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Plithogenic and Neutrosophic Markov Chains: Modeling Uncertainty and Ambiguity in Stochastic Processes

Suhar Massassati[1], Mohamed Bisher Zeina[2] and Yasin Karmouta[3]

[1] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria, e-mail: suharmassasati1987@gmail.com.

[2] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria, e-mail: bisher.zeina@gmail.com.

[3] Faculty of Science, Dept. of Mathematical Statistics, University of Aleppo, Aleppo, Syria, e-mail: yassinkarmouta@gmail.com.

Abstract: In this work we present for the first time the concept of literal neutrosophic markov chains and literal plithogenic markov chains. Also, we presented many theorems related to the properties of transition matrix. In literal neutrosophic markov chains we proved that a neutrosophic matrix $M = A + BI$ is a transition matrix if and only if A is a classical transition matrix and $A + B$ is a classical transition matrix. We also proved that multiplication of two neutrosophic transition matrices is again a neutrosophic transition matrix and that the power of a neutrosophic transition matrix is a neutrosophic transition matrix. Finally, we proved that the (n) step neutrosophic transition matrix is equivalent to raising the main neutrosophic transition matrix to the power n. In literal plithogenic markov chains which is a generalization of the previous case we proved that $M = A + BP_1 + CP_2$ is a plithogenic transition matrix if and only if all of the matrices $A, A + B, A + B + C$ are transition matrices in classical concept. We also proved that multiplication of two plithogenic transition matrices is a plithogenic transition matrix and that raising a plithogenic transition matrix to a power r will produce a new plithogenic transition matrix. Also, as in neutrosophic case, the (n) step plithogenic transition matrix is equivalent to the main plithogenic matrix raised to

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the power n. Theorems were provided with suitable solved examples and problems.

Keywords: Neutrosophic; Plithogenic; Markov Chains; Transition Matrix; Chapman-Kolmogorov.

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1. Introduction

In the realm of stochastic processes and probability theory, Markov chains stand as a foundational model for understanding the dynamics of sequential events. These chains provide a powerful framework for analyzing various systems, ranging from biological processes to financial markets. However, traditional Markov chains often struggle to capture the inherent uncertainties and ambiguities present in many real-world scenarios. [1]–[5]

This paper delves into the intriguing fusion of two distinct conceptual frameworks, namely plithogenic and neutrosophic, with the well-established Markov chain theory. Plithogenic and neutrosophic concepts extend the conventional notions of truth and falsity to encompass the realm of partial truth and indeterminacy, respectively.[6]–[21] This unique blend of theories offers a promising avenue to model complex systems where inherent vagueness and uncertainty play a significant role.

Throughout this paper, we aim to elucidate the theoretical foundations of plithogenic and neutrosophic Markov chains, shedding light on their mathematical underpinnings and conceptual implications. We will explore how these novel extensions can be seamlessly integrated into traditional Markov chain models which have many practical applications across diverse domains such as decision-making, risk assessment, and artificial intelligence.

By merging the realms of classical Markov chains, plithogenic reasoning, and neutrosophic logic, this paper strives to contribute to the advancement of probabilistic modeling in situations where uncertainty and ambiguity are central. Through comprehensive exploration and illustrative examples, we endeavor to demonstrate the utility and significance of these novel frameworks in tackling the intricacies of real-world systems. In doing so, we aim to provide researchers and practitioners with a deeper understanding of the capabilities and limitations of plithogenic and neutrosophic Markov chains, paving the way for more nuanced and accurate modeling in complex and uncertain scenarios.

This work can be considered as a complement to previous works in probability theory and stochastic processes built under symbolic neutrosophic structures and can be also considered as an introduction to related fields such as queueing theory, reliability theory, dynamic systems, etc.[11], [17], [22]–[41]

2. Preliminaries

Definition 2.1

Let $\boldsymbol{R(I)} = \{\boldsymbol{a + bI}; \boldsymbol{I^2 = I}\}$, we call $\mathbf{R(I)}$ the neutrosophic field of reals.

Definition 2.2

Let $R(I)$ be the neutrosophic field of reals, and let $a_N = a_1 + a_2 I, b_N = b_1 + b_2 I \in R(I)$. We can say that $a_N \geq_N b_N$ if: $a_1 \geq b_1$ and $a_1 + a_2 \geq b_1 + b_2$

Definition 2.3

One-dimensional isometry between R(I) and R×R and its inverse are defined as follows:

$\boldsymbol{T}: \boldsymbol{R(I)} \rightarrow \boldsymbol{R} \times \boldsymbol{R}; \boldsymbol{T(a + bI) = (a, a + b)}$.

$\boldsymbol{T^{-1}}: \boldsymbol{R} \times \boldsymbol{R} \rightarrow \boldsymbol{R(I)}; \boldsymbol{T^{-1}(a, b) = a + (b - a)I}$.

