:

DMA	B'_1	B'_2	$\cdots$	B'_q
V'_1	(V'_1, B'_1)	(V'_1, B'_2)	$\cdots$	(V'_1, B'_q)
V'_2	(V'_2, B'_1)	(V'_2, B'_2)	$\cdots$	(V'_2, B'_q)
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
V'_p	(V'_p, B'_1)	(V'_p, B'_2)	$\cdots$	(V'_p, B'_q)

Phase -2: Determining attribute weights

Verifying the weights for each of the attributes is an important part of the MADM strategy. It is possible for DM to use compromise functions to compute the weights for each characteristic when the details of the weights are unknown.

The compromise function of Γ''_j for each V'_i is interpreted as follows:

$$\Gamma''_j = \sum_{i=1}^{p} \left(3 + \xi_{ij}(V'_i, B'_j) + \zeta_{ij}(V'_i, B'_j) - \vartheta_{ij}(V'_i, B'_j) - \Phi_{ij}(V'_i, B'_j) - \chi_{ij}(V'_i, B'_j)\right) / 5 \quad (3)$$

Then the weight of the j-the characteristic is obtained by $\omega_j'' = \dfrac{\Gamma_j''}{\sum_{j=1}^{q}\Gamma_j''}$ (4)

Here, $\sum_{j=1}^{q}\omega_j'' = 1.$

Phase -3: Evaluation of a Positive Ideal Alternative (PIA)

This step involves constructing the PIA for all attributes by using the maximum operator.
In the following, PIA is represented by the letter *I* and is defined as:

$$I = \left(\xi_1'', \xi_2'', ..., \xi_q''\right), \quad (5)$$

where $\xi_j'' = \left(\max\{\xi_{ij}(V'_i, B'_j) : i = 1(1)p\}, \max\{\zeta_{ij}(V'_i, B'_j) : i = 1(1)p\}, \min\{\vartheta_{ij}(V'_i, B'_j) : i = 1(1)p\},\right.$ (6)

Phase -4: Compute the Accumulated Measure Value (AMV)

Let SVPNHTSM for each of the alternatives be aggregated using the AMV. Here, $P_{AMV}(V_i')$ denotes AMV and $P_{AMV}(V_i')$ is defined by

$$P_{AMF}(V_i') = \sum_{j=1}^{q}\omega_j''.P_{SVPNHTSM}\left((V_i', B_j'), \xi_j''\right), \quad (7)$$

where $\left(V_i', B_j'\right) = \left(\xi_{ij}(V_i', B_j'), \zeta_{ij}(V_i', B_j'), \vartheta_{ij}(V_i', B_j'), \Phi_{ij}(V_i', B_j'), \chi_{ij}(V_i', B_j')\right).$

Phase -5: Analyze the alternatives and rank them

Using a descending order of AMVs, the ranking order is determined. The highest value of AMV corresponds the best option.

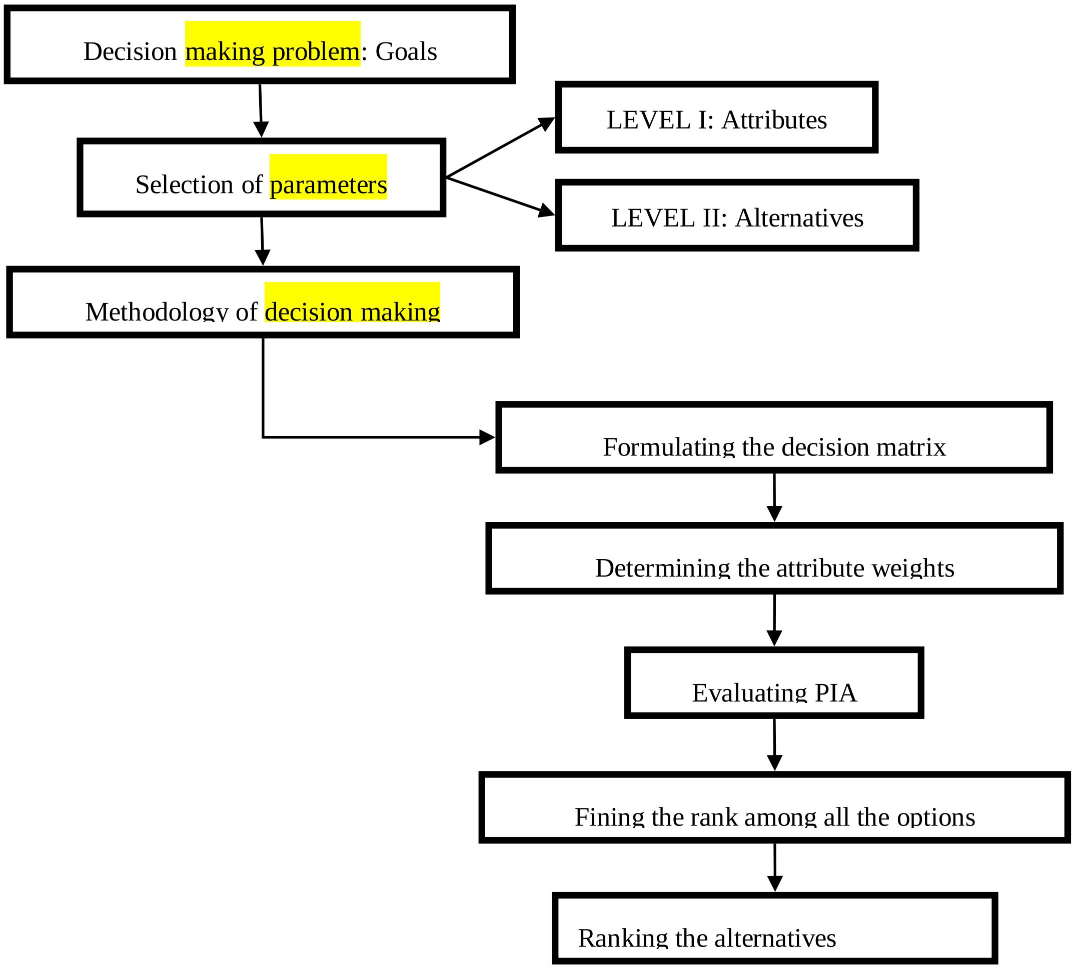

Figure 1. Flowchart of the proposed MADM strategy

5. MADM Strategy Implementation for Identifying the Most Serious Environmental Risk Factors During the COVID-19 Pandemic

The whole human race is in danger of extinction in the present scenario of the COVID-19 pandemic. As more than 6,34,000 people have died till now because of the virus thus it is very much clear that the virus will have a great effect on our lifestyle as well as the biological condition of mother earth [5].All the technological and scientific developments are proved meaningless in front of the SARS-COV2. The virus has affected every country (i.e., 213) present on earth in a drastic manner.
Most of the countries have taken massive screening measures and establishing public policies to fight the pandemic e.g., China has strictly taken the policy of self-quarantine, Britain has taken the method of herd immunity, India has taken the method of massive lockdown, etc. But still, the policies are not enough to meet the challenges presented by the virus. In the present scenario, the whole world is stuck in such a situation where economic and technical growth is too much affected.
No doubt the virus has affected our environment in a very good manner as the CO_2 and NO_2 emission has been drastically decreased due to the less usage of vehicles and as a result the temperature of earth has also decreased. Due to the halt of industries the air pollution as well as the noise pollution also came under control[5].
But still, there are also some bad impacts of the virus are there on the environment especially on the soil, water, and air sectors. e.g., the number of medical wastes coming from the hospitals has increased by at least 5 times which is quite difficult to recycle. However, a crucial topic of concern remains to be the proper waste management & recycling as recycling is considered to be an

efficacious way to obviate pollution & minimize energy wastage, using natural resources sustainably. Considering the present situation USA have put a halt in some recycling centers for minimizing the risk of escalation of the corona virus there.
Moreover, the production of organic and inorganic wastes effects the environment in a very wide manner e.g., deforestation, soil erosion, air as well as water pollution are frequently spotted. Additional to that due to the quarantine process the usage of inorganic plastic material has increased [40] .
Thus, the major goal of the task is to determine the most important option that has the least impact on the environmental criteria using the MADM technique. The alternatives are wisely selected based on various disaster management department's reports & are again established by the experts. So, generation of inorganic waste (β_1),organic waste (β_2) and medical waste (β_3) are considered as attributes in the present study. Since all the attributes have impact on deforestation (ν_1),water pollution (ν_2), air pollution (ν_3), and soil erosion (ν_4), so in the present study these factors are considered as the feasible alternatives. The Figure 2 shows the decision hierarchy for the present problem.

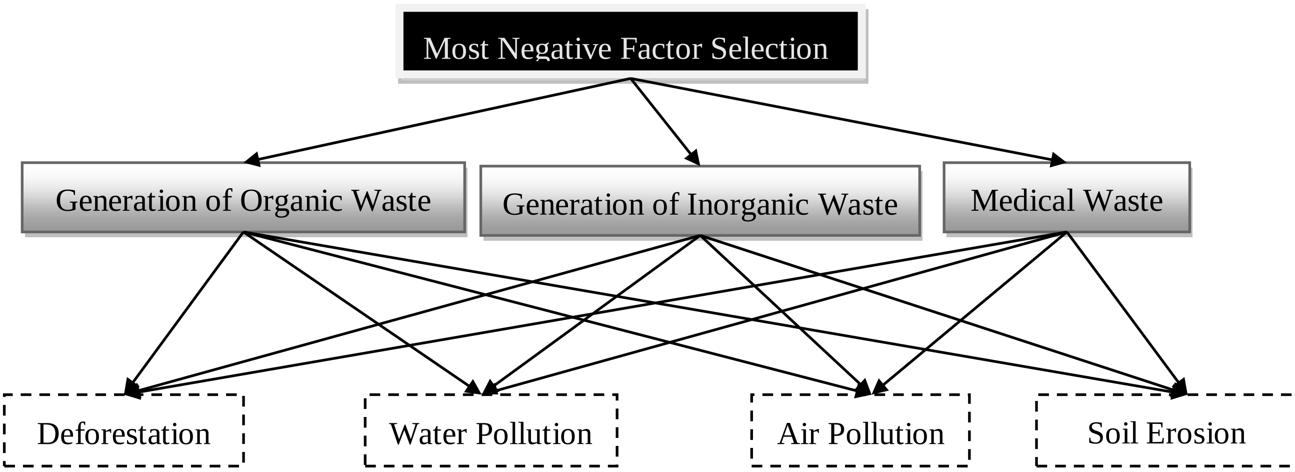

Figure 2. Hierarchical structure of the considered problem

Prepare the decision matrix in Table-2 using data pertaining to all possibilities offered by the DM. The PIA (***I***) for the decision matrix in Table-3 can be calculated using equation (3).

Table 2. Decision Matrix

	A^{β_1}	A^{β_2}	A^{β_3}
ν_1	(.0.75, .0.5, .0.3, 0.2, 0.6)	(0.9, 0.7, 0.3, 0.1, 0.4)	(0.75, 0.54, 0.23, 0.4, 0.13)
ν_2	(0.9, 0.8, 0.3, 0.2, 0.3)	(0.9, 0.4, 0.45, 0.2, 0.3)	(0.65, 0.45, 0.28, 0.3, 0.23)
ν_3	(0.8, 0.7, 0.3, 0.3, 0.4)	(0.8, 0.5, 0.3, 0.1, 0.2)	(0.86, 0.54, 0.4, 0.23, 0.12)

v_4	(0.8, 0.7, 0.5, 0.1, 0.2)	(0.9, 0.6, 0.3, 0.1, 0.3)	(0.76, 0.67, 0.34, 0.32, 0.5)

Table 3. Positive Ideal Solution

	A^{β_1}	A^{β_2}	A^{β_3}
I	(.0.9, .0.8, .0.3, .0.1, .0.2)	(0.9, 0.7, 0.3, 0.1, 0.2)	(0.86, 0.67, 0.23, 0.23, 0.12)

Using equations (4) and (5), the weights of the attributes are determined as:

$\omega_1'' = 0.335$, $\omega_2'' = 0.341$, $\omega_3'' = 0.323$.

Using equation (2), the SVPNHTSM of similarity between the PIS and the decision components belonging to the decision matrix are obtained as:

$P_{SVPNHTSM}(v_1, I) = 0.147938$, $P_{SVPNHTSM}(v_2, I) = 0.156656$, $P_{SVPNHTSM}(v_3, I) = 0.124575$

$P_{SVPNHTSM}(v_4, I) = 0.134095$.

SVPNWDSM ascends between the PIS and the decision elements from the DM in the following order:

$$P_{SVPNHTSM}(v_3, I) < P_{SVPNHTSM}(v_4, I) < P_{SVPNHTSM}(v_1, I) < P_{SVPNHTSM}(v_2, I)$$

Water pollution is more impacted by COVID-19 under the SVPNS environment. Figure 3 illustrates numerical results with graphical representations.

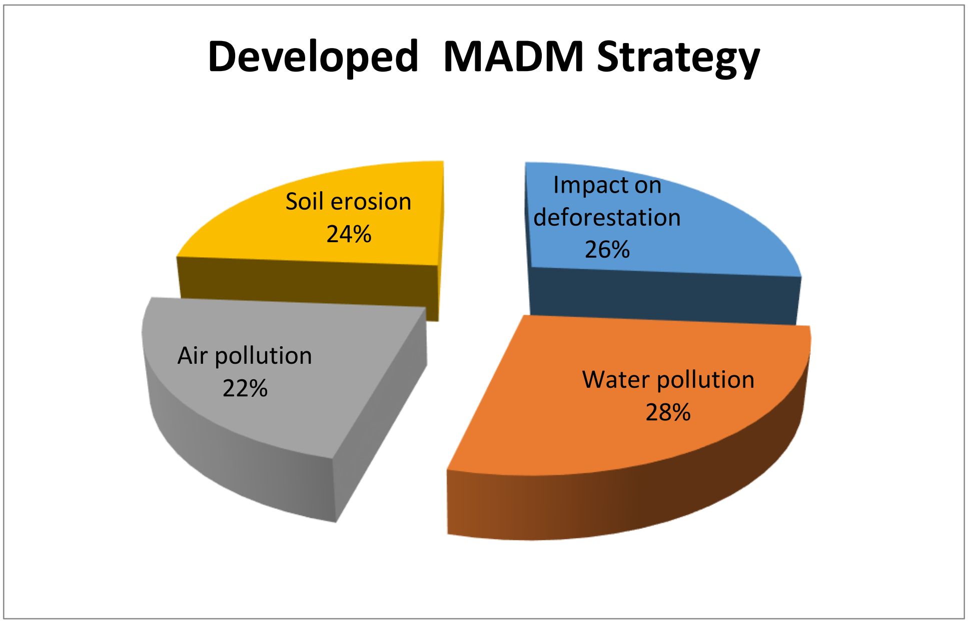

Figure 3. Result of MADM Strategy

6. Comparative study

Comparing the existing strategies with the proposed strategy (see Table 4), it can be seen that they obtain water pollution as the best alternative. Table-4 shows that the weighted values for all attributes are significantly closer to the two existing techniques. The weighted values of the similarity measures in the proposed strategy are not closed like the existing strategies, which enables a better decision for considering attributes. Compared to MADM strategies, this form of weighting supports a better decision. An illustration of a comparative study is shown in Figure 4.

Table 4-. Comparison among the existing strategies and the developed strategy

Methods	v_1	v_2	v_3	v_4	Order of Preference
MADM strategy based on cosine similarity measure [25]	0.672339	0.67277	0.66963	0.670349	$v_3 < v_4 < v_1 < v_2$
MADM strategy Weighted Dice SM [39]	0.206911	0.208836	0.208706	0.199668	$v_4 < v_3 < v_1 < v_2$
Developed MADM Strategy	0.147938	0.156656	0.124575	0.134095	$v_3 < v_4 < v_1 < v_2$

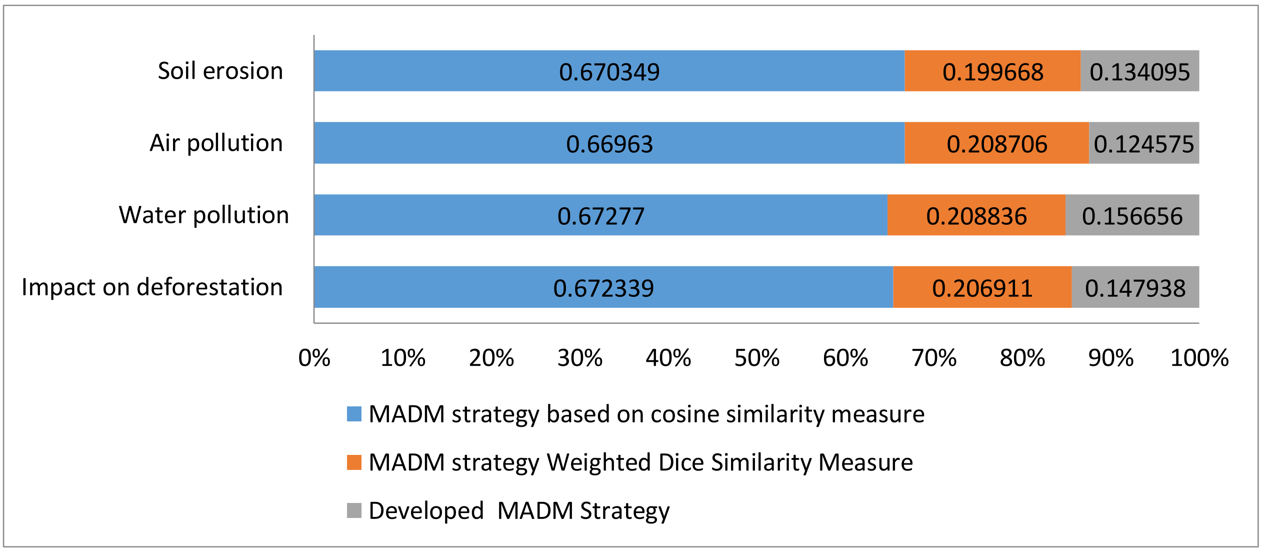

Figure-4: Result of Comparative study

7. Analyzing Sensitivity

To validate the predictions, this step aims to confirm the significance of the alternative as estimated using the developed MADM strategy. A change in the secondary criteria's magnitude can determine the sensitivity of the MADM strategy. Some vital alternatives remain unchanged. Therefore, if the rank changes, the strategy is understood to be under radar of sensitivity, and conversely. Hamby [17] proposed this type of sensitivity analysis and it is known as rank relative sensitivity analysis.
In studies, estimation is done by sensitivity analysis where a numerical model is required to validate the estimated output. Table 5 denotes the output obtained by sensitivity analysis. As per the results

obtained, 'water pollution' is found to have a Swing^2 value of 30.80% whereas 'deforestation' was found to have a Swing^2 value of 27.20 %. Thus, the 'water pollution' is considered as the most sensitive parameter. The less sensitive parameter is obtained as 'Soil erosion' which has Swing^2 value heaving 19.50 %. It suggests water pollution's being the most sensitive factor which gets followed by the bad effect of the corona virus in accordance with the weights gained by SVPNHTSM. Figure 5 illustrates the result of the sensitivity analysis.

Table 5- Result of sensitivity analysis

	Corresponding Input Value			Output Value				Percent
Input Variable	Low Output	Base Case	High Output	Low	Base	High	Swing	Swing^2
water pollution	0	0.5	1	0.20303	0.281358	0.359686	0.156656	30.8%
Deforestation	0	0.5	1	0.207663	0.281358	0.355053	0.14739	27.2%
Air pollution	0	0.5	1	0.2143105	0.281358	0.3484055	0.134095	22.5%
Soil erosion	0	0.5	1	0.2190705	0.281358	0.3436455	0.124575	19.5%

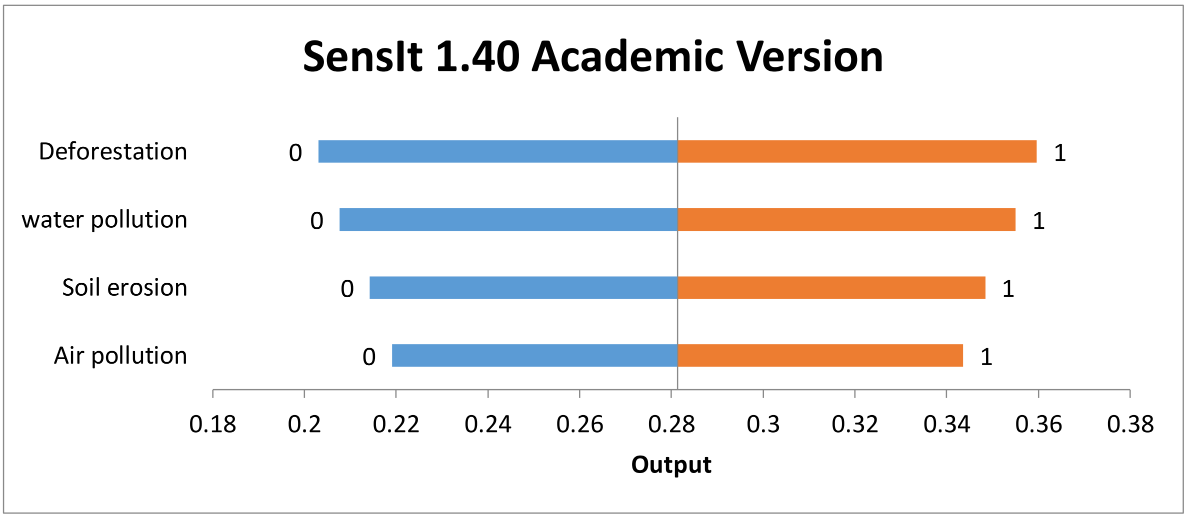

Figure-5: Result of sensitivity analysis

8. Advantage and Disadvantage of the study

Advantages: An MADM strategy based on SVPNHTSM has a four-time greater gradient of tanh than a sigmoid-based strategy. By using the tanh activation function, the gradient during training will be higher and the weights will be updated more frequently. Uncertainty has been comprehensively dealt with using SVPNSs as SVPNSs deal with degrees of contradiction, ignorance and unknown which are more realistic in decision making situation.

Disadvantages: SVPNSs have more components than fuzzy sets and IFSs. Therefore, more times are required to solve the mathematical model involving SVPNSs.

9. Conclusions

This paper develops a new MADM strategy to determine the most significant risk factor in the environment. Based on the proposed strategy, we create a range of alternatives, and obtain ranking order. The novel MADM method's findings are compared to those of other existing approaches. Under the SVPNS environment, it is evident that COVID-19 negatively impacts Water Pollution more than other alternatives. A major weakness of the study is that it doesn't ensure that, as the number of parameters increase, the most significant parameters remain ranked in the same order. In this study, there is no scenario analysis, which is another drawback. It is possible to extend the newly defined SVPNHTSM and SVPNWHTSM operators to other uncertain environments to address uncertainties in decision-making. In addition to clay-brick selection [29], air surveillance, and multiple target tracking [13],watershed hydrological system [15], the approach suggested here could be used to address other MCDM problems as well.

Conflict of Interest: There is no conflict of interest on the part of the authors.

Authors Contribution: The authors contributed equally to develop the paper.

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Neutrosophic Hesitant Fuzzy Techniques and its Application to Structural Design

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Abstract:

Structural optimization in construction has attracted significant attention to sustainable development. In reality, structural model is associated with different imprecise parameters. Several factors influence the uncertain framework for optimization structural models. To tackle such structural difficulties, an effective design and optimization configuration is required. In this proposed work, we have created a solution procedure to solve multi objective problems under neutrosophic-hesitant fuzzy (NHF) environment in context of structural design. The suggested procedure is based on the NHF decision-making set that assigns a set of potential values for each objective function's membership, non-membership, and indeterminacy degrees in a NHF environment. The efficiency, applicability, and utility of the proposed technique are presented here by using a three-bar truss design model.

Keywords- Multi objective structural problem; Hesitant fuzzy optimization; Neotrosophic-hesitant fuzzy optimization; Pareto optimal solution; Indeterminacy hesitant membership function

1. Introduction

When it comes to tackling optimization challenges, optimization techniques have a big impact in real life. When dealing with real-life situations with various problems, various sorts of mathematical models exist. As a result, the mathematical models are formed with single or multi objective function/functions along with a branch of constraints. In multi-objective optimization problem (MOOP), objective functions are conflicting in nature. The objective functions of this mathematical models are maximization type or minimization type or mixed type. In this type of problems, it is very difficult to identify the suitable feasible solutions. That is why, decision maker (DM) prefers a compromise programming (CP) approach that currently meets each goal function is available. As a result, the idea of CP approach has a significant impact on the global optimality criterion. A large amount of research has been presented in the past era on the topic of MOOP. In MOOP, the difficult task as a DM is to discover an appropriate compromised solution set from a set of possible Pareto-optimal solutions.

Due to local and global optimal, multi-objective nonlinear programming problem (MONLPP) is a complex problem as compare to linear multi-objective programming problem. Professor Zadeh pioneered [2] the new idea of fuzzy set (FS) to address the uncertainty in 1965 and Professor Zimmermann [4] proposed a fuzzy programming technique (FPT) for several objective mathematical problem based on fuzzy set. The FPT was only concerned with the degree of acceptance, but it may be required to address the function of rejection in order to obtain more practical outcomes.

The FS (fuzzy set) theory was used in structural model as well. A new concept was implemented a sequence of optimal solution (OS) for structure with fuzzy constraints based on alpha-cut method by Wang et al. [3]. Rao [5] discussed a four–bar generating mechanism with a fuzzy goal function and fuzzy constraints. Yeh et al. [6] created structural optimization using imprecise parameters. Xu [7] solved a nonlinear structural model using fuzzy two-phase method. Shih et al. [8,9] suggested a novel approach to discover a unique solution using alpha-cut approaches of the 1 and 2 types to the structural model in fuzzy environment farther they had developed another alternative approach based on alpha-cut method to obtain the OS of a nonlinear structural problem. Dey et al. [10] addressed multi-objective structural design issues using generalized fuzzy programming. Also, a computational algorithm was developed by Dey et al. [11] for a structural model with three bar using basic triangular norm in fuzzy environment. The extension of ordinary fuzzy set (FS) or hesitant fuzzy set (HFS) was introduced by Torra et al. [12]. It provided an opportunity to allow more feasible values of an element to a set. The potential values of an element in HFS is a subinterval of [0,1]. Many research scholars have recently investigated HFSs and used them in different domains of research. In 2016, a computational programming technique based on HFS was developed by G.L. Xu, et al. [13] for hybrid MCGDM model. In the same domain another paper was published in 2017 by S.-P. Wan [14] based on hesitant fuzzy programming method. L. Dymova, [15] created a user-friendly computer application using a fuzzy MCDM technique. Farther, they [16] had applied this fuzzy MCDM technique in a rolled-steel heat treatment metallurgical plant in 2021. But in structural design optimization, hesitant fuzzy set is likewise not extensively utilized.

In 1986 [17], intuitionistic fuzzy set (IFS) was developed by Prof. Atanassov. IFS is an advanced version of FS. In FS, the membership degree is only consideration whereas in IFS, both the level of membership and non-membership are considered with the condition that the sum both membership values is not greater than one. P. P. Angelov [18] used the optimization for the first time in a widespread intuitionist fuzzy environment in 1997. B. Singh et al. [19] proposed an intuitionistic fuzzy optimization technique based on structural model. M. Sarkar et al. [20] proposed a new computational algorithm based on triangular-norm and triangular-conorm in intuitionistic fuzzy environment to solve a welded beam design issue. Kabiraj *et al.* [21] gave the utility of fuzzy logic has been used in linear programming in 2019. In 2019, S.F.Zhang, et al. [22] proposed GRA based IFMCGDM method for personnel selection. Kizilaslan et al. [23] proposed intuitionistic fuzzy function approaches utilizing ordinary least square estimation rather than ridge regression in 2019. Ahmadini and Ahmad [24] proposed intuitionistic fuzzy goal programming with preference relations to address a multi-objective problem in 2021. A. Ebrahimnejad, [25] introduced a novel approach to solve data envelopment analysis (DEA) models characterized by intuitionistic fuzzy data. Recently, many researchers have worked with intuitionistic hesitant fuzzy (IHF) sets and implemented them to many domains. S.K Bharati [26] in 2018 introduced hesitant fuzzy algorithm to solve multi objective linear optimization problem (MOLOP).K.B. Shailendra, [27] introduced IHF algorithm for MOOP in 2021. But in structural design optimization, IHF set is likewise not extensively utilized. The concept of neutrosophic theory was revealed to address the importance of indeterminacy in real life. In generalized FS and IFS were discussed about membership and non-membership function only but there is no information about the indeterminacy. New concept of neutrosophic theory was presented in front of researcher by Prof. Smarandache in 1995 [28], which is a dialectics extension. The neutrosophic set (NS) can manage both uncertain and partial information, whereas IFSs can only manage partial information. The word neutrosophic is derived from two words: neutron (neutral in French) and Sophia (skill or wisdom in Greek). The NS is described by using three functions namely belonging (truth) function, belonging to a certain point (indeterminacy) function, and not belonging (falsity) function. The Neutrosophical Programming Approach (NPA), based on NS, was implemented, and is now widely utilized in real-world applications. M. Sarkar et al. [35] applied neutrosophic fuzzy numbers in the area of structural design and application. Abdel-Basset et al. [1] offered a new technique for solving a completely neutral linear programming problem (LPP) that applies to production planning. In 2018, Ye et al. [29] suggested an effective technique for addressing the issue of non-linear programming of the neutrosophic number in neutrosophic numerical environments. An approach for solving MONLPPs in IFS was introduced by Rani et al. [39]. The develop method has been was compared with other existing methods that are already includes. Zhou and Xu [30] developed a novel portfolio selection and investment technique at risk in a widespread and faltering environment. All the sets mentioned above have their limits with respect to the presence of each component in the set. A new optimization technique based on a single-valued neutrosophic hesitant fuzzy set (SVNHFS) was proposed by Ahmad et al. [31]. This set includes the concept of truth hesitancy degrees, falsity hesitant degrees as well as indeterminacy hesitant degrees for various objective functions. The neutrosophic set of indeterminacy concepts examines potential future lines of research in the field of real-life application. Many researchers have contributed to the field of neutrosophic optimization techniques and real-world applications, including [36, 38]. In 2020, F. Ahmad, et al. [37] were developed a computational approach based on modified neutrosophic fuzzy set (NFS) to optimize a supply chain decision making problem. According to Giri et al. [40], TOPSIS for MADM has been extended through the use of single valued neutrosophic fuzzy sets (SVNFS). B. Tanuwijaya et al [41] developed fuzzy time series (FTS) model based on SVNFSs in 2020. In 2021, F. Ahmad [42] proposed interactive NPA based on Type-2 fuzzy in domain supplier selection problem. In order to tackle a PP issue, Khan et al. [43] studied the IVTN value and employed NS and IFS approach. S. Gupta et al [44] introduced Dash diet model and optimized the calorie consumption and minimized diet cost under neutrosophic goal programming (NGP). The multi objective NGP was used to solve the diet model, satisfy daily nutrient needs, and compared various approaches.

A wide range of methods have been used in the literature in order to solve the uncertainty in structural design problems, such as fuzzy, intuitionistic fuzzy and neutrosophic fuzzy optimization. But combination with HFS and NFS is very rare in literature survey in context of structural design.

This research is prompted by NHF emerging as a novel field of study with the capacity to attract the individuals responsible for making decisions. The subsequent are the impacts of the study:

- It serves as a supplementary addition to the existing literature on MOSOP.
- A case study is presented in which solution processes for MOSOP methodologies are documented.
- In this work, a novel technique based on NHF under various membership functions has been used.
- The method is contrasted with HFS and IHFS, and the findings indicate that the proposed study is effective.
- The proposed neutrosophic hesitant fuzzy programming approaches (NHFPAs) utilizing the neutrosophic fuzzy decision set is quite simple and easy.

The synopsis of rest of the manuscript is highlighted below: Section 2: we have highlighted the multi-objective structural optimization model (MOSOM). In section 3, we give some basic concepts about FS, IFS, SVNS, HFS, and SVNHFS. Section 4 proposes a computational algorithm to solve a

MOOP using neutrosophic hesitant fuzzy optimization technique (NHFOT). Section 5 outlines the approach for resolving the multi-objective structural model using NHFOT. An illustrative example is studies in section 6 which reflects the applicability and validity of the proposed method effectively. Finally, section 7 highlights the concluding remarks and finding based on the present work.

2. Mathematical Form of Multi-Objective Structural Optimization Problem (MOSOP)

In structural model, the basic parameters of a bar truss structure system (such as Young's modulus, material density, maximum permissible stress, and so on) are established, and the objective is to find the cross section area of the bar truss so that we can find the lightest weight of the structure and smallest node displacement under loading condition.

The MOSOP is formulated as follows:

$$\begin{aligned} & Minimize\ W(A) \\ & Minimize\ \rho(A) \\ & \text{s.t}\ \ \sigma(A) \le [\sigma] \\ & A \in [A_{\min}, A_{\max}] \end{aligned} \tag{1}$$

where n number design variables $A = [A_1, A_2, ..., A_n]^T$ are considered. The design parameters are the cross-sectional area of the truss bar, the total structural weight is $W(A) = \sum_{i=1}^{n} \delta_i A_i L_i$, the deflection of loaded joint is $\rho(A)$, length of bar= L_i, cross section area= A_i, and the i^{th} group bars density= δ_i, respectively. Under different conditions, the stress constraint= $\sigma(A)$ and maximum allowable stress of the group bars= $[\sigma]$, cross section area (minimum)= $A_{\min}$ and cross section area (maximum) = $A_{\max}$ respectively.

3. Preliminaries

Definition 1. [32] (Neutrosophic Set (NS)) Assume, U be the universe discourse such that $x \in U$. A NS $\tilde{A}$ in U is characterized by the membership functions as, truth $Tf_{\tilde{A}}(x)$, indeterminacy $If_{\tilde{A}}(x)$ and a falsity $Ff_{\tilde{A}}(x)$ and is denoted by the following form:

$$\tilde{A} = \{(x, Tf_{\tilde{A}}(x), If_{\tilde{A}}(x), Ff_{\tilde{A}}(x)) : x \in U\}$$

Where the subsets $Tf_{\tilde{A}}(x)$, $If_{\tilde{A}}(x)$ and $Ff_{\tilde{A}}(x)$ are truth, indeterminacy and falsity membership function lies in $\mathrm{E} =]0^-, 1^+[$,also given as, $Tf_{\tilde{A}}(x) : U \to \mathrm{E}, If_{\tilde{A}}(x) : U \to \mathrm{E}, \textit{and}\ Ff_{\tilde{A}}(x) : U \to \mathrm{E}$ .There is no restriction on the sum of $Tf_{\tilde{A}}(x)$, $If_{\tilde{A}}(x)$ and $Ff_{\tilde{A}}(x)$,so we have,

$$0^- \le \sup Tf_{\tilde{A}}(x) + If_{\tilde{A}}(x) + \sup Ff_{\tilde{A}}(x) \le 3^+$$

Definition 2. [32] Let U be a universe set. A single valued neutrosophic set (SVNS) $\tilde{A}$ over U is given by $\tilde{A} = \{(x, Tf_{\tilde{A}}(x), If_{\tilde{A}}(x), Ff_{\tilde{A}}(x)) : x \in U\}$

Where $Tf_{\tilde{A}}(x)$, $If_{\tilde{A}}(x)$ and $Ff_{\tilde{A}}(x)$ lies in [0,1] and $0 \le Tf_{\tilde{A}}(x) + If_{\tilde{A}}(x) + Ff_{\tilde{A}}(x) \le 3$ for every $x \in U$.

Definition 3. [33] (Hesitant Fuzzy Set (HFS) Torra et al. [12], created a new tool called HFSs and which allow the acceptance degree to the set of various possible values. The HFS is as follows:

Let U be a universe set, then a HFS on U is expressed as $\tilde{Y} = \{< x_j, h_{\tilde{Y}}(x_j) > | x_j \in U\}$, where $h_{\tilde{Y}}(x_j)$ is set of possible degree of acceptance of the element $x_j \in U$ in [0,1] .Also, we call $h_{\tilde{Y}}(x_j)$, a hesitant fuzzy element.

Definition 4. [34] (SVNHFS) Let's say there's a fixed set U ; an SVNHFS on U is represented as: $S_{\tilde{Y}} = \{(x, Tf_{\tilde{Y}}(x), If_{\tilde{Y}}(x), Ff_{\tilde{Y}}(x)) : x \in U\}$ where set of possible values of $Tf_{\tilde{Y}}(x)$, $If_{\tilde{Y}}(x)$ and $Ff_{\tilde{Y}}(x)$ are lies in $[0,1]$,indicating the possible truth, indeterminacy hesitant degree of acceptance and the falsehood hesitant degree of rejection of the element $x \in U$ to the set $S_{\tilde{Y}}$ accordingly with the conditions $0 \le \mu, \kappa, \gamma \le 1$ and $0 \le \mu^+, \kappa^+, \gamma^+ \le 3$, where $\mu \in Tf_{\tilde{Y}}(x)$, $\kappa \in If_{\tilde{Y}}(x)$, $\gamma \in Ff_{\tilde{Y}}(x)$ with $\mu^+ \in Tf_C{}^+(x) = U_{\mu \in Tf_{\tilde{Y}}(x)} \max\{\mu\}$, $\kappa^+ \in If_{\tilde{Y}}{}^+(x) = U_{\kappa \in If_{\tilde{Y}}(x)} \max\{\kappa\}$, $\gamma^+ \in Ff_{\tilde{Y}}{}^+(x) = U_{\gamma \in Ff_{\tilde{Y}}(x)} \max\{\gamma\}$ for all $x \in U$.

For ease, the three-tuple $S_{\tilde{Y}} = \{Tf_{\tilde{Y}}(x), If_{\tilde{Y}}(x), Ff_{\tilde{Y}}(x)\}$ is known as a single-valued neutrosophic hesitant fuzzy element (SVNHFE) or triple hesitant fuzzy element.

According to Definition 6, the SVNHFS has three types of membership functions: truth $Tf_{\tilde{A}}(x)$, indeterminacy $If_{\tilde{A}}(x)$ and a falsity $Ff_{\tilde{A}}(x)$ membership function, resulting in a more dependable structure and providing flexible options to allocate values for every element in the field, and may handle three types of uncertainty at the same time. As a result, FSs, IFSs, SVNFSs, and HFSs can be considered as specific instances of SVNHFSs.([33]

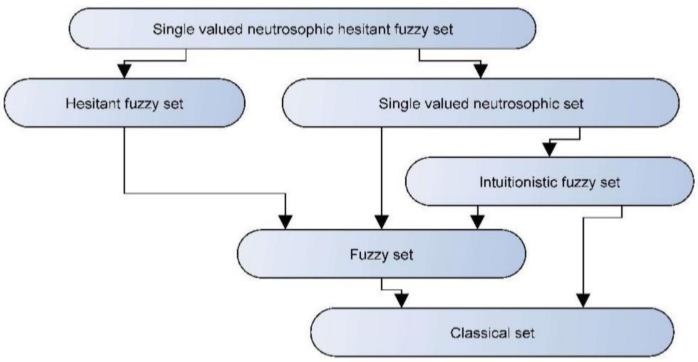

Figure 1: Dialogistic coverage of classical set to SVNHFS.

Definition 5. [34] Let there be two SVNHFSs, $S_{\tilde{Y}_1}$ and $S_{\tilde{Y}_2}$ in a universal set U . Then the union of $S_{\tilde{Y}_1}$ and $S_{\tilde{Y}_2}$ is described as:

$$S_{\tilde{Y}_1} \cup S_{\tilde{Y}_2} = \begin{Bmatrix} Tf_{\tilde{Y}} \in \left(Tf_{\tilde{Y}_1} \cup Tf_{\tilde{Y}_2} \right) : Tf_{\tilde{Y}} \geq \max\left(\min\left\{ Tf_{\tilde{Y}_1} \cup Tf_{\tilde{Y}_2} \right\} \right), \\ If_{\tilde{Y}} \in \left(If_{\tilde{Y}_1} \cup If_{\tilde{Y}_2} \right) : If_{\tilde{Y}} \leq \min\left(\max\left\{ If_{\tilde{Y}_1} \cup If_{\tilde{Y}_2} \right\} \right), \\ Ff_{\tilde{Y}} \in \left(Ff_{\tilde{Y}_1} \cup Ff_{\tilde{Y}_2} \right) : Ff_{\tilde{Y}} \leq \min\left(\max\left\{ Ff_{\tilde{Y}_1} \cup Ff_{\tilde{Y}_2} \right\} \right) \end{Bmatrix} \quad (2)$$

Definition 6. [34] Let there be two SVNHFSs, $S_{\tilde{Y}_1}$ and $S_{\tilde{Y}_2}$ in a universal set U . Then the intersection of $S_{\tilde{Y}_1}$ and $S_{\tilde{Y}_2}$ is described as:

$$S_{\tilde{Y}_1} \cap S_{\tilde{Y}_2} = \begin{Bmatrix} Tf_{\tilde{Y}} \in \left(Tf_{\tilde{Y}_1} \cap Tf_{\tilde{Y}_2} \right) : Tf_{\tilde{Y}} \geq \min\left(\max\left\{ Tf_{\tilde{Y}_1} \cap Tf_{\tilde{Y}_2} \right\} \right), \\ If_{\tilde{Y}} \in \left(If_{\tilde{Y}_1} \cap If_{\tilde{Y}_2} \right) : If_{\tilde{Y}} \leq \max\left(\min\left\{ If_{\tilde{Y}_1} \cap If_{\tilde{Y}_2} \right\} \right), \\ Ff_{\tilde{Y}} \in \left(Ff_{\tilde{Y}_1} \cap Ff_{\tilde{Y}_2} \right) : Ff_{\tilde{Y}} \leq \max\left(\min\left\{ Ff_{\tilde{Y}_1} \cap Ff_{\tilde{Y}_2} \right\} \right) \end{Bmatrix} \quad (3)$$

Definition 7. Assume that there is a set of feasible solution Λ of MOSOP (1). Then a point x^* taken into consideration to be a Pareto optimal solution of (1) iff there is no such point $x \in \Lambda$ such that $O_k(x^*) \geq O_k(x)\ \forall k$ as well as $O_k(x^*) > O_k(x)$ for as a minimum k.

Definition 8. A point $x^* \in \Lambda$ is called a weak Pareto OS of (1) iff there is not a point $x \in \Lambda$ such that $O_k(x^*) \geq O_k(x)\ \forall k$.

4. Proposed Algorithm

***4.1** To Solve MONLPPs using NHFPA*

One may take a MONLPP with k objectives.

$$Min.\ \ O(x) = \left[O_1(x), O_2(x), ...O_k(x) \right]^T \quad (4)$$

Subject to

$x \in U = \left\{ x \in \mathbb{R}^n \mid g_{ij}(x) \leq or = or \geq b_j\,,\ j = 1\ to\ m \in N \right\}$ and $L_i \leq x_i \leq U_i$ $\left(i = 1\ to\ n \in N, natural\ no. \right)$

Zimmermann [4] demonstrated that the MOOP can be resolved using fuzzy programming techniques.

The MONLPP is solved using the procedures listed below.

Step 1: The MONLPP (4) may be solved as a single objective nonlinear programming problem (SONLPP) by focusing on one objective at a time and overlooking the other objective goals which are called ideal solutions.

Step 2: The result achieved in step 1, the pay-off matrix may be created by identifying the corresponding listed values for every goal in the following manner:

$$\begin{matrix} & O_1(x) & O_2(x) & \dots & O_k(x) \\ \begin{matrix} x_1 \\ x_2 \\ \dots \\ x_k \end{matrix} & \multicolumn{4}{c}{\begin{bmatrix} O_1^*(x_1) & O_2^*(x_1) & \dots & O_k^*(x_1) \\ O_1^*(x_2) & O_2^*(x_2) & \dots & O_k^*(x_2) \\ \dots & \dots & \dots & \dots \\ O_1^*(x_k) & O_2^*(x_k) & \dots & O_k^*(x_k) \end{bmatrix}} \end{matrix}$$

In this case, the ideal solutions are $x_1, x_{2,}..........., x_k$ of the objective functions $O_1(x), O_2(x), \ldots..., O_k(x)$ accordingly.

Step-3: In each column the highest possible value U_k denotes upper tolerance, or upper bound, for the k^{th} objective function $O_k(x)$, where $U_k = \max\{O_k(x_1), O_k(x_2), ..., O_k(x_k)\}$ and the minimum value of each column L_k gives lower tolerance or lower limit for the k^{th} goal function $O_k(x)$, where $L_k = \min\{O_k(x_1), O_k(x_2), ..., O_k(x_K)\}$ for $k = 1, 2, ..., K$.

$$U_k^T = U_k, L_k^T = L_k \quad \text{for truth membership}$$

$$L_k^I = U_k^T - s_k, U_k^I = U_k^T \quad \text{for indeterminacy membership}$$

$$U_k^F = U_k^T, L_k^F = L_k^T + t_k \quad \text{for falsity membership}$$

Where $0 \le s_k \le (U_k - L_k)$ and $0 \le t_k \le (U_k - L_k)$ are specific real numbers in $(0,1)$.

Step-4: Under a NHF environment, we can now define the various hesitant membership functions as linear, exponential, and hyperbolic. Each of them is specified for the membership functions truth, uncertainty, and falsehood, which appears to be more accurate.

4.1.1. Linear-type hesitant membership functions approach (LTHMFA)

The linear type truth membership $Tf_k^{L_i}(O_k(x))$, indeterminacy membership $If_k^{L_i}(O_k(x))$ and a falsehood membership $Ff_k^{L_i}(O_k(x))$ functions under NHF context can be described as below

For truth hesitant fuzzy membership functions:

$$Tf_k^{L_1}(O_k(x)) = \begin{cases} 1 & \text{if } O_k(x) \le L_k^T \\ \mu_1\left[\dfrac{(U_k^T)^t - (O_k(x))^t}{(U_k^T)^t - (L_k^T)^t}\right] & \text{if } L_k^T \le O_k(x) \le U_k^T \\ 0 & \text{if } O_k(x) \ge U_k^T \end{cases}$$

$$Tf_k^{L_2}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^T\\ \mu_2\left[\dfrac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right] & \text{if } L_k^T\le O_k\left(x\right)\le U_k^T\\ 0 & \text{if } O_k\left(x\right)\ge U_k^T\end{cases}$$

....

$$Tf_k^{L_n}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^T\\ \mu_n\left[\dfrac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right] & \text{if } L_k^T\le O_k\left(x\right)\le U_k^T\\ 0 & \text{if } O_k\left(x\right)\ge U_k^T\end{cases}$$

For indeterminacy hesitant fuzzy membership functions:

$$If_k^{L_1}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^I\\ \kappa_1\left[\dfrac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right] & \text{if } L_k^I\le O_k\left(x\right)\le U_k^I\\ 0 & \text{if } O_k\left(x\right)\ge U_k^I\end{cases}$$

$$If_k^{L_2}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^I\\ \kappa_2\left[\dfrac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right] & \text{if } L_k^I\le O_k\left(x\right)\le U_k^I\\ 0 & \text{if } O_k\left(x\right)\ge U_k^I\end{cases}$$

...

$$If_k^{L_n}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^I\\ \kappa_n\left[\dfrac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right] & \text{if } L_k^I\le O_k\left(x\right)\le U_k^I\\ 0 & \text{if } O_k\left(x\right)\ge U_k^I\end{cases}$$

For Falsehood hesitant fuzzy membership functions

$$Ff_k^{L_1}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_1\left[\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

$$Ff_k^{L_2}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_2\left[\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

....

$$Ff_k^{L_n}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_n\left[\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

The following is a mathematical explanation of objective functions.

$$\begin{aligned}&\text{Max}\ \min_{k=1,2,..,K} Tf_k^{L_i}(O_k(x))\\ &\text{Max}\ \min_{k=1,2,..,K} If_k^{L_i}(O_k(x))\\ &\text{Min}\ \max_{k=1,2,..,K} Ff_k^{L_i}(O_k(x))\end{aligned} \tag{5}$$

$(i=1,2,.....,n)$, subject to all constraints of (4).

Assume that $Tf_k^{L_i}\left(O_k(x)\right)\ge\mu_i, If_k^{L_i}\left(O_k(x)\right)\ge\kappa_i$ and $Ff_k^{L_i}\left(O_k(x)\right)\le\gamma_i(i=1,2,.....,n)$, for all k

Where the parameter $t>0$. Utilizing additional variables μ_i,κ_i and γ_i, the following problem (5) can be transformed to the problem (6)

LTNHMFA $\quad M\text{ ax } \left(\sum_i\mu_i+\sum_i\kappa_i-\sum_i\gamma_i\right)$

Subject to

$$\alpha_1\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\ge\mu_1;\alpha_2\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\ge\mu_2,..,\alpha_n\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\ge\mu_n,$$

$$\beta_1\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\ge\kappa_1,\beta_2\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\ge\kappa_2,...,\beta_n\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\ge\kappa_n,$$

$$\lambda_1\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\le\gamma_1,\lambda_2\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\le\gamma_2,...,\lambda_n\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\le\gamma_n \qquad ...(6)$$

$\mu_i \geq \kappa_i, \mu_i \geq \gamma_i, 0 \leq \mu_i + \kappa_i + \gamma_i \leq 3, \mu_i, \kappa_i, \gamma_i \in (0,1)$ and $\alpha_i, \beta_i, \lambda_i \in (0,1)$, for all $(i = 1\, to\, n \in N)$

All the constraints of (4).

Theorem 1: There is only one OS $(x^*, \mu^*, \kappa^*, \gamma^*)$ of problem (6) (LTNHMFA) which is likewise an efficient solution of (4) where $\mu^* = (\mu_1^*, \mu_2^*, ..., \mu_n^*)$, $\kappa^* = (\kappa_1^*, \kappa_2^*, ..., \kappa_n^*)$ and $\gamma^* = (\gamma_1^*, \gamma_2^*, ..., \gamma_n^*)$.

Proof: Suppose that $(x^*, \mu^*, \kappa^*, \gamma^*)$ be the only OS of problem (6) which is an inefficient to solving the problem (4). Then there exist different feasible alternative $x'(x' \neq x^*)$ of the problem (4), so that $O_k(x^*) \leq O_k(x')\ \forall\, k$, $O_k(x^*) < O_k(x')$ for at least one k.

We have $\frac{(U_k^T)^t - (O_k(x^*))^t}{(U_k^T)^t - (L_k^T)^t} \leq \frac{(U_k^T)^t - (O_k(x'))^t}{(U_k^T)^t - (L_k^T)^t}$ $\forall\, k$ and $\frac{(U_k^T)^t - (O_k(x^*))^t}{(U_k^T)^t - (L_k^T)^t} < \frac{(U_k^T)^t - (O_k(x'))^t}{(U_k^T)^t - (L_k^T)^t}$ for at least one k.

Hence, $\underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x^*))^t}{(U_k^T)^t - (L_k^T)^t}\right) \leq \underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x'))^t}{(U_k^T)^t - (L_k^T)^t}\right)$,

$\underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x^*))^t}{(U_k^T)^t - (L_k^T)^t}\right) < \underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x'))^t}{(U_k^T)^t - (L_k^T)^t}\right)$ for at least one k

Similarly, $\underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x^*))^t}{(U_k^I)^t - (L_k^I)^t}\right) \leq \underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x'))^t}{(U_k^I)^t - (L_k^I)^t}\right)$ and

$\underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x^*))^t}{(U_k^I)^t - (L_k^I)^t}\right) < \underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x'))^t}{(U_k^I)^t - (L_k^I)^t}\right)$

Again,

$\underset{k}{Min}\left(\frac{(U_k^F)^t - (O_k(x^*))^t}{(U_k^F)^t - (L_k^F)^t}\right) \geq \underset{k}{Min}\left(\frac{(U_k^F)^t - (O_k(x'))^t}{(U_k^F)^t - (L_k^F)^t}\right)$, $\underset{k}{Min}\left(\frac{(U_k^F)^t - (O_k(x^*))^t}{(U_k^F)^t - (L_k^F)^t}\right) > \underset{k}{Min}\left(\frac{(U_k^F)^t - (O_k(x'))^t}{(U_k^F)^t - (L_k^F)^t}\right)$

for at least one k

Now, assume that $\mu^* = \underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x^*))^t}{(U_k^T)^t - (L_k^T)^t}\right)$ and $\mu' = \underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x'))^t}{(U_k^T)^t - (L_k^T)^t}\right)$,

$\kappa^* = \underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x^*))^t}{(U_k^I)^t - (L_k^I)^t}\right)$, $\kappa' = \underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x'))^t}{(U_k^I)^t - (L_k^I)^t}\right)$, $\gamma^* = \underset{k}{Min}\left(\frac{(U_k^F)^t - (O_k(x^*))^t}{(U_k^F)^t - (L_k^F)^t}\right)$ and

$\lambda' = \underset{k}{Min}\left(\frac{\left(U_k^F\right)^t - \left(O_k(x')\right)^t}{\left(U_k^F\right)^t - \left(L_k^F\right)^t}\right)$. Then $\mu^* \leq (<)\mu'$, $\kappa^* \leq (<)\kappa'$ and $\gamma^* \geq (>)\gamma'$ which gives $\left(\mu^* + \kappa^* - \gamma^*\right) < \left(\mu' + \kappa' - \gamma'\right)$ that implies the solution is not optimal which contradicts that $\left(x^*, \mu^*, \kappa^*, \gamma^*\right)$ is a unique OS of (6). As a result, it is a successful problem-solving strategy (6). Thus, the proof is finished.

4.1.2. Exponential-type hesitant membership functions approach (ETHMFA)

The truth membership function of exponential type $T_k^{E_i}(O_k(x))$, indeterminacy membership of exponential type $I_k^{E_i}(O_k(x))$ and a falsehood membership of exponential type $F_k^{E_i}(O_k(x))$ functions under NHF context can be can be described as follows

For truth hesitant fuzzy membership functions:

$$Tf_k^{E_1}\left(O_k(x)\right) = \begin{cases} 1 & \text{if } O_k(x) \leq L_k^T \\ \mu_1\left[1 - \exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t - \left(O_k(x)\right)^t}{\left(U_k^T\right)^t - \left(L_k^T\right)^t}\right)\right\}\right] & \text{if } L_k^T \leq O_k(x) \leq U_k^T \\ 0 & \text{if } O_k(x) \geq U_k^T \end{cases}$$

$$Tf_k^{E_2}\left(O_k(x)\right) = \begin{cases} 1 & \text{if } O_k(x) \leq L_k^T \\ \mu_2\left[1 - \exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t - \left(O_k(x)\right)^t}{\left(U_k^T\right)^t - \left(L_k^T\right)^t}\right)\right\}\right] & \text{if } L_k^T \leq O_k(x) \leq U_k^T \\ 0 & \text{if } O_k(x) \geq U_k^T \end{cases}$$

$$\cdots$$

$$Tf_k^{E_n}\left(O_k(x)\right) = \begin{cases} 1 & \text{if } O_k(x) \leq L_k^T \\ \mu_n\left[1 - \exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t - \left(O_k(x)\right)^t}{\left(U_k^T\right)^t - \left(L_k^T\right)^t}\right)\right\}\right] & \text{if } L_k^T \leq O_k(x) \leq U_k^T \\ 0 & \text{if } O_k(x) \geq U_k^T \end{cases}$$

For indeterminacy hesitant fuzzy membership functions:

$$If_k^{E_1}\left(O_k(x)\right) = \begin{cases} 1 & \text{if } O_k(x) \leq L_k^I \\ \kappa_1\left[1 - \exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t - \left(O_k(x)\right)^t}{\left(U_k^I\right)^t - \left(L_k^I\right)^t}\right)\right\}\right] & \text{if } L_k^I \leq O_k(x) \leq U_k^I \\ 0 & \text{if } O_k(x) \geq U_k^I \end{cases}$$

$$If_k^{E_2}\left(O_k(x)\right)=\begin{cases}1 & \text{if } O_k(x)\le L_k^I\\ \kappa_2\left[1-\exp\left\{-\psi\left(\dfrac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right] & \text{if } L_k^I\le O_k(x)\le U_k^I\\ 0 & \text{if } O_k(x)\ge U_k^I\end{cases}$$

$$\ldots$$

$$If_k^{E_n}\left(O_k(x)\right)=\begin{cases}1 & \text{if } O_k(x)\le L_k^I\\ \kappa_n\left[1-\exp\left\{-\psi\left(\dfrac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right] & \text{if } L_k^I\le O_k(x)\le U_k^I\\ 0 & \text{if } O_k(x)\ge U_k^I\end{cases}$$

For Falsity hesitant fuzzy membership functions

$$Ff_k^{E_1}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_1\left[1-\exp\left\{-\psi\left(\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

$$Ff_k^{E_2}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_2\left[1-\exp\left\{-\psi\left(\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

$$\ldots$$

$$Ff_k^{E_n}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_n\left[1-\exp\left\{-\psi\left(\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

Where ψ is the measure of ambiguity degree or shape parameter which designated by the DM.

Assume that $T_k^{E_i}\left(O_k(x)\right)\ge\mu_i, I_k^{E_i}\left(O_k(x)\right)\ge\kappa_i$ and $F_k^{E_i}\left(O_k(x)\right)\le\gamma_i\left(\left(i=1\, to\, n\in N\right)\right)$, for all k

Where the parameter $t>0$. Utilizing additional variables μ_i,κ_i and γ_i, the given problem (5) can be converted to (7).

ETNHMFA $\quad Max \left(\sum_i \mu_i + \sum_i \kappa_i - \sum_i \gamma_i\right)$

Subject to

$$\alpha_1\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)\ge \mu_1;\alpha_2\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)\ge \mu_2$$

$$,...,\alpha_n\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)\ge \mu_n,\beta_1\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)\ge \kappa_1,$$

$$\beta_2\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)\ge \kappa_2,...,\beta_n\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)\ge \kappa_n$$

$$\lambda_1\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)\le \gamma_1,\lambda_2\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)\le \gamma_2, \qquad (7)$$

$$...,\lambda_n\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)\le \gamma_n$$

$\mu_i \ge \kappa_i, \mu_i \ge \gamma_i, 0 \le \mu_i + \kappa_i + \gamma_i \le 3, \mu_i, \kappa_i, \gamma_i \in (0,1)$ and $\alpha_i, \beta_i, \lambda_i \in (0,1)$, for all $(i = 1\, to\, n \in N)$

All the constraints of (4).

Theorem 2: There is only one OS $\left(x^*,\mu^*,\kappa^*,\gamma^*\right)$ of problem (7) (ETNHMFA) which is likewise an efficient solution for (4) where $\mu^* = \left(\mu_1^*,\mu_2^*,.......,\mu_n^*\right)$, $\kappa^* = \left(\kappa_1^*,\kappa_2^*,.......,\kappa_n^*\right)$ and $\gamma^* = \left(\gamma_1^*,\gamma_2^*,.......,\gamma_n^*\right)$.

Proof: Suppose that $\left(x^*,\mu^*,\kappa^*,\gamma^*\right)$ be the only OS of (7) which is an inefficient to solving the problem (4). Then there exist different feasible alternative $x'\left(x' \ne x^*\right)$ of the problem (4), so that $O_k\left(x^*\right) \le O_k\left(x'\right)\ \ \forall\ k$ and $O_k\left(x^*\right) < O_k\left(x'\right)$ for at least one k.

Therefore, $1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\le 1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x')\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\ \ \forall\ \ k$ and

$1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}< 1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x')\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}$ for at least one k.

Hence, $\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)\le \underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x')\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)$,

$\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)< \underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x')\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)$ for at least one k

Similarly, $\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)\le \underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x')\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)$ and

$\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)< \underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x')\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)$ for at least one k

Again, $\underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)\ge \underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x')\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)$,

$\underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)> \underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x')\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)$ for at least one k

Now, assume that

$\mu^*=\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)$ and $\mu'=\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x')\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)$,

$\kappa^*=\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)$, $\kappa'=\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x')\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)$,

$\gamma^*=\underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)$ and $\lambda'=\underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x')\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)$.

Then $\mu^*\le(<)\mu'$, $\kappa^*\le(<)\kappa'$ and $\gamma^*\ge(>)\gamma'$ which gives $\left(\mu^*+\kappa^*-\gamma^*\right)<\left(\mu'+\kappa'-\gamma'\right)$ that implies the solution is not optimal which contradicts that $\left(x^*,\mu^*,\kappa^*,\gamma^*\right)$ is a unique OS of (7). As a result, it is a successful problem-solving strategy (7). Thus, the proof is finished.

4.1.3. Hyperbolic-type hesitant membership functions approach (HTHMFA)

The truth membership function of hyperbolic type $T_k^{H_i}(O_k(x))$, indeterminacy membership of hyperbolic $I_k^{H_i}(O_k(x))$ and a falsity membership of hyperbolic $F_k^{H_i}(O_k(x))$ functions under NHF context can be can be described as follows

For truth hesitant fuzzy membership functions:

$$Tf_k^{H_1}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^T\\ \mu_1\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\left(\dfrac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^T\le O_k\left(x\right)\le U_k^T\\ 0 & \text{if } O_k\left(x\right)\ge U_k^T\end{cases}$$

$$Tf_k^{H_2}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^T\\ \mu_2\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\left(\dfrac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^T\le O_k\left(x\right)\le U_k^T\\ 0 & \text{if } O_k\left(x\right)\ge U_k^T\end{cases}$$

$$\cdots$$

$$Tf_k^{H_n}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^T\\ \mu_n\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\left(\dfrac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^T\le O_k\left(x\right)\le U_k^T\\ 0 & \text{if } O_k\left(x\right)\ge U_k^T\end{cases}$$

For indeterminacy hesitant fuzzy membership functions:

$$If_k^{H_1}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^I\\ \kappa_1\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\left(\dfrac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^I\le O_k\left(x\right)\le U_k^I\\ 0 & \text{if } O_k\left(x\right)\ge U_k^I\end{cases}$$

$$If_k^{H_2}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^I\\ \kappa_2\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\left(\dfrac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^I\le O_k\left(x\right)\le U_k^I\\ 0 & \text{if } O_k\left(x\right)\ge U_k^I\end{cases}$$

$$....$$

$$If_k^{H_n}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^I\\ \kappa_n\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\left(\dfrac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^I\le O_k\left(x\right)\le U_k^I\\ 0 & \text{if } O_k\left(x\right)\ge U_k^I\end{cases}$$

For Falsity hesitant fuzzy membership functions

$$Ff_k^{H_1}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_1\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x)\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

$$Ff_k^{H_2}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_2\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x)\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

....

$$Ff_k^{H_n}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_n\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x)\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

where $\tau_k=\frac{6}{U_k-L_k}$ is the measure of ambiguity degree or shape parameter which designated by the DM. Assume that $T_k^{H_i}\left(O_k(x)\right)\ge\mu_i, I_k^{H_i}\left(O_k(x)\right)\ge\kappa_i$ and $F_k^{H_i}\left(O_k(x)\right)\le\gamma_i((i=1\, to\, n\in N))$, for all k

Where the parameter $t>0$. Utilizing additional variables μ_i,κ_i and γ_i, the given problem (5) can be converted to the problem (8)

HTNHMFA $\quad Max\ \left(\sum_i\mu_i+\sum_i\kappa_i-\sum_i\gamma_i\right)$

Subject to

$$\alpha_1\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\mu_1;\alpha_2\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\mu_2$$

$$,..,\alpha_n\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\mu_n,\beta_1\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\kappa_1,$$

$$\beta_2\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\kappa_2,...,\beta_n\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\kappa_n$$

$$\lambda_1\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left((O_k(x))^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right)\le\gamma_1,\lambda_2\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left((O_k(x))^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right)\le\gamma_2,$$

$$...,\lambda_n\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left((O_k(x))^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right)\le\gamma_n \qquad \mu_i\ge\kappa_i,\mu_i\ge\gamma_i,0\le\mu_i+\kappa_i+\gamma_i\le 3 \qquad ,$$

$\mu_i,\kappa_i,\gamma_i\in(0,1)$ and $\alpha_i,\beta_i,\lambda_i\in(0,1)$, for all $(i=1\,to\,n\in N)$ (8)

$\tau_k=\frac{6}{U_k-L_k}$. All the constraints of (4).

Theorem 3: There is only one OS $(x^*,\mu^*,\kappa^*,\gamma^*)$ of problem (8) (HTNHMFA) which is likewise an efficient solution for the issue (4) where $\mu^*=(\mu_1^*,\mu_2^*,.......,\mu_n^*)$, $\kappa^*=(\kappa_1^*,\kappa_2^*,.......,\kappa_n^*)$ and $\gamma^*=(\gamma_1^*,\gamma_2^*,.......,\gamma_n^*)$.

Proof: Suppose that $(x^*,\mu^*,\kappa^*,\gamma^*)$ be the only OS of (8) which is an inefficient to solving the problem (4). Then there exist different feasible alternative $x'(x'\ne x^*)$ of (4), so that $O_k(x^*)\le O_k(x')$ $\forall$ k and $O_k(x^*)<O_k(x')$ for at least one k.

Therefore, $\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\le\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}$ $\forall$ k and $\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}<\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}$ for at least one k.

Hence,

$$\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right)\le\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right),$$

$$\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right)<\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right)$$ for at least one k

Similarly,

$$\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right)\le\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right)$$ and

$$\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right)<\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right) \text{ for}$$

at least one k.

Again

$$\underset{k}{Min}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x^*)\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right)\geq\underset{k}{Min}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x')\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right),$$

$$\underset{k}{Min}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x^*)\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right)>\underset{k}{Min}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x')\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right) \text{ for}$$

at least one k

Now, assume that

$$\mu^*=\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right)\text{and}$$

$$\mu'=\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right),$$

$$\kappa^*=\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right),$$

$$\kappa'=\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right),$$

$$\gamma^*=\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right)\text{and}$$

$$\lambda'=\underset{k}{Min}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x')\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right).$$

Then $\mu^*\leq(<)\mu'$, $\kappa^*\leq(<)\kappa'$ and $\gamma^*\geq(>)\gamma'$ which gives $\left(\mu^*+\kappa^*-\gamma^*\right)<\left(\mu'+\kappa'-\gamma'\right)$ that implies the solution is not optimal which contradicts that $\left(x^*,\mu^*,\kappa^*,\gamma^*\right)$ is a unique OS of (8). As a result, it is a successful problem-solving strategy (8). Thus, the proof is finished.

A numerical example is given in Appendix A.

5.1. Solution procedure for MOSOP using NHFPA.

Step 1. The MOSOP (1) may be solved as a single objective by focusing on one objective at a time subject to the constraints given. Determine the values of the decision variables (DVs) and goal functions.

Step 2. Calculate the values of the remaining objectives based on the values of these DVs.

Step 3. For the remaining objective functions, repeat Steps 1 and 2.

Step 4: Then, according to step 3, the pay-off matrix may be shown as follows:

$$\begin{array}{c} \\ A^1 \\ A^2 \end{array}\begin{array}{cc} W(A) & \rho(A) \\ \left[\begin{array}{cc} W(A^1) & \rho(A^1) \\ W(A^2) & \rho(A^2) \end{array}\right] \end{array}$$

Step 5: The upper and lower bounds are $U_1 = \max\{W(A^1), W(A^2)\}$, $L_1 = \min\{W(A^1), W(A^2)\}$ for weight function $W(A)$,where $W(A) \in [L_1, U_1]$ and the upper and lower limits of objective are $U_2 = \max\{\rho(A^1), \rho(A^2)\}$, $L_2 = \min\{\rho(A^1), \rho(A^2)\}$ for deflection function $\rho(A)$,where $\rho(A) \in [L_2, U_2]$ are identified.

Step 6: Now the NHFPA for MOSOP with linear (or exponential or hyperbolic) accuracy, uncertainty, and falsehood neutrosophic membership functions yield equivalent MONLPP as.

$$\begin{aligned} &Max \min T_k^{\Omega_i}(W(A)); Max \min T_k^{\Omega_i}(\rho(A)); \\ &Max \min I_k^{\Omega_i}(W(A)), Max \min I_k^{\Omega_i}(\rho(A)); \\ &Min \max F_k^{\Omega_i}(W(A)); Min \max F_k^{\Omega_i}(\rho(A)) \end{aligned} \tag{9}$$

Subject to $\sigma(A) \le [\sigma_0]$ $A \in [A_{\min}, A_{\max}], \Omega_i = L_i, H_i, E_i; i = 1, 2, ..., n$ and $x_i \in [L_i, U_i]$ $(i = 1 \, to \, n \in N)$.

Now, by utilizing the arithmetic aggregation operator, the equation (9) can be expressed in the subsequent manner:

$$Max \zeta = \frac{\mu_1 + \mu_2 + ... + \mu_n}{n} + \frac{\kappa_1 + \kappa_2 + ... + \kappa_n}{n} - \frac{\gamma_1 + \gamma_2 + ... + \gamma_n}{n}$$

Subject to

$$\begin{aligned} &T_k^{\Omega_1}(W(A)) \ge \mu_1, T_k^{\Omega_2}(W(A)) \ge \mu_2, ..., T_k^{\Omega_n}(W(A)) \ge \mu_n \\ &I_k^{\Omega_1}(W(A)) \ge \kappa_1, I_k^{\Omega_2}(W(A)) \ge \kappa_2, ..., I_k^{\Omega_n}(W(A)) \ge \kappa_n \\ &F_k^{\Omega_1}(W(A)) \le \gamma_1, F_k^{\Omega_2}(W(A)) \le \gamma_2, ..., F_k^{\Omega_n}(W(A)) \le \gamma_n \\ &T_k^{\Omega_1}(\rho(A)) \ge \mu_1, T_k^{\Omega_2}(\rho(A)) \ge \mu_2, ..., T_k^{\Omega_n}(\rho(A)) \ge \mu_n \\ &I_k^{\Omega_1}(\rho(A)) \ge \kappa_1, I_k^{\Omega_2}(\rho(A)) \ge \kappa_2, ..., I_k^{\Omega_n}(\rho(A)) \ge \kappa_n \\ &F_k^{\Omega_1}(\rho(A)) \le \gamma_1, F_k^{\Omega_2}(\rho(A)) \le \gamma_2, ..., F_k^{\Omega_n}(\rho(A)) \le \gamma_n \end{aligned} \tag{10}$$

Subject to $\sigma(A) \le [\sigma_0]$ $A \in [A_{\min}, A_{\max}], \Omega_i = L_i, H_i, E_i; i = 1, 2, ..., n$ and $x_i \in [L_i, U_i]$ $(i = 1 \, to \, n \in N)$.

$A \ge 0,\ \mu_n, \kappa_n, \gamma_n \in (0,1); \mu_n + \kappa_n + \gamma_n \le 3, \mu_n \ge \kappa_n, \mu_n \ge \gamma_n, \forall n.$

Step 8: An appropriate mathematical programming algorithm can easily solve the above non-linear programming problem (10).

5.2. Numerical solution of a three-bar truss MOSOP

A well-known planar truss framework of three bars is depicted in Figure 2 in order to decrease the mass of the structure $W(A_1, A_2)$ and decrease the vertical bending at loading point $\rho(A_1, A_2)$ of a statically loaded three-bar planar truss under stress $\sigma_i(A_1, A_2)$ limitations on each of the truss elements $i = 1,2,3$.

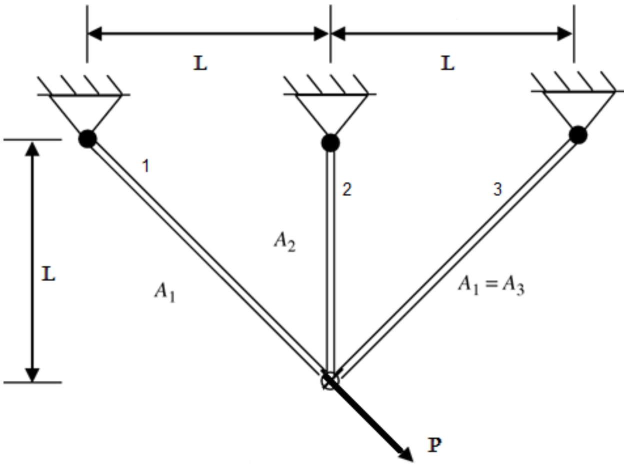

Figure 2. Design of the three-bar planar truss

In the following way, the MOSOP may be expressed:

$$\begin{aligned}
&\text{Minimize } W(A_1, A_2) = \delta L\left(2\sqrt{2}A_1 + A_2\right),\\
&\text{Minimize } \rho(A_1, A_2) = \frac{PL}{Y\left(A_1 + \sqrt{2}A_2\right)}\\
&\text{subject to } \sigma_1(A_1, A_2) \equiv \frac{P\left(\sqrt{2}A_1 + A_2\right)}{2A_1A_2 + \sqrt{2}A_1^2} \le \left[\sigma_1^T\right],\\
&\sigma_2(A_1, A_2) \equiv \frac{P}{A_1 + \sqrt{2}A_2} \le \left[\sigma_2^T\right],\\
&\sigma_3(A_1, A_2) \equiv \frac{PA_2}{2A_1A_2 + \sqrt{2}A_1^2} \le \left[\sigma_3^C\right],\\
&A_i \in \left[A_i^{\min}, A_i^{\max}\right], i = 1,2.
\end{aligned} \quad ..(11)$$

where, applied load= P ; material density= δ , L= Length of each bar, $\left[\sigma_i^T\right]$=maximum tensile stress limit for $i = 1,2$. $\left[\sigma_3^C\right]$=maximum compressive stress limit, Y = Young's modulus, A_1= cross sections of bar 1 and bar 3 and A_2 = cross section of bar 2.

The input information for MOSOP (11) are as follows:

P (applied force) = 20 KN, δ (material density)= 100 KN/m^3, L (bar length) = 1 m, $[\sigma_T]$ (maximum tensile stress limit for bars 1 and 3)= 20 KN/m^2, $[\sigma_C]$ (maximum limit of compressive stress for bar 2)= 15 KN/m^2, Y (Young's modulus)= 2×10^8 KN/m^2, range of bar cross section $0.1\times10^{-4}m^2 \le A_1, A_2 \le 5\times10^{-4}m^2$.

Solution: The tabulated values obtained in payoff matrix according to step 2 is as follows:

$$\begin{array}{c} \\ A^1 \\ A^2 \end{array}\begin{array}{cc} W(A_1,A_2) & \rho(A_1,A_2) \\ \left[\begin{array}{cc} 2.638958 & 14.64102 \\ 19.14214 & 1.656854 \end{array}\right] \end{array}$$

Here, $W_U^T = W_U = 19.12412$, $W_L^T = W_L = 2.638958$, $\rho_U^T = \rho_U = 14.64102$, $\rho_L^T = \rho_L = 1.656854$,

$$W_L^I = 19.12412 - s_1, W_U^I = W_U^T = 19.12412, \rho_L^I = 14.64102 - s_2, \rho_U^I = \rho_U^T = 14.64102,$$
$$W_U^F = W_U^T = 19.12412, W_L^F = 2.638958 + t_1, \ \rho_U^\upsilon = \rho_U^\mu = 14.64102, \rho_L^\upsilon = 1.656854 + t_2$$

where, $s_1, t_1 \in (19.12412 - 2.638958)$ and $s_2, t_2 \in (14.64102 - 1.656854)$.

Using the Linear type hesitant membership functions approach (LTHMFA) (6) the problem (11) equivalent to the following (12)

$$Maximize \ \xi = \frac{1}{3}\left(\sum_{i=1}^{3}\mu_i + \sum_{i=1}^{3}\kappa_i - \sum_{i=1}^{3}\gamma_i\right) \tag{12}$$

Subject to

$$\left(2\sqrt{2}A_1 + A_2\right)^t + \left((19.14214)^t - (2.63896)^t\right)\mu_1 / 0.98 \le (19.14214)^t$$
$$\left(2\sqrt{2}A_1 + A_2\right)^t + \left((19.14214)^t - (2.63896)^t\right)\mu_2 / 0.99 \le (19.14214)^t$$
$$\left(2\sqrt{2}A_1 + A_2\right)^t + \left((19.14214)^t - (2.63896)^t\right)\mu_3 \le (19.14214)^t$$

$$\left(2\sqrt{2}A_1 + A_2\right)^t + (s_1)^t \kappa_1 / 0.98 \le (19.14214)^t$$
$$\left(2\sqrt{2}A_1 + A_2\right)^t + (s_1)^t \kappa_2 / 0.99 \le (19.14214)^t$$
$$\left(2\sqrt{2}A_1 + A_2\right)^t + (s_1)^t \kappa_3 \le (19.14214)^t$$

$$\left(2\sqrt{2}A_1 + A_2\right)^t - (2.63896)^t - (t_1)^t \le \left((19.14214)^t - (2.63896)^t - (t_1)^t\right)\gamma_1 / 0.98$$
$$\left(2\sqrt{2}A_1 + A_2\right)^t - (2.63896)^t - (t_1)^t \le \left((19.14214)^t - (2.63896)^t - (t_1)^t\right)\gamma_2 / 0.99$$
$$\left(2\sqrt{2}A_1 + A_2\right)^t - (2.63896)^t - (t_1)^t \le \left((19.14214)^t - (2.63896)^t - (t_1)^t\right)\gamma_3$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left((14.64102)^t-(1.65685)^t\right)\mu_1/0.98\le(14.64102)^t$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left((14.64102)^t-(1.65685)^t\right)\mu_2/0.99\le(14.64102)^t$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left((14.64102)^t-(1.65685)^t\right)\mu_3\le(14.64102)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+(s_2)^t\kappa_1/0.98\le(14.64102)^t$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+(s_2)^t\kappa_2/0.99\le(14.64102)^t$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+(s_2)^t\kappa_3\le(14.64102)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-(1.65685)^t-(t_2)^t\le\left((14.64102)^t-(1.65685)^t-(t_2)^t\right)\gamma_1/0.98$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-(1.65685)^t-(t_2)^t\le\left((14.64102)^t-(1.65685)^t-(t_2)^t\right)\gamma_2/0.99$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-(1.65685)^t-(t_2)^t\le\left((14.64102)^t-(1.65685)^t-(t_2)^t\right)\gamma_3$$

$\mu_i\ge\kappa_i,\mu_i\ge\gamma_i,\mu_i+\kappa_i+\gamma_i\le3,\mu_i,\kappa_i,\gamma_i\in(0,1)$ for $i=1,2,3$ and all the constraints of (11).

Using the Exponential type hesitant membership functions approach (ETHMFA) (7) the problem (11) equivalent to the following (13)

$$Maximize\ \xi=\frac{1}{3}\left(\sum_{i=1}^{3}\mu_i+\sum_{i=1}^{3}\kappa_i-\sum_{i=1}^{3}\gamma_i\right) \tag{13}$$

Subject to

$$\left(2\sqrt{2}A_1+A_2\right)^t-\left((19.14214)^t-(2.63896)^t\right)\ln\left(1-\frac{\mu_1}{0.98}\right)/\psi\le(19.14214)^t$$
$$\left(2\sqrt{2}A_1+A_2\right)^t-\left((19.14214)^t-(2.63896)^t\right)\ln\left(1-\frac{\mu_2}{0.99}\right)/\psi\le(19.14214)^t$$
$$\left(2\sqrt{2}A_1+A_2\right)^t-\left((19.14214)^t-(2.63896)^t\right)\ln(1-\mu_3)/\psi\le(19.14214)^t$$

$$\left(2\sqrt{2}A_1+A_2\right)^t+(s_1)^t\ln\left(1-\frac{\kappa_1}{0.98}\right)/\psi\le(19.14214)^t$$
$$\left(2\sqrt{2}A_1+A_2\right)^t+(s_1)^t\ln\left(1-\frac{\kappa_2}{0.99}\right)/\psi\le(19.14214)^t$$
$$\left(2\sqrt{2}A_1+A_2\right)^t+(s_1)^t\ln(1-\kappa_3)/\psi\le(19.14214)^t$$

$$\left(2\sqrt{2}A_1+A_2\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\le\left(\left(19.14214\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\right)\left(-\ln\left(1-\frac{\gamma_1}{0.98}\right)/\psi\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\le\left(\left(19.14214\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\right)\left(-\ln\left(1-\frac{\gamma_2}{0.99}\right)/\psi\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\le\left(\left(19.14214\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\right)\left(-\ln\left(1-\gamma_3\right)/\psi\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(\left(14.64102\right)^t-\left(1.65685\right)^t\right)\ln\left(1-\frac{\mu_1}{0.98}\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(\left(14.64102\right)^t-\left(1.65685\right)^t\right)\ln\left(1-\frac{\mu_2}{0.99}\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(\left(14.64102\right)^t-\left(1.65685\right)^t\right)\ln\left(1-\mu_3\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left(s_1\right)^t\ln\left(1-\frac{\kappa_1}{0.98}\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left(s_1\right)^t\ln\left(1-\frac{\kappa_2}{0.99}\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left(s_1\right)^t\ln\left(1-\kappa_3\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\le\left(\left(14.64102\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\right)\left(-\ln\left(1-\frac{\gamma_1}{0.98}\right)/\psi\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\le\left(\left(14.64102\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\right)\left(-\ln\left(1-\frac{\gamma_2}{0.98}\right)/\psi\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\le\left(\left(14.64102\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\right)\left(-\ln\left(1-\gamma_3\right)/\psi\right)$$

$\mu_i\ge\kappa_i,\mu_i\ge\gamma_i,\mu_i+\kappa_i+\gamma_i\le 3,\mu_i,\kappa_i,\gamma_i\in(0,1)$ for $(i=1\,to\,n\in N)$ and all the constraints of (11).

Using the Hyperbolic type hesitant membership functions approach (HTHMFA) (8) the problem (11) equivalent to the following (14)

$$Maximize\ \xi=\frac{1}{3}\left(\sum_{i=1}^{3}\mu_i+\sum_{i=1}^{3}\kappa_i-\sum_{i=1}^{3}\gamma_i\right)\tag{14}$$

Subject to

$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(\frac{2\mu_1}{0.98}-1\right)\le\frac{\tau_{W(A)}}{2}\left(\left(19.14214\right)^t+\left(2.63896\right)^t\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(\frac{2\mu_2}{0.99}-1\right)\le\frac{\tau_{W(A)}}{2}\left(\left(19.14214\right)^t+\left(2.63896\right)^t\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(2\mu_3-1\right)\le\frac{\tau_{W(A)}}{2}\left(\left(19.14214\right)^t+\left(2.63896\right)^t\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(\frac{2\kappa_1}{0.98}-1\right)\le\frac{\tau_{W(A)}}{2}\left(2\times(19.14214)^t-(s_1)^t\right)$$
$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(\frac{2\kappa_2}{0.99}-1\right)\le\frac{\tau_{W(A)}}{2}\left(2\times(19.14214)^t-(s_1)^t\right)$$
$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(2\kappa_3-1\right)\le\frac{\tau_{W(A)}}{2}\left(2\times(19.14214)^t-(s_1)^t\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}-\tanh^{-1}\left(\frac{2\gamma_1}{0.98}-1\right)\le\frac{\tau_{W(A)}}{2}\left((19.14214)^t+(2.63896)^t+(t_1)^t\right)$$
$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}-\tanh^{-1}\left(\frac{2\gamma_2}{0.99}-1\right)\le\frac{\tau_{W(A)}}{2}\left((19.14214)^t+(2.63896)^t+(t_1)^t\right)$$
$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}-\tanh^{-1}\left(2\gamma_3-1\right)\le\frac{\tau_{W(A)}}{2}\left((19.14214)^t+(2.63896)^t+(t_1)^t\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(\frac{2\mu_1}{0.98}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(\frac{2\mu_2}{0.99}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(2\mu_3-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(\frac{2\kappa_1}{0.98}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left(2\times(14.64102)^t-(s_2)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(\frac{2\kappa_2}{0.99}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left(2\times(14.64102)^t-(s_2)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(2\kappa_3-1\right)\le\frac{\tau_{\rho(A)}}{2}\left(2\times(14.64102)^t-(s_2)^t\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}-\tanh^{-1}\left(\frac{2\gamma_1}{0.98}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t+(t_2)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}-\tanh^{-1}\left(\frac{2\gamma_2}{0.99}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t+(t_2)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}-\tanh^{-1}\left(2\gamma_3-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t+(t_2)^t\right)$$

Where $\tau_{W(A)}=\dfrac{6}{19.14214-2.638958}$ and $\tau_{\rho(A)}=\dfrac{6}{14.64102-1.656854}$

$\mu_i\ge\kappa_i,\mu_i\ge\gamma_i,\mu_i+\kappa_i+\gamma_i\le3,\mu_i,\kappa_i,\gamma_i\in(0,1)$ for $(i=1\,to\,n\in N)$ and all the constraints of (11).
On solving the neutrosophic hesitant optimization model (12), (13) and (14) the solution outcomes are outlined in Table 2 and Table 3.

Table 1. Input data for MOSOP (11)

P (KN) (Applied load)	$\delta(KN/m^3)$ (Material density)	L (Length)	$[\sigma_T]\,(KN/m^2)$ (maximum limit of tensile stress)	$[\sigma_C]\,(KN/m^2)$ (maximum limit of compressive stress)	Y (KN/m^2) (Young's modulus)	$A_i^{\min}\left(10^{-4}m^2\right)$ and $A_i^{\max}\left(10^{-4}m^2\right)$ (Cross section of bars)
20	100	1	20	15	2×10^8	$A_1^{\min}=0.1$ $A_1^{\max}=5$ $A_2^{\min}=0.1$ $A_2^{\max}=5$

A Comparative result of MOSOP (11) on basis of different membership function is given in table 2.

Table 2: A comparative optimal results on structural weight and deflection for t=2

Membership functions	Methods	$A_1\times10^{-4}m^2$	$A_2\times10^{-4}m^2$	$W(A_1,A_2)10^2KN$	$\rho(A_1,A_2)10^{-7}m$
Linear Type	FO [10]	0.5995887	3.789761	5.485654	3.356200
	IFO [45]	0.5766526	3.694181	5.325201	3.447673
	NFO	0.581611	3.462786	5.140011	3.628012
	Proposed NHFT	0.5932745	3.391146	5.069180	3.711209
Exponential Type	FO	0.5985788	3.779858	5.472895	3.364678
	IFO	0.5765578	3.678758	5.309510	3.460728
	NFO	0.5965065	3.437476	5.124651	3.664459
	Proposed NHFT	0.5934251	3.399846	5.078306	3.702652
Hyperbolic Type	FO	0.8535467	5.000000	7.414195	2.523782
	IFO	0.8354725	5.000000	7.363073	2.529551
	NFO	0.7994567	5.000000	7.261205	2.541127
	Proposed NHFT	0.7934604	5.000000	7.244245	2.543064

FO: Fuzzy Optimization; IFO: Intuitionistic Fuzzy Optimization; NFO: Neutrosophic fuzzy optimization

A comparative analysis for MOSOP based on several techniques using different membership functions as linear, exponential, hyperbolic types are shown in the Table 2. For all membership functions, it is obvious that the objective values are much superior to other current methods. Furthermore, the proposed NHFT performance measurements for different membership functions may be represented as Hyperbolic>Exponential>Linear. (9.787309 > 8.947535 > 8.780389(sum of weight and deflection)). However, the maximum acceptance degree of our suggested NHFT approach is better attained, demonstrating its superiority over other existing methods.

Table 3: Result comparison with minimizing the indeterminacy membership and maximizing the indeterminacy membership under proposed method at t=2

	Membership functions	$A_1 \times 10^{-4} m^2$	$A_2 \times 10^{-4} m^2$	$W(A_1, A_2) 10^2 KN$	$\rho(A_1, A_2) 10^{-7} m$
Maximize indeterminacy membership	Linear	0.5932745	3.391146	5.069180	3.711209
	Exponential	0.5934251	3.399846	5.078306	3.702652
	Hyperbolic	0.7934604	5.000000	7.244245	2.543064
Minimize indeterminacy membership	Linear	0.5925640	3.350599	5.026623	3.751623
	Exponential	0.5927716	3.362361	5.038972	3.739808
	Hyperbolic	0.7942572	5.000000	7.246499	2.542807

The comparison of proposed method under maximize and minimize indeterminacy membership function are displayed in the Table 3. From the above table, it is evident that the objective values under maximizing the indeterminacy membership are quite better than minimizing the indeterminacy membership under proposed method. However, our suggested approach's highest attainment of acceptance level is more effectively reached and demonstrates its superiority over reducing uncertainty membership level.

Table 4: Optimal results of different acceptance tolerance on Structural Weight and Deflection for t=2

Acceptance tolerance	Membership functions	$A_1 \times 10^{-4} m^2$	$A_2 \times 10^{-4} m^2$	$W(A_1, A_2) 10^2 KN$	$\rho(A_1, A_2) 10^{-7} m$
s_1=0.96, s_2=0.98, t_1=0.78, t_2=0.86	Linear	0.5932745	3.391145	5.069179	3.711208
	Exponential	0.5928106	3.364578	5.041298	3.737591
	Hyperbolic	0.7943072	5.000000	7.246640	2.542790
s_1=0.95, s_2=0.98, t_1=0.88, t_2=0.86	Linear	0.5929295	3.371356	5.048413	3.730824
	Exponential	0.5929295	3.371356	5.048414	3.730824
	Hyperbolic	0.7934604	5.000000	7.244245	2.543064

5.3 Sensitivity Analysis

A comparative study for MOSOP based on various acceptance tolerances was conducted using the suggested NHFP approach using linear, exponential, and hyperbolic membership functions. The compromise solution based on various membership functions is presented in Table 2. This result is showing sensitivity in Table 4 with different tolerances. It also shows that the neutrosophic optimization technique with exponential membership functions gives the lightest structural weight and the hyperbolic membership functions gives the least deflection at loading point.

6. Conclusion and Future Research Scope

We developed a MOSOM in a NHF fuzzy environment in this article. A computational algorithm for solving multi-objective structural models using neutrosophic hesitant fuzzy optimization has been developed. We have discussed a comparative study to identify the best optimal result using different membership functions. A three-bar truss numerical example, it shows that exponential membership function gives lightest structural weight whereas hyperbolic membership function gives least deflection in loading point. This method is simple and easy to use.

Our proposed approach might be used in the following fields of research:

- Our proposed method may be used in linear optimization problems with hesitantly and uncertainty.
- It may use in real life decision making of multi objective transportations and assignment problems with interval values.
- It can be expanded to handle issues involving multi objective fractional programming.
- For better decision making, it might be applied in game theory as well as goal programming problem with uncertainty and hesitation.
- It may be implemented in multi objective stochastic linear programming problem.

Our suggested computational technique can be further enhanced for the agricultural, industrial and health management as well, and it may be successfully applied in the variety of field like aircraft control system development, chemical engineering where in multiple objectives with multiple objectives, supply chain management, and industrial neural network architecture.

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Appendix A

Experimental Study

To demonstrate the effectiveness and validity of the suggested approach, we illustrate the numerical instance of formulating a MONLPP as presented below:

$$M_1: \quad Minimize \quad f_1(x) = x_1^{-1} x_2^{-2}$$
$$Minimize \quad f_2(x) = 2x_1^{-2} x_2^{-3}$$
$$\text{s.t } x_1 + x_2 \leq 1, x_1, x_2 \geq 0.$$

By solving each objective function separately as stated in (M_1), we obtain the subsequent optimal solution, lower and upper limit for each objective. $X^1 = (0.333, 0.667), X^2 = (0.4, 0.6)$ along with $L_1 = 6.75, U_1 = 6.94, L_2 = 57.87$ and $U_2 = 60.75$.

Linear type membership functions

For f_1: The membership functions of first objective as.

$$Tf_k^{L_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ 0.98\left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{L_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ 0.99\left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{L_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ \left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{L_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ 0.98\left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{L_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ 0.99\left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{L_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ \left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{L_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.98\left[\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{L_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.99\left[\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{L_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ \left[\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

For f_2: The membership functions of second objective as. (Linear type)

$$Tf_k^{L_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 57.87\\ 0.98\left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{L_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_{21}(x)\le 57.87\\ 0.99\left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{L_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 57.87\\ \left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{L_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ 0.98\left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{L_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ 0.99\left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{L_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ \left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{L_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.98\left[\dfrac{\left(2x_1^{-2}x_2^{-3}\right)^t-(57.87)^t-(t_2)^t}{(60.75)^t-(57.87)^t-(t_2)^t}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{L_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.99\left[\dfrac{\left(2x_1^{-2}x_2^{-3}\right)^t-(57.87)^t-(t_2)^t}{(60.75)^t-(57.87)^t-(t_2)^t}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{L_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ \left[\dfrac{\left(2x_1^{-2}x_2^{-3}\right)^t-(57.87)^t-(t_2)^t}{(60.75)^t-(57.87)^t-(t_2)^t}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

Which is transformed into an equivalent MONLPP with linear type as:

$$Max\ \zeta_1=\frac{\mu_1+\mu_2+\mu_3}{3}+\frac{\kappa_1+\kappa_2+\kappa_3}{3}-\frac{\gamma_1+\gamma_2+\gamma_3}{3}$$

$$Tf_k^{Li}(f_i)\ge\mu_i,\ If_k^{Li}(f_i)\ge\kappa_i,\ Ff_k^{Li}(f_i)\le\gamma_i$$

$$x_1+x_2\le 1,\ x_1,x_2\ge 0;\ 0\le\mu_i,\kappa_i,\gamma_i\le 1;\ 0\le t_1,t_2\le 1,\ 0\le s_1,s_2\le 1$$

$$\mu_i\ge\kappa_i,\ \mu_i\ge\gamma_i,\ \mu_i+\kappa_i+\gamma_i\le 3 \text{ for } i=1,2,3;\ k=1,2$$

Exponential type membership functions

For f_1: The membership functions of first objective as.

$$Tf_k^{E_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right)\right\}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{E_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right)\right\}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{E_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ \left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right)\right\}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{E_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right)\right\}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{E_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right)\right\}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{E_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ \left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right)\right\}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{E_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right)\right\}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{E_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right)\right\}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{E_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ \left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right)\right\}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

For f_2: The membership functions of second objective as. (Exponential type)

$$Tf_k^{E_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 57.87\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right)\right\}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{E_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 57.87\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right)\right\}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{E_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 57.87\\ \left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right)\right\}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{E_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right)\right\}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{E_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right)\right\}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{E_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ \left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right)\right\}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{E_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(57.87)^t-(t_2)^t}{(6.94)^t-(57.87)^t-(t_2)^t}\right)\right\}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{E_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(57.87)^t-(t_2)^t}{(6.94)^t-(57.87)^t-(t_2)^t}\right)\right\}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{E_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ \left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(57.87)^t-(t_2)^t}{(6.94)^t-(57.87)^t-(t_2)^t}\right)\right\}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

Which is transformed into an equivalent MONLPP with exponential type as:

$$Max\ \zeta_2=\frac{\mu_1+\mu_2+\mu_3}{3}+\frac{\kappa_1+\kappa_2+\kappa_3}{3}-\frac{\gamma_1+\gamma_2+\gamma_3}{3}$$

$$Tf_k^{Ei}(f_i)\ge\mu_i, If_k^{Ei}(f_i)\ge\kappa_i, Ff_k^{Ei}(f_i)\le\gamma_i$$

$$x_1+x_2\le 1, x_1,x_2\ge 0; 0\le\mu_i,\kappa_i,\gamma_i\le 1; 0\le t_1,t_2\le 1, 0\le s_1,s_2\le 1$$

$$\mu_i\ge\kappa_i,\mu_i\ge\gamma_i,\mu_i+\kappa_i+\gamma_i\le 3,\psi=4 \quad \text{for } i=1,2,3; k=1,2$$

Hyperbolic type membership functions

For f_1: The membership functions of first objective as.

$$Tf_k^{H_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75\\ 0.98\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\dfrac{(6.94)^t+(6.75)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{H_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75\\ 0.99\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\dfrac{(6.94)^t+(6.75)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{H_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75\\ \left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\dfrac{(6.94)^t+(6.75)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{H_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.94-s_1\\ 0.98\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\dfrac{2(6.94)^t-(s_1)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{H_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.94-s_1\\ 0.99\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\dfrac{2(6.94)^t-(s_1)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{H_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.94-s_1\\ \left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\frac{2(6.94)^t-(s_1)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{H_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.98\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(x_1^{-1}x_2^{-2}\right)^t-\frac{(6.94)^t+(6.75)^t+(t_1)^t}{2}\right\}\tau_{f_1(x)}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{H_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.99\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(x_1^{-1}x_2^{-2}\right)^t-\frac{(6.94)^t+(6.75)^t+(t_1)^t}{2}\right\}\tau_{f_1(x)}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{H_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ \left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(x_1^{-1}x_2^{-2}\right)^t-\frac{(6.94)^t+(6.75)^t+(t_1)^t}{2}\right\}\tau_{f_1(x)}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

For f_2: The membership functions of second objective as. (Hyperbolic type)

$$Tf_k^{H_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87\\ 0.98\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\frac{(60.75)^t+(57.87)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{H_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87\\ 0.99\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\frac{(60.75)^t+(57.87)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{H_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87\\ \left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\frac{(60.75)^t+(57.87)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{H_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 60.75-s_2\\ 0.98\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\dfrac{2(60.75)^t-(s_2)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{H_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 60.75-s_2\\ 0.99\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\dfrac{2(60.75)^t-(s_2)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{H_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 60.75-s_2\\ \left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\dfrac{2(60.75)^t-(s_2)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{H_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.98\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\left(2x_1^{-2}x_2^{-3}\right)^t-\dfrac{(60.75)^t+(57.87)^t+(t_2)^t}{2}\right\}\tau_{f_2(x)}\right] & \text{if } 57.87+t_2\le f_2(x)\le 6.94\\ 1 & \text{if } f_2(x)\ge 6.94\end{cases}$$

$$Ff_k^{H_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.99\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\left(2x_1^{-2}x_2^{-3}\right)^t-\dfrac{(60.75)^t+(57.87)^t+(t_2)^t}{2}\right\}\tau_{f_2(x)}\right] & \text{if } 57.87+t_2\le f_2(x)\le 6.94\\ 1 & \text{if } f_2(x)\ge 6.94\end{cases}$$

$$Ff_k^{H_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ \left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\left(2x_1^{-2}x_2^{-3}\right)^t-\dfrac{(60.75)^t+(57.87)^t+(t_2)^t}{2}\right\}\tau_{f_2(x)}\right] & \text{if } 57.87+t_2\le f_2(x)\le 6.94\\ 1 & \text{if } f_2(x)\ge 6.94\end{cases}$$

Which is reduced to equivalent MONLPP with hyperbolic type as:

$$Max\ \zeta_3=\frac{\mu_1+\mu_2+\mu_3}{3}+\frac{\kappa_1+\kappa_2+\kappa_3}{3}-\frac{\gamma_1+\gamma_2+\gamma_3}{3}$$

$$Tf_k^{Hi}(f_i)\ge\mu_i,\ If_k^{Hi}(f_i)\ge\kappa_i,\ Ff_k^{Hi}(f_i)\le\gamma_i$$

$$x_1+x_2\le 1,\ x_1,x_2\ge 0;\ 0\le\mu_i,\kappa_i,\gamma_i\le 1;\ 0\le t_1,t_2\le 1,\ 0\le s_1,s_2\le 1$$

$$\mu_i\ge\kappa_i,\ \mu_i\ge\gamma_i,\ \mu_i+\kappa_i+\gamma_i\le 3,\quad \tau_{f_k(x)}=\frac{6}{U_k-L_k}\quad \text{for } i=1,2,3;\ k=1,2$$

At t = 2, the OS of the MONLPP using the suggested NHFPA under various membership functions are as given below:

Method	Membership function type	x_1	x_2	$f_1(x)$	$f_2(x)$
Proposed NHFT	Linear	0.3659222	0.6340778	6.797139	58.59018
	Exponential	0.3656531	0.6343469	6.796371	58.60181
	Hyperbolic	0.3595183	0.6373039	6.848346	59.77892

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Generalized Symmetric Neutrosophic Fuzzy Matrices

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Abstract –We develop the concept of range symmetric Neutrosophic Fuzzy Matrix and Kernel symmetric Neutrosophic Fuzzy Matrix analogous to that of an EP –matrix in the complex field. First we present equivalent characterizations of a range symmetric matrix and then derive equivalent conditions for a Neutrosophic Fuzzy Matrix to be kernel symmetric matrix and study the relation between range symmetric and kernel symmetric Neutrosophic Fuzzy Matrices. The idea of Kernel and k-Kernel Symmetric (k-KS) Neutrosophic Fuzzy Matrices (NFM) are introduced with an example. We present some basic results of kernel symmetric matrices. We show that k-symmetric implies k-Kernel symmetric but the converse need not be true. The equivalent relations between kernel symmetric, k-kernel symmetric and Moore-Penrose inverse of NFM are explained with numerical results.

Keywords: Range symmetric, Kernel symmetric, k-Kernel Symmetric, Moore-penrose inverse

1. Introduction

The concept of fuzzy set was introduced by Zadeh [1] in 1965. The traditional fuzzy sets are characterized by the membership value or the grade of membership value. Some- times it may be very difficult to assign the membership value for fuzzy sets. An intuitionistic fuzzy set introduced by Atanassov [2] is appropriate for such a situation. The intuitionistic fuzzy sets can only handle the incomplete information considering both the truth membership (or simply membership) and falsity-membership(or nonmembership)values. It does not handle the indeterminate and inconsistent information which exists in belief system. Smarandache [3] introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data.

For a fuzzy matrix P, if P^+ exists, then it coincide with P^T, Kim and Roush [4] have studied Generalized fuzzy matrices. A Fuzzy matrix P is range symmetric if $R[P] = R[P^T]$ implies and kernel symmetric $N(P)=N(P^T)$. It is well known that for complex matrices, the concept of range symmetric and kernel symmetric is identical. For Neutrosophic Fuzzy matrix $P \in (IF)_n$, is range symmetric R[P] = R[P^T] implies N(P) = N(P^T) but the converse need not be true. Meenakshi [5] introduced the notion of fuzzy matrix. Let k-be a fixed product of disjoint transpositions in Sn = 1, 2,...,n and K be the associated permutation matrix. Hill and Waters [6] have introduced on k-real and k-hermitian matrices. Baskett and Katz [7] have studied theorems on products of EPr matrices. Schwerdtfeger [8] has studied the notion of introduction to Linear Algebra and the Theory of matrices. Meenakshi and Jayashri [9] have studied k-Kernel Symmetric Matrices. Riyaz Ahmad Padder and Murugadas [10-12] introduced on idempotent intuitionistic fuzzy Matrices of T-type, reduction of a nilpotent intuitionistic fuzzy matrix using implication operator and determinant theory for intuitionistic fuzzy matrices. Atanassov has studied [13] generalized index matrices. Meenakshi and Krishnamoorthy introduced on k-EP matrices. Ben and Greville [14] developed the concept of range symmetric fuzzy matrix and kernel symmetric fuzzy matrix analogues to that of an EP matrix in the complex field. Sumathi, Arockiarani [15] have discussed new operations on fuzzy neutrosophic soft matrices. Sumathi, Arockiarani , Jency,[16] have studied Fuzzy neutrosophic soft opological spaces. Abdel-Monem, Nabeeh and Abouhawwash [17] have studied An Integrated Neutrosophic Regional Management Ranking Method for Agricultural Water Management. Ahmed Abdelhafeez , Hoda Mohamed, Nariman Khalil [18] have discussed Rank and Analysis Several Solutions of Healthcare Waste to Achieve Cost Effectiveness and Sustainability Using Neutrosophic MCDM Model. Manas Karak, Animesh Mahata, Mahendra Rong, Supriya Mukherjee, Sankar Prasad Mondal, Said Broumi, Banamali Roy [19] have introduced A Solution Technique of Transportation Problem in Neutrosophic Environment. Meenakshi and Krishnamoorthy [20] have discussed on κ-EP matrices.

As mentioned in the above introduction section, Meenakshi introduced the concept of Range symmetric and Meenakshi and Jayashri developed the notion of kernel symmetric in fuzzy matrix. Here, we have applied the concept of range symmetric and kernel symmetric in Neutrosophic fuzzy matrix (NFM). Both this concept plays a significant role in hybrid fuzzy structure and we have applied the same in NFM and studied some of the results in detail. First we present equivalent characterizations of a range and kernel symmetric matrix and then, derive equivalent conditions for an Neutrosophic fuzzy matrix to be kernel symmetric Neutrosophic fuzzy matrix and study the relation between range symmetric and kernel symmetric Neutrosophic fuzzy matrices. Equivalent condition for varies g-inverses of a kernel symmetric matrix to be kernel symmetric are determined.

2. PRELIMINARIES AND NOTATIONS

PRELIMINARIES

Let the function be defined as $\kappa(x)=(x_{k[1]}, x_{k[2]}, x_{k[3]},\ldots, x_{k[n]}) \in F_{n\times1}$ for $x = x_1, x_2,\ldots,x_n \in F_{[1\times n]}$, where K is involuntary, the following conditions are satisfied. The associated permutation matrix, where K is a permutation matrix, $KK^T = K^TK= I_n$ then $K^T = K$.

(P_1) $K = K^T$, $K^2 = I$ and $\kappa(x) = Kx$ for all $P \in (IF)_n$,

(P_2) N(P) = N(PK) = N(KP)

(P_3) $(PK)^+ = KP^+$ and $(KP)^+ = P^+K$ exists, if P^+ exists .

(P_4) P^T is a g-inverse of P iff P^+ exist

Notations: For NFM of P ∈ $(NF)_n$,

P^T : Transpose of P, R(P) : Row space of P, C(P) : Column space of P,N(P) : Null Space of P , P^+ : Moore-Penrose inverse of P , $(NF)_n$: Square Neutrosophic Fuzzy Matrix. $F_{[1\times n]}$: The matrix one row n columns. $F_{[n\times 1]}$:

3. DEFINITIONS AND THEOREMS

Definition: 1 Let P be a NFM, if R [P] = R [P^T] then P is called as range symmetric.

Example: 1 Let us consider NFM $P = \begin{bmatrix} (0.2,0.5,0.7) & (0,0,0) & (0.6,0.4,0.2) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ <0.6,0.4,0.2> & (0,0,0) & <0.3,0.5,0.7> \end{bmatrix}$,

The following matrices are not range symmetric

$$P = \begin{bmatrix} (1,1,0) & (1,1,0) & (0,0,0) \\ (0,0,0) & (1,1,0) & (1,1,0) \\ (0,0,0) & (0,0,0) & (1,1,0) \end{bmatrix}, \; P^T = \begin{bmatrix} (1,1,0) & (0,0,0) & (0,0,0) \\ (1,1,0) & (1,1,0) & (0,0,0) \\ (0,0,0) & (1,1,0) & (1,1,0) \end{bmatrix}$$

$$[(1,1,0)\ (1,1,0)\ (0,0,0)] \in R(\mathrm{P}) \text{ , } [(1,1,0)\ (1,1,0)\ (0,0,0)] \in R(\mathrm{P}^T)$$

$$[(0,0,0)\ (1,1,0)\ (1,1,0)] \in R(\mathrm{P}) \text{ , } [(0,0,0)\ (1,1,0)\ (1,1,0)] \in R(\mathrm{P}^T)$$

$$[(0,0,0)\ (0,0,0)\ (1,1,0)] \in R(\mathrm{P}) \text{ ,} [(0,0,0)\ (0,0,0)\ (1,1,0)] \notin R(\mathrm{P}^T)$$

$$R(\mathrm{P}) \notin R(\mathrm{P}^T)$$

Definition : 2 Let P ∈ F_n be a Neutrosophic fuzzy matrix , if N(P) = N(P^T) then P is called kernel symmetric NFM where N(P)= {x/xP = (0,0,0) and x ∈ $F_{1\times n}$},

Example: 2 Let us consider NFM $P = \begin{bmatrix} (0.4,0.5,0.6) & (0,0,0) & (0.6,0.4,0.8) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0.4,0.5,0.7) & (0,0,0) & (0.4,0.3,0.6) \end{bmatrix}$,

$$N(\mathrm{P}) = N(\mathrm{P}^T) = (0,0,0)$$

Definition 3. Unit Neutrosophic fuzzy matrix (UNFM)

If $(NF)_n$ is said to be UNFM if a_{ii} = (1,1,0) and a_{ij} = (0,1,1) $i \neq j$, for all $i = j$. It is denoted by I.

Example: 3 Let us consider NFM, $I = \begin{bmatrix} (1,1,0) & (0,1,1) & (0,1,1) \\ (0,1,1) & (1,1,0) & (0,1,1) \\ (0,1,1) & (0,1,1) & (1,1,0) \end{bmatrix}$

Definition 4. Symmetric Neutrosophic fuzzy matrix

If $(NF)_n$ is said to be symmetric Neutrosophic fuzzy matrix if $a_{ij} = a_{ji}$

Example: 4 Let us consider NFM $P = \begin{bmatrix} (0.3, 0.5, 0.8) & (0,0,0) & (0.5, 0.3, 0.1) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ <0.5, 0.3, 0.1> & (0,0,0) & <0.3, 0.5, 0.7> \end{bmatrix}$,

Definition 5. Permutation neutrosophic fuzzy matrix (PNFM)

Every row single (1,1,0) with (0,0,1) 's everywhere else is called PNFM.

Example: 5 Let us consider NFPM, $K = \begin{bmatrix} (0,0,1) & (0,0,1) & (1,1,0) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (1,1,0) & (0,0,1) & (0,0,1) \end{bmatrix}$

Definition 6 (Null neutrosophic fuzzy matrix) Neutrosophic fuzzy matrix is said to be Null if all its entries are zero, i.e., all elements are (0,0,0).

Example: 6 Let us consider NFM $P = \begin{bmatrix} (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) \end{bmatrix}$,

Note:1 For Neutrosophic fuzzy matrix $P \in F_n$ with det P > <0,0,0>, has non- zero rows and non-columns, hereafter N(P) = <0,0,0> = N(P^T). Furthermore, a symmetric matrix P = P^T, that is N(P)= N(P^T).

Theorem:1 For P,Q∈$(NF)_n$ and K be a Neutrosophic fuzzy permutation matrix , N(P) = N(Q) ⇔ N(KPK^T) = N(KQK^T)

Proof: Let $w \in N(KPK^T)$

$\Rightarrow w(KPK^T) = (0,0,0)$

$\Rightarrow zK^T = (0,0,0)$ where $z = wKP$

$\Rightarrow z \in N(K^T)$

Since, $\det K = \det K^T > (0,0,0)$ (By Note:1)

Therefore, $N(K^T) = (0,0,0)$

Hence, $z = (0,0,0)$

$\Rightarrow wKP = (0,0,0)$

$\Rightarrow wK \in N(\mathrm{P}) = N(\mathrm{Q})$

$\Rightarrow wKQK^T = (0,0,0)$

$\Rightarrow w \in N(KQK^T)$

$N(KPK^T) \subseteq N(KQK^T)$

Similarly, $N(KQK^T) \subseteq N(KPK^T)$

Therefore, N (KPK^T) = N (KQK^T)

Conversely, if N (KPK^T) = N (KQK^T), then by the above proof,

N (P) = N (K^T (KPK^T) K)

= N (K^T (KQK^T) K)

N (P) = N(Q).

Example: 5 Let us consider NFM

$$P = \begin{bmatrix} (0,0,0.5) & (0,0,0) & (0.3,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0.7,0,0) & (0,0,0) & (0.3,0.2,0) \end{bmatrix}, \mathrm{Q} = \begin{bmatrix} (0.3,0.4,0.2) & (0,0,0) & (0.4,0.2,0.6) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0.5,0.3,0.4) & (0,0,0) & (0.5,0.3,0.6) \end{bmatrix},$$

$$K = \begin{bmatrix} (0,0,1) & (0,0,1) & (1,1,0) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (1,1,0) & (0,0,1) & (0,0,1) \end{bmatrix}$$

Theorem: 2 For Neutrosophic P ∈ $(NF)_n$, the following statements are equivalent

(i) N(P) = N(P^T)

(ii) N (KPK^T) = N ($KP^T K^T$) for some permutation NFM K.

(iii) Neutrosophic fuzzy permutation matrices K such that $KPK^T = \begin{bmatrix} D & (0,0,0) \\ (0,0,0) & (0,0,0) \end{bmatrix}$

with det D > $(0,0,0)$

Proof: (i) iff (ii).This equivalence follows from the theorem (1)

(i) iff (iii): Let N(P) = N(P^T)

If det P > $(0,0,0)$ then P has no zero row and columns,

Hence (iii) holds by taking K = I and D = P itself.

Suppose det P = (0, 0, 0) then N (P) = N (P^T) then N (P) = N (P^T) $\neq (0,0,0)$

For $x \neq (0,0,0)$, $x \in N(P)$ equivalent to each non-zero co efficient x_i of x, the fuzzy sums

$\sum x_i a_{ik} = (0,0,0)$ and $\sum x_i a_{ki} = (0,0,0)$ for all k.

