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On Parametric Divergence Measure of Neutrosophic Sets with its Application in Decision-making Models

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Abstract: In various decision-making models the divergence measure is found to be a useful information measure in handling impreciseness and uncertainty among the qualitative and quantitative factors of the decision-making process. In the proposed work, a novel parametric divergence measure for neutrosophic sets has been proposed along with its various properties. On the basis of the proposed parametric divergence measure, we have outlined some methodologies along with its implementing procedural steps for classification problem (pattern recognition problem, medical diagnosis problem) and multi criteria decision making problem. Also, numerical examples for the application problems have been provided for illustration of the proposed methodologies. Comparative remarks along with necessary observations and advantages have also been presented in view of the existing appraoches.

Keywords: Neutrosophic set; Divergence measure; Decision-making; Medical diagnosis; Pattern recognition.

1. Introduction

In the applications of expert system, fusion of information and belief system, the notion of truth-membership of fuzzy set (FS) [1] is not the only parameter to be supported by the evident but there is need of falsity-membership against by the evident. The intuitionistic fuzzy sets (IFSs) [2] consider both types of memberships and can manage the incomplete and imprecise information except the indeterminate/inconsistent information which may exist in case of a belief system. The concept of FSs and IFSs have been widely applied to model such uncertainties and hesitancy inherent in many practical circumstances having a comprehensive application in the area of decision processes, classification problems, econometrics, selection processes etc.

The notion of a neutrosophic set (NS) introduced by Smarandache [3] is a more generalized platform for handling and presenting the uncertainty, impreciseness, incompleteness and inconsistency inherited in a real world problem. As per the statement of Smarandache - "*Neutrosophy is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra*"[3]. From a philosophical point of view, the neutrosophic set can be understood as a formal generalized framework of the crisp set, fuzzy set, intuitionistic fuzzy set (IFS) etc. A special case of neutrosophic set is single valued neutrosophic set (SVNS) which has been given by Wang et al. [4]. In literature, various extensions of SVNSs have been available with a hybrid approach such as soft set analogous to NS, rough NS, neutrosophic hesitant fuzzy set, etc.

Various researchers have extensively studied different information measures (similarity measures, entropy, distance measures, divergence measures etc.) for different types of fuzzy sets/intuitionistic fuzzy sets because of their wider applicability in the different fields of science and engineering. In 1993, Bhandari and Pal [5] first studied the directed divergence based on the mutual information measure given by Kullback and Leibler [6]. Fan and Xie [7] provided a divergence measure based on exponential operation and established relation with the fuzzy exponential entropy. Further, Montes et al. [8] studied the special classes of measures of divergence in connecting with fuzzy and probabilistic uncertainty. Next, Ghosh et al. [9] have successfully implemented the fuzzy divergence measure in automated leukocyte recognition. Besides this, four fuzzy directed divergence measures were proposed by Bhatia and Singh [10] with important properties and particular cases.

Vlachos and Sergiadis [11] successfully presented an intuitionistic fuzzy directed divergence measure analogous to Shang and Jiang [12]. Further, a set of axioms for the distance measure of IFSs is provided by Wang and Xin [13] and then Hung and Yang [14] proposed a set of axioms for intuitionistic divergence measure by applying Hausdorff metric. Li [15] provided the intuitionistic fuzzy divergence measure and Hung and Yang [16] proposed intuitionistic *J*-divergence measure with its application in pattern recognition. In intuitionistic fuzzy setup, Montes et. al. [17] established some important relationships among divergence measures, dissimilarity measures and distance measures. In literature, the fuzzy divergence measures and intuitionistic fuzzy divergence measures have been widely applied in various applications – decision-making problems [18, 19], medical diagnosis [20], logical reasoning [21] and pattern recognition [22, 23] etc. Kaya and Kahraman [24] have provided comparison of fuzzy multi-criteria decision-making methods for intelligent building assessment along with detailed ranking results.

It may be noted that the degree of indeterminacy/hesitancy in case of IFSs is dependent on the other two uncertainty parameters of membership degree and non-membership degree. This gives a sense of limitation and boundedness for the decision makers to quantify the impreciseness factors. To overcome such limitations, the NS theory found to be more advantageous and effective tool in the field of information science and applications. Broumi and Smarandache [25] studied various types of similarity measures for neutrosophic sets. On the basis of the distance measure between two single valued neutrosophic sets, Majumdar and Samanta [26] proposed some similarity measures and studied their characteristics. Ye [27] studied various similarity measures for interval neutrosophic sets (INSs) on the basis of distance measures and used them in group decision-making [28]. Further, by using distance based similarity measures for single valued neutrosophic multisets, Ye et al. [29] solved the medical diagnosis problem. Also, Ye [30] studied various measures of similarity measures on the basis of cotangent function for SVNSs & utilized to solve MCDM problem and fault detection. Dhivya and Sridevi [31] studied a new single valued neutrosophic exponential similarity measure and its weighted form to overcome some drawbacks of existing measures and applied in decision making and medical diagnosis problem. Wu et al. [32] established a kind of relationship among entropy, similarity measure and directed divergence based on the three axiomatic definitions of information measure by involving a cosine function. Also, a new multi-attribute decision making method has been developed based on the proposed information measures with a numerical example of city pollution evaluation. Thao and Smarandache [33] proposed new divergence measure for neutrosophic set with some properties and utilized to solve the medical diagnosis problem and the classification problem.

Recently, the notion of NSs theory and its various generalizations have been explored in various field of research by different researchers. Abdel-Basset et al. [34] developed a new model to handle the hospital medical care evaluation system based on plithogenic sets and also studied intelligent medical decision support model [35] based on soft computing and internet of things. In addition to this, a hybrid plithogenic approach [36] by utilizing the quality function in the supply chain management has also been developed. Further, a new systematic framework for providing aid and support to the cancer patients by using neutrosophic sets has been successfully suggested by Abdel-Basset et al. [37]. Based on neutrosophic sets, some new decision-making models have also been successfully presented for project selection [38] and heart disease diagnosis [39] with advantages and defined limitations. In subsequent research, Abdel-Basset et al. [40] have proposed a modified forecasting model based on neutrosophic time series analysis and a new model for linear fractional programming based on triangular neutrosophic numbers [41]. Also, Yang et al. [42] have studied some new similarity and entropy measures of the interval neutrosophic sets on the basis of new axiomatic definition along with its application in MCDM problem.

In view of the above discussions on the recent trends in the field of neutrosophic set theory, it may be observed that the neutrosophic information measures such as distance measures, similarity measures, entropy, divergence measures, have been successfully utilized and implemented to handle the issues related to uncertainty and vagueness. For the sake of wider applicability and the desired flexibility, we need to develop some parametric information measures for SVNSs. These parametric measures will give rise to a family of information measures and we can have selections based on the desired requirements. Subsequently, they can be utilized in various soft computing applications. This approach is novel in its kind where we propose a parametric divergence measure for the neutrosophic sets with various properties so that these can be well utilized in different classification problem and decision-making problems.

