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Neutrosophic Set Approach to Algebraic Structures

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$2+3I$	$2+1I$	$2+2I$	$2+2I$	$2+1I$	$2+2I$	$2+2I$	$2+1I$	$2+2I$	$2+2I$
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$1+1I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+2I$	$1+3I$	$1+1I$	$1+2I$	$2+2I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$1+3I$	$1+2I$	$1+3I$	$1+1I$	$2+3I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$2+1I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$
$2+3I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$3+1I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$3+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$3+3I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$

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Preface

Real world is featured with complex phenomenons. As uncertainty is inevitably involved in problems arise in various fields of life and classical methods failed to handle these type of problems. Dealing with imprecise, uncertain or imperfect information was a big task for many years. Many models were presented in order to properly incorporate uncertainty into system description, Lotfi A.Zadeh in 1965 introduced the idea of a fuzzy set. Zadeh replaced conventional characteristic function of classical crisp sets which takes on its values in $\{0, 1\}$ by membership function which takes on its values in closed interval $[0, 1]$. Fuzzy set theory is conceptually a very powerful technique to deal with another aspect or vision of imperfect information related to vagueness and is a modelling tool for complex systems that can be controlled by human but very tough to define exactly. It also reduces the chances of failures in modelling. Until 1960's uncertainty was considered solely in terms of probability theory and understood as randomness but Zadeh discovered the relationships of probability and fuzzy set theory which has appropriate approach to deal with uncertainties. Fuzzy set theory not only formulate imprecise information into model but it helps us in problem solving and decision making. In fact a fuzzy set approaches are suitable when it is needed to model human knowledge or evaluation. Moreover a fuzzy logic is that branch of mathematics that allows a computer to model the real world in the same way as that of people do.

Many authors have applied the fuzzy set theory to generalize the basic theories of Algebra. Mordeson et al. has discovered the grand exploration of fuzzy semigroups, where theory of fuzzy semigroups is explored along with the applications of fuzzy semigroups in fuzzy coding, fuzzy finite state mechanics and fuzzy languages etc. and fuzzy approach is also applied to the problem of integrated design of high speed planar mechanism. But at a point when we talk about degree of non-membership or falsehood then fuzzy set theory does not work properly and we need something new to deal with it more properly. Krassimir T. Atanassov introduced the degree of non-membership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. An intuitionistic fuzzy set is basically a generalization or extension of fuzzy set and can be viewed in the perspective as an approach to fuzzy set in case when we are not provided with sufficient information. Use of intuitionistic fuzzy sets is helpful in the introduction of additional degrees of freedom (non-membership and hesitation margins) into set description and is extensively use as a tool of intensive research by scholars and scientists from over the so many years.

Various theories like theory of probability, fuzzy set theory, intutionistic fuzzy sets, rough set theory etc., are consistently being used as powerful constructive tools to deal with multiform uncertainties and imprecision enclosed in complex systems. But all these above theories do not model undetermined information adequately. Therefore, due to the existence of indeterminacy in various world problems, neutrosophy founds its way into the modern research. Neutrosophy is a generalization of fuzzy set, where the models represented by three types concepts that is truthfulness, falsehood and neutrality. Neutrosophy is a Latin world "neuter" - neutral, Greek "sophia" - skill/wisdom). Neutrosophy is a branch of philosophy, introduced by Florentin Smarandache which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy considers a proposition, theory, event, concept, or entity, "A" in relation to its opposite, "Anti-A" and that which is not A, "Non-A", and that which is neither "A" nor "Anti-A", denoted by "Neut-A". Neutrosophy is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, and neutrosophic statistics.

Inspiring from the realities of real life phenomenons like sport games (winning/ tie/ defeating), votes (yes/ NA/ no) and decision making (making a decision/ hesitating/ not making), F. Smrandache introduced a new concept of a neutrosophic set (NS in short) in 1995, which is the generalization of a fuzzy sets and intutionistic fuzzy set. NS is described by membership degree, indeterminate degree and non-membership degree. The idea of NS generates the theory of neutrosophic sets by giving representation to indeterminates. This theory is considered as complete representation of almost every model of all real-world problems. Therefore, if uncertainty is involved in a problem we use fuzzy theory while dealing indeterminacy, we need neutrosophic theory. In fact this theory has several applications in many different fields like control theory, databases, medical diagnosis problem and decision making problems.

This book consists of seven chapters. In chapter one we introduced neutrosophic ideals (bi, quasi, interior, (m,n) ideals) and discussed the properties of these ideals. Moreover we characterized regular and intra-regular AG-groupoids using these ideals.

In chapter two we introduced neutrosophic minimal ideals in AG-groupoids and discussed several properties.

In chapter three, we introduced different neutrosophic regularities of AG-groupoids. Further we discussed several condition where these classes are equivalent.

In chapter four, we introduced neutrosophic M-systems and neutrosophic p-systems in non-associative algebraic structure and discussed their relations with neutrosophic ideals.

In chapter five, we introduced neutrosophic strongly regular AG-groupoids and characterized this structure using neutrosophic ideals.

In chapter six, we introduced the concept of neutrosophic ideal, neutrosophic prime ideal, neutrosophic bi-ideal and neutrosophic quasi ideal of a neutrosophic semigroup. With counter example we have shown that the union and product of two neutrosophic quasi-ideals of a neutrosophic semigroup need not be a neutrosophic quasi-ideal of neutrosophic semigroup.

We have also shown that every neutrosophic bi-ideal of a neutrosophic semigroup need not be a neutrosophic quasi-ideal of a neutrosophic semigroup. We have also characterized the regularity and intra-rgularity of a neutrosophic semigroup.

In chapter seven, we introduced neutrosophic left almost rings and discussed several properties using their neutrosophic ideals.

1
Neutrosophic Sets in AG-groupoids

In this chapter we have defined neutrosophic ideals, neutrosophic interior ideals, neutrosophic quasi-ideals and neutrosophic bi-ideals (neutrosophic generalized bi-ideals) and proved some results related to them. Furthermore, we have done some characterization of a neutrosophic LA-semigroup by the properties of its neutrosophic ideals. It has been proved that in a neutrosophic intra-regular LA-semigroup neutrosophic left, right, two-sided, interior, bi-ideal, generalized bi-ideal and quasi-ideals coincide and we have also proved that the set of neutrosophic ideals of a neutrosophic intra-regular LA-semigroup forms a semilattice structure.

Introduction

It is well known fact that common models with their limited and restricted boundaries of truth and falsehood are insufficient to detect the reality so there is a need to discover and introduce some other phenomenon that address the daily life problems in a more appropriate way. In different fields of life many problems arise which are full of uncertainties and classical methods are not enough to deal and solve them. In fact, reality of real life problems can not be represented by models with just crisp assumptions with only yes or no because of such certain assumptions may lead us to completely wrong solutions. To overcome this problem, Lotfi A.Zadeh in 1965 introduced the idea of a fuzzy set which help to describe the behavior of systems that are too complex or are ill-defined to admit precise mathematical analysis by classical methods. He discovered the relationships of probability and fuzzy set theory which has appropriate approach to deal with uncertainties. According to him every set is not crisp and fuzzy set is one of the example that is not crisp. This fuzzy set help us to reduce the chances of failures in modelling.. Many authors have applied the fuzzy set theory to generalize the basic theories of Algebra. Mordeson et al. has discovered the grand exploration of fuzzy semigroups, where theory of fuzzy semigroups is explored along with the applications of fuzzy semigroups in fuzzy coding, fuzzy finite state mechanics and fuzzy languages etc.

Recently, several theories have been presented to dispute with uncertainty, vagueness and imprecision . Theory of probability, fuzzy set theory, intutionistic fuzzy sets, rough set theory etc., are consistently being used as actively operative tools to deal with multiform uncertainties and imprecision enclosed in a system. But all these above theories failed to deal with indeterminate and inconsistent information. Therefore, due to the existence of indeterminancy in various world problems, neutrosophy founds its way into the modern research. Neutrosophy was developed in attempt to generalize fuzzy logic. Neutrosophy is a Latin world "neuter" - neutral, Greek "sophia" - skill/wisdom). Neutrosophy is a branch of philosophy, introduced by Florentin Smarandache which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy considers a proposition, theory, event, concept, or entity, "A" in relation to its opposite, "Anti-A" and that which is not A, "Non-A", and that which is neither "A" nor "Anti-A", denoted by "Neut-A". Neutrosophy is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, and neutrosophic statistics.

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Preliminaries

Abel-Grassmann's Groupoid (abbreviated as an AG-groupoid or LA-semigroup) was first introduced by Naseeruddin and Kazim in 1972. LA-semigroup is a groupoid S whose elements satisfy the left invertive law $(ab)c = (cb)a$ for all a, b, $c \in S$. LA-semigroup generalizes the concept of commutative semigroups and have an important application within the theory of flocks. In addition to applications, a variety of properties have been studied for AG-groupoids and related structures. An LA-semigroup is a non-associative algebraic structure that is generally considered as a midway between a groupoid and a commutative semigroup but is very close to commutative semigroup because most of their properties are similar to commutative semigroup. Every commutative semigroup is an AG-groupoid but not vice versa. Thus AG-groupoids can also be non-associative, however, they do not necessarily have the Latin square property. An LA-semigroup S can have left identity e (unique) i.e $ea = a$ for all $a \in S$ but it can not have a right identity because if it has, then S becomes a commutative semigroup. An element s of LA-semigroup S is called idempotent if $s^2 = s$ and if holds for all elements of S then S is called idempotent LA-semigroup.

Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. In 1995, Florentin Smarandache introduced the idea of neutrosophy. Neutrosophic logic is an extension of fuzzy logic. In 2003 W.B Vasantha Kandasamy and Florentin Smarandache introduced algebraic structures (such as neutrosophic semigroup, neutrosophic ring, etc.). Moreover $SUI = \{a+bI$: where a, $b \in S$ and I is literal indeterminacy such that $I^2 = I\}$ becomes neutrosophic LA-semigroup under the operation defined as:

$(a + bI) * (c + dI) = ac + bdI$ for all $(a + bI)$, $(c + dI) \in SUI$. That is $(SUI, *)$ becomes neutrosophic LA-semigroup. They represented it by $N(S)$.

$$[(a_1 + a_2I)(b_1 + b_2I)](c_1 + c_2I) = [(c_1 + c_2I)(b_1 + b_2I)](a_1 + a_2I), \tag{1}$$

holds for all $(a_1 + a_2I)$, $(b_1 + b_2I)$, $(c_1 + c_2I) \in N(S)$.

It is since than called the neutrosophic left invertive law. A neutrosophic groupoid satisfying the left invertive law is called a neutrosophic left almost semigroup and is abbreviated as neutrosophic LA-semigroup.

In a neutrosophic LA-semigroup $N(S)$ medial law holds i.e

$$[(a_1+a_2I)(b_1+b_2I)][(c_1+c_2I)(d_1+d_2I)] = [(a_1+a_2I)(c_1+c_2I)][(b_1+b_2I)(d_1+d_2I)], \tag{2}$$

holds for all $(a_1 + a_2I)$, $(b_1 + b_2I)$, $(c_1 + c_2I)$, $(d_1 + d_2I) \in N(S)$.

There can be a unique left identity in a neutrosophic LA-semigroup. In a neutrosophic LA-semigroup $N(S)$ with left identity $(e + eI)$ the following laws hold for all $(a_1 + a_2I)$, $(b_1 + b_2I)$, $(c_1 + c_2I)$, $(d_1 + d_2I) \in N(S)$.

$$[(a_1+a_2I)(b_1+b_2I)][(c_1+c_2I)(d_1+d_2I)] = [(d_1+d_2I)(b_1+b_2I)][(c_1+c_2I)(a_1+a_2I)], \tag{3}$$

$$[(a_1+a_2I)(b_1+b_2I)][(c_1+c_2I)(d_1+d_2I)] = [(d_1+d_2I)(c_1+c_2I)][(b_1+b_2I)(a_1+a_2I)], \tag{4}$$

and

$$(a_1 + a_2I)[(b_1 + b_2I)(c_1 + c_2I)] = (b_1 + b_2I)[(a_1 + a_2I)(c_1 + c_2I)]. \tag{5}$$

for all $(a_1 + a_2I)$, $(b_1 + b_2I)$, $(c_1 + c_2I) \in N(S)$.

(3) is called neutrosophic paramedial law and a neutrosophic LA semigroup satisfies (5) is called neutrosophic AG**-groupoid.

Now, $(a + bI)^2 = a + bI$ implies $a + bI$ is idempotent and if holds for all $a+bI \in N(S)$ then $N(S)$ is called idempotent neutrosophic LA-semigroup.

Example 1 *Let $S = \{1, 2, 3\}$ with binary operation "$\cdot$" is an LA-semigroup with left identity 3 and has the following Cayley's table:*

$\cdot$	1	2	3
1	3	1	2
2	2	3	1
3	1	2	3

then $N(S) = \{1{+}1I, 1{+}2I, 1{+}3I, 2{+}1I, 2{+}2I, 2{+}3I, 3{+}1I, 3{+}2I, 3{+}3I\}$ is an example of neutrosophic LA-semigroup under the operation "$*$" and has the following Cayley's table:

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$
$1+2I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$1+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$2+1I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$2+2I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$
$2+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$3+1I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$3+2I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$

It is important to note that if $N(S)$ contains left identity $3 + 3I$ then $(N(S))^2 = N(S)$.

Lemma 2 *If a neutrosophic LA-semigroup $N(S)$ contains left identity $e + Ie$ then the following conditions hold.*
(i) $N(S)N(L) = N(L)$ for every neutrosophic left ideal $N(L)$ of $N(S)$.
(ii) $N(R)N(S) = N(R)$ for every neutrosophic right ideal $N(R)$ of $N(S)$

Proof. (i) Let $N(L)$ be the neutrosophic left ideal of $N(S)$ implies that $N(S)N(L) \subseteq N(L)$. Let $a{+}bI \in N(L)$ and since $a{+}bI = (e + eI)(a + bI) \in N(S)N(L)$ which implies that $N(L) \subseteq N(S)N(L)$. Thus $N(L) = N(S)N(L)$.

(ii) Let $N(R)$ be the neutrosophic right ideal of $N(S)$. Then $N(R)N(S) \subseteq N(R)$. Now,let $a + bI \in N(R)$. Then

$$\begin{aligned} a + bI &= (e + eI)(a + bI) \\ &= [(e + eI)(e + eI)](a + bI). \\ &= [(a + bI)(e + eI)](e + eI) \\ &\in (N(R)N(S))N(S) \\ &\subseteq N(R)N(S). \end{aligned}$$

Thus $N(R) \subseteq N(R)N(S)$. Hence $N(R)N(S) = N(R)$. ■

A subset $N(Q)$ of an neutrosophic LA-semigroup is called neutrosophic quasi-ideal if $N(Q)N(S) \cap N(S)N(Q) \subseteq N(Q)$. A subset $N(I)$ of an LA-semigroup $N(S)$ is called idempotent if $(N(I))^2 = N(I)$.

Lemma 3 *The intersection of a neutrosophic left ideal $N(L)$ and a neutrosophic right ideal $N(R)$ of a neutrosophic LA-semigroup $N(S)$ is a neutrosophic quasi-ideal of $N(S)$.*

Proof. Let $N(L)$ and $N(R)$ be the neutrosophic left and right ideals of neutrosophic LA-semigroup $N(S)$ resp.

Since $N(L) \cap N(R) \subseteq N(R)$ and $N(L) \cap N(R) \subseteq N(L)$ and $N(S)N(L) \subseteq N(L)$ and $N(R)N(S) \subseteq N(R)$. Thus

$$\begin{aligned}(N(L)\cap N(R))\,N(S)\cap N(S)\,(N(L)\cap N(R)) &\subseteq N(R)\,N(S)\cap N(S)\,N(L)\\ &\subseteq N(R)\cap N(L)\\ &= N(L)\cap N(R)\,.\end{aligned}$$

■

Hence, $N(L) \cap N(R)$ is a neutrosophic quasi-ideal of $N(S)$.

A subset(neutrosophic LA-subsemigroup) $N(B)$ of a neutrosophic LA-semigroup $N(S)$ is called neutrosophic generalized bi-ideal(neutosophic bi-ideal) of $N(S)$ if $(N(B)N(S))\,N(B) \subseteq N(B)$.

Lemma 4 *If $N(B)$ is a neutrosophic bi-ideal of a neutrosophic LA-semigroup $N(S)$ with left identity $e+eI$, then $((x_1+Iy_1)N(B))(x_2+Iy_2)$ is also a neutrosophic bi-ideal of $N(S)$, for any x_1+Iy_1 and x_2+Iy_2 in $N(S)$.*

Proof. Let $N(B)$ be a neutrosophic bi-ideal of $N(S)$, now we get

$$\begin{aligned}
& [\{\{(x_1+y_1I)\,N(B)\}\,(x_2+y_2I)\}N(S)][\{(x_1+y_1I)\,N(B)\}\,(x_2+y_2I)] \\
= & [\{N(S)(x_2+y_2I)\}\{(x_1+y_1I)N(B)\}][\{(x_1+y_1I)N(B)\}(x_2+y_2I)] \\
= & [\{\{(x_1+y_1I)N(B)\}(x_2+y_2I)\}\{(x_1+y_1I)N(B)\}][N(S)(x_2+y_2I)] \\
= & [\{\{(x_1+y_1I)N(B)\}(x_1+y_1I)\}\{(x_2+y_2I)N(B)\}][N(S)(x_2+y_2I)] \\
= & [\{\{(x_1+y_1I)N(B)\}(x_1+y_1I)\}N(S)][\{(x_2+y_2I)N(B)\}(x_2+y_2I)] \\
= & [\{N(S)(x_1+y_1I)\}\{(x_1+y_1I)N(B)\}][\{(x_2+y_2I)N(B)\}(x_2+y_2I)] \\
= & [\{N(B)(x_1+y_1I)\}\{(x_1+y_1I)N(S)\}][\{(x_2+y_2I)N(B)\}(x_2+y_2I)] \\
= & [\{N(B)(x_1+y_1I)\}\{(x_2+y_2I)N(B)\}][\{(x_1+y_1I)N(S)\}(x_2+y_2I)] \\
\subseteq & [\{N(B)(x_1+y_1I)\}\{x_2+y_2I)N(B)\}]N(S) \\
= & [\{N(B)(x_1+y_1I)\}\{(x_2+y_2I)N(B)\}][(e+eI)N(S)] \\
= & [\{N(B)(x_1+y_1I)\}(e+eI)][\{(x_2+y_2I)N(B)\}N(S)] \\
= & [\{(e+eI)(x_1+y_1I)\}N(B)][\{N(S)N(B)\}(x_2+y_2I)] \\
= & [(x_2+y_2I)\{N(S)N(B)\}][N(B)(x_1+y_1I)] \\
= & [\{(e+eI)(x_2+y_2I)\}\,(N(S)N(B))][N(B)(x_1+y_1I)] \\
= & [\{N(B)N(S)\}\{(x_2+y_2I)(e+eI)\}][N(B)(x_1+y_1I)] \\
= & [(N(B)N(S))\,N(B)][\{(x_2+y_2I)(e+eI)\}(x_1+y_1I)] \\
\subseteq & N(B)[\{(x_2+y_2I)(e+eI)\}(x_1+y_1I)] \\
= & [(x_2+y_2I)(e+eI)][N(B)(x_1+y_1I)] \\
= & [(x_1+y_1I)N(B)][(e+eI)(x_2+y_2I] \\
= & [(x_1+y_1I)N(B)](x_2+y_2I).
\end{aligned}$$

■

A subset $N(I)$ of a neutrosophic LA-semigroup $N(S)$ is called a neutrosophic interior ideal if $(N(S)N(I))N(S) \subseteq N(I)$.

A subset $N(M)$ of a neutrosophic LA-semigroup $N(S)$ is called a neutrosophic minimal left (right, two sided, interior, quasi- or bi-) ideal if it does not contains any other neutrosophic left (right, two sided, interior, quasi- or bi-) ideal of $N(S)$ other than itself.

Lemma 5 *If $N(M)$ is a minimal bi-ideal of $N(S)$ with left identity and $N(B)$ is any arbitrary neutrosophic bi-ideal of $N(S)$, then $N(M) = ((x_1 + Iy_1)N(B))(x_2 + Iy_2)$, for every $(x_1 + y_1I)$, $(x_1 + y_2I) \in N(M)$.*

Proof. Let $N(M)$ be a neutrosophic minimal bi-ideal and $N(B)$ be any neutrosophic bi-ideal of $N(S)$, then, $[(x_1 + y_1I)N(B)](x_2 + y_2I)$ is a neutrosophic bi-ideal of $N(S)$ for every $(x_1 + y_1I)$, $(x_2 + y_2I) \in N(S)$. Let

$(x_1 + y_1 I)$, $(x_2 + y_2 I) \in N(M)$, we have

$$\begin{aligned}
[(x_1 + y_1 I)N(B)](x_2 + y_2 I) &\subseteq [N(M)N(B)]N(M) \\
&\subseteq [N(M)N(S)]N(M) \\
&\subseteq N(M).
\end{aligned}$$

But $N(M)$ is a neutrosophic minimal bi-ideal, so $[(x_1+y_1 I)N(B)](x_{2+}y_2 I) = N(M)$. ■

Lemma 6 *In a neutrosophic LA-semigroup $N(S)$ with left identity, every idempotent neutrosophic quasi-ideal is a neutrosophic bi-ideal of $N(S)$.*
Proof. *Let $N(Q)$ be an idempotent neutrosophic quasi-ideal of $N(S)$, then clearly $N(Q)$ is a neutrosophic LA-subsemigroup too.*

$$\begin{aligned}
(N(Q)\,N(S))\,N(Q) &\subseteq (N(Q)\,N(S))\,N(S) \\
&= (N(S)N(S))\,N(Q) \\
&= N(S)N(Q), \text{ and} \\
(N(Q)N(S))\,N(Q) &\subseteq (N(S)N(S))\,N(Q) \\
&= (N(S)N(S))\,(N(Q)N(Q)) \\
&= (N(Q)N(Q))\,(N(S)N(S)) \\
&= N(Q)N(S).
\end{aligned}$$

Thus $(N(Q)N(S))\,N(Q) \subseteq (N(Q)N(S)) \cap (N(S)N(Q)) \subseteq N(Q)$. Hence, $N(Q)$ is a neutrosophic bi-ideal of $N(S)$. ■

Lemma 7 *If $N(A)$ is an idempotent neutrosophic quasi-ideal of a neutrosophic LA-semigroup $N(S)$ with left identity $e + eI$, then $N(A)N(B)$ is a neutrosophic bi-ideal of $N(S)$, where $N(B)$ is any neutrosophic subset of $N(S)$.*
Proof. *Let $N(A)$ be the neutrosophic quasi-ideal of $N(S)$ and $N(B)$ be any subset of $N(S)$.*

$$\begin{aligned}
((N(A)N(B))\,N(S))\,(N(A)N(B)) &= ((N(S)N(B))\,N(A))\,(N(A)N(B)) \\
&\subseteq ((N(S)N(S))\,N(A))\,(N(A)N(B)) \\
&= (N(S)N(A))\,(N(A)N(B)) \\
&= (N(B)N(A))\,(N(A)N(S)) \\
&= ((N(A)N(S))\,N(A))\,N(B) \\
&\subseteq N(A)N(B)
\end{aligned}$$

Hence $N(A)N(B)$ is neutrosophic bi-ideal of $N(S)$. ■

Lemma 8 *If $N(L)$ is a neutrosophic left ideal and $N(R)$ is a neutrosophic right ideal of a neutrosophic LA-semigroup $N(S)$ with left identity $e + eI$ then $N(L) \cup N(L)N(S)$ and $N(R) \cup N(S)N(R)$ are neutrosophic two sided ideals of $N(S)$.*

Proof. Let $N(R)$ be a neutrosophic right ideal of $N(S)$ then we have

$$\begin{aligned}
[N(R) \cup N(S)N(R)]N(S) &= N(R)N(S) \cup [N(S)N(R)]N(S) \\
&\subseteq N(R) \cup [N(S)N(R)][N(S)N(S)] \\
&= N(R) \cup [N(S)N(S)][N(R)N(S)] \\
&= N(R) \cup N(S)[N(R)N(S)] \\
&= N(R) \cup N(R)[N(S)N(S)] \\
&= N(R) \cup N(R)N(S) \\
&= N(R) \subseteq N(R) \cup N(S)N(R).
\end{aligned}$$

and

$$\begin{aligned}
N(S)[N(R) \cup N(S)N(R)] &= N(S)N(R) \cup N(S)[N(S)N(R)] \\
&= N(S)N(R) \cup [N(S)N(S)][N(S)N(R)] \\
&= N(S)N(R) \cup [N(R)N(S)][N(S)N(S)] \\
&\subseteq N(S)N(R) \cup N(R)[N(S)N(S)] \\
&= N(S)N(R) \cup N(R)N(S) \\
&\subseteq N(S)N(R) \cup N(R) \\
&= N(R) \cup N(S)N(R).
\end{aligned}$$

Hence $[N(R) \cup N(S)N(R)]$ is a neutrosophic two sided ideal of $N(S)$. Similarly we can show that $[N(L) \cup N(S)N(L)]$ is a neutrosophic two-sided ideal of $N(S)$. ■

Lemma 9 *A subset $N(I)$ of a neutrosophic LA-semigroup $N(S)$ with left identity $e + eI$ is a neutrosophic right ideal of $N(S)$ if and only if it is a neutrosophic interior ideal of $N(S)$.*

Proof. Let $N(I)$ be a neutrosophic right ideal of $N(S)$

$$\begin{aligned}
N(S)N(I) &= [N(S)N(S)]N(I) \\
&= [N(I)N(S)]N(S) \\
&\subseteq N(I)N(S) \\
&\subseteq N(I).
\end{aligned}$$

So $N(I)$ is a neutrosophic two-sided ideal of $N(S)$, so is a neutrosophic interior ideal of $N(S)$.

Conversely, assume that $N(I)$ is a neutrosophic interior ideal of $N(S)$, then we have

$$\begin{aligned} N(I)N(S) &= N(I)[N(S)N(S)] \\ &= N(S)[N(I)N(S)] \\ &= [N(S)N(S)][N(I)N(S)] \\ &= [N(S)N(I)][N(S)N(S)] \\ &= [N(S)N(I)]N(S) \\ &\subseteq N(I). \end{aligned}$$

■

If $N(A)$ and $N(M)$ are neutrosophic two-sided ideals of a neutrosophic LA-semigroup $N(S)$, such that $(N(A))^2 \subseteq N(M)$ implies $N(A) \subseteq N(M)$, then $N(M)$ is called neutrosophic semiprime.

Theorem 10 *In a neutrosophic LA-semigroup $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*

(i) If $N(A)$ and $N(M)$ are neutrosophic two-sided ideals of $N(S)$, then $(N(A))^2 \subseteq N(M)$ implies $N(A) \subseteq N(M)$.

(ii) If $N(R)$ is a neutrosophic right ideal of $N(S)$ and $N(M)$ is a neutrosophic two-sided ideal of $N(S)$ then $(N(R))^2 \subseteq N(M)$ implies $N(R) \subseteq N(M)$.

(iii) If $N(L)$ is a neutrosophic left ideal of $N(S)$ and $N(M)$ is a neutrosophic two-sided ideal of $N(S)$ then$(N(L))^2 \subseteq N(M)$ implies $N(L) \subseteq N(M)$.

Proof. $(i) \Rightarrow (iii)$

Let $N(L)$ be a left ideal of $N(S)$ and $[N(L)]^2 \subseteq N(M)$, then $N(L) \cup N(L)N(S)$ is a neutrosophic two sided ideal of $N(S)$, therefore by assumption (i) ,we have $[N(L) \cup N(L)N(S)]^2 \subseteq N(M)$ which implies $[N(L) \cup N(L)N(S)] \subseteq N(M)$ which further implies that $N(L) \subseteq N(M)$.

$(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are obvious. ■

Theorem 11 *A neutrosophic ideal $N(M)$ of an LA-semigroup $N(S)$ with left identity $e+eI$ is neutrosophic semiprime if and only if $(a_1+b_1I)^2 \in N(M)$ implies $a_1+b_1I \in N(M)$.*

Proof. Let $N(M)$ be a neutrosophic semiprime left ideal of $N(S)$ and $(a_1+b_1I)^2 \in N(M)$. Since $N(S)(a_1+b_1I)^2$ is a neutrosophic left ideal of $N(S)$ containing $(a_1+Ib_1)^2$, also $(a_1+b_1I)^2 \in N(M)$, therefore we have $(a_1+b_1I)^2 \in N(S)(a_1+b_1I)^2 \subseteq N(M)$. But we have

$$\begin{aligned} N(S)[a_1+b_1I]^2 &= N(S)[(a_1+b_1I)(a_1+b_1I)]. \\ &= [N(S)N(S)][(a_1+b_1I)(a_1+b_1I)] \\ &= [N(S)(a_1+b_1I)][N(S)(a_1+b_1I)] \\ &= [N(S)(a_1+b_1I)]^2. \end{aligned}$$

Therefore $[N(S)(a_1+b_1I)]^2 \subseteq N(M)$, but $N(M)$ is neutrosophic semiprime ideal so $N(S)(a_1+b_1I) \subseteq N(M)$. Since $(a_1+b_1I) \in N(S)(a_1+b_1I)$, therefore $(a_1+b_1I) \in N(M)$.

Conversely, assume that $N(I)$ is an ideal of $N(S)$ and let $(N(I))^2 \subseteq N(M)$ and $(a_1+b_1I) \in N(I)$

implies that $(a_1+b_1I)^2 \in (N(I))^2$, which implies that $(a_1+b_1I)^2 \in N(M)$ which further implies

that $(a_1+b_1I) \in N(M)$. Therefore $(N(I))^2 \subseteq N(M)$ implies $N(I) \subseteq N(M)$. Hence $N(M)$ is a

neutrosophic semiprime ideal. ■

A neutrosophic LA-semigroup $N(S)$ is called neutrosophic left (right) quasi-regular if every neutrosophic left (right) ideal of $N(S)$ is idempotent.

Theorem 12 *A neutrosophic LA-semigroup $N(S)$ with left identity is neutrosophic left quasi-regular if and only if $a+bI \in [N(S)(a+bI)][N(S)(a+bI)]$.*

Proof. Let $N(L)$ be any left ideal of $N(S)$ and $a+bI \in [N(S)(a+bI)][N(S)(a+bI)]$. Now for each $l_1+l_2I \in N(L)$, we have

$$\begin{aligned} l_1+l_2I &\in [N(S)(l_1+l_2I)][N(S)(l_1+l_2I)] \\ &\subseteq [N(S)N(L)][N(S)N(L)] \\ &\subseteq N(L)N(L) = (N(L))^2. \end{aligned}$$

Therefore, $N(L) = (N(L))^2$.

Conversely, assume that $N(A) = (N(A))^2$ for every neutrosophic left ideal $N(A)$ of $N(S)$. Since $N(S)(a+bI)$ is a neutrosophic left ideal of $N(S)$. So,

$a+bI \in N(S)(a+bI) = [N(S)(a+bI)][N(S)(a+bI)]$. ■

Theorem 13 *The subset $N(I)$ of a neutrosophic left quasi-regular LA-semigroup $N(S)$ is a neutrosophic left ideal of $N(S)$ if and only if it is a neutrosophic right ideal of $N(S)$.*

Proof. Let $N(L)$ be a neutrosophic left ideal of $N(S)$ and $s_1+s_2I \in N(S)$ therefore by

(1), we have

$$\begin{aligned} (l_1+l_2I)(s_1+s_2I) &= [\{(x_1+x_2I)(l_1+l_2I)\}\{(y_1+y_2I)(l_1+l_2I)\}](s_1+s_2I) \\ &= [\{(s_1+s_2I)\{(y_1+y_2I)(l_1+l_2I)\}\}][(x_1+x_2I)(l_1+l_2I)] \\ &\in [\{N(S)\{N(S)N(L)\}\}][N(S)N(L)] \\ &= [N(S)N(L)][N(S)N(L)] \\ &\subseteq N(L)N(L) = N(L). \end{aligned}$$

Conversely, assume that $N(I)$ is a neutrosophic right ideal of $N(S)$, as $N(S)$ is itself a neutrosophic left ideal and by assumption $N(S)$ is idempotent, therefore we have

$$\begin{aligned} N(S)N(I) &= [N(S)N(S)]N(I) \\ &= [N(I)N(S)]N(S) \\ &\subseteq N(I)N(S) \subseteq N(I). \end{aligned}$$

implies $N(I)$ is neutrosophic left bi-ideal too. ■

Lemma 14 *The intersection of any number of neutrosophic quasi-ideals of $N(S)$ is either empty or quasi-ideal of $N(S)$.*

Proof. Let $N(Q_1)$ and $N(Q_2)$ be two neutrosophic quasi ideals of neutrosophic LA-semigroup $N(S)$. If $N(Q_1)$ and $N(Q_2)$ are distinct then their intersection must be empty but if not then

$$\begin{aligned} & N(S)[N(Q_1) \cap N(Q_2)] \cap [N(Q_1) \cap N(Q_2)]N(S) \\ = & [N(S)N(Q_1) \cap N(S)N(Q_2)] \cap [N(Q_1)N(S) \cap N(Q_2)N(S)] \\ = & [N(S)N(Q_1) \cap N(Q_1)N(S)] \cap [N(S)N(Q_2) \cap N(Q_2)N(S)] \\ \subseteq & N(Q_1) \cap N(Q_2). \end{aligned}$$

Therefore $N(Q_1) \cap N(Q_2)$ is a neutrosophic quasi-ideal.

Now, generalizing the result and let $N(Q_1), N(Q_2), ..., N(Q_n)$ be the n-number of neutrosophic quasi ideals of neutrosophic quasi-ideals of $N(S)$ and assume that their intersection is not empty then

$$\begin{aligned} & N(S)[N(Q_1) \cap N(Q_2) \cap ... \cap N(Q_n)] \cap [N(Q_1) \cap N(Q_2) \cap ... \cap N(Q_n)]N(S) \\ = & [N(S)N(Q_1) \cap N(S)N(Q_2) \cap ... \cap N(S)N(Q_n)] \cap \\ & [N(Q_1)N(S) \cap N(Q_2)N(S) \cap ... \cap N(Q_n)N(S)] \\ = & [N(S)N(Q_1) \cap N(Q_1)N(S)] \cap [N(S)N(Q_2) \cap \\ & N(Q_2)N(S)]...[N(S)N(Q_n) \cap N(Q_n)N(S)] \\ \subseteq & N(Q_1) \cap N(Q_2) \cap ... \cap N(Q_n). \end{aligned}$$

Hence $N(Q_1) \cap N(Q_2) \cap ... \cap N(Q_n)$ is a neutrosophic quasi-ideal.

Therefore, the intersection of any number of neutrosophic quasi-ideals of $N(S)$ is either empty or quasi-ideal of $N(S)$. ■

An element $a+bI$ of a neutrosophic LA-semigroup $N(S)$ is called regular if there exists $x+yI \in N(S)$ such that $a+bI = [(a+bI)(x+yI)](a+bI)$, and $N(S)$ is called neutrosophic regular LA-semigroup if every element of $N(S)$ is regular.

Example 15 *Let $S = \{1, 2, 3\}$ with binary operation "$\cdot$" given in the following Cayley's table, is a regular LA-semigroup with left identity* 4

$\cdot$	1	2	3	4
1	3	4	1	2
2	2	1	4	3
3	4	3	2	1
4	1	2	3	4

then $N(S) = \{1+1I, 1+2I, 1+3I, 2+1I, 2+2I, 2+3I, 3+1I, 3+2I, 3+3I, 4+1I, 4+2I, 4+3I, 4+4I\}$ is an example of neutrosophic regular LA-semigroup under the operation "$*$" and has the following Cayley's table:

$*$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$
$1+1I$	$3+3I$	$3+4I$	$3+1I$	$3+2I$	$4+3I$	$4+4I$	$4+1I$	$4+2I$	$1+3I$	$1+4I$	$1+1I$	$1+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$
$1+2I$	$3+2I$	$3+1I$	$3+4I$	$3+3I$	$4+2I$	$4+1I$	$4+4I$	$4+3I$	$1+2I$	$1+1I$	$1+4I$	$1+3I$	$2+2I$	$2+1I$	$2+4I$	$2+3I$
$1+3I$	$3+4I$	$3+3I$	$3+2I$	$3+1I$	$4+4I$	$4+3I$	$4+2I$	$4+1I$	$1+4I$	$1+3I$	$1+2I$	$1+1I$	$2+4I$	$2+3I$	$2+2I$	$2+1I$
$1+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$
$3+1I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$1+3I$	$1+4I$	$1+1I$	$1+2I$	$4+3I$	$4+4I$	$4+1I$	$4+2I$	$3+3I$	$3+4I$	$3+1I$	$3+2I$
$2+2I$	$2+2I$	$2+1I$	$2+4I$	$2+3I$	$1+2I$	$1+1I$	$1+4I$	$1+3I$	$4+2I$	$4+1I$	$4+4I$	$4+3I$	$3+2I$	$3+1I$	$3+4I$	$3+3I$
$2+3I$	$2+4I$	$2+3I$	$2+2I$	$2+1I$	$1+4I$	$1+3I$	$1+2I$	$1+1I$	$4+4I$	$4+3I$	$4+2I$	$4+1I$	$3+4I$	$3+3I$	$3+2I$	$3+1I$
$2+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$
$3+1I$	$4+3I$	$4+4I$	$4+1I$	$4+2I$	$3+3I$	$3+4I$	$3+1I$	$3+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$1+3I$	$1+4I$	$1+1I$	$1+2I$
$3+2I$	$4+2I$	$4+1I$	$4+4I$	$4+3I$	$3+2I$	$3+1I$	$3+4I$	$3+3I$	$2+2I$	$2+1I$	$2+4I$	$2+3I$	$1+2I$	$1+1I$	$1+4I$	$1+3I$
$3+3I$	$4+4I$	$4+3I$	$4+2I$	$4+1I$	$3+4I$	$3+3I$	$3+2I$	$3+1I$	$2+4I$	$2+3I$	$2+2I$	$2+1I$	$1+4I$	$1+3I$	$1+2I$	$1+1I$
$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$
$4+1I$	$1+3I$	$1+4I$	$1+1I$	$1+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$3+3I$	$3+4I$	$3+1I$	$3+2I$	$4+3I$	$4+4I$	$4+1I$	$4+2I$
$4+2I$	$1+2I$	$1+1I$	$1+4I$	$1+3I$	$2+2I$	$2+1I$	$2+4I$	$2+3I$	$3+2I$	$3+1I$	$3+4I$	$3+3I$	$4+2I$	$4+1I$	$4+4I$	$4+3I$
$4+3I$	$1+4I$	$1+3I$	$1+2I$	$1+1I$	$2+4I$	$2+3I$	$2+2I$	$2+1I$	$3+4I$	$3+3I$	$3+2I$	$3+1I$	$4+4I$	$4+3I$	$4+2I$	$4+1I$
$4+4I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$

Clearly $N(S)$ is a neutrosophic LA-semigroup also $[(1+1I)(4+4I)](2+3I) \neq (1+1I)[(4+4I)(2+3I)]$, so $N(S)$ is non-associative and is regular because $(1+1I) = [(1+1I)(2+2I)](1+1I)$, $(2+2I) = [(2+2I)(3+3I)](2+2I)$, $(3+2I) = [(3+2I)(1+3I)](3+2I)$, $(4+1I) = [(4+1I)(4+2I)](4+1I)$, $(4+4I) = [(4+4I)(4+4I)](4+4I)$ etc.

Note that in a neutrosophic regular LA-semigroup, $[N(S)]^2 = N(S)$.

Lemma 16 *If $N(A)$ is a neutrosophic bi-ideal(generalized bi-ideal) of a regular neutrosophic LA-semigroup $N(S)$ then $[N(A)N(S)]N(A) = N(A)$.*

Proof. Let $N(A)$ be a bi-ideal(generalized bi-ideal) of $N(S)$, then $[N(A)N(S)]N(A) \subseteq N(A)$.

Let $a+bI \in N(A)$, since $N(S)$ is neutrosophic regular LA-semigroup so there exists an element $x+yI \in N(S)$ such that $a+bI = [(a+bI)(x+yI)](a+bI)$, therefore,

$a+bI = [(a+bI)(x+bI)](a+bI) \in [N(A)N(S)]N(A)$. This implies that $N(A) \subseteq [N(A)N(S)]N(A)$. Hence $[N(A)N(S)]N(A) = N(A)$. ■

Lemma 17 *If $N(A)$ and $N(B)$ are any neutrosophic ideals of a neutrosophic regular LA-semigroup $N(S)$, then $N(A) \cap N(B) = N(A)N(B)$.*

Proof. Assume that $N(A)$ and $N(B)$ are any neutrosophic ideals of $N(S)$ so $N(A)N(B) \subseteq N(A)N(S) \subseteq N(A)$ and $N(A)N(B) \subseteq N(S)N(B) \subseteq N(B)$. This implies that $N(A)N(B) \subseteq N(A) \cap N(B)$. Let $a+bI \in N(A) \cap$

$N(B)$, then $a+bI \in N(A)$ and $a+bI \in N(B)$. Since $N(S)$ is a neutrosophic regular AG-groupoid, so there exist $x+yI$ such that $a+bI = [(a+bI)(x+yI)](a+bI) \in [N(A)N(S]N(B) \subseteq N(A)N(B)$, which implies that $N(A) \cap N(B) \subseteq N(A)N(B)$. Hence $N(A)N(B) = N(A) \cap N(B)$. ∎

Lemma 18 *If $N(A)$ and $N(B)$ are any neutrosophic ideals of a neutrosophic regular LA-semigroup $N(S)$, then $N(A)N(B) = N(B)N(A)$.*

Proof. Let $N(A)$ and $N(B)$ be any neutrosophic ideals of a neutrosophic regular LA-semigroup $N(S)$. Now, let $a_1+a_2I \in N(A)$ and $b_1+b_2I \in N(B)$. Since, $N(A) \subseteq N(S)$ and $N(B) \subseteq N(S)$ and $N(S)$ is a neutrosophic

regular LA-semigroup so there exist x_1+x_2I, $y_1+y_2I \in N(S)$ such that $a_1+a_2I = [(a_1+a_2I)(x_1+x_2I)](a_1+a_2I)$ and $b_1+b_2I = [(b_1+b_2I)(y_1+y_2I)](b_1+b_2I)$.

Now, let $(a_1+a_2I)(b_1+b_2I) \in N(A)N(B)$ but

$$\begin{aligned}
(a_1+a_2I)(b_1+b_2I) &= [\{(a_1+a_2I)(x_1+x_2I)\}(a_1+a_2I)] \\
&\quad [\{(b_1+b_2I)(y_1+y_2I)\}(b_1+b_2I)] \\
&\in [\{N(A)N(S)\}N(A)][\{N(B)N(S)\}N(B)] \\
&\subseteq [N(A)N(A)][N(B)N(B)] \\
&= [N(B)N(B)][N(A)N(A)] \\
&\subseteq N(B)N(A) \\
N(A)N(B) &\subseteq N(B)N(A).
\end{aligned}$$

Now, let $(b_1+b_2I)(a_1+a_2I) \in N(B)N(A)$ but

$$\begin{aligned}
(b_1+b_2I)(a_1+a_2I) &= [\{(b_1+b_2I)(y_1+y_2I)\}(b_1+b_2I)] \\
&\quad [\{(a_1+a_2I)(x_1+x_2I)\}(a_1+a_2I)] \\
&\in [\{N(B)N(S)\}N(B)][\{N(A)N(S)\}N(A)] \\
&\subseteq [N(B)N(B)][N(A)N(A)] \\
&= [N(A)N(A)][N(B)N(B)] \\
&\subseteq N(A)N(B).
\end{aligned}$$

Since $N(B)N(A) \subseteq N(A)N(B)$. Hence $N(A)N(B) = N(B)N(A)$. ∎

Lemma 19 *Every neutrosophic bi-ideal of a regular neutrosophic LA-semigroup $N(S)$ with left identity $e+eI$ is a neutrosophic quasi-ideal of $N(S)$.*

Proof. Let $N(B)$ be a bi-ideal of $N(S)$ and $(s_1+s_2I)(b_1+b_2I) \in N(S)N(B)$, for $s_1+s_2I \in N(S)$ and $b_1+b_2I \in N(B)$. Since $N(S)$ is a neutrosophic regular LA-semigroup, so there exists x_1+x_2I

in $N(S)$ such that $b_1+b_2I = [(b_1+b_2I)(x_1+x_2I)](b_1+b_2I)$, then by using (4) and (1), we have

$$\begin{aligned}
& (s_1 + s_2I)(b_1 + b_2I) \\
= \ & (s_1 + s_2I)[\{(b_1 + b_2I)(x_1 + x_2I)\}(b_1 + b_2I)] \\
= \ & [(b_1 + b_2I)(x_1 + x_2I)][(s_1 + s_2I)(b_1 + b_2I)] \\
= \ & [\{(s_1 + s_2I)(b_1 + b_2I)\}(x_1 + x_2I)](b_1 + b_2I) \\
= \ & [(s_1 + s_2I)\{\{(b_1 + b_2I)(x_1 + x_2I)\}(b_1 + b_2I)\}(x_1 + x_2I)](b_1 + bI) \\
= \ & [[\{(b_1 + b_2I)(x_1 + x_2I)\}\{(s_1 + s_2I)(b_1 + b_2I)\}](x_1 + x_2I)](b_1 + b_2I) \\
= \ & [\{(x_1 + x_2I)((s_1 + s_2I)(b_1 + b_2I))\}\{(b_1 + b_2I)(x_1 + x_2I)\}](b_1 + b_2I) \\
= \ & [(b_1 + b_2I)[\{(x_1 + x_2I)\{(s_1 + s_2I)(b_1 + b_2I)\}\}](x_1 + x_2I)\}](b_1 + b_2I) \\
\in \ & [N(B)N(S)]N(B) \\
\subseteq \ & N(B).
\end{aligned}$$

Therefore, $N(B)N(S) \cap N(S)N(B) \subseteq N(S)N(B) \subseteq N(B)$. ■

Lemma 20 *In a neutrosophic regular LA-semigroup $N(S)$, every neutrosophic ideal is idempotent.*

Proof. Let $N(I)$ be any neutrosophic ideal of neutrosophic regular LA-semigroup $N(S)$. As we know, $(N(I))^2 \subseteq N(I)$ and let $a+bI \in N(I)$, since $N(S)$ is regular so there exists an element $x + yI \in N(S)$ such that

$$\begin{aligned}
a + bI &= [(a + bI)(x + yI)](a + bI) \\
&\in [N(I)N(S)]N(I) \\
&\subseteq N(I)N(I) = (N(I))^2.
\end{aligned}$$

This implies $N(I) \subseteq (N(I))^2$. Hence,$(N(I))^2 = N(I)$.

As $N(I)$ is the arbitrary neutrosophic ideal of $N(S)$. So, every ideal of neutrosophic regular AG-groupoid is idempotent. ■

Corollary 21 *In a neutrosophic regular LA-semigroup $N(S)$, every neutrosophic right ideal is idempotent.*

Proof. Let $N(R)$ be any neutrosophic right ideal of neutrosophic regular LA-semigroup $N(S)$ then $N(R)N(S) \subseteq N(R)$ and $(N(R))^2 \subseteq N(R)$.Now,let $a + bI \in N(R)$,

As $N(S)$ Is regular implies for $a+bI \in N(R)$,there exists $x+yI \in N(S)$ such that

$$\begin{aligned}
a + bI &= [(a + bI)(x + yI)](a + bI) \\
&\in [N(R)N(S)]N(I) \\
&\subseteq N(R)N(R) \\
&= (N(R))^2.
\end{aligned}$$

Thus $(N(R))^2 = N(R)$. Hence, $(N(R))^2 = N(R)$. So every neutrosophic right ideal of neutrosophic regular LA-semigroup $N(S)$ is idempotent. ■

Corollary 22 *In a neutrosophic regular LA-semigroup $N(S)$, every neutrosophic ideal is semiprime.*

Proof. Let $N(P)$ be any neutrosophic ideal of neutrosophic regular LA-semigroup $N(S)$

and let $N(I)$ be any other neutrosophic ideal such that $[N(I)]^2 \subseteq N(P)$.

Now as every ideal of $N(S)$ is idempotent. So,$[N(I)]^2 = N(I)$ implies $N(I) \subseteq N(P)$. Hence, every neutrosophic ideal of $N(S)$ is semiprime. ■

An LA-semigroup $N(S)$ is called neutrosophic intra-regular if for each element $a_1 + a_2I \in N(S)$ there exist elements $(x_1 + x_2I)$, $(y_1 + y_2I) \in N(S)$ such that $a_1 + a_2I = [(x_1 + x_2I)(a_1 + a_2I)^2](y_1 + y_2I)$.

Example 23 *Let $S = \{1, 2, 3\}$ with binary operation "$\cdot$" given in the following Cayley's table, is an intra-regular LA-semigroup with left identity* 2.

$\cdot$	1	2	3
1	2	3	1
2	1	2	3
3	3	1	2

then $N(S) = \{1+1I, 1+2I, 1+3I, 2+1I, 2+2I, 2+3I, 3+1I, 3+2I, 3+3I\}$ is an example of neutrosophic intra-regular LA-semigroup under the operation "$*$" and has the following Cayley's table:

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$
$1+2I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$1+3I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$2+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$2+2I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$2+3I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$3+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$3+2I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$3+3I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$

Clearly $N(S)$ is a neutrosophic LA-semigroup and is non-associative because $[(1+1I)*(2+2I)]*(2+3I) \neq (1+1I)*[(2+2I)*(2+3I)]$ and $N(S)$ is intra-regular as

$(1+1I) = [(1+3I)(1+1I)^2](2+31)$, $(2+3I) = [(1+1I)(2+3I)^2](3+1I)$, $(3+1I) = [(2+3I)(3+1I)^2](3+3I)$ etc.

Note that if $N(S)$ is a neutrosophic intra-regular LA-semigroup then $[N(S)]^2 = N(S)$.

Lemma 24 *In a neutrosophic intra-regular LA-semigroup $N(S)$ with left identity $e+eI$, every neutrosophic ideal is idempotent.*

Proof. Let $N(I)$ be any neutrosophic ideal of a neutrosophic intra-regular LA-semigroup $N(S)$ implies $[N(I)]^2 \subseteq N(I)$.Now, let $a_1+a_2I \in N(I)$ and since $N(I) \subseteq N(S)$ implies $a_1+a_2I \in N(S)$ Since, $N(S)$ is a neutrosophic intra-regular LA-semigroup, so there exist (x_1+x_2I), $(y_1+y_2I) \in N(S)$ such that

$$\begin{aligned}
(a_1+a_2I) &= [(x_1+x_2I)(a_1+a_2I)^2](y_1+y_2I) \\
&\in [N(S)(N(I))^2]N(S) \\
&= [N(S)(N(I)N(I))]N(S) \\
&= (N(I)(N(S)N(I)))N(S) \\
&\subseteq (N(I)N(I))N(S) \\
&= (N(S)N(I))N(I) \\
&\subseteq N(I)N(I) \\
&= [N(I)]^2.
\end{aligned}$$

Hence $[N(I)]^2 = N(I)$. As, $N(I)$ is arbitrary so every neutrosophic ideal of is idempotent in a neutrosophic intra-regular LA-semigroup $N(S)$ with left identity. ■

Lemma 25 *In a neutrosophic intra-regular LA-semigroup $N(S)$ with left identity $e+eI$, $N(I)N(J) = N(I) \cap N(J)$, for every neutrosophic ideals $N(I)$ and $N(J)$ in $N(S)$.*

Proof. Let $N(I)$ and $N(J)$ be any neutrosophic ideals of $N(S)$, then obviously $N(I)N(J) \subseteq N(I)N(S)$ and $N(I)N(J) \subseteq N(S)N(J)$ implies $N(I)N(J) \subseteq N(I) \cap N(J)$. Since $N(I) \cap N(J) \subseteq N(I)$ and $N(I) \cap N(J) \subseteq N(J)$, then $[N(I) \cap N(J)]^2 \subseteq N(I)N(J)$. Also $N(I) \cap N(J)$ is a neutrosophic ideal of $N(S)$, so we have $N(I) \cap N(J) = [N(I) \cap N(J)]^2 \subseteq N(I)N(J)$. Hence $N(I)N(J) = N(I) \cap N(J)$. ■

Theorem 26 *For neutrosophic intra-regular AG-groupoid with left identity $e+eI$, the following statements are equivalent.*
(i) $N(A)$ is a neutrosophic left ideal of $N(S)$.
(ii) $N(A)$ is a neutrosophic right ideal of $N(S)$.
(iii) $N(A)$ is a neutrosophic ideal of $N(S)$.
(iv) $N(A)$ is a neutrosophic bi-ideal of $N(S)$.
(v) $N(A)$ is a neutrosophic generalized bi-ideal of $N(S)$.
(vi) $N(A)$ is a neutrosophic interior ideal of $N(S)$.
(vii) $N(A)$ is a neutrosophic quasi-ideal of $N(S)$.
$(viii)$ $N(A)N(S) = N(A)$ and $N(S)N(A) = N(A)$.

Proof. $(i) \Rightarrow (viii)$

Let $N(A)$ be a neutrosophic left ideal of $N(S)$. Thus, $N(S)N(A) = N(A)$. Now let $(a_1 + a_2I) \in N(A)$ and $(s_1 + s_2I) \in N(S)$, since $N(S)$ is a neutrosophic intra-regular LA-semigroup, so there exist $(x_1 + x_2I)$, $(y_1 + y_2I) \in N(S)$ such that $(a_1 + a_2I) = [(x_1 + x_2I)(a_1 + a_2I)^2](y_1 + y_2I)$, therefore we have

$$\begin{aligned}
(a_1 + a_2I)(s_1 + s_2I) &= [\{(x_1 + x_2I)(a_1 + a_2I)^2\}(y_1 + y_2I)](s_1 + s_2I) \\
&= [\{(x_1 + x_2I)\{(a_1 + a_2I)(a_1 + a_2I)\}\}(y_1 + y_2I)](s_1 + s_2I) \\
&\in [\{N(S)\{N(A)N(A)\}\}N(S)]N(S) \\
&\subseteq [\{N(S)\{N(S)N(A)\}\}N(S)]N(S) \\
&\subseteq [\{N(S)N(A)\}N(S)]N(S) \\
&= [N(S)N(S)][N(S)N(A)] \\
&= N(S)[N(S)N(A)] \subseteq N(S)N(A) = N(A).
\end{aligned}$$

which implies that $N(A)$ is a neutrosophic right ideal of $N(S)$ and so $N(A)N(S) = N(S)$.

$(viii) \Rightarrow (vii)$

Let $N(A)N(S) = N(A)$ and $N(S)N(A) = N(A)$ then $N(A)N(S) \cap N(S)N(A) = N(A)$, which clearly implies that $N(A)$ is a neutrosophic quasi-ideal of $N(S)$.

$(vii) \Rightarrow (vi)$

Let $N(A)$ be a quasi-ideal of $N(S)$. Now let $[(s_1 + s_2I)(a_1 + a_2I)](s_1 + s_2I) \in [N(S)N(A)]N(S)$, since $N(S)$ is neutrosophic intra-regular LA-semigroup so there exist $(x_1 + x_2I)$, $(y_1 + y_2I)$, $(p_1 + p_2I)$, $(q_1 + q_2I) \in N(S)$ such that $(s_1 + s_2I) = [(x_1 + x_2I)(s_1 + s_2I)^2](y_1 + y_2I)$ and $(a_1 + a_2I) = [(p_1 + p_2I)(a_1 + a_2I)^2](q_1 + q_2I)$. Therefore we have

$$\begin{aligned}
& [(s_1 + s_2I)(a_1 + a_2I)](s_1 + s_2I) \\
= \; & [(s_1 + s_2I)(a_1 + a_2I)][\{(x_1 + xI)(s_1 + s_2I)^2\}(y_1 + y_2I)] \\
= \; & [\{(s_1 + s_2I)\{(x_1 + x_2I)(s_1 + s_2I)^2\}\}][(a_1 + a_2I)(y_1 + y_2I)] \\
= \; & (a_1 + a_2I)[\{(s_1 + s_2I)\{(x_1 + x_2I)(s_1 + s_2I)^2\}\}(y_1 + y_2I)] \\
\in \; & N(A)N(S).
\end{aligned}$$

and

$$\begin{aligned}
& [(s_1+s_2I)(a_1+a_2I)](s_1+s_2I) \\
= \; & [(s_1+s_2I)\{\{(p_1+p_2I)(a_1+a_2I)^2\}(q_1+q_2I)\}](s_1+s_2I) \\
= \; & [\{(p_1+p_2I)(a_1+a_2I)^2\}\{(s_1+s_2I)(q_1+q_2I)\}](s_1+s_2I) \\
= \; & [\{(p_1+p_2I)\{(a_1+a_2I)(a_1+a_2I)\}\}\{(s_1+s_2I)(q_1+q_2I)\}](s_1+s_2I) \\
= \; & [\{(a_1+a_2I)\{(p_1+p_2I)(a_1+a_2I)\}\}\{(s_1+s_2I)(q_1+q_2I)\}](s_1+s_2I) \\
= \; & [\{(q_1+q_2I)(s_1+s_2I)\}\{\{(p_1+p_2I)(a_1+a_2I)\}(a_1+a_2I)\}](s_1+s_2I) \\
= \; & [\{(p_1+p_2I)(a_1+a_2I)\}\{\{(q_1+q_2I)(s_1+s_2I)\}(a_1+a_2I)\}](s_1+s_2I) \\
= \; & [\{(a_1+a_2I)\{(q_1+q_2I)(s_1+s_2I)\}\}\{(a_1+a_2I)(p_1+p_2I)\}](s_1+s_2I) \\
= \; & [(a_1+a_2I)\{\{(a_1+a_2I)\{(q_1+q_2I)(s_1+s_2I)\}\}(p_1+p_2I)\}](s_1+s_2I) \\
= \; & [(s_1+s_2I)\{\{(a_1+a_2I)\{(q_1+q_2I)(s_1+s_2I)\}\}(p_1+p_2I)\}](a_1+a_2I) \\
\in \; & N(S)N(A) \subseteq N(A).
\end{aligned}$$

which shows that $N(A)$ is a neutrosophic interior ideal of $N(S)$.