As a result, P's i^{th} column and i^{th} row are both filled with zeros.

Now, by appropriately permuting the rows and columns,

All of the zero rows and zero columns can be shifted to the bottom and right, respectively.

Therefore, P is of the form $KPK^T = \begin{bmatrix} D & (0,0,0) \\ (0,0,0) & (0,0,0) \end{bmatrix}$

Where D - square matrix, D has non- zero rows and non-zero columns.

Therefore, det D > (0,0,0)

Thus (iii) holds

(iii) implies (ii) : If det P > (0,0,0) by remark ,D is kernel symmetric,

$\begin{bmatrix} D & <0,0,0> \\ <0,0,0> & <0,0,0> \end{bmatrix}$ is also kernel symmetric (ii) holds.

Example : 6 Let us Consider NFM ,

$$P = \begin{bmatrix} (0,0,0.5) & (0,0,0) & (0.3,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0.7,0,0) & (0,0,0) & (0.3,0.2,0) \end{bmatrix}, K = \begin{bmatrix} (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \\ (0,0,1) & (1,1,0) & (0,0,1) \end{bmatrix}$$

$$PK^T = \begin{bmatrix} (0,0,0.5) & (0,0,0) & (0.3,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0.7,0,0) & (0,0,0) & (0.3,0.2,0) \end{bmatrix} \begin{bmatrix} (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \\ (0,0,1) & (1,1,0) & (0,0,1) \end{bmatrix} = \begin{bmatrix} (0,1,0.5) & (0.3,1,0) & (0,1,0) \\ (0,1,0) & (0,1,0) & (0,1,0) \\ (0.7,1,0) & (0.3,1,0) & (0,1,0) \end{bmatrix}$$

$$KPK^T = \begin{bmatrix} (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \\ (0,0,1) & (1,1,0) & (0,0,1) \end{bmatrix} \begin{bmatrix} (0,1,0.5) & (0.3,1,0) & (0,1,0) \\ (0,1,0) & (0,1,0) & (0,1,0) \\ (0.7,1,0) & (0.3,1,0) & (0,1,0) \end{bmatrix} = \begin{bmatrix} (0,0,0.5) & (0.3,0,0) & (0,0,0) \\ (0.7,0,0) & (0.3,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) \end{bmatrix}$$

$$\det(P) = (0,0,0), \ N(P) = N(P^T) = (0,0,0)$$

$KPK^T = \begin{bmatrix} D & (0,0,0) \\ (0,0,0) & (0,0,0) \end{bmatrix}$, Where $D = \begin{bmatrix} <0,0,0.5> & <0.3,0,0> \\ <0.7,0,0> & <0.3,0,0> \end{bmatrix}$, determined of D > (0,0,0)

Theorem:3 For P∈$(NF)_n$ is kernel symmetric Neutrosophic fuzzy matrix and K being a permutation matrix if and only if $N(KPK^T) = N(K P^T K^T)$

Proof: Let $x \in N(KPK^T)$

$\Rightarrow x(KAK^T) = (0,0,0)$

$\Rightarrow yK^T = (0,0,0)$ where $y = xKP$

$\Rightarrow y \in N(K^T)$

Since, $\det K = \det K^T >\ (0,0,0)$

Therefore, $N(K^T) =\ (0,0,0)$

Hence, $y = (0,0,0)$

$\Rightarrow xKP =\ (0,0,0)$

$\Rightarrow xK \in N(\mathrm{P}) = N(\mathrm{P}^T)$

$\Rightarrow xKA^T K^T = (0,0,0)$

$\Rightarrow x \in N(K\left(P^T\right)K^T)$

$N(KPK^T) \subseteq N(KP^T K^T)$

Similarly, $N(KP^T K^T) \subseteq N(KPK^T)$

Therefore, $N(KPK^T) = N(KP^T K^T)$

Conversely, if $N(KPK^T) = N(K P^T K^T)$, then by the above proof ,

$N(P) = N(K^T(KPK^T)K)$

$= N(K^T(K P^T K^T)K)$

$N(P) = N(P^T)$.

Example:7 Let us consider NFM

$$P=\begin{bmatrix}(0,0.2,0.5) & (0,0,0) & (0.3,0.4,0)\\ (0,0,0) & (0,0,0) & (0,0,0)\\ (0.7,0.2,0) & (0,0,0) & (0.3,0.2,0)\end{bmatrix},\ K=\begin{bmatrix}(0,0,1) & (0,0,1) & (1,1,0)\\ (0,0,1) & (1,1,0) & (0,0,1)\\ (1,1,0) & (0,0,1) & (0,0,1)\end{bmatrix}$$

Theorem: 4 For P∈$(NF)_n$, is kernel symmetric Neutrosophic fuzzy matrix, then N(PP^T) = N(P) = N(P^TP)

Proof: Let, $x \in N(\mathrm{P})$

$\Leftrightarrow$ xP = $(0,0,0)$

$\Leftrightarrow$ xPP^T = $(0,0,0)$

$\Leftrightarrow\ x \in N(PP^T)$

$\Leftrightarrow\ N(\mathrm{P}) \subseteq N(\mathrm{PP}^T)$

Similarly, $N(\mathrm{PP}^T) \subseteq N(\mathrm{P})$

Therefore, N (P) = N (PPT)

Similarly, $N(\mathrm{P}) = N(\mathrm{P}^T\mathrm{P})$

Therefore, N (PPT) = N (P) = N (PTP)

Example:8 Let us consider NFM

$$P=\begin{bmatrix}(0.4,0.5,0) & (0,0,0) & (0.6,0.4,0.2)\\ (0,0,0) & (0,0,0) & (0,0,0)\\ (0.4,0.5,0.3) & (0,0,0) & (0.4,0.3,0.5)\end{bmatrix},$$

Theorem: 5 Let P, Q be the NFM and K NFPM, R (P) = R (Q) ⇔ R (KPK^T) = R (KQK^T)

Proof: Let R (P) = R (Q)

Then, R (PK^T) = R (P) K^T

= R (P) K^T

= R (PK^T)

Let $z \in \{R(KPK^T)\}$

$z = w(KPK^T)$ for some $w \in V^n$

$z = rPK^T$, $r = wK$

$z \in R\left(PK^T\right) = R\left(Q\left(K^T\right)\right)$

$z = uQK^T$ for some $u \in V^n$

$z = \left(uK^T\right)KQK^T$

$z = vKQK^T$ for some $v \in V^n$

$z \in R(KQK^T)$

Therefore, $R(KPK^T) \subseteq R(KQK^T)$

Similarly, $R(KQK^T) \subseteq R(KPK^T)$

Therefore, $R(KPK^T) = R(KQK^T)$

Conversely, Let $R(KPK^T) = R(KQK^T)$. Then by above proof

$R(\mathrm{P}) = R[K^T(KPK^T)K]$

$= R[K^T(KQK^T)K]$

$= R(\mathrm{Q})$

$R(\mathrm{P}) = R(\mathrm{Q})$

Example:9 Let us consider NFM

$$P = \begin{bmatrix} (0.2,0.5,0.4) & (0,0,0) & (0.7,0.2,0.6) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0.7,0.2,0.6) & (0,0,0) & (0.3,0.2,0.4) \end{bmatrix}, \mathrm{Q} = \begin{bmatrix} (0.7,0.2,0.6) & (0,0,0) & (0.3,0.2,0.4) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0.2,0.5,0.4) & (0,0,0) & (0.7,0.2,0.6) \end{bmatrix}$$

$$K = \begin{bmatrix} (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \end{bmatrix}$$

$R(P) = R(Q) \Leftrightarrow R(KPK^T) = R(KQK^T)$

Theorem:6 For $P \in (NF)_n$ be the NFM and K NFPM, $R(P) = R(P^T) \Leftrightarrow R(KPK^T) = R(K\,P^T\,K^T)$

Example: 10 $P = \begin{bmatrix} (0.4,0.5,0.6) & (0,0,0) & (0.3,0.5,0.6) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0.3,0.5,0.6) & (0,0,0) & (0.3,0.2,0.4) \end{bmatrix}, K = \begin{bmatrix} (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \end{bmatrix}$

Theorem:7 Let P,Q be the Neutrosophic fuzzy matrix and K being a permutation matrix, C(P) = C(Q) ⇔ C(KPK^T) = C(KQK^T)

Example:11 Let us consider NFM

$$P = \begin{bmatrix} (0.2,0.5,0.6) & (0,0,0) & (0.7,0.2,0.8) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0.7,0.2,0.4) & (0,0,0) & (0.3,0.2,0.5) \end{bmatrix}, Q = \begin{bmatrix} (0.7,0.2,0.8) & (0,0,0) & (0.2,0.5,0.6) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0.3,0.2,0.5) & (0,0,0) & (0.7,0.2,0.4) \end{bmatrix}$$

C(P)= C(Q) ⇔ C(KPK^T) = C(KQK^T)

k-KERNEL SYMMETRIC NFM

Definition: 3 Let P be a NFM .If P belongs to $(NF)_n$ is called k-Kernel symmetric Neutrosophic fuzzy if N(P) = N(KP^{T}K)

Note:2 Let P is k-Symmetric NFM implies it is k-kernel symmetric NFM, for P = K(P^T)K spontaneously implies N(P) = N(KP^T K) .Example 12. shows that the if and only if need not be true.

Example: 12 Let us Consider NFM

$$P = \begin{bmatrix} (0,0,0.5) & (0,0,0.4) & (0.3,0.4,0.5) \\ (0.5,0.4,0.6) & (0.1,0.4,0.6) & (0,0,0.4) \\ (0.4,0.5,0.3) & (0.3,0.4,0.5) & (0,0,0.3) \end{bmatrix}, K = \begin{bmatrix} (0,0,1) & (0,0,1) & (1,1,0) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (1,1,0) & (0,0,1) & (0,0,1) \end{bmatrix}$$

$$KP^T K = \begin{bmatrix} (0,0,0.3) & (0,0,0.4) & (0.3,0.4,0.5) \\ (0.3,0,0.5) & (0.1,0,0.6) & (0,0.4,0.4) \\ (0.4,0,0.3) & (0.5,0,0.6) & (0,0.4,0.5) \end{bmatrix}$$

Therefore, P $\neq$ KP^T K

But, N (P) = N(KP^T K) = $(0,0,0)$

Theorem: 8 For Neutrosophic fuzzy matrix P∈ $(NF)_n$, the given statements are equivalent:

(i) N(P) = N(KP^{T}K)

(ii) N(KP) = N((KP)^T)

(iii) N(PK) = N((PK)^T)

(iv) N(P^T) = N(KP),

(v) $N(P) = N((PK)^T)$

(vi) P^+ is k-KSNFM

(vii) $N(P) = N(P^+K)$

(viii) $K P^+P = PP^+K$

(ix) $P^+PK = KPP^+$

Proof: (i) implies (ii)

$\Leftrightarrow$ $N(P) = N(KP^TK)$

$\Leftrightarrow$ $N(KP) = N(P^TK)$ (By P_2) ($K^2 = I$)

$\Leftrightarrow$ $N(KP) = N((KP)^T)$ (Because , $(KP)^T = P^TK^T = P^TK$)

$\Leftrightarrow$ KP is Kernel symmetric,

Therefore, (ii) holds

(i) Implies (iii)

$\Leftrightarrow$ $N(P) = N(KP^TK)$

$\Leftrightarrow$ $N(PK) = N(KP^T)$ (By P_2) ($K^2 = I$)

$\Leftrightarrow$ $N(PK) = N((PK)^T)$ (Because , $(PK)^T = K^TP^T = KP^T$)

PK is Kernel symmetric,

Therefore, (iii) holds

(ii) Implies (iv)

$\Leftrightarrow$ $N(KP) = N(KP)^T = N(P^TK)$

$\Leftrightarrow$ $N(KP) = N(P^T)$ (By P_2)

Therefore, (iv) holds

(iii) Implies (v)

$\Leftrightarrow$ $N(PK) = N((PK)^T)$

$\Leftrightarrow$ $N(P) = N((PK)^T)$ (By P_2)

(ii) Implies (vi)

$\Leftrightarrow$ $N(KP) = N(KP)^T$

$\Leftrightarrow$ $N(KP) = N(P^TK)$ (By P_2)

$\Leftrightarrow$ $N(KP) = N(P^+K)$ Since $N(KP^+K) = N(P^+K)$

$\Leftrightarrow$ $N(KP) = N(P^+)$

P^+ is k-Kernel symmetric IFM

(i) implies (vii)

$\Leftrightarrow$ $N(P) = N(KP^TK)$

$\Leftrightarrow$ $N(P) = N(KP^TK) = N(P^TK)$

$\Leftrightarrow$ $N(P) = N(KP)^T$

$\Leftrightarrow$ $N(P) = N(P^+K)$ (By P_2)

(i) Implies (viii)

PK is Kernel symmetric NFM

$\Leftrightarrow$ $(PK)(PK)^+ = (PK)^+(PK)$

$\Leftrightarrow$ $(PK)(KP^+) = (KP^+)(PK)$

$\Leftrightarrow$ $PP^+ = KP^+PK$

$\Leftrightarrow$ $PP^+K = KP^+P$

Thus equivalence of (iii) and (viii) is proved.

(viii) $\Leftrightarrow$ (ix): Since, by the property (P_1),$K^2=I$, this uniformity follows by pre- and post multiplying by K.

$\Leftrightarrow$ $KP^+P = PP^+ K$

$\Leftrightarrow$ $K^2 P^+ AK = KPP^+ K^2$

$\Leftrightarrow$ $P^+ PK = KPP^+$.

4. CONCLUSION

Here some Theorem is described regarding the properties of kernel and range symmetric Neutrosophic Fuzzy Matrices. We introduced the concept of Kernel and k-Kernel Symmetric Neutrosophic Fuzzy Matrices with suitable examples. In addition, we have investigated some results of κ − kernel symmetric Neutrosophic Fuzzy Matrices with examples. In future, we shall prove some related properties of g-inverse of k-Kernel Symmetric NFM.

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Augmented Latin Square Designs for Imprecise Data

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Abstract: This paper addresses a novel approach for analyzing augmented Latin square design with uncertain observations, the so-called neutrosophic augmented Latin square design (NALSD). The contribution of our work lies in estimating the effects of rows, columns, control and new treatments, as well as formulating their sums of squares. Moreover, by determining the neutrosophic hypotheses and decision rule, the F_N-statistic in NANOVA table is given. The performance of the proposed design is evaluated using a numerical example and simulation study. In light of the results observed, it can find that the NALSD performs better than the classic augmented Latin square design (ALSD) in the presence of uncertainty.
Keywords: Augmented Latin square design, neutrosophic statistics, imprecise data, neutrosophic ANOVA.

1. Introduction

In the field of experimental design, the Latin square is one of the most common designs to control systematic error by two-way blocking. In this design, each treatment occurs once, and only once, in each row and column. Thus, the number of treatments, rows, and columns are all equal. In this context, Fisher [1] was the first to apply Latin Square designs. Many studies have been published on this design; however, several problems arose when using large samples, such as many genotypes in the early stages of plant breeding. Researchers have devised an appropriate solution to this problem using augmented designs. The augmented design is appropriate since it incorporates many additional entries for various treatments. This design aims to compare new genotypes against standard treatments, known as checks. The first research on augmented design as a blocking design was conducted by Federer [2]. There have been several classes of augmented designs, including the augmented randomized complete block and augmented Latin squares [3, 4], augmented Lattice squares [5], and augmented row-column designs with a small number of checks [6]. A review of augmented designs has been given by Federer and Crossa [7]. In a newer study, an augmented design without replicating all treatments was discussed by Burgueño, et al. [8]. More about the augmented designs can be viewed in [9-15]. None of the above-mentioned researches is applicable if there is uncertainty in data set regarding to collected unreliable observation.

Recently, neutrosophic logic has been extensively studied by Smarandache [16]. Smarandache [17] developed the idea of basic neutrosophic statistics (NS) as an extension of classical basic statistics and suggested that these statistics can be used effectively in uncertain situations. The difference between fuzzy statistics, neutrosophic statistics, and classical statistics were explained by Aslam [18]. The concept of neutrosophic ANOVA was introduced by Aslam [19]. Neutrosophic analysis of covariance has been applied to completely randomized designs as well as randomized complete block designs and split-plot designs by AlAita and Aslam [20]. AlAita, et al. [21] provided a discussion on the application of neutrosophic statistical analysis in split-plot designs. AlAita and Talebi [22] furnished exact neutrosophic analysis of missing value in augmented randomized complete block design. Aslam and Albassam [23] proposed post hoc multiple comparison tests under NS. Salama, et al. [24] suggested neutrosophic correlation and simple linear regression. Nagarajan, et al. [25] discussed the analysis of neutrosophic multiple regression. Numerous neutrosophic statistical studies have been discussed in [26-33].

Based on our knowledge no research on augmented Latin square designs is in indeterminate environments. This study aims to solve problems associated with studies and experiments that use imprecise and uncertain data in augmented Latin square designs. Also, we developed our proposed design under NS to provide additional information on the indeterminacy measure that classic statistics cannot provide.

2. Neutrosophic Basic Definitions

This section provides some basic concepts about neutrosophic statistics that will be useful throughout of this paper. Throughout this paper, suppose that $X_N \in [X_L, X_U]$ is a neutrosophic random variable (NRV) that follows the neutrosophic normal distribution (NND).

Definition 1: Consider the neutrosophic random variable (NRV) $X_N = X_L + X_U I_N$, the neutrosophic population mean and variance can be found as follows:

$$\mu_N \in \left[\frac{\sum_{i=1}^{N} x_{Li}}{N}, \frac{\sum_{i=1}^{N} x_{Ui}}{N}\right];\ \mu_N \in [\mu_L, \mu_U\ ] \text{ and } \sigma_N^2 \in \left[\frac{\sum_{i=1}^{N}(x_{Li}-\mu_L)^2}{N}, \frac{\sum_{i=1}^{N}(x_{Ui}-\mu_U)^2}{N}\right];\ \sigma_N^2 \in [\sigma_L^2, \sigma_U^2],$$

where X_L and $X_U I_N$ are determinate and indeterminate parts, respectively, and $I_N \in [I_L, I_U]$ is the measure of uncertainty.

Definition 2: Suppose n be a neutrosophic random sample selected from a population of size N having indeterminate observations. The estimated neutrosophic sample mean $\bar{x}_N$ and the variance s_N^2, are expressed by

$$\bar{x}_N \in \left[\frac{\sum_{i=1}^{n} x_{Li}}{n}, \frac{\sum_{i=1}^{n} x_{Ui}}{n}\right];\ \bar{x}_N \in [\bar{x}_L, \bar{x}_U\ ] \text{ and } s_N^2 \in \left[\frac{\sum_{i=1}^{n}(x_{Li}-\bar{x}_L)^2}{n-1}, \frac{\sum_{i=1}^{n}(x_{Ui}-\bar{x}_U)^2}{n-1}\right];\ s_N^2 \in [s_L^2, s_U^2].$$

3. Neutrosophic Augmented Latin Square Design (NALSD)

3.1. Neutrosophic Model and Notations

Consider a $b \times b$ Latin square, the neutrosophic statistical model for a NALSD can be formulated as follows:

$$y_{Nhijkg} = \mu_N + \alpha_{Ni} + \beta_{Nj} + \tau_{Nqk} + \tau_{Nlijg} + \varepsilon_{Nhijkg}, \begin{cases} i = 1,2,\dots,b \\ j = 1,2,\dots,b \\ k = 1,2,\dots,b \\ g = 1,2,\dots,n_{(lij)} \end{cases} \quad (1)$$

The neutrosophic form of y_{Nhijkg} can be expressed as

$$y_{Nhijkg} = y_{Lhijkg} + y_{Uhijkg} I_N;\ I_N \in [I_L, I_U],$$

where $h = l\ or\ q$ stands for the neutrosophic effects associated with new treatments or checks, respectively, μ_N is a neutrosophic overall mean, α_{Ni} is the neutrosophic effect of the ith row, β_{Nj} is the neutrosophic effect of the jth column, τ_{Nqk} is the neutrosophic effect of the kth check, τ_{Nlijg} is the neutrosophic effect of the gth new treatment in ith row and jth column, and ε_{Nhijkg} is the neutrosophic random error assumed to have mean zero and variance σ_N^2. We denote $\text{v} = \sum_{i=1}^{b}\sum_{j=1}^{b} n_{(lij)}$ for the number of new treatments, c for the number of check treatments, a for the number of rows, and b for the number of columns; therefore, $e = \text{v} + b$ is the total number of new and check treatments and the total number of all plots in the blocks (rows and columns) is n; i.e., $n = \text{v} + b^2$. Throughout the paper in the context of neutrosophic ANOVA, the SS_{NT}, SS_{NR}, SS_{NC}, SS_{NTr}, and SS_{NE} stand for the neutrosophic sum of squares (NSS) total, row, column, treatment, and error, respectively and the subscript N denotes the neutrosophic context.

3.2. Estimation of Neutrosophic Parameters

To estimate the neutrosophic model parameters in a NALSD, first, the least squares normal equations (NE) are obtained and given below.

$\mu_N: (\mathrm{v} + b^2)\hat{\mu}_N + (b-1)\sum_{k=1}^{b}\hat{\tau}_{Nqk} + \sum_{i=1}^{b}\sum_{j=1}^{b} n_{(lij)}\,\hat{\alpha}_{Ni} + \sum_{j=1}^{b}\sum_{i=1}^{b} n_{(lij)}\,\hat{\beta}_{Nj} = y_{N....}$,

$\alpha_{Ni}: (b + \sum_{j=1}^{b} n_{(lij)})(\hat{\mu}_N + \hat{\alpha}_{Ni}) + \sum_{k=1}^{b}\hat{\tau}_{Nqk} + \sum_{j=1}^{b}\sum_{g=1}^{n(lij)}\hat{\tau}_{Nlijg} + \sum_{j=1}^{b} n_{(lij)}\hat{\beta}_{Nj} = y_{N.i..}$,

$\beta_{Nj}: (b + \sum_{i=1}^{b} n_{(lij)})(\hat{\mu}_N + \hat{\beta}_{Nj}) + \sum_{k=1}^{b}\hat{\tau}_{Nqk} + \sum_{i=1}^{b}\sum_{g=1}^{n(lij)}\hat{\tau}_{Nlijg} + \sum_{i=1}^{b} n_{(lij)}\hat{\alpha}_{Ni} = y_{N..j.}$,

$\tau_{Nqk}: b(\hat{\mu}_N + \hat{\tau}_{Nqk}) = y_{Nq..k}$,

$\tau_{Nlijg}: \hat{\mu}_N + \hat{\alpha}_{Ni} + \hat{\beta}_{Nj} + \hat{\tau}_{Nlijg} = y_{Nlijg}$.

By solving the above NE using the constraints $\sum_{i=1}^{b}\hat{\alpha}_{Ni} = 0$, $\sum_{j=1}^{b}\hat{\beta}_{Nj} = 0$, and $\sum_{k=1}^{b}\hat{\tau}_{Nqk} + \sum_{i=1}^{b}\sum_{j=1}^{b}\sum_{g=1}^{n(lij)}\hat{\tau}_{Nlijg} = 0$, the estimates of the neutrosophic parameters of the model (1) are

$\hat{\mu}_N = \frac{1}{(\mathrm{v}+b)}\left(y_{N....} - (b-1)\sum_{k=1}^{b}\bar{y}_{Nq..k}\right) = \frac{1}{(\mathrm{v}+b)}(y_{N....} - (b-1)m_N)$; $\hat{\mu}_N \in [\hat{\mu}_L, \hat{\mu}_U]$,

$\hat{\alpha}_{Ni} = \frac{1}{b}\left(y_{Nqi..} - \sum_{k=1}^{b}\bar{y}_{Nq..k}\right) = \frac{1}{b}\left(y_{Nqi..} - m_N\right)$; $\hat{\alpha}_{Ni} \in [\hat{\alpha}_{Li}, \hat{\alpha}_{Ui}]$,

$\hat{\beta}_{Nj} = \frac{1}{b}\left(y_{Nq.j.} - \sum_{k=1}^{b}\bar{y}_{Nq..k}\right) = \frac{1}{b}\left(y_{Nq.j.} - m_N\right)$; $\hat{\beta}_{Nj} \in [\hat{\beta}_{Lj}, \hat{\beta}_{Uj}]$,

$\hat{\tau}_{Nqk} = \frac{y_{Nq..k}}{b} - \hat{\mu}_N$; $\hat{\tau}_{Nqk} \in [\hat{\tau}_{Lqk}, \hat{\tau}_{Uqk}]$,

$\hat{\tau}_{Nlijg} = y_{Nlijg} - \hat{\alpha}_{Ni} - \hat{\beta}_{Nj} - \hat{\mu}_N$; $\hat{\tau}_{Nlijg} \in [\hat{\tau}_{Llijg}, \hat{\tau}_{Ulijg}]$,

where $i, j, k = 1,2,\ldots,b$, $g = 1,2,\ldots,n_{(lij)}$ and $m_N = \sum_{i=1}^{b}\bar{y}_{Nqi..} = \sum_{j=1}^{b}\bar{y}_{Nq.j.} = \sum_{k=1}^{b}\bar{y}_{Nq..k}$.

In the same manner, the estimation of the parameters in corresponding neutrosophic treatment-reduced, row-reduced, and column-reduced models can be obtained.

3.3. Neutrosophic Testing of Parameters

Under the normality assumption of the data, it can use the ANAVO method to test neutrosophic parameters in NALSD. Therefore, we need to formulate the SS_{NT} and neutrosophic adjusted (adj) and unadjusted (unadj) sums of squares for rows, columns, treatments (new and check), and the NSS for error. Following, the calculated sums of squares are given.

$SS_{NT} = \sum_{i=1}^{b}\sum_{j=1}^{b}\sum_{k=1}^{b} y_{Nqijk}^2 + \sum_{i=1}^{b}\sum_{j=1}^{b}\sum_{g=1}^{n(lij)} y_{Nlijg}^2 - \frac{y_{N....}^2}{n}$; $SS_{NT} \in [SS_{LT}, SS_{UT}]$,

$SS_{NR(\text{unadj})} = \sum_{i=1}^{b}\frac{y_{N.i..}^2}{b+n_{(lij)}} - \frac{y_{N....}^2}{n}$; $SS_{NR(\text{unadj})} \in \left[SS_{LR(\text{unadj})}, SS_{UR(\text{unadj})}\right]$,

$SS_{NC(\text{unadj})} = \sum_{j=1}^{b}\frac{y_{N..j.}^2}{b+n_{(lij)}} - \frac{y_{N....}^2}{n}$; $SS_{NC(\text{unadj})} \in \left[SS_{LC(\text{unadj})}, SS_{UC(\text{unadj})}\right]$,

$SS_{NTr(\text{unadj})} = \frac{1}{b}\sum_{k=1}^{b} y_{Nq..k}^2 + \sum_{i=1}^{b}\sum_{j=1}^{b}\sum_{g=1}^{n(lij)} y_{Nlijg}^2 - \frac{y_{N....}^2}{n}$; $SS_{NTr(\text{unadj})} \in \left[SS_{LTr(\text{unadj})}, SS_{UTr(\text{unadj})}\right]$,

$SS_{NR(\text{adj.})} = \frac{1}{b}\left[\sum_{i=1}^{b}(y_{Nqi..} - m_N)\,y_{N.i..} - \sum_{i=1}^{b}\sum_{j=1}^{b}\sum_{g=1}^{n(lij)}(y_{Nqi..} - m_N)y_{Nlijg}\right]$; $SS_{NR(\text{adj})}\left[SS_{LR(\text{adj})}, SS_{UR(\text{adj})}\right]$,

$SS_{NC(\text{adj.})} = \frac{1}{b}\left[\sum_{j=1}^{b}(y_{Nq.j.} - m_N)\,y_{N..j.} - \sum_{j=1}^{b}\sum_{i=1}^{b}\sum_{g=1}^{n(lij)}(y_{Nq.j.} - m_N)y_{Nlijg}\right]$; $SS_{NC(\text{adj})}\left[SS_{LC(\text{adj})}, SS_{UC(\text{adj})}\right]$,

$SS_{NTr(\text{adj})} = \frac{1}{b}\left(\sum_{i=1}^{b} y_{Nqi..}^2 + \sum_{j=1}^{b} y_{Nq.j.}^2 + \sum_{k=1}^{b} y_{Nq..k}^2\right) - \frac{\left(\sum_{i=1}^{b} y_{N.i..}^2 + \sum_{j=1}^{b} y_{N..j.}^2\right)}{(b+\mathrm{v})} + \sum_{i=1}^{b}\sum_{j=1}^{b}\sum_{g=1}^{n(lij)} y_{Nlijg}^2 - 2m_N^2 + \frac{y_{N....}^2}{n}$; $SS_{NTr(\text{adj})} \in \left[SS_{LTr(\text{adj})}, SS_{UTr(\text{adj})}\right]$,

$SS_{NCheck} = \frac{1}{b}\sum_{k=1}^{b} y_{Nq..k}^2 - \frac{y_{Nq....}^2}{b^2}$; $SS_{NCheck} \in [SS_{LCheck}, SS_{UCheck}]$,

$SS_{Nnew} = \sum_{i=1}^{b}\sum_{j=1}^{b}\sum_{g=1}^{n(lij)} y_{Nlijg}^2 - \frac{y_{Nl....}^2}{\mathrm{v}}$; $SS_{Nnew} \in [SS_{Lnew}, SS_{Unew}]$,

$SS_{N\text{new and new}\times\text{ch}} = SS_{NTr(\text{adj})} - SS_{NCheck}$; $SS_{N\text{new and new}\times\text{ch}} \in [SS_{L\text{new and new}\times\text{ch}}, SS_{U\text{new and new}\times\text{ch}}]$,

$SS_{N\text{new}\times\text{check}} = SS_{NTr(\text{unadj})} - SS_{NCheck} - SS_{Nnew}$; $SS_{N\text{new}\times\text{check}} \in [SS_{L\text{new}\times\text{check}}, SS_{U\text{new}\times\text{check}}]$, and

$SS_{NE} = SS_{NT} - SS_{NTr(\text{adj})} - SS_{NR(\text{unadj})} - SS_{NC(\text{unadj})}$.

Neutrosophic mean squares for all source of variations are obtained in the ranges of the form $[MS_{L(.)}, MS_{U(.)}]$. Based on the calculated MSEs, the neutrosophic test statistics F_N are:

$F_{NTr(\text{adj})} = \frac{MS_{NTr(\text{adj})}}{MS_{NE}}$; $F_{NTr(\text{adj})} \in [F_{LTr(\text{adj})}, F_{UTr(\text{adj})}]$,
$F_{NR(\text{adj})} = \frac{MS_{NR(\text{adj})}}{MS_{NE}}$; $F_{NR(\text{adj})} \in [F_{LR(\text{adj})}, F_{UR(\text{adj})}]$,
$F_{NC(\text{adj})} = \frac{MS_{NC(\text{adj})}}{MS_{NE}}$; $F_{NC(\text{adj})} \in [F_{LC(\text{adj})}, F_{UC(\text{adj})}]$,
$F_{NCheck} = \frac{MS_{NCheck}}{MS_{NE}}$; $F_{NCheck} \in [F_{LCheck}, F_{UCheck}]$,
$F_{Nnew} = \frac{MS_{Nnew}}{MS_{NE}}$; $F_{Nnew} \in [F_{Lnew}, F_{Unew}]$,
$F_{N\text{new and new} \times \text{ch}} = \frac{MS_{N\text{new and new} \times \text{ch}}}{MS_{NE}}$; $F_{N\text{new and new} \times \text{ch}} \in [F_{L\text{new and new} \times \text{ch}}, F_{U\text{new and new} \times \text{ch}}]$, and
$F_{N\text{new} \times \text{check}} = \frac{MS_{N\text{new} \times \text{check}}}{MS_{NE}}$; $F_{N\text{new} \times \text{check}} \in [F_{L\text{new} \times \text{check}}, F_{U\text{new} \times \text{check}}]$.

The neutrosophic form of F_N is $F_N = F_L + F_U I_{F_N}$; $I_{F_N} \in [I_{F_L}, I_{F_U}]$, where F_L and $F_U I_{F_N}$ are determinate and indeterminate parts of each proposed test. This test reduces to a test under classic statistics if $I_{F_N} = 0$.

3.4. Neutrosophic Hypotheses and Decision Rules

In order to test the rows, columns, checks, and new treatments, the null and alternative hypotheses are as follows, respectively:

$H_{N0}: \alpha_{Ni} = 0$ vs $H_{N1}: at\ least\ one\ \alpha_{Ni} \neq 0,\ i = 1.2.,,,.b,$
$H_{N0}: \beta_{Nj} = 0$ vs $H_{N1}: at\ least\ one\ \beta_{Nj} \neq 0,\ j = 1.2.,,,.b,$
$H_{N0}: \tau_{Nqk} = 0$ vs $H_{N1}: at\ least\ one\ \tau_{Nqk} \neq 0,\ k = 1.2.,,,.b,$
$H_{N0}: \tau_{Nlijg} = 0$ vs $H_{N1}: at\ least\ one\ \tau_{Nlijg} \neq 0,\ g = 1.2.,,,.n_{(lij)}.$

The null hypothesis does not reject if $min\{p_N - value\} > \alpha$, where α is a level of significance. Meanwhile, we reject the null hypothesis if $max\{p_N - value\} \leq \alpha$.
All the above testing process are summarized in the NANOVA Tables 1 and 2 for the NALSD under NS.

Table 1 NANOVA Table (A) for NALSD

Sources of variation	Ndf	NSS	NMS	F_N-value
Rows (unadj)	$b-1$	$SS_{NB(\text{unadj})}$	$\frac{SS_{NR(\text{unadj})}}{b-1}$	
Columns (unadj)	$b-1$	$SS_{NB(\text{unadj})}$	$\frac{SS_{NC(\text{unadj})}}{b-1}$	
Treatments (adj)	$b+\text{v}-1$	$SS_{NTr(\text{adj})}$	$\frac{SS_{NTr(\text{adj})}}{b+\text{v}-1}$	$\frac{MS_{NTr(\text{adj})}}{MS_{NE}}$
Checks	$b-1$	SS_{NCheck}	$\frac{SS_{NCheck}}{b-1}$	$\frac{MS_{NCheck}}{MS_{NE}}$
New and New × Check	v	$SS_{N\text{new and new} \times \text{ch}}$	$\frac{SS_{N\text{new and new} \times \text{ch}}}{\text{v}}$	$\frac{MS_{N\text{new and new} \times \text{ch}}}{MS_{NE}}$
Error	$(b-1)(b-2)$	SS_{NE}	$\frac{SS_{NE}}{(b-1)(b-2)}$	
Total	$n-1$	SS_{NT}		

Table 2 NANOVA Table (B) for NALSD

Sources of variation	Ndf	NSS	NMS	F_N-value

Rows (adj)	$b-1$	$SS_{NB(\text{adj})}$	$\frac{SS_{NR(\text{adj})}}{b-1}$	$\frac{MS_{NR(\text{adj})}}{MS_{NE}}$
Columns (adj)	$b-1$	$SS_{NB(\text{adj})}$	$\frac{SS_{NC(\text{adj})}}{b-1}$	$\frac{MS_{NC(\text{adj})}}{MS_{NE}}$
Treatments (unadj)	$c+\text{v}-1$	$SS_{NTr(\text{unadj})}$	$\frac{SS_{NTr(\text{unadj})}}{c+\text{v}-1}$	
Checks	$c-1$	SS_{NCheck}	$\frac{SS_{NCheck}}{c-1}$	$\frac{MS_{NCheck}}{MS_{NE}}$
New treatments	$\text{v}-1$	SS_{Nnew}	$\frac{SS_{Nnew}}{\text{v}-1}$	$\frac{MS_{Nnew}}{MS_{NE}}$
New × Check	1	$SS_{N\text{new} \times \text{check}}$	$\frac{SS_{N\text{new} \times \text{check}}}{1}$	$\frac{MS_{N\text{new} \times \text{check}}}{MS_{NE}}$
Error	$(b-1)(b-2)$	SS_{NE}	$\frac{SS_{NE}}{(b-1)(b-2)}$	
Total	$n-1$	SS_{NT}		

4. Numerical Examples and Simulation

In this section, the performance of the proposed design is numerically assessed by an example and a simulation study. For assessing the efficiency of the proposed methods, the proposed tests $F_N \in [F_L, F_U]$ of the proposed design under NS are calculated and compared with the existing tests under classic statistics.

4.1. Numerical Example

In this example, we have generated neutrosophic data for NALSD. Five neutrosophic check treatments named A, B, C, D and E, and 50 neutrosophic new treatments, denoted by 1,2,…,50, are arranged in an augmented Latin square with 5 rows and 5 columns. The neutrosophic data are given in Table 5.
Using the computational software R, we can obtain neutrosophic data randomly for this example by running the following code

```
y_L<-rnorm(75,40,10)
z<-length(y_L)
I<-rnorm(75,3,0.5)
y_U<-c()
for(i in 1:z){
 y_U[i]<-y_L[i]+I[i]}
```

We applied the proposed method to calculate F_N-tests, where $F_N \in [F_L, F_U]$. The corresponding NANOVA results for the NALSD are presented in Tables 3 and 4.

4.2. Simulation Study

This section evaluates the quality of the proposed F test for NALSD using simulated data from the Monte Carlo (MC) procedure for the proposed model (1). In this study, MC simulations have been performed 10,000 times. The data have been generated using neutrosophic normal standard distribution. Furthermore, the neutrosophic variances have been assumed to be homogeneous, and the NALSDs are balanced. Also, to simulate type I error, the significance levels of 0.05 and 0.01 have been chosen as the initial values. Moreover, it has been assumed that the treatments all have zero mean under the null hypothesis. It has

Table 3 ANOVA Table (A) for the NALSD

Sources of variation	Ndf	NSS	NMS	F_N	Neutrosophic form F_N	p_N-value
Rows (unadj)	4	[373.906, 409.200]	[93.476, 102.300]			
Columns (unadj)	4	[355.476, 347.226]	[88.869, 86.806]			
Treatments (adj)	54	[6312.786, 6486.414]	[116.903, 120.119]	[1.054,1.062]	$1.054 + 1.062I_{F_N};\ I_{F_N} \in [0, 0.007]$	[0.493, 0.486]
Checks	4	[163.869, 181.964]	[40.967, 45.491]	[0.369,0.402]	$0.369 + 0.402I_{F_N};\ I_{F_N} \in [0, 0.082]$	[0.826, 0.804]
New and New × Check	50	[6148.917, 6304.450]	[122.978, 126.089]	[1.109,1.115]	$1.109 + 1.115I_{F_N};\ I_{F_N} \in [0, 0.005]$	[0.449, 0.444]
Error	12	[1331.077, 1357.583]	[110.923, 113.132]			
Total	74	[8373.244, 8600.422]				

Table 4 ANOVA Table (B) for the NALSD

Sources of variation	Ndf	NSS	NMS	F_N	Neutrosophic form F_N	p_N-value
Rows (adj)	4	[591.937, 613.179]	[147.984, 153.295]	[1.334,1.355]	$1.334 + 1.355I_{F_N};\ I_{F_N} \in [0, 0.015]$	[0.313, 0.306]
Columns (adj)	4	[193.643, 198.487]	[48.411, 49.622]	[0.436,0.439]	$0.436 + 0.439I_{F_N};\ I_{F_N} \in [0, 0.007]$	[0.780, 0.778]
Treatments (unadj)	54	[6256.587, 6431.174]	[115.863, 119.096]			
Checks	4	[163.869, 181.964]	[40.967, 45.491]	[0.369,0.402]	$0.369 + 0.402I_{F_N};\ I_{F_N} \in [0, 0.082]$	[0.826, 0.804]
New treatments	49	[6085.317, 6239.574]	[124.190, 127.338]	[1.120,1.126]	$1.120 + 1.126I_{F_N};\ I_{F_N} \in [0, 0.005]$	[0.440, 0.436]
New × Check	1	[7.401, 9.637]	[7.401, 9.637]	[0.067,0.085]	$0.067 + 0.085I_{F_N};\ I_{F_N} \in [0,0.212]$	[0.801, 0.775]
Error	12	[1331.077, 1357.583]	[110.923, 113.132]			
Total	74	[8373.244, 8600.422]				

been compared the significance levels and power of the test for the proposed test with the existing test under classic statistics.

To calculate the neutrosophic empirical Type I error rate and the test power for an MC experiment; the following steps need to be completed:

MC simulation for computing $\alpha_{\text{Empirical}}$

Step 1: We generate the random sample $x_{N1}^{(i)}, x_{N2}^{(i)}, \dots, x_{Nn}^{(i)}$ from the neutrosophic normal standard distribution under H_{N0}, $i = 1,2, \dots, 10000$.

Step 2: We compute the F_{Ni}-test under H_{N0}.

Step 3: We record the results by recording $I_{Ni} = 1$ when the H_{N0} is rejected, and $I_{Ni} = 0$ otherwise.

Step 4: We compute the ratio $\frac{1}{10000}\sum_{i=1}^{10000} I_{Ni}$ and take it as $\alpha_{\text{Empirical}}$.

MC simulation for computing $Power_{Empirical}$

Step 1: We generate the random sample $x_{N1}^{(i)}, x_{N2}^{(i)}, \dots, x_{Nn}^{(i)}$ from the neutrosophic normal standard distribution under H_{N1}, $i = 1,2, \dots, 10000$. For instance, $(\mu_{N1}, \mu_{N2}, \mu_{N3}, \mu_{N4}) = (1,2,3,4)$.

Step 2: We compute the F_{Ni}-test under H_{N1}.

Step 3: We record the results by recording $I_{Ni} = 1$ when the H_{N1} is rejected, and $I_{Ni} = 0$ otherwise.

Step 4: We compute the ratio $\frac{1}{10000}\sum_{i=1}^{10000} I_{Ni}$ and take it as $Power_{Empirical}$.

Table 5 Data for NALSD

Row	Column 1			Column 2			Column 3			Column 4			Column 5		
1	C [41.82, 44.92]	19 [39.79, 43.76]	37 [36.57, 39.68]	44 [25.12, 28.5]	8 [28.33, 31.23]	A [46.65, 50.38]	25 [37.95, 41.12]	B [35.98, 38.8]	35 [44.6, 47.91]	E [35.84, 38.27]	2 [42.52, 45.4]	26 [33.44, 36.42]	10 [32.99, 36.1]	D [31.88, 34.88]	24 [26.79, 29.21]
2	46 [36.41, 39.49]	12 [47.62, 50.68]	E [32.62, 36.32]	39 [28.09, 31.2]	C [55.38, 58.39]	4 [44.2, 47.73]	A [38.73, 42.47]	16 [60.19, 63.64]	11 [41.38, 44.3]	27 [38.9, 41.03]	D [38, 41.06]	28 [36.38, 39.33]	5 [45.34, 48.56]	45 [33.31, 36.51]	B [31.93, 34.42]
3	34 [39.22, 41.81]	D [36, 38.08]	21 [32.41, 36.03]	15 [39.74, 42.45]	1 [20.84, 23.21]	E [30, 32.73]	C [36.74, 39.25]	42 [29.98, 33.29]	36 [39.55, 42.18]	7 [39.95, 42.29]	B [20.34, 22.76]	32 [38.65, 40.58]	20 [36.98, 40.51]	38 [49.72, 52.26]	A [46.27, 50.09]
4	17 [47.01, 49.78]	B [49.68, 52.08]	6 [11.91, 14.94]	D [34.69, 37.77]	13 [53.81, 57.21]	31 [52.44, 56.24]	E [22.18, 24.44]	30 [40.75, 43.62]	9 [36.94, 39.56]	22 [38.69, 41.36]	A [29.62, 33.42]	41 [51.11, 54.6]	C [17.42, 20.22]	49 [33.59, 36.65]	33 [27.33, 30.05]
5	A [33.52, 36.34]	23 [19.12, 22.91]	43 [36.9, 39.8]	18 [11.39, 14.4]	48 [63.66, 66.76]	B [44.54, 47.56]	47 [43.05, 45.73]	D [48.56, 51.31]	29 [43.98, 47.66]	3 [49.46, 52.71]	50 [56.28, 60.34]	C [52.67, 55.82]	40 [17.73, 19.86]	E [45.29, 48.38]	14 [43.91, 47.75]

Table 6 Simulation results for NALSD with parameters $(b = 4, \mathrm{v} = 32, n = 48)$ *for Check treatment means* $(\mu_{N1}, \mu_{N2}, \mu_{N3}, \mu_{N4})$ *and different values of new treatments* $(\mu_{Ni} = 0, \mu_{Nj} = 0, \mu_{Nk} = 1, \mu_{Nl} = 2)$*,* $i = 1, \ldots, 10, j = 11, \ldots, 20, k = 21, \ldots, 30, l = 31, \ldots, 40$.

Test	α	Mean $\alpha_{\text{Empirical}}$	Mean $Power_{Empirical}$ $\delta_1 = (0,1,1,2)$	$\delta_2 = (1,2,2,3)$	$\delta_3 = (1,1,3,3)$	$\delta_4 = (0,1,2,3)$	$\delta_5 = (0,1,3,4)$	$\delta_6 = (0,3,4,4)$	$\delta_7 = (0,3,4,5)$	$\delta_8 = (0,2,4,6)$
NALSD	0.01	[0.0088, 0.0093]	[0.0412,0.0420]	[0.0688, 0.0734]	[0.0840, 0.0905]	[0.1072, 0.1188]	[0.1314, 0.1502]	[0.2130, 0.2366]	[0.2821, 0.3144]	[0.3514, 0.3940]
	0.05	[0.0476, 0.0477]	[0.1647, 0.1800]	[0.2463, 0.2732]	[0.2977, 0.3203]	[0.3495, 0.3857]	[0.4202, 0.4550]	[0.5702, 0.6135]	[0.6696, 0.7036]	[0.7493, 0.7884]

Table 7 Simulation results for NALSD with parameters $(b = 5, \mathrm{v} = 50, n = 75)$ *for Check treatment means* $(\mu_{N1}, \mu_{N2}, \mu_{N3}, \mu_{N4}, \mu_{N5})$ *and different values of new treatments* $(\mu_{Ni} = 0, \mu_{Nj} = 0, \mu_{Nk} = 1, \mu_{Nl} = 1, \mu_{Nu} = 2)$*,* $i = 1, \ldots, 10, j = 11, \ldots, 20, k = 21, \ldots, 30, l = 31, \ldots, 40, u = 41, \ldots, 50$.

Test	$\boldsymbol{\alpha}$	Mean $\alpha_{\text{Empirical}}$	Mean $Power_{Empirical}$ $\delta_1 = (0,0,1,1,1)$	$\delta_2 = (0,1,1,2,2)$	$\delta_3 = (0,1,2,2,2)$	$\delta_4 = (1,1,2,2,3)$	$\delta_5 = (2,2,3,3,3)$	$\delta_6 = (2,3,3,3,4)$	$\delta_7 = (0,1,3,4,4)$	$\delta_8 = (0,2,4,5,6)$
NALSD	0.01	[0.0094,0.0099]	[0.0581,0.0683]	[0.0820,0.0980]	[0.0980, 0.1207]	[0.1159, 0.1442]	[0.2317, 0.2865]	[0.3651, 0.4467]	[0.4485, 0.5319]	[0.8435, 0.9010]
	0.05	[0.0485,0.0494]	[0.2006, 0.2314]	[0.2668, 0.3080]	[0.2975, 0.3495]	[0.3616, 0.4112]	[0.5460, 0.6190]	[0.7164, 0.7826]	[0.7885, 0.8474]	[0.9845, 0.9945]

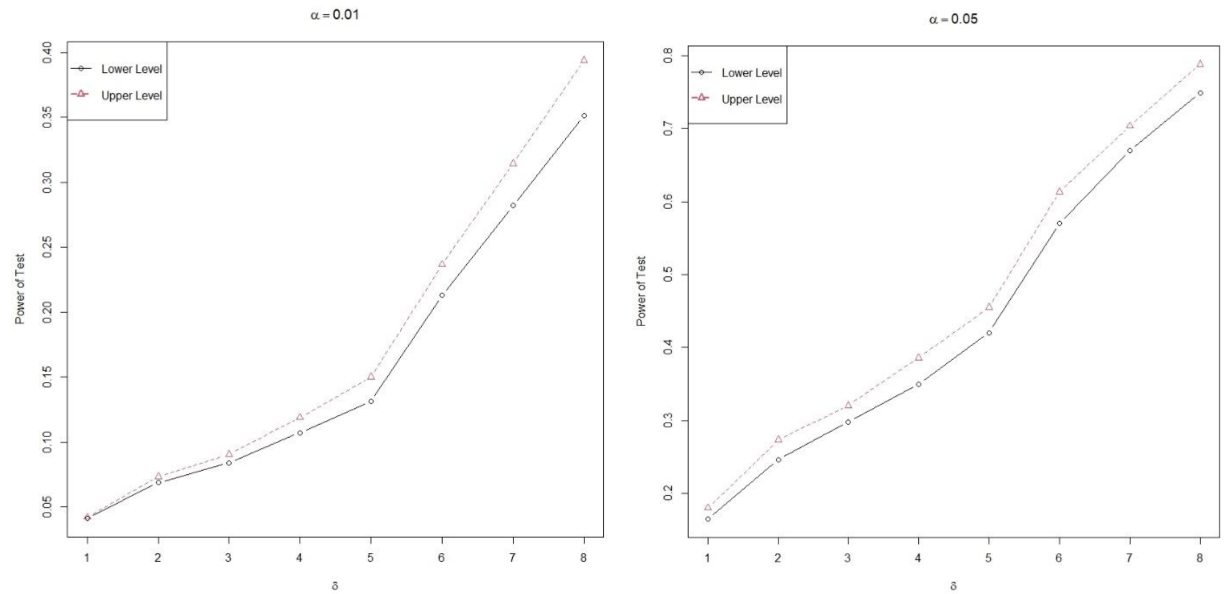

Figure 1 Power curves of the new and existing tests for NALSD with parameters $(b = 4, \mathrm{v} = 32, n = 48)$

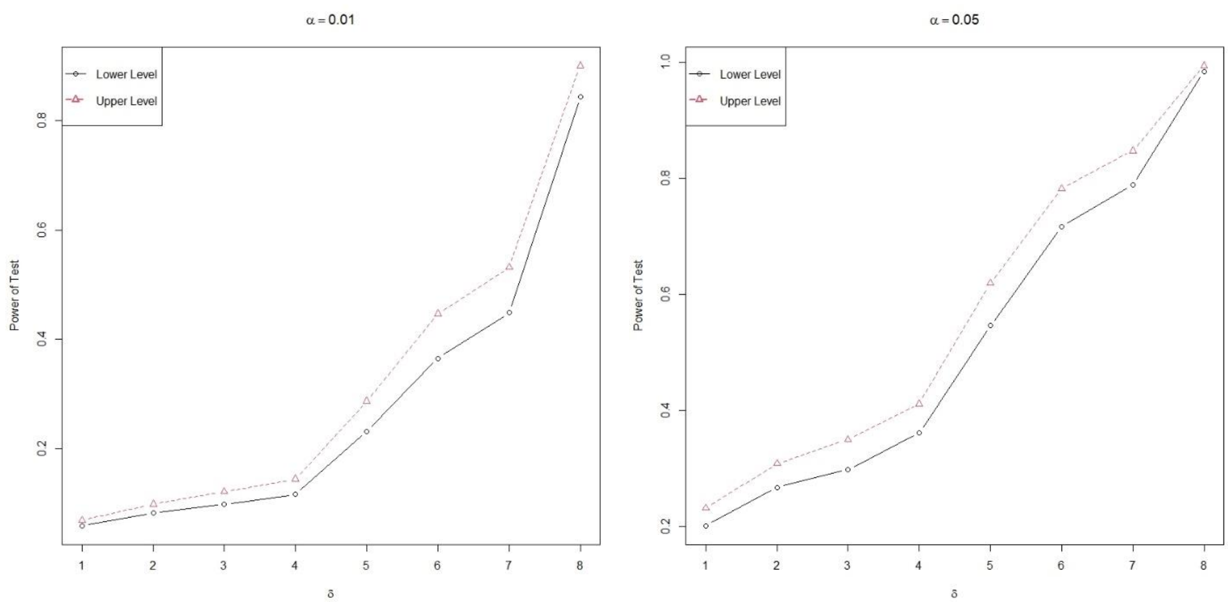

Figure 2 Power curves of the new and existing tests for NALSD with parameters $(b = 5, \mathrm{v} = 50, n = 75)$

The power of the test for the treatment effects through the NAN was calculated for neutrosophic data. The results are given in Tables 6 and 7 for NALSD with $(b = 4, \mathrm{v} = 32, n = 48)$ and $(b = 5, \mathrm{v} = 50, n = 75)$, for different sets of neutrosophic check means, $(\mu_{N1}, \mu_{N2}, \mu_{N3}, \mu_{N4})$ and $(\mu_{N1}, \mu_{N2}, \mu_{N3}, \mu_{N4}, \mu_{N5})$. The power of the test for the proposed and existing approaches' performance in Tables 6 and 7 is displayed in Figures 1 and 2.

Without loss of generality, the powers were plotted in ascending order. Evidently, the power of the test for the indeterminate part is higher than the power for the determinate part; so, the proposed approach performs better than the existing one in testing the treatment effects.

5. Comparative Study

As mentioned earlier, the proposed design is a generalization of the augmented Latin square design under classical statistics. The proposed F_N-test for NALSD reduces to the existing F-test for ALSD when all observations in the data are exact, determined and certain. Throughout this section, the proposed F_N-test is compared to the existing F-test in terms of the measure of indeterminacy, accuracy, flexibility, and information. For the purpose of comparison, the neutrosophic form of the F_N-test for the proposed design of the effects of treatments can be expressed as follows:

$$1.054 + 1.062 I_{F_N};\ I_{F_N} \in [0, 0.007]$$

Note that the neutrosophic form can be reduced to a statistic under classical statistics when $I_{F_N} = 0$; So, the first part of the neutrosophic form 1.054 describes the value of the test statistic under classical statistics. The second part $1.062 I_{F_N}$ illustrates the indeterminate portion of the neutrosophic form. Additionally, this test has a measure of indeterminacy of 0.007. According to the proposed test, the values of the F_N-test, $F_N \in [F_L, F_U]$ are flexible and lie in the indeterminate interval that is $F_N \in [1.054, 1.062]$. Based on the proposed test, it is expected that $F_N \in [F_L, F_U]$ may range from 1.054 to 1.062 under an uncertain environment. This range distinguishes the proposed test from the existing test under classical statistics, which is based on the determined value, which does not appropriate under uncertain conditions. Additionally, this test provides additional information about the testing approach when indeterminacy is present; namely, it provides additional information about the testing procedure which is the measure of indeterminacy. To illustrate the numerical example, for testing H_{N0} (means of treatment are equal), the probability that it will be accepted is 0.95, the probability that it will be rejected when true is 0.05, and the probability of uncertainty about it is 0.007.

Moreover, Tables 6 and 7 provide a comparative evaluation of the relative effectiveness of the proposed test in terms of the $\alpha_{\text{Empirical}}$ and $Power_{Empirical}$. The results indicate that the $\alpha_{\text{Empirical}}$ of the proposed test is close to 0.05 under NS. In addition, Figures 1 and 2 indicate that the curves of the power of the test for the indeterminate part are higher than those for the determinate part. This emphasizes that the indeterminate part plays an important role in uncertain environments. According to the results of the study, the proposed test for NALSD under NS is more informative, accuracy, and flexible than the test for ALSD under classical statistics.

6. Conclusion

This article introduces neutrosophic augmented Latin square design as a generalization to the existing augmented Latin square design. In this context, the statistical model and a NANOVA approach have been presented for the proposed design to deal with neutrosophic hypotheses and the decision rule about the treatment effects in the design. Besides, the performance of the proposed design has been evaluated using a numerical example and a simulation study. According to the results, the proposed design led to more accuracy in analyzing practical problems in uncertainty. It is conjectured that, based on the proposed design, many new investigations will be carried out in the future. Moreover, in practical experiments using the proposed design with uncertain data will be analyzed more precisely.

Declarations

Conflicts of Interest: The authors declare no conflict of interest.

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Multi-attribute group decision-making based on the Neutrosophic Bonferroni mean operator

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Abstract

In this study, a Neutrosophic technique and arithmetic ranking operations are employed to group decision-making problems with many qualities. The outcome is contrasted with the current approach. When compared to the current method, the proposed method is much more manageable and useful for solving group decision making problems involving several qualities. All of the information provided by the decision makers (DMs) in the Neutrosophic Multi-Attribute Group Decision Making (NMAGDM) problems.

Keywords: Neutrosophic possibility mean, Neutrosophic operator.

1.Introduction

Employed the TOPSIS method's expansion [1]. [2] and [3] created a method for choosing configuration items by using the software development. The aforesaid problem (FMAGDM), the aggregating function known as fuzzy weighted minkowski distance is utilised, as it was first developed by [4]. [6] employed a maximising deviation approach to tackle the aforesaid problem (FMAGDM) in a linguistic context. [7] approach of analysis is ad hoc. [8] have presented a computational coordination approach to resolve the above method (FMAGDM). [9] demonstrate how the multi-granularity linguistic method (FMAGDM) is employed to tackle the aforementioned issue. [10] Different distance values have been measured using non-homogeneous information, and a new method (FMAGDM) has been created to overcome the aforesaid problem. The above methods, however, all rely on type-1 fuzzy sets. [11] was the first to suggest that type-1 fuzzy sets may be extended to type-2 fuzzy sets. In [12] introduction, it is said that type-2 fuzzy sets were able to resolve more uncertainty than type-1 fuzzy sets by using type-1 fuzzy sets' clear membership values. [13] and utilised in many practical applications is presented in [14],[15] and [16]. This is due to the complexity of employing type-2 fuzzy sets. [17] work, a brand-new approach known as the FMAGDM—a linguistic weighted average method—is applied to interval type-2 fuzzy sets in order to tackle the aforementioned issue. According to [18] the FMAGDM is resolved utilising the ranking approach and arithmetic operations in interval type-2 fuzzy sets. [19] the TOPSIS approach is also employed to solve the FMAGDM using interval type-2 fuzzy sets. Even if the attribute weights are only partially known, [20] explanation of how the interval type-2 fuzzy set is utilised to characterise the attribute values is comprehensive. [21] presented a ranking approach that is used in an interval type-2 fuzzy set to resolve the FMAGDM method. [22],[23] and [24] developed a novel approach to solve the FMAGDM utilising a trapezoidal interval type-2 fuzzy set. [25] explanation of the possibility degree approach utilised to solve the FMAGDM problem. In our daily life the multiple criteria decision making problem achieved a vital role and it was elaborately researched by many scholars [26],[27] and [28] were used gained and lost dominance score method [29] and [30]. But the information they were given was incomplete while using the fuzzy set and it is necessary to investigate further to solve the group decision making problem well. The highlights of the paper is by improving the possibility degree, a better solution has been given compare to the existing method and the accuracy of the rank is increased in the proposed method compare to the existing method. Neutrosophic multicriteria is a decision-making technique that combines a number of criteria or elements, sometimes with sparse or ambiguous information, in order to arrive at a conclusion [31]. The expression of the students is assessed using real-time data obtained by taking pictures of the students in relation to various themes using a mathematical model built using a double bounded rough neutrosophic set [32]. The primary medical domains that NIP can produce for image segmentation from DICOM photos are mentioned in the suggested study. It has been discovered to be a better approach because of how it manages unclear information [33]. With the exception of placing more emphasis on Neutrosophic voice recognition, existing methods are utilised. the development of formulas that compute, classify, or distinguish between various stress conditions. The objectives of this research are to comprehend stress and develop methods to mitigate its impacts on voice recognition and human-computer interaction systems [34]. In this article, we offer an approach for estimating a system's anticipated expenses under various circumstances. The trapezoidal bipolar neutrosophic numbers are used to manage the uncertainties that are present in the various model parameters [35]. The dynamic programming method is used in this article to address complex group decision-making scenarios where the preference data is represented by linguistic variables. The complexity and ambiguity of reality make it challenging for decision-makers to draw judgements using precise data [36]. The advantage of the method is that it may be handled without a lower membership function for falsehood, which allows for significant calculation time savings [37]. In order to address the traffic issue, this paper attempted to give a general summary of each method. The suggested study is anticipated to be beneficial to numerous researchers studying traffic flow, traffic accident diagnostics, and its hybridization in the future [38]. This study demonstrates that, in contrast to standard regression models, neurosophic multiple regression is the most effective model for uncertainty [39]. The triangular interval type-2 fuzzy soft weighted arithmetic operator (TIT2FSWA) with the requisite mathematical features has been proposed in this research. Additionally, the proposed methodology has been applied to a decision-making problem for profit analysis [40]. For the purpose of demonstrating the originality of the suggested graphical representation, the proposed distance measure and several trapezoidal fuzzy neutrosophic number forms have been given out [41]. In this study, we will write the issue text suitably for such a situation before building the suitable mathematical model to achieve the lowest inspection cost possible [42]. The elements of Industry 5.0 are considered in this framework. By first examining the pertinent experts and body of published research, it is possible to discover the most crucial associated aspects and tactics [43]. For the region's economic and environmental wellbeing, it is crucial to reduce HCWT through suitable treatment. This research develops a novel multi-criteria decision-making strategy to address single-valued neutrosophic group decision-making problems with lacking weight data. [44].

2. **Preliminary:**

Definition 2.1:

The upper and lower trapezoidal Neutrosophic set is defined as

$$(TN_1, IN_1, FN_1) = \left((TN_1^U, IN_1^U, FN_1^U), (TN_1^L, IN_1^L, FN_1^L)\right)$$
$$= (\left((Ta_{11}^U, Ia_{11}^U, Fa_{11}^U), (Ta_{12}^U, Ia_{12}^U, Fa_{12}^U), (Ta_{13}^U, Ia_{13}^U, Fa_{13}^U), (Ta_{14}^U, Ia_{14}^U, Fa_{14}^U), (Th_1^U, Ih_1^U, Fh_1^U)\right),$$
$$((Ta_{11}^L, Ia_{11}^L, Fa_{11}^L), (Ta_{12}^L, Ia_{12}^L, Fa_{12}^L), (Ta_{13}^L, Ia_{13}^L, Fa_{13}^L), (Ta_{14}^L, Ia_{14}^L, Fa_{14}^L), (Th_1^L, Ih_1^L, Fh_1^L)))$$

(1)

Definition 2.2:

The upper and lower triangular Neutrosophic set is defined as

$$(TN_1, IN_1, FN_1) = \left((TN_1^U, IN_1^U, FN_1^U), (TN_1^L, IN_1^L, FN_1^L)\right)$$
$$= (\left((Ta_{11}^U, Ia_{11}^U, Fa_{11}^U), (Ta_{12}^U, Ia_{12}^U, Fa_{12}^U), (Ta_{12}^U, Ia_{12}^U, Fa_{12}^U), (Ta_{13}^U, Ia_{13}^U, Fa_{13}^U), (Th_1^U, Ih_1^U, Fh_1^U)\right),$$
$$((Ta_{11}^L, Ia_{11}^L, Fa_{11}^L), (Ta_{12}^L, Ia_{12}^L, Fa_{12}^L), (Ta_{12}^L, Ia_{12}^L, Fa_{12}^L), (Ta_{13}^L, Ia_{13}^L, Fa_{13}^L), (Th_1^L, Ih_1^L, Fh_1^L)))$$

(2)

Definition 2.3:

The additive operation of two upper and lower trapezoidal Neutrosophic set is defined as

$$(TN_1, IN_1, FN_1) \oplus (TN_2, IN_2, FN_2)$$
$$= \left((TN_1^U, IN_1^U, FN_1^U), (TN_1^L, IN_1^L, FN_1^L)\right) \oplus \left((TN_2^U, IN_2^U, FN_2^U), (TN_2^L, IN_2^L, FN_2^L)\right)$$
$$= (\left((Ta_{11}^U, Ia_{11}^U, Fa_{11}^U), (Ta_{12}^U, Ia_{12}^U, Fa_{12}^U), (Ta_{13}^U, Ia_{13}^U, Fa_{13}^U), (Ta_{14}^U, Ia_{14}^U, Fa_{14}^U), (Th_1^U, Ih_1^U, Fh_1^U)\right),$$
$$((Ta_{11}^L, Ia_{11}^L, Fa_{11}^L), (Ta_{12}^L, Ia_{12}^L, Fa_{12}^L), (Ta_{13}^L, Ia_{13}^L, Fa_{13}^L), (Ta_{14}^L, Ia_{14}^L, Fa_{14}^L), (Th_1^L, Ih_1^L, Fh_1^L)))$$
$$\oplus (\left((Ta_{21}^U, Ia_{21}^U, Fa_{21}^U), (Ta_{22}^U, Ia_{22}^U, Fa_{22}^U), (Ta_{23}^U, Ia_{23}^U, Fa_{23}^U), (Ta_{24}^U, Ia_{24}^U, Fa_{24}^U), (Th_2^U, Ih_2^U, Fh_2^U)\right),$$

$$((Ta_{21}^{L}, Ia_{21}^{L}, Fa_{21}^{L}), (Ta_{22}^{L}, Ia_{22}^{L}, Fa_{22}^{L}), (Ta_{23}^{L}, Ia_{23}^{L}, Fa_{23}^{L}), (Ta_{24}^{L}, Ia_{24}^{L}, Fa_{24}^{L}), (Th_{2}^{L}, Ih_{2}^{L}, Fh_{2}^{L})))$$

$$= \Big(\Big(\big((Ta_{11}^{U} - Ta_{21}^{U} + Ta_{11}^{U} * Ta_{21}^{U}), (Ia_{11}^{U} * Ia_{21}^{U}), (Fa_{11}^{U} * Fa_{21}^{U})\big), \big((Ta_{12}^{U} - Ta_{22}^{U} + Ta_{12}^{U} * Ta_{22}^{U}), (Ia_{12}^{U} * Ia_{22}^{U}), (Fa_{12}^{U} * Fa_{22}^{U})\big), \big((Ta_{13}^{U} - Ta_{23}^{U} + Ta_{13}^{U} * Ta_{23}^{U}), (Ia_{13}^{U} * Ia_{23}^{U}), (Fa_{13}^{U} * Fa_{23}^{U})\big), \big((Ta_{14}^{U} - Ta_{24}^{U} + Ta_{14}^{U} * Ta_{24}^{U}), (Ia_{14}^{U} * Ia_{24}^{U}), (Fa_{14}^{U} * Fa_{24}^{U})\big), \big((Th_{1}^{U} - Th_{2}^{U} + Th_{1}^{U} * Th_{2}^{U}), (Ih_{1}^{U} * Ih_{2}^{U}), (Fh_{1}^{U} * Fh_{2}^{U})\big)\Big), \Big(\big((Ta_{11}^{L} - Ta_{21}^{L} + Ta_{11}^{L} * Ta_{21}^{L}), (Ia_{11}^{L} * Ia_{21}^{L}), (Fa_{11}^{L} * Fa_{21}^{L})\big), \big((Ta_{12}^{L} - Ta_{22}^{L} + Ta_{12}^{L} * Ta_{22}^{L}), (Ia_{12}^{L} * Ia_{22}^{L}), (Fa_{12}^{L} * Fa_{22}^{L})\big), \big((Ta_{13}^{L} - Ta_{23}^{L} + Ta_{13}^{L} * Ta_{23}^{L}), (Ia_{13}^{L} * Ia_{23}^{L}), (Fa_{13}^{L} * Fa_{23}^{L})\big), \big((Ta_{14}^{L} - Ta_{24}^{L} + Ta_{14}^{L} * Ta_{24}^{L}), (Ia_{14}^{L} * Ia_{24}^{L}), (Fa_{14}^{L} * Fa_{24}^{L})\big), \big((Th_{1}^{L} - Th_{2}^{L} + Th_{1}^{L} * Th_{2}^{L}), (Ih_{1}^{L} * Ih_{2}^{L}), (Fh_{1}^{L} * Fh_{2}^{L})\big)\Big)\Big) \quad (3)$$

Definition 2.4:

The multiplicative operation of two upper and lower trapezoidal Neutrosophic set is defined as

$$(TN_1, IN_1, FN_1) \otimes (TN_2, IN_2, FN_2)$$
$$= \big((TN_1^U, IN_1^U, FN_1^U), (TN_1^L, IN_1^L, FN_1^L)\big) \otimes \big((TN_2^U, IN_2^U, FN_2^U), (TN_2^L, IN_2^L, FN_2^L)\big)$$
$$= \big(\big((Ta_{11}^{U}, Ia_{11}^{U}, Fa_{11}^{U}), (Ta_{12}^{U}, Ia_{12}^{U}, Fa_{12}^{U}), (Ta_{13}^{U}, Ia_{13}^{U}, Fa_{13}^{U}), (Ta_{14}^{U}, Ia_{14}^{U}, Fa_{14}^{U}), (Th_{1}^{U}, Ih_{1}^{U}, Fh_{1}^{U})\big),$$

$$((Ta_{11}^{L}, Ia_{11}^{L}, Fa_{11}^{L}), (Ta_{12}^{L}, Ia_{12}^{L}, Fa_{12}^{L}), (Ta_{13}^{L}, Ia_{13}^{L}, Fa_{13}^{L}), (Ta_{14}^{L}, Ia_{14}^{L}, Fa_{14}^{L}), (Th_{1}^{L}, Ih_{1}^{L}, Fh_{1}^{L})))$$
$$\otimes \big(\big((Ta_{21}^{U}, Ia_{21}^{U}, Fa_{21}^{U}), (Ta_{22}^{U}, Ia_{22}^{U}, Fa_{22}^{U}), (Ta_{23}^{U}, Ia_{23}^{U}, Fa_{23}^{U}), (Ta_{24}^{U}, Ia_{24}^{U}, Fa_{24}^{U}), (Th_{2}^{U}, Ih_{2}^{U}, Fh_{2}^{U})\big),$$

$$((Ta_{21}^{L}, Ia_{21}^{L}, Fa_{21}^{L}), (Ta_{22}^{L}, Ia_{22}^{L}, Fa_{22}^{L}), (Ta_{23}^{L}, Ia_{23}^{L}, Fa_{23}^{L}), (Ta_{24}^{L}, Ia_{24}^{L}, Fa_{24}^{L}), (Th_{2}^{L}, Ih_{2}^{L}, Fh_{2}^{L})))$$

$$= \Big(\Big(\big((Ta_{11}^{U} * Ta_{21}^{U}), (Ia_{11}^{U} * Ia_{21}^{U}), (Fa_{11}^{U} * Fa_{21}^{U})\big), \big((Ta_{12}^{U} * Ta_{22}^{U}), (Ia_{12}^{U} * Ia_{22}^{U}), (Fa_{12}^{U} * Fa_{22}^{U})\big), \big((Ta_{13}^{U} * Ta_{23}^{U}), (Ia_{13}^{U} * Ia_{23}^{U}), (Fa_{13}^{U} * Fa_{23}^{U})\big), \big((Ta_{14}^{U} * Ta_{24}^{U}), (Ia_{14}^{U} * Ia_{24}^{U}), (Fa_{14}^{U} * Fa_{24}^{U})\big), \big((Th_{1}^{U} * Th_{2}^{U}), (Ih_{1}^{U} * Ih_{2}^{U}), (Fh_{1}^{U} * Fh_{2}^{U})\big)\Big), \Big(\big((Ta_{11}^{L} * Ta_{21}^{L}), (Ia_{11}^{L} * Ia_{21}^{L}), (Fa_{11}^{L} * Fa_{21}^{L})\big), \big((Ta_{12}^{L} * Ta_{22}^{L}), (Ia_{12}^{L} * Ia_{22}^{L}), (Fa_{12}^{L} *$$

$Fa_{22}^{L})\big),\big((Ta_{13}^{L} * Ta_{23}^{L}),(Ia_{13}^{L} * Ia_{23}^{L}),(Fa_{13}^{L} * Fa_{23}^{L})\big),\big((Ta_{14}^{L} * Ta_{24}^{L}),(Ia_{14}^{L} * Ia_{24}^{L}),(Fa_{14}^{L} * Fa_{24}^{L})\big),\big((Th_{1}^{L} * Th_{2}^{L}),(Ih_{1}^{L} * Ih_{2}^{L}),(Fh_{1}^{L} * Fh_{2}^{L})\big)\big)\Big)$ (4)

Definition 2.5:

The arithmetic operation of upper and lower trapezoidal Neutrosophic set is defined as

$$k(TN_1, IN_1, FN_1) = k\big((TN_1^U, IN_1^U, FN_1^U),(TN_1^L, IN_1^L, FN_1^L)\big)$$
$$= \left(\begin{pmatrix} ((Ta_{11}^U)^k,(Ia_{11}^U)^k,(Fa_{11}^U)^k),((Ta_{12}^U)^k,(Ia_{12}^U)^k,(Fa_{12}^U)^k),((Ta_{13}^U)^k,(Ia_{13}^U)^k,(Fa_{13}^U)^k), \\ ((Ta_{14}^U)^k,(Ia_{14}^U)^k,(Fa_{14}^U)^k),((Th_1^U)^k,(Ih_1^U)^k,(Fh_1^U)^k) \end{pmatrix}, \begin{pmatrix} ((Ta_{11}^L)^k,(Ia_{11}^L)^k,(Fa_{11}^L)^k),((Ta_{12}^L)^k,(Ia_{12}^L)^k,(Fa_{12}^L)^k),((Ta_{13}^L)^k,(Ia_{13}^L)^k,(Fa_{13}^L)^k), \\ ((Ta_{14}^L)^k,(Ia_{14}^L)^k,(Fa_{14}^L)^k),((Th_1^L)^k,(Ih_1^L)^k,(Fh_1^L)^k) \end{pmatrix},\right)$$

(5)

Definition 2.6:

The score function for Neutrosophic triangular set is given by

$$\dot{S}^*\big(T_A(x), I_A(x), F_A(x)\big) = \frac{1}{2}\big(1 + T_A(x) - 2 * I_A(x) + F_A(x)\big) \quad (6)$$

Definition 2.7:

The proposed score function for Neutrosophic trapezoidal set is given by

$$\dot{\dot{S}}^*\big(T_A(x), I_A(x), F_A(x)\big) = \frac{1}{2}\big(1 + n * T_A(x) - I_A(x) + F_A(x)\big) \quad (7)$$

Where n represents number of terms in the matrices.

If a single value Neutrosophic number is $(TN, IN, FN) = \big((TN^U, IN^U, FN^U),(TN^L, IN^L, FN^L)\big)$, where$(TN^U, IN^U, FN^U)$ is the upper Neutrosophic member function and (TN^L, IN^L, FN^L) is the lower Neutrosophic member function having the level set as $(TN_\alpha^U, IN_\alpha^U, FN_\alpha^U) = \big[\big(TN_1^U(\alpha), IN_1^U(\alpha), FN_1^U(\alpha)\big),\big(TN_2^U(\alpha), IN_2^U(\alpha), FN_2^U(\alpha)\big)\big], \alpha \in [(0,0,0),(Th_U, Ih_U, Fh_U)]$ and $\big(TN_\beta^L, IN_\beta^L, FN_\beta^L\big) = \big[\big(TN_1^L(\beta), IN_1^L(\beta), FN_1^L(\beta)\big),\big(TN_2^L(\beta), IN_2^L(\beta), FN_2^L(\beta)\big)\big], \alpha \in [(0,0,0),(Th_L, Ih_L, Fh_L)]$ where (Th_U, Ih_U, Fh_U) is the highest membership Neutrosophic function of N^U and (Th_L, Ih_L, Fh_L) is the lower membership Neutrosophic function of N^L .

Definition 2.8:

The lower Neutrosophic possibility mean for $N = (N^U, N^L)$ is given by

$$\left(T\widetilde{M}_*(N), I\widetilde{M}_*(N), F\widetilde{M}_*(N)\right) = \left(\int_0^{Th_U}\left(TN_1^U(\alpha)\right)^{\alpha} d\alpha + \int_0^{Th_L}\left(TN_1^L(\beta)\right)^{\beta} d\beta, \int_0^{Ih_U}\left(IN_1^U(\alpha)\right)^{\alpha} d\alpha + \int_0^{Ih_L}\left(IN_1^L(\beta)\right)^{\beta} d\beta, \int_0^{Fh_U}\left(FN_1^U(\alpha)\right)^{\alpha} d\alpha + \int_0^{Fh_L}\left(FN_1^L(\beta)\right)^{\beta} d\beta\right) \quad (8)$$

Where $\left(T\widetilde{M}_*(N), I\widetilde{M}_*(N), F\widetilde{M}_*(N)\right)$ is the arithmetic mean of members of the Neutrosophic membership function.

Definition 2.9:

The upper Neutrosophic possibility mean for $(TN, IN, FN) = \left((TN^U, IN^U, FN^U), (TN^L, IN^L, FN^L)\right)$ is given by

$$\left(T\widetilde{M}^*(N), I\widetilde{M}^*(N), F\widetilde{M}^*(N)\right) = \left(\int_0^{Th_U}\left(TN_2^U(\alpha)\right)^{\alpha} d\alpha + \int_0^{Th_L}\left(TN_2^L(\beta)\right)^{\beta} d\beta, \int_0^{Ih_U}\left(IN_2^U(\alpha)\right)^{\alpha} d\alpha + \int_0^{Ih_L}\left(IN_2^L(\beta)\right)^{\beta} d\beta, \int_0^{Fh_U}\left(FN_2^U(\alpha)\right)^{\alpha} d\alpha + \int_0^{Fh_L}\left(FN_2^L(\beta)\right)^{\beta} d\beta\right) \quad (9)$$

Where $\left(T\widetilde{M}^*(N), I\widetilde{M}^*(N), F\widetilde{M}^*(N)\right)$ is the arithmetic mean of members of the Neutrosophic membership function.

Definition 2.100:

The closed bounded interval of Neutrosophic lower and upper mean value is given by the notation

$$\left(T\widetilde{M}(N), I\widetilde{M}(N), F\widetilde{M}(N)\right) = \left[\left(T\widetilde{M}_*(N), I\widetilde{M}_*(N), F\widetilde{M}_*(N)\right), \left(T\widetilde{M}^*(N), I\widetilde{M}^*(N), F\widetilde{M}^*(N)\right)\right].$$

Definition 2.11:

Similarly, the Neutrosophic mean value of (TN_1, IN_1, FN_1) and (TN_2, IN_2, FN_2) is given by $\left(T\widetilde{M}(N_1), I\widetilde{M}(N_1), F\widetilde{M}(N_1)\right) = \left[\left(T\widetilde{M}_*(N_1), I\widetilde{M}_*(N_1), F\widetilde{M}_*(N_1)\right), \left(T\widetilde{M}^*(N_1), I\widetilde{M}^*(N_1), F\widetilde{M}^*(N_1)\right)\right]$ and $\left(T\widetilde{M}(N_2), I\widetilde{M}(N_2), F\widetilde{M}(N_2)\right) = \left[\left(T\widetilde{M}_*(N_2), I\widetilde{M}_*(N_2), F\widetilde{M}_*(N_2)\right), \left(T\widetilde{M}^*(N_2), I\widetilde{M}^*(N_2), F\widetilde{M}^*(N_2)\right)\right]$

Definition 2.12:

The possibility Neutrosophic degree is given as

$$\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = \left(\begin{array}{c} min\left\{\max\left(\frac{T\widetilde{M}^*(N_1) - T\widetilde{M}_*(N_2)}{T\widetilde{M}^*(N_1) - T\widetilde{M}_*(N_1) + T\widetilde{M}^*(N_2) - T\widetilde{M}_*(N_2)}, 0\right), 1\right\}, min\left\{\max\left(\frac{I\widetilde{M}^*(N_1) - I\widetilde{M}_*(N_2)}{I\widetilde{M}^*(N_1) - I\widetilde{M}_*(N_1) + I\widetilde{M}^*(N_2) - I\widetilde{M}_*(N_2)}, 0\right), 1\right\}, \\ min\left\{\max\left(\frac{F\widetilde{M}^*(N_1) - F\widetilde{M}_*(N_2)}{F\widetilde{M}^*(N_1) - F\widetilde{M}_*(N_1) + F\widetilde{M}^*(N_2) - F\widetilde{M}_*(N_2)}, 0\right), 1\right\} \end{array} \right) \quad (10)$$

Definition 2.13:

The possibility Neutrosophic degree$\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right)$ has to satisfy the following property

$(0,0,0) \leq \left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) \leq (1,1,1)$ and $(0,0,0) \leq \left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) \leq (1,1,1)$

If$\left(T\widetilde{M}_*(N_1), I\widetilde{M}_*(N_1), F\widetilde{M}_*(N_1)\right) = \left(T\widetilde{M}_*(N_2), I\widetilde{M}_*(N_2), F\widetilde{M}_*(N_2)\right)$ and $\left(T\widetilde{M}^*(N_1), I\widetilde{M}^*(N_1), F\widetilde{M}^*(N_1)\right) = \left(T\widetilde{M}^*(N_2), I\widetilde{M}^*(N_2), F\widetilde{M}^*(N_2)\right)$, then $\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (0.5,0.5,0.5)$

For a Neutrosophic member $(TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), (TN_3, IN_3, FN_3)$, If $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (0.5,0.5,0.5)$ and $\left(p(TN_2 \geqslant TN_3), p(IN_2 \geqslant IN_3), p(FN_2 \geqslant FN_3)\right) = (0.5,0.5,0.5)$ then $\left(p(TN_1 \geqslant TN_3), p(IN_1 \geqslant IN_3), p(FN_1 \geqslant FN_3)\right) = (0.5,0.5,0.5)$.

For a Neutrosophic member $(TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), (TN_3, IN_3, FN_3)$, If $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (0.5,0.5,0.5)$ and $\left(p(TN_2 \geqslant TN_3), p(IN_2 \geqslant IN_3), p(FN_2 \geqslant FN_3)\right) = (0.5,0.5,0.5)$ then $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) + \left(p(TN_2 \geqslant TN_3), p(IN_2 \geqslant IN_3), p(FN_2 \geqslant FN_3)\right) = 2\left(p(TN_1 \geqslant TN_3), p(IN_1 \geqslant IN_3), p(FN_1 \geqslant FN_3)\right)$.

Definition 2.14:

For the Neutrosophic trapezoidal number $(TN, IN, FN) = \left((TN^U, IN^U, FN^U), (TN^L, IN^L, FN^L)\right) = \left(\left((Ta_1^U, Ia_1^U, Fa_1^U), (Ta_2^U, Ia_2^U, Fa_2^U), (Ta_3^U, Ia_3^U, Fa_3^U), (Ta_4^U, Ia_4^U, Fa_4^U), (Th^U, Ih^U, Fh^U)\right)\right.$,

$((Ta_1^L, Ia_1^L, Fa_1^L), (Ta_2^L, Ia_2^L, Fa_2^L), (Ta_3^L, Ia_3^L, Fa_3^L), (Ta_4^L, Ia_4^L, Fa_4^L), (Th^L, Ih^L, Fh^L)))$, the lower Neutrosophic possibility mean is calculated by,

$$\left(T\widetilde{M}_*(N), I\widetilde{M}_*(N), F\widetilde{M}_*(N)\right) = \begin{pmatrix} \int_0^{Th_U}\left(Ta_1^U + \frac{Ta_2^U - Ta_1^U}{Th_U}\right)^\alpha d\alpha + \int_0^{Th_L}\left(Ta_1^L + \frac{Ta_2^L - Ta_1^L}{Th_L}\right)^\beta d\beta\,, \\ \int_0^{Ih_U}\left(Ia_1^U + \frac{Ia_2^U - Ia_1^U}{Ih_U}\right)^\alpha d\alpha + \int_0^{Ih_L}\left(Ia_1^L + \frac{Ia_2^L - Ia_1^L}{Ih_L}\right)^\beta d\beta\,, \\ \int_0^{Fh_U}\left(Fa_1^U + \frac{Fa_2^U - Fa_1^U}{Fh_U}\right)^\alpha d\alpha + \int_0^{Fh_L}\left(Fa_1^L + \frac{Fa_2^L - Fa_1^L}{Fh_L}\right)^\beta d\beta\,, \end{pmatrix}$$

(11)

$$= \left(\left(\tfrac{1}{6}(Ta_1^U + 2Ta_2^U)Th_U^2 + \tfrac{1}{6}(Ta_1^L + 2Ta_2^L)Th_L^2\right), \left(\tfrac{1}{6}(Ia_1^U + 2Ia_2^U)Ih_U^2 + \tfrac{1}{6}(Ia_1^L + 2Ia_2^L)Ih_L^2\right), \left(\tfrac{1}{6}(Fa_1^U + 2Fa_2^U)Fh_U^2 + \tfrac{1}{6}(Fa_1^L + 2Fa_2^L)Fh_L^2\right)\right) \quad (12)$$

Definition 2.15:

For the Neutrosophic trapezoidal number $(TN, IN, FN) = \left((TN^U, IN^U, FN^U), (TN^L, IN^L, FN^L)\right) = \left(\left((Ta_1^U, Ia_1^U, Fa_1^U), (Ta_2^U, Ia_2^U, Fa_2^U), (Ta_3^U, Ia_3^U, Fa_3^U), (Ta_4^U, Ia_4^U, Fa_4^U), (Th^U, Ih^U, Fh^U)\right),\right.$

$\left.((Ta_1^L, Ia_1^L, Fa_1^L), (Ta_2^L, Ia_2^L, Fa_2^L), (Ta_3^L, Ia_3^L, Fa_3^L), (Ta_4^L, Ia_4^L, Fa_4^L), (Th^L, Ih^L, Fh^L))\right))$, the upper Neutrosophic possibility mean is calculated by,

$$\left(T\widetilde{M}^*(N), I\widetilde{M}^*(N), F\widetilde{M}^*(N)\right) = \begin{pmatrix} \int_0^{Th_U}\left(Ta_4^U + \frac{Ta_3^U - Ta_4^U}{Th_U}\right)^\alpha d\alpha + \int_0^{Th_L}\left(Ta_4^L + \frac{Ta_3^L - Ta_4^L}{Th_L}\right)^\beta d\beta\,, \\ \int_0^{Ih_U}\left(Ia_4^U + \frac{Ia_3^U - Ia_4^U}{Ih_U}\right)^\alpha d\alpha + \int_0^{Ih_L}\left(Ia_4^L + \frac{Ia_3^L - Ia_4^L}{Ih_L}\right)^\beta d\beta\,, \\ \int_0^{Fh_U}\left(Fa_4^U + \frac{Fa_3^U - Fa_4^U}{Fh_U}\right)^\alpha d\alpha + \int_0^{Fh_L}\left(Fa_4^L + \frac{Fa_3^L - Fa_4^L}{Fh_L}\right)^\beta d\beta\,, \end{pmatrix}$$

(13)

$$= \left(\left(\tfrac{1}{6}(Ta_4^U + 2Ta_3^U)Th_U^2 + \tfrac{1}{6}(Ta_4^L + 2Ta_3^L)Th_L^2\right), \left(\tfrac{1}{6}(Ia_4^U + 2Ia_3^U)Ih_U^2 + \tfrac{1}{6}(Ia_4^L + 2Ia_3^L)Ih_L^2\right), \left(\tfrac{1}{6}(Fa_4^U + 2Fa_3^U)Fh_U^2 + \tfrac{1}{6}(Fa_4^L + 2Fa_3^L)Fh_L^2\right)\right) \quad (14)$$

Definition 2.16:

The neutrosopic preference matrix (TP, IP, FP) is given as

$$(TP, IP, FP) = \begin{pmatrix} (p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)) & (p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)) & \dots & (p(TN_1 \geqslant TN_n), p(IN_1 \geqslant IN_n), p(FN_1 \geqslant FN_n)) \\ (p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)) & (p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)) & \dots & (p(TN_1 \geqslant TN_n), p(IN_1 \geqslant IN_n), p(FN_1 \geqslant FN_n)) \\ \vdots & \vdots & \vdots & \vdots \\ (p(TN_n \geqslant TN_1), p(IN_n \geqslant IN_1), p(FN_n \geqslant FN_1)) & (p(TN_n \geqslant TN_2), p(IN_n \geqslant IN_2), p(FN_n \geqslant FN_2)) & \dots & (p(TN_n \geqslant TN_n), p(IN_n \geqslant IN_n), p(FN_n \geqslant FN_n)) \end{pmatrix} \quad (15)$$

Definition 2.17:

The Neutrosophic ranking value$\mathcal{R}(TN, IN, FN)$ is given by

$$\mathcal{R}(TN, IN, FN) = \left(\frac{1}{n(n-1)}\left(\bigoplus_{k=1}^{n} p(TN_1 \geqslant TN_k) + \frac{n}{2} - 1\right), \frac{1}{n(n-1)}\left(\bigoplus_{k=1}^{n} p(IN_1 \geqslant IN_k) + \frac{n}{2} - 1\right), \frac{1}{n(n-1)}\left(\bigoplus_{k=1}^{n} p(FN_1 \geqslant FN_k) + \frac{n}{2} - 1\right)\right) \quad (16)$$

Step 1: Consider the problem in (27), and convert it into Neutrosophic trapezoidal number as

$$N_1 = \begin{pmatrix} ((0.7,0.2,0.1)(1.4,0.4,0.2)(2.8,0.8,0.4)(4.9,1.4,0.7)(0.7,0.2,0.1)), \\ ((1.05,0.3,0.15)(2.1,0.6,0.3)(2.1,0.6,0.3)(4.9,1.4,0.7)(0.56,0.16,0.08)) \end{pmatrix}$$ and

$$N_2 = \begin{pmatrix} ((1.05,0.3,0.15)(2.1,0.6,0.3)(4.2,1.2,0.6)(4.2,1.2,0.6)(0.7,0.2,0.1)) \\ , ((1.05,0.3,0.15)(2.31,0.66,0.33)(3.15,0.9,0.45)(3.5,1,0.5)(0.56,0.16,0.08)) \end{pmatrix}$$

Step 2:

Figure 1 represents the graphical representation of Neutrosophic trapezoidal number.

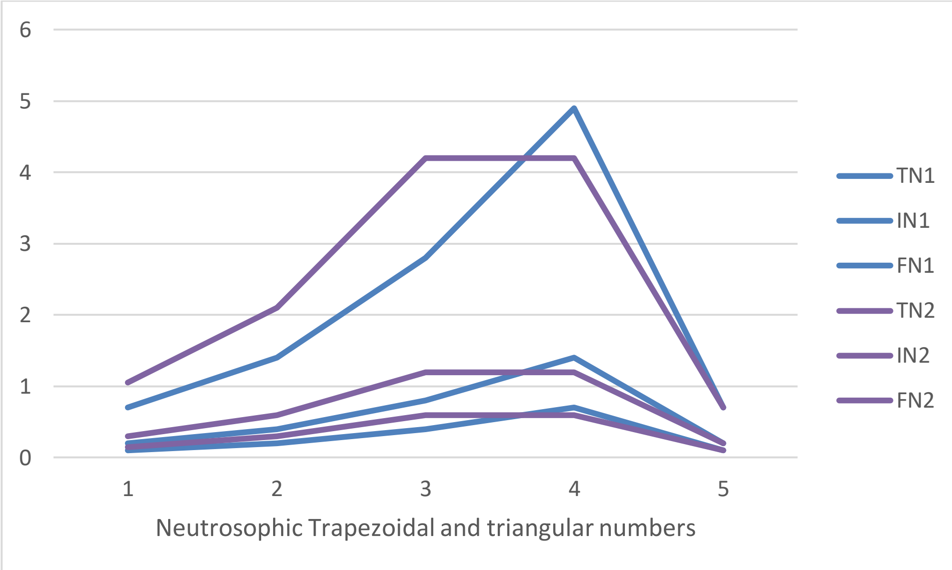

Figure 1: Neutrosophic Trapezoidal and triangular numbers

Step 3: For the above Neutrosophic member, the upper and lower possibility Neutrosophic mean value is given as

Case 1:

For $N_1 \& N_2$,

The closed bounded interval of Neutrosophic lower and upper mean value is given by

$$\left(T\widetilde{M}(N_1), I\widetilde{M}(N_1), F\widetilde{M}(N_1)\right) = \left[\left(T\widetilde{M}_*(N_1), I\widetilde{M}_*(N_1), F\widetilde{M}_*(N_1)\right), \left(T\widetilde{M}^*(N_1), I\widetilde{M}^*(N_1), F\widetilde{M}^*(N_1)\right)\right]$$

$$= [(-2.74, 0.01, 0.01), (0.78, 0.01, 0)]$$

$$\left(T\widetilde{M}(N_2), I\widetilde{M}(N_2), F\widetilde{M}(N_2)\right) = \left[\left(T\widetilde{M}_*(N_2), I\widetilde{M}_*(N_2), F\widetilde{M}_*(N_2)\right), \left(T\widetilde{M}^*(N_2), I\widetilde{M}^*(N_2), F\widetilde{M}^*(N_2)\right)\right]$$

$$= [(-5.39, 0.02, 0.01), (0.94, 0.01, 0)]$$

Case 2:

For $N_1 \& N_1$

The closed bounded interval of Neutrosophic lower and upper mean value is given by

$$\left(T\widetilde{M}(N_1), I\widetilde{M}(N_1), F\widetilde{M}(N_1)\right) = \left[\left(T\widetilde{M}_*(N_1), I\widetilde{M}_*(N_1), F\widetilde{M}_*(N_1)\right), \left(T\widetilde{M}^*(N_1), I\widetilde{M}^*(N_1), F\widetilde{M}^*(N_1)\right)\right]$$

$$= [(-2.74, 0.01, 0.01), (0.78, 0.01, 0)]$$

$$\left(T\widetilde{M}(N_1), I\widetilde{M}(N_1), F\widetilde{M}(N_1)\right) = \left[\left(T\widetilde{M}_*(N_1), I\widetilde{M}_*(N_1), F\widetilde{M}_*(N_1)\right), \left(T\widetilde{M}^*(N_1), I\widetilde{M}^*(N_1), F\widetilde{M}^*(N_1)\right)\right]$$

$$= [(-2.74, 0.01, 0.01), (0.78, 0.01, 0)]$$

Case 3:

For $N_2 \& N_2$

The closed bounded interval of Neutrosophic lower and upper mean value is given by

$$\left(T\widetilde{M}(N_2), I\widetilde{M}(N_2), F\widetilde{M}(N_2)\right) = \left[\left(T\widetilde{M}_*(N_2), I\widetilde{M}_*(N_2), F\widetilde{M}_*(N_2)\right), \left(T\widetilde{M}^*(N_2), I\widetilde{M}^*(N_2), F\widetilde{M}^*(N_2)\right)\right]$$

$$= [(-5.39, 0.02, 0.01), (0.94, 0.01, 0)]$$

$$\left(T\widetilde{M}(N_2), I\widetilde{M}(N_2), F\widetilde{M}(N_2)\right) = \left[\left(T\widetilde{M}_*(N_2), I\widetilde{M}_*(N_2), F\widetilde{M}_*(N_2)\right), \left(T\widetilde{M}^*(N_2), I\widetilde{M}^*(N_2), F\widetilde{M}^*(N_2)\right)\right]$$

$$= [(-5.39, 0.02, 0.01), (0.94, 0.01, 0)]$$

Case 4

For $N_2 \& N_1$

The closed bounded interval of Neutrosophic lower and upper mean value is given by

$$\left(T\widetilde{M}(N_2), I\widetilde{M}(N_2), F\widetilde{M}(N_2)\right) = \left[\left(T\widetilde{M}_*(N_2), I\widetilde{M}_*(N_2), F\widetilde{M}_*(N_2)\right), \left(T\widetilde{M}^*(N_2), I\widetilde{M}^*(N_2), F\widetilde{M}^*(N_2)\right)\right]$$

$$= [(-5.39, 0.02, 0.01), (0.94, 0.01, 0)]$$

$$\left(T\widetilde{M}(N_1), I\widetilde{M}(N_1), F\widetilde{M}(N_1)\right) = \left[\left(T\widetilde{M}_*(N_1), I\widetilde{M}_*(N_1), F\widetilde{M}_*(N_1)\right), \left(T\widetilde{M}^*(N_1), I\widetilde{M}^*(N_1), F\widetilde{M}^*(N_1)\right)\right]$$

$$= [(-2.74, 0.01, 0.01), (0.78, 0.01, 0)]$$

Step 4:

For the above cases, the Neutrosophic possibility degree is given by

For case 1, $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (0.38,0.35,0.36)$

For case 2, $\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (0.5,0.5,0.5)$

For case 3, $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (0.5,0.5,0.5)$

For case 4, $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (0.63,0.66,0.65)$

Step 5:

The Neutrosophic preference matrix is given by

$$(TP, IP, FP) = \begin{bmatrix} (0.5,0.5,0.5) & (0.38,0.35,0.36) \\ (0.63,0.66,0.65) & (0.5,0.5,0.5) \end{bmatrix}$$

Step 6:

The Neutrosophic ranking value is given as

$\mathcal{R}(TN_1, IN_1, FN_1) = (0.25,0.42,0.42)$ and $\mathcal{R}(TN_2, IN_2, FN_2) = (0.91,0.58,0.57)$

Step 7:

The rank of the alternatives $\mathcal{R}(N_1) = 0.9938$ and $\mathcal{R}(N_2) = 2.3027$

Step 8:

The rank is given in descending order

$$\mathcal{R}(N_2) > R(N_1)$$

Step 9:

The above result is compared with thirteen sets of trapezoidal and triangular Neutrosophic number in (20) is discussed in the next section.

4. Comparison result of trapezoidal and triangular Neutrosophic number:

A trapezoidal Neutrosophic member becomes the triangular Neutrosophic number, when the middle value is equal. Here we are taking the example of thirteen different sets in (20) to compare the result with the proposed method. Algorithm for this is same as the previous section but only in step 6, the scorefunction for deneutrosophic the triangular Neutrosophic is different. We use (6) for triangular Neutrosophic member.Also we give the graphical representation of each set also.

Table 1 represents the Neutrosophic member of thirteen set

S.No	Set	Neutrosophic values
1	I	$N_1 = ((0.245,0.07,0.035)(0.28,0.08,0.04)(0.28,0.08,0.04)(0.7,0.2,0.1)(0.7,0.2,0.1))$ $N_2 = ((0.105,0.03,0.015)(0.49,0.14,0.07)(0.49,0.14,0.07)(0.56,0.16,0.08)(0.7,0.2,0.1))$
2	II	$N_1 = \left((0,0,0)(0.07,0.02,0.01)(0.35,0.1,0.05)(0.7,0.2,0.1)(0.7,0.2,0.1)\right)$ $N_2 = ((0.05,0.42,0.12)(0.06,0.42,0.12)(0.06,0.49,0.14)(0.07,0.7,0.2)(0.1,0.7,0.2))$
3	III	$N_1 = ((0,0,0)(0.07,0.02,0.01)(0.35,0.1,0.05)(0.7,0.2,0.1)(0.7,0.2,0.1))$ $N_2 = ((0.42,0.12,0.06)(0.49,0.14,0.07)(0.49,0.14,0.07)(0.56,0.16,0.08)(0.7,0.2,0.1))$
4	IV	$N_1 = \left((0.28,0.08,0.04)(0.63,0.18,0.09)(0.63,0.18,0.09)(0.7,0.2,0.1)(0.7,0.2,0.1)\right)$ $N_2 = ((0.28,0.08,0.04)(0.49,0.14,0.07)(0.49,0.14,0.07)(0.7,0.2,0.1)(0.7,0.2,0.1))$ $N_3 = ((0.28,0.08,0.04)(0.35,0.1,0.05)(0.35,0.1,0.05)(0.7,0.2,0.1)(0.7,0.2,0.1))$
5	V	$N_1 = \left((0.35,0.1,0.05)(0.49,0.14,0.07)(0.49,0.14,0.07)(0.63,0.18,0.09)(0.7,0.2,0.1)\right)$ $N_2 = ((0.21,0.06,0.03)(0.49,0.14,0.07)(0.49,0.14,0.07)(0.63,0.18,0.09)(0.7,0.2,0.1))$ $N_3 = ((0.21,0.06,0.03)(0.28,0.08,0.04)(0.49,0.14,0.07)(0.63,0.18,0.09)(0.7,0.2,0.1))$
6	VI	$N_1 = \left((0.21,0.06,0.03)(0.35,0.1,0.05)(0.56,0.16,0.08)(0.63,0.18,0.09)(0.7,0.2,0.1)\right)$ $N_2 = ((0.21,0.06,0.03)(0.35,0.1,0.05)(0.35,0.1,0.05)(0.63,0.18,0.09)(0.7,0.2,0.1))$ $N_3 = ((0.21,0.06,0.03)(0.35,0.1,0.05)(0.35,0.1,0.05)(0.49,0.14,0.07)(0.7,0.2,0.1))$
7	VII	$N_1 = ((0.14,0.04,0.02)(0.35,0.1,0.05)(0.35,0.1,0.05)(0.56,0.16,0.08)(0.7,0.2,0.1))$ $N_2 = ((0.28,0.08,0.04)(0.35,0.1,0.05)(0.35,0.1,0.05)(0.42,0.12,0.06)(0.7,0.2,0.1))$

8	VIII	$N_1 = \big((0,0,0)(0.28,0.08,0.04)(0.42,0.12,0.06)(0.56,0.16,0.08)(0.7,0.2,0.1)\big)$ $N_2 = \big((0.14,0.04,0.02)(0.35,0.1,0.05)(0.35,0.1,0.05)(0.63,0.18,0.09)(0.7,0.2,0.1)\big)$ $N_3 = ((0.14,0.04,0.02)(0.42,0.12,0.06)(0.49,0.14,0.07)(0.56,0.16,0.08)(0.7,0.2,0.1))$
9	IX	$N_1 = \big((0,0,0)(0.14,0.04,0.02)(0.14,0.04,0.02)(0.28,0.08,0.04)(0.7,0.2,0.1)\big)$ $N_2 = ((0.42,0.12,0.06)(0.56,0.16,0.08)(0.56,0.16,0.08)(0.7,0.2,0.1)(0.7,0.2,0.1))$
10	X	$N_1 = \big((0.28,0.08,0.04)(0.42,0.12,0.06)(0.42,0.12,0.06)(0.56,0.16,0.08)(0.7,0.2,0.1)\big)$ $N_2 = ((1.26,0.36,0.18)(1.33,0.38,0.19)(1.33,0.38,0.19)(1.4,0.4,0.2)(0.7,0.2,0.1))$
11	XI	$N_1 = \big((0,0,0)(0.14,0.04,0.02)(0.14,0.04,0.02)(0.28,0.08,0.04)(0.7,0.2,0.1)\big)$ $N_2 = ((0.42,0.12,0.06)(0.56,0.16,0.08)(0.56,0.16,0.08)(0.7,0.2,0.1)(0.7,0.2,0.1))$
12	XII	$N_1 = \big((0.14,0.04,0.02)(0.42,0.12,0.06)(0.42,0.12,0.06)(0.7,0.2,0.1)(0.7,0.2,0.1)\big)$ $N_2 = ((0.14,0.04,0.02)(0.42,0.12,0.06)(0.42,0.12,0.06)(0.7,0.2,0.1)(0.14,0.04,0.02))$
13	XIII	$N_1 = \big((0.42,0.12,0.06)(0.7,0.2,0.1)(0.7,0.2,0.1)(0.7,0.2,0.1)(0.7,0.2,0.1)\big)$ $N_2 = ((0.56,0.16,0.08)(0.7,0.2,0.1)(0.7,0.2,0.1)(0.7,0.2,0.1)(0.14,0.04,0.02))$

Table 1: Neutrosophic member of thirteen set

The Neutrosophic possibility degree of thirteen set is given in below table 2

S.No	Set	Neutrosophic possibility degree
1	I	$\big(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\big) = (0,0,0.94)$ $\big(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\big) = (1,1,1)$ $\big(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\big) = (1,1,1)$ $\big(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\big) = (0.77,0,0.98)$

2	II	$\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (0,0.72,0.95)$ $\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$ $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (0.67,0,0.98)$
3	III	$\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (0,1.29,0.95)$ $\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$ $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (0.78,0,0.99)$
4	IV	$\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (1,0,1)$ $\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$ $\left(p(TN_1 \geqslant TN_3), p(IN_1 \geqslant IN_3), p(FN_1 \geqslant FN_3)\right) = (1,0,1)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$ $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (0.2,0,0.95)$ $\left(p(TN_2 \geqslant TN_3), p(IN_2 \geqslant IN_3), p(FN_2 \geqslant FN_3)\right) = (1,0,1)$ $\left(p(TN_3 \geqslant TN_2), p(IN_3 \geqslant IN_2), p(FN_3 \geqslant FN_2)\right) = (0,0,0.92)$ $\left(p(TN_3 \geqslant TN_1), p(IN_3 \geqslant IN_1), p(FN_3 \geqslant FN_1)\right) = (0,0,0.91)$ $\left(p(TN_3 \geqslant TN_3), p(IN_3 \geqslant IN_3), p(FN_3 \geqslant FN_3)\right) = (1,1,1)$
5	V	$\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$ $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (1,0,1)$ $\left(p(TN_1 \geqslant TN_3), p(IN_1 \geqslant IN_3), p(FN_1 \geqslant FN_3)\right) = (1,0,1)$

		$\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (1,0,1)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$ $\left(p(TN_2 \geqslant TN_3), p(IN_2 \geqslant IN_3), p(FN_2 \geqslant FN_3)\right) = (1,0,1)$ $\left(p(TN_3 \geqslant TN_1), p(IN_3 \geqslant IN_1), p(FN_3 \geqslant FN_1)\right) = (0,0,0.96)$ $\left(p(TN_3 \geqslant TN_2), p(IN_3 \geqslant IN_2), p(FN_3 \geqslant FN_2)\right) = (0.25,0,0.96)$ $\left(p(TN_3 \geqslant TN_3), p(IN_3 \geqslant IN_3), p(FN_3 \geqslant FN_3)\right) = (1,1,1)$
6	VI	$\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$ $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (1,0,1)$ $\left(p(TN_1 \geqslant TN_3), p(IN_1 \geqslant IN_3), p(FN_1 \geqslant FN_3)\right) = (1,0,1)$ $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (0,0,0.9)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$ $\left(p(TN_2 \geqslant TN_3), p(IN_2 \geqslant IN_3), p(FN_2 \geqslant FN_3)\right) = (1,0,1)$ $\left(p(TN_3 \geqslant TN_1), p(IN_3 \geqslant IN_1), p(FN_3 \geqslant FN_1)\right) = (0,0,0.84)$ $\left(p(TN_3 \geqslant TN_2), p(IN_3 \geqslant IN_2), p(FN_3 \geqslant FN_2)\right) = (0.5,0,0.94)$ $\left(p(TN_3 \geqslant TN_3), p(IN_3 \geqslant IN_3), p(FN_3 \geqslant FN_3)\right) = (1,1,1)$
7	VII	$\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$ $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (1,0,1)$ $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (0.34,0,0.96)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$
8	VIII	$\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$

		$\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (0.5,0,0.98)$ $\left(p(TN_1 \geqslant TN_3), p(IN_1 \geqslant IN_3), p(FN_1 \geqslant FN_3)\right) = (0.25,0,0.97)$ $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (0.75,0,0.99)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$ $\left(p(TN_2 \geqslant TN_3), p(IN_2 \geqslant IN_3), p(FN_2 \geqslant FN_3)\right) = (0.25,0,0.95)$ $\left(p(TN_3 \geqslant TN_1), p(IN_3 \geqslant IN_1), p(FN_3 \geqslant FN_1)\right) = (1,0,1)$ $\left(p(TN_3 \geqslant TN_2), p(IN_3 \geqslant IN_2), p(FN_3 \geqslant FN_2)\right) = (0.84,0,0.99)$ $\left(p(TN_3 \geqslant TN_3), p(IN_3 \geqslant IN_3), p(FN_3 \geqslant FN_3)\right) = (1,1,1)$
9	IX	$\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$ $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (0,0.39,0.89)$ $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (1,0,1)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$
10	X	$\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$ $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (0,1.63,0.88)$ $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (1,0,1)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$
11	XI	$\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$ $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (0,0.39,0.89)$ $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (1,0,1)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$

12	XII	$\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$ $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (1,0,1))$ $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (0,0.15,0.72)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$
13	XIII	$\left(p(TN_1 \geqslant TN_1), p(IN_1 \geqslant IN_1), p(FN_1 \geqslant FN_1)\right) = (1,1,1)$ $\left(p(TN_1 \geqslant TN_2), p(IN_1 \geqslant IN_2), p(FN_1 \geqslant FN_2)\right) = (1,0,1))$ $\left(p(TN_2 \geqslant TN_1), p(IN_2 \geqslant IN_1), p(FN_2 \geqslant FN_1)\right) = (0,0.15,0.75)$ $\left(p(TN_2 \geqslant TN_2), p(IN_2 \geqslant IN_2), p(FN_2 \geqslant FN_2)\right) = (1,1,1)$

Table 2: Neutrosophic possibility degree of thirteen set

Table 3 represents the Neutrosophic preference matrix of 13 sets

S.No	Set	Neutrosophic preference matrix
1	I	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & (0,0,0.94) \\ (0.77,0,0.98) & (1,1,1) \end{pmatrix}$
2	II	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & (0,0.72,0.95) \\ (0.67,0,0.98) & (1,1,1) \end{pmatrix}$
3	III	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & (0,1.29,0.95) \\ (0.78,0,0.99) & (1,1,1) \end{pmatrix}$
4	IV	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & (1,0,1) & (1,0,1) \\ (0.2,0,0.95) & (1,1,1) & (1,0,1) \\ (0,0,0.91) & (0,0,0.92) & (1,1,1) \end{pmatrix}$
5	V	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & (1,0,1) & (1,0,1) \\ (1,0,1) & (1,1,1) & (1,0,1) \\ (0,0,0.96) & (0.25,0,0.96) & (1,1,1) \end{pmatrix}$
6	VI	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & (1,0,1) & (1,0,1) \\ (0,0,0.9) & (1,1,1) & (1,0,1) \\ (0,0,0.84) & (0.5,0,0.94) & (1,1,1) \end{pmatrix}$

7	VII	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & (1,0,1) \\ (0.34,0,0.96) & (1,1,1) \end{pmatrix}$
8	VIII	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & (0.5,0,0.98) & (0.25,0,0.97) \\ (0.75,0,0.99) & (1,1,1) & (0.25,0,0.95) \\ (1,0,1) & (0.84,0,0.99) & (1,1,1) \end{pmatrix}$
9	IX	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & (0,0.39,0.89) \\ (1,0,1) & (1,1,1) \end{pmatrix}$
10	X	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & (0,1.63,0.88) \\ (1,0,1) & (1,1,1) \end{pmatrix}$
11	XI	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & (0,0.39,0.89) \\ (1,0,1) & (1,1,1) \end{pmatrix}$
12	XII	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & 1,0,1) \\ (0,0.15,0.72) & (1,1,1) \end{pmatrix}$
13	XIII	$(TP, IP, FP) = \begin{pmatrix} (1,1,1) & 1,0,1) \\ (0,0.15,0.75) & (1,1,1) \end{pmatrix}$

Table 4: Neutrosophic preference matrix of thirteen set

Table 4: represents the Neutrosophic ranking value of thirteen sets

S.No	Set	Neutrosophic ranking value
1	I	$\mathcal{R}(TN_1, IN_1, FN_1) = (1,0,0.97), \mathcal{R}(TN_2, IN_2, FN_2) = (1,0,0.99)$
2	II	$\mathcal{R}(TN_1, IN_1, FN_1) = (1,0.85,0.98), \mathcal{R}(TN_2, IN_2, FN_2) = (1,0,0.99)$
3	III	$\mathcal{R}(TN_1, IN_1, FN_1) = (1,1.14,0.97), \mathcal{R}(TN_2, IN_2, FN_2) = (1,0,1)$
4	IV	$\mathcal{R}(TN_1, IN_1, FN_1) = (1,0,1), \mathcal{R}(TN_2, IN_2, FN_2) = (1,0,0.99), \mathcal{R}(TN_3, IN_3, FN_3) = (1,0,0.96)$
5	V	$\mathcal{R}(TN_1, IN_1, FN_1) = (1,0,1), \mathcal{R}(TN_2, IN_2, FN_2) = (1,0,1), \mathcal{R}(TN_3, IN_3, FN_3) = (1,0,0.98)$
6	VI	$\mathcal{R}(TN_1, IN_1, FN_1) = (1,0,1), \mathcal{R}(TN_2, IN_2, FN_2) = (1,0,1), \mathcal{R}(TN_3, IN_3, FN_3) = (1,0,0.98)$
7	VII	$\mathcal{R}(TN_1, IN_1, FN_1) = (1,0,1), \mathcal{R}(TN_2, IN_2, FN_2) = (1,0,0.98)$
8	VIII	$\mathcal{R}(TN_1, IN_1, FN_1) = (1,0,0.98), \mathcal{R}(TN_2, IN_2, FN_2) = (1,0,0.98), \mathcal{R}(TN_3, IN_3, FN_3) = (1,0,1)$

9	IX	$\mathcal{R}(TN_1, IN_1, FN_1) = (1{,}0.63{,}0.94), \mathcal{R}(TN_2, IN_2, FN_2) = (1{,}0{,}1)$
10	X	$\mathcal{R}(TN_1, IN_1, FN_1) = (1{,}1.28{,}0.94), \mathcal{R}(TN_2, IN_2, FN_2) = (1{,}0{,}1)$
11	XI	$\mathcal{R}(TN_1, IN_1, FN_1) = (1{,}0.63{,}0.94), \mathcal{R}(TN_2, IN_2, FN_2) = (1{,}0{,}1)$
12	XII	$\mathcal{R}(TN_1, IN_1, FN_1) = (1{,}0{,}1), \mathcal{R}(TN_2, IN_2, FN_2) = (1{,}0.38{,}0.85)$
13	III	$\mathcal{R}(TN_1, IN_1, FN_1) = (1{,}0{,}1), \mathcal{R}(TN_2, IN_2, FN_2) = (1{,}0.38{,}0.87)$

Table 5: Neutrosophic ranking value of thirteen sets

Table 6 represents the rank of the alternatives of thirteen sets

The rank of the alternatives $\mathcal{R}(N_1) = 0.9938$ and $\mathcal{R}(N_2) = 2.3027$

S.No	Set	rank of the alternatives
1	I	$\mathcal{R}(N_1) = 1.484$, $\mathcal{R}(N_2) = 1.495$
2	II	$\mathcal{R}(N_1) = 0.642$, $\mathcal{R}(N_2) = 1.494$
3	III	$\mathcal{R}(N_1) = 0.351$, $\mathcal{R}(N_2) = 1.497$
4	IV	$\mathcal{R}(N_1) = 1.5$, $\mathcal{R}(N_2) = 1.494, \mathcal{R}(N_3) = 1.478$
5	V	$\mathcal{R}(N_1) = 1.5$, $\mathcal{R}(N_2) = 1.5, \mathcal{R}(N_3) = 1.489$
6	VI	$\mathcal{R}(N_1) = 1.5$, $\mathcal{R}(N_2) = 1.486, \mathcal{R}(N_3) = 1.470$
7	VII	$\mathcal{R}(N_1) = 1.5$, $\mathcal{R}(N_2) = 1.5$
8	VIII	$\mathcal{R}(N_1) = 1.49$, $\mathcal{R}(N_2) = 1.49, \mathcal{R}(N_3) = 1.5$
9	IX	$\mathcal{R}(N_1) = 0.8$, $\mathcal{R}(N_2) = 1.5$
10	X	$\mathcal{R}(N_1) = 0.2, \mathcal{R}(N_2) = 1.5$
11	XI	$\mathcal{R}(N_1) = 0.8$, $\mathcal{R}(N_2) = 1.5$
12	XII	$\mathcal{R}(N_1) = 1.5$, $\mathcal{R}(N_2) = 1$

13	XIII	$\mathcal{R}(N_1) = 1.5$, $\mathcal{R}(N_2) = 1.1$

Table 6: Rank of the alternatives of thirteen sets

Table 7 represents the comparison results with the previous methods

S.No	Set	Alternatives	Kerre (30)	Lee (31) Uni form	Lee (31) Propo rtional	Bass (32)	Chang (33) $\alpha = 0.1$, $\beta = 0.9$	Chang (33) $\alpha = 0.5$, $\beta = 0.5$	Chan (20)	The Proposed Method
1	I	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$	0.96 0.89	0.58 0.55	0.54 0.59	0.84 1	0.417 0.462	0.519 0.544	0.52 0.48	1.484 , 1.495
2	II	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$	0.51 0.89	0.41 0.60	0.38 0.60	0.82 1	0.158 0.554	0.45 0.55	0.4 0.6	0.642 , 1.494
3	III	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$	0.42 0.95	0.41 0.70	0.38 0.70	0.66 1	0.158 0.644	0.45 0.6	0.36 0.64	0.351 , 1.497
4	IV	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$, $\mathcal{R}(N_3)$	1 0.86 0.76	0.77 0.70 0.63	0.80 0.70 0.60	1 0.74 0.6	0.878 0.788 0.698	0.65 0.6 0.55	0.39 0.33 0.28	1.5 , 1.494, 1.478
5	V	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$, $\mathcal{R}(N_3)$	1 0.91 0.75	0.70 0.63 0.58	0.70 0.65 0.57	1 1 1	0.752 0.743 0.73	0.6 0.575 0.538	0.4 0.32 0.28	1.5 , 1.5, 1.489
6	VI	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$, $\mathcal{R}(N_3)$	1 0.85 0.75	0.62 0.57 0.50	0.63 0.55 0.50	1 1 1	0.775 0.653 0.572	0.563 0.525 0.5	0.39 0.34 0.27	1.5 , 1.486, 1.470
7	VII	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$	0.91 0.91	0.50 0.50	0.50 0.50	1 1	0.608 0.536	0.5 0.5	0.5 0.5	1.5 , 1.5
8	VIII	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$, $\mathcal{R}(N_3)$	0.76 0.92 0.96	0.44 0.53 0.56	0.46 0.53 0.58	1 0.88 1	0.635 0.649 0.694	0.475 0.513 0.538	0.28 0.35 0.37	1.49 , 1.49, 1.5

9	IX	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$	0.64 1	0.20 0.80	0.20 0.80	0 0.8	0.158 0.688	0.35 0.6	0.28 0.72	0.8 , 1.5
10	X	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$	0.78 1	0.60 0.90	0.60 0.90	0 0.2	0.518 0.784	0.55 0.5	0.49 0.51	0.2, 1.5
11	XI	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$	0.89 0.88	0.20 0.80	0.20 0.80	0 0.2	0.118 0.698	0.15 0.65	0.25 0.75	0.8 , 1.5
12	XII	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$	0.72 0.97	0.60 0.60	0.60 0.60	0.2 0.2	0.446 0.406	0.55 0.35	0.63 0.37	1.5 , 1
13	XIII	$\mathcal{R}(N_1)$, $\mathcal{R}(N_2)$	0.82 1	0.87 0.95	0.90 0.95	0.2 0.2	0.932 0.901	0.7 0.525	0.63 0.37	1.5 , 1.1

Table 7: comparison results with the previous methods

From the above table7, the proposed method is comparatively better than the previous methods because it is giving the accurate result then the previous methods.