The rest of the paper is structured as - In Section 2, some fundamental preliminaries of the neutrosophic sets, information measures are presented with its properties. In Section 3, a new parametric divergence measure for neutrosophic sets has been introduced with its proof. In Section 4, various properties of the proposed divergence measure have also been discussed along with their proofs. Further, in Section 5, application examples of classification problems and decision-making problem have been solved by providing the necessary steps of the proposed methodologies based on the proposed parametric divergence measure. In view of the results obtained in contrast with the existing methodologies related to these fields, some comparative remarks have also been stated for the problems under consideration. The presented work and its results have been summarized in Section 6 with scope for the future work.

2. Preliminaries

Here, some basic definitions and fundamental notions in reference with neutrosophic set, information measures and its properties are presented. Smarandache [3] introduced the notion of neutrosophic set as follows:

Definition 1. [3] Let X be a fixed class of points (objects) with a generic element x in X. A neutrosophic set M in X is specified by a truth-membership function $T_M(x)$, an indeterminacy-membership function $I_M(x)$ and a falsity-membership function $F_M(x)$, where $T_M(x), I_M(x)$ and $F_M(x)$ are real standard or nonstandard subsets of the interval $\left({}^{-}0,1^{+}\right)$ such that $T_M(x): X \to \left({}^{-}0,1^{+}\right), I_M(x): X \to \left({}^{-}0,1^{+}\right), F_M(x): X \to \left({}^{-}0,1^{+}\right)$ and the sum of these functions viz. $T_M(x)+I_M(x)+F_M(x)$ satisfies the requirement ${}^{-}0 \le \sup T_M(x)+\sup I_M(x)+\sup F_M(x) \le 3^{+}$. We denote the neutrosophic set $M=\left\{(x, T_M(x), I_M(x), F_M(x) \mid x \in X\right\}$.

In case of neutrosophic set, indeterminacy gets quantified in an explicit way, while truth-membership, indeterminacy-membership and falsity-membership are independent terms. Such framework is found to be very useful in the applications of information fusion where the data are logged from different sources. For scientific and engineering applications, Wang et al. [4] defined a single valued neutrosophic set (SVNS) as an instance of a neutrosophic set as follows:

Definition 2 *[4] Let X be a fixed class of points (objects) with a generic element x in X. A single valued neutrosophic set M in X is characterized by a truth-membership function $T_M(x)$, an indeterminacy membership function $I_M(x)$ and a falsity-membership function $F_M(x)$. For each point $x \in X$, $T_M(x)$, $I_M(x)$, $F_M(x) \in [0,1]$. A single valued neutrosophic set M can be denoted by*

$M = \{ < T_M(x), I_M(x), F_M(x) \mid x \in X \}$.

It may be noted that $T_M(x) + I_M(x) + F_M(x) \in [0,3]$.

We denote $SVNS(X)$ as the set of all the SVNSs on X. For any two SVNSs $M, N \in SVNS(X)$, some of the basic and important operations and relations may be defined as follows (Refer [4]):

- **Union of M and N:** $M \cup N = \{ (x, T_{M\cup N}(x), I_{M\cup N}(x), F_{M\cup N}(x) \mid x \in X \}$;
  where $T_{M\cup N}(x) = \max\{T_M(x), T_N(x)\}$, $I_{M\cup N}(x) = \min\{I_M(x), I_N(x)\}$ and
  $F_{M\cup N}(x) = \min\{F_M(x), F_N(x)\}$; *for all* $x \in X$.
- **Intersection of M and N:** $M \cap N = \{ (x, T_{M\cap N}(x), I_{M\cap N}(x), F_{M\cap N}(x) \mid x \in X \}$;
  where $T_{M\cap N}(x) = \min\{T_M(x), T_N(x)\}$, $I_{M\cap N}(x) = \max\{I_M(x), I_N(x)\}$ and
  $F_{M\cap N}(x) = \max\{F_M(x), F_N(x)\}$; *for all* $x \in X$.
- **Containment:** $M \subseteq N$ if and only if
  $T_M(x) \le T_N(x), I_M(x) \ge I_N(x), F_M(x) \ge F_N(x)$, *for all* $x \in X$.
- **Complement:** The complement of a neutrosophic set M, denoted by $\overline{M}$, defined by
  $T_{\overline{M}}(x) = 1 - T_M(x), I_{\overline{M}}(x) = 1 - I_M(x), T_{\overline{M}}(x) = 1 - T_M(x)$; *for all* $x \in X$.

Definition 3. *[32] Consider M and N be two single-valued neutrosophic sets, then the cross entropy between M and N must satisfy the following two axioms:*

- $C(M,N) \ge 0$;
- $C(M,N) = 0$ if $M = N$.

Based on the above stated axioms, Wu et al. [32] proposed the divergence measure for two SVNS M and N, given by

$$C_1(M,N) = 1 - \frac{1}{3(\sqrt{2}-1)} \sum_{t=1}^{3} (\sqrt{2} \cos\left(\frac{M_t - N_t}{4}\right)\pi - 1).$$

Also, Thao and Smarandache [33] have put forward various properties and axiomatic definition for divergence measure of single valued neutrosophic sets M and N with four axioms as follows:

- DivAxiom 1: $D(M,N) = D(N,M)$;
- DivAxiom 2: $D(M,N) \ge 0$; and $D(M,N) = 0$ if $M = N$.
- DivAxiom 3: $D(M \cap P, N \cap P) \le D(M,N) \, \forall P \in SVNS(X)$.
- DivAxiom 4: $D(M \cup P, N \cup P) \le D(M,N) \, \forall P \in SVNS(X)$.

3. Parametric Divergence Measure of Neutrosophic Sets

In this section, we present a new parametric divergence measure for two arbitrary SVNSs and discuss its properties. Recently, Ohlan et al. [43] proposed the generalized Hellinger's divergence measure for fuzzy sets A and B as follows:

$$h_\alpha(A,B) = \sum_{i=1}^{n} \left(\frac{\left(\sqrt{\mu_A(x_i)} - \sqrt{\mu_B(x_i)}\right)^{2(\alpha+1)}}{\sqrt{\mu_A(x_i)\mu_B(x_i)}} + \frac{\left(\sqrt{\mu_{\overline{A}}(x_i)} - \sqrt{\mu_{\overline{B}}(x_i)}\right)^{2(\alpha+1)}}{\sqrt{\mu_{\overline{A}}(x_i)\mu_{\overline{B}}(x_i)}} \right), \alpha \in \mathbb{N}. \quad (1)$$

Analogous to the above proposed divergence measure for fuzzy sets given by Equation (1), we propose the following parametric divergence measure for single valued neutrosophic set:

$$Div_\alpha(M,N)=\sum_{i=1}^{n}2^\alpha\left[\frac{\left(\sqrt{T_M(x_i)}-\sqrt{T_N(x_i)}\right)^{2(\alpha+1)}}{\left(T_M(x_i)+T_N(x_i)\right)^\alpha}+\frac{\left(\sqrt{1-T_M(x_i)}-\sqrt{1-T_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_M(x_i)-T_N(x_i)\right)^\alpha}\right]$$

$$+\sum_{i=1}^{n}2^\alpha\left[\frac{\left(\sqrt{I_M(x_i)}-\sqrt{I_N(x_i)}\right)^{2(\alpha+1)}}{\left(I_M(x_i)+I_N(x_i)\right)^\alpha}+\frac{\left(\sqrt{1-I_M(x_i)}-\sqrt{1-I_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_M(x_i)-I_N(x_i)\right)^\alpha}\right]$$

$$+\sum_{i=1}^{n}2^\alpha\left[\frac{\left(\sqrt{F_M(x_i)}-\sqrt{F_N(x_i)}\right)^{2(\alpha+1)}}{\left(F_M(x_i)+F_N(x_i)\right)^\alpha}+\frac{\left(\sqrt{1-F_M(x_i)}-\sqrt{1-F_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_M(x_i)-F_N(x_i)\right)^\alpha}\right],\alpha\in\mathbb{N}. \quad (2)$$

Next, we need to prove that the proposed parametric divergence measure for single valued neutrosophic sets is a valid information measure.