$(vi) \Rightarrow (v)$

Let $N(A)$ be a neutrosophic interior ideal of a neutrosophic intra-regular LA-semigroup $N(S)$

and $[(a_1+a_2I)(s_1+s_2I)](a_1+a_2I) \in [N\,(A)N(S)]N(A)$. Now we get

$$\begin{aligned}
& [(a_1+a_2I)(s_1+s_2I)](a_1+a_2I) \\
= \; & [(a_1+a_2I)(s_1+s_2I)][\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)] \\
= \; & [(x_1+x_2I)(a_1+a_2I)^2][\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)] \\
= \; & [(x_1+x_2I)\{(a_1+a_2I)(a_1+a_2I)\}][\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)] \\
= \; & [\{\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)\}\{(a_1+a_2I)(a_1+a_2I)\}](x_1+x_2I) \\
= \; & [(a_1+a_2I)\{\{\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)\}(a_1+a_2I)\}](x_1+x_2I) \\
= \; & [(a_1+a_2I)\{\{(a_1+a_2I)(y_1+y_2I)\}\{(a_1+a_2I)(s_1+s_2I)\}\}](x_1+x_2I) \\
= \; & [\{(a_1+a_2I)(y_1+y_2I)\}\{(a_1+a_2I)\{(a_1+a_2I)(s_1+s_2I)\}\}](x_1+x_2I) \\
= \; & [\{\{(a_1+a_2I)\{(a_1+a_2I)(s_1+s_2I)\}\}(y_1+y_2I)\}(a_1+a_2I)](x_1+x_2I) \\
\in \; & [N\,(S)N(A)]N(S) \subseteq N(A).
\end{aligned}$$

$(v) \Rightarrow (iv)$

Let $N(A)$ be a neutrosophic generalized bi-ideal of $N(S)$. Let $a_1+a_2I \in N(A)$, and since $N(S)$ is neutrosophic intra-regular LA-semigroup so there exist (x_1+x_2I), (y_1+y_2I) in $N(S)$ such that $a_1+a_2I = [(x_1+x_2I)(a_1+$

$a_2I)^2](y_1 + y_2I)$, then using (3) and (4), we have

$$\begin{aligned}
& (a_1 + a_2I)(a_1 + a_2I) \\
= \; & [\{(x_1 + x_2I)(a_1 + a_2I)^2\}(y_1 + y_2I)](a_1 + a_2I) \\
= \; & [\{(x_1 + x_2I)(a_1 + a_2I)^2\}\{(e_1 + e_2I)(y_1 + y_2I)\}](a_1 + a_2I) \\
= \; & [\{(y_1 + y_2I)(e_1 + e_2I)\}\{(a_1 + a_2I)^2(x_1 + x_2I)\}](a_1 + a_2I) \\
= \; & [(a_1 + a_2I)^2\{\{(y_1 + y_2I)(e_1 + e_2I)\}(x_1 + x_2I)\}](a_1 + a_2I) \\
= \; & [\{(a_1 + a_2I)(a_1 + a_2I)\}\{\{(y_1 + y_2I)(e_1 + e_2I)\}(x_1 + x_2I)\}](a_1 + a_2I) \\
= \; & [\{(x_1 + x_2I)\{(y_1 + y_2I)(e_1 + e_2I)\}\}\{(a_1 + a_2I)(a_1 + a_2I)\}](a_1 + a_2I) \\
= \; & [(a_1 + a_2I)\{\{(x_1 + x_2I)\{(y_1 + y_2I)(e_1 + e_2I)\}\}(a_1 + a_2I)\}](a_1 + a_2I) \\
\in \; & [N(A)N(S)]N(A) \subseteq N(A).
\end{aligned}$$

Hence $N(A)$ is a neutrosophic bi-ideal of $N(S)$.

$(iv) \Rightarrow (iii)$

Let $N(A)$ be any neutrosophic bi-ideal of $N(S)$ and let $(a_1 + a_2I)(s_1 + s_2I) \in N(A)N(S)$. Since $N(S)$ is neutrosophic intra-regular LA-semigroup, so there exist $(x_1 + x_2I)$, $(y_1 + y_2I) \in N(S)$ such that $(a_1 + a_2I) = [(x_1 + x_2I)(a_1 + a_2I)^2](y_1 + y_2I)$. Therefore we have

$$
\begin{aligned}
& (a_1 + a_2 I)(s_1 + s_2 I) \\
= \;& [\{(x_1 + x_2 I)(a_1 + a_2 I)^2\}(y_1 + y_2 I)](s_1 + s_2 I) \\
= \;& [(s_1 + s_2 I)(y_1 + y_2 I)][(x_1 + x_2 I)(a_1 + a_2 I)^2] \\
= \;& [(a_1 + a_2 I)^2(x_1 + x_2 I)][(y_1 + y_2 I)(s_1 + s_2 I)] \\
= \;& [\{\{(y_1 + y_2 I)(s_1 + s_2 I)\}(x_1 + x_2 I)\}(a_1 + a_2 I)^2] \\
= \;& [\{(y_1 + y_2 I)(s_1 + s_2 I)\}(x_1 + x_2 I)][(a_1 + a_2 I)(a_1 + a_2 I)] \\
= \;& [(a_1 + a_2 I)(a_1 + a_2 I)][(x_1 + x_2 I)\{(y_1 + y_2 I)(s_1 + s_2 I)\}] \\
= \;& [\{(x_1 + x_2 I)\{(y_1 + y_2 I)(s_1 + s_2 I)\}\}(a_1 + a_2 I)](a_1 + a_2 I) \\
= \;& \{(x_1 + x_2 I)\{(y_1 + y_2 I)(s_1 + s_2 I)\}\} \\
& \{\{(x_1 + x_2 I)(a_1 + a_2 I)^2\}(y_1 + y_2 I)\}(a_1 + a_2 I) \\
= \;& [\{(x_1 + x_2 I)(a_1 + a_2 I)^2\}\{\{(x_1 + x_2 I)\{(y_1 + y_2 I)(s_1 + s_2 I)\}\} \\
& (y_1 + y_2 I)\}](a_1 + a_2 I) \\
= \;& [\{(y_1 + y_2 I)\{(x_1 + x_2 I)\{(y_1 + y_2 I)(s_1 + s_2 I)\}\}\} \\
& \{(a_1 + a_2 I)^2(x_1 + x_2 I)\}](a_1 + a_2 I) \\
= \;& [(a_1 + a_2 I)^2\{\{(y_1 + y_2 I)\{(x_1 + x_2 I)\{(y_1 + y_2 I)(s_1 + s_2 I)\}\}\} \\
& (x_1 + x_2 I)\}](a_1 + a_2 I) \\
= \;& [\{(a_1 + a_2 I)(a_1 + a_2 I)\}\{\{(y_1 + y_2 I)\{(x_1 + x_2 I) \\
& \{(y_1 + y_2 I)(s_1 + s_2 I)\}\}\}(x_1 + x_2 I)\}](a_1 + a_2 I) \\
= \;& [\{(x_1 + x_2 I)\{(y_1 + y_2 I)\{(x_1 + x_2 I)\{(y_1 + y_2 I)(s_1 + s_2 I)\}\}\}\} \\
& \{(a_1 + a_2 I)(a_1 + a_2 I)\}](a_1 + a_2 I) \\
\in \;& [N(A)N(S)]N(A) \subseteq N(A).
\end{aligned}
$$

$$\begin{aligned}
& (s_1+s_2I)(a_1+a_2I) \\
= & (s_1+s_2I)[\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)] \\
= & [(x_1+x_2I)(a_1+a_2I)^2][(s_1+s_2I)(y_1+y_2I)] \\
= & [(x_1+x_2I)\{(a_1+a_2I)(a_1+a_2I)\}][(s_1+s_2I)(y_1+y_2I)] \\
= & [(a_1+a_2I)\{(x_1+x_2I)(a_1+a_2I)\}][(s_1+s_2I)(y_1+y_2I)] \\
= & [\{(s_1+s_2I)(y_1+y_2I)\}\{(x_1+x_2I)(a_1+a_2I)\}](a_1+a_2I) \\
= & [\{(a_1+a_2I)(x_1+x_2I)\}\{(y_1+y_2I)(s_1+s_2I)\}](a_1+a_2I) \\
= & [\{\{(y_1+y_2I)(s_1+s_2I)\}(x_1+x_2I)\}(a_1+a_2I)](a_1+a_2I) \\
= & [\{\{(y_1+y_2I)(s_1+s_2I)\}(x_1+x_2I)\} \\
& \{\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)\}](a_1+a_2I) \\
= & [\{\{(y_1+y_2I)(s_1+s_2I)\}\{(x_1+x_2I)(a_1+a_2I)^2\}\} \\
& \{(x_1+x_2I)(y_1+y_2I)\}](a_1+a_2I) \\
= & [\{\{(a_1+a_2I)^2(x_1+x_2I)\}\{(s_1+s_2I)(y_1+y_2I)\}\} \\
& \{(x_1+x_2I)(y_1+y_2I)\}](a_1+a_2I) \\
= & [\{\{\{(a_1+a_2I)(a_1+a_2I)\}(x_1+x_2I)\}\{(s_1+s_2I)(y_1+y_2I)\}\} \\
& \{(x_1+x_2I)(y_1+y_2I)\}](a_1+a_2I) \\
= & [\{\{(x_1+x_2I)(y_1+y_2I)\}\{(s_1+s_2I)(y_1+y_2I)\}\} \\
& \{\{(a_1+a_2I)(a_1+a_2I)\}(x_1+x_2I)\}](a_1+a_2I) \\
= & [\{\{(x_1+x_2I)(y_1+y_2I)\}\{(a_1+a_2I)(a_1+a_2I)\}\} \\
& \{\{(s_1+s_2I)(y_1+y_2I)\}(x_1+x_2I)\}](a_1+a_2I) \\
= & [\{\{(a_1+a_2I)(a_1+a_2I)\}\{(y)(x_1+x_2I)\}\} \\
& \{\{(s_1+s_2I)(y_1+y_2I)\}(x_1+x_2I)\}](a_1+a_2I) \\
= & [\{\{\{(s_1+s_2I)(y_1+y_2I)\}(x_1+x_2I)\} \\
& \{(y_1+y_2I)(x_1+x_2I)\}\} \\
& \{(a_1+a_2I)(a_1+a_2I)\}](a_1+a_2I) \\
= & [(a_1+a_2I)\{\{\{\{(s_1+s_2I)(y_1+y_2I)\}(x_1+x_2I)\} \\
& \{(y_1+y_2I)(x_1+x_2I)\}\}(a_1+a_2I)\}](a_1+a_2I) \\
\in & [N(A)N(S)]N(A) \\
\subseteq & N(A).
\end{aligned}$$

Therefore $N(A)$ is a neutrosophic ideal of $N(S)$.

$(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are obvious. ■

Lemma 27 *A neutrosophic LA-semigroup $N(S)$ with left identity $(e+eI)$ is intra-regular if and only if every neutrosophic bi-ideal of $N(S)$ is idempotent.*

Proof. Assume that $N(S)$ is a neutrosophic intra-regular LA-semigroup with left identity $(e+eI)$ and $N(B)$ is a neutrosophic bi-ideal of $N(S)$. Let

$(b+bI)\in N(B)$, and since $N(S)$ is intra-regular so there exist (c_1+c_2I), (d_1+d_2I) in $N(S)$ such that $(b_1+b_2I)=[(c_1+c_2I)(b_1+b_2I)^2](d_1+d_2I)$, then we have

$$
\begin{aligned}
& (b_1+b_2I) \\
= \ & [(c_1+c_2I)(b_1+b_2I)^2](d_1+d_2I) \\
= \ & [\{(c_1+c_2I)(b_1+b_2I)^2\}\{(e+eI)(d_1+d_2I)\}] \\
= \ & [\{(d_1+d_2I)(e+eI)\}\{\big(b_1+b_2I)^2(c_1+c_2I\big)\}] \\
= \ & [(b_1+b_2I)^2\{\{(d_1+d_2I)(e+eI)\}(c_1+c_2I)\}] \\
= \ & [\{(b_1+b_2I)(b_1+b_2I)\}\{\{(d_1+d_2I)(e+eI)\}(c_1+c_2I)\}] \\
= \ & [\{\{(d_1+d_2I)(e+eI)\}(c_1+c_2I)\}(b_1+b_2I)](b_1+b_2I) \\
= \ & [\{\{(d_1+d_2I)(e+eI)\}(c_1+c_2I)\} \\
& \{\{(c_1+c_2I)(b_1+b_2I)^2\}(d_1+d_2I)\}](b_1+b_2I) \\
= \ & [\{\{(c_1+c_2I)(b_1+b_2I)^2\}\{\{(d_1+d_2I)(e+eI)\}(c_1+c_2I)\} \\
& (d_1+d_2I)\}](b_1+b_2I) \\
= \ & [\{(c_1+c_2I)\{(b_1+b_2I)(b_1+b_2I)\}\}\{\{(d_1+d_2I)(e+eI)\} \\
& (c_1+c_2I)\}(d_1+d_2I)](b_1+b_2I) \\
= \ & [\{(b_1+b_2I)\{(c_1+c_2I)(b_1+b_2I)\}\}\{\{\{(d_1+d_2I)(e+eI)\} \\
& (c_1+c_2I)\}(d_1+d_2I)\}](b_1+b_2I) \\
= \ & [\{\{\{\{(d_1+d_2I)(e+eI)\}(c_1+c_2I)\}(d_1+d_2I)\}\} \\
& \{(c_1+c_2I)(b_1+b_2I)\}(b_1+b_2I)](b_1+b_2I) \\
= \ & [\{\{\{\{(d_1+d_2I)(e+eI)\}(c_1+c_2I)\}(d_1+d_2I)\} \\
& \{\{(c_1+c_2I)\{\{(c_1+c_2I)(b_1+b_2I)^2\} \\
& (d_1+d_2I)\}\}(b_1+b_2I)](b_1+b_2I) \\
= \ & [\{\{\{\{(d_1+d_2I)(e+eI)\}(c_1+c_2I)\}(d_1+d_2I)\}\}\{(c_1+c_2I) \\
& \{\{(c_1+c_2I)\{(b_1+b_2I)(b_1+b_2I)\}\}(d_1+d_2I))\}\} \\
& (b_1+b_2I)\}](b_1+b_2I) \\
= \ & [\{\{\{\{(d_1+d_2I)(e+eI)\}(c_1+c_2I)\}(d_1+d_2I)\}\}\{(c_1+c_2I)\{\{(b_1+b_2I) \\
& \{(c_1+c_2I)(b_1+b_2I)\}\}(d_1+d_2I)\}\}(b_1+b_2I)](b_1+b_2I) \\
= \ & [\{\{\{\{(d_1+d_2I)(e+eI)\}(c_1+c_2I)\}(d_1+d_2I)\}\}\{(b_1+b_2I) \\
& \{\{(c_1+c_2I)\{(c_1+c_2I)(b_1+b_2I)\}\}(d_1+d_2I)\}\}(b_1+b_2I)](b_1+b_2I) \\
= \ & [(b_1+b_2I)\{\{\{\{\{(d_1+d_2I)(e+eI)\}(c_1+c_2I)\}(d_1+d_2I)\}\} \\
& \{\{(c_1+c_2I)\{(c_1+c_2I)(b_1+b_2I)\}\}(d_1+d_2I)\}\}(b_1+b_2I)](b_1+b_2I) \\
\in \ & [\{N(B)N(S)\}N(B)]N(B)\subseteq N(B)N(B).
\end{aligned}
$$

Hence $[N(B)]^2=N(B)$.

Conversely, since $N(S)(a+bI)$ is a neutrosophic bi-ideal of $N(S)$, and by assumption $N(S)(a+bI)$ is idempotent, so we have

$$\begin{aligned}(a+bI) &\in [N(S)(a+bI)][N(S)(a+bI)]\\ &= [\{N(S)(a+bI)\}\{N(S)(a+bI)\}][N(S)(a+bI)]\\ &= [\{N(S)N(S)\}\{(a+bI)(a+bI)\}][N(S)(a+bI)]\\ &\subseteq [N(S)(a+bI)^2][N(S)N(S)]\\ &= [N(S)(a+bI)^2]N(S).\end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular LA-semigroup. ■

Theorem 28 *In a neutrosophic LA-semigroup $N(S)$ with left identity $e+eI$, the following statements are equivalent.*
(i) $N(S)$ is intra-regular.
(ii) Every neutrosophic two sided ideal of $N(S)$ is semiprime.
(iii) Every neutrosophic right ideal of $N(S)$ is semiprime.
(iv) Every neutrosophic left ideal of $N(S)$ is semiprime.

Proof. $(i) \Rightarrow (iv)$
Let $N(S)$ is intra-regular, then every neutrosophic left ideal of $N(S)$ is semiprime.
$(iv) \Rightarrow (iii)$
Let $N(R)$ be a neutrosophic right ideal and $N(I)$ be any neutrosophic ideal of $N(S)$ such that $[N(I)]^2 \subseteq N(R)$. Then clearly $[N(I)]^2 \subseteq N(R) \cup N(S)N(R)$. Now $N(R) \cup N(S)N(R)$ is a neutrosophic two-sided ideal of $N(S)$, so is neutrosophic left. Then by (iv) we have $N(I) \subseteq N(R) \cup N(S)N(R)$. Now we have

$$\begin{aligned}N(S)N(R) &= [N(S)N(S)]N(R)\\ &= [N(R)N(S)]N(S)\\ &\subseteq N(R)N(S) \subseteq N(R).\end{aligned}$$

This implies that $N(I) \subseteq N(R) \cup N(S)N(R) \subseteq N(R)$. Hence $N(R)$ is semiprime.
It is clear that $(iii) \Rightarrow (ii)$.
Now $(ii) \Rightarrow (i)$
Since $(a+bI)^2N(S)$ is a neutrosophic right ideal of $N(S)$ containing $(a+bI)^2$ and clearly it is a neutrosophic two sided ideal so by assumption (ii), it is semiprime, therefore, $(a+bI) \in (a+bI)^2N(S)$. Thus we have

$$\begin{aligned}a+bI &\in (a+bI)^2N(S)\\ &= (a+bI)^2[N(S)N(S)]\\ &= N(S)[(a+bI)^2N(S)]\\ &= [N(S)N(S)][(a+bI)^2N(S)]\\ &= [N(S)(a+bI)^2)][N(S)N(S)]\\ &= [N(S)(a+bI)^2]N(S).\end{aligned}$$

Hence $N(S)$ is intra-regular. ■

Theorem 29 *An LA-semigroup $N(S)$ with left identity $e + eI$ is intra-regular if and only if every neutrosophic left ideal of $N(S)$ is idempotent.*

Proof. Let $N(S)$ be a neutrosophic intra-regular LA-semigroup then every neutrosophic ideal of $N(S)$ is idempotent.

Conversely, assume that every neutrosophic left ideal of $N(S)$ is idempotent. Since $N(S)(a+bI)$ is a neutrosophic left ideal of $N(S)$, so we have

$$\begin{aligned} a+bI &\in N(S)(a+bI) \\ &= [N(S)(a+bI)][N(S)(a+bI)] \\ &= [\{N(S)(a+bI)\}\{N(S)(a+bI)\}]\{N(S)(a+bI)\} \\ &= [\{N(S)N(S)\}\{(a+bI)(a+bI)\}]\{N(S)(a+bI)\} \\ &\subseteq [N(S)(a+bI)^2][N(S)N(S)] \\ &= [N(S)(a+bI)^2]N(S). \end{aligned}$$

■

Theorem 30 *A neutrosophic LA-semigroup $N(S)$ with left identity $e + eI$ is intra-regular if and only if $N(R) \cap N(L) \subseteq N(R)N(L)$, for every neutrosophic semiprime right ideal $N(R)$ and every neutrosophic left ideal $N(L)$ of $N(S)$.*

Proof. Let $N(S)$ be an intra-regular LA-semigroup, so $N(R)$ and $N(L)$ become neutrosophic ideals of $N(S)$, therefore $N(R)\cap N(L) \subseteq N(L)N(R)$, for every neutrosophic ideal $N(R)$ and $N(L)$ and $N(R)$ is semiprime.

Conversely, assume that $N(R) \cap N(L) \subseteq N(R)N(L)$ for every neutrosophic right ideal $N(R)$, which is semiprime and every neutrosophic left ideal $N(L)$ of $N(S)$. Since $(a + bI)^2 \in (a + bI)^2N(S)$, which is a neutrosophic right ideal of $N(S)$ so is semiprime which implies that $(a + bI) \in (a + bI)^2N(S)$. Now clearly $N(S)(a + bI)$ is a neutrosophic left ideal of $N(S)$ and $(a + bI) \in N(S)(a + bI)$ Therefore using (3),we have

$$\begin{aligned} a+bI &\in [(a+bI)^2N(S)] \cap [N(S)(a+bI)] \\ &\subseteq [(a+bI)^2N(S)][N(S)(a+bI)] \\ &\subseteq [(a+bI)^2N(S)][N(S)N(S)] \\ &= [(a+bI)^2N(S)]N(S) \\ &= [\{(a+bI)(a+bI)\}N(S)]N(S) \\ &= [\{(a+bI)(a+bI)\}\{N(S)N(S)\}]N(S) \\ &= [\{N(S)N(S)\}\{(a+bI)(a+bI)\}]N(S) \\ &= [N(S)\{(a+bI)(a+bI)\}]N(S) \\ &= [N(S)(a+bI)^2]N(S). \end{aligned}$$

Therefore $N(S)$ is a neutrosophic intra-regular LA-semigroup. ■

Theorem 31 *For a neutrosophic LA-semigroup $N(S)$ with left identity $e+eI$, the following statements are equivalent.*

(i) $N(S)$ is intra-regular.

(ii) $N(L)\cap N(R)\subseteq N(L)N(R)$, for every right ideal $N(R)$, which is neutrosophic semiprime and every neutrosophic left ideal $N(L)$ of $N(S)$.

(iii) $N(L)\cap N(R)\subseteq [N(L)N(R)]N(L)$, for every neutrosophic semiprime right ideal $N(R)$ and every neutrosophic left ideal $N(L)$.

Proof. $(i)\Rightarrow(iii)$

Let $N(S)$ be intra-regular and $N(L)$, $N(R)$ be any neutrosophic left and right ideals of $N(S)$ and let $a_1+a_2I\in N(L)\cap N(R)$, which implies that $a_1+a_2I\in N(L)$ and $a_1+a_2I\in N(R)$. Since $N(S)$ is intra-regular so there exist (x_1+x_2I), (y_1+y_2I) in $N(S)$, such that $a_1+a_2I=[(x_1+x_2I)(a_1+a_2I)^2](y_1+y_2I)$, then by using (4), (1) and (3), we have

$$\begin{aligned}
a_1+a_2I &= [(x_1+x_2I)(a_1+a_2I)^2](y_1+y_2I)\\
&= [(x_1+x_2I)\{(a_1+a_2I)(a_1+a_2I)\}](y_1+y_2I)\\
&= [(a_1+a_2I)\{(x_1+x_2I)(a_1+a_2I)\}](y_1+y_2I)\\
&= [(y_1+y_2I)\{(x_1+x_2I)(a_1+a_2I)\}](a_1+a_2I)\\
&= [(y_1+y_2I)\{(x_1+x_2I)\{\{(x_1+x_2I)(a_1+a_2I)^2\}\\
&\quad (y_1+y_2I)\}\}](a_1+a_2I)\\
&= [(y_1+y_2I)\{\{(x_1+x_2I)(a_1+a_2I)^2\}\{(x_1+x_2I)(y)\}\}](a_1+a_2I)\\
&= [\{(x_1+x_2I)(a_1+a_2I)^2\}\{(y_1+y_2I)\\
&\quad \{(x_1+x_2I)(y_1+y_2I)\}\}](a_1+a_2I)\\
&= [\{(x_1+x_2I)\{(a_1+a_2I)(a_1+a_2I)\}\}\{(y_1+y_2I)\\
&\quad \{(x_1+x_2I)(y_1+y_2I)\}\}](a_1+a_2I)\\
&= [\{(a_1+a_2I)\{(x_1+x_2I)(a_1+a_2I)\}\}\{(y_1+y_2I)\\
&\quad \{(x_1+x_2I)(y_1+y_2I)\}\}](a_1+a_2I)\\
&\in [\{N(R)\{N(S)N(L))\}\}N(S)]N(L)\\
&\subseteq [\{N(R)N(L)\}N(S)]N(L)\\
&= [N(L)N(S)][N(R)N(L)]\\
&= [N(L)N(R)][N(S)N(L)]\\
&\subseteq [N(L)N(R)]N(L),
\end{aligned}$$

which implies that $N(L)\cap N(R)\subseteq [N(L)N(R)]N(L)$. Also $N(L)$ is semiprime.

$(iii)\Rightarrow(ii)$

Let $N(R)$ and $N(L)$ be neutrosophic left and right ideals of $N(S)$ and $N(R)$ is semiprime, then by assumption (iii) and by (3), (4) and (1), we have

$$\begin{aligned} N(R)\cap N(L) &\subseteq [N(R)N(L)]N(R) \\ &\subseteq [N(R)N(L)]N(S) \\ &= [N(R)N(L)][N(S)N(S)] \\ &= [N(S)N(S)][N(L)N(R)] \\ &= N(L)[\{N(S)N(S)\}N(R)] \\ &= N(L)[\{N(R)N(S)\}N(S)] \\ &\subseteq N(L)[N(R)N(S)] \\ &\subseteq N(L)N(R). \end{aligned}$$

$(ii) \Rightarrow (i)$

Since $e+eI \in N(S)$ implies $a+bI \in N(S)(a+bI)$, which is a neutrosophic left ideal of $N(S)$, and $(a+bI)^2 \in (a+bI)^2N(S)$, which is a semiprime neutrosophic right ideal of $N(S)$, therefore, $a+bI \in (a+bI)^2N(S)$. Now using (3) we have

$$\begin{aligned} a+bI &\in [N(S)(a+bI)] \cap [\big(a+bI)^2N(S)\big)] \\ &\subseteq [N(S)(a+bI)][(a+bI)^2N(S)] \\ &\subseteq [N(S)N(S)][(a+bI)^2N(S)] \\ &= [N(S)(a+bI)^2][N(S)N(S)] \\ &= [N(S)(a+bI)^2]N(S). \end{aligned}$$

Hence $N(S)$ is intra-regular. ■

A neutrosophic LA-semigroup $N(S)$ is called totally ordered under inclusion if $N(P)$ and $N(Q)$ are any neutrosophic ideals of $N(S)$ such that either $N(P) \subseteq N(Q)$ or $N(Q) \subseteq N(P)$.

A neutrosophic ideal $N(P)$ of a neutrosophic LA-semigroup $N(S)$ is called strongly irreducible if $N(A) \cap N(B) \subseteq N(P)$ implies either $N(A) \subseteq N(P)$ or $N(B) \subseteq N(P)$, for all neutrosophic ideals $N(A)$, $N(B)$ and $N(P)$ of $N(S)$.

Lemma 32 *Every neutrosophic ideal of a neutrosophic intra-regular LA-semigroup $N(S)$ is prime if and only if it is strongly irreducible.*

Proof. Assume that every ideal of $N(S)$ is neutrosophic prime. Let $N(A)$ and $N(B)$ be any neutrosophic ideals of $N(S)$ so, $N(A)N(B) = N(A) \cap N(B)$, where $N(A) \cap N(B)$ is neutrosophic ideal of $N(S)$. Now, let $N(A) \cap N(B) \subseteq N(P)$ where $N(P)$ is a neutrosophic ideal of $N(S)$ too. But by assumption every neutrosophic ideal of a neutrosophic intra-regular LA-semigroup $N(S)$ is prime so is neutrosophic prime, therefore, $N(A)N(B) = N(A) \cap N(B) \subseteq N(P)$ implies $N(A) \subseteq N(P)$ or $N(B) \subseteq N(P)$.Hence, $N(S)$ is strongly irreducible.

Conversely, assume that $N(S)$ is strongly irreducible. Let $N(A)$, $N(B)$ and $N(P)$ be any neutrosophic ideals of $N(S)$ such that $N(A) \cap N(B) \subseteq N(P)$ implies $N(A) \subseteq N(P)$ or $N(B) \subseteq N(P)$. Now, let $N(A) \cap N(B) \subseteq N(P)$ but $N(A)N(B) = N(A) \cap N(B)$ and $N(A)N(B) \subseteq N(P)$ implies $N(A) \subseteq N(P)$ or $N(B) \subseteq N(P)$. Since, $N(P)$ is arbitrary neutrosophic ideal of $N(S)$ so, Every neutrosophic ideal of a neutrosophic intra-regular LA-semigroup $N(S)$ is prime. ■

Theorem 33 *Every neutrosophic ideal of a neutrosophic intra-regular LA-semigroup $N(S)$ is neutrosophic prime if and only if $N(S)$ is totally ordered under inclusion.*

Proof. Assume that every ideal of $N(S)$ is neutrosophic prime. Let $N(P)$ and $N(Q)$ be any neutrosophic ideals of $N(S)$, so, $N(P)N(Q) = N(P) \cap N(Q)$, where $N(P) \cap N(Q)$ is neutrosophic ideal of $N(S)$, so is neutrosophic prime, therefore $N(P)N(Q) \subseteq N(P) \cap N(Q)$, which implies that $N(P) \subseteq N(P) \cap N(Q)$ or $N(Q) \subseteq N(P) \cap N(Q)$, which implies that $N(P) \subseteq N(Q)$ or $N(Q) \subseteq N(P)$. Hence $N(S)$ is totally ordered under inclusion.

Conversely, assume that $N(S)$ is totally ordered under inclusion. Let $N(I)$, $N(J)$ and $N(P)$ be any neutrosophic ideals of $N(S)$ such that $N(I)N(J) \subseteq N(P)$. Now without loss of generality assume that $N(I) \subseteq N(J)$ then

$$\begin{aligned} N(I) &= [N(I)]^2 = N(I)N(I) \\ &\subseteq N(I)N(J) \subseteq N(P). \end{aligned}$$

Therefore either $N(I) \subseteq N(P)$ or $N(J) \subseteq N(P)$, which implies that $N(P)$ is neutrosophic prime. ■

Theorem 34 *The set of all neutrosophic ideals $N(I)_s$ of a neutrosophic intra-regular $N(S)$ with left identity $e + eI$, forms a semilattice structure.*

Proof. Let $N(A), N(B) \in N(I)_s$, since $N(A)$ and $N(B)$ are neutrosophic ideals of $N(S)$ so we have

$$\begin{aligned} [N(A)N(B)]N(S) &= [N(A)N(B)][N(S)N(S)] \\ &= [N(A)N(S)][N(B)N(S)] \\ &\subseteq N(A)N(B). \\ \text{Also } N(S)[N(A)N(B)] &= [N(S)N(S)][N(A)N(B)] \\ &= [N(S)N(A)][N(S)N(B)] \\ &\subseteq N(A)N(B). \end{aligned}$$

Thus $N(A)N(B)$ is a neutrosophic ideal of $N(S)$. Hence $N(I)_s$ is closed. Also, we have,

$$N(A)N(B) = N(A) \cap N(B) = N(B) \cap N(A) = N(B)N(A)$$

which implies that $N(I)_s$ is commutative, so is associative. Now $[N(A)]^2 = N(A)$, for all $N(A) \in N(I)_s$. Hence $N(I)_s$ is semilattice. ■

1.1 Neutrosophic Quasi Ideals

In this section, we introduce neutrosophic ideals in neutrosophic Abel Grassmann groupoids and we also introduce a new class namely neutrosophic intra-regular AG-groupoids. We characterize neutrosophic intra-regular Abel Grassmann groupoids using the properties of their neutrosophic quasi-ideals and neutrosophic semiprime ideals.

Now $(a+bI)^2 = a+bI$ implies $a+bI$ is idempotent and if holds for all $a+bI \in N(S)$ then $N(S)$ is called idempotent neutrosophic AG-groupoid.

An element $a+bI$ of a neutrosophic AG-groupoid $N(S)$ is called **neutrosophic intra-regular** if there exist (x_1+x_2I), $(y_1+y_2I) \in N(S)$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$ and $N(S)$ is called **neutrosophic intra-regular**, if every element of $N(S)$ is neutrosophic intra-regular.

A **neutrosophic AG-subgroupoid** of a neutrosophic AG-groupoid $N(S)$, is a non-empty neutrosophic subset $N(A)$ of $N(S)$ such that $(N(A))^2 \subseteq N(A)$.

A non-empty neutrosophic subset $N(A)$ of a neutrosophic AG-groupoid $N(S)$ is called a **neutrosophic left (right) ideal** of $N(S)$ if $N(S)N(A) \subseteq N(A)$ $(N(A)N(S) \subseteq N(A))$ and it is called a **neutrosophic two-sided ideal** if it is both neutrosophic left and a neutrosophic right ideal of $N(S)$.

A non-empty neutrosophic subset $N(A)$ of an AG-groupoid $N(S)$ is called a **neutrosophic generalized bi-ideal** of $N(S)$ if $(N(A)N(S))N(A) \subseteq N(A)$ and a neutrosophic AG-subgroupoid $N(A)$ of $N(S)$ is called a neutrosophic bi-ideal of $N(S)$ if $(N(A)N(S))N(A) \subseteq N(A)$.

A non-empty neutrosophic subset $N(A)$ of a neutrosophic AG-groupoid $N(S)$ is called **neutrosophic semiprime** if $(a+bI)^2 \in N(A)$ implies $(a+bI) \in N(A)$.

A neutrosophic subset $N(A)$ of a neutrosophic AG-groupoid $N(S)$ is called a neutrosophic generalized interior ideal of $N(S)$ if $[N(S)A]N(S) \subseteq N(A)$. A neutrosophic AG-subgroupoid $N(A)$ of a neutrosophic AG-groupoid $N(S)$ is called a neutrosophic interior ideal of $N(S)$ if $(N(S)A)N(S) \subseteq N(A)$.

A neutrosophic subset $N(A)$ of an AG-groupoid $N(S)$ is called a neutrosophic quasi-ideal of $N(S)$ if $(N(S)N(A)) \cap (N(A)N(S)) \subseteq N(A)$.

If $N(S)$ is a neutrosophic AG**-groupoid such that $N(S) = (N(S))^2$. Then it is simple to show that $N(S)(a+bI)$ becomes neutrosophic quasi-ideal and neutrosophic left ideal of $N(S)$. Also every neutrosophic right ideal of $N(S)$ becomes neutrosophic left ideal.

Example 35 *Let $S = \{1,2,3\}$ with binary operation "$\cdot$" given in the fol-*

lowing Cayley's table, is an intra-regular AG-groupoid with left identity 2.

$\cdot$	1	2	3
1	2	3	1
2	1	2	3
3	3	1	2

then $N(S) = \{1+1I, 1+2I, 1+3I, 2+1I, 2+2I, 2+3I, 3+1I, 3+2I, 3+3I\}$ is an example of neutrosophic intra-regular AG-groupoid under the operation "$*$" and has the following Cayley's table:

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$
$1+2I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$1+3I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$2+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$2+2I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$2+3I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$3+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$3+2I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$3+3I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$

Clearly $N(S)$ is a neutrosophic AG-groupoid and is non-associative because $[(1+1I)*(2+2I)]*(2+3I) \neq (1+1I)*[(2+2I)*(2+3I)]$ and $N(S)$ is intra-regular as

$(1+1I) = [(1+3I)(1+1I)^2](2+31)$, $(2+3I) = [(1+1I)(2+3I)^2](3+1I)$, $(3+1I) = [(2+3I)(3+1I)^2](3+3I)$ etc.

Lemma 36 *If a neutrosophic AG-groupoid $N(S)$ contains left identity $e+eI$ then the following conditions hold.*

(i) $N(S)N(L) = N(L)$ for every neutrosophic left ideal $N(L)$ of $N(S)$.

(ii) $N(R)N(S) = N(R)$ for every neutrosophic right ideal $N(R)$ of $N(S)$.

Theorem 37 *For neutrosophic intra-regular AG-groupoid with left identity $e+eI$, the following statements are equivalent.*

(i) $N(A)$ is a neutrosophic left ideal of $N(S)$.

(ii) $N(A)$ is a neutrosophic right ideal of $N(S)$.

(iii) $N(A)$ is a neutrosophic ideal of $N(S)$.

(iv) $N(A)$ is a neutrosophic bi-ideal of $N(S)$.

(v) $N(A)$ is a neutrosophic generalized bi-ideal of $N(S)$.

(vi) $N(A)$ is a neutrosophic interior ideal of $N(S)$.

(vii) $N(A)$ is a neutrosophic quasi-ideal of $N(S)$.

(viii) $N(A)N(S) = N(A)$ and $N(S)N(A) = N(A)$.

Proof. $(i) \Rightarrow (viii)$

Let $N(A)$ be a neutrosophic left ideal of $N(S)$. Thus, $N(S)N(A) = N(A)$. Now let $(a_1 + a_2I) \in N(A)$ and $(s_1 + s_2I) \in N(S)$, since $N(S)$ is a neutrosophic intra-regular

AG-groupoid, so there exist $(x_1 + x_2I)$, $(y_1 + y_2I) \in N(S)$ such that $(a_1 + a_2I) = [(x_1 + x_2I)(a_1 + a_2I)^2](y_1 + y_2I)$, therefore by (1), we have

$$\begin{aligned}
(a_1 + a_2I)(s_1 + s_2I) &= [\{(x_1 + x_2I)(a_1 + a_2I)^2\}(y_1 + y_2I)](s_1 + s_2I) \\
&= [\{(x_1 + x_2I)\{(a_1 + a_2I)(a_1 + a_2I)\}\}(y_1 + y_2I)](s_1 + s_2I) \\
&\in [\{N(S)\{N(A)N(A)\}\}N(S)]N(S) \\
&\subseteq [\{N(S)\{N(S)N(A)\}\}N(S)]N(S) \\
&\subseteq [\{N(S)N(A)\}N(S)]N(S) \\
&= [N(S)N(S)][N(S)N(A)] \\
&= N(S)[N(S)N(A)] \subseteq N(S)N(A) = N(A).
\end{aligned}$$

which implies that $N(A)$ is a neutrosophic right ideal of $N(S)$, thus $N(A)N(S) = N(S)$.

$(viii) \Rightarrow (vii)$

Let $N(A)N(S) = N(A)$ and $N(S)N(A) = N(A)$ then $N(A)N(S) \cap N(S)N(A) = N(A)$, which clearly implies that $N(A)$ is a neutrosophic quasi-ideal of $N(S)$.

$(vii) \Rightarrow (vi)$

Let $N(A)$ be a quasi-ideal of $N(S)$ and $[(s_1 + s_2I)(a_1 + a_2I)](s_1 + s_2I)$ belongs to $[N(S)N(A)]N(S)$. Since $N(S)$ is neutrosophic intra-regular AG-groupoid so there exist $(x_1 + x_2I)$, $(y_1 + y_2I)$, $(p_1 + p_2I)$, $(q_1 + q_2I) \in N(S)$ such that $(s_1 + s_2I) = [(x_1 + x_2I)(s_1 + s_2I)^2](y_1 + y_2I)$ and $(a_1 + a_2I) = [(p_1 + p_2I)(a_1 + a_2I)^2](q_1 + q_2I)$. Therefore using (2), (4), (3) and (1), we have

$$\begin{aligned}
& [(s_1 + s_2I)(a_1 + a_2I)](s_1 + s_2I) \\
=\ & [(s_1 + s_2I)(a_1 + a_2I)][\{(x_1 + xI)(s_1 + s_2I)^2\}(y_1 + y_2I)] \\
=\ & [\{(s_1 + s_2I)\{(x_1 + x_2I)(s_1 + s_2I)^2\}\}][(a_1 + a_2I)(y_1 + y_2I)] \\
=\ & (a_1 + a_2I)[\{(s_1 + s_2I)\{(x_1 + x_2I)(s_1 + s_2I)^2\}\}(y_1 + y_2I)] \\
\in\ & N(A)N(S).
\end{aligned}$$

and

$$\begin{aligned}
& [(s_1+s_2I)(a_1+a_2I)](s_1+s_2I) \\
= \quad & [(s_1+s_2I)\{\{(p_1+p_2I)(a_1+a_2I)^2\}(q_1+q_2I)\}](s_1+s_2I) \\
= \quad & [\{(p_1+p_2I)(a_1+a_2I)^2\}\{(s_1+s_2I)(q_1+q_2I)\}](s_1+s_2I) \\
= \quad & [\{(p_1+p_2I)\{(a_1+a_2I)(a_1+a_2I)\}\}\{(s_1+s_2I)(q_1+q_2I)\}](s_1+s_2I) \\
= \quad & [\{(a_1+a_2I)\{(p_1+p_2I)(a_1+a_2I)\}\}\{(s_1+s_2I)(q_1+q_2I)\}](s_1+s_2I) \\
= \quad & [\{(q_1+q_2I)(s_1+s_2I)\}\{\{(p_1+p_2I)(a_1+a_2I)\}(a_1+a_2I)\}](s_1+s_2I) \\
= \quad & [\{(p_1+p_2I)(a_1+a_2I)\}\{\{(q_1+q_2I)(s_1+s_2I)\}(a_1+a_2I)\}](s_1+s_2I) \\
= \quad & [\{(a_1+a_2I)\{(q_1+q_2I)(s_1+s_2I)\}\}\{(a_1+a_2I)(p_1+p_2I)\}](s_1+s_2I) \\
= \quad & [(a_1+a_2I)\{\{(a_1+a_2I)\{(q_1+q_2I)(s_1+s_2I)\}\}(p_1+p_2I)\}](s_1+s_2I) \\
= \quad & [(s_1+s_2I)\{\{(a_1+a_2I)\{(q_1+q_2I)(s_1+s_2I)\}\}(p_1+p_2I)\}](a_1+a_2I) \\
\in \quad & N(S)N(A) \subseteq N(A).
\end{aligned}$$

which shows that $N(A)$ is a neutrosophic interior ideal of $N(S)$.
$(vi) \Rightarrow (v)$
Let $N(A)$ be a neutrosophic interior ideal of a neutrosophic intra-regular AG-groupoid $N(S)$
and $[(a_1+a_2I)(s_1+s_2I)](a_1+a_2I) \in [N(A)N(S)]N(A)$. Now using (4) and (1), we get

$$\begin{aligned}
& [(a_1+a_2I)(s_1+s_2I)](a_1+a_2I) \\
= \quad & [(a_1+a_2I)(s_1+s_2I)][\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)] \\
= \quad & [(x_1+x_2I)(a_1+a_2I)^2][\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)] \\
= \quad & [(x_1+x_2I)\{(a_1+a_2I)(a_1+a_2I)\}][\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)] \\
= \quad & [\{\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)\}\{(a_1+a_2I)(a_1+a_2I)\}](x_1+x_2I) \\
= \quad & [(a_1+a_2I)\{\{\{(a_1+a_2I)(s_1+s_2I)\}(y_1+y_2I)\}(a_1+a_2I)\}](x_1+x_2I) \\
= \quad & [(a_1+a_2I)\{\{(a_1+a_2I)(y_1+y_2I)\}\{(a_1+a_2I)(s_1+s_2I)\}\}](x_1+x_2I) \\
= \quad & [\{(a_1+a_2I)(y_1+y_2I)\}\{(a_1+a_2I)\{(a_1+a_2I)(s_1+s_2I)\}\}](x_1+x_2I) \\
= \quad & [\{\{(a_1+a_2I)\{(a_1+a_2I)(s_1+s_2I)\}\}(y_1+y_2I)\}(a_1+a_2I)](x_1+x_2I) \\
\in \quad & [N(S)N(A)]N(S) \subseteq N(A).
\end{aligned}$$

$(v) \Rightarrow (iv)$
Let $N(A)$ be a neutrosophic generalized bi-ideal of $N(S)$. Let $a_1+a_2I \in N(A)$, and since $N(S)$ is neutrosophic intra-regular AG-groupoid so there exist (x_1+x_2I), (y_1+y_2I) in $N(S)$ such that $a_1+a_2I = [(x_1+x_2I)(a_1+

$a_2I)^2](y_1 + y_2I)$, then using (3) and (4), we have

$$\begin{aligned}
& (a_1 + a_2I)(a_1 + a_2I) \\
= \;& [\{(x_1 + x_2I)(a_1 + a_2I)^2\}(y_1 + y_2I)](a_1 + a_2I) \\
= \;& [\{(x_1 + x_2I)(a_1 + a_2I)^2\}\{(e + eI)(y_1 + y_2I)\}](a_1 + a_2I) \\
= \;& [\{(y_1 + y_2I)(e_1 + e_2I)\}\{\big(a_1 + a_2I)^2(x_1 + x_2I\big)\}](a_1 + a_2I) \\
= \;& [(a_1 + a_2I)^2\{\{(y_1 + y_2I)(e + eI)\}(x_1 + x_2I)\}](a_1 + a_2I) \\
= \;& [\{(a_1 + a_2I)(a_1 + a_2I)\}\{\{(y_1 + y_2I)(e + eI)\}(x_1 + x_2I)\}](a_1 + a_2I) \\
= \;& [\{(x_1 + x_2I)\{(y_1 + y_2I)(e + eI)\}\}\{(a_1 + a_2I)(a_1 + a_2I)\}](a_1 + a_2I) \\
= \;& [(a_1 + a_2I)\{\{(x_1 + x_2I)\{(y_1 + y_2I)(e + eI)\}\}(a_1 + a_2I)\}](a_1 + a_2I) \\
\in \;& [N\,(A)N(S)]N(A) \subseteq N(A).
\end{aligned}$$

Hence $N(A)$ is a neutrosophic bi-ideal of $N(S)$.

$(iv) \Rightarrow (iii)$

Let $N(A)$ be any neutrosophic bi-ideal of $N(S)$ and let $(a_1 + a_2I)(s_1 + s_2I) \in N(A)N(S)$. Since $N(S)$ is neutrosophic intra-regular AG-groupoid, so there exist $(x_1 + x_2I)$, $(y_1 + y_2I) \in N(S)$ such that $(a_1 + a_2I) = [(x_1 + x_2I)(a_1 + a_2I)^2](y_1 + y_2I)$. Therefore using (1), (3), (4) and (2), we have

$$
\begin{aligned}
& (a_1 + a_2I)(s_1 + s_2I) \\
= \;& [\{(x_1 + x_2I)(a_1 + a_2I)^2\}(y_1 + y_2I)](s_1 + s_2I) \\
= \;& [(s_1 + s_2I)(y_1 + y_2I)][(x_1 + x_2I)(a_1 + a_2I)^2] \\
= \;& [(a_1 + a_2I)^2(x_1 + x_2I)][(y_1 + y_2I)(s_1 + s_2I)] \\
= \;& [\{\{(y_1 + y_2I)(s_1 + s_2I)\}(x_1 + x_2I)\}(a_1 + a_2I)^2] \\
= \;& [\{(y_1 + y_2I)(s_1 + s_2I)\}(x_1 + x_2I)][(a_1 + a_2I)(a_1 + a_2I)] \\
= \;& [(a_1 + a_2I)(a_1 + a_2I)][(x_1 + x_2I)\{(y_1 + y_2I)(s_1 + s_2I)\}] \\
= \;& [\{(x_1 + x_2I)\{(y_1 + y_2I)(s_1 + s_2I)\}\}(a_1 + a_2I)](a_1 + a_2I) \\
= \;& \{(x_1 + x_2I)\{(y_1 + y_2I)(s_1 + s_2I)\}\}\{\{(x_1 + x_2I)(a_1 + a_2I)^2\} \\
& (y_1 + y_2I)\}(a_1 + a_2I) \\
= \;& [\{(x_1 + x_2I)(a_1 + a_2I)^2\}\{\{(x_1 + x_2I)\{(y_1 + y_2I)(s_1 + s_2I)\}\} \\
& (y_1 + y_2I)\}](a_1 + a_2I) \\
= \;& [\{(y_1 + y_2I)\{(x_1 + x_2I)\{(y_1 + y_2I)(s_1 + s_2I)\}\}\} \\
& \{(a_1 + a_2I)^2(x_1 + x_2I)\}](a_1 + a_2I) \\
= \;& [(a_1 + a_2I)^2\{\{(y_1 + y_2I)\{(x_1 + x_2I)\{(y_1 + y_2I)(s_1 + s_2I)\}\}\} \\
& (x_1 + x_2I)\}](a_1 + a_2I) \\
= \;& [\{(a_1 + a_2I)(a_1 + a_2I)\}\{\{(y_1 + y_2I)\{(x_1 + x_2I) \\
& \{(y_1 + y_2I)(s_1 + s_2I)\}\}\}(x_1 + x_2I)\}](a_1 + a_2I) \\
= \;& [\{(x_1 + x_2I)\{(y_1 + y_2I)\{(x_1 + x_2I)\{(y_1 + y_2I)(s_1 + s_2I)\}\}\}\} \\
& \{(a_1 + a_2I)(a_1 + a_2I)\}](a_1 + a_2I) \\
\in \;& [N(A)N(S)]N(A) \subseteq N(A).
\end{aligned}
$$

$$
\begin{aligned}
& (s_1+s_2I)(a_1+a_2I) \\
= \;& (s_1+s_2I)[\{(x_1+x_2I)(a_1+a_2I)^2\}(y_1+y_2I)] \\
= \;& [(x_1+x_2I)(a_1+a_2I)^2][(s_1+s_2I)(y_1+y_2I)] \\
= \;& [(x_1+x_2I)\{(a_1+a_2I)(a_1+a_2I)\}][(s_1+s_2I)(y_1+y_2I)] \\
= \;& [(a_1+a_2I)\{(x_1+x_2I)(a_1+a_2I)\}][(s_1+s_2I)(y_1+y_2I)] \\
= \;& [\{(s_1+s_2I)(y_1+y_2I)\}\{(x_1+x_2I)(a_1+a_2I)\}](a_1+a_2I) \\
= \;& [\{(a_1+a_2I)(x_1+x_2I)\}\{(y_1+y_2I)(s_1+s_2I)\}](a_1+a_2I) \\
= \;& [\{\{(y_1+y_2I)(s_1+s_2I)\}(x_1+x_2I)\}(a_1+a_2I)](a_1+a_2I) \\
= \;& [\{\{(y_1+y_2I)(s_1+s_2I)\}(x_1+x_2I)\}\{\{(x_1+x_2I) \\
& (a_1+a_2I)^2\}(y_1+y_2I)\}](a_1+a_2I) \\
= \;& [\{\{(y_1+y_2I)(s_1+s_2I)\}\{(x_1+x_2I)(a_1+a_2I)^2\}\} \\
& \{(x_1+x_2I)(y_1+y_2I)\}](a_1+a_2I) \\
= \;& [\{\{(a_1+a_2I)^2(x_1+x_2I)\}\{(s_1+s_2I)(y_1+y_2I)\}\} \\
& \{(x_1+x_2I)(y_1+y_2I)\}](a_1+a_2I) \\
= \;& [\{\{\{(a_1+a_2I)(a_1+a_2I)\}(x_1+x_2I)\}\{(s_1+s_2I)(y_1+y_2I)\}\} \\
& \{(x_1+x_2I)(y_1+y_2I)\}](a_1+a_2I) \\
= \;& [\{\{(x_1+x_2I)(y_1+y_2I)\}\{(s_1+s_2I)(y_1+y_2I)\}\} \\
& \{\{(a_1+a_2I)(a_1+a_2I)\}(x_1+x_2I)\}](a_1+a_2I) \\
= \;& [\{\{(x_1+x_2I)(y_1+y_2I)\}\{(a_1+a_2I)(a_1+a_2I)\}\} \\
& \{\{(s_1+s_2I)(y_1+y_2I)\}(x_1+x_2I)\}](a_1+a_2I) \\
= \;& [\{\{(a_1+a_2I)(a_1+a_2I)\}\{(y)(x_1+x_2I)\}\} \\
& \{\{(s_1+s_2I)(y_1+y_2I)\}(x_1+x_2I)\}](a_1+a_2I) \\
= \;& [\{\{\{(s_1+s_2I)(y_1+y_2I)\}(x_1+x_2I)\} \\
& \{(y_1+y_2I)(x_1+x_2I)\}\}\{(a_1+a_2I)(a_1+a_2I)\}](a_1+a_2I) \\
= \;& [(a_1+a_2I)\{\{\{\{(s_1+s_2I)(y_1+y_2I)\}(x_1+x_2I)\} \\
& \{(y_1+y_2I)(x_1+x_2I)\}\}(a_1+a_2I)\}](a_1+a_2I) \\
\in \;& [N(A)N(S)]N(A) \\
\subseteq \;& N(A).
\end{aligned}
$$

Therefore $N(A)$ is a neutrosophic ideal of $N(S)$.
$(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are obvious. ■

Lemma 38 *Intersection of two neutrosophic ideals of a neutrosophic AG-groupoid $N(S)$ is a neutrosophic ideal of $N(S)$ too.*
Proof. *Let $N(A)$ and $N(B)$ are two neutrosophic ideals of $N(S)$ implies $N(S)N(A) \subseteq N(A)$, $N(A)N(S) \subseteq N(A)$ and $N(S)N(B) \subseteq N(B)$,*

$N(B)N(S) \subseteq N(B)$ respectively. Now

$$\begin{aligned}
N(S)[N(A)\cap N(B)] &= N(S)N(A)\cap N(S)N(B)\\
&\subseteq N(A)\cap N(B).\ \textit{And}\\
[N(A)\cap N(B)]N(S) &= N(A)N(S)\cap N(B)N(S)\\
&\subseteq N(A)\cap N(B).
\end{aligned}$$

Hence $N(A)\cap N(B)$ is a neutrosophic ideal of $N(S)$. ■

Lemma 39 *Product of a neutrosophic bi-ideal and a neutrosophic subset of a neutrosophic AG-groupoid $N(S)$ with left identity is a neutrosophic bi-ideal.*

Proof. *Let $N(A)$ and $N(B)$ be two neutrosophic bi-ideal and a neutrosophic subset of $N(S)$ respectively. Now by using neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have*

$$\begin{aligned}
[\{N(A)N(B)\}N(S)]\{N(A)N(B)\} &= [\{N(S)N(B)\}N(A)]\{N(A)N(B)\}\\
&\subseteq [\{N(S)N(S)\}N(A)]\{N(A)N(B)\}\\
&\subseteq [N(S)N(A)]\{N(A)N(B)\}\\
&= \{N(B)N(A)\}[N(A)N(S)]\\
&= \{[N(A)N(S)]N(A)\}N(B)\\
&\subseteq N(A)N(B).
\end{aligned}$$

Hence $N(A)N(B)$ is a neutrosophic bi-ideal of $N(S)$. ■

Theorem 40 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*

(i) $N(S)$ is neutrosophic intra-regular.

(ii) $N(R)\cap N(L) = N(R)N(L)$, for every neutrosophic semiprime right ideal $N(R)$ and every neutrosophic left ideal $N(L)$.

(iii) $N(A) = [N(A)N(S)]N(A)$, for every neutrosophic quasi-ideal $N(A)$.

Proof. *$(i)\Rightarrow(iii)$: Let $N(A)$ be a neutrosophic quasi ideal of $N(S)$ then $N(A)$ is a neutrosophic ideal of $N(S)$, thus $[N(A)N(S)]N(A)\subseteq N(A)$.*

Now let $a+bI\in N(A)$, and since $N(S)$ is neutrosophic intra-regular so there exist elements (x_1+x_2I), (y_1+y_2I) in $N(S)$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Now by using neutrosophic medial law with left identity $e+eI$, neutrosophic left invertive law, neutrosophic medial law *and neutrosophic paramedial law we have*

$$
\begin{aligned}
& a+bI \\
= \; & [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= \; & [(x_1+x_2I)\{(a+bI)(a+bI)\}](y_1+y_2I) \\
= \; & [(a+bI)\{(x_1+x_2I)(a+bI)\}](y_1+y_2I) \\
= \; & [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}](a+bI) \\
= \; & [(y_1+y_2I)\{(x_1+x_2I)\{\{(x_1+x_2I)(a+bI)^2\}(y_1+y_2I)\}\}](a+bI) \\
= \; & [(y_1+y_2I)\{\{(x_1+x_2I)(a+bI)^2\}\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI) \\
= \; & [\{(x_1+x_2I)(a+bI)^2\}\{(y_1+y_2I)\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI) \\
= \; & [\{(x_1+x_2I)\{(a+bI)(a+bI)\}\}\{(y_1+y_2I)\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI) \\
= \; & [\{(a+bI)\{(x_1+x_2I)(a+bI)\}\}\{(y_1+y_2I)\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI) \\
= \; & [\{(a+bI)(y_1+y_2I)\}\{\{(x_1+x_2I)(a+bI)\}\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI) \\
= \; & [\{(x_1+x_2I)(a+bI)\}\{\{(a+bI)y\}\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI) \\
= \; & [\{(x_1+x_2I)(a+bI)\}\{\{(a+bI)(x_1+x_2I)\}(y_1+y_2I)^2\}](a+bI) \\
= \; & [\{(y_1+y_2I)^2\{(a+bI)(x_1+x_2I)\}\}\{(a+bI)(x_1+x_2I)\}](a+bI) \\
= \; & [(a+bI)\{\{(y_1+y_2I)^2\{(a+bI)(x_1+x_2I)\}\}(x_1+x_2I)\}](a+bI) \\
\in \; & [N(A)N(S)]N(A).
\end{aligned}
$$

Hence $N(A) = [N(A)\,N(S)]N(A)$.

$(iii) \Rightarrow (ii)$: *Clearly* $N(R)N(L) \subseteq N(R) \cap N(L)$ *holds. Now*

$$
\begin{aligned}
& N(S)[N(R)\cap N(L)] \cap [N(R)\cap N(L)]N(S) \\
= \; & N(S)N(R) \cap N(S)N(L) \cap N(R)N(S) \cap N(L)N(S) \\
= \; & N(R)N(S) \cap N(S)N(L) \cap N(S)N(R) \cap N(L)N(S) \\
\subseteq \; & N(R) \cap N(L) \cap [N(S)N(R) \cap N(L)N(S)] \\
\subseteq \; & N(R) \cap N(L). \textit{ And} \\
& N(R)\cap N(L) \\
= \; & [\{N(R)\cap N(L)\}N(S)][N(R)\cap N(L)] \\
= \; & [N(R)N(S) \cap N(L)N(S)][N(R)\cap N(L)] \\
\subseteq \; & [N(R)\cap N(L)N(S)][N(R)\cap N(L)] \\
\subseteq \; & N(R)N(L).
\end{aligned}
$$

Hence $N(R) \cap N(L) = N(R)N(L)$.

$(ii) \Rightarrow (i)$: *Assume that* $N(R) \cap N(L) = N(R)N(L)$ *for every neutrosophic right ideal* $N(R)$ *and every neutrosophic left ideal* $N(L)$ *of* $N(S)$. *Since* $(a+bI)^2 \in (a+bI)^2N(S)$, *which is a neutrosophic right ideal of* $N(S)$ *and as by given assumption* $(a+bI)^2N(S)$ *is neutrosophic semiprime which implies that* $a+bI \in (a+bI)^2N(S)$. *Now clearly* $(a+bI) \cup N(S)(a+bI)$

is a neutrosophic principal left ideal, therefore

$$\begin{aligned} a+bI &\in [(a+bI)\cup N(S)(a+bI)]\cap(a+bI)^2N(S) \\ &\subseteq N(S)[(a+bI)^2N(S)] \\ &= [N(S)N(S)][(a+bI)^2N(S)] \\ &= [N(S)(a+bI)^2][N(S)N(S)] \\ &= [N(S)(a+bI)^2]N(S). \end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 41 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*
(i) $N(S)$ is neutrosophic intra-regular.
(ii) For a neutrosophic ideal $N(I)$ and neutrosophic quasi-ideal $N(Q)$, $N(I)\cap N(Q)=N(I)N(Q)$ and $N(I)$ is neutrosophic semiprime.
(iii) For neutrosophic quasi-ideals $N(Q_1)$ and $N(Q_2)$, $N(Q_1)\cap N(Q_2)$ $=N(Q_1)$
$N(Q_2)$ and $N(Q_1)$ and $N(Q_2)$ are neutrosophic semiprime.