5. Conclusion:

Neutrosophic environments are more suited to portray the decision-makers uncertainty, indeterminacy, and ambiguity than trapezoidal and triangular ones. In comparison to the current way, the proposed method will provide the decision maker with the optimal attribute with greater accuracy. To demonstrate the NMAGDM process of the proposed technique, we additionally provide numerical examples. The result shows that the offered strategy provides us with a workable way to address NMAGDM problems based on trapezoidal and triangular Neutrosophic settings. Future research will involve using the suggested methods to address various other plithogenic environment-related decision-making problems.

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Applications of neutrosophic complex numbers in triangles

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Abstract: It may be difficult for researchers to memorize or remember the trigonometric ratios of any neutrosophic angle, and this is what prompted us to activate the role of the neutrosophic complex numbers for that. In this paper we presented neutrosophic Euler's formulas and neutrosophic De Moivre's formula. Also, we benefited from that by finding the trigonometric ratios of the multiples of neutrosophic angle in terms of the trigonometric ratios of the neutrosophic angle $(\ddot{\theta}+\ddot{\varphi}I)$ and convert trigonometric ratios from formula $sin^n(\ddot{\theta}+\ddot{\varphi}I)$, or formula $cos^m(\ddot{\theta}+\ddot{\varphi}I)$, into a linear expression for the multiples of the neutrosophic angle $(\ddot{\theta}+\ddot{\varphi}I)$, which made it easier for us to find integrals of the neutrosophic trigonometric functions by other methods.

Keywords: neutrosophic Euler's formulas, neutrosophic complex numbers, neutrosophic De Moivre's formula.

1. Introduction

As Smarandache proposed the Neutrosophic Logic as an alternative to the existing logics to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction. Smarandache introduced the concept of neutrosophy as a new school of philosophy [4][8]. He presented the definition of the standard form of neutrosophic real number and conditions for the division of two neutrosophic real numbers to exist [3][5], studying the concept of the Neutrosophic probability [11][6], the Neutrosophic statistics [5], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1][8]. Y.Alhasan presented the definition of the concept of neutrosophic complex numbers and its properties including the conjugate of neutrosophic complex number, division of neutrosophic complex numbers, the inverted neutrosophic complex number and the absolute value of a neutrosophic complex number and Theories related to the conjugate of neutrosophic complex numbers, the product of a neutrosophic complex number by its conjugate equals the absolute value of number and he studied the general exponential form of a neutrosophic complex number [2-4]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [10]. An algebraic approach to neutrosophic euclidean geometry is presented [7].

Complex numbers play a significant role in daily life because they make it much easier to perform mathematical operations and give us a way to solve equations for which there are no real-number-group solutions. The electrical engineering field makes extensive use of complex numbers to calculate electric voltage and measure alternating current.

Paper is divided into four pieces. provides an introduction in the first portion, which includes a review of neutrosophic science. A few definitions of a neutrosophic complex number are covered in the second section. The third section defined neutrosophic Euler's formulas, neutrosophic De Moivre's formula and discuses applications of neutrosophic complex numbers in triangles. The paper's conclusion is provided in the fourth section.

2. Preliminaries

2.1 The general Trigonometric form of a neutrosophic complex number [4]

Definition 1

The following formula:

$$z = \mathrm{r}\ \ (\cos(\theta + \vartheta I) + sin\ \ (\theta + \vartheta I)\ \ i)$$

is called the general trigonometric form of a neutrosophic complex number

Definition 2 [7]

Let f: $R(I) \ \rightarrow \ R(I)$; $f = f(X)$ and $X = x + yI \in R(I)$ the f is called a neutrosophic real function with one neutrosophic variable. a neutrosophic real function $f(X)$ written as follows:

$$f(X) = f(x + yI) = f(x) + I[f(x + y) - f(x)]$$

3. Neutrosophic Euler's formulas

Let:

$$e^{i(\ddot{\theta}+\ddot{\varphi}I)} = cos\big(\ddot{\theta} + \ddot{\varphi}I\big) + i\ \ sin(\ddot{\theta} + \ddot{\varphi}I)$$

$$e^{-i(\ddot{\theta}+\ddot{\varphi}I)} = cos\big(\ddot{\theta} + \ddot{\varphi}I\big) - i\ \ sin(\ddot{\theta} + \ddot{\varphi}I)$$

by additional:

$$cos\big(\ddot{\theta} + \ddot{\varphi}I\big) = \frac{e^{i(\ddot{\theta}+\ddot{\varphi}I)} + e^{-i(\ddot{\theta}+\ddot{\varphi}I)}}{2}$$

by subtraction:

$$sin(\ddot{\theta} + \ddot{\varphi}I) = \frac{e^{i(\ddot{\theta}+\ddot{\varphi}I)} - e^{-i(\ddot{\theta}+\ddot{\varphi}I)}}{2i}$$

They are neutrosophic Euler's formulas.

3.1 Neutrosophic De Moivre's formula

$$z = \acute{r}e^{i(\ddot{\theta}+\ddot{\varphi}I)}$$

$$\mathrm{z}^{n} = \left(\acute{r}e^{i(\ddot{\theta}+\ddot{\varphi}I)}\right)^{n}$$

$$\acute{r}^{n}e^{i(n\ddot{\theta}+n\ddot{\varphi}I)} = \acute{r}^{n}\ \ cos\big(n\ddot{\theta} + n\ddot{\varphi}I\big) + i\ \ \acute{r}^{n}\ \ sin\big(n\ddot{\theta} + n\ddot{\varphi}I\big) \qquad ; n \in \mathbb{Z}$$

$$= \acute{r}^n \ \left[cos(n\ddot{\theta}) + I\left(cos(n\ddot{\theta} + n\ddot{\varphi}) - cos(n\ddot{\theta})\right)\right] + i \ \acute{r}^n \ \left[sin(n\ddot{\theta}) + I\left(sin(n\ddot{\theta} + n\ddot{\varphi}) - sin(n\ddot{\theta})\right)\right]$$

Example1

$$\left(1 + \left(\frac{\sqrt{2}}{2} - 1\right)I + \left(\frac{\sqrt{2}}{2} \ \ I\right)i\right)^{24} = \left(e^{i\left(\frac{\pi}{4}I\right)}\right)^{24} = cos24\left(\frac{\pi}{4}I\right) + i \ \ sin24\left(\frac{\pi}{4}I\right)$$

$$= cos(6\pi I) + i \ \ sin(6\pi I)$$

$$= cos(0) + I\left[\cos\left(0 + (6\pi)\right) - \cos(0)\right] + i \ \ (\sin(0) + I[sin(0 + 6\pi) - \sin(0)])$$

$$= 1 + 0I + 0i = 1$$

Theorem1

Let $(\ddot{\theta} + \ddot{\varphi}I)$ neutrosophic real number, then the solution of the equation:

$$e^{i(\ddot{\theta}+\ddot{\varphi}I)} = e^{i(\ddot{\vartheta}+\ddot{\omega}I)}$$

by unknown $(\ddot{\vartheta} + \ddot{\omega}I)$, is:

$$\left\{\ddot{\theta} + \ddot{\varphi}I + 2\pi k \qquad ; k \in \mathrm{Z}\right\}$$

Proof:

multiply:

$$e^{i(\ddot{\theta}+\ddot{\varphi}I)} = e^{i(\ddot{\vartheta}+\ddot{\omega}I)}$$

by:

$$e^{-i(\ddot{\vartheta}+\ddot{\omega}I)} \neq 1$$

we find:

$$e^{i(\ddot{\theta}+\ddot{\varphi}I)}e^{-i(\ddot{\vartheta}+\ddot{\omega}I)} = 1$$

$$e^{i(\ddot{\theta}-\ddot{\vartheta}+(\ddot{\omega}-\ddot{\varphi})I)} = 1$$

$$cos\left(\ddot{\theta} - \ddot{\vartheta} + (\ddot{\omega} - \ddot{\varphi})I\right) + i \ \ sin\left(\ddot{\theta} - \ddot{\vartheta} + (\ddot{\omega} - \ddot{\varphi})I\right) = 1$$

then:

$$cos\left(\ddot{\theta} - \ddot{\vartheta} + (\ddot{\omega} - \ddot{\varphi})I\right) = 1 \qquad and \qquad sin\left(\ddot{\theta} - \ddot{\vartheta} + (\ddot{\omega} - \ddot{\varphi})I\right) = 0$$

hence:

$$\ddot{\vartheta} + \ddot{\omega}I = \ddot{\theta} + \ddot{\varphi}I + 2\pi k \qquad ; k \in \mathrm{Z}$$

3.2. Applications of neutrosophic complex numbers in triangles

3.2.1 Finding the trigonometric ratios of the multiples of neutrosophic angle in terms of the trigonometric ratios of the angle neutrosophic $(\ddot{\theta} + \ddot{\varphi}I)$

The trigonometric ratios of angle neutrosophic $\mathbf{2}(\ddot{\theta} + \ddot{\boldsymbol{\varphi}}I)$ in terms of the trigonometric ratios of angle $(\ddot{\theta} + \ddot{\boldsymbol{\varphi}}I)$:

by using De Moivre's formula:

$$\left(cos\left(\ddot{\theta} + \ddot{\varphi}I\right) + i \ \ \sin(\ddot{\theta} + \ddot{\varphi}I)\right)^2 = \cos 2\left(\ddot{\theta} + \ddot{\varphi}I\right) + i\,sin\,2\left(\ddot{\theta} + \ddot{\varphi}I\right)$$

$$cos^2\left(\ddot{\theta} + \ddot{\varphi}I\right) - sin^2(\ddot{\theta} + \ddot{\varphi}I) + 2i \ \ cos\left(\ddot{\theta} + \ddot{\varphi}I\right) \ \ sin(\ddot{\theta} + \ddot{\varphi}I) = cos\,2\left(\ddot{\theta} + \ddot{\varphi}I\right) + i\,sin\,2(\ddot{\theta} + \ddot{\varphi}I)$$

by equating the two real parts of both sides of the equality:

$$cos\,2(\ddot{\theta}+\ddot{\varphi}I) = cos^2(\ddot{\theta}+\ddot{\varphi}I) - sin^2(\ddot{\theta}+\ddot{\varphi}I)$$

by equating the two imaginary parts on both sides of the equality:

$$sin\,2(\ddot{\theta}+\ddot{\varphi}I) = 2\ \ cos(\ddot{\theta}+\ddot{\varphi}I)\ \ sin(\ddot{\theta}+\ddot{\varphi}I)$$

where:

$$cos(\ddot{\theta}+\ddot{\varphi}I) = cos(\ddot{\theta}) + I\left(cos(\ddot{\theta}+\ddot{\varphi}) - cos(\ddot{\theta})\right)$$

$$sin(\ddot{\theta}+\ddot{\varphi}I) = sin(\ddot{\theta}) + I\left(sin(\ddot{\theta}+\ddot{\varphi}) - sin(\ddot{\theta})\right)$$

to find $tan\,2(\ddot{\theta}+\ddot{\varphi}I)$:

$$tan\,2(\ddot{\theta}+\ddot{\varphi}I) = \frac{sin\,2(\ddot{\theta}+\ddot{\varphi}I)}{cos\,2(\ddot{\theta}+\ddot{\varphi}I)}$$

$$= \frac{2\ \ cos(\ddot{\theta}+\ddot{\varphi}I)\ \ \sin(\ddot{\theta}+\ddot{\varphi}I)}{cos^2(\ddot{\theta}+\ddot{\varphi}I) - sin^2(\ddot{\theta}+\ddot{\varphi}I)}$$

$$= \frac{\frac{2\ \ cos(\ddot{\theta}+\ddot{\varphi}I)\ \ \sin(\ddot{\theta}+\ddot{\varphi}I)}{cos^2(\ddot{\theta}+\ddot{\varphi}I)}}{\frac{cos^2(\ddot{\theta}+\ddot{\varphi}I) - sin^2(\ddot{\theta}+\ddot{\varphi}I)}{cos^2(\ddot{\theta}+\ddot{\varphi}I)}}$$

$$\Longrightarrow \qquad tan\,2(\ddot{\theta}+\ddot{\varphi}I) = \frac{2\,tan(\ddot{\theta}+\ddot{\varphi}I)}{1 - tan(\ddot{\theta}+\ddot{\varphi}I)}$$

where:

$$tan(\ddot{\theta}+\ddot{\varphi}I) = tan(\ddot{\theta}) + I\left(tan(\ddot{\theta}+\ddot{\varphi}) - tan(\ddot{\theta})\right)$$

Example2

Write the trigonometric ratios of angle neutrosophic $4(\ddot{\theta}+\ddot{\varphi}I)$ in terms of the trigonometric ratios of angle $(\ddot{\theta}+\ddot{\varphi}I)$.

by using De Moivre's formula:

$$\left(cos(\ddot{\theta}+\ddot{\varphi}I) + i\ \ sin(\ddot{\theta}+\ddot{\varphi}I)\right)^4 = cos\,4(\ddot{\theta}+\ddot{\varphi}I) + i\,sin\,4(\ddot{\theta}+\ddot{\varphi}I)$$

$$cos^4(\ddot{\theta}+\ddot{\varphi}I) + 4i\ \ cos^3(\ddot{\theta}+\ddot{\varphi}I)\ \ sin(\ddot{\theta}+\ddot{\varphi}I) + 6i^2\ \ cos^2(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^2(\ddot{\theta}+\ddot{\varphi}I) + 4i^3\ \ cos(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^3(\ddot{\theta}+\ddot{\varphi}I) + i^4 sin^4(\ddot{\theta}+\ddot{\varphi}I) = cos\,4(\ddot{\theta}+\ddot{\varphi}I) + i\,sin\,4(\ddot{\theta}+\ddot{\varphi}I)$$

$$cos^4(\ddot{\theta}+\ddot{\varphi}I) + 4i\ \ cos^3(\ddot{\theta}+\ddot{\varphi}I)\ \ sin(\ddot{\theta}+\ddot{\varphi}I) - 6\ \ cos^2(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^2(\ddot{\theta}+\ddot{\varphi}I) - 4i\ \ cos(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^3(\ddot{\theta}+\ddot{\varphi}I) + sin^4(\ddot{\theta}+\ddot{\varphi}I) = cos\,4(\ddot{\theta}+\ddot{\varphi}I) + i\,sin\,4(\ddot{\theta}+\ddot{\varphi}I)$$

$$\left[cos^4(\ddot{\theta}+\ddot{\varphi}I) - 6\ \ cos^2(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^2(\ddot{\theta}+\ddot{\varphi}I) + sin^4(\ddot{\theta}+\ddot{\varphi}I)\right] + i\left[4\ \ cos^3(\ddot{\theta}+\ddot{\varphi}I)\ \ sin(\ddot{\theta}+\ddot{\varphi}I) - 4\ \ cos(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^3(\ddot{\theta}+\ddot{\varphi}I)\right] = cos\,4(\ddot{\theta}+\ddot{\varphi}I) + i\,sin\,4(\ddot{\theta}+\ddot{\varphi}I)$$

by equating the two real parts of both sides of the equality:

$$cos\,4(\ddot{\theta}+\ddot{\varphi}I)=cos^4(\ddot{\theta}+\ddot{\varphi}I)-6\ \ cos^2(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^2(\ddot{\theta}+\ddot{\varphi}I)+sin^4(\ddot{\theta}+\ddot{\varphi}I)$$

by equating the two imaginary parts on both sides of the equality:

$$sin\,4(\ddot{\theta}+\ddot{\varphi}I)=4\ \ cos^3(\ddot{\theta}+\ddot{\varphi}I)\ \ sin(\ddot{\theta}+\ddot{\varphi}I)-4\ \ cos(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^3(\ddot{\theta}+\ddot{\varphi}I)$$

where:

$$cos(\ddot{\theta}+\ddot{\varphi}I)=cos(\ddot{\theta})+I\left(cos(\ddot{\theta}+\ddot{\varphi})-cos(\ddot{\theta})\right)$$

$$sin(\ddot{\theta}+\ddot{\varphi}I)=sin(\ddot{\theta})+I\left(sin(\ddot{\theta}+\ddot{\varphi})-sin(\ddot{\theta})\right)$$

to find $tan\,4(\ddot{\theta}+\ddot{\varphi}I)$:

$$tan\,4(\ddot{\theta}+\ddot{\varphi}I)=\frac{sin\,4(\ddot{\theta}+\ddot{\varphi}I)}{cos\,4(\ddot{\theta}+\ddot{\varphi}I)}$$

$$=\frac{4\ \ cos^3(\ddot{\theta}+\ddot{\varphi}I)\ \ sin(\ddot{\theta}+\ddot{\varphi}I)-4\ \ cos(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^3(\ddot{\theta}+\ddot{\varphi}I)}{cos^4(\ddot{\theta}+\ddot{\varphi}I)-6\ \ cos^2(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^2(\ddot{\theta}+\ddot{\varphi}I)+sin^4(\ddot{\theta}+\ddot{\varphi}I)}$$

$$=\frac{\dfrac{4\ \ cos^3(\ddot{\theta}+\ddot{\varphi}I)\ \ sin(\ddot{\theta}+\ddot{\varphi}I)-4\ \ cos(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^3(\ddot{\theta}+\ddot{\varphi}I)}{cos^4(\ddot{\theta}+\ddot{\varphi}I)}}{\dfrac{cos^4(\ddot{\theta}+\ddot{\varphi}I)-6\ \ cos^2(\ddot{\theta}+\ddot{\varphi}I)\ \ sin^2(\ddot{\theta}+\ddot{\varphi}I)+sin^4(\ddot{\theta}+\ddot{\varphi}I)}{cos^4(\ddot{\theta}+\ddot{\varphi}I)}}$$

$$\Longrightarrow \qquad tan\,4(\ddot{\theta}+\ddot{\varphi}I)=\frac{4\,tan(\ddot{\theta}+\ddot{\varphi}I)-4\,tan^3(\ddot{\theta}+\ddot{\varphi}I)}{1-6tan^2(\ddot{\theta}+\ddot{\varphi}I)+tan^4(\ddot{\theta}+\ddot{\varphi}I)}$$

where:

$$tan(\ddot{\theta}+\ddot{\varphi}I)=tan(\ddot{\theta})+I\left(tan(\ddot{\theta}+\ddot{\varphi})-tan(\ddot{\theta})\right)$$

3.2.2 Convert trigonometric ratios from Formula $sin^n(\ddot{\theta}+\ddot{\varphi}I)$, or Formula $cos^m(\ddot{\theta}+\ddot{\varphi}I)$, into a linear expression for the multiples of the neutrosophic angle $(\ddot{\theta}+\ddot{\varphi}I)$

Example3

Write $cos^3(\ddot{\theta}+\ddot{\varphi}I)$ in the form of the sum of the trigonometric ratios of the multiples of the neutrosophic angle $(\ddot{\theta}+\ddot{\varphi}I)$

by using neutrosophic Euler's formulas:

$$cos^3(\ddot{\theta}+\ddot{\varphi}I)=\left(\frac{e^{i(\ddot{\theta}+\ddot{\varphi}I)}+e^{-i(\ddot{\theta}+\ddot{\varphi}I)}}{2}\right)^3$$

$$=\frac{1}{8}\left(e^{3i(\ddot{\theta}+\ddot{\varphi}I)}+3e^{2i(\ddot{\theta}+\ddot{\varphi}I)}e^{-i(\ddot{\theta}+\ddot{\varphi}I)}+3e^{i(\ddot{\theta}+\ddot{\varphi}I)}e^{-2i(\ddot{\theta}+\ddot{\varphi}I)}+e^{-3i(\ddot{\theta}+\ddot{\varphi}I)}\right)$$

$$=\frac{1}{8}\left[\left(e^{3i(\ddot{\theta}+\ddot{\varphi}I)}+e^{-3i(\ddot{\theta}+\ddot{\varphi}I)}\right)+3\left(e^{i(\ddot{\theta}+\ddot{\varphi}I)}+e^{-i(\ddot{\theta}+\ddot{\varphi}I)}\right)\right]$$

by using neutrosophic Euler's formulas to return to the trigonometric ratios:

$$cos^3(\ddot{\theta}+\ddot{\varphi}I)=\frac{1}{8}[2cos3(\ddot{\theta}+\ddot{\varphi}I)+6cos(\ddot{\theta}+\ddot{\varphi}I)]$$

$$=\frac{1}{4}cos3(\ddot{\theta}+\ddot{\varphi}I)+\frac{3}{4}cos(\ddot{\theta}+\ddot{\varphi}I)$$

$$=\frac{1}{4}\left[cos(3\ddot{\theta})+I\left(cos(3\ddot{\theta}+3\ddot{\varphi})-cos(3\ddot{\theta})\right)\right]+\frac{3}{4}\left[cos(\ddot{\theta})+I\left(cos(\ddot{\theta}+\ddot{\varphi})-cos(\ddot{\theta})\right)\right]$$

Example4

Write $sin^6(\ddot{\theta}+\ddot{\varphi}I)$ in the form of the sum of the trigonometric ratios of the multiples of the neutrosophic angle $(\ddot{\theta}+\ddot{\varphi}I)$, then find based on what you find:

$$\int sin^6(\ddot{\theta}+\ddot{\varphi}I)\ \ d(\ddot{\theta}+\ddot{\varphi}I)$$

Solution:

by using neutrosophic Euler's formulas:

$$sin^6(\ddot{\theta}+\ddot{\varphi}I)=\left(\frac{e^{i(\ddot{\theta}+\ddot{\varphi}I)}-e^{-i(\ddot{\theta}+\ddot{\varphi}I)}}{2i}\right)^6$$

$$=\frac{-1}{64}\left(e^{i(\ddot{\theta}+\ddot{\varphi}I)}-e^{-i(\ddot{\theta}+\ddot{\varphi}I)}\right)^6$$

$$=\frac{-1}{64}\big(e^{6i(\ddot{\theta}+\ddot{\varphi}I)}+6e^{5i(\ddot{\theta}+\ddot{\varphi}I)}e^{-i(\ddot{\theta}+\ddot{\varphi}I)}+15e^{4i(\ddot{\theta}+\ddot{\varphi}I)}e^{-2i(\ddot{\theta}+\ddot{\varphi}I)}-20e^{3i(\ddot{\theta}+\ddot{\varphi}I)}e^{-3i(\ddot{\theta}+\ddot{\varphi}I)}$$
$$+15e^{2i(\ddot{\theta}+\ddot{\varphi}I)}e^{-4i(\ddot{\theta}+\ddot{\varphi}I)}-6e^{i(\ddot{\theta}+\ddot{\varphi}I)}e^{-5i(\ddot{\theta}+\ddot{\varphi}I)}e^{-6i(\ddot{\theta}+\ddot{\varphi}I)}\big)$$

$$=\frac{-1}{64}\left[\left(e^{6i(\ddot{\theta}+\ddot{\varphi}I)}+e^{-6i(\ddot{\theta}+\ddot{\varphi}I)}\right)-6\left(e^{4i(\ddot{\theta}+\ddot{\varphi}I)}+e^{4i(\ddot{\theta}+\ddot{\varphi}I)}\right)+15\left(e^{2i(\ddot{\theta}+\ddot{\varphi}I)}+e^{-2i(\ddot{\theta}+\ddot{\varphi}I)}\right)-20\right]$$

by using neutrosophic Euler's formulas to return to the trigonometric ratios:

$$cos^3(\ddot{\theta}+\ddot{\varphi}I)=\frac{-1}{64}[2cos6(\ddot{\theta}+\ddot{\varphi}I)-12\,cos\,4(\ddot{\theta}+\ddot{\varphi}I)+30\,cos\,2(\ddot{\theta}+\ddot{\varphi}I)-20]$$

$$\Rightarrow\qquad cos^3(\ddot{\theta}+\ddot{\varphi}I)=\frac{-1}{32}[cos6(\ddot{\theta}+\ddot{\varphi}I)-6\,cos\,4(\ddot{\theta}+\ddot{\varphi}I)+15\,cos\,2(\ddot{\theta}+\ddot{\varphi}I)-10]$$

$$=\frac{-1}{32}\Big(\left[cos(6\ddot{\theta})+I\left(cos(6\ddot{\theta}+6\ddot{\varphi})-cos(6\ddot{\theta})\right)\right]+6\left[cos(4\ddot{\theta})+I\left(cos(4\ddot{\theta}+4\ddot{\varphi})-cos(4\ddot{\theta})\right)\right]$$
$$+15\left[cos(2\ddot{\theta})+I\left(cos(2\ddot{\theta}+2\ddot{\varphi})-cos(2\ddot{\theta})\right)\right]-10\Big)$$

to find:

$$\int sin^6(\ddot{\theta}+\ddot{\varphi}I)\ \ d(\ddot{\theta}+\ddot{\varphi}I)$$
$$=\int\frac{-1}{32}[cos6(\ddot{\theta}+\ddot{\varphi}I)-6\,cos\,4(\ddot{\theta}+\ddot{\varphi}I)+15\,cos\,2(\ddot{\theta}+\ddot{\varphi}I)-10]\,d(\ddot{\theta}+\ddot{\varphi}I)$$

$$= \frac{-1}{32}\left[\frac{1}{6}sin\,6(\ddot{\theta}+\ddot{\varphi}I)-\frac{3}{2}sin\,4(\ddot{\theta}+\ddot{\varphi}I)+\frac{15}{2}sin\,2(\ddot{\theta}+\ddot{\varphi}I)-10(\ddot{\theta}+\ddot{\varphi}I)\right]+a+bI$$

$$= \frac{-1}{32}\left(\frac{1}{6}\left[sin(6\ddot{\theta})+I\left(sin(6\ddot{\theta}+6\ddot{\varphi})-sin(6\ddot{\theta})\right)\right]-\frac{3}{2}\left[sin(4\ddot{\theta})+I\left(sin(4\ddot{\theta}+4\ddot{\varphi})-sin(4\ddot{\theta})\right)\right]\right.$$
$$\left.+\frac{15}{2}\left[sin(2\ddot{\theta})+I\left(sin(2\ddot{\theta}+2\ddot{\varphi})-sin(2\ddot{\theta})\right)\right]-10(\ddot{\theta}+\ddot{\varphi}I)\right)+a+bI$$

4. Conclusions

The importance of this paper comes from the fact that we were able to find the formula of neutrosophic Euler's formulas and neutrosophic De Moivre's formula according to an accurate scientific method, which facilitated finding applications of neutrosophic complex numbers in triangles, and access to easy ways to calculate trigonometric integrals. One of the research most important on neutrosophic complex numbers is this paper.

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AHP Approach using Interval Neutrosophic Weighted Averaging (INWA) Operator for Ranking Flash Floods Contributing Factors

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Abstract: Aggregation operators are crucial in the process of multicriteria decision-making (MCDM) problems, as their main goal is to aggregate a collection of input to a single number. Analytic Hierarchy Process (AHP) has been used to solve a variety of MCDM situations in which crisp numbers are used to define linguistic assessment. The Interval-Valued Neutrosophic (IVN) number can consider the indeterminacy, fuzziness, and uncertainty in the real-world problem. A new combination of Interval Neutrosophic Weighted Averaging (INWA) aggregation operators into the AHP method is proposed in this study. The proposed combination method is applied to a case of factors affecting floods. In recent years, flood is one of the frequent natural disasters impacting Penang, one of the states that is famous for its tourism industry. Hence, an improved decision model is used to rank the factors of flash floods in Penang Malaysia. based on the INWA aggregated matrix implemented into the AHP approach is presented. The ranking order is determined after assessing the obtained data, with the highest score being the most important factor (rephrase ayat ni). Government and authorities can use the findings to establish early preparations and prevention strategies to deal with the flash floods problem.

Keywords: decision-making; fuzzy set theory; neutrosophic set theory; interval neutrosophic set theory; averaging operator

1. Introduction

Multi-criteria decision-making involves multiple decision makers and multiple deciding criteria. The issue is the use of a crisp number scale to describe the decision makers' opinions does not cater the fuzziness and indeterminacy during the evaluation process in real-world problems. To address flaws in real-number applications, fuzzy set theory was introduced by Lotfi Zadeh in 1965. A fuzzy set is a crisp set with a membership function that can take any value between 0 and 1. Several extensions of fuzzy set theories such as interval-valued fuzzy set, intuitionistic fuzzy set, neutrosophic set, and many more have been applied in various case studies. The neutrosophic set (NS) appears to be more reasonable and acceptable compared to these FSs [1]. Besides, the concept of neutrosophic sets introduced by Smarandache [2] is interesting and useful in modeling several real-life problems. The truth membership function, the indeterminacy membership function, and the falsity membership function, all of which are entirely independent, are related to the neutrosophic set theory (NS), which is a generalization of the intuitionistic fuzzy set (IFS) theory. This form clearly and successfully deals with not only missing information but also indeterminate and inconsistent information [3]. Hence, neutrosophic sets (NS) can help in dealing with the uncertainty that exists in real-world circumstances.

Wang et al. [4] established the concept of Interval-Valued Neutrosophic Sets theory (IVNS), which is a subset of neutrosophic sets [5]. This concept is characterized by a membership function, a non-membership function, and an indeterminacy function, whose values are intervals rather than real numbers. IVNS is considered a valuable and practical tool for dealing with indeterminate and inconsistent information in the real world since it is more powerful than NS in dealing with vagueness and uncertainty. In multi-criteria decision-making problems, multiple decision-makers must be aggregated using the appropriate aggregation operators.

Aggregation operators are an interesting area of research that plays an important role in group decision-making analysis. The traditional aggregation operators are usually based on the arithmetic and geometric mean approaches, often known as algebraic sum and algebraic product. The issue is the averaging method assumes a similar weight for all decision makers. In the real world, different weights may be assigned to different evaluations by multiple decision-makers [6]. Hence, Aczel and Saaty [7] proposed a weighted geometric (WG) mean aggregation operator for the synthesis of ratio judgments in the AHP method while Dong and Dong [8] later proposed a weighted arithmetic (WA) aggregation operator with a fuzzy set as its quantifier. In the neutrosophic environment, several neutrosophic aggregation operators were suggested, such as Interval Neutrosophic Weighted Averaging (INWA) and Interval Neutrosophic Weighted Geometric (INWG) [9]. In this study, the implementation of the INWA aggregation operator into the Multi-Criteria Decision-Making (MCDM) method is introduced.

Decision-making is a process of selecting the best alternatives based on certain criteria. MCDM also known as Multi-Criteria Decision Analysis (MCDA) is a method or process of decision-making involving multiple criteria that need to be considered to choose the best option between them. This method has been used in many fields such as engineering [10], management science [11], education [12], investment problem [13], and medical science [14]. There are many methods available to solve MCDM problems such as the Analytic Hierarchy Process (AHP), Technique for Order of Preference by Similarities to Ideal Solution (TOPSIS) [15,16], Preference Ranking Organization Method for Enrichment of Evaluations (PROMETHEE) [17], and Decision-Making Trial and Evaluation Laboratory (DEMATEL) [18]. Amongst these MCDM methods, AHP is a more flexible and realistic method to use because it produces a simple way to find the relationships between criteria and alternatives [19]. The AHP method was proposed by Saaty [20] as an easily justified, discriminating, and intentional MCDM technique. AHP has the ability to detangle a difficult problem by breaking it down into smaller parts with the hierarchical structure approach. Recently, most of the AHP methods have been extended based on fuzzy set and neutrosophic set theories. The application of the AHP method also has been diversified. In this study, the proposed AHP method with INWA operator is used to solve the flash floods problem in Penang.

Flash floods are the most terrifying natural disasters that can occur with little or no warning. A flash flood is a rapid-developing, short-duration flood that occurs within a few hours of the triggering event. Perhaps, flash floods are the most frequent disasters that happened and caused the greatest damage to the world. Flash floods occurred because of natural factors and human factors. Based on the literature review, eight factors are taken into consideration, which are rain intensity, rain duration, poor drainage system, dam and levee failure, urbanization, slow-moving thunderstorm, soil erosion, and land use pattern [21-29]. Five decision makers are invited to answer the questionnaire by pairwise comparison between factors. The findings of this study will be beneficial to the Drainage Irrigation Department (DID) or even the society as a source of reference that can be used to identify the most important factor in flash floods occurs. Hence, this research is important to help the Drainage Irrigation Department (DID), the in-charge agency of natural disasters in Penang, and the society recognized the most important factors that caused flash floods happened more accurately so that they are better prepared to deal with flash floods in the future. Section 2 goes over some preliminary concepts. Section 3 describes details the AHP's research methodology with the INWA operator. Section 4 discusses the proposed method's application to the problem of flash floods, and Section 5 concludes with remarks.

2. Preliminaries

In this section, we review some basic concepts related to INVS which will be used in the rest of the paper.

Definition 1: [4] Interval-Valued Neutrosophic (IVN) Sets

Let X be a universe of discourse and Int [0,1] be the set of all closed subsets of [0,1]. Then an interval neutrosophic set is defined as:

$$A=\left\{\left\langle x,u_A(x),p_A(x),v_A(x)\right\rangle : x\in X\right\} \tag{1}$$

where $u_A : X \rightarrow Int[0,1], p_A : X \rightarrow Int[0,1]$ and $v_A : X \rightarrow Int[0,1]$ with $0 \le \sup u_A^U(x) + \sup p_A^U + \sup v_A^U \le 3$ for all. The interval $u_A(x), p_A(x)$ and $v_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity-membership of x to A respectively.

For convenience, if let $u_A(x)=\left[u_A^L(x),u_A^U(x)\right], p_A(x)=\left[p_A^L(x),p_A^U(x)\right]$, and $v_A(x)=\left[v_A^L(x),v_A^U(x)\right]$,then $A=\left\{x,\left[u_A^L(x),u_A^U(x)\right],\left[p_A^L(x),p_A^U(x)\right],\left[v_A^L(x),v_A^U(x)\right]: x\in X\right\}$

with the condition, $0 \le \sup u_A^U(x) + \sup p_A^U + \sup v_A^U \le 3$ for all $x \in X$. Here, we only consider the sub-unitary interval of [0,1]. Therefore, an interval neutrosophic set is clearly a neutrosophic set [3]. Table 1 shows the IVN scales.

Table 1: Linguistic IVN Scales [3]

Linguistic Variables	**IVN**
Equal Importance (EI)	<[0.5,0.5],[0.5,0.5],[0.5,0.5]>
Equal Importance Complement (EI^C)	<[0.5,0.5],[0.5,0.5],[0.5,0.5]>
Weakly More Importance (WMI)	<[0.5,0.6],[0.35,0.45],[0.4,0.5]>
Weakly More Importance Complement (WMI^C)	<[0.4,0.5],[0.35,0.45],[0.5,0.6]>
Moderate Importance (MI)	<[0.55,0.65],[0.3,0.4],[0.35,0.45]>
Moderate Importance Complement (MI^C)	<[0.35,0.45],[0.3,0.4],[0.55,0.65]>
Moderately More Importance (MMI)	<[0.6,0.7],[0.25,0.35],[0.3,0.4]>
Moderately More Importance Complement (MMI^C)	<[0.3,0.4],[0.25,0.35],[0.6,0.7]>
Strong Importance (SI)	<[0.65,0.75],[0.2,0.3],[0.25,0.35]>
Strong Importance Complement (SI^C)	<[0.25,0.35],[0.2,0.3],[0.65,0.75]>
Strongly More Importance (SMI)	<[0.7,0.8],[0.15,0.25],[0.2,0.3]>
Strongly More Importance Complement (SMI^C)	<[0.2,0.3],[0.15,0.25],[0.7,0.8]>
Very Strong Importance (VSI)	<[0.75,0.85],[0.1,0.2],[0.15,0.25]>
Very Strong Importance Complement (VSI^C)	<[0.15,0.25],[0.1,0.2],[0.75,0.85]>
Very Strongly More Importance (VSMI)	<[0.8,0.9],[0.05,0.1],[0.1,0.2]>
Very Strongly More Importance Complement ($VSMI^C$)	<[0.1,0.2],[0.05,0.1],[0.8,0.9]>
Extreme Importance (EI)	<[0.9,0.95],[0,0.05],[0.05,0.15]>
Extreme Importance Complement (EI^C)	<[0.05,0.1],[0,0.05],[0.85,0.95]>
Extremely High Importance (EHI)	<[0.95,1],[0,0],[0,0.1]>
Extremely High Importance Complement (EHI^C)	<[0,0.05],[0,0],[0.9,1]>
Absolutely More Importance (AMI)	<[1,1],[0,0],[0,0]>
Absolutely More Importance Complement (AMI^C)	<[0,0],[0,0],[1,1]>

Definition 2: [30] Interval Neutrosophic Weighted Average (INWA) Operator

Let $A = \{A_1, A_2, ..., A_n\}$ be a collection of Interval Neutrosophic Set (INS), where $A_j = \left\langle \left[T_j^L, T_j^U\right], \left[I_j^L, I_j^U\right], \left[F_j^L, F_j^U\right] \right\rangle$ $(j = 1, 2, ..., n)$ in interval neutrosophic number and if

$INWA_W\left\{A_1, A_2,..., A_n\right\} = (w_1 A_1 \oplus w_2 A_2 \oplus ... \oplus w_n A_n)$, then INWA is called an interval neutrosophic weighted averaging (INWA) operator of dimension *n*, where $w = \left(w_1, w_2,..., w_n\right)^T$ is the weight vector of $A_j (j = 1, 2,..., n)$, weight $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

3. Methodology

3.1 Research Framework

The research framework of this study presents the workflow to determine the most important factor of flash floods as shown in Figure 1.

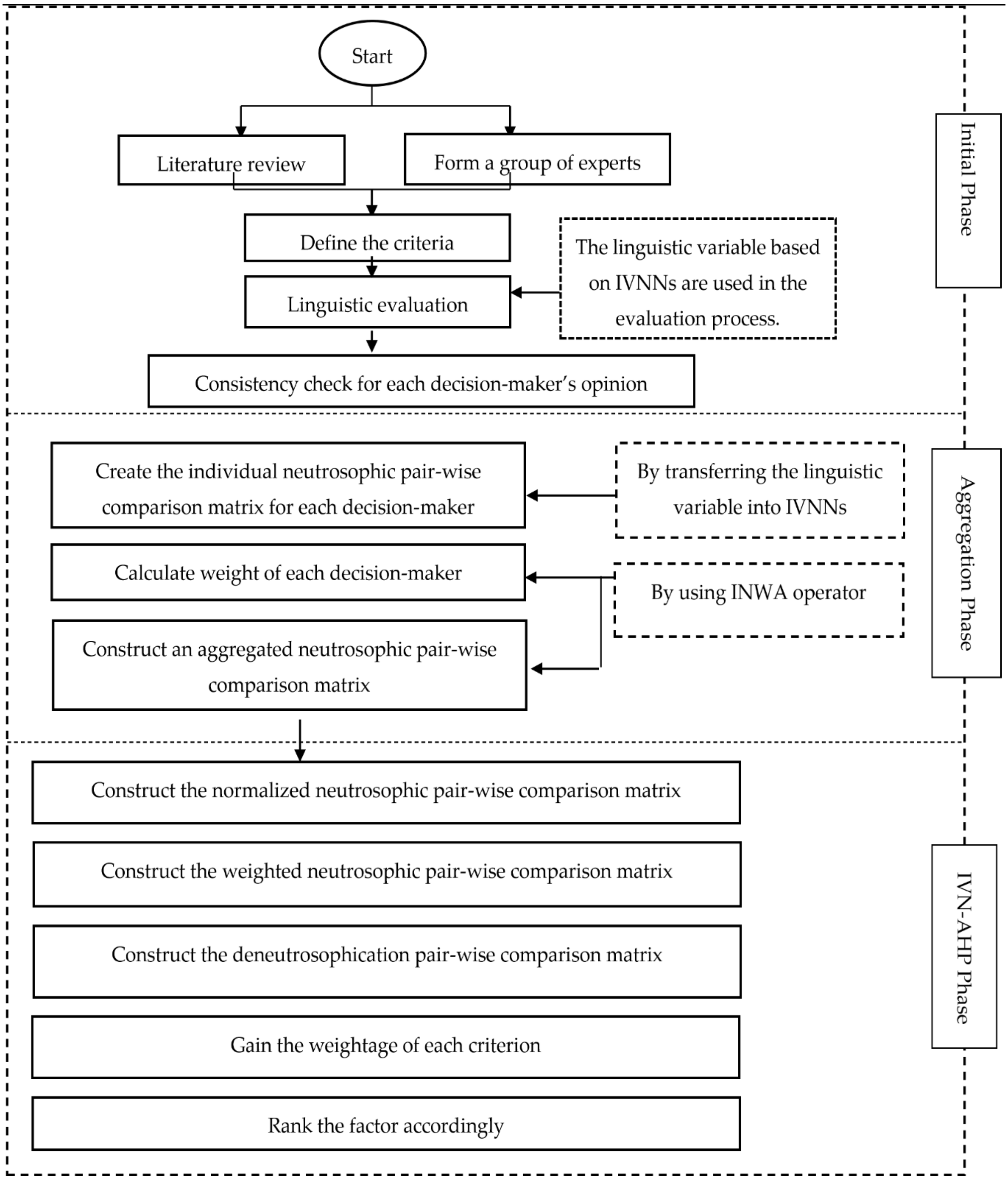

Figure 1: Research Framework

3.2 Phase 1: Data Collection

Data collection is the systematic process of acquiring and measuring information on variables of interest in order to answer research questions, test hypotheses, and evaluate outcomes. A survey was conducted to analyse the factor of flash floods that occurred in Penang. The questionnaires were given to five decision-makers at the Department of Irrigation and Drainage Seberang Perai Utara Pulau Pinang, who are experts in determining which factor is the most important. The decision-makers are required to give opinions on the evaluation of pair-wise comparison for factors. The data obtained is called primary data. The questionnaires contained two sections which are Section A and Section B.

Section A is about demographic profiles such as gender, position, and years of working experience while in Section B, decision-makers have to evaluate the pair-wise comparison between all the factors by using the linguistic scale. The decision makers' pair-wise comparisons are converted into the Interval-Valued Neutrosophic scale as in Step 3.2.

3.3 Phase 2: Data Evaluation – IVN-AHP Method

The AHP method is based on the logic of structuring a problem in hierarchies and then evaluating the components in the hierarchy through pairwise comparisons. Although AHP is a popular solution for MCDM problems, it does not always reflect human thought. Unlike traditional AHP, IVN-AHP can effectively integrate human cognition into decision-making by expressing uncertainty using three variables (*T, I, and F*). The weights of the factors affecting flash floods are calculated in this study using the IVN-AHP methodology. The steps of IVN-AHP are given below [3]:

Step 1: Identify the factors of the decision-making problem based on the literature.

Step 2: Decompose the complex problem into a hierarchical structure.

Step 3: Employs pair-wise comparison of the factors using a linguistic scale.

Step 3.1: Transform to crisp scale and calculate the Consistency Ratio.

i) Develop a pair-wise comparison matrix based on the decision-maker preference using a crisp number for each factor.

$$S_{ij} = \begin{matrix} c_1 \\ \vdots \\ c_n \end{matrix} \begin{pmatrix} 1 & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ 1/S_{1n} & \cdots & 1 \end{pmatrix} \tag{2}$$

ii) The resulting weights were estimated using Row Geometric Mean Method (RGMM) as proposed by Saaty (1980).

$$C_j = \left[\prod_{j=1}^{i} C_{ij} \right]^{\frac{1}{i}} \tag{3}$$

iii) Calculate the weight of each criterion

$$w_j = \frac{\left[\prod_{j=1}^{i} S_{ij} \right]^{\frac{1}{i}}}{\sum_{j=1}^{i} \left[\prod_{j=1}^{i} S_{ij} \right]^{\frac{1}{i}}} \tag{4}$$

iv) Find the eigenvector and using the equation as follows:

$$Sw = S \times w \tag{5}$$

$$\lambda_{\max} = \frac{Sw}{n \times w} \tag{6}$$

where,

S : comparison matrix,
w : eigenvector of the matrix S,
n : number of criteria,
$\lambda_{\max}$: largest eigenvalue.

v) Calculate the consistency index (CI)

$$CI = \frac{\lambda_{\max} - n}{n-1} \tag{7}$$

iv) To calculate the consistency ratio ($CR \leq 0.1$), divide the consistency index (CI) with random index (RI). We assume that the data obtained is consistent if CR for crisp number consistent. The random index (RI) value is selected based on the sample size of n matrix as shown in Table 2.

$$CR = \frac{CI}{RI} \tag{8}$$

Table 2: Random Inconsistency Index (RI) for n = 1,2,...,12 [20]

n	1	2	3	4	5	6	7	8	9	10	11	12
RI	0	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49	1.51	1.58

Step 3.2: Transform the pair-wise comparison based on linguistic scale to Interval Valued Neutrosophic Set scale introduced by Wang et al. [4] and evaluate the pair-wise comparison of the factors.

Step 4: Aggregation Process

In this phase, to aggregate all decision-makers' opinions, the INWA operator is employed to get the weight of decision-maker.

Step 4.1: Weight of Decision Maker

i. Develop a pair-wise comparison matrix based on the decision-maker position using an Interval-Valued Neutrosophic (IVN) Set.

ii.

$$P = \begin{matrix} DM_1 \\ DM_2 \\ \vdots \\ DM_n \end{matrix} \begin{bmatrix} [T_{11}^L, T_{11}^U], & [I_{11}^L, I_{11}^U], & [F_{11}^L, F_{11}^U] & \cdots & [T_{1n}^L, T_{1n}^U], & [I_{1n}^L, I_{1n}^U], & [F_{1n}^L, F_{1n}^U] \\ [T_{21}^L, T_{21}^U], & [I_{21}^L, I_{21}^U], & [F_{21}^L, F_{21}^U] & \ddots & & \vdots & \\ & \vdots & & \ddots & & \vdots & \\ [T_{n1}^L, T_{n1}^U], & [I_{n1}^L, I_{n1}^U], & [F_{n1}^L, F_{n1}^U] & \cdots & [T_{nn}^L, T_{nn}^U], & [I_{nn}^L, I_{nn}^U], & [F_{nn}^L, F_{nn}^U] \end{bmatrix} \tag{9}$$

ii. Converting the neutrosophic reference relations into their corresponding crisp preference relations by deneutrosophicated method.

$$D(x)=\left(\frac{T_{kj}^{L}(x)+T_{kj}^{U}(x)}{2}+I_{kj}^{U}(x)\left(1-\frac{I_{kj}^{L}(x)+I_{kj}^{U}(x)}{2}\right)-\left(1-F_{kj}^{U}(x)\right)\left(\frac{F_{kj}^{L}(x)+F_{kj}^{U}(x)}{2}\right)\right) \quad (10)$$

iii. Calculate the weight of matrix *P* by aggregating using the Row Geometric Mean Method (RGMM).

$$DM_{j}=\left[\prod_{j=1}^{i} DM_{ij}\right]^{\frac{1}{i}} \quad (11)$$

iv. Calculate the weight of each decision maker.

$$w_{j}=\frac{\left[\prod_{j=1}^{i} P_{ij}\right]^{\frac{1}{i}}}{\sum_{j=1}^{i}\left[\prod_{j=1}^{i} P_{ij}\right]^{\frac{1}{i}}} \quad (12)$$

Step 4.2: Aggregate with the pair-wise comparison obtained in Step 3.2 using INWA operator.

$$INWA_{w}\left\{A_{1},A_{2},\ldots,A_{n}\right\}=\left\langle \begin{array}{l}\left[1-\prod_{j=1}^{n}(1-T_{j}^{L}(x))^{w_j},1-\prod_{j=1}^{n}(1-T_{j}^{U}(x))^{w_j}\right],\\ \left[\prod_{j=1}^{n}(I_{j}^{L}(x))^{w_j},\prod_{j=1}^{n}(I_{j}^{U}(x))^{w_j}\right],\\ \left[\prod_{j=1}^{n}(F_{j}^{L}(x))^{w_j},\prod_{j=1}^{n}(F_{j}^{U}(x))^{w_j}\right]\end{array}\right\rangle \quad (13)$$

Step 5: Ranking Process

In this phase, the weight obtained will be used to rank the factor of flash floods. Then, the highest the weight will be the most critical factor.

Step 5.1: The constructed pair-wise comparison obtained from the aggregated using INWA is used.

Step 5.2: The importance weights, $\overline{N}_{ij}$ of the factors are normalized to make them comparable data and thus to rate and rank factors.

$$N_{ij}=\left[\frac{T_{kj}^{L}}{\sum_{k=1}^{n}T_{kj}^{U}},\frac{T_{kj}^{U}}{\sum_{k=1}^{n}T_{kj}^{U}}\right],\left[\frac{I_{kj}^{L}}{\sum_{k=1}^{n}I_{kj}^{U}},\frac{I_{kj}^{U}}{\sum_{k=1}^{n}I_{kj}^{U}}\right],\left[\frac{F_{kj}^{L}}{\sum_{k=1}^{n}F_{kj}^{U}},\frac{F_{kj}^{U}}{\sum_{k=1}^{n}F_{kj}^{U}}\right];\ j=1,2,\ldots,n \quad (14)$$

Step 5.3: The arithmetic mean of each row is calculated to obtain the neutrosophic importance weight, W_j vector of the factors by Equation (15).

$$W_j = \left[\frac{\sum_{k=1}^{n} \frac{T_{1j}^L}{\sum_{k=1}^{n} T_{kj}^U}}{n}, \frac{\sum_{k=1}^{n} \frac{T_{1j}^U}{\sum_{k=1}^{n} T_{kj}^U}}{n} \right], \left[\frac{\sum_{k=1}^{n} \frac{I_{1j}^L}{\sum_{k=1}^{n} I_{kj}^U}}{n}, \frac{\sum_{k=1}^{n} \frac{I_{1j}^U}{\sum_{k=1}^{n} I_{kj}^U}}{n} \right], \left[\frac{\sum_{k=1}^{n} \frac{F_{1j}^L}{\sum_{k=1}^{n} F_{kj}^U}}{n}, \frac{\sum_{k=1}^{n} \frac{F_{1j}^U}{\sum_{k=1}^{n} F_{kj}^U}}{n} \right] \quad (15)$$

Step 5.4: All the above steps are repeated for each factor.

Step 5.5: In order to obtain the crisp weights of the factors, the deneutrosophication formula in Equation (10) is used. **Step 5.6**: Rank the weight accordingly.

3.4 Phase 3: Comparison Analysis

The aggregation operator changes to INWG operator as Equation (16) follows then Step 5.1 until Step 5.6 repeated:

$$INWG_w \{A_1, A_2, ..., A_n\} = \left\langle \begin{array}{l} \left[\prod_{j=1}^{n} (T_j^L(x))^{w_j}, \prod_{j=1}^{n} (T_j^U(x))^{w_j} \right], \\ \left[1 - \prod_{j=1}^{n} (1 - I_j^L(x))^{w_j}, 1 - \prod_{j=1}^{n} (1 - I_j^U(x))^{w_j} \right], \\ \left[1 - \prod_{j=1}^{n} (1 - F_j^L(x))^{w_j}, 1 - \prod_{j=1}^{n} (1 - F_j^U(x))^{w_j} \right] \end{array} \right\rangle \quad (16)$$

Then compare the weight of each factor and the ranking based on different aggregation operator.

4. Application

4.1 Weight of decision-maker

The weight of the decision-maker is important in aggregating the pairwise comparison based on decision makers' opinions. In this study, the weight of decision-makers is calculated by using AHP method in interval neutrosophic environment. During the comparison phase, the decision-makers' weight is compared based on their positions. The decision-maker with higher position has more experience in handling the flash floods. Table 3 shows the comparison between the position of each decision maker in linguistic term.

Table 3: Pair-wise Judgement for Decision Makers' Weight in Linguistic Term

Decision Maker	DM1	DM2	DM3	DM4	DM5

DM1	EI	MI	SI	VSI	EHI
DM2	MI^C	EI	MI	SI	VSI
DM3	SI^C	MI^C	EI	MI	SI
DM4	VSI^C	SI^C	MI^C	EI	MI
DM5	EHI^C	VSI^C	SI^C	MI^C	EI

Then, the pair-wise comparison in linguistic term is converted into interval neutrosophic numbers by the conversion scale of IVNs (refer Table 1). Then, the pair-wise judgement of decision-makers' weight is calculated by using deneutrosophication formula to obtain crisp pair-wise comparison. Table 4 shows the result after deneutrosophication formula applied. The example of calculation for DM1 as follows:

$$D(DM1)_{DM1}=\left(\left(\frac{0.5+0.5}{2}\right)+0.5\left(\frac{0.5+0.5}{2}\right)-(1-0.5)\left(\frac{0.5+0.5}{2}\right)\right)=0.5$$

Table 4 Result of Deneutrosophication Calculation

Decision Maker	DM 1	DM 2	DM 3	DM 4	DM 5
DM 1	0.5	0.64	0.73	0.82	0.93
DM 2	0.45	0.5	0.64	0.73	0.82
DM 3	0.35	0.45	0.5	0.64	0.73
DM 4	0.25	0.35	0.45	0.5	0.64
DM 5	0.03	0.25	0.35	0.45	0.50

Then, Row Geometric Mean (RGM) formula is used to calculate weight vector as shown in Table 5. The example of calculation for row 1 as follows:

$$DM1=\left(0.5\times0.64\times0.73\times0.82\times0.93\right)^{\frac{1}{5}}=0.71$$

Table 5: Result of Weight Vector

Decision Maker	Weight Vector
DM 1	0.71
DM 2	0.61
DM 3	0.52
DM 4	0.42
DM 5	0.22
Total	2.47

Finally, the weight of each decision makers obtained as shown in Table 6. The example calculation as shown below:

$$W_{DM1} = \frac{0.71}{2.47} = 0.29$$

Table 6: Weight of each DM

Position	Decision Maker	Weight, *W*
District engineer	DM 1	0.29
Senior assistant engineer	DM 2	0.25
Assistant engineer	DM 3	0.21
Administrative engineer	DM 4	0.17
Officer	DM 5	0.09
	Total	1.00

4.2. Implementation of IVN-AHP based on INWA Operator

The decision makers' opinion gathered in pair-wise comparison. Then, the consistency ratio was check for each decision-makers' opinion and all decision-makers' opinions are consistent since the consistency ratio for each decision-maker's opinion is less than 0.10. For example, Table 7 shows the pair-wise comparison based on DM's 1 opinion in linguistic term.

Table 7: Pair-wise Comparison based on DM's 1 Opinion in Linguistic Term.

	C1	C2	C3	C4	C5	C6	C7	C8
C1	EI	SI	EI	WI	SI^C	EI	MP	MI
C2	SI^C	EI	WI^C	EI	MI^C	MI^C	WI	WI
C3	EI	WI	EI	MI	EI	EI	SI	WI
C4	WI^C	EI	MI^C	EI	MI^C	MI^C	WI	WI^C
C5	SI	MI	EI	MI	EI	EI	MI	WI
C6	EI	MI	EI	MI	EI	EI	VSI	SI
C7	MP^C	WI^C	SI^C	WI^C	MI^C	VSI^C	EI	SI^C
C8	MI^C	WI^C	WI^C	WI	WI^C	SI^C	SI	EI

Table 7 shows example of pair-wise comparison based on opinion from decision maker 1 in term of interval neutrosophic scale. Then, all the five decision makers' opinion is aggregated using INWA operator and the results are shown in Table 8. The following calculation is demonstrated for cell C11 by using INWA aggregation operator:

$$INWA_W\{DM_1, DM_2, DM_3, DM_4, DM_5\} = \begin{bmatrix} 1-\left((1-0.5)^{0.29}\times(1-0.5)^{0.25}\times(1-0.5)^{0.21}\times(1-0.5)^{0.17}\times(1-0.5)^{0.09}\right), \\ 1-\left((1-0.5)^{0.29}\times(1-0.5)^{0.25}\times(1-0.5)^{0.21}\times(1-0.5)^{0.17}\times(1-0.5)^{0.09}\right) \\ 0.5^{0.29}\times0.5^{0.25}\times0.5^{0.21}\times0.5^{0.17}\times0.5^{0.09}, 0.5^{0.29}\times0.5^{0.25}\times0.5^{0.21}\times0.5^{0.17}\times0.5^{0.09} \\ 0.5^{0.29}\times0.5^{0.25}\times0.5^{0.21}\times0.5^{0.17}\times0.5^{0.09}, 0.5^{0.29}\times0.5^{0.25}\times0.5^{0.21}\times0.5^{0.17}\times0.5^{0.09} \end{bmatrix}$$

$$INWA_W\{DM_1, DM_2, DM_3, DM_4, DM_5\} = [0.5, 0.5, 0.5, 0.5, 0.5, 0.5]$$

Table 8: INWA Aggregated Matrix

	C1						C2					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.6633	0.7552	0.1932	0.2921	0.2448	0.3367
C2	0.2889	0.3571	0.1932	0.2921	0.6429	0.7111	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
C3	0.5129	0.5423	0.4406	0.4731	0.4577	0.4871	0.6311	0.7292	0.2117	0.3006	0.2708	0.3689
C4	0.2886	0.3894	0.2209	0.3251	0.6106	0.7114	0.4664	0.4881	0.4406	0.4731	0.5119	0.5336
C5	0.5702	0.6528	0.3031	0.3893	0.3472	0.4298	0.6013	0.7111	0.2267	0.3222	0.2889	0.3987
C6	0.5239	0.5765	0.3941	0.4506	0.4235	0.4761	0.6114	0.7139	0.2336	0.3376	0.2861	0.3886
C7	0.2691	0.3692	0.2174	0.3178	0.6308	0.7309	0.4365	0.4876	0.3959	0.4581	0.5124	0.5635
C8	0.3817	0.4818	0.3304	0.4306	0.5182	0.6183	0.4489	0.5494	0.3500	0.4500	0.4506	0.5111
	C3						C4					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.4664	0.4881	0.4406	0.4731	0.5119	0.5336	0.6237	0.7266	0.2209	0.3251	0.2734	0.3763
C2	0.3260	0.4014	0.2117	0.3006	0.5986	0.6740	0.5129	0.5423	0.4406	0.4731	0.4577	0.4871
C3	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5599	0.6602	0.2895	0.3900	0.3398	0.4401
C4	0.3417	0.4418	0.2895	0.3900	0.5582	0.6583	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
C5	0.5239	0.5765	0.3941	0.4506	0.4235	0.4761	0.6328	0.7376	0.2079	0.3152	0.2624	0.3672
C6	0.5494	0.6271	0.3314	0.4126	0.3729	0.4506	0.6382	0.7404	0.2076	0.3109	0.2596	0.3618
C7	0.1972	0.2974	0.1388	0.2417	0.7026	0.8028	0.3875	0.4876	0.3364	0.4366	0.5124	0.6125
C8	0.3647	0.4648	0.3135	0.4137	0.5352	0.6353	0.5129	0.6130	0.3369	0.4371	0.3870	0.4871
	C5						C6					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.3654	0.4365	0.3031	0.3893	0.5635	0.6346	0.4350	0.4773	0.3941	0.4506	0.5227	0.5650
C2	0.3224	0.4234	0.2267	0.3222	0.5766	0.6776	0.2975	0.3979	0.2336	0.3376	0.6021	0.7025
C3	0.4350	0.4773	0.3941	0.4506	0.5227	0.5650	0.3881	0.4557	0.3314	0.4126	0.5443	0.6119
C4	0.2825	0.3832	0.2079	0.3152	0.6168	0.7175	0.2707	0.3712	0.2076	0.3109	0.6288	0.7293
C5	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
C6	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
C7	0.2487	0.3492	0.1814	0.2862	0.6508	0.7513	0.1732	0.2733	0.1203	0.2215	0.7267	0.8268
C8	0.3606	0.4607	0.3085	0.4089	0.5393	0.6394	0.2821	0.3825	0.2250	0.3264	0.6175	0.7179
	C7						C8					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.6320	0.7323	0.2174	0.3178	0.2677	0.3680	0.5194	0.6195	0.3304	0.4306	0.3805	0.4806
C2	0.5133	0.5720	0.3959	0.4581	0.4280	0.4867	0.4557	0.5562	0.3500	0.4500	0.4438	0.5443
C3	0.7076	0.8093	0.1388	0.2417	0.1907	0.2924	0.5362	0.6364	0.3135	0.4137	0.3636	0.4638
C4	0.5133	0.6135	0.3364	0.4366	0.3865	0.4867	0.3880	0.4881	0.3369	0.4371	0.5119	0.6120
C5	0.6626	0.7657	0.1814	0.2862	0.2343	0.3374	0.5410	0.6413	0.3085	0.4089	0.3587	0.4590
C6	0.7283	0.8289	0.1203	0.2215	0.1711	0.2717	0.6232	0.7242	0.2250	0.3264	0.2758	0.3768
C7	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.3589	0.4344	0.2972	0.3859	0.5656	0.6411
C8	0.5719	0.6576	0.2972	0.3859	0.3424	0.4281	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000

Table 9 shows the sum of each column. The example of calculation for sum of column 1 as shown below.

$$Total_{C1,T^L} = 0.5000 + 0.2889 + 0.5129 + 0.2886 + 0.5702 + 0.5239 + 0.2691 + 0.3817 = 3.3354$$

Table 9: Sum of each column

	C1						C2					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.6633	0.7552	0.1932	0.2921	0.2448	0.3367
C2	0.2889	0.3571	0.1932	0.2921	0.6429	0.7111	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
C3	0.5129	0.5423	0.4406	0.4731	0.4577	0.4871	0.6311	0.7292	0.2117	0.3006	0.2708	0.3689
C4	0.2886	0.3894	0.2209	0.3251	0.6106	0.7114	0.4664	0.4881	0.4406	0.4731	0.5119	0.5336
C5	0.5702	0.6528	0.3031	0.3893	0.3472	0.4298	0.6013	0.7111	0.2267	0.3222	0.2889	0.3987
C6	0.5239	0.5765	0.3941	0.4506	0.4235	0.4761	0.6114	0.7139	0.2336	0.3376	0.2861	0.3886
C7	0.2691	0.3692	0.2174	0.3178	0.6308	0.7309	0.4365	0.4876	0.3959	0.4581	0.5124	0.5635
C8	0.3817	0.4818	0.3304	0.4306	0.5182	0.6183	0.4489	0.5494	0.3500	0.4500	0.4506	0.5511
Total	3.3354	3.8691	2.5997	3.1787	4.1309	4.6646	4.3588	4.9345	2.5517	3.1336	3.0655	3.6412

	C3						C4					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.4664	0.4881	0.4406	0.4731	0.5119	0.5336	0.6237	0.7266	0.2209	0.3251	0.2734	0.3763
C2	0.3260	0.4014	0.2117	0.3006	0.5986	0.6740	0.5129	0.5423	0.4406	0.4731	0.4577	0.4871
C3	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5599	0.6602	0.2895	0.3900	0.3398	0.4401
C4	0.3417	0.4418	0.2895	0.3900	0.5582	0.6583	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
C5	0.5239	0.5765	0.3941	0.4506	0.4235	0.4761	0.6328	0.7376	0.2079	0.3152	0.2624	0.3672
C6	0.5494	0.6271	0.3314	0.4126	0.3729	0.4506	0.6382	0.7404	0.2076	0.3109	0.2596	0.3618
C7	0.1972	0.2974	0.1388	0.2417	0.7026	0.8028	0.3875	0.4876	0.3364	0.4366	0.5124	0.6125
C8	0.3647	0.4648	0.3135	0.4137	0.5352	0.6353	0.5129	0.6130	0.3369	0.4371	0.3870	0.4871
Total	3.2694	3.7971	2.6196	3.1823	4.2029	4.7306	4.3679	5.0077	2.5397	3.1880	2.9923	3.6321

	C5						C6					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.3654	0.4365	0.3031	0.3893	0.5635	0.6346	0.4350	0.4773	0.3941	0.4506	0.5227	0.5650
C2	0.3224	0.4234	0.2267	0.3222	0.5766	0.6776	0.2975	0.3979	0.2336	0.3376	0.6021	0.7025
C3	0.4350	0.4773	0.3941	0.4506	0.5227	0.5650	0.3881	0.4557	0.3314	0.4126	0.5443	0.6119
C4	0.2825	0.3832	0.2079	0.3152	0.6168	0.7175	0.2707	0.3712	0.2076	0.3109	0.6288	0.7293
C5	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
C6	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
C7	0.2487	0.3492	0.1814	0.2862	0.6508	0.7513	0.1732	0.2733	0.1203	0.2215	0.7267	0.8268
C8	0.3606	0.4607	0.3085	0.4089	0.5393	0.6394	0.2821	0.3825	0.2250	0.3264	0.6175	0.7179
Total	3.0145	3.5304	2.6218	3.1724	4.4696	4.9855	2.8467	3.3579	2.5122	3.0595	4.6421	5.1533

	C7						C8					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.6320	0.7323	0.2174	0.3178	0.2677	0.3680	0.5194	0.6195	0.3304	0.4306	0.3805	0.4806
C2	0.5133	0.5720	0.3959	0.4581	0.4280	0.4867	0.4557	0.5562	0.3500	0.4500	0.4438	0.5443
C3	0.7076	0.8093	0.1388	0.2417	0.1907	0.2924	0.5362	0.6364	0.3135	0.4137	0.3636	0.4638
C4	0.5133	0.6135	0.3364	0.4366	0.3865	0.4867	0.3880	0.4881	0.3369	0.4371	0.5119	0.6120
C5	0.6626	0.7657	0.1814	0.2862	0.2343	0.3374	0.5410	0.6413	0.3085	0.4089	0.3587	0.4590
C6	0.7283	0.8289	0.1203	0.2215	0.1711	0.2717	0.6232	0.7242	0.2250	0.3264	0.2758	0.3768
C7	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.3589	0.4344	0.2972	0.3859	0.5656	0.6411
C8	0.5719	0.6576	0.2972	0.3859	0.3424	0.4281	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
Total	4.8291	5.4793	2.1876	2.8478	2.5207	3.1709	3.9223	4.6000	2.6615	3.3526	3.4000	4.0777

Table 10 shows the normalized weight of each factor. As an example, the calculation for Factor 1 (C1) shown as followed:

$$N_{11} = \left[\frac{0.5000}{3.8691}, \frac{0.5000}{3.8691}\right], \left[\frac{0.5000}{3.1787}, \frac{0.5000}{3.1787}\right], \left[\frac{0.5000}{4.6646}, \frac{0.5000}{4.6646}\right]$$

$$N_{11} = [0.1292, 0.1292], [0.1573, 0.1573], [0.1072, 0.1072]$$

Table 10: Normalized Weight

	C1						C2					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.1292	0.1292	0.1573	0.1573	0.1072	0.1072	0.1344	0.1531	0.0616	0.0932	0.0672	0.0925
C2	0.0747	0.0923	0.0608	0.0919	0.1378	0.1525	0.1013	0.1013	0.1596	0.1596	0.1373	0.1373
C3	0.1326	0.1402	0.1386	0.1488	0.0981	0.1044	0.1279	0.1478	0.0676	0.0959	0.0744	0.1013
C4	0.0746	0.1006	0.0695	0.1023	0.1309	0.1525	0.0945	0.0989	0.1406	0.1510	0.1406	0.1465
C5	0.1474	0.1687	0.0954	0.1225	0.0744	0.0921	0.1219	0.1441	0.0724	0.1028	0.0793	0.1095
C6	0.1354	0.1490	0.1240	0.1418	0.0908	0.1021	0.1239	0.1447	0.0746	0.1077	0.0786	0.1067
C7	0.0696	0.0954	0.0684	0.1000	0.1352	0.1567	0.0885	0.0988	0.1263	0.1462	0.1407	0.1548
C8	0.0987	0.1245	0.1039	0.1355	0.1111	0.1325	0.0910	0.1113	0.1117	0.1436	0.1238	0.1514
	C3						C4					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.1228	0.1285	0.1384	0.1487	0.1082	0.1128	0.1245	0.1451	0.0693	0.1020	0.0753	0.1036
C2	0.0859	0.1057	0.0665	0.0944	0.1265	0.1425	0.1024	0.1083	0.1382	0.1484	0.1260	0.1341
C3	0.1317	0.1317	0.1571	0.1571	0.1057	0.1057	0.1118	0.1318	0.0908	0.1223	0.0935	0.1212
C4	0.0900	0.1164	0.0910	0.1225	0.1180	0.1391	0.0998	0.0998	0.1568	0.1568	0.1377	0.1377
C5	0.1380	0.1518	0.1238	0.1416	0.0895	0.1006	0.1264	0.1473	0.0652	0.0989	0.0723	0.1011
C6	0.1447	0.1651	0.1041	0.1296	0.0788	0.0953	0.1274	0.1479	0.0651	0.0975	0.0715	0.0996
C7	0.0519	0.0783	0.0436	0.0760	0.1485	0.1697	0.0774	0.0974	0.1055	0.1369	0.1411	0.1686
C8	0.0961	0.1224	0.0985	0.1300	0.1131	0.1343	0.1024	0.1224	0.1057	0.1371	0.1065	0.1341
	C5						C6					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.1035	0.1236	0.0956	0.1227	0.1130	0.1273	0.1295	0.1421	0.1288	0.1473	0.1014	0.1096
C2	0.0913	0.1199	0.0715	0.1016	0.1157	0.1359	0.0886	0.1185	0.0764	0.1103	0.1168	0.1363
C3	0.1232	0.1352	0.1242	0.1421	0.1048	0.1133	0.1156	0.1357	0.1083	0.1348	0.1056	0.1187
C4	0.0800	0.1086	0.0655	0.0994	0.1237	0.1439	0.0806	0.1105	0.0679	0.1016	0.1220	0.1415
C5	0.1416	0.1416	0.1576	0.1576	0.1003	0.1003	0.1489	0.1489	0.1634	0.1634	0.0970	0.0970
C6	0.1416	0.1416	0.1576	0.1576	0.1003	0.1003	0.1489	0.1489	0.1634	0.1634	0.0970	0.0970
C7	0.0704	0.0989	0.0572	0.0902	0.1305	0.1507	0.0516	0.0814	0.0393	0.0724	0.1410	0.1604
C8	0.1021	0.1305	0.0973	0.1289	0.1082	0.1283	0.0840	0.1139	0.0735	0.1067	0.1198	0.1393
	C7						C8					
	T^L	T^U	I^L	I^U	F^L	F^U	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.1154	0.1337	0.0764	0.1116	0.0844	0.1160	0.1129	0.1347	0.0985	0.1284	0.0933	0.1179
C2	0.0937	0.1044	0.1390	0.1609	0.1350	0.1535	0.0991	0.1209	0.1044	0.1342	0.1088	0.1335
C3	0.1291	0.1477	0.0487	0.0849	0.0601	0.0922	0.1166	0.1383	0.0935	0.1234	0.0892	0.1137
C4	0.0937	0.1120	0.1181	0.1533	0.1219	0.1535	0.0843	0.1061	0.1005	0.1304	0.1255	0.1501
C5	0.1209	0.1397	0.0637	0.1005	0.0739	0.1064	0.1176	0.1394	0.0920	0.1220	0.0880	0.1126
C6	0.1329	0.1513	0.0423	0.0778	0.0539	0.0857	0.1355	0.1574	0.0671	0.0974	0.0676	0.0924
C7	0.0913	0.0913	0.1756	0.1756	0.1577	0.1577	0.0780	0.0944	0.0886	0.1151	0.1387	0.1572
C8	0.1044	0.1200	0.1044	0.1355	0.1080	0.1350	0.1087	0.1087	0.1491	0.1491	0.1226	0.1226

Table 11 shows the neutrosophic weight of each factor. As an example, the calculation neutrosophic weight for Factor 1 (C1) shown as followed:

$$W_1 = \begin{bmatrix} \dfrac{0.1292+0.1344+0.1228+0.1245+0.1035+0.1295+0.1154+0.1129}{8}, \\ \dfrac{0.1292+0.1531+0.1285+0.1451+0.1236+0.1421+0.1337+0.1347}{8}, \\ \dfrac{0.1573+0.0616+0.1384+0.0693+0.0956+0.1288+0.0764+0.0985}{8}, \\ \dfrac{0.1573+0.0932+0.1487+0.1020+0.1227+0.1473+0.1116+0.1284}{8}, \\ \dfrac{0.1072+0.0672+0.1082+0.0753+0.1130+0.1014+0.0844+0.0933}{8}, \\ \dfrac{0.1072+0.0925+0.1128+0.1036+0.1273+0.1096+0.1160+0.1179}{8} \end{bmatrix}$$

$$W_1 = [0.1215, 0.1363, 0.1032, 0.1264, 0.0938, 0.1109]$$

Table 11: Neutrosophic Weight

	Weight					
	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.1215	0.1363	0.1032	0.1264	0.0938	0.1109
C2	0.0921	0.1089	0.1020	0.1252	0.1255	0.1407
C3	0.1236	0.1386	0.1036	0.1262	0.0914	0.1088
C4	0.0872	0.1066	0.1012	0.1272	0.1275	0.1456
C5	0.1328	0.1477	0.1042	0.1262	0.0843	0.1025
C6	0.1363	0.1507	0.0998	0.1216	0.0798	0.0974
C7	0.0723	0.0920	0.0881	0.1140	0.1417	0.1595
C8	0.0984	0.1192	0.1055	0.1333	0.1141	0.1347

The deneutrosophication formula was used to obtain crisp weight for each factor shown in Table 12. The example of calculation for Factor 1 (C1) shown below:

$$D(C1) = \left(\left(\frac{0.1215+0.1363}{2}\right) + (0.1264)\left(\frac{0.1032+0.1264}{2}\right) - (1-0.1109)\left(\frac{0.0938+0.1109}{2}\right)\right) = 0.1498$$

Table 12: Ranking of Flash Floods' Factors

Factor		Weight	Rank
C1	Poor Drainage System	0.1498	4
C2	Dam and Levee Failure	0.0971	6
C3	Urbanization	0.1535	3
C4	Land Use Pattern	0.0929	7
C5	Rain Intensity	0.1681	2
C6	Rain Duration	0.1717	1

C7	Slow Moving Thunderstorm	0.0581	8
C8	Soil Erosion	0.1185	5

4.3 Comparative Analysis

In this study, the comparative analysis of different aggregation operators which are interval neutrosophic weighted average (INWA), interval neutrosophic geometric average (INWG), interval neutrosophic average (INA), and interval neutrosophic geometric (ING) operators are presented for solving the flash floods problem. Linguistic terms are used to facilitate comparisons between subject factors because decision-makers are more familiar with using linguistic terms than providing exact crisp evaluations. Figures 2 and 3 show the ranking results using INWA and INWG respectively.

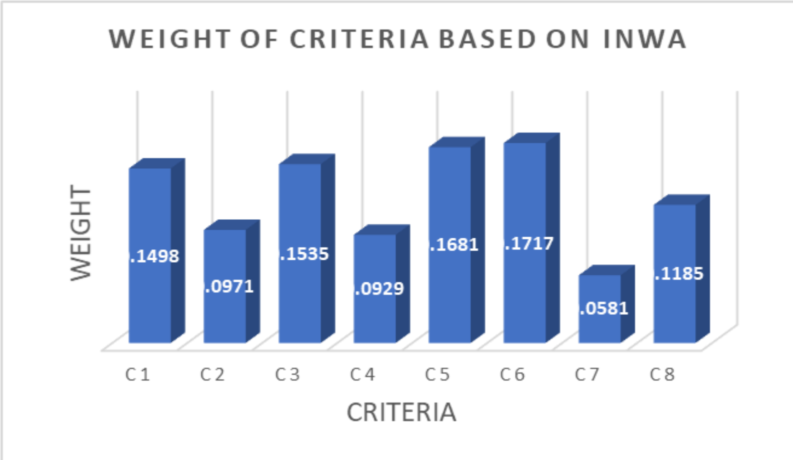

Figure 2: Weight of factors based on INWA

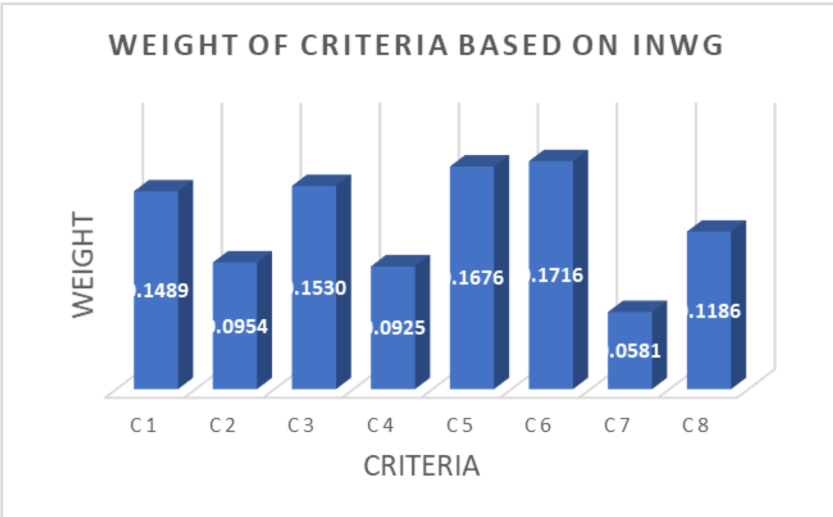

Figure 3: Weight of factors based on INWG

According to Figure 2, the results obtained by using the INWA operator show that the rain duration has the greatest weight (0.1717). This means that the duration of rain was the most important factor in causing flash floods in Penang. While slow-moving thunderstorms have the lowest weight (0.0581), they are the least important cause of flash floods in Penang. Surprisingly, when the ranking results of flash flood factors using the INWG operator are compared, the highest priority remains the same, which is the rain duration with a weightage of 0.1716. This proves that the rain duration is the most important factor in causing flash floods. Furthermore, we compare with the INA and ING, where the weights of decision makers are assumed to be the same. Besides that, we also compare the ranking of factors with the INA and ING where the weights of decision makers are assumed to be the same, which is $w_{DM} = (0.2, 0.2, 0.2, 0.2, 0.2)^T$. For both INA and ING operators, the obtained results show that the rain duration factor is the most important factor in causing the flash flood. Figures 4 and 5 show a bar chart of the obtained factor ranking order using INA and ING.

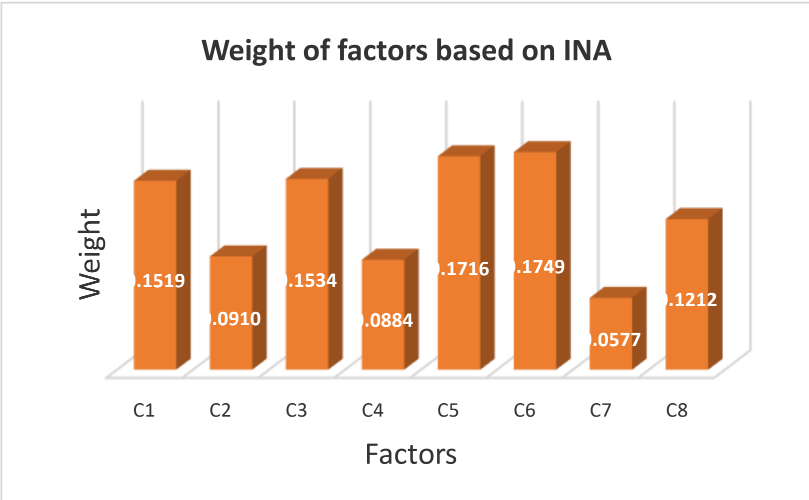

Figure 4: Weight of factors based on INA

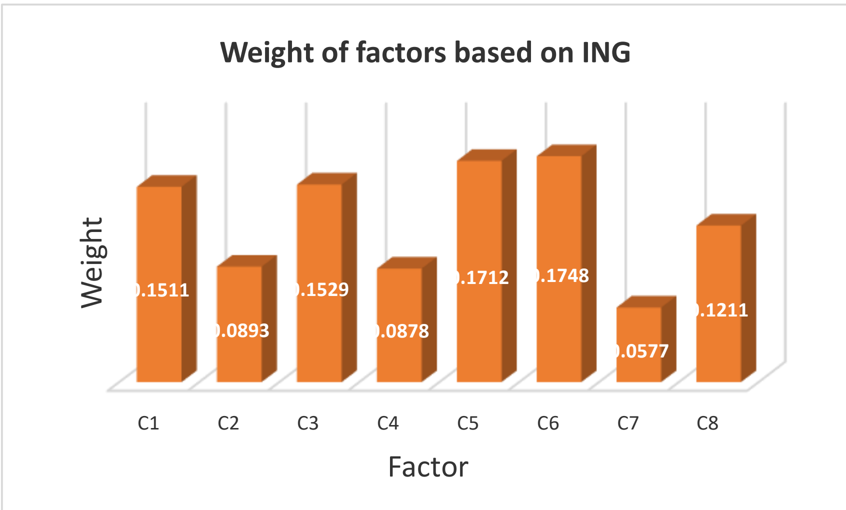

Figure 5: Weight of factors based on ING

The result of the comparative analysis of INWA, INWG, INA, and ING based on different weights and equal weights of each decision-maker are summarized in Table 12. Using INWA and INWG with different weights of decision-makers, the ranking of factors of flash floods in Penang has been determined. It can be noted that the weight of each decision-maker should be measured according to some characteristics such as their positions, working experience, and knowledge about that particular case study. This is an important point to emphasize that the weight of each decision-maker should be measured based on the characteristics mentioned so that the results obtained are more accurate and reliable.

The aggregation operator is also important, particularly in decision-making problems, because it provides a high-level view of data prior to analysis. One of the most effective and simple methods for decision-making problems is to use aggregation functions. The obtained ranking of factors is completely consistent when employing the IVN-AHP method with INWA operator and any other aggregation operators such as INWG, INA and ING operators. This validates the proposed method's applicability in solving decision making problems. In addition, the INWA operator is suitable to apply in this case study since it is easy to explore and understand. Therefore, the INWA aggregation operator is recommended to use in this study.

Table 12: Summary Table for Comparative Analysis

Criteria		Different Weight		Same Weight		Rank
		INWA	INWG	INA	ING	
C1	Poor Drainage System	0.1498	0.1489	0.1519	0.1511	4
C2	Dam and Levee Failure	0.0971	0.0954	0.0910	0.0893	6
C3	Urbanization	0.1535	0.1530	0.1534	0.1529	3
C4	Land Use Pattern	0.0929	0.0925	0.0884	0.0878	7
C5	Rain Intensity	0.1681	0.1676	0.1716	0.1712	2
C6	Rain Duration	0.1717	0.1716	0.1749	0.1748	1
C7	Slow Moving Thunderstorm	0.0581	0.0581	0.0577	0.0577	8
C8	Soil Erosion	0.1185	0.1186	0.1212	0.1211	5

5. Conclusion

As a conclusion, the interval neutrosophic AHP method based on the INWA operator has been proposed in this study to determine the most important factor of flash floods in Penang. Eight factors of the flash flood are considered in this study which are the rain intensity, rain duration, poor drainage system, dam and levee failure, urbanization, slow-moving thunderstorm, soil erosion, and land use pattern. By using the AHP method with the INWA operator, the following ranking order of factors is established: rain duration, rain intensity, urbanization, poor drainage system, soil erosion, dam and levee failure, land use pattern and slow-moving thunderstorms. The obtained results are consistent when evaluated with various aggregation operators such as INA, ING, and INWG.

The recommendation for future research is to consider the other factors of flash floods, as there are numerous factors that can be used, whether from a literature review or an expert's perspective. The other factors can provide additional information regarding the flash flood factor in Penang. In addition, future researchers can use this study as a reference to determine the factors contributing to flash floods in other states. Besides, as a further extension of this research, the implemented IVN-AHP method based on the INWA aggregation operator can be used for different types of case studies that involve the decision-making problem such as determining the ranking's factor of road accidents, analyzing the IT project prioritization for oil and gas company, and measuring patients' priorities. Plus, this research also can be extended by implementing another aggregation operator in the IVN-AHP method such as Interval Neutrosophic Ordered Weighted Averaging (INOWA), Interval Neutrosophic Ordered Weighted Geometric (INOWG), and Interval Neutrosophic

Prioritized Ordered Weighted Averaging (INPOWA) in the future.

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Solution and Analysis of System of Differential Equation with Initial Condition as $TrapN_{number}$

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Abstract: In this present study, we have analyzed the solution of system of first order simultaneous differential equations with initial condition as a neutrosophic environment. Here, we consider the initial values as Trapezoidal Neutrosophic Numbers ($TrapN_{number}$). The solution procedure of the system of first order ODE is developed using $TrapN_{number}$ and $(\alpha,\beta,\gamma)_{cut}$ of a TN_{number}. Furthermore, a numerical example is illustrated to validate the efficiency and feasibility of the proposed neutrosophic method, and the solutions are compared with the crisp values. The numerical solutions $x_1(t,\alpha), x_2(t,\alpha), x_1'(t,\beta), x_2'(t,\beta), x_1''(t,\gamma), x_2''(t,\gamma), y_1(t,\alpha)$, $y_2(t,\alpha), y_1'(t,\beta), y_2'(y,\beta), y_1''(t,\gamma)$ and $y_2''(t,\gamma)$ for the different values of α,β and γ at $t = 0.5$ are examined via tables and graphs. The numerical solutions delight the conditions of strong solution.

Keywords: *Difference equation; $TrapN_{number}$; $(\alpha,\beta,\gamma)_{cut}$ of a $TrapN_{number}$*

1. Introduction

The concept of Neutrosophic set (N_{set}) was introduced by Smarandache [1]. N_{set} is a wide-ranging context of the C_{set}, F_{set} [2], IF_{set} [3-6], and the IVF_{set} [7-10], respectively. N_{set} is used to characterize the uncertainty and indeterminacy in any multicriteria decision making problems. N_{set} is a proposition of three different components namely, T_{value}, $I_{value,}$ and F_{value}, and the grade of these membership values are defined within $^{-}]0,1[^{+}$. Researchers in different field of Engineers have applied N_{set} on various applications. The multifaceted factors of N_{logic}, SVN_{number}, $TriN_{number}$, and $TrapN_{number}$ have been applied in the differential equations are analyzed in [11-15]. The system of first order simultaneous differential equations (SFOSDE) with an initial condition plays an important role in various field of science and engineering like fluid mechanics, thermodynamics, heat and electromagnetism, rate of chemical reaction, bacteria/ plants and organisms growth rates, and population/economic growth rates, etc. Almost all the modern scientific analysis associates differential equations. There are many ways to solve SFOSD. Mondal et al [16] solving the system of differential equation and its application with intuitionistic fuzzy environment. Sadeghi et al. [17] discussed the necessary and sufficient condition for the existence of solution of fuzzy differential equations. Keshavarz et al. [18] investigated the application of differential equations in Newton's law of cooling, distribution of a drug in the human body and harmonic oscillator problem using fuzzy logic. Karpagappriya et al. [19] examined the solution of fuzzy initial value problems using cubic spline function. Several other researchers [21–23] have also discussed the significance of Neutrosophic in Agricultural Water Management, Healthcare Waste to Achieve Cost Effectiveness and Transportation Problem.

In the above literature studies, researchers explored several numerical/analytical solutions of differential equations. In almost all cases authors find the solutions of fuzzy differential equations using fuzzy environments. However, no attempt has been made to find the solutions of System of Differential Equation with Initial Condition as trapezoidal Neutrosophic number. The purpose of the present study is to investigate the Neutrosophic solutions of first order system of differential equations. The efficiency and feasibility of the present approach is illustrated by numerical examples and the results are for better perceptive of our investigation.

2. Preliminary

Definition 2.1. $\boldsymbol{N_{set}}$: [24] Let X be a universe set. A N_{set} $\widetilde{A_N}$ on X is defined as $\widetilde{A_N} = \{\langle x, T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x)\rangle : x \in X\}$, where $T_{\widetilde{A_N}}(x): X \to ^{-}]0,1[^{+}$ is said to be the $T_{membership}$, which represents the degree of confidence, $I_{\widetilde{A_N}}(x): X \to ^{-}]0,1[^{+}$is said to be the $I_{membership}$, which represents the degree of uncertainty, and $F_{\widetilde{A_N}}(x): X \to ^{-}]0,1[^{+}$ is said to be the $F_{membership}$, which represents the degree of skepticism, respectively of the element $x \in X$ in $\widetilde{A_N}$, such that $^{-}0 \leq T_{\widetilde{A_N}}(x) + I_{\widetilde{A_N}}(x) + F_{\widetilde{A_N}}(x) \leq 3^{+}$.

Definition 2.2. $\boldsymbol{SVN_{set}}$ [24] A N_{set} $\widetilde{A_N}$ on X (Definition 2.1) is said to be SVN_{set} ($\widetilde{SA_N}$) if x is a single-valued independent variable. $\widetilde{SA_N} = \{\langle x, T_{\widetilde{SA_N}}(x), I_{\widetilde{SA_N}}(x), F_{\widetilde{SA_N}}(x)\rangle : x \in X\}$, where $T_{\widetilde{SA_N}}(x), I_{\widetilde{SA_N}}(x), F_{\widetilde{SA_N}}(x): X \to ^{-}]0,1[^{+}$ represents the concept of $T_{membership}, I_{membership}, F_{membership}$, functions, respectively of the element $x \in X$ in $\widetilde{SA_N}$, such that $^{-}0 \leq T_{\widetilde{SA_N}}(x) + I_{\widetilde{SA_N}}(x) + F_{\widetilde{SA_N}}(x) \leq 3^{+}$.

Definition 2.3. $\boldsymbol{(\alpha, \beta, \gamma)_{cut}}$: [24] The $(\alpha, \beta, \gamma)_{cut}$ N_{set} is $\widetilde{A_N}$ defined as $\widetilde{A_N}_{(\alpha,\beta,\gamma)} = \{\langle T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x)\rangle : x \in X, T_{\widetilde{A_N}}(x) \geq \alpha, I_{\widetilde{A_N}}(x) \leq \beta, F_{\widetilde{A_N}}(x) \leq \gamma\}$, where $\alpha, \beta, \gamma \in [0,1]$, such that $\alpha + \beta + \gamma \leq 3$.

Definition 2.4. $\boldsymbol{N_{number}}$: [24] A N_{set} $\widetilde{A_N}$ over the real numbers set $\mathbb{R}$ is a neutrosophic number if it satisfies the following conditions:

i. $\widetilde{A_N}$ is normal: $\exists x_0 \in \mathbb{R}$ such that $T_{\widetilde{A_N}}(x_0) = 1$. $\left(I_{\widetilde{A_N}}(x_0) = F_{\widetilde{A_N}}(x_0) = 0 \right)$.

ii. $\widetilde{A_N}$ is convex for the truth function $T_{\widetilde{A_N}}(x)$.
(ie) $T_{\widetilde{A_N}}(\mu x_1 + (1-\mu)x_2) \geq \min\left(T_{\widetilde{A_N}}(x_1), T_{\widetilde{A_N}}(x_2)\right)$, for all $x_1, x_2 \in \mathbb{R}$ and $\mu \in [0,1]$.

iii. $\widetilde{A_N}$ is concave set for the falsity, indeterministic functions namely, $F_{\widetilde{A_N}}(x)$ and $I_{\widetilde{A_N}}(x)$.
(ie) $I_{\widetilde{A_N}}(\mu x_1 + (1-\mu)x_2) \geq \max\left(I_{\widetilde{A_N}}(x_1), I_{\widetilde{A_N}}(x_2)\right)$ and $F_{\widetilde{A_N}}(\mu x_1 + (1-\mu)x_2) \geq \max\left(F_{\widetilde{A_N}}(x_1), F_{\widetilde{A_N}}(x_2)\right)$, for all $x_1, x_2 \in \mathbb{R}$ and $\mu \in [0,1]$.

Definition 2.5. $\boldsymbol{TrapN_{number}}$: [25] A subset of N_{number} , $TrapN_{number}$, $\widetilde{A_N}$ in $\mathbb{R}$ with the following $T_{function}, I_{function}, and\ F_{function}$ is defined as

$$T_{\widetilde{A_N}}(x) = \begin{cases} \left(\dfrac{x-a_1}{a_2-a_1}\right) u_{\widetilde{A_N}} & for\ a_1 \leq x \leq a_2 \\ u_{\widetilde{A_N}} & for\ a_2 \leq x \leq a_3 \\ \left(\dfrac{a_4-x}{a_4-a_3}\right) u_{\widetilde{A_N}} & for a_3 \leq x \leq a_4 \\ 0 & otherwise \end{cases};$$

$$I_{\widetilde{A_N}}(x) = \begin{cases} \left(\frac{a_2-x}{a_2-a_1}\right) v_{\widetilde{A_N}} & for\ a_1 \leq x \leq a_2 \\ v_{\widetilde{A_N}} & for\ a_2 \leq x \leq a_3 \\ \left(\frac{a_4-x}{a_4-a_3}\right) v_{\widetilde{A_N}} & for a_3 \leq x \leq a_4 \\ 1 & otherwise \end{cases} \qquad (1)$$

$$F_{\widetilde{A_N}}(x) = \begin{cases} \left(\dfrac{a_2-x}{a_2-a_1}\right) w_{\widetilde{A_N}} & for\ a_1 \leq x \leq a_2 \\ w_{\widetilde{A_N}} & for\ a_2 \leq x \leq a_3 \\ \left(\dfrac{a_4-x}{a_4-a_3}\right) w_{\widetilde{A_N}} & for a_3 \leq x \leq a_4 \\ 1 & otherwise \end{cases}$$

where $^{-}0 \leq T_{\widetilde{A_N}}(x) + I_{\widetilde{A_N}}(x) + F_{\widetilde{A_N}}(x) \leq 3^{+}, x \in \widetilde{A_N}$.

Definition 2.6. $\boldsymbol{(\alpha,\beta,\gamma)_{cut}\ of\ a\ TrapN_{number}}$: The $(\alpha,\beta,\gamma)_{cut}$ $of\ a\ TrapN_{number}$ $\widetilde{A_N} = \langle (a_1, a_2, a_3, a_4); u_{\widetilde{A_N}}, v_{\widetilde{A_N}}, w_{\widetilde{A_N}} \rangle$ is defined as follows:

$(\widetilde{A_N})_{\alpha,\beta,\gamma} = \left[T_{\widetilde{A_{N1}}}(\alpha), T_{\widetilde{A_{N2}}}(\alpha); I_{\widetilde{A_{N1}}}(\beta), I_{\widetilde{A_{N2}}}(\beta); F_{\widetilde{A_{N1}}}(\gamma), F_{\widetilde{A_{N2}}}(\gamma)\right]$, where

$T_{\widetilde{A_{N1}}}(\alpha) = [a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}}, T_{\widetilde{A_{N2}}}(\alpha) = [a_4 - \alpha(a_4 - a_3)]u_{\widetilde{A_N}}$

$I_{\widetilde{AN1}}(\beta) = [a_2 - \beta(a_2 - a_1)]v_{\widetilde{A_N}}, I_{\widetilde{AN2}}(\beta) = [a_3 + \beta(a_4 - a_3)]v_{\widetilde{A_N}}$ (2)
$F_{\widetilde{AN1}}(\gamma) = [a_2 - \gamma(a_2 - a_1)]w_{\widetilde{A_N}}, F_{\widetilde{AN2}}(\gamma) = [a_3 + \gamma(a_4 - a_3)]w_{\widetilde{A_N}}$

here $0 < \alpha \leq 1, 0 < \beta \leq 1, 0 < \gamma \leq 1$ *and* $^{-}0 < \alpha + \beta + \gamma \leq 3^{+}$.

Definition 2.7. $\boldsymbol{Strong_{solution}}$: Let the solution of the neutrosophic differential equations be $x(t)$ and $y(t)$, and its $(\alpha,\beta,\gamma)_{cut}$ be $[x(t,\alpha,\beta,\gamma)] = [x_1(t,\alpha), x_2(t,\alpha), x_1'(t,\beta), x_2'(t,\beta), x_1''(t,\gamma), x_2''(t,\gamma)]$ and $[y(t,\alpha,\beta,\gamma)] = [y_1(t,\alpha), y_2(t,\alpha), y_1'(t,\beta), y_2'(t,\beta), y_1''(t,\gamma), y_2''(t,\gamma)]$.
The solution is a strong solution if,

i. $\frac{dx_1(t,\alpha)}{d\alpha} > 0, \frac{dx_2(t,\alpha)}{d\alpha} < 0$ *and* $\frac{dy_1(t,\alpha)}{d\alpha} > 0, \frac{dy_2(t,\alpha)}{d\alpha} < 0 \forall \alpha \in [0,1], x_1(t,1) \leq x_2(t,1)$ *and* $y_1(t,1) \leq y_2(t,1)$.
ii. $\frac{dx_1'(t,\beta)}{d\alpha} < 0, \frac{dx_2'(t,\beta)}{d\alpha} > 0$ *and* $\frac{dy_1'(t,\beta)}{d\alpha} < 0, \frac{dy_2'(t,\beta)}{d\alpha} > 0 \forall \beta \in [0,1], x_1'(t,0) \leq x_2'(t,0)$ *and* $y_1'(t,0) \leq y_2'(t,0)$. (3)
iii. $\frac{dx_1''(t,\gamma)}{d\alpha} < 0, \frac{dx_2''(t,\gamma)}{d\alpha} > 0$ *and* $\frac{dy_1''(t,\gamma)}{d\alpha} < 0, \frac{dy_2''(t,\gamma)}{d\alpha} > 0 \forall \gamma \in [0,1], x_1''(t,0) \leq x_2''(t,0)$ *and* $y_1''(t,0) \leq y_2''(t,0)$.

3. Solution of Neutrosophic boundary value problem

In this section, we discuss the solution of first-order ODE with a neutrosophic initial value conditions.

3.1 Solution of System of First-Order ODE using $TrapN_{number}$

Let us consider the system of first order differential equation

$$\frac{dx}{dt} = k_1 y \quad (4)$$

and

$$\frac{dy}{dt} = k_2 x \quad (5)$$

with boundary condition $x(t_0) = \tilde{a}$ and $y(t_0) = \tilde{b}$ where $\tilde{a} = \langle (a_1, a_2, a_3, a_4); u_{\widetilde{A_N}}, v_{\widetilde{A_N}}, w_{\widetilde{A_N}} \rangle$ and $\tilde{b} = \langle (b_1, b_2, b_3, b_4); u_{\widetilde{A_N}}, v_{\widetilde{A_N}}, w_{\widetilde{A_N}} \rangle$ are $TrapN_{number}$.

Case $(\boldsymbol{i})$ when k_1 *and* k_2 are positive constant $(ie) k_1, k_2 > 0$
The (α,β,γ)-cut of Eq. (4)& (5) are

$$\frac{d}{dt}[x_1(t,\alpha), x_2(t,\alpha); x_1'(t,\beta), x_2'(t,\beta); x_1''(t,\gamma), x_2''(t,\gamma)]$$
$$= k_1[y_1(t,\alpha), y_2(t,\alpha); y_1'(t,\beta), y_2'(t,\beta); y_1''(t,\gamma), y_2''(t,\gamma)] \quad (6)$$

$$\frac{d}{dt}[y_1(t,\alpha),y_2(t,\alpha);y_1'(t,\beta),y_2'(t,\beta);y_1''(t,\gamma),y_2''(t,\gamma)]$$
$$= k_2[x_1(t,\alpha),x_2(t,\alpha);x_1'(t,\beta),x_2'(t,\beta);x_1''(t,\gamma),x_2''(t,\gamma)] \quad (7)$$

with the initial condition

$$x(t_0;\alpha,\beta,\gamma) = \langle [a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}}, [a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}};\ [a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}}, [a_3 + \beta(a_4-a_3)]v_{\widetilde{A_N}}; [a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}}, [a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} \rangle \quad (8)$$

and

$$y(t_0;\alpha,\beta,\gamma) = \langle [b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}, [b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}};\ [b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}, [b_3 + \beta(b_4-b_3)]v_{\widetilde{A_N}}; [b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}}, [b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}} \rangle \quad (9)$$

From (6) we get

$$\left.\begin{array}{l} \frac{dx_1(t,\alpha)}{dt} = k_1y_1(t,\alpha); \frac{dx_2(t,\alpha)}{dt} = k_1y_2(t,\alpha) \\ \frac{dx_1'(t,\beta)}{dt} = k_1y_1'(t,\beta); \frac{dx_2'(t,\beta)}{dt} = k_1y_2'(t,\beta) \\ \frac{dx_1''(t,\gamma)}{dt} = k_1y_1''(t,\gamma); \frac{dx_2''(t,\gamma)}{dt} = k_1y_2''(t,\gamma) \end{array}\right\} \quad (10)$$

From (7) we get

$$\left.\begin{array}{l} \frac{dy_1(t,\alpha)}{dt} = k_2x_1(t,\alpha); \frac{dy_2(t,\alpha)}{dt} = k_2x_2(t,\alpha) \\ \frac{dy_1'(t,\beta)}{dt} = k_2x_1'(t,\beta); \frac{dy_2'(t,\beta)}{dt} = k_2x_2'(t,\beta) \\ \frac{dy_1''(t,\gamma)}{dt} = k_2y_1''(t,\gamma); \frac{dy_2''(t,\gamma)}{dt} = k_2x_2''(t,\gamma) \end{array}\right\} \quad (11)$$

with initial conditions

$$\left.\begin{array}{r} x_1(t_0,\alpha) = [a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}}; x_2(t_0,\alpha) = [a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}} \\ x_1'(t_0,\beta) = [a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}}; x_2'(t_0,\beta) = [a_3+\beta(a_4-a_3)]v_{\widetilde{A_N}} \\ x_1''(t_0,\gamma) = [a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}};\ x_2''(t_0,\gamma) = [a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} \end{array}\right\} \quad (12)$$

and

$$\left.\begin{array}{r} y_1(t_0,\alpha) = [b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}; y_2(t_0,\alpha) = [b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}} \\ y_1'(t_0,\beta) = [b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}; y_2'(t_0,\beta) = [b_3+\beta(b_4-b_3)]v_{\widetilde{A_N}} \\ y_1''(t_0,\gamma) = [b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}};\ b_2''(t_0,\gamma) = [b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}} \end{array}\right\} \quad (13)$$

From Eqs. (10) and (11) we have

$$\frac{d^2x_1(t,\alpha)}{dt^2} = k_1k_2x_1(t,\alpha) \quad (14)$$

The solution of Eq.(14) is

$$x_1(t,\alpha) = Ae^{\sqrt{k_1k_2}t} + Be^{-\sqrt{k_1k_2}t} \quad (15)$$

substituting Eq.(15) in Eq.(11) , we get

$$Ae^{\sqrt{k_1k_2}t} - Be^{-\sqrt{k_1k_2}t} = \sqrt{\frac{k_1}{k_2}}y_1(t,\alpha) \quad (16)$$

Using initial condition, we get

$$Ae^{\sqrt{k_1k_2}t} + Be^{-\sqrt{k_1k_2}t} = [a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}}$$

and $Ae^{\sqrt{k_1k_2}t} - Be^{-\sqrt{k_1k_2}t} = \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}$

Therefore,

$$\left.\begin{aligned} A &= \frac{1}{2}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}t_0} \\ B &= \frac{1}{2}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}t_0} \end{aligned}\right\} \quad (17)$$

substituting Eq.(17) in Eqs.(12) and (13), we get

$$x_1(t,\alpha) = \frac{1}{2}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$
$$+\frac{1}{2}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (18)$$

$$y_1(t,\alpha) = \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$
$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_1 + \alpha(a_2 - a_1)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_1 + \alpha(b_2 - b_1)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (19)$$

similarly,

$$x_2(t,\alpha) = \frac{1}{2}\left\{[a_4 - \alpha(a_4 - a_3)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_4 - \alpha(b_4 - b_3)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$
$$+\frac{1}{2}\left\{[a_4 - \alpha(a_4 - a_3)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_4 - \alpha(b_4 - b_3)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (20)$$

$$y_2(t,\alpha) = \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_4 - \alpha(a_4 - a_3)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_4 - \alpha(b_4 - b_3)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$
$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_4 - \alpha(a_4 - a_3)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_4 - \alpha(b_4 - b_3)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (21)$$

$$x_1'(t,\beta) = \frac{1}{2}\left\{[a_2 - \beta(a_2 - a_1)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2 - \beta(b_2 - b_1)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_2 - \beta(a_2 - a_1)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2 - \beta(b_2 - b_1)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (22)$$

$$y_1'(t,\beta) = \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2 - \beta(a_2 - a_1)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2 - \beta(b_2 - b_1)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2 - \beta(a_2 - a_1)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2 - \beta(b_2 - b_1)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (23)$$

$$x_2'(t,\beta) = \frac{1}{2}\left\{[a_3 + \beta(a_4 - a_3)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3 + \beta(b_4 - b_3)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_3 + \beta(a_4 - a_3)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3 + \beta(b_4 - b_3)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (24)$$

$$y_2'(t,\beta) = \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3 + \beta(a_4 - a_3)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3 + \beta(b_4 - b_3)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3 + \beta(a_4 - a_3)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3 + \beta(b_4 - b_3)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (25)$$

$$x_1''(t,\gamma) = \frac{1}{2}\left\{[a_2 - \gamma(a_2 - a_1)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2 - \gamma(b_2 - b_1)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_2 - \gamma(a_2 - a_1)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2 - \gamma(b_2 - b_1)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (26)$$

$$y_1''(t,\gamma) = \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2 - \gamma(a_2 - a_1)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2 - \gamma(b_2 - b_1)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2 - \gamma(a_2 - a_1)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2 - \gamma(b_2 - b_1)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (27)$$

$$x_2''(t,\gamma) = \frac{1}{2}\left\{[a_3 + \gamma(a_4 - a_3)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3 + \gamma(b_4 - b_3)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_3 + \gamma(a_4 - a_3)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3 + \gamma(b_4 - b_3)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (28)$$

$$y_2''(t,\gamma) = \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$-\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} (29)$$

Case $(\boldsymbol{ii})$ when $k_1 and\ k_2$ are negative constants $(ie)k_1, k_2 < 0$

The general solution of the system of solutions are as follows:

$$x_1(t,\alpha) = \frac{1}{2}\left\{[a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} (30)$$

$$y_2(t,\alpha) = -\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_1+\alpha(a_2-a_1)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_1+\alpha(b_2-b_1)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} (31)$$

$$x_2(t,\alpha) = \frac{1}{2}\left\{[a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} (32)$$

$$y_1(t,\alpha) = -\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_4-\alpha(a_4-a_3)]u_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_4-\alpha(b_4-b_3)]u_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} (33)$$

$$x_1'(t,\beta) = \frac{1}{2}\left\{[a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} (34)$$

$$y_2'(t,\beta) = -\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2-\beta(a_2-a_1)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2-\beta(b_2-b_1)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (35)$$

$$x_2'(t,\beta) = \frac{1}{2}\left\{[a_3+\beta(a_4-a_3)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3+\beta(b_4-b_3)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_3+\beta(a_4-a_3)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3+\beta(b_4-b_3)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (36)$$

$$y_1'(t,\beta) = -\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\beta(a_4-a_3)]v_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3+\beta(b_4-b_3)]v_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\beta(a_4-a_3)]v_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3+\beta(b_4-b_3)]v_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (37)$$

$$x_1''(t,\gamma) = \frac{1}{2}\left\{[a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (38)$$

$$y_2''(t,\gamma) = -\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_2-\gamma(a_2-a_1)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_2-\gamma(b_2-b_1)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (39)$$

$$x_2''(t,\gamma) = \frac{1}{2}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (40)$$

$$y_1''(t,\gamma) = -\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} - \sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{\sqrt{k_1k_2}(t-t_0)}$$

$$+\frac{1}{2}\sqrt{\frac{k_2}{k_1}}\left\{[a_3+\gamma(a_4-a_3)]w_{\widetilde{A_N}} + \sqrt{\frac{k_1}{k_2}}[b_3+\gamma(b_4-b_3)]w_{\widetilde{A_N}}\right\}e^{-\sqrt{k_1k_2}(t-t_0)} \quad (41)$$

3.2 Numerical Example:

Consider a system of differential equation $\frac{dx}{dt} = 4y$ and $\frac{dy}{dt} = 5x$ with initial conditions $x(t_0) = \tilde{a} = \langle(3,4,5,6); 0.7, 0.6, 0.4\rangle$ and $y(t_0) = \tilde{b} = \langle(5,6,7,8); 0.5, 0.3, 0.2\rangle$.

