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Neutrosophic Topology

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Abstract: In this paper, we redefine the neutrosophic set operations and, by using them, we introduce neutrosophic topology and investigate some related properties such as neutrosophic closure, neutrosophic interior, neutrosophic exterior, neutrosophic boundary and neutrosophic subspace.

Keywords: Neutrosophic set, neutrosophic topological space, neutrosophic open set, neutrosophic closed set, neutrosophic interior, neutrosophic exterior, neutrosophic boundary and neutrosophic subspace.

1 Introduction

The concept of neutrosophic sets was first introduced by Smarandache [13, 14] as a generalization of intuitionistic fuzzy sets [1] where we have the degree of membership, the degree of indeterminacy and the degree of non-membership of each element in X. After the introduction of the neutrosophic sets, neutrosophic set operations have been investigated. Many researchers have studied topology on neutrosophic sets, such as Smarandache [14] Lupianez [7–10] and Salama [12]. Various topologies have been defined on the neutrosophic sets. For some of them the De Morgan's Laws were not valid.

Thus, in this study, we redefine the neutrosophic set operations and investigate some properties related to these definitions. Also, we introduce for the first time the neutrosophic interior, neutrosophic closure, neutrosophic exterior, neutrosophic boundary and neutrosophic subspace. In this paper, we propose to define basic topological structures on neutrosophic sets, such that interior, closure, exterior, boundary and subspace.

2 Preliminaries

In this section, we will recall the notions of neutrosophic sets [13]. Moreover, we will give a new approach to neutrosophic set operations.

Definition 1 *[13] A neutrosophic set A on the universe of discourse X is defined as*

$$A = \big\{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\big\}$$

where $\mu_A, \sigma_A, \gamma_A : X \to]^-0, 1+[$ and $^-0 \leq \mu_A(x) + \sigma_A(x) + \gamma_A(x) \leq 3^+$ From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$. Set of all neutrosophic set over X is denoted by $\mathcal{N}(X)$.

Definition 2 *Let $A, B \in \mathcal{N}(X)$. Then,*

i. (Inclusion) If $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$, then A is neutrosophic subset of B and denoted by $A \sqsubseteq B$. (Or we can say that B is a neutrosophic super set of A.)

ii. (Equality) If $A \sqsubseteq B$ and $B \sqsubseteq A$, then $A = B$.

iii. (Intersection) Neutrosophic intersection of A and B, denoted by $A \sqcap B$, and defined by

$$A \sqcap B = \Big\{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\Big\}.$$

iv. (Union) Neutrosophic union of A and B, denoted by $A \sqcup B$, and defined by

$$A \sqcup B = \Big\{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\Big\}.$$

v. (Complement) Neutrosophic complement of A is denoted by A^c and defined by

$$A^c = \big\{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle : \ x \in X\big\}.$$

vi. (Universal Set) If $\mu_A(x) = 1$, $\sigma_A(x) = 0$ and $\nu_A(x) = 0$ for all $x \in X$, A is said to be neutrosophic universal set, denoted by $\tilde{X}$.

vii. (Empty Set) If $\mu_A(x) = 0$, $\sigma_A(x) = 1$ and $\nu_A(x) = 1$ for all $x \in X$, A is said to be neutrosophic empty set, denoted by $\tilde{\emptyset}$.

Remark 3 *According to Definition 2, $\tilde{X}$ should contain complete knowledge. Hence, its indeterminacy degree and non-membership degree are 0 and its membership degree is 1. Similarly, $\tilde{\emptyset}$ should contain complete uncertainty. So, its indeterminacy degree and non-membership degree are 1 and its membership degree is 0.*

Example 4 *Let $X = \{x, y\}$ and $A, B, C \in \mathcal{N}(X)$ such that*

$$\begin{aligned} A &= \{\langle x, 0.1, 0.4, 0.3\rangle, \langle y, 0.5, 0.7, 0.6\rangle\} \\ B &= \{\langle x, 0.9, 0.2, 0.3\rangle, \langle y, 0.6, 0.4, 0.5\rangle\} \\ C &= \{\langle x, 0.5, 0.1, 0.4\rangle, \langle y, 0.4, 0.3, 0.8\rangle\}. \end{aligned}$$