Theorem 1. *The divergence measure* $Div_\alpha(M,N)$ *given by Equation (2) is a valid divergence measure for two SVNSs.*

Proof: In order to prove the theorem, we need to show that the divergence measure given by Equation (2) satisfies the four axioms (Divaxiom (1) - (4) [33]) stated in Section 2.

• **Divaxiom 1:** Since Equation (2) is symmetric with respect to M and N, therefore it is quite obvious that $Div_\alpha(M,N)=Div_\alpha(N,M)$.

• **Divaxiom 2:** In view of Equation (2), we observe that $Div_\alpha(M,N)=0 \Leftrightarrow T_M(x)=T_N(x),\ I_M(x)=I_N(x),\ F_M(x)=F_N(x)\ for\ all\ x\in X$. It remains to show that $Div_\alpha(M,N)\geq 0$. For this, we first show the convexity of Div_α. Since Div_α is of the Csiszar's f -divergence type with generating mapping $f_\alpha:(0,\infty)\to\mathbb{R}^+$, defined by,

$$f_\alpha(t)=\frac{2^\alpha\left(\sqrt{t}-1\right)^{2(\alpha+1)}}{(t+1)^\alpha}\ with\ f_\alpha(1)=0. \quad (3)$$

Differentiating Equation (3) two times with respect to t and on simplification, we get

$$f_\alpha^{''}(t)=\left(\frac{2^\alpha}{2}\right)\frac{\left(2t+2\alpha\sqrt{t}+2\alpha t^{3/2}+4\alpha t+t^2+1\right)(\alpha+1)\left(\sqrt{t}-1\right)^{2\alpha}}{(t+1)^{\alpha+2}t^{3/2}}.$$

Since $\alpha\in\mathbb{N}$ and $t\in(0,\infty)$, therefore, $f_\alpha^{''}(t)\geq 0$ which proves the convexity of $f_\alpha(t)$. Thus, $Div_\alpha(M,N)\geq 0$.

• **Divaxiom 3:** For this purpose we decompose the collection X into two disjoint subsets X_1 and X_2 such that,

$$X_1=\{x_i\in X\,|\,T_M(x_i)\geq T_N(x_i)\geq T_P(x_i),I_M(x_i)\leq I_N(x_i)\leq I_P(x_i),F_M(x_i)\leq F_N(x_i)\leq F_P(x_i)\}; \quad (4)$$

and

$$X_2=\{x_i\in X\,|\,T_M(x_i)\leq T_N(x_i)\leq T_P(x_i),I_M(x_i)\geq I_N(x_i)\geq I_P(x_i),F_M(x_i)\geq F_N(x_i)\geq F_P(x_i)\}. \quad (5)$$

Using the definition of intersection of neutrosophic sets and Equation (2) in connection of Equations (4) and (5), the component terms with respect to X_1 will vanish while the component terms with respect to X_2 only will remain in left hand side. Therefore, the left hand side term will have only one term while the right hand side will have two regular terms. The detailed calculation may be shown easily. In view of this, Divaxiom 3 is satisfied.

• **Divaxiom 4:** This axiom can similarly be proved by using the definition of union on the basis of proof of Divaxiom 3. This implies that $Div_\alpha\left(M,N\right)$ is a valid divergence measure between the single valued neutrosophic sets M and N.

4. Properties of New Parameterized Neutrosophic Divergence Measure

In this section some important properties of the proposed parametric measures of neutrosophic fuzzy divergence are given and proved.

Theorem 2. *For any* $M, N \text{ and } P \in SVNS(X)$, *the proposed divergence measure (2) satisfies the following properties:*

1. $Div_\alpha\left(M\cup N, M\cap N\right) = Div_\alpha\left(M,N\right)$
2. $Div_\alpha\left(M\cup N, M\right) + Div_\alpha\left(M\cap N, M\right) = Div_\alpha\left(M,N\right)$
3. $Div_\alpha\left(M\cup N, P\right) + Div_\alpha\left(M\cap N, P\right) = Div_\alpha\left(M,P\right) + Div_\alpha\left(N,P\right)$
4. $Div_\alpha\left(M, M\cup N\right) = Div_\alpha\left(N, M\cap N\right)$
5. $Div_\alpha\left(M, M\cap N\right) = Div_\alpha\left(N, M\cup N\right)$.

Proof : For this purpose we decompose the collection X into two disjoint subsets X_1 & X_2 s.t.,

$$X_1 = \left\{x_i \in X \mid T_M\left(x_i\right) \le T_N\left(x_i\right), I_M(x_i) \ge I_N(x_i), F_M(x_i) \ge F_N(x_i)\right\}; \tag{6}$$

$$X_2 = \left\{x_i \in X \mid T_M\left(x_i\right) \ge T_N\left(x_i\right), I_M(x_i) \le I_N(x_i), F_M(x_i) \le F_N(x_i)\right\}. \tag{7}$$

1. $Div_\alpha(M\cup N, M\cap N)$

$$= \sum_{i=1}^{n} 2^\alpha \left[\frac{\left(\sqrt{T_{M\cup N}\left(x_i\right)} - \sqrt{T_{M\cap N}\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(T_{M\cup N}\left(x_i\right) + T_{M\cap N}\left(x_i\right)\right)^\alpha} + \frac{\left(\sqrt{1-T_{M\cup N}\left(x_i\right)} - \sqrt{1-T_{M\cap N}\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-T_{M\cup N}\left(x_i\right) - T_{M\cap N}\left(x_i\right)\right)^\alpha}\right]$$

$$+\sum_{i=1}^{n} 2^\alpha \left[\frac{\left(\sqrt{I_{M\cup N}\left(x_i\right)} - \sqrt{I_{M\cap N}\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(I_{M\cup N}\left(x_i\right) + I_{M\cap N}\left(x_i\right)\right)^\alpha} + \frac{\left(\sqrt{1-I_{M\cup N}\left(x_i\right)} - \sqrt{1-I_{M\cap N}\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-I_{M\cup N}\left(x_i\right) - I_{M\cap N}\left(x_i\right)\right)^\alpha}\right]$$

$$+\sum_{i=1}^{n} 2^\alpha \left[\frac{\left(\sqrt{F_{M\cup N}\left(x_i\right)} - \sqrt{F_{M\cap N}\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(F_{M\cup N}\left(x_i\right) + F_{M\cap N}\left(x_i\right)\right)^\alpha} + \frac{\left(\sqrt{1-F_{M\cup N}\left(x_i\right)} - \sqrt{1-F_{M\cap N}\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-F_{M\cup N}\left(x_i\right) - F_{M\cap N}\left(x_i\right)\right)^\alpha}\right], \alpha \in \mathbb{N}.$$