Solution:

The solution is given by the equations

$$x_1(t,\alpha) = \frac{1}{2}\left\{[3+\alpha]0.7 + \sqrt{\frac{4}{5}}[5+\alpha]0.5\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[3+\alpha]0.7 - \sqrt{\frac{4}{5}}[5+\alpha]0.5\right\}e^{-\sqrt{20}t}$$

$$y_1(t,\alpha) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[3+\alpha]0.7 + \sqrt{\frac{4}{5}}[5+\alpha]0.5\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[3+\alpha]0.7 - \sqrt{\frac{4}{5}}[5+\alpha]0.5\right\}e^{-\sqrt{20}t}$$

$$x_2(t,\alpha) = \frac{1}{2}\left\{[6-\alpha]0.7 + \sqrt{\frac{4}{5}}[8-\alpha]0.5\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[6-\alpha]0.7 - \sqrt{\frac{4}{5}}[8-\alpha]0.5\right\}e^{-\sqrt{20}t}$$

$$y_2(t,\alpha) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[6-\alpha]0.7 + \sqrt{\frac{4}{5}}[8-\alpha]0.5\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[6-\alpha]0.7 - \sqrt{\frac{4}{5}}[8-\alpha]0.5\right\}e^{-\sqrt{20}t}$$

$$x_1'(t,\beta) = \frac{1}{2}\left\{[4-\beta]0.6 + \sqrt{\frac{4}{5}}[6-\beta]0.3\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[4-\beta]0.6 - \sqrt{\frac{4}{5}}[6-\beta]0.3\right\}e^{-\sqrt{20}t}$$

$$y_1'(t,\beta) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[4-\beta]0.6 + \sqrt{\frac{4}{5}}[6-\beta]0.3\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[4-\beta]0.6 - \sqrt{\frac{4}{5}}[6-\beta]0.3\right\}e^{-\sqrt{20}t}$$

$$x_2'(t,\beta) = \frac{1}{2}\left\{[5+\beta]0.6 + \sqrt{\frac{4}{5}}[7+\beta]0.3\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[5+\beta]0.6 - \sqrt{\frac{4}{5}}[7+\beta]0.3\right\}e^{-\sqrt{20}t}$$

$$y_2'(t,\beta) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[5+\beta]0.6 + \sqrt{\frac{4}{5}}[7+\beta]0.3\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[5+\beta]0.6 - \sqrt{\frac{4}{5}}[7+\beta]0.3\right\}e^{-\sqrt{20}t}$$

$$x_1''(t,\gamma) = \frac{1}{2}\left\{[4-\gamma]0.4 + \sqrt{\frac{4}{5}}[6-\gamma]0.2\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[4-\gamma]0.4 - \sqrt{\frac{4}{5}}[6-\gamma]0.3\right\}e^{-\sqrt{20}t}$$

$$y_1''(t,\gamma) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[4-\gamma]0.4 + \sqrt{\frac{4}{5}}[6-\gamma]0.3\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[4-\gamma]0.4 - \sqrt{\frac{4}{5}}[6-\gamma]0.3\right\}e^{-\sqrt{20}t}$$

and

$$x_2''(t,\gamma) = \frac{1}{2}\left\{[5+\gamma]0.4 + \sqrt{\frac{4}{5}}[7+\gamma]0.2\right\}e^{\sqrt{20}t} + \frac{1}{2}\left\{[5+\gamma]0.4 - \sqrt{\frac{4}{5}}[7+\gamma]0.3\right\}e^{-\sqrt{20}t}$$

$$y_2''(t,\gamma) = \frac{1}{2}\sqrt{\frac{4}{5}}\left\{[5+\gamma]0.4 + \sqrt{\frac{4}{5}}[7-\gamma]0.3\right\}e^{\sqrt{20}t} - \frac{1}{2}\left\{[5+\gamma]0.4 - \sqrt{\frac{4}{5}}[7+\gamma]0.3\right\}e^{-\sqrt{20}t}$$

Table 1. $\boldsymbol{TrapN_{number}}$ **solution of** $\boldsymbol{x(t,\ \alpha,\beta,\gamma)}$ **at** $\boldsymbol{t = 0.5}$

$\boldsymbol{\alpha}$	$\boldsymbol{x_1(t,\alpha)}$	$\boldsymbol{x_2(t,\alpha)}$	$\boldsymbol{\beta}$	$\boldsymbol{x_1'(t,\beta)}$	$\boldsymbol{x_2'(t,\beta)}$	$\boldsymbol{\gamma}$	$\boldsymbol{x_1''(t,\gamma)}$	$\boldsymbol{x_2''(t,\gamma)}$
0.1	52.45255	90.30831	0.1	46.23416	58.51406	0.1	30.82277	39.00937
0.2	53.80454	88.95632	0.2	45.21083	59.53738	0.2	30.14056	39.69159
0.3	55.15653	87.60432	0.3	44.18751	60.56071	0.3	29.45834	40.3738
0.4	56.50853	86.25233	0.4	43.16419	61.58403	0.4	28.77612	41.05602
0.5	57.86052	84.90034	0.5	42.14086	62.60736	0.5	28.09391	41.73824
0.6	59.21251	83.54835	0.6	41.11754	63.63068	0.6	27.41169	42.42045
0.7	60.5645	82.19369	0.7	40.09421	64.654	0.7	26.72947	43.10267
0.8	61.91649	80.84437	0.8	39.07089	65.67733	0.8	26.04726	43.78489
0.9	63.26848	79.49238	0.9	38.04756	66.70065	0.9	25.36504	44.4671
1	64.62047	78.14039	1	37.02424	67.72398	1	24.68282	45.14932

Table 2. $\boldsymbol{TrapN_{number}}$ **solution of** $\boldsymbol{y(t,\ \alpha,\beta,\gamma)}$ **at** $\boldsymbol{t = 0.5}$

$\boldsymbol{\alpha}$	$\boldsymbol{y_1(t,\alpha)}$	$\boldsymbol{y_2(t,\alpha)}$	$\boldsymbol{\beta}$	$\boldsymbol{y_1'(t,\beta)}$	$\boldsymbol{y_2'(t,\beta)}$	$\boldsymbol{\gamma}$	$\boldsymbol{y_1''(t,\gamma)}$	$\boldsymbol{y_2''(t,\gamma)}$
0.1	46.91499	80.7742	0.1	41.35309	52.33656	0.1	33.13125	41.01924
0.2	48.12425	79.56495	0.2	40.4378	53.25185	0.2	32.42677	41.15803
0.3	49.3335	78.35569	0.3	39.52251	54.16714	0.3	31.7223	41.29683
0.4	50.54276	77.14643	0.4	38.60722	55.08243	0.4	31.01783	41.43562
0.5	51.75202	75.93717	0.5	37.69193	55.99772	0.5	30.31335	41.57441
0.6	52.96128	74.72792	0.6	36.77664	56.91301	0.6	29.60888	41.71321
0.7	54.17054	73.51866	0.7	35.86135	57.8283	0.7	28.90441	41.852
0.8	55.37979	72.3094	0.8	34.94606	58.74359	0.8	28.19994	41.99079
0.9	56.58905	71.10014	0.9	34.03077	59.65888	0.9	27.49546	42.12959
1	57.79831	69.89089	1	33.11548	60.57417	1	26.79099	42.26838

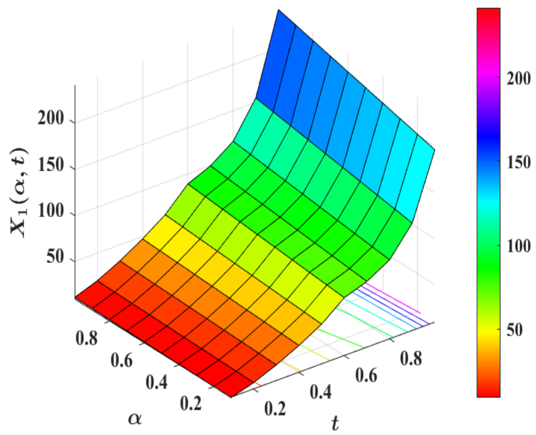

Fig. 1. $x_1(\alpha, t)$ **for increasing values of** t

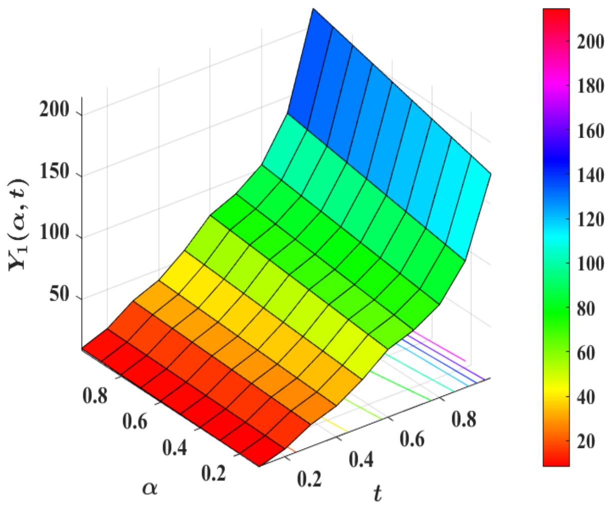

Fig. 2. $y_1(\alpha, t)$ **for increasing values of** t

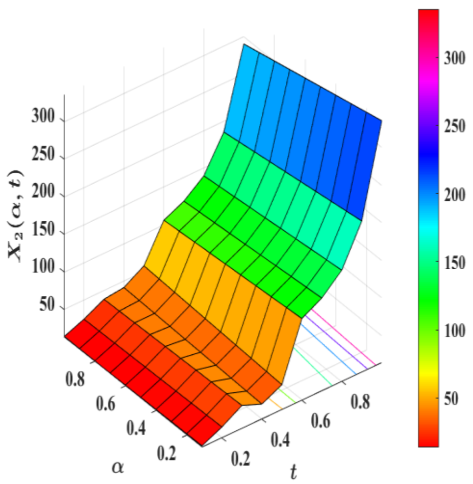

Fig. 3. $x_2(\alpha, t)$ **for increasing values of** t

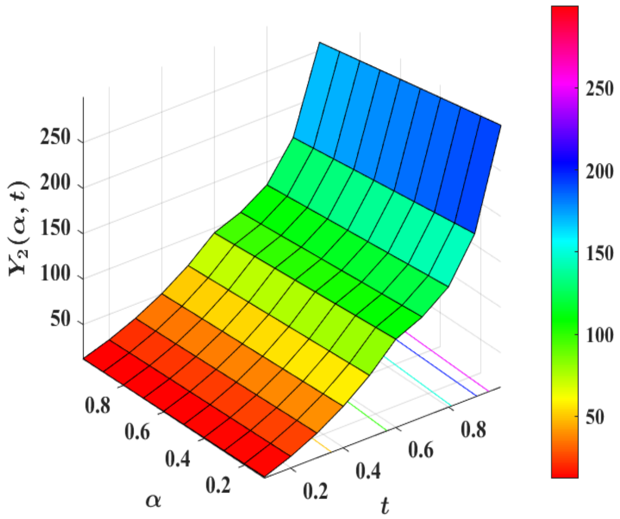

Fig. 4. $y_2(\alpha, t)$ **for increasing values of** t

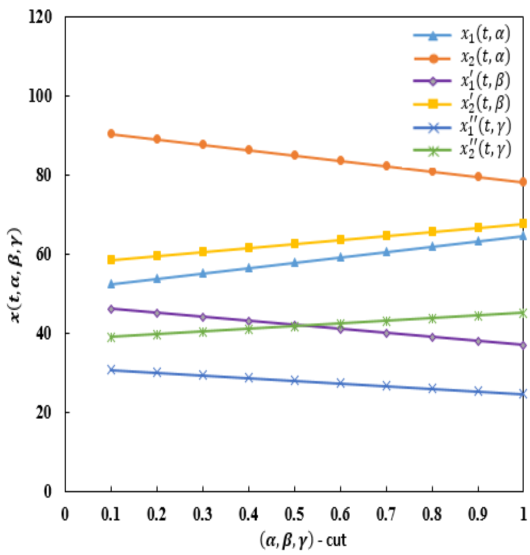

Fig. 5. Solution of $x(t,\ \alpha,\beta,\gamma)$ at $t = 0.5$

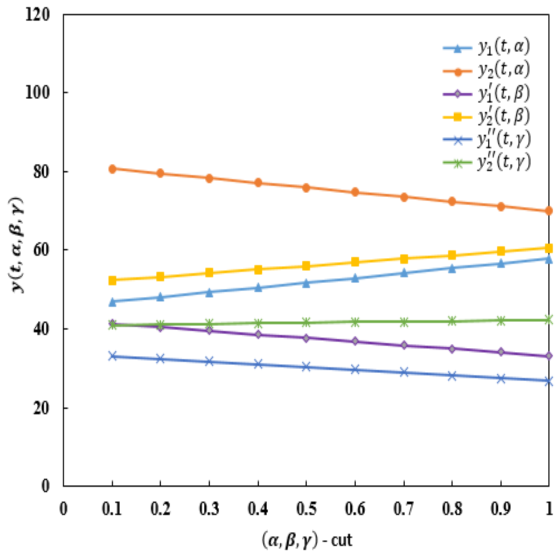

Fig. 6. Solution of $y(t,\ \alpha,\beta,\gamma)$ at $t = 0.5$

3.3 Results and Discussion

The system of first-order simultaneous differential equations (SFOSDE) with initial conditions is a powerful tool used in various scientific and engineering disciplines to model and analyze dynamic systems. These equations capture relationships between different variables and how they change over time, providing insights into the behavior of complex systems. SFOSDEs are used to describe the flow of fluids in various scenarios, like incompressible or compressible flows, turbulence, and boundary layer analysis. These equations can model the heat transfer, energy exchange, and temperature changes in systems governed by the laws of thermodynamics. SFOSDEs can be employed to model reaction rates, concentrations, and reaction pathways. Differential equations help in understanding growth rates, interactions between different species, population dynamics, and the spread of diseases in biological and ecological systems. Uncertainty is a common aspect in various scientific and engineering applications, and it's important to consider it when modeling real-world systems. In mathematical modeling, uncertainty can be dealt with in various ways, depending on the specific context and the type of uncertainty being considered. Solving SFOSDEs with neutrosophic inputs would involve extending traditional solution methods to handle neutrosophic values. This might involve developing new numerical techniques or analytical approaches that can accommodate the three aspects of membership in neutrosophic sets.

The solutions of a system of differential equation $\frac{dx}{dt} = 4y$ and $\frac{dy}{dt} = 5x$ with initial conditions $x(t_0) = \tilde{a} = \langle(3,4,5,6); 0.7, 0.6, 0.4\rangle$ and $y(t_0) = \tilde{b} =$

$\langle(5,6,7,8); 0.5, 0.3, 0.2\rangle$ for various t and $0 < \alpha, \beta, \gamma \leq 1$ are depicted in Tables 1 and 2 Figures 1-4. It is observed that, the increasing values of α, β, γ increase $x_1(\alpha, t)$, $y_1(\alpha, t)$ and decrease $x_2(\alpha, t)$, $y_2(\alpha, t)$ where as $x_1'(\alpha, t)$, $y_1'(\alpha, t)$and $x_1''(\alpha, t)$, $y_1''(\alpha, t)$ are diminishing functions and $x_2'(\alpha, t)$, $y_2'(\alpha, t)$and $x_2''(\alpha, t)$, $y_2''(\alpha, t)$ are escalation functions. Therefore, the numerical example of system of differential equation satisfies the conditions of the $Strong_{solution}$ of a neutrosophic difference equation. Hence the obtained solution is a strong solution. In addition, the graphs for different values for $T_{membership}$ $(T_{\widetilde{A_N}}(x))$, $I_{membership}$ $(I_{\widetilde{A_N}}(x))$, and $F_{membership}$ $(F_{\widetilde{A_N}}(x))$ functions with $(\alpha, \beta, \gamma)_{cut}$ at time $t = 0$ are displayed in Figures 5 and 6. As the α_{cut} value enhances and $\beta_{cut}, \gamma_{cut}$ values diminishes the solutions of $x(t,\ \alpha, \beta, \gamma)$ and $y(t,\ \alpha, \beta, \gamma)$ which approaches to the exact solution.

Conclusion

A technique for approximating the solution of system of first order simultaneous differential equations with initial condition as a neutrosophic environment is presented in this proposed work. We used the initial conditions as trapezoidal neutrosophic numbers. Furthermore, numerical examples are illustrated for better understanding of solving SFOSD utilizing $TrapN_{number}$ via MATLAB software. The simulation results are shown in Tables and Figures. From these it is noticed that the new technique using neutrosophic are effective and more flexible to estimate the solutions of SFOSD. This technique is used to develop the solution of highly nonlinear coupled ordinary differential equations in the field of Computational Fluid Mechanics.

Future Work:

In future, one can extend this technique to solve higher-order linear and nonlinear neutrosophic initial value problems. Also, we will focus on higher-order nonlinear coupled partial differential equations and their applications in neutrosophic environments.

Table 3. Notations

C_{set}	Classical set
F_{set}	Fuzzy Set
IF_{set}	Intuitionistic Fuzzy Set
IVF_{set}	Interval Valued Fuzzy Set
N_{set}	Neutrosophic set
N_{logic}	Neutrosophic logic

SVN_{set}	Single Values Neutrosophic set
SVN_{number}	Single Values Neutrosophic number
$TriN_{number}$	Triangular Neutrosophic number
$TrapN_{number}$	Trapezoidal Neutrosophic number
$T_{value}, I_{value,} and\ F_{value}$	membership, indeterminacy, and non-membership values
$T_{membership}, I_{membership}, F_{membership}$	truth, indeterminacy, and falsity membership
$T_{function}, I_{function}, and\ F_{functuion}$	truth, indeterminacy, and falsity function
$(\alpha,\beta,\gamma)_{cut}$	$(\alpha,\beta,\gamma) - cut$
$(\alpha,\beta,\gamma)_{cut}\ of\ a\ TrapN_{number}$	$(\alpha,\beta,\gamma) - cut$ of a Trapezoidal Neutrosophic number

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Optimal Neutrosophic Assignment and the Hungarian Method

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Abstract:
Operations research methods focus on formulating decision models, models that approximate the real work environment to mathematical models, and their optimal solution helps the decision maker to make optimal decisions that guarantee the greatest profit or the lowest cost. These models were built based on the data collected on the practical issue under study, and these data are classic values, specific values that are correct and accurate during the period in which these data are collected, which means that they remain ideal if the surrounding conditions remain the same as those in which they were collected. Data, and since the aim of this study is to make ideal decisions and to develop future plans through which we achieve the greatest profit or the least cost, It was necessary to search for a study that fits all the conditions in which the work environment passes in the present and the future. If we take data, neural values, indeterminate values - uncertain - leave nothing to chance, and take into account all conditions from worst to best. In this paper, we will present a complementary study to what we have done in previous research, the purpose of which was to reformulate some operational research methods using neutrosophic data. We will reformulate the assignment problem and one of the methods for its solution, which is the Hungarian method, using neutrosophic concepts, and we will explain the difference between using classical and neutrosophic data through examples.

Key words:
The optimal assignment problem - The Neutrosophic optimal assignment problem - The Hungarian method for solving assignment problems - linear models - Neutrosophic logic.

Introduction:
Assignment issues are a special case of linear programming issues that are concerned with the optimal assignment of various economic, productive and human resources for the various works to be accomplished, and we encounter them frequently in practical life in educational institutions - hospitals - construction projects…..etc. In order to obtain an optimal assignment that achieves the greatest profit and the least loss in all conditions that the work environment can pass through,

it was necessary to reformulate the issue of optimal assignment and one of the ways to solve it using the concepts of the science neutrosophic , the science that has proven its ability to provide ideal and appropriate solutions in all circumstances. And in many areas, as shown in research and studies presented by researchers interested in scientific development [1-15].Since the optimal assignment model is one of the important models in the field of operations research, where we have a machines or people, and we need to assignment them to do the required work, and the number of works is equal to the number of machines or people. We need an optimal assignment for each machine so that it performs only one work and achieves the greatest profit or the lowest cost depending on the nature of the issue. This issue was studied using classical data by building the mathematical models, when we solving it, we get the desired. In this research, we will reformulate the issue using neutrosophic data that takes into account all changes that may occur in the work environment by taking costs or profit neutrosophic values, meaning that the cost (or return profit) of assignment the machine or person i to perform work j is $Nc_{ij} \in c_{ij} \pm \varepsilon_{ij}$ where ε_{ij} is the indeterminacy and $\varepsilon_{ij} \in [\lambda_{\text{ij1}}, \lambda_{\text{ij2}}]$, it is any neighborhood to the value c_{ij} that we get while collecting data on the problem then the cost (or profit) matrix becomes $NC_{ij} = [c_{ij} \pm \varepsilon_{ij}]$.

Discussion:

Assignment issues are considered a special case of linear programming issues and are concerned with the optimal assignment of various economic, human and productive resources for the different work to be accomplished, based on what we have presented in the research Mysterious Neutrosophic Linear Models [16] and the classical formulation contained in the two references [17,18]. In this research, we will reformulate the problem of optimal assignment and the Hungarian method that is used to solve these problems using the concepts of neutrosophic science, that is, we will take the costs or profit from the neutrosophic values, so that the cost of doing job j by the machine or the person i is $Nc_{ij} \in c_{ij} \pm \varepsilon_{ij}$, where ε_{ij} is indeterminacy and $\varepsilon_{ij} \in [\lambda_{\text{ij1}}, \lambda_{\text{ij2}}]$, which is any neighborhood of the value c_{ij} which we get it while collecting the data and then the cost matrix becomes equal to $NC_{ij} = [c_{ij} \pm \varepsilon_{ij}]$ and the problem text is as follows:

Standard assignment issues:

In these issues, the number of machines or people equals the number of works, which we will address in this research.

Text of the minimum cost type neutrosophic normative assignment problem:

If we have $\boldsymbol{n}$ machines, we denote them by $M_1, M_2, \dots\dots, M_n$ and we have a set of works consisting of $\boldsymbol{n}$ different work we denote them by $N_1, N_2, \dots\dots, N_n$ we want to designate the machines to do these jobs, cost of doing any work j on the device i, it is $Nc_{ij} \in c_{ij} \pm \varepsilon_{ij}$. Assuming that any machine can do only one job, it is required to find the optimal assignment so that the cost is as small as possible.

Formulation of the mathematical model:
To formulate the linear mathematical model, we assume:

$$\boldsymbol{x_{ij}} = \begin{cases} \mathbf{1} & \boldsymbol{if\ job\ j\ was\ given\ to\ machine\ i} \\ \mathbf{0} & \boldsymbol{otherwise} \end{cases}$$

Then write the target function as follows:

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} \pm \varepsilon_{ij}) x_{ij}$$

Conditions for machines:
Since each machine accepts only one action, we find:

$$\sum_{j=1}^{n} \boldsymbol{x_{ij}} = \mathbf{1} \quad ; \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \dots\dots, \boldsymbol{n}$$

Business terms:
Since each work is assigned to only one machine, we find:

$$\sum_{i=1}^{n} \boldsymbol{x_{ij}} = \mathbf{1} \quad ; \boldsymbol{j} = \mathbf{1}, \mathbf{2}, \dots\dots, \boldsymbol{n}$$

Accordingly, the neutrosophic mathematical model is written as follows:
Find the minimum value:

$$\boldsymbol{Z} = \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} \pm \varepsilon_{ij}) \boldsymbol{x_{ij}}$$

Machine terms:

$$\sum_{j=1}^{n} \boldsymbol{x_{ij}} = \mathbf{1} \quad ; \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \dots\dots, \boldsymbol{n}$$

Business terms:

$$\sum_{i=1}^{n} \boldsymbol{x_{ij}} = \mathbf{1} \quad ; \boldsymbol{j} = \mathbf{1}, \mathbf{2}, \dots\dots, \boldsymbol{n}$$

Example 1: (The data are classic values).

Formulation of the mathematical model for the problem of standard assignment of minimum cost:

We want to find the optimal assignment for four jobs on four machines. The cost of assignment is given in the following table:

Business / The machines	N_1	N_2	N_3	N_4
M_1	**10**	**9**	**8**	**7**
M_2	**3**	**4**	**5**	**6**
M_3	**2**	**1**	**1**	**2**
M_4	**4**	**3**	**5**	**6**

Table No. (1) Table of Distribution cost table and classic values

To formulate the linear mathematical model:
We impose:

$$x_{ij} = \begin{cases} 1 & \text{if job } j \text{ was given to machine } i \\ 0 & \text{otherwise} \end{cases} \quad ; \ i,j = 1,2,3,4$$

Using the problem data, we get the following objective function:

$$Z = 10x_{11} + 9x_{12} + 8x_{13} + 7x_{14} + 3x_{21} + 4x_{22} + 5x_{23} + 6x_{24} + 2x_{31} + x_{32} + x_{33} + 2x_{34} + 4x_{41} + 3x_{42} + 5x_{43} + 6x_{44}$$

Machine terms:

$$x_{11} + x_{12} + x_{13} + x_{14} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$

Business terms:

$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$
$$x_{13} + x_{23} + x_{33} + x_{43} = 1$$
$$x_{14} + +x_{24} + x_{34} + x_{44} = 1$$

Therefore, the mathematical model is written as follows:
Find the minimum value of the function:

$$Z = 10x_{11} + 9x_{12} + 8x_{13} + 7x_{14} + 3x_{21} + 4x_{22} + 5x_{23} + 6x_{24} + 2x_{31} + x_{32} + x_{33} + 2x_{34} + 4x_{41} + 3x_{42} + 5x_{43} + 6x_{44}$$

Within the conditions:

$$x_{11} + x_{12} + x_{13} + x_{14} = 1$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 1$$
$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$
$$x_{13} + x_{23} + x_{33} + x_{43} = 1$$
$$x_{14} + +x_{24} + x_{34} + x_{44} = 1$$

Where x_{ij} it is either equal to zero or one.

In the previous model, there is some indeterminacy in the assignment process, as we do not know which machine will perform a certain work. In addition to that, we will also use neutrosophic data. We will take the cost of neutrosophic values, i.e. the cost of assignment machine i to perform work j is $Nc_{ij} \in c_{ij} \pm \varepsilon_{ij}$, where ε_{ij} is the indeterminacy and $\varepsilon_{ij} \in [\lambda_{\text{ij1}}, \lambda_{\text{ij2}}]$, which is any neighborhood to the value c_{ij} then the cost matrix becomes $NC_{ij} = [c_{ij} \pm \varepsilon_{ij}]$.

Example 2: (Cost is neutrosophic values):
We want to find the optimal assignment for four jobs on four machines. The cost of assignment is given in the following table**:**

Business / The machines	N_1	N_2	N_3	N_4
M_1	$10 + \varepsilon_{11}$	$9 + \varepsilon_{12}$	$8 + \varepsilon_{13}$	$7 + \varepsilon_{14}$
M_2	$3 + \varepsilon_{21}$	$4 + \varepsilon_{22}$	$5 + \varepsilon_{23}$	$6 + \varepsilon_{24}$
M_3	$2 + \varepsilon_{31}$	$1 + \varepsilon_{32}$	$1 + \varepsilon_{33}$	$2 + \varepsilon_{34}$
M_4	$4 + \varepsilon_{41}$	$3 + \varepsilon_{42}$	$5 + \varepsilon_{43}$	$6 + \varepsilon_{44}$

Table No. (2) Table of allocation cost of neutrosophic values

Where ε_{ij} is the limitation on the costs of assignment and it can be any neighborhood of the values contained in Table No. (1)

To formulate the linear mathematical model we assume:

$$x_{ij} = \begin{cases} 1 & \text{if job } j \text{ was given to machine } i \\ 0 & \text{otherwise} \end{cases} \quad ; i, j = 1, 2, 3, 4$$

Using the problem data, we get the following objective function:

$$Z \in \{(10 + \varepsilon_{11})x_{11} + (9 + \varepsilon_{12})x_{12} + (8 + \varepsilon_{13})x_{13} + (7 + \varepsilon_{14})x_{14} + (3 + \varepsilon_{21})x_{21} + (4 + \varepsilon_{22})x_{22} + (5 + \varepsilon_{23})x_{23} + (6 + \varepsilon_{24})x_{24} + (2 + \varepsilon_{31})x_{31} + (1 + \varepsilon_{32})x_{32} + (1 + \varepsilon_{33})x_{33} + (2 + \varepsilon_{34})x_{34} + (4 + \varepsilon_{41})x_{41} + (3 + \varepsilon_{42})x_{42} + (5 + \varepsilon_{43})x_{43} + (6 + \varepsilon_{44})x_{44}\}$$

Machine terms:

$$x_{11} + x_{12} + x_{13} + x_{14} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$

Business terms:

$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$

$$\boldsymbol{x_{12} + x_{22} + x_{32} + x_{42} = 1}$$
$$\boldsymbol{x_{13} + x_{23} + x_{33} + x_{43} = 1}$$
$$\boldsymbol{x_{14} + +x_{24} + x_{34} + x_{44} = 1}$$

Therefore, the mathematical model is written as follows:
Find the minimum value of the function:

$$Z \in \{(\mathbf{10} + \varepsilon_{11})x_{11} + (\mathbf{9} + \varepsilon_{12})x_{12} + (\mathbf{8} + \varepsilon_{13})x_{13} + (\mathbf{7} + \varepsilon_{14})x_{14} + (\mathbf{3} + \varepsilon_{21})x_{21} + (\mathbf{4} + \varepsilon_{22})x_{22} + (\ \mathbf{5} + \varepsilon_{23})x_{23} + (\mathbf{6} + \varepsilon_{24})x_{24} + (\mathbf{2} + \varepsilon_{31})x_{31} + (\mathbf{1} + \varepsilon_{32})x_{32} + (\mathbf{1} + \varepsilon_{33})x_{33} + (\mathbf{2} + \varepsilon_{34})x_{34} + (\mathbf{4} + \varepsilon_{41})x_{41} + (\mathbf{3} + \varepsilon_{42})x_{42} + (\mathbf{5} + \varepsilon_{43})x_{43} + (\mathbf{6} + \varepsilon_{44})x_{44}\}$$

Within the conditions:

$$\boldsymbol{x_{11} + x_{12} + x_{13} + x_{14} = 1}$$
$$\boldsymbol{x_{21} + x_{22} + x_{23} + x_{24} = 1}$$
$$\boldsymbol{x_{31} + x_{32} + x_{33} + x_{34} = 1}$$
$$\boldsymbol{x_{41} + x_{42} + x_{43} + x_{44} = 1}$$
$$\boldsymbol{x_{11} + x_{21} + x_{31} + x_{41} = 1}$$
$$\boldsymbol{x_{12} + x_{22} + x_{32} + x_{42} = 1}$$
$$\boldsymbol{x_{13} + x_{23} + x_{33} + x_{43} = 1}$$
$$\boldsymbol{x_{14} + +x_{24} + x_{34} + x_{44} = 1}$$

Where $\boldsymbol{x_{ij}}$ it is either equal to zero or one.

Since the number of works is equal to the number of machines, the issue is a standard assignment issue, and the optimal solution can be obtained using several methods, including the Hungarian method in this research.

This method was named after the scientist who created it, a mathematician D.Konig. Its principle depends on finding the total opportunity-cost matrix, references [16, 17].

Explanation of the method based on what was stated in the reference [18]:

This method is based on a mathematical property discovered by the scientist D.Konig,

If the cost is non-negative values, then subtracting or adding a fixed number of elements of any row or any column in the standard allocation cost matrix does not affect the optimal assignment, and specifically does not affect the optimal values $\boldsymbol{x_{ij}}$.

The algorithm begins by identifying the smallest element in each row, and subtracting it from all the elements of the row , or by selecting the smallest element in each column and subtracting it from all the elements of that column, we get a new cost matrix that includes at least one element equal to zero in each row or column. We do the assignment process using cells with a cost equal to zero. If possible, we have obtained the optimal allocation. For this assignment, the cost elements $(\boldsymbol{c_{ij}})$ are non-negative, so the minimum value of the objective function cannot be $\sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{c_{ij}\, x_{ij}}$ is less than zero.

We will use the above to find the optimal assignment for the problem in Example 2 based on the following information:

Taking the indeterminacy $\boldsymbol{\varepsilon}_{ij} = \boldsymbol{\varepsilon} \in [0 ,5]$, the problem becomes:

Example 3:

We want to find the optimal assignment for four jobs on four machines. The cost of assignment is given in the following table:

Business / The machines	$\boldsymbol{N_1}$	$\boldsymbol{N_2}$	$\boldsymbol{N_3}$	$\boldsymbol{N_4}$

M_1	$[10,15]$	$[9,14]$	$[8,13]$	$[7,12]$
M_2	$[3,8]$	$[4,9]$	$[5,10]$	$[6,11]$
M_3	$[2,7]$	$[1,6]$	$[1,6]$	$[2,7]$
M_4	$[4,9]$	$[3,8]$	$[5,10]$	$[6,11]$

Table No. (3) Table of Example data

To formulate the linear mathematical model:
We assume:

$$x_{ij} = \begin{cases} 1 & \textit{if job j was given to machine i} \\ 0 & \textit{otherwise} \end{cases} \qquad ; i,j = 1,2,3,4$$

Using the problem data, we get the following objective function:

$$Z \in \{[10,15]x_{11} + [9,14]x_{12} + [8,13]x_{13} + [7,12]x_{14} + [3,8]x_{21} + [4,9]x_{22} + [5,10]x_{23} + [6,11]x_{24} + [2,7]x_{31} + [1,6]x_{32} + [1,6]x_{33} + [2,7]x_{34} + [4,9]x_{41} + [3,8]x_{42} + [5,10]x_{43} + [6,11]x_{44}\}$$

Machine terms:

$$x_{11} + x_{12} + x_{13} + x_{14} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$

Business terms:

$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$
$$x_{13} + x_{23} + x_{33} + x_{43} = 1$$
$$x_{14} + +x_{24} + x_{34} + x_{44} = 1$$

Therefore, the mathematical model is written as follows:
Find the minimum value of the function:

$$Z \in \{[10,15]x_{11} + [9,14]x_{12} + [8,13]x_{13} + [7,12]x_{14} + [3,8]x_{21} + [4,9]x_{22} + [5,10]x_{23} + [6,11]x_{24} + [2,7]x_{31} + [1,6]x_{32} + [1,6]x_{33} + [2,7]x_{34} + [4,9]x_{41} + [3,8]x_{42} + [5,10]x_{43} + [6,11]x_{44}\}$$

Within the conditions:

$$x_{11} + x_{12} + x_{13} + x_{14} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$
$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$
$$x_{13} + x_{23} + x_{33} + x_{43} = 1$$
$$x_{14} + +x_{24} + x_{34} + x_{44} = 1$$

Where x_{ij} it is either equal to zero or one.

Solution using the Hungarian method:
From the table Number 3 we find the following:

Business / The machines	N_1	N_2	N_3	N_4
M_1	$[10, 15]$	$[9, 14]$	$[8, 13]$	$[7, 12]$
M_2	$[3, 8]$	$[4, 9]$	$[5, 10]$	$[6, 11]$
M_3	$[2, 7]$	$[1, 6]$	$[1, 6]$	$[2, 7]$
M_4	$[4, 9]$	$[3, 8]$	$[5, 10]$	$[6, 11]$

To form the opportunity cost matrix for the rows, we do the following:
We form the opportunity cost matrix for the rows as follows:

In the first row, the lowest cost is $[7, 12]$ we subtract it from all the elements of the first row.

In the second row, the lowest cost is $[3, 8]$, we subtract from all elements of the second row.

In the third row, the lowest cost is $[1, 6]$, we subtract it from all elements of the third row.

In the fourth row, the lowest cost is $[3, 8]$, we subtract from all elements of the fourth row.

We get the opportunity cost matrix for the following rows:

Business / The machines	N_1	N_2	N_3	N_4
M_1	**3**	**2**	**1**	**0**
M_2	**0**	**1**	**2**	**3**
M_3	**1**	**0**	**0**	**1**
M_4	**1**	**0**	**2**	**3**

Table No. (4) Table of Total opportunity cost matrix table

We try to make the assignment using cells with cost equal to zero we find:
Dedicate the machine M_1 to get the job N_4 done.

Dedicate the machine M_2 to get the job N_1 done.

Dedicate the machine M_3 to get the job N_3 done.

Dedicate the machine M_4 to get the job N_2 done.

Thus, we have obtained the optimal assignment and the minimum cost:

$$\begin{aligned} \boldsymbol{Z} \in \{[10, 15] \times 0 + [9, 14] \times 0 + [8, 13] \times 0 + [7, 12] \times 1 + [3, 8] \times 1 + [4, 9] \times 0 + [5, 10] \times 0 \\ + [6, 11] \times 0 + [2, 7] \times 0 + [1, 6] \times 0 + [1, 6] \times 1 + [2, 7] \times 0 + [4, 9] \times 0 \\ + [3, 8] \times 1 + [5, 10] \times 0 + [6, 11] \times 0\} \end{aligned}$$

$$\boldsymbol{Z} \in [7, 12] + [3, 8] + [1, 6] + [3, 8] = [14, 34]$$

That is, the optimal allocation is:
Dedicate the machine M_1 to get the job N_4 done.

Dedicate the machine M_2 to get the job N_1 done.

Dedicate the machine M_3 to get the job N_3 done.

Dedicate the machine M_4 to get the job N_2 done.

The cost:

$$Z \in [14 , 34].$$

The Hungarian method is summarized based on what was stated in the reference [17]:

1- We determine the smallest element in each row and subtract it from the rest of the elements of that row. Thus, we get a new matrix that is the opportunity cost matrix for the rows.

2- We determine the smallest element in each column of the opportunity cost matrix for the rows and Subtract it from the elements of that column. Thus, we get the total opportunity cost matrix.

3- We draw as few horizontal and vertical straight lines as possible to pass through all zero elements of the total opportunity cost matrix.

4- If the number of the straight lines drawn passing through the zero elements is equal to the number of rows (columns). Then we say that we have reached the optimal assignment**.**

5- If the number of straight lines passing through the zero elements is less than the number of rows (Columns).Then we move on to the next step.

6- We choose the lesser element from the elements that no straight line passed through and subtract it from all the elements that no straight line. Then we add it to all the elements that lie at the intersection of two lines. The elements that the straight lines passed through remain the same without any change. We get a new matrix that we call it the modified total opportunity cost matrix.

7- We draw vertical and horizontal straight lines passing through all the zero elements in the modified total opportunity cost matrix. If the number of straight lines drawn passing through the zero elements is equal to the number of rows (columns). Then we have reached the optimal assignment solution.

8- If the number of the lines is not equal to the number of rows (columns). We go back to step (1), we Repeat the previous steps until reaching the optimal assignment that makes the total opportunity cost equal to zero.

Example 4:

We have three machines M_1 , M_2, M_3 and three works N_1, N_2, N_3 and each work is done completely using any of the three machines and in return each machine can perform any of the three works as well. What is required is to allocate these mechanisms to the existing works so that we get the optimal assignment , i.e. the assignment that gives us here the minimum total cost, bearing in mind that the costs of completing these works vary according to the different mechanisms implemented for these works, and this cost is related to the performance of each work and is shown in the following table:

Business / The machines	N_1	N_2	N_3
M_1	$[20, 23]$	$[27, 30]$	$[30, 33]$

M_2	$[10, 13]$	$[18, 21]$	$[16, 19]$
M_3	$[14, 17]$	$[16, 19]$	$[12, 15]$

Table No. (5) Table of allocation cost neutrosophic values example data

Mathematical model:

Find the minimum value of the function:

$$Z \in \{[20, 23]x_{11} + [27, 30]x_{12} + [30, 33]x_{13} + [10, 13]x_{21} + [18, 21]x_{22} + [16, 19]x_{23} + [14, 17]x_{31} + [16, 19]x_{32} + [12, 15]x_{33}\}$$

Within the conditions:

$$x_{11} + x_{12} + x_{13} = 1$$
$$x_{21} + x_{22} + x_{23} = 1$$
$$x_{31} + x_{32} + x_{33} = 1$$
$$x_{11} + x_{21} + x_{31} = 1$$
$$x_{12} + x_{22} + x_{32} = 1$$
$$x_{13} + x_{23} + x_{33} = 1$$

Where x_{ij} it is either equal to zero or one.

Finding the optimal assignment using the Hungarian method:

We take Table No. (5)

Business / The machines	N_1	N_2	N_3
M_1	$[20, 23]$	$[27, 30]$	$[30, 33]$
M_2	$[10, 13]$	$[18, 21]$	$[16, 19]$
M_3	$[14, 17]$	$[16, 19]$	$[12, 15]$

1.

In the first row, the lowest cost is $[20, 23]$, which we subtract from all the elements of the first row.

In the second row, the least cost is $[10, 13]$ and we subtract it from all the elements of the second row.

In the third row, the lowest cost is $[12, 15]$, which we subtract from all the elements of the third row.

We get the opportunity cost table for the following rows:

Business / The machines	N_1	N_2	N_3
M_1	0	7	10
M_2	0	8	6
M_3	2	4	0

Table No. (6) Table of opportunity cost matrix for lines

2.

In the first column, the lowest cost is 0 we subtract it from all the items in the first column.

In the second column, the lowest cost is 4 we subtract it from all the items in the second column.

In the third column, the lowest cost is **4** we subtract it from all the items in the third column.

We get the table:

Business / The machines	N_1	N_2	N_3
M_1	0	3	10
M_2	0	4	6
M_3	2	0	0

Table No. (7) Table of total opportunity cost matrix

3. We draw as few horizontal and vertical straight lines as possible to pass through all zero elements of the total opportunity cost matrix.
4. If the number of straight lines drawn passing through the zero elements is equal to the number of rows (columns), then we say that we have reached the optimal assignment.
5. If the number of straight lines passing through the zero elements is less than the number of rows or columns, then we move on to the third step.

Business / The machines	N_1	N_2	N_3
M_1	0	3	10
M_2	0	4	6
M_3	2	0	0

Table No. (7) Total Opportunity Cost Matrix

We Note that the number of lines is less than the number of rows (columns). So we go to (6).

6.

a. We choose the lowest element through which no straight line has passed. Smallest element is (3).

b. We subtract it from the rest of the elements through which none of the lines drawn are passed.

c. We add it to all the elements at the intersection of two straight lines drawn.

d. Elements through which straight lines pass remain unchanged.

e. We draw vertical and horizontal straight lines passing through all zero elements of the adjusted total opportunity cost matrix, and we get:

Business / The machines	N_1	N_2	N_3
M_1	0	0	7
M_2	0	1	3
M_3	5	0	0

Table No. (8) Modified Total Opportunity Cost Matrix

7. If the number of drawn lines is equal to the number of rows (columns), then we have reached the optimal assignment.

   From Table No. (8), we note that the number of drawn lines is equal to the number of rows, meaning that we have obtained the optimal assignment, which is as follows:

   In the third column, we have zero only in the cell that is M_3N_3, so the third machine is used to perform the third work. $M_3 \to N_3$

   We delete the third row and the third column, we get the following table:

Business / The machines	N_1	N_2
M_1	**0**	**0**
M_2	**0**	**1**

In the same way. The first machine is used to perform the second work, $M_1 \to N_2$.

The second machine used to perform the first work, $M_2 \to N_1$.

The minimum total cost is:

$$Z \in \{[20,23] \times 0 + [27,30] \times 1 + [30,33] \times 0 + [10,13] \times 1 + [18,21] \times 0 + [16,19] \times 0 + [14,17] \times 0 + [16,19] \times 0 + [12,15] \times 1\}$$

$$Z \in [27,30] + [10,13] + [12,15] = [49,58]$$

That is, the optimal assignment is:

Machine M_1 is assigned to do N_2work.

Machine M_2 is assigned to perform N_1work.

Machine M_3 is assigned to perform N_3work.

The cost:

$$Z \in [49,58]$$

Important notes:

When we study the issues of optimal assignment, we come across the following:

1. **There is two types of assignment issues according to the objective function:**

The first type:

It is required to obtain a minimum value for the objective function, knowing that the cost of completing any work by a machine is a known value, and therefore the total cost is as small as possible.

The second type:

What is required is to obtain a maximum value for the objective function, and here it is known that we have the profit accruing from the completion of any work, by a machine, and therefore the total cost is the greatest possible.

In this type, we transform matter to the first type according to the following steps:

a. Multiply the elements of the cost matrix by the value (-1).
b. If some elements of the matrix are negative, we add enough positive numbers to the corresponding rows and columns so that all elements become non-negative.
c. Then the issue becomes a matter of assignment and we want to make the objective function smaller and all elements of the cost matrix are non-negative, so we can apply the Hungarian method.

2. **There are two types of customization issues according to the number of businesses and the number of machines**:

   In this research, we studied the standard assignment issues. It should be noted that there are non-standard assignment issues. In these issues, the number of works is not equal to the number of machines , and here we convert them into standard issues by adding a fictitious work or a fictitious machine and make the cost equal to zero So that the objective function does not change, then we build the mathematical model as it is in the standard models.

Conclusion and results:

Due to the importance of the issue of assignment in our practical life, it received great attention from scholars and researchers, as this issue was addressed on the basis that it is a transfer issue, and special methods were followed to solve transfer issues to find the optimal assignment ,but we find that many references deal with these issues according to the Hungarian method that It was found to be addressed. And through our study of these issues, we find that the use of the Hungarian method greatly helps to find the optimal assignment with less effort than other methods. On the other hand according to the results obtained when using the data Neutrosophic values , in issues of assignment and in all practical matters that are affected by the conditions surrounding the work environment so that decision makers can make appropriate decisions for all circumstances that achieve companies and institutions the greatest profit and the lowest cost.

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Neutrosophic Laplace Distribution with Application in Financial Data Analysis

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Abstract: The Laplace distribution, also known as the double exponential distribution, is a continuous probability distribution that is often used for modelling the data having heavy tails. In this paper, we proposed the Neutrosophic Laplace distribution which is the extension of the classical Laplace Distribution. We derived various statistical properties of the Neutrosophic Laplace Distribution such as mean, variance, skewness, rth moment, quartiles, and moment-generating function. The expressions for the estimation of the parameters are also derived using the maximum likelihood function of the distribution. A simulation study has been done to evaluate the performance of estimates. An application of the Neutrosophic Laplace Distribution is discussed to study the daily return of the NIFTY50 from Indian Stock Market. The analysis shows that the neutrosophic Laplace Model is acceptable, effective, and adequate for dealing with uncertainty in an unpredictable context.

Keywords: Neutrosophic Laplace Distribution, Estimation, Indeterminacy, Financial Data Analysis, Simulation, Stock Returns.

1. Introduction

The continuous random variables have commonly been described and analysed by continuous statistical probability models as they provide a framework for understanding the distribution of continuous data and making probabilistic predictions. These models have numerous applications in the fields such as physics and engineering, quality control and process improvement, environmental analysis, financial modeling, market research and consumer behavior, insurance and actuarial science, demography and epidemiology and reliability engineering. The Laplace Distribution (LD) has gained popularity due to its unique properties and its ability to model various phenomena. The LD arises naturally as the distribution of the difference between two independent random variables follows the exponential distribution, which makes it useful for modeling the behavior of certain stochastic processes. Laplace [1] employed this distribution to model the frequency of an error as an exponential function of its magnitude after the sign was ignored. The Laplace model is most well-suited for modelling the data with outliers or heavy-tailed behaviour. The comprehensive reference book [2] provides a detailed treatment of various continuous distributions, including the Laplace distribution. It covers theoretical aspects, properties, and applications. Everitt and Hand [3] explored the mixture models including the Laplace mixture model which is a combination of LDs. It covers estimation techniques and applications in statistical modelling. Rue et al. [4] discussed the use of the Laplace approximation for Bayesian inference in latent Gaussian models. It introduces the Integrated Nested Laplace Approximation (INLA) methodology, which has become popular in Bayesian statistics. Ghosh and Chaudhuri [5] discussed the Bayesian analysis of regression models with Laplace-distributed errors. It discusses the choice of priors, estimation methods and inference in the context of Laplace regression models. One of the reasons for the popularity of the Laplace distribution in research is its connection to the Laplace transform, which is a mathematical technique used in solving differential equations.

The finance sector plays a vital role in the economy by providing a range of financial services that facilitate the efficient allocation of capital, risk management and economic growth. Uncertainty in financial data refers to the inherent unpredictability and variability observed in financial markets and related economic variables. Fitting uncertain financial data involves developing statistical models or techniques to capture the characteristics and patterns present in such data. This process is essential for understanding and analysing financial variables that exhibit uncertainty such as stock prices, returns, volatility, or option prices. Fuzzy logic, a variant of neutrosophic logic, provides information solely about truth and falsity measures. In contrast, neutrosophic logic, an extension of fuzzy logic, also accounts for the degree of uncertainty. Neutrosophic statistics employ precise numbers to represent data within intervals. Smarandache [6] introduced the concept of neutrosophy to accurately represent and model the inherent indeterminacies present in data. It represents a novel domain in philosophy, serving as an expansion of fuzzy and intuitionistic fuzzy logics [7-11]. Smarandache [12-15] proposed the fundamental principles of neutrosophic sets across multiple domains.

Neutrosophic statistics offers greater flexibility compared to classical statistics. When both data and inference methods are definite, neutrosophic statistics aligns with classical statistics. However, given the prevalence of indeterminate data in real life situations, there is a greater demand for neutrosophic statistical procedures over classical ones. Numerous researchers have introduced highly valuable neutrosophic probability distributions for the analysis of such data sets. Alhasan and Smarandache [16] proposed several distributions under indeterminacy including "neutrosophic Rayleigh distribution, neutrosophic Weibull distribution, neutrosophic five-parameter Weibull distribution, neutrosophic three-parameter Weibull distribution, neutrosophic beta Weibull distribution and neutrosophic inverse Weibull distribution". Aslam [17] introduced the concept of the neutrosophic Raleigh distribution and employed it to model wind speed data. In their work, Alhabib et al. [18] introduced the concept of neutrosophic Uniform, neutrosophic exponential, and neutrosophic Poisson distributions. Khan et al. [19] extended the classical gamma distribution in neutrosophic environment and its application in the complex data analysis.

Albassam et al. [20] discussed the basic properties of the neutrosophic Weibull distribution and its application in the analysis of the wind speed data and LED manufacturing process. They utilized that the neutrosophic Weibull model is suitable, logical, and efficient when applied within an environment characterized by uncertainty. Sherwan et al. [21] extended the beta distribution under neutrosophic environment and proved the several properties for legitimate the proposed distribution. Jdid et al. [22] developed a mathematical model to minimize the inspection costs and demonstrated the study using both classical and neutrosophic values. Sleem et al. [22] described an integrated framework for assessing customer factors and product requirements in VR Metaverse design by merging CRITIC approach with SVNS. Hezam[23] proposed a strategy for machine tool selection using an innovative hybrid MCDM framework under neutrosophic environment.There is a vast body of literature encompassing various statistical distributions that can be utilized to model different kinds of data.

Classical distributions are only applicable when all data observations are exact in nature. However, real-world data is often imprecise, uncertain and have lack of exactness. The applications of existing classical distributions are not suitable for such cases. By an extensive exploration of the literature, no previous research has focused on examining the properties of the Laplace distribution in the context of uncertainty. To fill the research gap, in this paper, we introduced and analysed several properties of the Laplace distribution under conditions of indeterminacy. The Neutrosophic Laplace Distribution (NLD) is an extension of the Laplace Distribution (LD) that incorporates the concept of neutrosophic logic. The maximum likelihood estimation method has been used to estimate the parameters. The effectiveness of obtained estimators is evaluated through a simulation analysis. An application of the proposed distribution is discussed on the financial data analysis. The neutrosophic Laplace model is anticipated to be more effective in modelling stock return data compared to the traditional Laplace distribution used in classical statistics.

2. Neutrosophic Laplace (double exponential) Distribution

A Neutrosophic continuous random variable $z_n = z_l + z_u i_n$ is said to have Neutrosophic Laplace Distribution if it follows the following probability density function

$$f(z_n;\theta_n,\beta_n) = \begin{cases} \frac{1}{2\beta_l} e^{\left(-\frac{|z_l-\theta_l|}{\beta_l}\right)} + \left\{\frac{1}{2\beta_u} e^{\left(-\frac{|z_u-\theta_u|}{\beta_u}\right)}\right\} i_n, & -\infty < z_n < \infty \ ; -\infty < \theta_n < \infty \ , \beta_n > 0 \\ 0 & \end{cases} \quad \textbf{(1)}$$

Here, $\theta_n = \theta_l + \theta_u i_n$ is the location parameter, $\beta_n = \beta_l + \beta_u i_n$ is the scale parameter where $i_n \in (i_l, i_u)$. The shape of the curve of NLD depends upon the value of β_n .

Suppose that $z_l = z_u = z_n$, the pdf can be written as

$$f(z_n;\theta_n,\beta_n) = \left(\frac{1}{2\beta_n} e^{\left(-\frac{|z_n-\theta_n|}{\beta_n}\right)}\right)(1 + i_n) \quad \textbf{(2)}$$

If $i_l = 0$, the NLD will reduce to the classical Laplace distribution.

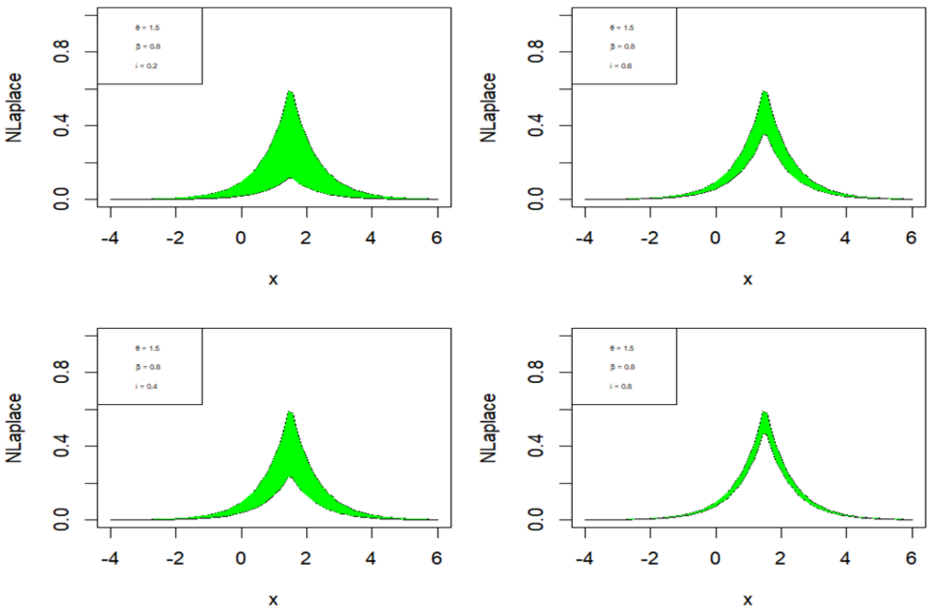

Figure. 1: The pdf graph representing the distribution of NLD with different level of indeterminant scale and shape parameters

The cumulative distribution function (cdf) is

$$F(z_n) = \begin{cases} \frac{1}{2} e^{\left(\frac{z_n-\theta_n}{\beta_n}\right)}(1 + i_n) \ , & z_n < \theta_n \\ 1 - \frac{1}{2} e^{\left(-\frac{(z_n-\theta_n)}{\beta_n}\right)}(1 + i_n) & , z_n \geq \theta_n \end{cases} \quad \textbf{(3)}$$

Where $\theta_n = \theta_l + \theta_u i_n$ is the location parameter, $\beta_n = \beta_l + \beta_u i_n$ is the scale parameter where $i_n \in (i_l, i_u)$.

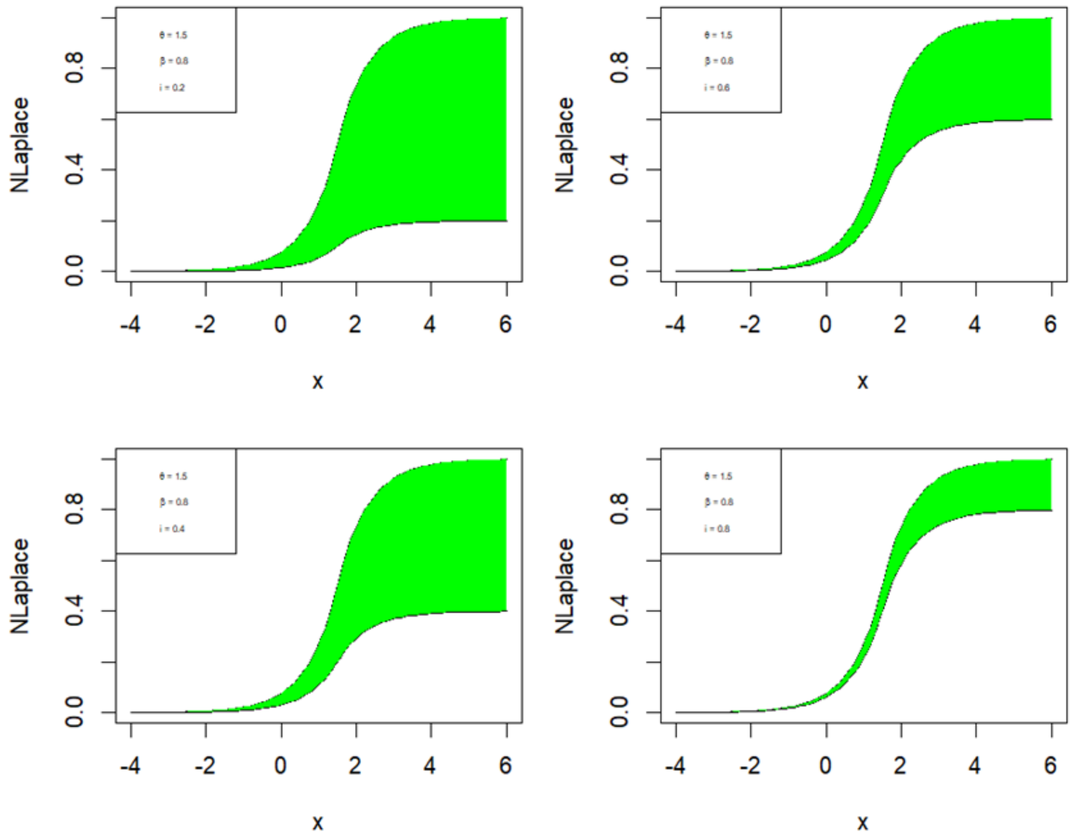

Fig. 2: The cdf plot representing the distribution of NLD with different level of indeterminant scale and shape parameters

3. Statistical Properties

Theorem 1: *Suppose 'z' is a random variable that conforms to the NLD. In that case, the neutrosophic rth moment can be expressed as follows:*

$$E(z_n^r) = \mu_r' = \frac{1}{2}\sum_{k=0}^{r}\binom{r}{k}\beta_n^k\theta_n^{r-k}\{1+(-1)^k\}k!\,(1+i_n), \quad \forall \ k = 0,1,2\ldots r \tag{4}$$

where μ_r' *is neutrosophic* r^{th} *moment of NLD.*

Proof: We know that **(5)**

$$E(z_n^r) = \int_{-\infty}^{\infty} z_n^r f(z_n) dz_n$$

$$E(z_n^r) = \int_{-\infty}^{\infty} z_n^r \left(\frac{1}{2\beta_n} e^{\left(-\frac{|z_n-\theta_n|}{\beta_n}\right)}\right)(1+i_n)dz_n$$

Putting $y = \frac{|z_n-\theta_n|}{\beta_n}$, we get

$$E(z_n^r) = \frac{1}{2}\int_{-\infty}^{\infty}(y\beta_n+\theta_n)^r\left(e^{-|y|}\right)(1+i_n)dy$$

$$E(z_n^r) = \frac{1}{2}\int_{-\infty}^{\infty}\left(\sum_{k=0}^{r}\binom{r}{k}(y\beta_n)^k(\theta_n)^{r-k}\right)\left(e^{-|y|}\right)(1+i_n)dy$$

We get the following expression after some algebraic simplification.

$$E(z_n^r) = \mu_r' = \frac{1}{2}\sum_{k=0}^{r}\binom{r}{k}\beta_n^k\theta_n^{r-k}\{1+(-1)^k\}k!\,(1+i_n) \quad \textbf{(6)}$$

The first four moments of the NLD is given by:

$$\mu_1' = E(z_n) = \theta_n(1+i_n) \quad \textbf{(7)}$$

$$\mu_2' = E(z_n^2) = (\theta_n^2 + 2\beta_n^2)(1+i_n) \quad \textbf{(8)}$$

$$\mu_3' = E(z_n^3) = (\theta_n^3 + 6\theta_n\beta_n^2)(1+i_n) \quad \textbf{(9)}$$

$$\mu_4' = E(z_n^4) = (\theta_n^4 + 12\theta_n^2\beta_n^2 + 24\beta_n^4)(1+i_n) \quad \textbf{(10)}$$

The NLD's mean, variance, skewness, and kurtoses are expressed as:

$$\text{Neutrosphic Mean} = \theta_n(1+i_n) \quad \textbf{(11)}$$

$$\text{Neutrosphic Variance} = (\theta_n^2 + 2\beta_n^2)(1+i_n) - \left(\theta_n(1+i_n)\right)^2 \quad \textbf{(12)}$$

$$\text{Skewness} = \frac{\mu_3^2}{\mu_2^3} = \frac{\mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3}{\mu_2' - {\mu_1'}^2} = 0 \quad \textbf{(13)}$$

$$\text{Kurtosis} = \frac{\mu_4}{\mu_2^2} = 6 \quad \textbf{(14)}$$

Theorem 2: *The first, second and third quartile of the NLD is given by* $Q_{n_1} = \theta_n + \beta_n.\log_e 0.5\,(1+i_n)^{-1}$,
$Q_{n_2} = \theta_n(1+i_n)$,$Q_{n_3} = \theta_n - \beta_n.\log_e 0.25\,(1+i_n)^{-1}$ *respectively.*

Proof: We know that, $F(Q_{n_i}) = P(z_n \le Q_{n_i}) = \frac{i}{4}$ where $i = 1,2,3$

First quartile $(Q_{n_1} < \theta_n)$ is give by

$$F(Q_{n_1}) = \frac{1}{2}e^{\left(\frac{Q_{n_1}-\theta_n}{\beta_n}\right)}(1+i_n) = \frac{1}{4}$$

$$e^{\left(\frac{Q_{n_1}-\theta_n}{\beta_n}\right)}(1+i_n) = \frac{1}{2}$$

$$\left(\frac{Q_{n_1}-\theta_n}{\beta_n}\right)(1+i_n) = \log_e 0.5$$

$$Q_{n_1} = \theta_n + \beta_n.\log_e 0.5\,(1+i_n)^{-1} \quad \textbf{(15)}$$

Second quartile $(Q_{n_2} \ge \theta_n)$ is give by

$$F(Q_{n_2}) = 1 - \frac{1}{2}e^{\left(-\frac{(Q_{n_2}-\theta_n)}{\beta_n}\right)}(1+i_n) = \frac{2}{4}$$

$$Q_{n_2} = \theta_n(1+i_n) \quad \textbf{(16)}$$

Second quartile $(Q_{n_3} \ge \theta_n)$ is give by

$$F(Q_{n_3}) = 1 - \frac{1}{2}e^{\left(-\frac{(Q_{n_3}-\theta_n)}{\beta_n}\right)}(1+i_n) = \frac{3}{4}$$

$$e^{\left(-\frac{(Q_{n_2}-\theta_n)}{\beta_n}\right)}(1+i_n) = \frac{1}{4}$$

$$\left(\frac{Q_{n_3}-\theta_n}{\beta_n}\right)(1+i_n) = \log_e 0.25$$

$$Q_{n_3} = \theta_n - \beta_n.\log_e 0.25\,(1+i_n)^{-1} \quad \textbf{(17)}$$

where $\theta_n$is the location parameter, β_n is the scale parameter and $i_n \in (i_l, i_u\ )\ is\ indeterminancy.$

Theorem 3: *The moment generating function of NLD is*

$$M_{z_n}(t) = \frac{e^{\theta_n t}}{(1-\beta_n^2 t^2)} \qquad for\ \ |t| < \frac{1}{\beta_n}$$

Where $M_{z_n}(t)$ = *moment generating function*

Proof : We know that

$$M_{z_n}(t) = \int_{-\infty}^{\infty} e^{tz_n} f(z_n) dz_n \qquad (18)$$

$$M_{z_n}(t) = \int_{-\infty}^{\infty} e^{t(z_n-\theta_n+\theta_n)} \left(\frac{1}{2\beta_n} e^{\left(-\frac{|z_n-\theta_n|}{\beta_n}\right)}\right)(1+i_n)dz_n$$

$$M_{z_n}(t) = \frac{e^{t\theta_n}}{2\beta_n}\int_{-\infty}^{\infty} e^{t(z_n-\theta_n)} \left(e^{\left(-\frac{|z_n-\theta_n|}{\beta_n}\right)}\right)(1+i_n)dz_n$$

Let $(z_n - \theta_n) = u$ such that $z_n = du$

$$M_{z_n}(t) = \frac{e^{t\theta_n}}{2\beta_n}\int_{-\infty}^{\infty} e^{tu} e^{\left(-\frac{|u|}{\beta_n}\right)}(1+i_n)du$$

$$M_{z_n}(t) = \frac{e^{t\theta_n}}{2\beta_n}\int_{-\infty}^{0} e^{\left(t+\frac{1}{\beta_n}\right)u}(1+i_n)du + \int_{0}^{\infty} e^{-\left(t-\frac{1}{\beta_n}\right)u}(1+i_n)du$$

$$M_{z_n}(t) = \frac{e^{t\theta_n}}{2\beta_n}\left\{\left[\frac{\beta_n}{1+t} - 0\right] + \left[0 - \frac{\beta_n}{1-t}\right]\right\}(1+i_n)$$

$$M_{z_n}(t) = \frac{e^{t\theta_n}(1+i_n)}{1-\beta_n^2 t^2} \qquad for\ \ |t| < \frac{1}{\beta_n}\ \ , 0 \le i_n \le 1 \qquad (19)$$

4. Parameter estimation and simulation

The maximum likelihood approach can be utilized to measure the parameters of the NLD. The likelihood function can be expressed as follows:

$$\prod_{k=1}^{n} f(z_{kn}) = \prod_{i=1}^{N}\left[\frac{1}{2\beta_n} e^{\left(-\frac{|z_{kn}-\theta_n|}{\beta_n}\right)} + \left\{\frac{1}{2\beta_n} e^{\left(-\frac{|z_{kn}-\theta_n|}{\beta_n}\right)}\right\} i_n\right] \qquad (20)$$

The log likelihood function is given by

$$L(\theta_n, \beta_n, i_n) = \prod_{i=1}^{N} \log\left[\frac{1}{2\beta_n} e^{\left(-\frac{|z_{kn}-\theta_n|}{\beta_n}\right)}\right](1+i_n)$$

$$L(\theta_n, \beta_n, i_n) = log\left(\left(\frac{1}{2\beta_n}\right)^N \exp\left(-\frac{\sum_{k=1}^{N}|z_{kn}-\theta_n|}{\beta_n}\right)\right)(1+i_n)^N$$

$$L(\theta_n, \beta_n, i_n) = -Nlog(2) - Nlog(\beta_n) - \frac{\sum_{k=1}^{N}|z_{kn}-\theta_n|}{\beta_n} + N(1+i_n)$$

By differentiating the $L(\theta_n, \beta_n, i_n)$ w.r.t. the parameters

We have

$$\frac{\partial L}{\partial \beta_n} = -\frac{N}{\beta_n} + \frac{\sum_{k=1}^{N}|z_{kn}-\theta_n|}{\beta_n^2} \qquad (21)$$

$$\frac{\partial L}{\partial \theta_n} = \frac{\sum_{k=1}^{N}(z_{kn}-\theta_n)}{\beta_n|z_{kn}-\theta_n|} \qquad (22)$$

The derived MLE of θ_n and β_n are $\widetilde{z_{kn}}$ i.e., median of the k neutrosophic observations and $\frac{1}{N}\sum_{k=1}^{N}|z_{kn} - \theta_n|$ respectively. Here, $\theta_n$is the location parameter, β_n is the scale parameter and $i_n \in (i_l, i_u\ )$ is indeterminacy.

Now, we presented a simulation analysis to assess the accuracy of the estimates. To conduct the simulation, we generate N=10000 random samples from the NLD with varying sizes, namely n = 30, 50, 100, 200, 300 and 1000.The Table 1 shows the average estimates (AEs) and mean square errors (MSEs) of $\widehat{\theta_n}$ and $\widehat{\beta_n}$. R studio software (version 2023.03.1+446) is used to generate the numerical findings.

Table 1. Results obtained from simulating the NLD estimates

Sample	Actual Value			Average estimates		Mean square Error	
n	$\boldsymbol{\theta_n}$	$\boldsymbol{\beta_n}$	$\boldsymbol{i_n}$	$\widehat{\boldsymbol{\theta_n}}$	$\widehat{\boldsymbol{\beta_n}}$	$\widehat{\boldsymbol{\theta_n}}$	$\widehat{\boldsymbol{\beta_n}}$
30	0.4	0.8	0	0.6392	1.0102	0.0612	0.1844
50				0.3733	0.9445	0.0350	0.1336
100				0.2972	0.8075	0.0259	0.0807
200				0.3961	0.8056	0.0105	0.0577
30	0.6	1.0	0.2	0.8319	1.6266	0.1958	0.2970
50				0.7526	1.0042	0.0732	0.1420
100				0.7295	1.0023	0.0959	0.1023
300				0.7100	1.0013	0.0380	0.065
30	1.0	1.0	0.3	1.4783	1.3674	0.1834	0.1789
50				1.3877	0.1797	0.1001	0.1668
100				1.2427	1.0713	0.0809	0.1471
300				1.1765	1.0652	0.0568	0.0753
30	3.0	5.0	0.5	6.5462	8.8494	0.6397	1.6157
50				6.1257	7.9343	0.5596	1.1221
100				6.1292	7.2815	0.4966	0.7282
300				4.1914	7.0772	0.2884	0.4086
1000				3.1789	7.0376	0.1965	0.2873

The simulation finding in table 1 shows that as sample size increases, the difference between the actual and estimated scale and shape parameters decreases i.e. (average bias reduces). It indicates that the compatibility between practice and theory improves as the sample size increases and the mean square errors of the estimators decreases. The resulting estimators are clearly asymptotically consistent and the MLE of the parameters performs worthily and provides asymptotically exact and correct results.

5. Application and Comparative Analysis

The Laplace Distribution is commonly used in finance to model asset returns. However, the financial data often exhibit indeterminacy and fat tails which means extreme events occur more frequently. The Neutrosophic Laplace Distribution's characteristics of handling indeterminacy and heavy tails in data makes it a suitable choice for modelling. Stock returns are influenced by various factors such as economic conditions, market sentiment, company-specific news, geopolitical events and investor behaviour which introduce uncertainty into the stock market. We have considered the data of daily returns (in %) of NIFTY50 from Indian Stock market. The dataset contains 827 observations i.e. (from 01-01-2020 to 28-03-2023). However, due to uncertainties and incomplete information, we applied the NLD to capture the indeterminacy and ambiguity associated with the returns. We also compared the fitness of distribution of the returns using the LD and NLD. The statistical summary of the data is given in table 2.

Table 2. The descriptive statistics of the daily returns of NIFTY50

Min	1st Q	Median	Mean	3rd Q	MAD	Var	Skewness	Kurtosis	Max
-6.818	-0.571	-0.0068	-0.0795	0.4508	0.718	1.1703	0.725	13.27992	9.306

The Bayesian Information Criterion (BIC) and Akaike Information Criterion (AIC) are model selection criteria used to compare the goodness of fit and complexity of different statistical models, including the Laplace distribution. Both AIC and BIC are calculated based on the likelihood function and the number of parameters in the model. The formulas for AIC and BIC are as follows:

$$AIC = -2 * logL + 2k$$

$$BIC = -2 * logL + klog(N)$$

Where:

$logL$: The logarithm of the likelihood functions of the model.

k: The number of parameters in the model.

n: The sample size.

Lower values of AIC and BIC indicate a better balance between model fit and complexity. The model with the lower AIC or BIC is generally preferred as it suggests a better trade-off between goodness of fit and model complexity. It is important to note that the values of AIC and BIC are not specific to the LD but can be applied to compare the models in general. The loglikelihood estimated value along with AIC and BIC corresponding to indeterminacy value (i_n) is given in table 3.

Table 3: The MLE, AIC and BIC measure of the daily returns of NIFTY50

	i_n	$logL$	AIC	BIC
LD	0	-1122.67	2249.34	2258.78
	0.1	-1043.85	2091.70	2101.13
	0.2	-971.89	1947.78	1957.22
	0.3	-905.69	1815.39	1824.82
	0.4	-844.41	1692.81	1702.25
	0.5	-787.35	1578.70	1588.14
NLD	0.6	-733.98	1471.95	1481.39
	0.7	-683.84	1371.68	1381.12
	0.8	-636.57	1277.14	1286.58
	0.9	-591.86	1187.71	1197.15
	1	-549.44	1102.87	1112.31

The goodness-of-fit metrics and MLEs for the classical LD and the NLD with varying indeterminacy parameter values are shown in table 3. In terms of goodness of fit, the neutrosophic Laplace distribution exceeds the standard Laplace distribution. The indeterminacy parameter is found to have a considerable impact on fitting quality. The AIC and BIC along with log likelihood values decreases as change in the value of indeterminacy parameter. The NLD fits better in the daily return of the financial data of NIFTY50 as compare to the classical LD.

6. Conclusion

Here, a neutrosophic Laplace distribution has been introduced as a generalization of the classical Laplace distribution by considering the interval form of data commonly encountered in real-life scenarios. We investigated various properties of the proposed distribution such as rth moment, mean, variance, skewness, kurtosis, first four moments, moment generating function and quartiles. The maximum likelihood estimation approach is employed to estimate the parameters and the performance of these estimators is evaluated via a simulation study. An application of the proposed distribution is discussed on the financial data analysis. From the comparative analysis, the indeterminacy parameter is found to have a considerable impact on fitting quality. The AIC and BIC along with log likelihood values decreases as change in the value of indeterminacy parameter. Therefore, it is concluded that the Neutrosophic Laplace Distribution fits better in the daily return of the financial data of NIFTY50 as compared to the classical Laplace Distribution. In future, this work could be extended for some other continuous distributions and mixture distributions.

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New Types of Topologies and Neutrosophic Topologies (Improved Version)

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Abstract: In this paper we recall the six new types of topologies, and their corresponding topological spaces, that we introduced in the last years (2019-20223), such as: NeutroTopology, AntiTopology, Refined Neutrosophic Topology, Refined Neutrosophic Crisp Topology, SuperHyperTopology, and Neutrosophic SuperHyperTopology.

The n[th]-PowerSets $P^n(H)$ and $P_*^n(H)$, that the SuperHyperTopology and respectively Neutrosophic SuperHyperTopology are based on, better describe our real world, since a system H (that may be a set, company, institution, country, region, etc.) is organized in sub-systems, which in their turn are organized each in sub-sub-systems, and so on.

Keywords: Classical Topology; Topological Space; NeutroTopology; AntiTopology; Refined Neutrosophic Topology; Refined Neutrosophic Crisp Topology; SuperHyperTopology; Neutrosophic SuperHyperTopology.

1. Introduction

We recall the classical definition of Topology, then the procedures of NeutroSophication and respectively AntiSophication of it, that result in adding in two new types of topologies: NeutroTopology and respectively AntiTopology.

Then we define topology on Refined Neutrosophic Set (2013), Refined Neutrosophic Crisp Set [3]. Afterwards, we extend the topology on the framework of SuperHyperAlgebra [6].

The corresponding neutrosophic topological spaces are presented.

This research is an improvement of paper [7].

2. *Classical Topology*

Let $\mathcal{U}$ be a non-empty set, and $P(\mathcal{U})$ the power set of $\mathcal{U}$.

Let $\tau \subseteq P(\mathcal{U})$ be a family of subsets of $\mathcal{U}$.

Then τ is called a Classical Topology on $\mathcal{U}$ if it satisfies the following axioms:

(CT-1) ϕ and $\mathcal{U}$ belong to τ.

(CT-2) The intersection of any finite number of elements in τ is in τ.

(CT-3) The union of any finite or infinite number of elements in τ is in τ.

All three axioms are totally (100%) true (or T = 1, I = 0, F = 0). We simply call them (classical) *Axioms*.

Then ($\mathcal{U}$, τ) is called a *Classical Topological Space* on $\mathcal{U}$.

3. *NeutroSophication of the Topological Axioms*

NeutroSophication of the topological axioms means that the axioms become partially true, partially indeterminate, and partially false. They are called *NeutroAxioms*.

(NCT-1) Either $\{\phi \notin \tau \text{ and } \mathcal{U} \in \tau\}$, or $\{\phi \in \tau \text{ and } \mathcal{U} \notin \tau\}$.

(NCT-2) There exist a finite number of elements in τ whose intersection belong to τ (degree of truth *T*); and a finite number of elements in τ whose intersection is indeterminate (degree of indeterminacy *I*); and a finite number of elements in τ whose intersection does not belong to τ (degree of falsehood *F*); where (*T, I, F*) $\notin \{(1,0,0), (0,0,1)\}$ since (1, 0, 0) represents the above Classical Topology, while (0, 0, 1) the below AntiTopology.

(NCT-3) There exist a finite or infinite number of elements in τ whose union belongs to τ (degree of truth *T*); and a finite or infinite number of elements in τ whose union is indeterminate (degree of indeterminacy *I*); and a finite or infinite number of elements in τ whose union does not belong to τ (degree of falsehood *F*); where of course (*T, I, F*) $\notin \{(1,0,0), (0,0,1)\}$.

4. *AntiSophication of the Topological Axioms*

AntiSophication of the topological axioms means to negate (anti) the axioms, the axioms become totally (100%) false (or T = 0, I = 0, F = 1). They are called *AntiAxioms*.

(ACT-1) $\phi \notin \tau$ and $\mathcal{U} \notin \tau$.

(ACT-2) The intersection of any finite number ($n \geq 2$) of elements in τ is not in τ.

(ACT-3) The union of any finite or infinite number ($n \geq 2$) of elements in τ is not in τ.

5. Neutrosophic Triplets related to Topology

As such, we have a neutrosophic triplet of the form:

<Axiom(1, 0, 0), NeutroAxiom(*T, I, F*), AntiAxiom(0, 0, 1)>,

where (*T, I, F*) $\neq$ (1, 0, 0) and (*T, I, F*) $\neq$ (0, 0, 1).

Correspondingly, one has:

<Topology, NeutroTopology, AntiTopology>.

Therefore, in general:

(Classical) Topology is a topology that has all axioms totally true. We simply call them *Axioms*.

NeutroTopology is a topology that has at least one *NeutroAxiom* and the others are all *classical Axioms* [therefore, no AntiAxiom].

AntiTopology is a topology that has one or more *AntiAxioms*, no matter what the others are (*classical Axioms*, or *NeutroAxioms*).

6. Theorem on the number of Structures/NeutroStructures/AntiStructures

If a Structure has *m* axioms, with $m \geq 1$, then after NeutroSophication and AntiSophication one obtains 3^m types of structures, categorized as follows:

1 *Classical Structure* + ($2^m - 1$) *NeutroStructures* + ($3^m - 2^m$) *AntiStructures*.

7. Consequence on the number of Topologies/NeutroTopologies/AntiTopologies

As a particular case of the previous theorem, from a Topology which has $m = 3$ axioms, one makes, after NeutrosSophication and AntiSophication, $3^3 = 27$ types of structures, as follows: 1 classical Topology, $2^3 - 1 = 7$ NeutroTopologies, and $3^3 - 2^2 = 19$ AntiTopologies.

1 *Classical Topology* + 7 *NeutroTopologies* + 19 *AntiTopologies*

are presented below:

There is 1 (one) type of Classical Topology, whose axioms are listed below:

1 *Classical Topology*

$$\begin{pmatrix} CT-1 \\ CT-2 \\ CT-3 \end{pmatrix}$$

8. Definition of NeutroTopology [4, 5]

It is a topology that has at least one topological axiom which is partially true, partially indeterminate, and partially false,

or *(T, I, F)*, where T = True, *I* = Indeterminacy, *F* = False,

and no topological axiom is totally false,

in other words: $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$, where *(1, 0, 0)* represents the classical Topology, while *(0, 0, 1)* represents the below AntiTopology.
Therefore, the NeutroTopology is a topology in between the classical Topology and the AntiTopology.

There are 7 types of different NeutroTopologies, whose axioms, for each type, are listed below:

7 *NeutroTopologies*

$$\begin{pmatrix} NCT-1 \\ CT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ CT-2 \\ NCT-3 \end{pmatrix},$$

$$\begin{pmatrix} NCT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ CT-2 \\ NCT-3 \end{pmatrix},$$

$$\begin{pmatrix} NCT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}.$$

9. Definition of AntiTopology [4, 5]

It is a topology that has at least one topological axiom that is 100% false (T, I, F) = (0, 0, 1).
The NeutroTopology and AntiTopology are particular cases of NeutroAlgebra and AntiAlgebra [4] and, in general, they all are particular cases of the NeutroStructure and AntiStructure respectively, since we consider "Structure" in any field of knowledge [5].

There are 19 types of different AntiTopologies, whose axioms, for each type, are listed below:

19 *AntiTopologies*

$$\begin{pmatrix} ACT-1 \\ CT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} ACT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} ACT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ CT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ CT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}.$$

10. Refined Neutrosophic Set

Let U be a universe of discourse, and a non-empty subset *R* of it,

$$R = \left\{ \begin{array}{c} x\left(T_1(x), T_2(x), \ldots, T_p(x)\right); \\ \left(I_1(x), I_2(x), \ldots, I_r(x)\right); \\ \left(F_1(x), F_2(x), \ldots, F_s(x)\right); \end{array} \right\}$$

with all $T_j, I_k, F_l \in [0,1]$, $1 \le j \le p, 1 \le k \le r, 1 \le l \le s$, and no restriction on their sums $0 \le T_m + I_m + F_m \le 3$, with $1 \le m \le \max\{p, r, s\}$, where *p, r, s* $\geq$ 0 are fixed integers, and at least one of them is $\geq$ *2,* in order to ensure the refinement (sub-parts) or multiplicity (multi-parts) – depending on the application, of at least one neutrosophic component amongst *T* (truth), *I* (indeterminacy), *F* (falsehood); **and of course** $x \in \mathcal{U}$.

By notation we consider that index zero means the empty-set, i.e. $T_0 = I_0 = F_0 = \phi$ (or zero), and the same for the missing sub-parts (or multi-parts).
For example, the below (2,3,1)-Refined Neutrosophic Set is identical to a (3,3,3)-Refined Neutrosophic Set: $(T_1, T_2; I_1, I_2, I_3; F_1) \equiv (T_1, T_2, 0; I_1, I_2, I_3; F_1, 0, 0)$,

where the missing components T_3, and F_2, F_3 were replaced each of them by 0 (zero)
R is called a *(p, r, s)-refined neutrosophic set* { or *(p, r, s)-RNT* }.

The neutrosophic set has been extended to the Refined Neutrosophic Set (Logic, and Probability) by Smarandache [1] in 2013, where there are multiple parts of the neutrosophic components, as such T was split into subcomponents $T_1, T_2, \ldots, T_p$, and I into $I_1, I_2, \ldots, I_r$, and F into $F_1, F_2, \ldots, F_s$, with $p + r + s = n \ge 2$ and integers p, r, s ≥ 0 and at least one of them is ≥ 2 in order to ensure the refinement (or multiplicity) of at least one neutrosophic component amongst T, I, and F.

Even more: the subcomponents T_j, I_k, and/or F_l can be countable or uncountable infinite subsets of [0, 1].

This definition also includes the *Refined Fuzzy Set*, when $r = s = 0$ and $p \geq 2$;

and the definition of the *Refined Intuitionistic Fuzzy Set*, when $r = 0$, and either $p \geq 2$ and $s \geq 1$, or $p \geq 1$ and $s \geq 2$.

All other fuzzy extension sets (Pythagorean Fuzzy Set, Spherical Fuzzy Set, Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, etc.) can be refined/multiplicated in a similar way.

11. Definition of Refined Neutrosophic Topology

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{P}(\mathcal{U})$ be the family of all (*p, r, s*)-refined neutrosophic subsets of $\mathcal{U}$.

Let $\tau_{RNT} \subseteq \mathcal{P}(\mathcal{U})$ be a family of (*p, r, s*)-refined neutrosophic subsets of $\mathcal{U}$.

Then τ_{RNT} is called a *Refined Neutrosophic Topology* (*RNT*) if it satisfies the axioms:

(RNT-1) ϕ and $\mathcal{U}$ belong to τ_{RNT};

(RNT-2) The intersection of any finite number of elements in τ_{RNT} is in τ_{RNT};

(RNT-3) The union of any finite or infinite number of elements in τ_{RNT} is in τ_{RNT}.

Then $(\mathcal{U}, \tau_{RNT})$ is called a Refined Neutrosophic Topological Space on $\mathcal{U}$.

The *Refined Neutrosophic Topology* is a topology defined on a Refined Neutrosophic Set.

{Similarly, the Refined Fuzzy Topology is defined on a Refined Fuzzy Set, while the Refined Intuitionistic Fuzzy Topology is defined on a Refined Intuitionistic Fuzzy Set, etc.

And, as a generalization, on any type of fuzzy extension set [such as: Pythagorean Fuzzy Set, Spherical Fuzzy Set, Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, etc.] one can define a corresponding fuzzy extension topology.}

12. Neutrosophic Crisp Set

The *Neutrosophic Crisp Set* was defined by Salama and Smarandache in 2014 and 2015.
Let X be a non-empty fixed space. And let *D* be a Neutrosophic Crisp Set [2], where
D = <*A, B, C*>, with *A, B, C* as subsets of *X*.

Depending on the intersections and unions between these three sets *A, B, C* one gets several:
Types of Neutrosophic Crisp Sets [2, 3]
The object having the form $D = <A, B, C>$ is called:

(a) A neutrosophic crisp set of Type 1 (NCS-Type1) if it satisfies:

$A\cap B = B\cap C = C\cap A = \phi$ (empty set).

(b) A neutrosophic crisp set of Type 2 (NCS-Type2) if it satisfies:

$A\cap B = B\cap C = C\cap A = \phi$ and $A\cup B \cup C = X$.

(c) A neutrosophic crisp set of Type 3 (NCS-Type3) if it satisfies:

$A\cap B \cap C = \phi$ and $A\cup B \cup C = X$.

Of course, more types of Neutrosophic Crisp Sets may be defined by modifying the intersections and unions of the subsets *A, B*, and *C*.

13. Refined Neutrosophic Crisp Set

The ***Refined Neutrosophic Crisp Set*** [3] was introduced by Smarandache in 2019, by refining/multiplication of *D* (and denoting it by *RD* = Refined D) by refining/multiplication of its sets *A, B, C* into sub-subsets/multi-sets as follows:

$RD = (A_1, ..., A_p; B_1, ..., B_r; C_1, ..., C_s)$, with $p, r, s \geq 1$ be positive integers and at least one of them be ≥ 2 in order to ensure the refinement/multiplication of at least one component amongs A, B, C, where

$$A = \bigcup_{i=1}^{p} A_i, B = \bigcup_{j=1}^{r} B_j, C = \bigcup_{k=1}^{s} C_k$$

and many types of Refined Neutrosophic Crisp Sets may be defined by modifying the intersections or unions of the subsets/multisets $A_i, B_j, C_k, 1 \leq i \leq p, 1 \leq j \leq r, 1 \leq k \leq s$,

depending on each application.

14. Definition of Refined Neutrosophic Crisp Topology

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{P}(\mathcal{U})$ be the family of all (*p, r, s*)-refined neutrosophic crisp subsets of $\mathcal{U}$.

Let $\tau_{RNCT} \subseteq \mathcal{P}(\mathcal{U})$ be a family of (*p, r, s*)-refined neutrosophic crisp subsets of $\mathcal{U}$.

Then τ_{RNCT} is called a *Refined Neutrosophic Crisp Topology* (*RNCT*) if it satisfies the axioms:

(RNCT-1) ϕ and $\mathcal{U}$ belong to τ_{RNCT};

(RNCT-2) The intersection of any finite number of elements in τ_{RNCT} is in τ_{RNCT};

(RNCT-3) The union of any finite or infinite number of elements in τ_{RNCT} is in τ_{RNCT}.

Then $(\mathcal{U}, \tau_{RNCT})$ is called a Refined Neutrosophic Crisp Topological Space on $\mathcal{U}$.

Therefore, the *Refined Neutrosophic Crisp Topology* is a topology defined on the Refined Neutrosophic Crisp Set.

15. SuperHyperOperation

We recall our 2016 concepts of SuperHyperOperation, SuperHyperAxiom, SuperHyperAlgebra, and their corresponding Neutrosophic SuperHyperOperation Neutrosophic SuperHyperAxiom and Neutrosophic SuperHyperAlgebra [6].

Let $P_*^n(H)$ be the n^th^-powerset of the set *H* such that none of *P(H)*, $P^2(H)$, …, $P^n(H)$ contain the empty set ϕ .

Also, let $P_n(H)$ be the n^th^-powerset of the set *H* such that at least one of the *P(H)*, $P^2(H)$, …, $P^n(H)$ contain the empty set ϕ . For any subset A, we identify {A} with A.

The SuperHyperOperations are operations whose codomain is either $P_*^n(H)$ and in this case one has classical-type SuperHyperOperations, or $P^n(H)$ and in this case one has Neutrosophic SuperHyperOperations, for integer $n \geq 2$.

16. The n^th^-PowerSet better describe our real world

The n^th^-PowerSets $P^n(H)$ and $P_*^n(H)$, that the SuperHyperTopology and respectively Neutrosophic SuperHyperTopology are based on, better describe our real world, since a system *H* (that may be a set, company, institution, country, region, etc.) is organized in sub-systems, which in their turn are organized each in sub-sub-systems, and so on.

17. SuperHyperAxiom

A classical-type SuperHyperAxiom or more accurately a *(m, n)*-SuperHyperAxiom is an axiom based on classical-type SuperHyperOperations.

Similarly, a Neutrosophic SuperHyperAxiom {or Neutrosphic (m, n)-SuperHyperAxiom} is an axiom based on Neutrosophic SuperHyperOperations.

There are:

- Strong SuperHyperAxioms, when the left-hand side is equal to the right-hand side as in non-hyper axioms,
- and Week SuperHyperAxioms, when the intersection between the left-hand side and the right-hand side is non-empty.

18. SuperHyperAlgebra and SuperHyperStructure

A SuperHyperAlgebra or more accurately *(m-n)*-SuperHyperAlgebra is an algebra dealing with SuperHyperOperations and SuperHyperAxioms.

Again, a Neutrosophic SuperHyperAlgebra {or Neutrosphic (m, n)-SuperHyperAlgebra} is an algebra dealing with Neutrosophic SuperHyperOperations and Neutrosophic SuperHyperOperations.

In general, we have SuperHyperStructures {or *(m-n)*-SuperHyperStructures}, and corresponding Neutrosophic SuperHyperStructures.

For example, there are SuperHyperGrupoid, SuperHyperSemigroup, SuperHyperGroup, SuperHyperRing, SuperHyperVectorSpace, etc.

19. Distinction between SuperHyperAlgebra vs. Neutrosophic SuperHyperAlgebra

i. If none of the power sets $P^k(H)$, $1 \le k \le n$, do not include the empty set ϕ, then one has a classical-type SuperHyperAlgebra;

ii. If at least one power set, $P^k(H)$, $1 \le k \le n$, includes the empty set ϕ, then one has a Neutrosophic SuperHyperAlgebra.

20. Definition of SuperHyperTopology (SHT) [6]

It is a topology designed on the n[th]-PowerSet of a given non-empty set H, that excludes the empty-set, denoted as $P_*^n(H)$, built as follows:

$P_*(H)$ is the first powerset of the set *H*, and the index $*$ means without the empty-set (Ø);

$P_*^2(H) = P_*(P_*(H))$ is the second powerset of *H* (or the powerset of the powerset of *H*), without the empty-sets; and so on,

the *n*-th powerset of *H*,

$P_*^n(H) = P_*(P_*^{n-1}(H)) = \underbrace{P_*(P_*(...P_*(H)...))}_{n}$, where P_* is repeated *n* time ($n \ge 2$), and without the empty-sets.

Let consider τ_{SHT} be a family of subsets of $P_*^n(H)$.

Then τ_{SHT} is called a Neutrosophic SuperHyperTopology on $P_*^n(H)$, if it satisfies the following axioms:

(SHT-1) ϕ and $P_*^n(H)$ belong to τ_{SHT}.

(SHT-2) The intersection of any finite number of elements in τ_{SHT} is in τ_{SHT}.

(SHT-3) The union of any finite or infinite number of elements in τ_{SHT} is in τ_{SHT}.

Then $(P_*^n(H), \tau_{SHT})$ is called a SuperHyperTopological Space on $P_*^n(H)$.

21. Definition of Neutrosophic SuperHyperTopology (NSHT) [6]

It is, similarly, a topology designed on the n-th PowerSet of a given non-empty set H, but includes the empty-sets [that represent indeterminacies] too.

As such, in the above formulas, $P_*(H)$ that excludes the empty-set, is replaced by $P(H)$ that includes the empty-set.

$P(H)$ is the first powerset of the set H, including the empty-set (∅);

$P^2(H) = P(P(H))$ is the second powerset of H (or the powerset of the powerset of H), that includes the empty-sets;

and so on, the n-th powerset of H,

$$P^n(H) = P(P^{n-1}(H)) = \underbrace{P(P(...P(H)...))}_{n}$$

where P is repeated n times ($n \geq 2$), and includes the empty-sets (∅).

Let consider τ_{NSHT} be a family of subsets of $P^n(H)$.

Then τ_{NSHT} is called a Neutrosophic SuperHyperTopology on $P^n(H)$, if it satisfies the following axioms:

(NSHT-1) ϕ and $P^n(H)$ belong to τ_{NSHT}.

(NSHT-2) The intersection of any finite number of elements in τ_{NSHT} is in τ_{NSHT}.

(NSHT-3) The union of any finite or infinite number of elements in τ_{NSHT} is in τ_{NSHT}.

Then $(P^n(H), \tau_{NSHT})$ is called a Neutrosophic SuperHyperTopological Space on $P^n(H)$.

22. Conclusion

These six new types of topologies, and their corresponding topological space, were introduced by Smarandache in 2019-2023, but they have not yet been much studied and applied, except the NeutroTopologies and AntiTopologies which got some attention from researchers.

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An application of neutrosophic theory on manifolds and their topological transformations

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Abstract. This paper presents an investigation into the mathematical concepts of neutrosophic folding and neuretraction on neutrosophic manifolds, specifically focusing on their application in hyperspace. Through the application of specific transformations on a neutrosophic manifold situated in hyperspace, we can obtain neutrosophic manifolds in lower dimensions. Based on our research, we can accurately establish the connection between neutrosophic folding and neuretraction on a neutrosophic manifold. Furthermore, we can determine the relationship between neuretraction and neutrosophic folding.

Keywords: neutrosophic folding; neuretraction; neutrosophic hyperspace; neutrosophic manifold.

1. Introduction

Neutrosophy is a scientific field that combines neutrality and philosophy. Samaransache founded various fields in 1980, such as set theory, probability, and logic, with numerous applications that highlight the deep interaction between mathematics and other scientific disciplines. [19]. The concept of fuzzy sets was introduced by Zadeh as a novel method for elucidating intricate concepts by including the concept of membership. Scholars in the fields of mathematics and computer science developed this theory, which possesses a broad spectrum of expedient applications [22]. Neutrosophy is basically rooted in the fundamental concepts of fuzzy set theory (NS) and intuitionistic fuzzy set theory (IFS) [8, 10, 13, 22]. The concept of neutrosophic sets was introduced by Smarandache with the aim of representing uncertain or vague information. This is achieved through the utilization of three distinct functions, namely truth, indeterminacy, and falsity. Unlike other theories, the function of indeterminacy is independent of the functions of truth and falsity [12,18,19]. Smarandache's (NS) theory expanded the scope of (IFS), offering novel perspectives on how to effectively manage uncertainty when making decisions based on personal experience, as stated in reference [20]. The values of the truth, indeterminacy, and falsity functions are within $]^{-}0,\ 1^{+}[$, making it difficult to apply to practical problems [17].

Due to this, Wang created the single-valued neutrosophic sets $(SVNS)$, such that the truth, indeterminacy, and falsity maps are real elements of the $[0,\ 1]$ space [12, 21]. Further investigation on a $(SVNG)$ and a neutrosophic topology was debated in [1, 4, 6, 7, 9, 11, 14, 16]. Additional insights on the applications of homotopy theory were provided in [2, 3]. The paper aims to contribute to the field of mathematics by exploring and providing a deeper understanding of the neutrosophic transformation in the context of neutrosophic manifold theory.

2. **Preliminaries**

Definition 2.1. [19] Assume that $\mathcal{W}$ is a finite set of objects, and that $(\mathfrak{t})$ stands for a generic component in $\mathcal{W}$. A (NS) E in $\mathcal{W}$ is comprised of three membership functions, a truth-membership function $\upsilon_{\mathrm{E}}(\mathfrak{t})$, an indeterminacy-membership function $\rho_{\mathrm{E}}(\mathfrak{t})$ and a falsity-membership function $\sigma_{\mathrm{E}}(\mathfrak{t})$. Also, $\upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_E(\mathfrak{t})$ and $\sigma_E(\mathfrak{t})$ are the elements of $]^{-}0,\ 1^{+}[$. E can be represented as

E$=\{\mathfrak{t},(\upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_{\mathrm{E}}(\mathfrak{t}),\ \sigma_{\mathrm{E}}(\mathfrak{t})) : \mathfrak{t}\in\mathcal{W},\ \upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_{\mathrm{E}}(\mathfrak{t}),\ \sigma_{\mathrm{E}}(\mathfrak{t})\in\ ]^{-}0,\ 1^{+}[\}$. Indeed, $^{-}0\leq\upsilon_{\mathrm{E}}(\mathfrak{t})+\rho_{\mathrm{E}}(\mathfrak{t})+\sigma_{\mathrm{E}}(\mathfrak{t})\leq 3^{+}$.

Definition 2.2. [21] Assume that $\mathcal{W}$ is a finite set of objects, and that $(\mathfrak{t})$ stands for a generic component in $\mathcal{W}$. A $(SVNS)$ E in $\mathcal{W}$ is comprised of three membership functions, a truth-membership function $\upsilon_{\mathrm{E}}(\mathfrak{t})$, an indeterminacy-membership function $\rho_{\mathrm{E}}(\mathfrak{t})$ and a falsity-membership function $\sigma_{\mathrm{E}}(\mathfrak{t})$. Also each, $\upsilon_{\mathrm{E}}(\mathfrak{t}), \rho_{\mathrm{E}}(\mathfrak{t})$ and $\sigma_{\mathrm{E}}(\mathfrak{t})$ are elements in $]0,\ 1[$. E can be represented as E$=\{\mathfrak{t},(\upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_{\mathrm{E}}(\mathfrak{t}),\ \sigma_{\mathrm{E}}(\mathfrak{t})) : \mathfrak{t}\in\mathcal{W},\ \upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_{\mathrm{E}}(\mathfrak{t}),\ \sigma_{\mathrm{E}}(\mathfrak{t})\in\ ]0,\ 1[\}$. In this approach, $0\leq\upsilon_{\mathrm{E}}(\mathfrak{t})+\rho_{\mathrm{E}}(\mathfrak{t})+\sigma_{\mathrm{E}}(\mathfrak{t})\leq 3$. In the interest of clarity and concision, we refer to a neutrosophic set $\langle\upsilon_{\mathrm{E}},\ \rho_{\mathrm{E}},\ \sigma_{\mathrm{E}}\rangle$ and $\langle\upsilon_{\mathrm{E}}(\mathfrak{t}),\ \rho_{\mathrm{E}}(\mathfrak{t}),\ \sigma_{\mathrm{E}}(\mathfrak{t})\rangle$ as ω and $\omega(\mathfrak{t})$ respectively.

Definition 2.3. [15] A topological space that satisfies the T_2 separation axiom and is locally homeomorphic to an open n-dimensional disk U^n is referred to as an n-dimensional manifold.

Definition 2.4. [5] Let X be a topological space and Cbe a subspace of X, where $i : C\to X$ is the inclusion. if there exists a continuous map $r : X\to C$ satisfying the condition $r\circ i = 1|_C$. Then, C is referred to a retract of X. The existence of a map r is denoted as a retraction of X into C.

Theorem 2.5. *[5] The n-dimensional closed disk $L^n = \{z \in \mathcal{R}^n : |z| \leq 1\}$ is a retract of $\mathcal{R}^n$.*

Definition 2.6. [15] Consider two topological spaces X_1 and X_2, and let φ_0 and $\varphi_1$denote continuous mappings from X_1 to X_1. The homotopy between φ_0 and $\varphi_1$is established when a continuous map $\varphi : X_1 \times I \to X_2$ exists, and satisfying the conditions $\varphi(s, 0) = \varphi_0(s)$ and $\varphi(s, 1) = \varphi_1(s)$ for all $s \in X_1$.

3. Neutrosophic manifolds and their transformations

Our study introduces a collection of important concepts that support our paper and enable us to arrive at significant conclusions.

Definition 3.1. A neutrosophic n-dimensional manifold is characterized as a pair $\langle M^n,\ \omega\rangle$ in which, M^n is n -dimensional manifold.

Example 3.2. A neutrosophic Euclidean n-space $\langle \mathcal{R}^n,\ \omega\rangle$ can be regarded as a neutrosophic n-dimensional manifold. Additionally, a neutrosophic unit n-dimensional sphere $\langle S^n,\ \omega\rangle$ can be considered as a neutrosophic n-dimensional manifold.

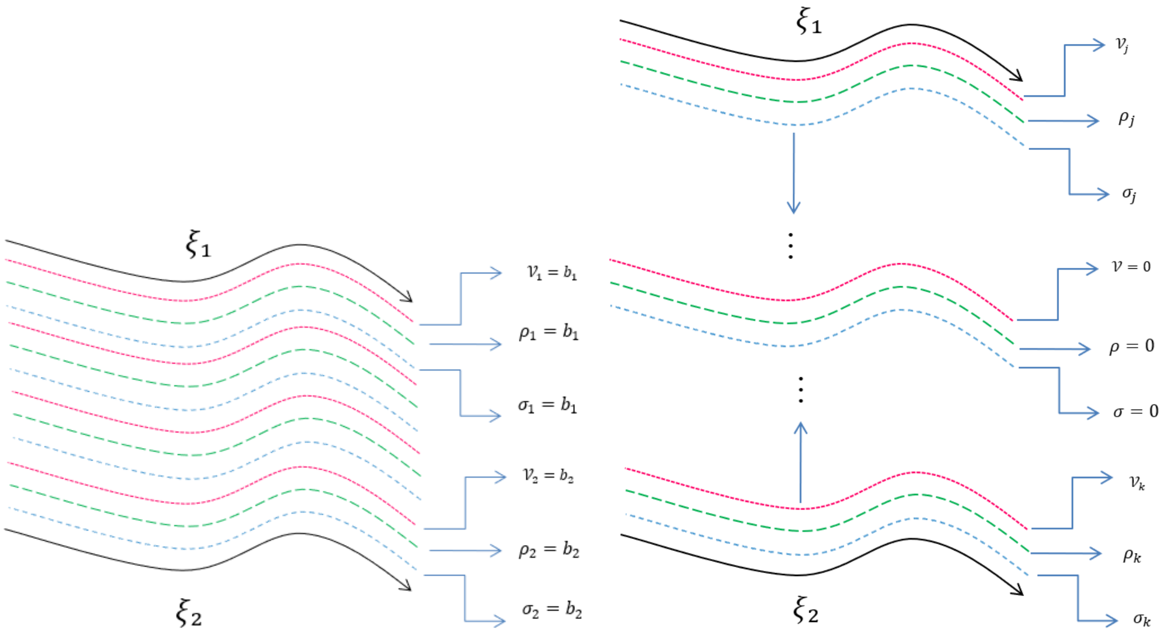
Definition 3.3. The neutrosophic arc $\zeta : [0,\ 1] \to \mathcal{R}^3$ is called a simple neutrosophic arc if, for each $\mathrm{z_j}$, $\mathrm{z_k} \in [0,\ 1]$, $\xi((\mathrm{z_j},\ \omega_\mathrm{j}) \neq \xi((\mathrm{z_k},\ \omega_\mathrm{k})$ whenever $(\mathrm{z_j}\ ,\ \omega_\mathrm{j}) \neq (\mathrm{z_k},\ \omega_\mathrm{k})$.

Now, we will delve into the notion of neutrosophy homotopic and describe two types of it.

Definition 3.4. A neutrosophic homotopy is a collection of neutrosophic maps $h_t : \langle M,\ \omega\rangle \to \langle N,\ \omega\rangle$, $t \in [0, 1]$, in which the associated neutrosophic map $\Phi : \langle M,\ \omega\rangle \times [0, 1] \to \langle N,\ \omega\rangle$ given by $\Phi((x,\ \omega), t) = h_t(x,\ \omega)$, and the two neutrosophic maps $h_0,\ h_1 : \langle M,\ \omega\rangle \to \langle N,\ \omega\rangle$ are called neutrosophy homotopic if there is a neutrosophic homotopy h_t that connects them and is represented by $h_0 \approx h_1$.

Theorem 3.5. *Let ξ_1 and ξ_2 be two neutrosophic arcs. Then, there are two types of neutrosophy homotopic arcs.*

Proof. The initial category encompasses a pair of neutrosophic arcs, ξ_1 and ξ_2 with specific values for ω_1 and ω_2 namely, $\omega_1 = \mathrm{b}_1$ and $\omega_2 = \mathrm{b}_2$ for all points of the arcs as shown in Fig.1a. The second category encompasses a pair of neutrosophic arcs, ξ_1 and ξ_2 with specific values for ω_1 and ω_2, where ξ_1 is a neutrosophic arc that has values for ω_j in the form of $\langle \upsilon_j,\ \rho_j,\ \sigma_j\ \rangle$ and ξ_2 is a neutrosophic arc that has values for ω_k in the form of $\langle \upsilon_k,\ \rho_k,\ \sigma_k\ \rangle$ for which max $\omega_j \longrightarrow 0$ or max $\omega_k \longrightarrow 0$ where, $\omega_j,\ \omega_k \in [0,\ 1]$, as shown in Fig.1b. □



(A) Neutrosophy homotopic of type (I) (B) Neutrosophy homotopic of type (II)

FIGURE 1. neutrosophy homotopic

Definition 3.6. Let $\langle M, \ \omega \rangle$ be a neutrosophic manifold with a neutrosophic submanifold $\langle \mathcal{C} \ , \ \omega \rangle$, and let us consider the existence of a continuous neutrosophic map $\mathfrak{d}$: $\langle M, \ \omega \rangle \longrightarrow \langle \mathcal{C} \ , \ \omega \rangle$ for which $\mathfrak{d}\,(\mathfrak{c}, \ \omega\,(\mathfrak{c})) = (\mathfrak{c} \ \omega\,(\mathfrak{c}))\,, \ \forall \mathfrak{c} \in \mathcal{C}$. Then, $\mathfrak{d}$ is called neuretraction.

Example 3.7. $\left\langle S^1, \ \omega \right\rangle$ is neuretraction of $\left\langle R^2 - \{0\}, \ \omega \right\rangle$.

Based on Definition 3.6, we can conclude that any of the following situations qualify as neuretraction:

Definition 3.8. (a) $\mathfrak{d}\,(\mathfrak{c}, \ \omega\,(\mathfrak{c})) = \ (\mathfrak{c}, \ \min(\upsilon), \ \min(\rho), \ \min(\sigma))$
(b) $\mathfrak{d}\,(\mathfrak{c}, \ \omega\,(\mathfrak{c})) = (\mathfrak{c}, \ \max(\upsilon), \ \max(\rho), \ \max(\sigma))$
(c) $\mathfrak{d}\,(\mathfrak{c}, \ \omega\,(\mathfrak{c})) = (\mathfrak{c}, \ \omega \in (0, \ 1))$. Now, for the rest of our discussion, and for simplicity, we shall denote the neutrosophic manifold $\langle M, \ \omega \rangle$ by the symbol M.

To show that isometry exists on both the upper and lower neutrosophic hypermanifolds, we shall use the potent framework of neutrosophic theory in the concept that follows.

Definition 3.9. A map $\mathcal{F} : \cup M \longrightarrow \cup M$ is said to be an isoneutrosophic folding if $\mathcal{F}\,(M) = M$ and for each member of the upper neutrosophic hypermanifold $\overline{M}_g$, there is a $\underline{M}_{\mathfrak{g}}$ lower M for which $\overline{\omega}_{\mathfrak{g}} = \underline{\omega}_{\mathfrak{g}}$ for any corresponding point, i.e., $\overline{\omega_{\mathfrak{g}}}(\overline{c}) = \underline{\omega}_{\mathfrak{g}}(\ \underline{c})$ as shown in Fig.2

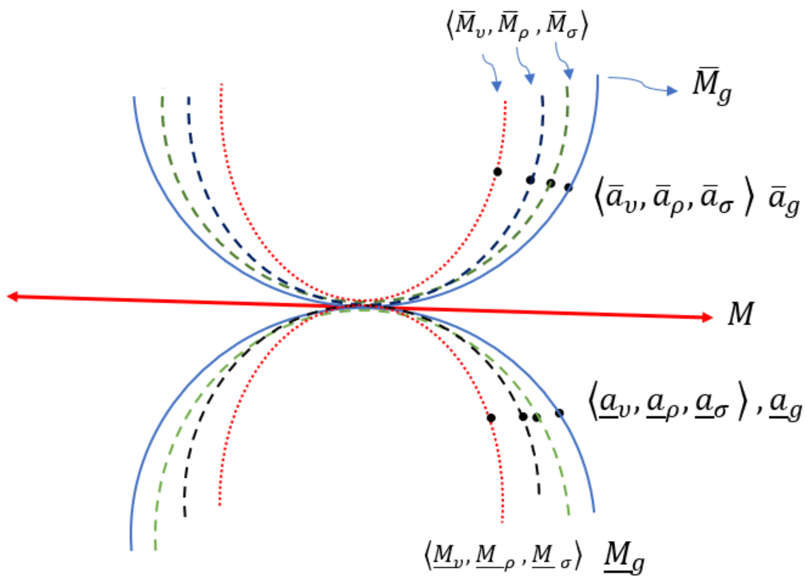


FIGURE 2. Isoneutrosophic folding

Theorem 3.10. *Assuming M is a neutrosophic hyperspace in $\mathcal{R}^{m+1}$. Then, we conclude that there are two types of neutrosophication that coincide in M.*

(a) For every $\mathfrak{c} \in M$, $\omega(\mathfrak{c}) = \langle 1,\ 1,\ 1\rangle$. Geometrically, parallel neutrosophic manifolds, which is known as the "crisp property."

(b) For each distinct point $\mathfrak{c}_{t_1},\ \mathfrak{c}_{t_2} \in M$ and, $\omega(\mathfrak{c}_{t_1}) \neq \omega(\mathfrak{c}_{t_2})$, there is a chain of homeomorphic neutrosophic manifolds connected at a common point.

