NEUTROSOPHIC CUBIC IDEALS

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ABSTRACT. Operational properties of neutrosophic cubic sets are investigated.The notion of neutrosophic cubic subsemigroups and neutrosophic cubic left(resp.right) ideals are introduced, and several properties are investigated. Relations between neutrosophic cubic subsemigroups and neutrosophic cubic left(resp.right) ideals are discussed. Characterizations of neutrosophic cubic left(resp. right)ideals are considered, and how the images or inverse images of neutrosophic cubic subsemigroups and cubic left (resp. right) ideals become neutrosophic cubic subsemigroups and neutrosophic cubic left (resp. right)ideals, respectively, are studied..

1. INTRODUCTION

Fuzzy sets are initiated by Zadeh [1]. In [2], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. In traditional fuzzy logic, to represent, e.g., the expert's degree of certainty in different statements, numbers from the interval [0, 1] are used. It is often difficult for an expert to exactly quantify his or her certainty,therefore, instead of a real number, it is more adequate to represent this degree of certainty by an interval or even by a fuzzy set. In the first case, we get an interval-valued fuzzy set. In the second case, we get a second-order fuzzy set. Interval-valued fuzzy sets have been actively used in real-life applications. For example, Sambuc [10] in Medical diagnosis in thyroidian pathology, Kohout [18] also in Medicine, in a system CLINAID, Gorzalczany in Approximate reasoning, Turksen [10, 11] in Interval-valued logic, in preferences modelling [12], etc. These works and others show the importance of these sets. Fuzzy sets deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [4] introduced a new notion, called a cubic set, and investigated several properties. Cubic set theory is applied to BCK/BCI-algebras (see [6, 7, 8, 9]) and Γ-semihypergroups (see [2]). Jun et al.[5] introduced the concept of cubic ideals in semigroup. They investigated operational properties of cubic sets, the notion of cubic subsemigroups and cubic left (resp. right) ideals, and investigate several properties. The concept of neutrosophic set (NS) developed by Smarandache[16], is a more general platform which extends the concepts of the classic set and fuzzy set[17], Neutrosophic set theory is applied to various parts (refer to the site http://fs.gallup.unm.edu/neutrosophy.htm).Haiban

extended the idea of neutrosophic sets to interval neutrosophic sets[14, 15]. Jun introduced the notion of neutrosophic cubic set[19].

In this paper we introduce the concept of neutrosophic cubic ideals in semigroup. We investigate some operational properties of neutrosophic cubic sets. The notion of neutrosophic cubic subsemigroups and neutrosophic cubic left (resp.right) ideals are introduced, and several properties are investigated. Relations between neutrosophic cubic subsemigroups and neutrosophic cubic left (resp. right) ideals are discussed. Characterizations of neutrosophic cubic left (resp. right)ideals are considered, and how the images or inverse images of neutrosophic cubic subsemigroups and cubic left (resp. right) ideals become neutrosophic cubic subsemigroups and neutrosophic cubic left (resp. right) ideals, respectively, are studied.

2. Preliminaries

A non-empty set S together with an associative binary operation "$\cdot$" is called a semigroup. A non-empty subset A of a semigroup S is called a subsemigroup if $AA \subseteq A$. A non empty subset A of S is left($right$) ideal of S if $SA \subseteq A$ $(AS \subseteq A)$ .Jun et al. [4] have defined the cubic set as follows: Let X be a non-empty set. A cubic set in X is a structure of the form: $C = \{\langle x, \widetilde{\mu}(x), \lambda(x)\rangle \,|x \in X\}$ where $\widetilde{\mu}$ is an interval-valued fuzzy set in X and λ is a fuzzy set in X. Let X be a non-empty set. A neutrosophic set (NS) in X (see [16,17]) is a structure of the form: $\lambda = \{\langle \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle \,|x \in X\}$ where $\lambda_T : X \to [0,1]$is a truth membership function, $\lambda_I : X \to [0,1]$ is an indeterminate membership function, and $\lambda_F : X \to [0,1]$ is a false membership function. Let X be a non-empty set. An interval neutrosophic set (INS) in X (see [14, 15]) is a structure of the form: $\widetilde{\mu}$ $= \{\langle \widetilde{\mu}_T(x), \widetilde{\mu}_I(x), \widetilde{\mu}_F(x)\rangle \,|x \in X\}$ where $\widetilde{\mu}_T, \widetilde{\mu}_I$ and $\widetilde{\mu}_F$ are interval-valued fuzzy sets[23] in X, which are called an interval truth membership function, an interval indeterminacy membership function and an interval falsity membership function, respectively. Jun et al.[19] considered the notion of neutrosophic cubic sets. Let X be a non-empty set. A neutrosophic cubic set (NCS) in X is a pair $A = (\widetilde{\mu}, \lambda)$ where $\widetilde{\mu} = \{\langle \widetilde{\mu}_T(x), \widetilde{\mu}_I(x), \widetilde{\mu}_F(x)\rangle \,|x \in X\}$ is an interval neutrosophic set in X and $\lambda = \{\langle \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle \,|x \in X\}$ is a neutrosophic set in X.

Jun et al.[5] inrodeced the idea of cubic ideals in semigroup and investigated several properties.

3. Operational Properties of neutrosophic cubic sets

Definition 1. *Let X be nonempty set. A NC set A in X is structure*

$$A = \{\langle x, \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F\rangle : x \in X\}$$

For any nonempty subset G of set X,the characteristic NC set of G in χ defined to be a structure

$$\chi_G = \{\langle x, \widetilde{\mu}_{TG}, \widetilde{\mu}_{IG}, \widetilde{\mu}_{FG}, \lambda_{TG}, \lambda_{IG}, \lambda_{FG}\rangle : x \in X\}$$

which is briefly denoted by

$$\chi_G = \langle \widetilde{\mu}_{TG}, \widetilde{\mu}_{IG}, \widetilde{\mu}_{FG}, \lambda_{TG}, \lambda_{IG}, \lambda_{FG}\rangle$$

where

$$\widetilde{\mu}_{TG} = \begin{cases} \widetilde{1} & if\ x \in G \\ \widetilde{0} & otherwise \end{cases} \qquad \lambda_{TG} = \begin{cases} 0 & if\ x \in G \\ 1 & otherwise \end{cases}$$

$$\widetilde{\mu}_{IG} = \begin{cases} \widetilde{1} & if\ x \in G \\ \widetilde{0} & otherwise \end{cases} \qquad \lambda_{IG} = \begin{cases} 0 & if\ x \in G \\ 1 & otherwise \end{cases}$$

$$\widetilde{\mu}_{FG} = \begin{cases} \widetilde{1} & if\ x \in G \\ \widetilde{0} & otherwise \end{cases} \qquad \lambda_{FG} = \begin{cases} 0 & if\ x \in G \\ 1 & otherwise \end{cases}$$

The whole NC set S in semigroup S is defined to be structure

$$S = \left\{ \left\langle x, \widetilde{1}_{TS}, \widetilde{1}_{IS}, \widetilde{1}_{FS}, 0_{TS}, 0_{IS}, 0_{FS} \right\rangle : x \in S \right\}$$

with

$\widetilde{1}_{TS} = [1\ 1],\ \widetilde{1}_{IS} = [1\ 1],\ \widetilde{1}_{FS} = [1\ 1],\ and\ 0_{TS} = 0,\ 0_{IS} = 0,\ 0_{FS} = 0\ for\ all\ x \in X.$

It will briefly denoted by

$$S = \left\langle \widetilde{1}_{TS}, \widetilde{1}_{IS}, \widetilde{1}_{FS}, 0_{TS}, 0_{IS}, 0_{FS}, \right\rangle$$

For two NC set $A = \langle x, \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ and $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_T, \eta_I, \eta_F \rangle$ in semigroup S. We define

$$A \ \sqsubseteq \ B \Longleftrightarrow \widetilde{\mu}_T \preceq \widetilde{\nu}_T, \widetilde{\mu}_I \preceq \widetilde{\nu}_I, \widetilde{\mu}_F \preceq \widetilde{\nu}_F,$$
$$and\ \lambda_T \ \geq \ \eta_T, \lambda_I \geq \eta_I, \lambda_F \geq \eta_F$$

and the NC product of $A = \langle x, \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ and $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_T, \eta_I, \eta_F \rangle$ is define to be NC set

$$A \otimes B = \left\{ \left\langle \begin{array}{c} x, (\widetilde{\mu}_T \circ \widetilde{\nu}_T)(x), (\widetilde{\mu}_I \circ \widetilde{\nu}_I)(x), (\widetilde{\mu}_F \circ \widetilde{\nu}_F)(x) \\ (\lambda_T \circ \eta_T)(x), (\lambda_I \circ \eta_I)(x), (\lambda_F \circ \eta_F)(x) : x \in S \end{array} \right\rangle \right\}$$

which briefly denoted by

$$A \otimes B = \left\langle \begin{array}{c} (\widetilde{\mu}_T \circ \widetilde{\nu}_T)(x), (\widetilde{\mu}_I \circ \widetilde{\nu}_I)(x), (\widetilde{\mu}_F \circ \widetilde{\nu}_F)(x) \\ (\lambda_T \circ \eta_T)(x), (\lambda_I \circ \eta_I)(x), (\lambda_F \circ \eta_F)(x) \end{array} \right\rangle$$

where $\widetilde{\mu}_T \circ \widetilde{\nu}_T, \widetilde{\mu}_I \circ \widetilde{\nu}_I, \widetilde{\mu}_F \circ \widetilde{\nu}_F$ and $\lambda_T \circ \eta_T, \lambda_I \circ \eta_I, \lambda_F \circ \eta_F$ are define as follows, respectively

$$(\widetilde{\mu}_T \circ \widetilde{\nu}_T)(x) = \begin{cases} r\sup_{x=yz} [r\min\{\widetilde{\mu}_T(y), \widetilde{\nu}_T(z)\}] & if\ x=yz\ for\ some\ y,z \in S \\ [0\ 0] & otherewise, \end{cases} \quad and$$

$$(\lambda_T \circ \eta_T)(x) = \begin{cases} \Lambda_{x=yz} \max\{\lambda_T(y), \eta_T(z)\} & if\ x=yz\ for\ some\ y,z \in S \\ 1 & otherewise, \end{cases}$$

$$(\widetilde{\mu}_I \circ \widetilde{\nu}_I)(x) = \begin{cases} r\sup_{x=yz} [r\min\{\widetilde{\mu}_I(y), \widetilde{\nu}_I(z)\}] & if\ x=yz\ for\ some\ y,z \in S \\ [0\ 0] & otherewise, \end{cases} \quad and$$

$$(\lambda_I \circ \eta_I)(x) = \begin{cases} \Lambda_{x=yz} \max\{\lambda_I(y), \eta_I(z)\} & if\ x=yz\ for\ some\ y,z \in S \\ 1 & otherewise, \end{cases}$$