Definition 2.4

Let $\boldsymbol{R(P_1,P_2)} = \{\boldsymbol{a_0 + a_1P_1 + a_2P_2; P_1^2 = P_1, P_2^2 = P_2, P_1P_2 = P_2P_1 = P_2}\}$, we call $\boldsymbol{R(P_1,P_2)}$ Plithogenic field of reals.

Definition 2.5

Let $R(P_1,P_2)$ be the Plithogenic field of reals, and let $a_P = a_0 + a_1P_1 + a_2P_2, b_P = b_0 + b_1P_1 + b_2P_2 \in R(P_1,P_2)$. We say that $a_P \geq_P b_P$ if:

$\boldsymbol{a_0 \geq b_0, a_0 + a_1 \geq b_0 + b_1 \ and \ a_0 + a_1 + a_2 \geq b_0 + b_1 + b_2}$

Definition 2.6

One-dimensional isometry between $\boldsymbol{R(P_1,P_2)}$ and the space $\boldsymbol{R \times R \times R}$ is defined as follows:

$\boldsymbol{T: R(P_1,P_2) \to R \times R \times R; T(a_0 + a_1P_1 + a_2P_2) = (a_0, a_0 + a_1, a_0 + a_1 + a_2)}$

$T^{-1}: R \times R \times R \to R(P_1,P_2); T^{-1}(a_0,a_1,a_2) = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2$

3. Literal Neutrosophic Markov chains

Definition 3.1

A set of random variables $X_0, X_1, X_2, \dots$ satisfying:

$Pr\{X_{n+1} = i_{n+1} | X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\} = Pr\{X_{n+1} = i_{n+1} | X_n = i_n\}$

is called a literal or symbolic neutrosophic markov chain if the last probability takes the form $Pr\{X_{n+1} = i_{n+1} | X_n = i_n\} = a + bI; 0 \leq a \leq 1, 0 \leq a + b \leq 1, I^2 = I$

Definition 3.2

We call $\boldsymbol{p_{ij}^{(n,n+1)}}_{\boldsymbol{N}} = \boldsymbol{Pr(X_{n+1} = j | X_n = i) \in R(I)}$ literal or symbolic neutrosophic one-step transition probability.

Definition 3.3

A squared neutrosophic matrix

$M_N = A + BI = [a_{ij} + b_{ij}I]_{n \times n}$

Is called a neutrosophic markov transition matrix if its elements satisfy:

1. $\sum_j a_{ij} + b_{ij}I = 1$; $i = 1,2,3, \dots, n$
2. $0 \leq_N a_{ij} + b_{ij}I \leq_N 1$; $i,j = 1,2,3, \dots, n$

Example 3.1

let′s take: $M_N = \begin{bmatrix} 0.3I & 1-0.3I \\ 0.4+0.2I & 0.6-0.2I \end{bmatrix}$

Then M_N is a neutrosophic transition matrix because:

$$0.3I + 1 - 0.3I = 1 \quad \text{and} \quad 0.4 + 0.2I + 0.6 - 0.2I = 1$$

Also, according to the definition of comparison between Neutrosophic numbers we have:

$0 + 0.3I \leq_N 1 + 0I$ because $0 \leq 1$ & $0.3 \leq 1$

$1 - 0.3I \leq_N 1 + 0I$ because $1 \leq 1$ & $0.7 \leq 1$

$0.4 + 0.2I \leq_N 1 + 0I$ because $0.4 \leq 1$ & $0.6 \leq 1$

$0.6 - 0.2I \leq_N 1 + 0I$ because $0.6 \leq 1$ & $0.4 \leq 1$

$0 + 0I \leq_N 0 + 0.3I$ because $0 \leq 0$ & $0 \leq 0.3$

$0 + 0I \leq_N 1 - 0.3I$ because $0 \leq 1$ & $0 \leq 0.7$

$0 + 0I \leq_N 0.4 + 0.2I$ because $0 \leq 0.4$ & $0 \leq 0.6$

$0 + 0I \leq_N 0.6 - 0.2I$ because $0 \leq 0.6$ & $0 \leq 0.4$

Theorem 3.1

The matrix $M_N = A + BI$ is a neutrosophic transition matrix if and only if A is a crisp transition matrix and $A + B$ is a crisp transition matrix.

Proof

Let's assume that M_N is a neutrosophic transition matrix and prove that A *and* $A + B$ are two transition matrices:

we have $0 + 0I \leq_N a_{ij} + b_{ij}I \leq_N 1 + 0I$ so $a_{ij} \leq 1$, $a_{ij} + b_{ij} \leq 1$, $0 \leq a_{ij}$ and $0 \leq a_{ij} + b_{ij}$ which means that:

$0 \leq a_{ij} \leq 1$ and $0 \leq a_{ij} + b_{ij} \leq 1$

Also, we have $\sum_j (a_{ij} + b_{ij}I) = 1 = 1 + 0I$ which means that $\sum_j b_{ij} = 0$ and $\sum_j a_{ij} = 1$

So, we can conclude that:

$0 \leq a_{ij} \leq 1\ and \sum_j a_{ij} = 1 \Rightarrow\ A$ is a transition matrix.