Then,

i. *We have that $A \sqsubseteq B$.*

ii. *Neurosophic union of B and C is*

$$\begin{aligned} B \sqcup C &= \Big\{\big\langle x, (0.9 \vee 0.5), (0.2 \wedge 0.1), (0.3 \wedge 0.4)\big\rangle, \\ &\quad \big\langle y, (0.6 \vee 0.4), (0.4 \wedge 0.3), (0.5 \wedge 0.8)\big\rangle\Big\} \\ &= \Big\{\langle x, 0.9, 0.1, 0.3\rangle, \langle y, 0.6, 0.3, 0.5\rangle\Big\}. \end{aligned}$$

iii. *Neurosophic intersection of A and C is*

$$\begin{aligned} A \sqcap C &= \Big\{\big\langle x, (0.1 \wedge 0.5), (0.4 \vee 0.1), (0.3 \vee 0.4)\big\rangle, \\ &\quad \big\langle y, (0.5 \wedge 0.4), (0.7 \vee 0.3), (0.6 \vee 0.8)\big\rangle\Big\} \\ &= \Big\{\langle x, 0.1, 0.4, 0.3, \rangle, \langle y, 0.5, 0.7, 0.6\rangle\Big\}. \end{aligned}$$

iv. *Neutrosophic complement of C is*

$$\begin{aligned} C^c &= \{\langle x, 0.5, 0.1, 0.4\rangle, \langle y, 0.4, 0.3, 0.8\rangle\}^c \\ &= \{\langle x, 0.4, 1 - 0.1, 0.5\rangle, \langle y, 0.8, 1 - 0.3, 0.4\rangle\} \\ &= \{\langle x, 0.4, 0.9, 0.5\rangle, \langle y, 0.8, 0.7, 0.4\rangle\}. \end{aligned}$$

Theorem 5 *Let $A, B \in \mathcal{N}(X)$. Then, followings hold.*

i. $A \sqcap A = A$ *and* $A \sqcup A = A$

ii. $A \sqcap B = B \sqcap A$ *and* $A \sqcup B = B \sqcup A$

iii. $A \sqcap \tilde{\emptyset} = \tilde{\emptyset}$ *and* $A \sqcap \tilde{X} = A$

iv. $A \sqcup \tilde{\emptyset} = A$ *and* $A \sqcup \tilde{X} = \tilde{X}$

v. $A \sqcap (B \sqcap C) = (A \sqcap B) \sqcap C$ *and* $A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$

vi. $(A^c)^c = A$

Proof. It is clear.

Theorem 6 *Let $A, B \in \mathcal{N}(X)$. Then, De Morgan's law is valid.*

i. $\left(\bigsqcup_{i \in I} A_i\right)^c = \bigsqcap_{i \in I} A_i^c$

ii. $\left(\bigsqcap_{i \in I} A_i\right)^c = \bigsqcup_{i \in I} A_i^c$

Proof.

i. From Definition 2 *v.*

$$\begin{aligned} \left(\bigsqcup_{i \in I} A_i\right)^c &= \Big\{\Big\langle x, \bigvee_{i \in I} \mu_{A_i}(x), \bigwedge_{i \in I} \sigma_{A_i}(x), \\ &\quad \bigwedge_{i \in I} \nu_{A_i}(x)\Big\rangle : x \in X\Big\}^c \\ &= \Big\{\Big\langle x, \bigwedge_{i \in I} \nu_{A_i}(x), 1 - \bigwedge_{i \in I} \sigma_{A_i}(x), \\ &\quad \bigvee_{i \in I} \mu_{A_i}(x)\Big\rangle : x \in X\Big\} \\ &= \bigsqcap_{i \in I} A_i^c \end{aligned}$$

ii. It can proved by similar way to *i.*

Theorem 7 *Let $B \in \mathcal{N}(X)$ and $\{A_i : i \in I\} \subseteq \mathcal{N}(X)$. Then,*

i. $B \sqcap \left(\bigsqcup_{i \in I} A_i\right) = \bigsqcup_{i \in I} (B \sqcap A_i)$

ii. $B \sqcup \left(\bigsqcap_{i \in i} A_i\right) = \bigsqcap_{i \in I} (B \sqcup A_i)$.