$$(\widetilde{\mu}_F \circ \widetilde{\nu}_F)(x) = \begin{cases} r\sup_{x=yz} [r\min\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(z)\}] & if\ x=yz\ for\ some\ y,z \in S \\ [0\ 0] & otherewise, \end{cases} \quad and$$

$$(\lambda_F \circ \eta_F)(x) = \begin{cases} \Lambda_{x=yz} \max\{\lambda_F(y), \eta_F(z)\} & if\ x=yz\ for\ some\ y,z \in S \\ 1 & otherewise, \end{cases}$$

for all $x \in S$ we also define the cap and union of two NC sets as follows. Let A and B be a twoNC setsin x. The intersection of A and B, denoted by $A \cap B$,is the NC set

$$A \sqcap B = \left\langle \widetilde{\mu}_T \widetilde{\cap} \widetilde{\nu}_T, \widetilde{\mu}_I \widetilde{\cap} \widetilde{\nu}_I, \widetilde{\mu}_F \widetilde{\cap} \widetilde{\nu}_F, \lambda_T \vee \eta_T, \lambda_I \vee \eta_I, \lambda_F \vee \eta_F \right\rangle$$

where $\left(\widetilde{\mu}_T \widetilde{\cap} \widetilde{\nu}_T\right)(x) = r\min\{\widetilde{\mu}_T(x), \widetilde{\nu}_T(x)\}$, $\left(\widetilde{\mu}_I \widetilde{\cap} \widetilde{\nu}_I\right)(x) = r\min\{\widetilde{\mu}_I(x), \widetilde{\nu}_I(x)\}$, $\left(\widetilde{\mu}_F \widetilde{\cap} \widetilde{\nu}_F\right)(x) = r\min\{\widetilde{\mu}_F(x), \widetilde{\nu}_F(x)\}$ and
$(\lambda_T \vee \eta_T)(x) = \max\{\lambda_T(x), \eta_T(x)\}$, $(\lambda_I \vee \eta_I)(x) = \max\{\lambda_I(x), \eta_I(x)\}$, $(\lambda_F \vee \eta_F)(x) = \max\{\lambda_F(x), \eta_F(x)\}$.
The union of A and B, denoted by $A \cup B$,is the NC set

$$A \sqcup B = \left\langle \widetilde{\mu}_T \widetilde{\cup} \widetilde{\nu}_T, \widetilde{\mu}_I \widetilde{\cup} \widetilde{\nu}_I, \widetilde{\mu}_F \widetilde{\cup} \widetilde{\nu}_F, \lambda_T \wedge \eta_T, \lambda_I \wedge \eta_I, \lambda_F \wedge \eta_F \right\rangle$$

where $\left(\widetilde{\mu}_T \widetilde{\cup} \widetilde{\nu}_T\right)(x) = r\max\{\widetilde{\mu}_T(x), \widetilde{\nu}_T(x)\}$, $\left(\widetilde{\mu}_I \widetilde{\cup} \widetilde{\nu}_I\right)(x) = r\max\{\widetilde{\mu}_I(x), \widetilde{\nu}_I(x)\}$, $\left(\widetilde{\mu}_F \widetilde{\cup} \widetilde{\nu}_F\right)(x) = r\max\{\widetilde{\mu}_F(x), \widetilde{\nu}_F(x)\}$
and $(\lambda_T \wedge \eta_T)(x) = \min\{\lambda_T(x), \eta_T(x)\}$, $(\lambda_I \wedge \eta_I)(x) = \min\{\lambda_I(x), \eta_I(x)\}$, $(\lambda_F \wedge \eta_F)(x) = \min\{\lambda_F(x), \eta_F(x)\}$.

Proposition 1. *For any NC sets $A = \langle x, \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$, $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_T, \eta_I, \eta_F \rangle$ and $C = \left\langle \widetilde{\zeta}_T, \widetilde{\zeta}_I, \widetilde{\zeta}_F, \delta_T, \delta_I, \delta_F \right\rangle$ in semigroup S. We have*

$$
\begin{array}{llll}
[1] & A \sqcup (B \sqcap C) & = & (A \sqcup B) \sqcap (A \sqcup C) \\
[2] & A \sqcap (B \sqcup C) & = & (A \sqcap B) \sqcup (A \sqcap C) \\
[3] & A \otimes (B \sqcup C) & = & (A \otimes B) \sqcup (A \otimes C) \\
[4] & A \otimes (B \sqcap C) & \sqsubseteq & (A \otimes B) \sqcap (A \otimes C)
\end{array}
$$

Proof. (1) and (2) are straightforwart.
(3) let x be any elemen of s. if x is not expressed as x=yz , then

$$
\begin{array}{rcl}
(\widetilde{\mu}_T \circ (\widetilde{\nu}_T \widetilde{\cup} \widetilde{\zeta}_T))(x) & = & [0\ 0] = ((\widetilde{\mu}_T \circ \widetilde{\nu}_T) \widetilde{\cup} (\widetilde{\mu}_T \circ \widetilde{\zeta}_T))(x), \\
(\widetilde{\mu}_I \circ (\widetilde{\nu}_I \widetilde{\cup} \widetilde{\zeta}_I))(x) & = & [0\ 0] = ((\widetilde{\mu}_I \circ \widetilde{\nu}_I) \widetilde{\cup} (\widetilde{\mu}_I \circ \widetilde{\zeta}_I))(x), \\
(\widetilde{\mu}_F \circ (\widetilde{\nu}_F \widetilde{\cup} \widetilde{\zeta}_F))(x) & = & [0\ 0] = ((\widetilde{\mu}_F \circ \widetilde{\nu}_F) \widetilde{\cup} (\widetilde{\mu}_F \circ \widetilde{\zeta}_F))(x)
\end{array}
$$

and

$$
\begin{array}{rcl}
(\lambda_T \circ (\eta_T \wedge \delta_T))(x) & = & 1 = ((\lambda_T \circ \eta_T) \wedge (\lambda_T \circ \delta_T))(x), \\
(\lambda_I \circ (\eta_I \wedge \delta_I))(x) & = & 1 = ((\lambda_I \circ \eta_I) \wedge (\lambda_I \circ \delta_I))(x), \\
(\lambda_F \circ (\eta_F \wedge \delta_F))(x) & = & 1 = ((\lambda_F \circ \eta_F) \wedge (\lambda_F \circ \delta_F))(x).
\end{array}
$$

therefore $A \otimes (B \sqcup C) = (A \otimes B) \sqcup (A \otimes C)$. Assume that xis expressed as x=yz. Then

$$
\begin{aligned}
(\widetilde{\mu}_T \circ (\widetilde{\nu}_T \widetilde{\cup} \widetilde{\zeta}_T))(x) &= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_T(y), (\widetilde{\nu}_T \widetilde{\cup} \widetilde{\zeta}_T)(z) \right\} \right] \\
&= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_T(y), r \max \left\{ \widetilde{\nu}_T(z), \widetilde{\zeta}_T(z) \right\} \right\} \right] \\
&= r \sup_{x=yz} \left[r \max \left\{ r \min\{\widetilde{\mu}_T(y), \widetilde{\nu}_T(z)\} r \min \left\{ \widetilde{\mu}_T(y), \widetilde{\zeta}_T(z) \right\} \right\} \right] \\
&= r \max \left\{ r \sup_{x=yz} \left\{ [r \min\{\widetilde{\mu}_T(y), \widetilde{\nu}_T(z)\} r \min \left\{ \widetilde{\mu}_T(y), \widetilde{\zeta}_T(z) \right\} \}] \right\} \right\} \\
&= ((\widetilde{\mu}_T \circ \widetilde{\nu}_T) \widetilde{\cup} (\widetilde{\mu}_T \circ \widetilde{\zeta}_T))(x) \\
(\widetilde{\mu}_I \circ (\widetilde{\nu}_I \widetilde{\cup} \widetilde{\zeta}_I))(x) &= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_I(y), (\widetilde{\nu}_I \widetilde{\cup} \widetilde{\zeta}_I)(z) \right\} \right] \\
&= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_I(y), r \max \left\{ \widetilde{\nu}_I(z), \widetilde{\zeta}_I(z) \right\} \right\} \right] \\
&= r \sup_{x=yz} \left[r \max \left\{ r \min\{\widetilde{\mu}_I(y), \widetilde{\nu}_I(z)\} r \min \left\{ \widetilde{\mu}_I(y), \widetilde{\zeta}_I(z) \right\} \right\} \right] \\
&= r \max \left\{ r \sup_{x=yz} \left\{ [r \min\{\widetilde{\mu}_I(y), \widetilde{\nu}_I(z)\} r \min \left\{ \widetilde{\mu}_I(y), \widetilde{\zeta}_I(z) \right\} \}] \right\} \right\} \\
&= ((\widetilde{\mu}_I \circ \widetilde{\nu}_I) \widetilde{\cup} (\widetilde{\mu}_I \circ \widetilde{\zeta}_I))(x) \\
(\widetilde{\mu}_F \circ (\widetilde{\nu}_F \widetilde{\cup} \widetilde{\zeta}_F))(x) &= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_F(y), (\widetilde{\nu}_F \widetilde{\cup} \widetilde{\zeta}_F)(z) \right\} \right] \\
&= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_F(y), r \max \left\{ \widetilde{\nu}_F(z), \widetilde{\zeta}_F(z) \right\} \right\} \right] \\
&= r \sup_{x=yz} \left[r \max \left\{ r \min\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(z)\} r \min \left\{ \widetilde{\mu}_F(y), \widetilde{\zeta}_F(z) \right\} \right\} \right] \\
&= r \max \left\{ r \sup_{x=yz} \left\{ [r \min\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(z)\} r \min \left\{ \widetilde{\mu}_F(y), \widetilde{\zeta}_F(z) \right\} \}] \right\} \right\} \\
&= ((\widetilde{\mu}_F \circ \widetilde{\nu}_F) \widetilde{\cup} (\widetilde{\mu}_F \circ \widetilde{\zeta}_F))(x)
\end{aligned}
$$