Proof. $(i)\Longrightarrow(iii)$: Let $N(Q_1)$ and $N(Q_2)$ be a quasi-ideal of $N(S)$. Now $N(Q_1)$ and $N(Q_2)$ become ideals of $N(S)$. Therefore $N(Q_1)Q_2\subseteq N(Q_1)\cap N(Q_2)$. Now let $a+bI\in N(Q_1)\cap N(Q_2)$ which implies that $a+bI\in N(Q_1)$ and $a+bI\in N(Q_2)$. For $a+bI\in N(S)$ there exists (x_1+x_2I), (y_1+y_2I) in $N(S)$ such that $a+bI=[(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Now by using (i), neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\ &= [(x_1+x_2I)\{(a+bI)(a+bI)\}](y_1+y_2I) \\ &= [(a+bI)\{(x_1+x_2I)(a+bI)\}](y_1+y_2I) \\ &= [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}](a+bI) \\ &\in [N(S)\{N(S)N(Q_1)\}]N(Q_2) \\ &\subseteq [N(S)N(Q_1)]N(Q_2) \\ &\subseteq N(Q_1)N(Q_2). \end{aligned}$$

This implies that $N(Q_1)\cap N(Q_2)\subseteq N(Q_1)N(Q_2)$. Hence $N(Q_1)\cap N(Q_2)=N(Q_1)N(Q_2)$. Now we will show that $N(Q_1)$ and $N(Q_2)$ are neutrosophic semiprime. For this let $(a+bI)^2\in N(Q_1)$, by (i) $a+bI=[(x_1+x_2I)(a+bI)^2](y_1+y_2I)\in[N(S)N(Q_1)]N(S)\subseteq N(Q_1)$. Similarly $N(Q_2)$ is neutrosophic semiprime.
$(iii)\Longrightarrow(ii)$ is obvious.
$(ii)\Longrightarrow(i)$: Obviously $N(S)(a+bI)$ is a neutrosophic quasi-ideal contains $a+bI$ and $N(S)(a+bI)^2$

is a neutrosophic ideal contains $(a+bI)^2$. By (ii) $N(S)(a+bI)^2$ is neutrosophic semiprime so $a+bI \in N(S)(a+bI)^2$. Therefore by (ii) we get

$$\begin{aligned} a+bI &\in N(S)(a+bI)^2 \cap N(S)(a+bI) \\ &= [N(S)(a+bI)^2][N(S)(a+bI)] \subseteq [N(S)(a+bI)^2]N(S). \end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 42 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*
(i) $N(S)$ is neutrosophic intra-regular.
(ii) For neutrosophic quasi-ideals $N(Q_1)$ and $N(Q_2)$, $N(Q_1) \cap N(Q_2) = [N(Q_1)N(Q_2)]N(Q_1)$.

Proof. $(i) \Longrightarrow (ii)$: Let $N(Q_1)$ and $N(Q_2)$ be neutrosophic quasi-ideals of $N(S)$. Now $N(Q_1)$ and $N(Q_2)$ become neutrosophic ideals of $N(S)$. Therefore

$$\begin{aligned} [N(Q_1)N(Q_2)]N(Q_1) &\subseteq [N(Q_1)N(S)]N(Q_1) \subseteq N(Q_1) \text{ and} \\ [N(Q_1)N(Q_2)]N(Q_1) &\subseteq [N(S)N(Q_2)]N(S) \subseteq N(Q_2). \end{aligned}$$

This implies that $[N(Q_1)N(Q_2)]N(Q_1) \subseteq N(Q_1) \cap N(Q_2)$. We can easily see that $N(Q_1) \cap N(Q_2)$ becomes an ideal. Now we get,

$$\begin{aligned} N(Q_1) \cap N(Q_2) &= [N(Q_1) \cap N(Q_2)]^2 \\ &= [N(Q_1) \cap N(Q_2)]^2[N(Q_1) \cap N(Q_2)] \\ &= [\{N(Q_1) \cap N(Q_2)\}\{N(Q_1) \cap N(Q_2)\}][N(Q_1) \cap N(Q_2)] \\ &\subseteq [N(Q_1)N(Q_2)]N(Q_1). \end{aligned}$$

Thus $N(Q_1) \cap N(Q_2) \subseteq [N(Q_1)N(Q_2)]N(Q_1)$. Hence $N(Q_1) \cap N(Q_2) = [N(Q_1)N(Q_2)]N(Q_1)$.

$(ii) \Longrightarrow (i)$: Let $N(Q)$ be a neutrosophic quasi-ideal of $N(S)$ then by (ii), we get $N(Q) = N(Q) \cap N(Q) = [N(Q)N(Q)]N(Q) \subseteq (N(Q))^2 N(Q) \subseteq N(Q)N(Q) = (N(Q))^2$. This implies that $N(Q) \subseteq (N(Q))^2$ therefore $(N(Q))^2 = N(Q)$. Since $N(S)(a+bI)$ is a neutrosophic quasi-ideal, therefore

$$a+bI \in N(S)(a+bI) = [N(S)(a+bI)]^2 = N(S)(a+bI)^2 = [N(S)(a+bI)^2]N(S).$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 43 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*
(i) $N(S)$ is neutrosophic intra-regular.
(ii) For neutrosophic quasi-ideal $N(Q)$ and neutrosophic ideal $N(J)$, $N(Q) \cap N(J) \subseteq N(J)N(Q)$ and $N(J)$ is neutrosophic semiprime.

Proof. $(i) \Longrightarrow (ii)$: Let us suppose that $N(Q)$ is a neutrosophic quasi-ideal and $N(J)$ is a neutrosophic

ideal of $N(S)$. Let $a+bI \in N(Q) \cap N(J)$ implies $a+bI \in N(Q)$ and $a+bI \in N(J)$. For each $a+bI \in N(S)$ there exists (x_1+x_2I), (y_1+y_2I) in $N(S)$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Then by using (i) and neutrosophic left invertive law we have

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\ &= [(x_1+x_2I)\{(a+bI)(a+bI)\}](y_1+y_2I) \\ &= [(a+bI)\{(x_1+x_2I)(a+bI)\}](y_1+y_2I) \\ &= [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}](a+bI) \\ &\in [N(S)\{N(S)N(J)\}]N(Q) \subseteq N(J)N(Q). \end{aligned}$$

Therefore $N(Q) \cap N(J) \subseteq N(J)N(Q)$. Next let $(a+bI)^2 \in N(J)$. Thus $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \in [N(S)N(J)]N(S) \subseteq N(J)$. Hence $N(J)$ is neutrosophic semiprime.

$(ii) \Longrightarrow (i)$: Since $N(S)(a+bI)$ is a neutrosophic quasi and $(a+bI)^2 N(S)$ is a a neutrosophic ideal

of $N(S)$ containing $a+bI$ and $(a+bI)^2$ respectively. Thus by (ii) $N(J)$ is neutrosophic semiprime so $a+bI \in (a+bI)^2 N(S)$. Therefore by hypothesis. neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned} a+bI &\in N(S)(a+bI) \cap (a+bI)^2 N(S) \\ &\subseteq [N(S)(a+bI)][(a+bI)^2 N(S)] \\ &= [N(S)(a+bI)^2][(a+bI)N(S)] \\ &\subseteq [N(S)(a+bI)^2]N(S). \end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 44 *If $N(A)$ is a neutrosophic interior ideal of neutrosophic AG-groupoid $N(S)$ with left identity, then $(N(A))^2$ is also neutrosophic interior ideal.*

Proof. Using neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned} [N(S)\{N(A)\}^2]N(S) &= [\{N(S)\}\{N(A)N(A)\}][N(S)N(S)] \\ &= [\{N(S)N(A)\}\{N(S)N(A)\}][N(S)N(S)] \\ &= [\{N(S)N(A)\}N(S)][\{N(S)N(A)\}N(S)] \\ &\subseteq N(A)N(A) = (N(A))^2. \end{aligned}$$

■

Theorem 45 *For $N(S)$ with left identity $e + eI$, the following conditions are equivalent.*
(i) $N(S)$ is neutrosophic intra-regular.
(ii) For neutrosophic quasi-ideal $N(Q)$, neutrosophic right ideal $N(R)$ and neutrosophic two sided ideal $N(I)$, $[N(Q) \cap N(R)] \cap N(I) \subseteq [N(Q)N(R)]N(I)$ and $N(R)$, $N(I)$ are neutrosophic semiprime.
(iii) For neutrosophic quasi-ideal $N(Q)$, neutrosophic right ideal $N(R)$ and neutrosophic right ideal $N(I)$, $[N(Q) \cap N(R)] \cap N(I) \subseteq [N(Q)R]N(I)$ and $N(R)$, $N(I)$ are neutrosophic semiprime.
(iv) For neutrosophic quasi-ideal $N(Q)$, neutrosophic right ideal $N(R)$ and neutrosophic interior ideal $N(I)$, $[N(Q) \cap N(R)] \cap N(I) \subseteq [N(Q)N(R)]N(I)$ and $N(R)$, $N(I)$ are neutrosophic semiprime.

Proof. $(i) \implies (iv)$: Let $a + bI \in [N(Q) \cap N(R)] \cap N(I)$. This implies that $a + bI \in N(Q)$, $a + bI \in N(R)$, $a + bI \in N(I)$. Since $N(S)$ is neutrosophic intra-regular therefore for each $a + bI \in N(S)$ there exists $x_1 + x_2I$, $y_1 + y_2I \in N(S)$ such that $a + bI = [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)$. Now by using (i), neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned}
& a + bI \\
= & [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
= & [(x_1 + x_2I)\{(a + bI)(a + bI)\}](y_1 + y_2I) \\
= & [(a + bI)\{(x_1 + x_2I)(a + bI)\}](y_1 + y_2I) \\
= & [(a + bI)\{(x_1 + x_2I)\{\{(x_1 + x_2I)(a + bI)^2\}(y_1 + y_2I)\}\}](y_1 + y_2I) \\
= & [(a + bI)\{\{(x_1 + x_2I)(a + bI)^2\}\{(x_1 + x_2I)(y_1 + y_2I)\}\}](y_1 + y_2I) \\
= & [(y_1 + y_2I)\{\{(x_1 + x_2I)(a + bI)^2\}\{(x_1 + x_2I)(y_1 + y_2I)\}\}](a + bI) \\
= & [(y_1 + y_2I)\{\{(x_1 + x_2I)\{(a + bI)(a + bI)\}\}\{(x_1 + x_2I)(y_1 + y_2I)\}\}](a + bI) \\
= & [(y_1 + y_2I)\{\{(a + bI)\{(x_1 + x_2I)(a + bI))\}\}\{(x_1 + x_2I)(y_1 + y_2I)\}\}](a + bI) \\
= & [\{(a + bI)\{(x_1 + x_2I)(a + bI)\}\}\{(y_1 + y_2I)\{(x_1 + x_2I)(y_1 + y_2I)\}\}](a + bI) \\
= & [\{\{(y_1 + y_2I)\{(x_1 + x_2I)(y_1 + y_2I)\}\}\{(x_1 + x_2I)(a + bI)\}\}(a + bI)](a + bI) \\
\in & [\{\{N(S)\{N(S)N(S)\}\}\{N(S)N(Q)\}\}N(R)]N(I) \subseteq [N(Q)N(R)]N(I).
\end{aligned}$$

Therefore, $[N(Q) \cap N(R)] \cap N(I) \subseteq [N(Q)N(R)]N(I)$. Next let $(a + bI)^2 \in N(R)$. Then using (1) and neutrosophic left invertive law, we get

$$\begin{aligned}
a + bI &= [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
&= [(x_1 + x_2I)\{(a + bI)(a + bI)\}](y_1 + y_2I) \\
&= [(a + bI)\{(x_1 + x_2I)(a + bI)\}](y_1 + y_2I) \\
&= [(y_1 + y_2I)\{(x_1 + x_2I)(a + bI)\}](a + bI) \in N(R).
\end{aligned}$$

This implies that $a + bI \in N(R)$. Similarly we can show that $N(I)$ is neutrosophic semiprime.

$(iv) \Longrightarrow (iii) \Longrightarrow (ii)$: are obvious.

$(ii) \Longrightarrow (i)$: We know that $N(S)(a+bI)$ is a neutrosophic quasi and $N(S)(a+bI)^2$ is neutrosophic right as well as neutrosophic two sided ideal of $N(S)(a+bI)^2$ and by (ii) $N(S)(a+bI)^2$ is neutrosophic semiprime so $a+bI \in N(S)(a+bI)^2$. Then by hypothesis and by using neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned}
a+bI &\in [N(S)(a+bI) \cap N(S)(a+bI)^2] \cap N(S)(a+bI)^2 \\
&= [\{N(S)(a+bI)\}\{N(S)(a+bI)^2\}]\{N(S)(a+bI)^2\} \\
&= [\{N(S)(a+bI)^2\}\{N(S)(a+bI)^2\}]\{N(S)(a+bI)\} \\
&\subseteq [\{N(S)(a+bI)^2\}N(S)]N(S) \\
&= [N(S)N(S)][N(S)(a+bI)^2] \\
&= [(a+bI)^2 N(S)][N(S)N(S)] \\
&= [N(S)(a+bI)^2]N(S).
\end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 46 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*

(i) $N(S)$ is neutrosophic intra-regular.

(ii) For every neutrosophic bi-ideal $N(B)$ and neutrosophic quasi-ideal $N(Q)$, $N(B) \cap N(Q) \subseteq N(B)N(Q)$.

(iii) For every neutrosophic generalized bi-ideal $N(B)$ and neutrosophic quasi-ideal $N(Q)$, $N(B) \cap N(Q) \subseteq N(B)N(Q)$.

Proof. $(i) \Longrightarrow (iii)$: Let $N(B)$ is a neutrosophic bi-ideal and $N(Q)$ is a neutrosophic quasi-ideal of $N(S)$. Let $a+bI \in N(B) \cap N(Q)$ which implies that $a+bI \in N(B)$ and $a+bI \in N(Q)$. Since $N(S)$ is neutrosophic intra-regular so for $a+bI \in N(S)$ there exists (x_1+x_2I), $(y_1+y_2I) \in N(S)$ such that $a+bI = [x(a+bI)^2]y$. Now, $N(B)$ and $N(Q)$ become ideals of $N(S)$. Then using (i), neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned}
a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
&= [(x_1+x_2I)\{(a+bI)(a+bI)\}](y_1+y_2I) \\
&= [(a+bI)\{(x_1+x_2I)(a+bI)\}](y_1+y_2I) \\
&= [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}](a+bI) \\
&\in [N(S)\{N(S)N(B)\}]N(Q) \subseteq N(B)N(Q).
\end{aligned}$$

Hence $N(B) \cap N(Q) \subseteq N(B)N(Q)$.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$: Using (ii) we get

$$\begin{aligned} a+bI &\in N(S)(a+bI)\cap N(S)(a+bI) \\ &\subseteq N(S)(a+bI)^2 = [N(S)(a+bI)^2]N(S). \end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 47 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*

(i) $N(S)$ is neutrosophic intra-regular.

(ii) For neutrosophic quasi-ideal $N(Q_1)$, neutrosophic two sided ideal $N(I)$ and neutrosophic quasi-ideal $N(Q_2)$, $[N(Q_1)\cap N(I)]\cap N(Q_2)\subseteq [N(Q_1)N(I)]N(Q_2)$ and $N(I)$ is neutrosophic semiprime.

(iii) For neutrosophic quasi-ideal $N(Q_1)$, neutrosophic right ideal $N(I)$ and neutrosophic quasi ideal $N(Q_2)$,$[N(Q_1)\cap N(I)]\cap N(Q_2)\subseteq [N(Q_1)N(I)]N(Q_2)$ and $N(I)$ is neutrosophic semiprime.

(iv) For neutrosophic quasi-ideal $N(Q_1)$, neutrosophic interior ideal $N(I)$ and neutrosophic quasi-ideal $N(Q_2)$, $[N(Q_1)\cap N(I)]\cap N(Q_2)\subseteq [N(Q_1)N(I)]N(Q_2)$ and $N(I)$ is neutrosophic semiprime.

Proof. $(i) \Longrightarrow (v)$: Let $N(Q_1)$ and $N(Q_2)$ be neutrosophic quasi-ideals and $N(I)$ be a neutrosophic interior ideal of $N(S)$ respectively. Let $a+bI\in [N(Q_1)\cap N(I)]\cap N(Q_2)$ this implies that $a+bI\in N(Q_1)$, $a+bI\in N(I)$ and $a+bI\in N(Q_2)$. For $a+bI\in N(S)$ there exists (x_1+x_2I), $(y_1+y_2I)\in N(S)$ such that $a+bI=[(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Thus, $N(Q_1), N(Q_2)$ and $N(I)$ become ideals of $N(S)$. Therefore by using (i), neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\ &= [(x_1+x_2I)\{(a+bI)(a+bI)\}](y_1+y_2I) \\ &= [(a+bI)\{(x_1+x_2I)(a+bI)\}](y_1+y_2I) \\ &= [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}](a+bI) \\ &= [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}][\{(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}\}(a+bI)] \\ &= [(a+bI)\{(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}\}][\{(x_1+x_2I)(a+bI)\}(y_1+y_2I)] \\ &\in [N(Q_1)\{N(S)\{N(S)N(I))\}\}][\{N(S)N(Q_2)\}N(S)] \\ &\subseteq [N(Q_1)N(I)]N(Q_2). \end{aligned}$$

Hence $[N(Q_1)\cap N(I)]\cap N(Q_2)\subseteq [N(Q_1)I]N(Q_2)$. Next let $(a+bI)^2\in N(I)$ then $a+bI=[(x_1+x_2I)(a+bI)^2](y_1+y_2I)=(N(I))^2\subseteq N(I)$ this implies that $a+bI\in N(I)$. So $N(I)$ is neutrosophic semiprime.

$(v)\Longrightarrow(iv)\Longrightarrow(iii)\Longrightarrow(ii)$ are obvious.

$(ii)\Longrightarrow(i)$: Since $N(S)(a+bI)$ is a neutrosophic quasi and $N(S)(a+bI)^2$ is a neutrosophic ideal of $N(S)$ containing $a+bI$. Also by (ii) $N(S)(a+bI)^2$

is neutrosophic semiprime so $a+bI \in N(S)(a+bI)^2$. Thus by using neutrosophic paramedial and neutrosophic medial laws, we have

$$\begin{aligned}
a+bI &\in [N(S)(a+bI) \cap N(S)(a+bI)^2] \cap N(S)(a+bI) \\
&\subseteq [\{N(S)(a+bI)\}\{N(S)(a+bI)^2\}][N(S)(a+bI)] \\
&= [\{(a+bI)^2 N(S)\}\{(a+bI)N(S)\}][N(S)(a+bI)] \\
&= [\{(a+bI)^2 N(S)\}\{N(S)N(S)\}][N(S)N(S)] \\
&= [\{(a+bI)^2 N(S)\}N(S)]N(S) \\
&= [\{N(S)N(S)\}(a+bI)^2)]N(S) = [N(S)(a+bI)^2]N(S).
\end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 48 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*
(i) $N(S)$ is neutrosophic intra-regular.
(ii) Every neutrosophic quasi-ideal is idempotent.
(iii) For neutrosophic quasi-ideals $N(A), N(B)$, $N(A) \cap N(B) = N(A)N(B) \cap N(B)N(A)$.

Proof. $(i) \Longrightarrow (iii)$: Let $N(A)$ and $N(B)$ be neutrosophic quasi-ideals of $N(S)$. Thus

$$\begin{aligned}
N(A)N(B) \cap N(B)N(A) &\subseteq N(A)N(B) \subseteq N(S)N(B) \subseteq N(B) \text{ and} \\
N(A)N(B) \cap N(B)N(A) &\subseteq N(B)N(A) \subseteq N(S)N(A) \subseteq N(A).
\end{aligned}$$

Hence $N(A)N(B) \cap N(B)N(A) \subseteq N(A) \cap N(B)$. Now let $a+bI \in N(A) \cap N(B)$ this implies that $a+bI \in N(A)$ and $a+bI \in N(B)$. Since $N(S)$ is neutrosophic intra-regular AG-groupoid so for

$a+bI$ in $N(S)$ there exists (x_1+x_2I), $(y_1+y_2I) \in N(S)$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$ and $(y_1+y_2I) = (u_1+u_2I)(v_1+v_2I)$ for some (u_1+u_2I), (v_1+v_2I) in $N(S)$. Then using (i), neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we

have

$$\begin{aligned}
a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
&= [(x_1+x_2I)\{(a+bI)(a+bI)\}](y_1+y_2I) \\
&= [(a+bI)\{(x_1+x_2I)(a+bI)\}][(u_1+u_2I)(v_1+v_2I)] \\
&= [(a+bI)(u_1+u_2I)][\{(x_1+x_2I)(a+bI)\}(v_1+v_2I)] \\
&\in [N(A)N(S)][\{N(S)N(B)\}N(S)] \subseteq N(A)N(B).
\end{aligned}$$

Similarly we can show that $a+bI \in N(B)N(A)$. Thus $N(A) \cap N(B) \subseteq N(A)N(B) \cap N(B)N(A)$. Therefore $N(A) \cap N(B) = N(A)N(B) \cap N(B)N(A)$.

$(iii) \Longrightarrow (ii)$: Let $N(Q)$ be a neutrosophic quasi-ideal of $N(S)$. Thus by (iii), $N(Q) \cap N(Q) = N(Q)N(Q) \cap N(Q)N(Q)$ implies $N(Q) = N(Q)N(Q)$.

$(ii) \Longrightarrow (i)$: Since $N(S)(a+bI)$ is a neutrosophic quasi-ideal of $N(S)$ contains $a+bI$ and by (ii),

it is idempotent therefore by using neutrosophic AG-groupoid laws, we have

$$\begin{aligned}
a+bI &\in N(S)(a+bI) \\
&= [N(S)(a+bI)]^2 \\
&= [N(S)(a+bI)][N(S)(a+bI)] \\
&= [\{N(S)(a+bI)\}(a+bI)]N(S) \\
&= [(a+bI)^2 N(S)]N(S) \\
&= [N(S)(a+bI)^2]N(S)
\end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 49 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*

(i) *$N(S)$ is neutrosophic intra-regular.*

(ii) *For neutrosophic bi-ideal $N(B)$, neutrosophic two sided ideal $N(I)$ and neutrosophic quasi-ideal $N(Q)$, $[N(B) \cap N(I)] \cap N(Q) \subseteq [N(B)N(I)]N(Q)$ and $N(I)$ is neutrosophic semiprime.*

(iii) *For neutrosophic bi-ideal $N(B)$, neutrosophic right ideal $N(I)$ and neutrosophic quasi-ideal $N(Q)$, $[N(B) \cap N(I)] \cap N(Q) \subseteq [N(B)N(I)]N(Q)$ and $N(I)$ is neutrosophic semiprime.*

(iv) *For neutrosophic generalized bi-ideal $N(B)$, neutrosophic interior ideal $N(I)$ and neutrosophic quasi-ideal $N(Q)$, $[N(B) \cap N(I)] \cap N(Q) \subseteq [N(B)N(I)]N(Q)$ and $N(I)$ is neutrosophic semiprime.*

Proof. $(i) \Longrightarrow (iv)$: Let $N(B)$ be a neutrosophic generalized bi-ideal, $N(I)$ be a neutrosophic interior ideal and $N(Q)$ be a neutrosophic quasi-ideal of $N(S)$ respectively. Let $a+bI \in [N(B) \cap N(I)] \cap N(Q)$. this implies that $a+bI \in N(B)$, $a+bI \in N(I)$ and $a+bI \in N(Q)$. Since $N(S)$ is neutrosophic intra-regular so for $a+bI \in N(S)$ there exists (x_1+x_2I), $(y_1+y_2I) \in N(S)$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Now, $N(B)$, $N(I)$

and $N(Q)$ become neutrosophic ideals of $N(S)$. Therefore by using (i), neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$
\begin{aligned}
& a+bI \\
= \; & [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= \; & [(x_1+x_2I)\{(a+bI)(a+bI)\}](y_1+y_2I) \\
= \; & [(a+bI)\{(x_1+x_2I)(a+bI)\}](y_1+y_2I) \\
= \; & [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}](a+bI) \\
= \; & [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}][\{(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}\}(a+bI)] \\
= \; & [(a+bI)\{(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}\}][\{(x_1+x_2I)(a+bI)\}(y_1+y_2I)] \\
\in \; & [N(B)\{N(S)\{N(S)N(I)\}\}][\{N(S)N(Q)\}N(S)] \\
\subseteq \; & [N(B)N(I)]N(Q).
\end{aligned}
$$

Therefore, $[N(B)\cap N(I)]\cap N(Q) \subseteq [N(B)N(I)]N(Q)$. Next let $(a+bI)^2 \in N(I)$ and since $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I) = (N(I))^2 \subseteq N(I)$ this implies that $a+bI \in N(I)$.

$(iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.

$(ii) \Longrightarrow (i)$: Clearly $N(S)(a+bI)$ is both neutrosophic quasi and neutrosophic bi-ideal containing $a+bI$ and $N(S)(a+bI)^2$ is neutrosophic two sided ideal contains $(a+bI)^2$ respectively. Now by (ii) $N(S)(a+bI)^2$ is semiprime so $a+bI \in N(S)(a+bI)^2$. Therefore by using neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$
\begin{aligned}
a+bI \; & \in [N(S)(a+bI)\cap N(S)(a+bI)^2]\cap N(S)(a+bI) \\
& \subseteq [\{N(S)(a+bI)\}\{N(S)(a+bI)^2\}][N(S)(a+bI)] \\
& \subseteq [\{(a+bI)^2N(S)\}\{(a+bI)N(S)\}][N(S)N(S)] \\
& \subseteq [\{(a+bI)^2N(S)\}\{N(S)N(S)\}][N(S)N(S)] \\
& = [\{(a+bI)^2N(S)\}N(S))]N(S) \\
& = [\{N(S)N(S)\}(a+bI)]N(S) = [N(S)(a+bI)^2]N(S).
\end{aligned}
$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 50 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*

(i) *$N(S)$ is neutrosophic intra-regular.*

(ii) *For neutrosophic quasi-ideals $N(Q)$ and neutrosophic bi-ideal $N(B)$, $N(Q)\cap N(B) \subseteq N(Q)N(B)$.*

(iii) *For neutrosophic quasi-ideal $N(Q)$ and neutrosophic generalized bi-ideal $N(B)$, $N(Q)\cap N(B) \subseteq N(Q)N(B)$.*

Proof. $(i) \Longrightarrow (iii)$: Let $N(Q)$ and $N(B)$ be neutrosophic quasi and neutrosophic generalized bi-ideal of $N(S)$. Let $a+bI \in N(Q)\cap N(B)$

this implies that $a + bI \in N(Q)$ and $a + bI \in N(B)$. Since $N(S)$ is neutrosophic intra-regular so for $a + bI \in N(S)$ there exists $(x_1 + x_2 I)$, $(y_1 + y_2 I) \in N(S)$ such that $a + bI = \left[(x_1 + x_2 I)(a + bI)^2\right](y_1 + y_2 I)$. Thus, $N(Q)$ and $N(B)$ becomes ideals of $N(S)$. Therefore by using (i), neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned} a + bI &= \left[(x_1 + x_2 I)(a + bI)^2\right](y_1 + y_2 I) \\ &= \left[(x_1 + x_2 I)\{(a + bI)(a + bI)\}\right](y_1 + y_2 I) \\ &= \left[(a + bI)\{(x_1 + x_2 I)(a + bI)\}\right](y_1 + y_2 I) \\ &= \left[(y_1 + y_2 I)\{(x_1 + x_2 I)(a + bI)\}\right](a + bI) \\ &\in \left[N(S)\{N(S)N(Q)\}\right]N(B) \subseteq N(Q)N(B). \end{aligned}$$

Thus $a + bI \in N(Q)N(B)$ implies $N(Q) \cap N(B) \subseteq N(Q)N(B)$.

$(iii) \Longrightarrow (ii)$ is obvious.

$(i) \Longrightarrow (ii)$: Clearly $(a + bI)N(S)$ is both neutrosophic quasi and bi-ideal of $N(S)$ containing $a + bI$. Therefore by using (i), neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned} a + bI &\in N(S)(a + bI) \cap N(S)(a + bI) \\ &\subseteq [N(S)(a + bI)][N(S)(a + bI)] \\ &= [N(S)(a + bI)^2] = [N(S)(a + bI)^2]N(S). \end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 51 *For $N(S)$ with left identity $e + eI$, the following conditions are equivalent.*

(i) *$N(S)$ is neutrosophic intra-regular.*

(ii) *For every neutrosophic quasi-ideal $N(Q)$ of $N(S)$, $N(Q) = [N(S)N(Q)]^2 \cap [N(Q)N(S)]^2$.*

Proof. $(i) \Longrightarrow (ii)$: Let $N(Q)$ be any neutrosophic quasi-ideal of $N(S)$. Now it becomes a neutrosophic ideal of $N(S)$. Now using (i), neutrosophic medial law and neutrosophic paramedial law we get

$$\begin{aligned} & [N(S)N(Q)]^2 \cap [N(Q)N(S)]^2 \\ = \; & [N(S)N(Q)][N(S)N(Q)] \cap [N(Q)N(S)][N(Q)N(S)] \\ = \; & N(Q)N(Q) \cap N(Q)N(Q) = N(Q)N(Q) \subseteq N(Q). \end{aligned}$$

Now let $a + bI \in N(Q)$ and since $N(S)$ is neutrosophic intra-regular so there exists $(x_1 + x_2 I)$, $(y_1 + y_2 I) \in N(S)$ such that $a + bI = \left[(x_1 + x_2 I)(a + bI)^2\right](y_1 + y_2 I)$. Then using (i), neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned}
& a+bI \\
= & \left[(x_1+x_2I)(a+bI)^2\right](y_1+y_2I) \\
= & \left[(a+bI)\{(x_1+x_2I)(a+bI)\}\right](y_1+y_2I) \\
= & \left[(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}\right](a+bI) \\
= & \left[(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}\right]\left[\{(x_1+x_2I)(a+bI)^2\}(y_1+y_2I)\right] \\
= & \left[(x_1+x_2I)(a+bI)^2\right]\left[\{(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}\}(y_1+y_2I)\right] \\
= & \left[(y_1+y_2I)\{(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}\}\right]\left[\{(a+bI)(a+bI)\}(x_1+x_2I))\right] \\
= & \left[(a+bI)(a+bI)\right]\left[\{(y_1+y_2I)\{(y_1+y_2I)\{(x_1+x_2I)(a+bI))\}\}\}(x_1+x_2I)\right] \\
= & \left[(x_1+x_2I)\{(y_1+y_2I)\{(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}\}\}\right]\left[(a+bI)(a+bI)\right] \\
\in & N(S)\left[N(Q)N(Q)\right] \\
= & \left[N(S)N(S)\right]\left[N(Q)N(Q)\right] \\
= & \left[N(S)N(Q)\right]\left[N(S)N(Q)\right]=\left[N(S)N(Q)\right]^2.
\end{aligned}$$

Thus $a+bI\in[N(S)N(Q))]^2$. Since $[N(S)N(Q)]^2=[N(Q)N(S)]^2$ by using neutrosophic medial

and neutrosophic paramedial laws. Therefore $a+bI\in[N(S)N(Q)]^2\cap[N(Q)N(S)]^2$. Thus $N(Q)\subseteq[N(S)N(Q)]^2\cap[N(Q)N(S)]^2$. Hence $[N(S)N(Q)]^2\cap[N(Q)N(S)]^2=N(Q)$.

$(ii)\Rightarrow(i)$: Clearly $N(S)(a+bI)$ is a neutrosophic quasi-ideal containing $a+bI$. Thus by (ii)

and neutrosophic paramedial law, neutrosophic medial law and neutrosophic left invertive law we

have

$$\begin{aligned}
a+bI &\in N(S)(a+bI) \\
&= \left[N(S)\{N(S)(a+bI)\}\right]^2 \\
&= \left[\{N(S)N(S)\}\{N(S)(a+bI)\}\right]^2 \\
&= \left[\{(a+bI)N(S)\}\{N(S)N(S)\}\right]^2 \\
&= \left[\{N(S)N(S)\}(a+bI)\right]^2 \\
&= \left[N(S)(a+bI)\right]^2 \\
&= [N(S)(a+bI)^2]=[N(S)(a+bI)^2]N(S).
\end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 52 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*

(i) $N(S)$ is neutrosophic intra-regular.

(ii) For every neutrosophic quasi-ideal of $N(S)$, $N(Q)=[N(S)N(Q)]^2N(Q)\cap[N(Q)N(S)]^2N(Q)$.

Proof. $(i) \Longrightarrow (ii)$: Let $N(Q)$ be a neutrosophic quasi-ideal of a neutrosophic intra-regular AG-groupoid $N(S)$ with left identity. Now it becomes a neutrosophic ideal of $N(S)$. Then

$$[N(S)N(Q)]^2 N(Q) \cap [N(Q)N(S)]^2 N(Q) \subseteq N(Q).$$

Now let $a \in N(Q)$ and since $N(S)$ is neutrosophic intra-regular so there exists $(x_1+x_2I), (y_1+y_2I) \in N(S)$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](x_1+x_2I)$. Then by using (i), neutrosophic left invertive law and neutrosophic medial law and neutrosophic paramedial laws, we have

$$\begin{aligned}
a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I)\\
&= [(a+bI)\{(x_1+x_2I)(a+bI)\}](y_1+y_2I)\\
&= [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}](a+bI)\\
&= [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}](a+bI)\\
&= [(y_1+y_2I)\{(x_1+x_2I)\{\{(x_1+x_2I)(a+bI)^2\}(y_1+y_2I)\}\}](a+bI)\\
&= [(y_1+y_2I)\{\{(x_1+x_2I)(a+bI)^2\}\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI)\\
&= [\{(x_1+x_2I)(a+bI)^2\}\{(y_1+y_2I)\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI)\\
&= [\{(x_1+x_2I)(y_1+y_2I)\}\{(a+bI)^2\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI)\\
&= [(a+bI)^2\{\{(x_1+x_2I)(y_1+y_2I)\}\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI)\\
&= [(a+bI)\{(x_1+x_2I)(y_1+y_2I)\}]^2(a+bI).
\end{aligned}$$

Thus $a+bI \in [\{N(Q)\{N(S)N(S)\}]^2 N(Q) = [N(Q)N(S)]^2 N(Q)$ implies that $a+bI \in [N(Q)N(S)]^2 N(Q)$ therefore, $N(Q) \subseteq [N(Q)N(S)]^2 N(Q)$. Now since $[N(Q)N(S)]^2 = [N(S)N(Q)]^2$ this also implies that $N(Q) \subseteq [N(S)Q]^2 N(Q)$. Hence $N(Q) \subseteq [N(S)N(Q)]^2 N(Q) \cap [N(Q)N(S)]^2 N(Q)$. Therefore, $N(Q) = [N(S)N(Q)]^2 N(Q) \cap [N(Q)N(S)]^2 N(Q)$.

$(ii) \Rightarrow (i)$: Clearly $N(S)(a+bI)$ is a neutrosophic quasi-ideal containing $a+bI$. Therefore by

(ii) we have

$$\begin{aligned}
a+bI &\in N(S)(a+bI)\\
&= [N(S)\{N(S)(a+bI)\}]^2[N(S)(a+bI)]\\
&\subseteq [N(S)(a+bI)]^2[N(S)(a+bI)]\\
&= [N(S)(a+bI)^2][N(S)(a+bI)] \subseteq [N(S)(a+bI)^2]N(S).
\end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 53 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*

(i) *$N(S)$ is neutrosophic intra-regular.*

(ii) *For any neutrosophic quasi-ideals $N(Q_1)$ and $N(Q_2)$ of $N(S)$, $N(Q_1)Q_2 \subseteq N(Q_2)N(Q_1)$ and $N(Q_1)$, $N(Q_2)$ are neutrosophic semiprime.*

Proof. $(i) \Longrightarrow (ii)$: Let $N(Q_1)$ and $N(Q_2)$ be any neutrosophic quasi-ideals of a neutrosophic intra-regular AG-groupoid $N(S)$ with left identity. Thus $N(Q_1)$ and $N(Q_2)$ become neutrosophic ideals of $N(S)$. Let $a+bI \in N(Q_1)N(Q_2)$. Then $a+bI = (u_1+u_2I)(v_1+v_2I)$ where $u_1+u_2I \in N(Q_1)$ and $v_1+v_2I \in N(Q_2)$. Now since $N(S)$ in neutrosophic intra-regular therefore for u_1+u_2I and v_1+v_2I in $N(S)$ there exists $(x_1+x_1'I)$, $(x_2+x_2'I)$, $(y_1+y_1'I)$, $(y_2+y_2'I) \in N(S)$ such that $a+bI = [\{\{(x_1+x_1'I)(u_1+u_2I)^2\}(y_1+y_1'I)\}\{\{(x_2+x_2'I)(v_1+v_2I)^2\}(y_2+y_2'I)\}]$. Then by using using (i), neutrosophic left invertive law, neutrosophic AG**-groupoid law, neutrosophic medial
and neutrosophic paramedial laws we have

$$
\begin{aligned}
& a+bI \\
= \; & [\{\{(x_1+x_1'I)(u_1+u_2I)^2\}(y_1+y_1'I))\}\{\{(x_2+x_2'I)(v_1+v_2I)^2\}(y_2+y_2'I)\}] \\
= \; & [\{(x_1+x_1'I)(u_1+u_2I)^2\}\{(x_2+x_2'I)(v_1+v_2I)^2\}][(y_2+y_2'I)(y_1+y_1'I)] \\
= \; & [\{(x_1+x_1'I)\{(u_1+u_2I)(u_1+u_2I)\}\}\{(x_2+x_2'I)\{(v_1+v_2I)(v_1+v_2I)\}\}][(y_2+y_2'I)(y_1+y_1'I)] \\
= \; & [\{(u_1+u_2I)\{(x_1+x_1'I)(u_1+u_2I)\}\}\{(v_1+v_2I)\{(x_2+x_2'I)(v_1+v_2I)\}\}] \\
& [(y_2+y_2'I)(y_1+y_1'I)] \\
= \; & [\{\{(x_2+x_2'I)(v_1+v_2I)\}\{(x_1+x_1'I)(u_1+u_2I)\}\}\{(v_1+v_2I)(u_1+u_2I)\}] \\
& [(y_2+y_2'I)(y_1+y_1'I)] \\
= \; & [\{\{(x_2+x_2'I)(x_1+x_1'I)\}\{(v_1+v_2I)(u_1+u_2I)\}\}\{v(u_1+u_2I)\}] \\
& [(y_2+y_2'I)(y_1+y_1'I)] \\
= \; & [\{\{(v_1+v_2I)(u_1+u_2I)\}\{(v_1+v_2I)(u_1+u_2I)\}\}\{(x_2+x_2'I)(x_1+x_1'I)\}] \\
& [(y_2+y_2'I)(y_1+y_1'I)] \\
= \; & [\{(y_2+y_2'I)(y_1+y_1'I)\}\{(x_2+x_2'I)(x_1+x_1'I)\}][\{\{(v_1+v_2I)(u_1+u_2I)\} \\
& \{(v_1+v_2I)(u_1+u_2I)\}\} \\
= \; & [\{(y_2+y_2'I)(y_1+y_1'I)\}\{(x_2+x_2'I)(x_1+x_1'I)\}][(v_1+v_2I)^2(u_1+u_2I)^2\} \\
= \; & [\{(y_2+y_2'I)(y_1+y_1'I)\}(v_1+v_2I)^2][\{(x_2+x_2'I)(x_1+x_1'I)\}(u_1+u_2I)^2] \\
\in \; & [\{N(S)N(S))\}(N(Q_1))^2][\{N(S)N(S)\}(N(Q_2))^2] \\
\subseteq \; & [N(S)N(Q_2)][N(S)N(Q_1)] \\
\subseteq \; & N(Q_2)N(Q_1).
\end{aligned}
$$

Thus $a+bI \in N(Q_2)N(Q_1)$. Hence $N(Q_1)N(Q_2) \subseteq N(Q_2)N(Q_1)$. Let $(a+bI)^2 \in N(Q_1)$. Then since $N(S)$ is neutrosophic intra-regular so for $a+bI \in N(S)$ there exists (x_1+x_2I), $(y_1+y_2I) \in N(S)$ such that, $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Then by using using (i), neutrosophic left invertive law, neutrosophic AG**-groupoid law, neutrosophic medial and neutrosophic paramedial laws we have

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\ &= [(x_1+x_2I)\{(a+bI)(a+bI)\}](y_1+y_2I) \\ &= [(a+bI)\{(x_1+x_2I)(a+bI)\}](y_1+y_2I) \\ &= [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}](a+bI) \\ &\in [\{N(S)N(S)\}N(Q_1)]N(Q_1) \subseteq N(Q_1). \end{aligned}$$

Similarly we can show that $N(Q_2)$ neutrosophic semiprime.

$(ii) \Longrightarrow (i)$: Let $N(S)(a+bI)$ be a neutrosophic quasi-ideal of $N(S)$ containing $a+bI$ then by (ii)

and by using neutrosophic left invertive law, neutrosophic AG**-groupoid law, neutrosophic medial and neutrosophic paramedial laws we have

$$\begin{aligned} a+bI &\in N(S)(a+bI) \cap N(S)(a+bI) \\ &= [N(S)(a+bI))][N(S)(a+bI)] \\ &= [N(S)(a+bI)^2] = [N(S)(a+bI)^2]N(S). \end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 54 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*

(i) $N(S)$ is neutrosophic intra-regular.

(ii) For any neutrosophic quasi-ideal $N(A)$ and neutrosophic two sided ideal $N(B)$ of $N(S)$, $N(A) \cap N(B) = [N(A)N(B)]N(A)$ and $N(B)$ is neutrosophic semiprime.

(iii) For any neutrosophic quasi-ideal $N(A)$ and neutrosophic right ideal $N(B)$ of $N(S)$, $N(A) \cap N(B) = [N(A)N(B)]N(A)$ and $N(B)$ is neutrosophic semiprime.

(iv) For any neutrosophic quasi-ideal $N(A)$ and neutrosophic interior ideal $N(B)$ of $N(S)$, $N(A)$, $N(B)$, $N(A) \cap N(B) = [N(A)N(B)]N(A)$ and $N(B)$ is neutrosophic semiprime.

Proof. $(i) \Rightarrow (iv)$: Let $N(A)$ and $N(B)$ be a neutrosophic quasi-ideal and a neutrosophic interior ideal of $N(S)$ respectively. Thus, $N(A)$ and $N(B)$ are neutrosophic ideals of $N(S)$. Then $[N(A)N(B)]N(A) \subseteq [N(A)N(S)]N(A) \subseteq N(A)$ and $[N(A)N(B)]N(A) \subseteq [N(S)N(B)]N(S) \subseteq N(B)$. Thus $[N(A)N(B))]N(A) \subseteq N(A) \cap N(B)$. Next let $a+bI \in N(A) \cap N(B)$, which implies that $a+bI \in N(A)$ and $a+bI \in N(B)$. Since $N(S)$ is neutrosophic intra-regular so for $a+bI$ there exists $(x_1+x_2I), (y_1+y_2I) \in N(S)$, such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Then by using using (i), neutrosophic left invertive law, neutrosophic AG**-groupoid law, neutrosophic medial and neutrosophic paramedial laws we have

$$
\begin{aligned}
& a+bI \\
= & [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= & [(a+bI)\{(x_1+x_2I)(a+bI)\}](y_1+y_2I) \\
= & [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}](a+bI) \\
= & [(y_1+y_2I)\{(x_1+x_2I)(a+bI)\}](a+bI) \\
= & [(y_1+y_2I)\{(x_1+x_2I)\{\{(x_1+x_2I)(a+bI)^2\}(y_1+y_2I)\}\}](a+bI) \\
= & [(y_1+y_2I)\{\{(x_1+x_2I)(a+bI)^2\}\{(x_1+x_2I)y\}\}](a+bI) \\
= & [\{(x_1+x_2I)(a+bI)^2\}\{(y_1+y_2I)\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI) \\
= & [\{(a+bI)\{(x_1+x_2I)(a+bI)\}\}\{(y_1+y_2I)\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI) \\
= & [\{\{(y_1+y_2I)\{(x_1+x_2I)(y_1+y_2I)\}\}\{(x_1+x_2I)(a+bI)\}\} \\
& (a+bI))](a+bI) \\
= & [(a+bI)(a+bI)][\{(y_1+y_2I)\{(x_1+x_2I)(y_1+y_2I)\}\} \\
& \{(x_1+x_2I)(a+bI)\}] \\
\subseteq & [N(A)N(B)][N(S)\{N(S)N(A)\}] \subseteq [N(A)N(B)]N(A).
\end{aligned}
$$

Thus $N(A)\cap N(B) = [N(A)N(B))]N(A)$. Now in order to show that $N(B)$ is neutrosophic semiprime let $(a+bI)^2 \in N(B)$. Therefore for each $a+bI \in N(S)$ there exists $(x_1+x_2I), (y_1+y_2I) \in N(S)$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \in N(B)N(B) \subseteq N(B)$. Thus $(a+bI)^2 \in N(B)$ implies that $a+bI \in N(B)$. Hence $N(B)$ is neutrosophic semiprime.

$(iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.

$(ii) \Longrightarrow (i)$: Since $N(S)(a+bI)$ is neutrosophic quasi-ideal and $N(S)(a+bI)^2$ be neutrosophic two sided ideal containing $a+bI$ and $(a+bI)^2$ respectively. And by (ii) $N(S)(a+bI)^2$ is neutrosophic semiprime so $a+bI \in N(S)(a+bI)^2$. Therefore using (ii), neutrosophic left invertive law, neutrosophic AG**-groupoid law, neutrosophic medial and neutrosophic paramedial laws we have

$$
\begin{aligned}
& N(S)(a+bI) \cap N(S)(a+bI)^2 \\
= & [\{N(S)(a+bI)\}\{N(S)(a+bI)^2\}][N(S)(a+bI)] \\
\subseteq & [\{N(S)N(S)\}\{N(S)(a+bI)^2\}][N(S)N(S)] \\
= & [\{(a+bI)^2 N(S)\}\{N(S)N(S)\}]N(S) \\
= & [\{(a+bI)^2 N(S)\}N(S)]N(S) \\
= & [\{N(S)N(S)\}(a+bI)^2)]N(S) \\
= & [N(S)(a+bI)^2]N(S).
\end{aligned}
$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 55 *For $N(S)$ the following conditions are equivalent.*

(i) $N(S)$ is neutrosophic intra-regular.

(ii) For every neutrosophic left ideal $N(A)$ and neutrosophic $N(B)$ of $N(S)$, $N(A) \cap N(B) = [N(A)N(B)] \cap [N(B)N(A)]$.

(iii) For every neutrosophic quasi ideal $N(A)$ and every neutrosophic left ideal $N(B)$ of $N(S)$, $N(A) \cap N(B) = [N(A)N(B)] \cap [N(B)N(A)]$.

(iv) For every neutrosophic quasi ideals $N(A)$ and $N(B)$ of $N(S)$, $N(A) \cap N(B) = [N(A)N(B)] \cap [N(B)N(A)]$.

Proof. $(i) \Longrightarrow (iv)$: Let $N(A)$ and $N(B)$ be any neutrosophic generalized bi-ideal of $N(S)$, then $N(A)$ and $N(B)$ are neutrosophic ideals of $N(S)$. Clearly $N(A)N(B) \subseteq N(A) \cap N(B)$, now, $N(A) \cap N(B)$ is a neutrosophic ideal and, $N(A) \cap N(B) = [N(A) \cap N(B)]^2$. Now $N(A) \cap N(B) = [N(A) \cap N(B)]^2 \subseteq AN(B)$. Thus $N(A) \cap N(B) = N(A)N(B)$ and then $N(A) \cap N(B) = N(B) \cap N(A) = N(B)N(A)$.

Hence $N(A) \cap N(B) = [N(A)N(B)] \cap [N(B)N(A)]$.

$(iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.

$(ii) \Rightarrow (i)$: Since $N(S)(a+bI)$ is a left ideal of $N(S)$ containing $a+bI$. Therefore by (ii) and neutrosophic medial law we get

$$\begin{aligned} N(S)(a+bI) \cap N(S)(a+bI) &= [N(S)(a+bI)][N(S)(a+bI)] \\ &= N(S)(a+bI)^2 \\ &= [N(S)(a+bI)^2]N(S). \end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 56 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*

(i) $N(S)$ is neutrosophic intra-regular.

(ii) For any neutrosophic quasi-ideals $N(Q)$ and neutrosophic two sided ideal $N(I)$ of $N(S)$, $N(Q) \cap N(I) = [N(Q)N(I)]N(Q)$ and $N(I)$ is neutrosophic semiprime.

(iii) For any neutrosophic quasi-ideals $N(Q)$ and neutrosophic right ideal $N(I)$ of $N(S)$, $N(Q) \cap N(I) = [N(Q)N(I)]N(Q)$ and $N(I)$ is neutrosophic semiprime.

(iv) For any neutrosophic quasi ideals $N(Q)$ and neutrosophic interior ideal $N(I)$ of $N(S)$, $N(Q) \cap N(I) = [N(Q)N(I)]N(Q)$ and $N(I)$ is neutrosophic semiprime.

Proof. $(i) \Rightarrow (v)$: Let $N(Q)$ and $N(I)$ be a neutrosophic quasi-ideal and a neutrosophic interior ideal of $N(S)$ respectively. Now, $N(Q)$ and $N(I)$ are neutrosophic ideals of $N(S)$. Then $[N(Q)I]N(Q) \subseteq [N(Q)N(S)]N(Q) \subseteq N(Q)$ and $[N(Q)N(I)]N(Q) \subseteq [N(S)N(I)]N(S) \subseteq N(I)$. Thus $[N(Q)I]N(Q) \subseteq N(Q) \cap N(I)$. Now let $a+bI \in N(Q) \cap N(I)$ implies that $a+bI \in N(Q)$ and $a+bI \in N(I)$. Since $N(S)$ is neutrosophic intra-regular so for $a+bI$ there exists (x_1+x_2I), $(y_1+y_2I) \in N(S)$, such that $a+bI =$

$[(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Then using (i) and neutrosophic left invertive law, neutrosophic medial and neutrosophic paramedial laws we have

$$\begin{aligned}
& a+bI \\
= & [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= & [(x_1+x_2I)\{(a+bI)(a+bI)\}](y_1+y_2I) \\
= & [(a+bI)\{(x_1+x_2I)(a+bI)\}](y_1+y_2I) \\
= & [(a+bI)\{(x_1+x_2I)\{\{(x_1+x_2I)(a+bI)^2\}(y_1+y_2I)\}\}](y_1+y_2I) \\
= & [(a+bI)\{\{(x_1+x_2I)(a+bI)^2\}\{(x_1+x_2I)(y_1+y_2I)\}\}](y_1+y_2I) \\
= & [(y_1+y_2I)\{\{(x_1+x_2I)(a+bI)^2\}\{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI) \\
= & [(y_1+y_2I)\{\{(x_1+x_2I)\{(a+bI)(a+bI)\}\} \\
& \{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI) \\
= & [(y_1+y_2I)\{\{(a+bI)\{(x_1+x_2I)(a+bI)\}\}\{(x_1+x_2I)y\}\}](a+bI) \\
= & [\{(a+bI)\{(x_1+x_2I)(a+bI)\}\}\{(y_1+y_2I) \\
& \{(x_1+x_2I)(y_1+y_2I)\}\}](a+bI) \\
= & [\{\{(y_1+y_2I)\{(x_1+x_2I)(y_1+y_2I)\}\}\{(x_1+x_2I)(a+bI)\}\} \\
& (a+bI)](a+bI) \\
= & [(a+bI)(a+bI)][\{(y_1+y_2I)\{(x_1+x_2I)(y_1+y_2I)\}\} \\
& \{(x_1+x_2I)(a+bI)\}] \\
\in & [N(Q)N(I)][N(S)\{N(S)N(S)\}\{N(S)N(Q)\}] \\
\subseteq & [N(Q)N(I)]N(Q).
\end{aligned}$$

Thus $N(Q)\cap N(I)=[N(Q)N(I)]N(Q)$. Next to show that $N(I)$ is neutrosophic semiprime let $(a+bI)^2\in N(I)$. Therefore for each $a+bI\in N(S)$ there exists (x_1+x_2I), $(y_1+y_2I)\in N(S)$ such that $a+bI=[(x_1+x_2I)(a+bI)^2](y_1+y_2I)\in[N(S)N(I)]N(S)\subseteq N(I)$. Thus $(a+bI)^2\in N(I)$ this implies that $a+bI\in N(I)$. Hence $N(I)$ is neutrosophic semiprime.

$(v)\Longrightarrow(iv)\Longrightarrow(iii)\Longrightarrow(ii)$ are obvious.

$(ii)\Longrightarrow(i)$: Since $N(S)(a+bI)$ is a neutrosophic quasi-ideal and $N(S)(a+bI)^2$ be a two sided ideal containing $a+bI$ and $(a+bI)^2$ respectively. And by (ii) $N(S)(a+bI)^2$ is neutrosophic semiprime so $a+bI\in N(S)(a+bI)^2$. Therefore using (ii), neutrosophic left invertive law, neutrosophic medial law and neutrosophic paramedial we have

$$\begin{aligned}
& N(S)\,(a+bI) \cap N(S)\,(a+bI)^2 \\
= \;& [\{N(S)\,(a+bI)\}\{N(S)\,(a+bI)^2\}][N(S)\,(a+bI)] \\
= \;& [\{N(S)N(S)\}\{N(S)\,(a+bI)^2\}][N(S)N(S)] \\
= \;& [\{(a+bI)^2\,N(S)\}\{N(S)N(S)\}]N(S) \\
= \;& [\{(a+bI)^2\,N(S)\}N(S)]N(S) \\
= \;& [\{N(S)N(S)\}\,(a+bI)^2)]N(S) \\
= \;& [N(S)\,(a+bI)^2]N(S).
\end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

1.2 Neutrosophic Generalized Ideals

In this section; we study (m,n)-ideals of an LA -semigroup in detail. We characterize $(0,2)$-ideals of an LA -semigroup $N(S)$ and prove that $N(A)$ is a $(0,2)$-ideal of $N(S)$ if and only if $N(A)$ is a left ideal of some left ideal of $N(S)$. We also show that an LA -semigroup $N(S)$ is $0-(0,2)$-bisimple if and only if $N(S)$ is right 0-simple. Furthermore we study 0-minimal (m,n)-ideals in an LA -semigroup $N(S)$ and prove that if $N(R)[N(L)]$ is a 0-minimal right (left) ideal of $N(S)$, then either $N(R)^m N(L)^n = \{0\}$ or $R^m L^n$ is a 0-minimal (m,n)-ideal of $N(S)$ for m,n 3: Finally we discuss (m,n)-ideals in an (m,n)-regular LA -semigroup $N(S)$ and show that $N(S)$ is $(0,1)$-regular if and only if $N(L) = N(S)N(L)$ where $N(L)$ is a $(0,1)$-ideal of $N(S)$.

In this chapter, we investigate two classes of ideals called the (m,n)-ideals and 0-minimal ideals of an LA-semigroup and their characterizations. First we study $(0,2)$-ideals of an LA -semigroup S and prove that A is a $(0,2)$-ideal of S if and only if A is a left ideal of some left ideal of S. Further, we characterize $(0,2)$-bi-ideals in unitary LA -semigroups and proceed to prove that A is a 0-minimal $(0,2)$-bi-ideal of a unitary LA -semigroup S with zero. Then either $A^2 = \{0\}$ or A is right 0-simple. We also study some interesting results in (m,n)-ideals and investigate that if A is an (m,n)-ideal of S and B is an (m,n)-ideal of A such that B is idempotent. Then B is an (m,n)-ideal of S. The concept of (m,n)-regular LA -semigroups is indeed an important and interesting part of the paper. In this respect, we prove that if S is a unitary (m,n)-regular LA -semigroup such that $m=n$. Then for every $R \in R_{(m,0)}$ and $L \in L_{(0,n)}$, $R \cap L = R^m L \cap RL^n$:

Preliminaries and examples

If $N(S)$ is an LA -semigroup with product . $N(S) \times N(S) \to N(S)$, then $ab \cdot c$ and $(ab)c$ both denote the product $(a \cdot b) \cdot c$.

If there is an element 0 of an LA -semigroup $(N(S), \cdot)$ such that $x \cdot 0 = 0 \cdot x = 0 \ \forall \ x \in N(S)$ we call 0 a zero element of $N(S)$.

Example 57 *Let $N(S) = \{a_1 + b_1I, a_2 + b_2I, a_3 + b_3I, a_4 + b_4I, e + eI\}$ with a left identity d. Then the following multiplication table shows that $(N(S), \cdot)$ is a unitary LA -semigroup with a zero element a.*

$\cdot$	$a_1 + b_1I$	$a_2 + b_2I$	$a_3 + b_3I$	$a_4 + b_4I$	$e + eI$
$a_1 + b_1I$	$a_1 + b_1I$	$a_1 + b_1I$	$a_1 + b_1I$	$a_1 + b_1I$	$a_1 + b_1I$
$a_2 + b_2I$	$a_1 + b_1I$	$e + eI$	$e + eI$	$a_3 + b_3I$	$e + eI$
$a_3 + b_3I$	$a_1 + b_1I$	$e + eI$	$e + eI$	$a_2 + b_2I$	$e + eI$
$a_4 + b_4I$	$a_1 + b_1I$	$a_2 + b_2I$	$a_3 + b_3I$	$a_4 + b_4I$	$e + eI$
$e + eI$	$a_1 + b_1I$	$e + eI$	$e + eI$	$e + eI$	$e + eI$

Example 58 *Let $N(S) = \{a_1 + b_1I, a_2 + b_2I, a_3 + b_3, a_4 + b_4I\}$. Then the following multiplication table shows that $(N(S), \cdot)$ is an LA -semigroup with a zero element a.*

$\cdot$	$a_1 + b_1I$	$a_2 + b_2I$	$a_3 + b_3$	$a_4 + b_4I$
$a_1 + b_1I$	$a_1 + b_1I$	$a_1 + b_1I$	$a_1 + b_1I$	$a_1 + b_1I$
$a_2 + b_2I$	$a_1 + b_1I$	$a_4 + b_4I$	$a_4 + b_4I$	$a_3 + b_3$
$a_3 + b_3$	$a_1 + b_1I$	$a_3 + b_3$	$a_3 + b_3$	$a_3 + b_3$
$a_4 + b_4I$	$a_1 + b_1I$	$a_3 + b_3$	$a_3 + b_3$	$a_3 + b_3$

The above LA -semigroup $N(S)$ has commutative powers, that is $[(a_1 + b_1I)(a_1 + b_1I)].(a_1 + b_1I) = (a + bI).[(a + bI)(a + bI)]$ for all $(a_1 + b_1I) \in N(S)$ which is called a locally associative LA -semigroup [7]. Note that $N(S)$ has no associative powers for all $a + bI \in N(S)$ because $(bb \cdot b)b \neq b(bb \cdot b)$ for $b \in N(S)$.

Assume that $N(S)$ is an LA -semigroup. Let us define $(a + bI)^1 = a + bI$ and $a^m = (...(((aa)a)a)...a)a = a^{m-1}a$ for all $a + bI \in N(S)$ where $m \geq 1$. It is easy to see that $a^m = a^{m-1}a = aa^{m-1}$ for all $a \in S$ and

$m \geq 3$ if $N(S)$ has a left identity. Also, we can show by induction, $(ab)^m = a^m b^m$ and $a^m a^n = a^{m+n}$ hold for all $a + bI, c + dI \in N(S)$ and $m, n \geq 3$.

A subset $N(A)$ of an LA -semigroup $N(S)$ is called an LA -subsemigroup of $N(S)$ if $[N(A)]^2 \subseteq N(A)$.

The concept of (m, n)-ideals of a semigroup and an LA -semigroup was given in [5] and [1] respectively.

An LA -subsemigroup $N(A)$ of an LA -semigroup $N(S)$ is said to be an (m, n)-ideal of $N(S)$ if $[N(A)]^m N(S)$.