Proof. It can be proved easily from Definition 2.

3 Neutrosophic topological spaces

In this section, we will introduce neutrosophic topological space and give their properties.

Definition 8 *Let $\tau \subseteq \mathcal{N}(X)$, then τ is called a neutrosophic topology on X if*

i. *$\tilde{X}$ and $\tilde{\emptyset}$ belong to τ,*

ii. *The union of any number of neutrosophic sets in τ belongs to τ,*

iii. *The intersection of any two neutrosophic sets in τ belongs to τ.*

The pair (X, τ) is called a neutrosophic topological space over X. Moreover, the members of τ are said to be neutrosophic open sets in X. If $A^c \in \tau$, then $A \in \mathcal{N}(X)$ is said to be neutrosophic closed set in X

Theorem 9 *Let (X, τ) be a neutrosophic topological space over X. Then*

i. *$\tilde{\emptyset}$ and $\tilde{X}$ are neutrosophic closed sets over X.*

ii. *The intersection of any number of neutrosophic closed sets is a neutrosophic closed set over X.*

iii. *The union of any two neutrosophic closed sets is a neutrosophic closed set over X.*

Proof. Proof is clear.

Example 10 *Let $\tau = \{\tilde{\emptyset}, \tilde{X}\}$ and $\sigma = \mathcal{N}(X)$. Then, (X,τ) and (X,σ) are two neutrosophic topological spaces over X. Moreover, they are called neutrosophic discrete topological space and neutrosophic indiscrete topological space over X, respectively.*

Example 11 *Let $X = \{a, b\}$ and $A \in \mathcal{N}(X)$ such that*

$$A = \{\langle a, 0.2, 0.4, 0.6\rangle, \langle b, 0.1, 0.3, 0.5\rangle\}.$$

Then, $\tau = \{\tilde{\emptyset}, \tilde{X}, A\}$ is a neutrosophic topology on X.

Theorem 12 *Let (X, τ_1) and (X, τ_2) be two neutrosophic topological spaces over X, then $(X, \tau_1 \cap \tau_2)$ is a neutrosophic topological space over X.*

Proof. Let (X, τ_1) and (X, τ_2) be two neutrosophic topological spaces over X. It can be seen clearly that $\tilde{\emptyset}, \tilde{X} \in \tau_1 \cap \tau_2$. If $A, B \in \tau_1 \cap \tau_2$ then, $A, B \in \tau_1$ and $A, B \in \tau_2$. It is given that $A \sqcap B \in \tau_1$ and $A \sqcap B \in \tau_2$. Thus, $A \sqcap B \in \tau_1 \cap \tau_2$. Let $\{A_i : i \in I\} \subseteq \tau_1 \cap \tau_2$. Then, $A_i \in \tau_1 \cap \tau_2$ for all $i \in I$. Thus, $A_i \in \tau_1$ and $A_i \in \tau_2$ for all $i \in I$. So, we have $\bigsqcup_{i\in I} A_i \in \tau_1 \cap \tau_2$.

Corollary 13 *Let $\{(X, \tau_i) : i \in I\}$ be a family of neutrosophic topological spaces over X. Then, $(X, \bigcap_{i\in I} \tau_i)$ is a neutrosophic topological space over X.*

Proof. It can proved similar way Theorem 12.