and

$$
\begin{aligned}
(\lambda_T \circ (\eta_T \wedge \delta_T))(x) &= \underset{x=yz}{\Lambda} \left[\max\left\{ \lambda_T(y), (\eta_T \wedge \delta_T)(z) \right\} \right] \\
&= \underset{x=yz}{\Lambda} \left[\max\left\{ \lambda_T(y), \min(\eta_T(z), \delta_T(z)) \right\} \right] \\
&= \underset{x=yz}{\Lambda} \left[\max\left\{ \min(\lambda_T(y), \eta_T(z)), \min(\lambda_T(y), \delta_T(z)) \right\} \right] \\
&= \min\left[\underset{x=yz}{\Lambda} \left\{ \max(\lambda_T(y), \eta_T(z)), \max(\lambda_T(y), \delta_T(z)) \right\} \right] \\
&= \min\left[\underset{x=yz}{\Lambda} \max(\lambda_T(y), \eta_T(z)), \underset{x=yz}{\Lambda} \max(\lambda_T(y), \delta_T(z)) \right] \\
&= \min\left[(\lambda_T \circ \eta_T, \lambda_T \circ \delta_T)(x) \right] \\
&= ((\lambda_T \circ \eta_T) \wedge (\lambda_T \circ \delta_T))(x) \\
(\lambda_I \circ (\eta_I \wedge \delta_I))(x) &= \underset{x=yz}{\Lambda} \left[\max\left\{ \lambda_I(y), (\eta_I \wedge \delta_I)(z) \right\} \right] \\
&= \underset{x=yz}{\Lambda} \left[\max\left\{ \lambda_I(y), \min(\eta_I(z), \delta_I(z)) \right\} \right] \\
&= \underset{x=yz}{\Lambda} \left[\max\left\{ \min(\lambda_I(y), \eta_I(z)), \min(\lambda_I(y), \delta_I(z)) \right\} \right] \\
&= \min\left[\underset{x=yz}{\Lambda} \left\{ \max(\lambda_I(y), \eta_I(z)), \max(\lambda_I(y), \delta_I(z)) \right\} \right] \\
&= \min\left[\underset{x=yz}{\Lambda} \max(\lambda_I(y), \eta_I(z)), \underset{x=yz}{\Lambda} \max(\lambda_I(y), \delta_I(z)) \right] \\
&= \min\left[(\lambda_I \circ \eta_I, \lambda_I \circ \delta_I)(x) \right] \\
&= ((\lambda_I \circ \eta_I) \wedge (\lambda_I \circ \delta_I))(x) \\
(\lambda_F \circ (\eta_F \wedge \delta_F))(x) &= \underset{x=yz}{\Lambda} \left[\max\left\{ \lambda_F(y), (\eta_F \wedge \delta_F)(z) \right\} \right] \\
&= \underset{x=yz}{\Lambda} \left[\max\left\{ \lambda_F(y), \min(\eta_F(z), \delta_F(z)) \right\} \right] \\
&= \underset{x=yz}{\Lambda} \left[\max\left\{ \min(\lambda_F(y), \eta_F(z)), \min(\lambda_F(y), \delta_F(z)) \right\} \right] \\
&= \min\left[\underset{x=yz}{\Lambda} \left\{ \max(\lambda_F(y), \eta_F(z)), \max(\lambda_F(y), \delta_F(z)) \right\} \right] \\
&= \min\left[\underset{x=yz}{\Lambda} \max(\lambda_F(y), \eta_F(z)), \underset{x=yz}{\Lambda} \max(\lambda_F(y), \delta_F(z)) \right] \\
&= \min\left[(\lambda_F \circ \eta_F, \lambda_F \circ \delta_F)(x) \right] \\
&= ((\lambda_F \circ \eta_F) \wedge (\lambda_F \circ \delta_F))(x)
\end{aligned}
$$

hence (3) holds.

(4) let $x \in S$.If x isnot expressed asx=yz , then it is clear . As

sume that there exsist y,z$\in S$ such that x=yz .Then

$$
\begin{aligned}
(\widetilde{\mu}_T \circ (\widetilde{\nu}_T \widetilde{\cap} \widetilde{\zeta}_T))(x) &= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_T(y), (\widetilde{\nu}_T \widetilde{\cap} \widetilde{\zeta}_T)(z) \right\} \right] \\
&= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_T(y), r \min \left\{ \widetilde{\nu}_T(z), \widetilde{\zeta}_T(z) \right\} \right\} \right] \\
&= r \sup_{x=yz} \left[r \max \left\{ r \min\{\widetilde{\mu}_T(y), \widetilde{\nu}_T(z)\} r \min \left\{ \widetilde{\mu}_T(y), \widetilde{\zeta}_T(z) \right\} \right\} \right] \\
&\preceq r \min \left\{ r \sup_{x=yz}[r \min\{\widetilde{\mu}_T(y), \widetilde{\nu}_T(z)\}], r \sup_{x=yz}[r \min \left\{ \widetilde{\mu}_T(y), \widetilde{\zeta}_T(z) \right\}] \right\} \\
&= ((\widetilde{\mu}_T \circ \widetilde{\nu}_T) \widetilde{\cap} (\widetilde{\mu}_T \circ \widetilde{\zeta}_T))(x) \\
(\widetilde{\mu}_I \circ (\widetilde{\nu}_I \widetilde{\cap} \widetilde{\zeta}_I))(x) &= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_I(y), (\widetilde{\nu}_I \widetilde{\cap} \widetilde{\zeta}_I)(z) \right\} \right] \\
&= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_I(y), r \min \left\{ \widetilde{\nu}_I(z), \widetilde{\zeta}_I(z) \right\} \right\} \right] \\
&= r \sup_{x=yz} \left[r \max \left\{ r \min\{\widetilde{\mu}_I(y), \widetilde{\nu}_I(z)\} r \min \left\{ \widetilde{\mu}_I(y), \widetilde{\zeta}_I(z) \right\} \right\} \right] \\
&\preceq r \min \left\{ r \sup_{x=yz}[r \min\{\widetilde{\mu}_I(y), \widetilde{\nu}_I(z)\}], r \sup_{x=yz}[r \min \left\{ \widetilde{\mu}_I(y), \widetilde{\zeta}_I(z) \right\}] \right\} \\
&= ((\widetilde{\mu}_I \circ \widetilde{\nu}_I) \widetilde{\cap} (\widetilde{\mu}_I \circ \widetilde{\zeta}_I))(x) \\
(\widetilde{\mu}_F \circ (\widetilde{\nu}_F \widetilde{\cap} \widetilde{\zeta}_F))(x) &= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_F(y), (\widetilde{\nu}_F \widetilde{\cap} \widetilde{\zeta}_F)(z) \right\} \right] \\
&= r \sup_{x=yz} \left[r \min \left\{ \widetilde{\mu}_F(y), r \min \left\{ \widetilde{\nu}_F(z), \widetilde{\zeta}_F(z) \right\} \right\} \right] \\
&= r \sup_{x=yz} \left[r \max \left\{ r \min\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(z)\} r \min \left\{ \widetilde{\mu}_F(y), \widetilde{\zeta}_F(z) \right\} \right\} \right] \\
&\preceq r \min \left\{ r \sup_{x=yz}[r \min\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(z)\}], r \sup_{x=yz}[r \min \left\{ \widetilde{\mu}_F(y), \widetilde{\zeta}_F(z) \right\}] \right\} \\
&= ((\widetilde{\mu}_F \circ \widetilde{\nu}_F) \widetilde{\cap} (\widetilde{\mu}_F \circ \widetilde{\zeta}_F))(x)
\end{aligned}
$$

and

$$
\begin{aligned}
(\lambda_T \circ (\eta_T \vee \delta_T))(x) &= \bigwedge_{x=yz} \max\left\{\lambda_T(y), (\eta_T \vee \delta_T)(z)\right\} \\
&= \bigwedge_{x=yz} \max\left\{\lambda_T(y), \max\{\eta_T(z), \delta_T(z)\}\right\} \\
&= \bigwedge_{x=yz} \max\left\{\max\{\lambda_T(y), \eta_T(z)\}, \max\{\lambda_T(y), \delta_T(z)\}\right\} \\
&\geq \max\left\{\bigwedge_{x=yz} \max\{\lambda_T(y), \eta_T(z)\}, \bigwedge_{x=yz} \max\{\lambda_T(y), \delta_T(z)\}\right\} \\
&= ((\lambda_T \circ \eta_T) \vee (\lambda_T \circ \delta_T))(x). \\
(\lambda_I \circ (\eta_I \vee \delta_I))(x) &= \bigwedge_{x=yz} \max\left\{\lambda_I(y), (\eta_I \vee \delta_I)(z)\right\} \\
&= \bigwedge_{x=yz} \max\left\{\lambda_I(y), \max\{\eta_I(z), \delta_I(z)\}\right\} \\
&= \bigwedge_{x=yz} \max\left\{\max\{\lambda_I(y), \eta_I(z)\}, \max\{\lambda_I(y), \delta_I(z)\}\right\} \\
&\geq \max\left\{\bigwedge_{x=yz} \max\{\lambda_I(y), \eta_I(z)\}, \bigwedge_{x=yz} \max\{\lambda_I(y), \delta_I(z)\}\right\} \\
&= ((\lambda_I \circ \eta_I) \vee (\lambda_I \circ \delta_I))(x). \\
(\lambda_F \circ (\eta_F \vee \delta_F))(x) &= \bigwedge_{x=yz} \max\left\{\lambda_F(y), (\eta_F \vee \delta_F)(z)\right\} \\
&= \bigwedge_{x=yz} \max\left\{\lambda_F(y), \max\{\eta_F(z), \delta_F(z)\}\right\} \\
&= \bigwedge_{x=yz} \max\left\{\max\{\lambda_F(y), \eta_F(z)\}, \max\{\lambda_F(y), \delta_F(z)\}\right\} \\
&\geq \max\left\{\bigwedge_{x=yz} \max\{\lambda_F(y), \eta_F(z)\}, \bigwedge_{x=yz} \max\{\lambda_F(y), \delta_F(z)\}\right\} \\
&= ((\lambda_F \circ \eta_F) \vee (\lambda_F \circ \delta_F))(x).
\end{aligned}
$$

Hence (4) holds. $\qquad\square$

Proposition 2. *For any NC sets* $A = \langle x, \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$, $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_T, \eta_I, \eta_F \rangle$ *and* $C = \left\langle \widetilde{\zeta}_T, \widetilde{\zeta}_I, \widetilde{\zeta}_F, \delta_T, \delta_I, \delta_F \right\rangle$ *in semigroup S, if* $A \sqsubseteq B$, *then* $A \otimes C \sqsubseteq B \otimes C$ *and* $C \otimes A \sqsubseteq C \otimes B$.