$[N(A)]^n \subseteq N(A)$ where m, n are non-negative integers such that $m = n \neq 0$: Here $[N(A)]^m$ or $[N(A)]^n$ are suppressed if $m = 0$ or $n = 0$, that is $[N(A)]^0 N(S) = N(S)$ or $N(S)[N(A)]^0 = N(S)$. Note that if $m = n = 1$,then an (m, n)-ideal $N(A)$ of an LA -semigroup $N(S)$ is called a bi-ideal of $N(S)$. If we take $m = 0$ or $n = 0$, then an (m,n)-ideal $N(A)$ of an LA -semigroup $N(S)$ becomes a left or a right ideal of $N(S)$.

An (m,n)-ideal $N(A)$ of an LA -semigroup $N(S)$with zero is said to be 0-minimal if $N(A) \neq \{0\}$ and $\{0\}$ is the only (m,n)-ideal of $N(S)$ properly contained in $N(A)$.

An LA -semigroup $N(S)$ with zero is said to be $0-(0,2)$-bisimple if $[N(S)]^2 \neq \{0\}$ and f$\{0\}$ is the only proper $(0,2)$-bi-ideal of $N(S)$.

An LA -semigroup $N(S)$ with zero is said to be nilpotent if $[N(S)]^1 = \{0\}$ for some positive integer l.

Let $m_1 + n_1 I, m_2 + n_2 I$ be non-negative integers and $N(S)$ be an LA -semigroup. We say that $N(S)$ is (m,n)-regular

if for every element $a + bI \in N(S)$ there exists some $x + yI \in N(S)$ such that $a + bI = [(a+bI)^m(x+yI)](a+bI)^n$. Note that $(a+bI)^0$ is defined as an operator element such that $(a+bI)^0(x+yI) = x+yI$ and $(z+sI)(a+yI)^0 = z+sI$ for any $x+yI, z+sI \in N(S)$.

3.0-minimal $(0,2)$-bi-ideals in unitary LA -semigroups

If $N(S)$ is a unitary LA -semigroup, then it is easy to see that $[N(S)]^2 = N(S)$, $N(S)[N(A)]^2 = [N(A)]^2 N(S)$ and $N(A) \subseteq N(S)N(A), \forall\ N(A) \subseteq N(S)$. Note that every right ideal of a unitary LA -semigroup $N(S)$ is a left ideal of $N(S)$ but the converse is not true in general. Example 1 shows that there exists a subset $\{a_1 + b_1 I, a_2 + b_2 I, e_1 + e_2 I\}$ of $N(S)$ which is a left ideal of $N(S)$ but not a right ideal of $N(S)$. It is easy to see that $N(S)N(A)$ and $N(S)[N(A)]^2$ are the left and right ideals of a unitary LA -semigroup $N(S)$. Thus $N(S)[N(A)]^2$ is an ideal of a unitary LA -semigroup $N(S)$.

Lemma 59 *Let $N(S)$ be a unitary LA -semigroup. Then $N(A)$ is a $(0,2)$-ideal of $N(S)$ if and only if $N(A)$ is an ideal of some left ideal of $N(S)$.*

Proof. Let $N(A)$ be a $(0,2)$-ideal of $N(S)$, then $N(S)N(A) \cdot N(A) = N(A)N(A) \cdot N(S) = N(S)[N(A)]^2 \subseteq N(A)$ and $N(A) \cdot N(S)N(A) = N(S) \cdot N(A)N(A) = N(S)N(S) \cdot N(A)N(A) = N(S)[N(A)]^2 \subseteq N(A)$. Hence $N(A)$ is an ideal of a left ideal $N(S)N(A)$ of $N(S)$.

Conversely, assume that $N(A)$ is a left ideal of a left ideal $N(L)$ of $N(S)$, then

$$N(S)[N(A)]^2 = N(A)N(A) \cdot N(S) = N(S)N(A) \cdot N(A) \subseteq N(S)N(L) \cdot N(A) \subseteq N(L)N(A) \subseteq N(A),$$

and clearly $N(A)$ is an LA -subsemigroup of $N(S)$, therefore $N(A)$ is a $(0,2)$-ideal of $N(S)$. ■

Corollary 60 *Let $N(S)$ be a unitary LA -semigroup. Then $N(A)$ is a $(0,2)$-ideal of $N(S)$ if and only if $N(A)$ is a left ideal of some left ideal of $N(S)$.*

Lemma 61 *Let $N(S)$ be a unitary LA -semigroup. Then $N(A)$ is a $(0,2)$-bi-ideal of $N(S)$ if and only if $N(A)$ is an ideal of some right ideal of $N(S)$.*

Proof. Let $N(A)$ be a$(0,2)$-bi-ideal of $N(S)$, then

$$\begin{aligned} N(S)N(A)^2 \cdot N(A) &= N(A)^2 N(S) \cdot N(A) \\ &= N(A)N(S) \cdot N(A)^2 \subseteq N(S)N(A)^2 \subseteq N(A) \end{aligned}$$

and

$$\begin{aligned} & N(A) \cdot N(S)N(A)^2 \\ = \; & N(S)N(S) \cdot N(A)N(A)^2 \\ = \; & N(A)^2 N(A) \cdot N(S)N(S) \\ = \; & N(S)N(A) \cdot N(A)^2 \\ \subseteq \; & N(S)N(A)^2 \subseteq N(A). \end{aligned}$$

Hence $N(A)$ is an ideal of some right ideal $N(S)[N(A)]^2$ of $N(S)$.

Conversely, assume that $N(A)$ is an ideal of a right ideal $N(R)$ of $N(S)$, then

$$\begin{aligned} N(S)N(A)^2 &= N(A) \cdot N(S)N(A) \\ &= N(A) \cdot [N(S)N(S)]N(A) \\ &= N(A) \cdot [N(A)N(S)]N(S) \\ &\subseteq N(A) \cdot [N(R)N(S)]N(R) \\ &\subseteq N(A)N(R) \subseteq N(A), \end{aligned}$$

and $[N(A)N(S)]N(A) \subseteq [N(R)N(S)]N(A) \subseteq N(R)N(A) \subseteq N(A)$, which shows that $N(A)$ is a $(0,2)$ bi-ideal of $N(S)$. ■

Theorem 62 *Let $N(S)$ be a unitary LA -semigroup. Then the following statements are equivalent.*

(i) $N(A)$ is a $(1,2)$-ideal of $N(S)$;
(ii) $N(A)$ is a left ideal of some bi-ideal of $N(S)$;
(iii) $N(A)$ is a bi-ideal of some ideal of $N(S)$;
(iv) $N(A)$ is a $(0,2)$-ideal of some right ideal of $N(S)$;
(v) $N(A)$ is a left ideal of some $(0,2)$-ideal of $N(S)$.

Proof. (i) $\Longrightarrow$ (ii). It is easy to see that $N(S)N(A)^2 \cdot S$ is a bi-ideal of $N(S)$. Let $N(A)$ be a $(1,2)$-ideal of $N(S)$, then

$$\begin{aligned}
& [N(S)N(A)^2 \cdot N(S)]N(A) \\
= & [N(S)N(A)^2 \cdot N(S)N(S)]N(A) \\
= & [N(S)N(S) \cdot N(A)^2 N(S)]N(A) \\
= & [N(S) \cdot N(A)^2 N(S)]N(A) \\
= & N(A)^2 N(S) \cdot N(A) \\
= & N(A)N(S) \cdot N(A)^2 \subseteq N(A),
\end{aligned}$$

which shows that $N(A)$ is a left ideal of a bi-ideal $N(S)[N(A)]^2 \cdot N(S)$ of $N(S)$.

(ii) $\Longrightarrow$(iii): Let $N(A)$ be a left ideal of a bi-ideal $N(B)$ of $N(S)$, then

$$\begin{aligned}
& [N(A) \cdot N(S)N(A)^2]N(A) \\
= & [N(S) \cdot N(A)N(A)^2]N(A) \\
\subseteq & [N(S)\{N(S)N(A) \cdot N(A)N(A)\}]N(A) \\
= & [N(S)\{N(A)N(A) \cdot N(A)N(S)\}]N(A) \\
= & [N(A)N(A) \cdot N(S)\{N(A)N(S)\}]N(A) \\
= & [[N(S)\{N(A)N(S)\} \cdot N(A)]N(A)]N(A) \\
= & [\{N(AN(S) \cdot N(A)\}N(A)]N(A) \\
\subseteq & [\{N(B)N(S) \cdot N(B)\}N(A)]N(A) \\
\subseteq & N(B)N(A).N(A) \subseteq N(A),
\end{aligned}$$

which shows that $N(A)$ is a bi-ideal of an ideal $N(S)[N(A)]^2$ of $N(S)$.

(iii) $\Longrightarrow$(iv). Let $N(A)$ be a bi-ideal of an ideal $N(J)$ of $N(S)$, then

$$\begin{aligned}
N(S)N(A)^2.N(A)^2 & = [N(A)^2 \cdot N(A)N(A)]N(S) \\
& = [N(A) \cdot N(A)^2 N(A)]N(S) \\
& \subseteq [N(A) \cdot \{N(A)N(J)\}N(A)]N(S) \\
& = N(A)N(A) \cdot N(S) \\
& = N(S)N(A) \cdot N(A) \\
& \subseteq N(S)N(I) \cdot N(S) \\
& \subseteq N(I),
\end{aligned}$$

which shows that $N(A)$ is a $(0,2)$-ideal of a right ideal $N(S)[N(A)]^2$ of $N(S)$.

(iv) $\Longrightarrow$(v). It is easy to see that $N(S)[N(A)]^3$ is a $(0,2)$-ideal of $N(S)$. Let $N(A)$ be a $(0,2)$-ideal of a right ideal $N(R)$ of $N(S)$, then

$$\begin{aligned} N(A)\cdot N(S)N(A)^3 &= N(A)[N(S)N(S)\cdot N(A)^2N(A)] \\ &= N(A)[N(A)N(A)^2\cdot N(S)] \\ &\subseteq N(A)[\{N(S)N(A)\cdot N(A)N(A)\}N(S)] \\ &= N(A)[\{N(A)N(A)\cdot N(A)N(S)\}N(S)] \\ &= [N(A)N(A)][\{N(A)\cdot N(A)N(S)\}N(S)] \\ &= [N(S)\cdot N(A)\{N(A)N(S)\}]N(A)^2 \\ &= [N(A)\cdot N(S)\{N(A)N(S)\}]N(A)^2 \\ &\subseteq N(R)N(S)\cdot N(A)^2 \subseteq N(R)N(A)^2 \subseteq N(A), \end{aligned}$$

which shows that $N(A)$ is a left ideal of a $(0,2)$-ideal $N(S)[N(A)]^2$ of $N(S)$.

(v) $\Longrightarrow$(i). Let $N(A)$ be a left ideal of a $(0,2)$-ideal $N(O)$ of $N(S)$, then

$$\begin{aligned} N(A)N(S)\cdot N(A)^2 &= [N(A)N(A)\cdot N(S)N(S)]N(A) \\ &= N(S)N(A)^2\cdot N(A) \\ &\subseteq N(S)N(O)^2\cdot N(A) \\ &\subseteq N(O)N(A) \subseteq N(A), \end{aligned}$$

which shows that $N(A)$ is a $(1,2)$-ideal of $N(S)$. ■

Lemma 63 *Let $N(S)$ be a unitary LA -semigroup and $N(A)$ be an idempotent subset of $N(S)$. Then $N(A)$ is a $(1,2)$-ideal of $N(S)$ if and only if there exist a left ideal $N(L)$ and a right ideal $N(R)$ of $N(S)$ such that $N(R)N(L) \subseteq N(A) \subseteq N(R) \cap N(L)$.*

Proof. Assume that $N(A)$ is a $(1,2)$-ideal of $N(S)$ such that $N(A)$ is idempotent: Setting $N(L) = N(S)N(A)$ and $N(R) = N(S)[N(A)]^2$, then

$$\begin{aligned} N(R)N(L) &= N(S)N(A)^2.N(S)N(A) \\ &= N(A)^2N(S)\cdot N(S)N(A) \\ &= [N(S)N(A)\cdot N(S)N(S)]N(A)^2 \\ &= [N(S)N(S)\cdot N(A)N(S)]N(A)^2 \\ &= [N(S)\{N(A)N(A)\cdot N(S)N(S)\}]N(A)^2 \\ &= [N(S)\{N(S)N(S)\cdot N(A)N(A)\}]N(A)^2 \\ &= [N(S)[N(A)\{N(S)N(S)\cdot N(A)\}]]N(A)^2 \\ &= [N(A)\{N(S)\cdot N(S)N(A)\}]N(A)^2 \\ &\subseteq N(A)N(S)\cdot N(A)^2 \subseteq N(A). \end{aligned}$$

It is clear that $N(A) \subseteq N(R) \cap N(L)$.

Conversely, let $N(R)$ be a right ideal and $N(L)$ be a left ideal of $N(S)$ such that $N(R)N(L) \subseteq N(A) \subseteq N(R) \cap N(L)$, then

$N(A)N(S) \cdot N(A)^2 = N(A)N(S) \cdot N(A)N(A) \subseteq N(R)N(S) \cdot N(S)N(L) \subseteq N(R)N(L) \subseteq N(A)$. Assume that $N(S)$ is a unitary LA -semigroup with zero. Then it is easy to see that every left (right) ideal of $N(S)$ is a $(0;2)$-ideal of $N(S)$. Hence if $N(O)$ is a 0-minimal $(0,2)$-ideal of $N(S)$ and $N(A)$ is a left (right) ideal of $N(S)$ contained in $N(O)$, then either $N(A) = \{0\}$ or $N(A) = N(O)$. ■

Lemma 64 *Let $N(S)$ be a unitary LA -semigroup with zero. Assume that $N(A)$ is a 0-minimal ideal of $N(S)$ and $N(O)$ is an LA -subsemigroup of $N(A)$. Then $N(O)$ is a $(0,2)$-ideal of $N(S)$ contained in $N(A)$ if and only if $[N(O)]^2 = \{0\}$ or $N(O) = N(A)$.*

Proof. Let $N(O)$ be a $(0,2)$-ideal of $N(S)$ contained in a 0-minimal ideal $N(A)$ of $N(S)$. Then $N(S)[N(O)]^2 \subseteq N(O) \subseteq N(A)$.

Since $N(S)[N(O)]^2$ is an ideal of $N(S)$, therefore by minimality of $N(A), N(S)[N(O)]^2 = \{0\}$or $N(S)[N(O)]^2 = N(A)$. If $N(S)[N(O)]^2 = N(A)$, then $N(A) = N(S)[N(O)]^2 \subseteq N(O)$ and therefore $N(O) = N(A)$. Let $N(S)[N(O)]^2 = \{0\}$, then $[N(O)]^2N(S) = N(S)[N(O)]^2 = \{0\} \subseteq [N(O)]^2$, which shows that $[N(O)]^2$ is a right ideal of $N(S)$, and hence an ideal of $N(S)$ contained in $N(A)$, therefore by minimality of $N(A)$, we have $[N(O)]^2 = \{0\}$ or $[N(O)]^2 = N(A)$. Now if $[N(O)]^2 = N(A)$, then $N(O) = N(A)$.

Conversely, let $[N(O)]^2 = \{0\}$, then $N(S)[N(O)]^2 = [N(O)]^2N(S) = \{0\}N(S) = \{0\} = [N(O)]^2$. Now if $N(O) = N(A)$, then

$$\begin{aligned} N(S)[N(O)]^2 &= N(S)N(S) \cdot N(O)N(O) \\ &= N(S)N(A) \cdot N(S)N(A) \subseteq N(A) = N(O), \end{aligned}$$

which shows that $N(O)$ is a $(0,2)$-ideal of $N(S)$ contained in $N(A)$. ■

Corollary 65 *2 Let $N(S)$ be a unitary LA -semigroup with zero. Assume that $N(A)$ is a 0-minimal left ideal of $N(S)$ and $N(O)$ is an LA -subsemigroup of $N(A)$. Then $N(O)$ is a $(0,2)$-ideal of $N(S)$ contained in $N(A)$ if and only if $[N(O)]^2 = \{0\}$ or $N(O) = N(A)$.*

Lemma 66 *Let $N(S)$ be a unitary LA -semigroup with zero and $N(O)$ be a 0-minimal $(0,2)$-ideal of $N(S)$. Then $N(O) = \{0\}$ or $N(O)$ is a 0-minimal right (left) ideal of $N(S)$.*

Proof. Let $N(O)$ be a 0-minimal $(0,2)$-ideal of $N(S)$, then

$$\begin{aligned} N(S)[N(O)^2]^2 &= N(S)N(S) \cdot [N(O)]^2[N(O)]^2 \\ &= [N(O)]^2[N(O)]^2 \cdot N(S) \\ &= N(S)[N(O)]^2 \cdot [N(O)]^2 \\ &\subseteq N(O)[N(O)]^2 \subseteq [N(O)]^2 \end{aligned}$$

which shows that $[N(O)]^2$ is a $(0,2)$-ideal of $N(S)$ contained in $N(O)$ therefore by minimality of $N(O)$,

$[N(O)]^2 = \{0\}$ or $[N(O)]^2 = N(O)$. Suppose that $[N(O)]^2 = N(O)$, then

$$N(O)N(S) = N(O)N(O) \cdot N(S)N(S) = N(S)[N(O)]^2 \subseteq N(O)$$

which shows that $N(O)$ is a right ideal of $N(S)$. Let $N(R)$ be a right ideal of $N(S)$ contained in $N(O)$, then $[N(R)]^2N(S) = N(R)N(R) \cdot N(S)$ $\subseteq N(R)$.Thus $N(R)$ is a $(0,2)$-ideal of $N(S)$ contained in $N(O)$ and again by minimality of $N(O)$, $N(R) = \{0\}$ or $N(R) = N(O)$. ■

The following Corollary follows from Lemma 4 and Corollary 2.

Corollary 67 *3 Let $N(S)$ be a unitary LA -semigroup. Then $N(O)$ is a minimal $(0,2)$-ideal of $N(S)$ if and only if $N(O)$ is a minimal left ideal of $N(S)$.*

Theorem 68 *Let $N(S)$ be a unitary LA -semigroup. Then $N(A)$ is a minimal $(2,1)$-ideal of $N(S)$ if and only if $N(A)$ is a minimal bi-ideal of $N(S)$.*

Proof. Let $N(A)$ be a minimal $(2,1)$-ideal of $N(S)$. Then

$$\begin{aligned}
& [\{N(A)^2N(S) \cdot N(A)\}^2N(S)][N(A)^2N(S) \cdot N(A)] \\
= & [[\{N(A)^2N(S) \cdot N(A)\}\{N(A)^2N(S) \cdot N(A)\}]N(S)][N(A)^2NS) \cdot N(A)] \\
\subseteq & [[\{N(A)N(S) \cdot N(A)\}\{N(A)N(S) \cdot N(A)\}]N(S)][N(A)N(S) \cdot N(A)] \\
= & [[\{N(A)N(S) \cdot N(A)N(S)\}\{N(A)N(A)\}]N(S)][N(A)N(S) \cdot N(A)] \\
= & [\{N(A)^2N(S) \cdot N(A)N(A)\}N(S)][N(A)N(S) \cdot N(A)] \\
\subseteq & [\{N(A)N(S) \cdot N(A)N(S)\}N(S)][N(A)N(S) \cdot N(A) \\
\subseteq & [N(A)^2N(S) \cdot N(S)][N(A)N(S) \cdot N(A)] \\
= & [N(A)N(S) \cdot N(A)N(S)][N(S)N(A)] \\
= & N(A)^2N(S) \cdot N(S)N(A) \\
= & N(A)N(S) \cdot N(S)N(A)^2 \\
= & [N(S)N(A)^2 \cdot N(S)]N(A) \\
= & [N(A)^2N(S) \cdot N(S)]N(A) \\
= & [N(S)N(S) \cdot N(A)N(A)]N(A) \\
= & N(A)^2N(S) \cdot N(A),
\end{aligned}$$

and similarly we can show that $[N(A)^2N(S) \cdot N(A)]^2 \subseteq N(A)^2N(S) \cdot N(A)$. Thus $N(A)^2N(S) \cdot N(A)$ is a $(2,1)$-ideal of $N(S)$ contained in $N(A)$, therefore by minimality of $N(A)$,$N(A)^2N(S)\cdot N(A)=N(A)$. Now

$$\begin{aligned}N(A)N(S)\cdot N(A) &= [N(A)N(S)][N(A)^2N(S)\cdot N(A)]\\ &= [\{N(A)^2N(S)\cdot N(A)\}N(S)]N(A)\\ &= [N(S)N(A)\cdot N(A)^2N(S)]N(A)\\ &= [N(A)^2\{N(S)N(A)\cdot N(S)]N(A)\\ &\subseteq N(A)^2N(S)\cdot N(A)=N(A),\end{aligned}$$

It follows that $N(A)$ is a bi-ideal of $N(S)$. Suppose that there exists a bi-ideal $N(B)$ of $N(S)$ contained in $N(A)$,

then $[N(B)]^2N(S)\cdot N(B)\subseteq N(B)N(S)\cdot N(B)$, so $N(B)$ is a $(2,1)$-ideal of $N(S)$ contained in $N(A)$, therefore $N(B)=N(A)$.

Conversely, assume that $N(A)$ is a minimal bi-ideal of $N(S)$ then it is easy to see that $N(A)$ is a $(2,1)$-ideal of $N(S)$. Let C be a $(2,1)$-ideal of $N(S)$ contained in $N(A)$, then

$$\begin{aligned}&[\{N(C)^2N(S)\cdot N(C)\}N(S)][N(C)^2N(S)\cdot N(C)]\\ =\;& [N(S)N(C)\cdot N(C)^2N(S)][N(C)^2N(S)\cdot N(C)]\\ =\;& [N(S)N(C)^2\cdot N(C)N(S)][N(C)^2N(S)\cdot N(C)]\\ =\;& [N(C)\{N(S)N(C)^2\cdot N(S)][N(C)^2N(S)\cdot N(C)]\\ =\;& [N(C)^2N(S)\cdot N(C)][N(S)N(C)^2\cdot N(S)N(S)]N(C)\\ =\;& [\{N(C)^2N(S)\cdot N(C)\}\{N(S)\cdot N(C)^2N(S)\}]N(C)\\ =\;& [\{N(C)^2N(S)\cdot N(C)\}\{N(C)^2N(S)\}]N(C)\\ =\;& [N(C)^2[\{N(C)^2N(S)\cdot N(C)\}N(S)]N(C)\\ \subseteq\;& N(C)^2N(S)\cdot N(C).\end{aligned}$$

This shows that $N(C)^2N(S)\cdot N(C)$ is a bi-ideal of $N(S)$ and by minimality of $N(A), N(C)^2N(S)\cdot N(C)=N(A)$ Thus $N(A)=N(C)^2N(S)\cdot N(C)\subseteq N(C)$, and therefore $N(A)$ is a minimal $(2,1)$-ideal of $N(S)$. ■

Theorem 69 *3.Let $N(A)$ be a 0-minimal $(0,2)$-bi-ideal of a unitary LA -semigroup $N(S)$ with zero. Then exactly one of the following cases occurs:*

(i) $N(A)=\{0,a+bI\},(a+bI)^2=0$;
(ii) $\forall a+bI\in N(A)\backslash\{0\},N(S)(a+bI)^2=N(A)$.

Proof. Assume that $N(A)$ is a 0-minimal $(0,2)$-bi-ideal of $N(S)$. Let $a+bI\in N(A)\backslash\{0\}$, then $N(S)(a+bI)^2\subseteq N(A)$. Also $N(S)(a+bI)^2$ is a $(0,2)$-bi-ideal of $N(S)$, therefore $N(S)(a+bI)^2=\{0\}$ or $N(S)(a+bI)^2=N(A)$.

Let $N(S)(a+bI)^2=\{0\}$. Since $(a+bI)^2\in N(A)$, we have either $(a+bI)^2=a+bI$ or $(a+bI)^2=0$ or $(a+bI)^2\in N(A)\backslash\{0,a+bI\}$. If $(a+bI)^2=a+bI$,

then $(a+bI)^3 = (a+bI)^2(a+bI) = (a+bI)$, which is impossible because $(a+bI)^3 \in (a+bI)^2N(S) = N(S)(a+bI)^2 = \{0\}$. Let $(a+bI)^2 \in N(A)\backslash\{0, a+bI\}$,we
have

$$\begin{aligned} & N(S)\cdot\{0,(a+bI)^2\}\{0,(a+bI)^2\} \\ = \; & N(S)N(S)\cdot(a+bI)^2(a+bI)^2 \\ = \; & N(S)(a+bI)^2\cdot N(S)(a+bI)^2 \\ = \; & \{0\}\subseteq\{0,(a+bI)^2\}, \end{aligned}$$

and

$$\begin{aligned} & [\{0,(a+bI)^2\}N(S)]\{0,(a+bI)^2\} \\ = \; & \{0,(a+bI)^2N(S)\}\{0,(a+bI)^2\} \\ = \; & (a+bI)^2S\cdot(a+bI)^2 \\ \subseteq \; & N(S)(a+bI)^2 = \{0\}\subseteq\{0,(a+bI)^2\}. \end{aligned}$$

Therefore $\{0,(a+bI)^2\}$ is a $(0,2)$-bi-ideal of $N(S)$ contained in $N(A)$. We observe that $\{0,(a+bI)^2\} \neq \{0\}$and $\{0,(a+bI)^2\} \neq N(A)$. This is a contradiction to the fact that $N(A)$ is a 0-minimal $(0,2)$-bi-ideal of $N(S)$.Therefore $(a+bI)^2 = 0$ and $N(A) = \{0, a+bI\}$. If $N(S)(a+bI)^2 \neq \{0\}$,then $N(S)(a+bI)^2 = N(A)$. ■

Corollary 70 *4. Let $N(A)$ be a 0-minimal $(0,2)$-bi-ideal of a unitary LA -semigroup $N(S)$ with zero such that $N(A)^2 \neq 0$. Then $N(A) = N(S)(a+bI)^2$ for every $a+bI \in N(A)\backslash\{0\}$.*

Lemma 71 *6. Let $N(S)$ be a unitary LA -semigroup. Then every right ideal of $N(S)$ is a $(0,2)$-bi-ideal of $N(S)$.*

Proof. Assume that $N(A)$ is a right ideal of $N(S)$, then

$$\begin{aligned} N(S)N(A)^2 &= N(A)N(A)\cdot N(S)N(S) \\ &= N(A)N(S)\cdot N(A)N(S) \\ &\subseteq N(A)N(A)\subseteq N(A)N(S) \\ &\subseteq N(A), N(A)N(S)\cdot N(A)\subseteq N(A), \end{aligned}$$

and clearly $N(A)^2 \subseteq N(A)$ therefore $N(A)$ is a $(0,2)$-bi-ideal of $N(S)$.

The converse of Lemma 6 is not true in general. Example 1 shows that there exists a $(0,2)$-bi-ideal $N(A) = \{a+bI, c+dI, e_1+e_2I\}$ of $N(S)$ which is not a right ideal of $N(S)$. ■

Theorem 72 *4. Let $N(S)$ be a unitary LA -semigroup with zero. Then $N(S)(a+bI)^2 = N(S)\forall(a+bI) \in N(S)\backslash\{0\}$ if and only if $N(S)$ is $0-(0,2)$-bisimple if and only if $N(S)$ is right 0-simple.*

Proof. Assume that $N(S)(a+bI)^2 = N(S)$ for every $a+bI \in N(S)\backslash\{0\}$. Let $N(A)$ be a $(0,2)$-bi-ideal of $N(S)$ such that $N(A) \neq \{0\}$. Let $a+bI \in N(A)\backslash\{0\}$, then $N(S) = N(S)(a+bI)^2 \subseteq N(S)N(A)^2 \subseteq N(A)$. Therefore $N(S) = N(A)$. Since $N(S) = N(S)(a+bI)^2 \subseteq N(S)N(S) =\subseteq N(S)^2$, we have $N(S)^2 = N(S) \neq \{0\}$.Thus $N(S)$ is $0-(0,2)$-bisimple. The converse statement follows from Corollary 4.

Let $N(R)$ be a right ideal of $0-(0,2)$-bisimple $N(S)$. Then by Lemma 6, $N(R)$ is a $(0,2)$-bi-ideal of $N(S)$ and so $N(R) = \{0\}$ or $N(R) = N(S)$.

Conversely, assume that $N(S)$ is right 0-simple. Let $a+bI \in N(S)\backslash\{0\}$, then $N(S)(a+bI)^2 = N(S)$. Hence $N(S)$ is $0-(0,2)$-bisimple. ∎

Theorem 73 *5. Let $N(A)$ be a 0-minimal $(0,2)$-bi-ideal of a unitary LA -semigroup $N(S)$ with zero. Then either $N(A)^2 = \{0\}$ or $N(A)$ is right 0-simple.*

Proof. . Assume that $N(A)$ is 0-minimal $(0,2)$-bi-ideal of $N(S)$ such that $N(A)^2 \neq \{0\}$ Then by using

Corollary 4, $N(S)(a+bI)^2 = N(A)$ for every $a+bI \in N(A)\backslash\{0\}$. Since $(a+bI)^2 \in N(A)\backslash\{0\}$ for every $a+bI \in N(A)\backslash\{0\}$, we have

$(a+bI)^4 = [(a+bI)^2]^2 \in N(A)\backslash\{0\}$ for every $a+bI \in N(A)\backslash\{0\}$. Let $a+bI \in N(A0\backslash\{0\}$, then

$$\begin{aligned}
& [N(A)(a+bI)^2]N(S) \cdot N(A)(a+bI)^2 \\
= \;& (a+bI)^2N(A) \cdot N(S)[N(A)(a+bI)^2] \\
= \;& [\{N(S) \cdot N(A)(a+bI)^2\}N(A)](a+bI)^2 \\
\subseteq \;& [\{N(S) \cdot N(A)\}N(A)](a+bI)^2 \\
= \;& [N(A)N(A) \cdot N(S)N(S)](a+bI)^2 \\
= \;& N(S)N(A)^2 \cdot (a+bI)^2 \subseteq N(A)(a+bI)^2,
\end{aligned}$$

and

$$\begin{aligned}
& N(S)[N(A)(a+bI)^2]^2 \\
= \;& N(S)[N(A)(a+bI)^2 \cdot N(A)(a+bI)^2] \\
= \;& N(S)[(a+bI)^2N(A) \cdot (a+bI)^2N(A)] \\
= \;& N(S)[(a+bI)^2\{(a+bI)^2N(A) \cdot N(A)\}] \\
= \;& (a+bI)(a+bI)[N(S)\{(a+bI)^2N(A) \cdot N(A)\}] \\
= \;& [\{(a+bI)^2N(A) \cdot N(A)\}N(S)](a+bI)^2 \\
\subseteq \;& [N(A)N(A) \cdot N(S)N(S)](a+bI)^2 \\
= \;& N(S)N(A)^2 \cdot (a+bI)^2 \subseteq N(A)(a+bI)^2,
\end{aligned}$$

which shows that $N(A)(a+bI)^2$ is a $(0,2)$-bi-ideal of $N(S)$ contained in $N(A)$. Hence $N(A)(a+bI)^2 = \{0\}$ or $N(A)(a+bI)^2 = N(A)$.

Since $(a+bI)^4 \in N(A)(a+bI)^2$ and $(a+bI)^4 \in N(A)\backslash\{0\}$, we get $N(A)(a+bI)^2 = N(A)$. Thus by using Theorem 4, $N(A)$ is right 0-simple. ■

1.3 (m,n)-ideals in unitary LA -semigroups

In this section, we characterize a unitary LA -semigroup in terms of (m,n)-ideals with the assumption that $m,n \geq 5$. If we take $m,n \geq 2$, then all the results of this section can be trivially followed for a locally associative unitary LA -semigroup. If $N(S)$ is a unitary LA -semigroup, then it is easy to see that $N(S)N(A)^m = N(A)^m N(S)$ and $N(A)^m N(A)^n = N(A)^n N(A)^m$ for $m,n \geq 3$ such that $N(A) = e$ if occurs, where e is a left identity of $N(S)$.

Lemma 74 *Let $N(S)$ be a unitary LA -semigroup. If $N(R)$ and $N(L)$ are the right and left ideals of $N(S)$ respectively, then $N(R)N(L)$ is an (m,n)-ideal of $N(S)$.*

Proof. Let $N(R)$ and $N(L)$ be the right and left ideals of $N(S)$ respectively,then

$$
\begin{aligned}
& [N(R)N(L)]^m N(S) \cdot [N(R)N(L)]^n \\
= \; & [N(R)^m N(L)^m \cdot N(S)][N(R)^n N(L)^n] \\
= \; & [N(R)^m N(L)^m \cdot N(R)^n][N(S)N(L)^n] \\
= \; & [N(L)^m N(R)^m \cdot N(R)^n][N(S)N(L)^n] \\
= \; & [N(R)^n N(R)^m \cdot N(L)^m][N(S)N(L)^n] \\
= \; & [N(R)^m N(R)^n \cdot N(L)^m][N(S)N(L)^n] \\
= \; & [N(R)^{m+n} N(L)^m][N(S)N(L)^n] \\
= \; & N(S)[N(R)^{m+n} N(L)^m \cdot N(L)^n] \\
= \; & N(S)[N(L)^n N(L)^m \cdot N(R)^{m+n}] \\
= \; & N(S)N(S) \cdot N(L)^{m+n} N(R)^{m+n} \\
= \; & N(S)N(L)^{m+n} \cdot N(S)N(R)^{m+n} \\
= \; & N(R)^{m+n} N(S) \cdot N(L)^{m+n} N(S) \\
= \; & N(S)N(R)^{m+n} \cdot N(S)N(L)^{m+n},
\end{aligned}
$$

and

$$\begin{aligned}
& N(S)N(R)^{m+n} \cdot N(S)N(L)^{m+n} \\
= \; & [N(S) \cdot N(R)^{m+n-1}N(R)][N(S) \cdot N(L)^{m+n-1}N(L)] \\
= \; & [N(S)\{N(R)^{m+n-2}N(R) \cdot N(R)\}][N(S)\{N(L)^{m+n-2}N(L) \cdot N(L)\}] \\
= \; & [N(S)\{N(R)N(R) \cdot N(R)^{m+n-2}\}][N(S)\{N(L)N(L) \cdot N(L)^{m+n-2}\}] \\
\subseteq \; & [N(S)N(S) \cdot N(R)N(R)^{m+n-2}][N(S)N(S) \cdot N(L)N(L)^{m+n-2}] \\
\subseteq \; & [N(S)N(R) \cdot N(S)N(R)^{m+n-2}][N(S)N(L) \cdot N(S)N(L)^{m+n-2}] \\
\subseteq \; & [N(R)^{m+n-2}N(S) \cdot N(R)N(S)][N(L) \cdot N(S)N(L)^{m+n-2}] \\
\subseteq \; & [N(R)^{m+n-2}N(S) \cdot N(R)][N(S) \cdot N(L)N(L)^{m+n-2}] \\
= \; & [N(R)N(S) \cdot N(R)^{m+n-2}][N(S)N(L)^{m+n-1}] \\
\subseteq \; & N(R)N(R)^{m+n-2} \cdot N(S)N(L)^{m+n-1} \\
\subseteq \; & N(S)N(R)^{m+n-1} \cdot N(S)N(L)^{m+n-1},
\end{aligned}$$

therefore

$$\begin{aligned}
& [N(R)N(L)]^m N(S) \cdot [N(R)N(L)]^n \\
\subseteq \; & N(S)N(R)^{m+n} \cdot N(S)N(L)^{m+n} \\
\subseteq \; & N(S)N(R)^{m+n-1} \cdot N(S)N(L)^{m+n-1} \subseteq ... \subseteq N(S)N(R) \cdot N(S)N(L) \\
\subseteq \; & [N(S)N(S) \cdot N(R)]N(L) \\
= \; & [N(R)N(S) \cdot N(S)]N(L) \\
\subseteq \; & N(R)N(L),
\end{aligned}$$

and also

$$\begin{aligned}
& N(R)N(L) \cdot N(R)N(L) \\
= \; & N(L)N(R) \cdot N(L)N(R) \\
= \; & [N(L)N(R) \cdot N(R)]N(L) \\
= \; & [N(R)N(R) \cdot N(L)]N(L) \\
\subseteq \; & [N(R)N(S) \cdot N(S)]N(L) \\
\subseteq \; & N(R)N(L),
\end{aligned}$$

This shows that $N(R)N(L)$ is an (m,n)-ideal of $N(S)$. ■

Theorem 75 *Let $N(S)$ be a unitary LA -semigroup with zero. If $N(S)$ has the property that it contains no non-zero nilpotent (m,n)-ideals and $R(L)$ is a $0-$minimal right (left) ideal of $N(S)$ then either $N(R)N(L) = \{0\}$ or $N(R)N(L)$ is a 0-minimal (m,n)-ideal of $N(S)$.*

Proof. Assume that $R(L)$ is a 0-minimal right (left) ideal of $N(S)$ such that $N(R)N(L) \neq \{0\}$, then by lemma 7, $N(R)N(L)$ is an (m,n)-ideal of $N(S)$. Now we show that $N(R)N(L)$ is a 0-minimal (m,n)-ideal of $N(S)$.

Let $\{0\} \neq M \subseteq N(R)N(L)$ be an (m,n)-ideal of $N(S)$. Note that since $N(R)N(L) \subseteq N(R) \cap N(L)$, we have $N(M) \subseteq N(R) \cap N(L)$. Hence $N(M) \subseteq N(R)$ and $N(M) \subseteq N(L)$. By hypothesis, $N(M)^m \neq \{0\}$ and $N(M)^n \neq \{0\}$. Since $\{0\} \neq N(S)N(M)^m = N(M)^m N(S)$,

therefore

$$\begin{aligned}
\{0\} &\neq N(M)^m N(S) \subseteq N(R)^m N(S) \\
&= N(R)^{m+1} N(R) \cdot N(S) \\
&= N(S)N(R) \cdot N(R)^{m-1} \\
&= N(S)N(R) \cdot N(R)^{m-2} N(R) \\
&= N(R)N(R)^{m-2} \cdot N(R)N(S) \\
&\subseteq N(R)N(R)^{m+2} \cdot N(R) = N(R)^m,
\end{aligned}$$

and

$$\begin{aligned}
N(R)^m &\subseteq N(S)N(R)^m = N(S)N(S) \cdot N(R)N(R)^{m-1} \\
&= N(R)^{m-1} N(R) \cdot N(S) \\
&= [N(R)^{m-2} N(R) \cdot N(R)]N(S) \\
&= [N(R)N(R) \cdot N(R)^{m-2}]N(S) \\
&= N(S)N(R)^{m-2} \cdot N(R)N(R) \\
&\subseteq N(S)N(R)^{m-2} \cdot N(R) \\
&= [N(S)N(S) \cdot N(R)^{m-3} N(R)]N(R) \\
&= [N(R)N(R)^{m-3} \cdot N(S)N(S)]N(R) \\
&= [N(R)N(S) \cdot N(R)^{m-3} N(S)]N(R) \\
&\subseteq [N(R) \cdot N(R)^{m-3} N(S)]N(R) \\
&= [N(R)^{m-3} \cdot N(R)N(S)]N(R) \\
&\subseteq N(R)^{m-3} N(R) \cdot N(R) = N(R)^{m-1},
\end{aligned}$$

therefore $\{0\} \neq N(M)^m N(S) \subseteq N(R)^m \subseteq N(R)^{m-1} \subseteq \cdots \subseteq N(R)$. It is easy to see that $N(M)^m N(S)$is a right ideal of $N(S)$.

Thus $N(M)^m N(S) = N(R)$ since $N(R)$ is 0-minimal. Also

$$\begin{aligned}
\{0\} &\neq N(S)N(M)^n \subseteq \{0\} \neq N(S)N(L)^n = N(S) \cdot N(L)^{n-1} N(L) \\
&= N(L)^{n-1} \cdot N(S)N(L) \subseteq N(L)^{n-1} N(L) = N(L)^n,
\end{aligned}$$

and

$$\begin{aligned} N(L)^n &\subseteq N(S)N(L)^n = N(S)N(S) \cdot N(L)N(L)^{n-1} \\ &= N(L)^{n-1}N(L) \cdot N(S) \\ &= N(L)^{n-2}N(L) \cdot N(L)]N(S) \\ &= N(S)N(L) \cdot N(L)^{n-2}N(L) \\ &\subseteq N(L) \cdot N(L)^{n-2}N(L) \\ &= N(L)^{n-2} \cdot N(L)N(L) \subseteq N(L)^{n-2}N(L) \\ &= N(L)^{n-1} \subseteq ... \subseteq N(L), \end{aligned}$$

therefore $\{0\} \neq N(S)N(M)^n \subseteq N(L)^n \subseteq N(L)^{n-1} \subseteq ... \subseteq N(L)$. It is easy to see that $N(S)N(M)^n$ is a left ideal of $N(S)$

Thus $N(S)N(M)^n = N(L)$ since $N(L)$ is 0-minimal. Therefore

$$\begin{aligned} N(M) &\subseteq N(R)N(L) = N(M)^m N(S) \cdot N(S)N(M)^n \\ &= N(M)^n N(S) \cdot N(S)N(M)^m \\ &= [N(S)N(M)^m \cdot N(S)]N(M)^n \\ &= [N(S)N(M)^m \cdot N(S)N(S)]N(M)^n \\ &= [N(S) \cdot N(M)^m N(S)]N(M)^n \\ &= [N(M)^m \cdot N(S)N(S)]N(M)^n \\ &= N(M)^m N(S) \cdot N(M)^n \subseteq N(M). \end{aligned}$$

Thus $N(M) = N(R)N(L)$, which means that $N(R)N(L)$ is a 0-minimal (m, n)-ideal of $N(S)$. ■

Theorem 76 *Let $N(S)$ be a unitary LA -semigroup. If $R(L)$ is a 0-minimal right (left) ideal of $N(S)$, then either $N(R)^m N(L)^n = \{0\}$ or $N(R)^m N(L)^n$ is a 0-minimal (m, n)-ideal of $N(S)$.*

Proof. Assume that $R(L)$ is a 0-minimal right (left) ideal of $N(S)$ such that $N(R)^m N(L)^n \neq \{0\}$ then

$N(R)^m \neq \{0\}$ and $N(L)^n \neq \{0\}$ Hence $\{0\} \neq N(R)^m \subseteq N(R)$ and $\{0\} \neq N(L)^n \subseteq N(L)$, which shows that $N(R)^m = N(R)$and $N(L)^n = N(L)$ since $R(L)$ is a 0-minimal right (left) ideal of $N(S)$. Thus by lemma 7, $N(R)^m N(L)^n = N(R)N(L)$ is an (m, n)-ideal of $N(S)$. Now we show that $N(R)^m N(L)^n$ is a 0-minimal (m, n)-ideal of $N(S)$. Let $\{0\} \neq N(M) \subseteq N(R)^m N(L)^n = N(R)N(L) \subseteq N(R) \cap N(L)$ be an (m, n)-ideal of $N(S)$. Hence

$$\begin{aligned} \{0\} &\neq N(S)N(M)^2 = N(M)N(M) \cdot N(S)N(S) \\ &= N(M)N(S) \cdot N(M)N(S) \\ &\subseteq N(R)N(S) \cdot N(R)N(S) \subseteq N(R) \end{aligned}$$

and $\{0\} \neq N(S)N(M) \subseteq N(S)N(L) \subseteq N(L)$. Thus

$$\begin{aligned} N(R) &= N(S)N(M)^2 = N(M)N(M) \cdot N(S)N(S) \\ &= N(S)N(M) \cdot N(M) \subseteq N(S)N(M) \end{aligned}$$

and $N(S)N(M) = N(L)$ since $R(L)$ is a 0-minimal right (left) ideal of $N(S)$. Therefore

$$\begin{aligned} N(M) &\subseteq N(R)^m N(L)^n \subseteq [N(S)N(M)]^m [N(S)N(M)]^n \\ &= N(S)^m N(M)^m \cdot N(S)^n N(M)^n \\ &= N(S)N(S) \cdot N(M)^m N(M)^n \\ &= N(M)^n N(M)^m \cdot N(S) = N(S)N(M)^m \cdot N(M)^n \\ &= N(M)^m N(S) \cdot N(M)^n \subseteq N(M), \end{aligned}$$

Thus $N(M) = N(R)^m N(L)^n$,which shows that $N(R)^m N(L)^n$ is a 0-minimal (m,n)-ideal of $N(S)$. ■

Theorem 77 *Let $N(S)$ be a unitary LA -semigroup with zero. Assume that $N(A)$ is an (m,n)-ideal of $N(S)$ and $N(B)$ is an (m,n)-ideal of $N(A)$ such that $N(B)$ is idempotent. Then $N(B)$ is an (m,n)-ideal of $N(S)$.*

Proof. It is trivial that $N(B)$ is an LA -subsemigroup $N(S)$. Secondly, since $N(A)^m N(S) \cdot N(A)^n \subseteq N(A)$ and $N(B)^m N(A) \cdot N(B)^n \subseteq N(B)$, then

$$
\begin{aligned}
& N(B)^m N(S) \cdot N(B)^n \\
= \;& [N(B)^m N(B)^m \cdot N(S)][N(B)^n N(B)^n] \\
= \;& [N(B)^n N(B)^n)][N(S) \cdot N(B)^m N(B)^m] \\
= \;& [\{N(S) \cdot N(B)^m N(B)^m\} N(B)^n] N(B)^n \\
= \;& [\{N(B)^n \cdot N(B)^m N(B)^m\}\{N(S)N(S)\}] N(B)^n \\
= \;& [\{N(B)^m \cdot N(B)^n N(B)^m\}\{N(S)N(S)\}] N(B)^n \\
= \;& [N(S)\{N(B)^n N(B)^m \cdot N(B)^m)] N(B)^n \\
= \;& [N(S)\{N(B)^n N(B)^m \cdot N(B)^{m-1} N(B)\}] N(B)^n \\
= \;& [N(S)\{N(B)N(B)^{m-1} \cdot N(B)^m N(B)^n\}] N(B)^n \\
= \;& [N(S)\{N(B)^m \cdot N(B)^m N(B)^n\}] N(B)^n \\
= \;& [N(B)^m\{N(S)N(S) \cdot N(B)^m N(B)^n)\}] N(B)^n \\
= \;& [N(B)^m\{N(B)^n N(B)^m \cdot N(S)N(S)\}] N(B)^n \\
= \;& [N(B)^m\{N(S)N(B)^m \cdot N(B)^n\}] N(B)^n \\
= \;& [N(B)^m[\{N(S)N(S) \cdot N(B)^{m-1} N(B)\} N(B)^n]] N(B)^n \\
= \;& [N(B)^m\{N(B)^m N(S) \cdot N(B)^n] N(B)^n \\
\subseteq \;& [N(B)^m\{N(A)^m N(S) \cdot N(A)^n\}] N(B)^n \\
\subseteq \;& N(B)^m A \cdot N(B)^n \subseteq B,
\end{aligned}
$$

which shows that $N(B)$ is an(m, n)-ideal of $N(S)$. ■

Lemma 78 *Let $\langle a \rangle_{(m,n)} = (a + bI)^m N(S) \cdot (a + bI)^n$, then $\langle a \rangle_{(m,n)}$ is an (m, n)-ideal of a unitary LA -semigroup $N(S)$.*

Proof. Assume that N(S) is a unitary LA -semigroup and m,n are non-negative integers, then

$$
\begin{aligned}
& \langle a \rangle_{(m,n)} N(S) \cdot \langle a \rangle_{(m,n)} \\
= \;& [\{(a + bI)^m S \cdot (a + bI)\} N(S)][(a + bI)^m N(S) \cdot (a + bI)^n] \\
= \;& [(a + bI)^n \cdot (a + bI)^m N(S)[N(S)\{(a + bI)^m N(S) \cdot (a + bI)^n\}]^n \\
= \;& [\{N(S)\{(a + bI)^m N(S) \cdot (a + bI)^n\}\}\{(a + bI)^m N(S)\}](a + bI) \\
= \;& [(a + bI)^m[\{N(S)\{(a + bI)^m N(S) \cdot (a + bI)^n\}\} N(S)]](a + bI)^n \\
\subseteq \;& (a + bI)^m N(S) \cdot (a + bI)^n = \langle a \rangle_{(m,n)},
\end{aligned}
$$

and similarly we can show that $(\langle a \rangle_{(m,n)})^2 \subseteq \langle a \rangle_{(m,n)}$. ■

Theorem 79 *9. Let $N(S)$ be a unitary LA -semigroup and $\langle a+bI\rangle_{(m,n)}$ be an (m,n)-ideal of $N(S)$. Then the following statements hold:*
(i)$(\langle a+bI\rangle_{(1,0)})^m N(S) = (a+bI)^m N(S)$;
(ii) $N(S)(\langle a+bI\rangle_{(0,1)})^n = N(S)(a+bI)^n$;
(iii) $(\langle a+bI\rangle_{(1,0)})^m N(S) \cdot (\langle a+bI\rangle_{(0,1)})^n = [(a+bI)^m N(S)](a+bI)^n$.

Proof. (i).As $\langle a+bI\rangle_{(1,0)} = (a+bI)N(S)$, we have

$$\begin{aligned}
(\langle a+bI\rangle_{(1,0)})^m N(S) &= [(a+bI)N(S)]^m N(S) \\
&= [(a+bI)N(S)\}^{m-1}[(a+bI)N(S)] \cdot N(S) \\
&= N(S)[(a+bI)N(S)] \cdot [(a+bI)N(S)]^{m-1} \\
&= [(a+bI)N(S)][(a+bI)N(S)]^{m-1} \\
&= [(a+bI)N(S)][\{(a+bI)N(S)\}^{m-2}\{(a+bI)N(S)\}] \\
&= [(a+bI)N(S)]^{m-2}[(a+bI)N(S) \cdot (a+bI)N(S)] \\
&= [(a+bI)N(S)]^{m-2}[(a+bI)^2 N(S)] \\
&= ... = [(a+bI)N(S)]^{m-(m-1)}[(a+bI)^{m-1}N(S) \text{ [if m is odd]} \\
&= ... = [(a+bI)^{m-1}N(S)][(a+bI)N(S)] \text{ [if m is even]} \\
&= (a+bI)^m N(S).
\end{aligned}$$

Analogously, we can prove (ii) and (iii) is simple. ■

Corollary 80 *Let $N(S)$ be a unitary LA -semigroup and let $\langle a\rangle_{(m,n)}$ be an (m,n)-ideal of $N(S)$. Then the following statements hold:*
(i) $(\langle a+bI\rangle_{(1,0)})^m N(S) = N(S)(a+bI)^m$;
(ii) $N(S)(\langle a+bI\rangle_{(1,0)})^n = (a+bI)^n N(S)$;
(iii) $(\langle a+bI\rangle_{(1,0)})^m N(S) \cdot (\langle a+bI\rangle_{(1,0)})^n = [N(S)(a+bI)^m][(a+bI)^n N(S)]$.

Let $L_{(0,n)}$,$R_{(m,0)}$ and $A_{(m,n)}$ denote the sets of $(0,n)$-ideal, $(m,0)$-ideals and (m,n)-ideals of an LA -semigroup $N(S)$ respectively.

Theorem 81 *If $N(S)$ is a unitary LA -semigroup; then the following statements hold:*

(i) $N(S)$ is $(0,1)$-regular if and only if $\forall\ N(L) \in N(L)_{(0,1)}$, $N(L) = N(S)N(L)$;
(ii) $N(S)$ is $(2,0)$-regular if and only if $\forall\ N(R) \in N(R)_{(2,0)}$, $N(R) = N(R)^2 N(S)$ such that every $N(R)$ is semiprime;
(iii) $N(S)$ is $(0,2)$-regular if and only if $\forall\ N(U) \in N(U)_{(0,2)}$, $N(U) = N(U)^2 N(S)$ such that every $N(U)$ is semiprime.

Proof. Let $N(S)$ be $(0,1)$-regular, then for $a \in S$ there exists $x \in S$ such that $a+bI = (x+yI)(a+bI)$.Since $N(L)$ is $(0,1)$-ideal, therefore $N(S)N(L) \subseteq N(L)$. Let $a \in L$, then $a+bI = (x+yI)(a+bI) \in N(S)N(L) \subseteq N(L)$.Hence $N(L) = N(S)N(L)$. Converse is simple.

(ii): Let $N(S)$ be $(2,0)$-regular and $N(R)$ be $(2,0)$-ideal of $N(S)$, then it is easy to see that $N(R) = N(R)^2N(S)$. Now for $a+bI \in N(S)$ there exists $x+yI \in N(S)$ such that $a+bI = (a+bI)^2(x+yI)$. Let $(a+bI)^2 \in N(R)$, then $a+bI = (a+bI)^2(x+yI) \in N(R)N(S) = N(R)^2N(S) \cdot N(S) = N(S)N(S) \cdot N(R)^2 = N(R)^2N(S) = N(R)$, which shows that every $(2,0)$-ideal is semiprime.

Conversely, let $N(R) = N(R)^2N(S)$ for every $N(R) \in N(R)_{(2,0)}$. Since $N(S)(a+bI)^2$ is a $(2,0)$-ideal of $N(S)$ such that $(a+bI)^2 \in N(S)(a+bI)^2$, therefore $(a+bI) \in N(S)(a+bI)^2$. Thus

$$
\begin{aligned}
a+bI &\in N(S)(a+bI)^2 \\
&= [N(S)(a+bI)^2]^2N(S) \\
&= [N(S)(a+bI)^2 \cdot N(S)(a+bI)^2]N(S) \\
&= [(a+bI)^2N(S) \cdot (a+bI)^2N(S)]N(S) \\
&= [(a+bI)^2\{(a+bI)^2N(S) \cdot N(S)\}]N(S) \\
&= [(a+bI)^2 \cdot N(S)(a+bI)^2]N(S) \\
&= [N(S) \cdot N(S)(a+bI)^2](a+bI)^2 \\
&\subseteq N(S)(a+bI)^2 = (a+bI)^2N(S).
\end{aligned}
$$

This implies that $N(S)$ is $(2,0)$-regular.

Analogously, we can prove (iii). ■

Lemma 82 *If $N(S)$ is a unitary LA -semigroup; then the following statements hold:*

(i) If $N(S)$ is $(0,n)$-regular, then $\forall\, N(L) \in N(L)_{(0,n)}, N(L) = N(S)N(L)^n$;

(ii) If $N(S)$is $(m,0)$-regular, then $\forall N(R) \in N(R)_{(m,0)}, N(R) = N(R)^mN(S)$;

(iii) If $N(S)$ is (m,n)-regular, then $\forall N(U) \in N(A)_{(m,n)}, N(U) = [N(U)^mN(S)]N(U)^n$.

Proof. It is simple. ■

Corollary 83 *If $N(S)$ is a unitary LA -semigroup; then the following statements hold:*

(i) If $N(S)$ is $(0,n)$-regular, then $\forall N(L) \in N(L)_{(0,n)}, N(L) = N(L)^nN(S)$;

(ii) If $N(S)$ is $(m,0)$-regular, then $\forall N(R) \in N(R)_{(m,0)}, N(R) = N(S)N(R)^m$;

(iii) If $N(S)$ is (m,n)-regular, then $\forall N(U) \in N(A)_{(m,n)}, N(U) = N(U)^{m+n}N(S) = N(S)N(U)^{m+n}$.

Theorem 84 *Let $N(S)$ be a unitary (m,n)-regular LA -semigroup such that $m=n$. Then for every $N(R)\in N(R)_{(m,0)}$ and $N(L)\in N(L)_{(0,n)}$, $N(R)\cap N(L)=N(R)^{m}N(L)\cap N(R)N(L)^{n}$.*

Proof. It is simple. ■

Theorem 85 *Let $N(S)$ be a unitary (m,n)-regular LA -semigroup. If $N(M)[N(J)]$ is a 0-minimal $(m,0)$-ideal ($(0,n)$-ideal) of $N(S)$ such that $N(M)N(J)\subseteq N(M)\cap N(J)$, then either $N(M)N(J)=\{0\}$ or $N(M)N(J)$ is a 0-minimal (m,n)-ideal of $N(S)$.*

Proof. Let $N(M)[N(J)]$ be a 0-minimal $(m,0)$-ideal ($(0,n)$-ideal) of $N(S)$. Let $N(O)=N(M)N(J)$, then clearly $N(O)^{2}\subseteq N(O)$. Moreover

$$\begin{aligned}
N(O)^{m}N(S)\cdot N(O)^{n} &= [N(M)N(J)]^{m}N(S)\cdot [N(M)N(J)]^{n}\\
&= [N(M)^{m}N(J)^{m}]N(S)\cdot N(M)^{n}N(J)^{n}\\
&\subseteq [N(M)^{m}N(S)]N(S)\cdot N(S)N(J)^{n}\\
&= N(S)N(M)^{m}\cdot N(S)N(J)^{n}\\
&= N(M)^{m}N(S)\cdot N(S)N(J)^{n}\\
&\subseteq N(M)N(J)=N(O),
\end{aligned}$$

which shows that $N(O)$ is an (m,n)-ideal of $N(S)$. Let $\{0\}\neq N(P)\subseteq N(O)$ be a non-zero (m,n)-ideal of $N(S)$.

Since $N(S)$ is (m,n)-regular, therefore by using Lemma 9, we have

$$\begin{aligned}
\{0\} &\neq N(P)=N(P)^{m}N(S)\cdot N(P)^{n}\\
&= [N(P)^{m}\cdot N(S)N(S)]N(P)^{n}\\
&= [N(S)\cdot N(P)^{m}N(S)]N(P)^{n}\\
&= [N(P)^{n}\cdot N(P)^{m}N(S)][N(S)N(S)]\\
&= [N(P)^{n}N(S)][N(P)^{m}N(S)\cdot N(S)]\\
&= N(P)^{n}N(S)\cdot N(S)N(P)^{m}\\
&= N(P)^{m}N(S)\cdot N(S)N(P)^{n}.
\end{aligned}$$

Hence $N(P)^{m}\neq\{0\}$ and $N(S)N(P)^{n}\neq\{0\}$. Further $N(P)\subseteq N(O)=N(M)N(J)\subseteq N(M)\cap N(J)$ implies that $N(P)\subseteq N(M)$ and $N(P)\subseteq N(J)$. Therefore $\{0\}\neq N(P)^{m}N(S)\subseteq N(M)^{m}N(S)\subseteq N(M)$ which shows that $N(P)^{m}N(S)=N(M)$ since $N(M)$ is 0-minimal.