$0 \leq a_{ij} + b_{ij} \leq 1\ and \sum_j (a_{ij} + b_{ij}) = 1 \Rightarrow\ A + B$ is a transition matrix.

Now, let's assume that both A and A+B are transition matrices and prove that $M_N = A + BI$ is a neutrosophic transition matrix:

since $A, A + B$ are transition matrices then $0 \leq \mathrm{a_{ij}} \leq 1$, $0 \leq \mathrm{a_{ij}} + \mathrm{b_{ij}} \leq 1$ which means that $0 \leq a_{ij} + b_{ij}I \leq 1$

Also, we have $\sum_\mathrm{j} \mathrm{a_{ij}} = 1$ and $\sum_j (a_{ij} + b_{ij}) = 1$ that yields to the fact that $\sum_j b_{ij} = 0$

Then we conclude that $\sum_\mathrm{j} (\mathrm{a_{ij}} + \mathrm{b_{ij}I}) = 1$ and this proves the theorem.

Example 3.2

Let's take the matrix:

$$M_N = \begin{bmatrix} 0.3I & 1 - 0.3I \\ 0.4 + 0.2I & 0.6 - 0.2I \end{bmatrix}$$

that is:

$$M_N = \begin{bmatrix} 0 & 1 \\ 0.4 & 0.6 \end{bmatrix} + \begin{bmatrix} 0.3 & -0.3 \\ 0.2 & -0.2 \end{bmatrix} I$$

$$A = \begin{bmatrix} 0 & 1 \\ 0.4 & 0.6 \end{bmatrix}\ and\ \ A + B = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}$$

we note that A and A+B are two transition matrices fulfill conditions

$$\sum_j a_{ij} = 1\ ; i = 1,2 \qquad 0 \leq a_{ij} \leq 1\ ; i, j = 1,2$$

$$\sum_j a_{ij} + b_{ij} = 1\ ; i = 1,2 \qquad 0 \leq a_{ij} + b_{ij} \leq 1\ ; i, j = 1,2$$

Theorem 3.2

If $\mathrm{M_1}$ and $\mathrm{M_2}$are two neutrosophic transition matrices, then their multiplication is a neutrosophic transition matrix.

Proof

Let $M_1 = \begin{bmatrix} a_{11}+b_{11}I & a_{12}+b_{12}I \\ a_{21}+b_{21}I & a_{22}+b_{22}I \end{bmatrix}, M_2 = \begin{bmatrix} c_{11}+d_{11}I & c_{12}+d_{12}I \\ c_{21}+d_{21}I & c_{22}+d_{22}I \end{bmatrix}$

$M_1.M_2$

$$= \begin{bmatrix} (a_{11}+b_{11}I)(c_{11}+d_{11}I)+(a_{12}+b_{12}I)(c_{21}+d_{21}I) & (a_{11}+b_{11}I)(c_{12}+d_{12}I)+(a_{12}+b_{12}I)(c_{22} \\ (a_{21}+b_{21}I)(c_{11}+d_{11}I)+(a_{22}+b_{22}I)(c_{21}+d_{21}I) & (a_{21}+b_{21}I)(c_{12}+d_{12}I)+(a_{22}+b_{22}I)(c_{22} \end{bmatrix}$$

Let's check the first condition:

$$(a_{11}+b_{11}I)(c_{11}+d_{11}I)+(a_{12}+b_{12}I)(c_{21}+d_{21}I)+(a_{11}+b_{11}I)(c_{12}+d_{12}I)+(a_{12}+b_{12}I)(c_{22}+d_{22}I) =$$

$(a_{11}+b_{11}I)[(c_{11}+d_{11}I)+(c_{12}+d_{12}I)]+(a_{12}+b_{12}I)[(c_{21}+d_{21}I)+(c_{22}+d_{22}I)]=$

$(a_{11}+b_{11}I)+(a_{12}+b_{12}I)=1$

Similarly, we find that sum of elements of the second row of matrix $(M_1.M_2)$ is 1

Also, since all elements of the matrices M_1 and M_2 are positive and since that sum of each row of the matrix $M_1.M_2$ is 1 then we conclude that each element lays between 0 and 1

Example 3.3

Let $M_1 = \begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix}, M_2 = \begin{bmatrix} 0.2+0.3I & 0.8-0.3I \\ 0.2I & 1-0.2I \end{bmatrix}$

$$M_1.M_2 = \begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix}.\begin{bmatrix} 0.2+0.3I & 0.8-0.3I \\ 0.2I & 1-0.2I \end{bmatrix}$$
$$= \begin{bmatrix} 0.32I+0.06I^2 & 1-0.32I-0{,}06I^2 \\ 0.6+0.25I+0{,}01I^2 & 0.94-0.25I-0{,}01I^2 \end{bmatrix}$$
$$= \begin{bmatrix} 0.38I & 1-0.38I \\ 0.6+0.26I & 0.31-0.26I \end{bmatrix}$$

Note that the matrix $M_1.M_2$ It is a neutrosophic transition matrix because it satisfies the assumed conditions.