Proof. (a) In "a crisp property." For all $y_{t_1},\ y_{t_2} \in M$, we have $\omega(y_{t_1}) = \omega(y_{t_2}) = \langle 1,\ 1,\ 1\rangle$ also, all neutrosophic hypermanifolds M_s are parallel, $\forall \mathfrak{c}_s \in \underline{M}_s$, $\omega(\mathfrak{c}_s) < \langle 1,\ 1,\ 1\rangle$ and $\forall \mathfrak{c}_1, \mathfrak{c}_2 \in M_s$, $\omega(\mathfrak{c}_1) = \omega(\mathfrak{c}_2)$, $M_s = \overline{M}_s$ or $M_s = \underline{M}_s$ as shown in Fig.3. In this situation, we can define ω as

$\omega = \langle \upsilon,\ \rho,\ \sigma\rangle$ where

$\langle \upsilon,\ \rho,\ \sigma\rangle = \left\langle \begin{cases} \frac{1}{1+l_1} & \text{if} \quad l_1 > 0 \\ \frac{1}{1-l_1} & \text{if} \quad l_1 < 0 \end{cases}, \begin{cases} \frac{1}{1+l_2} & \text{if} \quad l_2 > 0 \\ \frac{1}{1-l_2} & \text{if} \quad l_2 < 0 \end{cases}, \begin{cases} \frac{1}{1+l_3} & \text{if} \quad l_3 > 0 \\ \frac{1}{1-l_3} & \text{if} \quad l_3 < 0 \end{cases} \right\rangle$, the list $(l_i,\ i = 1,\ 2,\ 3)$ can be represented in Fig.3. Moreover, we have $\omega = \langle 0,\ 0,\ 0\rangle$ whenever $l_i \longrightarrow \pm\infty$. However, this illustrates the degree of neutrosophication in the crisp case of M. In fact, $\forall \gamma$ there is a neutrosophic strip at γ, specifically ζ_γ for which $\omega(\mathfrak{c}) < \langle 1,\ 1,\ 1\rangle$ whenever $\mathfrak{c} \in \zeta_\gamma$ and decreases if $l_i \longrightarrow \pm\infty$.

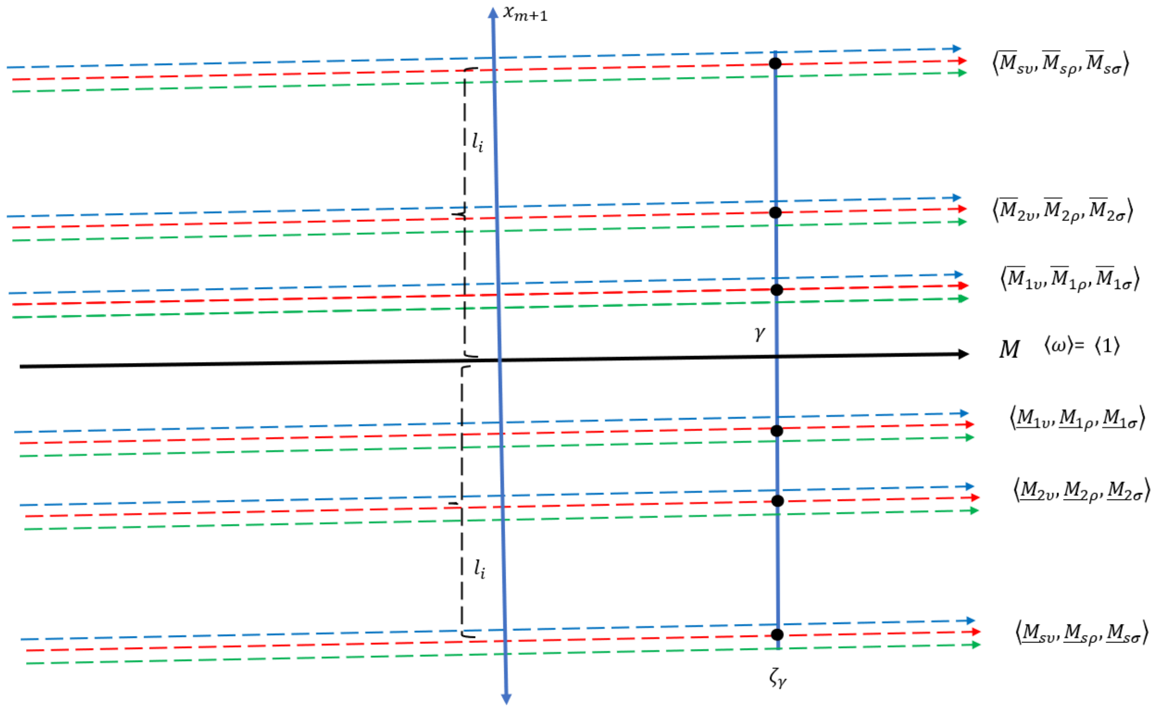


FIGURE 3. parallel hyperspaces and their isoneutrosophic folding

(b) Let M be a neutrosophic hyperspace for which $\omega(\mathfrak{c}_s) \neq \omega(\mathfrak{c}_t)$, $\mathfrak{c}_s \neq \mathfrak{c}_t$ in M, and suppose that $\mathfrak{q}$ is a point at which $\omega(\mathfrak{q}) = (\max \omega_s,\ s \in \mathbb{N})$. For all point $\gamma \in M$, $\exists$ neutrosophic strip ζ_γ such that $\omega(\mathfrak{c}_1) < \omega(\mathfrak{c}_2) < \omega(\gamma)$, whenever $\mathfrak{c}_1,\ \mathfrak{c}_2 \in \zeta_\gamma$. If there is no other common neutrosophic point than $\mathfrak{q}$, then $\exists$ a point $\mathfrak{c}_j \in M_j,\ j = 1,\ 2,\ 3$. For all horizontal neutrosophic strips, q has a maximum value (neutrosophic point) in the neutrosophic strip. □

The sequence of neuretraction within a neutrosophic hyperspace will be inferred from the data that follow.

Theorem 3.11. *If M is a neutrosophic hyperspace in $\mathcal{R}^{m+1}$, and $\mathfrak{d} : M \longrightarrow \mathcal{C}$, is a neuretraction. Then, there exists a sequence $\langle \mathfrak{d}_i : \cup M \longrightarrow \mathcal{C}_i,\ i = 1, 2, \ldots m \rangle$ of a neuretraction. Also, if we consider $dim\,(\cup M) = dim\ \mathcal{C}_i$, then all $\mathfrak{d}_i$ are special types of isoneutrosophic folding.*

Proof. Let $\mathcal{F}$ be a isoneutrosophic folding of $\cup \underline{M}$ into $\cup \overline{M}$ such that $\mathcal{F}\,(M) = M \quad \omega\,(\ \underline{M}) = \omega(\overline{M})$. Thus, we conclude $\mathcal{F}\,(\underline{M}) = \overline{M}$ as shown in Fig.4. Now for each neuretraction $\mathfrak{d} : M \longrightarrow \mathcal{C}$ (in a case of no common point) we obtain the induced neuretractions $\mathfrak{d}_i : M \longrightarrow \mathcal{C}_i$, $dim\ \mathcal{C}_i = dim M$. But if $\mathfrak{d} : M \longrightarrow p$, there are induced neuretractions $\underline{\mathfrak{d}}_i : \underline{M}_i \longrightarrow \underline{p_i}$ and $\overline{\mathfrak{d}_i} : \overline{M_i} \longrightarrow \overline{p_i}$. However, these neuretractions are not types of neutrosophic folding, because $dim\ \mathcal{C}_i \neq dim M_i$. For example, in Fig.5, $\exists$ an isoneutrosophic folding, whereas there is no isneutrosophic folding as a type of neuretraction in $\mathfrak{d} : \cup M \longrightarrow p$, since $dim\ p \neq dim M_i$. □

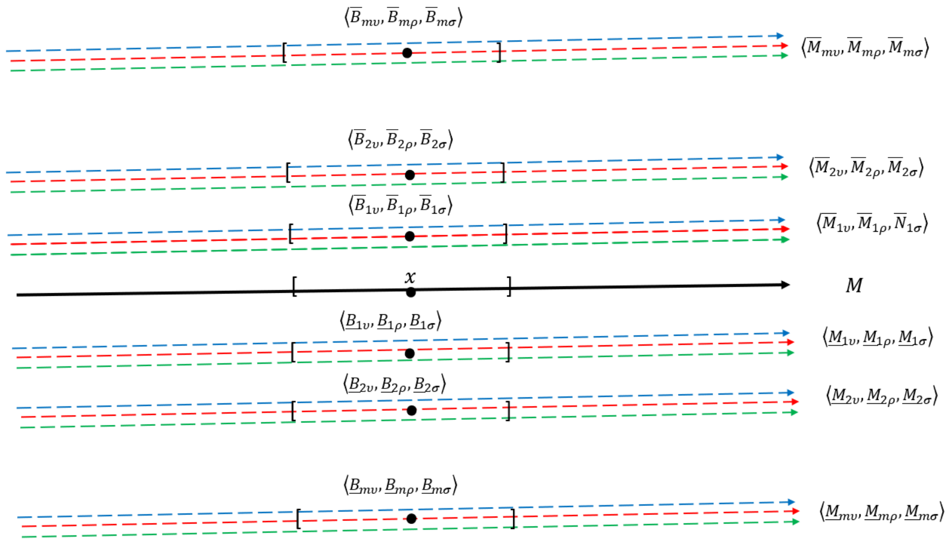


Figure 4. Isoneutrosophic folding on parallel hyperspaces

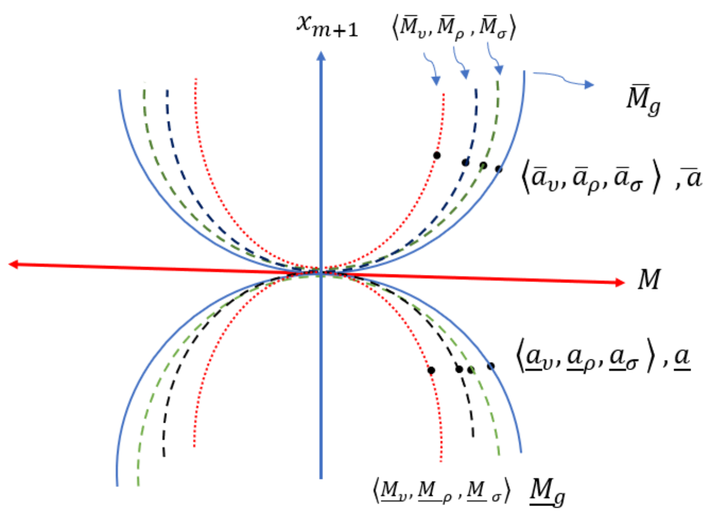


Figure 5. Neuretraction on hyperspaces with common point

The advanced results in this study reveal multiple occurrences of neuretractions corresponding to a set of neutrosophic manifolds that exhibit homeomorphism to a set of neutrosophic unit spheres with n dimensions, all of which possess a shared center.

Theorem 3.12. *Suppose that M is a neutrosophic manifold of dimension m, which is homeomorphic to a neutrosophic unit sphere, with $\omega(y_i) = 1$ for each $y_j \in \mathcal{S}^m$, else $\omega = \langle \upsilon, \rho, \sigma \rangle$ where*

$$\langle \upsilon, \rho, \sigma \rangle = \left\langle \begin{cases} l_1, & 0 < l_1 < 1 \\ \frac{1}{l_1}, & l_1 > 1 \end{cases}, \begin{cases} l_2, & 0 < l_2 < 1 \\ \frac{1}{l_2}, & l_2 > 1 \end{cases}, \begin{cases} l_3, & 0 < l_3 < 1 \\ \frac{1}{l_3}, & l_3 > 1 \end{cases} \right\rangle \text{ where, } y_j \in$$

$\cup\mathcal{S}_j^m$ as a union of m-dimensional neutrosophic spheres with a common center and let $H=\{(y,\ \omega):|y|\leq 1\}$ be an n-dimensional neutrosophic closed ball. Then, for every neuretraction of $(H-\mathfrak{q})$ onto $\mathcal{S}^{m-1}$ there are induced neuretractions of $H_j-\mathfrak{q}$ onto $\mathcal{S}_j^{m-1}$. Moreover, under the condition $\mathcal{F}(\mathcal{S}^m)=\mathcal{S}^m$, we get an isoneutrosophic folding $\mathcal{F}:\overline{\mathcal{S}}\,{}_j^m\longrightarrow\underline{\mathcal{S}}_j^m$.

Proof. Assume M is a neutrosophic manifold, $\mathcal{S}^n$ is a neutrosophic unit sphere, and M is homeomorphic to $\mathcal{S}^n$ as shown in Fig.6. If there is a neutrosophic sphere $\mathcal{S}^m$ inside the neutrosophic system, say $\underline{\mathcal{S}}_j^m$ (Neutrosophication will be reduced, $\omega=\langle\upsilon,\ \rho,\ \sigma\rangle\longrightarrow\langle 0,\ 0,\ 0\rangle$ if $l_i\longrightarrow 0$ and $\omega=\langle\upsilon,\ \rho,\ \sigma\rangle\longrightarrow\langle 0,\ 0,\ 0\rangle$ if $l_i\longrightarrow\infty$ for $i=1,2,3$. Indeed, for all neutrosophic points (c, $\omega=\langle 1,\ 1,\ 1\rangle$) $\exists$ (a neutrosophic) strip of neutrosophic points $(\overline{c}_j,\ \overline{\omega}_{\rm j}<\langle 1,\ 1,\ 1\rangle)\in\overline{\mathcal{S}}\,{}_j^m$, and $\left(\underline{c}_j,\ \underline{\omega}_{\rm j}<\langle 1,\ 1,\ 1\rangle\right)\in\underline{\mathcal{S}}_j^m$. However, for the isoneutrosophic folding $\mathcal{F}:\overline{\mathcal{S}}\,{}_j^m\longrightarrow\underline{\mathcal{S}}_j^m$, in which $\overline{\omega_j}=\underline{\omega_j}$ there is an induced isoneutrosophic folding $\overline{\mathcal{F}}:\overline{H}_j\longrightarrow\underline{H}_j$, as well as neuretractions $\overline{\mathfrak{d}}_j:\left(\overline{H}_j-\mathfrak{q}\right)\longrightarrow\overline{\mathcal{S}}\,{}_j^{m-1}$ and $\underline{\mathfrak{d}}_j:\left(\underline{H}_j-\mathfrak{q}\right)\longrightarrow\underline{\mathcal{S}}_j^{m-1}$. □

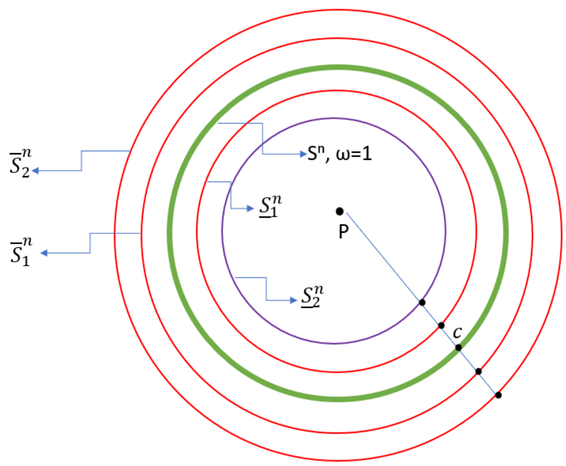


FIGURE 6. Neuretraction and neutrosophic folding on a spheres

Theorem 3.13. *Suppose that $\mathcal{N}$ is a neutrosophic manifold with $\mathfrak{d}:\mathcal{N}\longrightarrow\mathcal{M}$ is a neuretraction, then the geometric neuretraction $\mathfrak{d}_{\mathfrak{g}}$ induces a neuretractions $\mathfrak{d}_\upsilon,\ \mathfrak{d}_\rho,\ \mathfrak{d}_\sigma$. On the other hand, the converse is not true.*

Proof. Let us consider $\mathfrak{d}:\mathcal{N}\longrightarrow\mathcal{M}$ as a neuretraction, such that $\mathcal{N}=\langle\mathcal{N}_{\mathfrak{g}},\mathcal{N}_\upsilon,\ \mathcal{N}_\rho,\mathcal{N}_\sigma\rangle$ *and* $\mathcal{M}\subseteq\mathcal{N}$. Now, consider the geometric neuretraction of $\mathfrak{d}_{\mathfrak{g}}:\mathcal{N}_{\mathfrak{g}}\longrightarrow\mathcal{M}_{\mathfrak{g}}$ of $\mathcal{N}_{\mathfrak{g}}$ into $\mathcal{M}_{\mathfrak{g}}$, then we get the induced neuretractions $\mathfrak{d}_\upsilon:\mathcal{N}_\upsilon\longrightarrow\mathcal{M}_\upsilon$, $\mathfrak{d}_\rho:\mathcal{N}_\rho\longrightarrow\mathcal{M}_\rho$, $\mathfrak{d}_\sigma:\mathcal{N}_\sigma\longrightarrow\mathcal{M}_\sigma$ as shown in Fig.7. On the other hand, consider the neuretractions $\mathfrak{d}_\upsilon:\mathcal{N}_\upsilon\longrightarrow\mathcal{M}_\upsilon$, $\mathfrak{d}_\rho:\mathcal{N}_\rho\longrightarrow\mathcal{M}_\rho$, $\mathfrak{d}_\sigma:\mathcal{N}_\sigma\longrightarrow\mathcal{M}_\sigma$ as the identity neuretractions for all membership degrees, which have no impact on the geometric manifold $\mathcal{N}_{\mathfrak{g}}$ as shown in Fig.8 . □

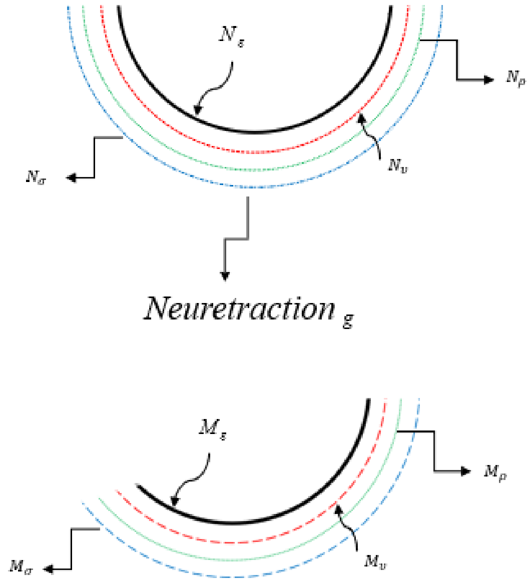


FIGURE 7. A neuretraction of type (I)

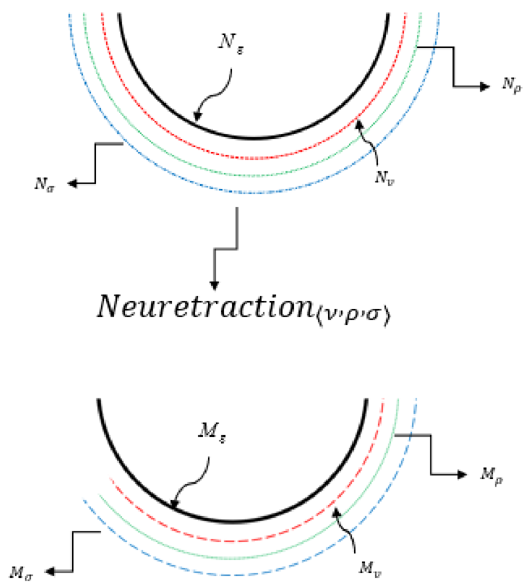


FIGURE 8. A neuretraction of type (II)

4. Conclusion

The present study aimed to develop a theoretical basis for neuretraction on a neutrosophic manifold. The neutrosophic folding and neuretraction on a neutrosophic manifold are achieved geometrically and topologically. The sequence of neuretractions in a neutrosophic hyperspace is obtained. The relationship between some types of transformations is deduced. An area that necessitates additional investigation pertains to the establishment and exploration of a fitting notion of neutrosophic homotopy groups, in conjunction with a thorough examination of their consequent neutrosophic homomorphism.

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On $\mathcal{S}_\theta$-summability in neutrosophic soft normed linear spaces

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Abstract. For any lacunary sequence $\theta = (k_s)$, the aim of the present paper is to define $\mathcal{S}_\theta$-convergence, $\mathcal{S}_\theta$-Cauchy and $\mathcal{S}_\theta$-completeness via neutrosophic soft norm. We study certain properties of these notions and give an important characterization of $\mathcal{S}_\theta$-convergence in neutrosophic soft normed linear spaces (briefly $NSNLS$). We provide examples that shows $\mathcal{S}_\theta$-convergence is a more general method of summability in these spaces.

Keywords: S_θ-convergence, S_θ-Cauchy, soft sets, soft normed linear spaces.

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1. **Introduction**

Statistical convergence was first introduced by Fast [8] and linked with the summability theory by Schoenberg [10]. Later, The idea is developed by Maddox [9], Fridy [12], Connor[13], Mursaleen and Edely [18], Šalát [32], Kumar and mursaleen [35] and many others.

Friday and Orhan [11] used lacunary sequences to define a new kind of statistical convergence as follows. "By a lacunary sequence we mean an increasing integer sequence $\theta = (k_s)$ with $k_0 = 0$ and $h_s = k_s - k_{s-1} \to \infty$ as $s \to \infty$. The intervals determined by θ will be denoted by $I_s = (k_{s-1}, k_s]$ and the ratio $\frac{k_s}{k_{s-1}}$ will be abbreviated as q_s. For $K \subseteq \mathbb{N}$, the number $\delta_\theta(K) = \lim_{s\to\infty} \dfrac{1}{h_s} |\{k \in I_s : k \in K\}|$ is called θ-density of K, provided the limit exists. A sequence $x = (x_k)$ of numbers is said to be lacunary statistically convergent (briefly S_θ-convergent) to x_0 if for every $\epsilon > 0$, $\lim_s \frac{1}{h_s} |\{k \in I_s : |x_k - x_0| \geq \epsilon\}| = 0$ or equivalently, the set $K(\epsilon)$ has θ-density zero, where $K(\epsilon) = \{k \in \mathbb{N} : |x_k - x_0| \geq \epsilon\}$. In this case, we write $\mathcal{S}_\theta - \lim_{k\to\infty} x_k = x_0$."Some further interesting works on lacunary statistical convergence can be found in [4], [19], [25], [34], [36], etc.

Zadeh [16] proposed the theory of fuzzy sets in 1965 as a more convenient tool for handling issues that cannot be modelled via crisp set theory. Atanassov [15] observed that fuzzy sets need more modification to handle problems in a time domain and therefore he introduced the intuitionistic fuzzy sets. After the introduction of intuitionistic fuzzy sets, a progressive development is made in this field. For instance, intuitionistic fuzzy metric spaces were introduced by Park [14], intuitionistic fuzzy topological spaces by Saadati and Park [26], etc.

The neutrosophic sets were initially introduced by Smarandache[7] as a generalization of fuzzy sets and intuitionistic fuzzy sets to avoid the complexity arising from uncertainty in settling many practical challenges in real-world activities. Kirişçi and Şimşek[17] defined neutrosophic norm and studied statistical convergence in neutrosophic normed spaces(NNS). For a broad view in this direction, we recommend to the reader [1], [2], [3], [20], [21], [22], [33].

Many approaches discussed above to minimize the uncertainty have their own drawbacks due to the inadequacy of the parametrization. In view of this, Molodtsov[6] proposed a new theory, called soft set theory to reduce the uncertainty during mathematical modelling. These sets turn out very useful tools in many areas of engineering and medical sciences. For instance: Maji et al [23] applied the theory of soft sets to decision-making problems. Kong et al.[39] presented a heuristic algorithm of normal parameter reduction of soft sets. Zou and Xiao[38] presented a data analysis approach of soft sets under incomplete information. Yuksel et al.[30] applied soft set theory to diagnose the prostate cancer risk in human beings whereas Çelik and Yamak[37] applied fuzzy soft set theory for medical diagnosis using fuzzy arithmetic operations.

Maji [24] presented a combined concept of Neutrosophic soft sets in 2013. Recently, Bera and Mahapatra [31] defined a generalized norm and called it a neutrosophic soft norm. They also studied some properties of $NSNLS$ and developed fundamental concepts of sequences in these spaces. In this article, we develop and study the concept of $\mathcal{S}_\theta$-convergence in $NSNLS$. We also introduce the concepts of $\mathcal{S}_\theta$-Cauchy sequence, $\mathcal{S}_\theta$-completeness and develop some of their properties.

2. **Preliminaries**

This section starts with a brief information on soft sets, soft vector spaces and neutrosophic soft normed spaces. We begin with the following notations and definitions.

Throughout this work, $\mathbb{N}$ will denote the set of positive integers, $\mathbb{R}$ the set of reals and $\mathbb{R}^+$ the set of positive real numbers.

Definition 2.1 [5] A binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $\circ$ satisfies the following conditions:

(i) $d \circ e = e \circ d$ and $d \circ (e \circ f) = (d \circ e) \circ f$.

(ii) $\circ$ is continuous.

(iii) $d \circ 1 = 1 \circ d = d$ for all $d \in [0,1]$.

(iv) $d \circ e \leq f \circ g$ if $d \leq f, e \leq g$ with $d, e, f, g \in [0,1]$.

Definition 2.2 [5] A binary operation $\diamond : [0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-conorm(s-norm) if $\diamond$ satisfies the following conditions:

(i) $d \diamond e = e \diamond d$ and $d \diamond (e \diamond f) = (d \diamond e) \diamond f$.

(ii) $\diamond$ is continuous.

(iii) $d \diamond 0 = 0 \diamond d =$ for all $d \in [0,1]$.

(iv) $d \diamond e \leq f \diamond g$ if $d \leq f, e \leq g$ with $d, e, f, g \in [0,1]$.

For any universe set U and the set E of the parameters, the soft set is defined as follows:

Definition 2.3 [6] A pair (H, E) is called a soft set over U if and only if H is a mapping of E into the set of all subsets of the set U. i.e., the soft set is a parametrized family of subsets of the set U.

Moreover, every set $H(\epsilon), \epsilon \in E$, from this family may be considered as the set of ϵ-elements of the soft set (H, E), or as the set of ϵ-approximate elements of the set.

Definition 2.4 [6] A soft set (H, E) over U is said to be absolute soft set if for all $\epsilon \in E$, $H(\epsilon) = U$. We will denote it by $\widetilde{U}$.

Definition 2.5 [27] Let $\mathbb{R}$ be the set of real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E taken as a set of parameters. Then a mapping $F : E \rightarrow B(\mathbb{R})$ is called a soft real set. If a soft real set is a singleton soft set, then it is called a soft real number and denoted by $\widetilde{r}, \widetilde{s}, \widetilde{t}$, etc. $\widetilde{0}, \widetilde{1}$ are the soft real numbers where $\widetilde{0}\ (e) = 0, \widetilde{1}\ (e) = 1$ for all $e \in E$ respectively.

Let $\mathbb{R}(E)$ and $\mathbb{R}^+(E)$ respectively denote the sets of all soft real numbers and all positive soft real numbers.

Definition 2.6 [28] Let (H, E) be a soft set over U. The set (H, E) is said to be a soft point, denoted by H_e^u if there is exactly one $e \in E$ s.t $H(e) = \{u\}$ for some $u \in U$ and $H(e') = \phi$ for all $e' \in E - \{e\}$.

Two soft points $H_e^u, H_{e'}^w$ are said to be equal if $e = e'$ and $u = w$. Let $\Delta_{\widetilde{U}}$ denotes the set of all soft points on $\widetilde{U}$.

In case U is a vector space over $\mathbb{R}$ and the parameter set $E = \mathbb{R}$, the soft point is called a soft vector.

Soft vector spaces are used to define soft norm as follows:

Definition 2.7 [29] Let $\widetilde{U}$ be a absolute soft vector space. Then a mapping $\| \cdot \| : \widetilde{U} \rightarrow \mathbb{R}^+(E)$ is said to be a soft norm on $\widetilde{U}$, if $\| \cdot \|$ satisfies the following conditions:

(i) $\|u_e\| \geq \widetilde{0}$ for all $u_e \in \widetilde{U}$ and $\|u_e\| = \widetilde{0} \Leftrightarrow u_e = \widetilde{\theta}_0$ where $\widetilde{\theta}_0$ denotes the zero element of $\widetilde{U}$.

(ii) $\| \widetilde{\alpha}\, u_e\| = |\widetilde{\alpha}| \|u_e\|$ for all $u_e \in \widetilde{U}$ and for every soft scalar $\widetilde{\alpha}$.

(iii) $\|u_e + v_{e'}\| \leq \|u_e\| + \|v_{e'}\|$ for all $u_e, v_{e'} \in \widetilde{U}$.

(iv) $\|u_e \cdot v_{e'}\| = \|u_e\| \, \|v_{e'}\|, \forall \, u_e, v_{e'} \in \widetilde{U}$.

The soft vector space $\widetilde{U}$ with a soft norm $\|\cdot\|$ on $\widetilde{U}$ is said to be a soft normed linear space and is denoted by $(\widetilde{U}, \|\cdot\|)$.

We now recall the definition of neutrosophic soft normed linear spaces and the convergence structure in these spaces.

Definition 2.8 [31] Let $\widetilde{U}$ be a soft linear space over the field F and $\mathbb{R}(E), \Delta_{\widetilde{U}}$ denote respectively, the set of all soft real numbers and the set of all soft points on $\widetilde{U}$. Then a neutrosophic subset N over $\Delta_{\widetilde{U}} \times \mathbb{R}(E)$ is called a neutrosophic soft norm on $\widetilde{U}$ if for $u_e, v_{e'} \in \widetilde{U}$ and $\widetilde{\alpha} \in F$ ($\widetilde{\alpha}$ being soft scalar), the following conditions hold.

(i) $0 \leq G_N(u_e, \widetilde{\eta_1}), B_N(u_e, \widetilde{\eta_1}), Y_N(u_e, \widetilde{\eta_1}) \leq 1, \forall \, \widetilde{\eta_1} \in \mathbb{R}(E)$.

(ii) $0 \leq G_N(u_e, \widetilde{\eta_1}) + B_N(u_e, \widetilde{\eta_1}) + Y_N(u_e, \widetilde{\eta_1}) \leq 3, \forall \, \widetilde{\eta_1} \in \mathbb{R}(E)$.

(iii) $G_N(u_e, \widetilde{\eta_1}) = 0$ with $\widetilde{\eta_1} \leq \widetilde{0}$.

(iv) $G_N(u_e, \widetilde{\eta_1}) = 1,$ with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.

(v) $G_N(\widetilde{\alpha} \, u_e, \widetilde{\eta_1}) = G_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall \, \widetilde{\alpha} \, (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$.

(vi) $G_N(u_e, \widetilde{\eta_1}) \circ G_N(v_{e'}, \widetilde{\eta_2}) \leq G_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2}), \forall \, \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$

(vii) $G_N(u_e, \cdot)$ is continuous non-decreasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim_{\widetilde{\eta_1} \to \infty} G_N(u_e, \widetilde{\eta_1}) = 1$.

(viii) $B_N(u_e, \widetilde{\eta_1}) = 1$ with $\widetilde{\eta_1} \leq \widetilde{0}$.

(ix) $B_N(u_e, \widetilde{\eta_1}) = 0,$ with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.

(x) $B_N(\widetilde{\alpha} \, u_e, \widetilde{\eta_1}) = B_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall \, \widetilde{\alpha} \, (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$.

(xi) $B_N(u_e, \widetilde{\eta_1}) \diamond B_N(v_{e'}, \widetilde{\eta_2}) \geq B_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2}) \; \forall \, \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$.

(xii) $B_N(u_e, \cdot)$ is continuous non-increasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim_{\widetilde{\eta_1} \to \infty} B_N(u_e, \widetilde{\eta_1}) = 0$.

(xiii) $Y_N(u_e, \widetilde{\eta_1}) = 0$ with $\widetilde{\eta_1} \leq \widetilde{0}$.

(xiv)$Y_N(u_e, \widetilde{\eta_1}) = 0,$ with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.

(xv) $Y_N(\widetilde{\alpha} \, u_e, \widetilde{\eta_1}) = Y_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall \, \widetilde{\alpha} \, (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$.

(xvi) $Y_N(u_e, \widetilde{\eta_1}) \diamond Y_N(v_{e'}, \widetilde{\eta_2}) \geq Y_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2}) \; \forall \, \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$.

(xvii) $Y_N(u_e, \cdot)$ is continuous non-increasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim_{\widetilde{\eta_1} \to \infty} B_N(u_e, \widetilde{\eta_1}) = 0$.

In this case, $\mathcal{N} = (G_N, B_N, Y_N)$ is called the neutrosophic soft norm and $(\widetilde{U}(F), G_N, B_N, Y_N, \circ, \diamond)$ is the neutrosophic soft normed linear space ($NSNLS$ briefly).

Let $(\widetilde{U}, \|\cdot\|)$ be a soft normed space. Take the operations $\circ$ and $\diamond$ as $x \circ y = xy$; $x \diamond y = x + y - xy$. For $\widetilde{\eta} > \widetilde{0}$, define

$$G_N(u_e, \widetilde{\eta}) = \begin{cases} \frac{\widetilde{\eta}}{\widetilde{\eta} + \|u_e\|} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise} \end{cases}$$

$$B_N(u_e, \widetilde{\eta}) = \begin{cases} \frac{\|u_e\|}{\widetilde{\eta}+\|u_e\|} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise} \end{cases}$$

$$Y_N(u_e, \widetilde{\eta}) = \begin{cases} \frac{\|u_e\|}{\widetilde{\eta}} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise,} \end{cases}$$

then $(\widetilde{U}\ (F), G_N, B_N, Y_N, \circ, \diamond)$ is an $NSNLS$. From now onwards, unless otherwise stated by $\widetilde{V}$ we shall denote the $NSNLS$ $(\widetilde{U}\ (F), G_N, B_N, Y_N, \circ, \diamond)$.

Definition 2.9 [31] A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be convergent to a soft point $v_e \in \widetilde{V}$ if for $0 < \epsilon < 1$ and $\widetilde{\eta} > \widetilde{0}$ $\exists$ $n_0 \in \mathbb{N}$ s.t $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1 - \epsilon,\ B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon,\ Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon$. In this case, we write $\lim_{k\to\infty} v^k_{e_k} = v_e$.

Definition 2.10 [31] A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be cauchy sequence if for $0 < \epsilon < 1$ and $\widetilde{\eta} > \widetilde{0}$ $\exists$ $n_0 \in \mathbb{N}$ s.t for all $k, p \geq n_0$ $G_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) > 1-\epsilon,\ B_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) < \epsilon,\ Y_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) < \epsilon$.

3. **Lacunary statistical convergence in NSNLS**

In this section, we define $\mathcal{S}_\theta$-convergence in neutrosophic soft normed linear spaces and develop some of its properties.

Definition 3.1 A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be lacunary statistical convergent or $\mathcal{S}_\theta$-convergent to a soft point v_e in $\widetilde{V}$ w.r.t neutrosophic soft norm-(G_N, B_N, Y_N) if for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$,

$$\lim_{s\to\infty} \frac{1}{h_s} \left| \Big\{ k \in I_s : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon \Big\} \right| = 0,$$

i.e., $\delta_\theta(A) = 0$ where

$$A = \{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\}.$$

In this case, we write $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$.

Let, $S_\theta(G_N, B_N, Y_N)$ denotes the set of all sequences of soft points in $\widetilde{V}$ which are S_θ-convergent with respect to the neutrosophic soft norm (G_N, B_N, Y_N).

Definition 3.1 together with the property of θ-density, we have the following lemma.

Lemma 3.1 For any sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$, the following statements are equivalent:

(i) $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$;

(ii) $\delta_\theta\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-\epsilon\} = \delta_\theta\{k \in \mathbb{N} : B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\} = \delta_\theta\{k \in \mathbb{N} : Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\} = 0$;

(iii) $\delta_\theta\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon$ and $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = 1$;

(iv)$\delta_\theta\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon\} = \delta_\theta\{k \in \mathbb{N} : B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = \delta_\theta\{k \in \mathbb{N} : Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = 1$;

(v) $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 1$ and $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 0$, $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 0$.

Theorem 3.1 Let $\theta = (k_s)$ be a lacunary sequence and $v = (v^k_{e_k})$ be any sequence in $\widetilde{V}$. If $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k}$ exists, then it is unique.

Proof. Suppose that $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{n\to\infty} v^k_{e_k} = v_{e_1}$ and $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{n\to\infty} v^k_{e_k} = v'_{e_2}$, where $v_{e_1} \neq v'_{e_2}$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$. Choose $\varrho > 0$ s.t.

$$(1-\varrho) \circ (1-\varrho) > 1-\epsilon \text{ and } \varrho \lozenge \varrho < \epsilon \tag{1}$$

Define the following sets:

$$H_{G_N,1}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : G_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \leq 1-\varrho\right\}.$$

$$H_{G_N,2}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : G_N\left(v^k_{e_k} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \leq 1-\varrho\right\}.$$

$$H_{B_N,1}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : B_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

$$H_{B_N,2}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : B_N\left(v^k_{e_k} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

$$H_{Y_N,1}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : Y_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

$$H_{Y_N,2}(\varrho, \widetilde{\eta}) = \left\{k \in \mathbb{N} : Y_N\left(v^k_{e_k} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

Since $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_{e_1}$, then using lemma 3.1, we have
$\delta_\theta\{H_{G_N,1}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H_{B_N,1}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H_{Y_N,1}(\varrho, \widetilde{\eta})\} = 0$ and therefore $\delta_\theta\{H^C_{G_N,1}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H^C_{B_N,1}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H^C_{Y_N,1}(\varrho, \widetilde{\eta})\} = 1$.
Further, $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v'_{e_2}$, so
$\delta_\theta\{H_{G_N,2}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H_{B_N,2}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H_{Y_N,2}(\varrho, \widetilde{\eta})\} = 0$ and therefore $\delta_\theta\{H^C_{G_N,2}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H^C_{B_N,2}(\varrho, \widetilde{\eta})\} = \delta_\theta\{H^C_{Y_N,2}(\varrho, \widetilde{\eta})\} = 1$ for all $\widetilde{\eta} > \widetilde{0}$. Now define

$$\begin{aligned} K_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta}) = \{&H_{G_N,1}(\varrho, \widetilde{\eta}) \cup H_{G_N,2}(\varrho, \widetilde{\eta})\} \\ &\cap\{H_{B_N,1}(\varrho, \widetilde{\eta}) \cup H_{B_N,2}(\varrho, \widetilde{\eta})\} \cap \{H_{Y_N,1}(\varrho, \widetilde{\eta}) \cup H_{Y_N,2}(\varrho, \widetilde{\eta})\}, \end{aligned}$$

then $\delta_\theta\{K_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\} = 0$ and therefore, $\delta_\theta\{K^C_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\} = 1$. Let $m \in K^C_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})$, then we have following possibilities.

1. $m \in \left\{H_{G_N,1}(\varrho,\widetilde{\eta}) \cup H_{G_N,2}(\varrho,\widetilde{\eta})\right\}^C$; or
2. $m \in \left\{H_{B_N,1}(\varrho,\widetilde{\eta}) \cup H_{B_N,2}(\varrho,\widetilde{\eta})\right\}^C$; or
3. $m \in \left\{H_{Y_N,1}(\varrho,\widetilde{\eta}) \cup H_{Y_N,2}(\varrho,\widetilde{\eta})\right\}^C$.

Case 1: Let $m \in \left\{H_{G_N,1}(\varrho,\widetilde{\eta}) \cup H_{G_N,2}(\varrho,\widetilde{\eta})\right\}^C$, then $m \in H^C_{G_N,1}(\varrho,\widetilde{\eta})$ and $m \in H^C_{G_N,2}(\varrho,\widetilde{\eta})$ and therefore,

$$G_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) > 1-\varrho \text{ and } G_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) > 1-\varrho. \tag{2}$$

Now

$$\begin{aligned} G_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= G_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\geq G_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \circ G_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\ &> (1-\varrho) \circ (1-\varrho) \qquad \text{by (2)} \\ &> 1-\epsilon. \qquad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $G_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 1$ for all $\widetilde{\eta} > \widetilde{0}$, which gives $v_{e_1} = v'_{e_2}$.

Case 2: Let $m \in \left\{H_{B_N,1}(\varrho,\widetilde{\eta}) \cup H_{B_N,2}(\varrho,\widetilde{\eta})\right\}^C$, then $m \in H^C_{B_N,1}(\varrho,\widetilde{\eta}))$ and $m \in H^C_{B_N,2}(\varrho,\widetilde{\eta})$ and therefore,

$$B_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) < \varrho \text{ and } B_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) < \varrho. \tag{3}$$

Now

$$\begin{aligned} B_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= B_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq B_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\ &< \varrho \diamond \varrho \qquad \text{by (3)} \\ &< \epsilon. \qquad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $B_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 0$ for all $\widetilde{\eta} > \widetilde{0}$, which gives $v_{e_1} = v'_{e_2}$.

Case 3: Let $m \in \left\{H_{Y_N,1}(\varrho,\widetilde{\eta}) \cup H_{Y_N,2}(\varrho,\widetilde{\eta})\right\}^C$, then $m \in H^C_{Y_N,1}(\varrho,\widetilde{\eta})$ and $m \in H^C_{Y_N,2}(\varrho,\widetilde{\eta})$ and therefore,

$$Y_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) < \varrho \text{ and } Y_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) < \varrho. \tag{4}$$

Now

$$
\begin{aligned}
Y_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= Y_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\
&\leq Y_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\
&< \varrho \diamond \varrho \qquad \text{by (4)} \\
&< \epsilon. \qquad\qquad \text{by (1)}
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, so we have $Y_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 0$ for all $\widetilde{\eta} > \widetilde{0}$, which gives $v_{e_1} = v'_{e_2}$. Hence, in all cases we have $v_{e_1} = v'_{e_2}$, i.e., $\mathcal{S}_\theta(G_N, B_N, Y_N)$-limit of $(v^k_{e_k})$ is unique.□

Theorem 3.2 Let $\theta = (k_s)$ be a lacunary sequence and $v = (v^k_{e_k})$ be any sequence in $\widetilde{V}$. If $(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$, then $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$.

Proof. Let $(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$. Then for each $\epsilon > 0$ and $\eta > 0, \exists$ positive integers $k_0 \in \mathbb{N}$ s.t $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1 - \epsilon$ and $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\ \forall\ k > k_0$. Hence, the set

$$
\begin{aligned}
A = \big\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or} \\
B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\big\}
\end{aligned}
$$

has finite number of terms. Since every finite subset of $\mathbb{N}$ has θ-density zero and hence

$$
\begin{aligned}
\delta_\theta\big(\big\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or} \\
B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\big\}\big) = 0.
\end{aligned}
$$

Therefore, $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$. □

The following example shows that the converse of the above theorem need not be true.

Example 3.1 Let $(\widetilde{\mathbb{R}}, \|\cdot\|)$ be a soft normed linear space. For v_e in $\widetilde{\mathbb{R}}$ and $\widetilde{\eta} > \widetilde{0}$, if we define

$$
G_N(v_e, \widetilde{\eta}) = \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus \|v_e\|},\ B_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta} \oplus \|v_e\|},\ Y_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta}},
$$

$x \circ y = xy$ and $x \diamond y = \min\{x+y, 1\}$, then it is easy to see that $\widetilde{V} = (\widetilde{\mathbb{R}}, G_N, B_N, Y_N, \circ, \diamond)\ \forall\ x, y \in [0, 1]$ is a neutrosophic soft normed linear space.

Now define a sequence $v = (v^k_{e_k})$ in $\widetilde{V}$ by

$$
v^k_{e_k} = \begin{cases} \widetilde{k} & \text{if } k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s, s \in \mathbb{N} \\ \widetilde{0} & \text{otherwise.} \end{cases}
$$

Now, for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, let

$$\begin{aligned} A(\epsilon,\widetilde{\eta}) &= \Big\{k \in I_s : G_N(v^k_{e_k},\widetilde{\eta}) \leq 1-\epsilon \text{ or } B_N(v^k_{e_k},\widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k},\widetilde{\eta}) \geq \epsilon\Big\} \\ &= \Big\{k \in I_s : \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus \|v^k_{e_k}\|} \leq 1-\epsilon \text{ or } \frac{\|v^k_{e_k}\|}{\widetilde{\eta} \oplus \|v^k_{e_k}\|} \geq \epsilon, \frac{\|v^k_{e_k}\|}{\widetilde{\eta}} \geq \epsilon\Big\} \\ &= \Big\{k \in I_s : \|v^k_{e_k}\| \geq \frac{\widetilde{\eta}\,\epsilon}{1-\epsilon} \text{ or } \|v^k_{e_k}\| \geq \widetilde{\eta}\,\epsilon\Big\} \\ &\subseteq \Big\{k \in I_s : v^k_{e_k} = \widetilde{k}\Big\} \\ &= \Big\{k \in I_s : k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s, s \in \mathbb{N}\Big\} \end{aligned}$$

and so we get

$$\frac{1}{h_s}|A(\epsilon,\widetilde{\eta})| \leq \frac{1}{h_s}|\{k \in I_s : k_s - [\sqrt{h_s}] + 1 \leq k \leq k_s\}| \leq \frac{\sqrt{h_s}}{h_s}.$$

Taking $s \to \infty$,
$\lim_{s\to\infty} \frac{1}{h_s}|A(\epsilon,\widetilde{\eta})| \leq \lim_{s\to\infty} \frac{\sqrt{h_s}}{h_s} = 0$, i.e., $\delta_\theta(A(\epsilon,\widetilde{\eta})) = 0$.
This shows that, $v = (v^k_{e_k})$ is $\mathcal{S}_\theta(G_N, B_N, Y_N)$- convergent to $\widetilde{0}$. But by the structure of the sequence, $v = (v^k_{e_k})$ is not convergent to $\widetilde{0}$ w.r.t (G_N, B_N, Y_N).

Theorem 3.3 Let $\theta = (k_s)$ be a lacunary sequence and let $u = (u^k_{e_k})$ and $v = (v^k_{e_k})$ be any two sequences in $\widetilde{V}$ s.t $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty}(u^k_{e_k}) = u_{e_1}$ and $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty}(v^k_{e_k}) = v_{e_2}$. Then

(i)$\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty}(u^k_{e_k} \oplus\ v^k_{e_k}) = u_{e_1} \oplus\ v_{e_2}$

(ii) $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty}(\widetilde{\alpha}\ u^k_{e_k}) = \widetilde{\alpha}\ u_{e_1}$, where $\widetilde{0} \neq \widetilde{\alpha} \in F$.

Proof. The proof of the theorem can be obtained as the proof of theorem 3.1, so omitted.□

Theorem 3.4 Let $\theta = (k_s)$ be a lacunary sequence. A sequence $v = (v^k_{e_k})$ in $\widetilde{V}$ is $\mathcal{S}_\theta(G_N, B_N, Y_N)$-convergent to v_e, if and only if $\exists$ a subset $K = \{k_1, k_2, ...\}$ of $\mathbb{N}$ s.t $\delta_\theta(K) = 1$ and $(G_N, B_N, Y_N) - \lim_{\substack{k\in K \\ k\to\infty}} v^k_{e_k} = v_e$.

Proof. First suppose that $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$. For $\widetilde{\eta} > \widetilde{0}$ and $\beta \in \mathbb{N}$, define the set

$$\begin{aligned} K_{G_N,B_N,Y_N}(\beta,\widetilde{\eta}) = \Big\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) > 1 - \frac{1}{\beta} \text{ and} \\ B_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) < \frac{1}{\beta}, Y_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) < \frac{1}{\beta}\Big\} \text{ and} \end{aligned}$$

$$\begin{aligned} K^C_{G_N,B_N,Y_N}(\beta,\widetilde{\eta}) = \Big\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \leq 1 - \frac{1}{\beta} \text{ or} \\ B_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \geq \frac{1}{\beta}, Y_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \geq \frac{1}{\beta}\Big\}. \end{aligned}$$

Since $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$, it follows that $\delta_\theta(K^C_{G_N,B_N,Y_N}(\beta, \widetilde{\eta})) = 0$. Furthermore, for $\widetilde{\eta} > \widetilde{0}$ and $\beta \in \mathbb{N}$, we observe $K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta}) \supset K_{G_N,B_N,Y_N}(\beta+1, \widetilde{\eta})$ and

$$\delta_\theta(K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta})) = 1. \tag{5}$$

Now, we have to show that, for $k \in K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta})$, $(G_N, B_N, Y_N) - \lim_{\substack{k\in K\\ k\to\infty}} v^k_{e_k} = v_e$. Suppose for $k \in K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta})$, $(v^k_{e_k})$ is not convergent to v_e w.r.t (G_N, B_N, Y_N). Then $\exists$ some $\xi > 0$ and a +ve integer k_0 s.t $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-\xi$ or $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \xi, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \xi \;\forall\; k > k_0$. Let $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\xi$ and $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \xi, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \xi \;\forall\; k < k_0$. Then

$$\delta_\theta\big(\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\xi \text{ and } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \xi, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \xi\}\big) = 0.$$

Since $\xi > \frac{1}{\beta}$ where $\beta \in \mathbb{N}$, we have $\delta_\theta(K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta})) = 0$. In this way we obtained a contradiction to (5) as $\delta_\theta(K_{G_N,B_N,Y_N}(\beta, \widetilde{\eta})) = 1$. Hence, $(G_N, B_N, Y_N) - \lim_{\substack{k\in K\\ k\to\infty}} v^k_{e_k} = v_e$.

Conversely, Suppose that $\exists$ a subset $K = \{k_1, k_2, ..., k_j, ...\}$ of $\mathbb{N}$ with $\delta_\theta(K) = 1$ and $(G_N, B_N, Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$ over K i.e., $(G_N, B_N, Y_N) - \lim_{\substack{k\in K\\ k\to\infty}} v^k_{e_k} = v_e$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, $\exists$ $k_{j_0} \in \mathbb{N}$ s.t for all $k_j \geq k_{j_0}$, $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon$ and $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon$. So if we consider the set

$$T_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta}) = \Big\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-\epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\Big\},$$

then it is easy to see that $T_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta}) \subset \mathbb{N} - \{k_{j_0+1}, k_{j_0+2}, ...\}$. This immediately implies that $\delta_\theta\big(T_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta})\big) \leq \delta_\theta(\mathbb{N}) - \delta_\theta(\{k_{j_0+1}, k_{j_0+2}, ...\}) = 1-1 = 0$ and therefore $\delta_\theta\big(T_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta})\big) = 0$ as $\delta_\theta\big(T_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta})\big)$ can not be negative. This shows that $\mathcal{S}_\theta(G_N, B_N, Y_N) - \lim_{n\to\infty} v^k_{e_k} = v_e$. □

4. **Lacunary statistical completeness in NSNLS**

Definition 4.1 A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be lacunary statistically cauchy (or $\mathcal{S}_\theta$-Cauchy) w.r.t neutrosophic soft norm (G_N, B_N, Y_N) if for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, $\exists$ $p \in \mathbb{N}$ s.t

$$\lim_{s\to\infty} \frac{1}{h_s} \left| \Big\{k \in I_s : G_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \leq 1-\epsilon \text{ or } B_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon\Big\} \right| = 0,$$

or equivalently, the θ-density of the set K is zero, i.e., $\delta_\theta(K)=0$ where

$$K=\{k\in\mathbb{N}: G_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon \text{ or } B_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon, Y_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon\}.$$

Theorem 4.1 Let $\theta=(k_s)$ be any lacunary sequence. If a sequence $v=(v^k_{e_k})$ of soft points in $\widetilde{V}$ is $\mathcal{S}_\theta(G_N,B_N,Y_N)$-convergent, then it is $\mathcal{S}_\theta(G_N,B_N,Y_N)$ cauchy.

Proof. Let $v=(v^k_{e_k})$ be any lacunary statistically convergent sequence with $\mathcal{S}_\theta(G_N,B_N,Y_N)-\lim_{k\to\infty} v^k_{e_k}=v_e$. Let $\epsilon>0$ and $\widetilde{\eta}>\widetilde{0}$. Choose $\varrho>0$ s.t (1) is satisfied. Define a set,

$$M(\varrho,\widetilde{\eta})=\left\{k\in\mathbb{N}: G_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})\leq 1-\varrho \text{ or } B_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})\geq\varrho, Y_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})\geq\varrho\right\},$$

then

$$M^C(\varrho,\widetilde{\eta})=\left\{k\in\mathbb{N}: G_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})> 1-\varrho \text{ and } B_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})<\varrho, Y_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})<\varrho\right\}.$$

Since $\mathcal{S}_\theta(G_N,B_N,Y_N)-\lim_{n\to\infty} v^k_{e_k}=v_e$, so $\delta_\theta(M(\varrho,\widetilde{\eta}))=0$ and $\delta_\theta(M^C(\varrho,\widetilde{\eta}))=1$. Let $p\in M^C(\varrho,\widetilde{\eta})$, then

$$G_N\left(v^p_{e_p}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)>1-\varrho \text{ and } B_N\left(v^p_{e_p}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)<\varrho, Y_N\left(v^p_{e_p}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)<\varrho. \tag{6}$$

Now, let $T(\epsilon,\widetilde{\eta})=\{k\in\mathbb{N}: G_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon$ or $B_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon, Y_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon\}$, then we have to show that $T(\epsilon,\widetilde{\eta})\subseteq M(\varrho,\widetilde{\eta})$. Let $m\in T(\epsilon,\widetilde{\eta})$, then

$$G_N(v^m_{e_m}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon \text{ or } B_N(v^m_{e_m}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon, Y_N(v^m_{e_m}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon. \tag{7}$$

Case 1: If $G_N(v^m_{e_m}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon$, then $G_N\left(v^m_{e_m}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)\leq 1-\varrho$ and therefore $m\in M(\varrho,\widetilde{\eta})$. As otherwise i.e., if $G_N\left(v^m_{e_m}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)>1-\varrho$, then by (1), (6) and (7) we get

$$\begin{aligned}1-\epsilon\geq G_N(v^m_{e_m}\ominus v^p_{e_p},\widetilde{\eta}) &= G_N\left(v^m_{e_m}\ominus v_e\oplus\ v_e\ominus v^p_{e_p},\frac{\widetilde{\eta}}{2}\oplus\frac{\widetilde{\eta}}{2}\right)\\ &\geq G_N\left(v^m_{e_m}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)\circ G_N\left(v^p_{e_p}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)\\ &>(1-\varrho)\circ(1-\varrho)\\ &>1-\epsilon,\end{aligned}$$

which is impossible. Thus, $T(\epsilon,\widetilde{\eta}) \subseteq M(\varrho,\widetilde{\eta})$.
Case 2: If $B_N(v^m_{e_m} \ominus v^p_{e_p},\widetilde{\eta}) \geq \epsilon$, then $B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \varrho$ and therefore $m \in M(\varrho,\widetilde{\eta})$. As otherwise i.e., if $B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) < \varrho$, then by (1), (6) and (7) we get

$$\begin{aligned}\epsilon \leq B_N(v^m_{e_m} \ominus v^p_{e_p},\widetilde{\eta}) = B_N\left(v^m_{e_m} \ominus v_e \oplus\ v_e \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ \leq B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ < \varrho \diamond \varrho \\ < \epsilon,\end{aligned}$$

which is impossible.
Also, If $Y_N(v^m_{e_m} \ominus v^p_{e_p},\widetilde{\eta}) \geq \epsilon$, then $Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \varrho$ and therefore $m \in M(\varrho,\widetilde{\eta})$. As otherwise i.e., if $Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) < \varrho$, then by (1), (6) and (7) we get

$$\begin{aligned}\epsilon \leq Y_N(v^m_{e_m} \ominus v^p_{e_p},\widetilde{\eta}) = Y_N\left(v^m_{e_m} \ominus v_e \oplus\ v_e \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ \leq Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ < \varrho \diamond \varrho \\ < \epsilon,\end{aligned}$$

which is impossible. Thus, $T(\epsilon,\widetilde{\eta}) \subseteq M(\varrho,\widetilde{\eta})$.
Hence in all cases, $T(\epsilon,\widetilde{\eta}) \subseteq M(\varrho,\widetilde{\eta})$. Since $\delta_\theta(M(\varrho,\widetilde{\eta})) = 0$, so $\delta_\theta(T(\epsilon,\widetilde{\eta})) = 0$, and therefore $v = (v^k_{e_k})$ is $\mathcal{S}_\theta(G_N, B_N, Y_N)$ Cauchy.□

Definition 4.2 A $NSNLS$ $\widetilde{V}$ is said to be $\mathcal{S}_\theta$-complete if every $\mathcal{S}_\theta$-Cauchy sequence in $\widetilde{V}$ w.r.t neutrosophic soft norm-(G_N, B_N, Y_N) is $\mathcal{S}_\theta$- convergent w.r.t neutrosophic soft norm-(G_N, B_N, Y_N).

Theorem 4.2 Let $\theta = (k_s)$ be any lacunary sequence. Then every $NSNLS$ $\widetilde{V}$ is $\mathcal{S}_\theta$-complete but not complete in general.

Proof. Let $v = (v^k_{e_k})$ be $\mathcal{S}_\theta$-Cauchy but not $\mathcal{S}_\theta$-convergent w.r.t neutrosophic soft norm-(G_N, B_N, Y_N). For a given $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$. Choose $\varrho > 0$ s.t (1) is satisfied. Now

$$\begin{aligned}G_N(v^k_{e_k} \ominus v^p_{e_p},\widetilde{\eta}) \geq G_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \circ G_N(v^p_{e_p} \ominus v_e,\widetilde{\eta}) > (1-\varrho) \circ (1-\varrho) > 1-\epsilon \\ B_N(v^k_{e_k} \ominus v^p_{e_p},\widetilde{\eta}) \leq B_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \diamond B_N(v^p_{e_p} \ominus v_e,\widetilde{\eta}) < \varrho \diamond \varrho < \epsilon \\ Y_N(v^k_{e_k} \ominus v^p_{e_p},\widetilde{\eta}) \leq Y_N(v^k_{e_k} \ominus v_e,\widetilde{\eta}) \diamond Y_N(v^p_{e_p} \ominus v_e,\widetilde{\eta}) < \varrho \diamond \varrho < \epsilon.\end{aligned}$$

Since $v = (v^k_{e_k})$ is not $\mathcal{S}_\theta$-convergent w.r.t neutrosophic soft norm-(G_N, B_N, Y_N). Therefore $\delta_\theta(H^C(\varrho, \widetilde{\eta})) = 0$, where

$$H(\varrho, \widetilde{\eta}) = \{k \in \mathbb{N} : \mathcal{B}_{v^k_{e_k} \ominus v^p_{e_p}}(\varrho) \leq 1 - \epsilon\}$$

and so $\delta_\theta(H(\varrho, \widetilde{\eta})) = 1$ which is a contradiction, since $v = (v^k_{e_k})$ was $\mathcal{S}_\theta$-cauchy w.r.t neutrosophic soft norm-(G_N, B_N, Y_N). So $v = (v^k_{e_k})$ must be $\mathcal{S}_\theta$-convergent w.r.t neutrosophic soft norm-(G_N, B_N, Y_N). Hence every $NSNLS$ $\widetilde{V}$ is $\mathcal{S}_\theta$-complete.

The following example demonstrates that $NSNLS$ is not complete in general:

Example 4.1[26] Let $\widetilde{U}= (0, 1]$ and $G_N(v, \widetilde{\eta}) = \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus |v|}$, $B_N(v, \widetilde{\eta}) = \frac{|v|}{\widetilde{\eta} \oplus |v|}$, $Y_N(v, \widetilde{\eta}) = \frac{|v|}{\widetilde{\eta}}$ for all $v \in \widetilde{U}$. Then $\widetilde{V}= (\widetilde{U}, G_N, B_N, Y_N, \min, \max)$ is $NSNLS$ but not complete, since the sequence of soft points $(\widetilde{\frac{1}{k}})$ is cauchy w.r.t (G_N, B_N, Y_N) but not convergent w.r.t (G_N, B_N, Y_N).

Theorem 4.3 If every $\mathcal{S}_\theta$-cauchy sequence of soft points in $\widetilde{V}$ has a S_θ-convergent subsequence then $\widetilde{V}$ is S_θ-complete.

Proof. Let $v = (v^k_{e_k})$ be any S_θ-cauchy sequence of soft points in $\widetilde{V}$ which has a $\mathcal{S}_\theta$-convergent subsequence $(v^{k(j)}_{e_{k(j)}})$ i.e., $\mathcal{S}_\theta - \lim_{j\to\infty} v^{k(j)}_{e_{k(j)}} = v_e$ for some v_e in $\widetilde{V}$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$. Choose $\varrho > 0$ s.t (1) is satisfied. Since $v = (v^k_{e_k})$ is $\mathcal{S}_\theta$-cauchy, so $\exists\ n_0 \in \mathbb{N}$ s.t $\forall\ k, p \geq n_0$ $\delta_\theta(A) = 0$ where

$$A = \left\{k \in \mathbb{N} : G_N\left(v^k_{e_k} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) \leq 1 - \varrho \text{ or } B_N\left(v^k_{e_k} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho, Y_N\left(v^k_{e_k} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

Again since $\mathcal{S}_\theta - \lim_{j\to\infty} v^{k(j)}_{e_{k(j)}} = v_e$. So we have $\delta_\theta(B) = 0$, where

$$B = \left\{k(j) \in \mathbb{N} : G_N\left(v^{k(j)}_{e_{k(j)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \leq 1 - \varrho \text{ or } B_N\left(v^{k(j)}_{e_{k(j)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \varrho, Y_N\left(v^{k(j)}_{e_{k(j)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \varrho\right\}.$$

Now define

$$D = \{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\}.$$

We now show that $A^C \cap B^C \subseteq D^C$. Let $m \in A^C \cap B^C$. As $m \in A^C$, so

$$G_N\left(v^m_{e_m} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) > 1 - \varrho \text{ and } B_N\left(v^m_{e_m} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) < \varrho, Y_N\left(v^m_{e_m} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) < \varrho, \tag{8}$$

and since $m \in B^C$, so $m = k(j_0)$ for $j_0 \in \mathbb{N}$ and

$$G_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus\ v_e, \frac{\widetilde{\eta}}{2}\right) > 1 - \varrho \text{ and}$$
$$B_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus\ v_e, \frac{\widetilde{\eta}}{2}\right) < \varrho, Y_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus\ v_e, \frac{\widetilde{\eta}}{2}\right) < \varrho. \tag{9}$$

Now

$$\begin{aligned} G_N(v_{e_m}^m \ominus v_e, \widetilde{\eta}) &= G_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)} \oplus v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\geq G_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)}, \frac{\widetilde{\eta}}{2}\right) \circ G_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &> (1-\varrho) \circ (1-\varrho) \quad \text{for } p = k(j_0) \\ &> 1 - \epsilon \end{aligned}$$

and

$$\begin{aligned} B_N(v_{e_m}^m \ominus v_e, \widetilde{\eta}) &= B_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)} \oplus v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq B_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)}, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \varrho \diamond \varrho \quad \text{for } p = k(j_0) \\ &< \epsilon, \end{aligned}$$

$$\begin{aligned} Y_N(v_{e_m}^m \ominus v_e, \widetilde{\eta}) &= Y_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)} \oplus v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq Y_N\left(v_{e_m}^m \ominus v_{e_{k(j_0)}}^{k(j_0)}, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v_{e_{k(j_0)}}^{k(j_0)} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \varrho \diamond \varrho \quad \text{for } p = k(j_0) \\ &< \epsilon, \qquad\qquad \text{by (1), (8) and (9)} \end{aligned}$$

which implies that $m \in D^C$, so $A^C \cap B^C \subseteq D^C$ or $D \subseteq A \cup B$. Therefore, $\delta_\theta(D) \leq \delta_\theta(A \cup B) = 0$. This shows that $v = (v_{e_k}^k)$ is $\mathcal{S}_\theta$-convergent and therefore, $\widetilde{V}$ is $\mathcal{S}_\theta$-complete.□

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Smart Assessment of Wheat Suppliers via MARCOS-based MCDM Modelling under a Neutrosophic Scenario

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Abstract: Wheat has had a substantial influence on the food security of a number of different nations. In addition, governments are grappling with a number of difficulties, such as fast population expansion, a lack of available water, growing urbanization, and a restricted amount of wheat production in agricultural settings. As a result, the majority of their wheat and wheat products come from outside sources. The purpose of this research is to discover the main wheat suppliers and rate them according to certain criteria by analyzing the different ways of supplier selection that are currently available. The type-2 neutrosophic numbers-Measurement of Alternatives and Ranking according to the Compromise Solution (T2NNs-MARCOS) methodology was used in order to evaluate, choose, and rank the most reliable wheat suppliers in the African and Middle Eastern regions. According to the data, Russia is the country that provides the highest quality wheat. Because of its proximity, its robust connections via official channels, and its adaptability, this provider is often regarded as being the most reliable and cost-effective option. Because wheat is a key commodity, importers, decision-makers, and anyone involved with wheat imports may find this research helpful in identifying and selecting suppliers.

Keywords: Wheat; Neutrosophic; Supplier; Supply chain; T2NNs; MARCOS.

1. Introduction

Wheat is a fundamental commodity in numerous nations, particularly in the regions of the Middle East and Africa, where dietary practices heavily rely on various wheat-based products. Wheat-based products such as bread, pasta, and sweets are commonly consumed as staple food items. Hence, wheat stands as the primary and fundamental commodity subject to governmental oversight, encompassing its importation, storage, and subsequent distribution. The quantity of tender is contingent upon factors such as the existing storage capacity, consumption rate, warehouse management practices, strategic plans for food security, and prevailing storage conditions. Hence, the tender encompasses the expenses associated with procurement, shipment, conveyance, handling, insurance, and additional charges and expenditures. The importation of wheat in the Middle East region exhibits a distinct process, wherein the relevant governmental authorities issue invitations to tender. Subsequently, applicants are required to select suppliers based on the specified conditions. This signifies that the government does not directly determine the supplier, but rather, the responsibility lies with the applicant or bidder to make the selection.

It is imperative to establish explicit terms and conditions for the tender process, encompassing various aspects such as specifications, quality requirements, timelines, supplier solvency, procedural requirements, contractual and financial considerations, as well as essential tests and acceptance criteria. After the tender has been awarded, it is imperative for the relevant authorities to adhere to the specified guidelines for the storage and distribution of wheat, in accordance with the established principles governing this process. Additionally, it is imperative to guarantee the presence of a strategic inventory of said product for specific timeframes, typically ranging from six months to a minimum of one year. The primary specifications for wheat encompass its origin, protein content, test weight, moisture level, purity, fall number, wet gluten content, presence of soft grain admixture, foreign matter, and grain admixture. The primary factors that determine the quality of processing are grain hardness, protein concentration and quality, and gluten strength.

Therefore, the primary objective of this study is to address this research gap by providing answers to the following research inquiries: The supplier selection process encompasses various approaches and stages. Which wheat suppliers offer high-quality products at the most competitive prices and provide flexible delivery options? What are the appropriate criteria for assessing suppliers? Based on the prevailing international environment and situation, an inquiry is made regarding the most reputable wheat suppliers in the Middle East.

One of the key challenges encountered in the process of decision-making is the identification and selection of the most optimal alternative, which necessitates the careful consideration of numerous selection criteria [1], [2]. Multi-criteria decision-making (MCDM) techniques are frequently employed to effectively manage a diverse range of decision-making criteria [3], [4]. The extensive utilization of these techniques in the supply chain domain can be attributed to their computational capabilities [5].

The primary purpose of this research was to determine the most important wheat suppliers in the Middle East and Africa via the use of MCDM methods and to rank those suppliers according to the features that were discovered. Wheat is an essential agricultural product across the nations that make up the Middle East, and the government is in charge of bringing it in, regulating it, and storing it. The purpose of this research was to investigate different wheat suppliers in light of established standards. This research gives a comprehensive framework for the selection of suppliers, which may be used to effectively find suppliers of wheat as well as other items, products, or materials and to reduce the risks associated with the selection process. The type-2 neutrosophic numbers-Measurement of Alternatives and Ranking according to the Compromise Solution (T2NNs-MARCOS) MCDM methodology was used throughout the evaluation, selection, and ranking of the most effective wheat providers [6], [7]. It was determined using a numerical case study which wheat suppliers were the most important, and then it was determined which wheat supplier was the best based on the features that were determined. The neutrosophic set applied in many applications like [8], [9][10]–[13]

The remainder parts of the research are planned as follows: Section 2 develops the applied approach for selecting a suitable supplier of wheat. Section 3 employs a real case study for applying the suggested methodology and analysis of the results. Section 5 concludes the research.

2. Methodology

In this section, the proposed methodology to solve the problem of selecting and determining the best wheat supplier is presented. The proposed methodology is based on the MARCOS method. The proposed methodology consists of several stages. The first stage presents the details of the study and the selection of experts. The second stage is related to determining the weights of the criteria used in the study. The third and final stage is related to the arrangement of the alternatives chosen in the study. Figure 1 provides details of the proposed methodology.

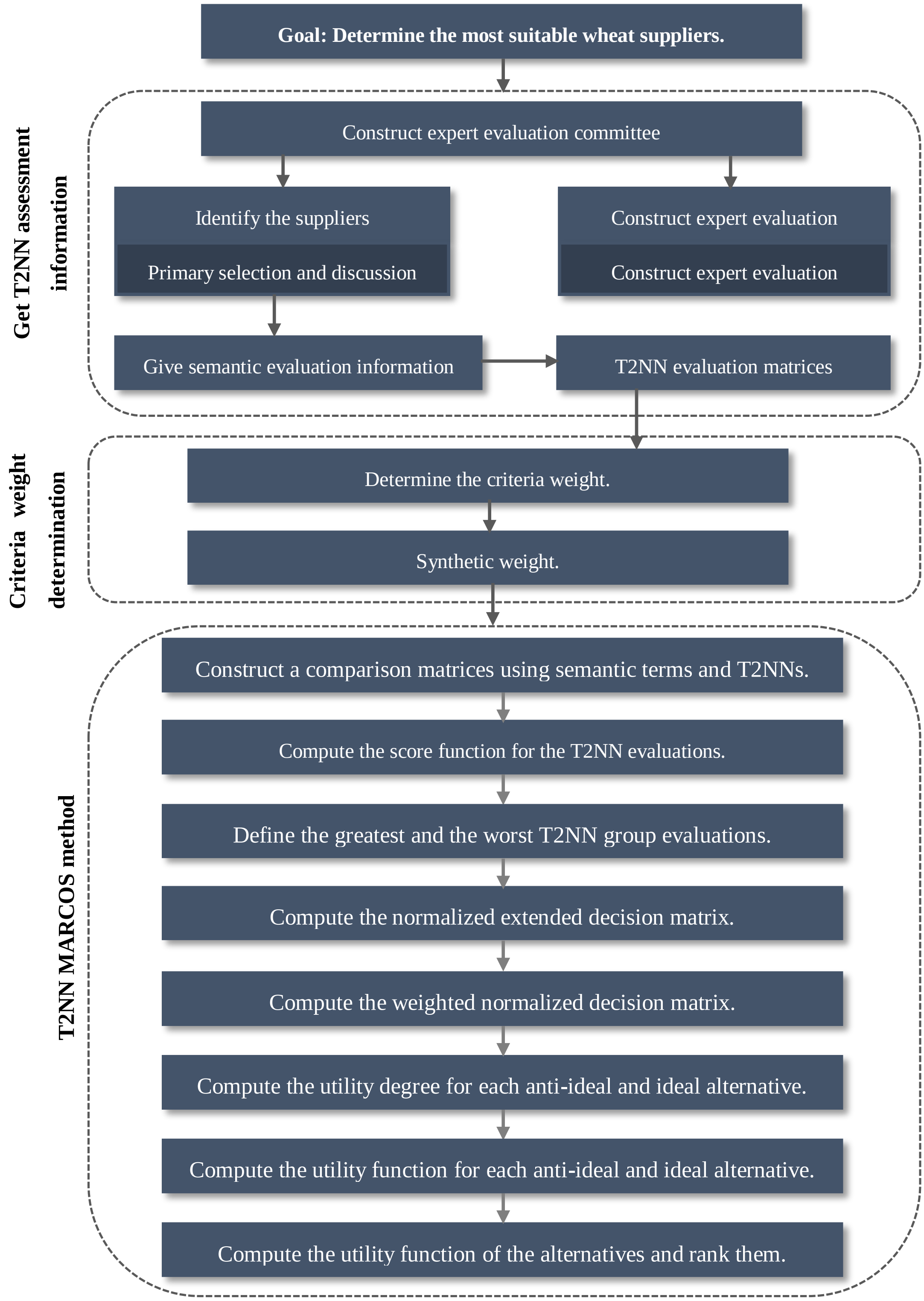

Figure 1. Details of the proposed methodology.

Step 1. The problem is studied in detail and the participating experts are identified as shown in Table 1. The participating experts give their opinions on the problem and define the criteria and available alternatives. Suppose a set of m alternatives is represented by $A = \{A_1, \ldots, A_i, \ldots \; A_m\}$ and a set of n criteria is denoted by $C = \{C_1, \ldots, C_n, \ldots, C_n\}$. Let experts = $\{E_1, \ldots, E_e, \ldots, E_k\}$ be a set of experts who offered their valuation report for each alternative $A_i(i = 1, 2 \ldots m)$ against their criteria $C_j(j = 1, 2 \ldots n)$. Let $w = (\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_e)^T$ be the weight vector for experts $E_e(e = 1, 2 \ldots k)$ such that $\sum_{j=1}^{n} \mathrm{w}_l = 1$.

Table 1. Details on the participants of the panel of experts.

Expert	Experience	Occupation	Profession	Gender
$Expert_1$	5	Industry	Government policy maker	Male
$Expert_2$	12	Academia		Male
$Expert_3$	11	Industry		Male
$Expert_4$	8	Academia		Male

Step 2. A set of variables and their corresponding T2NNs are identified as shown in Table 2, for experts to use in evaluating the selected criteria and alternatives.

Table 2. T2NN semantic terms for weighing dimensions and alternatives.

Semantic terms	Abridgements	Type-2 neutrosophic number
Exceedingly little	EXC	$\langle(0.20, 0.20, 0.10); (0.65, 0.80, 0.85); (0.45, 0.80, 0.70)\rangle$
Little	LLE	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$
Moderate little	MOL	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$
Moderate	MOD	$\langle(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)\rangle$
Moderate high	MOH	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$
High	HIG	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$
Exceedingly high	EXH	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$

Step 3. Construct a preference matrix of criteria by experts to show their preferences to determine the criteria weights using the linguistic terms, then by using T2NNs.

Step 4. Compute the score function for the T2NN assessments according to Eq. (1) [14].

$$S(\tilde{X}_{ij}) = \frac{1}{12}\left\langle 8 + \left(T_{T_{\tilde{X}_{ij}}}(y) + 2\left(T_{I_{\tilde{X}_{ij}}}(y)\right) + T_{F_{\tilde{X}_{ij}}}(y)\right) - \left(I_{T_{\tilde{X}_{ij}}}(y) + 2\left(I_{I_{\tilde{X}_{ij}}}(y)\right) + I_{F_{\tilde{X}_{ij}}}(y)\right) - \left(F_{T_{\tilde{X}_{ij}}}(y) + 2\left(F_{I_{\tilde{X}_{ij}}}(y)\right) + F_{F_{\tilde{X}_{ij}}}(y)\right)\right\rangle, \ i = 1, \ldots, m; \ j = 1, \ldots, n. \quad (1)$$

Step 5. Determine the best and the worst T2NN assessments according to the extended T2NN decision matrix for denoting the ideal (AI) and anti-ideal (AAI) alternatives, respectively according to Eqs. (2) and (3).

The anti-ideal substitute $A_0 = \{X_{01}, \ldots, X_{0j}, \ldots, X_{0n}\}$

$$A_{0j} = \begin{cases} \max\limits_{1\le i\le m} X_{ij} & |C_j \in C^- \\ \max\limits_{1\le i\le m} X_{ij} & |C_j \in C^+ \end{cases}, \ j = 1, \ldots, n.$$

(2)

where A_{0j}(j = 1, ..., n) designates anti-ideal group estimations under each criterion.

The ideal substitute $A_{m+1} = \{X_{m+1\ 1}, \ldots, X_{m+1\ j}, \ldots, X_{m+1\ n}\}$

$$A_{m+1\ j} = \begin{cases} \max\limits_{1\le i\le m} X_{ij} & |C_j \in C^- \\ \max\limits_{1\le i\le m} X_{ij} & |C_j \in C^+ \end{cases}, \ j = 1, \ldots, n.$$

(3)

where $A_{m+1\ j}$(j = 1,..., n) indicates ideal group evaluations under each criterion.

Step 6. Calculate the normalized decision matrix according to Eq. (4).

$$R_{ij} = \begin{cases} \frac{x_{ij}}{x_{m+ij}} & |C_j \in C^+ \\ \frac{x_{m+ij}}{x_{ij}} & |C_j \in C^- \end{cases}, \text{ i = 0, ..., m + 1; j = 1, ..., n.} \quad (4)$$

Step 7. Calculate the weighted normalized decision matrix according to Eq. (5).

$$S_{ij} = w_j \ S_{ij}, \text{ i = 0, ..., m + 1; j = 1, ..., n.} \quad (5)$$

Step 8. Compute the utility degree for each anti-ideal substitute according to Eq. (6).Then, compute the utility degree for each ideal substitute according to Eq. (7).

$$U^-_i = \frac{\sum_{j=1}^{n} s_{ij}}{\sum_{j=1}^{n} s_{0j}}, \text{ i = 0, ..., m + 1.} \quad (6)$$

$$U^+_i = \frac{\sum_{j=1}^{n} s_{ij}}{\sum_{j=1}^{n} s_{m+1\ j}}, \text{ i = 0, ..., m + 1.} \quad (7)$$

Step 9. Compute the utility function for each anti-ideal alternative according to Eq. (8).Then, compute the utility function for each ideal substitute according to Eq. (9).

$$f(U^-) = \frac{U^+_0}{U^-_0 + U^+_0} \quad (8)$$

$$f(U^+) = \frac{U^-_{m+1}}{U^-_{m+1} + U^+_{m+1}} \quad (9)$$

Step 10. Compute the utility function of the substitutes and rank them by employing Eq. (10). The optimal substitute has the highest utility function.

$$f(U_i) = \frac{(U^-_i + U^+_i)[f(U^-) \times f(U^+)]}{f(U^-) + f(U^+) - f(U^-) \times f(U^+)}, \text{ i = 1, ..., m.} \quad (10)$$

3. Application

3.1 Case Study

Most countries in the Middle East and Africa rely on wheat in their daily diet. Wheat, flour, and bread are staples that may be found on most people's dinner tables. The variety, quality, purchase prices from the source, transportation fees, loading and unloading charges, and other considerations such as delivery intervals all play a role in determining the source of wheat. Wheat production follows the cycles of the seasons, and storage capacity are often restricted or only enough for a range of time spans. Wheat prices fluctuate across the world based on the variety being purchased and the accepted level of quality. Wheat is normally divided into two categories: hard and soft. When choosing wheat suppliers for the Middle East and Africa, it is important to keep in mind that many nations in North America and Europe control the majority of the wheat supply chain. Wheat has been negatively affected by COVID-19 since it caused crop harvesting to be delayed, and the subsequent lockdown had an effect on both the supply chain and price. The extent to which wheat can be grown has a considerable bearing on the wheat supply chain's ability to continue operating profitably. As a result, the wheat supply chain has to commit to and actively engage in innovations that are sustainable via joint efforts. When doing an investigation to determine who the primary source of wheat is or how the various suppliers stack up against one another, each of these aspects should be taken into consideration. In addition, the identification and selection of the primary wheat suppliers in the Middle East and Africa may be impacted in the future by developments and risks that are both anticipated and unanticipated. In this study, we seek to assess four countries as suppliers of wheat. The four countries are Romania (A_1), Australia (A_2), Russia (A_3), and Ukraine (A_4).

3.2 Application of the proposed methodology

In this part, the steps of the proposed approach are applied to evaluate and select the most suitable country as a supplier of wheat for the countries of the Middle East and Africa.

Step 1. In the beginning, the problem and its main and subsidiary aspects were studied. In this regard, four experts were selected, as shown in Table 1, for the participation of the authors in expressing their views on the importance of the criteria, the arrangement of alternatives, and other matters related to the study.

Step 2. Seven semantic terms and their corresponding T2NNs were identified as in Table 2, to be used by experts in evaluating the criteria, determining their weights, and arranging the four selected alternatives.

Step 3. Seven criteria have been identified that have a direct impact on choosing the best country as a supplier of wheat. The seven selected criteria are Quality (C_1), Expenses (price and costs) (C_2), Delivery (time, place, and amount) (C_3), Origin (source country) (C_4), Flexibility (C_5), Communication (C_6), and Reliability/solvency of the importer (C_7). In addition, four alternatives were selected to be used in the evaluation process. The four alternatives selected are Romania (A_1), Australia (A_2), Russia (A_3), and Ukraine (A_4).

Step 4. An evaluation matrix was constructed by the four experts to show their preferences for the seven criteria using linguistic terms as in Table 3, then by using T2NNs as presented in Table 4.

Step 5. The T2NNs were converted to real values using Eq. (1), and the final weights for the seven criteria were determined as exhibited in Table 4 and Figure 2.

Table 3. Assessment matrix of criteria by the four experts using semantic terms.

Experts	Criteria						
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
Expert_1	MOL	EXH	HIG	LLE	MOL	EXH	HIG
Expert_2	EXC	MOL	EXH	LLE	LLE	LLE	EXC
Expert_3	EXH	EXH	MOH	HIG	LLE	LLE	MOL
Expert_4	HIG	EXH	EXH	MOH	EXH	MOH	EXH

Table 4. Assessment matrix of criteria by the four experts using T2NNs.

Experts	Criteria	
	C_1	C_2
Expert_1	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$
Expert_2	$\langle(0.20, 0.20, 0.10); (0.65, 0.80, 0.85); (0.45, 0.80, 0.70)\rangle$	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$
Expert_3	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$
Expert_4	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$
Weight	0.139	0.185
Experts	C_3	C_4
Expert_1	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$
Expert_2	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$
Expert_3	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$
Expert_4	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$
Weight	0.192	0.122
Experts	C_5	C_6
Expert_1	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$
Expert_2	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$
Expert_3	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$
Expert_4	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$
Weight	0.114	0.129
Experts	C_7	
Expert_1	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$	
Expert_2	$\langle(0.20, 0.20, 0.10); (0.65, 0.80, 0.85); (0.45, 0.80, 0.70)\rangle$	
Expert_3	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$	
Expert_4	$\langle(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)\rangle$	
Weight	0.119	

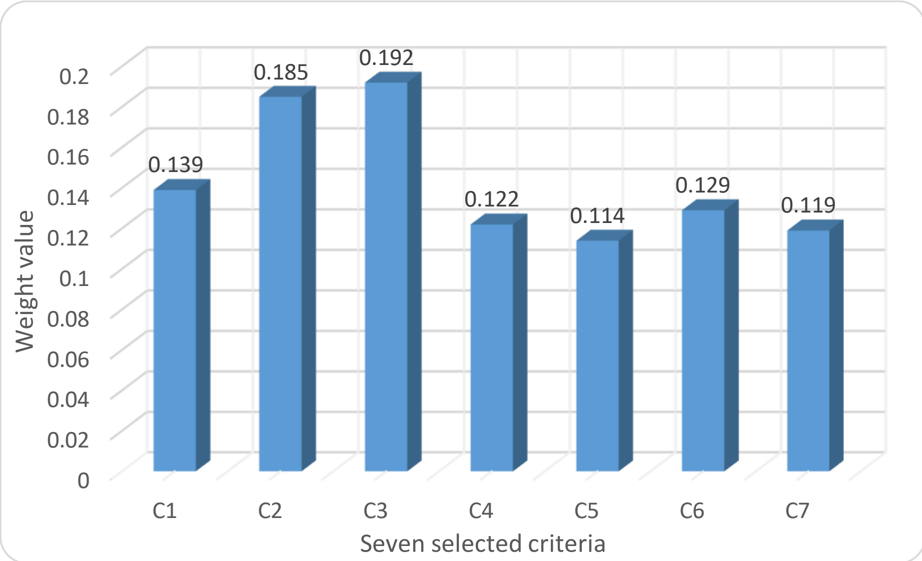

Figure 2. Weights of criteria.

Step 6. An evaluation matrix was constructed by the four experts to show their preferences for the four alternatives regarding the seven criteria using linguistic terms as in Table 5, then by using T2NNs as presented in Table 6.

Step 7. The T2NNs were converted to real values using Eq. (1).

Step 8. The best and the worst T2NN assessments according to the T2NN decision matrix were determined for denoting the AI and AAI substitutes, respectively according to Eqs. (2) and (3), as presented in Table 7.

Step 9. The normalized decision matrix was computed according to Eq. (4) as presented in Table 7.

Step 10. The weighted normalized decision matrix was computed according to Eq. (5) as presented in Table 8.

Step 11. The utility degree for each anti-ideal substitute was computed according to Eq. (6), as presented in Table 9. Then, the utility degree for each ideal substitute was computed according to Eq. (7), as presented in Table 9.

Step 12. The utility function for each anti-ideal alternative was computed according to Eq. (8), as presented in Table 9. Then, the utility function for each ideal substitute was computed according to Eq. (9), as presented in Table 9.

Step 13. The utility function of the substitutes was computed according to Eq. (10), as presented in Table 9. The alternatives were ranked as presented in Table 9 and shown in Figure 3.

Table 5. Assessment matrix of the four alternatives by the four experts using semantic terms.

Experts	Criteria						
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	EXH	MOL	EXH	HIG	HIG	MOH	EXH
A_2	HIG	MOH	EXH	MOH	HIG	HIG	MOH
A_3	MOL	MOH	LLE	EXC	EXC	MOD	LLE
A_4	MOH	MOD	MOD	MOH	LLE	HIG	MOD

Table 6. Assessment matrix of the four alternatives by the four experts using T2NNs.

Alternatives	Criteria	
	C_1	C_2
A_1	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$
A_2	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$

A_3	⟨(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)⟩	⟨(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)⟩
A_4	⟨(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)⟩	⟨(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)⟩
Alternatives	**Criteria**	
	C_3	C_4
A_1	⟨(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)⟩	⟨(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)⟩
A_2	⟨(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)⟩	⟨(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)⟩
A_3	⟨(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)⟩	⟨(0.20, 0.20, 0.10); (0.65, 0.80, 0.85); (0.45, 0.80, 0.70)⟩
A_4	⟨(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)⟩	⟨(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)⟩
Alternatives	**Criteria**	
	C_5	C_6
A_1	⟨(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)⟩	⟨(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)⟩
A_2	⟨(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)⟩	⟨(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)⟩
A_3	⟨(0.20, 0.20, 0.10); (0.65, 0.80, 0.85); (0.45, 0.80, 0.70)⟩	⟨(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)⟩
A_4	⟨(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)⟩	⟨(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)⟩
Alternatives	**Criteria**	
	C_7	
A_1	⟨(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)⟩	
A_2	⟨(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)⟩	
A_3	⟨(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)⟩	
A_4	⟨(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)⟩	

Table 7. Normalized matrix of the four alternatives according to all criteria.

Experts	Criteria						
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
AAI	0.495	0.648	0.333	0.296	0.296	0.716	0.333
A_1	0.495	1.000	0.333	0.296	0.296	0.817	0.333
A_2	0.568	0.648	0.333	0.338	0.296	0.716	0.437
A_3	1.000	0.648	1.000	1.000	1.000	1.000	1.000
A_4	0.648	0.793	0.534	0.338	0.774	0.716	0.534
AI	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 8. Weighted normalized matrix of the four alternatives according to all criteria.

Experts	Criteria						
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
AAI	0.069	0.120	0.064	0.036	0.034	0.092	0.040
A_1	0.069	0.185	0.064	0.036	0.034	0.105	0.040
A_2	0.079	0.120	0.064	0.041	0.034	0.092	0.052
A_3	0.139	0.120	0.192	0.122	0.114	0.129	0.119
A_4	0.090	0.147	0.103	0.041	0.088	0.092	0.064
AI	0.139	0.185	0.192	0.122	0.114	0.129	0.119

Table 9. Final ranking of the four alternatives.

Alternatives	O_i	U_i^-	U_i^+	$f(U^-)$	$f(U^-)$	$f(U_i)$	Rank
AAI	0.455						
A_1	0.533	1.172	0.533	0.313	0.687	0.466	3
A_2	0.482	1.061	0.482	0.313	0.687	0.422	4
A_3	0.935	2.057	0.935	0.313	0.687	0.819	1
A_4	0.625	1.375	0.625	0.313	0.687	0.547	2
AI	1.000						

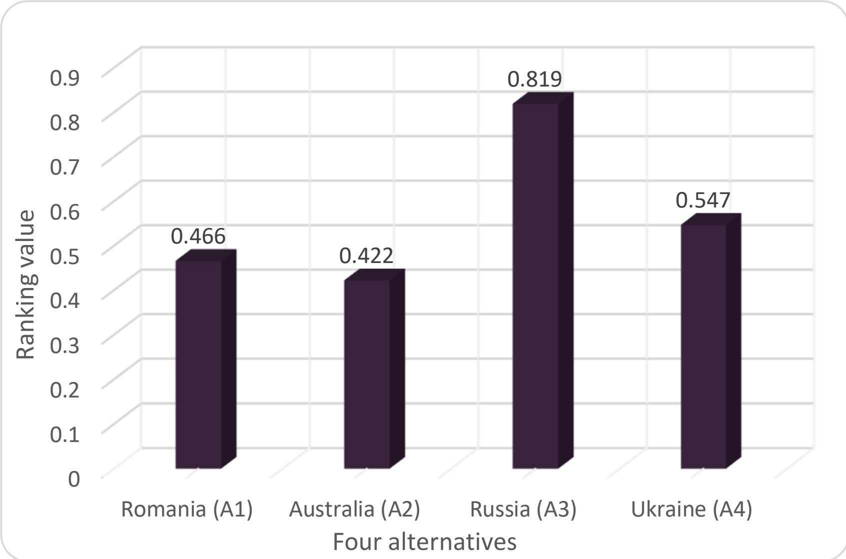

Figure 3. Final ranking of the four alternatives.

3.3 Results and discussion

In this part, the results obtained from the application of the proposed model to evaluate and determine the most suitable countries supplying wheat to the countries of the Middle East and Africa are discussed. The results are divided into two parts. The first part is concerned with evaluating the seven criteria and determining the weights. The seven criteria were evaluated through expert opinions as shown in Table 4. The results indicate that the Delivery criterion (time, place, and amount) (C_3), is the criterion with the highest weight by 0.192, followed by the Expenses criterion (price and costs) (C_2) with a weight of 0.185, while the Flexibility criterion (C_5) has the least weight by 0.114.

The second part is concerned with evaluating the four alternatives selected in the study. The four selected alternatives were arranged as shown in Table 9 and Figure 3. The results show that Russia (A_3) is the highest in the order, followed by Ukraine (A_4), while Australia (A_2) is the lowest in the order.

4. Conclusions

Wheat is a fundamental and significant product that is used in the majority of countries, including those in the Middle East and Africa, where derivatives of wheat are almost always present on dining tables. Because of this, the governments are able to maintain a consistent supply of wheat via the processes of importing, storing, and distributing it. The supply chain for wheat has a considerable influence not just on environmental sustainability but also on the safety of food supplies. In addition, nations are confronted with a number of issues, some of which include a fast-expanding population, considerable urbanization, a lack of water, and poor soil quality. Despite the ever-increasing need for food, agriculture is not a viable solution to the problem. In addition, the choice of supply is affected by a broad variety of variables, such as the price of the product at issue, the number of producers, the cost of inputs, technical advancements, the cost of alternative goods, and unpredictability in the form of the weather. This research addresses a knowledge gap regarding the ranking or selection of top wheat suppliers for the African area as well as the Middle Eastern region. This research examines alternatives to wheat suppliers based on recognized needs. This is important in light of the fact that wheat is seen as an essential food item in the Middle East. Given that governments are in charge of importing, managing, and storing wheat, this is of the utmost importance.

The main objective of the study is to identify and choose the most suitable wheat suppliers from the four countries used in the study. The four countries identified in the evaluation process are Russia, Romania, Ukraine, and Australia. Also, seven basic criteria were identified in selecting the most suitable suppliers. The evaluation process was conducted in a neutrosophic environment and by applying the MARCOS method to determine the most suitable countries for wheat supply.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Neutrosophic Multi-Criteria Decision Making for Sustainable Procurement in Food Business

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Abstract: Purchasing food in a way that minimizes negative effects on the environment, society, and the economy is a growing trend in the food industry. Sustainable procurement is discussed in this study, along with its significance, important criteria, and advantages in the food business. Businesses may aid sustainable development, lessen their impact on the environment, provide aid to local communities, and keep up with shifting consumer expectations for sustainably and ethically produced food when they prioritize responsible sourcing practices. To effectively implement sustainable procurement in the food sector, this article stresses the need for teamwork, openness, and a long-term commitment to sustainability. So, this paper ranks the best supplier in sustainable procurement in the food business to achieve sustainability. The concept of multi-criteria decision-making (MCDM) is used in this paper to deal with the various criteria. This paper used the TOPSIS method as an MCDM tool to compute the weights of criteria and rank the suppliers. The TOPSIS method is integrated with the single-valued neutrosophic set to deal with uncertain and vague information. There are seven criteria and 10 suppliers in the food business are evaluated and ranked in this study. We obtained the environmental impacts as the best criteria in seven criteria. The goal of environmental impact prioritizing suppliers and products that minimize negative environmental impact.

Keywords: Neutrosophic Set, MCDM, TOPSIS, Procurement, Sustainability

1. Introduction

As businesses become more aware of the environmental, social, and financial consequences of their supply chains, they are beginning to prioritize sustainable procurement practices within the food industry.

Businesses in the food industry, such as restaurants, caterers, and grocery stores, may have a significant impact on global sustainability by shifting to ethical purchasing policies. Sustainable food procurement involves making ethical and ecologically sound decisions throughout the manufacturing, distribution, consumption, and disposal of food items. This article delves into the topic of sustainable procurement in the food sector, discussing its value, obstacles, and recommendations for moving forward. Businesses in the food industry may improve their environmental impact, give back to their communities, and satisfy customer demand for sustainably and ethically sourced products by giving sustainable procurement first priority[1], [2].

Deforestation, greenhouse gas emissions, and water pollution are just some of the ways in which the food business is damaging the environment. Sustainable and sustainably sourced food items are in high demand as consumer knowledge of the environmental and social implications of food production increases. The key to environmentally, socially, and economically responsible food procurement is to take into account all stages of the supply chain, from raw material sourcing through final retail packaging. The focus is on long-term viability rather than short-term gains in efficiency or quality[3], [4].

The mitigation of negative effects on the environment is a major advantage of sustainable procurement in the food business. Organic farming and regenerative agriculture are two examples of sustainable agriculture that help companies reduce their chemical footprint, save biodiversity, and preserve scarce natural resources. Waste is reduced and landfill contributions are decreased because of sustainable procurement's emphasis on responsible waste management and the promotion of environmentally friendly packaging materials[5], [6].

The importance of social responsibility in sustainable food purchases cannot be overstated. Businesses may aid in the growth of their communities by investing in the agricultural sector. Farmers may be protected from exploitation and paid fairly for their goods with the help of fair trade practices and ethical sourcing. In addition, by prioritizing universal access to safe, healthy, and reasonably priced food, sustainable procurement may contribute to solving problems of food security and food justice[7], [8].

There are a number of obstacles that must be overcome before the food business can adopt sustainable buying practices. One major challenge is the proliferation of middlemen and international sourcing networks that characterize modern supply chains. It might be difficult to ensure traceability and transparency across the supply chain, but new tools like blockchain and digital tracking systems are making it easier than ever. It is important for firms to weigh the long-term advantages against the potential additional expenses of obtaining sustainable goods, and to explore opportunities for cooperation and partnership to take advantage of economies of scale[9], [10].

There are a variety of approaches that companies may take to sustainable buying in the food sector. Establishing connections with certified sustainable suppliers, developing explicit sustainability standards for suppliers, and performing frequent audits and evaluations are all crucial. Sustainable practices across the supply chain can only be driven by encouraging supplier participation and cooperation. Moreover, companies may guarantee that procurement choices are consistent with sustainable values by investing in staff training and education[11], [12].

Sustainable food procurement practices are mostly driven by consumer demand. Sustainable food enterprises may gain an advantage as consumers grow more aware of the ecological and social consequences of their purchases. Increased brand reputation and customer satisfaction may result from open communication regarding sustainable sourcing practices and certifications that have been earned by the company[13], [14].

In this paper, we improve the supply chain by selecting the best suppliers in sustainable procurement in the food business. There are various criteria for sustainable procurement in the food business so, we used the concept of multi-criteria decision-making (MCDM) to deal with various criteria[15], [16].

In light of these considerations, the proper handling of uncertainties or imprecision has emerged as a critical problem in MCDM analysis. The single-valued neutrosophic set (SVNS) suggested by Smarandache and Wang et al. is one such tool for capturing such uncertainties or imprecision information[17], [18]. The SVNS, a novel and practical extension of fuzzy sets, is distinguished by the strength of the relationships between its truth-member, indeterminacy-member, and falsity-member. The SVNS seems to be more successful at dealing with uncertain information than other fuzzy tools like the intuitionistic fuzzy set (IFS) and the Pythagorean fuzzy set (PFS), as it can deal with indeterminate information that IFS and PFS cannot. According to this new line of inquiry, SVNS theory may be used to MCDM issues even while facing ambiguity and complexity[19], [20].

The paper is organized as follows: section 2 provides the challenges in the food business. The proposed method in the neutrosophic TOPSIS method is organized in section 3. The results and discussion of the proposed method are presented in section 4. Section 5 presented the conclusions of this study.

2. Challenges in Food Business

Many obstacles might arise when companies strive to practice sustainable buying in the food sector. Some typical difficulties encountered by the food industry are listed below.

Tracing the origin and viability of food items may be difficult because of the food industry's notoriously complicated and worldwide supply networks, which sometimes include several middlemen. When working with several suppliers with different data availability, it may be challenging to maintain supply chain transparency and traceability[21], [22].

Consequences on Expenditures Sustainable product sourcing and working with certified suppliers may cost more than traditional product procurement in certain cases. Some organizations, particularly those with slim profit margins, may be put off by the initial investment or additional expenditures associated with sustainable buying practices.

Supply Chain Challenges It may be difficult for businesses to locate suppliers who match the requirements for sustainable procurement, especially if they need a big quantity of a certain product. An obstacle to implementation may be the scarcity of sustainable suppliers in a certain area or for a given component[23], [24].

The tastes and expectations of consumers change with time, and businesses must be prepared to respond by offering more and more sustainably and ethically based goods. Successfully navigating customers' ever-evolving expectations and communicating the company's commitment to sustainable sourcing is essential for gaining their confidence and loyalty.

Sustainable buying necessitates weighing several variables, including the effect on the environment, the impact on society, and the profitability of the business. It might be difficult to strike a balance between competing needs. Environmental concerns about transportation emissions, for instance, may collide to prioritize local sources[24], [25].

It might be difficult to get suppliers on board with adopting sustainable procedures and standards for both production and business. It calls for establishing reliable connections, inspecting suppliers, and encouraging cooperation for ongoing improvement. However, not all vendors can easily adapt to new conditions or fulfill stringent environmental standards.

The process of verifying and certifying a company's sustainability claims may be time-consuming and costly. Additional time, money, and knowledge may be needed to ensure that all sustainable practices and certifications are being adhered to.

The value of sustainability, the criteria for sustainable sourcing, and the advantages of sustainable procurement can only be fully realized if all personnel in an organization are educated and trained in these areas. In bigger organizations with more varied teams and stakeholders, it may be difficult to ensure that everyone has the same knowledge of and commitment to sustainability.

Collaboration among stakeholders, utilizing technology for traceability and transparency, seeking partnerships and collaborations, and incorporating sustainability concerns into the core business plan are just some of the avenues that may be pursued to address these difficulties. Food companies may set the path for good change in the food sector by overcoming these challenges to sustainable procurement[10], [13].

3. Neutrosophic TOPSIS Method

To deal with the MCDM difficulties precisely, the traditional TOPSIS developed by Hwang and Yoon has been widely used. Distance from the negative ideal solution (NIS) to the positive ideal solution (PIS) is used in this method to evaluate alternatives (and ultimately choose the best one). The optimal option(s) will be those that both minimize the travel time to the PIS and maximize the travel time to the NIS[26], [27] [28],[29]. The following steps explain how Ye modified the traditional TOPSIS approach to work in an SVNLS setting as shown in Figure 1.

Step 1. Build the decision matrix

$$X^e = \begin{bmatrix} x_{11}^{(e)} & \cdots & x_{1n}^{(e)} \\ \vdots & \ddots & \vdots \\ x_{m1}^{(e)} & \cdots & x_{mn}^{(e)} \end{bmatrix} \quad (1)$$

Where $x_{ij}^{(e)} =< \left(T_{x_{ij}}^{e}, I_{x_{ij}}^{e}, F_{x_{ij}}^{e}\right) > , i = 1,2,3 \ldots m \ (alternatives); j = 1,2,3, \ldots n \ (criteria)$

Step 2. Normalize the decision matrix

The normalization matrix is built based on positive and negative criteria.

$$L = \begin{bmatrix} l_{11}^{(e)} & \cdots & l_{1n}^{(e)} \\ \vdots & \ddots & \vdots \\ l_{m1}^{(e)} & \cdots & l_{mn}^{(e)} \end{bmatrix} \quad (2)$$

Step 3. Combined the decision matrix

$$L = \begin{bmatrix} l_{11} & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{m1} & \cdots & l_{mn} \end{bmatrix} \quad (3)$$

Where $l_{11} = \sum_{e=1}^{d} w_e * l_{ij}^{(e)}$, d refers to the number of decision makers.

Step 4. Compute the weights of criteria

Step 5. Compute the weighted decision matrix

$$G = \begin{bmatrix} w_1 l_{11} & \cdots & w_n l_{1n} \\ \vdots & \ddots & \vdots \\ w_1 l_{m1} & \cdots & w_n l_{mn} \end{bmatrix} \quad (4)$$

Step 6. Compute the distance between alternatives ($S_i(i = 1,2,3, \ldots . m)$) and positive and negative criteria

$$T(S_i, S^+) = \sum_{j=1}^{n} t\left(g_{ij}, g_i^+\right) \quad (5)$$

$$T(S_i, S^-) = \sum_{j=1}^{n} t\left(g_{ij}, g_i^-\right) \quad (6)$$

Step 7. Calculate the coefficient of closeness value

$$F(S_i) = \frac{T(S_i, S^-)}{T(S_i, S^+) + T(S_i, S^-)} \quad (7)$$

Step 8. Order the suppliers

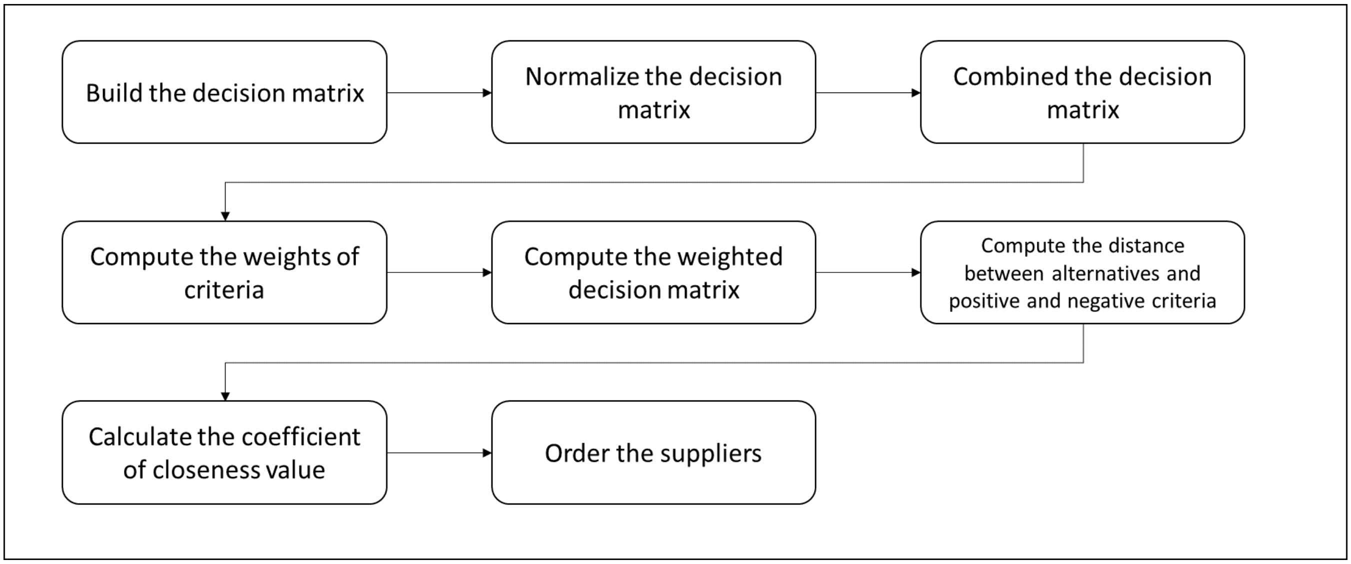

Figure 1. The Steps of the single valued neutrosophic TOPSIS method.

4. Results

This section introduces the results of the proposed method. This paper used single-valued neutrosophic numbers to evaluate the criteria and suppliers. There are various experts in the field of supply chain in the food business to evaluate the criteria and suppliers. This study gathered seven criteria from previous studies to evaluate it and ten suppliers. First, we compute the weights of these criteria, then rank and select suppliers in the food business to achieve sustainable procurement. There are seven criteria organized as:

Organic and regenerative farming practices have a positive effect on the environment because they improve soil quality, increase biodiversity, and reduce the need for synthetic chemicals.

You may help protect marine habitats and support local economies by purchasing seafood from sustainable fisheries and aquaculture businesses.

Favor vendors that have integrated water and energy-saving practices throughout their whole manufacturing operations.

Accountability to Society:

If you care about things like fair salaries, safe working conditions, and the absence of child labor, you should support businesses that source their goods ethically.

Priorities purchasing from regional farmers and manufacturers to bolster regional economies, cut down on carbon emissions from transportation, and foster growth in existing communities.

Support vendors that value diversity and inclusion in their workforce, and who seek to ensure that all of their workers are afforded the same respect and opportunity.

Purchase meat, dairy, and eggs from farms that place a premium on animal welfare and adhere to established industry guidelines for humane animal care.

Favor vendors that raise their animals without confining them in cages and instead provide them access to outdoor areas where they may forage and engage in other natural behaviors.

Sustainable packaging is selecting vendors whose packaging is either fully or partially recyclable, compostable, or biodegradable. This helps reduce landfill trash and supports the circular economy.

Reduce your impact on the environment by supporting businesses that recycle and compost food scraps and other organic waste, as well as packaging and other items.

Look for vendors that have supplier certifications like USDA Organic, Fair Trade, MSC (Marine Stewardship Council), or Rainforest Alliance to know that they engage in ethical and sustainable practices.

Make that your suppliers are abiding by all applicable laws and regulations about food quality and safety, as well as the environment and workers' rights.

Transparency and tractability

Seeing the whole supply chain: If you want to know where your food came from and how it was made, you need to find a supplier that can tell you.

Regular audits and inspections of suppliers are necessary to guarantee compliance with sustainability standards and maintain supply chain transparency.

Effortless Updating:

Inspire your suppliers to work together on sustainability projects and to brainstorm new ways to solve environmental and social problems so that everyone benefits.

To ensure ongoing development and accountability, it is important to set up systems for tracking supplier performance and encouraging frequent reporting on sustainability measures.

To motivate real change and advance sustainability in the food sector, organizations must set their sustainable procurement criteria, communicate them clearly to suppliers, and periodically analyze and evaluate supplier compliance.

Then we applied the SVNS TOPSIS method to show the weights of the criteria and rank the suppliers. There are seven criteria and ten suppliers in this study.

Step 1. Build the decision matrix

We used three decision-makers who have expertise in the food business to rank the criteria of sustainable procurement in the food business and suppliers. Then we built the decision matrix between criteria and suppliers based on the opinions of three decision-makers by using Eq. (1).

Step 2. Normalize the decision matrix

Then we normalized the decision matrix by using Eq. (2) as shown in Table 1.

Table 1. Normalized decision matrix

	$SPFB_1$	$SPFB_2$	$SPFB_3$	$SPFB_4$	$SPFB_5$	$SPFB_6$	$SPFB_7$
$SPFBS_1$	0.19916	0.412945	0.581344	0.550445	0.355256	0.455857	0.161866
$SPFBS_2$	0.216038	0.141239	0.357227	0.341767	0.332499	0.221326	0.248287
$SPFBS_3$	0.19916	0.216647	0.307107	0.152471	0.149182	0.14429	0.175584
$SPFBS_4$	0.533344	0.153209	0.160277	0.169914	0.149182	0.149792	0.312759
$SPFBS_5$	0.381442	0.372847	0.451764	0.152471	0.162014	0.14429	0.587109
$SPFBS_6$	0.44389	0.270868	0.17223	0.339829	0.233255	0.14429	0.508233
$SPFBS_7$	0.311398	0.512291	0.17386	0.402174	0.437433	0.523355	0.161866
$SPFBS_8$	0.105487	0.378234	0.17386	0.152471	0.610003	0.462338	0.253088
$SPFBS_9$	0.213506	0.128073	0.245849	0.421233	0.22883	0.382306	0.161866
$SPFBS_{10}$	0.305491	0.323773	0.24449	0.1641	0.149182	0.156517	0.253088

Step 3. Combined the decision matrix

We combined the decision matrix into one matrix by using Eq. (3)

Step 4. Compute the weights of criteria

Then the weights of criteria are computed as shown in Figure 2. The environmental impacts have the largest weight in all criteria.

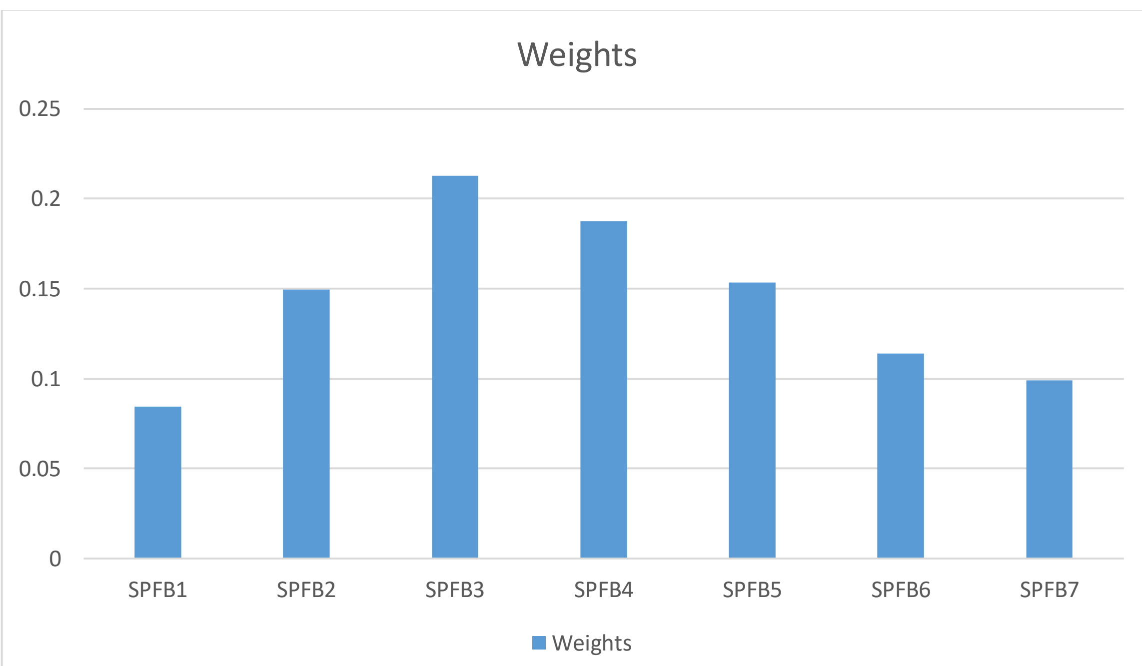

Figure 2. Weights of the criteria of sustainable procurement in food business.

Step 5. Compute the weighted decision matrix

Then we used Eq. (4) to compute the weighted decision matrix by multiplying the weights of criteria by the normalization matrix as shown in Table 2.