Proof. Strraitforward. $\qquad\square$

Proposition 3. *For any nonempty subsets G and H of semigroup S, we have*

$$[1] \quad \chi G \otimes \chi H = \chi GH, i.e.$$

$$\left\langle \widetilde{\mu}_{T\chi G} \widetilde{\circ} \widetilde{\mu}_{T\chi H}, \widetilde{\mu}_{I\chi G} \widetilde{\circ} \widetilde{\mu}_{I\chi H}, \widetilde{\mu}_{F\chi G} \widetilde{\circ} \widetilde{\mu}_{F\chi H}, \lambda_{T\chi G} \circ \lambda_{T\chi H}, \lambda_{I\chi G} \circ \lambda_{I\chi H}, \lambda_{F\chi G} \circ \lambda_{F\chi H} \right\rangle = \left\langle \widetilde{\mu}_{T\chi GH}, \widetilde{\mu}_{I\chi GH}, , \right.$$

$$[2] \quad \chi G \cap \chi H = \chi G \cap H, i.e.$$

$$\left\langle \widetilde{\mu}_{T\chi G} \cap \widetilde{\mu}_{T\chi H}, \widetilde{\mu}_{I\chi G} \cap \widetilde{\mu}_{I\chi H}, \widetilde{\mu}_{F\chi G} \cap \widetilde{\mu}_{F\chi H}, \lambda_{T\chi G} \vee \lambda_{T\chi H}, \lambda_{I\chi G} \vee \lambda_{I\chi H}, \lambda_{F\chi G} \vee \lambda_{F\chi H} \right\rangle = \left\langle \widetilde{\mu}_{T\chi G \cap H}, \widetilde{\mu}_{F\chi G \cap H} \right.$$

$$[3] \quad \chi G \cup \chi H = \chi G \cup H, i.e.$$

$$\left\langle \widetilde{\mu}_{T\chi G} \cup \widetilde{\mu}_{T\chi H}, \widetilde{\mu}_{I\chi G} \cup \widetilde{\mu}_{I\chi H}, \widetilde{\mu}_{F\chi G} \cup \widetilde{\mu}_{F\chi H}, \lambda_{T\chi G} \wedge \lambda_{T\chi H}, \lambda_{I\chi G} \wedge \lambda_{I\chi H}, \lambda_{F\chi G} \wedge \lambda_{F\chi H} \right\rangle = \left\langle \widetilde{\mu}_{T\chi G \cup H}, \widetilde{\mu}_{F\chi G \cup H} \right.$$

Proof. (1) Let $a \in S$. If $a \in GH$, then $\widetilde{\mu}_{T\chi GH}(a) = [1\,1], \widetilde{\mu}_{I\chi GH}(a) = [11], \widetilde{\mu}_{F\chi GH}(a) = [11], \lambda_{T\chi GH}(a) = 0, \lambda_{I\chi GH}(a) = 0, \lambda_{F\chi GH}(a) = 0$ and a=bc for some b$\in$ G and

c$\in H$ Thus

$$
\begin{aligned}
(\widetilde{\mu}_{T\chi G} \circ \widetilde{\mu}_{T\chi H})(a) &= \underset{a=xy}{r \sup} \left[r \min \left\{ \widetilde{\mu}_{T\chi G}(x), \widetilde{\mu}_{T\chi H}(y) \right\} \right] \\
&\succeq r \min \left\{ \widetilde{\mu}_{T\chi G}(b), \widetilde{\mu}_{T\chi H}(c) \right\} \\
&= [1\ 1] \\
(\widetilde{\mu}_{I\chi G} \circ \widetilde{\mu}_{I\chi H})(a) &= \underset{a=xy}{r \sup} \left[r \min \left\{ \widetilde{\mu}_{I\chi G}(x), \widetilde{\mu}_{I\chi H}(y) \right\} \right] \\
&\succeq r \min \left\{ \widetilde{\mu}_{I\chi G}(b), \widetilde{\mu}_{I\chi H}(c) \right\} \\
&= [1\ 1] \\
(\widetilde{\mu}_{F\chi G} \circ \widetilde{\mu}_{F\chi H})(a) &= \underset{a=xy}{r \sup} \left[r \min \left\{ \widetilde{\mu}_{F\chi G}(x), \widetilde{\mu}_{F\chi H}(y) \right\} \right] \\
&\succeq r \min \left\{ \widetilde{\mu}_{F\chi G}(b), \widetilde{\mu}_{F\chi H}(c) \right\} \\
&= [1\ 1]
\end{aligned}
$$

and

$$
\begin{aligned}
(\lambda_{T\chi G} \circ \lambda_{T\chi H})(a) &= \underset{a=xy}{\Lambda} [\max \left\{ \lambda_{T\chi G}(x), \lambda_{T\chi H}(y) \right\}] \\
&\leq \max \left\{ \lambda_{T\chi G}(b), \lambda_{T\chi H}(c) \right\} \\
&= 0 \\
(\lambda_{I\chi G} \circ \lambda_{I\chi H})(a) &= \underset{a=xy}{\Lambda} [\max \left\{ \lambda_{I\chi G}(x), \lambda_{I\chi H}(y) \right\}] \\
&\leq \max \left\{ \lambda_{I\chi G}(b), \lambda_{I\chi H}(c) \right\} \\
&= 0 \\
(\lambda_{F\chi G} \circ \lambda_{F\chi H})(a) &= \underset{a=xy}{\Lambda} [\max \left\{ \lambda_{F\chi G}(x), \lambda_{F\chi H}(y) \right\}] \\
&\leq \max \left\{ \lambda_{F\chi G}(b), \lambda_{F\chi H}(c) \right\} \\
&= 0
\end{aligned}
$$

It follows that $(\widetilde{\mu}_{T\chi G} \circ \widetilde{\mu}_{T\chi H})(a) = [1\ 1], (\widetilde{\mu}_{I\chi G} \circ \widetilde{\mu}_{I\chi H})(a) = [1\ 1], (\widetilde{\mu}_{F\chi G} \circ \widetilde{\mu}_{F\chi H})(a) = [1\ 1]$ and $(\lambda_{T\chi G} \circ \lambda_{T\chi H})(a) = 0, (\lambda_{I\chi G} \circ \lambda_{I\chi H})(a) = 0, (\lambda_{F\chi G} \circ \lambda_{F\chi H})(a) = 0.$Therefore

$$
\begin{aligned}
\langle \widetilde{\mu}_{T\chi G} \circ \widetilde{\mu}_{T\chi H}, \lambda_{T\chi G} \circ \lambda_{T\chi H} \rangle &= \langle \widetilde{\mu}_{T\chi GH}, \lambda_{T\chi GH} \rangle \\
\langle \widetilde{\mu}_{I\chi G} \circ \widetilde{\mu}_{I\chi H}, \lambda_{I\chi G} \circ \lambda_{I\chi H} \rangle &= \langle \widetilde{\mu}_{I\chi GH}, \lambda_{I\chi GH} \rangle \\
\langle \widetilde{\mu}_{F\chi G} \circ \widetilde{\mu}_{F\chi H}, \lambda_{F\chi G} \circ \lambda_{F\chi H} \rangle &= \langle \widetilde{\mu}_{F\chi GH}, \lambda_{F\chi GH} \rangle
\end{aligned}
$$

that is ,$\chi G \otimes \chi H = \chi GH$. Assume that a$\notin GH$. Then $\widetilde{\mu}_{T\chi GH}(a) = [0\ 0], \widetilde{\mu}_{I\chi GH}(a) = [0\ 0], \widetilde{\mu}_{F\chi GH}(a) = [0\ 0]$ and $\lambda_{T\chi GH}(a) = 1, \lambda_{I\chi GH}(a) = 1, \lambda_{F\chi GH}(a) = 1$. Let y,z$\in S$ be such that a$= yz$. Then we know that y$\notin G$ or z$\notin H$.Assume that

y$\notin G$.Then

$$
\begin{aligned}
(\widetilde{\mu}_{T\chi G} \circ \widetilde{\mu}_{T\chi H})(a) &= r\sup_{a=yz}[r\min\{\widetilde{\mu}_{T\chi G}(y),\widetilde{\mu}_{T\chi H}(z)\}] \\
&= r\sup_{a=yz}[r\min\{[0\ 0],\widetilde{\mu}_{T\chi H}(z)\}] \\
&= [0\ 0] = \widetilde{\mu}_{T\chi GH}(a) \\
(\widetilde{\mu}_{I\chi G} \circ \widetilde{\mu}_{I\chi H})(a) &= r\sup_{a=yz}\left[r\min\{\widetilde{\mu}_{I\chi G}(y),\widetilde{\mu}_{I\chi H}(z)\}\right] \\
&= r\sup_{a=yz}[r\min\{[0\ 0],\widetilde{\mu}_{I\chi H}(z)\}] \\
&= [0\ 0] = \widetilde{\mu}_{I\chi GH}(a) \\
(\widetilde{\mu}_{F\chi G} \circ \widetilde{\mu}_{F\chi H})(a) &= r\sup_{a=yz}[r\min\{\widetilde{\mu}_{F\chi G}(y),\widetilde{\mu}_{F\chi H}(z)\}] \\
&= r\sup_{a=yz}[r\min\{[0\ 0],\widetilde{\mu}_{F\chi H}(z)\}] \\
&= [0\ 0] = \widetilde{\mu}_{F\chi GH}(a)
\end{aligned}
$$

and

$$
\begin{aligned}
(\lambda_{T\chi G} \circ \lambda_{T\chi H})(a) &= \Lambda_{a=yz}[\max\{\lambda_{T\chi G}(y),\lambda_{T\chi H}(z)\}] \\
&= \Lambda_{a=yz}[\max\{1,\lambda_{T\chi H}(z)\}] \\
&= 1 = \lambda_{T\chi GH}(a) \\
(\lambda_{I\chi G} \circ \lambda_{I\chi H})(a) &= \Lambda_{a=yz}[\max\{\lambda_{I\chi G}(y),\lambda_{I\chi H}(z)\}] \\
&= \Lambda_{a=yz}[\max\{1,\lambda_{I\chi H}(z)\}] \\
&= 1 = \lambda_{I\chi GH}(a) \\
(\lambda_{F\chi G} \circ \lambda_{F\chi H})(a) &= \Lambda_{a=yz}[\max\{\lambda_{F\chi G}(y),\lambda_{F\chi H}(z)\}] \\
&= \Lambda_{a=yz}[\max\{1,\lambda_{F\chi H}(z)\}] \\
&= 1 = \lambda_{F\chi GH}(a)
\end{aligned}
$$

Similarly, if z$\notin H$,then $(\widetilde{\mu}_{T\chi G}\circ\widetilde{\mu}_{T\chi H})(a) = [0\ 0] = \widetilde{\mu}_{T\chi GH}(a), (\widetilde{\mu}_{I\chi G}\circ\widetilde{\mu}_{I\chi H})(a) = [0\ 0] = \widetilde{\mu}_{I\chi GH}(a), (\widetilde{\mu}_{F\chi G}\circ\widetilde{\mu}_{F\chi H})(a) = [0\ 0] = \widetilde{\mu}_{F\chi GH}(a)$ and $(\lambda_{T\chi G} \circ \lambda_{T\chi H})(a) = 1 = \lambda_{T\chi GH}(a), (\lambda_{I\chi G} \circ \lambda_{I\chi H})(a) = 1 = \lambda_{I\chi GH}(a), (\lambda_{F\chi G} \circ \lambda_{F\chi H})(a) = 1 = \lambda_{F\chi GH}(a)$. Therefore $\chi G \otimes \chi H = \chi GH$.(2) and (3) are straightforward. $\qquad\square$