Remark 14 *If we get the union operation instead of the intersection operation in Theorem 12, the claim may not be correct. This situation can be seen following example.*

Example 15 *Let $X = \{a, b\}$ and $A, B \in \mathcal{N}(X)$ such that*

$$\begin{aligned} A &= \{\langle a, 0.2, 0.4, 0.6\rangle, \langle b, 0.1, 0.3, 0.5\rangle\} \\ B &= \{\langle a, 0.4, 0.6, 0.8\rangle, \langle b, 0.3, 0.5, 0.7\rangle\}. \end{aligned}$$

Then, $\tau_1 = \{\tilde{\emptyset}, \tilde{X}, A\}$ and $\tau_2 = \{\tilde{\emptyset}, \tilde{X}, B\}$ are two neutrosophic topology on X. But, $\tau_1 \cup \tau_2 = \{\tilde{\emptyset}, \tilde{X}, A, B\}$ is not neutrosophic topology on X. Because, $A \sqcap B \notin \tau_1 \cup \tau_2$. So, $\tau_1 \cup \tau_2$ is not neutrosophic topological space over X.

Definition 16 *Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then, the neutrosophic interior of A, denoted by $\mathrm{int}(A)$ is the union of all neutrosophic open subsets of A. Clearly $\mathrm{int}(A)$ is the biggest neutrosophic open set over X which containing A.*

Theorem 17 *Let (X, τ) be a neutrosophic topological space over X and $A, B \in \mathcal{N}(X)$. Then*

i. $\mathrm{int}(\tilde{\emptyset}) = \tilde{\emptyset}$ *and* $\mathrm{int}(\tilde{X}) = \tilde{X}$.

ii. $\mathrm{int}(A) \sqsubseteq A$.

iii. *A is a neutrosophic open set if and only if $A = \mathrm{int}(A)$.*

iv. $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.

v. *$A \sqsubseteq B$ implies $\mathrm{int}(A) \sqsubseteq \mathrm{int}(B)$.*

vi. $\mathrm{int}(A) \sqcup \mathrm{int}(B) \sqsubseteq \mathrm{int}(A \sqcup B)$.

vii. $\mathrm{int}(A \sqcap B) = \mathrm{int}(A) \sqcap \mathrm{int}(B)$.

Proof. *i.* and *ii.* are obvious.

iii. If A is a neutrosophic open set over X, then A is itself a neutrosophic open set over X which contains A. So, A is the largest neutrosophic open set contained in A and $\mathrm{int}(A) = A$. Conversely, suppose that $\mathrm{int}(A) = A$. Then, $A \in \tau$.

iv. Let $\mathrm{int}(A) = B$. Then, $\mathrm{int}(B) = B$ from *iii.* and then, $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.

v. Suppose that $A \sqsubseteq B$. As $\mathrm{int}(A) \sqsubseteq A \sqsubseteq B$. $\mathrm{int}(A)$ is a neutrosophic open subset of B, so from Definition 16, we have that $\mathrm{int}(A) \sqsubseteq \mathrm{int}(B)$.

vi. It is clear that $A \sqsubseteq A \sqcup B$ and $B \sqsubseteq A \sqcup B$. Thus, $\mathrm{int}(A) \sqsubseteq \mathrm{int}(A \sqcup B)$ and $\mathrm{int}(B) \sqsubseteq \mathrm{int}(A \sqcup B)$. So, we have that $\mathrm{int}(A) \sqcup \mathrm{int}(B) \sqsubseteq \mathrm{int}(A \sqcup B)$ by *v.*

vii. It is known that $\mathrm{int}(A \sqcap B) \sqsubseteq \mathrm{int}(A)$ and $\mathrm{int}(A \sqcap B) \sqsubseteq \mathrm{int}(B)$ by *v.* so that $\mathrm{int}(A \sqcap B) \sqsubseteq \mathrm{int}(A) \sqcap \mathrm{int}(B)$. Also, from $\mathrm{int}(A) \sqsubseteq A$ and $\mathrm{int}(B) \sqsubseteq B$, we have $\mathrm{int}(A) \sqcap \mathrm{int}(B) \sqsubseteq A \sqcap B$. These imply that $\mathrm{int}(A \sqcap B) = \mathrm{int}(A) \sqcap \mathrm{int}(B)$.