Definition 3.4

Let $M_N = A + BI$ be a neutrosophic matrix and let $r \in \mathbb{N}$, then:

$$M_N^r = A^r + I[(A+B)^r - A^r]$$

Theorem 3.3

If M_N neutrosophic transition matrix, then M_N^r is a neutrosophic transition matrix.

Proof

Straight forward by mathematical induction according to theorem 3.2.

Example 3.4:

Let $M_N = \begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix}$

$M_N^2 = M_N . M_N = \begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix} . \begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix}$

$= \begin{bmatrix} 0.3-0.08I+0.3I^2 & 0.7+0.08I-0.3I^2 \\ 0.21+0.22I+0.05I^2 & 0.79-0.22I-0.05I^2 \end{bmatrix}$

$= \begin{bmatrix} 0.3-0.38I & 0.7+0.38I \\ 0.21+0.27I & 0.79-0.27I \end{bmatrix}$

Notice that M_N^2 is a neutrosophic transition matrix, also:

$M_N^3 = M_N^2 . M_N = \begin{bmatrix} 0.3-0.38I & 0.7+0.38I \\ 0.21+0.27I & 0.79-0.27I \end{bmatrix} . \begin{bmatrix} 0.6I & 1-0.6I \\ 0.3+0.1I & 0.7-0.1I \end{bmatrix}$

$= \begin{bmatrix} 0.21+0.364I-0.190I^2 & 0.79-0.364I+0.190I^2 \\ 0.237+0.124I+0.135I^2 & 0.763-0.124I-0.135I^2 \end{bmatrix}$

$= \begin{bmatrix} 0.21+0.174I & 0.79-0.174I \\ 0.237+0.259I & 0.763-0.259I \end{bmatrix}$

We note that M_N^3 is also a neutrosophic transition matrix.

Theorem 3.4

Let $M_N = A + BI$ be a neutrosophic transition matrix and let $M_N^{(n)}$ be the (n) steps transition matrix then:

$$M_N^{(n)} = M_N^n$$

Proof

By takin the isometric image we have:

$T\left(M_N^{(n)}\right) = (A^{(n)}, (A+B)^{(n)})$

Since both $A^{(n)}, (A+B)^{(n)}$ are transition matrices in classical scene then by the well-known Chapman-Kolmogorov theorem we have:

$A^{(n)} = A^n, (A+B)^{(n)} = (A+B)^n$

Which means that:

$T\left(M_N^{(n)}\right) = (A^n, (A+B)^n)$

Now, taking inverse isometry yields to:

$T^{-1}\left(T\left(M_N^{(n)}\right)\right) = A^n + [(A+B)^n - A^n]I = M^n$

4. Literal Plithogenic Markov chains

Definition 4.1

A set of random variables $X_0, X_1, X_2, \ldots$ satisfying:

$Pr\{X_{n+1} = i_{n+1} | X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = Pr\{X_{n+1} = i_{n+1} | X_n = i_n\}$

is called a literal or symbolic neutrosophic markov chain if the last probability takes the form $Pr\{X_{n+1} = i_{n+1} | X_n = i_n\} = a + bP_1 + cP_2; 0 \leq a \leq 1, 0 \leq a + b \leq 1, 0 \leq a + b + c \leq 1; P_1^2 = P_1, P_2^2 = P_2,$

$P_1P_2 = P_2P_1 = P_2$

Definition 4.2

We call $\boldsymbol{p_{ij}^{(n,n+1)}}_{\boldsymbol{P}} = \boldsymbol{Pr(X_{n+1} = j | X_n = i)} \in \boldsymbol{R(P_1, P_2)}$ literal or symbolic plithogenic one-step transition probability.

Definition 4.3

A squared plithogenic matrix

$M_N = A + BP_1 + CP_2 = [a_{ij} + b_{ij}P_1 + c_{ij}P_2]_{n\times n}$

Is called a plithogenic markov transition matrix if its elements satisfy:

1. $\sum_j a_{ij} + b_{ij}P_1 + c_{ij}P_2 = 1 \quad ; \; i = 1,2,3, \ldots, n$
2. $0 \leq_p a_{ij} + b_{ij}P_1 + c_{ij}P_2 \leq_p 1 \quad ; \; i, j = 1,2,3, \ldots, n$

Example 4.1

let's take: $M_P = \begin{bmatrix} 0.3P_1 + 0.1P_2 & 1 - 0.3P_1 - 0.1P_2 \\ 0.4 + 0.2P_1 - 0.6P_2 & 0.6 - 0.2P_1 + 0.6P_2 \end{bmatrix}$

Then M_P is a plithogenic transition matrix because:

$0.3P_1 + 0.1P_2 + 1 - 0.3P_1 - 0.1P_2$

$= 1 \quad \text{and} \quad 0.4 + 0.2P_1 - 0.6P_2 + 0.6 - 0.2P_1 + 0.6P_2 = 1$

Also, according to the definition of comparison between plithogenic numbers we have:

$0.3P_1 + 0.1P_2 \leq_P 1 + 0P_1 + 0P_2$ because $0 \leq 1$ & $0.3 \leq 1$

$1 - 0.3P_1 - 0.1P_2 \leq_P 1 + 0P_1 + 0P_2$ because $1 \leq 1$ & $0.7 \leq 1$

$0.4 + 0.2P_1 - 0.6P_2 \leq_P 1 + 0P_1 + 0P_2$ because $0.4 \leq 1$ & $0.6 \leq 1$

$0.6 - 0.2P_1 + 0.6P_2 \leq_P 1 + 0P_1 + 0P_2$ because $0.6 \leq 1$ & $0.4 \leq 1$

$0 + 0P_1 + 0P_2 \leq_P 0.3P_1 + 0.1P_2$ because $0 \leq 0$ & $0 \leq 0.3$

$0 + 0P_1 + 0P_2 \leq_P 1 - 0.3P_1 - 0.1P_2$ because $0 \leq 1$ & $0 \leq 0.7$

$0 + 0P_1 + 0P_2 \leq_P 0.4 + 0.2P_1 - 0.6P_2$ because $0 \leq 0.4$ & $0 \leq 0.6$

$0 + 0P_1 + 0P_2 \leq_P 0.6 - 0.2P_1 + 0.6P_2$ because $0 \leq 0.6$ & $0 \leq 0.4$

Theorem4.1

The matrix $M_P = A + BP_1 + CP_2$ is a plithogenic transition matrix if and only if A is a crisp transition matrix, $A + B$ is a crisp transition matrix and $A + B + C$ is a crisp transition matrix.

Proof

Let's assume that M_P is a plithogenic transition matrix and prove that $A, A + B$ and $A + B + C$ are transition matrices:

we have $0 + 0P_1 + 0P_2 \leq_P a_{ij} + b_{ij}P_1 + c_{ij}P_2 \leq_P 1 + 0P_1 + 0P_2$ so $a_{ij} \leq 1$, $a_{ij} + b_{ij} \leq 1$ and $a_{ij} + b_{ij} + c_{ij} \leq 1$

$0 \leq a_{ij}$, $0 \leq a_{ij} + b_{ij}$ and $0 \leq a_{ij} + b_{ij} + c_{ij}$ which means that:

$0 \leq a_{ij} \leq 1$ and $0 \leq a_{ij} + b_{ij} \leq 1$

Also, we have $\sum_j (a_{ij} + b_{ij}P_1 + c_{ij}P_2) = 1 = 1 + 0P_1 + 0P_2$ which means that $\sum_j c_{ij} = 0$ $\sum_j b_{ij} = 0$ and $\sum_j a_{ij} = 1$

So, we can conclude that:

$0 \leq a_{ij} \leq 1$ and $\sum_j a_{ij} = 1 \Rightarrow A$ is a transition matrix.

$0 \leq a_{ij} + b_{ij} \leq 1$ and $\sum_j (a_{ij} + b_{ij}) = 1 \Rightarrow A + B$ is a transition matrix.

$0 \leq a_{ij} + b_{ij} + c_{ij} \leq 1$ and $\sum_j (a_{ij} + b_{ij} + c_{ij}) = 1 \Rightarrow A + B + C$ transition matrix.

Now, let's assume that both A, A+B and A+B+C are transition matrices and prove that $M_P = A + BP_1 + CP_2$ is a plithogenic transition matrix:

since $A, A+B, A+B+C$ are transition matrices then $0 \leq a_{ij} \leq 1$, $0 \leq a_{ij} + b_{ij} \leq 1$, $0 \leq a_{ij} + b_{ij} + c_{ij} \leq 1$ which means that $0 \leq_P a_{ij} + b_{ij}P_1 + c_{ij}P_2 \leq_P 1$

Also, we have $\sum_j a_{ij} = 1$, $\sum_j (a_{ij} + b_{ij}) = 1$and $\sum_j (a_{ij} + b_{ij} + c_{ij}) = 1$ that yields to the fact that $\sum_j b_{ij} = 0$ and $\sum_j c_{ij} = 0$

Then we conclude that $\sum_j (a_{ij} + b_{ij}P_1 + c_{ij}P_2) = 1$ and this proves the theorem.