Likewise, we can show that $N(S)N(P)^n = N(J)$. Thus we have

$$\begin{aligned}
N(P) &\subseteq N(O) = N(M)N(J) \\
&= N(P)^m N(S) \cdot N(S)N(P)^n \\
&= N(P)^n N(S) \cdot N(S)N(P)^m \\
&= [N(S)N(P)^m \cdot N(S)N(S)N(P)^n \\
&= [N(S) \cdot N(P)^m N(S)]N(P)^n \\
&= N(P)^m N(S) \cdot N(P)^n \subseteq N(P).
\end{aligned}$$

This means that $N(P) = N(M)N(J)$ and hence $N(M)N(J)$ is 0-minimal.
■

Theorem 86 *Let $N(S)$ be a unitary (m,n)-regular LA -semigroup. If $N(M)[N(J)]$ is a 0-minimal $(m,0)$- ideal ($(0,n)$-ideal) of $N(S)$, then either $N(M) \cap N(J) = \{0\}$ or $N(M) \cap N(J)$ is a 0-minimal (m,n)-ideal of $N(S)$.*

Proof. Once we prove that $N(M) \cap N(J)$ is an (m,n)-ideal of $N(S)$, the rest of the proof is same as in

Theorem 11. Let $N(O) = N(M) \cap N(J)$, then it is easy to see that $N(O)^2 \subseteq N(O)$. Moreover

$$\begin{aligned}
N(O)^m N(S) \cdot N(O)^n &\subseteq N(M)^m N(S) \cdot N(N)^n \\
&\subseteq N(M)N(J)^n \subseteq N(S)N(J)^n \subseteq N(J).
\end{aligned}$$

But, we also have

$$\begin{aligned}
N(O)^m N(S) \cdot N(O)^n &\subseteq N(M)^m N(S) \cdot N(J)^n \\
&= [N(M)^m \cdot N(S)N(S)]N(J)^n \\
&= [N(S) \cdot N(M)^m N(S)]N(J)^n \\
&= [N(J) \cdot N(M)^m N(S)]N(S) \\
&= [N(M)^m \cdot N(J)^n N(S)][N(S)N(S)] \\
&= [N(M)^m N(S)][N(J)^n N(S) \cdot N(S)] \\
&= N(M)^m N(S) \cdot N(S)N(J)^n \\
&= N(M)^m N(S) \cdot N(J)^n N(S) \\
&= N(J)^n [N(M)^m N(S) \cdot N(S)] = N(J)^n \cdot N(S)N(M)^m \\
&= N(J)^n \cdot N(M)^m N(S) = N(M)^m \cdot N(J)^n N(S) \\
&= N(M)^m \cdot N(S)N(J)^n \subseteq N(M)^m N(J) \\
&\subseteq N(M)^m N(S) \subseteq N(M).
\end{aligned}$$

Thus $N(O)^m N(S) \cdot N(O)^n \subseteq N(M) \cap N(J) = N(O)$ and therefore $N(O)$ is an (m,n)-ideal of $N(S)$. ■

1.4 Generalized Neutrosophic (1,2) Ideals

In this section we introduce bi-ideals interior ideals (1,2) ideals, two sided ideals, minimal ideals in Abel-Grassmann groupoids. We characterize intra-regular Abel Grassmann groupoids using the properties of above mentioned ideals.

Some of the basic theory about meutrosophic presented, are discussed in this research paper and then some solutions for the concepts of these theories are also in part of discussion. The general motivations about this research is the development of an approach for automatically construction of such systems which work as adequately as possible

As we know that in modelling of real world problems, a designer comes across with various difficulties. Whenever it is needed for interpretation for the real world problems containing imprecise or uncertain data to mathematical formulation, then classical approach does not applicable. To handle such situations Zadeh in 1965 introduced the idea of fuzzy set and replaced the conventional characteristic function of classical crisp set by the fuzzy set [0,1] that is not crisp and represents membership to a degree. Fuzzy set theory is conceptually a very powerful modelling and solution technique to incorporate imprecise or uncertain information into system description. In 1970, Bellman and Zadeh presented the first application of fuzzy theory that was decision making process and this theory has been applied to various fields of modern society such as artificial intelligence, image processing, pattern recognition, robotics, psychology etc. Fuzzy sets and fuzzy logic are used in order to handle and model imprecise modes of reasoning that play a vital role in the conspicuous human abilities to make wise decisions in an environment related to ambiguousness and imprecision and can be used in wide range of domains in which there are paucity of information, as in bioinformatics.

But fuzzy set theory has certain limitations upon which it does not work properly that is, this theory only deals with degree of membership and does not deal with degree of non-membership or falsehood. To circumvent these limitations, we need a theory that is more generalized as compared to that of fuzzy set theory, Krassimir T. Atanassov introduced the degree of non-membership/falsehood (f) in 1986 and presented a generalized set named as intuitionistic fuzzy set. An intuitionistic fuzzy set is an extension of fuzzy set and can be viewed in the perspective as an approach to fuzzy set in case when we are not provided with sufficient information. Intuitionistic fuzzy set adds an additional degree of freedom (non-membership and hesitation margins) into set description and is extensively use as a tool of intensive research by scholars and scientists from over the so many years.

Another most convenient and effectual theory which has been introduced to solve uncertainty problems and indeterminacy issues in most appropriate way is known as "Neutrosophy" presented by Florentin Smarandache, reputed professor at University of New Mexico. Where neutrosophy is a new branch of philosophy, called neutrosophy, which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

It was Professor Florentin Smarandache who introduced neutrosophy in the right moment motivated in conditions of Logic started in Ancient with Classical Logic of Aristotle, developed and covered by Three Valued Logic of Lukasiewicz, next ring being Fuzzy Logic of Zadeh, and finally the comprehensive Neutrosophic Logic of Smarandache. Neutrosophy is very helpful in handling all neutralities. In 1995, Florentin Smarandache introduced the idea of neutrosophy. Neutrosophic logic is an extension of fuzzy logic. In 2003 W.B Vasantha Kandasamy and Florentin Smarandache introduced algebraic structures (such as neutrosophic semigroup, neutrosophic ring, etc.)

An element $(a_1 + a_2 I)$ of a *Neutrosophic* AG-groupoid $N(S))$ is called *Neutrosophic* **intra-regular** if there exist $(x_1 + x_2 I), (y_1 + y_2 I) \in N(S)$ such that $(a_1 + a_2 I) = ((x_1 + x_2 I)(a_1 + a_2 I)^2)(y_1 + y_2 I)$ and $N(S)$ is called **intra-regular**, if every element of $N(S)$ is intra-regular.

Theorem 87 *Every Neutrosophic AG-groupoid $N(S)$ with left identity is an intra-regular if $N(S)$ is left (right) invertible.*

Proof. Let $N(S)$ be a left invertible *Neutrosophic* AG-groupoid with left identity, then for $(a_1 + a_2 I) \in N(S)$ there exists $(a_1 + a_2 I)' \in N(S)$ such that $(a_1 + a_2 I)'(a_1 + a_2 I) = e + eI$. Now by using left invertive law, medial law with left identity and medial law, we have

$$
\begin{aligned}
& (a_1 + a_2 I) \\
= \; & (e + eI)(a_1 + a_2 I) \\
= \; & (e + eI)((e + eI)(a_1 + a_2 I)) \\
= \; & ((a_1 + a_2 I)'(a_1 + a_2 I))((e + eI)(a_1 + a_2 I)) \\
\in \; & (N(S)(a_1 + a_2 I))(N(S)(a_1 + a_2 I)) \\
= \; & (N(S)(a_1 + a_2 I))((N(S)N(S))(a_1 + a_2 I)) \\
= \; & (N(S)(a_1 + a_2 I))(((a_1 + a_2 I)N(S))N(S)) \\
= \; & ((a_1 + a_2 I)N(S))((N(S)(a_1 + a_2 I))N(S)) \\
= \; & ((a_1 + a_2 I)(N(S)(a_1 + a_2 I)))(N(S)N(S)) \\
= \; & ((a_1 + a_2 I)(N(S)(a_1 + a_2 I)))N(S) \\
= \; & (N(S)((a_1 + a_2 I)(a_1 + a_2 I)))N(S) \\
= \; & (N(S)(a_1 + a_2 I)^2)N(S).
\end{aligned}
$$

Which shows that $N(S)$ is intra-regular. Similarly in the case of right invertible. ■

Theorem 88 *An Neutrosophic AG-groupoid $N(S)$ is intra-regular if $N(S)(a_1+a_2I) = N(S)$ or $(a_1+a_2I)N(S) = N(S)$ holds for all $(a_1+a_2I) \in N(S)$.*

Proof. Let $N(S)$ be an *Neutrosophic* AG-groupoid such that $N(S)(a_1+a_2I) = N(S)$ holds for all $(a_1+a_2I) \in N(S)$, then $N(S) = N(S)^2$. Let $(a_1+a_2I) \in N(S)$, therefore by using medial law, we have

$$\begin{aligned}
(a_1+a_2I) &\in N(S) \\
&= (N(S)N(S))N(S) \\
&= ((N(S)(a_1+a_2I))(N(S)(a_1+a_2I)))N(S) \\
&= ((N(S)N(S))((a_1+a_2I)(a_1+a_2I)))N(S) \\
&\subseteq (N(S)(a_1+a_2I)^2)N(S).
\end{aligned}$$

Which shows that $N(S)$ is intra-regular.

Let $(a_1+a_2I) \in N(S)$ and assume that $(a_1+a_2I)N(S) = N(S)$ holds for all $(a_1+a_2I) \in N(S)$, then by using left invertive law, we have

$$\begin{aligned}
(a_1+a_2I) &\in N(S) \\
&= N(S)N(S) \\
&= ((a_1+a_2I)N(S))N(S) \\
&= (N(S)N(S))(a_1+a_2I) \\
&= N(S)(a_1+a_2I).
\end{aligned}$$

Thus $N(S)(a_1+a_2I) = N(S)$ holds for all $(a_1+a_2I) \in N(S)$, therefore it follows from above that $N(S)$ is intra-regular. ■

The converse is not true in general from Example above.

Corollary 89 *If $N(S)$ is an Neutrosophic AG-groupoid such that $(a_1+a_2I)N(S) = N(S)$ holds for all $(a_1+a_2I) \in N(S)$, then $N(S)(a_1+a_2I) = N(S)$ holds for all $(a_1+a_2I) \in N(S)$.*

Theorem 90 *If $N(S)$ is intra-regular Neutrosophic AG-groupoid with left identity, then $(B(S)N(S))B(S) = B(S) \cap N(S)$, where $B(S)$ is a bi-(generalized bi-) ideal of $N(S)$.*

Proof. Let $N(S)$ be an intra-regular *Neutrosophic* AG-groupoid with left identity, then clearly $(B(S)N(S))B(S) \subseteq B(S) \cap N(S)$. Now let $(b_1+b_2I) \in B(S) \cap N(S)$, which implies that $(b_1+b_2I) \in B(S)$ and $(b_1+b_2I) \in N(S)$. Since $N(S)$ is intra-regular so there exist $(x_1+x_2I), (y_1+y_2I) \in N(S)$ such that $(b_1+b_2I) = ((x_1+x_2I)(b_1+b_2I)^2)(y_1+y_2I)$. Now by using medial law with left identity, left invertive law, paramedial law and medial law, we have

$$
\begin{aligned}
& (b_1+b_2I) \\
= \; & ((x_1+x_2I)((b_1+b_2I)(b_1+b_2I)))(y_1+y_2I) \\
= \; & ((b_1+b_2I)((x_1+x_2I)(b_1+b_2I)))(y_1+y_2I) \\
= \; & ((y_1+y_2I)((x_1+x_2I)(b_1+b_2I)))(b_1+b_2I) \\
= \; & ((y_1+y_2I)((x_1+x_2I)(((x_1+x_2I)(b_1+b_2I)^2)(y_1+y_2I))))(b_1+b_2I) \\
= \; & ((y_1+y_2I)(((x_1+x_2I)(b_1+b_2I)^2)((x_1+x_2I)(y_1+y_2I))))(b_1+b_2I) \\
= \; & (((x_1+x_2I)(b_1+b_2I)^2)((y_1+y_2I)((x_1+x_2I)(y_1+y_2I))))(b_1+b_2I) \\
= \; & ((((x_1+x_2I)(y_1+y_2I))(y_1+y_2I))((b_1+b_2I)^2(x_1+x_2I)))(b_1+b_2I) \\
= \; & (((b_1+b_2I)(b_1+b_2I))((((x_1+x_2I)(y_1+y_2I))(y_1+y_2I))(x_1+x_2I))) \\
& (b_1+b_2I) \\
= \; & (((b_1+b_2I)(b_1+b_2I))(((x_1+x_2I)(y_1+y_2I))((x_1+x_2I)(y_1+y_2I)))) \\
& (b_1+b_2I) \\
= \; & (((b_1+b_2I)(b_1+b_2I))((x_1+x_2I)^2(y_1+y_2I)^2))(b_1+b_2I) \\
= \; & (((y_1+y_2I)^2(x_1+x_2I)^2)((b_1+b_2I)(b_1+b_2I)))(b_1+b_2I) \\
= \; & ((b_1+b_2I)(((y_1+y_2I)^2(x_1+x_2I)^2)(b_1+b_2I)))(b_1+b_2I) \\
\in \; & ((b_1+b_2I)N(S))(b_1+b_2I).
\end{aligned}
$$

This shows that $(B(S)N(S))B(S) = B(S) \cap N(S)$. ■

The converse is not true in general. For this, let us consider an *Neutrosophic* AG-groupoid $N(S)$ with left identity $e+eI$ in Example above. It is easy to see that $\{(a_1+a_2I),(b_1+b_2I),(f_1+f_2I)\}$ is a bi-(generalized bi-) ideal of $N(S)$ such that $(B(S)N(S))B(S) = B(S) \cap N(S)$ but $N(S)$ is not an intra-regular because $(d_1+d_2I) \in N(S)$ is not an intra-regular.

Corollary 91 *If $N(S)$ is intra-regular Neutrosophic AG-groupoid with left identity, then $(B(S)N(S))B(S) = B(S)$, where $B(S)$ is a bi-(generalized bi-) ideal of $N(S)$.*

Theorem 92 *If $N(S)$ is intra-regular Neutrosophic AG-groupoid with left identity, then $(N(S)B(S))N(S) = N(S) \cap B(S)$, where $B(S)$ is an interior ideal of $N(S)$.*

Proof. Let $N(S)$ be an intra-regular *Neutrosophic* AG-groupoid with left identity, then clearly $(N(S)B(S))N(S) \subseteq N(S) \cap B(S)$. Now let $(b_1+b_2I) \in N(S) \cap B(S)$, which implies that $(b_1+b_2I) \in N(S)$ and $(b_1+b_2I) \in B(S)$. Since $N(S)$ is an intra-regular so there exist $(x_1+x_2I),(y_1+y_2I) \in N(S)$ such that $(b_1+b_2I) = ((x_1+x_2I)(b_1+b_2I)^2)(y_1+y_2I)$. Now by using paramedial law and left invertive law, we have

$$\begin{aligned}
& (b_1+b_2I) \\
= \;& ((e+eI)(x_1+x_2I))((b_1+b_2I)(b_1+b_2I)))(y_1+y_2I) \\
= \;& (((b_1+b_2I)(b_1+b_2I))((x_1+x_2I)(e+eI))(y_1+y_2I) \\
= \;& ((((x_1+x_2I)(e+eI))(b_1+b_2I))(b_1+b_2I))(y_1+y_2I) \\
\in \;& (N(S)(b_1+b_2I))N(S).
\end{aligned}$$

Which shows that $(N(S)(b_1+b_2I))N(S) = N(S) \cap (b_1+b_2I)$. ■

The converse is not true in general. It is easy to see that form Example above that $\{(a_1+a_2I), (b_1+b_2I), (f_1+f_2I)\}$ is an interior ideal of an *Neutrosophic* AG-groupoid $N(S)$ with left identity $(e+eI)$ such that $(N(S)B(S))N(S) = B(S) \cap N(S)$ but $N(S)$ is not an intra-regular because $(d_1+d_2I) \in N(S)$ is not an intra-regular.

Corollary 93 *If $N(S)$ is intra-regular Neutrosophic AG-groupoid with left identity, then $(N(S)B(S))N(S) = B(S)$, where $B(S)$ is an interior ideal of $N(S)$.*

Let $N(S)$ be an *Neutrosophic* AG-groupoid, then $\emptyset \neq A(S) \subseteq N(S)$ is called semiprime if $(a_1+a_2I)^2 \in A(S)$ implies $(a_1+a_2I) \in A(S)$.

Theorem 94 *An Neutrosophic AG-groupoid $N(S)$ with left identity is intra-regular if $L(S) \cup R(S) = L(S)R(S)$, where $L(S)$ and $R(S)$ are the left and right ideals of $N(S)$ respectively such that $R(S)$ is semiprime.*

Proof. Let $N(S)$ be an *Neutrosophic* AG-groupoid with left identity, then clearly $N(S)(a_1+a_2I)$ and $(a_1+a_2I)^2N(S)$ are the left and right ideals of $N(S)$ such that $(a_1+a_2I) \in N(S)(a_1+a_2I)$ and $(a_1+a_2I)^2 \in (a_1+a_2I)^2N(S)$, because by using paramedial law, we have

$$\begin{aligned}
& (a_1+a_2I)^2N(S) \\
= \;& ((a_1+a_2I)(a_1+a_2I))(N(S)N(S)) \\
= \;& (N(S)N(S))((a_1+a_2I)(a_1+a_2I)) \\
= \;& N(S)(a_1+a_2I)^2.
\end{aligned}$$

Therefore by given assumption, $(a_1+a_2I) \in (a_1+a_2I)^2N(S)$. Now by using left invertive law, medial law, paramedial law and medial law with left identity, we have

$$\begin{aligned}
(a_1+a_2I) &\in N(S)(a_1+a_2I)\cup(a_1+a_2I)^2N(S)\\
&= (N(S)(a_1+a_2I))((a_1+a_2I)^2N(S))\\
&= (N(S)(a_1+a_2I))(((a_1+a_2I)(a_1+a_2I))N(S))\\
&= (N(S)(a_1+a_2I))((N(S)(a_1+a_2I))((e+eI)\\
&\quad (a_1+a_2I)))\\
&\subseteq (N(S)(a_1+a_2I))((N(S)(a_1+a_2I))(N(S)\\
&\quad (a_1+a_2I)))\\
&= (N(S)(a_1+a_2I))((N(S)N(S))((a_1+a_2I)\\
&\quad (a_1+a_2I)))\\
&\subseteq (N(S)(a_1+a_2I))((N(S)N(S))(N(S)\\
&\quad (a_1+a_2I)))\\
&= (N(S)(a_1+a_2I))(((a_1+a_2I)N(S))\\
&\quad (N(S)N(S)))\\
&= (N(S)(a_1+a_2I))(((a_1+a_2I)N(S))N(S))\\
&= ((a_1+a_2I)N(S))((N(S)(a_1+a_2I))N(S))\\
&= ((a_1+a_2I)(N(S)(a_1+a_2I)))(N(S)N(S))\\
&= ((a_1+a_2I)(N(S)(a_1+a_2I)))N(S)\\
&= (N(S)((a_1+a_2I)(a_1+a_2I)))N(S)\\
&= (N(S)(a_1+a_2I)^2)N(S).
\end{aligned}$$

Which shows that $N(S)$ is intra-regular. ■

The converse is not true in general. In Example above, the only left and right ideal of $N(S)$ is $\{(a_1+a_2I),(b_1+b_2I)\}$, where $\{(a_1+a_2I),(b_1+b_2I)\}$ is semiprime such that $\{(a_1+a_2I),(b_1+b_2I)\}\cup\{(a_1+a_2I),(b_1+b_2I)\}=\{(a_1+a_2I),(b_1+b_2I)\}\{(a_1+a_2I),(b_1+b_2I)\}$ but $N(S)$ is not an intra-regular because $(d_1+d_2I)\in N(S)$ is not an intra-regular.

Lemma 95 *If $N(S)$ is intra-regular regular Neutrosophic AG-groupoid, then $N(S)=N(S)^2$.*

Theorem 96 *For a left invertible Neutrosophic AG-groupoid $N(S)$ with left identity, the following conditions are equivalent.*

(i) $N(S)$ is intra-regular.

(ii) $R(S)\cap L(S)=R(S)L(S)$, where $R(S)$ and $L(S)$ are any left and right ideals of $N(S)$ respectively.

Proof. $(i)\Longrightarrow(ii)$: Assume that $N(S)$ is intra-regular *Neutrosophic* AG-groupoid with left identity and let $(a_1+a_2I)\in N(S)$, then there exist $(x_1+x_2I),(y_1+y_2I)\in N(S)$ such that $(a_1+a_2I)=((x_1+x_2I)(a_1+a_2I)^2)(y_1+y_2I)$. Let $R(S)$ and $L(S)$ be any left and right ideals of $N(S)$

respectively, then obviously $R(S)L(S) \subseteq R(S) \cap L(S)$. Now let $(a_1 + a_2I) \in R(S) \cap L(S)$ implies that $(a_1 + a_2I) \in R(S)$ and $(a_1 + a_2I) \in L(S)$. Now by using medial law with left identity, medial law and left invertive law, we have

$$\begin{aligned}
& (a_1 + a_2I) \\
= \; & ((x_1 + x_2I)(a_1 + a_2I)^2)(y_1 + y_2I) \\
\in \; & (N(S)(a_1 + a_2I)^2)N(S) \\
= \; & (N(S)((a_1 + a_2I)(a_1 + a_2I)))N(S) \\
= \; & ((a_1 + a_2I)(N(S)(a_1 + a_2I)))N(S) \\
= \; & ((a_1 + a_2I)(N(S)(a_1 + a_2I)))(N(S)N(S)) \\
= \; & ((a_1 + a_2I)N(S))((N(S)(a_1 + a_2I))N(S)) \\
= \; & (N(S)(a_1 + a_2I))(((a_1 + a_2I)N(S))N(S)) \\
= \; & (N(S)(a_1 + a_2I))((N(S)N(S))(a_1 + a_2I)) \\
= \; & (N(S)(a_1 + a_2I))(N(S)(a_1 + a_2I)) \\
\subseteq \; & (N(S)R(S))(N(S)L(S)) \\
= \; & ((N(S)N(S))R(S))(N(S)L(S)) \\
= \; & ((R(S)N(S))N(S))(N(S)L(S)) \\
\subseteq \; & R(S)L(S).
\end{aligned}$$

This shows that $R(S) \cap L(S) = R(S)L(S)$.

$(ii) \implies (i)$: Let $N(S)$ be a left invertible *Neutrosophic* AG-groupoid with left identity, then for $(a_1+a_2I) \in N(S)$ there exists $(a_1+a_2I)' \in N(S)$ such that $(a_1 + a_2I)'(a_1 + a_2I) = (e + eI)$. Since $(a_1 + a_2I)^2N(S)$ is a right ideal and also a left ideal of $N(S)$ such that $(a_1 + a_2I)^2 \in (a_1 + a_2I)^2N(S)$, therefore by using given assumption, medial law with left identity and left invertive law, we have

$$\begin{aligned}
(a_1 + a_2I)^2 \; & \in \; (a_1 + a_2I)^2N(S) \cap (a_1 + a_2I)^2N(S) \\
& = \; ((a_1 + a_2I)^2N(S))((a_1 + a_2I)^2N(S)) \\
& = \; (a_1 + a_2I)^2(((a_1 + a_2I)^2N(S))N(S)) \\
& = \; (a_1 + a_2I)^2((N(S)N(S))(a_1 + a_2I)^2) \\
& = \; ((a_1 + a_2I)(a_1 + a_2I))(N(S)(a_1 + a_2I)^2) \\
& = \; ((N(S)(a_1 + a_2I)^2)(a_1 + a_2I))(a_1 + a_2I).
\end{aligned}$$

Thus we get, $(a_1 + a_2I)^2 = (((x_1 + x_2I)(a_1 + a_2I)^2)(a_1 + a_2I))(a_1 + a_2I)$ for some $(x_1 + x_2I) \in N(S)$.

Now by using left invertive law, we have

$$
\begin{aligned}
& ((a_1+a_2I)(a_1+a_2I))(a_1+a_2I)^{'} \\
= \; & ((((x_1+x_2I)(a_1+a_2I)^2)(a_1+a_2I))(a_1+a_2I))(a_1+a_2I)^{'} \\
& ((a_1+a_2I)^{'}(a_1+a_2I))(a_1+a_2I) \\
= \; & ((a_1+a_2I)^{'}(a_1+a_2I))((((x_1+x_2I)(a_1+a_2I)^2)(a_1+a_2I)) \\
& (a_1+a_2I) \\
= \; & ((x_1+x_2I)(a_1+a_2I)^2)(a_1+a_2I).
\end{aligned}
$$

This shows that $N(S)$ is intra-regular. ■

Lemma 97 *Every two-sided ideal of an intra-regular Neutrosophic AG-groupoid $N(S)$ with left identity is idempotent.*

Theorem 98 *In a Neutrosophic AG-groupoid $N(S)$ with left identity, the following conditions are equivalent.*

(i) $N(S)$ is intra-regular.
(ii) $A(S)=(N(S)A(S))^2$, where $A(S)$ is any left ideal of S.
Proof. $(i) \Longrightarrow (ii)$: Let $A(S)$ be a left ideal of an intra-regular *Neutrosophic* AG-groupoid $N(S)$ with left identity, then $N(S)A(S) \subseteq A(S)$ and $(N(S)A(S))^2 = N(S)A(S) \subseteq A(S)$. Now $A(S) = A(S)A(S) \subseteq N(S)A(S) = (N(S)A(S))^2$, which implies that $A(S) = (N(S)A(S))^2$.
$(ii) \Longrightarrow (i)$: Let $A(S)$ be a left ideal of $N(S)$, then $A(S) = (N(S)A(S))^2 \subseteq (A(S))^2$, which implies that $A(S)$ is idempotent and $N(S)$ is intra-regular. ■

Theorem 99 *In an intra-regular Neutrosophic AG-groupoid $N(S)$ with left identity, the following conditions are equivalent.*

(i) $A(S)$ is a bi-(generalized bi-) ideal of $N(S)$.
(ii) $(A(S)N(S))A(S) = A(S)$ and $(A(S))^2 = A(S)$.
Proof. $(i) \Longrightarrow (ii)$: Let $A(S)$ be a bi-ideal of an intra-regular *Neutrosophic* AG-groupoid $N(S)$ with left identity, then $(A(S)N(S))A(S) \subseteq A(S)$. Let $(a_1+a_2I) \in A(S)$, then since $N(S)$ is intra-regular so there exist (x_1+x_2I), $(y_1+y_2I) \in N(S)$ such that $(a_1+a_2I) = ((x_1+x_2I)(a_1+a_2I)^2)(y_1+y_2I)$. Now by using medial law with left identity, left invertive law, medial law

and paramedial law, we have

$$
\begin{aligned}
& (a_1 + a_2I) \\
= \;& ((x_1 + x_2I)(a_1 + a_2I)^2)(y_1 + y_2I) \\
= \;& ((x_1 + x_2I)((a_1 + a_2I)(a_1 + a_2I)))(y_1 + y_2I) \\
= \;& ((a_1 + a_2I)((x_1 + x_2I)(a_1 + a_2I)))(y_1 + y_2I) \\
= \;& ((y_1 + y_2I)((x_1 + x_2I)(a_1 + a_2I)))(a_1 + a_2I) \\
= \;& ((y_1 + y_2I)((x_1 + x_2I)(((x_1 + x_2I)(a_1 + a_2I)^2) \\
& (y_1 + y_2I))))(a_1 + a_2I) \\
= \;& ((y_1 + y_2I)(((x_1 + x_2I)(a_1 + a_2I)^2)((x_1 + x_2I) \\
& (y_1 + y_2I))))(a_1 + a_2I) \\
= \;& (((x_1 + x_2I)(a_1 + a_2I)^2)((y_1 + y_2I)((x_1 + x_2I) \\
& (y_1 + y_2I))))(a_1 + a_2I) \\
= \;& (((x_1 + x_2I)((a_1 + a_2I)(a_1 + a_2I)))((y_1 + y_2I)((x_1 + x_2I) \\
& (y_1 + y_2I))))(a_1 + a_2I) \\
= \;& (((a_1 + a_2I)((x_1 + x_2I)(a_1 + a_2I)))((y_1 + y_2I)((x_1 + x_2I) \\
& (y_1 + y_2I))))(a_1 + a_2I) \\
= \;& (((a_1 + a_2I)(y_1 + y_2I))(((x_1 + x_2I)(a_1 + a_2I))((x_1 + x_2I) \\
& (y_1 + y_2I))))(a_1 + a_2I) \\
= \;& (((x_1 + x_2I)(a_1 + a_2I))(((a_1 + a_2I)(y_1 + y_2I))((x_1 + x_2I) \\
& (y_1 + y_2I))))(a_1 + a_2I) \\
= \;& (((x_1 + x_2I)(a_1 + a_2I))(((a_1 + a_2I)(x_1 + x_2I))(y_1 + y_2I)^2)) \\
& (a_1 + a_2I) \\
= \;& (((y_1 + y_2I)^2((a_1 + a_2I)(x_1 + x_2I)))((a_1 + a_2I) \\
& (x_1 + x_2I)))(a_1 + a_2I) \\
= \;& ((a_1 + a_2I)(((y_1 + y_2I)^2((a_1 + a_2I)(x_1 + x_2I))) \\
& (x_1 + x_2I)))(a_1 + a_2I) \\
\in \;& ((a_1 + a_2I)N(S))(a_1 + a_2I).
\end{aligned}
$$

Thus $(A(S)N(S))A(S) = A(S)$ holds. Now by using medial law with left identity, left invertive law, paramedial law and medial law, we have

$$
\begin{aligned}
& (a_1 + a_2I) \\
= \quad & ((x_1 + x_2I)(a_1 + a_2I)^2)(y_1 + y_2I) \\
= \quad & ((x_1 + x_2I)((a_1 + a_2I)(a_1 + a_2I)))(y_1 + y_2I) \\
= \quad & ((a_1 + a_2I)((x_1 + x_2I)(a_1 + a_2I)))(y_1 + y_2I) \\
= \quad & ((y_1 + y_2I)((x_1 + x_2I)(a_1 + a_2I)))(a_1 + a_2I) \\
= \quad & ((y_1 + y_2I)((x_1 + x_2I)(((x_1 + x_2I)(a_1 + a_2I)^2) \\
& (y_1 + y_2I))))(a_1 + a_2I) \\
= \quad & ((y_1 + y_2I)(((x_1 + x_2I)(a_1 + a_2I)^2) \\
& ((x_1 + x_2I)(y_1 + y_2I))))(a_1 + a_2I) \\
= \quad & (((x_1 + x_2I)(a_1 + a_2I)^2)((y_1 + y_2I) \\
& ((x_1 + x_2I)(y_1 + y_2I))))(a_1 + a_2I) \\
= \quad & (((x_1 + x_2I)((a_1 + a_2I)(a_1 + a_2I))) \\
& ((y_1 + y_2I)((x_1 + x_2I)(y_1 + y_2I))))(a_1 + a_2I) \\
= \quad & (((a_1 + a_2I)((x_1 + x_2I)(a_1 + a_2I)))((y_1 + y_2I) \\
& ((x_1 + x_2I)(y_1 + y_2I))))(a_1 + a_2I) \\
= \quad & ((((y_1 + y_2I)((x_1 + x_2I)(y_1 + y_2I)))((x_1 + x_2I) \\
& (a_1 + a_2I)))(a_1 + a_2I))(a_1 + a_2I) \\
= \quad & ((((a_1 + a_2I)(x_1 + x_2I))(((x_1 + x_2I)(y_1 + y_2I)) \\
& (y_1 + y_2I)))(a_1 + a_2I))(a_1 + a_2I)
\end{aligned}
$$

$$
\begin{aligned}
&= ((((a_1 + a_2I)(x_1 + x_2I))((y_1 + y_2I)^2(x_1 + x_2I))) \\
&\quad (a_1 + a_2I))(a_1 + a_2I) \\
&= ((((a_1 + a_2I)(y_1 + y_2I)^2)((x_1 + x_2I)(x_1 + x_2I))) \\
&\quad (a_1 + a_2I))(a_1 + a_2I) \\
&= ((((a_1 + a_2I)(y_1 + y_2I)^2)(x_1 + x_2I)^2)(a_1 + a_2I))(a_1 + a_2I) \\
&= ((((x_1 + x_2I)^2(y_1 + y_2I)^2)(a_1 + a_2I))(a_1 + a_2I))(a_1 + a_2I) \\
&= ((((x_1 + x_2I)^2(y_1 + y_2I)^2)(((x_1 + x_2I)((a_1 + a_2I)(a_1 + a_2I))) \\
&\quad (y_1 + y_2I)))(a_1 + a_2I))(a_1 + a_2I) \\
&= ((((x_1 + x_2I)^2(y_1 + y_2I)^2)(((a_1 + a_2I)((x_1 + x_2I)(a_1 + a_2I))) \\
&\quad (y_1 + y_2I)))(a_1 + a_2I))(a_1 + a_2I) \\
&= ((((x_1 + x_2I)^2((a_1 + a_2I)((x_1 + x_2I)(a_1 + a_2I)))) \\
&\quad ((y_1 + y_2I)^2(y_1 + y_2I)))(a_1 + a_2I))(a_1 + a_2I) \\
&= ((((a_1 + a_2I)((x_1 + x_2I)^2((x_1 + x_2I)(a_1 + a_2I)))) \\
&\quad (y_1 + y_2I)^3)(a_1 + a_2I))(a_1 + a_2I) \\
&= ((((a_1 + a_2I)(((x_1 + x_2I)(x_1 + x_2I))((x_1 + x_2I)(a_1 + a_2I)))) \\
&\quad (y_1 + y_2I)^3)(a_1 + a_2I))(a_1 + a_2I) \\
&= ((((a_1 + a_2I)(((a_1 + a_2I)(x_1 + x_2I))((x_1 + x_2I)(x_1 + x_2I)))) \\
&\quad (y_1 + y_2I)^3)(a_1 + a_2I))(a_1 + a_2I) \\
&= (((((a_1 + a_2I)(x_1 + x_2I))((a_1 + a_2I)(x_1 + x_2I)^2)) \\
&\quad (y_1 + y_2I)^3)(a_1 + a_2I))(a_1 + a_2I) \\
&= (((((a_1 + a_2I)(a_1 + a_2I))((x_1 + x_2I)(x_1 + x_2I)^2)) \\
&\quad (y_1 + y_2I)^3)(a_1 + a_2I))(a_1 + a_2I) \\
&= ((((y_1 + y_2I)^3(x_1 + x_2I)^3)((a_1 + a_2I)(a_1 + a_2I))) \\
&\quad (a_1 + a_2I))(a_1 + a_2I) \\
&= (((a_1 + a_2I)(((y_1 + y_2I)^3(x_1 + x_2I)^3)(a_1 + a_2I))) \\
&\quad (a_1 + a_2I))(a_1 + a_2I) \\
&\subseteq (((a_1 + a_2I)N(S))(a_1 + a_2I))(a_1 + a_2I) \\
&\subseteq (a_1 + a_2I)(a_1 + a_2I) \\
&= (a_1 + a_2I)^2.
\end{aligned}
$$

Hence $A(S) = (A(S))^2$ holds.
$(ii) \Longrightarrow (i)$ is obvious. ■

Theorem 100 *In an intra-regular Neutrosophic AG-groupoid $N(S)$ with left identity, the following conditions are equivalent.*

(i) $A(S)$ is a quasi ideal of $N(S)$.
(ii) $N(S)Q(S) \cap Q(S)N(S) = Q(S)$.

Proof. $(i) \Longrightarrow (ii)$: Let (q) be a quasi ideal of an intra-regular *Neutrosophic* AG-groupoid $N(S)$ with left identity, then $N(S)Q(S) \cap Q(S)N(S) \subseteq Q(S)$. Let $q \in Q(S)$, then since $N(S)$ is intra-regular so there exist $(x_1 + x_2I)$, $(y_1 + y_2I) \in N(S)$ such that $(q_1 + q_2I) = ((x_1 + x_2I)(q_1 + q_2I)^2)(y_1 + y_2I)$. Let $(p_1 + p_2I), (q_1 + q_2I) \in N(S)Q(S)$, then by using medial law with left identity, medial law and paramedial law, we have

$$\begin{aligned}
& (p_1 + p_2I)(q_1 + q_2I) \\
= \; & (p_1 + p_2I)(((x_1 + x_2I)(q_1 + q_2I)^2)(y_1 + y_2I)) \\
= \; & ((x_1 + x_2I)(q_1 + q_2I)^2)((p_1 + p_2I)(y_1 + y_2I)) \\
= \; & ((x_1 + x_2I)((q_1 + q_2I)(q_1 + q_2I)))((p_1 + p_2I) \\
& (y_1 + y_2I)) \\
= \; & ((q_1 + q_2I)((x_1 + x_2I)(q_1 + q_2I)))((p_1 + p_2I) \\
& (y_1 + y_2I)) \\
= \; & ((q_1 + q_2I)(p_1 + p_2I))(((x_1 + x_2I)(q_1 + q_2I)) \\
& (y_1 + y_2I)) \\
= \; & ((x_1 + x_2I)(q_1 + q_2I))(((q_1 + q_2I)(p_1 + p_2I)) \\
& (y_1 + y_2I)) \\
= \; & ((y_1 + y_2I)((q_1 + q_2I)(p_1 + p_2I)))((q_1 + q_2I) \\
& (x_1 + x_2I)) \\
= \; & (q_1 + q_2I)(((y_1 + y_2I)((q_1 + q_2I)(p_1 + p_2I))) \\
& (x_1 + x_2I)) \\
\in \; & (q_1 + q_2I)N(S).
\end{aligned}$$

Now let $(q_1 + q_2I)(y_1 + y_2I) \in Q(S)N(S)$, then by using left invertive law, medial law with left identity and paramedial law, we have

$$\begin{aligned}
& (q_1 + q_2I)(p_1 + p_2I) \\
= \; & (((x_1 + x_2I)(q_1 + q_2I)^2)(y_1 + y_2I))(p_1 + p_2I) \\
= \; & ((p_1 + p_2I)(y_1 + y_2I))((x_1 + x_2I)(q_1 + q_2I)^2) \\
= \; & ((p_1 + p_2I)(y_1 + y_2I))((x_1 + x_2I)((q_1 + q_2I)(q_1 + q_2I))) \\
= \; & (x_1 + x_2I)(((p_1 + p_2I)(y_1 + y_2I))((q_1 + q_2I)(q_1 + q_2I))) \\
= \; & (x_1 + x_2I)(((q_1 + q_2I)(q_1 + q_2I))((y_1 + y_2I)(p_1 + p_2I))) \\
= \; & ((q_1 + q_2I)(q_1 + q_2I))((x_1 + x_2I)((y_1 + y_2I)(p_1 + p_2I))) \\
= \; & (((x_1 + x_2I)((y_1 + y_2I)(p_1 + p_2I)))(q_1 + q_2I))(q_1 + q_2I) \\
\in \; & N(S)(q_1 + q_2I).
\end{aligned}$$

Hence $Q(S)N(S) = N(S)Q(S)$. As by using medial law with left identity and left invertive law, we have

$$\begin{aligned}
& (q_1+q_2I) \\
= \;& ((x_1+x_2I)(q_1+q_2I)^2)(y_1+y_2I) \\
= \;& ((x_1+x_2I)((q_1+q_2I)(q_1+q_2I)))(y_1+y_2I) \\
= \;& ((q_1+q_2I)((x_1+x_2I)(q_1+q_2I)))(y_1+y_2I) \\
= \;& ((y_1+y_2I)((x_1+x_2I)(q_1+q_2I)))(q_1+q_2I) \\
\in \;& N(S)(q_1+q_2I).
\end{aligned}$$

Thus $(q_1+q_2I) \in N(S)Q(S) \cap Q(S)N(S)$ implies that $N(S)Q(S) \cap Q(S)N(S) = Q(S)$.

$(ii) \Longrightarrow (i)$ is obvious. ■

Theorem 101 *In an intra-regular Neutrosophic AG-groupoid $N(S)$ with left identity, the following conditions are equivalent.*

(i) $A(S)$ is an interior ideal of $N(S)$.

(ii) $(N(S)A(S))N(S) = A(S)$.

Proof. $(i) \Longrightarrow (ii)$ Let $A(S)$ be an interior ideal of an intra-regular *Neutrosophic* AG-groupoid $N(S)$ with left identity, then $(N(S)A(S))N(S) \subseteq A(S)$. Let $(a_1+a_2I) \in A(S)$, then since $N(S)$ is intra-regular so there exist (x_1+x_2I), $(y_1+y_2I) \in N(S)$ such that $(a_1+a_2I) = ((x_1+x_2I)(a_1+a_2I)^2)(y_1+y_2I)$. Now by using medial law with left identity, left invertive law and paramedial law, we have

$$\begin{aligned}
& (a_1+a_2I) \\
= \;& ((x_1+x_2I)(a_1+a_2I)^2)(y_1+y_2I) \\
= \;& ((x_1+x_2I)((a_1+a_2I)(a_1+a_2I)))(y_1+y_2I) \\
= \;& ((a_1+a_2I)((x_1+x_2I)(a_1+a_2I)))(y_1+y_2I) \\
= \;& ((y_1+y_2I)((x_1+x_2I)(a_1+a_2I)))(a_1+a_2I) \\
= \;& ((y_1+y_2I)((x_1+x_2I)(a_1+a_2I)))(((x_1+x_2I) \\
& (a_1+a_2I)^2)(y_1+y_2I)) \\
= \;& ((((x_1+x_2I)(a_1+a_2I)^2)(y_1+y_2I))((x_1+x_2I) \\
& (a_1+a_2I)))(y_1+y_2I) \\
= \;& (((a_1+a_2I)(x_1+x_2I))((y_1+y_2I)((x_1+x_2I) \\
& (a_1+a_2I)^2)))(y_1+y_2I) \\
= \;& ((((y_1+y_2I)((x_1+x_2I)(a_1+a_2I)^2))(x_1+x_2I)) \\
& (a_1+a_2I))(y_1+y_2I) \\
\in \;& (N(S)(a_1+a_2I))N(S).
\end{aligned}$$

Thus $(N(S)A(S))N(S) = A(S)$.

$(ii) \Longrightarrow (i)$ is obvious. ■

Theorem 102 *In an intra-regular Neutrosophic AG-groupoid $N(S)$ with left identity, the following conditions are equivalent.*

(i) $A(S)$ is a $(1,2)$-ideal of $N(S)$.
(ii) $(A(S)N(S))(A(S))^2 = A(S)$ and $(A(S))^2 = A(S)$.
Proof. $(i) \Longrightarrow (ii)$: Let $A(S)$ be a $(1,2)$-ideal of an intra-regular *Neutrosophic* AG-groupoid $N(S)$ with left identity, then $(A(S)N(S))(A(S))^2 \subseteq A(S)$ and $(A(S))^2 \subseteq A(S)$. Let $(a_1 + a_2I) \in A(S)$, then since $N(S)$ is intra-regular so there exist $(x_1 + x_2I)$, $(y_1 + y_2I) \in N(S)$ such that $(a_1 + a_2I) = ((x_1 + x_2I)(a_1 + a_2I)^2)(y_1 + y_2I)$
. Now by using medial law with left identity, left invertive law and paramedial law, we have

$$
\begin{aligned}
& (a_1 + a_2I) \\
= \; & ((x_1 + x_2I)(a_1 + a_2I)^2)(y_1 + y_2I) \\
= \; & ((x_1 + x_2I)((a_1 + a_2I)(a_1 + a_2I)))(y_1 + y_2I) \\
= \; & ((a_1 + a_2I)((x_1 + x_2I)(a_1 + a_2I)))(y_1 + y_2I) \\
= \; & ((y_1 + y_2I)((x_1 + x_2I)(a_1 + a_2I)))(a_1 + a_2I) \\
= \; & ((y_1 + y_2I)((x_1 + x_2I)(((x_1 + x_2I)(a_1 + a_2I)^2) \\
& (y_1 + y_2I))))(a_1 + a_2I) \\
= \; & ((y_1 + y_2I)(((x_1 + x_2I)(a_1 + a_2I)^2)((x_1 + x_2I) \\
& (y_1 + y_2I))))(a_1 + a_2I) \\
= \; & (((x_1 + x_2I)(a_1 + a_2I)^2)((y_1 + y_2I)((x_1 + x_2I) \\
& (y_1 + y_2I))))(a_1 + a_2I) \\
= \; & ((((x_1 + x_2I)(y_1 + y_2I))(y_1 + y_2I))((a_1 + a_2I)^2 \\
& (x_1 + x_2I)))(a_1 + a_2I) \\
= \; & (((y_1 + y_2I)^2(x_1 + x_2I))((a_1 + a_2I)^2(x_1 + x_2I))) \\
& (a_1 + a_2I) \\
= \; & ((a_1 + a_2I)^2(((y_1 + y_2I)^2(x_1 + x_2I))(x_1 + x_2I))) \\
& (a_1 + a_2I) \\
= \; & ((a_1 + a_2I)^2((x_1 + x_2I)^2(y_1 + y_2I)^2))(a_1 + a_2I) \\
= \; & ((a_1 + a_2I)((x_1 + x_2I)^2(y_1 + y_2I)^2))(a_1 + a_2I)^2 \\
= \; & ((a_1 + a_2I)((x_1 + x_2I)^2(y_1 + y_2I)^2))((a_1 + a_2I) \\
& (a_1 + a_2I)) \\
\in \; & ((a_1 + a_2I)N(S))(a_1 + a_2I)^2.
\end{aligned}
$$

Thus $(A(S)N(S))(A(S))^2 = A(S)$. Now by using medial law with left identity, left invertive law, paramedial law and medial law, we have

$$
\begin{aligned}
& (a_1 + a_2I) \\
= \;& ((x_1 + x_2I)(a_1 + a_2I)^2)(y_1 + y_2I) \\
= \;& ((x_1 + x_2I)((a_1 + a_2I)(a_1 + a_2I)))(y_1 + y_2I) \\
= \;& ((a_1 + a_2I)((x_1 + x_2I)(a_1 + a_2I)))(y_1 + y_2I) \\
= \;& ((y_1 + y_2I)((x_1 + x_2I)(a_1 + a_2I)))(a_1 + a_2I) \\
= \;& ((y_1 + y_2I)((x_1 + x_2I)(a_1 + a_2I)))(((x_1 + x_2I) \\
& (a_1 + a_2I)^2)(y_1 + y_2I)) \\
= \;& ((x_1 + x_2I)(a_1 + a_2I)^2)(((y_1 + y_2I)((x_1 + x_2I) \\
& (a_1 + a_2I)))(y_1 + y_2I)) \\
= \;& ((x_1 + x_2I)((a_1 + a_2I)(a_1 + a_2I)))(((y_1 + y_2I) \\
& ((x_1 + x_2I)(a_1 + a_2I)))(y_1 + y_2I)) \\
= \;& ((a_1 + a_2I)((x_1 + x_2I)(a_1 + a_2I)))(((y_1 + y_2I) \\
& ((x_1 + x_2I)(a_1 + a_2I)))(y_1 + y_2I)) \\
= \;& ((((y_1 + y_2I)((x_1 + x_2I)(a_1 + a_2I)))(y_1 + y_2I)) \\
& ((x_1 + x_2I)(a_1 + a_2I)))(a_1 + a_2I) \\
= \;& (((a_1 + a_2I)(x_1 + x_2I))((y_1 + y_2I)((y_1 + y_2I) \\
& ((x_1 + x_2I)(a_1 + a_2I)))))(a_1 + a_2I) \\
= \;& (((((x_1 + x_2I)(a_1 + a_2I)^2)(y_1 + y_2I))(x_1 + x_2I)) \\
& ((y_1 + y_2I)((y_1 + y_2I)((x_1 + x_2I)(a_1 + a_2I)))))(a_1 + a_2I) \\
= \;& ((((x_1 + x_2I)(y_1 + y_2I))((x_1 + x_2I)(a_1 + a_2I)^2)) \\
& ((y_1 + y_2I)((y_1 + y_2I)((x_1 + x_2I)(a_1 + a_2I)))))(a_1 + a_2I)
\end{aligned}
$$

$$
\begin{aligned}
&= ((((x_1+x_2I)(y_1+y_2I))(y_1+y_2I))(((x_1+x_2I)(a_1+a_2I)^2)\\
&\quad ((y_1+y_2I)((x_1+x_2I)(a_1+a_2I)))))(a_1+a_2I)\\
&= (((y_1+y_2I)^2(x_1+x_2I))(((x_1+x_2I)((a_1+a_2I)(a_1+a_2I)))\\
&\quad ((y_1+y_2I)((x_1+x_2I)(a_1+a_2I)))))(a_1+a_2I)\\
&= (((y_1+y_2I)^2(x_1+x_2I))(((x_1+x_2I)(y_1+y_2I))(((a_1+a_2I)(a_1+a_2I))\\
&\quad ((x_1+x_2I)(a_1+a_2I)))))(a_1+a_2I)\\
&= (((y_1+y_2I)^2(x_1+x_2I))(((a_1+a_2I)(a_1+a_2I))(((x_1+x_2I)(y_1+y_2I))\\
&\quad ((x_1+x_2I)(a_1+a_2I)))))(a_1+a_2I)\\
&= (((a_1+a_2I)(a_1+a_2I))(((y_1+y_2I)^2(x_1+x_2I))(((x_1+x_2I)(y_1+y_2I))\\
&\quad ((x_1+x_2I)(a_1+a_2I)))))(a_1+a_2I)\\
&= (((a_1+a_2I)(a_1+a_2I))(((y_1+y_2I)^2(x_1+x_2I))(((x_1+x_2I)(x_1+x_2I))\\
&\quad ((y_1+y_2I)(a_1+a_2I)))))(a_1+a_2I)\\
&= (((((x_1+x_2I)(x_1+x_2I))((y_1+y_2I)(a_1+a_2I)))((y_1+y_2I)^2(x_1+x_2I)))\\
&\quad ((a_1+a_2I)(a_1+a_2I)))(a_1+a_2I)\\
&= (((((a_1+a_2I)(y_1+y_2I))((x_1+x_2I)(x_1+x_2I)))((y_1+y_2I)^2(x_1+x_2I)))\\
&\quad ((a_1+a_2I)(a_1+a_2I)))(a_1+a_2I)\\
&= (((((x_1+x_2I)^2(y_1+y_2I))(a_1+a_2I))((y_1+y_2I)^2(x_1+x_2I)))\\
&\quad ((a_1+a_2I)(a_1+a_2I)))(a_1+a_2I)\\
&= ((((x_1+x_2I)(y_1+y_2I)^2)((a_1+a_2I)((x_1+x_2I)^2(y_1+y_2I))))\\
&\quad ((a_1+a_2I)(a_1+a_2I)))(a_1+a_2I)\\
&= (((a_1+a_2I)(((x_1+x_2I)(y_1+y_2I)^2)((x_1+x_2I)^2(y_1+y_2I))))\\
&\quad ((a_1+a_2I)(a_1+a_2I)))(a_1+a_2I)\\
&= (((a_1+a_2I)((x_1+x_2I)^3(y_1+y_2I)^3))((a_1+a_2I)\\
&\quad (a_1+a_2I)))(a_1+a_2I)\\
&\in (((a_1+a_2I)N(S))(a_1+a_2I)^2)(a_1+a_2I)\\
&\subseteq (a_1+a_2I)(a_1+a_2I)\\
&= (a_1+a_2I)^2.
\end{aligned}
$$

Hence $(A(S))^2 = A(S)$.
$(ii) \Longrightarrow (i)$ is obvious. ■

Lemma 103 *Every non empty subset $A(S)$ of an intra-regular Neutrosophic AG-groupoid $N(S)$ with left identity is a left ideal of $N(S)$ if and only if it is a right ideal of $N(S)$.*

Theorem 104 *In an intra-regular Neutrosophic AG-groupoid $N(S)$ with left identity, the following conditions are equivalent.*

(i) $A(S)$ is a $(1,2)$-ideal of $N(S)$.
(ii) $A(S)$ is a two-sided ideal of $N(S)$.

Proof. $(i) \Longrightarrow (ii)$: Assume that $N(S)$ is intra-regular *Neutrosophic* AG-groupoid with left identity and let $A(S)$ be a $(1,2)$-ideal of $N(S)$ then, $(A(S)N(S))(a_1+a_2I)^2 \subseteq A(S)$. Let $(a_1+a_2I) \in A(S)$, then since $N(S)$ is intra-regular so there exist (x_1+x_2I), $(y_1+y_2I) \in N(S)$ such that $(a_1+a_2I) = ((x_1+x_2I)(a_1+a_2I)^2)(y_1+y_2I)$.

Now by using medial law with left identity, left invertive law and para-medial law, we have

$$
\begin{aligned}
& N(S)(a_1+a_2I) \\
= \; & N(S)(((x_1+x_2I)(a_1+a_2I)^2)(y_1+y_2I)) \\
= \; & ((x_1+x_2I)(a_1+a_2I)^2)(N(S)(y_1+y_2I)) \\
= \; & ((x_1+x_2I)((a_1+a_2I)(a_1+a_2I)))(N(S)(y_1+y_2I)) \\
= \; & ((a_1+a_2I)((x_1+x_2I)(a_1+a_2I)))(N(S)(y_1+y_2I)) \\
= \; & ((N(S)(y_1+y_2I))((x_1+x_2I)(a_1+a_2I)))(a_1+a_2I) \\
= \; & ((N(S)(y_1+y_2I))((x_1+x_2I)(a_1+a_2I))) \\
& (((x_1+x_2I)(a_1+a_2I)^2)(y_1+y_2I)) \\
= \; & ((x_1+x_2I)(a_1+a_2I)^2)(((N(S)(y_1+y_2I)) \\
& ((x_1+x_2I)(a_1+a_2I)))(y_1+y_2I)) \\
= \; & ((y_1+y_2I)((N(S)(y_1+y_2I))((x_1+x_2I)(a_1+a_2I)))) \\
& ((a_1+a_2I)^2(x_1+x_2I)) \\
= \; & (a_1+a_2I)^2(((y_1+y_2I)((N(S)(y_1+y_2I))((x_1+x_2I) \\
& (a_1+a_2I))))(x_1+x_2I)) \\
= \; & ((a_1+a_2I)(a_1+a_2I))(((y_1+y_2I)((N(S)(y_1+y_2I)) \\
& ((x_1+x_2I)(a_1+a_2I))))(x_1+x_2I)) \\
= \; & ((x_1+x_2I)((y_1+y_2I)((N(S)(y_1+y_2I))((x_1+x_2I) \\
& (a_1+a_2I)))))((a_1+a_2I)(a_1+a_2I))
\end{aligned}
$$

$$
\begin{aligned}
&= ((x_1+x_2I)((y_1+y_2I)(((a_1+a_2I)(x_1+x_2I)) \\
&\quad ((y_1+y_2I)N(S)))))((a_1+a_2I)(a_1+a_2I)) \\
&= ((x_1+x_2I)(((a_1+a_2I)(x_1+x_2I))((y_1+y_2I) \\
&\quad ((y_1+y_2I)N(S)))))((a_1+a_2I)(a_1+a_2I)) \\
&= (((a_1+a_2I)(x_1+x_2I))((x_1+x_2I)((y_1+y_2I) \\
&\quad ((y_1+y_2I)N(S)))))((a_1+a_2I)(a_1+a_2I)) \\
&= (((((x_1+x_2I)(a_1+a_2I)^2)(y_1+y_2I))(x_1+x_2I)) \\
&\quad ((x_1+x_2I)((y_1+y_2I)((y_1+y_2I)N(S)))))((a_1+a_2I)(a_1+a_2I)) \\
&= ((((x_1+x_2I)(y_1+y_2I))((x_1+x_2I)(a_1+a_2I)^2)) \\
&\quad ((x_1+x_2I)((y_1+y_2I)((y_1+y_2I)N(S)))))((a_1+a_2I)(a_1+a_2I)) \\
&= ((((a_1+a_2I)^2(x_1+x_2I))((y_1+y_2I)(x_1+x_2I)))((x_1+x_2I) \\
&\quad ((y_1+y_2I)((y_1+y_2I)N(S)))))((a_1+a_2I)(a_1+a_2I)) \\
&= (((((y_1+y_2I)(x_1+x_2I))(x_1+x_2I))(a_1+a_2I)^2)((x_1+x_2I)((y_1+y_2I) \\
&\quad ((y_1+y_2I)N(S)))))((a_1+a_2I)(a_1+a_2I)) \\
&= ((((y_1+y_2I)((y_1+y_2I)N(S)))(x_1+x_2I))((a_1+a_2I)^2 \\
&\quad (((y_1+y_2I)(x_1+x_2I))(x_1+x_2I))))((a_1+a_2I)(a_1+a_2I)) \\
&= ((((y_1+y_2I)((y_1+y_2I)N(S)))(x_1+x_2I))((a_1+a_2I)^2 \\
&\quad ((x_1+x_2I)^2(y_1+y_2I))))((a_1+a_2I)(a_1+a_2I)) \\
&= ((a_1+a_2I)^2((((y_1+y_2I)((y_1+y_2I)N(S)))(x_1+x_2I))((x_1+x_2I)^2 \\
&\quad (y_1+y_2I))))((a_1+a_2I)(a_1+a_2I)) \\
&= (((a_1+a_2I)(a_1+a_2I))((((y_1+y_2I)((y_1+y_2I)N(S)))(x_1+x_2I)) \\
&\quad ((x_1+x_2I)^2(y_1+y_2I))))((a_1+a_2I)(a_1+a_2I)) \\
&= ((((x_1+x_2I)^2(y_1+y_2I))(((y_1+y_2I)((y_1+y_2I)N(S)))(x_1+x_2I))) \\
&\quad ((a_1+a_2I)(a_1+a_2I)))((a_1+a_2I)(a_1+a_2I)) \\
&= ((a_1+a_2I)(((x_1+x_2I)^2(y_1+y_2I))((((y_1+y_2I)((y_1+y_2I)N(S))) \\
&\quad (x_1+x_2I))(a_1+a_2I))))((a_1+a_2I)(a_1+a_2I)) \\
&\in ((a_1+a_2I)N(S))(a_1+a_2I)^2 \\
&\subseteq (a_1+a_2I).
\end{aligned}
$$

Hence $A(S)$ is a left ideal of $N(S)$ and $A(S)$ is a two-sided ideal of $N(S)$.

$(ii) \Longrightarrow (i)$: Let $A(S)$ be a two-sided ideal of $N(S)$. Let $(y_1+y_2I) \in (A(S)N(S))(A(S))^2$, then $(y_1+y_2I) = ((a_1+a_2I)N(S))(b_1+b_2I)^2$ for some $(a_1+a_2I), (b_1+b_2I) \in A(S)$ and $(s_1+s_2I) \in N(S)$. Now by using medial law with left identity, we have

$$
\begin{aligned}
& (y_1 + y_2 I) \\
= \; & ((a_1 + a_2 I)(s_1 + s_2 I))(b_1 + b_2 I)^2 \\
= \; & ((a_1 + a_2 I)(s_1 + s_2 I))((b_1 + b_2 I)(b_1 + b_2 I)) \\
= \; & (b_1 + b_2 I)(((a_1 + a_2 I)(s_1 + s_2 I))(b_1 + b_2 I)) \\
\in \; & A(S)N(S) \\
\subseteq \; & A(S).
\end{aligned}
$$

Hence $(A(S)N(S))(A(S))^2 \subseteq A(S)$, therefore $A(S)$ is a $(1,2)$-ideal of $N(S)$. ■

Lemma 105 *Let $N(S)$ be an Neutrosophic AG-groupoid, then $N(S)$ is intra-regular if and only if every left ideal of $N(S)$ is idempotent.*

Lemma 106 *Every non empty subset $A(S)$ of an intra-regular Neutrosophic AG-groupoid $N(S)$ with left identity is a two-sided ideal of $N(S)$ if and only if it is a quasi ideal of $N(S)$.*

Theorem 107 *A two-sided ideal of an intra-regular Neutrosophic AG-groupoid $N(S)$ with left identity is minimal if and only if it is the intersection of two minimal two-sided ideals of $N(S)$.*

Proof. Let $N(S)$ be intra-regular *Neutrosophic* AG-groupoid and $Q(S)$ be a minimal two-sided ideal of $N(S)$, let $(a_1+a_2I) \in Q(S)$. As $N(S)(N(S)(a_1+a_2I)) \subseteq N(S)(a_1+a_2I)$ and $N(S)((a_1+a_2I)N(S)) \subseteq (a_1+a_2I)(N(S)N(S)) = (a_1+a_2I)N(S)$, which shows that $N(S)(a_1+a_2I)$ and $(a_1+a_2I)N(S)$ are left ideals of $N(S)$, so $N(S)(a_1+a_2I)$ and $(a_1+a_2I)N(S)$ are two-sided ideals of $N(S)$.