In view of the Equation (6) and Equation (7), we have

$$\Rightarrow Div_\alpha(M\cup N, M\cap N) = \sum_{x_i \in X_1}^{n} 2^\alpha \left[\frac{\left(\sqrt{T_N\left(x_i\right)} - \sqrt{T_M\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(T_N\left(x_i\right) + T_M\left(x_i\right)\right)^\alpha} + \frac{\left(\sqrt{1-T_N\left(x_i\right)} - \sqrt{1-T_M\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-T_N\left(x_i\right) - T_M\left(x_i\right)\right)^\alpha}\right]$$

$$+\sum_{x_i \in X_1}^{n} 2^\alpha \left[\frac{\left(\sqrt{I_N\left(x_i\right)} - \sqrt{I_M\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(I_N\left(x_i\right) + I_M\left(x_i\right)\right)^\alpha} + \frac{\left(\sqrt{1-I_N\left(x_i\right)} - \sqrt{1-I_M\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-I_N\left(x_i\right) - I_M\left(x_i\right)\right)^\alpha}\right]$$

$$+\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_N(x_i)}-\sqrt{F_M(x_i)}\right)^{2(\alpha+1)}}{\left(F_N(x_i)+F_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_N(x_i)}-\sqrt{1-F_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_N(x_i)-F_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_M(x_i)}-\sqrt{T_N(x_i)}\right)^{2(\alpha+1)}}{\left(T_M(x_i)+T_N(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_M(x_i)}-\sqrt{1-T_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_M(x_i)-T_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_M(x_i)}-\sqrt{I_N(x_i)}\right)^{2(\alpha+1)}}{\left(I_M(x_i)+I_N(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_M(x_i)}-\sqrt{1-I_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_M(x_i)-I_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_M(x_i)}-\sqrt{F_N(x_i)}\right)^{2(\alpha+1)}}{\left(F_M(x_i)+F_N(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_M(x_i)}-\sqrt{1-F_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_M(x_i)-F_N(x_i)\right)^{\alpha}}\right]$$

$$=Div_{\alpha}(M,N).$$

2. $Div_{\alpha}(M\cup N,M)+Div_{\alpha}(M\cap N,M)$

$$=\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_{M\cup N}(x_i)}-\sqrt{T_M(x_i)}\right)^{2(\alpha+1)}}{\left(T_{M\cup N}(x_i)+T_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_{M\cup N}(x_i)}-\sqrt{1-T_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_{M\cup N}(x_i)-T_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_{M\cap N}(x_i)}-\sqrt{T_M(x_i)}\right)^{2(\alpha+1)}}{\left(T_{M\cap N}(x_i)+T_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_{M\cap N}(x_i)}-\sqrt{1-T_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_{M\cap N}(x_i)-T_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_{M\cup N}(x_i)}-\sqrt{I_M(x_i)}\right)^{2(\alpha+1)}}{\left(I_{M\cup N}(x_i)+I_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_{M\cup N}(x_i)}-\sqrt{1-I_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_{M\cup N}(x_i)-I_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_{M\cap N}(x_i)}-\sqrt{I_M(x_i)}\right)^{2(\alpha+1)}}{\left(I_{M\cap N}(x_i)+I_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_{M\cap N}(x_i)}-\sqrt{1-I_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_{M\cap N}(x_i)-I_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_{M\cup N}(x_i)}-\sqrt{F_M(x_i)}\right)^{2(\alpha+1)}}{\left(F_{M\cup N}(x_i)+F_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_{M\cup N}(x_i)}-\sqrt{1-F_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_{M\cup N}(x_i)-F_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_{M\cap N}(x_i)}-\sqrt{F_M(x_i)}\right)^{2(\alpha+1)}}{\left(F_{M\cap N}(x_i)+F_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_{M\cap N}(x_i)}-\sqrt{1-F_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_{M\cap N}(x_i)-F_M(x_i)\right)^{\alpha}}\right].$$

$$\Rightarrow Div_{\alpha}(M\cup N,M)+Div_{\alpha}(M\cap N,M)$$

$$=\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_N(x_i)}-\sqrt{T_M(x_i)}\right)^{2(\alpha+1)}}{\left(T_N(x_i)+T_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_N(x_i)}-\sqrt{1-T_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_N(x_i)-T_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_M(x_i)}-\sqrt{T_M(x_i)}\right)^{2(\alpha+1)}}{\left(T_M(x_i)+T_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_M(x_i)}-\sqrt{1-T_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_M(x_i)-T_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_M(x_i)}-\sqrt{T_M(x_i)}\right)^{2(\alpha+1)}}{\left(T_M(x_i)+T_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_M(x_i)}-\sqrt{1-T_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_M(x_i)-T_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_N(x_i)}-\sqrt{T_M(x_i)}\right)^{2(\alpha+1)}}{\left(T_N(x_i)+T_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_N(x_i)}-\sqrt{1-T_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_N(x_i)-T_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_N(x_i)}-\sqrt{I_M(x_i)}\right)^{2(\alpha+1)}}{\left(I_N(x_i)+I_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_N(x_i)}-\sqrt{1-I_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_N(x_i)-I_M(x_i)\right)^{\alpha}}\right]+0+0$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_N(x_i)}-\sqrt{I_M(x_i)}\right)^{2(\alpha+1)}}{\left(I_N(x_i)+I_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_N(x_i)}-\sqrt{1-I_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_N(x_i)-I_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_N(x_i)}-\sqrt{F_M(x_i)}\right)^{2(\alpha+1)}}{\left(F_N(x_i)+F_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_N(x_i)}-\sqrt{1-F_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_N(x_i)-F_M(x_i)\right)^{\alpha}}\right]+0+0$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_N(x_i)}-\sqrt{F_M(x_i)}\right)^{2(\alpha+1)}}{\left(F_N(x_i)+F_M(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_N(x_i)}-\sqrt{1-F_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_N(x_i)-F_M(x_i)\right)^{\alpha}}\right]$$