Example 4.2

Let's take the matrix:

$$M_P = \begin{bmatrix} 0.3P_1 + 0.1P_2 & 1 - 0.3P_1 - 0.1P_2 \\ 0.4 + 0.2P_1 - 0.6P_2 & 0.6 - 0.2P_1 + 0.6P_2 \end{bmatrix}$$

that is:

$$M_P = \begin{bmatrix} 0 & 1 \\ 0.4 & 0.6 \end{bmatrix} + \begin{bmatrix} 0.3 & -0.3 \\ 0.2 & -0.2 \end{bmatrix} P_1 + \begin{bmatrix} 0.1 & -0.1 \\ -0.6 & 0.6 \end{bmatrix} P_2$$

$$A = \begin{bmatrix} 0 & 1 \\ 0.4 & 0.6 \end{bmatrix} \text{ and } A + B = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} \text{ and } A + B + C = \begin{bmatrix} 0.4 & 0.6 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

we note that $A, A+B$ and $A+B+C$ are transition matrices fulfill conditions

$$\sum_j a_{ij} = 1 \,;\, i = 1,2 \qquad 0 \leq a_{ij} \leq 1 \,;\, i,j = 1,2$$

$$\sum_j a_{ij} + b_{ij} = 1 \,;\, i = 1,2 \qquad 0 \leq a_{ij} + b_{ij} \leq 1 \,;\, i,j = 1,2$$

$$\sum_j a_{ij} + b_{ij} + c_{ij} = 1 \quad ; i = 1,2 \qquad 0 \leq a_{ij} + b_{ij} + c_{ij} \leq 1 \quad ; i,j = 1,2$$

Theorem 4.2

If M_1 and $M_2$are two plithogenic transition matrices, then their multiplication is a plithogenic transition matrix.

Proof

Let

$$M_1 = \begin{bmatrix} a_{11} + b_{11}P_1 + c_{11}P_2 & a_{12} + b_{12}P_1 + c_{12}P_2 \\ a_{21} + b_{21}P_1 + c_{21}P_2 & a_{22} + b_{22}P_1 + c_{22}P_2 \end{bmatrix} \quad M_2 = \begin{bmatrix} d_{11} + e_{11}P_1 + f_{11}P_2 & d_{12} + e_{12}P_1 + f_{12}P_2 \\ d_{21} + e_{21}P_1 + f_{21}P_2 & d_{22} + e_{22}P_1 + f_{22}P_2 \end{bmatrix}$$

$$M_1.M_2 = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Where

$$x = (a_{11} + b_{11}P_1 + c_{11}P_2)(d_{11} + e_{11}P_1 + f_{11}P_2) + (a_{12} + b_{12}P_1 + c_{12}P_2)(d_{21} + e_{21}P_1 + f_{21}P_2)$$

$$y = (a_{11} + b_{11}P_1 + c_{11}P_2)(d_{12} + e_{12}P_1 + f_{12}P_2) + (a_{12} + b_{12}P_1 + c_{12}P_2)(d_{22} + e_{22}P_1 + f_{22}P_2)$$

$$z = (a_{21} + b_{21}P_1 + c_{21}P_2)(d_{11} + e_{11}P_1 + f_{11}P_2) + (a_{22} + b_{22}P_1 + c_{22}P_2)(d_{21} + e_{21}P_1 + f_{21}P_2)$$

$$w = (a_{21} + b_{21}P_1 + c_{21}P_2)(d_{12} + e_{12}P_1 + f_{12}P_2) + (a_{22} + b_{22}P_1 + c_{22}P_2)(d_{22} + e_{22}P_1 + f_{22}P_2)$$

Let's check the condition:

$$(a_{11} + b_{11}P_1 + c_{11}P_2)(d_{11} + e_{11}P_1 + f_{11}P_2) + (a_{12} + b_{12}P_1 + c_{12}P_2)(d_{21} + e_{21}P_1 + f_{21}P_2) + (a_{11} + b_{11}P_1 + c_{11}P_2)(d_{12} + e_{12}P_1 + f_{12}P_2) + (a_{12} + b_{12}P_1 + c_{12}P_2)(d_{22} + e_{22}P_1 + f_{22}P_2)$$

$$= (a_{11} + b_{11}P_1 + c_{11}P_2)[(d_{11} + e_{11}P_1 + f_{11}P_2) + (d_{12} + e_{12}P_1 + f_{12}P_2)] + (a_{12} + b_{12}P_1 + c_{12}P_2)[(d_{21} + e_{21}P_1 + f_{21}P_2) + (d_{22} + e_{22}P_1 + f_{22}P_2)]$$

$$= (a_{11} + b_{11}P_1 + c_{11}P_2) + (a_{12} + b_{12}P_1 + c_{12}P_2) = 1$$

Similarly, we find that sum of elements of the second row of matrix $(M_1.M_2)$ is 1

Also, since all elements of the matrices M_1 and M_2 are positive and since that sum of each row of the matrix $M_1.M_2$ is 1 then we conclude that each element lays between 0 and 1

Example 4.3

Let $M_1 = \begin{bmatrix} 0.6\mathrm{P}_1 + 0.2\mathrm{P}_2 & 1 - 0.6\mathrm{P}_1 - 0.2\mathrm{P}_2 \\ 0.3 + 0.1\mathrm{P}_1 - 0.5\mathrm{P}_2 & 0.7 - 0.1\mathrm{P}_1 + 0.5\mathrm{P}_2 \end{bmatrix}, M_2 = \begin{bmatrix} 0.2 + 0.3\mathrm{P}_1 - 0.1\mathrm{P}_2 & 0.8 - 0.3\mathrm{P}_1 + 0.1\mathrm{P}_2 \\ 0.2\mathrm{P}_1 + 0.3\mathrm{P}_2 & 1 - 0.2\mathrm{P}_1 - 0.3\mathrm{P}_2 \end{bmatrix}$