Example 18 *Let $X = \{a, b\}$ and $A, B, C \in \mathcal{N}(X)$ such that*

$$\begin{aligned} A &= \{\langle a, 0.5, 0.5, 0.5\rangle, \langle b, 0.3, 0.3, 0.3\rangle\} \\ B &= \{\langle a, 0.4, 0.4, 0.4\rangle, \langle b, 0.6, 0.6, 0.6\rangle\} \\ C &= \{\langle a, 0.7, 0.7, 0.7\rangle, \langle b, 0.2, 0.2, 0.2\rangle\}. \end{aligned}$$

Then, $\tau = \{\tilde{\emptyset}, \tilde{X}, A\}$ is a neutrosophic soft topological space over X. Therefore, $\mathrm{int}(B) = \tilde{\emptyset}$, $\mathrm{int}(C) = \tilde{\emptyset}$ and $\mathrm{int}(B \sqcup C) = A$. So, $\mathrm{int}(B) \sqcup \mathrm{int}(C) \neq \mathrm{int}(B \sqcup C)$.

Definition 19 *Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then, the neutrosophic closure of A, denoted by $\mathrm{cl}(A)$ is the intersection of all neutrosophic closed super sets of A. Clearly $\mathrm{cl}(A)$ is the smallest neutrosophic closed set over X which contains A.*

Example 20 *In the Example 10, according to the neutrosophic topological space (X, σ), neutrosophic interior and neutrosophic closure of each element of $\mathcal{N}(X)$ is equal to itself.*

Theorem 21 *Let (X,τ) be a neutrosophic topological space over X and $A, B \in \mathcal{N}(X)$. Then*

i. $\text{cl}(\tilde{\emptyset}) = \tilde{\emptyset}$ *and* $\text{cl}(\tilde{X}) = \tilde{X}$.

ii. $A \sqsubseteq \text{cl}(A)$.

iii. *A is a neutrosophic closed set if and only if $A = \text{cl}(A)$.*

iv. $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.

v. $A \sqsubseteq B$ *implies* $\text{cl}(A) \sqsubseteq \text{cl}(B)$.

vi. $\text{cl}(A \sqcup B) = \text{cl}(A) \sqcup \text{cl}(B)$.

vii. $\text{cl}(A \sqcap B) \sqsubseteq \text{cl}(A) \sqcap \text{cl}(B)$.

Proof. *i.* and *ii.* are clear. Moreover, proofs of *vi.* and *vii.* are similar to Theorem 17 *vi.* and *vii.*.

iii. If A is a neutrosophic closed set over X then A is itself a neutrosophic closed set over X which contains A. Therefore, A is the smallest neutrosophic closed set containing A and $A = \text{cl}(A)$. Conversely, suppose that $A = \text{cl}(A)$. As A is a neutrosophic closed set, so A is a neutrosophic closed set over X.

iv. A is a neutrosophic closed set so by *iii.*, then we have $A = \text{cl}(A)$.

v. Suppose that $A \sqsubseteq B$. Then every neutrosophic closed super set of B will also contain A. This means that every neutrosophic closed super set of B is also a neutrosophic closed super set of A. Hence the neutrosophic intersection of neutrosophic closed super sets of A is contained in the neutrosophic intersection of neutrosophic closed super sets of B. Thus $\text{cl}(A) \sqsubseteq \text{cl}(B)$.