$$=Div_{\alpha}(M,N).$$

3. $Div_{\alpha}(M\cup N,P)+Div_{\alpha}(M\cap N,P)$

$$=\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_{M\cup N}\left(x_i\right)}-\sqrt{T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(T_{M\cup N}\left(x_i\right)+T_P\left(x_i\right)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_{M\cup N}\left(x_i\right)}-\sqrt{1-T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-T_{M\cup N}\left(x_i\right)-T_P\left(x_i\right)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_{M\cap N}\left(x_i\right)}-\sqrt{T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(T_{M\cap N}\left(x_i\right)+T_P\left(x_i\right)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_{M\cap N}\left(x_i\right)}-\sqrt{1-T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-T_{M\cap N}\left(x_i\right)-T_P\left(x_i\right)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_{M\cup N}\left(x_i\right)}-\sqrt{I_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(I_{M\cup N}\left(x_i\right)+I_P\left(x_i\right)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_{M\cup N}\left(x_i\right)}-\sqrt{1-I_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-I_{M\cup N}\left(x_i\right)-I_P\left(x_i\right)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_{M\cap N}\left(x_i\right)}-\sqrt{I_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(I_{M\cap N}\left(x_i\right)+I_P\left(x_i\right)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_{M\cap N}\left(x_i\right)}-\sqrt{1-I_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-I_{M\cap N}\left(x_i\right)-I_P\left(x_i\right)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_{M\cup N}\left(x_i\right)}-\sqrt{F_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(F_{M\cup N}\left(x_i\right)+F_P\left(x_i\right)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_{M\cup N}\left(x_i\right)}-\sqrt{1-F_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-F_{M\cup N}\left(x_i\right)-F_P\left(x_i\right)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_{M\cap N}\left(x_i\right)}-\sqrt{F_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(F_{M\cap N}\left(x_i\right)+F_P\left(x_i\right)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_{M\cap N}\left(x_i\right)}-\sqrt{1-F_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-F_{M\cap N}\left(x_i\right)-F_P\left(x_i\right)\right)^{\alpha}}\right]$$

$$=\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_N\left(x_i\right)}-\sqrt{T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(T_N\left(x_i\right)+T_P\left(x_i\right)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_N\left(x_i\right)}-\sqrt{1-T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-T_N\left(x_i\right)-T_P\left(x_i\right)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_M\left(x_i\right)}-\sqrt{T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(T_M\left(x_i\right)+T_P\left(x_i\right)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_M\left(x_i\right)}-\sqrt{1-T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-T_M\left(x_i\right)-T_P\left(x_i\right)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_M\left(x_i\right)}-\sqrt{T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(T_M\left(x_i\right)+T_P\left(x_i\right)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_M\left(x_i\right)}-\sqrt{1-T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-T_M\left(x_i\right)-T_P\left(x_i\right)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_N\left(x_i\right)}-\sqrt{T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(T_N\left(x_i\right)+T_P\left(x_i\right)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_N\left(x_i\right)}-\sqrt{1-T_P\left(x_i\right)}\right)^{2(\alpha+1)}}{\left(2-T_N\left(x_i\right)-T_P\left(x_i\right)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_N(x_i)}-\sqrt{I_P(x_i)}\right)^{2(\alpha+1)}}{\left(I_N(x_i)+I_P(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_N(x_i)}-\sqrt{1-I_P(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_N(x_i)-I_P(x_i)\right)^{\alpha}}\right]+0+0$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_M(x_i)}-\sqrt{I_P(x_i)}\right)^{2(\alpha+1)}}{\left(I_M(x_i)+I_P(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_M(x_i)}-\sqrt{1-I_P(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_M(x_i)-I_P(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_M(x_i)}-\sqrt{I_P(x_i)}\right)^{2(\alpha+1)}}{\left(I_M(x_i)+I_P(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_M(x_i)}-\sqrt{1-I_P(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_M(x_i)-I_P(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_N(x_i)}-\sqrt{I_P(x_i)}\right)^{2(\alpha+1)}}{\left(I_N(x_i)+I_P(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_N(x_i)}-\sqrt{1-I_P(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_N(x_i)-I_P(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_N(x_i)}-\sqrt{F_P(x_i)}\right)^{2(\alpha+1)}}{\left(F_N(x_i)+F_P(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_N(x_i)}-\sqrt{1-F_P(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_N(x_i)-F_P(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_M(x_i)}-\sqrt{F_P(x_i)}\right)^{2(\alpha+1)}}{\left(F_M(x_i)+F_P(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_M(x_i)}-\sqrt{1-F_P(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_M(x_i)-F_P(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_M(x_i)}-\sqrt{F_P(x_i)}\right)^{2(\alpha+1)}}{\left(F_M(x_i)+F_P(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_M(x_i)}-\sqrt{1-F_P(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_M(x_i)-F_P(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_N(x_i)}-\sqrt{F_P(x_i)}\right)^{2(\alpha+1)}}{\left(F_N(x_i)+F_P(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_N(x_i)}-\sqrt{1-F_P(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_N(x_i)-F_P(x_i)\right)^{\alpha}}\right]$$

$$=Div_{\alpha}(M,P)+Div_{\alpha}(N,P).$$

4. $Div_{\alpha}(M,M\cup N)$

$$=\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_M(x_i)}-\sqrt{T_{M\cup N}(x_i)}\right)^{2(\alpha+1)}}{\left(T_M(x_i)+T_{M\cup N}(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_M(x_i)}-\sqrt{1-T_{M\cup N}(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_M(x_i)-T_{M\cup N}(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_M(x_i)}-\sqrt{I_{M\cup N}(x_i)}\right)^{2(\alpha+1)}}{\left(I_M(x_i)+I_{M\cup N}(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_M(x_i)}-\sqrt{1-I_{M\cup N}(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_M(x_i)-I_{M\cup N}(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_M(x_i)}-\sqrt{F_{M\cup N}(x_i)}\right)^{2(\alpha+1)}}{\left(F_M(x_i)+F_{M\cup N}(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_M(x_i)}-\sqrt{1-F_{M\cup N}(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_M(x_i)-F_{M\cup N}(x_i)\right)^{\alpha}}\right]$$

$$=\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_M(x_i)}-\sqrt{T_N(x_i)}\right)^{2(\alpha+1)}}{\left(T_M(x_i)+T_N(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_M(x_i)}-\sqrt{1-T_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_M(x_i)-T_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_M(x_i)}-\sqrt{I_N(x_i)}\right)^{2(\alpha+1)}}{\left(I_M(x_i)+I_N(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_M(x_i)}-\sqrt{1-I_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_M(x_i)-I_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_M(x_i)}-\sqrt{F_N(x_i)}\right)^{2(\alpha+1)}}{\left(F_M(x_i)+F_N(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_M(x_i)}-\sqrt{1-F_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_M(x_i)-F_N(x_i)\right)^{\alpha}}\right]=Div_{\alpha}\left(N,M\cap N\right).$$

5. $Div_{\alpha}(M,M\cap N)$

$$=\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_M(x_i)}-\sqrt{T_{M\cap N}(x_i)}\right)^{2(\alpha+1)}}{\left(T_M(x_i)+T_{M\cap N}(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_M(x_i)}-\sqrt{1-T_{M\cap N}(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_M(x_i)-T_{M\cap N}(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_M(x_i)}-\sqrt{I_{M\cap N}(x_i)}\right)^{2(\alpha+1)}}{\left(I_M(x_i)+I_{M\cap N}(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_M(x_i)}-\sqrt{1-I_{M\cap N}(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_M(x_i)-I_{M\cap N}(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{i=1}^{n}2^{\alpha}\left[\frac{\left(\sqrt{F_M(x_i)}-\sqrt{F_{M\cap N}(x_i)}\right)^{2(\alpha+1)}}{\left(F_M(x_i)+F_{M\cap N}(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_M(x_i)}-\sqrt{1-F_{M\cap N}(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_M(x_i)-F_{M\cap N}(x_i)\right)^{\alpha}}\right]$$

$$=\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{T_M(x_i)}-\sqrt{T_N(x_i)}\right)^{2(\alpha+1)}}{\left(T_M(x_i)+T_N(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-T_M(x_i)}-\sqrt{1-T_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_M(x_i)-T_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n}2^{\alpha}\left[\frac{\left(\sqrt{I_M(x_i)}-\sqrt{I_N(x_i)}\right)^{2(\alpha+1)}}{\left(I_M(x_i)+I_N(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-I_M(x_i)}-\sqrt{1-I_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_M(x_i)-I_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{F_M(x_i)}-\sqrt{F_N(x_i)}\right)^{2(\alpha+1)}}{\left(F_M(x_i)+F_N(x_i)\right)^{\alpha}}+\frac{\left(\sqrt{1-F_M(x_i)}-\sqrt{1-F_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_M(x_i)-F_N(x_i)\right)^{\alpha}}\right]=Div_{\alpha}\left(N,M\cup N\right).$$