4. Neutrosophic cubic subsemigroups and ideals

Definition 2. *A NC set $A=\langle\widetilde{\mu}_T,\widetilde{\mu}_I,\widetilde{\mu}_F,\lambda_T,\lambda_I,\lambda_F\rangle$ in semigroug S is called a NC subsemigroup of S if it satisfies:*

$$
\begin{aligned}
(\forall x,y \in S) \quad \widetilde{\mu}_T(xy) &\succeq r\min\{\widetilde{\mu}_T(x),\widetilde{\mu}_T(y)\}, \widetilde{\mu}_I(xy) \succeq r\min\{\widetilde{\mu}_I(x),\widetilde{\mu}_I(y)\}, \widetilde{\mu}_F(xy) \succeq r\min\{\widetilde{\mu}_F(x),\widetilde{\mu}_I \\
\lambda_T(xy) &\leq \max\{\lambda_T(x),\lambda_T(y)\}, \lambda_I(xy) \leq \max\{\lambda_I(x),\lambda_I(y)\}, \lambda_F(xy) \leq \{\lambda_F(x),\lambda_F(y)\}
\end{aligned}
$$

Example 1. *Consider a semigroup $S=\{a,b,c\}$ with the following Caylay's table.*

.	a	b	c
a	c	c	c
b	c	c	a
c	c	b	c

Defins a NC set $A=\langle\widetilde{\mu}_T,\widetilde{\mu}_I,\widetilde{\mu}_F,\lambda_T,\lambda_I,\lambda_F\rangle$ in S by

S	$\widetilde{\mu}_T$	$\widetilde{\mu}_I$	$\widetilde{\mu}_F$	λ_T	λ_I	λ_F
a	$[0.3, 0.6]$	$[0.5, 0.7]$	$[0.6, 0.7]$	0.4	0.6	0.8
b	$[0.2, 0.4]$	$[0.3, 0.4]$	$[0.2, 0.5]$	0.6	0.7	0.9
c	$[0.7, 0.9]$	$[0.8, 0.9]$	$[0.7, 0.8]$	0.2	0.3	0.4

Theorem 1. *A NC set $A=\langle\widetilde{\mu}_T,\widetilde{\mu}_I,\widetilde{\mu}_F,\lambda_T,\lambda_I,\lambda_F\rangle$ in semigroug S is a NC sub-semigroup of S if and only if $A\otimes A \sqsubseteq A$.*

Proof. straightforward.   □

Definition 3. *A NC set $A=\langle\widetilde{\mu}_T,\widetilde{\mu}_I,\widetilde{\mu}_F,\lambda_T,\lambda_I,\lambda_F\rangle$ in a semigroug S is called a right(left) NC ideal of S if it satisfies:*

$$(\forall a,b \ \in \ S) \quad \widetilde{\mu}_T(xy) \succeq \widetilde{\mu}_T(x)\,(\widetilde{\mu}_T(xy) \succeq \widetilde{\mu}_T(y)) \ ,\widetilde{\mu}_I(xy) \succeq \widetilde{\mu}_I(x)\,(\widetilde{\mu}_I(xy) \succeq \widetilde{\mu}_I(y)) \ ,\widetilde{\mu}_F(xy) \succeq$$

and $\quad \lambda_T(xy) \ \leq \ \lambda_T(x)\,(\lambda_T(xy) \leq \lambda_T(y))\,,\lambda_I(xy) \leq \lambda_I(x)\,(\lambda_I(xy) \leq \lambda_I(y))\,,\lambda_F(xy) \leq \lambda_F(x)(\lambda_F(xy) \leq \lambda_F($

By a (two sided) NC ideal we mean a left and right NC ideal.

Example 2. *Consider a semigroup $S=\{a,b,c\}$ with the following Caylay's table.*

.	a	b	c
a	a	a	a
b	a	a	a
c	a	a	c

Defins a NC set $A=\langle\widetilde{\mu}_T,\widetilde{\mu}_I,\widetilde{\mu}_F,\lambda_T,\lambda_I,\lambda_F\rangle$ in S by

S	$\widetilde{\mu}_T$	$\widetilde{\mu}_I$	$\widetilde{\mu}_F$	λ_T	λ_I	λ_F
a	$[0.7, 0.9]$	$[0.8, 0.9]$	$[0.7, 0.8]$	0.2	0.3	0.4
b	$[0.2, 0.4]$	$[0.3, 0.4]$	$[0.2, 0.5]$	0.6	0.7	0.9
c	$[0.1, 0.3]$	$[0.5, 0.6]$	$[0.2, 0.4]$	0.7	0.5	0.8

It is easy to verify that $A = \langle\widetilde{\mu}_T,\widetilde{\mu}_I,\widetilde{\mu}_F,\lambda_T,\lambda_I,\lambda_F\rangle$ is a cubic ideal of S.
Obviously, every left (resp. right) cubic ideal is a cubic subsemigroup.
But the converse may not be true as seen in the following example

Example 3. *Consider a semigroup $S=\{a,b,c,d\}$ with the following Caylay's table.*

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Defins a NC set $A=\langle\widetilde{\mu}_T,\widetilde{\mu}_I,\widetilde{\mu}_F,\lambda_T,\lambda_I,\lambda_F\rangle$ in S by

S	$\widetilde{\mu}_T$	$\widetilde{\mu}_I$	$\widetilde{\mu}_F$	λ_T	λ_I	λ_F
a	$[0.5, 0.8]$	$[0.7, 0.9]$	$[0.6, 0.8]$	0.2	0.1	0.3
b	$[0.3, 0.6]$	$[0.4, 0.7]$	$[0.3, 0.5]$	0.6	0.7	0.4
c	$[0.5, 0.8]$	$[0.5, 0.6]$	$[0.5, 0.6]$	0.4	0.5	0.4
d	$[0.2, 0.4]$	$[0.2, 0.4]$	$[0.1, 0.3]$	0.6	0.8	0.5

It is easy to verify that $A = \langle\widetilde{\mu}_T,\widetilde{\mu}_I,\widetilde{\mu}_F,\lambda_T,\lambda_I,\lambda_F\rangle$ is a NC subsemigroup of S;
but it is not a left NC ideal of S since $\widetilde{\mu}_T(dc) = \widetilde{\mu}_T(b) =[0.3,0.6]\nsucceq [0.5,0.8] = \widetilde{\mu}_T(c)$
and/or $\lambda_T(dc) =\lambda_T(b) = 0.6 > 0.4 = \lambda_T(c)\,.$

Theorem 2. *For NC set $A = \langle\widetilde{\mu}_T,\widetilde{\mu}_I,\widetilde{\mu}_F,\lambda_T,\lambda_I,\lambda_F\rangle$ in semigroug S, the following are equivalent:*

(1) $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NC ideal of S.

(2) $S \otimes A \sqsubseteq A$.

Proof. Assume that $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NC ideal of S. Let a∈ S. If $(S \otimes A)(a) = \langle [0,0], 1 \rangle$, then it is clear that $S \otimes A \sqsubseteq A$. Otherewise, there exsists $x, y \in S$ such that $a = xy$. Since $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NC ideal of S, we have

$$
\begin{aligned}
\left(\widetilde{1}_S \circ \widetilde{\mu}_T \right)(a) &= \underset{a=xy}{r \sup} \left[r \min \left\{ \widetilde{1}_S(x), \widetilde{\mu}_T(y) \right\} \right] \\
&\preceq \underset{a=xy}{r \sup} \left[r \min \left\{ [1,1], \widetilde{\mu}_T(xy) \right\} \right] \\
&= r \sup \widetilde{\mu}_T(a) \\
&= \widetilde{\mu}_T(a) \\
\left(\widetilde{1}_S \circ \widetilde{\mu}_I \right)(a) &= \underset{a=xy}{r \sup} \left[r \min \left\{ \widetilde{1}_S(x), \widetilde{\mu}_I(y) \right\} \right] \\
&\preceq \underset{a=xy}{r \sup} \left[r \min \left\{ [1,1], \widetilde{\mu}_I(xy) \right\} \right] \\
&= r \sup \widetilde{\mu}_I(a) \\
&= \widetilde{\mu}_I(a) \\
\left(\widetilde{1}_S \circ \widetilde{\mu}_F \right)(a) &= \underset{a=xy}{r \sup} \left[r \min \left\{ \widetilde{1}_S(x), \widetilde{\mu}_F(y) \right\} \right] \\
&\preceq \underset{a=xy}{r \sup} \left[r \min \left\{ [1,1], \widetilde{\mu}_F(xy) \right\} \right] \\
&= r \sup \widetilde{\mu}_F(a) \\
&= \widetilde{\mu}_F(a)
\end{aligned}
$$

and

$$
\begin{aligned}
(0_S \circ \lambda_T)(a) &= \underset{a=xy}{\Lambda} \max \left\{ 0_S(x), \lambda_T(y) \right\} \\
&\geq \underset{a=xy}{\Lambda} \max \left\{ \lambda_T(xy) \right\} \\
&= \Lambda \lambda_T(a) = \lambda_T(a) \\
(0_S \circ \lambda_I)(a) &= \underset{a=xy}{\Lambda} \max \left\{ 0_S(x), \lambda_I(y) \right\} \\
&\geq \underset{a=xy}{\Lambda} \max \left\{ \lambda_I(xy) \right\} \\
&= \Lambda \lambda_I(a) = \lambda_I(a) \\
(0_S \circ \lambda_F)(a) &= \underset{a=xy}{\Lambda} \max \left\{ 0_S(x), \lambda_F(y) \right\} \\
&\geq \underset{a=xy}{\Lambda} \max \left\{ \lambda_F(xy) \right\} \\
&= \Lambda \lambda_F(a) = \lambda_F(a)
\end{aligned}
$$

Therfore $S \otimes A \sqsubseteq A$.