Example 22 *Let $X = \{a, b\}$ and $A, B \in \mathcal{N}(X)$ such that*

$$\begin{aligned} A &= \{\langle a, 0.5, 0.5, 0.5\rangle, \langle b, 0.4, 0.4, 0.4\rangle\} \\ B &= \{\langle a, 0.6, 0.6, 0.6\rangle, \langle b, 0.3, 0.3, 0.3\rangle\}. \end{aligned}$$

Then,

$$\tau = \{\tilde{\emptyset}, \tilde{X}, A, B, A \sqcap B, A \sqcup B\}$$

is a neutrosophic topology on X. Moreover, set of neutrosophic closed sets over X is

$$\{\tilde{X}, \tilde{\emptyset}, A^c, B^c, (A \sqcap B)^c, (A \sqcup B)^c\}.$$

Therefore

$$\begin{aligned} A^c &= \{\langle a, 0.5, 0.5, 0.5\rangle, \langle b, 0.4, 0.6, 0.4\rangle\} \\ B^c &= \{\langle a, 0.6, 0.4, 0.6\rangle, \langle b, 0.3, 0.7, 0.3\rangle\} \\ (A \sqcap B)^c &= \{\langle a, 0.6, 0.4, 0.5\rangle, \langle b, 0.4, 0.6, 0.4\rangle\} \\ (A \sqcup B)^c &= \{\langle a, 0.5, 0.5, 0.6\rangle, \langle b, 0.3, 0.7, 0.4\rangle\}. \end{aligned}$$

Thus, we have that

$$\begin{aligned} A \sqcap B &= \{\langle a, 0.5, 0.5, 0.6\rangle, \langle b, 0.3, 0.7, 0.4\rangle\} \\ \text{cl}(A) &= \tilde{X} \\ \text{cl}(B) &= \tilde{X} \\ \text{cl}(A \sqcap B) &= (A \sqcup B)^c \\ \text{cl}(A \sqcap B) &\sqsubseteq \text{cl}(A) \sqcap \text{cl}(B). \end{aligned}$$

Remark 23 *Example 18 and Example 22 show that there is not equality in Theorem 17* vi. *and Theorem 21* vii.

Theorem 24 *Let (X,τ) be a neutrosophic topological space over X and $A, B \in \mathcal{N}(X)$. Then*

i. $\text{int}(A^c) = (\text{cl}(A))^c$,

ii. $\text{cl}(A^c) = (\text{int}(A))^c$.

Proof. Let $A, B \in \mathcal{N}(X)$. Then,

i. It is known that

$$\text{cl}(A) = \bigsqcap_{\substack{B^c \in \tau \\ A \sqsubseteq B}} B.$$

Therefore, we have that

$$(\text{cl}(A))^c = \bigsqcup_{\substack{B^c \in \tau \\ B^c \sqsubseteq A^c}} B^c.$$

Right hand of above equality is $\text{int}(A^c)$, thus $\text{int}(A^c) = (\text{cl}(A))^c$.

ii. If it is taken A^c instead of A in *i.*, then it can be seen clearly that $(\text{cl}(A^c))^c = \text{int}((A^c)^c) = \text{int}(A)$. So, $\text{cl}(A^c) = (\text{int}(A))^c$.

Definition 25 *Let (X,τ) be a neutrosophic topological space over X then the neutrosophic exterior of a neutrosophic set A over X is denoted by $\text{ext}(A)$ and is defined as $\text{ext}(A) = \text{int}(A^c)$.*

Theorem 26 *Let (X,τ) be a neutrosophic topological space over X and $A, B \in \mathcal{N}(X)$. Then*

i. $\text{ext}(A \sqcup B) = \text{ext}(A) \sqcap \text{ext}(B)$

ii. $\text{ext}(A) \sqcup \text{ext}(B) \sqsubseteq \text{ext}(A \sqcap B)$

Proof. Let $A, B \in \mathcal{N}(X)$. Then,

i. By Definition 25, Theorem 6 and Theorem 17 *vii.*

$$\begin{aligned} \text{ext}(A \sqcup B) &= \text{int}((A \sqcup B)^c) \\ &= \text{int}(A^c \sqcap B^c) \\ &= \text{int}(A^c) \sqcap \text{int}(B^c) \\ &= \text{ext}(A) \sqcap \text{ext}(B) \end{aligned}$$

ii. It is similar to *i.*

Definition 27 *Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then, the neutrosophic boundary of a neutrosophic set A over X is denoted by $\mathrm{fr}(A)$ and is defined as $\mathrm{fr}(A) = \mathrm{cl}(A) \sqcap \mathrm{cl}(A^c)$. It must be noted that $\mathrm{fr}(A) = \mathrm{fr}(A^c)$.*