Table 2. Weighted normalized decision matrix

	$SPFB_1$	$SPFB_2$	$SPFB_3$	$SPFB_4$	$SPFB_5$	$SPFB_6$	$SPFB_7$
$SPFBS_1$	0.016807	0.061654	0.123603	0.103175	0.054523	0.05187	0.016024
$SPFBS_2$	0.018231	0.021087	0.075952	0.06406	0.05103	0.025184	0.024579
$SPFBS_3$	0.016807	0.032346	0.065296	0.028579	0.022896	0.016418	0.017382
$SPFBS_4$	0.045008	0.022874	0.034078	0.031849	0.022896	0.017044	0.030961
$SPFBS_5$	0.032189	0.055667	0.096052	0.028579	0.024865	0.016418	0.05812
$SPFBS_6$	0.037459	0.040441	0.036619	0.063697	0.035799	0.016418	0.050312
$SPFBS_7$	0.026278	0.076486	0.036965	0.075383	0.067135	0.05955	0.016024
$SPFBS_8$	0.008902	0.056471	0.036965	0.028579	0.09362	0.052607	0.025054
$SPFBS_9$	0.018017	0.019122	0.052271	0.078955	0.03512	0.043501	0.016024
$SPFBS_{10}$	0.02578	0.04834	0.051983	0.030759	0.022896	0.017809	0.025054

Step 6. Compute the distance between alternatives ($S_i(i = 1,2,3, \dots . m)$) and positive and negative criteria.

All criteria are positive criteria, so we compute the distance of each suppliers and positive criteria as shown in Eq. (5).

Step 7. Calculate the coefficient of closeness value

Then compute the closeness value by using Eq. (7).

Step 8. Order the suppliers

The suppliers are ranked according to the largest value in closeness coefficient. The supplier 1 is the best and supplier 4 is the worst as shown in figure 3.

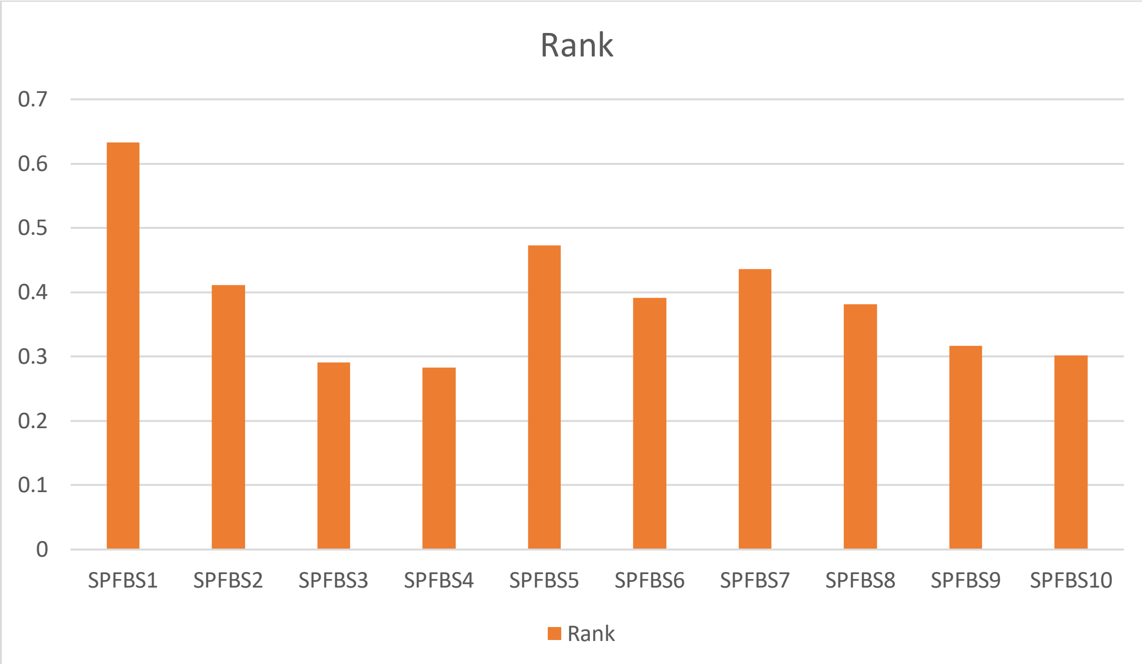

Figure 3. The rank of suppliers in food business.

5. Conclusions

Fostering a more sustainable and resilient food system requires a focus on sustainable procurement in the food industry. Sustainable development may be greatly aided by the food industry if it takes into account environmental implications, social responsibility, animal welfare, packaging and waste management, certifications and standards, traceability and transparency, and the need for continual improvement. Reduced environmental degradation, greater community development, enhanced brand reputation, and the satisfaction of customer expectations are all possible thanks to sustainable procurement practices. However, sustainable procurement isn't without its obstacles, such as convoluted supply networks and unknown financial consequences. Overcoming these challenges, this paper introduces the framework to show the importance of sustainable procurement in food business criteria and select the best supplier in the food business. This paper used the TOPSIS MCDM method to rank these suppliers. The TOPSIS is integrated with a single valued neutrosophic set to deal with uncertain data. The main results show that environmental impacts have the highest importance in all criteria. Food companies may play a crucial role in ensuring the long-term viability of the food industry and society at large by adopting sustainable procurement practices.

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On Hypersoft Semi-open Sets

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Abstract. A generalisation of soft sets called a hypersoft set incorporates a multiargument function. The major goal of this study is to provide appropriate examples for the introduction of hypersoft semi-open sets (SOS) and hypersoft semi-closed sets (SCS). Additionally, we investigate the definition and characteristics of hypersoft semi-open sets in hypersoft topological spaces (TS). The hypersoft semi interior and hypersoft semi closure of the hypersoft set are defined at the end.

Keywords: hypersoft set; hypersoft topology; hypesoft semi-open and closed set; hypersoft semi-interior; hypersoft semi-closure.

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1. **Introduction**

Molodstov [14] established the concept of a soft set in 1999 to handle difficult problems in finance, technical education, and ecological science when no mathematical instruments could effectively address the many types of uncertainty. [13] constructed a number of soft set theory operators and carried out a more thorough conceptual analysis.

Numerous applications of topology, a subfield of mathematics, may be found in the computer and physical sciences. Soft topology is determined on soft sets in two different ways, one by Shabir [20] and the other by Cagman et al. [5].

Soft SOS and soft SCS were first presented in soft TS by Sasikala, V., E. and Sivaraj, D., [19]. Soft semi connected and soft locally semi connected characteristics in soft TS were established by Krishnaveni, J., and Sekar, C., [12].

In 2018, Florentin Smarandache [22] extended the concept of a soft set to a hypersoft set and the hypersoft topology was introduced by Musa and Assad, [15].

The hypersoft sets have been utilised in the Covid-19 Decision Making Model by Inthumathi et al., who cited [9]. Hypersoft subspace topology, hypersoft basis, hypersoft limit point, and hypersoft Hausdorff space were also introduced by Inthumathi et al. in 10. Neutrosophic hypersoft TS and Neutrosophic Semi-open hypersoft sets were produced by Ajay et al. [2] with an illustration to the MAGDM in the Covid-19 Scenario.

When there exist inconclusive data, uncertain functions, or ambiguous sets, Florentin Smarandache [6] recently invented the IndetermHypersoft set as an enhancement of the hypersoft set. He [7] as the company that created the TreeSoft set as an addition to the Multisoft set. It can be seen that the level 2 TreeSoft set resembles the hypersoft set.

The framework of the manuscript is as follows. The preliminary information relevant to this article is provided in section 2. Hypersoft SOS and hypersoft SCS are introduced in section 3 of the paper. We present the idea of hypersoft semi-interior and hypersoft semi-closure in section 4 and demonstrate some of its features.

2. Preliminaries

The preceding definitions are crucial to understanding the content of this manuscript.

Definition 2.1. [14] "Let U be an initial universe and E be a set of parameters. Let P(U) denote the power set of U. The pair (F, E) or simply F_E, is called a soft set over U, where F is a mapping given by $F : E \to P(U)$".

Definition 2.2. [22] "Let U be a universe of discourse, P(U) the power set of U and $E_1, E_2, \cdots, E_n$ the pairwise disjoint sets of parameters. Let A_i be the nonempty subset of E_i for each i=1,2,...,n. A hypersoft set can be identified by the pair $(\Omega, A_1 * A_2 * \cdots * A_n)$, where $\Omega : A_1 * A_2 * \cdots * A_n \to P(U)$. For sake of simplicity, we write the symbols $\mathscr{T}$ for $E_1 * E_2 * \cdots * E_n$, P for $A_1 * A_2 * \cdots * A_n$".

Definition 2.3. [1] "Let $(\Omega, \mathscr{Q})$ and $(\mathscr{G}, \mathfrak{R})$ be two hypersoft sets over U. Then union of $(\Omega, \mathscr{Q})$ and $(\mathscr{G}, \mathfrak{R})$ is denoted by $(\mathscr{H}, \mathscr{S}) = (\Omega, \mathscr{Q}) \cup (\mathscr{G}, \mathfrak{R})$ with $\mathscr{S} = D_1 * D_2 * \cdots * D_n$, where $D_i = Q_i \cup R_i$ for i=1,2,...,n, and $\mathscr{H}$ is defined by

$$\mathscr{H}(\alpha) = \begin{cases} \Omega(\alpha), & if \alpha \in \mathscr{Q} - \mathfrak{R} \\ \mathscr{G}(\alpha), & if \alpha \in \mathfrak{R} - \mathscr{Q} \\ \Omega(\alpha) \cup \mathscr{G}(\alpha), & if \alpha \in \mathscr{Q} \cap \mathfrak{R} \\ 0, & else, \end{cases}$$

where $\alpha = (d_1, d_2, ..., d_n) \in \mathscr{S}$".

Definition 2.4. [1] "Let $(\Omega, \mathscr{Q})$ and $(\mathscr{G}, \mathfrak{R})$ be two hypersoft sets over U. Then intersection of $(\Omega, \mathscr{Q})$ and $(\mathscr{G}, \mathfrak{R})$ is denoted by $(\mathscr{H}, \mathscr{S}) = (\Omega, \mathscr{Q}) \cap (\mathscr{G}, \mathfrak{R})$ with $\mathscr{S}=D_1 * D_2 * \cdots * D_n$, is such that $D_i=Q_i \cap R_i$ for i=1,2,...,n, and $\mathscr{H}$ is defined as $\mathscr{H}(\alpha) = \Omega(\alpha) \cap \mathscr{G}(\alpha)$, where $\alpha = (d_1, d_2, ..., d_n) \in \mathscr{S}$. If D_i is an empty for some i, then $(\Omega, \mathscr{Q}) \cap (\mathscr{G}, \mathfrak{R})$ is defined to be a null hypersoft set".

Definition 2.5. [15] "Let τ be a collection of hypersoft sets over U, then τ is said to be a hypersoft topology over U if

(1) $(\emptyset, \mathsf{P})$ and(Ω, P) belongs to τ,
(2) The intersection of any two hypersoft sets in τ belongs to τ,
(3) The union of any number of a hypersoft sets in τ belongs to τ.
Then $((\Omega, \mathsf{P}), \tau)$ is called a hypersoft TS over U".

Proposition 2.6. *[15] "Let $((\Omega, \mathsf{P}), \tau)$ be a hypersoft space over U. Then*

(1) $(\emptyset, \mathsf{P})$ *and* (Ω, P) *are hypersoft closed sets over U,*
(2) *The union of any two hypersoft closed sets is a hypersoft closed set over U,*
(3) *The intersection of any number of hypersoft closed sets is a hypersoft closed set over U".*

Definition 2.7. [15] "Let $((\Omega, \mathsf{P}), \tau)$ be a hypersoft TS and let $(\Omega, \mathscr{Q})$ be a hypersoft set then

(1) The hypersoft interior of $(\Omega, \mathscr{Q})$ is the hypersoft set
$h\text{-}int(\Omega, \mathscr{Q}) = \bigcup\{(\Omega, \mathfrak{R}) : (\Omega, \mathfrak{R}) \text{ is hypersoft open and } (\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q})\}$.
(2) The hypersoft closure of $(\Omega, \mathscr{Q})$ is the hypersoft set
$h\text{-}cl(\Omega, \mathscr{Q}) = \bigcap\{(\Omega, \mathfrak{R}) : (\Omega, \mathfrak{R}) \text{ is hypersoft closed and } (\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})\}$".

Proposition 2.8. *[15] "Let $((\Omega, \mathsf{P}), \tau)$ be a hypersoft TS and let $(\Omega, \mathscr{Q})$ be a hypersoft set then*

(1) *$h\text{-}cl(\Omega, \mathscr{Q})$ is the smallest hypersoft closed set containing $(\Omega, \mathscr{Q})$.*
(2) *$(\Omega, \mathscr{Q})$ is a hypersoft closed set if and only if $(\Omega, \mathscr{Q})$=$h\text{-}cl(\Omega, \mathscr{Q})$".*

Proposition 2.9. *[15] "Let $((\Omega, \mathsf{P}), \tau)$ be a hypersoft TS and let $(\Omega, \mathscr{Q})$ be a hypersoft set then*

(1) *$h\text{-}int(\Omega, \mathscr{Q})$ is the largest hypersoft open set contained in $(\Omega, \mathscr{Q})$.*
(2) *$(\Omega, \mathscr{Q})$ is a hypersoft open set if and only if $(\Omega, \mathscr{Q})$=$h\text{-}int(\Omega, \mathscr{Q})$".*

Proposition 2.10. *[15] "Let $((\Omega, \mathsf{P}), \tau)$ be a hypersoft TS and let $(\Omega, \mathscr{Q})$,$(\Omega, \mathfrak{R})$ be a hypersoft sets over U. Then*

(1) *$h\text{-}int(h\text{-}int(\Omega, \mathscr{Q}))$=$h\text{-}int((\Omega, \mathscr{Q})$.*
(2) *$(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})$ implies $h\text{-}int(\Omega, \mathscr{Q})\subseteq h\text{-}int(\Omega, \mathfrak{R})$.*

(3) $h\text{-}int((\Omega,\mathscr{Q})\cup(\Omega,\mathfrak{R}))\subseteq h\text{-}int(\Omega,\mathscr{Q})\cup h\text{-}int(\Omega,\mathfrak{R})$.
(4) $h\text{-}int((\Omega,\mathscr{Q})\cap(\Omega,\mathfrak{R}))=h\text{-}int(\Omega,\mathscr{Q})\cap h\text{-}int(\Omega,\mathfrak{R})$".

Proposition 2.11. *[15] "Let $((\Omega,\mathsf{P}),\tau)$ be a hypersoft TS and let $(\Omega,\mathscr{Q})$,$(\Omega,\mathfrak{R})$ be a hypersoft sets over U. Then*

(1) $(\Omega,\mathscr{Q})\subseteq(\Omega,\mathfrak{R})$ *implies* $h\text{-}cl(\Omega,\mathscr{Q})\subseteq h\text{-}cl(\Omega,\mathfrak{R})$.
(2) $h\text{-}cl((\Omega,\mathscr{Q})\cup(\Omega,\mathfrak{R}))=h\text{-}cl(\Omega,\mathscr{Q})\cup h\text{-}cl(\Omega,\mathfrak{R})$.
(3) $h\text{-}cl((\Omega,\mathscr{Q})\cap(\Omega,\mathfrak{R}))\subseteq h\text{-}cl(\Omega,\mathscr{Q})\cap h\text{-}cl(\Omega,\mathfrak{R})$.
(4) $h\text{-}cl(h\text{-}cl(\Omega,\mathscr{Q}))=h\text{-}cl((\Omega,\mathscr{Q})$".

3. Hypersoft semi-open sets and hypersoft semi-closed sets

In this segment we produce the notion of hypersoft SOS, hypersoft SCS and examine a few of its properties.

Definition 3.1. Let $(\Omega,\mathscr{Q})$ be a hypersoft set of a hypersoft TS $((\Omega,\mathsf{P}),\tau)$. $(\Omega,\mathscr{Q})$ is known as a hypersoft SOS if $(\Omega,\mathscr{Q})\subseteq h\text{-}cl(h\text{-}int(\Omega,\mathscr{Q}))$.

Definition 3.2. A hypersoft set $(\Omega,\mathscr{Q})$ in a hypersoft TS $((\Omega,\mathsf{P}),\tau)$ is called a hypersoft SCS if its relative complement is a hypersoft SOS.

Example 3.3. Let $U=\{h_1,h_2\}$, $Q_1=\{\ell_1,\ell_2\}$, $Q_2=\{\ell_3\}$, $Q_3=\{\ell_4\}$ and let Ω is a function from $\mathsf{P}\to$ P(U). Then the hypersoft sets are classified as follows.
$(\Omega,\mathsf{P})_1=\{((\ell_1,\ell_3,\ell_4),\emptyset),((\ell_2,\ell_3,\ell_4),\emptyset)\}$,
$(\Omega,\mathsf{P})_2=\{((\ell_1,\ell_3,\ell_4),\emptyset),((\ell_2,\ell_3,\ell_4),\{h_1\})\}$,
$(\Omega,\mathsf{P})_3=\{((\ell_1,\ell_3,\ell_4),\emptyset),((\ell_2,\ell_3,\ell_4),\{h_2\})\}$,
$(\Omega,\mathsf{P})_4=\{((\ell_1,\ell_3,\ell_4),\emptyset),((\ell_2,\ell_3,\ell_4),\{h_1,h_2\})\}$,
$(\Omega,\mathsf{P})_5=\{((\ell_1,\ell_3,\ell_4),\{h_1\}),((\ell_2,\ell_3,\ell_4),\emptyset)\}$,
$(\Omega,\mathsf{P})_6=\{((\ell_1,\ell_3,\ell_4),\{h_1\}),((\ell_2,\ell_3,\ell_4),\{h_1\})\}$,
$(\Omega,\mathsf{P})_7=\{((\ell_1,\ell_3,\ell_4),\{h_1\}),((\ell_2,\ell_3,\ell_4),\{h_2\})\}$,
$(\Omega,\mathsf{P})_8=\{((\ell_1,\ell_3,\ell_4),\{h_1\}),((\ell_2,\ell_3,\ell_4),\{h_1,h_2\})\}$,
$(\Omega,\mathsf{P})_9=\{((\ell_1,\ell_3,\ell_4),\{h_2\}),((\ell_2,\ell_3,\ell_4),\emptyset)\}$,
$(\Omega,\mathsf{P})_{10}=\{((\ell_1,\ell_3,\ell_4),\{h_2\}),((\ell_2,\ell_3,\ell_4),\{h_1\})\}$,
$(\Omega,\mathsf{P})_{11}=\{((\ell_1,\ell_3,\ell_4),\{h_2\}),((\ell_2,\ell_3,\ell_4),\{h_2\})\}$,
$(\Omega,\mathsf{P})_{12}=\{((\ell_1,\ell_3,\ell_4),\{h_2\}),((\ell_2,\ell_3,\ell_4),\{h_1,h_2\})\}$,
$(\Omega,\mathsf{P})_{13}=\{((\ell_1,\ell_3,\ell_4),\{h_1,h_2\}),((\ell_2,\ell_3,\ell_4),\emptyset)\}$,
$(\Omega,\mathsf{P})_{14}=\{((\ell_1,\ell_3,\ell_4),\{h_1,h_2\}),((\ell_2,\ell_3,\ell_4),\{h_1\})\}$,
$(\Omega,\mathsf{P})_{15}=\{((\ell_1,\ell_3,\ell_4),\{h_1,h_2\}),((\ell_2,\ell_3,\ell_4),\{h_2\})\}$,
$(\Omega,\mathsf{P})_{16}=\{((\ell_1,\ell_3,\ell_4),\{h_1,h_2\}),((\ell_2,\ell_3,\ell_4),\{h_1,h_2\})\}$,
$\tau=\{(\Omega,\mathsf{P})_1,(\Omega,\mathsf{P})_5,(\Omega,\mathsf{P})_7,(\Omega,\mathsf{P})_8,(\Omega,\mathsf{P})_{16}\}$.

Then $((\Omega,\mathsf{P}),\tau)$ is a hypersoft TS.
The collection of all hypersoft open sets is
$\{(\Omega,\mathsf{P})_1,(\Omega,\mathsf{P})_5,(\Omega,\mathsf{P})_7,(\Omega,\mathsf{P})_8,(\Omega,\mathsf{P})_{16}\}$.
The set of all hypersoft closed sets is
$\{(\Omega,\mathsf{P})_1,(\Omega,\mathsf{P})_9,(\Omega,\mathsf{P})_{10},(\Omega,\mathsf{P})_{12},(\Omega,\mathsf{P})_{16}\}$.

The collection of hypersoft SOS is
$\{(\Omega,\mathsf{P})_1,(\Omega,\mathsf{P})_5,(\Omega,\mathsf{P})_6,(\Omega,\mathsf{P})_7,(\Omega,\mathsf{P})_8,(\Omega,\mathsf{P})_{13},(\Omega,\mathsf{P})_{14},(\Omega,\mathsf{P})_{15},(\Omega,\mathsf{P})_{16}\}$.
The collection of hypersoft SCS is
$\{(\Omega,\mathsf{P})_1,(\Omega,\mathsf{P})_2,(\Omega,\mathsf{P})_3,(\Omega,\mathsf{P})_4,(\Omega,\mathsf{P})_9,(\Omega,\mathsf{P})_{10},(\Omega,\mathsf{P})_{11},(\Omega,\mathsf{P})_{12},(\Omega,\mathsf{P})_{16}\}$.

Theorem 3.4. *Every hypersoft open set in a hypersoft TS $((\Omega,\mathsf{P}),\tau)$ is a hypersoft SOS.*

Proof:
Let $(\Omega,\mathscr{Q})$ be a hypersoft open set. Then $h\text{-}int(\Omega,\mathscr{Q})=(\Omega,\mathscr{Q})$. we know that, $(\Omega,\mathscr{Q})\subseteq h\text{-}cl(\Omega,\mathscr{Q})$. Thus $(\Omega,\mathscr{Q})\subseteq h\text{-}cl(h\text{-}int(\Omega,\mathscr{Q}))$.

The preceding Ex. 3.5 demonstrate that the reverse implification of Thm. 3.4 is not true.

Example 3.5. Consider the hypersoft TS of Ex. 3.3.
Here $(\Omega,\mathsf{P})_6,(\Omega,\mathsf{P})_{13},(\Omega,\mathsf{P})_{14},(\Omega,\mathsf{P})_{15}$ are hypersoft semi-open set but not hypersoft open sets, since$(\Omega,\mathsf{P})_6,(\Omega,\mathsf{P})_{13},(\Omega,\mathsf{P})_{14},(\Omega,\mathsf{P})_{15}\notin\tau$.

Remark 3.6. $(\emptyset,\mathsf{P})$ and (Ω,P) are always hypersoft SCS and hypersoft SOS.

Proposition 3.7. *A hypersoft set $(\Omega,\mathscr{Q})$ in a hypersoft TS $((\Omega,\mathsf{P}),\tau)$ is a hypersoft SOS iff $\exists$ a hypersoft open set $(\Omega,\mathfrak{R})$ such that $(\Omega,\mathfrak{R})\subseteq(\Omega,\mathscr{Q})\subseteq h\text{-}cl(\Omega,\mathfrak{R})$.*

Proof: Assume that $(\Omega,\mathscr{Q})\subseteq h\text{-}cl(h\text{-}int(\Omega,\mathscr{Q}))$. Then for $(\Omega,\mathfrak{R})=h\text{-}int(\Omega,\mathscr{Q})$, we have $(\Omega,\mathfrak{R})\subseteq(\Omega,\mathscr{Q})\subseteq h\text{-}cl(\Omega,\mathfrak{R})$. Therefore, the condition holds. Conversely, suppose that $(\Omega,\mathfrak{R})\subseteq(\Omega,\mathscr{Q})\subseteq h\text{-}cl(\Omega,\mathfrak{R})$ for some hypersoft open set $(\Omega,\mathfrak{R})$. Since $(\Omega,\mathfrak{R})\subseteq h\text{-}int(\Omega,\mathscr{Q})$, and so $h\text{-}cl(\Omega,\mathfrak{R})\subseteq h\text{-}cl(h\text{-}int(\Omega,\mathscr{Q}))$. Hence $(\Omega,\mathscr{Q})\subseteq h\text{-}cl(\Omega,\mathfrak{R})\subseteq h\text{-}cl(h\text{-}int(\Omega,\mathscr{Q}))$. Hence $(\Omega,\mathscr{Q})$ is hypersoft SOS.

Theorem 3.8. *Let $((\Omega,\mathsf{P}),\tau)$ be a hypersoft TS and $\{(\Omega,\mathscr{Q})_\alpha:\alpha\in\Delta\}$ be a set of hypersoft SOS in $((\Omega,\mathsf{P}),\tau)$. Then $\cup_{\alpha\in\Delta}(\Omega,\mathscr{Q})_\alpha$ is also a hypersoft SOS.*

Proof: Let $\{(\Omega,\mathscr{Q})_\alpha:\alpha\in\Delta\}$ be a set of hypersoft SOS in $((\Omega,\mathsf{P}),\tau)$. Then $\forall\ \alpha\in\Delta$, we have a hypersoft open set $(\Omega,\mathfrak{R})_\alpha\subseteq(\Omega,\mathscr{Q})_\alpha$ such that $(\Omega,\mathfrak{R})_\alpha\subseteq(\Omega,\mathscr{Q})_\alpha\subseteq h\text{-}cl(\Omega,\mathfrak{R})_\alpha$. Then $\cup_{\alpha\in\Delta}(\Omega,\mathfrak{R})_\alpha\subseteq\cup_{\alpha\in\Delta}(\Omega,\mathscr{Q})_\alpha\subseteq\cup_{\alpha\in\Delta}h\text{-}cl(\Omega,\mathfrak{R})_\alpha\subseteq$
$h\text{-}cl(\cup_{\alpha\in\Delta}(\Omega,\mathfrak{R})_\alpha)$.

Theorem 3.9. *Every hypersoft closed set in a hypersoft TS $((\Omega, \mathsf{P}), \tau)$ is a hypersoft SCS.*

Proof: Let $(\Omega, \mathscr{Q})$ be a hypersoft closed set. Then $h\text{-}cl(\Omega, \mathscr{Q}) = (\Omega, \mathscr{Q})$. we know that, $(\Omega, \mathscr{Q}) \supseteq h\text{-}int(\Omega, \mathscr{Q})$. Thus $(\Omega, \mathscr{Q}) \supseteq h\text{-}int(h\text{-}cl(\Omega, \mathscr{Q}))$.

The opposite simplification of Thm. 3.9 cannot be true, as demonstrated by Ex. 3.10 before it.

Example 3.10. Here $(\Omega, \mathsf{P})_2, (\Omega, \mathsf{P})_3, (\Omega, \mathsf{P})_4$ and $(\Omega, \mathsf{P})_{11}$ are hypersoft SCS but not hypersoft closed sets.

Theorem 3.11. $(\Omega, \mathfrak{R})$ *be a hypersoft semi-closed in a hypersoft TS* $((\Omega, \mathsf{P}), \tau)$ *iff* $h\text{-}int(\Omega, \mathscr{T}) \subseteq (\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{T})$ *for some hypersoft closed set* $(\Omega, \mathscr{T})$.

Proof: $(\Omega, \mathfrak{R})$ is hypersoft semi-closed iff $(\Omega, \mathfrak{R})^c$ is hypersoft semi-open iff there is a hypersoft open set $(\Omega, \mathscr{S})$ s.t. $(\Omega, \mathscr{S}) \subseteq (\Omega, \mathfrak{R})^c \subseteq h\text{-}cl(\Omega, \mathscr{S})$, by proposition 3.7 iff there is a hypersoft open set $(\Omega, \mathscr{S})$ s.t. $(h\text{-}cl(\Omega, \mathscr{S}))^c \subseteq (\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{S})^c$ iff there is a hypersoft open set $(\Omega, \mathscr{S})$ s.t. $h\text{-}int(\Omega, \mathscr{S})^c \subseteq (\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{S})^c$ iff there is a hypersoft closed set $(\Omega, \mathscr{S})$ s.t. $h\text{-}int(\Omega, \mathscr{T}) \subseteq (\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{T})$, where $(\Omega, \mathscr{T}) = (\Omega, \mathscr{S})^c$.

Theorem 3.12. *A hypersoft set* $(\Omega, \mathscr{Q})$ *in a hypersoft TS* $((\Omega, \mathsf{P}), \tau)$ *is hypersoft semi-closed iff* $h\text{-}int(h\text{-}cl(\Omega, \mathscr{Q})) \subseteq (\Omega, \mathscr{Q})$.

Proof: $(\Omega, \mathscr{Q})$ is hypersoft semi-closed iff $(\Omega, \mathscr{Q})^c$ is hypersoft semi-open iff $(\Omega, \mathscr{Q})^c \subseteq h\text{-}cl(h\text{-}int(\Omega, \mathscr{Q})^c)$ iff $(\Omega, \mathscr{Q})^c \subseteq h\text{-}cl((h\text{-}cl(\Omega, \mathscr{Q}))^c)$, by definition iff $(\Omega, \mathscr{Q})^c \subseteq (h\text{-}int(h\text{-}cl(\Omega, \mathscr{Q})))^c$, iff $h\text{-}int(h\text{-}cl(\Omega, \mathscr{Q})) \subseteq (\Omega, \mathscr{Q})$.

Theorem 3.13. *Let* $((\Omega, \mathsf{P}), \tau)$ *be a hypersoft TS and* $\{(\Omega, \mathscr{Q})_\alpha : \alpha \in \Delta\}$ *be a set of hypersoft SCS in* $((\Omega, \mathsf{P}), \tau)$. *Then* $\cap_{\alpha \in \Delta}(\Omega, \mathscr{Q})_\alpha$ *is also a hypersoft SCS.*

Proof: Let $\{(\Omega, \mathscr{Q})_\alpha : \alpha \in \Delta\}$ be a set of hypersoft SCS in $((\Omega, \mathsf{P}), \tau)$. Then $\forall\ \alpha \in \Delta$, we have a hypersoft soft closed set $(\Omega, \mathfrak{R})_\alpha$ s.t. $h\text{-}int(\Omega, \mathfrak{R})_\alpha \subseteq (\Omega, \mathscr{Q})_\alpha \subseteq (\Omega, \mathfrak{R})_\alpha$. Then $h\text{-}int(\cap_{\alpha \in \Delta}(\Omega, \mathfrak{R}_\alpha)) \subseteq \cap_{\alpha \in \Delta} h\text{-}int(\Omega, \mathfrak{R}_\alpha) \subseteq \cap_{\alpha \in \Delta}(\Omega, \mathscr{Q})_\alpha \subseteq \cap_{\alpha \in \Delta}(\Omega, \mathfrak{R})_\alpha$. Because $\cap_{\alpha \in \Delta}(\Omega, \mathfrak{R})_\alpha = (\Omega, \mathfrak{R})$ is hypersoft closed set by prop 2.6(3), then $\cap_{\alpha \in \Delta}(\Omega, \mathscr{Q})_\alpha$ is hypersoft SCS.

4. Hypersoft semi-interior and hypersoft semi-closure

Definition 4.1. Let $((\Omega, \mathsf{P}), \tau)$ be a hypersoft TS and let $(\Omega, \mathscr{Q})$ be a hypersoft set in (Ω, P).

(1) The hypersoft semi-interior of $(\Omega, \mathscr{Q})$ is the hypersoft set $\bigcup\{(\Omega, \mathfrak{R}) : (\Omega, \mathfrak{R})$ is hypersoft semi-open and $(\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q})\}$ and it is identified by $h\text{-}sint(\Omega, \mathscr{Q})$.

(2) The hypersoft semi-closure of $(\Omega, \mathscr{Q})$ is the hypersoft set $\bigcap\{(\Omega, \mathfrak{R}) : (\Omega, \mathfrak{R})$ is hypersoft semi-closed and $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})\}$ and it is identified by $h\text{-}scl(\Omega, \mathscr{Q})$.

Clearly, $h\text{-}scl(\Omega, \mathscr{Q})$ is the smallest hypersoft SCS containing $(\Omega, \mathscr{Q})$ and $h\text{-}sint(\Omega, \mathscr{Q})$ is the largest hypersoft SOS $\subseteq (\Omega, \mathscr{Q})$. By Thm. 3.8 and 3.13, we have $h\text{-}sint(\Omega, \mathscr{Q})$ is hypersoft SOS and $h\text{-}scl(\Omega, \mathscr{Q})$ is hypersoft SCS.

Example 4.2. Let the hypersoft TS $((\Omega, \mathsf{P}), \tau)$ and the hypersoft set $(\Omega, \mathsf{P})_8=\{((\ell_1, \ell_3, \ell_4), \{h_1\}), ((\ell_2, \ell_3, \ell_4), \{h_1, h_2\})\}$ be the same as in Example 3.3, we get $h\text{-}sint(\Omega, \mathscr{Q})_8 = (\Omega, \mathscr{Q})_8$.

Example 4.3. Let the hypersoft TS $((\Omega, \mathsf{P}), \tau)$ and the hypersoft set $(\Omega, \mathsf{P})_{14}=\{((\ell_1, \ell_3, \ell_4), \{h_1, h_2\}), ((\ell_2, \ell_3, \ell_4), \{h_1\})\}$ be the same as in Example 3.3, we get $h\text{-}scl(\Omega, \mathscr{Q})_{14} = (\Omega, \mathscr{Q})_{16}$.

Theorem 4.4. *Let $((\Omega, \mathsf{P}), \tau)$ be a hypersoft TS and $(\Omega, \mathscr{Q})$ be a hypersoft set in (Ω, P). Then $h\text{-}int(\Omega, \mathscr{Q}) \subseteq h\text{-}sint(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathscr{Q}) \subseteq h\text{-}scl(\Omega, \mathscr{Q}) \subseteq h\text{-}cl(\Omega, \mathscr{Q})$.*

Proof: The proof follows from the following facts that every hypersoft open set is hypersoft SOS and every hypersoft closed set is hypersoft SCS.

Theorem 4.5. *Let $((\Omega, \mathsf{P}), \tau)$ be a hypersoft TS and let $(\Omega, \mathscr{Q})$ be a hypersoft set in (Ω, P). Then the succeeding conditions holds.*

(1) $(h\text{-}scl(\Omega, \mathscr{Q}))^c = h\text{-}sint(\Omega, \mathscr{Q})^c$.
(2) $(h\text{-}sint(\Omega, \mathscr{Q}))^c = h\text{-}scl(\Omega, \mathscr{Q})^c$.

Proof:

(1) $(h\text{-}scl(\Omega, \mathscr{Q}))^c$
$=(\bigcap\{(\Omega, \mathfrak{R}) : (\Omega, \mathfrak{R})$ is hypersoft semi-closed and $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})\})^c$,
$=\bigcup\{(\Omega, \mathfrak{R})^c : (\Omega, \mathfrak{R})$ is hypersoft semi-closed and $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})\}$,
$=\bigcup\{(\Omega, \mathfrak{R})^c : (\Omega, \mathfrak{R})^c$, is hypersoft semi-open and $(\Omega, \mathfrak{R})^c \subseteq (\Omega, \mathscr{Q})^c\}$,
$=h\text{-}sint(\Omega, \mathscr{Q})^c$.

(2) $(h\text{-}sint(\Omega, \mathscr{Q}))^c$
$=(\bigcup\{(\Omega, \mathfrak{R}) : (\Omega, \mathfrak{R})$ is hypersoft semi-open and $(\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q})\})^c$,
$=\bigcap\{(\Omega, \mathfrak{R})^c : (\Omega, \mathfrak{R})$ is hypersoft semi-open and $(\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q})\}$,
$=\bigcup\{(\Omega, \mathfrak{R})^c : (\Omega, \mathfrak{R})^c$ is hypersoft semi-closed and $(\Omega, \mathscr{Q})^c \subseteq (\Omega, \mathfrak{R})^c\}$,
$=h\text{-}scl(\Omega, \mathscr{Q})^c$.

Theorem 4.6. *Let $((\Omega, \mathsf{P}), \tau)$ be a hypersoft TS and let $(\Omega, \mathscr{Q})$ and $(\Omega, \mathfrak{R})$ be a hypersoft sets in (Ω, P). Then the preceding condition holds.*

(1) *h-scl*$(\emptyset, \mathsf{P})$=$(\emptyset, \mathsf{P})$ *and h-scl*(Ω, P)=(Ω, P)

(2) $(\Omega, \mathscr{Q})$ *is hypersoft semi-closed set iff* $(\Omega, \mathscr{Q})$=*h-scl*$(\Omega, \mathscr{Q})$.

(3) *h-scl*(*h-scl*$(\Omega, \mathscr{Q})$)=*h-scl*$(\Omega, \mathscr{Q})$.

(4) $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})$ *implies h-scl*$(\Omega, \mathscr{Q}) \subseteq$ *h-scl*$(\Omega, \mathfrak{R})$.

(5) *h-scl*$((\Omega, \mathscr{Q}) \cap (\Omega, \mathfrak{R})) \subseteq$ *h-scl*$(\Omega, \mathscr{Q}) \cap$ *h-scl*$(\Omega, \mathfrak{R})$.

(6) *h-scl*$((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R})) =$ *h-scl*$(\Omega, \mathscr{Q}) \cup$ *h-scl*$(\Omega, \mathfrak{R})$.

Proof:

(1) The proof is obvious.

(2) If $(\Omega, \mathscr{Q})$ is hypersoft SCS, then $(\Omega, \mathscr{Q})$ is itself a hypersoft SCS in (Ω, P) which $\subset (\Omega, \mathscr{Q})$. So, *h-scl*$(\Omega, \mathscr{Q})$ is the smallest hypersoft SCS $\subset (\Omega, \mathscr{Q})$ and $(\Omega, \mathscr{Q}) =$ *h-scl*$(\Omega, \mathscr{Q})$. Conversely, suppose that $(\Omega, \mathscr{Q}) =$ *h-scl*$(\Omega, \mathscr{Q})$. Since *h-scl*$(\Omega, \mathscr{Q})$ is a hypersoft SCS, so $(\Omega, \mathscr{Q})$ is hypersoft SCS.

(3) Since *h-scl*$(\Omega, \mathscr{Q})$ is a hypersoft SCS therefore by part(2) we obtain *h-scl*(*h-scl*$(\Omega, \mathscr{Q})$)=*h-scl*$(\Omega, \mathscr{Q})$.

(4) Suppose that $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})$. Then every hypersoft semi-closed super set of $(\Omega, \mathfrak{R})$ will also $\subset (\Omega, \mathscr{Q})$. That is every hypersoft semi-closed super set of $(\Omega, \mathfrak{R})$ is also a hypersoft semi-closed super set of $(\Omega, \mathscr{Q})$. Thus *h-scl*$(\Omega, \mathscr{Q}) \subseteq$ *h-scl*$(\Omega, \mathfrak{R})$.

(5) Since $(\Omega, \mathscr{Q}) \cap (\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q})$ and $(\Omega, \mathscr{Q}) \cap (\Omega, \mathfrak{R}) \subseteq (\Omega, \mathfrak{R})$ and so by part(4) *h-scl*$((\Omega, \mathscr{Q}) \cap (\Omega, \mathfrak{R})) \subseteq$ *h-scl*$(\Omega, \mathscr{Q})$ and *h-scl*$((\Omega, \mathscr{Q}) \cap (\Omega, \mathfrak{R})) \subseteq$ *h-scl*$(\Omega, \mathfrak{R})$. Thus *h-scl*$((\Omega, \mathscr{Q}) \cap (\Omega, \mathfrak{R})) \subseteq$ *h-scl*$(\Omega, \mathscr{Q}) \cap$ *h-scl*$(\Omega, \mathfrak{R})$.

(6) Since $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R})$ and $(\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R})$. So by part(iv) $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})$ implies *h-scl*$(\Omega, \mathscr{Q}) \subseteq$ *h-scl*$(\Omega, \mathfrak{R})$. Then *h-scl*$(\Omega, \mathscr{Q}) \subseteq$ *h-scl*$((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R}))$ and *h-scl*$(\Omega, \mathfrak{R}) \subseteq$ *h-scl*$((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R}))$, which is implies *h-scl*$(\Omega, \mathscr{Q}) \cup$ *h-scl*$(\Omega, \mathfrak{R}) \subseteq$ *h-scl*$((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R}))$. Now, *h-scl*$(\Omega, \mathscr{Q})$, *h-scl*$(\Omega, \mathfrak{R})$ is belong to hypersoft SCS in (Ω, P) which is implies that *h-scl*$(\Omega, \mathscr{Q}) \cup$ *h-scl*$(\Omega, \mathfrak{R})$ is belong to hypersoft SCS in (Ω, P). Then $(\Omega, \mathscr{Q}) \subseteq$ *h-scl*$(\Omega, \mathscr{Q})$ and $(\Omega, \mathfrak{R}) \subseteq$ *h-scl*$(\Omega, \mathfrak{R})$ imply $(\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R}) \subseteq$ *h-scl*$(\Omega, \mathscr{Q}) \cup$ *h-scl*$(\Omega, \mathfrak{R})$. That is *h-scl*$(\Omega, \mathscr{Q}) \cup$ *h-scl*$(\Omega, \mathfrak{R})$ is a hypersoft SCS containing $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})$. Hence *h-scl*$((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R})) \subseteq$ *h-scl*$(\Omega, \mathscr{Q}) \cup$ *h-scl*$(\Omega, \mathfrak{R})$. So, *h-scl*$((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R})) =$ *h-scl*$(\Omega, \mathscr{Q}) \cup$ *h-scl*$(\Omega, \mathfrak{R})$.

Theorem 4.7. *Let* $((\Omega, \mathsf{P}), \tau)$ *be a hypersoft TS and let* $(\Omega, \mathscr{Q})$ *and* $(\Omega, \mathfrak{R})$ *be a hypersoft sets in* (Ω, P). *Then the succeeding condition holds.*

(1) *h-sint*$(\emptyset, \mathsf{P})$=$(\emptyset, \mathsf{P})$ *and h-sint*(Ω, P)=(Ω, P).

(2) $(\Omega, \mathscr{Q})$ *is hypersoft SOS iff* $(\Omega, \mathscr{Q})$=*h-sint*$(\Omega, \mathscr{Q})$.

(3) *h-sint*(*h-sint*$(\Omega, \mathscr{Q})$)=*h-sint*$(\Omega, \mathscr{Q})$.

(4) $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})$ *implies* $h\text{-}sint(\Omega, \mathscr{Q}) \subseteq h\text{-}sint(\Omega, \mathfrak{R})$.

(5) $h\text{-}sint(\Omega, \mathscr{Q}) \cap h\text{-}sint(\Omega, \mathfrak{R}) \subseteq h\text{-}sint((\Omega, \mathscr{Q}) \cap (\Omega, \mathfrak{R}))$.

(6) $h\text{-}sint((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R})) = h\text{-}sint(\Omega, \mathscr{Q}) \cup h\text{-}sint(\Omega, \mathfrak{R})$.

Proof:

(1) The proof is obvious.

(2) If $(\Omega, \mathscr{Q})$ is hypersoft SOS, then $(\Omega, \mathscr{Q})$ is itself a hypersoft SOS in $(\Omega, \mathsf{P}) \subset (\Omega, \mathscr{Q})$. So, $h\text{-}sint(\Omega, \mathscr{Q})$ is the largest hypersoft SOS contained in $(\Omega, \mathscr{Q})$ and $(\Omega, \mathscr{Q}) = h\text{-}sint(\Omega, \mathscr{Q})$. Conversely, suppose that
$(\Omega, \mathscr{Q}) = h\text{-}sint(\Omega, \mathscr{Q})$. Since $h\text{-}sint(\Omega, \mathscr{Q})$ is a hypersoft SOS, so $(\Omega, \mathscr{Q})$ is hypersoft semi-open set in (Ω, P).

(3) Since $h\text{-}sint(\Omega, \mathscr{Q})$ is a hypersoft SOS therefore by part(2) we have
$h\text{-}sint(h\text{-}sint(\Omega, \mathscr{Q}))$=$h\text{-}sint(\Omega, \mathscr{Q})$.

(4) Suppose that $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})$. Since $h\text{-}sint(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})$. $h\text{-}sint(\Omega, \mathscr{Q})$ is a hypersoft semi-open subset of $(\Omega, \mathfrak{R})$, so by defn. of $h\text{-}sint(\Omega, \mathfrak{R})$,
$h\text{-}sint(\Omega, \mathscr{Q}) \subseteq h\text{-}sint(\Omega, \mathfrak{R})$.

(5) Since $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathscr{Q}) \cap ((\Omega, \mathfrak{R})$ and $(\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q}) \cap ((\Omega, \mathfrak{R})$ and so by part(4), $h\text{-}sint(\Omega, \mathscr{Q}) \subseteq h\text{-}sint((\Omega, \mathscr{Q}) \cap (\Omega, \mathfrak{R}))$ and $h\text{-}sint(\Omega, \mathfrak{R}) \subseteq h\text{-}sint((\Omega, \mathscr{Q}) \cap (\Omega, \mathfrak{R}))$. So that $h\text{-}sint(\Omega, \mathscr{Q}) \cap h\text{-}sint(\Omega, \mathfrak{R}) \subseteq h\text{-}sint((\Omega, \mathscr{Q}) \cap (\Omega, \mathfrak{R}))$, since $h\text{-}sint((\Omega, \mathscr{Q}) \cap (\Omega, \mathfrak{R}))$ is a hypersoft semi-open set.

(6) Since $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R})$ and $(\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R})$ and So by part(4) $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})$ implies $h\text{-}sint(\Omega, \mathscr{Q}) \subseteq h\text{-}sint(\Omega, \mathfrak{R})$. Then $h\text{-}sint(\Omega, \mathscr{Q}) \subseteq$
$h\text{-}sint((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R}))$ and $h\text{-}sint(\Omega, \mathfrak{R}) \subseteq h\text{-}sint((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R}))$ which is implies $h\text{-}sint(\Omega, \mathscr{Q}) \cup h\text{-}sint(\Omega, \mathfrak{R}) \subseteq h\text{-}sint((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R}))$. Now, $h\text{-}sint(\Omega, \mathscr{Q})$,
$h\text{-}sint(\Omega, \mathfrak{R})$ is belong to hypersoft SOS in (Ω, P) which is implies that $h\text{-}sint(\Omega, \mathscr{Q}) \cup h\text{-}sint(\Omega, \mathfrak{R})$ is belong to hypersoft SOS in (Ω, P). Then $(\Omega, \mathscr{Q}) \subseteq h\text{-}sint(\Omega, \mathscr{Q})$ and $(\Omega, \mathfrak{R}) \subseteq h\text{-}sint(\Omega, \mathfrak{R})$ imply $(\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R}) \subseteq$
$h\text{-}sint(\Omega, \mathscr{Q}) \cup h\text{-}sint(\Omega, \mathfrak{R})$. That is $h\text{-}sint(\Omega, \mathscr{Q}) \cup h\text{-}sint(\Omega, \mathfrak{R})$. is a hypersoft SOS containing $(\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R})$. Hence $h\text{-}sint((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R})) \subseteq$
$h\text{-}sint(\Omega, \mathscr{Q}) \cup h\text{-}sint(\Omega, \mathfrak{R})$. So, $h\text{-}sint((\Omega, \mathscr{Q}) \cup (\Omega, \mathfrak{R}))$=$h\text{-}sint(\Omega, \mathscr{Q}) \cup$
$h\text{-}sint(\Omega, \mathfrak{R})$.

Theorem 4.8. *Let* $((\Omega, \mathsf{P}), \tau)$ *be a hypersoft TS and let* $(\Omega, \mathscr{Q})$ *be a hypersoft set in* (Ω, P). *Then the preceding conditions holds.*

(1) $h\text{-}scl(h\text{-}cl(\Omega, \mathscr{Q})) = h\text{-}cl(h\text{-}scl(\Omega, \mathscr{Q})) = h\text{-}cl(\Omega, \mathscr{Q})$.

(2) $h\text{-}sint(h\text{-}int(\Omega, \mathscr{Q})) = h\text{-}int(h\text{-}sint(\Omega, \mathscr{Q})) = h\text{-}int(\Omega, \mathscr{Q})$.

Proof:

(1) Let $h\text{-}cl(\Omega, \mathscr{Q})$ is hypersoft closed set, then $h\text{-}cl(\Omega, \mathscr{Q})$ is hyersoft SCS by Thm. 3.9. So we can get $h\text{-}scl(h\text{-}cl(\Omega, \mathscr{Q})) = h\text{-}cl(\Omega, \mathscr{Q})$ by Theorem 4.6(2). By Thm. 4.4, we have $(\Omega, \mathscr{Q}) \subseteq h\text{-}scl(\Omega, \mathscr{Q}) \subseteq h\text{-}cl(\Omega, \mathscr{Q})$, then we can get $h\text{-}cl(\Omega, \mathscr{Q}) \subseteq h\text{-}cl(h\text{-}scl(\Omega, \mathscr{Q})) \subseteq h\text{-}cl(\Omega, \mathscr{Q})$ and so $h\text{-}cl(h\text{-}scl(\Omega, \mathscr{Q})) = h\text{-}cl(\Omega, \mathscr{Q})$. This completes the proof.

(2) Let $((\Omega, \mathsf{P}), \tau)$ be a hypersoft TS and because $h\text{-}int(\Omega, \mathscr{Q})$ is hypersoft open set, we have $h\text{-}int(\Omega, \mathscr{Q})$ is hypersoft SOS by Thm. 3.4. So we can get $h\text{-}sint(h\text{-}int(\Omega, \mathscr{Q})) = h\text{-}int(\Omega, \mathscr{Q})$ by Thm. 4.7(2). By Thm. 4.4, we have $h\text{-}int(\Omega, \mathscr{Q}) \subseteq h\text{-}sint(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathscr{Q})$, then we can get $h\text{-}int(\Omega, \mathscr{Q}) \subseteq h\text{-}int(h\text{-}sint(\Omega, \mathscr{Q})) \subseteq h\text{-}int(\Omega, \mathscr{Q})$ and so $h\text{-}int(h\text{-}sint(\Omega, \mathscr{Q})) = h\text{-}int(\Omega, \mathscr{Q})$.

Theorem 4.9. *Let $((\Omega, \mathsf{P}), \tau)$ be a hypersoft TS and let $(\Omega, \mathscr{Q})$ be a hypersoft set in (Ω, P). Then the succeeding are equivalent.*

(1) *$(\Omega, \mathscr{Q})$ is hypersoft SCS.*
(2) $h\text{-}int(h\text{-}cl(\Omega, \mathscr{Q})) \subseteq (\Omega, \mathscr{Q})$.
(3) $h\text{-}cl(h\text{-}int((\Omega, \mathscr{Q})^c) \supseteq (\Omega, \mathscr{Q})^c$.
(4) *$(\Omega, \mathscr{Q})^c$ is hypersoft SOS.*

Proof:

(1)⇒(2): If $(\Omega, \mathscr{Q})$ is hypersoft SCS, then $\exists$ hypersoft closed set $(\Omega, \mathfrak{R})$ s.t. $h\text{-}int(\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R}) \Rightarrow h\text{-}int(\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q}) \subseteq h\text{-}cl(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})$. By the property of interior, we get $h\text{-}int(h\text{-}cl(\Omega, \mathscr{Q})) \subseteq h\text{-}int(\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q})$.

(2)⇒(3): $h\text{-}int(h\text{-}cl(\Omega, \mathscr{Q})) \subseteq (\Omega, \mathscr{Q}) \Rightarrow (\Omega, \mathscr{Q})^c \subseteq h\text{-}int(h\text{-}cl(\Omega.\mathscr{Q}))^c=$
$h\text{-}cl(h\text{-}int(\Omega, \mathscr{Q})^c) \supseteq (\Omega, \mathscr{Q})^c$.

(3)⇒(4): $(\Omega, \mathfrak{R}) = h\text{-}int((\Omega, \mathscr{Q})^c)$ is an hypersoft open set s.t. $h\text{-}int((\Omega, \mathscr{Q})^c) \subseteq (\Omega, \mathscr{Q})^c \subseteq h\text{-}cl(h\text{-}int((\Omega, \mathscr{Q})^c)$, hence $(\Omega, \mathscr{Q})^c$ is hypersoft SOS.

(4)⇒(1): As $(\Omega, \mathscr{Q})^c$ is hypersoft SOS, $\exists$ an hypersoft open set $(\Omega, \mathfrak{R})$ s.t. $(\Omega, \mathfrak{R}) \subseteq (\Omega, \mathscr{Q})^c \subseteq h\text{-}cl(\Omega, \mathfrak{R}) \Rightarrow (\Omega, \mathfrak{R})^c$ is a hypersoft closed set such that $(\Omega, \mathscr{Q}) \subseteq (\Omega, \mathfrak{R})^c$ and $(\Omega, \mathscr{Q})^c \subseteq h\text{-}cl(\Omega, \mathfrak{R}) \Rightarrow h\text{-}int(\Omega, \mathfrak{R})^c \subseteq (\Omega, \mathscr{Q})$. Hence $(\Omega, \mathscr{Q})$ is hypersoft SCS.

5. **Conclusion**

We have introduced hypersoft semi-open sets in hypersoft TS which are identified over an initial universe with a fixed set of parameters. We then define hypersoft semi-interior and hypersoft semi-closure with suitable example. The concept of open sets produced in this work may be developed to α-open hypersoft sets and β-open hypersoft sets. Based on the works of [6] and [7], our future research may be on IndetermHypersoft semi-open sets and on TreeSoft semi-open sets.

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An Effective Decision-Making Framework for Evaluating the Intelligent Logistics Development Scenarios Performance

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Abstract: This research was conducted with the intention of determining which scenario for the construction of an intelligent reverse logistics system had the most potential for success. This selection would then be used as a reference point for decision-making throughout the process of constructing environmentally friendly closed supply chains and circular economies. The research includes the definition of four different development scenarios, each of which is then reviewed by representatives of the key stakeholders based on a comprehensive list of eight sub-indicators that are categorized under the four primary dimensions. In order to tackle the issue that was specified, a brand new multi-criteria decision-making (MCDM) framework was constructed. This model included using the Combinative Distance based Assessment (CODAS) technique in a neutrosophic environment and applying the type-2 neutrosophic numbers (T2NNs). The utilization of the developed framework led to the identification of the scenario that optimally reconciles the widespread implementation of Industry 4.0 technologies with the requisite resources.

Keywords: Reverse Logistics; Intelligent Logistics; Industry 4.0; MCDM; Supply chain; T2NNs; CODAS.

1. Introduction

The absence of raw resources, the increasing degradation of the environment, a rising degree of social responsibility, environmental restrictions, and shifting market conditions have brought the topic of reverse logistics to the forefront of many ongoing research on sustainability [1]. The development of reverse logistics systems is the most important requirement and a precondition for the establishment of a closed-loop supply chain, which is a type of supply chain that is analogous to the idea of a circular economy. In the beginning, public awareness was the driving force behind research on closed-loop supply chains and reverse logistics, which means that the difficulties caused by return flows to ordinary people and their environment were the impetus for this line of inquiry [2]. These issues become the focus of the legislative authorities, which pass a variety of laws and directives to regulate this area as a response to the growing consumer society, the decrease in the product lifetime, and the pressure from the general public to find solutions to the problems caused by end-of-life products [3]. Finally, reverse logistics and closed-loop supply chains are seen as the regions in which many actors in the supply chain have the potential to make money for themselves. A market that is centered on reverse logistics is now in the process of developing as a result of the emergence of new demands for the supply of services as well as new suppliers of those services [4]. As a result, the construction of a sustainable reverse logistics system that is in accordance with the objectives and interests of the primary stakeholders, including the service providers, service users, administrations, and people, becomes the goal. They gain advantages as a result of this type of system as a result of a decrease in the amount of waste that is disposed of, an increase in the amount of product value and energy that is recovered, an extension of the product's life cycle, the extraction of materials and their subsequent recycling, the generation of competitive advantage, an acceleration of the return on investment, an improvement in customer relations, and a decrease in the amount of emissions produced by transportation [5,6]. As a result, the focus of this research is on the formulation and analysis of potential futures for the development of intelligent reverse logistics systems [4]. These futures are to be conceived with the level of advancement of Industry 4.0 technologies, the scope of their potential applications, and the current state of social, economic, technical, service quality, and environmental trends in mind.

As a consequence of this, the scenario that offers the greatest balance between the extensive use of Industry 4.0 technologies and the required resources for its development and implementation is chosen as the optimal option. It is feasible to draw the conclusion from the findings that the broadest possible use of technologies related to Industry 4.0 does not necessarily guarantee the most acceptable development scenarios and that the choice ought to be taken by reaching a compromise between the interests of all stakeholders. In this work, a unique multi-criteria decision-making (MCDM) model has been constructed in order to answer the issue that has been specified [7–9]. This model incorporates the Combinative Distance based Assessment (CODAS) [10] approach inside a neutrosophic environment. The neutrosophic set is applied in various filed as: [11-17]

The residue sections of the study are organized as follows: Section 2 develops the suggested framework for determining the suitable reverse logistics development scenario. Section 3 applies the suggested framework and analysis of the findings. Section 4 concludes the study.

2. Suggested Framework

In this section, the proposed approach to solve the problem of selecting and defining the best scenarios for the development of intelligent logistics services is introduced. The proposed framework is divided into three stages. The first stage is related to studying the problem and defining the main goal. In addition, identifying the committee involved with the authors in studying the problem and expressing their opinions on the main dimensions, and evaluating the alternatives. The second stage is related to evaluating the dimensions and determining the weights, whether for the main dimensions or the sub-indicators. The third stage is related to evaluating and ranking the alternatives selected for the study using the CODAS method. Also, the study and its details were conducted in a neutrosophic environment and by applying T2NNs. Figure 1 presents the steps of the proposed approach and details of the study procedure.

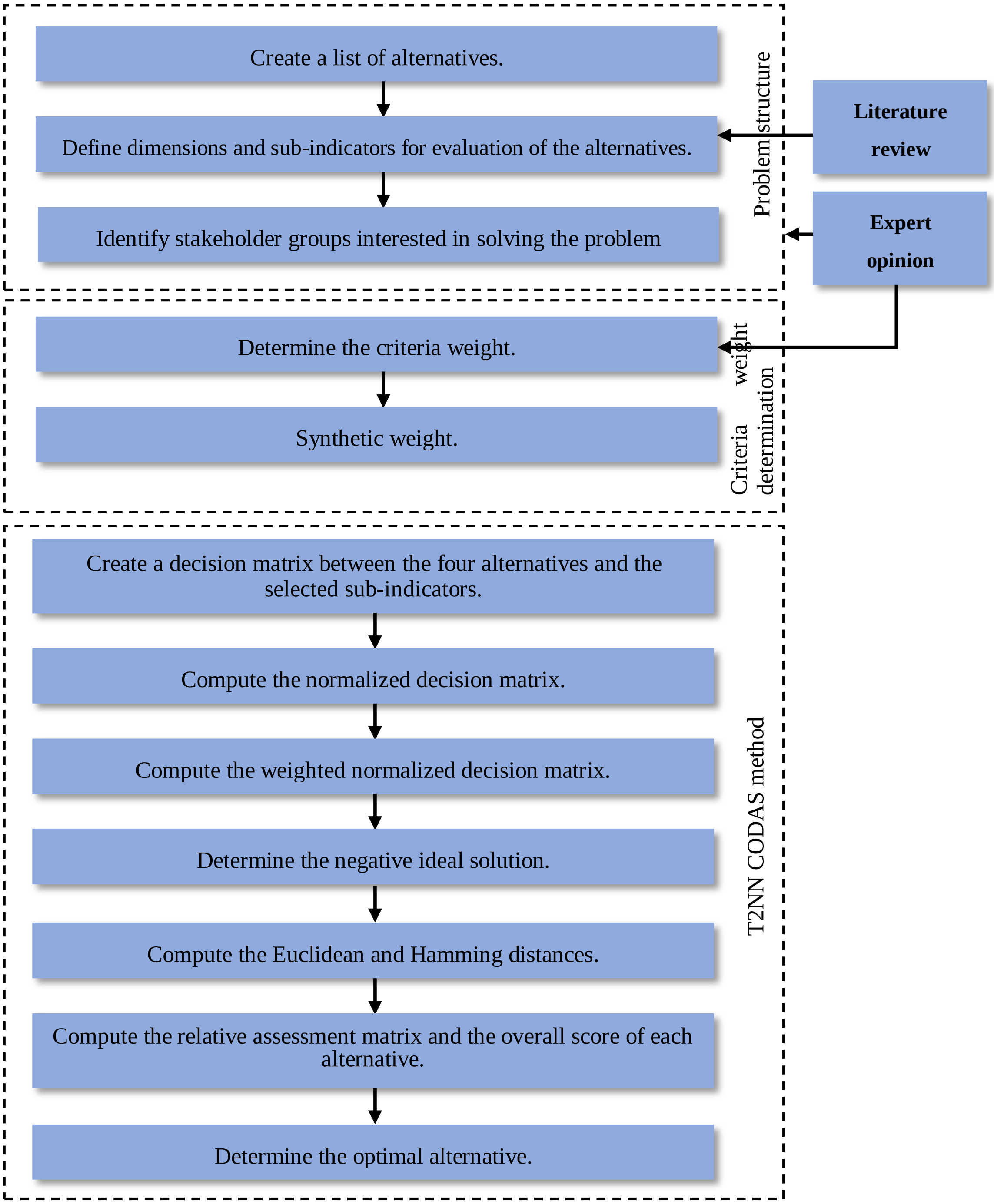

Figure 1. Research framework.

Step 1. The issue is considered in detail and the participating consultants are determined as shown in Table 1. The participating consultants give their opinions on the problem and define the dimensions and available scenarios. Suppose a set of m substitutes is represented by $A = \{A_1, \dots, A_i, \dots\ A_m\}$ and a set of n dimensions is denoted by $D = \{D_1, \dots, D_n, \dots, D_n\}$. Let consultants $= \{Consultant_1, \dots, Consultant_e, \dots, Consultant_k\}$ be a set of consultants who offered their valuation report for each substitute $A_i\,(i = 1, 2\dots\ m)$ against their dimensions $D_j\,(j = 1, 2\dots\ n)$. Let $w = (\mathrm{w}_1, \mathrm{w}_2, \dots, \mathrm{w}_e)^T$ be the weight vector for consultants $Consultant_e(e = 1, 2\dots\ k)$ such that $\sum_{j=1}^{n} \mathrm{w}_l = 1$.

Step 2. The issue is considered in detail and the participating consultants are determined as shown in Table 1. The participating consultants give their opinions on the problem and define the dimensions and available scenarios.

Table 1. Particulars on the members of the panel of consultants.

Consultants	Experience	Occupation	Academic degree	Gender
Consultant_1	18	Industry	M.Sc.	Male
Consultant_2	25	Academia	Ph.D.	Male
Consultant_3	18	Industry	M.Sc.	Female
Consultant_4	25	Industry	Ph.D.	Female

Step 2. A set of linguistic variables and their equivalent T2NNs are identified as presented in Table 2, for consultants to use in evaluating the main dimensions and their sub-indicators, in addition to evaluating and arranging the selected alternatives.

Table 2. T2NN linguistic terms for weighing dimensions and alternatives.

Linguistic terms	Abridgements	Type-2 neutrosophic number
Exceedingly little	ELE	$\langle(0.20, 0.20, 0.10); (0.65, 0.80, 0.85); (0.45, 0.80, 0.70)\rangle$
Little	LLE	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$
Moderate little	MEE	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$
Moderate	MOE	$\langle(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)\rangle$
Moderate high	HHH	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$
High	HIH	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$
Exceedingly high	ELG	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$

Step 3. Construct a decision matrix of main dimensions or their sub-indicators by consultants to show their preferences to identify the main dimensions and their sub-indicators weights using the linguistic terms, then by using T2NNs.

Step 4. Calculate the score function for the T2NN valuations according to Eq. (1)[18].

$$S(\tilde{Z}_{ij}) = \frac{1}{12}\left\langle 8 + \left(T_{T_{\tilde{Z}_{ij}}}(a) + 2\left(T_{I_{\tilde{Z}_{ij}}}(a)\right) + T_{F_{\tilde{Z}_{ij}}}(a)\right) - \left(I_{T_{\tilde{Z}_{ij}}}(a) + 2\left(I_{I_{\tilde{Z}_{ij}}}(a)\right) + I_{F_{\tilde{Z}_{ij}}}(a)\right) - \left(F_{T_{\tilde{Z}_{ij}}}(a) + 2\left(F_{I_{\tilde{Z}_{ij}}}(a)\right) + F_{F_{\tilde{Z}_{ij}}}(a)\right)\right\rangle, \ i = 1, \ldots, m;\ j = 1, \ldots, n. \quad (1)$$

Step 5. Determine the local weights for the main dimensions and their sub-indicators based on the opinions of the consultants. In this regard, the global weights of the sub-indicators are determined, which are used in evaluating and arranging the selected alternatives.

Step 6. Building a decision matrix between the selected alternatives and sub-indicators according to the opinions of consultants to express their preferences for these alternatives using linguistic terms, then using the T2NNs in Table 2.

Step 7. Convert the T2NNs to real values using Eq. (1).

Step 8. Calculate the normalized decision matrix according to Eq. (2) for benefit indicators *B*, and for cost indicators *C*.

$$y_{ij} = \begin{cases} \dfrac{y_{ij}}{\max_i y_{ij}} & if\ j \in B \\ \dfrac{\min_i y_{ij}}{y_{ij}} & if\ j \in C \end{cases}$$

(2)

Step 9. Compute the weighted normalized decision matrix according to Eq. (3). Here $(w_j)_{1\times n}$ introduces weight of j^{th} indicator.

$$G = [g_{ij}]_{n\times m} = w_j \times y_{ij} \tag{3}$$

Step 10. Identify the negative ideal solution NS_j for each indicator according to Eq. (4).

$$NS = [ns_j]_{1\times m} = \min_i g_{ij} \tag{4}$$

Step 11. Compute the Euclidean and Taxicab distances of substitutes from negative ideal solution by employing the Eqs. (5) and (6).

$$E_i = \sqrt{\sum_{j=1}^{n}\left(g_{ij} - NS_j\right)^2} \tag{5}$$

$$T_i = \sum_{j=1}^{n}\left|g_{ij} - NS_j\right| \tag{6}$$

Step 12. Construct the comparative valuation matrix $[h_{is}]_{n\times n}$ according to Eq. (7).

$$h_{is} = (E_i - E_s) + (\gamma(E_i - E_s) \times (T_i - T_s)) \tag{7}$$

where $s \in \{1, 2\ldots m\}$ and γ designates a threshold function to identify the equality of the Euclidean distances of two substitutes.

Step 13. Compute the valuation score of each substitute according to Eq. (8). Rank the substitutes according to the greatest valuation score is the one that is measured to be the optimum substitute.

$$F_i = \sum_{k=1}^{n} h_{is} \tag{8}$$

3. Application

In this section, the steps of the proposed methodology T2NN-CODAS are applied to evaluate and determine the most appropriate scenarios for the development of intelligent reverse logistics services. This section is divided into three parts. The first part is about identifying experts, main dimensions, and their sub-indicators. In addition, the alternatives selected for the study are defined. In the second part, the steps of the proposed methodology are applied. The third part discusses and analyzes the results of the study.

Step 1. The problem was studied and the main objective was determined, which is to choose the most appropriate scenario among the scenarios for developing smart reverse logistics services. Also, four consultants were identified for the participation of the authors in conducting the study and evaluating the four main dimensions and their eight sub-indicators that have an impact on choosing the most appropriate scenario for the development of reverse logistics services.

Step 2. Seven terms and their corresponding T2NNs were identified, as shown in Table 2, for use by the participants in the assessment process, whether for the main dimensions and their sub-indicators or the selected alternatives.

Step 3. The evaluation dimensions and their sub-indicators have been identified. The four evaluation dimensions that have been identified are economic (D_1), technical (D_2), environmental (D_3), and service quality (D_4). In addition, each main dimension includes two sub-indicators. The eight sub-indicators in a row are investment and logistics costs (D_{1_1}), conservation of property value (D_{1_1}), developmental level (D_{2_1}), complexity and compatibility (D_{2_2}), waste and emissions reduction (D_{3_1}), protection of energy sources (D_{3_2}), reliability and flexibility (D_{4_1}), and time efficiency (D_{4_2}). In addition, the selected alternatives are four scenarios. The four scenarios are defined as follows:

- *Scenario 1 (A_1)*
  The Internet of Things, cloud computing, and electronic and mobile markets are the three technologies that are most suited to Industry 4.0, and the first scenario assumes that they will be used to accomplish some of the fundamental tasks.
- *Scenario 2 (A_2)*
  The second scenario contains the technologies and their applications from the first scenario, as well as some more applications of the same technologies, as well as the applications of autonomous vehicles, artificial intelligence, big data, and data mining in reverse logistics

networks. In addition, this scenario also includes some further applications of the same technologies.

- *Scenario 3 (A_3)*
  This scenario encompasses the technologies utilized in previous instances, while also incorporating supplementary ones. Another potential application of the Internet of Things is the implementation of a control system, which involves a shift from traditional pushed flows to pulled flows. The system offers data regarding the timing, geographical coordinates, and quantity of waste that necessitates collection, thereby streamlining waste management within an extensive physical region encompassed by the reverse logistics network.
- *Scenario 4 (A_4)*
  This scenario represents the highest level of complexity, involving the integration of all relevant Industry 4.0 technologies and their established or potential uses within the reverse logistics network. In this scenario, the utilization of IoT technology is extended to encompass the development of a comprehensive reverse logistics information management system. The primary objective of this system is to gather precise and dependable real-time data regarding the evolving characteristics of products. The system facilitates the monitoring, gathering, and administration of data, as well as the decision-making process pertaining to the handling of reverse flow materials and products and the utilization of resources.

Step 4. A decision matrix of main dimensions was created by the four consultants to show their preferences of the main dimensions weights using the linguistic terms as presented in Table 3, then by using T2NNs as exhibited in Table 4. The final weights of the main dimensions are presented in Table 4 and shown in Figure 2.

Table 3. Assessment matrix of main dimensions by the four consultants using semantic terms.

Consultants	Criteria			
	D_1	D_2	D_3	D_4
$Consultant_1$	ELG	LLE	HHH	ELE
$Consultant_2$	MEE	MEE	HIH	HIH
$Consultant_3$	MOE	HIH	ELG	MOE
$Consultant_4$	HIH	HHH	MOE	HHH

Table 4. Assessment matrix of main dimensions by the four consultants using T2NNs.

Consultants	Main dimensions	
	D_1	D_2
$Consultant_1$	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$
$Consultant_2$	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$
$Consultant_3$	$\langle(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)\rangle$	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$
$Consultant_4$	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$
Weight	0.266	0.219
Consultants	D_3	D_4
$Consultant_1$	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$	$\langle(0.20, 0.20, 0.10); (0.65, 0.80, 0.85); (0.45, 0.80, 0.70)\rangle$
$Consultant_2$	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$
$Consultant_3$	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$	$\langle(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)\rangle$
$Consultant_4$	$\langle(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)\rangle$	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$
Weight	0.290	0.224

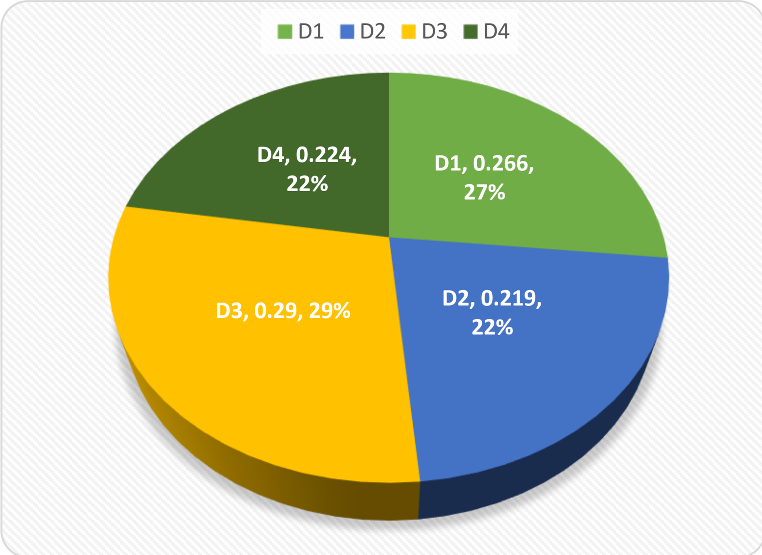

Figure 2. Weights of main dimensions.

Step 5. A decision matrix of all sub-indicators was created by the four consultants to show their preferences of the sub-indicators weights using the linguistic terms as presented in Table 5, then by using T2NNs as exhibited in Table 6. The final weights of all sub-indicators are presented in Tables 6-9. Also, the global weights of the sub-indicators were calculated based on the weights of the main dimensions and the weights of the local sub-indicators as presented in Table 10 and shown in Figure 3.

Table 5. Assessment matrix of all sub-indicators using semantic terms.

Consultants	**All sub-indicators**							
	D_{1_1}	D_{1_2}	D_{2_1}	D_{2_2}	D_{3_1}	D_{3_2}	D_{4_1}	D_{4_2}
Consultant_1	ELG	LLE	HHH	ELE	HHH	ELG	HIH	ELE
Consultant_2	MEE	MEE	HIH	HIH	HIH	MEE	MEE	LLE
Consultant_3	MOE	HIH	ELG	MOE	ELE	MOE	MOE	HHH
Consultant_4	HIH	HHH	MOE	HHH	LLE	ELG	MEE	HIH

Table 6. Assessment matrix of economic dimension's sub-indicators using T2NNs.

Consultants	**Economic dimension's sub-indicators**	
	D_{1_1}	D_{1_2}
Consultant_1	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$
Consultant_2	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$
Consultant_3	$\langle(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)\rangle$	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$
Consultant_4	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$
Weight	0.548	0.452

Table 7. Assessment matrix of technical dimension's sub-indicators using T2NNs.

Consultants	**Technical dimension's sub-indicators**	
	D_{2_1}	D_{2_2}
Consultant_1	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$	$\langle(0.20, 0.20, 0.10); (0.65, 0.80, 0.85); (0.45, 0.80, 0.70)\rangle$
Consultant_2	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$
Consultant_3	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$	$\langle(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)\rangle$
Consultant_4	$\langle(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)\rangle$	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$

Weight	0.564	0.436

Table 8. Assessment matrix of environmental dimension's sub-indicators using T2NNs.

Consultants	**Environmental dimension's sub-indicators**	
	$\mathbf{D_{3\_1}}$	$\mathbf{D_{3\_2}}$
Consultant_1	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$
Consultant_2	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$
Consultant_3	$\langle(0.20, 0.20, 0.10); (0.65, 0.80, 0.85); (0.45, 0.80, 0.70)\rangle$	$\langle(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)\rangle$
Consultant_4	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$	$\langle(0.95, 0.90, 0.95); (0.10, 0.10, 0.05); (0.05, 0.05, 0.05)\rangle$
Weight	0.416	0.584

Table 9. Assessment matrix of service quality dimension's sub-indicators using T2NNs.

Consultants	**Service quality dimension's sub-indicators**	
	$\mathbf{D_{4\_1}}$	$\mathbf{D_{4\_2}}$
Consultant_1	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$	$\langle(0.20, 0.20, 0.10); (0.65, 0.80, 0.85); (0.45, 0.80, 0.70)\rangle$
Consultant_2	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$	$\langle(0.35, 0.35, 0.10); (0.50, 0.75, 0.80); (0.50, 0.75, 0.65)\rangle$
Consultant_3	$\langle(0.50, 0.45, 0.50); (0.40, 0.35, 0.50); (0.35, 0.30, 0.45)\rangle$	$\langle(0.60, 0.45, 0.50); (0.20, 0.15, 0.25); (0.10, 0.25, 0.15)\rangle$
Consultant_4	$\langle(0.40, 0.30, 0.35); (0.50, 0.45, 0.60); (0.45, 0.40, 0.60)\rangle$	$\langle(0.70, 0.75, 0.80); (0.15, 0.15, 0.25); (0.10, 0.15, 0.15)\rangle$
Weight	0.527	0.473

Table 10. Final global weights of main dimensions and their sub-indicators.

Main dimensions	Weights of dimensions	Sub-indicators	Global weight of sub-indicators
Economic D_1	0.266	Investment and logistics costs D_{1_1}	0.146
		Conservation of property value D_{1_2}	0.120
Technical D_2	0.219	Developmental level D_{2_1}	0.124
		Complexity and compatibility D_{2_2}	0.095
Environmental D_3	0.290	Waste and emissions reduction D_{3_1}	0.121
		Protection of energy sources D_{3_2}	0.169
Service quality D_4	0.224	Reliability and flexibility D_{4_1}	0.118
		Time efficiency TPB_{4_2}	0.106

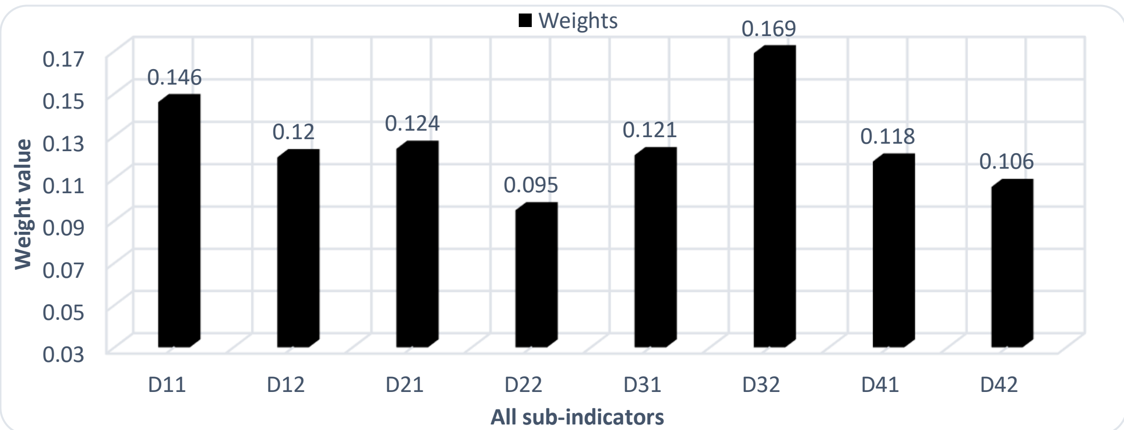

Figure 3. Final global weights of all sub-barriers.

Step 6. A decision matrix between the selected four scenarios and the eight sub-indicators was created according to the opinions of consultants to express their preferences for these scenarios using linguistic terms as exhibited in Table 11. The T2NNs was converted to real values according to Eq. (1).

Step 7. The normalized decision matrix was computed according to Eq. (2) for all sub-indicators as benefit indicators as presented in Table 12.

Step 8. The weighted normalized decision matrix was calculated according to Eq. (3), as presented in Table 13.

Step 9. The negative ideal solution was determined for each indicator according to Eq. (4), as displayed in Table 13.

Step 10. The Euclidean and Taxicab distances of substitutes from negative ideal solution were identified by employing the Eqs. (5) and (6), as presented in Table 14.

Step 11. The comparative valuation matrix was computed according to Eq. (7), as presented in Table 15. The four scenarios were ranked according to Eq. (8) as displayed in Table 15 and shown in Figure 4.

Table 11. Assessment matrix of the four scenarios according to the eight indicators using semantic terms.

Scenarios	All sub-indicators							
	D_{1_1}	D_{1_2}	D_{2_1}	D_{2_2}	D_{3_1}	D_{3_2}	D_{4_1}	D_{4_2}
$Scenario_1$	MOE	LLE	HHH	ELE	HHH	HHH	HIH	MOE
$Scenario_2$	MEE	ELE	HIH	ELG	MOE	MEE	MEE	LLE
$Scenario_3$	MOE	HIH	ELG	MOE	ELE	MOE	MOE	HIH
$Scenario_4$	HIH	MEE	MOE	HHH	LLE	ELG	MEE	HIH

Table 12. Normalized matrix of the four scenarios according to the eight indicators.

Scenarios	All sub-indicators							
	D_{1_1}	D_{1_2}	D_{2_1}	D_{2_2}	D_{3_1}	D_{3_2}	D_{4_1}	D_{4_2}
$Scenario_1$	0.716	0.383	0.763	0.258	1.000	0.763	1.000	0.716
$Scenario_2$	0.568	0.296	0.871	1.000	0.817	0.495	0.568	0.383
$Scenario_3$	0.716	1.000	1.000	0.624	0.338	0.624	0.716	1.000
$Scenario_4$	1.000	0.568	0.624	0.763	0.437	1.000	0.568	1.000

Table 13. Weighted normalized matrix of the four scenarios according to the eight indicators.

Scenarios	All sub-indicators							
	D_{1_1}	D_{1_2}	D_{2_1}	D_{2_2}	D_{3_1}	D_{3_2}	D_{4_1}	D_{4_2}
$Scenario_1$	0.105	0.046	0.095	0.025	0.121	0.129	0.118	0.076
$Scenario_2$	0.083	0.036	0.108	0.095	0.099	0.084	0.067	0.041
$Scenario_3$	0.105	0.120	0.124	0.059	0.041	0.105	0.084	0.106
$Scenario_4$	0.146	0.068	0.077	0.073	0.053	0.169	0.067	0.106
Negative ideal	0.083	0.036	0.077	0.025	0.041	0.084	0.067	0.041

Table 14. Euclidean and taxicab distances of the four scenarios.

Alternatives	E_i	T_i
$Scenario_1$	0.115	0.261

$Scenario_2$	0.096	0.159
$Scenario_3$	0.127	0.292
$Scenario_4$	0.138	0.306

Table 15. Comparative valuation matrix and ranking of the four scenarios.

Alternatives	$Scenario_1$	$Scenario_2$	$Scenario_3$	$Scenario_4$	F_i	Rank
$Scenario_1$	0.000	0.019	-0.012	-0.023	-0.016	3
$Scenario_2$	-0.018	0.000	-0.030	-0.041	-0.088	4
$Scenario_3$	0.012	0.031	0.000	-0.011	0.032	2
$Scenario_4$	0.023	0.043	0.011	0.000	0.078	1

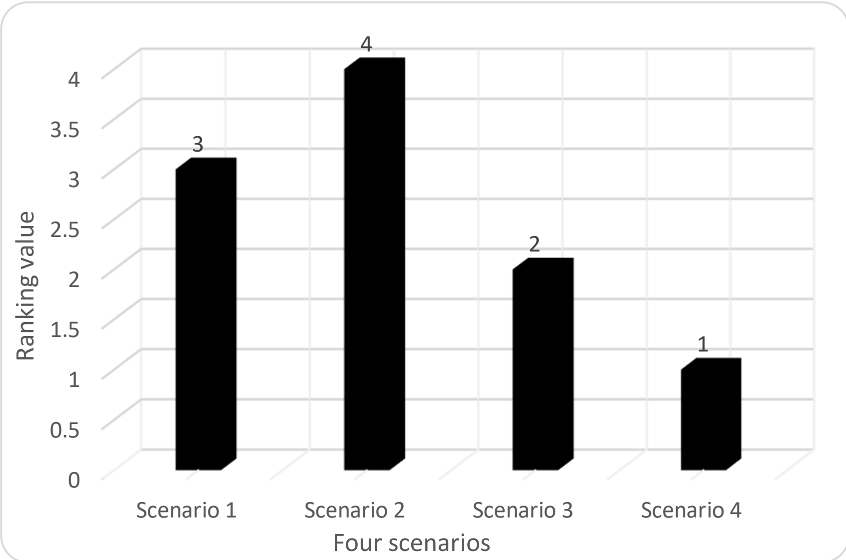

Figure 4. Final ranking of the four intelligent development logistics systems.

3.1 Results and discussion

In this part, the results obtained from the application of the proposed model to evaluate and determine the most suitable intelligent logistics development scenarios are discussed. The results are divided into two parts. The first part is concerned with evaluating the four main dimensions and their eight sub-indicators and determining the weights. The four main dimensions were evaluated through expert opinions as shown in Table 4.

The results indicate that the environmental dimension (D_3), is the dimension with the highest weight by 0.290, followed by the economic dimension (D_1) with a weight of 0.266, while the technical dimension (D_2) has the least weight by 0.219.

The second part is concerned with evaluating the four scenarios selected in the study. The four selected scenarios were arranged as shown in Table 15 and Figure 4. The results show that Scenario 4 (A_4) is the highest in the order, followed by Scenario 3 (A_3), while Scenario 2 (A_2) is the lowest in the order.

4. Conclusions

The primary objective of this study was to ascertain the framework for the advancement of smart reverse logistics, which can guide decision-making at both strategic and tactical levels. Additionally, the study aimed to develop a reverse logistics system that is universally accepted by all key stakeholders, thereby facilitating its extensive implementation and maximizing the associated positive outcomes. In light of this consideration, the study formulated four development scenarios. The individuals were assessed based on four dimensions that considered the objectives and concerns of the primary stakeholders. A proposed solution to address the defined problem involves the utilization of a T2NN-CODAS MCDM method. The formulation of the complete smart reverse logistics development scenarios as well as the framework for their assessment and ranking is the primary addition that this research makes to the existing body of literature dealing with reverse logistics and Industry 4.0.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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On Derived Superhyper BE-Algebras

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Abstract. Florentin Smarandache, introduced a creative idea named superhyper algebras as a popularization of algebras, via the nested power sets, superhyper operation, and superhyper axiom. This paper used the significance of superhyper BEsubalgebras as a popularization of BE subalgebras and investigated attributes of this pioneer significance in logic algebra. We prove that every BE algebras can be extended to a superhyper BE subalgebra and show how a superhyper BE algebra is a popularization of a hyper BE algebra and a BE algebra. The identity element in superhyper BE algebras acts a main designation in the form, attribute, analysis, and communication of other elements in these types of superhyper logic algebras.

Keywords: BE algebra, hyper BE algebra, superhyper BE algebra, generalized operation.

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1. **Introduction**

Logic algebra is one of the superlative important algebraic interdisciplinary structures that is used in various engineering sciences, especially computer science and all related branches. In this theory, on each finite and arbitrary collection, axiom principles are defined according to the ruling logic and its application in the actual religion, which gives this collection a special rule and rule. Based on these principles, a logical algebra is formed that all elements under this algebra must follow a certain rule. The importance of logical algebras is so great that many researchers in different fields are investigating its attribute and importance. During the research, they also found some deficiencies in the field of these algebras, which they solved with new definitions and popularizations of its principles. One of the ways to fix the defects in these algebras is to generalize them to logical algebraic significances and superstructures, which leads to the popularization and correction of the subject principles in logical algebras. Logical algebraic hyperstructures by covering the shortcomings of logical algebras can have more applications in the actual universe, especially when the proportion amongst a set of objects is discussed. One of the important logic algebras is the class of BE algebras, which is applied in the computer sciences. An mentioned, the BE algebras as a special of logic algebras, has and still has shortcomings, so the of hyper BE algebras is started to solve the defects in these logic algebras. The hyper BE algebras extended in recent years and has many applications. In this theory, for every given two elements, is created a set of elements that are related to axiom principles and must follow their principles. In this theory, just two elements can be combined and related to a set. But if we want to have more than two elements, we can't follow the axiom principles in hyper BE algebras. Therefore, this causes a defect in the application of these logical hyperalgebras and it is necessary to eliminate this defect. Based on these defects, Smarandache presented a new significance titled superhyper algebras as a popularization of hyperalgebras which have disparate attribute and are relevant with the actual universe [12–14]. In this theory, we can for every given more than two elements consider a set of sets of elements that are related under axiom principles and so we cover the defects in the of hyperalgebras. After then and based on these unprecedented significances, some researchers investigated some varius of superhyperalgebras. Hamidi, et al. in the realm of indeterminate logic (hyper) algebra, bring forward the significance of neutro BCK subalgebras. Beside, Rahmati et al. introduced the significance of superhyper BCK algebras as a popularization of BCK algebras and investigated some attribute of this unprecedented significance [6]. They published the significance of eextension of G algebras to superhyper G Algebras, wherethrough has nice outcomes in superhyper logic algebras. Some kinds of literature in this scops wherethrough we use for our work is such as on hyper K algebras [1], systems of propositional calculi [8], on hyper BE algebras [9], on fuzzy subalgebras of BE algebras [10], extension of G algebras to superhyper G algebras [11], popularizations and alternatives of classical algebraic structures to neutroalgebraic structures and antialgebraic structures [15], Implicative ideals of BCK algebras based on MBJ neutrosophic sets, on commutative BE algebras [17], extended fuzzy BCK Subalgebras [18], compactness and neutrosophic topological space via grills, neutrosophic systems with applications [2], separation axioms in neutrosophic topological spaces, neutrosophic systems with applications [3], separation axioms in neutrosophic topological spaces [4] and new types of topologies and neutrosophic topologies [16]. In this research, we try to apply the axioms in superhyper algebras and present the notation of superhyper BE algebras and analyze the connection between of BE algebras and superhyper BE algebras.

Motivation: Considering the ideas and creativity presented in new mathematics, especially Smarandache mathematics, we are looking to increase the communication of mathematics with applied and interdisciplinary topics. Development and expansion of logical algebra of superhypr BE algebras, can be the basis for discussions related to the communication of engineering sciences, especially computer sciences. Our whole motive for presenting this research is that superhypr BE algebras considering the nested super actions, it can study and research the sets of elements of whole-to-whole and part-to-whole communication elements in the first, middle and outer layers.

2. Preliminaries

In what lies ahead, we recollect some significances that require at follows.

Definition 2.1. [7] A system $(X;\sigma,\iota)$ known as a BE algebra provided,

$(BE1)$ $x\sigma x=\iota$,

$(BE2)$ $x\sigma\iota=\iota$,

$(BE3)$ $\iota\sigma x=x$,

$(BE4)$ $x\sigma(y\sigma z)=y\sigma(x\sigma z)$,

wherethrough $x,y,z\in X$ are arbitrary. An arbitrary BE algebra (X,σ,ι) is commutative, provided for each $x,y\in X,(x\sigma y)\sigma y=(y\sigma x)\sigma x$.

Proposition 2.2. [7] *Assume X is an arbitrary BE algebra. Afterwards for all $x,y\in X$*

(i) $x\sigma(y\sigma x)=\iota$,

(ii) $y\sigma((y\sigma x)\sigma x)=\iota$.

Definition 2.3. [10] Assume H is an arbitrary nondevoid set, $\epsilon\in X$ and $\varrho:H^2\to\mathcal{P}^*(H)$ be a map. Afterwards (H,ϱ,ι) known as a hyper BE–algebra, provided

(HBE_1) $\iota\in x\varrho\iota$ and $\iota\in x\varrho x$,

(HBE_2) $x\varrho(y\varrho z)=y\varrho(x\varrho z)$,

(HBE_3) $x\in\iota\varrho x$,

(HBE_4) provided $\iota\in\iota\varrho x$, so $x=\iota$,

wherethrough $x,y,z\in H$.

Theorem 2.4. [10] *In every hyper BE algebra H,*

(i) $A\varrho(B\varrho C)=B\varrho(A\varrho C)$,

(ii) $\iota\in A\varrho A$,

(iii) *provided* $\iota\in\iota\varrho A$, *afterwards* $\iota\in A$,

(iv) $\iota\in x\varrho(y\varrho x)$,

(v) *provided* $\iota\in x\varrho(y\varrho z)$, *afterwards* $\iota\in y\varrho(x\varrho z)$,

(vi) $\iota\in(x\varrho y)\varrho y$,

(vii) *provided* $z\in x\varrho y$, *afterwards* $\iota\in x\varrho(z\varrho y)$,

(viii) *provided* $y\in\iota\varrho x$, *afterwards* $\iota\in y\varrho x$,

wherethrough $A,B,C\subseteq H$.

Definition 2.5. [12,13] Let X be an arbitrary nondevoid set and $\iota \in X$. For a map $\vartheta^*_{\{m\to n\}}: X^m \to \mathcal{P}^n_*(X)$, $(X, \vartheta^*_{\{m\to n\}})$ known as an $\{m \to n\}$-super hyperalgebra, that $\mathcal{P}^n_*(X)$ is the n^{th} nested powerset of X and $\emptyset \notin \mathcal{P}^n_*(X)$.

3. On development of BE subalgebra

In what follows, present the significance of superhyper BE subalgebras based on the popularization of axioms of BE algebras and make a connection between of these logic superhyper algebras and BE algebras.

In the following, we prove a proposition that is fundamental in our work.

Lemma 3.1. *Assume (X, σ, ι) is a BE algebra. Afterwards $x\sigma(y\sigma z) = \iota\sigma(x\sigma(y\sigma z))$, whence $x, y, z \in X$.*

Proof. Whereof for every $x \in X$,, we have $x\varrho\iota = x$, we get $x\sigma(y\sigma z) = \iota\sigma(x\sigma(y\sigma z))$, whence $x, y, z \in X$. □

According to Lemma 3.1, we describe the significance of $\{m \to n\}$ superhyper BE subalgebras.

Definition 3.2. Let $m - 2 = \kappa$, X be a nonavoided set and $\iota \in X$. Afterwards $(X, \vartheta^*_{\{m\to n\}}, \iota)$ known as an $\{m \to n\}$ superhyper BE subalgebra, provided

(i) $\iota \in \vartheta^*_{\{m\to n\}}\underbrace{(x, x, \ldots, x, x)}_{m-times}$,

(ii) $\iota \in \vartheta^*_{\{m\to n\}}(x, \underbrace{(\iota, \iota, \ldots, \iota, \iota)}_{(\lambda)-times})$,

(iii) $x \in \vartheta^*_{\{m\to n\}}(\underbrace{(\iota, \iota, \ldots, \iota, \iota)}_{(\lambda)-times}, x)$,

(iv)

$$\vartheta^*_{\{m\to n\}}(\underbrace{(\iota, \iota, \ldots, \iota, \iota)}_{(\kappa)-times}, x, \vartheta^*_{\{m\to n\}}(y, x_1, \ldots, x_{m-1})) = \vartheta^*_{\{m\to n\}}(\underbrace{(\iota, \iota, \ldots, \iota, \iota)}_{(\kappa)-times}, y, \vartheta^*_{\{m\to n\}}(x, x_1, \ldots, x_{m-1})).$$

From now on, we will use $\{m \to n\}$ S.H BE algebra instead of $\{m \to n\}$ superhyper BE subalgebra, for simplify.

Example 3.3. (i) Suppose $(X, \vartheta^*_{\{m\to n\}})$ is an $\{m \to n\}$ S.H BE algebra. Afterwards $(X, \vartheta^*_{(2,0)})$ is a BE subalgebra.

(ii) Assume $(X, \vartheta^*_{\{m\to n\}})$ is an $\{m \to n\}$ S.H BE algebra. Afterwards $(X, \vartheta^*_{\{2\to 1\}})$ is a hyper BE subalgebra.

Example 3.4. Let $X = \{\iota, s\}$.

(i) Afterwards (X, ϑ^*) is a $\{3 \to 3\}$ S.H BE algebra what comes next:

$$\vartheta^*_{(3,3)}(r,r,r) = \mathcal{P}^3_*(\{\iota, r\}), \vartheta^*_{(3,3)}(r,s,\iota) = \mathcal{P}^3_*(\{\iota\}),$$
$$\vartheta^*_{(3,3)}(\iota,s,t) = \mathcal{P}^3_*(\{r\}), \text{and, for each another cases } \vartheta^*_{(3,3)}(r,s,t) = \mathcal{P}^3_*(\{r,s,t\}),$$

where

$$\begin{aligned}
&\mathcal{P}_*(\{s\}) = \mathcal{P}^2_*(\{s\}) = \mathcal{P}^3_*(\{s\}) = \{s\},\\
&\mathcal{P}_*(\{\iota, s\}) = \{\iota, s, \{\iota, s\}\},\\
&\mathcal{P}^2_*(\{\iota, s\}) = \{\iota, s, \{\iota, s\}, \{\iota, \{\iota, s\}\}, \{s, \{\iota, s\}\}\},\\
&\mathcal{P}^3_*(\{\iota, s\}) = \big\{\iota, s, \{\iota, s\}, \{\iota, \{\iota, s\}\},\\
&\{s, \{\iota, s\}\}, \{\iota, \{\iota, \{\iota, s\}\}, \{\iota, \{s, \{\iota, s\}\}, \{s, \{\iota, \{\iota, s\}\},\\
&\{s, \{s, \{\iota, s\}\}, \{\{\iota, s\}, \{\iota, \{\iota, s\}\}\}, \{\{\iota, s\}, \{s, \{\iota, s\}\}, \{\{\iota, \{\iota, s\}\}, \{s, \{\iota, s\}\}\}\big\}.
\end{aligned}$$

(i) By definition, $\iota \in \vartheta^*_{\{3\to3\}}(r,r,r) = \mathcal{P}^3_*(\{\iota, r\})$.

(ii) By definition, $\iota \in \vartheta^*_{\{3\to3\}}(r,\iota,\iota) = \mathcal{P}^3_*(\{\iota\})$.

(iii) By definition, $x \in \vartheta^*_{\{3\to3\}}(\iota,\iota,r) = \mathcal{P}^3_*(\{r\})$.

(iv) By definition,

$$\begin{aligned}
\vartheta^*_{\{3\to3\}}(\iota, x, \vartheta^*_{\{3\to3\}}(y,z,w)) &= \vartheta^*_{\{3\to3\}}(\iota, r, \mathcal{P}^3_*(\{u,v,w\}))\\
&= \mathcal{P}^3_*(\{r,u,v,w\})\\
&= \vartheta^*_{\{3\to3\}}(\iota, u, \vartheta^*_{\{3\to3\}}(r,v,w)).
\end{aligned}$$

(ii) Afterwards (X, ϑ^*) is a $\{3 \to 0\}$ S.H BE subalgebra what comes next:

$$\vartheta^*_{(3,\iota)}(r,r,r) = \{\iota\}, \text{and for another cases}, \vartheta^*_{(3,\iota)}(s,r,t) = \{t\}.$$

Theorem 3.5. *Assume $(X, \vartheta^*_{\{m\to n\}})$ is an $\{m \to n\}$ S.H BE algebra. Afterwards for every $k \geq n$, $(X, \vartheta^*_{\{m\to n\}})$ is an $\{m \to k\}$ superhyper BE subalgebra.*

Proof. Suppose $(X, \vartheta^*_{\{m\to n\}})$ is an $\{m \to n\}$ S.H BE algebra and $k \geq n$. Whereof $\mathcal{P}^n_*(X) \subseteq \mathcal{P}^k_*(X)$, for every $y_1, y_2, \ldots, y_m \in X, \vartheta^*_{\{m\to n\}}(y_1, y_2, \ldots, y_m) \subseteq \vartheta^*_{(m,k)}(y_1, y_2, \ldots, y_m)$. Thus $\iota \in \vartheta^*_{\{m\to n\}}(y_1, y_2, \ldots, y_m)$ implies that $\iota \in \vartheta^*_{(m,k)}(y_1, y_2, \ldots, y_m)$ and every principles are accurate. □

Example 3.6. Let $X = \{0, s\}$. Afterwards for each $n \geq 3$, using above Theorem, (X, ϑ^*) is a $\{3 \to n\}$ S.H BE algebra what comes next:

$$\vartheta^*_{(3,3)}(u,u,u) = \mathcal{P}^n_*(\{\iota, u\}), \vartheta^*_{(3,3)}(u,v,\iota) = \mathcal{P}^n_*(\{\iota\}),$$
$$\vartheta^*_{(3,3)}(\iota,v,w) = \mathcal{P}^n_*(\{u\}),$$
$$\text{and for another cases } \vartheta^*_{(3,3)}(u,v,w) = \mathcal{P}^n_*(\{u,v,w\}).$$

Let $k \in \mathbb{N}$, $m = 2k$, $(X, \vartheta^*_{\{m\to n\}})$ be an $\{m \to n\}$ S.H BE algebra. For every given $c_1, c_2, \ldots, c_m \in X$, define $(c_1, c_2, \ldots, c_{\frac{m}{2}}) \leq (c_{\frac{m}{2}+1}, c_{\frac{m}{2}+2}, \ldots, c_m)$ iff $\iota \in \vartheta^*_{\{m\to n\}}(c_1, c_2, \ldots, c_m)$.

Theorem 3.7. *Let $k \in \mathbb{N}$, $m = 2k$, $c_1, c_2, \ldots, c_m \in X$. Afterwards $(X, \vartheta^*_{\{m\to n\}})$ is an $\{m \to n\}$ S.H BE algebra provided,*

(i) $\vartheta^*_{\{m\to n\}}(y, x, \ldots, x) \leq \vartheta^*_{\{m\to n\}}(y, x, \ldots, x)$,
(ii) $\vartheta^*_{\{m\to n\}}(c_1, c_2, \ldots, c_m) \leq \vartheta^*_{\{m\to n\}}(c_1, c_2, \ldots, c_{m-1}, \iota)$,
(iii) $\vartheta^*_{\{m\to n\}}(c_1, c_2, \ldots, c_m) \leq \vartheta^*_{\{m\to n\}}(\iota, c_1, \ldots, c_{m-1}, c_m)$.

Proof. Occurring by definition. □

Theorem 3.8. *Assume $(X, \vartheta^*_{\{m\to n\}})$ is an $\{m \to n\}$ S.H BE algebra. Afterwards $\iota \in \vartheta^*_{\{m\to n\}}(\underbrace{(\iota, \iota, \ldots, \iota, \iota)}_{(\kappa)-times}, x, \vartheta^*_{\{m\to n\}}(y, x, \ldots, x))$, when $m - 2 = \kappa$*

Proof. Assume $(X, \vartheta^*_{\{m\to n\}})$ is an $\{m \to n\}$ S.H BE algebra and $x, y \in X$. Afterwards

$$\begin{aligned} & \iota \in \vartheta^*_{\{m\to n\}}(\underbrace{(\iota, \iota, \ldots, \iota, \iota)}_{(\kappa)-times}, y, \iota) \\ \subseteq\ & \vartheta^*_{\{m\to n\}}(\underbrace{(\iota, \iota, \ldots, \iota, \iota)}_{(\kappa)-times}, y, \vartheta^*_{\{m\to n\}}(x, x, \ldots, x)) \\ =\ & \vartheta^*_{\{m\to n\}}(\underbrace{(\iota, \iota, \ldots, \iota, \iota)}_{(\kappa)-times}, x, \vartheta^*_{\{m\to n\}}(y, x, \ldots, x)). \end{aligned}$$

□

Theorem 3.9. *Suppose $(X, \vartheta^*_{\{m\to n\}})$ is an $\{m \to n\}$ S.H BE algebra. Afterwards for $m-2 = \kappa$,*

$$\iota \in \vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, x, \vartheta^*_{\{m\to n\}}(\vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, x, y), \underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, y).$$

Proof. Assume $x, y \in X$. Afterwards

$$\begin{aligned}
\iota &\in \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, \vartheta^*_{\{m\to n\}}((\underbrace{\iota,\iota,\ldots,\iota,\iota)}_{(\kappa)-times}, x, y)), \vartheta^*_{\{m\to n\}}((\underbrace{\iota,\iota,\ldots,\iota,\iota)}_{(\kappa)-times}, x, y)) \\
&= \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, x, \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, x, y), y)).
\end{aligned}$$

It concludes that $\iota \in \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, x, \vartheta^*_{\{m\to n\}}(\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, x, y), \underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, y)$. □

Definition 3.10. Let $(X, \vartheta^*_{\{m\to n\}})$ be an $\{m \to n\}$ S.H BE algebra. Afterwards $(X, \vartheta^*_{\{m\to n\}})$ known as distributive, provided

$$\begin{aligned}
&\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, x, \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, y, z)) \\
= \ &\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, x, y), \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, x, z)),
\end{aligned}$$

when $m - 2 = \kappa$.

Theorem 3.11. *Assume $(X, \vartheta^*_{\{m\to n\}})$ is a distributive $\{m \to n\}$ S.H BE algebra. Afterwards for when $m - 2 = \kappa$,*

(i) *If $x \le y$, afterwards $\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, z, x) \le \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, z, y)$,*

(ii) $\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, y, z) \le \vartheta^*_{\{m\to n\}}(\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, x, y), \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, x, z))$,

(iii) $\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, y, x) \le \vartheta^*_{\{m\to n\}}(\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, z, y), \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, z, x))$.

Proof. (i) Let $m - 2 = \kappa$, $x, y, z \in X$. Afterwards $x \le y$, imples that $\iota \in \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, x, y)$. Whereof $(X, \vartheta^*_{\{m\to n\}})$ is a distributive $\{m \to n\}$ S.H BE algebra, we get that

$$\begin{aligned}
\iota &\in \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, z, \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, x, y)) \\
&= \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, z, x), \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, z, y)).
\end{aligned}$$

It follows that $\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, z, x) \le \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times}, z, y)$.

(ii) Let $x, y, z \in X$. Afterwards for $\beta = \underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}$, we get that

$$\begin{aligned}
&\iota \in \vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, \vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, y, z)\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times})\\
\subseteq\ & \vartheta^*_{\{m\to n\}}(\beta, \vartheta^*_{\{m\to n\}}(\beta, \vartheta^*_{\{m\to n\}}(\beta, y, z), x), \vartheta^*_{\{m\to n\}}(\beta, \vartheta^*_{\{m\to n\}}(\beta, y, z), \vartheta^*_{\{m\to n\}}(\beta, y, z))\\
\subseteq\ & \vartheta^*_{\{m\to n\}}(\beta, \vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, y, z), \vartheta^*_{\{m\to n\}}(\beta, x, \vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, y, z)))\\
\subseteq\ & \vartheta^*_{\{m\to n\}}(\beta, \vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, y, z), \vartheta^*_{\{m\to n\}}(\beta, (\vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, x, y),\\
&\vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, x, z)).
\end{aligned}$$

It follows that

$$\vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, y, z) \leq \vartheta^*_{\{m\to n\}}(\vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, x, y), \vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, x, z)).$$

(iii) It is similar to (ii). □

Definition 3.12. Assume $(X, \vartheta^*_{\{m\to n\}})$ is an $\{m \to n\}$ S.H BE algebra. Afterwards $(X, \vartheta^*_{\{m\to n\}})$ known as commutative, provided

$$\begin{aligned}
&\vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, \vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, x, y), y)\\
=\ & \vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, \vartheta^*_{\{m\to n\}}(\underbrace{\iota, \iota, \ldots, \iota, \iota}_{(\kappa)-times}, y, x), x),
\end{aligned}$$

when $m - 2 = \kappa$.

Example 3.13. Let $X = \{-1, 1, 2, 3, \ldots\}$.

(i) Afterwards (X, ϑ^*) is a commutative $\{3 \to 3\}$ S.H BE subalgebra what comes next:

$$\vartheta^*_{(3,3)}(x, y, z) = \begin{cases}
\mathcal{P}^3_*(\{-1\}) & \text{provided } z \leq y \leq x\\
\mathcal{P}^3_*(\{-1, y-z, x-z\}) & \text{provided } z < y \leq x\\
\mathcal{P}^3_*(\{-1, y-z, x-z\}) & \text{provided } z < x \leq y\\
\mathcal{P}^3_*(\{-1, z-x, y-x\}) & \text{provided } x < z \leq y\\
\mathcal{P}^3_*(\{-1, z-x, y-x\}) & \text{provided } x < y \leq z\\
\mathcal{P}^3_*(\{-1, z-y, x-y\}) & \text{provided } y < z \leq x\\
\mathcal{P}^3_*(\{-1, z-y, x-y\}) & \text{provided } y < x \leq z\\
\mathcal{P}^3_*(\{x-y, x-z, y-z\}) & o.w
\end{cases},$$

Theorem 3.14. *Assume* $(X, \vartheta^*_{\{m\to n\}})$ *is a commutative* $\{m \to n\}$ *S.H BE algebra. Afterwards for when* $m-2=\kappa$,

$$\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},x,y)$$
$$\subseteq \vartheta^*_{\{m\to n\}}\big(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},x,y),y),y\big) \text{ and } m-2=\kappa.$$

Proof. Let $x, y \in X$. Afterwards for $\alpha = m-1$, we get

$$\begin{aligned}
&\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},x,y)\\
\subseteq\ &\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\alpha)-times},\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},x,y))\\
\subseteq\ &\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\epsilon,\iota}_{(\kappa)-times},\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\lambda)-times},y,\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},x,y),\\
&\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},x,y)))\\
\subseteq\ &\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},x,y),y),y).
\end{aligned}$$

Thus

$$\begin{aligned}
&\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},x,y)\\
\subseteq\ &\vartheta^*_{\{m\to n\}}\big(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\kappa)-times},x,y),y),y\big).
\end{aligned}$$

□

Corollary 3.15. *Assume* $(X, \vartheta^*_{\{m\to n\}})$ *is a commutative* $\{m \to n\}$ *S.H BE algebra. Afterwards* $(X, \vartheta^*_{\{m\to n\}})$ *is distributive iff for every* $x, y \in X$,

$$\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\lambda)-times},x,y) = \vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\lambda)-times},x,\vartheta^*_{\{m\to n\}}(\underbrace{\iota,\iota,\ldots,\iota,\iota}_{(\lambda)-times},x,y)) \text{ and } m-1=.$$

4. Conclusion and discussion

One of the aims of this article is to consider the relationship between some arbitrary elements that are dependent on the principles of axiom. Investigating this important issue is a more complicated task compared to the connection of only two elements, wherethrough creates a certain limitation for our article. Of course, the advantage of this work compared to the structural mode is that we have no restrictions on the selection of elements, and for any number of elements you can create a targeted connection. Indeed, the significance of $\{m \to n\}$ S.H BE algebras can cover the defects of hyper BE algebras and so can be applied to more actual problems. We try to define it in such a way that it outcomes in BE algebras and hyper BE algebras significances and at the same time we can connect more elements and eliminate the limitation of two elements in application. In the next investigations, attempt to achieve more outcomes concerning Neutro super (hyper)EQ subalgebras and utilizations in actual-universe problems. We demand to expand the significance of EQ algebras concerning networks and complex hypernetworks.

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Calculation of shortest path on Fermatean Neutrosophic Networks

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Abstract:

The shortest path (SP) problem (SPP) has several applications in graph theory. It can be used to calculate the distance between the provided initial and final vertex in a network. In this paper, we employed the Fermatean neutrosophic number as the appropriate edge weight of the network to estimate the SP connecting the start and end vertex. This technique is highly useful in establishing the shortest path for the decision-maker under uncertainty. We also investigated its effectiveness in comparison to several existing methods. Finally, a few numerical tests were performed to demonstrate the validity and stability of this new technique, as well as to compare different types of shortest paths with different networks.

Keywords: Fermatean neutrosophic graph; Fermatean neutrosophic number, shortest path problem, Uncertainty.

1. Introduction:

The SPP is mainly a significant and essential combinatorial network optimization decision-making problem, and we can also say it is the heart of network flows. For example, it has a broad area of applications and is used in routing [1], transportation [2], supply chain management [3], communications [4], wireless networks [5], etc. Basically, this focuses on finding the SP between a particular initial node and the final node. Nowadays, it has important applications in engineering and research. In terms of competence, efficiency, and analytical techniques, the SPP has been extremely well researched.

In a traditional problem, the length between two vertices is assumed to be a real number in a certain environment. But in case of uncertainty, fuzzy numbers can be used to obtain the best result. The SPP is an essential component of the transport management system, connecting a specific source vertex to a destination vertex. Where the variety of research papers published regarding SPP in an uncertain environment where vertex and edges are stimulated to represent transportation cost or time.

However, in a real-world situation, different types of ambiguity are usually caused like failures, deficient data, or other factors such as weather or traffic conditions. In these cases, evaluating the particular optimal path in given networks may be difficult, so here we take a fuzzy number. In real-world problems involving scheduling, transportation, vehicle green routing, etc. that use SPP, the edge weights need not be certain due to fluctuations in parameters such as weather and traffic conditions. In those circumstances, experts emphasize the use of probability concepts to handle randomness arising due to the uncertainties of the SPP. Zadeh [6] introduced the all-important uncertainty theory for dealing with imprecise data in many real-world problems. In real life, we have a tendency to find the optimum (maximum or minimum) solution to any problem. Since a large volume of the available data is imprecise and inconsistent, the results produced are inconsistent, which paves the way for the discovery of uncertainty theory. Fuzzy optimization has been a significantly popular topic among researchers in the last couple of decades due to its extensive usage in various areas involving network flow problems [7], production problems [8], Automotive Industry [9,10,11,12,13,14,15,16,17], pick-up and delivery problems [18], travelling salesman problems [19], and traffic assignment problems [20].

Over the past 20 years, numerous researchers have conducted extensive research on SPP using various fuzzy number types, such as edge and vertex weights. "Fuzzy SPP" is the name given to this type of SPP in fuzzy scenarios. In the course of time, numerous studies on the FSPP have been conducted [21,22,23,24,25,26,27].There are different research paper already done where edge weights are neutrosophic numbers (NNs), which can be single, interval, or bipolar valued, one can use a neutrosophic set (NS) to find the network's shortest path [28, 29]. In situations where the theory of fuzzy logic is not useful when handling imprecise, uncertain and indeterminate issues, Smarandache introduced neutrosophic in 1995 and proposed a tool called the neutrosophic set (NS) theory. Truth (T), indeterminacy (I), and falsity (F) are three autonomous mappings that give rise to NS and have values between [0, 1]. It is extremely challenging to use NS directly.