Conversely, suppose that $S \otimes A \sqsubseteq A$. For any elements x and y of S, let $a = xy$. Then

$$
\begin{aligned}
\widetilde{\mu}_T(xy) &= \widetilde{\mu}_T(a) \succeq \left(\widetilde{1}_S \circ \widetilde{\mu}_T\right)(a) \\
&= r\sup_{a=bc}\left[r\min\left\{\widetilde{1}_S(b), \widetilde{\mu}_T(c)\right\}\right] \\
&\succeq r\min\left\{\widetilde{1}_S(x), \widetilde{\mu}_T(y)\right\} = \widetilde{\mu}_T(y) \\
\widetilde{\mu}_I(xy) &= \widetilde{\mu}_I(a) \succeq \left(\widetilde{1}_S \circ \widetilde{\mu}_I\right)(a) \\
&= r\sup_{a=bc}\left[r\min\left\{\widetilde{1}_S(b), \widetilde{\mu}_I(c)\right\}\right] \\
&\succeq r\min\left\{\widetilde{1}_S(x), \widetilde{\mu}_I(y)\right\} = \widetilde{\mu}_I(y) \\
\widetilde{\mu}_F(xy) &= \widetilde{\mu}_F(a) \succeq \left(\widetilde{1}_S \circ \widetilde{\mu}_F\right)(a) \\
&= r\sup_{a=bc}\left[r\min\left\{\widetilde{1}_S(b), \widetilde{\mu}_F(c)\right\}\right] \\
&\succeq r\min\left\{\widetilde{1}_S(x), \widetilde{\mu}_F(y)\right\} = \widetilde{\mu}_F(y)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_T(xy) &= \lambda_T(a) \leq (0_S \circ \lambda_T)(a) \\
&= \bigwedge_{a=bc} \max\{0_S(b), \lambda_T(c)\} \\
&\leq \max\{0_S(x), \lambda_T(y)\} = \lambda_T(y) \\
\lambda_I(xy) &= \lambda_I(a) \leq (0_S \circ \lambda_I)(a) \\
&= \bigwedge_{a=bc} \max\{0_S(b), \lambda_I(c)\} \\
&\leq \max\{0_S(x), \lambda_I(y)\} = \lambda_I(y) \\
\lambda_F(xy) &= \lambda_F(a) \leq (0_S \circ \lambda_F)(a) \\
&= \bigwedge_{a=bc} \max\{0_S(b), \lambda_F(c)\} \\
&\leq \max\{0_S(x), \lambda_F(y)\} = \lambda_F(y)
\end{aligned}
$$

Hence $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NC ideal of S. $\square$

Similarly, we can induces the following theorem.

Theorem 3. *For a NS cubic st $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in asemigroup S, the following are equivalent:*
(1) $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ *is a right NS cubic ideal of S.*
(2) $A \otimes S \sqsubseteq A$.

Theorem 4. *If $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a NS cubic set in a semigroup S, then $S \otimes A$ (resp. $A \otimes S$) is a left (resp. right) NS cubic ideal of S.*

Proof. Since $S \otimes (S \otimes A) = (S \otimes S) \otimes A \sqsubseteq S \otimes A$, it follows from theorem 2 that $S \otimes A$ is a left NS cubic ideal of S. Similarly $A \otimes S$ is a right NS cubic ideal of S. $\square$

Now we will consider conditions for a left (resp. right) NS cubic ideal to be constant.

Proposition 4. *Let U be a left zero subsemigroup of a semigroup S. If $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NS cubic ideal of , then $A(x) = A(y)$ for all $x, y \in U$.*

Proof. Let $x, y \in U$. Then $xy = x$ and $yx = y$. Thus

$$
\begin{aligned}
\widetilde{\mu}_T(x) &= \widetilde{\mu}_T(xy) \succeq \widetilde{\mu}_T(y) = \widetilde{\mu}_T(yx) \succeq \widetilde{\mu}_T(x) \\
\widetilde{\mu}_I(x) &= \widetilde{\mu}_I(xy) \succeq \widetilde{\mu}_I(y) = \widetilde{\mu}_I(yx) \succeq \widetilde{\mu}_I(x) \\
\widetilde{\mu}_F(x) &= \widetilde{\mu}_F(xy) \succeq \widetilde{\mu}_F(y) = \widetilde{\mu}_F(yx) \succeq \widetilde{\mu}_F(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_T(x) &= \lambda_T(xy) \leq \lambda_T(y) = \lambda_T(xy) \leq \lambda_T(x) \\
\lambda_I(x) &= \lambda_I(xy) \leq \lambda_I(y) = \lambda_I(xy) \leq \lambda_I(x) \\
\lambda_F(x) &= \lambda_F(xy) \leq \lambda_F(y) = \lambda_F(xy) \leq \lambda_F(x)
\end{aligned}
$$

Therefore $A(x) = A(y)$ for all $x, y \in U$. $\square$

Similarly, we have the following proposition.

Proposition 5. *Let U be a right zero subsemigroup of a semigroup S. If $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a right NS cubic ideal of , then $A(x) = A(y)$ for all $x, y \in U$.*

Theorem 5. *Let $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a left NC ideal of a semigroup S. If the set of all idempotent elements of S forms a left zero subsemigroup of S, then $A(u) = A(v)$ for all idempotent elements u and v of S.*

Proof. Let $Idm(S)$ be the set of all idempotent elements of S and assume that $Idm(S)$ is a left zero subsemigroup of S. For any $u, v \in Idm(S)$, we have $uv = u$ and $uv = v$. Hence

$$
\begin{aligned}
\widetilde{\mu}_T(u) &= \widetilde{\mu}_T(uv) \succeq \widetilde{\mu}_T(v) = \widetilde{\mu}_T(uv) \succeq \widetilde{\mu}_T(u) \\
\widetilde{\mu}_I(u) &= \widetilde{\mu}_I(uv) \succeq \widetilde{\mu}_I(v) = \widetilde{\mu}_I(uv) \succeq \widetilde{\mu}_I(u) \\
\widetilde{\mu}_F(u) &= \widetilde{\mu}_F(uv) \succeq \widetilde{\mu}_F(v) = \widetilde{\mu}_F(uv) \succeq \widetilde{\mu}_F(u)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_T(u) &= \lambda_T(uv) \leq \lambda_T(v) = \lambda_T(uv) \leq \lambda_T(u) \\
\lambda_I(u) &= \lambda_I(uv) \leq \lambda_I(v) = \lambda_I(uv) \leq \lambda_I(u) \\
\lambda_F(u) &= \lambda_F(uv) \leq \lambda_F(v) = \lambda_F(uv) \leq \lambda_F(u)
\end{aligned}
$$

Therefore $A(u) = A(v)$ for all $u, v \in Idm(S)$. $\square$

Similarly, we have the following theorem.

Theorem 6. *Let $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a right NC ideal of a semigroup S. If the set of all idempotent elements of S forms a right zero subsemigroup of S, then $A(u) = A(v)$ for all idempotent elements u and v of S.*

Theorem 7. *Let S be a semigroup. Then the following properties hold.*
(1) *The intersection of two NC subsemigroups of S is a NC subsemigroup of S.*
(2) *The intersection of two left (resp. right) NC ideals of S is a left (resp. right) NC ideals of S*

Proof. (1) Let $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ and $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ be two NC subsemigroups of S. Let x and y be any elements of S. Then

$$
\begin{aligned}
(\widetilde{\mu}_T \cap \widetilde{\nu}_T)(xy) &= r\min\{\widetilde{\mu}_T(xy), \widetilde{\nu}_T(xy)\} \\
&\succeq r\min\{r\min\{\widetilde{\mu}_T(x), \widetilde{\mu}_T(y)\}, r\min\{\widetilde{\nu}_T(x), \widetilde{\nu}_T(y)\}\} \\
&= r\min\{r\min\{\widetilde{\mu}_T(x), \widetilde{\nu}_T(x)\}, r\min\{\widetilde{\mu}_T(y), \widetilde{\nu}_T(y)\}\} \\
&= r\min\{(\widetilde{\mu}_T \cap \widetilde{\nu}_T)(x), (\widetilde{\mu}_T \cap \widetilde{\nu}_T)(y)\} \\
(\widetilde{\mu}_I \cap \widetilde{\nu}_I)(xy) &= r\min\{\widetilde{\mu}_I(xy), \widetilde{\nu}_I(xy)\} \\
&\succeq r\min\{r\min\{\widetilde{\mu}_I(x), \widetilde{\mu}_I(y)\}, r\min\{\widetilde{\nu}_I(x), \widetilde{\nu}_I(y)\}\} \\
&= r\min\{r\min\{\widetilde{\mu}_I(x), \widetilde{\nu}_I(x)\}, r\min\{\widetilde{\mu}_I(y), \widetilde{\nu}_I(y)\}\} \\
&= r\min\{(\widetilde{\mu}_I \cap \widetilde{\nu}_I)(x), (\widetilde{\mu}_I \cap \widetilde{\nu}_I)(y)\} \\
(\widetilde{\mu}_F \cap \widetilde{\nu}_F)(xy) &= r\min\{\widetilde{\mu}_F(xy), \widetilde{\nu}_F(xy)\} \\
&\succeq r\min\{r\min\{\widetilde{\mu}_F(x), \widetilde{\mu}_F(y)\}, r\min\{\widetilde{\nu}_F(x), \widetilde{\nu}_F(y)\}\} \\
&= r\min\{r\min\{\widetilde{\mu}_F(x), \widetilde{\nu}_F(x)\}, r\min\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(y)\}\} \\
&= r\min\{(\widetilde{\mu}_F \cap \widetilde{\nu}_F)(x), (\widetilde{\mu}_F \cap \widetilde{\nu}_F)(y)\}
\end{aligned}
$$

and

$$
\begin{aligned}
(\lambda_T \vee \eta_T)(xy) &= \max\{\lambda_T(xy), \eta_T(xy)\} \\
&\leq \max\{\max\{\lambda_T(x), \lambda_T(y)\}, \max\{\eta_T(x), \eta_T(y)\}\} \\
&= \max\{\max\{\lambda_T(x), \eta_T(x)\}, \max\{\lambda_T(y), \eta_T(y)\}\} \\
&= \max\{(\lambda_T \vee \eta_T)(x), (\lambda_T \vee \eta_T)(y)\} \\
(\lambda_I \vee \eta_I)(xy) &= \max\{\lambda_I(xy), \eta_I(xy)\} \\
&\leq \max\{\max\{\lambda_I(x), \lambda_I(y)\}, \max\{\eta_I(x), \eta_I(y)\}\} \\
&= \max\{\max\{\lambda_I(x), \eta_I(x)\}, \max\{\lambda_I(y), \eta_I(y)\}\} \\
&= \max\{(\lambda_I \vee \eta_I)(x), (\lambda_I \vee \eta_I)(y)\} \\
(\lambda_F \vee \eta_F)(xy) &= \max\{\lambda_F(xy), \eta_F(xy)\} \\
&\leq \max\{\max\{\lambda_F(x), \lambda_F(y)\}, \max\{\eta_F(x), \eta_F(y)\}\} \\
&= \max\{\max\{\lambda_F(x), \eta_F(x)\}, \max\{\lambda_F(y), \eta_F(y)\}\} \\
&= \max\{(\lambda_F \vee \eta_F)(x), (\lambda_F \vee \eta_F)(y)\}
\end{aligned}
$$

Therefore $A \sqcap B = \langle \widetilde{\mu}_T \cap \widetilde{\nu}_T, \widetilde{\mu}_I \cap \widetilde{\nu}_I, \widetilde{\mu}_F \cap \widetilde{\nu}_F, \lambda_T \vee \eta_T, \lambda_I \vee \eta_I, \lambda_F \vee \eta_F \rangle$ is a NC subsemigroup of S.