Now

$$
\begin{aligned}
& N(S)(N(S)(a_1 + a_2 I) \cap (a_1 + a_2 I)N(S)) \cap (N(S)(a_1 + a_2 I) \\
& \cap (a_1 + a_2 I)N(S))N(S) \\
= \; & N(S)(N(S)(a_1 + a_2 I)) \cap N(S)((a_1 + a_2 I)N(S)) \\
& \cap (N(S)(a_1 + a_2 I))N(S) \cap ((a_1 + a_2 I)N(S))N(S) \\
\subseteq \; & (N(S)(a_1 + a_2 I) \cap (a_1 + a_2 I)N(S)) \\
& \cap (N(S)(a_1 + a_2 I))N(S) \cap N(S)(a_1 + a_2 I) \\
\subseteq \; & N(S)(a_1 + a_2 I) \cap (a_1 + a_2 I)N(S).
\end{aligned}
$$

This implies that $N(S)(a_1 + a_2 I) \cap (a_1 + a_2 I)N(S)$ is a quasi ideal of $N(S)$, so, $N(S)(a_1 + a_2 I) \cap (a_1 + a_2 I)N(S)$ is a two-sided ideal of $N(S)$.

Also since $(a_1 + a_2 I) \in Q(S)$, we have

$$\begin{aligned} & N(S)(a_1 + a_2 I) \cap (a_1 + a_2 I)N(S) \\ \subseteq \;& N(S)Q(S) \cap Q(S)N(S) \\ \subseteq \;& Q(S) \cap Q(S) \\ \subseteq \;& Q(S). \end{aligned}$$

Now since $Q(S)$ is minimal, so $N(S)(a_1 + a_2 I) \cap (a_1 + a_2 I)N(S) = Q(S)$, where $N(S)(a_1 + a_2 I)$ and $(a_1 + a_2 I)N(S)$ are minimal two-sided ideals of $N(S)$, because let $l_1 + I_2 I$ be an two-sided ideal of $N(S)$ such that $(I_1 + I_2 I) \subseteq N(S)(a_1 + a_2 I)$, then $(I_1 + I_2 I) \cap (a_1 + a_2 I)N(S) \subseteq N(S)(a_1 + a_2 I) \cap (a_1 + a_2 I)N(S) \subseteq Q(S)$, which implies that $(I_1 + I_2 I) \cap (a_1 + a_2 I)N(S) = Q(S)$. Thus $Q(S) \subseteq (I_1 + I_2 I)$. Therefore, we have

$$\begin{aligned} & N(S)(a_1 + a_2 I) \\ \subseteq \;& N(S)Q(S) \\ \subseteq \;& N(S)(I_1 + I_2 I) \\ \subseteq \;& (I_1 + I_2 I), \text{ gives } N(S)(a_1 + a_2 I) = I_1 + I_2 I. \end{aligned}$$

Thus $N(S)(a_1 + a_2 I)$ is a minimal two-sided ideal of $N(S)$. Similarly $(a_1 + a_2 I)N(S)$ is a minimal two-sided ideal of $N(S)$.

Conversely, let $Q(S) = (I_1 + I_2 I) \cap (J_1 + J_2 I)$ be a two-sided ideal of $N(S)$, where I and J are minimal two-sided ideals of $N(S)$, then, $Q(S)$ is a quasi ideal of $N(S)$, that is $N(S)Q(S) \cap Q(S)N(S) \subseteq Q(S)$. Let $Q(S)^{'}$ be a two-sided ideal of $N(S)$ such that $Q(S)^{'} \subseteq Q(S)$, then

$$\begin{aligned} & N(S)Q(S)^{'} \cap Q(S)^{'} N(S) \\ \subseteq \;& N(S)Q(S) \cap Q(S)N(S) \\ \subseteq \;& Q(S), \\ \text{also } N(S)Q(S)^{'} \subseteq \;& N(S)(I_1 + I_2 I) \subseteq (I_1 + I_2 I) \\ & \text{and } Q(S)^{'} N(S) \\ \subseteq \;& (J_1 + J_2 I)N(S) \\ \subseteq \;& (J_1 + J_2 I) \end{aligned}$$

Now

$$\begin{aligned} & N(S)(N(S)Q(S)^{'}) \\ = \;& (N(S)N(S))\,(N(S)Q(S)^{'}) \\ = \;& (Q(S)^{'} N(S))\,(N(S)N(S)) \\ = \;& (Q(S)^{'} N(S))N(S) \\ = \;& (N(S)N(S))\,Q(S)^{'} \\ = \;& N(S)Q(S)^{'}, \end{aligned}$$

which implies that $N(S)Q(S)^{'}$ is a left ideal and hence a two-sided ideal. Similarly $Q(S)^{'}N(S)$ is a two-sided ideal of $N(S)$. Since $(I_1 + I_2I)$ and $(J_1 + J_2I)$ are minimal two-sided ideals of $N(S)$, therefore $N(S)Q(S)^{'} = (I_1 + I_2I)$ and $Q(S)^{'}N(S) = (J_1 + J_2I)$. But $Q(S) = (I_1 + I_2I) \cap (J_1 + J_2I)$, which implies that, $Q(S) = N(S)Q(S)^{'} \cap Q(S)^{'}N(S) \subseteq Q(S)^{'}$. This give us $Q(S) = Q(S)^{'}$ and hence $Q(S)$ is minimal. ■

2

Neutrosophic Minimal Ideals of AG-groupoids

In this chapter we discuss neutrosophic minimal ideals of neutrosophic Abel-Grassmann's groupoid and we will prove some results related to neutrosophic simple (left and right) Abel-Grassmann's groupoid.

The world is the combination of complex phenomenons because it has various fields and each field has certain type of problems which cannot be determined by using classical methods. Main hindrance to handle these problems is its imprecision, imperfect and uncertain nature. Different models in different eras are hatched to handle this uncertainty into system description. The popular names among those who worked in the field of uncertainty is Lofti A.Zadeh who in 1965 presented a fuzzy set that was infact serving as the substitute of conventional crisp set. Theory provided by him was very strong and convincing because this theory have ability to deal with additional characteristic or imperfect data connected to imprecision. This model provide us modeling tool to deal with complex systems this system is controlled but its explanation is very difficult. By using this model we can minimize failure of modelling. Before fuzzy theory vagueness was measured exclusively in terms of probability theory and understood as haphazardness. Zadeh exposed the associations of probability and fuzzy set. This work provides us suitable methodology to pact with uncertainties. Fuzzy set theory is important because it provide us a way to verbalize vague material into model it also helps us in problem explaining. These methods are suitable when it is required to model human acquaintance. Fuzzy theory is very interesting branch of mathematics which allow a computer to model the actual world in the same technique as that of people do.

To simplify basic theories of algebra different authors used fuzzy set theory. Another theory was proposed by Mordeson et al he revealed the impressive investigation of fuzzy semigroups. Fuzzy semigroups was applied in fuzzy coding, fuzzy finite state and fuzzy languages, and problem of integrated design of high speed planar mechanism. Shortcoming of Fuzzy test was that it does not explain properly about gradation of non-membership or fiction. So we need something new to overcome this short coming. This problem was attempted to solve by Krassimir T. Atanassov in 1986 defined the intuitionistic fuzzy set. This theory is based on fuzzy test actually it is extended form of fuzzy test it solve the problems if sufficient information is not available. By using this theory degrees of freedom in case of non-membership and hesitation margins increases. It is broadly practice as an implement of rigorous research by scientists from many years. Many other theories like theory of probability and rough set theory are steadily actuality used as commanding productive tackles to pact with multiform uncertainties and imprecision. These theories do not explain unsettled evidence effectively.

This problem was try to solve by neutrosophy which is modern approach. Neutrosophy is derived from Latin world "neuter" means neutral and Greek language word "sophia" meaning skill. Neutrosophy is a subdivision of philosophy it was presented by Florentin Smarandache. Florentin studies the source, nature, and possibility of neutralities and their interactions. Neutrosophy is also based on fuzzy set it used three concepts that is truthfulness, falsehood and neutrality. Neutrosophy studies a proposal, concept, occasion, idea, or object, "A" in relation to its opposite, "Anti-A" and that which is not A "Non-A" and that which is neither "A" nor "Anti-A" denoted by "Neut-A".

F. Smrandache presented a novel idea of a neutrosophic set known as NS in short in 1995. This simplify fuzzy sets and intutionistic fuzzy set. This was based on inspiration from the realisms of actual life wonders like sport games have three options winning/ tie/ defeating and conclusion production like making a decision/ hesitating/ not making. NS is defined by affiliation degree, unspecified degree. This knowledge of NS produces theory of neutrosophic sets. This theory very efficiently and beautifully explain virtually all model of all material world hitches. If we are dealing with uncertainty we can practice fuzzy theory. Whereas dealing with indeterminacy we can use neutrosophic theory. Neutrosopic theory have many other uses like control theory, databases, medical diagnosis problem and decision making problems.

Vasantha Kandasmy and Florentin in 2003 introduced the concept of neutrosophic algebraic structures. These algebraic structures include neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and neutrosophic AG-groupoids.

LA-semigroup generalizes the concept of commutative semigroups and have an important application within the theory of flocks. In addition to applications, a variety of properties have been studied for AG-groupoids and related structures. An LA-semigroup is a non-associative algebraic structure that is generally considered as a midway between a groupoid and a commutative semigroup but is very close to commutative semigroup because most of their properties are similar to commutative semigroup. Every commutative semigroup is an AG-groupoid but not vice versa. Thus AG-groupoids can also be non-associative, however, they do not necessarily have the Latin square property. An LA-semigroup S can have left identity e (unique) i.e $ea = a$ for all $a \in S$ but it can not have a right identity because if it has, then S becomes a commutative semigroup. An element s of LA-semigroup S is called idempotent if $s^2 = s$ and if holds for all elements of S then S is called idempotent LA-semigroup. An element $0 \in S$ is called zero of S if $0a = a0 = 0$ for all $a \in S$.

In a neutrosophic commutative semigroup $N(S)$, $(a + bI)^2(c + dI)^2 = (a + bI)^2(c + dI)^2$ holds for all $(a + bI), (c + dI) \in N(S)$. Also, if $N(S)$ is a neutrosophic AG-groupoid with left identity $e + eI$, then the equation $(a+bI)^2(c+dI)^2 = (a+bI)^2(c+dI)^2$ holds for all $(a+bI), (c+dI) \in N(S)$. If $\{(a+bI), (c+dI)\}$ is any neutrosophic subset of a neutrosophic AG-groupoid $N(S)$, with left identity $e+eI$, then $(a+bI)(c+dI) = [(c+dI)(a+bI)](e+eI)$ holds for all $(a + bI), (c + dI) \in N(S)$. It is most interesting to see the applications of this neutrosophic non-associative structure in different fields as compare to a neutrosophic commutative semigroup and this motivate us to study a neutrosophic AG-groupoid.

An element $0 + 0I \in N(S)$ is called neutrosophic zero of $N(S)$ if $(0 + 0I)(a + bI) = (a + bI)(0 + 0I) = 0 + 0I$ for all $a + bI \in N(S)$.

Example 108 *Let $S = \{1, 2, 3\}$ with binary operation "$\cdot$" is an LA-semigroup with left identity 3 and has the following Cayley's table:*

$\cdot$	1	2	3
1	3	1	2
2	2	3	1
3	1	2	3

then $N(S) = \{1+1I, 1+2I, 1+3I, 2+1I, 2+2I, 2+3I, 3+1I, 3+2I, 3+3I\}$ is an example of neutrosophic LA-semigroup under the operation "$*$" with left identity $3 + 3I$ and has the following Cayley's table:

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$
$1+2I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$1+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$2+1I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$2+2I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$
$2+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$3+1I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$3+2I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$

A subset $N(I)$ of a neutrosophic AG-groupoid $N(S)$ is said to be a neutrosophic left (right) ideal of $N(S)$ if $N(S)N(I) \subseteq N(I)$ $(N(I)N(S) \subseteq N(I))$, and a neutrosophic left (right) ideal.

A neutrosophic subset $N(M)$ of a neutrosophic AG-groupoid $N(S)$ is said to be a neutrosophic minimal left (right) ideal if it does not contain any other neutrosophic left (right) ideal other than itself.

The neutrosophic kernel $N(K)$ may be described as the intersection of all neutrosophic two sided ideals of a neutrosophic AG-groupoid $N(S)$. If $N(M)$ is the neutrosophic minimal ideal of a neutrosophic AG-groupoid $N(S)$ then $N(M)$ is the neutrosophic kernel of $N(S)$ and if $N(S)$ has a neutrosophic kernel $N(K)$.

Example 109 *Let $S = \{1, 2, 3\}$ with binary operation "$\cdot$" is an LA-semigroup and has the following Cayley's table:*

$\cdot$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

then $N(S) = \{1+1I, 1+2I, 1+3I, 2+1I, 2+2I, 2+3I, 3+1I, 3+2I, 3+3I\}$ is an example of neutrosophic LA-semigroup under the operation "$*$" with left identity $3 + 3I$ and has the following Cayley's table:

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$1+3I$	$2+1I$	$2+2I$	$2+1I$	$2+1I$	$2+2I$	$2+2I$	$2+1I$	$2+2I$	$2+2I$
$2+1I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$2+3I$	$2+1I$	$2+2I$	$2+2I$	$2+1I$	$2+2I$	$2+2I$	$2+1I$	$2+2I$	$2+2I$
$3+1I$	$1+2I$	$1+2I$	$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$3+2I$	$1+2I$	$1+2I$	$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$3+3I$	$1+1I$	$1+2I$	$1+2I$	$2+1I$	$2+2I$	$2+2I$	$2+1I$	$2+2I$	$2+2I$

In this example $\{1, 2\}$ is an ideal of S and $\{2\}$ is a minimal ideal and is a kernel of S. While $\{1 + 1I, 1 + 2I, 2 + 1I, 2 + 2I\}$ and $\{2 + 2I\}$ are neutrosophic ideals of $N(S)$ and $\{2 + 2I\}$ is a neutrosophic minimal ideal and is a neutrosophic kernel of $N(S)$.

2.1 Neutrosophic Minimal Ideals and Neutrosophic Kernel

Lemma 110 *If a neutrosophic AG-groupoid $N(S)$ contains a neutrosophic minimal ideal $N(M)$, then $N(M)$ is the neutrosophic kernel of $N(S)$.*

Proof. Suppose $N(A)$ and $N(M)$ be a neutrosophic ideal and neutrosophic minimal ideal of $N(S)$ respectively. Then $N(A)N(M) \subseteq N(A)N(S) \subseteq N(A)$ and $N(A)N(M) \subseteq N(S)N(M) \subseteq N(M)$ implies $N(A)N(M) \subseteq N(A) \cap N(M)$. Therefore $N(A) \cap N(M)$ is non empty. But $N(M)$ is neutrosophic minimal ideal so $N(M)$ does not contain $N(A)N(M)$ properly. So

$N(M) = N(A)N(M) \subseteq N(A)$ which implies that $N(M) \subseteq N(A)$ thus $N(M)$ is contained in every neutrosophic ideal of $N(S)$ that is $N(M)$ is the kernel of $N(S)$. ■

A neutrosophic ideal $N(M)$ in a neutrosophic AG-groupoid $N(S)$ with zero is called neutrosophic zero-minimal if it is minimal in the set of all non-zero neutrosophic ideals. It is important to note that a neutrosophic AG-groupoid $N(S)$ with left identity $e + eI$ has atmost one neutrosophic minimal ideal.

Union of all the neutrosophic minimal left ideals of $N(S)$ is called the class sum of all the neutrosophic minimal left ideals of $N(S)$ and we denote it by $N(\Sigma)$.

Example 111 *Let $S = \{0, 1, 2, 3\}$ with binary operation "$\cdot$" is an LA-semigroup with zero 0 and has the following Cayley's table:*

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	3	1	2
3	0	2	3	1

then $N(S) = \{0+0I, 0+1I, 0+2I, 0+3I, 1+0I, 1+1I, 1+2I, 1+3I, 2+0I, 2+1I, 2+2I, 2+3I, 3+0I, 3+0I, 3+1I, 3+2I, 3+3I\}$ is an example of neutrosophic LA-semigroup under the operation "$*$" with neutrosophic zero $0 + 0I$ and has the following Cayley's table:

$*$	$0+0I$	$0+1I$	$0+2I$	$0+3I$	$1+0I$	$1+1I$	$1+2I$	$1+3I$	$2+0I$	$2+1I$	$2+2I$	$2+3I$	$3+0I$	$3+1I$	$3+2I$	$3+3I$
$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$0+1I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$0+2I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$0+3I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$1+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$1+0I$	$1+0I$	$1+0I$	$1+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$3+0I$	$3+0I$	$3+0I$	$3+0I$
$1+1I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$1+0I$	$1+1I$	$1+2I$	$1+3I$	$2+0I$	$2+1I$	$2+2I$	$2+3I$	$3+0I$	$3+1I$	$3+2I$	$3+3I$
$1+2I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$1+0I$	$1+3I$	$1+1I$	$1+2I$	$2+0I$	$2+3I$	$2+1I$	$2+2I$	$3+0I$	$3+3I$	$3+1I$	$3+2I$
$1+3I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$1+0I$	$1+2I$	$1+3I$	$1+1I$	$2+0I$	$2+2I$	$2+3I$	$2+1I$	$3+0I$	$3+2I$	$3+3I$	$3+1I$
$2+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$3+0I$	$3+0I$	$3+0I$	$3+0I$	$1+0I$	$1+0I$	$1+0I$	$1+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$
$2+1I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$3+0I$	$3+1I$	$3+2I$	$3+3I$	$1+0I$	$1+1I$	$1+2I$	$1+3I$	$2+0I$	$2+1I$	$2+2I$	$2+3I$
$2+2I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$3+0I$	$3+3I$	$3+1I$	$3+2I$	$1+0I$	$1+3I$	$1+1I$	$1+2I$	$2+0I$	$2+3I$	$2+1I$	$2+2I$
$2+3I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$3+0I$	$3+2I$	$3+3I$	$3+1I$	$1+0I$	$1+2I$	$1+3I$	$1+1I$	$2+0I$	$2+2I$	$2+3I$	$2+1I$
$3+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$3+0I$	$3+0I$	$3+0I$	$3+0I$	$1+0I$	$1+0I$	$1+0I$	$1+0I$
$3+1I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+1I$	$2+2I$	$2+3I$	$3+0I$	$3+0I$	$3+0I$	$3+0I$	$1+0I$	$1+1I$	$1+2I$	$1+3I$
$3+2I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+3I$	$2+1I$	$2+2I$	$3+0I$	$3+0I$	$3+0I$	$3+0I$	$1+0I$	$1+3I$	$1+1I$	$1+2I$
$3+3I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+2I$	$2+3I$	$2+1I$	$3+0I$	$3+0I$	$3+0I$	$3+0I$	$1+0I$	$1+2I$	$1+3I$	$1+1I$

Example 112 *Let $S = \{0, 1, 2, 3\}$ with binary operation "$\cdot$" is an LA-semigroup and has the following Cayley's table:*

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	2	2	2
2	0	2	2	2
3	0	1	2	2

then $N(S) = \{0+0I, 0+1I, 0+2I, 0+3I, 1+0I, 1+1I, 1+2I, 1+3I, 2+0I, 2+1I, 2+2I, 2+3I, 3+0I, 3+0I, 3+1I, 3+2I, 3+3I\}$ is an example of neutrosophic LA-semigroup under the operation "$*$" with neutrosophic zero $0+0I$ and has the following Cayley's table:

$*$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$1+0I$	$1+1I$	$1+2I$	$1+3I$	$2+0I$	$2+1I$	$2+2I$	$2+3I$	$3+0I$	$3+1I$	$3+2I$	$3+3I$
$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$0+1I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$0+2I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$0+3I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$1+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$
$1+1I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$
$1+2I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$
$1+3I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+1I$	$2+2I$	$2+2I$	$2+0I$	$2+1I$	$2+2I$	$2+2I$	$2+0I$	$2+1I$	$2+2I$	$2+2I$
$2+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$
$2+1I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$
$2+2I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$
$2+3I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$2+0I$	$2+1I$	$2+2I$	$2+2I$	$2+0I$	$2+1I$	$2+2I$	$2+2I$	$2+0I$	$2+1I$	$2+2I$	$2+2I$
$3+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$1+0I$	$1+0I$	$1+0I$	$1+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$	$2+0I$
$3+1I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$1+0I$	$1+2I$	$1+2I$	$1+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$
$3+2I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$1+0I$	$1+2I$	$1+2I$	$1+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$	$2+0I$	$2+2I$	$2+2I$	$2+2I$
$3+3I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$1+0I$	$1+1I$	$1+2I$	$1+2I$	$2+0I$	$2+1I$	$2+2I$	$2+2I$	$2+0I$	$2+1I$	$2+2I$	$2+2I$

In this example $\{1,2\}$ is an ideal of S and $\{2\}$ is a minimal ideal and is a kernel of S. While $\{1+1I, 1+2I, 2+1I, 2+2I\}$ and $\{2+2I\}$ are neutrosophic ideals of $N(S)$ and $\{2+2I\}$ is a neutrosophic zero minimal ideal of $N(S)$.

Lemma 113 *Let $N(S)$ be a neutrosophic AG-groupoid with left identity $e+eI$ then every neutrosophic two sided ideal of $N(S)$ contains every neutrosophic minimal left ideal of $N(S)$.*

Proof. Let $N(L)$ be a neutrosophic minimal left ideal of $N(S)$. Suppose $N(A)$ be any neutrosophic two sided ideal of $N(S)$ thus we have $N(L)N(A) \subseteq N(A)$, $N(A)N(L) \subseteq N(A)$ and $N(A)N(L) \subseteq N(L)$. Clearly $N(A)N(L)$ is the neutrosophic left ideal of $N(S)$ contained in $N(A)$ and $N(L)$. But since $N(L)$ is the neutrosophic minimal left ideal of $N(S)$ therefore, we conclude that $N(A)N(L) = N(L)$ which implies that $N(L) \subseteq N(A)$. ■

Lemma 114 *Every neutrosophic left ideal of neutrosophic kernel $N(K)$ of $N(S)$ is also a neutrosophic left ideal of $N(S)$.*

Proof. Suppose $N(K)$ be the neutrosophic kernel of $N(S)$ and $N(A)$ be the neutrosophic left ideal of $N(K)$ that is $N(K)N(A) \subseteq N(A)$. Since each element $a + bI$ of $N(A)$ belongs to some neutrosophic minimal left ideal $N(L)$ of $N(S)$. Thus $N(K)(a + bI)$ is the neutrosophic minimal left ideal of $N(S)$ contained in $N(L)$ for every $a + bI \in N(A)$ but since $N(L)$ is the neutrosophic minimal left ideal of $N(S)$ therefore we have $N(K)(a + bI) = N(L)$ which implies that $a + bI \in N(K)(a + bI)$ which further implies that $N(A) \subseteq N(K)N(A)$ thus we conclude that $N(A) = N(K)N(A)$. ∎

Remark 115 *Every neutrosophic minimal left ideal of $N(S)$ is also a neutrosophic minimal left ideal of $N(K)$ and vice versa.*

Lemma 116 *Every neutrosophic left ideal of $N(S)$ contains atleast one neutrosophic minimal left ideal of $N(S)$.*

Proof. Let $N(K)$ be the class sum of all the neutrosophic minimal left ideals of $N(S)$. Let $N(A)$ be any neutrosophic left ideal of $N(S)$ then we know that $N(K)N(A)$ is also a neutrosophic left ideal of $N(S)$ contained in $N(K)$. But $N(K)N(A) = N(A)$ therefore, we conclude that $N(A) \subseteq N(K)$ which implies that $N(A)$ contains atleast one neutrosophic left ideal of $N(S)$. ∎

A neutrosophic AG-groupoid $N(S)$ is called neutrosophic left (right) simple if it does not contains any proper neutrosophic left (right) ideal. If $N(S)$ is a neutrosophic AG-groupoid with left identity $e + eI$ then $N(S)(a + bI)$ is a neutrosophic principal left ideal of $N(S)$ generated by $a + bI$ for all $a + bI \in N(S)$ and $N(S)(a + bI)$ is a neutrosophic ideal of $N(S)$ for all $(a + bI) \in N_E(S)$, where $N_E(S)$ is the set of all the idempotent elements of $N(S)$.

Lemma 117 *If a neutrosophic AG-groupoid $N(S)$ with left identity $e+eI$ has a kernel $N(K)$. Then $N(K)$ is a neutrosophic simple AG-groupoid.*

Proof. Suppose $N(K)$ be the neutrosophic kernel of $N(S)$. Let $N(A)$ be a neutrosophic ideal of $N(K)$ then $[\{N(K)N(A)\}N(K)]^2$ is a neutrosophic ideal of $N(S)$ contained in $N(K)$. Now since $N(K)$ is the intersection of all the neutrosophic ideals of $N(S)$, therefore, $N(K) \subseteq [\{N(K)N(A)\}N(K)]^2$, thus $[\{N(K)N(A)\}N(K)]^2 = N(K)$. Since $[\{N(K)N(A)\}N(K)\}]^2 \subseteq N(A) \subseteq N(K)$. Thus $N(A) = N(K)$. Hence $N(K)$ is a neutrosophic simple AG-groupoid. ∎

Proposition 118 *A neutrosophic AG-groupoid $N(S)$ with left identity $e+eI$ is neutrosophic left simple if and only if $N(S)(a+bI) = N(S)$, for any $(a + bI)$ in $N(S)$.*

Proof. Let $N(S)$ be a neutrosophic left simple AG-groupoid. Then $N(S)(N(S)(a+bI)) \subseteq N(S)(a + bI)$. Hence $N(S) = N(S)(a + bI)$.

Conversely, if $N(S)(a+bI) = N(S)$ and let $N(A)$ is a neutrosophic left ideal of $N(S)$, then $N(S)N(A) \subseteq N(A)$, hence $N(S) = N(S)N(A) \subseteq N(A)$ imply that $N(S) \subseteq N(A)$. Hence $N(S)$ has no proper neutrosophic left ideal in other words $N(S)$ is neutrosophic left simple AG-groupoid. ■

2.2 Neutrosophic Simple AG-groupoids

Proposition 119 *A neutrosophic AG-groupoid $N(S)$ with left identity $e+eI$ is neutrosophic right simple if and only if $(a+bI)^2N(S) = N(S)$, for any $(a+bI)$ in $N(S)$.*

Proof. Let $N(S)$ be a right simple AG-groupoid. Then by neutrosophic left invertive law and (4), we get

$$\begin{aligned} & [(a+bI)^2N(S))]N(S) \\ = \; & [N(S)N(S)](a+bI)^2 \\ = \; & [N(S)N(S)][(a+bI)(a+bI)] \\ = \; & (a+bI)^2N(S). \end{aligned}$$

Hence $N(S) = (a+bI)^2N(S)$.

Conversely, if $(a+bI)^2N(S) = N(S)$ and let $N(A)$ is a neutrosophic right ideal of $N(S)$, then $N(A)N(S) \subseteq N(A)$, hence $N(S) = (a+bI)^2N(S) \subseteq [N(A)N(A)]N(S) \subseteq N(A)$ imply that $N(S) \subseteq N(A)$. Hence $N(S)$ has no proper neutrosophic right ideal in other words $N(S)$ is neutrosophic right simple. ■

Theorem 120 *If $N(S)$ is a neutrosophic AG-groupoid with left identity $e+eI$ and $0+0I \in N(S)$, then $N(S)$ is neutrosophic zero-simple if and only if $[N(S)(a+bI)^2]N(S) = N(S)$, for every non-zero element $a+bI$ in $N(S)$.*

Proof. If $N(S)$ is a neutrosophic zero-simple AG-groupoid, then $(N(S))^2$ is a neutrosophic ideal of $N(S)$ and so $(N(S))^2 \neq \{0+0I\}$, imply that $(N(S))^2 = N(S)$, hence $(N(S))^3 = N(S)$. Now for every $a+bI$ in $N(S)\backslash\{0+0I\}$ the neutrosophic subset $[N(S)(a+bI)^2]N(S)$ of $N(S)$ becomes a neutrosophic ideal of $N(S)$. Therefore, either $[N(S)(a+bI)^2]N(S) = N(S)$ or $[N(S)(a+bI)^2]N(S) = \{0+0I\}$. If $[N(S)(a+bI)^2]N(S) = \{0+0I\}$, then the neutrosophic set $N(I) = \{(x+yI) \in N(S) : [N(S)(x+yI)]N(S) = \{0+0I\}\}$, contains an element $(a+bI)^2$ other than zero and becomes a neutrosophic ideal of $N(S)$. As $N(S)$ is neutrosophic zero-simple so $N(I)N(S)$, that is, $[N(S)(x)]N(S) = \{0+0I\}$, for every $(x+yI)$ in $N(S)$, implying that $(N(S))^3 = \{0+0I\}$. But this is a contradiction to the fact that $N(S) = (N(S))^3$. Hence $[N(S)(a+bI)^2]N(S) = N(S)$.

Conversely assume that, $[N(S)(a+bI)^2]N(S) = N(S)$, for every $a+bI$ in $N(S)\backslash\{0+0I\}$, then certainly, $a+bI \neq \{0+0I\}$. Also if $N(A)$ is a neutrosophic ideal of $N(S)$ containing $a+bI$, then $[N(S)\,(N(A))^2]N(S) \subseteq N(A)$, implying that $[N(S)(a+bI)^2]N(S) \subseteq N(A)$ and so $N(S) \subseteq N(A)$.

■

Corollary 121 *$N(S)$ is neutrosophic simple AG-groupoid if and only if $[N(S)(a+bI)^2]N(S) = N(S)$.*

Proof. If $N(S)$ is a neutrosophic simple AG-groupoid with left identity $e+eI$, then $[N(S)(a+bI)^2]N(S)$ is a neutrosophic ideal of $N(S)$ and so $[N(S)(a+bI)^2]N(S) = N(S)$.

Conversely, if $[N(S)(a+bI)^2]N(S) = N(S)$, for every $a+bI$ in $N(S)$, then we need to show that $N(S)$ is a neutrosophic simple AG-groupoid. Let $N(A)$ be a neutrosophic ideal of $N(S)$ and $a+bI \in N(A)$. Then $[N(S)(N(A))^2]N(S) \subseteq N(A)$ implies that $[N(S)(a+bI)^2]N(S) \subseteq N(A)$. But $[N(S)(a+bI)^2]N(S) = N(S)$ implies that $N(A) = N(S)$. Now if $\{0+0I\} \in N(S)$, then $[N(S)\{0+0I\}]N(S) = \{0+0I\} \neq N(S)$. As $[N(S)(a+bI)^2]N(S) = N(S)$, holds for every $a+bI$ in $N(S)$, it means that $0+0I \notin N(S)$. Hence $N(S)$ without zero has no neutrosophic ideal except $N(S)$ itself which implies that $N(S)$ is a neutrosophic simple AG-groupoid. ■

Theorem 122 *A neutrosophic minimal left ideal $N(L)$ of $N(S)$ is a neutrosophic left simple AG-subgroupoid.*

Proof. Suppose $N(L)$ be a neutrosophic minimal left ideal of $N(S)$. Now let $(a+bI) \in N(L)$ then $N(L)(a+bI)$ is also a neutrosophic minimal left ideal of $N(S)$ contained in $N(L)$. But since $N(L)$ is neutrosophic minimal so $N(L)(a+bI) = N(L)$ which implies that $N(L)$ is a neutrosophic left simple AG-subgroupoid. ■

In the rest by $N(L)$ we shall mean the neutrosophic minimal left ideal of $N(S)$, by $N(R)$ we mean the neutrosophic minimal right ideal of $N(S)$, where $N(S)$ is a neutrosophic AG**-groupoid, and $N(R)N(L)$ is the neutrosophic minimal left ideal of $N(S)$ contained in $N(L)$ thus $N(R)N(L) = N(L)$.

Lemma 123 *If $a_1+a_2I \in N(R)$ and $b_1+b_2I \in N(R)N(L)$, then the equation $(a_1+a_2I)^2(x+yI) = b_1+b_2I$ has solution $x+yI$ in $N(R)N(L)$.*

Proof. Let $a_1+a_2I \in N(R)$ then $(a_1+a_2I)^2 \in (N(R))^2 \subseteq N(R)$. Now by using neutrosophic left invertive law, neutrosophic medial law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned}[(a_1+a_2I)^2 N(R)]N(S) &= [N(S)N(R)]\,(a_1+a_2I)^2 \\ &= (a_1+a_2I)^2\,[N(R)N(S)] \subseteq (a_1+a_2I)^2\,N(R).\end{aligned}$$

This implies $(a_1 + a_2I)^2 N(R)$ is a neutrosophic right ideal of $N(S)$ contained in $N(R)$ but since $N(R)$ is neutrosophic minimal right ideal so $(a_1 + a_2I)^2 N(R) = N(R)$. Therefore

$$
\begin{aligned}
N(R) &= (a_1 + a_2I)^2 N(R) \text{ implies} \\
N(R)N(L) &= [(a_1 + a_2I)^2 N(R)]N(L) \\
&= [N(L)N(R)] (a_1 + a_2I)^2 \\
&= [N(L)N(R)][(a_1 + a_2I)(a_1 + a_2I)] \\
&= (a_1 + a_2I)^2 [N(R)N(L)].
\end{aligned}
$$

Thus the equation $(a_1 + a_2I)^2 (x + yI) = b_1 + b_2I$ has solution $x + yI$ in $N(R)N(L)$ for all $b_1 + b_2I \in N(R)N(L)$. ∎

Lemma 124 *If $a_1 + a_2I \in N(L)$ and $b_1 + b_2I \in N(R)N(L)$, then the equation $(x + yI)(a_1 + a_2I) = b_1 + b_2I$ has a solution $x + yI$ in $N(R)N(L)$ for all $b_1 + b_2I \in N(R)N(L)$.*

Proof. Let $N(L)$ be the neutrosophic minimal ideal of $N(S)$. Let $a_1 + a_2I \in N(L)$ then by using neutrosophic left invertive law, neutrosophic medial law, neutrosophic medial law and neutrosophic paramedial law we have

$$
\begin{aligned}
N(S)[N(L)(a_1 + a_2I)] &= [(a_1 + a_2I) N(L)]N(S) \\
&= [N(S)N(L)] (a_1 + a_2I) \\
&\subseteq N(L)(a_1 + a_2I).
\end{aligned}
$$

This implies $N(L)(a_1 + a_2I)$ is a neutrosophic left ideal of $N(S)$, but as $N(L)$ is the neutrosophic minimal ideal of $N(S)$ so $N(L) = N(L)(a_1 + a_2I)$. Now

$$
\begin{aligned}
N(L) &= N(L)(a_1 + a_2I) \text{ implies} \\
N(R)N(L) &= N(R)[N(L)(a_1 + a_2I)] \\
&= [N(R)N(R)][N(L)(a_1 + a_2I)] \text{ since } (N(R))^2 = N(R). \\
&= [(a_1 + a_2I) N(L)]N(R) \\
&= [N(R)N(L)] (a_1 + a_2I).
\end{aligned}
$$

Therefore for $a_1 + a_2I \in N(L)$ and $b_1 + b_2I \in N(R)N(L)$, then the equation $(x + yI)(a_1 + a_2I) = b_1 + b_2I$ has a solution $x + yI$ in $N(R)N(L)$ for all $b_1 + b_2I \in N(R)N(L)$. ∎

Corollary 125 *If $a_1 + a_2I \in N(L)$ and $b_1 + b_2I \in N(R)N(L)$, then the equation $(x + yI)(a_1 + a_2I)^2 = b_1 + b_2I$ has solution $x + yI$ in $N(R)N(L)$ for all $b_1 + b_2I \in N(R)N(L)$.*

Proof. Let $N(R)$ and $N(L)$ be the neutrosophic minimal right and neutrosophic minimal left ideal of $N(S)$ respectively. Let $a_1 + a_2I \in N(L)$ implies

$(a_1 + a_2 I)^2 \in N(L)$. Now by using neutrosophic left invertive law, neutrosophic medial law, neutrosophic medial law and neutrosophic paramedial law we have

$$\begin{aligned}
N(S)[N(L)(a_1 + a_2 I)^2] &= [N(S)N(S)][N(L)(a_1 + a_2 I)^2] \\
&= [(a_1 + a_2 I)^2 N(L)][N(S)N(S)] \\
&= [(a_1 + a_2 I)^2 N(L)](N(S) \\
&= [N(S)N(L)](a_1 + a_2 I)^2 \\
&\subseteq N(L)(a_1 + a_2 I)^2 .
\end{aligned}$$

This implies $N(L)(a_1 + a_2 I)^2$ is a neutrosophic left ideal of $N(S)$ contained in $N(L)$ but since $N(L)$ is neutrosophic minimal right ideal so $(a_1 + a_2 I)^2 N(L) = N(L)$. As $(N(R))^2 \subseteq N(R)$ is neutrosophic is a neutrosophic right ideal of $N(S)$ contained in $N(R)$, but $N(R)$ be the neutrosophic minimal right of $N(S)$ implies $(N(R))^2 = N(R)$. Now

$$\begin{aligned}
N(L) &= N(L)(a_1 + a_2 I)^2 \text{ implies} \\
N(R)N(L) &= N(R)[N(L)(a_1 + a_2 I)^2] \\
&= [N(R)]^2[N(L)(a_1 + a_2 I)^2] \\
&= [(a_1 + a_2 I)^2 N(L)][N(R)N(R)] \\
&= [(a_1 + a_2 I)^2 N(L)]N(R) \\
&= [N(R)N(L)](a_1 + a_2 I)^2 .
\end{aligned}$$

Therefore, for $a_1 + a_2 I \in N(L)$ and $b_1 + b_2 I \in N(R)N(L)$, there exists $x + yI$ in $N(R)N(L)$ such that $(x + yI)(a_1 + a_2 I)^2 = b_1 + b_2 I$. ■

Definition 126 *Let $N(S)$ be a neutrosophic AG-groupoid containing zero '$0+0I$'. A neutrosophic left or neutrosophic right ideal $N(A)$ of $N(S)$ will be called nilpotent if $(N(A))^n = 0 + 0I$ for some positive integer n. A neutrosophic ideal $N(A)$(neutrosophic left or neutrosophic right) of $N(S)$ is called neutrosophic nil-ideal if for each $a+bI \in N(A)$ there exist some $n \in N$ such that $(a + bI)^n = \underbrace{[...\{\{\{(a + bI)(a + bI)\}(a + bI)\}(a + bI)\}...](a + bI)}_{n-times} = 0 + 0I$.*

Theorem 127 *Let $N(S)$ be without neutrosophic nilpotent ideal, then every neutrosophic minimal ideal of $N(S)$ is neutrosophic simple.*

Proof. Let $N(M)$ be a neutrosophic minimal ideal of $N(S)$. Suppose a neutrosophic ideal $N(A)$ properly contained in $N(M)$. And $(N(M)N(A))N(M) \subseteq N(M)$ is the neutrosophic left ideal of $N(M)$. Then $[N(M)N(A)N(M)]^2$ is a neutrosophic ideal of $N(S)$ contained in $N(M)$, but $N(M)$ is neutrosophic minimal ideal therefore either $[N(M)N(A)N(M)]^2 = \{0 + 0I\}$

or $[N(M)N(A)N(M)]^2 = N(M)$. Since $N(S)$ is without neutrosophic nilpotent ideal so the case which is not possible is

$$[\{N(M)N(A)\}N(M)]^2 = \{0+0I\}.$$

Therefore

$$[\{N(M)N(A)\}N(M)]^2 = N(M)\,, \text{ but } [\{N(M)N(A)\}N(M)]^2 \subseteq N(A).$$

This implies that $N(M) \subseteq N(A)$. Thus $N(M) = N(A)$ and hence $N(M)$ is a neutrosophic simple AG-groupoid. ■

Theorem 128 *If every neutrosophic ideal $N(A)$(neutrosophic left or neutrosophic right) of $N(S)$ contains an idempotent element then $N(S)$ has no neutrosophic nil-ideal.*

Proof. Suppose $N(L)$ be the neutrosophic left ideal of $N(S)$ containing an idempotent element $a+bI$. Now we assume that $N(L)$ be the neutrosophic nil-ideal of $N(S)$, that is for each element of $N(L)$ there exist some positive integer n such that $(a+bI)^n = \underbrace{[...\{\{\{(a+bI)(a+bI)\}(a+bI)\}(a+bI)\}...](a+bI)}_{n-times} =$ $0+0I$ but we know from hypothesis that $(a+bI)^2 = a+bI$ for some $a+bI \in N(A)$, a contradiction. Therefore, $N(S)$ can not have any neutrosophic nil-ideal. ■

Theorem 129 *Let $N(M)$ be the neutrosophic minimal ideal of $N(S)$. Then every neutrosophic left ideal of $N(M)$ is a neutrosophic minimal left ideal of $N(S)$.*

Proof. Let $N(M)$ be a neutrosophic minimal ideal of $N(S)$, then we know that $N(M)$ contains atleast one neutrosophic minimal left ideal $N(L)$ of $N(S)$. Now let $N(B) \neq \{0+0I\} \subseteq N(L)$ be a neutrosophic left ideal of $N(M)$ that is $N(M)N(B) \subseteq N(B)$ and $N(M)N(B)$ is the neutrosophic left ideal of $N(S)$. So $N(M)N(B) \subseteq N(L)$, but $N(L)$ is the neutrosophic minimal left ideal of $N(S)$ therefore $N(L) = N(M)N(B) \subseteq N(B)$ which implies that $N(L) \subseteq N(B)$. Thus $N(L) = N(B)$. ■

Lemma 130 *If $N(S)$ be a neutrosophic AG^{**}-groupoid such that $N(S) = (N(S))^2$. Then $[N(S)(a+bI)^2]N(S) = N(S)(a+bI)^2$ for all $a+bI \in N(S)$.*

Proof. By the use of neutrosophic left invertive law, neutrosophic medial, neutrosophic paramedial and neutrosophic AG**-groupoid law we get

$$\begin{aligned}
[N(S)(a+bI)]^2 &= [N(S)(a+bI)][N(S)(a+bI)] \\
&= [N(S)N(S)](a+bI)^2 \\
&= [(a+bI)(a+bI)][N(S)N(S)] \\
&= N(S)[\{(a+bI)(a+bI)\}N(S)] \\
&= [N(S)N(S)][\{(a+bI)(a+bI)\}N(S)] \\
&= [N(S)(a+bI)^2][N(S)N(S)] = [N(S)(a+bI)^2]N(S).
\end{aligned}$$

■

Theorem 131 *Let $N(L)$ be a neutrosophic 0-minimal left ideal of a neutrosophic AG-groupoid containing zero element $0+0I$ and $(N(L))^2 \neq \{0+0I\}$ then $N(L) = N(S)(a+bI)$ for all $a+bI \in N(L) \backslash \{0+0I\}$.*

Proof. Suppose $N(L)$ be a neutrosophic 0-minimal left ideal of $N(S)$ and $(N(L))^2 \neq \{0+0I\}$. Now let $a+bI$ be any non-zero element of $N(L)$. Then $N(S)(a+bI) \in N(S)N(L) \subseteq N(L)$ is the neutrosophic left ideal of $N(S)$. But $N(L)$ is a neutrosophic 0-minimal left ideal of $N(S)$ so either $N(S)(a+bI) = \{0+0I\}$ or $N(L) = N(S)(a+bI)$ for all $a+bI \in N(L)\backslash\{0+0I\}$. If $N(S)(a+bI) = \{0+0I\}$ then $a+bI = 0+0I$ which is impossible. Thus $N(L) = N(S)(a+bI)$ for all $a+bI \in N(L)\backslash\{0+0I\}$. ■

Corollary 132 *If $N(L)$ is the neutrosophic 0-minimal left ideal of $N(S)$ containing an idempotent element then $N(L)$ is a neutrosophic ideal of $N(S)$ and $N(L) = N(S)(a+bI)$ for all $a+bI \in N_E(L)$.*

Proof. Suppose $a+bI \neq 0$ belonging to $N(L)$ be an idempotent. Then $N(S)(a+bI)^2 = N(S)(a+bI)$ is a neutrosophic ideal of $N(S)$ and $N(S)(a+bI) \in N(S)N(L) \subseteq N(L)$ but since $N(L)$ is neutrosophic 0-minimal left ideal of $N(S)$ therefore, either $N(L) = N(S)(a+bI)$ or $N(S)(a+bI) = \{0+0I\}$ for all $a+bI \in N(L)\backslash\{0+0I\}$. If $N(S)(a+bI) = \{0+0I\}$ then $a+bI = 0+0I$, a contradiction to the hypothesis. Thus $N(L) = N(S)(a+bI)$ for all $a+bI \in N_E(L)$. ■

Corollary 133 *Every neutrosophic left ideal of $N(S)$ containing an idempotent element contains a neutrosophic ideal of $N(S)$.*

Proof. Suppose $N(L)$ be a neutrosophic left ideal of $N(S)$. Let $a+bI$ belonging to $N(L)$ be an idempotent. Then clearly $N(S)(a+bI) = N(S)(a+bI)^2$ is a neutrosophic ideal of $N(S)$ and $N(S)(a+bI) \in N(S)N(L) \subseteq N(L)$. ■

Lemma 134 *If $N(M)$ is a neutrosophic 0-minimal ideal of $N(S)$ and $(N(M))^2 \neq \{0+0I\}$. Let $N(L) \neq \{0+0I\}$ contained in $N(M)$ be a neutrosophic left ideal of $N(S)$ then either $(N(L))^2 = N(M)$ or $(N(L))^2 = \{0+0I\}$.*

Proof. Suppose $N(M)$ be a neutrosophic 0-minimal ideal of $N(S)$ and $(N(M))^2 \neq \{0+0I\}$. Let $N(L) \neq \{0+0I\}$ be a neutrosophic left ideal of $N(S)$ contained in $N(M)$ so, $(N(L))^2 \subseteq N(M)$ is a neutrosophic ideal of $N(S)$. But since $N(M)$ is neutrosophic minimal therefore either $(N(L))^2 = N(M)$ or $(N(L))^2 = \{0+0I\}$. ■

Theorem 135 *Let $N(M)$ be a neutrosophic 0-minimal ideal of $N(S)$ containing atleast one neutrosophic 0-minimal left ideal of $N(S)$. Then $N(M)$ is the union of all the neutrosophic 0-minimal left ideals of $N(S)$ contained in $N(M)$.*

Proof. Let $N(M)$ be a neutrosophic 0-minimal ideal of $N(S)$ and $N(M)$ contains neutrosophic 0-minimal left ideal $N(L)$ of $N(S)$. Now suppose $N(A) \subseteq N(M)$ be the union of all neutrosophic 0-minimal left ideals of $N(S)$ contained in $N(M)$. Clearly $N(A)$ is neutrosophic left ideal of $N(S)$. Now let $a+bI \in N(A) \setminus \{0\}$ and $c+dI \in N(S)$, by definition of $N(A)$ we know that $a+bI$ belong to some neutrosophic 0-minimal left ideal $N(L)$ of $N(S)$, that is $(a+bI) \in N(L)$, and $(a+bI)(c+dI) \in N(L)c$. Since $N(L)$ is neutrosophic 0-minimal left ideal of $N(S)$ therefore $N(L)(c+dI)$ is also 0-minimal left ideal of $N(S)$. Thus $N(L)(c+dI) \subseteq N(M)$ which implies that $(a+bI)(c+dI) \in N(M)$ and $(a+bI)(c+dI) \in N(A)$ as well which further implies that $N(A)(c+dI) \subseteq N(A)$ for all $c+dI \in N(S)$. Hence $N(A)N(S) \subseteq N(A)$, that is, $N(A)$ is the neutrosophic right ideal of $N(S)$ and hence the neutrosophic ideal of $N(S)$. As $N(A) \subseteq N(M)$ but $N(M)$ is neutrosophic 0-minimal therefore $N(M) = N(A)$. ■

If $N(A)$ is any non vacuous subset of a neutrosophic groupoid $N(S)$. The intersection $N(K)$ of all the neutrosophic left ideals $N(L)$ of $N(S)$ containing $N(A)$ is a neutrosophic left ideal. Hence $N(A) \subseteq N(K)$ as well.

Proposition 136 *Let $N(S)$ be a neutrosophic AG-groupoid and $N(A)$ be any neutrosophic subset of $N(S)$. Then $N(A) \cup N(S)N(A)$ is neutrosophic left ideal of $N(S)$ containing $N(A)$.*

Proof. Let $N(A)$ be any neutrosophic subset of $N(S)$, we see that

$$\begin{aligned}
& N(S)[N(A) \cup N(S)N(A)] \\
= \; & N(S)N(A) \cup N(S)[N(S)N(A)] \\
= \; & [N(S)N(A)] \cup [N(S)N(S)][N(S)N(A)] \\
= \; & [N(S)N(A)] \cup [N(A)N(S)][N(S)N(S)] \\
= \; & [N(S)N(A)] \cup [N(A)N(S)]N(S) \\
= \; & [N(S)N(A)] \cup [N(S)N(S)]N(A) \\
= \; & N(S)N(A) \cup N(S)N(A) \\
= \; & N(S)N(A) \subseteq N(A) \cup N(S)N(A).
\end{aligned}$$

Hence $N(A) \cup N(S)N(A)$ is the neutrosophic left ideal of $N(S)$ containing

$N(A)$, and we call $N(A) \cup N(S)N(A)$ a neutrosophic left ideal of $N(S)$ generated by $N(A)$. ■

Remark 137 *$N(A) \cup [N(S)N(A)] \cup [N(A)N(S)] \cup [N(S)N(A)N(S)]$ is a neutrosophic ideal of $N(S)$ generated by $N(A)$. If $N(A)$ contains only one element that is $N(A) = \{a + bI\}$ then we have $N(A) \cup N(S)N(A) = N(S)(a + bI)$.*

Theorem 138 *For $N(S)$ with left identity $e + eI$, the following conditions are equivalent.*

(i) $N(S)$ is neutrosophic intra-regular.

(ii) $N(R) \cap N(L) = N(R)N(L)$, for every neutrosophic semiprime right ideal $N(R)$ and every neutrosophic left ideal $N(L)$.

(iii) $N(A) = [N(A)N(S)]N(A)$, for every neutrosophic quasi-ideal $N(A)$.

Proof. *(i) $\Rightarrow$ (iii) : Let $N(A)$ be a neutrosophic quasi ideal of $N(S)$ then $N(A)$ is a neutrosophic ideal of $N(S)$, thus $[N(A)N(S)]N(A) \subseteq N(A)$.*

Now let $a + bI \in N(A)$, and since $N(S)$ is neutrosophic intra-regular so there exist elements $(x_1 + x_2I)$, $(y_1 + y_2I)$ in $N(S)$ such that $a + bI = [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)$. Now by using neutrosophic left invertive law, neutrosophic medial law, neutrosophic medial law and neutrosophic paramedial law we have

$$
\begin{aligned}
a + bI &= [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
&= [(x_1 + x_2I)\{(a + bI)(a + bI)\}](y_1 + y_2I) \\
&= [(a + bI)\{(x_1 + x_2I)(a + bI)\}](y_1 + y_2I) \\
&= [(y_1 + y_2I)\{(x_1 + x_2I)(a + bI)\}](a + bI) \\
&= [(y_1 + y_2I)\{(x_1 + x_2I)\{\{(x_1 + x_2I)(a + bI)^2\}(y_1 + y_2I)\}\}](a + bI) \\
&= [(y_1 + y_2I)\{\{(x_1 + x_2I)(a + bI)^2\}\{(x_1 + x_2I)(y_1 + y_2I)\}\}](a + bI) \\
&= [\{(x_1 + x_2I)(a + bI)^2\}\{(y_1 + y_2I)\{(x_1 + x_2I)(y_1 + y_2I)\}\}](a + bI) \\
&= [\{(x_1 + x_2I)\{(a + bI)(a + bI)\}\}\{(y_1 + y_2I)\{(x_1 + x_2I)(y_1 + y_2I)\}\}](a + bI) \\
&= [\{(a + bI)\{(x_1 + x_2I)(a + bI)\}\}\{(y_1 + y_2I)\{(x_1 + x_2I)(y_1 + y_2I)\}\}](a + bI) \\
&= [\{(a + bI)(y_1 + y_2I)\}\{\{(x_1 + x_2I)(a + bI)\}\{(x_1 + x_2I)(y_1 + y_2I)\}\}](a + bI) \\
&= [\{(x_1 + x_2I)(a + bI)\}\{\{(a + bI)y\}\{(x_1 + x_2I)(y_1 + y_2I)\}\}](a + bI) \\
&= [\{(x_1 + x_2I)(a + bI)\}\{\{(a + bI)(x_1 + x_2I)\}(y_1 + y_2I)^2\}](a + bI) \\
&= [\{(y_1 + y_2I)^2\{(a + bI)(x_1 + x_2I)\}\}\{(a + bI)(x_1 + x_2I)\}](a + bI) \\
&= [(a + bI)\{\{(y_1 + y_2I)^2\{(a + bI)(x_1 + x_2I)\}\}(x_1 + x_2I)\}](a + bI) \\
&\in [N(A)N(S)]N(A).
\end{aligned}
$$

Hence $N(A) = [N(A)N(S)]N(A)$.

$(iii) \Rightarrow (ii)$: Clearly $N(R)N(L) \subseteq N(R) \cap N(L)$ holds. Now

$$\begin{aligned}
& N(S)[N(R) \cap N(L)] \cap [N(R) \cap N(L)]N(S) \\
= \; & N(S)N(R) \cap N(S)N(L) \cap N(R)N(S) \cap N(L)N(S) \\
= \; & N(R)N(S) \cap N(S)N(L) \cap N(S)N(R) \cap N(L)N(S) \\
\subseteq \; & N(R) \cap N(L) \cap [N(S)N(R) \cap N(L)N(S)] \\
\subseteq \; & N(R) \cap N(L). \text{ and}
\end{aligned}$$

$$\begin{aligned}
& N(R) \cap N(L) \\
= \; & [\{N(R) \cap N(L)\}N(S)][N(R) \cap N(L)] \\
= \; & [N(R)N(S) \cap N(L)N(S)][N(R) \cap N(L)] \\
\subseteq \; & [N(R) \cap N(L)N(S)][N(R) \cap N(L)] \\
\subseteq \; & N(R)N(L).
\end{aligned}$$

Hence $N(R) \cap N(L) = N(R)N(L)$.

$(ii) \Rightarrow (i)$: Assume that $N(R) \cap N(L) = N(R)N(L)$ for every neutrosophic right ideal $N(R)$ and every neutrosophic left ideal $N(L)$ of $N(S)$. Since $(a+bI)^2 \in (a+bI)^2N(S)$, which is a neutrosophic right ideal of $N(S)$ and as by given assumption $(a+bI)^2N(S)$ is neutrosophic semiprime which implies that $a+bI \in (a+bI)^2N(S)$. Now clearly $(a+bI) \cup N(S)(a+bI)$ is a neutrosophic principal left ideal, therefore

$$\begin{aligned}
a+bI \;\in\; & [(a+bI) \cup N(S)(a+bI)] \cap (a+bI)^2N(S) \\
\subseteq \; & N(S)[(a+bI)^2N(S)] \\
= \; & [N(S)N(S)][(a+bI)^2N(S)] \\
= \; & [N(S)(a+bI)^2][N(S)N(S)] = [N(S)(a+bI)^2]N(S).
\end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 139 *For $N(S)$ with left identity $e+eI$, the following conditions are equivalent.*

(i) $N(S)$ is neutrosophic intra-regular.

(ii) For a neutrosophic ideal $N(I)$ and neutrosophic quasi-ideal $N(Q)$, $N(I) \cap N(Q) = N(I)N(Q)$ and $N(I)$ is semiprime.

(iii) For neutrosophic quasi-ideals $N(Q_1)$ and $N(Q_2)$, $N(Q_1) \cap N(Q_2) = N(Q_1)N(Q_2)$ and $N(Q_1)$ and $N(Q_2)$ are semiprime.

Proof. $(i) \Longrightarrow (iii)$: Let $N(Q_1)$ and $N(Q_2)$ be a neutrosophic quasi-ideal of $N(S)$. Now $N(Q_1)$ and $N(Q_2)$ become neutrosophic ideals of $N(S)$. Therefore, $N(Q_1)N(Q_2) \subseteq N(Q_1) \cap N(Q_2)$. Now let $(a+bI) \in N(Q_1) \cap N(Q_2)$ which implies that $(a+bI) \in N(Q_1)$ and $(a+bI) \in N(Q_2)$. For $(a+bI) \in N(S)$ there exists $(x_1+x_2I), (y_1+y_2I)$ in $N(S)$ such that

$a+bI = [(x_1+x_2I)\,(a+bI)^2](y_1+y_2I)$. Now using (1) and neutrosophic left invertive law, we get

$$\begin{aligned}
& a+bI \\
= \;& [(x_1+x_2I)\,(a+bI)^2](y_1+y_2I) \\
= \;& [(x_1+x_2I)\,\{(a+bI)(a+bI)\}](y_1+y_2I) \\
= \;& [(a+bI)\{(x_1+x_2I)\,(a+bI)\}](y_1+y_2I) \\
= \;& [(y_1+y_2I)\{(x_1+x_2I)\,(a+bI)\}](a+bI) \\
\in \;& [N(S)\{N(S)N(Q_1)\}]N(Q_2) \\
\subseteq \;& [N(S)N(Q_1)]N(Q_2) \\
\subseteq \;& N(Q_1)N(Q_2).
\end{aligned}$$

This implies that $N(Q_1)\cap N(Q_2) \subseteq N(Q_1)N(Q_2)$. Hence $N(Q_1)\cap N(Q_2) = N(Q_1)N(Q_2)$. Next we will show that $N(Q_1)$ and $N(Q_2)$ are neutrosophic semiprime. For this let $(a+bI)^2 \in N(Q_1)$. Therefore, $a+bI = [(x_1+x_2I)\,(a+bI)^2](y_1+y_2I) \in [N(S)N(Q_1)]N(S) \subseteq N(Q_1)$. Similarly $N(Q_2)$ is neutrosophic semiprime.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$: Obviously $N(S)(a+bI)$ is a neutrosophic quasi-ideal contains $a+bI$ and $N(S)(a+bI)^2$ is a neutrosophic ideal contains $(a+bI)^2$. By (ii) $N(S)(a+bI)^2$ is neutrosophic semiprime so $a+bI \in N(S)(a+bI)^2$. Therefore by (ii) we get

$$\begin{aligned}
a+bI &\in N(S)(a+bI)^2 \cap [(a+bI) \cup \{N(S)(a+bI) \cap (a+bI)N(S)\}] \\
&= [N(S)(a+bI)^2][a+bI \cup \{N(S)(a+bI) \cap (a+bI)N(S)\}] \\
&\subseteq [N(S)(a+bI)^2]N(S).
\end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular AG-groupoid. ■

Theorem 140 *For neutrosophic AG-groupoid $N(S)$ with left $e+eI$, the following conditions are equivalent.*

(i) *$N(S)$ is neutrosophic intra-regular.*

(ii) *For neutrosophic quasi-ideals $N(Q_1)$ and $N(Q_2)$, $N(Q_1)\cap N(Q_2) = [N(Q_1)N(Q_2)]N(Q_1)$.*

Proof. $(i) \Longrightarrow (ii)$: Let $N(Q_1)$ and $N(Q_2)$ be neutrosophic quasi-ideals of $N(S)$. Now $N(Q_1)$ and $N(Q_2)$ become neutrosophic ideals of $N(S)$. Therefore, $[N(Q_1)N(Q_2)]N(Q_1) \subseteq [N(Q_1)N(S)]N(Q_1) \subseteq N(Q_1)$ and $[N(Q_1)N(Q_2)]Q_1 \subseteq [N(S)N(Q_2)]N(S) \subseteq N(Q_2)$. This implies that $[N(Q_1)N(Q_2)]N(Q_1) \subseteq N(Q_1)\cap N(Q_2)$. We can easily see that $N(Q_1)\cap N(Q_2)$ becomes a neutrosophic ideal. Then

$$\begin{aligned}
& N(Q_1) \cap N(Q_2) \\
= \; & [N\,(Q_1) \cap N(Q_2)]^2 \\
= \; & [N\,(Q_1) \cap N(Q_2)]^2 [N\,(Q_1) \cap N(Q_2)] \\
= \; & [\{N(Q_1) \cap N(Q_2)\}\{N\,(Q_1) \cap N(Q_2)\}]\{N\,(Q_1) \cap N(Q_2)\} \\
\subseteq \; & [N(Q_1)N(Q_2)]N(Q_1).
\end{aligned}$$

Thus $N(Q_1) \cap N(Q_2) \subseteq [N(Q_1)N(Q_2)]N(Q_1)$. Hence $N(Q_1) \cap N(Q_2) = [N(Q_1)N(Q_2)]N(Q_1)$.