Fuzzy graphs can be used for SPP while there is uncertainty in the vertices and edges, but neutrosophic concepts may be better able to manage uncertainty because indeterminacy is also taken seriously [30]. Since it can able to manage uncertain, inconsistent, and indeterminate information, NS is almost indispensable when it connecting with real-world issues in science and engineering [31]. In intelligent transportation systems, maintaining routes or providing uncertain supplies is of utmost importance.

Some days before, Antony Crispin Sweety and Jansi R [32] introduced a new idea called the Fermatean neutrosophic set (FNS) by mixing the concept of two sets i.e Fermatean fuzzy sets and neutrosophic sets. After FNs, some new concepts were introduced by Said Broumi et al. [33], such as regular and Fermatean neutrosophic graphs, Cartesian, composition, and lexicographic products of FNG.

The primary goal of this paper is to present an efficient algorithmic strategy for SPP that can be adapted to an uncertain environment arising in practical situations. The following are the main contributions of this paper:

- We used the Fermatean neutrosophic number FNSP as a vertex and then calculated the arc length of a Fermatean neutrosophic network. Because a graph or network may contain ambiguity or imperfection in its relationships or connections in some cases.
- Fermatean neutrosophic number (FNSP) is an extension of the neutrosophic number (NS) that addresses uncertainty by allowing the representation and analysis of uncertain or ambiguous information.
- Here we introduce a new algorithm to handle the SPP in an uncertain environment that calculates the length of the shortest path connecting two given nodes.

The remaining portion of the paper is prearranged in the following way: In Section 1, the literature analysis has been compiled. In Section 2, an overview of a Fermatean neutrosophic set is available. In Section 3, with the help of the suggested score function, novel algorithms are proposed. Section 4 provides a numerical example of finding the FNSP in Fermatean neutrosophic environments. Section 5 discussed the comparison of the shortest path with different networks and with different parameters, along with the benefits of the suggested approach. Section 6 provides the presented work's conclusion.

2. Preliminaries:

In this part of the article, the fundamental ideas and definitions of the neutrosophic set, Fermatean neutrosophic set, Fermatean neutrosophic relation, and score function of the Fermatean neutrosophic number are presented.

Definition 2.1 [35]

A neutrosophic set (NS) A in a universal set X is defined as $A = \langle x, \left(\breve{T}_A(x), \breve{I}_A(x), \breve{F}_A(x) \right) : x \in X$. where $\breve{T}_A(x)$,$\breve{I}_A(x)$,$\breve{F}_A(x)$ is truth, Indeterminacy and falsity membership degree. and sum of these three degrees of membership is written as $0^{-} \leq \check{T}_A(x) + \check{I}_A(x) + \check{F}_A(x) \leq 3^{+}$.

Definition 2.2 [31]

A Fermatean neutrosophic set (FNS) A in universal set X is characterized by $= \{x, \left(\check{T}_A(x), \check{I}_A(x), \check{F}_A(x) \right) : x \in X\}$.

Where $\check{T}_A(x)$ shows the membership degree, $\check{I}_A(x)$ indicates the indeterminacy-membership degree, and $\check{F}_A(x)$ shows the non-membership degree.

Where sum cube of membership and falsity membership degree lies in between [0,1] i.e. $0 \leq (\check{T}_A(x))^3 + (\check{F}_A(x))^3 \leq 1$ and cube of indeterminacy degree lies in between[0,1] i.e.

$0 \leq (\check{I}_A(x))^3 \leq 1$.

Finally, the sum cube of this three membership degree lies in between [0, 2] i.e. $0 \leq ((\check{T}_A(x))^3 + (\check{I}_A(x))^3 + \left(\check{F}_A(x)\right)^3 \leq 2$.

Definition 2.3 [31]

Let X is a universal set and a mapping $S = ((\check{T}_S, \check{I}_S, \check{F}_S) : X \times X \rightarrow [0,1])$ is called a Fermatean Neutrosophic relation on X such that $(\check{T}_S(u,v), \check{I}_S(u,v), \check{F}_S(u,v) \in [0,1]$ for all $u,v \in X$.

Definition 2.4[31]

Let $S = (\check{T}_S, \check{I}_S, \check{F}_S)$ and $R = (\check{T}_R, \check{I}_R, \check{F}_R)$ be Fermatean Neutrosophic number on a vertices

.then the edge length from R to S is defined as

$$\check{T}_R(u,v) = min\{\check{T}_S(u), \check{T}_S(v)\}$$

$$\check{I}_R(u,v) = max\{\check{I}_S(u), \check{I}_S(v)\}$$

$$\check{F}_R(u,v) = max\{\check{F}_S(u), \check{F}_S(v)\}$$

if $\check{T}_R(u,v), \check{I}_R(u,v), \check{F}_R(u,v) \in [0,1]$

Definition 2.5[31]

Let the vertices $T_S(u,v), I_S(u,v), F_S(u,v)$ be Fermatean neutrosophic number then the score function is defined as $S = \frac{\check{T}_R(u,v)+\check{I}_R(u,v)+1-\check{F}_R(u,v)}{3}$

3. Proposed Shortest Path algorithm based on Fermatean neutrosophic number

In this section, the proposed Shortest Path algorithm based on Fermatean neutrosophic number aims to address the limitations of existing path finding algorithms by incorporating the concept of Fermatean neutrosophic numbers

- Step-1: Choose one vertex as the initial and one vertex as the final point of the associated network.
- Step-2: Find the total possible path from initial node to final node of the associated network.
- Step-3: By using definition -2.4 to find the edge weights of a network from nodes.
- Step-4: After getting all edge value now convert it to crisp number by using score function (Definition-2.5).
- Step-5: Calculate the average value of a path by adding all the edges. Arrange the path in ascending order from lowest to highest path.

4. Illustrative Example

Consider a Fermatean neutrosophic network, where the vertex weights are defined by Fermatean numbers. The source vertex is 1 and the destination vertex is 6. By utilizing Fermatean neutrosophic values, we can explore complex systems, decision-making processes, or social networks with incomplete or uncertain information. Neutrosophic graph theory and related techniques aim to handle indeterminacy and provide a more accurate representation of uncertain realities.

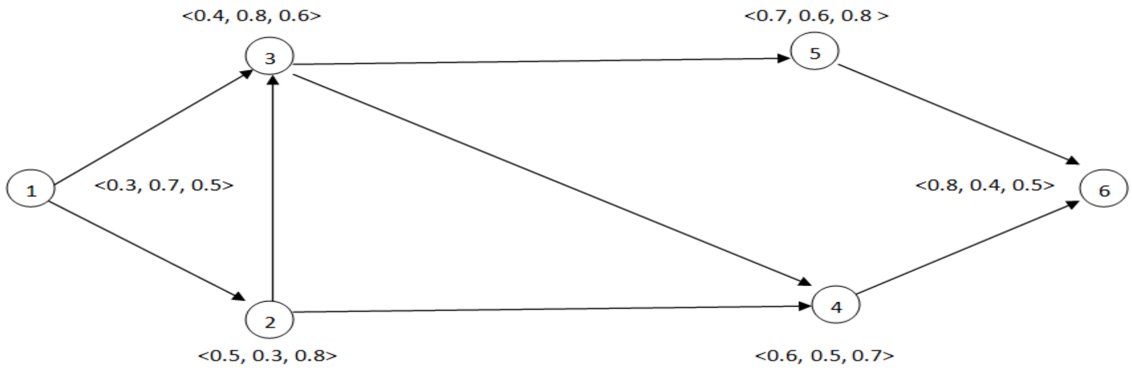

Figure1.Fermatean neutrosophic network

Here, Fermatean neutrosophic values on reality represent the recognition and acknowledgment of the existence of multiple perspectives and uncertainties in any given situation.

Vertices	Fermatean neutrosophic number
1	<0.3.0.7,0.5 >
2	<0.5,0.3.0.8>
3	<0.4,0.8,0.6>
4	<0.6,0.5,0.7>
5	<0.7,0.6,0.8>
6	<0.8,0.4,0.5>

Table-1: Vertices Weights

Implementation of Algorithm

In this section, it involves the conversion of a theoretical algorithm, which is a series of logical instructions, into a practical and executable solution that solves a specific problem.

Step-1:

From Figure1. Source vertex weight is 1 and destination vertex weight is 6.

Step-2:

The consequent of all probable paths connecting from source vertex to destination vertex are shown in Table-2.

Sl.no	Path
1	1-2-4-6
2	1-3-5-6
3	1-3 -4-6
4	1-2-3-4-6

Table-2: Path of a Network

Step-3:

In this given network, the nodes' weights are given. We can find the edge length of a given network by using definition 5.

From node 1 nodes weight is <0.3, 0.7, 0.5 >

From node 2 nodes weight is <0.5, 0.3, 0.8>

Then, the edge length from node 1 to node 2 is

$$\breve{T}_R(u,v) = min\{\breve{T}_S(u), \breve{T}_S(v)\}$$

$$\breve{I}_R(u,v) = max\{\breve{I}_S(u), \breve{I}_S(v)\}$$

$$\breve{F}_R(u,v) = max\{\breve{F}_S(u), \breve{F}_S(v)\}$$

i.e

$\breve{T}_R(u,v) = min\{0.3,0.5\} = 0.3$

$\breve{I}_R(u,v) = max\{0.7,0.3\} = 0.7$

$\breve{F}_R(u,v) = max\{0.5,0.8\} = 0.8$

So, the edge length from node-1 to node-2 is <0.3, 0.7, 0.8>

Similarly, we can find the edge lengths of all the nodes in the network given in the table.

Edges	Edge weights
1-2	<0.3,0.7,0.8>
1-3	<0.3,0.8,0.6>
2-3	<0.4,0.8,0.8>

2-4	<0.5,0.5,0.8>
3-4	<0.3,0.7,0.8>
3-5	<0.4,0.8,0.8>
4-6	<0.6,0.5,0.7>
5-6	<0.7,0.6,0.8>

Table-3:Edge weights

Step-4:

Now we convert Fermatean neutrosophic edge weight to crisp edge weight by using score function $S = \frac{\check{T}_R(u,v)+\check{I}_R(u,v)+1-\check{F}_R(u,v)}{3}$.

Edge weight from node-1 to node-2 is

$$S = \frac{0.3 + 0.7 + 1 - 0.8}{3} = \frac{1.2}{3} = 0.40$$

Similarly to convert all Fermatean neutrosophic edge weight to crisp edge in Table-4

Edges	Edge weights in crisp number
1-2	0.40
1-3	0.53
2-3	0.46
2-4	0.30
3-4	0.30
3-5	0.46
4-6	0.46
5-6	0.50

Table-4:Edge weights in crisp number

Step-5:

$$\text{Average weights of a Path} = \frac{\text{Sum of edges of path}}{\text{number of edges}}$$

$$\text{Average weights of a Path}(1\text{-}2\text{-}4\text{-}6) = \frac{(1-2)+(2-4)+(4-6)}{3}$$

$$\text{Average weights of a Path}(1\text{-}2\text{-}4\text{-}6) = \frac{0.40+0.30+0.46}{3} = 0.38$$

Similarly, to calculate average weights of all possible paths from source to destination.

Possible paths	Average weights of a path
1-2-4-6	0.38
1-3-5-6	0.49
1-3 -4-6	0.43
1-2-3-4-6	0.40

Table-5: average weights of all possible paths

Here, the shortest path 1-2-4-6 and shortest path value is 0.3

5. Comparison study:

In this segment, we evaluate our algorithm with Fermatean neutrosophic environment and with some existing methodology [30] and [31].

Shortest path with different network	**Path**	**Shortest path Length**
shortest path with IVNNs [34]	$1 \to 2 \to 4 \to 6$	[0.35,0.60],[0.01,0.04],[0.008,0.75]
shortest path with trapezoidal and triangular neutrosophic numbers [35]	$1 \to 2 \to 4 \to 6$	0.485
Our proposed algorithm	$1 \to 2 \to 4 \to 6$	0.38

Table-6: Shortest path with different network

and also we do a comparisons study for evaluation of SPP with different parameter as shown in Table-7

Evaluating SPP with Different parameter	**Arc lengths/vertices**	**Indeterminacy**	**Ambiguity**	**Uncertainty**	**Advantages**	**Limitations**
Crisp parameters	crisp Number	insufficient to handle	insufficient to handle	insufficient to handle	Able to determine easily information	unable to fully express the uncertain information
Fuzzy parameters	Fuzzy Number	Not able to manage	Not able to manage	capable to manage with uncertainty	Able to determine easily uncertain in formation	Able determine only membership information but not for non membership
Intuitionistic fuzzy parameters	Intuitionistic Fuzzy Number	insufficient to handle	able to manage	able to manage	Able determine both membership and non membership information	Not able to determine sum of membership and non membership value greater than one or not
Neutrosophic parameters	Neutrosophic Number	sufficient to handle	able to manage	May or may not be able to manage	Able determine both Truth, Indeterminacy and Falsity membership information	Not able to determine the cube sum of Truth, Indeterminacy and falsity membership value > 2
Fermatean neutrosophic parameters	Fermatean neutrosophic number	sufficient to handle	able to manage	able to manage	Able to determine the cube sum of Truth, Indeterminacy and falsity membership value in between 0 to 3	Not able to determine the interval data

Table-7: Shortest path with different parameter

6. Conclusion:

Uncertainty is essential to all scientific and engineering concerns. Fuzzy theory, intuitionistic fuzzy theory, and neutrosophic theory are the most valuable tools for determining the best answer to multi-criteria decision-making situations like the shortest path problem in a network.

In this paper, we study the advantages of employing the fermatean neutrosophic number in NSP. It incorporates uncertainty with the help of the Fermatean neutrosophic number edge weight between the source and destination vertex in a Fermatean neutrosophic environment. We are also expanding our research on this novel idea to include Interval-valued fermatean neutrosophic numbers, Interval-valued fermatean triangles, and trapezoidal neutrosophic numbers, as well as their applications, in future work.

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Evaluation the Impact Potentials of Materials and Systems with Specific Criteria under Neutrosophic Set

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Abstract: Energy, healthcare, electronics, transportation, ecology, and infrastructure are just a few of the areas that might greatly benefit from the use of new materials and systems. This study delves into the factors that should be taken into account when determining the possible effect of a certain substance or system. This analysis takes into account ecological, monetary, technical, health and safety, regulatory, commercial, and cooperative factors. Stakeholders may make more well-informed decisions and ensure that materials and systems are used to their maximum capacity if they take these into account. This paper used the concept of multi-criteria decision-making (MCDM) to deal with various criteria and factors. The VIKOR method is used as an MCDM method to rank the materials according to various criteria. The VIKOR method is integrated with the single-valued neutrosophic set to overcome uncertain information. This paper used eight criteria and ten materials to select the best one. The results show the cost criteria is the height weight in all criteria.

Keywords: Neutrosophic Set, Materials, Evaluation, MCDM, Uncertainty.

1. Introduction

Our contemporary world owes much to the materials and processes that drive technological progress and revolutionize several sectors. These materials and systems have the potential to have a huge impact on a variety of fields. Understanding and utilizing the potential of materials and systems is essential for tackling societal concerns, promoting sustainability, and supporting innovation in fields ranging from energy and healthcare to transportation and infrastructure[1], [2].

Potential impacts are evaluated using a multifaceted set of criteria and characteristics. Materials and systems should aim to decrease carbon emissions, energy consumption, and waste creation over their entire lifespan to minimize their environmental effect. Improving people's quality of life, their sense of security, and their ability to access resources are all examples of social effects. Material and system development should be cost-effective and promote economic growth to have a positive economic impact[3], [4].

The effect possibilities of materials and systems are being pushed forward by technological development. New materials with enhanced performance qualities like strength, durability, and conductivity may be created because of developments in materials science. These developments pave the way for the development of novel systems that are more functional and have more capacities[5], [6].

When calculating impact probabilities, health, and safety must take precedence. Human health and safety must be the priority while designing and developing new materials and systems. Promoting sustainability and aligning with global objectives, such as cutting carbon emissions or bolstering circular economy practices, necessitates compliance with policies and regulations.

Considerable thought must be given to the market's prospective interest and uptake. Market demands, client specifications, and competitive advantages should all be taken into account when designing materials and systems. Determining the economic feasibility and long-term success of materials and systems requires an analysis of market size, growth potential, and obstacles to entry[7], [8].

Last but not least, maximizing the effect of materials and systems requires teamwork and the participation of key stakeholders. To maximize the impact potentials of materials and systems, it is essential to encourage multidisciplinary cooperation among academics, industrial partners, policymakers, and end-users[9], [10].

In this work, we explore the factors and criteria used to calculate the potential effect of various materials and systems. We hope that by investigating these issues, we may shed light on the advantages and disadvantages of different materials and methods. To make smart choices, spur innovation, and build a future that is both sustainable and technologically advanced, it is essential to have a firm grasp on these impact potentials. So, the concept of MCDM is used to deal with these criteria and factors. The VIKOR method used an MCDM method to compute the weights of these factors and rank the materials. The VIKOR method is integrated with the neutrosophic set to deal with uncertain information.

In such contexts, decision-maker preferences are seldom quantifiable, stable, and consistent. Memberships in traditional set theory are inadequate here[11]. Zadeh proposed using fuzzy sets (FSs) for dealing with uncertainty in cognitive processes, with memberships of the proposed technique ranging from 0 to 1 [12], [13].

Atanassov developed the intuitionistic fuzzy set (IFS) as a solution to the hesitancy issue of decision-makers since incorporating fuzziness into DM problems is not sufficient. Each element's degree of non-membership is represented by a number between 0 and 1. IFSs have seen widespread use in DM issues[14], [15].

To properly describe the indeterminacy, Smarandache has proposed the neutrosophic set (NS), an enlarged and generic variant of the classical fuzzy set and the intuitionistic fuzzy set. Neutrosophic set elements include T for truth, I for indeterminacy, and F for falsehood[16], [17].

2. Examples of Materials Process and its Impacts

A few sectors that have benefited greatly from new materials and methods are listed below.

Modern materials, such as high-efficiency solar cells, have caused a paradigm shift in the renewable energy industry. Materials like perovskite solar cells have the potential to boost solar energy production due to their high light absorption and conversion rates. Similarly, the widespread adoption of electric cars and grid-scale energy storage solutions has been made possible by breakthroughs in battery technology and the utilization of materials like lithium-ion[18], [19].

Materials and systems have had a significant influence on healthcare, especially in the area of medical devices. Orthopedic implants made from biocompatible materials, such as titanium alloys, have improved patient outcomes and prolonged implant longevity. Smart materials, such as shape-memory alloys, have also been developed, allowing for less intrusive surgical techniques and enhancing patient comfort during medical treatments.

Improvements in materials and processes have allowed the electronics industry to shrink gadgets and boost their performance. Silicon and other similar materials have been crucial to the development of integrated circuits and microprocessors in the semiconductor industry. New opportunities for wearable electronics, flexible displays, and electronic textiles have emerged because of the advancement of flexible and stretchy electronics made possible by materials such as graphene and conductive polymers[20], [21].

The transportation sector has been profoundly affected by the use of lightweight materials like carbon fiber composites and aluminum alloys. The high strength-to-weight ratios of these materials let vehicles go further on a single tank of gas and produce less greenhouse gas emissions. Battery technology improvements, such as those seen in lithium-ion and solid-state batteries, have also contributed to increased electric vehicle range and a rise in their popularity.

Improvements in structural integrity, energy efficiency, and sustainability have resulted from advances in materials and technologies used in the building industry. Fiber-reinforced composites and ultra-high-strength concrete are two examples of high-performance materials that may benefit infrastructure projects with longer lifespans and lower maintenance costs. Building automation and energy management systems are only two examples of smart technologies that have improved building efficiency and occupant comfort.

The environmental sector has made significant progress in tackling environmental concerns thanks in large part to the use of sustainable materials and technologies. Waste has been cut down and the environmental effect of the packaging and building sectors has been lessened thanks to the use of recycled and eco-friendly materials. Water and air purification have benefited greatly from the use of cutting-edge filtration technologies and materials, which in turn has helped to enhance environmental quality[22], [23].

These cases show how materials and systems have had far-reaching effects in a variety of sectors, fostering innovation, enhancing performance, and resolving social issues. Future sustainable development and profoundly reshaping our planet may be enabled by further developments in materials science and engineering[24], [25].

3. Challenges of Impact Potentials of Materials and Systems

Although materials and systems might have far-reaching effects, several obstacles stand in the way of reaping such rewards. Key difficulties include, among others:

The environmental effect of materials and systems is a major obstacle that must be overcome. Many economic sectors depend heavily on nonrenewable resources or produce enormous amounts of trash and pollutants during their operations. To overcome these difficulties, it is crucial to implement circular economy concepts and seek sustainable alternatives, as well as to reduce energy consumption, waste production, and waste disposal costs[26], [27].

The price tag associated with creating and deploying cutting-edge infrastructure might be prohibitive for certain people. Large sums of money are often needed for R&D, production, and increasing output. Especially in areas where cost is a major factor in adoption, like healthcare and renewable energy, making these technologies inexpensive and accessible is vital.

Technological Readiness: It might be difficult to transition materials and systems from the research and development phase to practical applications. Scalability, dependability, and backward compatibility with current infrastructure are all technical hurdles that must be cleared. For materials and systems to be widely used, their long-term performance and durability must be guaranteed[28], [29].

There is a correlation between regulatory and policy frameworks and the pace of innovation in materials and systems. Promoting safety, sustainability, and market acceptance requires the establishment of suitable rules, standards, and incentives. However, complying with many local, state, federal, and international regulations may be difficult.

The degree to which the public embraces and uses novel technologies depends greatly on how they are received in the marketplace. Safety issues, ethical considerations, and the potential for industry upheaval are all valid concerns. To alleviate these worries and restore confidence among stakeholders, open dialogue, public participation, and clear information sharing are essential.

The full effect potential of materials and systems is seldom realized without the combined efforts of academics, industry professionals, policymakers, and end-users from a wide range of backgrounds and disciplines. Bridging knowledge gaps, aligning interests, and coordinating efforts across multiple industries and disciplines are all obstacles that may make interdisciplinary cooperation difficult. To face these difficulties, it is essential to construct efficient networks and platforms for cooperation[29], [30].

Taking into account a product's whole lifespan, from mining for basic materials to final disposal, is essential when assessing its potential effect. It is a difficult effort to evaluate resource depletion, waste management, and social equality at every step of the process. It is crucial to develop whole lifecycle assessment methods and embed sustainability concepts into all stages of the lifecycle.

Research and development, legislative interventions, industry cooperation, and public awareness are all necessary to effectively tackle these issues. If we can get beyond these roadblocks, we can use materials and systems to their fullest extent, promoting long-term growth and social progress[28], [31].

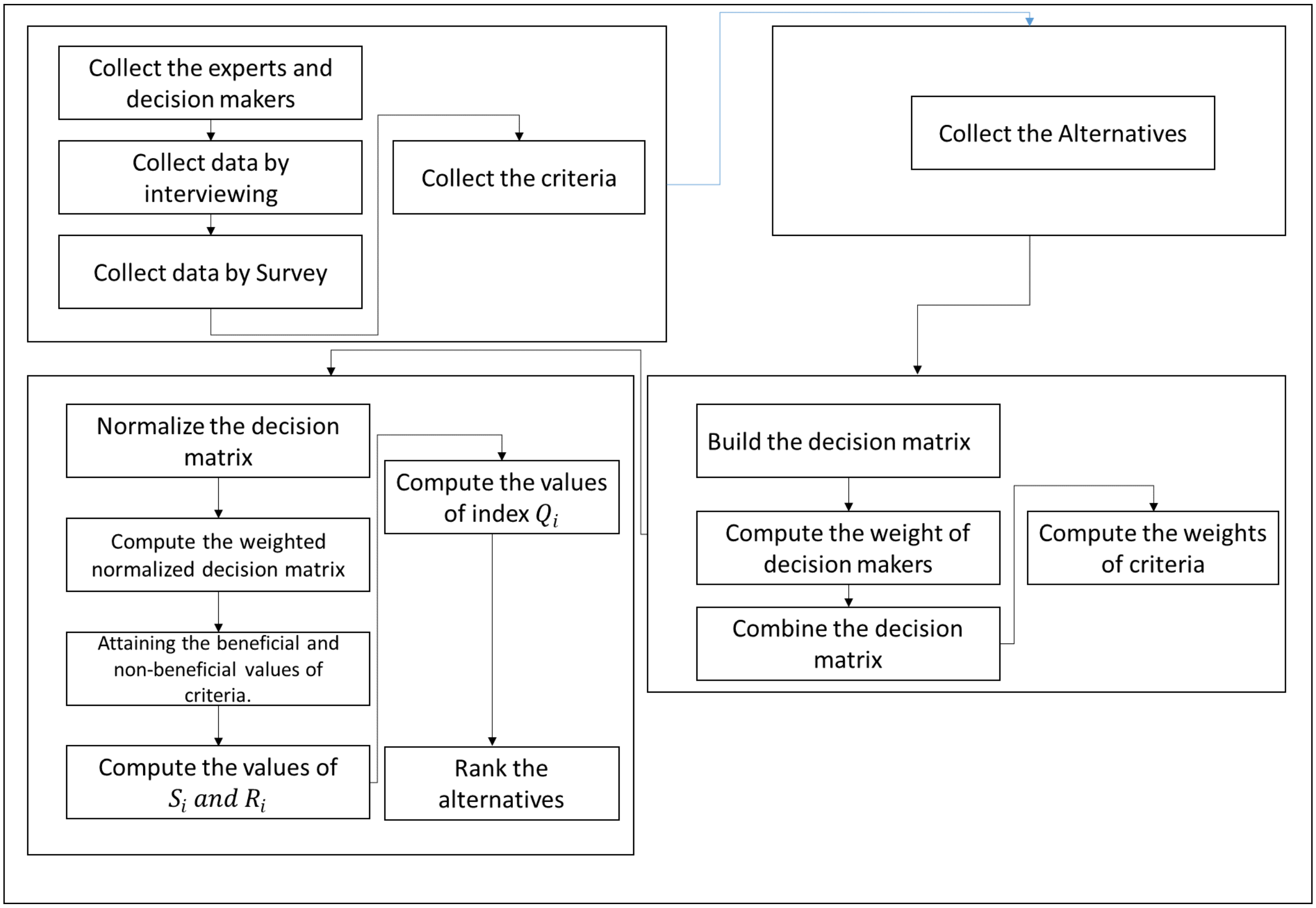

Figure 1. The framework of single valued neutrosophic set.

4. Neutrosophic VIKOR Method

In 2004, Opricovic and Tzeng were the first to publish the VIKOR technique. The fundamental idea behind the strategy is to choose the best option by ranking them according to competing criteria[32], [33] as shown in Figure 1.

Step 1. Build the decision matrix

The whole matrix of VIKOR method can be computed as:

$$T_{pj} = \left(\sum_{i=1}^{n}\left(w_i \frac{f_i^* - f_{ij}}{f_i^* - f_i^-}\right)\right)^{\frac{1}{p}} \quad (1)$$

Step 2. Compute the weight of decision makers

$A^e = (a_{ij}^e)$

Where e refers to the decision maker and experts.

Step 3. Combine the decision matrix

$$a_{ij} = < 1 - \prod_{e=1}^{k}\left(1 - T_{ij}^{(e)}\right), \prod_{e=1}^{k} I_{ij}^{(e)}, \prod_{e=1}^{k} F_{ij}^{(e)} > \quad (2)$$

Step 4. Compute the weights of criteria

Step 5. Normalize the decision matrix

Step 6. Compute the weighted normalized decision matrix

$$d = A_{ij} * w_j \quad (3)$$

Step 7. Attaining the beneficial and non-beneficial values of criteria.

$$T_j^{w+} = \left\{\left(\max T_{ij}^{w}\right), \left(\min T_{ij}^{w}\right)\right\} \quad (4)$$

$$I_j^{w+} = \left\{\left(\max I_{ij}^{w}\right), \left(\min I_{ij}^{w}\right)\right\} \quad (5)$$

$$F_j^{w+} = \left\{\left(\max F_{ij}^{w}\right), \left(\min F_{ij}^{w}\right)\right\} \quad (6)$$

$$T_j^{w-} = \left\{\left(\max T_{ij}^{w}\right), \left(\min T_{ij}^{w}\right)\right\} \quad (7)$$

$$I_j^{w-} = \left\{\left(\max I_{ij}^{w}\right), \left(\min I_{ij}^{w}\right)\right\} \quad (8)$$

$$F_j^{w-} = \left\{\left(\max F_{ij}^{w}\right), \left(\min F_{ij}^{w}\right)\right\} \quad (9)$$

Step 8. Compute the values of $S_i \; and \; R_i$

$$S_i = \sum_{j=1}^{n} w_j \frac{|| w_j^{+} - w_{ij} ||}{|| w_j^{+} - w_j^{-} ||} \quad (10)$$

$$R_i = \max w_j \frac{|| w_j^{+} - w_{ij} ||}{|| w_j^{+} - w_j^{-} ||} \quad (11)$$

Step 9. Compute the values of index Q_i

$$Q_i = \partial \left[\frac{S_i - S^{+}}{S^{-} - S^{+}}\right] + (1 - \partial) \left[\frac{R_i - R^{+}}{R^{-} - R^{+}}\right] \quad (12)$$

Step 10. Rank the alternatives

The alternatives are ranked according to the lowest value of Q_i

5. Results

This section introduces the results of the proposed method. This paper use eight criteria and ten materials. The following are eight criteria.

Material and system impact assessment entails considering both the positive and negative outcomes that might result from using the material or system. When assessing the possible effects, it is important to keep in mind the following:

Evaluate the environmental effects of materials and systems across their whole lifespan, from the gathering of raw materials to the final disposal of waste. Think about your impact on the environment in terms of things like your carbon footprint, energy use, trash output, and pollution risks. The potential effect of materials and systems is greater if they have a low environmental impact and help promote sustainability, such renewable materials or recyclable systems.

Consider the effects that materials and systems may have on people, groups, and the larger society. Think on how things can be better in terms of price, accessibility, and quality of life. Think about how you can help local economies and create employment. The potential influence of a material or system increases when it is used to solve social problems, broaden participation, or better people's lives.

Assess the financial effects of proposed materials and systems by analysing their costs, benefits, scalability, marketability, and growth prospects. Think about how you can save money, open up new markets, and stimulate creativity. The potential influence of a material or system increases as it provides economic advantages, stimulates industries, and aids in long-term economic growth.

Technological Breakthroughs and Improvements Evaluate the Possibilities Opened Up by Materials and Systems. Think of ways to boost performance, add new features, and create new capabilities. Determine whether there is a chance that a new technology will significantly improve upon current methods. High-impact materials and systems are those that propel technological progress and usher in new possibilities.

Assess how materials and setups could endanger people's health and safety. Think about the manufacture, using, and disposal stages from the perspective of toxicity, exposure hazards, and possible damage. Examine the opportunities for better health, medical progress, and increased safety in a variety of contexts. The health and safety impacts of materials and systems that pose low threats to human health and actually promote health and safety are greater.

Think about how well your products and systems conform to any applicable laws, regulations, and industry standards. Think about how they may help achieve policy goals like lowering waste or increasing energy efficiency, or achieving environmental goals. The potential effect of a material or system increases when it is both regulatory compliant and helps to achieve policy goals.

Examine the potential for materials and systems to be adopted in the market, as well as the demand for them. Think about the current size of the market, its potential for expansion, the level of competition, and any obstacles to entrance. Determine whether there is a chance to meet market demands, please customers, and give your company a leg up on the competition. The potential influence of a material or system increases as its market demand and rate of adoption increase.

Collaboration and Stakeholder Engagement: Analyze the Prospects for Working Together and Involving Stakeholders in the Design and Implementation of Resources and Tools. Think about how to include a wide range of people, such as companies, universities, government agencies, and end consumers. Think about how you may increase your effect by working together with others and sharing what you know.

Impact assessment is difficult and context-dependent, so keep that in mind. There may be varying priorities or requirements depending on the application or industry. To evaluate materials and systems in their whole, it is necessary to take into account a wide variety of criteria, adapt them to the individual situation, and include several points of view.

We applied the single valued neutrosophic VIKOR method on the eight criteria and ten alternatives.

Step 1. Build the decision matrix

The decision matrix is built based on the opinions of experts and single valued neutrosophic numbers.

Step 2. Compute the weight of decision makers

We compute the weights of criteria. The weights of criteria are equal.

Step 3. Combine the decision matrix

There are three experts and decision makers evaluate the criteria and alternatives, so we have three decision matrices. So, we combined these matrices to obtain the one matrix as shown in Table 1.

Step 4. Compute the weights of criteria

Then compute the weights of criteria as shown in Figure 2.

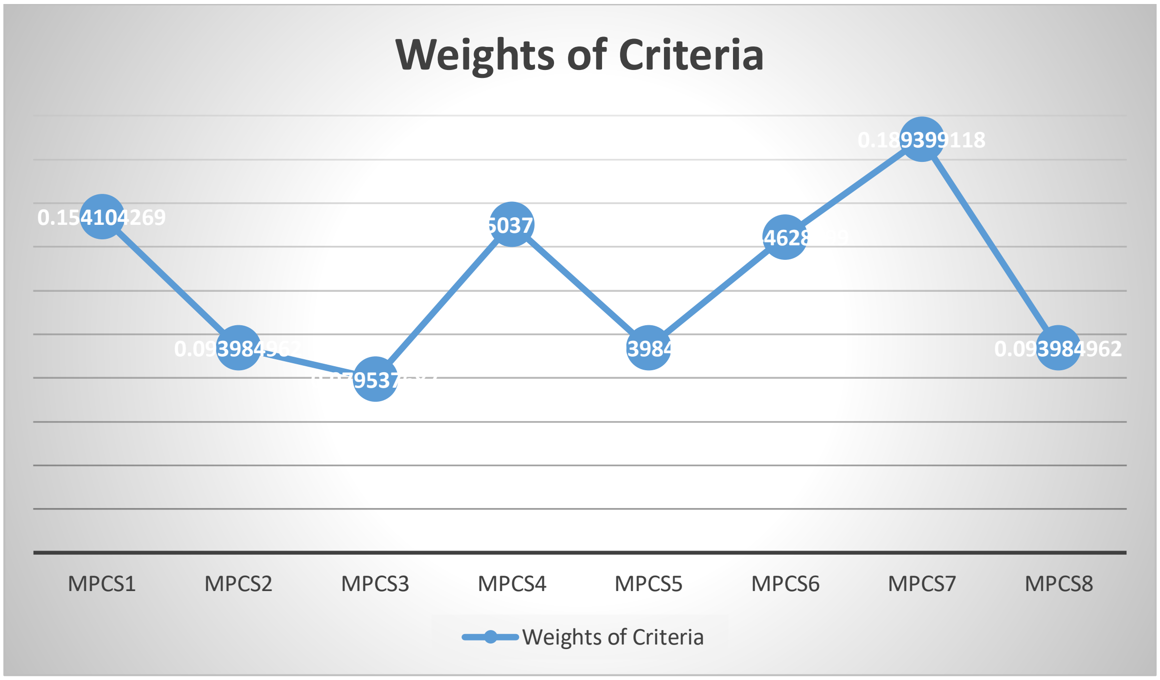

Figure 2. The weights of the eight criteria.

Step 5. Normalize the decision matrix

Then normalize the decision matrix.

Table 1. The combined decision matrix by the VIJOR method

	$MPCS_1$	$MPCS_2$	$MPCS_3$	$MPCS_4$	$MPCS_5$	$MPCS_6$	$MPCS_7$	$MPCS_8$
$MPMS_1$	0.369	0.236	0.256	0.452	0.369	0.562	0.9632	0.369
$MPMS_2$	0.623	0.369	0.256	0.516	0.236	0.369	0.256	0.236
$MPMS_3$	0.745	0.369	0.962	0.632	0.256	0.513	0.369	0.2563
$MPMS_4$	0.8526	0.7456	0.526	0.962	0.256	0.369	0.256	0.245
$MPMS_5$	0.369	0.856	0.826	0.625	0.263	0.2563	0.369	0.562
$MPMS_6$	0.512	0.963	0.963	0.256	0.26	0.856	0.526	0.2563
$MPMS_7$	0.369	0.236	0.526	0.256	0.562	0.962	0.263	0.526
$MPMS_8$	0.852	0.563	0.2569	0.695	0.256	0.856	0.236	0.632
$MPMS_9$	0.369	0.369	0.756	0.856	0.96	0.856	0.596	0.752
$MPMS_{10}$	0.256	0.256	0.963	0.256	0.415	0.526	0.236	0.523

Step 6. Compute the weighted normalized decision matrix

$$d = A_{ij} * w_j \tag{3}$$

We compete the weighted normalized decision matrix as shown in Table 2 by sing Eq. (3)

Table 2. The weighted normalization decision matrix by the VIJOR method

	$MPCS_1$	$MPCS_2$	$MPCS_3$	$MPCS_4$	$MPCS_5$	$MPCS_6$	$MPCS_7$	$MPCS_8$
$MPMS_1$	0.124916	0.093985	0.079538	0.108629	0.07672	0.062651	0	0.06976
$MPMS_2$	0.059307	0.076791	0.079538	0.094997	0.093985	0.023097	0.18419	0.093985
$MPMS_3$	0.027794	0.076791	0.000113	0.070289	0.091389	0.052609	0.154759	0.090287
$MPMS_4$	0	0.028105	0.049163	0	0.091389	0.023097	0.18419	0.092346
$MPMS_5$	0.124916	0.013833	0.015413	0.07178	0.09048	0	0.154759	0.034607
$MPMS_6$	0.087978	0	0	0.150376	0.090869	0.122904	0.113869	0.090287
$MPMS_7$	0.124916	0.093985	0.049163	0.150376	0.051666	0.144628	0.182367	0.041164
$MPMS_8$	0.000155	0.051711	0.079436	0.05687	0.091389	0.122904	0.189399	0.021857
$MPMS_9$	0.124916	0.076791	0.023288	0.022578	0	0.122904	0.095637	0
$MPMS_{10}$	0.154104	0.091399	0	0.150376	0.070748	0.055273	0.189399	0.04171

Step 7. Attaining the beneficial and non-beneficial values of criteria.

Obtain the beneficial and non- beneficial criteria by using Eqs. (4-9). The cost criterion is non-beneficial and other criteria are beneficial.

Step 8. Compute the values of $S_i \; and \; R_i$

Compute the values of $S_i \; and \; R_i$ by using Eqs. (10 and 11)

Step 9. Compute the values of index Q_i

Then compute the value of Q_i by using Eq. (12)

Step 10. Rank the alternatives

Then alternatives are ranked according to the lowest value of Q_i as shown in Figure 3. The best material is 9, and the worst material is 7.

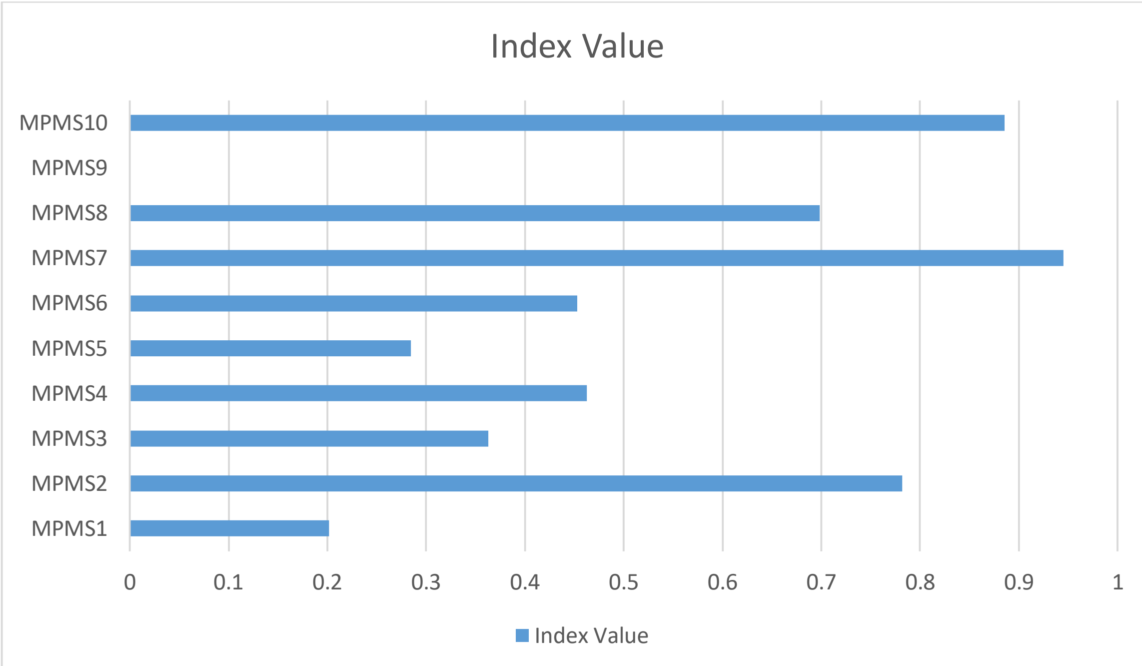

Figure 3. The value of Q_i.

6. Conclusions

Materials and systems may have far-reaching effects in a wide variety of ways. The environmental, social, economic, technical, health and safety, policy, market, and collaborative factors must all be taken into account in order to provide an accurate assessment of these possibilities. By evaluating these factors, stakeholders may learn more about the pros and cons of various materials and systems, increasing the likelihood of effective choices being made. Fostering innovation, tackling difficult issues, and promoting sustainable development are all possible via multidisciplinary cooperation and stakeholder involvement, which are essential for realizing their full potential. This paper used the concept of MCDM to deal with various factors and criteria. The VIKOR is an MCDM method used to rank the various materials based on various criteria. The VIKOR method is integrated with the single-valued neutrosophic set to deal with uncertain information. We applied the proposed method to eight criteria and ten materials. We obtained the cost criteria as the highest important of all criteria.

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Calculation of Fuzzy shortest path problem using Multi-valued Neutrosophic number under fuzzy environment

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Abstract: The most well-known subject in graph theory is the shortest path problem (SPP), which has real-world applications in several different fields of study, including transportation, emergency services, network communications, fire station services, etc. The arc weights of the applicable SP problems are typically represented by fuzzy numbers in real-world applications. In this paper, we discussed the process of finding the shortest distance in a connected graph network in which the arc weights are multi-valued neutrosophic numbers (MNNs). Moreover, here we compare our method with some of the existing results and illustrate one implementation of our method with the help of one numerical example.

Keywords: Directed graph network; Multi-valued neutrosophic numbers; selection sort technique; shortest path problem.

1. Introduction

Smarandache designed the perception of a neutrosophic set first, and most importantly, it is able to manage situations that are ambiguous, insufficient, inconsistent, and unspecified by applying additional correct ways. The indication of neutrosophic sets (NS) [1] is like that of normal fuzzy sets [2], intuitionistic fuzzy associate number sets [3], and interval-valued intuitionistic fuzzy sets [4], giving basic ideas for relations and operations over sets. On a personal level, the variety-class degree of the neutrosophic sets theory is represented by an indeterminacy-category degree and a falsity-category degree. To utilize it in genuine scientific and technical areas, the idea of a neutrosophic set is proposed by Qiuping, N. [5] based on neutrosophic logic to make it more applicable to real-world circumstances. In reality, the degree of verity class, indeterminacy category, and falsity category of a handful of sure statements can't be written precisely inside the vitality items but expressed by various alternative interval values, which necessitated the use of the multi-valued neutrosophic set (MVNS). For this reason, Peng [6] suggested the idea of a multi-valued neutrosophic set (MVNS), which is superfluous exact, and more adaptable than an inter-valued neutrosophic (IVN) set. Multi-valued neutrosophic sets (MVNs) are like single-valued neutrosophic sets with three class functions, and the unit interval contains their values (0, 1).

The SPP is a fundamental and remarkable connectional optimization issue that arises in a variety of engineering and scientific fields, including road networks, transport, and other technologies. The SPP issue in a given network seeks the optimal path between two given nodes whose arc length weight is less as possible. The weight assigned to edges of a given network can reflect necessary life elements such as time, value, and others. The call maker is supposed to be confident with the parameters (length, duration, etc.) among distinct nodes in the traditional shortest route issue. However, there is always ambiguity regarding the parameters across distinct nodes in real-life conditions. Many approaches have been established for determining the shortest path (SP) in different kinds of input files with respect to fuzzy sets (FS), intuitionistic fuzzy sets (IFs), and ambiguous sets,neutrosophic and fermatean neutrosophic sets[7–16].

To find SPP in a fuzzy environment, triangular, trapezoidal, and pentagonal numbers [17, 18] are already used as the arc length in many real-world problems, and in some cases, neutrosophic numbers [19] are used to describe the uncertain behavior in the neutrosophic environment and then interval-valued neutrosophic numbers [20] are used to evaluate the path. But in this case, we used multi-valued neutrosophic numbers as the arc length to solve the SPP.

The primary purpose of this research is to identify the SPP for a given network with arc length weights determined by MVNNs. The constitution of the remaining article is as follows: In Section -2, we discuss some fundamental concepts related to neutrosophic sets, specifically single-valued, neutrosophic (SVN) sets. In Section -3 we present an approach for determining SP with connected edges with respect to neutrosophic data. Section -4 shows a realistic case solved by the suggested approach. Section -5 contains the comparison study with some of the existing methods, and Section -6 includes the conclusions and recommendations for additional research.

2. **Motivation and Contribution:**

The most important motivation of this paper is to initiate a method for SPP in an uncertain atmosphere that has a broad area of application in the real world.

The following are the contributions of this paper's

- We use MVNNs as the arc length instead of the real number.
- A new methodology is used to evaluate the (SP) problem in an uncertain environment.
- Various algorithms already exist (Table 1) to solve FSPP in an uncertain environment, but here we use a new methodology to evaluate the FSPP.

Compare our methodology with the existing methodology.

Author	Evaluation of SPP using different method	Year
Broumi, S. et al. [22]	NSPP for interval-based data was evaluated using the Dijkstra method.	2016
Broumi, S. et al. [23]	SP was found using SV-TpNNs.	2016
Broumi, S. et al. [24]	SPP was evaluated using single-valued neutrosophic graphs.	2016
Broumi, S. et al. [25]	SPP was assessed using a neutrosophic setup and the trapezoidal number.	2016
Broumi, S. et al. [26]	SPP was evaluated in a bipolar neutrosophic environment.	2017
Broumi, S. et al. [27]	SPP was evaluated using an interval-valued neutrosophic number.	2017
Broumi, S. et al. [28]	The neutrosophic version of Bellman's algorithm is introduced.	2017
Our method	Evaluating SPP under Multi-value neutrosophic number	2019

3. Preliminaries

This section contains the literature studied for the basic notions and definitions of neutrosophic sets (NSs) and MVNSs

Definition 3.1:

Assume $\tilde{X}$ is a set of space points (objects), and $\tilde{x}$ represents the associated generic elements in $\tilde{X}$; then the element in neutrosophic set $\tilde{A}$ has the form

$$\tilde{A} = \{< \tilde{x}: \tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(x), \tilde{F}_{\tilde{A}}(x) > \tilde{x} \epsilon \tilde{X}\}$$

Here the function takes the form $T, I, F : \tilde{X} \to [0^-, 1^+]$ where $\tilde{T}$ is called the truth-membership function, $\tilde{I}$ is called indeterminacy-membership function, and $\tilde{F}$ is called falsity membership function of the element $\tilde{x} \in \tilde{X}$.

$$0^- \leq \{\tilde{T}_{\tilde{A}}(x) + \tilde{I}_{\tilde{A}}(x) + \tilde{F}_{\tilde{A}}(x)\} \leq 3^+$$

Now $\tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(x), \tilde{F}_{\tilde{A}}(x)$ are representing subsets of the interval $[0^-, 1^+]$ hence it's challenging to implement NSs to real-world situations.

Definition 3.2:

If $\tilde{X}$ is a point in space and $\tilde{x}$ represents generic elements defined in $\tilde{X}$. Then truth, indeterminacy and the falsity-membership function differentiate $\tilde{A}$ in $\tilde{X}$. The multi-valued neutrosophic (MVN) set is defined as.

$$\tilde{A} = \{\tilde{x} \ , \tilde{T}_{\tilde{A}}(\tilde{x}), \tilde{I}_{\tilde{A}}(\tilde{x}), \tilde{F}_{\tilde{A}}(\tilde{x}), \tilde{x} \in \widetilde{X} \}$$

Here both $\tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(\tilde{x})$ and $\tilde{F}_{\tilde{A}}(\tilde{x}) \in [0,1]$ are the collection of discrete values that satisfy the criterion $0 \leq \alpha, \beta, \gamma \leq 1, 0 \leq \alpha^+, \beta^+, \gamma^+ \leq 3, \alpha \in \tilde{T}_{\tilde{A}}(x), \beta \in \tilde{I}_{\tilde{A}}(x), \gamma \in \tilde{F}_{\tilde{A}}(x)$.

$$\alpha^+ = Sup\tilde{T}_{\tilde{A}}(x), \beta^+ = Sup \ \tilde{I}_{\tilde{A}}(x), \gamma^+ = Sup\tilde{F}_{\tilde{A}}(x) --------(2)$$

For the simplicity $\tilde{A} = \{\tilde{T}_{\tilde{A}}, \tilde{I}_{\tilde{A}}, \tilde{F}_{\tilde{A}}\}$ is referred to as a multi-valued neutrosophic (MVN) number.

The multi-valued neutrosophic (MVN) sets are termed as the single valued neutrosophic sets if $\tilde{A} = \{\tilde{T}_{\tilde{A}}, \tilde{I}_{\tilde{A}}, \tilde{F}_{\tilde{A}}\}$ has just one value.

Definition 3.3:

Assume that $\tilde{A}_1 = \{\tilde{T}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_1}, \tilde{F}_{\tilde{A}_1}\}$ and $\tilde{A}_2 = \{\tilde{T}_{\tilde{A}_2}, \tilde{I}_{\tilde{A}_2}, \tilde{F}_{\tilde{A}_2}\}$ are two sets of the neutrosophic numbers with multiple values. Then the functions for SVNNs are specified as follows:

(a) $\tilde{A}_1 + \tilde{A}_2 = \{\tilde{T}_{\tilde{A}_1} + \tilde{T}_{\tilde{A}_2} - \tilde{T}_{\tilde{A}_1}\tilde{T}_{\tilde{A}_2}, \tilde{I}_{\tilde{A}_1}\tilde{I}_{\tilde{A}_2}, \tilde{F}_{\tilde{A}_1}\tilde{F}_{\tilde{A}_2}\}$

(b) $\tilde{A}_1 \times \tilde{A}_2 = \{\tilde{T}_{\tilde{A}_1}\tilde{T}_{\tilde{A}_2}, \tilde{I}_{\tilde{A}_1} + \tilde{I}_{\tilde{A}_2} - \tilde{I}_{\tilde{A}_1}\tilde{I}_{\tilde{A}_2}, \tilde{F}_{\tilde{A}_1} + \tilde{F}_{\tilde{A}_2} - \tilde{F}_{\tilde{A}_1}\tilde{F}_{\tilde{A}_2}\}$

(c) $\lambda\tilde{A}_1 = \{1 - (1 - \tilde{T}_{\tilde{A}_1})^\lambda, {\tilde{I}_{\tilde{A}_1}}^\lambda, {\tilde{F}_{\tilde{A}_1}}^\lambda\}$

(d) ${\tilde{A}_1}^\lambda = \{{\tilde{T}_{\tilde{A}_1}}^\lambda, 1 - (1 - \tilde{I}_{\tilde{A}_1})^\lambda, 1 - (1 - \widetilde{F}_{\tilde{A}_1})^\lambda\}$

With $\lambda > 0$

Definition 3.4:

If $\tilde{A}_1 = \{\tilde{T}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_1}, \tilde{F}_{\tilde{A}_1}\}$ be a neutrosophic number of single value. Then,

The Score function are defined as the value $S\tilde{A}_1 = \frac{2 + \tilde{T}_{\tilde{A}_1} - \tilde{I}_{\tilde{A}_1} - \tilde{F}_{\tilde{A}_1}}{3}$

The Accuracy function takes the value as $a(\tilde{A}_1) = \{\tilde{T}_{\tilde{A}_1} - \tilde{F}_{\tilde{A}_1}\}$

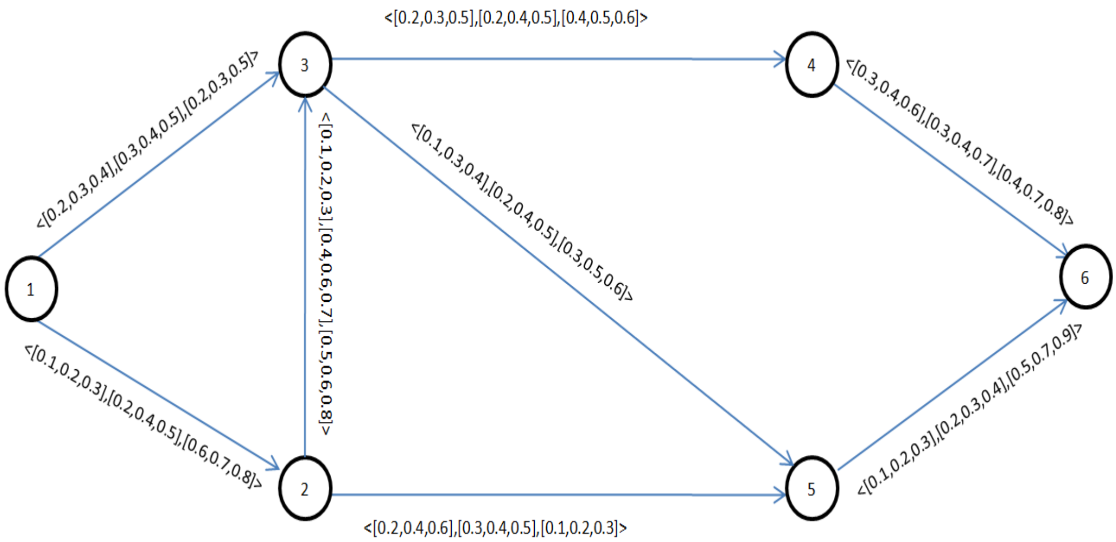
and certainty function is defined as $c(\tilde{A}_1) = \tilde{T}_{\tilde{A}_1}$

4. Algorithm to find the shortest path with respect to a multi-valued neutrosophic number

- Step 1: Select any vertex as the source and destination point of the given network.
- Step 2: Find every path that connects to the source node to the destination node.
- Step 3: Determine all possible edge values from discrete multi-valued neutrosophic numbers to simplify the MMNN to SVN by using the fuzzy simplicity method (equation 2.0) and using the Score Function to convert SVN to a crisp number (definition 2.4).
- Step 4: After obtaining all edge values (the Crisp number), calculate the path's average.
- After getting all path values, arrange them using the selection sort technique, and finally, get the shortest path.

5. Numerical Example:

Evaluation of the shortest path (SP) using multi-valued neutrosophic numbers



Step-1:

Let us look at a multi-valued neutrosophic network with source nodes 1 and destination nodes 6, with edge weights represented by MVNNs.

Step-2:

This table shows multi-valued neutrosophic distances.

Edges	**MVN distance**
1-2	<[0.1,0.2,0.3],[0.2,0.4,0.5],[0.6,0.7,0.8]>
1-3	<[0.2,0.3,0.4],[0.3,0.4,0.5],[0.2,0.3,0.5]>
2-3	<[0.1,0.2,0.3],[0.4,0.6,0.7],[0.5,0.6,0.8]>
2-5	<[0.2,0.4,0.6],[0.3,0.4,0.5],[0.1,0.2,0.3]>
3-4	<[0.2,0.3,0.5],[0.2,0.4,0.5],[0.4,0.5,0.6]>
3-5	<[0.1,0.3,0.4],[0.2,0.4,0.5],[0.3,0.5,0.6]>
4-6	<[0.3,0.4,0.6],[0.3,0.4,0.7],[0.4,0.7,0.8]>
5-6	<[0.1,0.2,0.3],[0.2,0.3,0.4],[0.5,0.7,0.9]>

Table-1

Step-3:

Table-1: Edge information in terms of multivalued neutrosophic number. Here $\tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(\tilde{x})$ and $\tilde{F}_{\tilde{A}}(\tilde{x}) \in [0,1]$, are the finite discrete value that satisfy the conditions $0 \le \alpha, \beta, \gamma \le 1, 0 \le \alpha^{+}, \beta^{+}, \gamma^{+} \le 3$

So now we apply equation (2) we get the Sake of simplicity of multi-value neutrosophic number

i.e. $\tilde{A} = \{\tilde{T}_{\tilde{A}}, \tilde{I}_{\tilde{A}}, \tilde{F}_{\tilde{A}}\}$

Then the edges From 1-2 is becomes [0.3,0.5,0.8] similarly all the edges value changes. here Table -2 represents the Simplicity of Multi-value neutrosophic number.

Edges	MVN distance
1-2	[0.3,0.5,0.8]
1-3	[0.4,0.5,0.5]
2-3	[0.3,0.7,0.8]
2-5	[0.6,0.5,0.3]
3-4	[0.5,0.5,0.6]
3-5	[0.4,0.5,0.6]
4-6	[0.6,0.7,0.8]
5-6	[0.3,0.4,0.9]

Table-2

Now by using Definition-2.4 we find the score function

$$S(A_1)=\frac{2+\overline{T}_1-\overline{I}_1-\overline{F}_1}{3}$$

- $$S(A_1)=\frac{2+0.3-0.5-0.8}{3}$$
- *S(A₁)=0.33*

Similarly to find all the values of edge (i,j) in table-3

edges	Score function
1-2	0.33
1-3	0.46
2-3	0.26
2-5	0.60
3-4	0.46
3-5	0.43
4-6	0.36
5-6	0.33

Table-3

Step-4:

Path from source to destination is

Path $1-3-4-6=\frac{0.33+0.46+0.36}{3}$

$$=0.42$$

Similarly all the values of path distance in Table-2

Path	**Distance**
1-3-4-6	0.42
1-3-5-6	0.40
1-2-3-5-6	0.43
1-2-5-6	0.31

Table- 4

Step-5:

Input:

```python
# Python compiler
# Find the shortest path
def selectionSort( itemsList ):
    n = len( itemsList )
    for i in range( n - 1 ):
        minValueIndex = i

        for j in range( i + 1, n ):
            if itemsList[j] < itemsList[minValueIndex] :
                minValueIndex = j

        if minValueIndex != i :
            temp = itemsList[i]
            itemsList[i] = itemsList[minValueIndex]
            itemsList[minValueIndex] = temp

    return itemsList

# Arrange the path according to the path values
p1 = [0.42,0.30,0.43,0.31]

print(selectionSort(p1))
```

Output:

```
[0.31, 0.4, 0.42, 0.43]
```

Step-6:

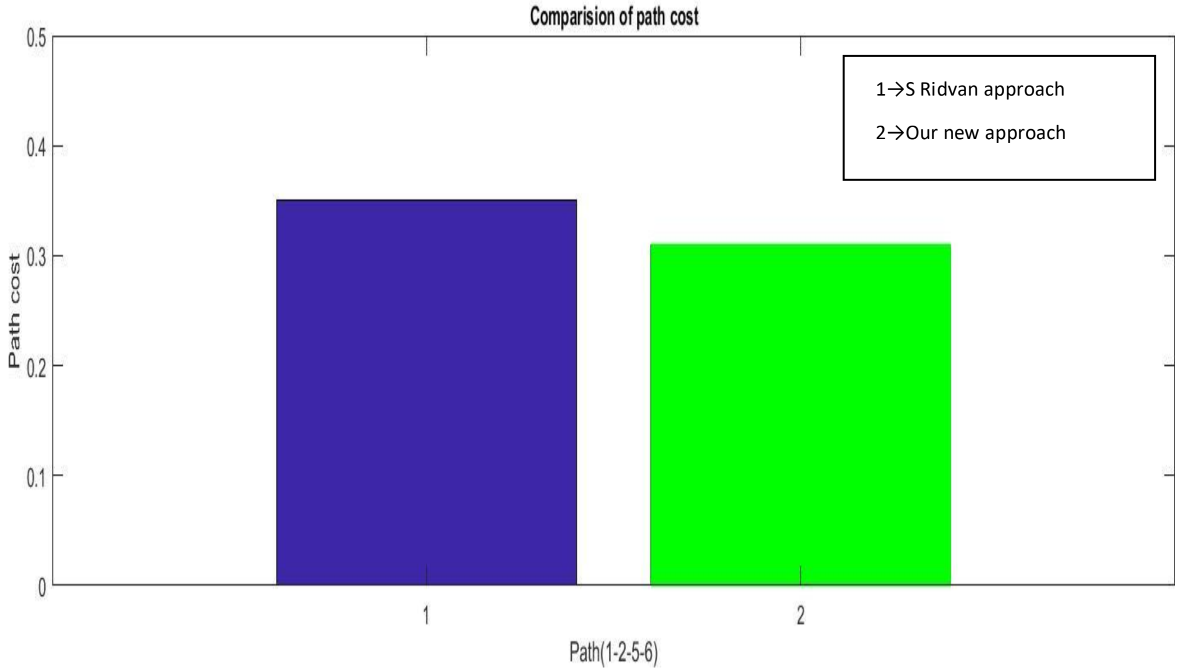
Minimum ranking value is 0.31. Hence the shortest path is 1-2-5-6

6. Comparison with existing Algorithm

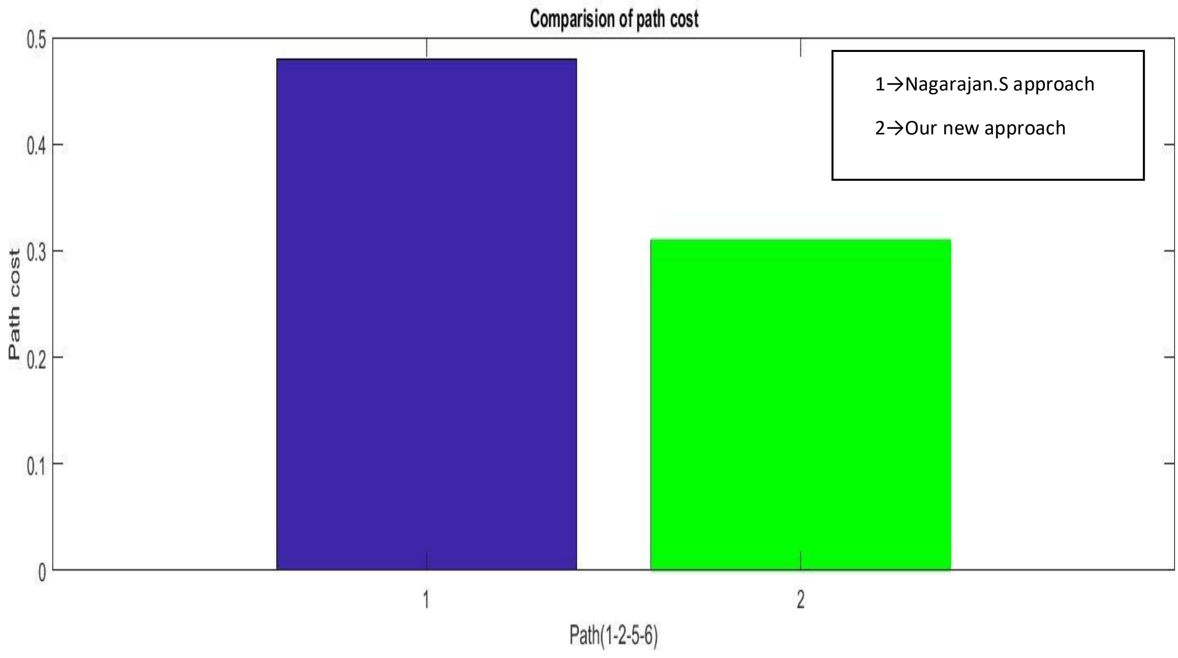
Here in this section we compare our proposed method with some of the existing method of for neutrosophic shortest path problems

Authors	Path sequence	Path length(Crisp)
Ridvan.S [29]	1-2-5-6	0.35
Nagarajan.S[30]	1-2-5-6	0.48
Broumi.S[31]	1-2-5-6	[0.35,0.60][0.01,0.04][0.008,0.075]
Our new approach	1-2-5-6	0.31

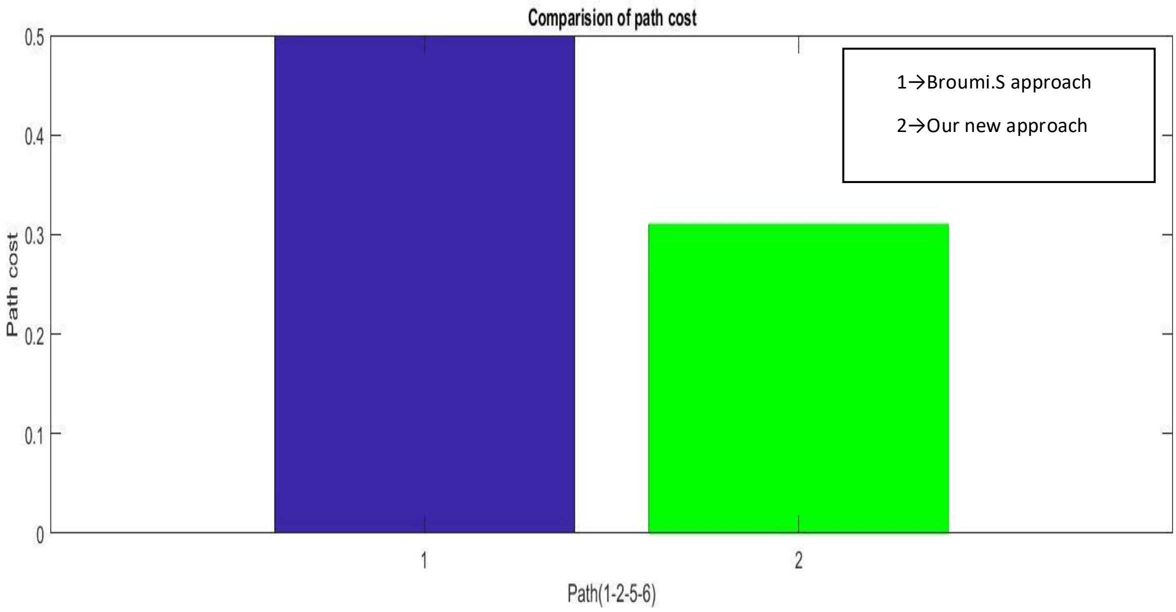
The results shows that our proposed algorithm is giving the crisp path length



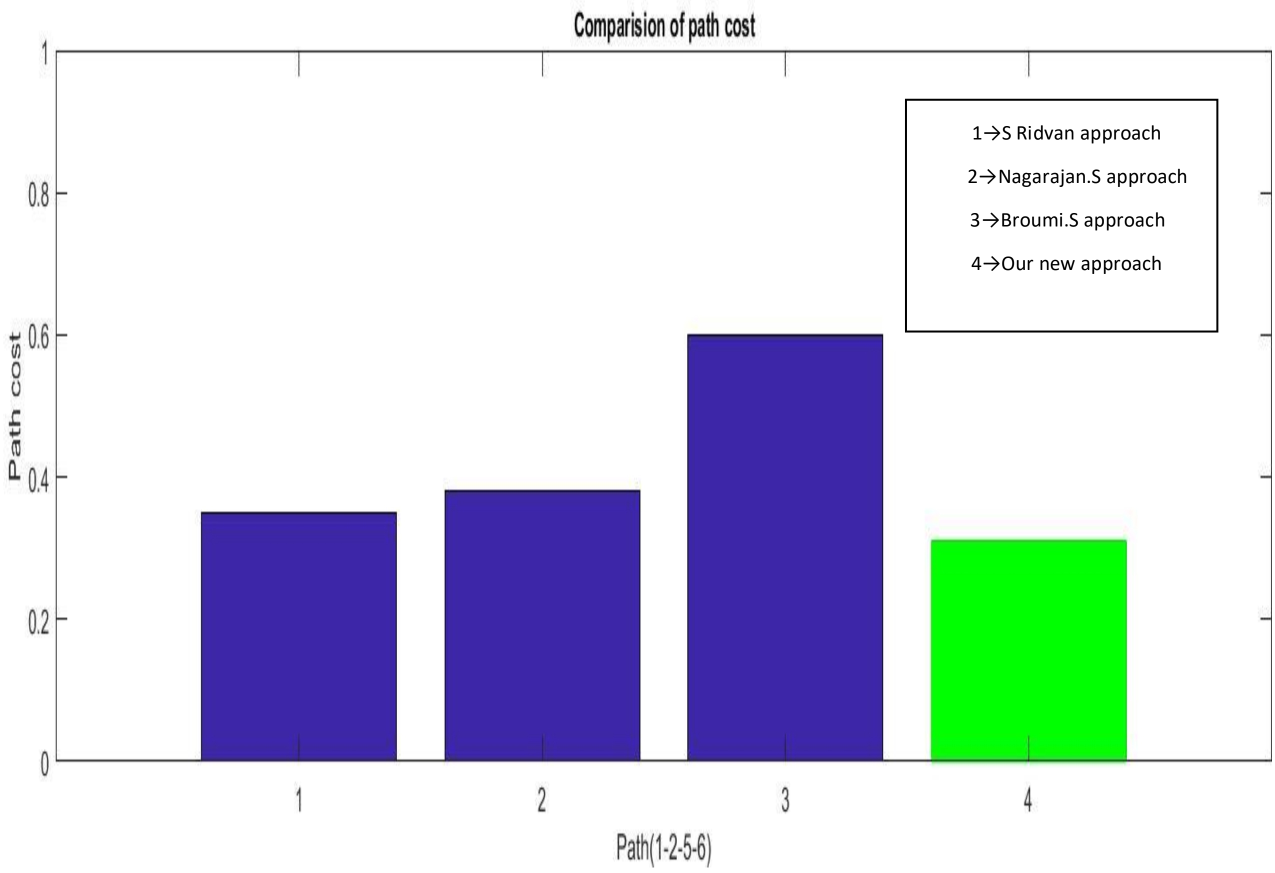
Comparison of our method with S Ridvan [21]



Comparison of our method with S.Nagarajan [22]



Comparison of our method with S broumi [23]



Final graph

Where the shortest neutrosophic path remains the same namely 1-2-5-6.

7. Conclusion:

This paper describes the NSP using edge weights represented by MVNS and the benefits of using MVNS with the NSP. The traditional new method is used in MVNS to integrate uncertainty between the destination and source nodes. To express the effectiveness of the suggested approach, we use numerical examples. The primary purpose of this study is to explain the NSP algorithm in a neutrosophic environment using MVNS as edge weights. For real-world issues, the suggested technique is quite successful. In future studies, it will be important to investigate a large-scale and realistic shortest-path issue using the suggested method.

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Value and Ambiguity Index-based Ranking Approach for Solving Neutrosophic Data Envelopment Analysis

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Abstract. The Neutrosophic set (NS), is a generalization of the fuzzy set and its extension sets, is a revolutionary type of fuzzy set that enables decision-makers (DMs) to express their level of uncertainty independently by assigned the truth, indeterminacy and falsity degrees of each element of NS. This article purposes a novel type of ranking function based on value and ambiguity index of a single value triangular neutrosophic number (SVTNN), which associated with DM's preference level and risk factor that show the attitude of the DM towards taking risk. Also, this article purposes a novel technique for solving the Neutrosophic DEA (Neu-DEA) model having multiple input-outputs are the SVTNNs. The proposed ranking function is used to converts the Neu-DEA model into a corresponding crisp DEA model which is solved to measure the efficiency of the decision making units (DMUs) in Neutrosophic environment. The efficiency scores of the DMUs are calculated based on the DM's preference level by taking a specific risk ($\lambda \in [0, 1]$). A numerical example is provided to demonstrate the proposed model's validity and existence, and to compare efficiency scores with Yang et al.'s ranking approach. Finally, How the DM's preference level and risk factor affect the efficiency score of DMUs are discussed in details.

Keywords: Efficiency Analysis; Neutrosophic Data Envelopment Analysis; Single value neutrosophic number; Value Index; Ambiguity Index; Ranking Function

MSC 2020: 90C90, 90C70, 90C08.

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1. Introduction

Evaluating the performance of any public or commercial organization is one of the most challenging jobs for DMs to ensure progress, expansion, and sustainability. DEA is the most practical, trustworthy, and reliable way to analyze the performance/efficiency of DMUs. DEA is a data-driven, non-parametric, linear programming-based approach that evaluates piecewise linear production functions to determine the efficiency score of homogeneous DMUs with multiple inputs and outputs. Based on DEA results, DMUs are divided into efficient and inefficient group and also ranked them. This MCDM technique is extensively used in a wide range of fields to evaluate the relative efficiency of DMUs. Charnes et al. [10] introduced the concept of DEA, which is based on Farrell's earlier work [18] on measuring the efficiency of DMUs with multiple inputs and outputs. This method is generally known as CCR model which assumes that the production technology of all DMUs demonstrates constant returns to scale (CRS). Banker et al. [8] added the convexity condition in CCR Model [10] and developed a mathematical model is called the BCC model which assumes that the production technology of all DMUs demonstrates variable returns to scale (VRS). According to the best practice frontier, DMUs are either in the efficient group or inefficient group. Those DMUs are in efficient group have an efficiency score of one and are found on the frontier. Those DMUs are in inefficient group have an efficiency score ranges from 0 to 1 and are not found on the frontier. The inefficient DMUs can improve their efficiency to approach the frontier by reducing current inputs while maintaining outputs or increasing current outputs while keeping inputs unchanged. After the CCR and BCC models, several DEA models, including Additive, SBM, Super Efficiency, Undesirable, and others, have been created to evaluate the relative efficiency of DMUs. DEA has become a popular performance evaluation technique adopted by numerous industries to measure their relative efficiencies, including agriculture, insurance, operations management, banking, healthcare, education, and environmental management [11, 22, 28, 42, 44]. In real-world applications, it is not always possible to provide the clear input and output data that demands conventional/traditional DEA models. However, in practical applications, the observed values might occasionally be confusing, insufficient, inconsistent, and imprecise. This kind of uncertain data may be handled using probability or fuzzy theory. Also, the interval DEA and stochastic DEA models are frequently used to address this problem [34, 35].

The idea of a fuzzy set (FS) was established by Zadeh [46] in 1965, in which each element of the FS is associated with a membership degree (μ) that lies in [0,1], and the non-membership degree is defined as $1 - \mu$. FS theory has been widely used in practical applications of uncertainty modeling. In 1992, Sengupta [38] was the first to introduce the concept of using fuzzy numbers to represent the inputs and outputs of DMUs in the DEA model. Following this work, many authors became interested in developing various approaches for solving the fuzzy DEA model. These approaches are categorized into six types such as "the tolerance technique, the α-level-based approach, the fuzzy ranking approach, the possibility approach, the fuzzy arithmetic, and the fuzzy random/type-2 fuzzy set" [17, 19]. In 2020, Zhou and Xu [49] provided a comprehensive overview of the present state, growth prospects, practical implementations, and future research directions of fuzzy DEA studies. However, only the fuzzy set has a single-membership degree of unclear and vague information, which is often insufficient to describe evidence of support and opposition together. Atanassov [7] expanded Zadeh's fuzzy set notion to the intuitionistic fuzzy set (IFS) in 1986, and its membership and non-membership degrees are defined separately, with their sum lying between [0, 1]. Several research papers using intuitionistic fuzzy sets have been published in DEA using various techniques, such as the weighted approach, (α, β)-cut approach, optimistic-pessimistic efficiency approach, hybrid TOPSIS-DEA, parametric approach, and composition approach, and alphabetical approach [6,37]. IFS is capable of addressing missing data for a wide range of real-world problems, but it can't deal with other types of uncertainty, such as indeterminate information.

Smarandache [40] proposed the Neutrosophic Logic and Neutrosophic Set (NS) as a generalization of FSs and IFSs in 1999. Each element of the Neutrosophic set has three independent degrees, namely "truth, indeterminacy, and falsity", and their sum ranges from 0 to 3. In order to represent uncertainty in different areas, various extensions of the fuzzy set, such as Pythagorean and spherical fuzzy sets, have been developed (as shown in Figure 1). The DM always tries to increase the truth membership degree while decreasing the indeterminacy and falsity membership degrees of each NS element, which are independently assigned. Currently, the study of neutrosophic set theory is highly popular, and its applications are widespread across a range of disciplines, including mathematics, computer science, engineering, medicine, economics, social science, and environmental science [1,4,12,20,25,26,39,48]. In 2018, Edalatpanah [14] presented the initial theoretical advancement of the Neutrosophic DEA (Neu-DEA) model. As a result, many other authors became interested in developing the Neu-DEA model in various neutrosophic environments and proposing novel techniques for solving it. Kahraman et al. [23] developed a novel Neutrosophic Analytic Hierarchy Process (NAHP), which was subsequently combined with the Neu-DEA model to evaluate the efficiency of 15 private universities. Abdelfattah [2] created a Neu-DEA model with triangular neutrosophic inputs and outputs and developed a unique technique that converts the Neu-DEA model into an interval DEA model that evaluates the efficiency of the DMUs in interval form. The input-oriented Neu-DEA model is proposed by considering the inputs and outputs as simplified neutrosophic numbers, which is a nonlinear model turned into an LP model by utilizing a natural logarithm to measure the efficiency of the DMUs [16]. Subsequently, several approaches for solving Neu-DEA models were utilized to measure the efficiency of the DMUs [15, 43]. In 2020, Mao et al. [24] proposed a novel approach to solving the Neu-DEA model with undesirable outputs, where all data is considered as SVNNs, in order to assess the efficiency of the DMUs. Yang et al. [45] developed the Neutrosophic DEA (Neu-DEA) model to evaluate the efficiency of 13 hospitals affiliated with Tehran University of Medical Sciences in Iran. The proposed model considered single-value triangular neutrosophic numbers (SVTNNs) for both inputs and outputs. In 2021, Abdelfattah [3] proposed ranking and parametric approaches to solving the Neu-DEA model in order to evaluate the efficiency of 32 regional hospitals located in Tunisia. The possibility mean [29] and ranking [33] approaches are developed to solve the Neu-DEA model to measure the performance of AIIMS in India [32] and major sea ports in India [30]. Recently, there has been a growing interest among authors in developing the DEA model by incorporating various extensions of fuzzy sets. One such technique is the Fermatean fuzzy DEA (FFDEA), developed to solve the Fermatean fuzzy multi-objective transportation problem (FFMOTP) [5]. Other novel solution techniques have been developed for solving the spherical fuzzy DEA model in the presence of spherical fuzzy inputs and outputs [27, 31]. Additionally, the plithogenic set has been utilized in the DEA model to evaluate the efficiency of 20 bank branches and the performance of hotel industries [21, 36].

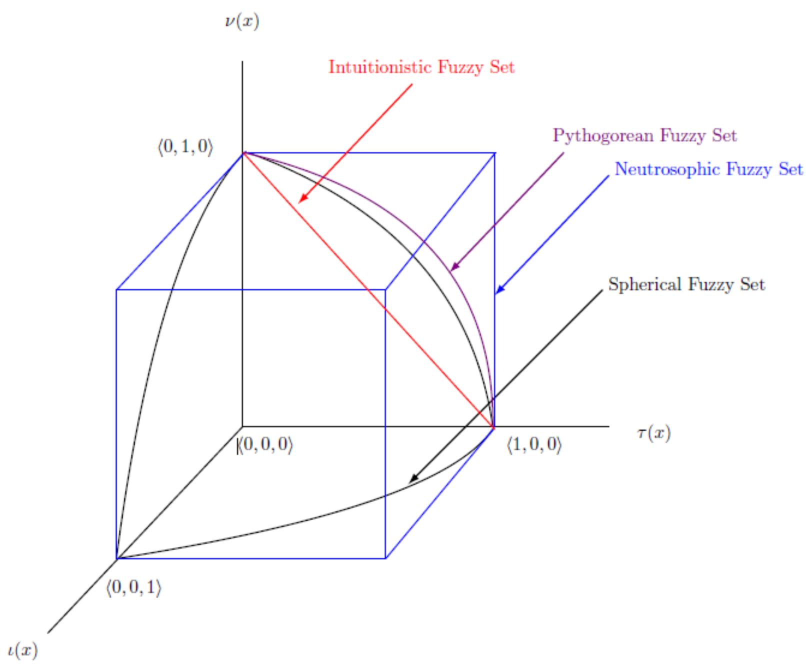

FIGURE 1. Representation of extended fuzzy sets

The primary contribution of this paper is the development of a novel ranking function based on the value and ambiguity index of the single-value triangular neutrosophic number (SVTNN). This ranking function is associated with risk sensitivity and DM's preferences. The suggested ranking function has been used to convert the Neu-DEA model with SVTNN inputs and outputs into the corresponding crisp DEA model. The efficiency scores of DMUs are measured using different risk factors according to the DM's specific preference level. This paper investigates how the DM's preference level affects the efficiency of the DMUs. A numerical example is presented to compare the efficiency scores to the model suggested by Yang et al. [45].

The rest of the manuscript is organized as follows: Section 2 provides some useful notation for this study and details comparison of the suggested model with other existing Nue-DEA model. Section 3 covers various aspects related to SVTNN, including Neutrosophic set, properties of SVTNN, Value and Ambiguity Index for SVTNN, as well as the proposed ranking function for SVTNN along with its properties. Section 4 discusses the Neu-DEA model based on SVTNN and how to obtain an equivalent crisp DEA model by applying the provided ranking function. Section 5 discusses step-by-step technique for solving Neu-DEA model. In Section 6, a case study is given to demonstrate the validity and applicability of the proposed model. Section 7 discusses the advantage and limitation of this study. Finally, the conclusion and recommendations for further study are presented in Section 8.

2. Notation and Comparison Study

To make it easier for readers to understand, the manuscript has been updated with additional notation that is presented in Table 1. The purpose of including this notation is to simplify the content and make it more accessible to a wider researcher.

TABLE 1. Notation used in this Study

Symbol	Description
$\widehat{X}$	Single Value Triangular Neutrosophic Number (SVTNN)
$\mathbb{V}$	Value Index function
$\mathcal{A}$	Ambiguity Index function
δ, ρ, η	Preference parameter for the DM
λ	Risk parameter
$\Re$	Ranking function
A	Input Matrix
B	Output Matrix
a_{ij}	i^{th} crisp input for DMU_j
b_{ki}	k^{th} crisp output for DMU_j
$\widehat{a_{ij}}$	i^{th} SVTNN input for DMU_j $\left(\text{i.e., } \widehat{a_{ij}} = \left\langle (\overbrace{a^L_{ij}}, \overbrace{a^M_{ij}}, \overbrace{a^U_{ij}}); (a^L_{ij}, a^M_{ij}, a^U_{ij}); (\underbrace{a^L_{ij}}, \underbrace{a^M_{ij}}, \underbrace{a^U_{ij}}) \right\rangle\right)$
$\widehat{b_{ki}}$	k^{th} SVTNN output for DMU_j $\left(\text{i.e., } \widehat{b_{ki}} = \left\langle (\overbrace{b^L_{ki}}, \overbrace{b^M_{ki}}, \overbrace{b^U_{ki}}); (b^L_{ki}, b^M_{ki}, b^U_{ki}); (\underbrace{b^L_{ki}}, \underbrace{b^M_{ki}}, \underbrace{b^U_{ki}}) \right\rangle\right)$

In this part of the section, we compare our novel contribution to existing publications, as shown in Table 2. Several researchers have developed various approaches for solving Neu-DEA models without utilizing DM's preference level and risk parameter. The preference level of a DM plays a significant role in uncertainty modeling, indicating the desired decision approach: pessimistic, optimistic, or neutral. Additionally, the risk parameter reflects the DM's attitude towards risk, whether they are a risk taker, risk averse, or neutral. In this

Table 2. Comparing our proposed work to already published Nue-DEA work

Researcher	Inputs and Outputs	Concept	DEA model	Risk Factor
Edalatpanah [14]	Single Valued Triangular Neutrosophic Number (SVTNN)	Score and Accuracy function	CCR	No
Kahraman et al. [23]	Linguistic interval-valued neutrosophic number	Deneutrosophication	Hybrid AHP-CCR	No
Abdelfattah [2]	Triangular neutrosophic numbers	Ranking [1] & parametric approach	CCR	No
Edalatpanah and Smarandache [16]	Simplified neutrosophic numbers (SNN)	Logarithm approach	BCC	No
Edalatpanah [15]	Triangular neutrosophic number (TNN)	Ranking approach [1]	Dual CCR	No
Mao et al. [24]	Simplified neutrosophic numbers (SNN)	Logarithm approach	Undesirable	No
Yang et al. [45]	Single Valued Triangular Neutrosophic Number (SVTNN)	Simple Ranking approach	CCR	No
Tapia [43]	Interval-valued neutrosophic numbers	Robust tolerance approach	CCR	No
Mohanta and Sharanappa [30]	Trapezoidal Neutrosophic Number (TrNN)	Ranking Approach	CCR	No
Mohanta and Toragay [33]	Pentagonal Neutrosophic Number (PNN)	Ranking Approach	CCR	No
Proposed Work	Single Valued Triangular Neutrosophic Number (SVTNN)	Ranking approach based on Value and Ambiguity index	CCR	Yes

article, we incorporate the DM's preference level and risk factor into the value and ambiguity index to create a ranking function for SVTNN. This ranking function effectively compares SVTNNs by taking into account the DM's preference level and risk attitude. This ranking function is then used to convert the Neu-DEA model into a corresponding crisp DEA model, from which the efficiency score of the DMUs can be obtained. The preference level (δ, ρ, η) and risk parameter $(\lambda \in [0, 1])$ are important factors in the performance assessment process because they increase the DM's freedom to express their own risk while making decisions.

3. Single Value Triangular Neutrophic Number (SVTNN) and Its Properties

This section introduces the Neutrosophic set, including the mathematical features of SVTNNs, as well as the value and ambiguity index of SVTNNs, which are defined to create a new ranking function. Additionally, a new ranking algorithm is introduced that utilizes the value and ambiguity index of SVTNNs.

Definition 3.1. [40] A neutrosophic set $\widehat{A}$ in a universe of discourse Ω is given by

$$\widehat{A} = \{\langle x; \tau(x), \iota(x), \nu(x) |\ x \in \Omega\rangle\} \tag{1}$$

where $\tau(x), \iota(x)$, and $\nu(x)$ are called truth, indeterminacy and falsity membership functions, respectively. These membership functions are defined as $\tau : \Omega \rightarrow [0,1]$, $\iota : \Omega \rightarrow [0,1]$ and $\nu : \Omega \rightarrow [0,1]$ such that $0 \leq \tau(x), \iota(x), \nu(x) \leq 3$.

Definition 3.2. [13] The single value triangular neutrophic number (SVTNN) is defined as $\widehat{X} = \left\langle (\overbrace{x^L}, \overbrace{x^M}, \overbrace{x^U}); (x^L, x^M, x^U); (\underbrace{x^L}, \underbrace{x^M}, \underbrace{x^U}) \right\rangle$ where the truth, indeterminacy, and falsehood membership degrees of x are defined as :

$$\tau(x) = \begin{cases} \dfrac{x - \overbrace{x^L}}{\overbrace{x^M} - \overbrace{x^L}}, & x \in [\overbrace{x^L}, \overbrace{x^M}] \\ \dfrac{\overbrace{x^U} - x}{\overbrace{x^U} - \overbrace{x^M}}, & x \in [\overbrace{x^M}, \overbrace{x^U}] \\ 0, & otherwise \end{cases} \qquad \iota(x) = \begin{cases} \dfrac{x - x^L}{x^M - x^L}, & x \in [x^L, x^M] \\ \dfrac{x^U - x}{x^U - x^M} & x \in [x^M, x^U] \\ 1, & otherwise \end{cases}$$

$$\nu(x) = \begin{cases} \dfrac{x - \underbrace{x^L}}{\underbrace{x^M} - \underbrace{x^L}}, & x \in [\underbrace{x^L}, \underbrace{x^M}] \\ \dfrac{\underbrace{x^U} - x}{\underbrace{x^U} - \underbrace{x^M}} & x \in [\underbrace{x^M}, \underbrace{x^U}] \\ 1, & otherwise \end{cases}$$

where $0 \leq \tau(x) + \iota(x) + \nu(x) \leq 3, \ \forall \ x \in \mathbb{R}$.

Definition 3.3. [13] Suppose $\widehat{X}_1 = \left\langle (\overbrace{x_1^L}, \overbrace{x_1^M}, \overbrace{x_1^U}); (x_1^L, x_1^M, x_1^U); (\underbrace{x_1^L}, \underbrace{x_1^M}, \underbrace{x_1^U}) \right\rangle$ and $\widehat{X}_2 = \left\langle (\overbrace{x_2^L}, \overbrace{x_2^M}, \overbrace{x_2^U}); (x_2^L, x_2^M, x_2^U); (\underbrace{x_2^L}, \underbrace{x_2^M}, \underbrace{x_2^U}) \right\rangle$ two SVTNNs. The arithmetic relations in SVTNNs are defined as

(1) $\widehat{X}_1 \oplus \widehat{X}_2 = \left\langle (\overbrace{x_1^L} + \overbrace{x_2^L}, \overbrace{x_1^M} + \overbrace{x_2^M}, \overbrace{x_1^U} + \overbrace{x_2^U}); (x_1^L + x_2^L, x_1^M + x_2^M, x_1^U + x_2^U); (\underbrace{x_1^L} + \underbrace{x_2^L}, \underbrace{x_1^M} + \underbrace{x_2^M}, \underbrace{x_1^U} + \underbrace{x_2^U}) \right\rangle$.

(2) $\widehat{X}_1 - \widehat{X}_2 = \left\langle (\overbrace{x_1^L} - \overbrace{x_2^L}, \overbrace{x_1^M} - \overbrace{x_2^M}, \overbrace{x_1^U} - \overbrace{x_2^U}); (x_1^L + x_2^L, x_1^M - x_2^M, x_1^U + x_2^U); (\underbrace{x_1^L} - \underbrace{x_2^L}, \underbrace{x_1^M} - \underbrace{x_2^M}, \underbrace{x_1^U} - \underbrace{x_2^U}) \right\rangle$.

(3) $\widehat{X}_1 \otimes \widehat{X}_2 = \left\langle (\overbrace{x_1^L}\overbrace{x_2^L}, \overbrace{x_1^M}\overbrace{x_2^M}, \overbrace{x_1^U}\overbrace{x_2^U}); (x_1^L x_2^L, x_1^M x_2^M, x_1^U x_2^U); (\underbrace{x_1^L}\underbrace{x_2^L}, \underbrace{x_1^M}\underbrace{x_2^M}, \underbrace{x_1^U}\underbrace{x_2^U}) \right\rangle$.

(4) $$\alpha\widehat{X}_1 = \begin{cases} \left\langle (\alpha \overbrace{x_1^L}, \alpha \overbrace{x_1^M}, \alpha \overbrace{x_1^U}); (\alpha x_1^L, \alpha x_1^M, \alpha x_1^U); (\alpha \underbrace{x_1^L}, \alpha \underbrace{x_1^M}, \alpha \underbrace{x_1^U}) \right\rangle, & \alpha > 0 \\ \left\langle (\alpha \overbrace{x_1^U}, \alpha \overbrace{x_1^M}, \alpha \overbrace{x_1^L}); (\alpha x_1^U, \alpha x_1^M, \alpha x_1^L); (\alpha \underbrace{x_1^U}, \alpha \underbrace{x_1^M}, \alpha \underbrace{x_1^L}) \right\rangle, & \alpha < 0 \end{cases}$$

Definition 3.4. Let $\widehat{X} = \left\langle (\overbrace{x^L}, \overbrace{x^M}, \overbrace{x^U}), (x^L, x^M, x^U), (\underbrace{x^L}, \underbrace{x^M}, \underbrace{x^U}) \right\rangle$ be a SVTNN. The (α, β, γ)-cut of $\widehat{X}$ is defined as

$$\widehat{X}_{(\alpha,\beta,\gamma)} = \{x : \tau(x) \geq \alpha, \iota(x) \leq \beta, \nu(x) \leq \gamma\} \tag{2}$$

such that $0 \leq \alpha \leq \tau(x), \iota(x) \leq \beta \leq 1$ and $\nu(x) \leq \gamma \leq 1$.

From Definition 3.2 and equation (2), the lower and upper limit of (α, β, γ)-cut of the SVTNN $\widehat{X}$ are defined as

$$\widehat{X}_\alpha = \Big[L(\alpha), U(\alpha)\Big] = \Big[\overbrace{x^L} + \alpha(\overbrace{x^M} - \overbrace{x^L}), \overbrace{x^U} - \alpha(\overbrace{x^U} - \overbrace{x^M})\Big]$$
$$\widehat{X}_\beta = [L(\beta), U(\beta)] = \Big[x^L + \beta(x^M - x^L), x^U - \beta(x^U - x^M)\Big]$$
$$\widehat{X}_\gamma = [L(\gamma), U(\gamma)] = \Big[\underbrace{x^L} + \gamma(\underbrace{x^M} - \underbrace{x^L}), \underbrace{x^U} - \gamma(\underbrace{x^U} - \underbrace{x^M})\Big]$$

3.1. *The Proposed Ranking Function for Single Value Triangular Neutrosophic Number (SVTNN)*

The Value index and Ambiguity index play an important role to ranked the fuzzy numbers in decision making problem [9, 47]. This subsection focuses on the development of value and ambiguity index for SVTNN. Furthermore, a new ranking function is established by incorporating value and ambiguity index.

Definition 3.5 (Value Index)**.** The value index $\mathbb{V}_\tau(\widehat{X}), \mathbb{V}_\iota(\widehat{X})$, and $\mathbb{V}_\nu(\widehat{X})$ with respect to the truth $\tau(x)$, indeterminacy $\iota(x)$, and falsehood $\nu(x)$ membership degrees are defined as

$$\mathbb{V}_\tau(\widehat{X}) = \int_0^1 \Big(L(\alpha) + U(\alpha)\Big) f(\alpha) d\alpha,\ \mathbb{V}_\iota(\widehat{X}) = \int_0^1 \Big(L(\beta) + U(\beta)\Big) g(\beta) d\beta,$$
$$\mathbb{V}_\nu(\widehat{X}) = \int_0^1 \Big(L(\gamma) + U(\gamma)\Big) h(\gamma) d\gamma. \tag{3}$$

where $f(\alpha) = \alpha,\ g(\beta) = 1 - \beta$ and $h(\gamma) = 1 - \gamma$ can be configured to reflect the nature of decision making in real-world scenarios.

Definition 3.6 (Ambiguity Index)**.** The ambiguity index $\mathcal{A}_\tau(\widehat{X}), \mathcal{A}_\iota(\widehat{X})$, and $\mathcal{A}_\nu(\widehat{X})$, with respect to the truth $\tau(x)$, indeterminacy $\iota(x)$, and falsehood $\nu(x)$ membership degrees are defined as

$$\mathcal{A}_\tau(\widehat{X}) = \int_0^1 \Big(U(\alpha) - L(\alpha)\Big) f(\alpha) d\alpha,\ \mathcal{A}_\iota(\widehat{X}) = \int_0^1 \Big(U(\beta) - L(\beta)\Big) g(\beta) d\beta,$$
$$\mathcal{A}_\nu(\widehat{X}) = \int_1^0 \Big(U(\gamma) - L(\gamma)\Big) h(\gamma) d\gamma, \tag{4}$$

where $L(\alpha), L(\beta)$ and $L(\gamma)$ are lower and $U(\alpha), U(\beta)$ and $U(\gamma)$ are upper limits of SVTNN $\widehat{X}$.

Thus from equations (3) and (4), the value and ambiguity for the truth, indeterminacy, and falsehood membership degrees are calculated as follows:

$$\mathbb{V}_\tau(\widehat{X}) = \frac{\overbrace{x^L} + \overbrace{x^U} + 4\overbrace{x^M}}{6}, \mathbb{V}_\iota(\widehat{X}) = \frac{x^L + x^U + 4x^M}{6},\ \mathbb{V}_\nu(\widehat{X}) = \frac{\underbrace{x^L} + \underbrace{x^U} + 4\underbrace{x^M}}{6}$$

$$\mathcal{A}_\tau(\widehat{X}) = \frac{\overbrace{x^U} - \overbrace{x^L}}{6}, \mathcal{A}_\iota(\widehat{X}) = \frac{x^U - x^L}{6}, \mathcal{A}_\nu(\widehat{X}) = \frac{\underbrace{x^U} - \underbrace{x^L}}{6}$$

Definition 3.7. Suppose $\widehat{X} = \left\langle (\overbrace{x^L}, \overbrace{x^M}, \overbrace{x^U}); (x^L, x^M, x^U); (\underbrace{x^L}, \underbrace{x^M}, \underbrace{x^U}) \right\rangle$ be a SVTNN. Then, for $\widehat{X}$, the value and ambiguity index are as follows:

$$\mathbb{V}_{\delta,\rho,\eta}(\widehat{X}) = \delta\mathbb{V}_\tau(\widehat{X}) + \rho\mathbb{V}_\iota(\widehat{X}) + \eta\mathbb{V}_\nu(\widehat{X}) \tag{5}$$

$$\mathcal{A}_{\delta,\rho,\eta}(\widehat{X}) = \delta\mathcal{A}_\tau(\widehat{X}) + \rho\mathcal{A}_\iota(\widehat{X}) + \eta\mathcal{A}_\nu(\widehat{X}) \tag{6}$$

where the DMs' preference value is represented by the co-efficients δ, ρ, η of $\mathbb{V}_{\delta,\rho,\eta}$ and $\mathcal{A}_{\delta,\rho,\eta}$ with the condition $\delta + \rho + \eta = 1$. In an uncertain situation, the DM may want to make pessimistic decisions for $\delta \in [0, 1/3]$ and $\rho + \eta \in [1/3, 1]$. For $\delta \in [1/3, 1]$ and $\rho + \eta \in [0, 1/3]$, on the other hand, the DM may seek to make an optimistic decision in an uncertain situation. The impact of three membership degrees are same to the DM for $\delta = \rho = \eta = 1/3$. As a result, the value index and ambiguity index may indicate how DMs think about SVTNNs.

Lemma 3.8. *Let* $\widehat{X}_1 = \left\langle (\overbrace{x_1^L}, \overbrace{x_1^M}, \overbrace{x_1^U});\ (x_1^L, x_1^M, x_1^U);\ (\underbrace{x_1^L}, \underbrace{x_1^M}, \underbrace{x_1^U}) \right\rangle$ *and* $\widehat{X}_2 = \left\langle (\overbrace{x_2^L}, \overbrace{x_2^M}, \overbrace{x_2^U}); (x_2^L, x_2^M, x_2^U); (\underbrace{x_2^L}, \underbrace{x_2^M}, \underbrace{x_2^U}) \right\rangle$ *be two SVTNNs in* $\mathbb{R}$. *Then for* $\delta, \rho, \eta \in [0, 1]$ *and* $\phi \in \mathbb{R}$, *the following are satisfy*

(1) $\mathbb{V}_{\delta,\rho,\eta}(\widehat{X_1} + \widehat{X_2}) = \mathbb{V}_{\delta,\rho,\eta}(\widehat{X_1}) + \mathbb{V}_{\delta,\rho,\eta}(\widehat{X_2})$.

(2) $\mathbb{V}_{\delta,\rho,\eta}(\phi\widehat{X_1}) = \phi\mathbb{V}_{\delta,\rho,\eta}(\widehat{X_1})$.

Proof. **(1)** From Definition 3.3, the sum of $\widehat{X_1}$ and $\widehat{X_2}$ is defined as

$$\begin{aligned}\widehat{X}_1 \oplus \widehat{X}_2 = \Big\langle &(\overbrace{x_1^L} + \overbrace{x_2^L}, \overbrace{x_1^M} + \overbrace{x_2^M}, \overbrace{x_1^U} + \overbrace{x_2^U}); (x_1^L + x_2^L, x_1^M + x_2^M, x_1^U + x_2^U);\\ &(\underbrace{x_1^L} + \underbrace{x_2^L}, \underbrace{x_1^M} + \underbrace{x_2^M}, \underbrace{x_1^U} + \underbrace{x_2^U})\Big\rangle\end{aligned} \tag{7}$$

From equation (5), we have

$$\begin{aligned}\mathbb{V}_{\delta,\rho,\eta}(\widehat{X}_1 \oplus \widehat{X}_2) &= \delta\mathbb{V}_\tau(\widehat{X}_1 \oplus \widehat{X}_2) + \rho\mathbb{V}_\iota(\widehat{X}_1 \oplus \widehat{X}_2) + \eta\mathbb{V}_\nu(\widehat{X}_1 \oplus \widehat{X}_2)\\ &= \delta\Big(\frac{\overbrace{x_1^L} + \overbrace{x_2^L} + \overbrace{x_1^U} + \overbrace{x_2^U} + 4(\overbrace{x_1^M} + \overbrace{x_2^M})}{6}\Big)\\ &\quad + \rho\Big(\frac{x_1^L + x_2^L + x_1^U + x_2^U + 4(x_1^M + x_2^M)}{6}\Big)\end{aligned}$$

$$+\eta\Big(\frac{\underbrace{x_1^L}+\underbrace{x_2^L}+\underbrace{x_1^U}+\underbrace{x_2^U}+4(\underbrace{x_1^M}+\underbrace{x_2^M})}{6}\Big)$$

$$=\delta\Big(\frac{\overbrace{x_1^L}+\overbrace{x_1^U}+4\overbrace{x_1^M}}{6}\Big)+\rho\Big(\frac{x_1^L+x_1^U+4x_1^M}{6}\Big)+\eta\Big(\frac{\underbrace{x_1^L}+\underbrace{x_1^U}+4\underbrace{x_1^M}}{6}\Big)$$

$$+\delta\Big(\frac{\overbrace{x_2^L}+\overbrace{x_2^U}+4\overbrace{x_2^M}}{6}\Big)+\rho\Big(\frac{x_2^L+x_2^U+4x_2^M}{6}\Big)+\eta\Big(\frac{\underbrace{x_2^L}+\underbrace{x_2^U}+4\underbrace{x_2^M}}{6}\Big)$$

$$=\mathbb{V}_{\delta,\rho,\eta}(\widehat{X}_1)+\mathbb{V}_{\delta,\rho,\eta}(\widehat{X}_2)$$

(2) From Definition 3.3, the scalar ($\phi\in\mathbb{R}$) multiplication of $\widehat{X}_1=\Big\langle(\overbrace{x_1^L},\overbrace{x_1^M},\overbrace{x_1^U});(x_1^L,x_1^M,x_1^U);(\underbrace{x_1^L},\underbrace{x_1^M},\underbrace{x_1^U})\Big\rangle$ is defined as

$$\phi\widehat{X}_1=\Big\langle(\phi\overbrace{x_1^L},\phi\overbrace{x_1^M},\phi\overbrace{x_1^U});(\phi x_1^L,\phi x_1^M,\phi x_1^U);(\phi\underbrace{x_1^L},\phi\underbrace{x_1^M},\phi\underbrace{x_1^U})\Big\rangle$$

From equation (5), we have

$$\mathbb{V}_{\delta,\rho,\eta}(\phi\widehat{X}_1)=\delta\mathbb{V}_\tau(\phi\widehat{X}_1)+\rho\mathbb{V}_\iota(\phi\widehat{X}_1)+\eta\mathbb{V}_\nu(\phi\widehat{X}_1)$$

$$=\delta\Big(\frac{\phi\overbrace{x_1^L}+\phi\overbrace{x_1^U}+4\phi\overbrace{x_1^M}}{6}\Big)+\rho\Big(\frac{\phi x_1^L+\phi x_1^U+4\phi x_1^M}{6}\Big)+\eta\Big(\frac{\phi\underbrace{x_1^L}+\phi\underbrace{x_1^U}+4\phi\underbrace{x_1^M}}{6}\Big)$$

$$=\phi\Big[\delta\Big(\frac{\overbrace{x_1^L}+\overbrace{x_1^U}+4\overbrace{x_1^M}}{6}\Big)+\rho\Big(\frac{x_1^L+x_1^U+4x_1^M}{6}\Big)+\eta\Big(\frac{\underbrace{x_1^L}+\underbrace{x_1^U}+4\underbrace{x_1^M}}{6}\Big)\Big]$$

$$=\phi\,\mathbb{V}_{\delta,\rho,\eta}(\widehat{X}_1)$$

This complete the proof. □

Lemma 3.9. *Let* $\widehat{X}_1=\Big\langle(\overbrace{x_1^L},\overbrace{x_1^M},\overbrace{x_1^U});(x_1^L,x_1^M,x_1^U);(\underbrace{x_1^L},\underbrace{x_1^M},\underbrace{x_1^U})\Big\rangle$ *and* $\widehat{X}_2=\Big\langle(\overbrace{x_2^L},\overbrace{x_2^M},\overbrace{x_2^U});(x_2^L,x_2^M,x_2^U);(\underbrace{x_2^L},\underbrace{x_2^M},\underbrace{x_2^U})\Big\rangle$ *be two SVTNNs in* $\mathbb{R}$. *Then for* $\delta,\rho,\eta\in[0,1]$ *and* $\phi\in\mathbb{R}$ *be a real number,*

(1) $\mathcal{A}_{\delta,\rho,\eta}(\widehat{X_1}+\widehat{X_2})=\mathcal{A}_{\delta,\rho,\eta}(\widehat{X_1})+\mathcal{A}_{\delta,\rho,\eta}(\widehat{X_2})$.

(2) $\mathcal{A}_{\delta,\rho,\eta}(\phi\widehat{X_1})=\phi\mathcal{A}_{\delta,\rho,\eta}(\widehat{X_1})$.

Proof.

(1) From Definition 3.3, the sum of two SVTNNs $\widehat{X_1}$ and $\widehat{X_2}$ is written as follows:

$$\widehat{X}_1\oplus\widehat{X}_2=\Big\langle(\overbrace{x_1^L}+\overbrace{x_2^L},\overbrace{x_1^M}+\overbrace{x_2^M},\overbrace{x_1^U}+\overbrace{x_2^U});(x_1^L+x_2^L,x_1^M+x_2^M,x_1^U+x_2^U);$$
$$(\underbrace{x_1^L}+\underbrace{x_2^L},\underbrace{x_1^M}+\underbrace{x_2^M},\underbrace{x_1^U}+\underbrace{x_2^U})\Big\rangle \quad (8)$$

From equation (6), we have

$$\mathcal{A}_{\delta,\rho,\eta}(\widehat{X}_1\oplus\widehat{X}_2)=\delta\mathcal{A}_\tau(\widehat{X}_1\oplus\widehat{X}_2)+\rho\mathcal{A}_\iota(\widehat{X}_1\oplus\widehat{X}_2)+\eta\mathcal{A}_\nu(\widehat{X}_1\oplus\widehat{X}_2)$$

$$
\begin{aligned}
&= \delta\Big(\frac{\overbrace{x_1^U} + \overbrace{x_2^U} - (\overbrace{x_1^L} + \overbrace{x_2^L})}{6}\Big) + \rho\Big(\frac{x_1^U + x_2^U - (x_1^L + x_2^L)}{6}\Big) \\
&\quad + \eta\Big(\frac{\underbrace{x_1^U} + \underbrace{x_2^U} - (\underbrace{x_1^L} + \underbrace{x_2^L})}{6}\Big) \\
&= \delta\Big(\frac{\overbrace{x_1^U} - \overbrace{x_1^L}}{6}\Big) + \rho\Big(\frac{x_1^U - x_1^L}{6}\Big) + \eta\Big(\frac{\underbrace{x_1^U} - \underbrace{x_1^L}}{6}\Big) \\
&\quad + \delta\Big(\frac{\overbrace{x_2^U} - \overbrace{x_2^L}}{6}\Big) + \rho\Big(\frac{x_2^U - x_2^L}{6}\Big) + \eta\Big(\frac{\underbrace{x_2^U} - \underbrace{x_2^L}}{6}\Big) \\
&= \mathcal{A}_{\delta,\rho,\eta}(\widehat{X}_1) + \mathcal{A}_{\delta,\rho,\eta}(\widehat{X}_2)
\end{aligned}
$$

(2) From Definition 3.3, we have

$$\phi\widehat{X}_1 = \Big\langle(\phi\overbrace{x_1^L}, \phi\overbrace{x_1^M}, \phi\overbrace{x_1^U}), (\phi x_1^L, \phi x_1^M, \phi x_1^U), (\phi\underbrace{x_1^L}, \phi\underbrace{x_1^M}, \phi\underbrace{x_1^U})\Big\rangle$$

From equation (6), we have

$$
\begin{aligned}
\mathcal{A}_{\delta,\rho,\eta}(\phi\widehat{X}_1) &= \delta\mathcal{A}_\tau(\phi\widehat{X}_1) + \rho\mathcal{A}_\iota(\phi\widehat{X}_1) + \eta\mathcal{A}_\nu(\phi\widehat{X}_1) \\
&= \delta\Big(\frac{\phi\overbrace{x_1^U} - \phi\overbrace{x_1^L}}{6}\Big) + \rho\Big(\frac{\phi x_1^U - \phi x_1^L}{6}\Big) + \eta\Big(\frac{\phi\underbrace{x_1^U} - \phi\underbrace{x_1^L}}{6}\Big) \\
&= \phi\Big[\delta\Big(\frac{\overbrace{x_1^U} - \overbrace{x_1^L}}{6}\Big) + \rho\Big(\frac{x_1^U - x_1^L}{6}\Big) + \eta\Big(\frac{\underbrace{x_1^U} - \underbrace{x_1^L}}{6}\Big)\Big] \\
&= \phi\ \mathcal{A}_{\delta,\rho,\eta}(\widehat{X}_1)
\end{aligned}
$$

This complete the proof. □

Definition 3.10. Let $\widehat{X} = \Big\langle(\overbrace{x^L}, \overbrace{x^M}, \overbrace{x^U}); (x^L, x^M, x^U); (\underbrace{x^L}, \underbrace{x^M}, \underbrace{x^U})\Big\rangle$ be a SVTNN. Then, the ranking function is defined as

$$\Re(\widehat{X}) = \lambda\mathbb{V}_{\delta,\rho,\eta}(\widehat{X}) + (1-\lambda)\mathcal{A}_{\delta,\rho,\eta}(\widehat{X}). \tag{9}$$

The variable λ represents the perspective of the DM regarding risk.

(1) If λ belongs to the range [0, 0.5), then the DM is willing to take risks and prefers uncertainty.

(2) If λ equals 0.5, then the DM has a neutral stance towards risk when making parameter selections.

(3) If λ belongs to the range (0.5, 1], then the DM is sensitive to taking risks when making decisions.

Theorem 3.11. *Let $\widehat{X}_1$ and $\widehat{X}_2$ be two SVTNNs in $\mathbb{R}$. Then for $\delta, \rho, \eta \in [0,1]$ and $\phi \in \mathbb{R}$*

(1) $\Re(\widehat{X_1} + \widehat{X_2}) = \Re(\widehat{X_1}) + \Re(\widehat{X_2})$.

(2) $\Re(\phi\widehat{X_1}) = \phi\Re(\widehat{X_1})$.

Proof.