Theorem 3. *For any* $M,N\in SVNS(X)$*, the proposed divergence measure (2) satisfies the following properties:*

1. $Div_{\alpha}\left(\overline{M},\overline{N}\right)=Div_{\alpha}\left(M,N\right)$
2. $Div_{\alpha}\left(\overline{M\cup N},\overline{M\cap N}\right)=Div_{\alpha}\left(\overline{M}\cap\overline{N},\overline{M}\cup\overline{N}\right)=Div_{\alpha}\left(M,N\right)$
3. $Div_{\alpha}\left(M,\overline{N}\right)=Div_{\alpha}\left(\overline{M},N\right)$
4. $Div_{\alpha}\left(M,\overline{N}\right)+Div_{\alpha}\left(\overline{M},\overline{N}\right)=Div_{\alpha}\left(M,N\right)+Div_{\alpha}\left(\overline{M},N\right)$

Proof:

1. As per the definition of the *complement* given in Section 2, this result holds.
2. In view of the Equation (6) and Equation (7), we get $Div_{\alpha}(\overline{M\cup N},\overline{M\cap N})$

$$=\sum_{x_i\in X_1}^{n} 2^{\alpha}\frac{\left(\sqrt{1-T_N(x_i)}-\sqrt{1-T_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_N(x_i)-T_M(x_i)\right)^{\alpha}}+\sum_{x_i\in X_1}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{T_N(x_i)}-\sqrt{T_M(x_i)}\right)^{2(\alpha+1)}}{\left(T_N(x_i)+T_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n} 2^{\alpha}\frac{\left(\sqrt{1-T_M(x_i)}-\sqrt{1-T_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_M(x_i)-T_N(x_i)\right)^{\alpha}}+\sum_{x_i\in X_2}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{T_M(x_i)}-\sqrt{T_N(x_i)}\right)^{2(\alpha+1)}}{\left(T_M(x_i)+T_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n} 2^{\alpha}\frac{\left(\sqrt{1-I_N(x_i)}-\sqrt{1-I_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_N(x_i)-I_M(x_i)\right)^{\alpha}}+\sum_{x_i\in X_1}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{I_N(x_i)}-\sqrt{I_M(x_i)}\right)^{2(\alpha+1)}}{\left(I_N(x_i)+I_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n} 2^{\alpha}\frac{\left(\sqrt{1-I_M(x_i)}-\sqrt{1-I_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_M(x_i)-I_N(x_i)\right)^{\alpha}}+\sum_{x_i\in X_2}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{I_M(x_i)}-\sqrt{I_N(x_i)}\right)^{2(\alpha+1)}}{\left(I_M(x_i)+I_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n} 2^{\alpha}\frac{\left(\sqrt{1-F_N(x_i)}-\sqrt{1-F_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_N(x_i)-F_M(x_i)\right)^{\alpha}}+\sum_{x_i\in X_1}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{F_N(x_i)}-\sqrt{F_M(x_i)}\right)^{2(\alpha+1)}}{\left(F_N(x_i)+F_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n} 2^{\alpha}\frac{\left(\sqrt{1-F_M(x_i)}-\sqrt{1-F_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_M(x_i)-F_N(x_i)\right)^{\alpha}}+\sum_{x_i\in X_2}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{F_M(x_i)}-\sqrt{F_N(x_i)}\right)^{2(\alpha+1)}}{\left(F_M(x_i)+F_N(x_i)\right)^{\alpha}}\right]=Div_{\alpha}(M,N).$$

On the other hand, $Div_{\alpha}(\overline{M}\cap\overline{N},\overline{M}\cup\overline{N})$

$$=\sum_{x_i\in X_1}^{n} 2^{\alpha}\frac{\left(\sqrt{1-T_N(x_i)}-\sqrt{1-T_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_M(x_i)-T_N(x_i)\right)^{\alpha}}+\sum_{x_i\in X_1}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{T_N(x_i)}-\sqrt{T_M(x_i)}\right)^{2(\alpha+1)}}{\left(T_N(x_i)+T_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n} 2^{\alpha}\frac{\left(\sqrt{1-T_M(x_i)}-\sqrt{1-T_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_M(x_i)-T_N(x_i)\right)^{\alpha}}+\sum_{x_i\in X_2}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{T_M(x_i)}-\sqrt{T_N(x_i)}\right)^{2(\alpha+1)}}{\left(T_M(x_i)+T_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n} 2^{\alpha}\frac{\left(\sqrt{1-I_N(x_i)}-\sqrt{1-I_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_N(x_i)-I_M(x_i)\right)^{\alpha}}+\sum_{x_i\in X_1}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{I_N(x_i)}-\sqrt{I_M(x_i)}\right)^{2(\alpha+1)}}{\left(I_N(x_i)+I_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n} 2^{\alpha}\frac{\left(\sqrt{1-I_M(x_i)}-\sqrt{1-I_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_M(x_i)-I_N(x_i)\right)^{\alpha}}+\sum_{x_i\in X_2}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{I_M(x_i)}-\sqrt{I_N(x_i)}\right)^{2(\alpha+1)}}{\left(I_M(x_i)+I_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n} 2^{\alpha}\frac{\left(\sqrt{1-F_N(x_i)}-\sqrt{1-F_M(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_N(x_i)-F_M(x_i)\right)^{\alpha}}+\sum_{x_i\in X_1}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{F_N(x_i)}-\sqrt{F_M(x_i)}\right)^{2(\alpha+1)}}{\left(F_N(x_i)+F_M(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_2}^{n} 2^{\alpha}\frac{\left(\sqrt{1-F_M(x_i)}-\sqrt{1-F_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_M(x_i)-F_N(x_i)\right)^{\alpha}}+\sum_{x_i\in X_2}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{F_M(x_i)}-\sqrt{F_N(x_i)}\right)^{2(\alpha+1)}}{\left(F_M(x_i)+F_N(x_i)\right)^{\alpha}}\right]=Div_{\alpha}(M,N).$$

Therefore, $Div_{\alpha}\left(\overline{M\cup N},\overline{M\cap N}\right)=Div_{\alpha}\left(\overline{M}\cap\overline{N},\overline{M}\cup\overline{N}\right)=Div_{\alpha}(M,N)$.