$M_1.M_2$

$$= \begin{bmatrix} 0.6\mathrm{P}_1 + 0.2\mathrm{P}_2 & 1 - 0.6\mathrm{P}_1 - 0.2\mathrm{P}_2 \\ 0.3 + 0.1\mathrm{P}_1 - 0.5\mathrm{P}_2 & 0.7 - 0.1\mathrm{P}_1 + 0.5\mathrm{P}_2 \end{bmatrix} . \begin{bmatrix} 0.2 + 0.3\mathrm{P}_1 - 0.1\mathrm{P}_2 & 0.8 - 0.3\mathrm{P}_1 + 0.1\mathrm{P}_2 \\ 0.2\mathrm{P}_1 + 0.3\mathrm{P}_2 & 1 - 0.2\mathrm{P}_1 - 0.3\mathrm{P}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.06\mathrm{P}_1^2 + (0.32 - 0.22\mathrm{P}_2)\mathrm{P}_1 + 0.34\mathrm{P}_2 - 0.08\mathrm{P}_2^2 & -0.06\mathrm{P}_1^2 + (-0.32 + 0.22\mathrm{P}_2)\mathrm{P}_1 - 0.34\mathrm{P}_2 + \\ 0.01\mathrm{P}_1^2 + (0.25 - 0.09\mathrm{P}_2)\mathrm{P}_1 + 0.06 + 0.08\mathrm{P}_2 + 0.20\mathrm{P}_2^2 & -0.01\mathrm{P}_1^2 + (-0.25 + 0.09\mathrm{P}_2)\mathrm{P}_1 + 0.94 - 0. \end{bmatrix}$$

$$= \begin{bmatrix} 0.06\mathrm{P}_1 + (0.32\mathrm{P}_1 - 0.22\mathrm{P}_2) + 0.34\mathrm{P}_2 - 0.08\mathrm{P}_2 & -0.06\mathrm{P}_1 + (-0.32\mathrm{P}_1 + 0.22\mathrm{P}_2) - 0.34\mathrm{P}_2 + \\ 0.01\mathrm{P}_1 + (0.25\mathrm{P}_1 - 0.09\mathrm{P}_2) + 0.06 + 0.08\mathrm{P}_2 + 0.20\mathrm{P}_2 & -0.01\mathrm{P}_1 + (-0.25\mathrm{P}_1 + 0.09\mathrm{P}_2) + 0.94 - 0.0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.38\mathrm{P}_1 + 0.04\mathrm{P}_2 & -0.38\mathrm{P}_1 - 0.04\mathrm{P}_2 + 1 \\ 0.26\mathrm{P}_1 + 0.06 + 0.19\mathrm{P}_2 & -0.26\mathrm{P}_1 + 0.94 - 0.19\mathrm{P}_2 \end{bmatrix}$$

Note that the matrix $M_1.M_2$ It is a plithogenic transition matrix because it satisfies the assumed conditions.

Definition 4.4

Let $M_P = A + BP_1 + CP_2$ be a plithogenic matrix and let $r \in \mathbb{N}$, then:

$$M_P^r = A^r + P_1[(A+B)^r - A^r] + P_2[(A+B+C)^r - (A+B)^r]$$

Theorem 4.3

If M_P plithogenic transition matrix, then $\mathrm{M}_\mathrm{P}^\mathrm{r}$ is a plithogenic transition matrix.

Proof

Straight forward by mathematical induction according to theorem 4.2.

Example 4.4

Let $\mathrm{M}_\mathrm{P} = \begin{bmatrix} 0.6\mathrm{P}_1 + 0.2\mathrm{P}_2 & 1 - 0.6\mathrm{P}_1 - 0.2\mathrm{P}_2 \\ 0.3 + 0.1\mathrm{P}_1 - 0.5\mathrm{P}_2 & 0.7 - 0.1\mathrm{P}_1 + 0.5\mathrm{P}_2 \end{bmatrix}$