The second property can be proved in a similar manner. $\qquad\square$

Proposition 6. *If $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is right NC ideal and $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ is a left NC ideal of a semigroup S, then $A \otimes B \sqsubseteq A \sqcap B$.*

Proof. Let $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is right NC ideal and $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ is any NC ideal of S. Then by Theorem (2) and Theorem (3) we have $A \otimes B \sqsubseteq A \otimes S \sqsubseteq A$ and $A \otimes B \sqsubseteq S \otimes B \sqsubseteq B$. Thus $A \otimes B \sqsubseteq A \sqcap B$. $\qquad\square$

Proposition 7. *If S is a regular semigroup, then $A \otimes B = A \sqcap B$ for every right NC ideal $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ and every left NC ideal $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ of S.*

Proof. Let a be any element of S. Since S is regular, there exist an element $x \in S$ such that $a = axa$. Hence we have

$$
\begin{aligned}
(\widetilde{\mu}_T \circ \widetilde{\nu}_T)(a) &= r \sup_{a=yz} \{r \min\{\widetilde{\mu}_T(y), \widetilde{\nu}_T(z)\}\} \\
&\succeq r \min\{\widetilde{\mu}_T(ax), \widetilde{\nu}_T(a)\} \\
&\succeq r \min\{\widetilde{\mu}_T(a), \widetilde{\nu}_T(a)\} \\
&= (\widetilde{\mu}_T \cap \widetilde{\nu}_T)(a) \\
(\widetilde{\mu}_I \circ \widetilde{\nu}_I)(a) &= r \sup_{a=yz} \{r \min\{\widetilde{\mu}_I(y), \widetilde{\nu}_I(z)\}\} \\
&\succeq r \min\{\widetilde{\mu}_I(ax), \widetilde{\nu}_I(a)\} \\
&\succeq r \min\{\widetilde{\mu}_I(a), \widetilde{\nu}_I(a)\} \\
&= (\widetilde{\mu}_I \cap \widetilde{\nu}_I)(a) \\
(\widetilde{\mu}_F \circ \widetilde{\nu}_F)(a) &= r \sup_{a=yz} \{r \min\{\widetilde{\mu}_F(y), \widetilde{\nu}_F(z)\}\} \\
&\succeq r \min\{\widetilde{\mu}_F(ax), \widetilde{\nu}_F(a)\} \\
&\succeq r \min\{\widetilde{\mu}_F(a), \widetilde{\nu}_F(a)\} \\
&= (\widetilde{\mu}_F \cap \widetilde{\nu}_F)(a)
\end{aligned}
$$

and

$$
\begin{aligned}
(\lambda_T \circ \eta_T)(a) &= \bigwedge_{a=yz} \max\{\lambda_T(y), \eta_T(z)\} \\
&\leq \max\{\lambda_T(ax), \eta_T(a)\} \\
&\leq \max\{\lambda_T(a), \eta_T(a)\} \\
&= (\lambda_T \cap \eta_T)(a) \\
(\lambda_I \circ \eta_I)(a) &= \bigwedge_{a=yz} \max\{\lambda_I(y), \eta_I(z)\} \\
&\leq \max\{\lambda_I(ax), \eta_I(a)\} \\
&\leq \max\{\lambda_I(a), \eta_I(a)\} \\
&= (\lambda_I \cap \eta_I)(a) \\
(\lambda_F \circ \eta_F)(a) &= \bigwedge_{a=yz} \max\{\lambda_F(y), \eta_F(z)\} \\
&\leq \max\{\lambda_F(ax), \eta_F(a)\} \\
&\leq \max\{\lambda_F(a), \eta_F(a)\} \\
&= (\lambda_F \cap \eta_F)(a)
\end{aligned}
$$

and so $A \otimes B \sqsupseteq A \sqcap B$. It follows from Proposition (6) that $A \otimes B = A \sqcap B$. □

We now discuss the converse of Proposition (7). We first consider the following lemmas.

Lemma 1. [20]. *For a semigroup S, the following conditions are equivalent.*
(1) *S is regular.*
(2) *$R \cap L = RL$ for every right ideal R of S and every left ideal L of S.*

Lemma 2. *For a non-empty subset G of a semigroup S, we have*
(1) *G is a subsemigroup of S if and only if the characteristic NC set $\chi = \langle \widetilde{\mu}_{T\chi}, \widetilde{\mu}_{I\chi}, \widetilde{\mu}_{F\chi}, \lambda_{T\chi}, \lambda_{I\chi}, \lambda_{F\chi} \rangle$ of G in S is a NC subsemigroup of S.*

(2) *G is a left (right) ideal of S if and only if the characteristic NC set* $\chi = \langle \widetilde{\mu}_{T\chi}, \widetilde{\mu}_{I\chi}, \widetilde{\mu}_{F\chi}, \lambda_{T\chi}, \lambda_{I\chi}, \lambda_{F\chi} \rangle$ *of G in S is a left (resp. right) NC ideal of S.*

Proof. Straight forward   □

Theorem 8. *For every right NC ideal* $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ *and every left NC ideal* $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ *of a semigroup S, if* $A \otimes B = A \cap B$, *then S is regular.*

Proof. Assume that $A \otimes B = A \cap B$ for every right NC ideal $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ and every left NC ideal $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle$ of a semigroup S. Let R and L be any right ideal and left ideal of S, respectively. In order to see that $R \cap L \subseteq RL$ holds, let a be any element of $R \cap L$. Then the characteristic NC sets $\chi_{_R} = \langle \widetilde{\mu}_{T\chi_R}, \widetilde{\mu}_{I\chi_R}, \widetilde{\mu}_{F\chi_R}, \lambda_{T\chi_R}, \lambda_{I\chi_R}, \lambda_{F\chi_R} \rangle$ and $\chi_{_L} = \langle \widetilde{\mu}_{T\chi_L}, \widetilde{\mu}_{I\chi_L}, \widetilde{\mu}_{F\chi_L}, \lambda_{T\chi_L}, \lambda_{I\chi_L}, \lambda_{F\chi_L} \rangle$ are a right NC ideal and a left NC ideal of S, respectively, by Lemma (2). It follows from the hypothesis and proposition (3) that

$$
\begin{aligned}
\widetilde{\mu}_{T\chi_{RL}}(a) &= \left(\widetilde{\mu}_{T\chi_R} \circ \widetilde{\mu}_{T\chi_L} \right)(a) \\
&= \left(\widetilde{\mu}_{T\chi_R} \cap \widetilde{\mu}_{T\chi_L} \right)(a) \\
&= \widetilde{\mu}_{T\chi_{R\cap L}}(a) = [1,1] \\
\widetilde{\mu}_{T\chi_{RL}}(a) &= \left(\widetilde{\mu}_{T\chi_R} \circ \widetilde{\mu}_{T\chi_L} \right)(a) \\
&= \left(\widetilde{\mu}_{T\chi_R} \cap \widetilde{\mu}_{T\chi_L} \right)(a) \\
&= \widetilde{\mu}_{T\chi_{R\cap L}}(a) = [1,1] \\
\widetilde{\mu}_{T\chi_{RL}}(a) &= \left(\widetilde{\mu}_{T\chi_R} \circ \widetilde{\mu}_{T\chi_L} \right)(a) \\
&= \left(\widetilde{\mu}_{T\chi_R} \cap \widetilde{\mu}_{T\chi_L} \right)(a) \\
&= \widetilde{\mu}_{T\chi_{R\cap L}}(a) = [1,1]
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_{T\chi_{RL}}(a) &= \left(\lambda_{T\chi_R} \circ \lambda_{T\chi_L} \right)(a) \\
&= \left(\lambda_{T\chi_R} \cap \lambda_{T\chi_L} \right)(a) \\
&= \lambda_{T\chi_{R\cap L}}(a) = [1,1] \\
\lambda_{I\chi_{RL}}(a) &= \left(\lambda_{I\chi_R} \circ \lambda_{I\chi_L} \right)(a) \\
&= \left(\lambda_{I\chi_R} \cap \lambda_{I\chi_L} \right)(a) \\
&= \lambda_{I\chi_{R\cap L}}(a) = [1,1] \\
\lambda_{F\chi_{RL}}(a) &= \left(\lambda_{F\chi_R} \circ \lambda_{F\chi_L} \right)(a) \\
&= \left(\lambda_{F\chi_R} \cap \lambda_{F\chi_L} \right)(a) \\
&= \lambda_{F\chi_{R\cap L}}(a) = [1,1]
\end{aligned}
$$

and so that $a \in RL$. Thus $R \cap L \subseteq RL$. Since the inclusion in the otherdirection always holds, we obtain that $R \cap L = RL$. It follows from Lemma (1) that S is regular. $\qquad \square$

Let $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ be a NC set in X. For any $s, o, g \in [0, 1]$ and $\widetilde{t}, \widetilde{i}, \widetilde{f} \in D[0, 1]$, we define $U(A; [s, t], r)$ as follows:

$$U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right) = \left\{ x \in X \middle| \left\langle \widetilde{\mu}_T(x) \succeq \widetilde{t}, \widetilde{\mu}_I(x) \succeq \widetilde{i}, \widetilde{\mu}_F(x) \succeq \widetilde{f}, \lambda_T(x) \le s, \lambda_I(x) \le o, \lambda_F(x) \le g \right\rangle \right\}$$

and we say it is a NC level set of $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ $(see [8])$.

Theorem 9. *For a NC set* $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ *in a semigroup S, the following are equivalent:*

    (1) $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ *is a NC subsemigroup of S.*

    (2)   *Every non-empty NC level set of* $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ *is a subsemigroup of S.*

Proof. Assume that $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a NC subsemigroup of S. let x,y$\in U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right)$ for all $s, o, g \in [0, 1]$ and $\widetilde{t}, \widetilde{i}, \widetilde{f} \in D[0, 1]$. Then $\widetilde{\mu}_T(x) \succeq \widetilde{t}, \widetilde{\mu}_I(x) \succeq \widetilde{i}, \widetilde{\mu}_F(x) \succeq \widetilde{f}$ *and* $\lambda_T(x) \le s, \lambda_I(x) \le o, \lambda_F(x) \le g$, $\widetilde{\mu}_T(y) \succeq \widetilde{t}, \widetilde{\mu}_I(y) \succeq \widetilde{i}, \widetilde{\mu}_F(y) \succeq \widetilde{f}$ *and* $\lambda_T(y) \le s, \lambda_I(y) \le o, \lambda_F(y) \le g$. It follows form Definition (2) that $\widetilde{\mu}_T(xy) \succeq r \min \{\widetilde{\mu}_T(x), \widetilde{\mu}_T(y)\} \succeq \widetilde{t}, \widetilde{\mu}_I(xy) \succeq r \min \{\widetilde{\mu}_I(x), \widetilde{\mu}_I(y)\} \succeq \widetilde{i}, \widetilde{\mu}_F(xy) \succeq r \min \{\widetilde{\mu}_F(x), \widetilde{\mu}_F(y)\} \succeq \widetilde{f}$ and $\lambda_T(xy) \le \max\{\lambda_T(x), \lambda_T(y)\} \le s, \lambda_I(xy) \le \max\{\lambda_I(x), \lambda_I(y)\} \le o, \lambda_F(xy) \le \max\{\lambda_F(x), \lambda_F(y)\} \le g$.Hence $x, y \in U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right)$ and thus $U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right)$ is a subsemigroup of S.