$(ii) \Longrightarrow (i)$: Let $N(Q)$ be a neutrosophic quasi-ideal of $N(S)$, then by (ii), we get $N(Q) = N(Q) \cap N(Q) = [N(Q)N(Q)]N(Q) \subseteq [N(Q)]^2 N(Q) \subseteq N(Q)N(Q) = [N(Q)]^2$. This implies

that $N(Q) \subseteq [N(Q)]^2$ therefore $[N(Q)]^2 = N(Q)$. Now since $(a + bI) \cup [N(S)(a + bI) \cap (a + bI)N(S)]$

is a neutrosophic quasi-ideal,

$$\begin{aligned}
& (a + bI) \\
\in \; & [((a + bI)) \cup \{N(S)(a + bI) \cap (a + bI)N(S)\}] \\
= \; & ([(a + bI) \cup \{N(S)(a + bI) \cap (a + bI)N(S)\}])^2 \\
\subseteq \; & [(a + bI) \cup N(S)(a + bI)]^2 \\
= \; & [((a + bI)) \cup N(S)((a + bI))][((a + bI)) \cup N(S)((a + bI))] \\
= \; & (a + bI)^2 \cup ((a + bI))[N(S)(a + bI)] \cup [N(S)(a + bI)](a + bI) \\
& \cup [N(S)(a + bI)][N(S)(a + bI)] \\
= \; & ((a + bI))^2 \cup ((a + bI))^2 N(S) \cup N(S)((a + bI))^2.
\end{aligned}$$

Hence $N(S)$ is neutrosophic intra-regular AG-groupoid. ■

3

Some Classes of Neutrosophic AG-groupoids

In this chapter we have introduced the notion of neutrosophic $(2, 2)$-regular, neutrosophic strongly regular neutrosophic $\mathcal{AG}$-groupoids and investigated these structures. We have shown that neutrosophic regular, neutrosophic intra-regular and neutrosophic strongly regular $\mathcal{AG}$-groupoid are the only generalized classes of neutrosophic $\mathcal{AG}$-groupoid. Further we have shown that non-associative regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2, 2)$-regular and strongly regular neutrosophic $\mathcal{AG}^*$-groupoids do not exist.

We know that in every branch of science there is lots of complications and problems appear which affluence the uncertainties and impaction. Most of these problems and complications are concerning with human life. These problems also play pivotal role for being subjective and classical. For Instance, methods which are commonly are not sufficient to apply on these problems. Because problems can not handle various ambiguities involved in it. To solve these complications, concept of fuzzy sets was published by Lotfi A.Zadeh in 1965, which has a wide range of applications in various fields such as engineering, artificial intelligence, control engineering, operation research, management science, robotics and many more. Many papers on fuzzy sets have been appeared which shows the importance and its applications to the set theory, algebra, real analysis,measure theory and topology etc., fuzzy set theory is applied in many real applications to handle uncertainty.

Zadeh introduced fuzzy sets to address uncertainties. By use of fuzzy sets the manipulate data and information of uncertainties can be processed. The idea of fuzzy sets was particularly designed to characterize uncertainty and vagueness and to present dignified tools in order to deal with the ambiguity intrinsic to the various problems. Fuzzy logic gives a conjecture morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. The concept of fuzzy logic gives a mathematical potency to deal with the uncertainties associated with the human intellectual processes, such as reasoning and judgment.

In literature, a lot of theories have been developed to contend with uncertainty, imprecision and vagueness. In which, theory of probability, rough set theory fuzzy set theory, intiutionistic fuzzy sets etc, have played imperative role to cope with diverse types of uncertainties and imprecision entrenched in a system. But all these above theories were not sufficient tool to deal with indeterminate and inconsistent information in believe system. F.Samrandache noticed that the law of excluded middle are presently inactive in the modern logics and getting inspired with sport games (winning/tie/defeating), voting system (yes/ NA/no), decision making (making a decision/hesitating/not making) etc, he developed a new concept called neutrosophic set (NS) which is basically generalization of fuzzy sets and intiutionistic fuzzy sets. NS can be described by membership degree, and indeterminate degree and non-membership degree. This theory with its hybrid structures have proven efficient tool in different fields such as control theory, databases, medical diagnosis problem, decision making problem, physics and topology etc.

The fundamental theory of neutrosophic set, proposed by Smarandache. Salama et al. provide a natural basis for trating mathematically the neutrosophic phenomena which presents pervasively in our real world and for developing new branches of neutrosophic mathematics. The neutrosophic logic is an extended idea of neutrosophy. By giving representation to indeterminates, the introduction of neutrosophic theory has put forth a significant concept. Uncertainty or indeterminacy proved to be one of the most important factor in approximately all real-world problems. Fuzzy theory is used when uncertainty is modeled and when there is indeterminancy involved we use neutrosophic theory. Most of fuzzy models dealing with the analysis and study of unsupervised data, make use of the directed graphs or bipartite graphs. Thus the use of graphs in fuzzy models becomes inevitable. The neutrosophic models are basically fuzzy models that authorize the factor of indeterminancy.

The neutrosophic algebraic structures have defined very recently. Basically, Vasantha K andasmy and Florentin Smarandache present the concept of neutrosophic algebraic structures by using neutrosophic theory. A number of the neutrosophic algebraic structures introduced and considered include neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and neutrosophic AG-groupoids.

Now, $(a+bI)^2 = a+bI$ implies $a+bI$ is idempotent and if holds for all $a+bI \in N(S)$ then $N(S)$ is called idempotent neutrosophic LA-semigroup.

This structure is closely related with a neutrosophic commutative semigroup, because if a Neutrosophic $\mathcal{AG}$-groupoid contains a right identity, then it becomes a commutative semigroup. Define the binary operation "$\bullet$" on a commutative inverse semigroup $N(\mathcal{S})$ as

$$(a_1 + a_2 I) \bullet (b_1 + b_2 I) = (b_1 + b_2 I)(a_1 + a_2 I)^{-1}$$

for all $a_1 + a_2 I, b_1 + b_2 I \in N(\mathcal{S})$

then $(N(\mathcal{S}), \bullet)$ becomes an $\mathcal{AG}$-groupoid.

A neutrosophic $\mathcal{AG}$-groupoid $(\mathcal{S}, \cdot)$ with neutrosophic left identity becomes a neutrosophic semigroup $\mathcal{S}$ under new binary operation "$\circ$" defined as

$$(x_1 + x_2I) \circ (y_1 + y_2I) = [(x_1 + x_2I)(a_1 + a_2I)](y_1 + y_2I)$$

for all $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$.

It is easy to show that "$\circ$" is associative

$$\begin{aligned}
& [(x_1 + x_2I) \circ (y_1 + y_2I)] \circ (z_1 + z_2I) \\
= \; & [[[(x_1 + x_2I)(a_1 + a_2I)](y_1 + y_2I)](a_1 + a_2I)](z_1 + z_2I) \\
= \; & [[(z_1 + z_2I)(a_1 + a_2I)][[(x_1 + x_2I)(a_1 + a_2I)](y_1 + y_2I)]] \\
= \; & [(x_1 + x_2I)(a_1 + a_2II)[[(z_1 + z_2I)(a_1 + a_2I)](y_1 + y_2I)] \\
= \; & [(x_1 + x_2I)(a_1 + a_2I)[(y_1 + y_2I)(a_1 + a_2I)](z_1 + z_2I)] \\
= \; & (x_1 + x_2I) \circ [(y_1 + y_2I) \circ (z_1 + z_2I)].
\end{aligned}$$

Hence $(\mathcal{S}, \circ)$ is a neutrosophic semigroup. The Connections discussed above make this non-associative structure interesting and useful.

3.1 Regularities in Neutrosophic $\mathcal{AG}$-groupoids

An element $a+bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **regular** element of $N(\mathcal{S})$ if there exists $x_1 + x_2I \in N(\mathcal{S})$ such that $a + bI = [(a + bI) * (x_1 + x_2I)](a + bI)$ and $\mathcal{S}$ is called regular if all elements of $\mathcal{S}$ are regular.

An element $a + bI$ of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a **weakly regular** element of $\mathcal{S}$ if there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$ such that $a + bI = [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)]$ and $N(\mathcal{S})$ is called weakly regular if all elements of $\mathcal{S}$ are weakly regular.

An element $a + bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called an **intra-regular** element of $N(\mathcal{S})$ if there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$ such that $a + bI = [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)$ and $N(\mathcal{S})$ is called intra-regular if all elements of $N(\mathcal{S})$ are intra-regular.

An element $a+bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **right regular** element of $N(\mathcal{S})$ if there exists $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = (a+bI)^2(x_1+x_2I) = [(a+bI)(a+bI)](x_1+x_2I)$ and $N(\mathcal{S})$ is called right regular if all elements of $N(\mathcal{S})$ are right regular.

An element $a + bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **left regular** element of $\mathcal{S}$ if there exists $x_1 + x_2I \in N(\mathcal{S})$ such that $a + bI = (x_1 + x_2I)(a + bI)^2 = (x_1 + x_2I)[(a + bI)(a + bI)]$ and $N(\mathcal{S})$ is called left regular if all elements of $N(\mathcal{S})$ are left regular.

An element $a + bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **left quasi regular** element of $N(\mathcal{S})$ if there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$

such that $a+bI=[(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$ and $N(\mathcal{S})$ is called left quasi regular if all elements of $\mathcal{S}$ are left quasi regular.

An element $a+bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **completely regular** element of $N(\mathcal{S})$ if $a+bI$ is regular, left regular and right regular. $N(\mathcal{S})$ is called completely regular if it is regular, left and right regular.

An element $a+bI$ of a Neutrosopic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **(2,2)-regular** element of $N(\mathcal{S})$ if there exists $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)^2(x_1+x_2I)](a+bI)^2$ and $N(\mathcal{S})$ is called $(2,2)$-regular $\mathcal{AG}$-groupoid if all elements of $N(\mathcal{S})$ are $(2,2)$-regular.

An element $a+bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **strongly regular** element of $N(\mathcal{S})$ if there exists $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI=[(a+bI)(x_1+x_2I)](a+bI)$ and $(a+bI)(x_1+x_2I)=(x_1+x_2I)(a+bI)$. $N(\mathcal{S})$ is called strongly regular Neutrosophic $\mathcal{AG}$-groupoid if all elements of $N(\mathcal{S})$ are strongly regular.

A Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called Neutrosophic $\mathcal{AG}^*$-groupoid if the following holds

$$[(a_1+a_2I)(b_1+b_2I)](c_1+c_2I)=(b_1+b_2I)[(a_1+a_2I)(c_1+c_2I)] \qquad (5)$$

for all $a_1+a_2I, b_1+b_2I, c_1+c_2I \in N(\mathcal{S})$.

In Neutrosophic $\mathcal{AG}^*$-groupoid $\mathcal{S}$, the following law holds

$$(x_1x_2)(x_3x_4)=(x_{p(1)}x_{p(2)})(x_{p(3)}x_{p(4)}), \qquad (6)$$

where $\{p(1),p(2),p(3),p(4)\}$ means any permutation on the set $\{1,2,3,4\}$. It is an easy consequence that if $\mathcal{S}=\mathcal{S}^2$, then $\mathcal{S}$ becomes a commutative semigroup.

A Neutrosophic $\mathcal{AG}$-groupoid may or may not contains a left identity. The left identity of a Neutrosophic $\mathcal{AG}$-groupoid allow us to introduce the inverses of elements in a Neutrosophic $\mathcal{AG}$-groupoid. If a Neutrosophic $\mathcal{AG}$-groupoid contains a left identity, then it is unique [14].

Example 141 *Let us consider a Neutrosophic $\mathcal{AG}$-groupoid*
$N(\mathcal{S})=\{1+1I,\ 1+2I,\ 1+3I,\ 2+1I,\ 2+2I,\ 2+3I,\ 3+1I,\ 3+2I,\ 3+3I\}$
in the following multiplication table.

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+2I$	$1+3I$	$1+1I$	$1+2I$	$2+2I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$1+3I$	$1+2I$	$1+3I$	$1+1I$	$2+3I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$2+1I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$
$2+3I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$3+1I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$3+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$3+3I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$

Clearly $\mathcal{S}$ is non-commutative and non-associative, because $bc \neq cb$ and $(cc)a \neq c(ca)$. Note that $\mathcal{S}$ has no left identity.

Lemma 142 *If $N(\mathcal{S})$ is a regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular or strongly regular neutrosophic$\mathcal{AG}$-groupoid, then $N(S)$= $N(\mathcal{S})^2$.*
Proof. *Let $N(\mathcal{S})$ be a neutrosophic regular $\mathcal{AG}$-groupoid, then $N(\mathcal{S})^2 \subseteq N(\mathcal{S})$ is obvious. Let $a + bI \in N(\mathcal{S})$, then since $N(\mathcal{S})$ is regular so there exists $x + yI \in N(\mathcal{S})$ such that $a + bI = [(a + bI)(x + yI)](a + bI)$. Now*

$$a + bI = [(a + bI)(x + yI)](a + bI) \in N(\mathcal{S})\mathcal{N}(\mathcal{S}) = N(\mathcal{S})^2.$$

Similarly if $N(\mathcal{S})$ is weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular or strongly regular, then we can show that $N(\mathcal{S}) = N(\mathcal{S})^2$. ■

The converse is not true in general, because in Example above, $N(\mathcal{S}) = N(\mathcal{S})^2$ holds but $N(\mathcal{S})$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular, because $d_1 + d_2I \in N(\mathcal{S})$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular.

3.2 Some Characterizations

Theorem 143 *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then $N(\mathcal{S})$ is intra-regular if and only if for all $a + bI \in N(\mathcal{S})$, $a + bI = [(x_1 + x_2I)(a + bI)][(a + bI)(z_1 + z_2I)]$ holds for some $x_1 + x_2I, z_1 + z_2I \in N(\mathcal{S})$.*
Proof. *Let $N(\mathcal{S})$ be an intra-regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$ such that $a + bI = [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)$.*

Now $y_1 + y_2I = (u_1 + u_2I)(v_1 + v_2I)$ *for some* $u_1 + u_2I, v_1 + v_2I \in N(\mathcal{S})$.

$$\begin{aligned}
& a + bI \\
= \;& [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
= \;& [(x_1 + x_2I)[(a + bI)(a + bI)]](y_1 + y_2I) \\
= \;& [(a + bI)[(x_1 + x_2I)(a + bI)]](y_1 + y_2I) \\
= \;& [(y_1 + y_2I)[(x_1 + x_2I)(a + bI)]](a + bI) \\
= \;& [(y_1 + y_2I)[(x_1 + x_2I)(a + bI)]][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
= \;& [[(u_1 + u_2I)(v_1 + v_2)][(x + yI)(a + bI)]][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I), \\
= \;& [[(a + bI)(x_1 + x_2I)][(v_1 + v_2)(u_1 + u_2I)]][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
= \;& [[(a + bI)(x_1 + x_2I)](t_1 + t_2I)][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
= \;& [[(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)](t_1 + t_2I)][(a + bI)(x_1 + x_2I)] \\
= \;& [[(t_1 + t_2I)(y_1 + y_2I)][(x_1 + x_2I)(a + bI)^2]][(a + bI)(x_1 + x_2I)] \\
= \;& [[(a + bI)^2(x_1 + x_2I)][(y_1 + y_2I)(t_1 + t_2I)]][(a + bI)(x_1 + x_2I)] \\
= \;& [[(a + bI)^2(x_1 + x_2I)][(s_1 + s_2I)][(a + bI)(x_1 + x_2I)], \\
= \;& [[(s_1 + s_2I)(x_1 + x_2I)](a + bI)^2][(a + bI)(x_1 + x_2I)] \\
= \;& [[(s_1 + s_2I)(x_1 + x_2I)][(a + bI)(a + bI)]][(a + bI)(x_1 + x_2I)] \\
= \;& [[(a + bI)(a + bI)][(x_1 + x_2I)(s_1 + s_2I)]][(a + bI)(x_1 + x_2I)] \\
= \;& [[(a + bI)(a + bI)](w_1 + w_2I)[(a + bI)(x_1 + x_2I)], \\
= \;& [[(w_1 + w_2I)(a + bI)](a + bI)][(a + bI)(x_1 + x_2I)] \\
= \;& [(z_1 + z_2I)(a + bI)][(a + bI)(x_1 + x_2I)], \\
= \;& [(x_1 + x_2I)(a + bI)][(a + bI)(z_1 + z_2I)]
\end{aligned}$$

where $(w_1+w_2I)(a+bI) = (z_1+z_2I) \in N(S)$ *where* $(x_1+x_2I)(s_1+s_2I) = (w_1 + w_2I) \in N(S)$ *where* $(y_1 + y_2I)(t_1 + t_2I) = (s_1 + s_2I) \in N(S)$ *where* $(v_1+v_2)(u_1+u_2I) = (t_1+t_2I) \in N(S)$ *where* $[(u_1+u_2I)(v_1+v_2) = (y_1+y_2I) \in N(S)$ ■

Proof. *Conversely, let for all* $a + bI \in N(\mathcal{S})$, $a + bI = [(x_1 + x_2I)(a + bI)][(a + bI)(z_1 + z_2I)]$ *holds for some* $x_1 + x_2I, z_1 + z_2I \in N(\mathcal{S})$. *Now by* *using* (4), (1), (2) *and* (3), *we have*

$$
\begin{aligned}
& a+bI \\
= {} & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)] \\
= {} & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= {} & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= {} & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= {} & [[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)]](a+bI) \\
= {} & [[(x_1+x_2I)(a+bI)](z_1+z_2I)]^2(a+bI) \\
= {} & [[(x_1+x_2I)(a+bI)]^2(z_1+z_2I)^2](a+bI) \\
= {} & [[(x_1+x_2I)^2(a+bI)^2][(z_1+z_2I)(z_1+z_2I)]](a+bI) \\
= {} & [[(x_1+x_2I)^2(z_1+z_2I)][(a+bI)^2(z_1+z_2I)]](a+bI) \\
= {} & [(a+bI)^2[[(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= {} & [[(a+bI)(a+bI)][(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= {} & [[(a+bI)(a+bI)][[(z_1+z_2I)(z_1+z_2I)](x_1+x_2I)^2]](a+bI) \\
= {} & [[(a+bI)(a+bI)][(z_1+z_2I)^2(x_1+x_2I)^2]](a+bI) \\
= {} & [[(x_1+x_2I)^2(z_1+z_2I)^2][(a+bI)(a+bI)]](a+bI) \\
= {} & [(t_1+t_2I)[(a+bI)(a+bI)]](a+bI), \\
= {} & [(t_1+t_2I)(a+bI)^2](u_1+u_2I)
\end{aligned}
$$

where $[(x_1+x_2I)^2(z_1+z_2I)^2] = (t_1+t_2I) \in N(S)$ *and* $(a+bI) = (u_1+u_2I) \in N(S)$ *where* $(a+bI) = (u_1+u_2I) \in N(S)$ ■

Proof. Thus $N(\mathcal{S})$ is intra-regular. ■

Theorem 144 *If* $N(\mathcal{S})$ *is a Neutrosophic* $\mathcal{AG}$*-groupoid with left identity* ($\mathcal{AG}^{**}$*-groupoid*), *then the following are equivalent.*

(i) $N(\mathcal{S})$ is weakly regular.
(ii) $N(\mathcal{S})$ is intra-regular.
Proof. $(i) \implies (ii)$ Let $N(\mathcal{S})$ be a weakly regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$ and , $x_1+x_2I = (u_1+u_2I)(v_1+v_2I)$ for some $u_1+u_2I, v_1+v_2I \in N(\mathcal{S})$. Let $(v_1+v_2I)(u_1+u_2I) = t_1+t_2I \in N(\mathcal{S})$.

Now by using (3), (1), (4) and (2), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= (x_1+x_2I)[(a+bI)^2(y_1+y_2I)] \\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)^2(y_1+y_2I)] \\
&= [(y_1+y_2I)(a+bI)^2][(v_1+v_2I)(u_1+u_2I)] \\
&= [(y_1+y_2I)(a+bI)^2](t_1+t_2I)
\end{aligned}$$

Thus $N(\mathcal{S})$ is intra-regular.

$(ii) \Longrightarrow (i)$ Let $N(S)$ be a intra regular Neutrosophic AG-groupoid with left identity (Neutrosophic $\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(S)$

$$\begin{aligned}
a+bI &= [(y_1+y_2I)(a+bI)^2](t_1+t_2I) \\
&= [(y_1+y_2I)(a+bI)^2][(v_1+v_2I)(u_1+u_2I)] \\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)^2(y_1+y_2I)] \\
&= (x_1+x_2I)[(a+bI)^2(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]
\end{aligned}$$

$\Longrightarrow$Thus $N(S)$ is weakly regular. ■

Theorem 145 *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid* $(Neutrosophic\mathcal{AG}^{**}$*-groupoid*$)$, *then the following are equivalent.*

(i) $N(\mathcal{S})$ is weakly regular.

(ii) $N(\mathcal{S})$ is right regular.

Proof. $(i) \Longrightarrow (ii)$ Let $N(\mathcal{S})$ be a weakly regular Neutrosophic $\mathcal{AG}$-groupoid ($\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = (a+bI)(x_1+x_2I)(a+bI)(y_1+y_2I)$ and let $(x_1+x_2I)(y_1+y_2I) = t_1+t_2I$ for some $t+tI \in N(\mathcal{S})$. Now by using (2), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= (a+bI)^2(t_1+t_2I)
\end{aligned}$$

Thus $N(\mathcal{S})$ is right regular.

$(ii) \Longrightarrow (i)$ It follows from (2).

$$\begin{aligned}
a+bI &= (a+bI)^2(t_1+t_2I) \\
&= [(a+bI)(a+bI)](t_1+t_2I) \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]
\end{aligned}$$

where $(t_1 + t_2I) = (x_1 + x_2I)(y_1 + y_2I) \in N(\mathcal{S})$. Thus $N(S)$ is weakly regular. $\blacksquare$

Theorem 146 *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*
(i) $N(\mathcal{S})$ is weakly regular.
(ii) $N(\mathcal{S})$ is left regular.

Proof. $(i) \Longrightarrow (ii)$ Let $N(\mathcal{S})$ be a weakly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$ such that $a + bI = [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)]$. Now by using (2) and (3), we have

$$\begin{aligned} a + bI &= [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)] \\ &= [(a + bI)(a + bI)][(x_1 + x_2I)(y_1 + y_2I)] \\ &= [(y_1 + y_2I)(x_1 + x_2I)][(a + bI)(a + bI)] \\ &= [(y_1 + y_2I)(x_1 + x_2I)](a + bI)^2 \\ &= (t_1 + t_2I)(a + bI)^2, \end{aligned}$$

where $[(y_1 + y_2I)(x_1 + x_2I)] = (t_1 + t_2I)$ for some $(t_1 + t_2I) \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is left regular.

$(ii) \Longrightarrow (i)$ It follows from (3) and (2).

$$\begin{aligned} a + bI &= (t_1 + t_2I)(a + bI)^2 \\ &= [(y_1 + y_2I)(x_1 + x_2I)](a + bI)^2 \\ &= [(y_1 + y_2I)(x_1 + x_2I)][(a + bI)(a + bI)] \\ &= [(a + bI)(a + bI)][(x_1 + x_2I)(y_1 + y_2I)] \\ &= [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)], \end{aligned}$$

where($y_1 + y_2I)(x_1 + x_2I) = t_1 + t_2I$ for some $t_1 + t_2I \in N(\mathcal{S})$. Thus $N(S)$ is weakly regular. $\blacksquare$

Theorem 147 *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*
(i) $N(\mathcal{S})$ is weakly regular.
(ii) $N(\mathcal{S})$ is left quasi regular

Proof. $(i) \Longrightarrow (ii)$Let $N(\mathcal{S})$ be a weakly regular $Neutrosophic\mathcal{AG}$-groupoid with left identity, then for $a+bI \in N(\mathcal{S})$ there exists $(x_1+x_2I), (y_1+y_2I) \in N(\mathcal{S})$ such that $a + bI = [(a + b)(x_1 + x_2I)][(a + b)(y_1 + y_2I)]$

$$\begin{aligned} a + bI &= [(a + bI)(x_1 + x_2I)][(a + b)(y_1 + y_2I)] \\ &= [(y_1 + y_2I)(a + bI)][(x_1 + x_2I)(a + bI)]. \end{aligned}$$

Thus $N(S)$ is left quasi regular.

$(ii) \Longrightarrow (i)$ Let $N(\mathcal{S})$ be a left quasi regular Neutrosophic $\mathcal{AG}$-groupoid with left identity then for $a+bI \in N(\mathcal{S})$there exists (x_1+x_2I), $(y_1+y_2I) \in N(\mathcal{S})$ such that $a+bI = [(y_1+y_2I)(a+b)][(x_1+x_2I)(a+b)]$

$$\begin{aligned} a+bI &= [(y_1+y_2I)(a+b)][(x_1+x_2I)(a+b)] \\ &= [(a+b)(x_1+x_2I)][(a+b)(y_1+y_2I)]. \end{aligned}$$

Thus $N(S)$ is weakly regular. ■

Theorem 148 *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity, then the following are equivalent.*

(i) $N(\mathcal{S})$ is $(2,2)$-regular.

(ii) $N(\mathcal{S})$ is completely regular.

Proof. $(i) \Longrightarrow (ii)$ Let $N(\mathcal{S})$ be a $(2,2)$-regular $Neutrosophic\mathcal{AG}$-groupoid with left identity, then for $a+bI \in N(\mathcal{S})$ there exists $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)^2(x+yI)](a+bI)^2$. Now

$$\begin{aligned} a+bI &= [(a+bI)^2(x_1+x_2I)](a+bI)^2 \\ &= [(y_1+y_2I)(a+bI)^2], \end{aligned}$$

where $(a+bI)^2(x_1+x_2I) = y_1+y_2I \in N(\mathcal{S})$, and by using (3), we have

$$\begin{aligned} a+bI &= [(a+bI)^2(x_1+x_2I)][(a+bI)(a+bI)] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)[(a+bI)^2] \\ &= (a+bI)^2(z_1+z_2I), \end{aligned}$$

where $(x_1+x_2I)(a+bI)^2 = z_1+z_2I \in N(\mathcal{S})$. And by using (3), (1) and (4), we have

$$\begin{aligned} & a+bI \\ =\ & [(a+bI)^2[(x_1+x_2I)[(a+bI)(a+bI)]] \\ =\ & [(a+bI)(a+bI)][(x_1+x_2I)(a+bI)^2] \\ =\ & [(a+bI)(a+bI)][[(e_1+e_2I)(x_1+x_2I)][a+bI)(a+bI)]] \\ =\ & [(a+bI)(a+bI)][[(a+bI)(a+bI)][(x_1+x_2I)(e_1+e_2I)]] \\ =\ & [(a+bI)(a+bI)][(a+bI)^2(t_1+t_2I)] \\ =\ & [[(a+bI)^2(t_1+t_2I)](a+bI)](a+bI) \\ =\ & [[[(a+bI)(a+bI)](t_1+t_2I)](a+bI)](a+bI) \\ =\ & [[(t_1+t_2I)(a+bI)][(a+bI)](a+bI)]](a+bI) \\ =\ & [[(a+bI)(a+bI)][(t_1+t_2I)(a+bI)]](a+bI) \\ =\ & [[(a+bI)(t_1+t_2I)][(a+bI)(a+bI)]](a+bI) \\ =\ & [(a+bI)[[(a+bI)(t_1+t_2I)](a+bI)]](a+bI) \\ =\ & [(a+bI)(u_1+u_2I)](a+bI) \end{aligned}$$

where $t_1 + t_2I = (x_1 + x_2I)(e_1 + e_2I) \in N(\mathcal{S})$ & where $u_1 + u_2I = (a+bI)^2(t_1 + t_2I). \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is neutrosophic left regular, right regular and regular, so $N(\mathcal{S})$ is completely regular.

$(ii) \Longrightarrow (i)$ Assume that $N(\mathcal{S})$ is a completely regular neutrosophic $\mathcal{AG}$-groupoid with left identity, then for any $a + bI \in N(\mathcal{S})$ there exist $x + xI, y + yI, z + zI \in N(\mathcal{S})$ such that $a + bI = [(a + bI)(x + xI)](a + bI)$, $a + bI = (a + bI)^2(y + yI)$ and $a + bI = (z + zI)(a + bI)^2$. Now by using (1), (4) and (3), we have

$$\begin{aligned}
& a+bI \\
= & [(a+bI)(x+xI)](a+bI) \\
= & [[(a+bI)^2(y+yI)](x+xI)][(z_1+z_2I)(a+bI)^2] \\
= & [[(x_1+x_2I)(y_1+y_2I)](a+bI)^2][(z_1+z_2I)(a+bI)^2] \\
= & [[(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]][(z_1+z_2I)(a+bI)^2] \\
= & [[(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)]][(z_1+z_2I)(a+bI)^2] \\
= & [(a+bI)^2[(y_1+y_2I)(x_1+x_2I)]][(z_1+z_2I)(a+bI)^2] \\
= & [[(z_1+z_2I)(a+bI)^2][(y_1+y_2I)(x_1+x_2I)]](a+bI)^2 \\
= & [[(z_1+z_2I)(y_1+y_2I)][(a+bI)^2(x_1+x_2I)]](a+bI)^2 \\
= & [(a+bI)^2[[(z_1+z_2I)(y_1+y_2I)](x_1+x_2I)]](a+bI)^2 \\
= & [(a+bI)^2(v_1+v_2I)](a+bI)^2,
\end{aligned}$$

where $[(z_1+z_2I)(y_1+y_2I)][(x_1+x_2I)] = (v_1+v_2I) \in N(\mathcal{S})$. This shows that $N(\mathcal{S})$ is $(2,2)$-regular. ■

Lemma 149 *Every weakly regular neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosopic\mathcal{AG}^{**}$-groupoid) is regular.*

Proof. Assume that $N(\mathcal{S})$ is a weakly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$. Let $(x_1+x_2I)(y_1+y_2I) = t_1+t_2I \in N(\mathcal{S})$ and $[(t_1+t_2I)[(y_1+y_2I)(x_1+x_2I)]](a+bI) = u_1+u_2I \in N(\mathcal{S})$. Now by using

(1), (2), (3) and (4), we have

$$\begin{aligned}
& a+bI \\
= \;& [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
= \;& [[(a+bI)(y_1+y_2I)](x_1+x_2I)](a+bI) \\
= \;& [[(x_1+x_2I)(y_1+y_2I)](a+bI)](a+bI) \\
= \;& [(t_1+t_2I)(a+bI)](a+bI) \\
= \;& [(t_1+t_2I)[[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]]](a+bI) \\
= \;& [(t_1+t_2I)[[(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)]]](a+bI) \\
= \;& [(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)]]](a+bI) \\
= \;& [(t_1+t_2I)[(a+bI)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]]](a+bI) \\
= \;& [(a+bI)[(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]]](a+bI) \\
= \;& [(a+bI)(u_1+u_2I)](a+bI),
\end{aligned}$$

where $[(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]] = u_1+u_2I \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is regular. ■

Theorem 150 *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*
(i) $N(\mathcal{S})$ is weakly regular.
(ii) $N(\mathcal{S})$ is completely regular.

Proof. $(i) \Longrightarrow (ii)$
Let $N(\mathcal{S})$ be a weakly regular Neutrosophic $\mathcal{AG}$-groupoid ($\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = (a+bI)(x_1+x_2I)(a+bI)(y_1+y_2I)$ and let $(x_1+x_2I)(y_1+y_2I) = t_1+t_2I$ for some $t+tI \in N(\mathcal{S})$. Now by using (2), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= (a+bI)^2(t_1+t_2I),
\end{aligned}$$

where $(x_1+x_2I)(y_1+y_2I) = t_1+t_2I \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is right regular.
Let $N(\mathcal{S})$ be a weakly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$. Now by using (2) and (3), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= [(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)] \\
&= [(y_1+y_2I)(x_1+x_2I)](a+bI)^2 \\
&= (t_1+t_2I)(a+bI)^2,
\end{aligned}$$

where($y_1 + y_2I)(x_1 + x_2I) = t_1 + t_2I$ for some $t_1 + t_2I \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is left regular.

Assume that $N(\mathcal{S})$ is a weakly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$ such that $a + bI = [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)]$. Let $(x_1 + x_2I)(y_1 + y_2I) = t_1 + t_2I \in N(\mathcal{S})$ and $[(t_1 + t_2I)[(y_1 + y_2I)(x_1 + x_2I)]](a + bI) = u_1 + u_2I \in N(\mathcal{S})$. Now by using (1), (2), (3) and (4), we have

$$\begin{aligned}
a + bI &= [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)] \\
&= [[(a + bI)(y_1 + y_2I)](x_1 + x_2I)](a + bI) \\
&= [[(x_1 + x_2I)(y_1 + y_2I)](a + bI)](a + bI) \\
&= [(t_1 + t_2I)(a + bI)](a + bI) \\
&= [(t_1 + t_2I)[[(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)]]](a + bI) \\
&= [(t_1 + t_2I)[[(a + bI)(a + bI)][(x_1 + x_2I)(y_1 + y_2I)]]](a + bI) \\
&= [(t_1 + t_2I)[[(y_1 + y_2I)(x_1 + x_2I)][(a + bI)(a + bI)]]](a + bI) \\
&= [(t_1 + t_2I)[(a + bI)[[(y_1 + y_2I)(x_1 + x_2I)](a + bI)]]](a + bI) \\
&= [(a + bI)[(t_1 + t_2I)[[(y_1 + y_2I)(x_1 + x_2I)](a + bI)]]](a + bI) \\
&= [(a + bI)(u_1 + u_2I)](a + bI),
\end{aligned}$$

where $[(t_1 + t_2I)[[(y_1 + y_2I)(x_1 + x_2I)](a + bI)]] = (u_1 + u_2I) \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is regular. Thus $N(S)$ is completely regular.

$(ii) \Longrightarrow (i)$

Assume that $N(S)$ is completely regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $t_1 + t_2I \in N(\mathcal{S})$such that $a + bI = (a + bI)^2(x_1 + x_2I)$, $a + bI = (y_1 + y_2I)(a + bI)^2$, $a + bI = [(a + bI)(z_1 + z_2I)](a + bI)$

$$\begin{aligned}
a + bI &= (a + bI)^2(x_1 + x_2I) \\
&= [(a + bI)(a + bI)][(v_1 + v_2I)(u_1 + u_2I)] \\
&= [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)],
\end{aligned}$$

where $(x_1 + x_2I) = (v_1 + v_2I)(u_1 + u_2I) \in N(\mathcal{S})$. Thus $N(S)$ is neutrosophic weakly regular

$$\begin{aligned}
a + bI &= (x_1 + x_2I)(a + bI)^2 \\
&= [(u_1 + u_2I)(v_1 + v_2I)](a + bI)^2 \\
&= [(u_1 + u_2I)(v_1 + v_2I)[(a + bI)(a + bI)] \\
&= [(a + bI)(a + bI)][(u_1 + u_2I)(v_1 + v_2I)] \\
&= [(a + bI)(u_1 + u_2I)][(a + bI)(v_1 + v_2I)],
\end{aligned}$$

where$(x_1 + x_2I) = (u_1 + u_2I)(v_1 + v_2I)$ for some $(x_1 + x_2I) \in N(\mathcal{S})$.

Thus $N(S)$ is weakly regular.

$$\begin{aligned}
& a+bI \\
= \; & [(a+bI)(z_1+z_2I)](a+bI) \\
= \; & [[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2] \\
= \; & [[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]] \\
= \; & [(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]] \\
= \; & [(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)],
\end{aligned}$$

where $t_1+t_2I = [[(z_1+z_2I)(x_1+x_2I)](a+bI)]] \in N(\mathcal{S})$ & $(w_1+w_2I) = [(y_1+y_2I)(a+bI)] \in N(S)$. Thus $N(\mathcal{S})$ is weakly regular. ■

Lemma 151 *Every strongly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid) is completely regular.*

Proof. Assume that $N(\mathcal{S})$ is a strongly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exists $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)](a+bI)$ and $(a+bI)(x_1+x_2I) = (x_1+x_2I)(a+bI)$. Now by using (1), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)](a+bI) \\
&= [(x_1+x_2I)(a+bI)](a+bI) \\
&= [(a+bI)(a+bI)](x_1+x_2I) \\
&= (a+bI)^2(x_1+x_2I).
\end{aligned}$$

This shows that $N(\mathcal{S})$ is right regular and it is clear to see that $N(\mathcal{S})$ is completely regular. ■

Note that a completely regular Neutrosophic$\mathcal{AG}$-groupoid need not to be a strongly regular Neutrosophic $\mathcal{AG}$-groupoid, as can be seen from the following example.

Theorem 152 *In a Neutrosophic $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), the following are equivalent.*
(i) $N(\mathcal{S})$ is weakly regular.
(ii) $N(\mathcal{S})$ is intra-regular.
(iii) $N(\mathcal{S})$ is right regular.
(iv) $N(\mathcal{S})$ is left regular.
(v) $N(\mathcal{S})$ is left quasi regular.
(vi) $N(\mathcal{S})$ is completely regular.
(vii) For all $a+bI \in N(\mathcal{S})$, there exist $x+xI, y+yI \in N(\mathcal{S})$ such that $a+bI = [(x_{11}+x_2I)(a+bI)][(a+bI)(y_1+y_2I)]$.

Proof. $(i) \Longrightarrow (ii)$ Let $N(\mathcal{S})$ be weakly regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in$

$N(\mathcal{S})$ there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$ such that $a + bI = [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)]$ and $x_1 + x_2I = (u_1 + u_2I)(v_1 + v_2I)$ for some $u_1 + u_2I, v_1 + v_2I \in N(\mathcal{S})$. Let $(v_1 + v_2I)(u_1 + u_2I) = t_1 + t_2I \in N(\mathcal{S})$. Now by using (3), (1), (4) and (2), we have

$$\begin{aligned} a + bI &= [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)] \\ &= [(a + bI)(a + bI)][(x_1 + x_2I)(y_1 + y_2I)] \\ &= (x_1 + x_2I)[(a + bI)^2(y_1 + y_2I)] \\ &= [(u_1 + u_2I)(v_1 + v_2I)][(a + bI)^2(y_1 + y_2I)] \\ &= [(y_1 + y_2I)(a + bI)^2][(v_1 + v_2I)(u_1 + u_2I)] \\ &= [(y_1 + y_2I)(a + bI)^2](t_1 + t_2I) \end{aligned}$$

where $(v_1 + v_2I)(u_1 + u_2I) = (t_1 + t_2I) \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is intra-regular

.$(ii) \Longrightarrow (iii)$ Let $N(\mathcal{S})$ be a weakly regular $Neutrosophic\ \mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$ such that $a + bI = [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)$

$$\begin{aligned} a + bI &= [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\ &= [[(u_1 + u_2I)(v_1 + v_2I)][(a + bI)(a + bI)]](y_1 + y_2I) \\ &= [(a + bI)^2((v_1 + v_2I)(u_1 + u_2I))](y_1 + y_2I) \\ &= [(y_1 + y_2I)((v_1 + v_2I)(u_1 + u_2I))](a + bI)^2 \\ &= [(y_1 + y_2I)((v_1 + v_2I)(u_1 + u_2I))][(a + bI)(a + bI)] \\ &= [(a + bI)(a + bI)][((v_1 + v_2I)(u_1 + u_2I))(y_1 + y_2I)] \\ &= (a + bI)^2(s_1 + s_2I), \end{aligned}$$

where $x_1 + x_2I = (u_! + u_2I)(v_1 + v_2I) \in N(\mathcal{S})$ & $s_1 + s_2I = [(y_1 + y_2I)[v_1 + v_2I)(u_1 + u_2I)]] \in N(\mathcal{S})$. Thus $N(S)$ is right regular

$(iii) \Longrightarrow (iv)$ Let $N(\mathcal{S})$ be a right regular $Neutrosophic\ \mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $x_1 + x_2I \in N(\mathcal{S})$ such that $a + bI = (a + b)^2(x_1 + x_2I)$

$$\begin{aligned} a + bI &= (a + b)^2(x_1 + x_2I) \\ &= [(a + bI)(a + bI)](x_1 + x_2I) \\ &= [(a + bI)(a + bI)][(u_1 + u_2I)(v_1 + v_2I)] \\ &= [(v_1 + v_2I)(u_1 + u_2I)][a + bI)(a + bI)] \\ &= (y_1 + y_2I)(a + bI)^2 \end{aligned}$$

where $y_1 + y_2I = [(v_1 + v_2I)(u_1 + u_2I)] \in N(\mathcal{S})$. Thus $N(S)$ is left regular

$(iv) \Longrightarrow (v)$ Let $N(\mathcal{S})$ be a left regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $x_1 + x_2I \in N(\mathcal{S})$ such that $a + bI = (x_1 + x_2I)(a + bI)^2$

$$\begin{aligned} a + bI &= (x_1 + x_2I)(a + bI)^2 \\ &= [(u_1 + u_2I)(v_1 + v_2I)][(a + bI)(a + bI)] \\ &= [(u_1 + u_2I)(a + bI)][(v_1 + v_2I)(a + bI)] \end{aligned}$$

Thus $N(S)$ is left quasi regular

$(v) \Longrightarrow (vi)$ Let $N(\mathcal{S})$ be a left quasi regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $x_1 + x_2I \in N(\mathcal{S})$ such that $a + bI = [(x_1 + x_2I)(a + bI)][(y_1 + y_2I)(a + bI)]$

$$\begin{aligned} a + bI &= [(x_1 + x_2I)(a + bI)][(y_1 + y_2I)(a + bI)] \\ &= [(a + bI)(a + bI)][(y_1 + y_2I)(x_1 + x_2I)] \\ &= (a + bI)^2(v_1 + v_2I) \end{aligned}$$

where $v_1 + v_2I = [(y_1 + y_2I)(x_1 + x_2I)] \in N(\mathcal{S})$

Thus $N(S)$ is right regular $\Longrightarrow (1)$

Let $N(\mathcal{S})$ be a left quasi regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $x_1 + x_2I \in N(\mathcal{S})$ such that $a + bI = [(x_1 + x_2I)(a + bI)][(y_1 + y_2I)(a + bI)]$

$$\begin{aligned} a + bI &= [(x_1 + x_2I)(a + bI)][(y_1 + y_2I)(a + bI)] \\ &= [(x_1 + x_2I)(y_1 + y_2I)][(a + bI)(a + bI)] \\ &= (u_1 + u_2I)(a + bI)^2 \end{aligned}$$

where $(u_1 + u_2I) = [(x_1 + x_2I)(y_1 + y_2I)] \in N(\mathcal{S})$

Thus $N(S)$ is left regular$\Longrightarrow (2)$

Let $N(\mathcal{S})$ be a left quasi regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $x_1 + x_2I \in N(\mathcal{S})$ such that $a + bI = [(x_1 + x_2I)(a + bI)][(y_1 + y_2I)(a + bI)]$

$$\begin{aligned}
a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\
&= [(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)] \\
&= [((y_1+y_2I)(x_1+x_2I))(a+bI)](a+bI) \\
&= [(v_1+v_2I)(a+bI)](a+bI), \\
&= [(v_1+v_2I)[(x_1+x_2I)(a+bI))((y_1+y_2I)(a+bI)]](a+bI) \\
&= [(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]](a+bI) \\
&= [(v_1+v_2I)[(a+bI)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\
&= [(a+bI)[(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\
&= [(a+bI)(t_1+t_2I)](a+bI)
\end{aligned}$$

where $(v_1+v_2I)=(y_1+y_2I)(x_1+x_2I)\in N(\mathcal{S})$ & where $t_1+t_2I=[(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]\in N(\mathcal{S})$

Thus $N(S)$ is regular $\Longrightarrow$ (3)

By (1).(2) & (3) $N(S)$ is completely regular.

$(vi)\Longrightarrow(i)$ Let $N(\mathcal{S})$ be a complete regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI\in N(\mathcal{S})$ there exist $x_1+x_2I\in N(\mathcal{S})$ such that $a+bI=(a+bI)^2(x_1+x_2I)$, $a+bI=(y_1+y_2I)(a+bI)^2$, $a+bI=[(a+bI)(z_1+z_2I)]\ (a+bI)$

$$\begin{aligned}
a+bI &= (a+bI)^2(x_1+x_2I) \\
&= [(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I)] \\
&= [(a+bI)(u_1+u_2I)][(a+bI)(v_1+v_2I)]
\end{aligned}$$

where $(x_1+x_2I)=[(u_1+u_2I)(v_1+v_2I)]\in N(\mathcal{S})$. Thus $N(S)$ is weakly regular.

$$\begin{aligned}
a+bI &= (y_1+y_2I)(a+bI)^2 \\
&= [(v_1+v_2I)(u_1+u_2I)][(a+bI)(a+bI)] \\
&= [(a+bI)(u_1+u_2I)][(a+bI)(v_1+v_2I)]
\end{aligned}$$

where $(y_1+y_2I)=[(v_1+v_2I)(u_1+u_2I)]\in N(\mathcal{S})$. Thus $N(S)$ is neutrosophic weakly regular.

$$\begin{aligned}
&a+bI \\
=\;& [(a+bI)(z_1+z_2I)](a+bI) \\
=\;& [[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2] \\
=\;& [[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]] \\
=\;& [(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]] \\
=\;& [(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)]
\end{aligned}$$

where $t_1 + t_2I = [[(z_1 + z_2I)(x_1 + x_2I)](a + bI)]] \in N(\mathcal{S})$ & $(w_1 + w_2I) = [(y_1 + y_2I)(a + bI)] \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is neutrosophic weakly regular.

$(ii) \Longrightarrow (vii)$ Let $N(\mathcal{S})$ be an intra-regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then for any $a + bI \in N(\mathcal{S})$ there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$ such that $a + bI = [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)$. Now $y_1 + y_2I = (u_1 + u_2I)(v_1 + v_2I)$ for some $u_1 + u_2I, v_1 + v_2I \in N(\mathcal{S})$. Thus by using (4), (1) and (3), we have

$$
\begin{aligned}
& a + bI \\
= \; & [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
= \; & [(x_1 + x_2I)[(a + bI)(a + bI)]](y_1 + y_2I) \\
= \; & [(a + bI)[(x_1 + x_2I)(a + bI)]](y_1 + y_2I) \\
= \; & [(y_1 + y_2I)[(x_1 + x_2I)(a + bI)]](a + bI) \\
= \; & [(y_1 + y_2I)[(x_1 + x_2I)(a + bI)]][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
= \; & [[(u_1 + u_2I)(v_1 + v_2)][(x + yI)(a + bI)]][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
= \; & [[(a + bI)(x_1 + x_2I)][(v_1 + v_2)(u_1 + u_2I)]][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
= \; & [[(a + bI)(x_1 + x_2I)](t_1 + t_2I)][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
= \; & [[(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)](t_1 + t_2I)][(a + bI)(x_1 + x_2I)] \\
= \; & [[(t_1 + t_2I)(y_1 + y_2I)][(x_1 + x_2I)(a + bI)^2]][(a + bI)(x_1 + x_2I)] \\
= \; & [[(a + bI)^2(x_1 + x_2I)][(y_1 + y_2I)(t_1 + t_2I)]][(a + bI)(x_1 + x_2I)] \\
= \; & [[(a + bI)^2(x_1 + x_2I)][(s_1 + s_2I)][(a + bI)(x_1 + x_2I)] \\
= \; & [[(s_1 + s_2I)(x_1 + x_2I)](a + bI)^2][(a + bI)(x_1 + x_2I)] \\
= \; & [[(s_1 + s_2I)(x_1 + x_2I)][(a + bI)(a + bI)]][(a + bI)(x_1 + x_2I)] \\
= \; & [[(a + bI)(a + bI)][(x_1 + x_2I)(s_1 + s_2I)]][(a + bI)(x_1 + x_2I)] \\
= \; & [[(a + bI)(a + bI)](w_1 + w_2I)[(a + bI)(x_1 + x_2I)] \\
= \; & [[(w_1 + w_2I)(a + bI)](a + bI)][(a + bI)(x_1 + x_2I)] \\
= \; & [(z_1 + z_2I)(a + bI)][(a + bI)(x_1 + x_2I)] \\
= \; & [(x_1 + x_2I)(a + bI)][(a + bI)(z_1 + z_2I)]
\end{aligned}
$$

where $[(u_1 + u_2I)(v_1 + v_2) = (y_1 + y_2I) \in N(S)$ & where $(v_1 + v_2)(u_1 + u_2I) = (t_1 + t_2I) \in N(S)$ & where $(y_1 + y_2I)(t_1 + t_2I) = (s_1 + s_2I) \in N(S)$ & where $(x_1 + x_2I)(s_1 + s_2I) = (w_1 + w_2I) \in N(S)$ & where $(w_1 + w_2I)(a + bI) = (z_1 + z_2I) \in N(S)$

$(vii) \Longrightarrow (ii)$ let for all $a + bI \in N(\mathcal{S})$, $a + bI = [(x_1 + x_2I)(a + bI)][(a + bI)(z_1 + z_2I)]$ holds for some $x + xI, z + zI \in N(\mathcal{S})$. Now by using (4), (1),

(2) and (3), we have

$$\begin{aligned}
& a+bI \\
= \; & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)] \\
= \; & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= \; & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= \; & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= \; & [[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)]](a+bI) \\
= \; & [[(x_1+x_2I)(a+bI)](z_1+z_2I)]^2(a+bI) \\
= \; & [[(x_1+x_2I)(a+bI)]^2(z_1+z_2I)^2](a+bI) \\
= \; & [[(x_1+x_2I)^2(a+bI)^2][(z_1+z_2I)(z_1+z_2I)]](a+bI) \\
= \; & [[(x_1+x_2I)^2(z_1+z_2I)][(a+bI)^2(z_1+z_2I)]](a+bI) \\
= \; & [(a+bI)^2[[(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= \; & [[(a+bI)(a+bI)][(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= \; & [[(a+bI)(a+bI)][[(z_1+z_2I)(z_1+z_2I)](x_1+x_2I)^2]](a+bI) \\
= \; & [[(a+bI)(a+bI)][(z_1+z_2I)^2(x_1+x_2I)^2]](a+bI) \\
= \; & [[(x_1+x_2I)^2(z_1+z_2I)^2][(a+bI)(a+bI)]](a+bI) \\
= \; & [(t_1+t_2I)[(a+bI)(a+bI)]](a+bI), \\
= \; & [(t_1+t_2I)(a+bI)^2](u_1+u_2I)
\end{aligned}$$

where $[(x_1+x_2I)^2(z_1+z_2I)^2] = (t_1+t_2I) \in N(S)$ and $(a+bI) = (u_1+u_2I) \in N(S)$ where $(a+bI) = (u_1+u_2I) \in N(S)$. Thus $N(S)$ is neutrosophic intra regular. ■

Remark 153 *Every intra-regular, right regular, left regular, left quasi regular and completely regular $\mathcal{AG}$-groupoids with left identity ($\mathcal{AG}^{**}$-groupoids) are regular.*

The converse of above is not true in general. Indeed, from Example above, regular $\mathcal{AG}$-groupoid with left identity is not necessarily intra-regular.

Theorem 154 *In a Neutrosophic $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the following are equivalent.*
(i) $N(\mathcal{S})$ is weakly regular.
(ii) $N(\mathcal{S})$ is intra-regular.
(iii) $N(\mathcal{S}$ is right regular.
(iv) $N(\mathcal{S})$ is left regular.
(v) $N(\mathcal{S})$ is left quasi regular.
(vi) $N(\mathcal{S})$ is completely regular.
(vii) For all $a+bI \in N(\mathcal{S})$, there exist $x+xI, y+yI \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(a+bI)(y_1+y_2I)]$.
(viii)$N(\mathcal{S})$ is $(2,2)$-regular.

Proof. $(i) \implies (ii)$ Let $N(\mathcal{S})$ be a weakly regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$ and $x_1+x_2I = (u_1+u_2I)(v_1+v_2I)$ for some $u_1+u_2I, v_1+v_2I \in N(\mathcal{S})$. Let $(v_1+v_2I)(u_1+u_2I) = t_1+t_2I \in N(\mathcal{S})$. Now by using (3), (1), (4) and (2), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= (x_1+x_2I)[(a+bI)^2(y_1+y_2I)] \\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)^2(y_1+y_2I)] \\
&= [(y_1+y_2I)(a+bI)^2][(v_1+v_2I)(u_1+u_2I)] \\
&= [(y_1+y_2I)(a+bI)^2](t_1+t_2I)
\end{aligned}$$

where $(v_1+v_2I)(u_1+u_2I) = (t_1+t_2I) \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is intra-regular.

$(ii) \implies (iii)$ Let $N(\mathcal{S})$ be a intra regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$

$$\begin{aligned}
a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
&= [[(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)]](y_1+y_2I), \\
&= [(a+bI)^2((v_1+v_2I)(u_1+u_2I))](y_1+y_2I) \\
&= [(y_1+y_2I)((v_1+v_2I)(u_1+u_2I))](a+bI)^2 \\
&= [(y_1+y_2I)((v_1+v_2I)(u_1+u_2I))][(a+bI)(a+bI)] \\
&= [(a+bI)(a+bI)][((v_1+v_2I)(u_1+u_2I))(y_1+y_2I)] \\
&= (a+bI)^2(s_1+s_2I)
\end{aligned}$$

where $x_1+x_2I = (u_! + u_2I)(v_1+v_2I) \in N(\mathcal{S})$ & where $s_1+s_2I = [(y_1+y_2I)[v_1+v_2I)(u_1+u_2I)]] \in N(\mathcal{S})$. Thus $N(S)$ is right regular

$(iii) \implies (iv)$ Let $N(\mathcal{S})$ be a right regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = (a+b)^2(x_1+x_2I)$

$$\begin{aligned}
a+bI &= (a+b)^2(x_1+x_2I) \\
&= [(a+bI)(a+bI)](x_1+x_2I) \\
&= [(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I)] \\
&= [(v_1+v_2I)(u_1+u_2I)][a+bI)(a+bI)] \\
&= (y_1+y_2I)(a+bI)^2
\end{aligned}$$

where $y_1+y_2I=[(v_1+v_2I)(u_1+u_2I)]\in N(\mathcal{S})$. Thus $N(S)$ is left regular

$(iv)\Longrightarrow(v)$ Let $N(\mathcal{S})$ be a left regular $Neutrosophic\ \mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI\in N(\mathcal{S})$ there exist $x_1+x_2I\in N(\mathcal{S})$ such that $a+bI=(x_1+x_2I)(a+bI)^2$

$$\begin{aligned}
a+bI &= (x_1+x_2I)(a+bI)^2\\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)]\\
&= [(u_1+u_2I)(a+bI)][(v_1+v_2I)(a+bI)]
\end{aligned}$$

Thus $N(S)$ is left quasi regular

$(v)\Longrightarrow(vi)$ Let $N(\mathcal{S})$ be a left quasi regular $Neutrosophic\ \mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI\in N(\mathcal{S})$ there exist $x_1+x_2I\in N(\mathcal{S})$ such that $a+bI=[(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned}
a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]\\
&= [(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)]\\
&= (a+bI)^2(v_1+v_2I)
\end{aligned}$$

where $v_1+v_2I=[(y_1+y_2I)(x_1+x_2I)]\in N(\mathcal{S})$. Thus $N(S)$ is neutrosophic right regular. Let $N(\mathcal{S})$ be a left quasi regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI\in N(\mathcal{S})$ there exist $x_1+x_2I\in N(\mathcal{S})$ such that $a+bI=[(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned}
a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]\\
&= [(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]\\
&= (u_1+u_2I)(a+bI)^2
\end{aligned}$$

where $(u_1+u_2I)=[(x_1+x_2I)(y_1+y_2I)]\in N(\mathcal{S})$. Thus $N(S)$ is neutrosophic left regular.

Let $N(\mathcal{S})$ be a neutrosophic left quasi regular $Neutrosophic\ \mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI\in N(\mathcal{S})$ there exist $x_1+x_2I\in N(\mathcal{S})$ such that $a+bI=[(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned}
& a+bI \\
= \; & [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\
= \; & [(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)] \\
= \; & [((y_1+y_2I)(x_1+x_2I))(a+bI)](a+bI) \\
= \; & [(v_1+v_2I)(a+bI)](a+bI) \\
= \; & [(v_1+v_2I)[(x_1+x_2I)(a+bI))((y_1+y_2I)(a+bI)]](a+bI) \\
= \; & [(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]](a+bI) \\
= \; & [(v_1+v_2I)[(a+bI)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\
= \; & [(a+bI)[(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\
= \; & [(a+bI)(t_1+t_2I)](a+bI)
\end{aligned}$$

where $(v_1+v_2I)=(y_1+y_2I)(x_1+x_2I)\in N(\mathcal{S})$ & where $t_1+t_2I = [(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]\in N(\mathcal{S})$. Thus $N(S)$ is regular $\Longrightarrow$ (3)

By (1).(2) & (3) $N(S)$ is neutrosophic completely regular.

$(vi)\Longrightarrow(i)$ Assume that $N(S)$ is neutrosophic completely regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI\in N(\mathcal{S})$ there exist $t_1+t_2I\in N(\mathcal{S})$such that $a+bI=(a+bI)^2(x_1+x_2I)$, $a+bI=(y_1+y_2I)(a+bI)^2$, $a+bI=[(a+bI)(z_1+z_2I)](a+bI)$

$$\begin{aligned}
& a+bI \\
= \; & [(a+bI)(z_1+z_2I)](a+bI) \\
= \; & [[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2] \\
= \; & [[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]] \\
= \; & [(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]] \\
= \; & [(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)]
\end{aligned}$$

Thus $N(S)$ is neutrosophic weakly regular.