Example 28 *Let consider the neutrosophic sets A and B in the Example 22. According to the neutrosophic topology in Example 11 we have $\mathrm{fr}(A) = \tilde{\emptyset}$ and $\mathrm{fr}(C) = (A \sqcap B)^c$.*

Theorem 29 *Let (X, τ) be a neutrosophic topological space over X and $A, B \in \mathcal{N}(X)$. Then*

i. $(\mathrm{fr}(A))^c = \mathrm{ext}(A) \sqcup \mathrm{int}(A)$.

ii. $\mathrm{cl}(A) = \mathrm{int}(A) \sqcup \mathrm{fr}(A)$.

Proof. Let $A, B \in \mathcal{N}(X)$. Then,

i. By Theorem 24 *i.*, we have

$$\begin{aligned}(\mathrm{fr}(A))^c &= (\mathrm{cl}(A) \sqcap \mathrm{fr}(A^c))^c \\ &= (\mathrm{cl}(A))^c \sqcup (\mathrm{fr}(A^c))^c \\ &= (\mathrm{cl}(A))^c \sqcup ((\mathrm{int}(A))^c)^c \\ &= \mathrm{ext}(A) \sqcup \mathrm{int}(A).\end{aligned}$$

ii. By Theorem 24 *i.*, we have

$$\begin{aligned}\mathrm{int}(A) \sqcup \mathrm{fr}(A) &= \mathrm{int}(A) \sqcup (\mathrm{cl}(A) \sqcap \mathrm{fr}(A^c)) \\ &= (\mathrm{int}(A) \sqcup \mathrm{cl}(A)) \sqcap (\mathrm{int}(A) \sqcup \mathrm{fr}(A^c)) \\ &= \mathrm{cl}(A) \sqcap (\mathrm{int}(A) \sqcup (\mathrm{int}(A))^c) \\ &= \mathrm{cl}(A) \sqcap \tilde{X} \\ &= \mathrm{cl}(A).\end{aligned}$$

Theorem 30 *Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then*

i. A is a neutrosophic open set over X if and only if $A \sqcap \mathrm{fr}(A) = \tilde{\emptyset}$.

ii. A is a neutrosophic closed set over X if and only if $\mathrm{fr}(A) \sqsubseteq A$.

Proof. Let $A \in \mathcal{N}(X)$. Then

i. Assume that A is a neutrosophic open set over X. Thus $\mathrm{int}(A) = A$. By Theorem 24, $\mathrm{fr}(A) = \mathrm{cl}(A) \sqcap \mathrm{fr}(A^c) = \mathrm{cl}(A) \sqcap (\mathrm{int}(A))^c$. So,

$$\begin{aligned}\mathrm{fr}(A) \sqcap \mathrm{int}(A) &= \mathrm{cl}(A) \sqcap (\mathrm{int}(A))^c \sqcap \mathrm{int}(A) \\ &= \mathrm{cl}(A) \sqcap A^c \sqcap A \\ &= \tilde{\emptyset}.\end{aligned}$$

Conversely, let $A \sqcap \mathrm{fr}(A) = \tilde{\emptyset}$. Then, $A \sqcap \mathrm{cl}(A) \sqcap \mathrm{fr}(A^c) = \tilde{\emptyset}$ or $A \sqcap \mathrm{fr}(A^c) = \tilde{\emptyset}$ or $\mathrm{cl}(A) \sqsubseteq A^c$ which implies A^c is a neutrosophic set and so A is a neutrosophic open set.