3. $$Div_{\alpha}(M,\overline{N})=\sum_{x_i\in X_1}^{n} 2^{\alpha}\frac{\left(\sqrt{T_M(x_i)}-\sqrt{1-T_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-T_N(x_i)-T_M(x_i)\right)^{\alpha}}+\sum_{x_i\in X_2}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{1-T_M(x_i)}-\sqrt{T_N(x_i)}\right)^{2(\alpha+1)}}{\left(1-T_M(x_i)+T_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n} 2^{\alpha}\frac{\left(\sqrt{I_M(x_i)}-\sqrt{1-I_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-I_N(x_i)-I_M(x_i)\right)^{\alpha}}+\sum_{x_i\in X_2}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{1-I_M(x_i)}-\sqrt{I_N(x_i)}\right)^{2(\alpha+1)}}{\left(1-I_M(x_i)+I_N(x_i)\right)^{\alpha}}\right]$$

$$+\sum_{x_i\in X_1}^{n} 2^{\alpha}\frac{\left(\sqrt{F_M(x_i)}-\sqrt{1-F_N(x_i)}\right)^{2(\alpha+1)}}{\left(2-F_N(x_i)-F_M(x_i)\right)^{\alpha}}+\sum_{x_i\in X_2}^{n} 2^{\alpha}\left[\frac{\left(\sqrt{1-F_M(x_i)}-\sqrt{F_N(x_i)}\right)^{2(\alpha+1)}}{\left(1-F_M(x_i)+F_N(x_i)\right)^{\alpha}}\right]=Div_{\alpha}\left(M,\overline{N}\right).$$

4. Using (a) and (c), $Div_{\alpha}\left(M,\overline{N}\right)+Div_{\alpha}\left(\overline{M},\overline{N}\right)=Div_{\alpha}\left(M,N\right)+Div_{\alpha}\left(\overline{M},N\right)$ holds.

5. Application of Parametric Divergence Measure in Decision Making Problems

We study some important applications of the proposed divergence measure for neutrosophic sets in the area of classication problems and decision-making.

5.1 Pattern Recognition

In order to illustrate an application of the proposed divergence measure in the field of pattern recognition, we refer to a well posed problem which has been discussed in literature [33]. Consider three existing patterns A_1, A_2 and A_3 representing the classes C_1, C_2 and C_3 respectively and being described by the following SVNSs in $X=\{x_1,x_2,x_3\}$:

$$A_1=\{(x_1,0.7,0.7,0.2),(x_2,0.7,0.8,0.4),(x_3,0.6,0.8,0.2)\};$$

$$A_2=\{(x_1,0.5,0.7,0.3),(x_2,0.7,0.7,0.5),(x_3,0.8,0.6,0.1)\};$$

$$A_3=\{(x_1,0.9,0.5,0.1),(x_2,0.7,0.6,0.4),(x_3,0.8,0.5,0.2)\}.$$

Consider an unknown sample pattern B which is given by

$$B=\{(x_1,0.7,0.8,0.4),(x_2,0.8,0.5,0.3),(x_3,0.5,0.8,0.5)\}.$$

Now, the main objective of the problem is to find out the class to which *B* belongs. As per the principle of minimum divergence measure [44], the procedure for allocation of *B* to C_{β^*} is determined by

$$\beta^*=arg\min_{\beta}(Div_{\alpha}(A_k,B)). \tag{8}$$

Table 1: Values of $Div_{\alpha}(A_k,B)$ with $\beta\in\{1,2,3\}$

	α	A_1	A_2	A_3
B	1	**0.035200913**	0.109091158	0.116197599
B	4	**0.0001939**	0.010382714	0.003212291

Clearly, from the Table 1, it may be observed that *B* has to get into the class C_1. The obtained result is based on the proposed parametric divergence measure and is perfectly consistent with the results achieved by [33].

5.2 Medical Diagnosis

In a classical problem of medical diagnosis, assume that if a doctor needs to diagnose some of patients' "$P = \{Al, Bob, Joe, Ted\}$" under some defined diagnoses

"$D = \{Viral\ fever,\ Malaria,\ Typhoid,\ Stomach\ problem,\ Cough,\ Chest\ problem\}$",

& a set of symptoms "$S = \{Temperature,\ Headache,\ Stomach\ pain,\ Cough,\ Chest\ pain\}$".

The following tables (Refer Table 2 & Table 3) serve the purpose of the proposed computational application:

Table 2: Symptoms characteristic for the diagnoses considered [33]

	"Viral Fever"	*"Malaria"*	*"Typhoid"*	*"Stomach Problem"*	*"Chest Problem"*
"Temperature"	(0.7,0.5,0.6)	(0.7,0.9,0.1)	(0.3,0.7,0.2)	(0.1,0.6,0.7)	(0.1,0.9,0.8)
"Headache"	(0.8,0.2,0.9)	(0.4,0.5,0.5)	(0.6,0.9,0.2)	(0.7,0.4,0.3)	(0.1,0.6,0.7)
"Stomach Pain"	(0.8,0.1,0.1)	(0.5,0.9,0.2)	(0.2,0.5,0.5)	(0.7,0.7,0.8)	(0.5, 0. 7, 0.6)
"Cough"	(0.45,0.8,0.7)	(0.7,0.8,0.6)	(0.2,0.5,0.5)	(0.2,0.8,0.65)	(0.2,0.8,0.6)
"Chest Pain"	(0.2,0.6,0.5)	(0.1,0.6,0.8)	(0.1,0.8,0.8)	(0.5,0.8,0.6)	(0.8,0.8,0.2)

Table 3: Symptoms for the diagnose under consideration

	"Temperature"	*"Headache"*	*"Stomach pain"*	*"Cough"*	*"Chest pain"*
"Al"	(0.7,0.6,0.5)	(0.6,0.3,0.5)	(0.5,0.5,0.75)	(0.8,0.75,0.5)	(0.7,0.2,0.6)
"Bob"	(0.7,0.3,0.5)	(0.5,0.5,0.8)	(0.6,0.5,0.5)	(0.65,0.4,0.75)	(0.2,0.85,0.65)
"Joe"	(0.75,0.5,0.5)	(0.2,0.85,0.7)	(0.7,0.6,0.4)	(0.7,0.55,0.5)	(0.5,0.9,0.64)
"Ted"	(0.4,0.7,0.6)	(0.7,0.5,0.7)	(0.6,0.7,0.5)	(0.5,0.9,0.65)	(0.6,0.5,0.85)

In order to have a proper diagnose, we evaluate the value of the proposed divergence measure $Div_\alpha(P, d_\beta)$ between the patient's symptoms & the defined symptoms for each diagnose $d_\beta \in D$, with $\beta = \{1, 2, ..., 5\}$. Similar to the Equation (8), the proper diagnose d_β for the patient P may be based on the following equation:

$$\beta^* = \arg\min_\beta \left(Div_\alpha \left(P, d_\beta \right) \right). \tag{9}$$

Table 4: Values of $Div_\alpha(A_k, B)$, with $\beta \in \{1,2,3\}$

	"Viral Fever"	*"Malaria"*	*"Typhoid"*	*"Stomach Problem"*	*"Chest Problem"*
"Al"	0.29738	0.27867	0.41362	**0.26433**	0.37028

"Bob"	**0.09767**	0.23291	0.13969	0.18798	0.41526
"Joe"	0.29928	0.16506	**0.13089**	0.26385	0.26343
"Ted"	0.20289	0.16876	0.20367	**0.03168**	0.27893

In view of Table 4, it is being concluded that the patient *Al* and *Ted* are suffering from the *stomach problem*, *Bob* is suffering from *viral fever* and *Joe* is suffering from *Typhoid*.