Conversly, let $s, o, g \in [0, 1]$ and $\widetilde{t}, \widetilde{i}, \widetilde{f} \in D[0, 1]$ be a such that $U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right) \ne \emptyset$, and $U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right)$ is a subsemigroup of S. Suppose that Definition (2) is false. Then there exist $a, b \in S$ such that $\widetilde{\mu}_T(ab) \not\succeq r \min \{\widetilde{\mu}_T(a), \widetilde{\mu}_T(b)\}$ or $\widetilde{\mu}_I(ab) \not\succeq r \min \{\widetilde{\mu}_I(a), \widetilde{\mu}_I(b)\}$or $\widetilde{\mu}_F(ab) \not\succeq r \min \{\widetilde{\mu}_F(a), \widetilde{\mu}_F(b)\}$ or $\lambda_T(ab) \not\le \max\{\lambda_T(a), \lambda_T(b)\}$.or $\lambda_I(ab) \not\le \max\{\lambda_I(a), \lambda_I(b)\}$ or$\lambda_F(ab) \not\le \max\{\lambda_F(a), \lambda_F(b)\}$. If $\widetilde{\mu}_T(ab) \not\succeq r \min \{\widetilde{\mu}_T(a), \widetilde{\mu}_T(b)\}$, then $\widetilde{\mu}_T(ab) \prec \widetilde{t} \preceq r \min \{\widetilde{\mu}_T(a), \widetilde{\mu}_T(b)\}$ for some $\widetilde{t} \in D[0, 1]$. Hence $a, b \in U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right)$, but

$$ab \notin U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right)$$

This gives a contradiction. Similar results can be deduced for any component $U\left(A; \left\langle \widetilde{t}, \widetilde{i}, \widetilde{f}, s, o, g \right\rangle\right)$.

Hence Definition (2) is valid, and therefore $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ is a NC subsemigroup of S. $\qquad \square$

Theorem 10. *For a NC set* $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ *in a semigroup S, the following are equivalent:*

    (1) $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ *is a left (resp. right) NC ideal of S.*

    (2)   *Every non-empty NC level set of* $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ *is a left (resp. right) NC ideal of S.*

Proof. It can be easily verified by the similar way to the proof of Theorem(9). $\quad \square$

Denote by $N(X)$ the family of NC sets in a sets X. Let X and Y be given classical sets. A mapping $h : X \longrightarrow Y$ induces two mapping $N_h : N(X) \longrightarrow N(Y)$, $A \longmapsto N_h(A)$, and $N_h^{-1} : N(Y) \longrightarrow N(X)$, $B \longmapsto N_h^{-1}(B)$, where $N_h(A)$ is given by

$$N_h(\widetilde{\mu}_T)(y) = \begin{cases} r\sup\limits_{y=h(x)} \widetilde{\mu}_T(x) & if \ \ h^{-1}(y) \neq 0 \\ [0,0] & otherewise \end{cases}$$

$$N_h(\widetilde{\mu}_I)(y) = \begin{cases} r\sup\limits_{y=h(x)} \widetilde{\mu}_I(x) & if \ \ h^{-1}(y) \neq 0 \\ [0,0] & otherewise \end{cases}$$

$$N_h(\widetilde{\mu}_F)(y) = \begin{cases} r\sup\limits_{y=h(x)} \widetilde{\mu}_F(x) & if \ \ h^{-1}(y) \neq 0 \\ [0,0] & otherewise \end{cases}$$

$$N_h(\lambda_T)(y) = \begin{cases} \inf\limits_{y=h(x)} \lambda_T(x) & if \ \ h^{-1}(y) \neq 0 \\ 1 & otherewise \end{cases}$$

$$N_h(\lambda_I)(y) = \begin{cases} \inf\limits_{y=h(x)} \lambda_I(x) & if \ \ h^{-1}(y) \neq 0 \\ 1 & otherewise \end{cases}$$

$$N_h(\lambda_F)(y) = \begin{cases} \inf\limits_{y=h(x)} \lambda_F(x) & if \ \ h^{-1}(y) \neq 0 \\ 1 & otherewise \end{cases}$$

for all $y \in Y$; and $N_h^{-1}(B)$ is defined by $N_h^{-1}(\widetilde{\nu}_T)(x) = \widetilde{\nu}_T(h(x))$, $N_h^{-1}(\widetilde{\nu}_I)(x) = \widetilde{\nu}_I(h(x))$, $N_h^{-1}(\widetilde{\nu}_F)(x) = \widetilde{\nu}_F(h(x))$ and $N_h^{-1}(\eta_T)(x) = \eta_T(h(x))$, $N_h^{-1}(\eta_I)(x) = \eta_I(h(x))$, $N_h^{-1}(\eta_F)(x) = \eta_F(h(x))$ for all $x \in X$. Then the mapping N_h (resp. N_h^{-1}) is called a NC transformation (resp. inverse NC transformation) induced by h. A NC set $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle$ in X has the NC property if for any subset C of X there exists $x_0 \in C$ such that $\widetilde{\mu}_T(x_0) = r\sup\limits_{x \in C} \widetilde{\mu}_T(x)$, $\widetilde{\mu}_I(x_0) = r\sup\limits_{x \in C} \widetilde{\mu}_I(x)$, $\widetilde{\mu}_F(x_0) = r\sup\limits_{x \in C} \widetilde{\mu}_F(x)$ and $\lambda_T(x_0) = \inf\limits_{x \in C} \lambda_T(x)$, $\lambda_I(x_0) = \inf\limits_{x \in C} \lambda_I(x)$, $\lambda_F(x_0) = \inf\limits_{x \in C} \lambda_F(x)$.

Theorem 11. *For a homomorphsim $h : X \longrightarrow Y$ of semigroups, let $N_h : N(X) \longrightarrow N(Y)$ and $N_h^{-1} : N(Y) \longrightarrow N(X)$ be the NC transformation and inverse NC transformation, respectively, induced by h.*

(1) If $A = \langle \widetilde{\mu}_T, \widetilde{\mu}_I, \widetilde{\mu}_F, \lambda_T, \lambda_I, \lambda_F \rangle \in N(X)$ is a NC subsemigroup of X which has the NC property, then $N_h(A)$ is a NC subsemigroup of Y.

(2) If $B = \langle \widetilde{\nu}_T, \widetilde{\nu}_I, \widetilde{\nu}_F, \eta_{T,}, \eta_I, \eta_F \rangle \in N(Y)$ is a NC subsemigroup of Y, then $N_h^{-1}(B)$ is a NC subsemigroup of X.

Proof. (1) Given $h(x).h(y) \in h(X)$, let $x_0 \in h^{-1}(h(x))$ and $y_0 \in h^{-1}(h(y))$ be such that

$$\widetilde{\mu}_T(x_0) = r\sup\limits_{a \in h^{-1}(h(x))} \widetilde{\mu}_T(a), \widetilde{\mu}_I(x_0) = r\sup\limits_{a \in h^{-1}(h(x))} \widetilde{\mu}_I(a), \widetilde{\mu}_F(x_0) = r\sup\limits_{a \in h^{-1}(h(x))} \widetilde{\mu}_F(a)$$

$$\lambda_T(x_0) = \inf\limits_{a \in h^{-1}(h(x))} \lambda_T(a), \lambda_I(x_0) = \inf\limits_{a \in h^{-1}(h(x))} \lambda_I(a), \lambda_F(x_0) = \inf\limits_{a \in h^{-1}(h(x))} \lambda_F(a)$$

and

$$\widetilde{\mu}_T(y_0) = \underset{b \in h^{-1}(h(y))}{r \sup} \widetilde{\mu}_T(b), \widetilde{\mu}_I(y_0) = \underset{b \in h^{-1}(h(y))}{r \sup} \widetilde{\mu}_I(b), \widetilde{\mu}_F(y_0) = \underset{b \in h^{-1}(h(y))}{r \sup} \widetilde{\mu}_F(b)$$

$$\lambda_T(y_0) = \underset{b \in h^{-1}(h(y))}{\inf} \lambda_T(b), \lambda_I(y_0) = \underset{b \in h^{-1}(h(y))}{\inf} \lambda_I(b), \lambda_F(y_0) = \underset{b \in h^{-1}(h(y))}{\inf} \lambda_F(b)$$

respectively. Then

$$
\begin{aligned}
N_h(\widetilde{\mu}_T)(h(x)h(y)) &= \underset{z \in h^{-1}(h(x)h(y))}{r \sup} \widetilde{\mu}_T(z) \\
&\succeq \widetilde{\mu}_T(x_0 y_0) \succeq r \min\{\widetilde{\mu}_T(x_0), \widetilde{\mu}_T(y_0)\} \\
&= r \min\left\{ \underset{a \in h^{-1}(h(x))}{r \sup} \widetilde{\mu}_T(a), \underset{b \in h^{-1}(h(y))}{r \sup} \widetilde{\mu}_T(b) \right\} \\
&= r \min\{N_h(\widetilde{\mu}_T)(h(x)), N_h(\widetilde{\mu}_T)(h(y))\} \\
N_h(\widetilde{\mu}_I)(h(x)h(y)) &= \underset{z \in h^{-1}(h(x)h(y))}{r \sup} \widetilde{\mu}_I(z) \\
&\succeq \widetilde{\mu}_I(x_0 y_0) \succeq r \min\{\widetilde{\mu}_I(x_0), \widetilde{\mu}_I(y_0)\} \\
&= r \min\left\{ \underset{a \in h^{-1}(h(x))}{r \sup} \widetilde{\mu}_I(a), \underset{b \in h^{-1}(h(y))}{r \sup} \widetilde{\mu}_I(b) \right\} \\
&= r \min\{N_h(\widetilde{\mu}_I)(h(x)), N_h(\widetilde{\mu}_I)(h(y))\} \\
N_h(\widetilde{\mu}_F)(h(x)h(y)) &= \underset{z \in h^{-1}(h(x)h(y))}{r \sup} \widetilde{\mu}_F(z) \\
&\succeq \widetilde{\mu}_F(x_0 y_0) \succeq r \min\{\widetilde{\mu}_F(x_0), \widetilde{\mu}_F(y_0)\} \\
&= r \min\left\{ \underset{a \in h^{-1}(h(x))}{r \sup} \widetilde{\mu}_F(a), \underset{b \in h^{-1}(h(y))}{r \sup} \widetilde{\mu}_F(b) \right\} \\
&= r \min\{N_h(\widetilde{\mu}_F)(h(x)), N_h(\widetilde{\mu}_F)(h(y))\}
\end{aligned}
$$

$\square$

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