ii. Let A be a neutrosophic closed set. Then, $\mathrm{cl}(A) = A$. By Definition 27, $\mathrm{fr}(A) = \mathrm{cl}(A) \sqcap \mathrm{fr}(A^c) \sqsubseteq \mathrm{cl}(A) = A$. Therefore, $\mathrm{fr}(A) \sqsubseteq A$. Conversely, $\mathrm{fr}(A) \sqsubseteq A$. Then $\mathrm{fr}(A) \sqcap A^c = \tilde{\emptyset}$. From $\mathrm{fr}(A) = \mathrm{fr}(A^c)$, $\mathrm{fr}(A^c) \sqcap A^c = \tilde{\emptyset}$. By *i.*, A^c is a neutrosophic open set and so A is a neutrosophic closed set.

Theorem 31 *Let (X, τ) be a neutrosophic topological space over X and $A \in \mathcal{N}(X)$. Then*

i. $\mathrm{fr}(A) \sqcap \mathrm{int}(A) = \tilde{\emptyset}$

ii. $\mathrm{fr}(\mathrm{int}(A)) \sqsubseteq \mathrm{fr}(A)$

Proof. Let $A \in \mathcal{N}(X)$. Then,

i. From Theorem 30 *i.*, it is clear.

ii. By Theorem 24 *ii.*,

$$\begin{aligned}\mathrm{fr}(\mathrm{int}(A)) &= \mathrm{cl}(\mathrm{int}(A)) \sqcap \mathrm{cl}(\mathrm{int}(A)) \\ &= \mathrm{cl}(\mathrm{int}(A)) \sqcap \mathrm{fr}(A^c) \\ &\sqsubseteq \mathrm{cl}(A) \sqcap \mathrm{fr}(A^c) \\ &= \mathrm{fr}(A).\end{aligned}$$

Definition 32 *Let (X, τ) be a neutrosophic topological space and Y be a non-empty subset of X. Then, a neutrosophic relative topology on Y is defined by*

$$\tau_Y = \{A \sqcap \tilde{Y} : A \in \tau\}$$

where

$$\tilde{Y}(x) = \begin{cases} \langle 1, 0, 0 \rangle, & x \in Y \\ \langle 0, 1, 1 \rangle, & \textit{otherwise}. \end{cases}$$

Thus, (Y, τ_Y) is called a neutrosophic subspace of (X, τ).

Example 33 *Let $X = \{a, b, c\}$, $Y = \{a, b\} \subseteq X$ and $A, B \in \mathcal{N}(X)$ such that*

$$\begin{aligned}A &= \{\langle a, 0.4, 0.2, 0.2 \rangle, \langle b, 0.5, 0.4, 0.6 \rangle, \langle c, 0.2, 0.5, 0.7 \rangle\} \\ B &= \{\langle a, 0.4, 0.5, 0.3 \rangle, \langle b, 0.5, 0.6, 0.5 \rangle, \langle c, 0.3, 0.7, 0.8 \rangle\}.\end{aligned}$$

Then,

$$\tau = \{\tilde{\emptyset}, \tilde{X}, A, B, A \sqcap B, A \sqcup B\}$$

is a neutrosophic topology on X. Therefore

$$\tau_Y = \{\tilde{\emptyset}, \tilde{Y}, C, M, L, K\}$$

is a neutrosophic relative topology on Y such that $C = \tilde{Y} \sqcap A$, $M = \tilde{Y} \sqcap B$, $L = \tilde{Y} \sqcap (A \sqcap B)$ and $K = \tilde{Y} \sqcap (A \sqcup B)$.

4 Conclusion

In this work, we have redefined the neutrosophic set operations in accordance with neutrosophic topological structures. Then, we have presented some properties of these operations. We have also investigated neutrosophic topological structures of neutrosophic sets. Hence, we hope that the findings in this paper will help researchers enhance and promote the further study on neutrosophic topology.

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