It may be observed that the result obtained through the proposed method is perfectly consistent with the results achieved by [33].

Comparative Remarks: It may be observed that the proposed method is found to be perfectly competent to provide the desired result with an added advantage of the parameters involvement in the proposed divergence measure. The parameters may provide a better variability in the selection of a divergence measure for achieving a better specificity and accuracy.

5.2 Multi-criteria Decision-Making Problem

The main purpose of MCDM problem is to identify the alternative from the available alternatives under consideration. Here, on the basis of the proposed parametric divergence measure for neutrosophic sets, an algorithm for solving MCDM problem is being outlined. Consider the available m-alternatives, i.e., $Z=\{Z_1,Z_2,\ldots,Z_m\}$ and n-criterion, i.e., $O=\{o_1,o_2,\ldots,o_n\}$. The target of a decision maker is to pick the optimal alternative out of the available *m*-alternatives fulfilling the *n*-criterion. The perspectives/opinions of decision makers have been taken in the form of a matrix $A=[a_{ij}]_{m\times n}$ called neutrosophic decision matrix where $a_{ij}=(T_{ij},\, I_{ij},\, F_{ij})$.

Procedural Steps of Algorithm for MCDM Problem:

Step 1: Construct the neutrosophic decision matrix based on the available data.

Step 2: Sometimes heterogeneity in the type of criterions in a MCDM problem is observed. In order to resolve this issue, it is required to make them homogeneous before applying any methodology. Mainly, the criteria may be categorized into two types: benefit criteria and cost criteria. We need to transform the decision matrix, for this we transform the cost criteria into the benefit criteria. Thus the decision matrix $A=[a_{ij}]_{m\times n}$ is converted into a new decision matrix, say, $B=[b_{ij}]_{m\times n}$ where b_{ij} is given by

$$b_{ij}=(T_{ij},\, I_{ij},\, F_{ij})=\begin{cases} a_{ij} & \textit{for benefits criteria}\,; \\ a_{ij}^{c} & \textit{for cost criteria}\,; \end{cases} \tag{10}$$

where $B=[b_{ij}]_{m\times n}$ representing the alternatives in the form of

$$Z_i=\{(o_j, 1-T_{ij},\, 1-I_{ij},\, 1-F_{ij})\,|\, o_j\in O\};\ i=1,2,\ldots,m \textit{ and } j=1,2,\ldots,n. \tag{11}$$

Step 3: Evaluate the best preferred solution as

$$Z^{+}=\{\sup(T_{ij}(Z_i)), \inf(I_{ij}(Z_i)), \inf(F_{ij}(Z_i))\}\ i=1,2,\ldots,m \textit{ and } j=1,2,\ldots,n. \tag{12}$$

Step 4: Determine the value of divergence measure of alternatives $Z_i^{'}s$ from Z^{+} using Equation (2).

Step 5. Now, sorting the computed values of the divergence measure, we can find the preference order of the alternatives $Z_i^{'}s$. The best alternative is the one which corresponds to the least value of the divergence measure.

For the sake of illustration of the proposed methodology, a multi-criteria decision-making problem [45] related to a manufacturing company which needs to hire the best supplier. Assume that there are four available suppliers $Z = \{Z_1, Z_2, Z_3, Z_4\}$ whose capabilities and competencies have been evaluated with the help of four laid down criteria $O = \{o_1, o_2, o_3, o_4\}$. Based on the information available about the suppliers w.r.t. the individual criteria, we determine a neutrosophic decision matrix as given below:

1. In the given MCDM problem, all criterions are of same kind. Therefore, we need not to transform the cost criteria into the benefit criteria or vice versa by using Equation (10). The constructed neutrosophic decision matrix based on the available information is in the following Table 5.

Table 5: Neutrosophic Decision Matrix

	o_1	o_2	o_3	o_4
Z_1	(0.5, 0.1, 0.3)	(0.5, 0.1, 0.4)	(0.7, 0.1, 0.2)	(0.3, 0.2, 0.1)
Z_2	(0.4, 0.2, 0.3)	(0.3, 0.2, 0.4)	(0.9,0.0,0.1)	(0.5, 0.3, 0.2)
Z_3	(0.4, 0.3, 0.1)	(0.5, 0.1, 0.3)	(0.5, 0.0, 0.4)	(0.6, 0.2, 0.2)
Z_4	(0.6,0.1,0.2)	(0.2, 0.2, 0.5)	(0.4, 0.3, 0.2)	(0.7, 0.2, 0.1)

2. The best preferred solution obtained by using equation (12) is given by

$$Z^+ = \{(0.6, 0.1, 0.1), (0.5, 0.1, 0.3), (0.9, 0.0, 0.1), (0.7, 0.2, 0.1)\}.$$

3. Compute the values of divergence measure between $Z_i^{'}s\ (i = 1, 2, 3, 4)$ and Z^+ using Equation (2) and tabulate them in the following Table 6.

Table 6: Values of Proposed Divergence Measure between $Z_i^{'}s$ and Z^+

4. Now, the ranking of the alternatives can be performed. The best alterative is one which has the lowest value of the divergence measure from the best preferred solution. The sequence of the alternatives has been particularly obtained as: $Z_2 > Z_1 > Z_3 > Z_4$.

Divergence Measure	(Z_1, Z^+)	(Z_2, Z^+)	(Z_3, Z^+)	(Z_4, Z^+)
Proposed Divergence Measure	0.42409	0.27570	0.4791	0.80810
Ye's Divergence Measure [33]	1.1101	1.1801	0.9962	1.2406

Hence, among all the four suppliers, Z_2 is supposed to be the best one.

6. Conclusions and Scope for Future Work

The parametric divergence measure for SVNSs has been successfully proposed along with discussions on its various properties. In literature, this parametric measure for the neutrosophic set is for the first time where the applications of the proposed divergence measure have been successfully utilized in the computational fields of pattern analysis, medical diagnosis & MCDM problem. The procedural steps of the proposed methodologies for solving these application problems have been well illustrated with numerical examples for each. The results hence obtained are found to be equally and firmly consistent in comparison with the existing methodologies.

In order to have a direction for the scope of future work, it has been observed that there is a notion of another set called rough set, which do not conflict the concept of neutrosophic set, can be mutually incorporated. Sweety and Arockiarani [46] combined the mathematical tools of fuzzy sets, rough sets and neutrosophic sets and introduced a new notion termed as fuzzy neutrosophic rough sets. In future, the following important research contributions can be systematically carried out

- The study on the various information measures - entropy, similarity measures and divergence measures for fuzzy neutrosophic rough sets can be done with their various possible applications.
- In recent years, various researchers have duly utilized the notion of neutrosophic sets to relations, theory of groups and rings, theory of soft sets and so on. On the basis of this, the theoretical contribution related to fuzzy neutrosophic rough sets in the field of algebra may be proposed.

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