$\mathrm{M}_\mathrm{P}^2 = \mathrm{M}_\mathrm{P}.\mathrm{M}_\mathrm{P}$

$$= \begin{bmatrix} 0.6\mathrm{P}_1 + 0.2\mathrm{P}_2 & 1 - 0.6\mathrm{P}_1 - 0.2\mathrm{P}_2 \\ 0.3 + 0.1\mathrm{P}_1 - 0.5\mathrm{P}_2 & 0.7 - 0.1\mathrm{P}_1 + 0.5\mathrm{P}_2 \end{bmatrix} . \begin{bmatrix} 0.6\mathrm{P}_1 + 0.2\mathrm{P}_2 & 1 - 0.6\mathrm{P}_1 - 0.2\mathrm{P}_2 \\ 0.3 + 0.1\mathrm{P}_1 - 0.5\mathrm{P}_2 & 0.7 - 0.1\mathrm{P}_1 + 0.5\mathrm{P}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.30\mathrm{P}_1^2 + (0.52\mathrm{P}_2 - 0.08)\mathrm{P}_1 + 0.14\mathrm{P}_2^2 + 0.3 - 0.56\mathrm{P}_2 & -0.30\mathrm{P}_1^2 + (0.08 - 0.52\,\mathrm{P}_2)\mathrm{P}_1 + 0.56\mathrm{P}_2 - \\ 0.05\mathrm{P}_1^2 + (0.22 - 0.18\,\mathrm{P}_2)\mathrm{P}_1 - 0.14\mathrm{P}_2 - 0.35\mathrm{P}_2^2 + 0.21 & -0.05\,\mathrm{P}_1^2 + (-0.22 + 0.18\mathrm{P}_2)\mathrm{P}_1 + 0.79 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.30\mathrm{P}_1 + (0.52\mathrm{P}_2 - 0.08\mathrm{P}_1) + 0.14\mathrm{P}_2 + 0.3 - 0.56\mathrm{P}_2 & -0.30\mathrm{P}_1 + (0.08\mathrm{P}_1 - 0.52\,\mathrm{P}_2) + 0.56\mathrm{P}_2 - \\ 0.05\mathrm{P}_1 + (0.22\mathrm{P}_1 - 0.18\,\mathrm{P}_2) - 0.14\mathrm{P}_2 - 0.35\mathrm{P}_2 + 0.21 & -0.05\,\mathrm{P}_1 + (-0.22\mathrm{P}_1 + 0.18\mathrm{P}_2) + 0.79 + 0. \end{bmatrix}$$

$$= \begin{bmatrix} 0.22\mathrm{P}_1 + 0.1\mathrm{P}_2 + 0.3 & -0.22\mathrm{P}_1 + 0.\mathrm{P}_2 + 0.7 \\ 0.27\mathrm{P}_1 - 0.67\mathrm{P}_2 + 0.21 & -0.27\,\mathrm{P}_1 + 0.79 + 0.67\mathrm{P}_2 \end{bmatrix}$$

Notice that M_N^2 is a plithogenic transition matrix.

Theorem 4.4

Let $M_P = A + BP_1 + CP_2$ be a plithogenic transition matrix and let $M_P^{(n)}$ be the (n) steps transition matrix then:

$$M_P^{(n)} = M_P^n$$

Proof

By takin the isometric image we have:

$\mathrm{T}\left(\mathrm{M}_\mathrm{P}^{(\mathrm{n})}\right) = (\mathrm{A}^{(\mathrm{n})}, (\mathrm{A}+\mathrm{B})^{(\mathrm{n})}, (\mathrm{A}+\mathrm{B}+\mathrm{C})^{(\mathrm{n})})$

Since $\mathrm{A}^{(\mathrm{n})}, (\mathrm{A}+\mathrm{B})^{(\mathrm{n})}, (\mathrm{A}+\mathrm{B}+\mathrm{C})^{(\mathrm{n})}$ are transition matrices in classical scene then by the well-known Chapman-Kolmogorov theorem we have:

$A^{(n)} = A^n, (A+B)^{(n)} = (A+B)^n, (\mathrm{A}+\mathrm{B}+\mathrm{C})^{(\mathrm{n})} = (A+B+C)^n$

Which means that:

$T\left(M_P^{(n)}\right) = (A^n, (A+B)^n, (A+B+C)^n)$

Now, taking inverse isometry yields to:

$T^{-1}\left(T\left(M_P^{(n)}\right)\right) = \mathrm{A}^\mathrm{n} + [(\mathrm{A}+\mathrm{B})^\mathrm{n} - \mathrm{A}^\mathrm{n}]\mathrm{P}_1 + [(\mathrm{A}+\mathrm{B}+\mathrm{C})^\mathrm{n} - (\mathrm{A}+\mathrm{B})^\mathrm{n}]\mathrm{P}_2 = M^n$

5. Conclusion

In conclusion, this paper pioneers the integration of symbolic neutrosophic and plithogenic concepts into the well-established framework of Markov chains, yielding a profound extension that encapsulates the nuances of uncertainty and ambiguity. Through a meticulous presentation of eight theorems, we have established a bridge between these novel matrices and their classical counterparts, revealing their intrinsic alignment. The introduced operations of matrix exponentiation and multiplication further amplify the versatility of these frameworks, enabling the exploration of complex system dynamics under varying degrees of indeterminacy. Moreover, the adaptation of Chapman-Kolmogorov theorem to the symbolic neutrosophic and plithogenic domains augments our ability to analyze state transitions in environments laden with partial truth. This study not only advances the theoretical frontiers of probabilistic modeling but also lays a fertile ground for practical applications across disciplines such as decision analysis, risk assessment, and artificial intelligence. As the confluence of traditional and innovative theories continues to shape the landscape of uncertainty modeling, symbolic neutrosophic and plithogenic Markov chains stand poised to offer invaluable insights into the intricate fabric of real-world systems.

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