(1) From Definition 3.10, we have

$$\Re(\widehat{X_1}+\widehat{X_2}) = \lambda\mathbb{V}_{\delta,\rho,\eta}(\widehat{X_1}+\widehat{X_2}) + (1-\lambda)\mathcal{A}_{\delta,\rho,\eta}(\widehat{X_1}+\widehat{X_2})$$

From Lemma 3.8 and Lemma 3.9, we have

$$\begin{aligned}\Re(\widehat{X_1}+\widehat{X_2}) &= \lambda\Big(\mathbb{V}_{\delta,\rho,\eta}(\widehat{X_1}) + \mathbb{V}_{\delta,\rho,\eta}(\widehat{X_2})\Big) + (1-\lambda)\Big(\mathcal{A}_{\delta,\rho,\eta}(\widehat{X_1}) + \mathcal{A}_{\delta,\rho,\eta}(\widehat{X_2})\Big)\\ &= \Big[\lambda\mathbb{V}_{\delta,\rho,\eta}(\widehat{X_1}) + (1-\lambda)\mathcal{A}_{\delta,\rho,\eta}(\widehat{X_1})\Big] + \Big[\lambda\mathbb{V}_{\delta,\rho,\eta}(\widehat{X_2}) + (1-\lambda)\mathcal{A}_{\delta,\rho,\eta}(\widehat{X_2})\Big]\\ &= \Re(\widehat{X_1}) + \Re(\widehat{X_2})\end{aligned}$$

(2) From Definition 3.10, we have

$$\Re(\phi\widehat{X_1}) = \lambda\mathbb{V}_{\delta,\rho,\eta}(\phi\widehat{X_1}) + (1-\lambda)\mathcal{A}_{\delta,\rho,\eta}(\phi\widehat{X_1})$$

From Lemma 3.8 and Lemma 3.9, we have

$$\begin{aligned}\Re(\phi\widehat{X_1}) &= \lambda\Big(\phi\mathbb{V}_{\delta,\rho,\eta}(\widehat{X_1})\Big) + (1-\lambda)\Big(\phi\mathcal{A}_{\delta,\rho,\eta}(\widehat{X_1})\Big)\\ &= \phi\Big[\lambda\mathbb{V}_{\delta,\rho,\eta}(\widehat{X_1}) + (1-\lambda)\mathcal{A}_{\delta,\rho,\eta}(\widehat{X_1})\Big]\\ &= \phi\Re(\widehat{X_1})\end{aligned}$$

This complete the proof. □

Corollary 3.12. *Let* $\widehat{X}_i = \Big\langle(\overbrace{x_i^L}, \overbrace{x_i^M}, \overbrace{x_i^U}); (x_i^L, x_i^M, x_i^U); (\underbrace{x_i^L}, \underbrace{x_i^M}, \underbrace{x_i^U})\Big\rangle$ *be* n *SVTNNs in* $\mathbb{R}$ *and* $\phi_i \in \mathbb{R}$ *be the scalars where* $i = 1,2,3,\cdots,n$. *Then*

$$\begin{aligned}\Re\Big(\sum_{i=1}^{n}\phi_i\widehat{X}_i\Big) = \sum_{i=1}^{n}\Big[\lambda\Big(\delta(\frac{\overbrace{x_i^L}+\overbrace{x_i^U}+4\overbrace{x_i^M}}{6}) + \rho(\frac{x_i^L+x_i^U+4x_i^M}{6}) + \eta(\frac{\underbrace{x_i^L}+\underbrace{x_i^U}+4\underbrace{x_i^M},}{6})\Big)\\ + (1-\lambda)\Big(\delta(\frac{\overbrace{x_i^U}-\overbrace{x_i^L}}{6}) + \rho(\frac{x_i^U-x_i^L}{6}) + \eta(\frac{\underbrace{x_i^U}-\underbrace{x_i^L}}{6})\Big)\Big]\phi_i \qquad (10)\end{aligned}$$

Proof. This is proved by using Theorem 3.11, Lemma 3.8 and Lemma 3.9. □

Definition 3.13. Suppose $\widehat{X_1}$ and $\widehat{X_2}$ be two SVTNNs, then two SVTNNs are compared by

(1) $\widehat{X_1} \leq \widehat{X_2}$ if and only if $\Re(\widehat{X_1}) \leq \Re(\widehat{X_2})$,

(2) $\widehat{X_1} < \widehat{X_2}$ if and only if $\Re(\widehat{X_1}) < \Re(\widehat{X_2})$,

where $\Re(.)$ is the ranking function.

Note: If $\widehat{X} = a \in \mathbb{R}$ be a real crisp number so that it is independent of the risk factor λ, then $\Re(\widehat{X}) = a$.

4. Neutrosophic Data Envelopment Analysis

Consider $a_i = (a_{1i}, a_{2i}, \cdots, a_{mi}) \in \mathbb{R}^m$ and $b_i = (b_{1i}, b_{2i}, \cdots, b_{ri}) \in \mathbb{R}^r$ are the input and output vector of DMU_i for $i = 1, 2, \cdots, n$, respectively. The input matrix A and and the output matrix B are defined as $A = [a_1, \cdots, a_n] \in \mathbb{R}^{m\times n}$, and $B = [b_1, \cdots, b_n] \in \mathbb{R}^{r\times n}$ such that $A > 0$ and $B > 0$. Charnes et al. [10] developed the following LP model for measuring the efficiency of DMU_o

$$\begin{aligned} \max_{\omega,\mu} \theta &= \sum_{k=1}^{r} \omega_k b_{ko}, \\ \text{subject to } &\sum_{j=1}^{m} \mu_j a_{jo} = 1, \\ &\sum_{k=1}^{r} \omega_k b_{ki} \le \sum_{j=1}^{m} \mu_j a_{ji}, \ i = 1, 2, \cdots n, \\ \text{and } &\omega_k \ge 0, \ k = 1, 2, \cdots, r, \\ &\mu_j \ge 0, \ j = 1, 2, \cdots, m. \end{aligned} \tag{11}$$

This is popularly known CCR model.

In the classical DEA model, the efficiency score of the $DMUo$ will be erroneous if the input and output data of the DMUs are inaccurate, imprecise, or ambiguous. The application of Neutrosophic set theory is a powerful strategy for dealing with this type of data.

Assuming inputs and outputs of the DMUs are SVTNNs while the weights $\mu_j \in \mathbb{R}$ and $\omega_k \in \mathbb{R}$. Then, the Neutrosophic CCR (Nue-CCR) model can be defined as

$$\begin{aligned} \max_{\omega,\mu} \theta &= \sum_{k=1}^{r} \omega_k \widehat{b_{ko}}, \\ \text{subject to } &\sum_{j=1}^{m} \mu_j \widehat{a_{jo}} = 1, \\ &\sum_{k=1}^{r} \omega_k \widehat{b_{ki}} \le \sum_{j=1}^{m} \mu_j \widehat{a_{ji}}, \ i = 1, 2, \cdots n, \\ \text{and } &\omega_k \ge 0, \ k = 1, 2, \cdots, r, \\ &\mu_j \ge 0, \ j = 1, 2, \cdots, m, \end{aligned} \tag{12}$$

where $\widehat{a_{ji}} = \left\langle (\overbrace{a_{ji}^L}, \overbrace{a_{ji}^M}, \overbrace{a_{ji}^U}); (a_{ji}^L, a_{ji}^M, a_{ji}^U); (\underbrace{a_{ji}^L}, \underbrace{a_{ji}^M}, \underbrace{a_{ji}^U}) \right\rangle$ and $\widehat{b_{ki}} = \left\langle (\overbrace{b_{ki}^L}, \overbrace{b_{ki}^M}, \overbrace{b_{ki}^U}); (b_{ki}^L, b_{ki}^M, b_{ki}^U); (\underbrace{b_{ki}^L}, \underbrace{b_{ki}^M}, \underbrace{b_{ki}^U}) \right\rangle$ are the SVTNNs. The efficiency score of the Nue-CCR model is $\theta^* \in [0, 1]$.

Definition 4.1. A DMU is said to be efficient, if its efficiency score is 1; Otherwise it is consider as inefficient DMU.

Applying the ranking function ($\Re$) in the Neu-CCR model given in equation (12).

$$\begin{aligned}
&\max_{\omega,\mu}\ \theta = \Re\Big(\sum_{k=1}^{r}\omega_k\widehat{b_{ko}}\Big),\\
&\text{subject to } \Re\Big(\sum_{j=1}^{m}\mu_j\widehat{a_{jo}}\Big) = \Re\Big(1\Big),\\
&\Re\Big(\sum_{k=1}^{r}\omega_k\widehat{b_{ki}}\Big) \le \Re\Big(\sum_{j=1}^{m}\mu_j\widehat{a_{ji}}\Big),\ i = 1,2,\cdots n,\\
&\text{and } \omega_k \ge 0,\ k = 1,2,\cdots,r,\\
&\mu_j \ge 0,\ j = 1,2,\cdots,m.
\end{aligned} \tag{13}$$

Using Definition 3.3, The equation (13) can be written as

$$\begin{aligned}
&\max_{\omega,\mu}\ \theta = \Re\Bigg(\Big\langle(\sum_{k=1}^{r}\omega_k\overbrace{b_{ko}^{L}},\sum_{k=1}^{r}\omega_k\overbrace{b_{ko}^{M}},\sum_{k=1}^{r}\omega_k\overbrace{b_{ko}^{U}});(\sum_{k=1}^{r}\omega_k b_{ko}^{L},\sum_{k=1}^{r}\omega_k b_{ko}^{M},\sum_{k=1}^{r}\omega_k b_{ko}^{U});\\
&(\sum_{k=1}^{r}\omega_k\underbrace{b_{ko}^{L}},\sum_{k=1}^{r}\omega_k\underbrace{b_{ko}^{M}},\sum_{k=1}^{r}\omega_k\underbrace{b_{ko}^{U}})\Big\rangle\Bigg),\\
&\text{s. t } \Re\Bigg(\Big\langle(\sum_{j=1}^{m}\mu_j\overbrace{a_{jo}^{L}},\sum_{j=1}^{m}\mu_j\overbrace{a_{jo}^{M}},\sum_{j=1}^{m}\mu_j\overbrace{a_{jo}^{U}});(\sum_{j=1}^{m}\mu_j a_{jo}^{L},\sum_{j=1}^{m}\mu_j a_{jo}^{M},\sum_{j=1}^{m}\mu_j a_{jo}^{U});\\
&(\sum_{j=1}^{m}\mu_j\underbrace{a_{jo}^{L}},\sum_{j=1}^{m}\mu_j\underbrace{a_{jo}^{M}},\sum_{j=1}^{m}\mu_j\underbrace{a_{jo}^{U}})\Big\rangle\Bigg) = 1,\\
&\Re\Bigg(\Big\langle(\sum_{k=1}^{r}\omega_k\overbrace{b_{ki}^{L}},\sum_{k=1}^{r}\omega_k\overbrace{b_{ki}^{M}},\sum_{k=1}^{r}\omega_k\overbrace{b_{ki}^{U}});(\sum_{k=1}^{r}\omega_k b_{ki}^{L},\sum_{k=1}^{r}\omega_k b_{ki}^{M},\sum_{k=1}^{r}\omega_k b_{ki}^{U});\\
&(\sum_{k=1}^{r}\omega_k\underbrace{b_{ki}^{L}},\sum_{k=1}^{r}\omega_k\underbrace{b_{ki}^{M}},\sum_{k=1}^{r}\omega_k\underbrace{b_{ki}^{U}})\Big\rangle\Bigg) \le \Re\Bigg(\Big\langle(\sum_{j=1}^{m}\mu_j\overbrace{a_{ji}^{L}},\sum_{j=1}^{m}\mu_j\overbrace{a_{ji}^{M}},\sum_{j=1}^{m}\mu_j\overbrace{a_{ji}^{U}});\\
&(\sum_{j=1}^{m}\mu_j a_{ji}^{L},\sum_{j=1}^{m}\mu_j a_{ji}^{M},\sum_{j=1}^{m}\mu_j a_{ji}^{U});(\sum_{j=1}^{m}\mu_j\underbrace{a_{ji}^{L}},\sum_{j=1}^{m}\mu_j\underbrace{a_{ji}^{M}},\sum_{j=1}^{m}\mu_j\underbrace{a_{ji}^{U}})\Big\rangle\Bigg), i = 1,2,\cdots,n,\\
&\text{and } \omega_k \ge 0,\ k = 1,2,\cdots,r,\ \mu_j \ge 0,\ j = 1,2,\cdots,m.
\end{aligned}$$

Now from Theorem 3.11 and Corollary 3.12, we have

$$\begin{aligned}
&\max_{\omega,\mu}\ \theta = \sum_{k=1}^{r}\Big[\lambda\Big(\delta(\frac{\overbrace{b_{ko}^{L}}+\overbrace{b_{ko}^{U}}+4\overbrace{b_{ko}^{M}}}{6})+\rho(\frac{b_{ko}^{L}+b_{ko}^{U}+4b_{ko}^{M}}{6})+\eta(\frac{\underbrace{b_{ko}^{L}}+\underbrace{b_{ko}^{U}}+4\underbrace{b_{ko}^{M}},}{6})\Big)\\
&+(1-\lambda)\Big(\delta(\frac{\overbrace{b_{ko}^{U}}-\overbrace{b_{ko}^{L}}}{6})+\rho(\frac{b_{ko}^{U}-b_{ko}^{L}}{6})+\eta(\frac{\underbrace{b_{ko}^{U}}-\underbrace{b_{ko}^{L}}}{6})\Big)\Big]\omega_k \qquad (14)\\
&\text{s.t } \sum_{j=1}^{m}\Big[\lambda\Big(\delta(\frac{\overbrace{a_{jo}^{L}}+\overbrace{a_{jo}^{U}}+4\overbrace{a_{jo}^{M}}}{6})+\rho(\frac{a_{jo}^{L}+a_{jo}^{U}+4a_{jo}^{M}}{6})+\eta(\frac{\underbrace{a_{jo}^{L}}+\underbrace{a_{jo}^{U}}+4\underbrace{a_{jo}^{M}},}{6})\Big)
\end{aligned}$$

$$+ (1-\lambda)\Big(\delta(\frac{\overbrace{a_{jo}^{U}} - \overbrace{a_{jo}^{L}}}{6}) + \rho(\frac{a_{jo}^{U} - a_{jo}^{L}}{6}) + \eta(\frac{\underbrace{a_{jo}^{U}} - \underbrace{a_{jo}^{L}}}{6})\Big)\Big]\mu_j = 1$$

$$\sum_{k=1}^{r}\Big[\lambda\Big(\delta(\frac{\overbrace{b_{ki}^{L}} + \overbrace{b_{ki}^{U}} + 4\overbrace{b_{ki}^{M}}}{6}) + \rho(\frac{b_{ki}^{L} + b_{ki}^{U} + 4b_{ki}^{M}}{6}) + \eta(\frac{\underbrace{b_{ki}^{L}} + \underbrace{b_{ki}^{U}} + 4\underbrace{b_{ki}^{M}},}{6})\Big)$$

$$+ (1-\lambda)\Big(\delta(\frac{\overbrace{b_{ki}^{U}} - \overbrace{b_{ki}^{L}}}{6}) + \rho(\frac{b_{ki}^{U} - b_{ki}^{L}}{6}) + \eta(\frac{\underbrace{b_{ki}^{U}} - \underbrace{b_{ki}^{L}}}{6})\Big)\Big]\omega_k$$

$$\leq \sum_{j=1}^{m}\Big[\lambda\Big(\delta(\frac{\overbrace{a_{ji}^{L}} + \overbrace{a_{ji}^{U}} + 4\overbrace{a_{ji}^{M}}}{6}) + \rho(\frac{a_{ji}^{L} + a_{ji}^{U} + 4a_{ji}^{M}}{6}) + \eta(\frac{\underbrace{a_{ji}^{L}} + \underbrace{a_{ji}^{U}} + 4\underbrace{a_{ji}^{M}},}{6})\Big)$$

$$+ (1-\lambda)\Big(\delta(\frac{\overbrace{a_{ji}^{U}} - \overbrace{a_{ji}^{L}}}{6}) + \rho(\frac{a_{ji}^{U} - a_{ji}^{L}}{6}) + \eta(\frac{\underbrace{a_{ji}^{U}} - \underbrace{a_{ji}^{L}}}{6})\Big)\Big]\mu_j, \ i = 1, 2, \cdots, n,$$

$$\text{and } \omega_k \geq 0, \ k = 1, 2, \cdots, r, \ \mu_j \geq 0, \ j = 1, 2, \cdots, m.$$

This is the equivalent crisp LP model of the Neu-CCR model defined in equation (12).

Theorem 4.2. *The Neutrosophic CCR model presented in equation (12) and the corresponding crisp LP model presented in equation (14) of equal significance.*

Proof. By utilizing the ranking formula put forth in Definition 3.10 of the Neu-CCR model, which is illustrated in equation (13), it becomes straightforward to observe that the optimal feasible solution derived in equation (12) for every Neu-CCR model is also an optimal feasible solution of equation (14), and similarly, the converse holds true. □

5. Method for Solving Neutrosophic DEA model

Fuzzification is the process of converting precise input-output data into fuzzy input-output data by utilizing information from a knowledge base. Fuzzification is considered important and advantageous in the early stages of uncertainty theory because the fuzzifier is defined as a mapping from a crisp data space to a fuzzy data space within a particular discourse universe. This article utilizes triangular neutrosophic membership functions during the fuzzification process since they can be effectively implemented by embedded controllers in highly uncertain environments. In this particular case, a single value neutrosophic set is employed during the fuzzification process based on the observed data. The Neu-DEA model may be solved by performing the procedures listed below.

Step 1: Convert the DEA model into the Neu-DEA model by considering the input-output data are SVTNNs as shown in equation (12).

Step 2: Applying the ranking function ($\Re$) in the Neu-DEA model given in equation (12).

Step 3: Convert the Nue-DEA model into equivalent crisp LP model as shown in equation (14).

Step 4: Solve the given crisp LP model effectively and determine the optimal solution θ^* with different DM's preference parameters $(\delta,\ \rho,\ \eta\ )$ for each risk factor $\lambda \in [0, 1]$ which represents the risk taking attitude of the DM, given in Definition 3.10.

Step 5: The DMUs are ranked according to the average of each DMU's efficiency scores for each DM's preference level.

The flowchart depicted in Figure 2 illustrates the step-by-step approach utilized to solve the Neu-DEA model. This technique serves as a visual representation of the process employed to address the model's complexities and arrive at a solution.

FIGURE 2. Flowchart for solving Neu-DEA model

6. Numerical Example

In order to demonstrate the validity and applicability of the suggested model, this section consider a real case study of hospital performance assessment provided by Yang et al. [45] in a neutrosophic environment. The input parameters, which consist of "the number of doctors and the number of nurses", are displayed in Table 3. Correspondingly, the output parameters, including "days of hospitalization, patient satisfaction, and the number of outpatients", are shown in Table 4.

Table 3. The SVTNNs input data

DMU	Number of Doctors	Number of Nurses
D1	$\langle(404, 540, 674); (350, 440, 560); (420, 645, 700)\rangle$	$\langle(520, 530, 535); (520, 525, 530); (532, 534, 540)\rangle$
D2	$\langle(119, 136, 182); (122, 125, 137); (125, 178, 200)\rangle$	$\langle(177, 180, 188); (173, 175, 179); (185, 189, 195)\rangle$
D3	$\langle(139, 145, 158); (139, 140, 147); (146, 155, 167)\rangle$	$\langle(208, 214, 218); (195, 209, 215); (210, 217, 230)\rangle$
D4	$\langle(86, 93, 151); (83, 85, 87); (89, 138, 160)\rangle$	$\langle(114, 116, 118); (114, 115, 117); (116, 118, 125)\rangle$
D5	$\langle(84, 93, 143); (84, 89, 120); (90, 140, 155)\rangle$	$\langle(110, 117, 121); (105, 112, 120); (113, 119, 128)\rangle$
D6	$\langle(101, 113, 170); (110, 112, 115); (112, 120, 177)\rangle$	$\langle(101, 107, 111); (95, 100, 104); (108, 112, 115)\rangle$
D7	$\langle(561, 694, 864); (510, 640, 750), (582, 857, 930)\rangle$	$\langle(492, 495, 508); (492, 494, 500); (493, 506, 520)\rangle$
D8	$\langle(123, 179, 199); (122, 125, 130); (195, 200, 205)\rangle$	$\langle(66, 68, 73); (63, 67, 69); (68, 70, 78)\rangle$
D9	$\langle(101, 153, 155); (140, 145, 150); (145, 149, 167)\rangle$	$\langle(192, 195, 198); (185, 193, 197); (194, 196, 205)\rangle$
D10	$\langle(147, 164, 170); (147, 160, 167); (165, 169, 180)\rangle$	$\langle(333, 340, 357); (335, 338, 350); (338, 347, 364)\rangle$
D11	$\langle(130, 158, 192); (110, 144, 173); (146, 177, 205)\rangle$	$\langle(96, 100, 114); (97, 99, 103); (99, 110, 129)\rangle$
D12	$\langle(128, 137, 187); (128, 133, 164); (134, 184, 199)\rangle$	$\langle(213, 220, 224); (208, 215, 223); (216, 222, 231)\rangle$
D13	$\langle(151, 160, 210); (151, 156, 187); (157, 207, 222)\rangle$	$\langle(320, 327, 331); (315, 322, 330); (323, 329, 338)\rangle$

Table 4. The SVTNNs output data

DMU	Days of Hospitalization	Patient satisfaction	Numbers of Outpatient
D1	$\langle(121.13, 139.24, 140.04); (138.64, 139.14, 139.81); (139.14, 140.02, 141.17)\rangle$	$\langle(38, 41, 45); (38, 40, 43); (41, 44, 49)\rangle$	$\langle(104.23, 114.04, 278.51); (102.37, 109.15, 235.72); (104.81, 275.25, 279.88)\rangle$
D2	$\langle(31.54, 34.15, 38.27); (31.54, 34.93, 38.89); (34.86, 38.15, 39.83)\rangle$	$\langle(40, 44, 47); (35, 42, 45); (41, 46, 50)\rangle$	$\langle(34.54, 36.98, 54.82); (36.45, 36.80, 41.57); (47.61, 54.25, 55.35)\rangle$
D3	$\langle(81.62, 82.07, 85.51); (81.41, 81.94, 83.35); (81.78, 85.49, 88.16)\rangle$	$\langle(18, 20, 29); (19, 21, 23); (28, 30, 35)\rangle$	$\langle(157.75, 177.57, 264.52); (157.75, 176.68, 250.75); (180.29, 263.98, 272.16)\rangle$
D4	$\langle(19.54, 20.41, 20.59); (20.15, 20.25, 20.32); (20.54, 20.58, 20.70)\rangle$	$\langle(18, 21, 25); (15, 19, 23); (20, 24, 30)\rangle$	$\langle(32.89, 35.56, 87.74); (35.25, 35.50, 35.61); (87.50, 87.94, 88.30)\rangle$

D5	$\langle(23.89, 24.60, 26.09);$ $(23.56, 23.60, 23.68);$ $(25.97, 26.35, 26.72)\rangle$	$\langle(30, 36, 41); (34, 35, 37);$ $(35, 40, 57)\rangle$	$\langle(63.23, 69.58, 120.73);$ $(63, 65.17, 94.93);$ $(64.47, 118.75, 124.75)\rangle$
D6	$\langle(21.33, 21.49, 23.31);$ $(20.94, 24.25, 22.68);$ $(21.38, 23.14, 23.94)\rangle$	$\langle(50, 55, 60); (50, 53, 57);$ $(56, 59, 70)\rangle$	$\langle(72.84, 82.84, 94.18);$ $(82.15, 82.68, 84.89);$ $(85.75, 93.50, 97.18)\rangle$
D7	$\langle(145.77, 148.28, 169.01);$ $(145.77, 147.16, 168.31);$ $(150.69, 168.95, 175.18)\rangle$	$\langle(40, 44, 46); (42, 43, 45);$ $(43, 44, 55)\rangle$	$\langle(147.59, 150.37, 227.12);$ $(147.30, 147.45, 148.25);$ $(218.24, 224.61, 229.63)\rangle$
D8	$\langle(11.56, 11.74, 12.96);$ $(11.42, 11.61, 11.98);$ $(11.58, 12.64, 13.16)\rangle$	$\langle(60, 75, 80); (55, 60, 62);$ $(78, 83, 85)\rangle$	$\langle(189.37, 202.08, 284.99);$ $(189.37, 200.52, 281.63);$ $(270.16, 284.55, 289.12)\rangle$
D9	$\langle(57.55, 62.67, 63.03);$ $(62.15, 62.50, 62.93);$ $(62.50, 62.97, 63.61)\rangle$	$\langle(32, 35, 38); (32, 33, 35);$ $(34, 36, 45)\rangle$	$\langle(14.63, 14.85, 29.40);$ $(14.70, 14.75, 15.25);$ $(24.75, 28.36, 32.64)\rangle$
D10	$\langle(73.21, 76.03, 81.90);$ $(75.76, 76.05, 76.25);$ $(81.67, 82.27, 82.64)\rangle$	$\langle(22, 25, 40); (20, 24, 27);$ $(23, 25, 29)\rangle$	$\langle(96.77, 97.27, 110.39);$ $(96.77, 96.89, 105.14);$ $(99.76, 108.62, 115.27)\rangle$
D11	$\langle(22.90, 27.71, 35.56);$ $(22.90, 26.45, 31.28);$ $(27.92, 34.62, 39.41)\rangle$	$\langle(20, 23, 26); (21, 22, 24);$ $(22, 25, 30)\rangle$	$\langle(171.53, 182.46, 384.99);$ $(171.12, 178.65, 210.34);$ $(175.59, 270.65, 400.12)\rangle$
D12	$\langle(58.41, 59.12, 60.61);$ $(58.08, 58.12, 58.20);$ $(60.49, 60.87, 61.24)\rangle$	$\langle(25, 31, 37); (29, 30, 32);$ $(30, 35, 52)\rangle$	$\langle(59.87, 66.22, 117.37);$ $(59.64, 61.81, 91.57);$ $(61.11, 115.39, 121.39)\rangle$
D13	$\langle(66.97, 67.68, 69.17);$ $(66.64, 66.68, 66.76);$ $(69.05, 69.43, 69.80)\rangle$	$\langle(20, 27, 31); (23, 26, 28);$ $(24, 30, 46)\rangle$	$\langle(96.97, 103.32, 154.47);$ $(96.74, 98.91, 128.67);$ $(98.21, 152.49, 158.50)\rangle$

TABLE 5. Efficiency Score of the DMUs

DM's Preference	DMUs	Efficiency Score						Ranking
		$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$	Mean	
$(\delta, \rho, \eta) = (1, 0, 0)$	D1	1	0.703	0.6827	0.6763	0.6731	0.74702	10
	D2	0.8808	0.8957	0.916	0.9126	0.9095	0.90292	5
	D3	1	1	1	1	1	1	1
	D4	1	0.6057	0.6456	0.6532	0.6566	0.71222	11
	D5	0.582	0.9747	1	1	1	0.91134	4
	D6	0.4806	0.948	1	1	1	0.88572	6
	D7	1	0.8354	0.8061	0.7978	0.7937	0.8466	8
	D8	1	1	1	1	1	1	1
	D9	1	0.936	0.9892	1	1	0.98504	2

	D10	1	1	0.9501	0.9306	0.919	0.95994	3
	D11	1	1	1	1	1	1	1
	D12	0.6187	0.8863	0.9223	0.9316	0.9336	0.8585	7
	D13	0.582	0.7698	0.8055	0.8165	0.8206	0.75888	9
$(\delta,\rho,\eta)=(\frac{1}{2},\frac{1}{4},\frac{1}{4})$	D1	1	0.6929	0.6781	0.6736	0.6713	0.74318	10
	D2	1	0.8486	0.8483	0.8474	0.8467	0.8782	5
	D3	1	1	1	1	1	1	1
	D4	0.8889	0.5962	0.6006	0.6047	0.607	0.65948	11
	D5	0.6375	0.9041	0.915	0.9185	0.9202	0.85906	6
	D6	0.6327	1	1	1	1	0.92654	3
	D7	1	0.8525	0.816	0.8037	0.7979	0.85402	7
	D8	1	1	1	1	1	1	1
	D9	0.8183	0.9418	0.9585	0.9639	0.9666	0.92982	2
	D10	1	0.9175	0.8957	0.8888	0.8854	0.91748	4
	D11	1	1	1	1	1	1	1
	D12	0.6646	0.8427	0.8657	0.8754	0.8808	0.82584	8
	D13	0.6646	0.7452	0.7656	0.7741	0.7788	0.74566	9
$(\delta,\rho,\eta)=(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	D1	1	0.6895	0.6766	0.6726	0.6707	0.74188	10
	D2	1	0.8219	0.8221	0.8217	0.8214	0.85742	6
	D3	1	1	1	1	1	1	1
	D4	1	0.5869	0.5865	0.5876	0.5893	0.67006	12
	D5	0.6182	0.8696	0.8834	0.8882	0.8905	0.82998	8
	D6	0.7943	1	1	1	1	0.95886	3
	D7	1	0.8583	0.8199	0.807	0.8005	0.85714	7
	D8	1	1	1	1	1	1	1
	D9	0.7916	0.9381	0.9466	0.9493	0.9507	0.91526	4
	D10	1	0.8912	0.8794	0.8757	0.874	0.90406	5
	D11	1	1	1	1	0.9955	0.9991	2
	D12	0.6353	0.8223	0.8455	0.8553	0.8606	0.8038	9
	D13	0.6525	0.7335	0.7518	0.7596	0.7639	0.73226	11
$(\delta,\rho,\eta)=(0,0,1)$	D1	1	0.6747	0.6712	0.67	0.6694	0.73706	9
	D2	0.9612	0.6921	0.7016	0.7063	0.7088	0.754	7
	D3	1	1	1	1	1	1	1
	D4	0.6007	0.5211	0.5332	0.5386	0.5412	0.54696	10
	D5	1	0.8298	0.8077	0.8036	0.8017	0.84856	4
	D6	1	1	1	1	1	1	1
	D7	1	0.8883	0.8581	0.8479	0.8428	0.88742	3
	D8	1	1	1	1	1	1	1
	D9	1	0.9575	0.9315	0.9231	0.9189	0.9462	2
	D10	0.5714	0.8732	0.8783	0.88	0.8808	0.81674	6
	D11	1	1	1	1	1	1	1
	D12	1	0.7787	0.7705	0.7681	0.767	0.81686	5

	D13	1	0.695	0.6862	0.6853	0.6854	0.75038	8
$(\delta, \rho, \eta) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$	D1	0.8799	0.6959	0.6829	0.6785	0.6763	0.7227	7
	D2	1	0.8793	0.8663	0.8616	0.8592	0.89328	3
	D3	1	1	1	1	1	1	1
	D4	1	0.6968	0.671	0.6617	0.657	0.7373	6
	D5	0.1528	0.7717	0.857	0.891	0.9102	0.71654	9
	D6	1	1	1	1	1	1	1
	D7	1	0.8499	0.8002	0.7838	0.7757	0.84192	4
	D8	1	1	1	1	1	1	1
	D9	0.2363	0.901	0.9133	0.9185	0.9211	0.77804	5
	D10	0.2158	0.8411	0.8559	0.8609	0.8634	0.72742	6
	D11	1	0.9837	0.9435	0.9314	0.9257	0.95686	2
	D12	0.1528	0.7988	0.8613	0.8846	0.8968	0.71886	8
	D13	0.2026	0.7333	0.7748	0.79	0.7978	0.6597	10
$(\delta, \rho, \eta) = (0, 1, 0)$	D1	1	0.6899	0.678	0.674	0.672	0.74278	10
	D2	1	0.836	0.8324	0.8309	0.8301	0.86588	6
	D3	1	1	1	1	1	1	1
	D4	1	0.6015	0.6017	0.6025	0.6032	0.68178	12
	D5	0.4948	0.8477	0.8775	0.8884	0.8941	0.8005	8
	D6	0.743	1	1	1	1	0.9486	3
	D7	1	0.8562	0.814	0.7998	0.7931	0.85262	7
	D8	1	1	1	1	1	1	1
	D9	0.699	0.9294	0.9389	0.9421	0.9436	0.8906	5
	D10	1	0.8803	0.8743	0.8726	0.8718	0.8998	4
	D11	1	1	1	0.9912	0.9791	0.99406	2
	D12	0.5075	0.8157	0.849	0.8621	0.8691	0.78068	9
	D13	0.5456	0.7328	0.7572	0.7669	0.772	0.7149	11

The relative efficiencies of the 13 DMUs (or hospitals) are evaluated within a neutrosophic environment by solving the crisp LP model, as described in equation (14), which corresponds to the Neu-DEA model. Table 5 Table 5 shows the outcomes obtained by solving the suggested Neu-DEA model in MATLAB R2021a., which determines the relative efficiency of all DMUs across all degrees of risk.

The performance of DMUs is calculated by varying the risk parameter according to each DM's preference level. Table 5 demonstrates how the DMUs' performance is influenced by the DM's preference level and risk factor. The DMUs are ranked according to their efficiency by calculating the average efficiency for different risk factors corresponding to each DM's preference level. When the mean efficiency of a DMU is 1, it implies that the DMU is performing efficiently across all risk factors and is marked in blue color. Such DMUs are referred to as "fully efficient". On the other hand, when a DMU performs inefficiently across all risk factors, it is labeled in red color and called "fully inefficient". However, some DMUs may perform efficiently under some risk factors but not all. These DMUs are considered "partially efficient" or "partially inefficient." When considering optimistic, neutral, or pessimistic decisions, it is observed that DMUs D3 and D8 consistently maintain their efficiency across different risk factors.

As a result, $D3$ and $D8$ can be considered the top-performing DMUs among all the others. This finding holds true even when analyzing the neutral DM's preference level.

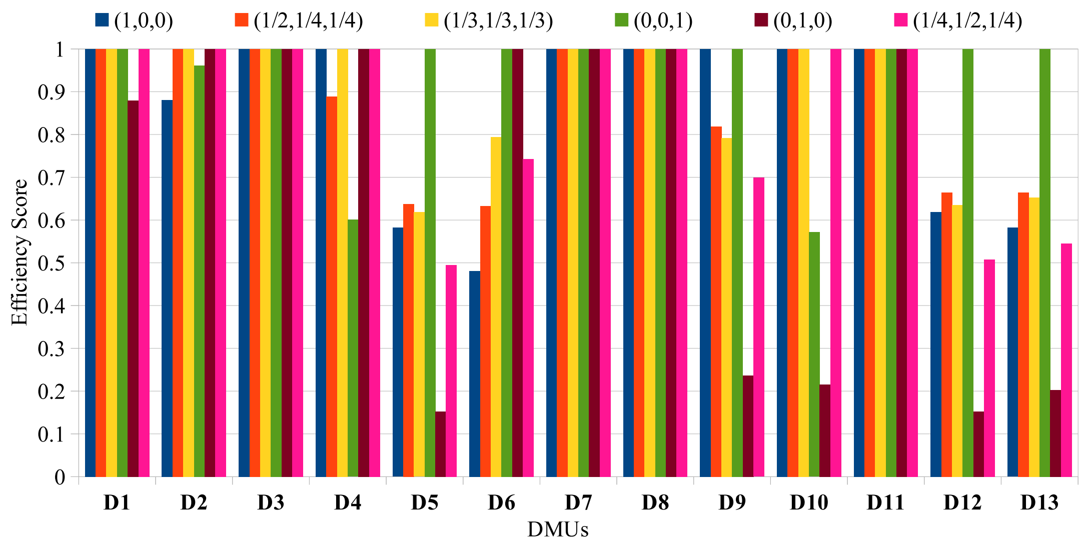

FIGURE 3. Efficiency Score of different preference level with risk factor $\lambda = 0$

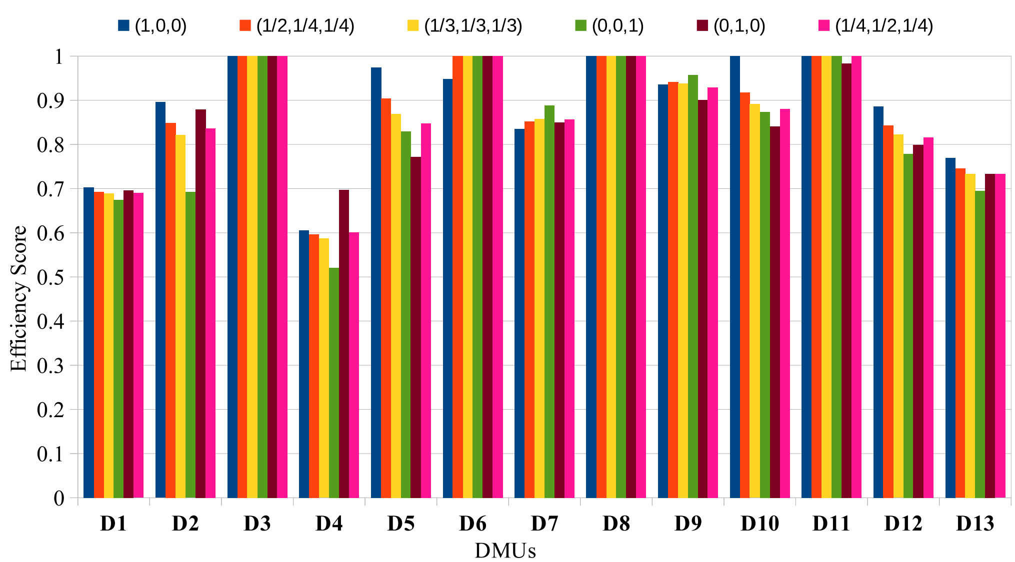

FIGURE 4. Efficiency Score of different preference level with risk factor $\lambda = 0.25$

In Figure 3, the efficiency of DMUs is compared across different preference levels, with a fixed risk factor of $\lambda = 0$. Approximately 60% of the DMUs are found to be efficient. Figure 4 examines the efficiency of DMUs across different preference levels, using a fixed risk factor of $\lambda = 0.25$. It is observed that approximately 29% of the DMUs are efficient. Similarly, Figure 5 analyzes the efficiency of DMUs with various preference levels, keeping the risk factor fixed at $\lambda = 0.5$. Approximately 30% of the DMUs

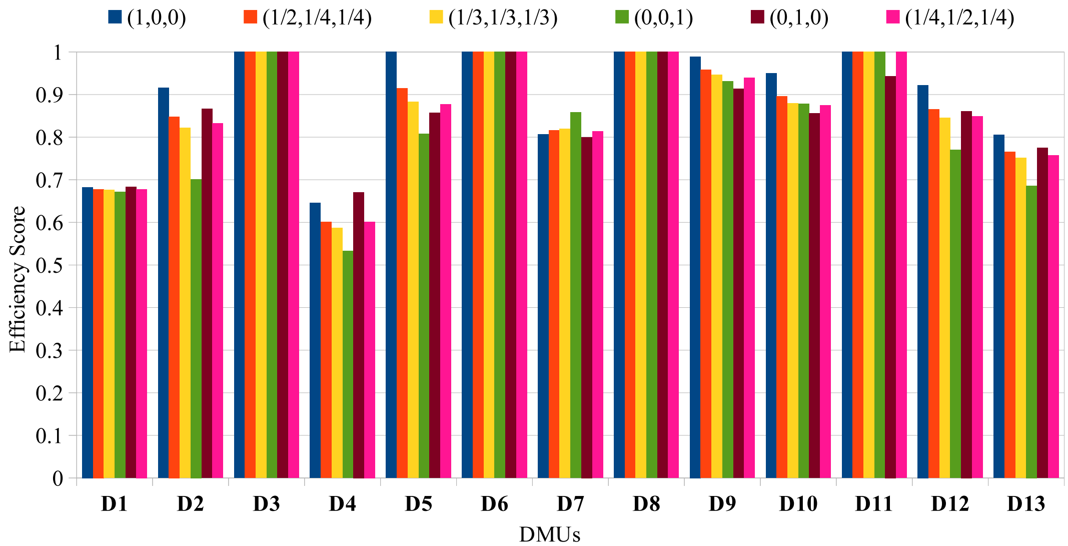

FIGURE 5. Efficiency Score of different preference level with risk factor $\lambda = 0.5$

are determined to be fully efficient. Figure 6 compares the efficiency of DMUs at various preference

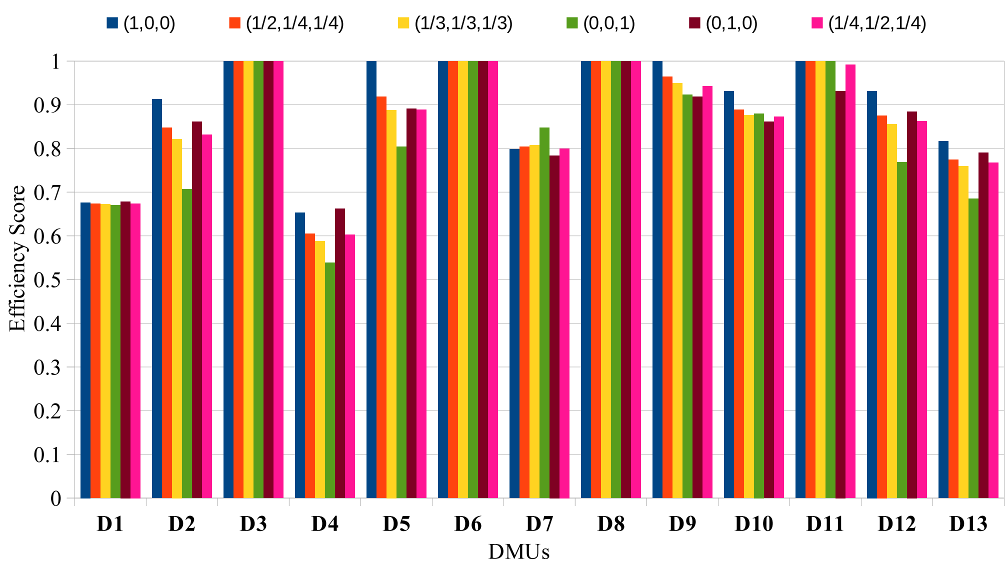

FIGURE 6. Efficiency Score of different preference level with risk factor $\lambda = 0.75$

levels while keeping a constant risk factor of $lambda = 0.75$. It has been found that around 30% of the DMUs are fully efficient. Figure 7 compares the efficiency of DMUs across various preference levels with a constant risk factor of $lambda = 1$. It has been shown that around 29% of DMUs are fully efficient. When considering the mean efficiency of the DMUs, it is found that approximately 22% of the DMUs are fully efficient. These results demonstrate the impact of both the risk factor and preference level on efficiency of DMUs. Notably, as the risk factor increases, the number of efficient DMUs tends to decrease.

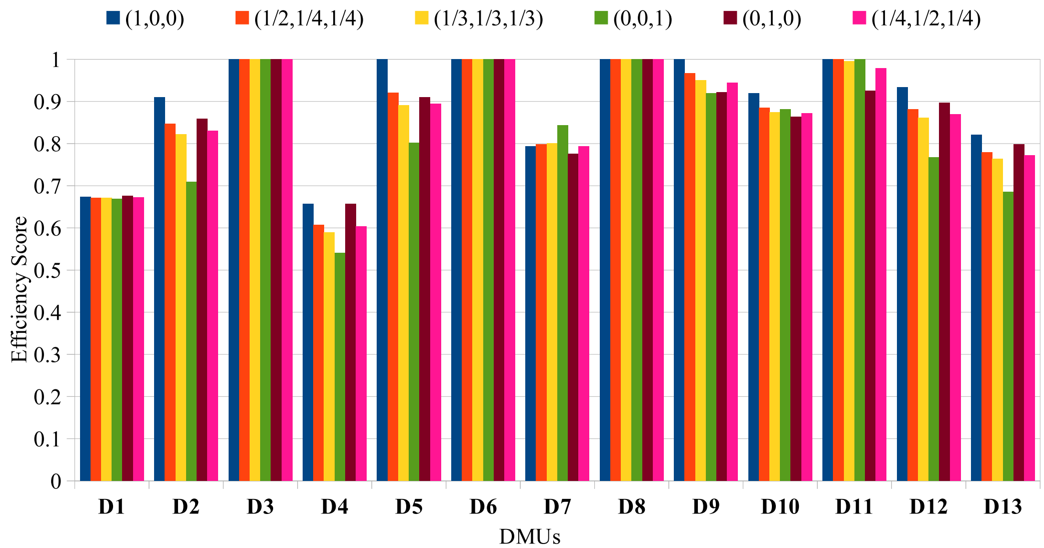

FIGURE 7. Efficiency Score of different preference level with risk factor $\lambda = 1$

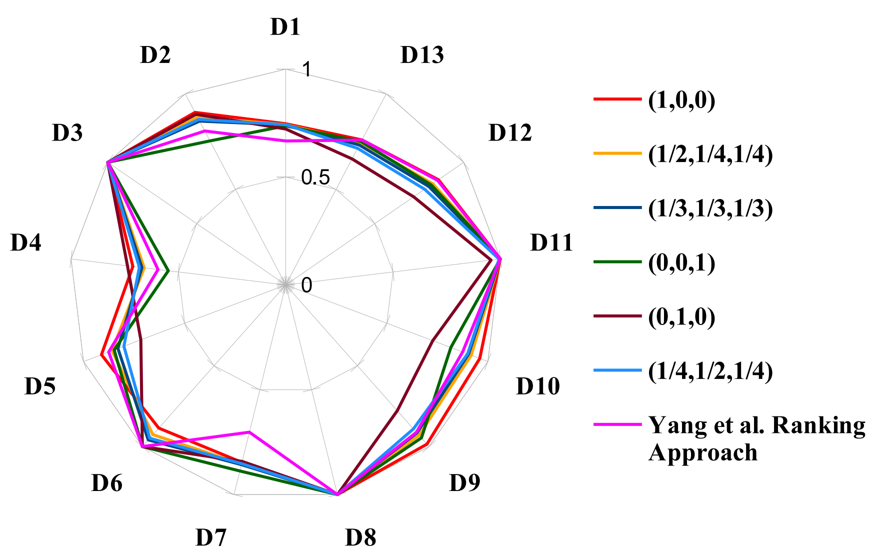

FIGURE 8. Compare the mean efficiency scores of DMUs in different preference level with Yang et al. [45] ranking approach

The DMUs are evaluated and ranked based on their mean efficiency scores while considering a fixed DM's preference level. The DMUs that achieve an efficiency score of 1 are identified as the most efficient and are assigned a rank of 1, indicating their top performance. Conversely, DMUs with efficiency scores below 1 are deemed inefficient, and their rankings are determined by their relative efficiency scores.

DMUs with higher relative efficiency scores receive higher rankings, whereas those with lower relative efficiency scores are ranked lower (See Table 5).

Figure 8 presents a comparison between the mean efficiency of DMUs obtained through the proposed solution technique and the efficiency scores derived from the method developed by Yang et al. [45]. The results show that DMUs D3 and D8 have the same efficiency score in both methods. However, for other DMUs, their efficiency scores differ based on the DM's preference level. The proposed solution technique is deemed more effective than the existing ranking approach since it allows the DM to obtain the efficiency of the DMUs according to their own preference level and risk factor. This gives the DM greater freedom to set their own preferences level and risk parameter while making a decision.

7. Advantages and Limitations of this Study

The key advantage of the suggested ranking function is that it enables DMs to evaluate their own degree of risk while making a decision. The ranking function is established by incorporating the value and ambiguity index, and it is related with the preference level of the DM, which indicates whether the DM prefers a pessimistic, optimistic, or neutral a decision. Also, the ranking function is associated with the risk factor which show the risk taking attitude of the DM. The suggested ranking function is utilised to solve the Neu-DEA model by converting it into an equivalent crisp DEA model that can be solved using existing LP approaches. The efficiency score of the DMUs is defined by a specific preference and risk level. This preference level and risk factor provide DMs with more freedom when analysing performance and making decisions. As a result, the suggested approach to solving the Neu-DEA model gives DMs greater freedom to consider their degree of risk and preferences when making decisions, allowing them to make better informed choices that are consistent with their objectives and values.

The present study is a case analysis that aims to measure the performance of hospitals. The investigation focuses on a small number of input and output variables, specifically two inputs and three outputs. However, the overall performance of hospitals is affected by a variety of variables that relate to either their inputs or their outputs. Therefore, the efficiency score may be changed by increasing or decreasing the number of inputs or outputs. The DMUs are ranked according to the mean efficiency scores of the DMUs with various risk factors for each DM's preference level, but this approach does not completely rank the DMUs, which is shown in the Table 5. Furthermore, the inputs of the DMUs may be interconnected with various internal structures rather than having a direct relationship with the final output. This suggested model only depends on the initial inputs and ultimate outputs, disregarding the internal structure of the DMUs. Therefore, while evaluating the performance of the DMUs, it is essential to take into account every relevant factor and internal DMU structures since these factors may have a big influence on the DMUs' rankings and efficiency evaluations. The suggested model is valuable, but it has limits in terms of its abilities to rank DMUs completely and accurately as well as the model does not consider the complex internal structures of DMUs.

8. Conclusions and Future Directions

The Neutrosophic Set is a generalized version of ordinary fuzzy sets, Intuitionistic fuzzy sets, Pythagorean fuzzy sets, and Spherical fuzzy sets [41]. It is becoming a famous scholarly topic because it is used to solve many problems in decision-making, data analysis, and artificial intelligence. The neutrosophic set has developed as a valuable technique for dealing with imprecise, indeterminate and inconsistent data. This set gives a more flexible way to analyse data, where data can be described using multiple combinations of truth, indeterminacy, and falsity membership functions. The neutrosophic set may reflect the complexities of real-world situations and give a more realistic representation of the underlying data by including these three variables. This research focuses on the construction of a ranking function for a SVTNN that integrates both the value and ambiguity indexes, as well as the DM's preference level and risk variables. Additionally, a novel solution technique is developed for the Neutrosophic DEA models with SVTNN inputs and outputs. The proposed ranking function converts the Neu-DEA model into an equivalent crisp LP model which is solved with different preference levels and risk parameters to measure the relative efficiency of the DMUs. The study examines how the DM's preference level and risk factor together affect the efficiency of the DMUs. Finally, to demonstrate the applicability and validity of the proposed model, a numerical example is presented and the efficiency scores are compared with Yang et al.'s ranking approach [45], as shown in Figure 8. It has been observed that DMU D3 and D8 are the most efficient DMUs, whereas D12 and D13 are the least efficient DMUs, and the number of efficient DMUs decreases as the risk factor increases. For a fixed DM's preference level, DMUs are ranked according to the mean efficiency score with different risk factors.

Future research directions can focus on addressing the limitations identified in the previous section. One potential area of investigation is the increase of the number of inputs and outputs to improve the accuracy of efficiency measurement. Additionally, there is a need to develop complete ranking techniques that can be provide a complete ranking of DMUs in the proposed work. Furthermore, the application of the network DEA model in a neutrosophic setting can be explored to calculate the efficiency of DMUs while considering the internal structure of the system. This novel technique gives encouraging outcomes and can be utilized to solve various additional DEA models, including "BCC, Super Efficiency, and Undesirable DEA, and Dynamic DEA models", by incorporating SVTNN inputs and outputs. Also as an application of this proposed approach employed for addressing MCDM, LP, agricultural economic, assignment problem, banking and finance, manufacturing and production, and transportation problems in neutrosophic environments.

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On Neutro Variable Q-subalgebras

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Abstract. This paper presents the unprecedented opinion of neutro Q algebra. Neutro Q algebras have a complex structure and are based on the three axioms of neutro algebras. We investigated the important properties of neutro Q algebras of orders 2 and 3. we try to find a connection between neutro Q algebras and groups and neutro groups. The opinions of very thin neutro Q algebras and valued-strong neutro Q algebras are introduced in this study and are computed in the numbers of 3-strong neutro Q algebras.

Keywords: Neutro Q algebra, very thin neutro Q algebra, valued-strong Q algebra.

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1. Introduction

Florentin Smarandache [9], according to the basic realism in the application of mathematical problems in the real world, to the introduction of the theory of neuto algebra the payment. From his point of view, whichever is a real point of view, it was noticed that all the real problems of the world are not based on pure mathematical rules. They look at this issue in such a way that in every finite or infinite set, there are elements that remain unknown or cannot accept or apply some mathematical principles. In classical algebra, a set of elements must apply to a series of specific rules, and this issue is mandatory, and this issue contradicts the real world because, in the real world, there are elements that do not follow any rules or always remain in an unknown state. They stay so the theory of neuto algebra it as it may bea new step in this field so that we know that we don't have to look at the real issues of the world in a pure and forced way. The neutro algebras are very important in the real world and some researchers have investigated these soups such as neutro groups [1], neutro BCK algebras [2], neutro hyper BCK subalgebras [3], on neutro D subalgebras [4], semihypergroups [6], on neutro BE algebras and anti BE algebras [7], CI algebra [8], neutro algebraic structures [10] antialgebraic and semigroups [11].

In this survey, we introduce a new extension of Q algebras, whichever is a generalization of groups. Our goal in presenting this topic is to design the principles of the topic in such a way as to challenge the topic of group theory. Our idea is to suggest that in neutro Q algebras, we can confuse the number of null elements and the number of invertible elements and check that it is not necessary entire elements to be participative. In final, we define the notation of very thin neutro Q algebras and k-strong neutro Q algebras and bring up the relation between groups, neutro groups, and neutro Q algebras.

2. **Preliminaries**

In what follows, we present the topics that we need in our research.

Definition 2.1. [9] Let $X \neq \emptyset$. Then (X, κ) is a neutro-algebra, if κ be a neutro operation or an operation, whichever is satisfied in the neutro axioms.

Definition 2.2. [5] Let $X \neq \emptyset$, $\kappa : X^2 \to X$ and ι be a steady. Then, (X, κ, ι) is titled a *Q-algebra* if,

(D-1) $\kappa(x, x) = \iota$,
(D-2) $\kappa(x, \iota) = x$,
(D-3) $\kappa(\kappa(x, y), z) = \kappa(\kappa(x, z), y)$.

3. **Neutro Q algebras**

In this section, characterize neutro Q algebras and inspect their properties.

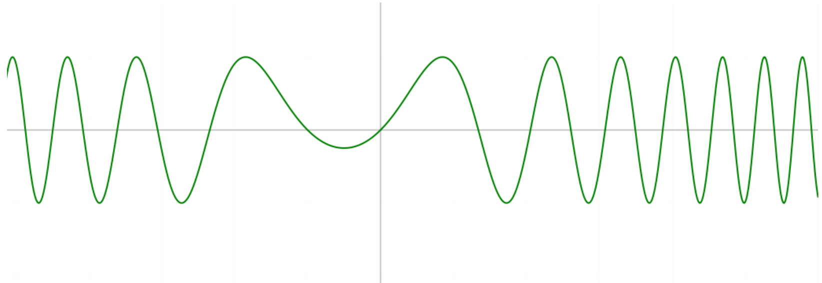
Definition 3.1. Let $X \neq \emptyset$, $\iota \in X$ be a steady and $\kappa : X^2 \to X$. Then $(X, \kappa\ , \iota)$ is a neutro Q algebra, if

(NQ-1) $\big(\exists x \in X$ with the aim that $x\kappa\ x = \iota\big)$ and $\big(\exists y \in X$ with the aim that $y\kappa\ y \neq \iota$ or indeterminate $\big)$;
(NQ-2) $\big(\exists x \in X$ with the aim that $x\kappa\ \iota = x\big)$ and $\big(\exists y \in X$ with the aim that $y\kappa\ \iota \neq y$ or indeterminate $\big)$;
(NQ-3) $\big(\exists x, y, z \in X$, with the aim that $(x\kappa\ y)\kappa\ z = (x\kappa\ z)\kappa\ y\big)$ and $\big(\exists r, s, t \in X$, with the aim that $(r\kappa\ s)\kappa\ t \neq (r\kappa\ t)\kappa\ s$ or indeterminate $\big)$.

Example 3.2. (i) $(\mathbb{R}, -, \iota)$ is a not a neutro Q algebra, while $(\mathbb{R}, \kappa\ , \iota)$ is a neutro Q algebra, where for $x, y \in \mathbb{R}$, characterize $x\kappa\ y = Sin(x + xy)$ as shown in Figure 1.

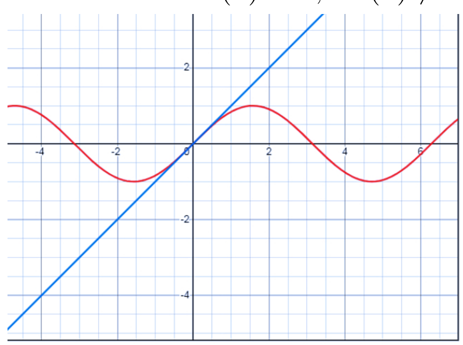
By Figure 1, one can see that there would be $x \in \mathbb{R}$, with the aim that $x\kappa\ x = \iota$ and there would be $y \in \mathbb{R}$, with the aim that $y\kappa\ y \neq \iota$. In addition, for any given $x, y, z \in \mathbb{R}, (x\kappa\ y)\kappa\ z = (x\kappa\ z)\kappa\ y$ if and only if $sin((sin(x+xy)+zsin(x+xy)) = sin((sin(x+xz)+ysin(x+xz))$. It

FIGURE 1. $x\kappa\ x = sin(x + x^2)$



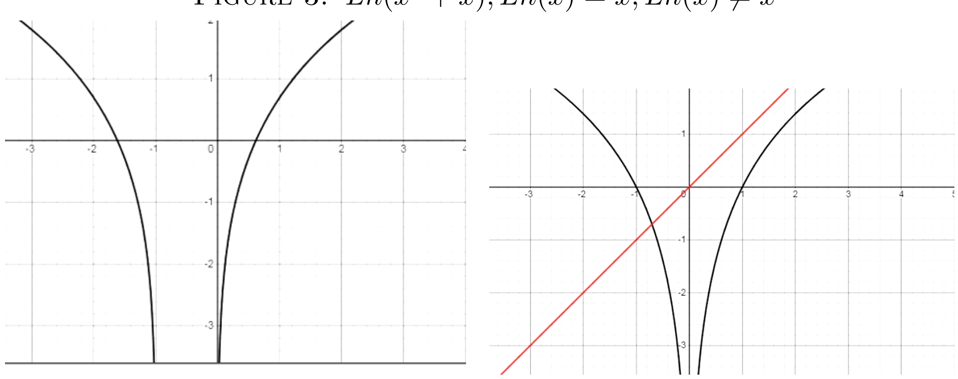
follows that if $z = y$, then $(x\kappa\ y)\kappa\ z = (x\kappa\ z)\kappa\ y$ and one can find $x, y, z \in \mathbb{R}$ with the aim that $(x\kappa\ y)\kappa\ z \neq (x\kappa\ z)\kappa\ y$. In addition, $x\kappa\ \iota = \iota$, implies that $sinx = \iota$, whichever $x = k\pi, k \in \mathbb{Z}$ and $x\kappa\ \iota \neq \iota$, implies that $x \neq k\pi, k \in \mathbb{Z}$. Moreover, $x\kappa\ \iota = x$, implies that $sinx = x$ and $x\kappa\ \iota \neq x$, implies that $sinx \neq x$. The axiom NQ-2 is valid by Figure 2.

FIGURE 2. $sin(x) = x, sin(x) \neq x$



(ii) $(\mathbb{R}, \kappa\ , \iota)$ is a neutro Q algebra, whichever for $x, y \in \mathbb{R}$, characterize $x\kappa\ y = Ln(x^2 + y)$. It is clear that $\iota\kappa\ \iota$ is indeterminate, $x\kappa\ \iota = Ln(x^2)$ and $x\kappa\ x = Ln(x^2 + x)$.

FIGURE 3. $Ln(x^2 + x), Ln(x) = x, Ln(x) \neq x$



By Figure 3, the axioms NQ-1 and NQ-2 are valid. In addition, for $x, y, z \in \mathbb{R}, (x\kappa\ y)\kappa\ z = (x\kappa\ z)\kappa\ y$ iff $Ln(Ln^2(x^2 + y) + z) = Ln(Ln^2(x^2 + z) + z)$. It is clear that for $y = z$, we have $(x\kappa\ y)\kappa\ z = (x\kappa\ z)\kappa\ y$ and for $y = z$, we have $(x\kappa\ y)\kappa\ z \neq (x\kappa\ z)\kappa\ y$ or indeterminate.

(iii) Let $X = \{\iota, 1, 2, 3, 4, 5\}$. Characterize κ on X in kind:

κ	ι	1	2	3	4	5
ι	ι	ι	ι	ι	ι	5
1	1	ι	1	1	1	5
2	2	2	ι	2	2	3
3	3	3	3	ι	3	ι
4	4	4	4	4	ι	1
5	ι	2	ι	2	ι	5

Computations show that $(\{\iota, 1, 2, 3, 4\}, \kappa\ )$ is a Q algebra and $(\{\iota, 1, 2, 3, 4, 5\}, \kappa\ )$ is a neutro Q algebra.

Theorem 3.3. *Any Q algebra, as it may be lengthen to a neutro Q algebra.*

Proof. Let $(X, \kappa\ , \iota)$ be a Q algebra and $\alpha \notin X$. Then $(X \cup \{\alpha\}, \kappa\ ', \iota)$ is a neutro Q algebra, whichever entire $x, y \in X, x\kappa\ 'y = x\kappa\ y$, $\alpha\kappa\ '\alpha = \alpha, \alpha\kappa\ '\iota = x$, whichever $x \notin \{\iota, \alpha\}$ and $x\kappa\ '\alpha \neq x$. Since $(\alpha\kappa\ '\iota)\kappa\ '\alpha = x\kappa\ '\alpha$, $(\alpha\kappa\ '\alpha)\kappa\ '\iota = \alpha\kappa\ '\iota = x$ and $x\kappa\ '\alpha \neq x$, we acquire that $(\alpha\kappa\ '\iota)\kappa\ '\alpha \neq (\alpha\kappa\ '\alpha)\kappa\ '\iota$. Hence $(X \cup \{\alpha\}, \kappa\ ', \iota)$ is a neutro Q algebra. □

Theorem 3.4. *Let $|X| = 2$, $\iota \in X$ and $(X, \kappa\ , \iota)$ is a neutro Q algebra.*

(i) If $\iota\kappa\ \iota = \iota$, then exists $x \in X$ with the aim that $\iota\kappa\ x = x$ or is indeterminate.

(ii) If $\iota\kappa\ \iota \neq \iota$, then one can find $x \in X$ with the aim that $\iota\kappa\ x \in \{\iota, x\}$ or is indeterminate.

Proof. (i) Since, $\iota \in X$, by NQ-1 one can find $x \in X$, with the aim that $x\kappa\ x = \iota$ and one can find $y \in X$ with the aim that $y\kappa\ y \neq \iota$ or is indeterminate. Assume that $\iota\kappa\ \iota = \iota$. In this case one can find just one $\iota \neq x \in X$ with the aim that $x\kappa\ x \neq \iota$ or is indeterminate. Since $|X| = 2$, we acquire $x\kappa\ x = x$ or is indeterminate. Moreover, by NQ-2, one can find $x \in X$ with the aim that $x\kappa\ \iota \neq x$, because of $\iota\kappa\ \iota = \iota$. Since $|X| = 2$, we acquire $x\kappa\ \iota = \iota$ or is indeterminate. In addition by NQ-3, $\iota = \iota\kappa\ \iota = (\iota\kappa\ \iota)\kappa\ \iota = (\iota\kappa\ \iota)\kappa\ \iota = \iota$, because of $\iota\kappa\ \iota = \iota$. Now bring up the consecutive cases:

Case 1:

$$(x\kappa\ x)\kappa\ \iota = (x\kappa\ \iota)\kappa\ x \Rightarrow x\kappa\ \iota = \iota\kappa\ x \Rightarrow \iota\kappa\ x = \iota \text{ or is indeterminate.}$$

Case 2:

$$(\iota\kappa\ x)\kappa\ \iota = (\iota\kappa\ \iota)\kappa\ x \Rightarrow (\iota\kappa\ x)\kappa\ \iota = \iota\kappa\ x \text{ or is indeterminate.}$$

In this case, if $\iota\kappa\ x = \iota$, then $\iota = \iota\kappa\ \iota = \iota\kappa\ x$ and so $\iota\kappa\ x = \iota$ or is indeterminate. If $\iota\kappa\ x = x$, then $x\kappa\ \iota = \iota\kappa\ x$ and so $\iota\kappa\ x = \iota$ or is indeterminate. They follow that NQ-3 is not valid and it is a contradication. Hence we must bring up the consecutive cases:

Case 1:

$$(x\kappa\ x)\kappa\ x \neq (x\kappa\ \iota)\kappa\ x \Rightarrow x\kappa\ \iota \neq \iota\kappa\ x \Rightarrow \iota\kappa\ x \neq \iota \Rightarrow \iota\kappa\ x = x \text{ or is indeterminate.}$$

Case 2:

$$(\iota\kappa\ x)\kappa\ \iota \neq (\iota\kappa\ \iota)\kappa\ x \Rightarrow (\iota\kappa\ x)\kappa\ \iota \neq \iota\kappa\ x \text{ or is indeterminate.}$$

In this case, if $\iota\kappa\ x = \iota$, then $\iota = \iota\kappa\ \iota \neq \iota\kappa\ x$ and so $\iota\kappa\ x \neq \iota$ or ($\iota\kappa\ x = x$ or is indeterminate). If $\iota\kappa\ x = x$, then $x\kappa\ \iota \neq \iota\kappa\ x$ and so $\iota\kappa\ x \neq \iota$ or ($\iota\kappa\ x = x$ or is indeterminate). They concludee that $\iota\kappa\ x = x$ or is indeterminate.

(ii) Since $|X| = 2$ and $\iota\kappa\ \iota \neq \iota$, we acquire there would be just one $\iota \neq x \in X$ such that $\iota\kappa\ \iota = x$ and so $x\kappa\ x = \iota$ or is indeterminate. Moreover, by NQ-2, there would be $x \in X$ with the aim that $x\kappa\ \iota = x$, because of $\iota\kappa\ \iota \neq \iota$. Since $|X| = 2$, we acquire $x\kappa\ \iota = x$ or is indeterminate. In addition by NQ-3, $(\iota\kappa\ \iota)\kappa\ \iota = x\kappa\ \iota = x$, because of $\iota\kappa\ \iota = x$. Now, bring up the consecutive cases:

Case 1:

$$(x\kappa\ x)\kappa\ \iota = (x\kappa\ \iota)\kappa\ x \Rightarrow \iota\kappa\ \iota = x\kappa\ x \Rightarrow x = \iota, \text{whichever is contradication.}$$

Case 2:

$$(\iota\kappa\ x)\kappa\ \iota = (\iota\kappa\ \iota)\kappa\ x \Rightarrow (\iota\kappa\ x)\kappa\ \iota = x\kappa\ x \Rightarrow (\iota\kappa\ x)\kappa\ \iota = \iota \text{ or is indeterminate.}$$

In this case, if $\iota\kappa\ x = \iota$, then $\iota\kappa\ \iota = \iota$, whichever is a contradication. If $\iota\kappa\ x = x$, then $x\kappa\ \iota = \iota$, whichever is a contradication. Thus, $(x\kappa\ x)\kappa\ \iota \neq (x\kappa\ \iota)\kappa\ x$ and $(\iota\kappa\ x)\kappa\ \iota \neq (\iota\kappa\ \iota)\kappa\ x$. It concludes that $\iota\kappa\ x \in \{\iota, x\}$ or is indeterminate. □

Corollary 3.5. *Let X be a set and $(X, \kappa\ , \iota)$ be a neutro Q algebra. Then $|X| \geq 2$.*

Theorem 3.6. *Let $(X, \kappa\ )$ be a neutro Q algebra. Then there would be $x, y \in X$ with the aim that $x\kappa\ (x\kappa\ y) \neq x\kappa\ y$ or indeterminate.*

Proof. Let entire $x, y \in X, x\kappa\ (x\kappa\ y) = x\kappa\ y$. Then by $x = y$, we acquire $x\kappa\ (x\kappa\ x) = x\kappa\ x$. Since $(X, \kappa\ )$ is a neutro Q algebra, using NQ-2, we obtain that $x = x\kappa\ \iota = x\kappa\ (x\kappa\ x) = x\kappa\ x = \iota$. It follows that $|X| = 1$, whichever is a contradication by Corollary 3.5. □

Theorem 3.7. *Let $(X, \kappa\ , \iota)$ be a neutro Q algebra.*

(i) *If there would be $x, y \in X$ with the aim that $x\kappa\ (x\kappa\ y) \neq x\kappa\ y$ or indeterminate and $\iota\kappa\ \iota = \iota$, then $|X| \geq 3$.*

(*ii*) *If one can find* $x, y \in X$ *with the aim that* $x\kappa\ (x\kappa\ y) = x\kappa\ y$ *or indeterminate and* $\iota\kappa\ \iota \neq \iota$*, then* $|X| \geq 3$.

Proof. Since $(X, \kappa\ )$ is a neutro Q algebra, by Corollary 3.5, we acquire $|X| \geq 2$. Suppose that $|X| = 2$ and $X = \{\iota, x\}$.

(*i*) Because $\iota\kappa\ \iota = \iota$, we acquire $x\kappa\ x = x$ and $x\kappa\ \iota = \iota$. It follows that $x\kappa\ (x\kappa\ x) \neq x\kappa\ x$.Thus $x\kappa\ x \neq x\kappa\ x$, whichever is a contradication and so $|X| \geq 3$.

(*ii*) Because $\iota\kappa\ \iota \neq \iota$, we acquire $x\kappa\ x = \iota$ and $x\kappa\ \iota = x$. It follows that $x\kappa\ (x\kappa\ x) = x\kappa\ x$.Thus $x\kappa\ \iota = \iota$ and so $x = \iota$, whichever is a contradication and so $|X| \geq 3$. □

Definition 3.8. Let $(X, \kappa\ , \iota)$ be a neutro Q algebra. Then we will call it is a neutro-commutative, if $\big(\exists x, y \in X$ with the aim that $x\kappa\ y = y\kappa\ x\big)$ and $\big(\exists r, s \in X$ with the aim that $r\kappa\ s \neq s\kappa\ r$ or indeterminate $\big)$.

Theorem 3.9. *Let* $(X, \kappa\ , \iota)$ *be a neutro* Q *algebra and* $|X| = 2$.

(*i*) *If* $\iota\kappa\ \iota = \iota$*, then* $(X, \kappa\ , \iota)$ *is neutro-commutative.*

(*ii*) *If* $\iota\kappa\ \iota \neq \iota$*, then* $(X, \kappa\ , \iota)$ *is not neutro-commutative, necessarily.*

(*iii*) *If* $\iota\kappa\ \iota \neq \iota$ *and* $\iota\kappa\ x = \iota$*, then* $(X, \kappa\ , \iota)$ *is neutro-commutative.*

Proof. Let $x \in X$.

(*i*) If $\iota\kappa\ \iota = \iota$, then by Theorem 3.4, we acquire $\iota\kappa\ x = x$ and $x\kappa\ \iota = \iota$.

(*ii, iii*) If $\iota\kappa\ \iota = x$, then by Theorem 3.4, we acquire $x\kappa\ \iota = x$ and $\iota\kappa\ x \in \{\iota, x\}$. Now, if $\iota\kappa\ x = x$, $(X, \kappa\ , \iota)$, then is not neutro-commutative and if $\iota\kappa\ x = \iota$, then $(X, \kappa\ , \iota)$ is neutro-commutative. □

Theorem 3.10. *Let* X *be nonevoid set,* $\iota \in X$ *and* $|X| \geq 2$*. Then* $(X, \kappa\ , \iota)$ *is a neutro* Q *algebra iff* $(X, \kappa\ , \iota)$ *is not a group.*

Proof. Let $(X, \kappa\ , \iota)$ is a group. Then for any $x \in X, x\kappa\ \iota = x$, and so the axiom ($NQ$-2) is not valid. Thus $(X, \kappa\ , \iota)$ is a not neutro Q algebra.

Conversly, let $(X, \kappa\ , \iota)$ is a neutro Q algebra. Then using the axiom (NQ-2), the structure $(X, \kappa\ , \iota)$ has't the identity elemen, so $(X, \kappa\ , \iota)$ is not a group. □

Theorem 3.11. *Every group with involution, as it may be lengthen to a neutro* Q *algebra.*

Proof. Let $(X, \kappa\ , \iota)$ be a group with involution and $\alpha \notin X$. Then $(X \cup \{\alpha\}, \kappa\ ', \iota)$ is a neutro Q algebra, whichever entire $x, y \in X, x\kappa\ 'y = x\kappa\ y$, $\alpha\kappa\ 'x \neq x\kappa\ '\alpha$ and $\alpha\kappa\ '\iota$ is indeterminate. Suppose that $z \in X$ is an involution. Then $z\kappa\ '(\alpha\kappa\ 'z) = (z\kappa\ '\alpha)\kappa\ 'z = (z\kappa\ 'z)\kappa\ '\alpha = \iota\kappa\ '\alpha = \alpha$. It follows that $z\kappa\ '(z\kappa\ '(\alpha\kappa\ 'z)) = z\kappa\ '\alpha$ and so $(z\kappa\ 'z)\kappa\ '(\alpha\kappa\ 'z) = z\kappa\ '\alpha$, whichever is a contadication. Thus one can find $\alpha, z \in X$ with the aim that $(z\kappa\ '\alpha)\kappa\ 'z \neq (z\kappa\ 'z)\kappa\ '\alpha$ and so the axiom NQ-3 is valid. In addition, $z\kappa\ 'z = \iota$, implies that NQ-1 is valid Hence $(X \cup \{\alpha\}, \kappa\ ', \iota)$ is a neutro Q algebra. □

Example 3.12. Bring up the abelian group $(\mathbb{Z}_6, \oplus)$ and $X = \{\bar{\iota}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \sqrt{2}\}$. For every $x, y \in \mathbb{Z}_6$, characterize $x\kappa\ y = x \oplus y$ and $\sqrt{2}\kappa\ \bar{\iota}$ is an indeterminate. It is easey to see that $(X, \kappa\ , \bar{\iota})$ is a neutro Q algebra.

S. Dus et al. introduced the opinion of neutro groups. We recall this consept in kind:

Let X be a non-empty set, $\iota \in X$ be a steady and "κ " be a map on X. An algebra $(X, \kappa\ , \iota)$ of type $(2, \iota)$ is a neutro group, if

(NG-1) $\big(\exists x, y, z \in X$ with the aim that $(x\kappa\ y)\kappa\ z = x\kappa\ (y\kappa\ z)\big)$ and $\big(\exists r, s, t \in X$ with the aim that $(r\kappa\ s)\kappa\ t \neq r\kappa\ (s\kappa\ t)$ or indeterminate $\big)$;

(NG-2) $\big(\exists x \in X$ with the aim that $x\kappa\ \iota = \iota\kappa\ x = x\big)$ and $\big(\exists y \in X$ with the aim that $y\kappa\ \iota \neq y$ or indeterminate $\big)$;

(NG-3) $\big(\exists x, y \in X$, with the aim that $x\kappa\ y = y\kappa\ x = \iota\big)$ and $\big(\exists r \in X$, with the aim that entire $s \in X$, $r\kappa\ s \neq \iota$ or indeterminate $\big)$.

Each above neutro axiom has a degree of equality (T), degree of non-equality (F), and degree of indeterminacy (I), where $(T, I, F) \notin (1, \iota, \iota), (\iota, \iota, 1)$.

Theorem 3.13. *Let $(X, \kappa\ , \iota)$ be a neutro Q algebra whichever satisfies in (NG-1). Then $(X, \kappa\ , \iota)$ is a neutro group.*

Proof. Since $(X, \kappa\ , \iota)$ is a neutro Q algebra, one can find $x \in X$ with the aim that $x\kappa\ \iota = \iota$ and one can find $y \in X$ with the aim that $y\kappa\ y = \iota$. Using (NG-1), we acquire $\iota\kappa\ x = (y\kappa\ y)\kappa\ x = y\kappa\ (y\kappa\ x) = x\kappa\ \iota = \iota$ and so $\iota\kappa\ x = \iota$. It follows that one can find $x \in X$ with the aim that $x\kappa\ \iota = \iota = \iota\kappa\ x$. □

Theorem 3.14. *Let $(X, \kappa\ , \iota)$ be a neutro Q algebra and $|X| = 2$. Then*

(i) *if $\iota\kappa\ \iota = \iota$, then $(X, \kappa\ , \iota)$ is not a neutro group.*
(ii) *if $\iota\kappa\ \iota \neq \iota$ and $\iota\kappa\ x = x$, then $(X, \kappa\ , \iota)$ is a neutro group.*
(iii) *if $\iota\kappa\ \iota \neq \iota$ and $\iota\kappa\ x = \iota$, then $(X, \kappa\ , \iota)$ is not a neutro group.*

Proof. (i) Since $\iota\kappa\ \iota = \iota$, by Theorem 3.4, for $x \in X$, we acquire $x\kappa\ x = x, x\kappa\ \iota = \iota$, and $\iota\kappa\ x = x$. It follows that the axioms NG-1 and NG-2 are valid. Let $x \in X$, then have the consecutive cases:

Case 1:

$$(x\kappa\ \iota)\kappa\ x = \iota\kappa\ x = x = x\kappa\ x = x\kappa\ (\iota\kappa\ x).$$

Case 2:

$$(x\kappa\ x)\kappa\ \iota = x\kappa\ \iota = x\kappa\ (x\kappa\ \iota).$$

Case 3:

$$(\iota\kappa\ x)\kappa\ x = x\kappa\ x = x = \iota\kappa\ x = \iota\kappa\ (x\kappa\ x).$$

Case 4:

$$(\iota\kappa\ \iota)\kappa\ x = \iota\kappa\ x = x = \iota\kappa\ x = \iota\kappa\ (\iota\kappa\ x).$$

Case 5:

$$(\iota\kappa\ x)\kappa\ \iota = x\kappa\ \iota = \iota = \iota\kappa\ \iota = \iota\kappa\ (x\kappa\ \iota).$$

They follow that the axiom NG-3 is not valid and so $(X, \kappa\ , \iota)$ is not a neutro group.

(ii) Let $\iota\kappa\ \iota \neq \iota$. Using theorem 3.4, for $x \in X$, acquire $x\kappa\ x = \iota, x\kappa\ \iota = x$ and $\iota\kappa\ x \in \{\iota, x\}$. If $\iota\kappa\ x = x$, then the axioms NG-2 and NG-3 are valid. Now, bring up the consecutive cases:

Case 1:

$$(x\kappa\ \iota)\kappa\ x = x\kappa\ x = \iota = x\kappa\ x = x\kappa\ (\iota\kappa\ x).$$

Case 2:

$$(x\kappa\ x)\kappa\ \iota = \iota\kappa\ \iota = x \neq \iota = x\kappa\ x = x\kappa\ (x\kappa\ \iota).$$

They follow that NG-1 is valid and so $(X, \kappa\ , \iota)$ is a neutro group.

(iii) Let $\iota\kappa\ \iota \neq \iota$. Using theorem 3.4, for $x \in X$, acquire $x\kappa\ x = \iota, x\kappa\ \iota = x$ and $\iota\kappa\ x \in \{\iota, x\}$. If $\iota\kappa\ x = \iota$, then the axiom NG-2 is not valid and so $(X, \kappa\ , \iota)$ is not a neutro group. □

Definition 3.15. Let $(X, \kappa\ , \iota)$ be a neutro Q algebra. We say that $(X, \kappa\ , \iota)$ is a very thin neutro Q algebra, if for any $\iota \neq x \in X$, $\iota\kappa\ \iota \neq \iota, x\kappa\ x = \iota$ and $x\kappa\ \iota = x$.

Example 3.16. Let $X = \{\iota, 1, 2, 3, 4, 5\}$. Then (X, κ_1, ι) is a very thin neutro Q algebra and (X, κ_2, ι) is a very strong neutro Q algebra in kind:

κ_1	ι	1	2	3	4	5
ι	1	ι	ι	ι	ι	ι
1	1	ι	1	1	1	1
2	2	2	ι	2	2	2
3	3	3	3	ι	3	3
4	4	4	4	4	ι	4
5	5	5	5	5	5	ι

and

κ_2	ι	1	2	3	4	5
ι	ι	ι	ι	ι	ι	ι
1	5	5	1	1	1	1
2	4	2	5	2	2	2
3	2	3	3	5	3	3
4	3	4	4	4	5	4
5	1	5	5	5	5	5

Theorem 3.17. *Let $(X, \kappa\ , \iota)$ be a neutro commutative very thin neutro Q algebra and $|X| \geq 3$. Then $(X, \kappa\ , \iota)$ is a neutro group.*

Proof. Let $x \in X$. Since $(X, \kappa\ , \iota)$ is a very thin neutro Q algebra, acquire entire $\iota \neq x \in X, x\kappa\ x = \iota$ and $x\kappa\ \iota = x$. Since $(X, \kappa\ , \iota)$ is neutro commutative, one can find $y \in X$ with the aim that $\iota\kappa\ y \neq y$. They follow that NG-2 and NG-3 are valid. Because $(x\kappa\ \iota)\kappa\ x = x\kappa\ x = \iota = x\kappa\ x = x\kappa\ (\iota\kappa\ x)$ and $(x\kappa\ x)\kappa\ \iota = \iota\kappa\ \iota \neq \iota = x\kappa\ x = x\kappa\ (x\kappa\ \iota)$, acquire NG-1 is valid. Hence $(X, \kappa\ , \iota)$ is a neutro group. □

Definition 3.18. Let $(X, \kappa\ , \iota)$ be a neutro Q algebra and $k \in \mathbb{N}$. We say that $(X, \kappa\ , \iota)$ is k-strong neutro Q algebra, if one can find $x_1, x_2, \ldots, x_{k-1} \in X$ with the aim that $\iota\kappa\ \iota = x_i\kappa\ x_i = \iota, x_i\kappa\ \iota = \iota\kappa\ x_i = x_i$, implies that $i \in \{1, 2, \ldots, k-1\}$ and for any $i \notin \{1, 2, \ldots, k-1\}, x_i\kappa\ x_i \neq \iota, x_i\kappa\ \iota \neq x_i, \iota\kappa\ x_i \neq x_i$ and are't indeterminate.

Let X be a set and $k \in \mathbb{N}$. Denote $\mathcal{N}(X, Q, k)$ by the set of all k-strong neutro Q algebras $(X, \kappa\ , \iota)$ and $\mathcal{N}(X, G, k)$ by the set of all k-strong neutro groups $(X, \kappa\ , \iota)$.

Theorem 3.19. *Let* $|X| = 3$. *Then* $|\mathcal{N}(X, Q, 2)| = 2^3 \times 3^2$.

Proof. Suppose that $X = \{\iota, a, b\}$. If $(X, \kappa\ , \iota)$ is a 2-strong neutro Q algebras, we have $\iota\kappa\ \iota = a\kappa\ a = \iota$ and $a\kappa\ \iota = \iota\kappa\ a = a$. Noe, characterize a general kayley table in kind:

$\kappa\ _i$	ι	a	b
ι	ι	a	$c(\iota, b)$
a	a	ι	$c(a, b)$
b	$c(b, \iota)$	$c(b, a)$	$c(b, b)$

,

whichever for any $x, y \in X, c(x, y)$ is the number of possible cases for choosing of the $x\kappa\ _i y$. Simple cimputations show that $c(\iota, b) = 2, c(a, b) = 3, c(b, \iota) = 2, c(b, a) = 3$ and $c(b, b) = 2$. Thus $|\mathcal{N}(X, Q, 2)| = 2^3 \times 3^2$. □

Corollary 3.20. *Let* $X \neq \emptyset$. *Then* $\mathcal{N}(X, G, 3) \subseteq \mathcal{N}(X, Q, 3)$.

4. Conclusions

As an important result of this article, we can mention that we have dealt with the connection of the real world of objects with each other under a series of imagined principles of logic and we have shown that even for a finite number of elements we can create an infinite system of rules. Since the structure of neutro Q subalgebra is complex, we investigated the neutro Q subalgebras with order $2, 3$. We show that the Q subalgebras can't be groups and try to make some conditions to be neutro groups. After completing this research, which is an interesting start in the field of neutro algebras, we intend to discuss the relationship of sets of elements with other sets in a real way and under the principles of neutro. In fact, we want to apply these types of algebras in the real world, and by modeling compatible and non-compatible systems with these types of algebras, we want to show the importance of publishing these types of articles.

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Research on a Class of Special Quasi TA-Neutrosophic Extended Triplet: TA-Groups

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Abstract. Tarski associative groupoid (TA-groupoid) and Tarski associative neutrosophic extended triplet groupoid (TA-NET-groupoid) are two interesting structures in non-associative algebra. In this paper, a new concept of TA-group is proposed based on TA-groupoid, as a special quasi TA-Neutrosophic extended triplet, its related properties are investigated and the relationship between TA-group and regular TA-groupoid is described in more detail. Moreover, the decomposition theorem of inverse TA-groupoid is proved. Finally, some concrete examples are provided to reveal that the relations among all kinds of TA-groupoids.

Keywords: Tarski associative groupoid (TA-groupoid); Tarski associative neutrosophic extended triplet groupoid (TA-NET-groupoid); Tarski associative group (TA-group)

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1. **Introduction**

Associative law is a kind of operation law describing symmetry in algebraic systems. Groups and semigroups are two typical algebraic systems which satisfy associative law ([1–3]). In recent years, with the wide application of algebraic systems in various fields ([4–6]), many kinds of non-associative algebraic structures have been studied in order to explore more generalized symmetries and operation laws in algebras. Among them, Abel-Grassmann's groupoid (AG-groupoid), Cyclic associative groupoid (CA-groupoid) and TA-groupoid have been widely discussed.

AG-groupoid([7]) is a non-associative groupoid satisfying the condition $(xy)z = (zy)x$. Based on this research, a series of AG-groupoid satisfying different conditions have been proposed([8–10]). In 1954, the term "cyclic associative law" was used in Sholander's article([11]) to represent the operating law: $(xy)z = (yz)x$. Subsequently, many scholars systematically studied the relevant algebraic structures satisfying cyclic associative law.

If a semigroup satisfies $x(yz) = x(zy)$, it can be called right commutative. On this basis, the association law is added, then the following equation holds: $(xy)z = x(zy)$ (Tarski associative law). Tarski associative law is actually a special case of generalized associative law proposed by Suschkewitsch ([12]) as early as 1929. Accordingly, Xiaohong Zhang proposed the concept of TA-groupoid in 2020 and studied its related properties([13]).

In addition, based on the relevant theory of Neutrosophic set([14]) proposed in 1995, Smarandache put forward a new algebraic structure of the neutrosophic extended triplet group (NETG) ([15]). Subsequently, domestic and foreign scholars carried out a lot of research on this basis. Among them, Xiaohong Zhang clarified some theoretical knowledge of TA-NET-groupoid in 2020 by adding local identity elements and local inverse elements to the NETGs and combining with TA-groupoid, laying a theoretical foundation for the research of related algebraic structures. Xiaogang An et al. concluded that TA-NET-groupoid is a semigroup, and explained the relationship between regular TA-groupoid and Tarski associative neutrosophic extended triplet groupoid (TA-NET-groupoid)([16]). The research of these scholars greatly promoted the further development of algebra. In order to further study the structure of regular TA-groupoid, the TA-group and inverse TA-groupoid are proposed, properties of TA-group and the relations between different TA-groupoids are studied in detail in this paper.

This paper is organized as follows. In Section 2, some basic definitions and properties of TA-groupoid and TA-NET-groupoid are recalled. In TA-groupoid, there is a special class of groupoid with several right identity elements, which we call TA-group. Thus, the concept of TA-group is put forward first in Section 3, which is followed with the discussion of some structural properties of TA-group and the relationship between TA-group and other algebraic structures. We then summarize our paper and indicate the next research direction at last.

2. Preliminaries

Definition 2.1 ([13])**.** If a groupoid $(S, *)$ satisfies Tarski associative law: $\forall x, y, z \in S$, $(x * y) * z = x * (z * y)$, then S is said as a Tarski associative groupoid (shortly TA-groupoid).

A TA-groupoid $(S, *)$ is called locally associative ([13]) if $\forall m \in S$, $(m*m)*m = m*(m*m)$. Then TA-groupoid is locally associative.

Proposition 2.1 ([13])**.** Let $(S, *)$ be a TA-groupoid. Then $\forall a, b, c, d, e, f \in S$,
(1) $(a * b) * (c * d) = (a * d) * (c * b)$;
(2) $((a * b) * (c * d)) * (e * f) = (a * d) * ((e * f) * (c * b))$.

Definition 2.2 ([15, 17])**.** If S is a non-empty set under the binary operation $*$, for any $m \in S$, there are neut(m) and anti(m), s.t. $neut(m) \in S$, $anti(m) \in S$, and $m * neut(m) =$

$neut(m) * m = m$; $m * anti(m) = anti(m) * m = neut(m)$. Then S is said as a neutrosophic extended triplet set.

Annotation: For any $m \in S$, neither $neut(m)$ nor $anti(m)$ is unique. Thus $\{neut(m)\}$ and $\{anti(m)\}$ are used to denote the sets of $neut(m)$ and $anti(m)$, respectively.

Definition 2.3 ([13]). Let $(G, *)$ be a neutrosophic extended triplet set. If
(1) $\forall x, y \in G$, $x * y \in G$;
(2) $\forall x, y, z \in G$, $(x * y) * z = x * (z * y)$.

Then, we say that $(G, *)$ is a Tarski associative neutrosophic extended triplet groupoid (or TA-NET-groupoid). A TA-NET-groupoid satisfying the commutative law is a commutative TA-NET-groupoid.

Theorem 2.1 ([13]). Let $(G, *)$ be a TA-NET-groupoid. Then $\forall w \in G$,
(1) $neut(w) * neut(w) = neut(w)$;
(2) $neut(neut(w)) = neut(w)$;
(3) $anti(neut(w)) \in \{anti(neut(w))\}, w = anti(neut(w)) * w$.

Theorem 2.2 ([13]). Let $(G, *)$ be a TA-NET-groupoid. Then $\forall w \in G$, $\forall p, q \in \{anti(w)\}, \forall anti(w) \in \{anti(w)\}$,
(1) $p * (neut(w)) = neut(w) * q$;
(2) $anti(neut(w)) * anti(w) \in \{anti(w)\}$;
(3) $neut(w) * anti(q) = w * neut(q)$;
(4) $neut(p) * neut(w) = neut(w) * neut(p) = neut(w)$;
(5) $(q * neut(w)) * w = w * (neut(w) * q) = neut(w)$;
(6) $neut(q) * w = w$.

Theorem 2.3 ([13]). If $(G, *)$ is a TA-NET-groupoid. $E(G)$ represents the set composed of all different neutral elements in G, for all $e \in E(G)$, $G(e) = \{a \in G | neut(a) = e\}$. Then,
(1) $G(e)$ is a subgroup of G.
(2) for $\forall e_1, e_2 \in E(G), e_1 \neq e_2 \Rightarrow G(e_1) \cap G(e_2) = \varnothing$.
(3) $G = \cup_{e \in E(G)} G(e)$.

Theorem 2.4 ([16]). A TA-NET-groupoid is a semigroup.

Definition 2.4 ([13]). A TA-groupoid $(G, *)$ is said to be left cancellative, if $x \in G$, $a, b \in G$, $x * a = x * b$ implies $a = b$.

Definition 2.5 ([13]). A TA-groupoid $(G, *)$ is said to be a right cancellative TA-groupoid, if $x \in G$, $a, b \in G$, $a * x = b * x$ implies $a = b$. A groupoid is a cancellative TA-groupoid which is both a left and right cancellative.

Theorem 2.5 ([13]). Let $(G, *)$ be a TA-groupoid. Then

(1) A left cancellative element is a right cancellative element;

(2) Two left cancellative elements are still left cancellative after * operation;

(3) A left cancellative element and a right cancellative element are left cancellative after * operation;

(4) For any $x, y \in G$, if $x * y$ is right cancellative, then y is right cancellative.

Definition 2.6 ([16]). Assume that $(G, *)$ is a TA-groupoid, $a \in G$. Then a is a regular element of G if there exists $x \in G$ such that $a * (x * a) = a$. The TA-groupoid G is said to be regular if all its elements are regular.

Definition 2.7 ([2]). A semigroup S is said to be an inverse semigroup, if there is a unary operation $a \mapsto a^{-1}$ satisfying

$$(a^{-1})^{-1} = a, aa^{-1}a = a,$$

and for all $x, y \in S$,

$$(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}).$$

Theorem 2.6 ([2]). Let S be a semigroup. It is an inverse semigroup iff all its elements have a unique inverse.

3. TA-Group and Inverse TA-Groupoid

In the following, we propose two new concepts of TA-group and inverse TA-groupoid, and investigate their properties and structures.

Definition 3.1. Let $(S, *)$ be a TA-groupoid. Then, S is called a TA-(r,l)-loop, if for any $a \in S$, exist two elements $neut_{(r,l)}(a)$ and $anti_{(r,l)}(a)$ in S satisfying the condition: $a * neut_{(r,l)}(a) = a$, $anti_{(r,l)}(a) * a = neut_{(r,l)}(a)$. That is, $a * (anti_{(r,l)}(a) * a) = a$.

Definition 3.2. Assume that $(G, *)$ is a TA-groupoid. G is said to be a Tarski associative group (or simply TA-group), if

(1) there is a right identity element in G, that is to say, $\exists e \in G$, for all element $a \in G$, $a * e = a$;

(2) there is a certain right identity element $e \in G$, for any $a \in G$, there exists an element $a' \in G$ such that $a' * a = e$.

Obviously, by definition 3.1 and 3.2 we know that TA-group is a special TA-(r,l)-loop.

Exmaple 3.1. Let $G = \{1, 2, 3, 4\}$. In Table 1, the TA-group $(G, *)$ is given. And

$$1 * 1 = 1, 2 * 1 = 2, 3 * 1 = 3, 4 * 1 = 4;$$

$$1 * 1 = 1, 1 * 2 = 1, 3 * 3 = 1, 3 * 4 = 1.$$

At this time, right identity element is 1.

$$1*2=1, 2*2=2, 3*2=3, 4*2=4;$$

$$2*1=2, 2*2=2, 4*3=2, 4*4=2.$$

At this time, right identity element is 2.

Table 1. This is a TA-group.

$*$	**1**	**2**	**3**	**4**
1	1	1	3	3
2	2	2	4	4
3	3	3	1	1
4	4	4	2	2

Theorem 3.1. Let $(G,*)$ be a TA-group, e is a right identity element in G. Then
(1) $(a, a' \in G,\ a' * a = e) \Rightarrow e * a' = a'$;
(2) $(a, a' \in G,\ a' * a = e) \Rightarrow (a * a') * (a * a') = a * a'$;
(3) $(a, a' \in G,\ a' * a = e) \Rightarrow x * (a * a' = x)$ for all $x \in G$.

Proof. (1) In order to obtain the conclusion, suppose that $a, a' \in G$, $a' * a = e$. We have

$$e * a' = (a' * a) * a' = a' * (a' * a) = a' * e = a'.$$

(2) If $a, a' \in G$, $a' * a = e$. Then by (1),

$$(a*a)*a' = (a*a)*(e*a') = (a*a')*(e*a) = ((a*a')*a)*e = (a*a')*a = a*(a*a');$$

$$a = a*e = a*(a'*a) = (a*a)*a'.$$

It follows that $a = (a*a)*a' = a*(a*a')$. On the other hand,

$$a'*a = a'*(a*(a*a')) = (a'*(a*a'))*a;$$

$$\begin{aligned} a' &= e*a' = (a'*a)*a' = ((a'*(a*a'))*a)*a' \\ &= (a'*(a*a'))*(a'*a) = (a'*(a*a'))*e \\ &= a'*(a*a'). \end{aligned}$$

Therefore,

$$\begin{aligned} a*a' &= ((a*a)*a')*(a'*(a*a')) = ((a*a)*(a*a'))*(a'*a') \\ &= ((a*a')*(a*a))*(a'*a') = (a*a')*((a'*a')*(a*a)) \\ &= (a*a')*((a'*a)*(a*a')) = (a*a')*(e*(a*a')) \\ &= ((a*a')*(a*a'))*e = (a*a')*(a*a'). \end{aligned}$$

(3) Assume that $a, a' \in G$, $a' * a = e$. For any $x \in G$, applying (1) we get that $x * (a * a') = (x * (a * a')) * e = x * (e * (a * a')) = x * ((e * a') * a) = x * (a' * a) = x * e = x$. □

Theorem 3.2. Let $(G, *)$ be a TA-groupoid with right identity element. Then it is a semigroup.

Proof. Let $(G, *)$ be a TA-groupoid with right identity element. e is right identity element in G, for any $a, b, c \in G$, there is,

$$\begin{aligned} a * (b * c) &= [a * (b * c)] * e \\ &= a * [e * (b * c)] \\ &= (a * b) * (e * c) \\ &= (a * c) * (e * b) (By Proposition 2.1) \\ &= a * [(e * b) * c] \\ &= a * [e * (c * b)] \\ &= [a * (c * b)] * e \\ &= a * (c * b). \end{aligned}$$

Then according to Tarski associative law, $a * (b * c) = a * (c * b) = (a * b) * c$. That is to say, G satisfies associative law. So G is a semigroup. □

Theorem 3.3. Let $(G, *)$ be a TA-group. Then it is a regular semigroup.

Proof. Because TA-group is a TA-groupoid with right identity element, according to Theorem 3.2, G is a semigroup. Then according to definition of TA-group, there exists $x \in G$ such that $a * (x * a) = a$. So G is regular semigroup. □

But not every regular semigroup is TA-group, see Example 3.2.

Exmaple 3.2. Let $G = \{1, 2, 3, 4\}$. In Table 2, a regular semigroup $(G, *)$ is given.

Table 2. This is a regular semigroup.

*	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	2	3	4
3	3	3	3	3
4	4	4	4	4

But it isn't TA-group since $(2 * 1) * 3 = 1 \neq 3 = 2 * (3 * 1)$.

Theorem 3.4. TA-group is TA-NET-groupoid.

Proof. Let G be a TA-group and e is right identity element of G. Then for all $a \in G$, there exists $x \in G$ such that $x * a = e$. That is, $a * (x * a) = a$. On the basis of Theorem 3.2, $a * (a * x) = a$. Assume that $a * x = neut(a)$ and $x = anti(a)$, there is, $a * neut(a) = a$ and $a * anti(a) = neut(a) = anti(a) * a$. Then $neut(a) * a = (a * x) * a = a * (a * x) = a$. So $neut(a) * a = a = a * neut(a)$ and $anti(a) * a = neut(a) = a * anti(a)$. So G is a TA-NET-groupoid. □

But not every TA-NET-groupoid is TA-group, see Example 3.3.

Exmaple 3.3. Let $G = \{1, 2, 3, 4\}$. Consider a TA-NET-groupoid in Table 3.

$$neut(1) = 1, anti(1) = 1; neut(2) = 2, anti(2) = 2;$$

$$neut(3) = 3, anti(3) = 3; neut(4) = 4, anti(4) = 4.$$

TABLE 3. This is a TA-NET-groupoid.

$*$	**1**	**2**	**3**	**4**
1	1	1	1	1
2	2	2	3	2
3	4	4	3	4
4	4	4	4	4

But it isn't TA-group since there isn't right identity element.

Theorem 3.5. Let $(S, *)$ be a TA-group, $a, b, c, d, f \in S$, e is the right identity element in S. There is,

(1) if $a * b = e$, e is identity element of a;

(2) $((a * b) * c) * d = a * (d * (c * b))$;

(3) if $a * b = c * d$, then $a * (d^{-1} * b) = c$.

Proof. (1) If $a * b = e$, i.e. a is left inverse element of b. Then

$$a = a * e = a * (a * b) = (a * b) * a = e * a.$$

That means that e is an identity element of a.

(2) According to Proposition 2.1, $((a * b) * c) * d = (a * b) * (d * c) = (a * c) * (d * b) = a * ((d * b) * c) = a * (d * (b * c))$.

(3) If $a*b = c*d$, there exists $d^{-1} \in S$ s.t. $d^{-1} * d = e$. Then according to Tarski associative law,

$$(a * b) * d^{-1} = (c * d) * d^{-1} = c * (d^{-1} * d) = c * e = c.$$

That is to say, $a * (d^{-1} * b) = c$. □

But right identity element of TA-group isn't unique, see Example 3.4.

Exmaple 3.4. Let $G = \{1, 2, 3, 4\}$. Consider a TA-group in Table 1.

Right identity elements are 1 and 2.

Theorem 3.6. Let $(G, *)$ be a TA-group, e be right identity element of G. Then for any $a \in G$, left inverse element of a relative to e is unique.

Proof. Let $a \in G$ and e is right identity element in G. Assume that left inverse element of a relative to e isn't unique, that is, there exist $b, c \in G$ s.t. $b * a = e$ and $c * a = e$. Then

$$b = b * e = b * (c * a) = (b * a) * c = (c * a) * c = c * (c * a) = c * e = c.$$

So $b = c$ and left inverse element is unique. □

Proposition 3.1. Assume that $(G, *)$ is a TA-group. There is,

(1) G is right cancellative;

(2) if $a * b = e$ is a right identity element, then $b * a = e_1$ also is a right identity element.

Proof. (1) Assume that $(G, *)$ is a TA-group and e is a right identity element in G. For any $a, b \in G$ and there exists $y \in G$ s.t. $a * y = b * y$. And there exists $y' \in G$ s.t. $a * e = a, b * e = b, y * e = y, y' * y = e$. Then

$$a = a * e = a * (y' * y) = (a * y) * y' = (b * y) * y' = b * (y' * y) = b * e = b.$$

So G is right cancellative.

(2) According to Theorem 3.2, TA-group satisfies right commutative law, then for any $c \in G$, there is,

$$c * (b * a) = c * (a * b) = c * e = c.$$

Then $b * a$ is a right identity element in G, that is, $b * a = e_1$ is a right identity element in G. □

Theorem 3.7. Let G be semigroup. Then it is a TA-group if and only if it satisfies:

(1) $\forall a, b \in G$, there is unique solution to equation $x * a = b$;

(2) $\forall a, b, c \in G$, $(a * b) * c = a * (c * b)$.

Proof. ($\Rightarrow$) Assume that c, d are solutions to equation $x * a = b$, then $c * a = d * a$. Because G is TA-group, according to Proposition 3.1(1), $c = d$. So there is unique a solution to equation $x * a = b$.

(⇐) For given $a \in G$, there exists $e \in G$ s.t. $e * a = a$. Then $e^2 * a = e * (e * a) = e * a = a$, because $x * a = b$ has a unique solution, and $e^2 = e$. If there exists e' s.t. $e' * a = a * e$, then $(e' * a) * e = (a * e) * e = a * e^2 = a * e$. So $e' * a = a * e = (e' * a) * e$, then $a = e' * a = a * e$. For all $b \in G$, because $x * a = e$ has unique solution, there exists $c \in G$ s.t. $c * a = b$. And $b * e = (c * a) * e = c * (a * e) = c * a = b$, so e is right identity element of G.

Let $b = e$, there exists $c \in G$ s.t. $c * a = e$, that is, c is left inverse element of a relative to a. So G is TA-group. □

Theorem 3.8. Assume that G is TA-groupoid. Then it is TA-group if and only if it satisfies:
(1) e is right identity element in G, that is, $\forall a \in G$, there is, $a * e = a$;
(2) e is right identity element in G, and there exists right inverse element $b \in G$ such that $a * b = e$.

Proof. (⇒) Let G be TA-groupoid. According to the definition of TA-group and e is right identity element in G, $\forall a \in G$, there exist $a' \in G$, s.t. $a * e = a, a' * a = e$. According to Proposition 3.1, $a * a' = e_1$ is also a right identity element. So for any $a \in G$, there exist a', e_1 s.t. $a * a' = e_1$. According to Theorem 3.3, for all $a, b, c \in G$, there is, $(a * b) * c = a * (b * c)$.

(⇐) Let G be a TA-groupoid. Then for any $a \in G$, there exist $e, c \in G$ s.t. $a*e = a, a*c = e$. Accoring to Theorem 3.2, it satisfies right commutative law.

$$a * (c * a) = a * (a * c) = a * e = a.$$

That is to say, $c * a$ is local right identity element of a. And $\forall b \in G$,

$$b * (c * a) = b * (a * c) = b * e = a.$$

So $c * a$ is right identity element in G, and c is left inverse element of a relative to $c * a$. Thus G is TA-group. □

In the following, we proposed the notion of TA-subgroup and gave the equivalent characterization of TA-subgroup.

Definition 3.3. Let $(G, *)$ be TA-group and S be non-empty subset of G. If S is a TA-group under operation $*$ on G, then S is called TA-subgroup of G.

Theorem 3.9. The non-empty subset S of G is TA-subgroup if and only if
(1) $\forall a, b \in S$, there is, $a * b \in S$;
(2) e is a right identity element of S, and for all $a \in S$, there is $a' \in S$ s.t. $a' * a = e$.

Proof. (⇒) According to Definition 3.3, (1) and (2) hold.

(⇐) $\forall a, b \in S$, there is, $a * b \in S$, then S is a TA-groupoid. Because e is right identity element of S, then for any $a \in S$, there is, $a * e = a$. And there exists $a' \in S$ such that $a' * a = e$. So S is TA-group. Thus, S is TA-subgroup of G. □

Theorem 3.10. Commutative TA-group is Abelian group.

Proof. Let $(G, *)$ be a TA-group, e is a right identity element in G. According to Theorem 3.3, G is a commutative semigroup. Then for any $a \in G$, there is $x \in G$ s.t. $a * e = a, x * a = e$. Then $e * a = a * e = a$ and $a * x = x * a = e$. So e is identity element of G and x is inverse element of a. Assume that e' also is identity element in G, there is, $e = e * e' = e'$. That is to say, identity element is unique. Assume that there exist $x, y \in G$ such that $x * a = e = y * a$, according to Proposition 3.1, G is right cancellative, and $x = y$, thus the inverse element is unique. So G is Abelian group. □

Theorem 3.11. If right identity element of TA-group is unique, then

(1) left inverse element is right inverse element;

(2) right identity element is left identity element;

(3) identity element is unique;

(4) inverse element is unique;

(5) it is a group.

Proof. (1) Suppose that $(G, *)$ is a TA-group. $\forall a \in G$, $\exists a', a'' \in G$, s.t. $a * e = a$, $a' * a = e$. $a' * e = a'$, $a'' * a' = e$. $a'' * e = a''$. Then

$$e * a = a'' * a' * a = a'' * (a' * a) = a'' * e = a''.$$

Then

$$e * a' = a'' * a' * a' = e * a * a' * a',$$

And

$$e * a' = e * a * a' * a'.$$

$$e * a' * a = e * (a' * a) = e * e = e.$$

$$e * a * a' * a' * a = e * a * a' * e = e * a * (a' * e) = e * a * a'.$$

So $e = e * a * a'$. Because e is unique, then $a * a' = e$.

(2) By(1), $e * a = (a * a') * a = a * (a * a') = a * e = a$.

(3) According to (2), right identity element is left identity element and right identity element is unique, then there exists identity element and it is unique.

(4) By (1), left inverse element and right inverse element are unique and they are equivalent to each other. Assume that inverse element of a isn't unique and there exists $y \in G$ s.t. $a * y = e$. So $a' = a' * e = a' * (a * y) = (a' * a) * y = e * y = y$. That is, inverse element of a is unique.

(5) By (3) and (4), there are identity element and inverse element in G, and they are unique. And TA-group satiefies associative law, then G is a group. □

Example 3.5 shows that TA-group whose right identity element is unique, and it is a group.

Exmaple 3.5. Let $G = \{a, b, c, d\}$, in Table 4, the operation $*$ on G is given. It is both a TA-group and a group. And

$$a * a = a, b * a = b = a * b, c * a = c = a * c, d * a = d = a * d.$$

$$a * a = a, b * b = a, c * d = a = d * c.$$

TABLE 4. This is a group.

*	**a**	**b**	**c**	**d**
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

Remark 3.1. TA-group is regular TA-groupoid.

But not every regular TA-groupoid is TA-group, see Example 3.6.

Exmaple 3.6. Let $G = \{a, b, c, d\}$, consider the regular TA-groupoid in Table 5. Since there is no right identity element, so it isn't TA-group.

TABLE 5. This is a regular TA-groupoid.

*	**a**	**b**	**c**	**d**
a	a	a	b	b
b	a	a	b	b
c	c	c	c	c
d	d	d	d	d

In the next part, the concept and property of inverse TA-groupoid are given.

Definition 3.4. Let G be a regular TA-groupoid. Then for any $a \in G$, there exists $x \in G$ s.t. $a * (x * a) = a$. G is called an inverse TA-groupoid if x is unique.

The following Theorem shows that the decomposition theorem of inverse TA-groupoid.

Theorem 3.12. Let G be inverse TA-groupoid. Then it is disjoint union of groups.

Proof. Let $x * a = e$, then $a * e = a$. Assume that G_e is composed of element whose local right identity element is e.

For any $a, b, c \in G_e$, there is,

$$\begin{aligned} a * (b * c) &= [a * (b * c)] * e = a * [e * (b * c)] \\ &= (a * b) * (e * c) = (a * c) * (e * b)(By Proposition 2.1) \\ &= a * [(e * b) * c] = a * [e * (c * b)] \\ &= [a * (c * b)] * e = a * (c * b) \end{aligned}$$

According to Tarski associative law, $(a * b) * c = a * (c * b) = a * (b * c)$. Then it satisfies associative law.

For any $a \in G_e$, there exists $x \in G$ such that $a * (x * a) = a$. Assume that $x * a = e$, then $a * e = a$, $a * ((x * e) * a) = a * (x * (a * e)) = a * (x * a) = a$. Because x is unique, then $x * e = x$. That is to say, $x \in G_e$.

For any $a \in G_e$, $a * e = a$. Then $a * (e * e) = (a * e) * e = a * e = a$. And there exists $x \in G_e$ such that $x * a = e$. $(e * e) * e = (e * e) * (x * a) = (e * a) * (x * e) = (e * a) * x = e * (x * a) = e * e$. That is to say, $e * e \in G_e$ and $e * e$ is right identity element of G_e. According to Theorem 3.2, $e * e = (x * a) * (x * a) = (x * a) * (a * x) = x * ((a * x) * a) = x * (a * (x * a)) = x * a = e$. So $e \in G_e$.

For any $a, b \in G_e$, according to associative law, there is, $(a * b) * e = a * (b * e) = a * b$. So $a * b \in G_e$. That is to say, G_e is TA-groupoid.

Above all, G_e satisfies associative law, $e \in G_e$ and for any $a \in G_e$, there exists $x \in G_e$ such that $x * a = e$.

Because x is unique and e is unique, we know G_e is a TA-group with unique right identity element, then it is a group.

Then G is union of group, and x is unique and $x * a = e$ is unique. That is to say, for any local right identity element $e \in G$, every subgroup G_e of G is disjoint, G is disjoint union of groups. □

According to Definition 2.3 and Theorem 3.12, we know inverse TA-groupoid is TA-NET-groupoid, but whether a TA-NET-groupoid is a inverse TA-groupoid? see Example 3.7. The example shows that not every TA-NET-groupoid is inverse TA-groupoid.

Exmaple 3.7. Let $G = \{1, 2, 3, 4\}$. In Table 6, the TA-NET-groupoid $(G, *)$ is shown. And

$$1 * 1 = 1; 2 * 2 = 2;$$

$$1 * 3 = 3 = 3 * 1, 3 * 3 = 1;$$

$$4 * 4 = 4; 5 * 5 = 5.$$

Table 6. This is a TA-NET-groupoid.

*	**1**	**2**	**3**	**4**	**5**
1	1	1	3	4	4
2	1	2	3	4	5
3	3	3	1	4	4
4	4	4	4	4	4
5	4	5	4	4	5

It isn't inverse TA-groupoid since $1 * (1 * 1) = 1$, $1 * (2 * 1) = 1$ and $1 \neq 2$.

We know both completely regular semigroup and inverse TA-groupoid are disjoint union of groups, and completely regular semigroup satisfies associative law, whether a completely regular semigroup is a TA-groupoid? See Example 3.8. The example shows that not every completely regular semigroup is TA-groupoid.

Exmaple 3.8. Let $G = \{1, 2, 3, 4\}$. In Table 7, a completely regular semigroup $(G, *)$ is given.

Table 7. This is a completely regular semigroup.

*	**1**	**2**	**3**	**4**
1	1	4	4	4
2	1	2	3	4
3	1	3	3	4
4	1	4	4	4

It isn't TA-groupoid since $(1 * 1) * 2 = 4 \neq 1 = 1 * (2 * 1)$.

Because TA-NET-groupoid is semigroup and inverse TA-groupoid is TA-NET-groupoid, inverse TA-groupoid is semigroup. According to Definition 2.7, Theorem 2.6 and Definition 3.4, inverse TA-groupoid is inverse semigroup.

The following figure shows relationships among various TA-groupoid.

4. Conclusions

In this paper, we proposed the concept of TA-group and inverse TA-groupoid. Some results are obtained as follows: (1) if right identity element of TA-group is unique, then it is a group; (2) the equivalent characterization of TA-group is given; (3) inverse TA-groupoid is disjoint union of group; (4) commutative TA-groupoid is group. Figure 1 shows their relations.

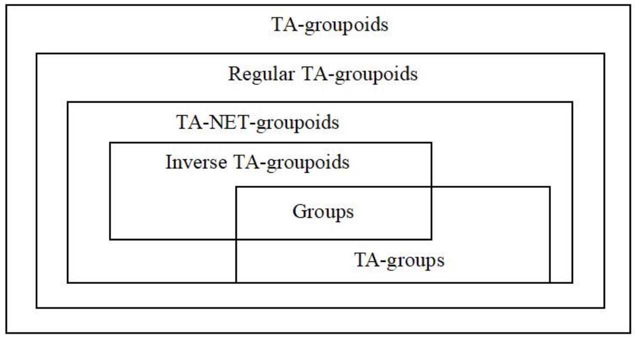

Figure 1. The relationships among various TA-groupoid.

As a future direction for further research, we can discuss the relationships among TA-groupoid, AG-groupoid and hyper logical algebras(see [18–20]).

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