$(ii)\Longrightarrow(vii)$Let $N(\mathcal{S})$ be a neutrosophic intra-regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then for any $a+bI\in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I\in N(\mathcal{S})$ such that $a+bI=[(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Now $y+yI=(u_1+u_2I)(v_1+v_2I)$ for some

$u+uI, v+vI \in N(\mathcal{S})$.

$$\begin{aligned}
& a+bI \\
= & [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= & [(x_1+x_2I)[(a+bI)(a+bI)]](y_1+y_2I) \\
= & [(a+bI)[(x_1+x_2I)(a+bI)]](y_1+y_2I) \\
= & [(y_1+y_2I)[(x_1+x_2I)(a+bI)]](a+bI) \\
= & [(y_1+y_2I)[(x_1+x_2I)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= & [[(u_1+u_2I)(v_1+v_2)][(x+yI)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I), \\
= & [[(a+bI)(x_1+x_2I)][(v_1+v_2)(u_1+u_2I)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= & [[(a+bI)(x_1+x_2I)](t_1+t_2I)][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= & [[(x_1+x_2I)(a+bI)^2](y_1+y_2I)](t_1+t_2I)][(a+bI)(x_1+x_2I)] \\
= & [[(t_1+t_2I)(y_1+y_2I)][(x_1+x_2I)(a+bI)^2]][(a+bI)(x_1+x_2I)] \\
= & [[(a+bI)^2(x_1+x_2I)][(y_1+y_2I)(t_1+t_2I)]][(a+bI)(x_1+x_2I)] \\
= & [[(a+bI)^2(x_1+x_2I)][(s_1+s_2I)][(a+bI)(x_1+x_2I)], \\
= & [[(s_1+s_2I)(x_1+x_2I)](a+bI)^2][(a+bI)(x_1+x_2I)] \\
= & [[(s_1+s_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(a+bI)(x_1+x_2I)] \\
= & [[(a+bI)(a+bI)][(x_1+x_2I)(s_1+s_2I)]][(a+bI)(x_1+x_2I)] \\
= & [[(a+bI)(a+bI)](w_1+w_2I)[(a+bI)(x_1+x_2I)], \\
= & [[(w_1+w_2I)(a+bI)](a+bI)][(a+bI)(x_1+x_2I)] \\
= & [(z_1+z_2I)(a+bI)][(a+bI)(x_1+x_2I)], \\
= & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]
\end{aligned}$$

where $(w_1+w_2I)(a+bI) = (z_1+z_2I) \in N(S)$ where $(x_1+x_2I)(s_1+s_2I) = (w_1+w_2I) \in N(S)$ where $(y_1+y_2I)(t_1+t_2I) = (s_1+s_2I) \in N(S)$ where $(v_1+v_2)(u_1+u_2I) = (t_1+t_2I) \in N(S)$ where $[(u_1+u_2I)(v_1+v_2) = (y_1+y_2I) \in N(S)$

$(vii) \Longrightarrow (ii)$let for all $a+bI \in N(\mathcal{S})$, $a+bI = [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]$ holds for some $x_1+x_2I, z_1+z_2I \in N(\mathcal{S})$. Now by using (4), (1), (2) and (3), we have

$$\begin{aligned}
& a+bI \\
= \ & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)] \\
= \ & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= \ & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= \ & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= \ & [[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)]](a+bI) \\
= \ & [[(x_1+x_2I)(a+bI)](z_1+z_2I)]^2(a+bI) \\
= \ & [[(x_1+x_2I)(a+bI)]^2(z_1+z_2I)^2](a+bI) \\
= \ & [[(x_1+x_2I)^2(a+bI)^2][(z_1+z_2I)(z_1+z_2I)]](a+bI) \\
= \ & [[(x_1+x_2I)^2(z_1+z_2I)][(a+bI)^2(z_1+z_2I)]](a+bI) \\
= \ & [(a+bI)^2[[(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= \ & [[(a+bI)(a+bI)][(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= \ & [[(a+bI)(a+bI)][[(z_1+z_2I)(z_1+z_2I)](x_1+x_2I)^2]](a+bI) \\
= \ & [[(a+bI)(a+bI)][(z_1+z_2I)^2(x_1+x_2I)^2]](a+bI) \\
= \ & [[(x_1+x_2I)^2(z_1+z_2I)^2][(a+bI)(a+bI)]](a+bI) \\
= \ & [(t_1+t_2I)[(a+bI)(a+bI)]](a+bI), \\
= \ & [(t_1+t_2I)(a+bI)^2](u_1+u_2I)
\end{aligned}$$

where $[(x_1+x_2I)^2(z_1+z_2I)^2] = (t_1+t_2I) \in N(S)$ and $(a+bI) = (u_1+u_2I) \in N(S)$ where $(a+bI) = (u_1+u_2I) \in N(S)$. Thus $N(S)$ is intra regular.

$(vi) \implies (viii)$ Assume that $N(S)$ is neutrosophic completely regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $t_1+t_2I \in N(\mathcal{S})$ such that $a+bI = (a+bI)^2(x_1+x_2I)$, $a+bI = (y_1+y_2I)(a+bI)^2$, $a+bI = [(a+bI)(z_1+z_2I)](a+bI)$

$$\begin{aligned}
a+bI &= (a+bI)^2(x_1+x_2I) \\
&= [(a+bI)(a+bI)][(v_1+v_2I)(u_1+u_2I)] \\
&= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]
\end{aligned}$$

where $(x_1+x_2I) = (v_1+v_2I)(u_1+u_2I) \in N(\mathcal{S})$. Thus $N(S)$ is weakly regular

$$\begin{aligned}
a+bI &= (x_1+x_2I)(a+bI)^2 \\
&= [(u_1+u_2I)(v_1+v_2I)](a+bI)^2 \\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)] \\
&= [(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I))] \\
&= [(a+bI)(u_1+u_2I)][(a+bI)(v_1+v_2I)]
\end{aligned}$$

where$(x_1+x_2I)=(u_1+u_2I)(v_1+v_2I)$ for some $(x_1+x_2I)\in N(S)$. Thus $N(S)$ is weakly regular.

$$
\begin{aligned}
& a+bI \\
= \ & [(a+bI)(z_1+z_2I)](a+bI) \\
= \ & [[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2] \\
= \ & [[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]] \\
= \ & [(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]] \\
= \ & [(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)]
\end{aligned}
$$

where $t_1+t_2I=[[(z_1+z_2I)(x_1+x_2I)](a+bI)]]\in N(\mathcal{S})$ & $(w_1+w_2I)=[(y_1+y_2I)(a+bI)]\in N(\mathcal{S})$. Thus $N(S)$ is weakly regular.

$(viii)\Longrightarrow(vi)$ Let $N(\mathcal{S})$ be a neutrosophic $(2,2)$-regular $Neutrosophic\mathcal{AG}$-groupoid with left identity, then for $a+bI\in N(\mathcal{S})$ there exists $x_1+x_2I\in N(\mathcal{S})$ such that $a+bI=[(a+bI)^2(x+yI)](a+bI)^2$. Now

$$
\begin{aligned}
a+bI &= [(a+bI)^2(x_1+x_2I)](a+bI)^2 \\
&= [(y_1+y_2I)(a+bI)^2]
\end{aligned}
$$

where $(a+bI)^2(x_1+x_2I)=(y_1+y_2I)\in N(\mathcal{S})$ and by using (3), we have

$$
\begin{aligned}
a+bI &= (a+bI)^2[(x_1+x_2I)[(a+bI)(a+bI)]] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)[(a+bI)^2]] \\
&= (a+bI)^2(z_1+z_2I)
\end{aligned}
$$

where $(x_1+x_2I)(a+bI)^2=(z_1+z_2I)\in N(\mathcal{S})$. And by using (3), (1) and (4), we have

$$
\begin{aligned}
& a+bI \\
= \ & [(a+bI)^2[(x_1+x_2I)[(a+bI)(a+bI)]] \\
= \ & [(a+bI)(a+bI)][(x_1+x_2I)(a+bI)^2] \\
= \ & [(a+bI)(a+bI)][[(e_1+e_2I)(x_1+x_2I)][a+bI)(a+bI)]] \\
= \ & [(a+bI)(a+bI)][[(a+bI)(a+bI)][(x_1+x_2I)(e_1+e_2I)]] \\
= \ & [(a+bI)(a+bI)][(a+bI)^2(t_1+t_2I)] \\
= \ & [[(a+bI)^2(t_1+t_2I)](a+bI)](a+bI) \\
= \ & [[[(a+bI)(a+bI)](t_1+t_2I)](a+bI)](a+bI) \\
= \ & [[(t_1+t_2I)(a+bI)][(a+bI)](a+bI)]](a+bI) \\
= \ & [[(a+bI)(a+bI)][(t_1+t_2I)(a+bI)]](a+bI) \\
= \ & [[(a+bI)(t_1+t_2I)][(a+bI)(a+bI)]](a+bI) \\
= \ & [(a+bI)[[(a+bI)(t_1+t_2I)](a+bI)]](a+bI) \\
= \ & [(a+bI)(u_1+u_2I)](a+bI)
\end{aligned}
$$

where $t_1 + t_2I = (x_1 + x_2I)(e_1 + e_2I) \in N(\mathcal{S})$
& $u_1 + u_2I = (a + bI)^2(t_1 + t_2I). \in N(\mathcal{S})$

Thus $N(\mathcal{S})$ is left regular, right regular and regular, so $N(\mathcal{S})$ is completely regular.

$(vi) \Longrightarrow (viii)$ Assume that $N(\mathcal{S})$ is a completely regular Neutrosophic $\mathcal{AG}$-groupoid with left identity, then for any $a + bI \in N(\mathcal{S})$ there exist $x+xI, y+yI, z+zI \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)](a+bI)$, $a+bI = (a+bI)^2(y_1+y_2I)$ and $a+bI = (z_1+z_2I)(a+bI)^2$. Now by using (1), (4) and (3), we have

$$\begin{aligned}
& a + bI \\
= \;& [(a + bI)(x_1 + x_2I)](a + bI) \\
= \;& [[(a + bI)^2(y_1 + y_2I)](x_1 + x_2I)][(z_1 + z_2I)(a + bI)^2] \\
= \;& [[(x_1 + x_2I)(y_1 + y_2I)](a + bI)^2][(z_1 + z_2I)(a + bI)^2] \\
= \;& [[(x_1 + x_2I)(y_1 + y_2I)][(a + bI)(a + bI)]][(z_1 + z_2I)(a + bI)^2] \\
= \;& [[(a + bI)(a + bI)][(y_1 + y_2I)(x_1 + x_2I)]][(z_1 + z_2I)(a + bI)^2] \\
= \;& [(a + bI)^2[(y_1 + y_2I)(x_1 + x_2I)]][(z_1 + z_2I)(a + bI)^2] \\
= \;& [[(z_1 + z_2I)(a + bI)^2][(y_1 + y_2I)(x_1 + x_2I)]](a + bI)^2 \\
= \;& [[(z_1 + z_2I)(y_1 + y_2I)][(a + bI)^2(x_1 + x_2I)]](a + bI)^2 \\
= \;& [(a + bI)^2[[(z_1 + z_2I)(y_1 + y_2I)](x_1 + x_2I)]](a + bI)^2 \\
= \;& [(a + bI)^2(v_1 + v_2I)](a + bI)^2
\end{aligned}$$

where $[(z_1+z_2I)(y_1+y_2I)][(x_1+x_2I)] = (v_1+v_2I) \in N(\mathcal{S})$. Thus $N(S)$ is $(2, 2)$-regular. ■

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4

M-Systems in Neutrosophic AG-groupoids

In this chapter we have studied M-systems, p-systems and ideals in Neutrosophic AG-groupoids. We have proved that if S is an Neutrosophic AG-groupoid with left identity e, then the set of all ideals K form an Neutrosophic AG-groupoid. The set K of ideals is a locally associative Neutrosophic AG-groupoid if S is fully idempotent and is a commutative semigroup containing left identity e. We have shown that I^n, for $n \geq 2$, is an ideal for each I in Y. Also we have shown that $(AB)^n$ is an ideal and $(AB)^n = A^n B^n$, for all ideals A, B in Y. We have proved that a left ideal P of an Neutrosophic AG-groupoid S with left identity is quasi-prime if and only if $S \backslash P$ is an M-system. A left ideal I of S with left identity is quasi-semiprime if and only if $S \backslash I$ is a p-system. Specifically, we have shown that every right ideal is an M-system and every M-system is a p-system.

We know that in every branch of science there is lots of complications and problems appear which affluence the uncertainties and impaction. Most of these problems and complications are concerning with human life. These problems also play pivotal role for being subjective and classical. For Instance, methods which are commonly are not sufficient to apply on these problems. Because problems can not handle various ambiguities involved in it. To solve these complications, concept of fuzzy sets was published by Lotfi A.Zadeh in 1965, which has a wide range of applications in various fields such as engineering, artificial intelligence, control engineering, operation research, management science, robotics and many more. Many papers on fuzzy sets have been appeared which shows the importance and its applications to the set theory, algebra, real analysis,measure theory and topology etc., fuzzy set theory is applied in many real applications to handle uncertainty.

Zadeh introduced fuzzy sets to address uncertainties. By use of fuzzy sets the manipulate data and information of uncertainties can be processed. The idea of fuzzy sets was particularly designed to characterize uncertainty and vagueness and to present dignified tools in order to deal with the ambiguity intrinsic to the various problems. Fuzzy logic gives a conjecture morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. The concept of fuzzy logic gives a mathematical potency to deal with the uncertainties associated with the human intellectual processes, such as reasoning and judgment.

In literature, a lot of theories have been developed to contend with uncertainty, imprecision and vagueness. In which, theory of probability, rough set theory fuzzy set theory, intiutionistic fuzzy sets etc, have played imperative role to cope with diverse types of uncertainties and imprecision entrenched in a system. But all these above theories were not sufficient tool to deal with indeterminate and inconsistent information in believe system. F.Samrandache noticed that the law of excluded middle are presently inactive in the modern logics and getting inspired with sport games (winning/tie/defeating), voting system (yes/ NA/no), decision making (making a decision/hesitating/not making) etc, he developed a new concept called neutrosophic set (NS) which is basically generalization of fuzzy sets and intiutionistic fuzzy sets. NS can be described by membership degree, and indeterminate degree and non-membership degree. This theory with its hybrid structures have proven efficient tool in different fields such as control theory, databases, medical diagnosis problem, decision making problem, physics and topology etc.

The neutrosophic algebraic structures have defined very recently. Basically, Vasantha K andasmy and Florentin Smarandache present the concept of neutrosophic algebraic structures by using neutrosophic theory. A number of the neutrosophic algebraic structures introduced and considered include neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and neutrosophic AG-groupoids.

Now, $(a+bI)^2 = a+bI$ implies $a+bI$ is idempotent and if holds for all $a+bI \in N(S)$ then $N(S)$ is called idempotent Neutrosophic AG-groupoid.

This structure is closely related with a Neutrosophic commutative semigroup, because if a Neutrosophic $\mathcal{AG}$-groupoid contains a right identity, then it becomes a commutative semigroup. Define the binary operation "$\bullet$" on a commutative inverse Neutrosophic semigroup $N(\mathcal{S})$ as

$$(a_1 + a_2I) \bullet (b_1 + b_2I) = (b_1 + b_2I)(a_1 + a_2I)^{-1}$$

for all $a_1 + a_2I, b_1 + b_2I \in N(\mathcal{S})$

then $(N(\mathcal{S}), \bullet)$ becomes an $\mathcal{AG}$-groupoid.

A Neutrosophic $\mathcal{AG}$-groupoid $(N(\mathcal{S}), \cdot)$ with Neutrosophic left identity becomes a neutrosophic semigroup $\mathcal{S}$ under new binary operation "$\circ$" defined as

$$(x_1 + x_2I) \circ (y_1 + y_2I) = [(x_1 + x_2I)(a_1 + a_2I)](y_1 + y_2I)$$

for all $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$.

It is easy to show that "$\circ$" is associative

$$\begin{aligned}
& [(x_1+x_2I)\circ(y_1+y_2I)]\circ(z_1+z_2I) \\
= \; & [[[(x_1+x_2I)(a_1+a_2I)](y_1+y_2I)](a_1+a_2I)](z_1+z_2I) \\
= \; & [[(z_1+z_2I)(a_1+a_2I)][[(x_1+x_2I)(a_1+a_2I)](y_1+y_2I)]] \\
= \; & [(x_1+x_2I)(a_1+a_2II)[[(z_1+z_2I)(a_1+a_2I)](y_1+y_2I)] \\
= \; & [(x_1+x_2I)(a_1+a_2I)[(y_1+y_2I)(a_1+a_2I)](z_1+z_2I)] \\
= \; & (x_1+x_2I)\circ[(y_1+y_2I)\circ(z_1+z_2I)].
\end{aligned}$$

Hence $(\mathcal{S},\circ)$ is a neutrosophic semigroup. The Connections discussed above make this non-associative structure interesting and useful.

A subset $N(J)$ of an Neutrosophic AG-groupoid $N(S)$ is called a Neutrosophic right (left) ideal if $N(J)N(S)\subseteq N(J)$ $(N(S)N(J)\subseteq N(J))$, and is called a Neutrosophic ideal if it is a two-sided neutrosophic ideal.

A Neutrosophic AG-groupoid is called neutrosophic fully idempotent if all neutrosophic ideals of $N(S)$ are idempotent.

If $N(S)$ is a Neutrosophic AG-groupoid with left identity $e+eI$ then the Neutrosophic principal ideal generated by a fixed element $a+bI$ is defined as $\langle a+bI\rangle = N[S](a+bI) = \{(x+yI)(a+bI) : (x+yI)\in N(S)\}$. Clearly, $\langle a+bI\rangle$ is a left ideal of $N[S]$. Also it is worth mentioning that if $N(A)$ is a neutrosophic left ideal of $N(S)$ then $N(A)=\langle N(A)\rangle$, because by definition $N[S]N(A)\subseteq N(A)$ and $N(A)=(e+eI)N(A)\subseteq N[S]N(A)$.

4.1 Neutrosophic Semilattice Structures

Proposition 156 *If $N(S)$ is a Neutrosophic AG-groupoid with left identity $e+eI$ and $N(A)$, $N(B)$ are Neutrosophic ideals of $N(S)$, then the following assertions are equivalent.*

(i) $N(S)$ is neutrosophic fully idempotent,

(ii) $N(A)\cap N(B)=\langle N(A)N(B)\rangle$, and

(iii) the Neutrosophic ideals of $N(S)$ form a semilattice $(N[L],\wedge)$ where $N(A)\wedge N(B)=\langle N(A)N(B)\rangle$.

Proof. $(i)\Rightarrow(ii)$: Always $N(A)N(B)\subseteq N(A)\cap N(B)$ for ideals $N(A)$ and $N(B)$ of $N(S)$. Hence $\langle N(A)N(B)\rangle\subseteq N(A)\cap N(B)$. For the reverse inclusion, let $x+yI\in N(A)\cap N(B)$. If $\langle x+yI\rangle$ denotes the neutrosophic principal left ideal generated by $x+yI$, then $x+yI\in\langle x+yI\rangle=\langle x+yI\rangle\langle x+yI\rangle\subseteq\langle N(A)N(B)\rangle$. Thus $N(A)\cap N(B)\subseteq\langle N(A)N(B)\rangle$. Hence $N(A)\cap N(B)=\langle N(A)N(B)\rangle$.

$(ii)\Rightarrow(iii)$: $N(A)\wedge N(B)=\langle N(A)N(B)\rangle=N(A)\cap N(B)=N(B)\cap N(A)=N(B)\wedge N(A)$ and $N(A)N(A)=\langle N(A)N(A)\rangle=N(A)\cap N(A)=N(A)$. Similarly, associativity follows. Hence $(N[L],\wedge)$ is a semilattice.

$(iii) \Rightarrow (i)$:

$$N(A) = N(A) \wedge N(A) = \langle N(A)N(A)\rangle = N(A)N(A),$$

implies that $N(S)$ is Neutrosophic fully idempotent. ∎

A Neutrosophic AG-groupoid $N(S)$ is called Neutrosophic regular if for each $a + bI$ in $N(S)$ there exists $x + yI$ in $N(S)$ such that $a + bI = [(a + bI)(x + yI)](a + bI)$. A Neutrosophic AG-groupoid $N(S)$ is called inverse Neutrosophic AG-groupoid if for each $a + bI$ in $N(S)$ there exists a unique $x + yI$ in $N(S)$ such that, $a + bI = [(a + bI)(x + yI)](a + bI)$ and $x + yI = [(x + yI)(a + bI)](x + yI)$. If $N(S)$ is a regular Neutrosophic AG-groupoid then it is an easy consequence that $N(S) = [N(S)]^2$. Therefore, the following corollary is an immediate consequence of Proposition 1.

Corollary 157 *If $N(S)$ is a regular Neutrosophic AG-groupoid then $N(S)$ is Neutrosophic fully idempotent.*

Proof. It is always true that $[N(J)]^2 \subseteq N(J)$. For the reverse inclusion, if $a + bI \in N(J)$, then since $N(S)$ is regular, there exists $c + dI$ in $N(S)$ such that $a + bI = [(a + bI)(c + dI)](a + bI) \in [N(J)]^2$. Thus $N(J) \subseteq [N(J)]^2$, and hence $N(S)$ is fully idempotent. ∎

Proposition 158 *If $N(S)$ is a Neutrosophic AG-groupoid with left identity $e + eI$ then every neutrosophic right ideal is a neutrosophic left ideal.*

Proof. It is straight forward. ∎

By $N(K)$ we shall mean the set of all Neutrosophic ideals of a Neutrosophic AG-groupoid $N(S)$.

Lemma 159 *If $N(S)$ is a Neutrosophic AG-groupoid with left identity $e + eI$, then $N(K)$ forms a Neutrosophic AG-groupoid.*

Proof. let $N(A), N(B) \in N(K)$ then by identity (2) and definition of neutrosophic ideal we have $N(A)N(B)$

$$\begin{aligned} [N(A)N(B)]N(S) &= [N(A)N(B)][(e + eI)(N(S)] \\ &= [N(A)(e + eI)][N(B)N(S)] \subseteq N(A)N(B), \end{aligned}$$

which shows that $N(A)N(B)$ is a neutrosophic right ideal of $N(S)$ and so by Proposition 2, it becomes an ideal of $N(S)$. Also $[N(A)N(B)]N(C) = [N(C)N(B)]N(A)$, holds for all ideals $N(A)$, $N(B)$ and $N(C)$. Hence $N(K)$ is an Neutrosophic AG-groupoid. ∎

Note that the sets of neutrosophic left and neutrosophic right ideals of $N(S)$ are Neutrosophic AG-groupoids.

It is interesting to note that if $N(S)$ is a Neutrosophic AG-groupoid with left identity $e + eI$ and $N(J_1), N(J_2), N(J_3)$ are proper Neutrosophic ideals of $N(S)$, then $[N(J)N(J_2)]N(J_3)$ is a neutrosophic ideal of $N(S)$.

It can be generalized, that is, $[...[[N(J_1)N(J_2)]N(J_3)]...]N(J_n)$ is also a Neutrosophic ideal.

The set $N(K)$ is called a locally associative Neutrosophic AG-groupoid if $(JJ)J = J(JJ)$ for each ideal J. We shall denote this locally associative Neutrosophic AG-groupoid by $N(Y)$.

Lemma 160 *If $N(S)$ is Neutrosophic fully idempotent, then $N(K)$ is a locally associative Neutrosophic AG-groupoid.*

Proof. Let $N(A), N(B) \in N(K)$. Then using identity (2) and the definition of ideals we get

$$\begin{aligned}(AB)S &= (AB)(SS) = (AS)(BS) \subseteq AB, \\ S(AB) &= (SS)(AB) = (SA)(SB) \subseteq AB.\end{aligned}$$

Hence $N(A)N(B) \in N(K)$. Also $[N(A)N(B)]N(C) = [N(C)N(B)]N(A)$ holds for all ideals $N(A), N(B)$ and $N(C)$ of $N(S)$. Since $N(S)$ is fully idempotent, $[N(A)N(A)]N(A) = N(A)[N(A)N(A)] = N(A)$, for each ideal $N(A)$ of $N(S)$. Hence $N(K)$ is a locally associative Neutrosophic AG-groupoid. ■

Lemma 161 *If a fully idempotent Neutrosophic AG-groupoid $N(S)$ contains the left identity $e + eI$ and $e + eI \in N(Y)$, then $N(Y)$ becomes a Neutrosophic semilattice structure.*

Proof. It is straight forward. ■

The set of all neutrosophic ideals of a Neutrosophic AG-groupoid $N(S)$ is called a strong locally associative Neutrosophic AG-groupoid if $[(N(A)N(B)]N(A) = N(A)[N(B)N(A)]$ for all ideals $N(A), N(B)$.

Note that if $N(S)$ is fully idempotent Neutrosophic AG-groupoid then the set of Neutrosophic ideals of $N(S)$ form a strong locally associative Neutrosophic AG-groupoid.

Proposition 162 *For $N(Y)$ in a Neutrosophic AG-groupoid $N(S)$ with left identity $e + eI$, the following assertions are true:*
(i) $N(Y)$ has associative powers,
(ii) $[N(J)]^m[N(J)]^n = [N(J)]^{m+n}$, $\forall\ N(J) \in N(Y)$,
(iii) $[N(J)^m]^n = J^{mn}$, $\forall\ N(J) \in N(Y)$ and positive integers m, n, and
(iv) $[N(A)N(B)]^n = [N(A)]^n[N(B)]^n$ for $n \geq 1$, $\forall\ N(A), N(B) \in N(Y)$.

Proof. It is same as in [5]. ■

Lemma 163 *If $N(I) \in N(Y)$ in an Neutrosophic AG-groupoid $N(S)$ with left identity $e + eI$, then $[N(I)]^n$ for $n \geq 2$ is a Neutrosophic ideal.*

Proof. Clearly $[N(I)]^2$ is a Neutrosophic ideal. Now suppose the result is true for $n = k - 1$. Then using identity (1), we get

$$\begin{aligned}[N(J)]^n N(S) &= [N(J)^{n-1}N(J)]N(S) = [N(S)N(J)][N(J)]^{n-1} \\ &\subseteq N(J)[N(J)]^{n-1} = [N(J)]^n.\end{aligned}$$

This implies that $[N(I)]^n$ is a Neutrosophic right ideal of $N(S)$ and by Proposition 2, $[N(J)]^n$ becomes an ideal. ■

Lemma 164 *If $N(S)$ is an Neutrosophic AG-groupoid with left identity $e+eI$ and $N(A), N(B) \in N(Y)$, then $[N(A)N(B)]^n \in N(Y)$.*

Proof. By Proposition 3 (iv), $[N(A)N(B)]^n = [N(A)]^n[N(B)]^n$. Also by Lemma 4, $[N(A)]^n$ and $[N(B)]^n$ are Neutrosophic ideals of $N(S)$ and the product of two Neutrosophic ideals is a neutrosophic ideal, so $[(N(A)N(B)]^n \in N(Y)$. ■

If $N(S)$ is a Neutrosophic AG-groupoid with left identity $e+eI$, then $N(S)[(N(S)(a+bI)] \subseteq N(S)(a+bI)$ in [7].

Also $[(a+bI)N(S)]N(S) \subseteq (a+bI)N(S)$, if $[(x+yI)(e+eI)]N(S) = (x+yI)N(S)$, for all $(x+yI)$ in $N(S)$.

An Neutrosophic AG-groupoid $N(S)$ is called Neutrosophic right (left) normal if $a+bI \in (a+bI)N(S)$ $(N(S)(a+bI))$ for each $a+bI \in N(S)$.

It is an easy fact that every Neutrosophic AG-groupoid $N(S)$ with left identity $e+eI$ is Neutrosophic left normal.

Example 165 *Let $N(S) = \{1+1I, 1+2I, 1+3I\ , 2+1I, 2+2I, 2+3I, 3+1I, 3+2I, 3+3I\}$ and a binary operation $(\cdot)$ be defined in $N(S)$ as follows.*

$\circ$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$1+3I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$2+1I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$
$2+3I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$3+1I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$3+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$3+3I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$

Then $(N(S), .)$ is a Neutrosophic AG-groupoid with left identity a. Clearly $N(S)$ is Neutrosophic left normal. Also $N(S)$ is neutrosophic right normal.

A subset $N(M)$ of a Neutrosophic AG-groupoid $N(S)$ with left identity $e+eI$ is called a Neutrosophic M-system if for $a+bI, c+dI \in N(M)$ there exists $x+yI$ in $N(S)$ such that $(a+bI)[(x+yI)(c+dI)] \in N(M)$.

$N(A) \in N(K)$ is called a left zero if $N(A)N(B) = N(A)$ for every $N(B)$ in $N(K)$ and is called a right zero if $N(B)N(A) = N(A)$ for every $N(B)$ in $N(K)$.

If every $N(A) \in N(K)$ is a left zero or a right zero then every non-empty subset of $N(K)$ is a Neutrosophic M-system. Let $N(P)$ be a Neutrosophic left ideal of a Neutrosophic AG-groupoid $N(S)$, $N(P)$ is called Neutrosophic quasi-prime if for left ideals $N(A)$,$N(B)$ of $N(S)$ such that $N(A)N(B) \subseteq N(P)$, we have $N(A) \subseteq N(P)$ or $N(B) \subseteq N(P)$. $N(P)$ is called Neutrosophic quasi-semiprime if for any left ideal $N(J)$ of $N(S)$ such that $[N(J)]^2 \subseteq N(P)$, we have $N(J) \subseteq N(P)$.

4.2 Relations between Ideals and M-systems

Proposition 166 *Let $N(P)$ be a Neutrosophic left ideal of a Neutrosophic AG-groupoid $N(S)$ with left identity $e + eI$, then the following are equivalent,*

(i) $N(P)$ is a Neutrosophic quasi-prime ideal.

(ii) For all Neutrosophic left ideals $N(A)$ and $N(B)$ of $N(S)$: $N(A)N(B) = \langle N(A)N(B)\rangle \subseteq N(P) \Rightarrow N(A) \subseteq N(P)$ or $N(B) \subseteq N(P)$.

(iii) For all left ideals $N(A)$ and $N(B)$ of $N(S)$: $N(A) \nsubseteq N(P)$ and $N(B) \nsubseteq N(P) \Rightarrow N(A)N(B) \nsubseteq N(P)$.

(iv) For all $a + bI, c + dI \in N(S)$: $a + bI \notin N(P)$ and $c + dI \notin P \Rightarrow \langle a + bI\rangle\langle c + dI\rangle \nsubseteq P$.

(v) For all $a+bI, c+dI \in N(S)$: $(a+bI)[N(S)(c+dI)] \subseteq N(P)$ implies either $a + bI \in N(P)$ or $c + dI \in N(P)$.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ is trivial.

$(i) \Rightarrow (iv)$, Let $\langle a + bI\rangle\langle c + dI\rangle \subseteq N(P)$, then by (i) either $\langle a + bI\rangle \subseteq N(P)$ or $\langle c + dI\rangle \subseteq N(P)$, which implies that either $a + bI \in N(P)$ or $c + dI \in N(P)$.

$(iv) \Rightarrow (ii)$, Let $N(A)N(B) \subseteq N(P)$, if $a+bI \in N(A)$ and $c+dI \in N(B)$, then $\langle a + bI\rangle\langle c + dI\rangle \subseteq N(P)$, now by (iv) either $a + bI \in N(P)$ or $c + dI \in N(P)$, which implies that either $N(A) \subseteq N(P)$ or $N(B) \subseteq N(P)$.

$(i) \Leftrightarrow (v)$, Let $N(P)$ be a left ideal of a Neutrosophic LA semigroup $N(S)$ with identity $e+eI$. Now suppose that $(a+bI)[N(S)(c+dI) \subseteq N(P)$. Then by (2), (3) and (1), we get

$$\begin{aligned}
N(S)[(a+bI)[N(S)(c+dI)]] &\subseteq N(S)N(P) \subseteq N(P), \text{ that is} \\
& \quad N(S)[(a+bI)[N(S)(c+dI)]] \\
&= [N(S)N(S)[(a+bI)[N(S)(c+dI)]]] \\
&= [N(S)(a+bI)[N(S)[N(S)(c+dI)]]] \\
&= [N(S)(a+bI)][[N(S)N(S)][N(S)(c+dI)]] \\
&= [N(S)(a+bI)][(c+dI)N(S)][N(S)N(S)] \\
&= [N(S)(a+bI)][[N(S)N(S)](c+dI)] \\
&= [N(S)(a+bI)][N(S)(c+dI)].
\end{aligned}$$

Since $N(S)(a+bI)$ is a left ideal for all $(a+bI)$ in $N(S)$ [7]. Hence, either $(a + bI) \in N(P)$ or $(c + dI) \in N(P)$.

Conversely, assume that $N(A)N(B) \subseteq N(P)$ where $N(A)$ and $N(B)$ are left ideals of $N(S)$ such that $N(A) \nsubseteq N(P)$. Then there exists $x_1 + y_1I \in N(A)$ such that $x_2 + y_2I \notin N(P)$. Now $(x_1 + y_1I)[N(S)(x_2 + y_2I)] \subseteq$

$N(A)[N(S)N(B)] \subseteq N(A)N(B) \subseteq N(P)$, for all $x_2 + y_2 I \in N(B)$. Hence by hypothesis, $x_2 + y_2 I \in N(P)$ for all $x_2 + y_2 I \in N(B)$. This shows that $N(P)$ is Neutrosophic quasi-prime. ■

Proposition 167 *A left ideal $N(P)$ of a Neutrosophic AG-groupoid $N(S)$ with left identity e+eI is Neutrosophic quasi-prime if and only if $N(S) \backslash N(P)$ is a Neutrosophic M-system.*

Proof. Let $N(S) \backslash N(P)$ be a Neutrosophic M-system and $(a+bI)[N(S)(c+dI)] \subseteq N(P)$ with $(a+bI) \notin N(P)$ and $c+dI \notin N(P)$. Then $a+bI, c+dI \in N(S) \backslash N(P)$ and there exists $x+yI$ in $N(S)$ such that $(a+bI)[(x+yI)(c+dI)] \in N(S) \backslash N(P)$. This implies that $(a+bI)[(x+yI)(c+dI)] \notin N(P)$, which is a contradiction. Hence either $a+bI \in N(P)$ or $c+dI \in N(P)$.

Conversely, assume that $N(P)$ is Neutrosophic quasi-prime. Let $a+bI, c+dI \in N(S) \backslash N(P)$. We show that there exists $x+yI$ in $N(S)$ such that $(a+bI)[(x+yI)(c+dI)] \in N(S) \backslash N(P)$. Suppose there does not exist $x+yI$ such that $(a+bI)[(x+yI)(c+dI)] \in N(S) \backslash N(P)$. This implies that $(a+bI)[(x+yI)(c+dI)] \in N(P)$, which further implies that either $a+bI \in N(P)$ or $c+dI \in N(P)$. But this is a contradiction. Hence $(a+bI)[(x+yI)(c+dI)] \in N(S) \backslash N(P)$. ■

A subset $N(B)$ of a Neutrosophic AG-groupoid $N(S)$ is called a Neutrosophic p-system if for every $b_1 + b_2 I$ in $N(B)$ there exists $x+yI$ in $N(S)$ such that $(b_1 + b_2 I)[(x+yI)(b_1 + b_2 I)] \in N(B)$.

Proposition 168 *Let $N(A)$ be a Neutrosophic left ideal of a Neutrosophic AG-groupoid $N(S)$ with left identity $e+eI$, then the following are equivalent,*

(i) $N(A)$ is Neutrosophic quasi-semiprime.

(ii) For any Neutrosophic left ideals $N(J)$ of $N(S)$: $[N(J)]^2 = \langle [N(J)]^2 \rangle \subseteq N(A) \Rightarrow N(J) \subseteq N(A)$.

(iii) For any Neutrosophic left ideals $N(J)$ of $N(S)$: $N(J) \nsubseteq N(A) \Rightarrow [N(J)]^2 \nsubseteq N(A)$.

(iv) For all $a+bI \in N(S)$: $\langle a+bI \rangle^2 \subseteq N(A) \Rightarrow a+bI \in N(A)$.

(v) For all $a+bI \in N(S)$: $(a+bI)[N(S)(a+bI)] \subseteq N(A) \Rightarrow a+bI \in N(A)$.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ is trivial.

$(i) \Rightarrow (iv)$, Let $\langle a+bI \rangle^2 \subseteq N(A)$, then by (i) either $\langle a+bI \rangle \subseteq N(A)$, which implies that $a+bI \in N(A)$.

$(iv) \Rightarrow (ii)$, Let $[N(J)]^2 \subseteq N(A)$, if $a+bI \in N(J)$, then $\langle a+bI \rangle^2 \subseteq N(P)$, now by (iv) $a+bI \in N(A)$, which implies that $N(A) \subseteq N(P)$.

$(i) \Leftrightarrow (v)$, is easy. ■

Proposition 169 (a) *Each Neutrosophic M-system is a Neutrosophic p-system.*

(b) *A Neutrosophic left ideal $N(J)$ of a Neutrosophic AG-groupoid $N(S)$ with left identity is Neutrosophic quasi-semiprime if and only if $N(S) \backslash N(J)$*

is Neutrosophic a p-system.

Proof. (a) Let $a+bI \in N(M)$, then there exists $x+yI$ in $N(S)$ such that $(a+bI)[(x+yI)(a+bI)] \in N(M)$ implying that $N(M)$ is a neutrosophic p-system.

(b) Let $N(J)$ be a Neutrosophic quasi-semiprime ideal and $a+bI \in N(S)\backslash N(I)$. Then $a+bI \notin N(J)$ implies that $(a+bI)[(x+yI)(a+bI)] \notin N(J)$. Thus $(a+bI)[(x+yI)(a+bI)] \in N(S)\backslash N(J)$ and hence $N(S)\backslash N(J)$ is a Neutrosophic p-system.

Conversely, suppose that $N(S)\backslash N(J)$ is a Neutrosophic p-system and $(a+bI)[N(S)(a+BI)] \subseteq N(J)$. We need to show that $a+bI \in N(J)$. Assume that $a+bI \notin N(J)$. Then $a+bI \in N(S)\backslash N(J)$ implies that $(a+bI)[(x+yI)(a+bI)] \in N(S)\backslash N(J)$. Thus $(a+bI)[N(S)(a+bI)] \nsubseteq N(J)$. But this is a contradiction. Hence $a+bI \in N(J)$ and so $N(J)$ is Neutrosophic quasi-semiprime. ■

Lemma 170 *Every Neutrosophic right ideal of a Neutrosophic AG-groupoid $N(S)$ with left identity $e+eI$ is a Neutrosophic p-system.*

Proof. Let $N(J)$ be a Neutrosophic right ideal of $N(S)$. Then by Proposition 2, I becomes a Neutrosophic ideal of $N(S)$. If $i+jI \in N(J)$ then $(i+jI)[(x+yI)(i+jI)] \in N(J)$ for all $x+yI \in N(S)$. Hence $N(J)$ is a Neutrosophic p-system. ■

A Neutrosophic ideal $N(P)$ of a Neutrosophic AG-groupoid $N(S)$ is called Neutrosophic prime if $N(A)N(B) \subseteq N(S)$.implies that either $N(A) \subseteq N(S)$ or $N(B) \subseteq N(S)$ A Neutrosophic ideal $N(P)$ of $N(S)$ is called Neutrosophic semiprime if $[N(J)]^2 \subseteq N(P)$ implies that $N(J) \subseteq N(P)$, for every Neutrosophic ideal $N(J)$ of $N(S)$.

If $N(J)$ is a Neutrosophic ideal of a Neutrosophic AG-groupoid $N(S)$, we call $\Gamma(N(J)) = \{\bigcap_{N(P)\supseteq N(J)} N(P) : N(P)$ is a Neutrosophic prime ideal$\}$ the Neutrosophic prime radical of $N(J)$. Of course, $\Gamma(N(J)$ is a Neutrosophic semiprime ideal of $N(S)$.

Theorem 171 *Let $N(J)$ be a Neutrosophic right ideal of a Neutrosophic AG-groupoid $N(S)$ with left identity $e+eI$. Then the following are equivalent.*

(i) $N(J)$ is Neutrosophic semiprime if and only if $\Gamma(nN(J)) = N(J)$,

(ii) If $N(J)$ is Neutrosophic semiprime then $N(J)$ is the intersection of all Neutrosophic prime ideals containing $N(J)$, and

(iii) $\Gamma(N(J))$ is the intersection of all Neutrosophic semiprime ideals containing $N(J)$.

Proof. The theorem can be easily proved. ■

5

Neutrosophic Strongly Regular AG-groupoids

Present-day all the fields of science and technology are facing complex processes and phenomenons for which we are not always provided entire information to properly solve these phenomenons. For these cases and situations, some useful mathematical models are developed to hold various types of systems containing uncertainty elements. A basic part of these models is basically an extension of the recently proposed ordinary set theory, namely, the so-called fuzzy sets. Now giving overview about fuzzy sets I firstly discuss the role of fuzzy set in fields of uncertainty. Fuzzy sets (FSs, for short) were firstly introduced by L.A. Zadeh in 1965. The main interest in this theory is to address with uncertainty. The use of fuzzy set theory in various fields of modern society is at a wide range. Fuzzy sets are a useful tools for the operation research analyst facing uncertainty and subjectivity. Fundamental to the fuzzy set is the extension of the characteristic function taking the value of 0 or 1 to the membership function which can take any value from the closed interval [0,1]. Zadeh attained this goal by replacing the conventional characteristic function of the classical "crisp" set, which takes on its values in {0,1} by the so-called membership function, which takes on the values in the interval [0,1], allowing for the representation of membership to a degree However, the membership function is basically only a single-valued function, which cannot be used to express the evidences of support and objection simultaneously in many practical situations. For example, in a voting event, sometimes non participation occurred in accumulation to support and objection which shows hesitation and indeterminacy of the voter to the object. As in such a problem the fuzzy set cannot be used to completely express all the information, it faces a lot of limits in actual applications. Atanassov (1983) extends the fuzzy set characterized by a membership function to

the intuitionistic fuzzy set (IFS), which is characterized by a membership function, a non-membership function and a hesitancy function. As a result, the IFS can describe the fuzzy characters of things more thoroughly and comprehensively, which is proved to be more effectual to deal with vagueness and uncertainty.

Although FS and IFS are very important tool to address uncertainty problems but still there is lack of perfection solution i.e researchers were trying to develop new affective theory to process complex problems therefore for this purpose an attempt was made by Samarandache Florentin to in form of neutrosophic theory.. Therefore, neutrosophic set theory was introduced by Samarandache[] as a generalization of fuzzy sets and intutionistic fuzzy sets. Neutrosophic theory based on neutrosophy, which is a branch of philosophy, In 2005,Smarandache showed that neutrosophic set is generalization of paraconsistent set and intutionistic fuzzy set. Neutrosophy appears as the incidence as the application of a law, of an axiom, of an idea, of a conceptual accredited construction on an unclear, indeterminate phenomenon, contradictory to the purpose of making it intelligible.. The concept of neutrosophic set and logic came into being due to neutrosophy, where each proposition is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. This mathematical tool is used to handle problems with imprecise, indeterminate, incomplete and inconsistent etc. Kandasamy and Smarandache apply this concept in algebraic structures in a slight different manner by using the indeterminate/unknown element I, which they call neutrosophic element. The neutrosophic element I is then combine to the elements of the algebraic structure by taking union and link with respect to the binary operation * of the algebraic structure. Therefore, a neutrosophic algebraic structure is generated in this way. In fields of uncertainty and indeterminancy Neutrosophy was proved to be significantly useful to address mathematical models.

Note that a commutative Neutrosophic AG-groupoid $N(S)$ with left identity becomes a commutative semigroup because if $(a_1+a_2I),(b_1+b_2I)$ and $(c_1+c_2I)\in N(S)$. Then using left invertive law and commutative law, we get

$$\begin{aligned} & ((a_1+a_2I)(b_1+b_2I))(c_1+c_2I) \\ = & ((c_1+c_2I)(b_1+b_2I))(a_1+a_2I) \\ = & (a_1+a_2I)((c_1+c_2I)(b_1+b_2I)) \\ = & (a_1+a_2I)((b_1+b_2I)(c_1+c_2I)). \end{aligned}$$

An Neutrosophic AG-groupoid is a non-associative algebraic structure lies in between a groupoid and a commutative semigroup. Although it is non-associative but mostly it works like a commutative semigroup. In a commutative semigroup $N(S)$, $(a_1+a_2I)^2(b_1+b_2I)^2=(b_1+b_2I)^2(a_1+a_2I)^2$ holds for all $(a_1+a_2I),(b_1+b_2I)\in N(S)$. Also, if $N(S)$ is an Neutrosophic AG-groupoid with left identity $(e+eI)$, then the equation $(a_1+a_2I)^2(b_1+b_2I)^2=(b_1+b_2I)^2(a_1+a_2I)^2$ also holds for all $(a_1+a_2I),(b_1+b_2I)\in N(S)$. If $\{(a_1+a_2I),(b_1+b_2I)\}$ is any subset of an Neutrosophic AG-groupoid $N(S)$, with left identity $(e+eI)$, then $(a_1+a_2I)(b_1+b_2I)=((b_1+b_2I)(a_1+a_2I))(e+eI)$ holds for all $(a_1+a_2I),(b_1+b_2I)\in N(S)$. It is most interesting to see the applications of this non-associative structure in different fields as compare to a commutative semigroup and this motivate us to study an Neutrosophic AG-groupoid.

Our first aim in this paper is to explore the newly introduced class namely strongly regular Neutrosophic AG-groupoid. First we give the relation of strongly regular Neutrosophic AG-groupoids with regular and intra-regular Neutrosophic AG-groupoids and then we further explore this new non associative class.

5.1 Classes of Neutrosophic AG-groupoids

Let $N(S)$ be an Neutrosophic AG-groupoid. By **Neutrosophic AG-subgroupoid** of $N(S)$, we means a non-empty subset $(a_1 + a_2I)$ of $N(S)$ such that $(a_1 + a_2I)^2 \subseteq (a_1 + a_2I)$.

A non-empty subset (a_1+a_2I) of an Neutrosophic AG-groupoid $N(S)$ is called a neutrosophic **left (right) ideal** of $N(S)$ if $N(S)(a_1+a_2I)N(S) \subseteq (a_1 + a_2I)N(S)$ $((a_1 + a_2I)(S)N(S) \subseteq (a_1 + a_2I)(S))$ and it is called a neutrosophic **two-sided ideal** if it is both left and a right ideal of $N(S)$.

A non-empty subset $(a_1 + a_2I)(N(S))$ of an Neutrosophic AG-groupoid $N(S)$ is called a neutrosophic **generalized bi-ideal** of $N(S)$ if $((a_1 + a_2I)N(S)N(S))(a_1+a_2I)N(S) \subseteq (a_1+a_2I)N(S)$ and an AG-subgroupoid $(a_1+a_2I)N(S)$ of $N(S)$ is called a neutrosophic bi-ideal of $N(S)$ if $((a_1 + a_2I)N(S))N(S))(a_1 + a_2I)N(S) \subseteq (a_1 + a_2I)N(S)$.

A non-empty subset $(a_1 + a_2I)N(S)$ of a Neutrosophic AG-groupoid $N(S)$ is called a neutrosophic **quasi-ideal** of $N(S)$ if $N(S)(a_1+a_2I)N(S))\cap ((a_1 + a_2I)N(S))N(S) \subseteq (a_1 + a_2I)N(S)$.

Note that every one sided ideal of an Neutrosophic AG-groupoid $N(S)$ is a neutrosophic quasi-ideal and neutrosophic right ideal of $N(S)$ is neutrosophic bi-ideal of $N(S)$.

If $N(S)$ is an Neutrosophic AG-groupoid with left identity $(e + eI)$ then neutrosophic **principal left ideal** generated by a fixed element "(a_1+a_2I)" is defined as $\langle(a_1 + a_2I)\rangle = N(S)(a_1 + a_2I) = \{N(S)(a_1 + a_2I) : N(S) \in N(S)\}$. Clearly, $\langle(a_1+a_2I)\rangle$ is a left ideal of $N(S)$ contains (a_1+a_2I). Note that if $(a_1 + a_2I)$ is an ideal of $N(S)$, then $((a_1 + a_2I)N(S))^2$ is an ideal of $N(S)$. Also it is easy to verify that $(a_1 + a_2I)N(S) = \langle(a_1 + a_2I)N(S)\rangle$ and $((a_1 + a_2I)N(S))^2 = \langle((a_1 + a_2I)N(S)^2)\rangle$.

If an Neutrosophic AG-groupoid $N(S)$ contains left identity $(e+eI)$ then $N(S) = (e + eI)N(S) \subseteq N(S)^2$.Therefore $N(S) = N(S)^2$. Also $N(S)(a_1 + a_2I)$ becomes neutrosophic bi-ideal and neutrosophic quasi-ideal of $N(S)$. Using paramedial, medial and left invertive law we get

$$\begin{aligned}
& ((N(S)(a_1 + a_2I))N(S))N(S)(a_1 + a_2I) \\
\subseteq \;& (N(S)N(S))(N(S)(a_1 + a_2I)) \\
= \;& ((a_1 + a_2I)N(S))(N(S)N(S)) \\
= \;& ((a_1 + a_2I)N(S))N(S) \\
= \;& (N(S)N(S))(a_1 + a_2I) \\
= \;& N(S)(a_1 + a_2I)
\end{aligned}$$

It is easy to show that $(N(S)(a_1+a_2I))(N(S)(a_1+a_2I)) \subseteq N(S)(N(S)(a_1+a_2I))$. Hence $N(S)(a_1+a_2I)$ is a bi-ideal of $N(S)$. Also

$$\begin{aligned} & N(S)(N(S)(a_1+a_2I)) \cap (N(S)(a_1+a_2I))N(S) \\ \subseteq\ & N(S)(N(S)(a_1+a_2I)) \\ \subseteq\ & N(S)(a_1+a_2I). \end{aligned}$$

Therefore $N(S)(a_1+a_2I)$ is a quasi-ideal of $N(S)$. Also using medial and paramedial laws and (1), we get

$$\begin{aligned} & (N(S)(a_1+a_2I))^2 \\ =\ & (N(S)(a_1+a_2I))(N(S)(a_1+a_2I)) \\ =\ & (N(S)N(S))(a_1+a_2I)^2 \\ =\ & ((a_1+a_2I)(a_1+a_2I))(N(S)N(S)) \\ =\ & N(S)(((a_1+a_2I)(a_1+a_2I))N(S)) \\ =\ & (N(S)N(S))(((a_1+a_2I)(a_1+a_2I))N(S)) \\ =\ & (N(S)(a_1+a_2I)^2)N(S)N(S) \\ =\ & (N(S)(a_1+a_2I)^2)N(S). \end{aligned}$$

Therefore $N(S)(a_1+a_2I)^2 = (a_1+a_2I)^2N(S) = (N(S)(a_1+a_2I)^2)N(S)$.

An Neutrosophic AG-groupoid $N(S)$ is said to be neutrosophic regular if for every (a_1+a_2I) in $N(S)$ there exists some (x_1+x_2I) in $N(S)$ such that $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$.

An Neutrosophic AG-groupoid $N(S)$ is said to be neutrosophic intra-regular if for every (a_1+a_2I) in $N(S)$ there exists some $(x_1+x_2I), (y_1+y_2I)$ in $N(S)$ such that $(a_1+a_2I) = ((x_1+x_2I)(a_1+a_2I)^2)(y_1+y_2I)$.

An Neutrosophic AG-groupoid $N(S)$ is said to be neutrosophic strongly regular if for every (a_1+a_2I) in $N(S)$ there exists some (x_1+x_2I) in $N(S)$ such that $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$ and $(a_1+a_2I)(x_1+x_2I) = (x_1+x_2I)(a_1+a_2I)$.

Here we begin with examples of Neutrosophic AG-groupoids.

Example 172 *Let $S = \{1,2,3\}$, the binary operation "$\cdot$" be defined on $N(S)$ as follows:*

$\cdot$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	1

Clearly $(S,\cdot)$ is an AG-groupoid without left identity.

then $N(S) = \{1+1I, 1+2I, 1+3I, 2+1I, 2+2I, 2+3I, 3+1I, 3+2I, 3+3I\}$ is an example of neutrosophic LA-semigroup without left identity under the operation "$*$" and has the following Cayley's table:

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$1+3I$	$2+1I$	$2+2I$	$2+1I$	$2+1I$	$2+2I$	$2+1I$	$2+1I$	$2+2I$	$2+1I$
$2+1I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$2+3I$	$2+1I$	$2+2I$	$2+1I$	$2+1I$	$2+2I$	$2+1I$	$2+1I$	$2+2I$	$2+1I$
$3+1I$	$1+2I$	$1+2I$	$1+2I$	$2+2I$	$2+2I$	$2+2I$	$1+2I$	$1+2I$	$1+2I$
$3+2I$	$1+2I$	$1+2I$	$1+2I$	$2+2I$	$2+2I$	$2+2I$	$1+2I$	$1+2I$	$1+2I$
$3+3I$	$1+1I$	$1+2I$	$1+1I$	$2+1I$	$2+2I$	$2+1I$	$1+1I$	$1+2I$	$1+1I$

Example 173 *Let $N(S) = \{1, 2, 3, 4\}$, the binary operation "$\cdot$" be defined on $N(S)$ as follows:*

$\cdot$	1	2	3	4
1	1	2	3	4
2	4	3	3	3
3	3	3	3	3
4	2	3	3	3

Clearly $(N(S), \cdot)$ is an Neutrosophic AG-groupoid with left identity 1.

Example 174 *1. Let $N(S) = \{1, 2, 3\}$, the binary operation "$\cdot$" be defined on $N(S)$ as follows:*

$\cdot$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Clearly $(S, \cdot)$ is a Neutrosophic strongly regular AG-groupoid with left identity 1.

then $N(S) = \{1+1I, 1+2I, 1+3I, 2+1I, 2+2I, 2+3I, 3+1I, 3+2I, 3+3I\}$ is an example of Neutrosophic Strongly regular AG-groupoid with left identity 1 *under the operation "$*$" and has the following Cayley's table:*

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$1+3I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$2+1I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$
$2+3I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$3+1I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$3+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$3+3I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$

Note that every strongly regular Neutrosophic AG-groupoid is regular, but converse is not true, for converse consider the following example.

Example 175 *Let $N(S) = \{1, 2, 3\}$, the binary operation "$\cdot$" be defined on $N(S)$ as follows:*

$\cdot$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Clearly $(S, \cdot)$ is regular Neutrosophic AG-groupoid, but not strongly regular.
then $N(S) = \{1+1I, 1+2I, 1+3I, 2+1I, 2+2I, 2+3I, 3+1I, 3+2I, 3+3I\}$ is an example of neutrosophic LA-semigroup under the operation "$*$" and has the following Cayley's table:

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$1+1I$	$1+1I$	$1+1I$	$1+1I$	$1+1I$	$1+1I$	$1+1I$	$1+1I$	$1+1I$
$1+2I$	$1+1I$	$1+1I$	$1+3I$	$1+1I$	$1+1I$	$1+3I$	$1+1I$	$1+1I$	$1+3I$
$1+3I$	$1+1I$	$1+2I$	$1+1I$	$1+1I$	$1+2I$	$1+1I$	$1+1I$	$1+2I$	$1+1I$
$2+1I$	$1+1I$	$1+1I$	$1+1I$	$1+1I$	$1+1I$	$1+1I$	$3+1I$	$3+1I$	$3+1I$
$2+2I$	$1+1I$	$1+1I$	$1+3I$	$1+1I$	$1+1I$	$1+3I$	$3+1I$	$3+1I$	$3+3I$
$2+3I$	$1+1I$	$1+2I$	$1+1I$	$1+1I$	$1+2I$	$1+1I$	$3+1I$	$3+2I$	$3+1I$
$3+1I$	$1+1I$	$1+1I$	$1+1I$	$2+1I$	$2+1I$	$2+1I$	$1+1I$	$1+1I$	$1+1I$
$3+2I$	$1+1I$	$1+1I$	$1+3I$	$2+1I$	$2+1I$	$2+3I$	$1+1I$	$1+1I$	$1+3I$
$3+3I$	$1+1I$	$1+2I$	$1+1I$	$2+1I$	$2+2I$	$2+1I$	$1+1I$	$1+2I$	$1+1I$

Clearly $(N(S), \cdot)$ is regular Neutrosophic AG-groupoid, but not strongly regular.

Theorem 176 *Every strongly regular Neutrosophic AG-groupoid is intra-regular.*
Proof. *Let $N(S)$ be strongly regular Neutrosophic AG-groupoid,then for every $(a_1 + a_2I) \in N(S)$ there exists some $(x_1 + x_2I) \in N(S)$ such that $(a_1 + a_2I) = ((a_1 + a_2I)(x_1 + x_2I))(a_1 + a_2I)$ and $(a_1 + a_2I)(x_1 + x_2I) =$*

$(x_1 + x_2I)(a_1 + a_2I)$, *then using left invertive law we get*

$$\begin{aligned}
& (a_1 + a_2I) \\
= \ & ((a_1 + a_2I)(x_1 + x_2I))(a_1 + a_2I) \\
= \ & ((a_1 + a_2I)(x_1 + x_2I))[((a_1 + a_2I)(x_1 + x_2I))(a_1 + a_2I)] \\
= \ & ((a_1 + a_2I)(x_1 + x_2I))[((x_1 + x_2I)(a_1 + a_2I))(a_1 + a_2I)] \\
= \ & ((a_1 + a_2I)(x_1 + x_2I))((a_1 + a_2I)^2(x_1 + x_2I)) \\
= \ & [((a_1 + a_2I)^2(x_1 + x_2I))(a_1 + a_2I)](x_1 + x_2I) \\
= \ & [((a_1 + a_2I)(x_1 + x_2I))(a_1 + a_2I)^2](x_1 + x_2I) \\
= \ & ((u_1 + u_2I)(a_1 + a_2I)^2)(x_1 + x_2I), \text{ where } (u_1 + u_2I) \\
= \ & (a_1 + a_2I)(x_1 + x_2I)
\end{aligned}$$

Hence $N(S)$ is intra-regular.

Converse of above theorem is not true, for converse consider the following example. ■

Example 177 *Let $N(S) = \{1, 2, 3, 4, 5, 6, 7\}$, the binary operation "$\cdot$" be defined on $N(S)$ as follows:*

$\cdot$	1	2	3	4	5	6	7
1	1	3	5	7	2	4	6
2	4	6	1	3	5	7	2
3	7	2	4	6	1	3	5
4	3	5	7	2	4	6	1
5	6	1	3	5	7	2	4
6	2	4	6	1	3	5	7
7	5	7	2	4	6	1	3

Clearly $(N(S), \cdot)$ is intra-regular Neutrosophic AG-groupoid, but not strongly regular.

5.2 Characterizations of Neutrosophic Strongly Regular AG-groupoids

Theorem 178 *For an Neutrosophic AG-groupoid $N(S)$ with left identity the following are equivalent,*

(i) $N(S)$ is strongly regular,

(ii) $L(S) \cap (a_1 + a_2I)(S) \subseteq L(S)(a_1 + a_2I)(S)$ and $L(S)$ is strongly regular AG-subgroupoid, where $L(S)$ is any left ideal and $(a_1 + a_2I)(S)$ is any subset of $N(S)$.

Proof. $(i) \Longrightarrow (ii)$

Let $N(S)$ be a strongly regular Neutrosophic AG-groupoid with left identity. Let $(a_1+a_2I) \in L(S) \cap (a_1+a_2I)(S)$, now since $N(S)$ is strongly regular so there exists some $(x_1+x_2I) \in N(S)$ such that $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$ and $(a_1+a_2I)(x_1+x_2I) = (x_1+x_2I)(a_1+a_2I)$. Then

$$\begin{aligned}
& (a_1+a_2I) \\
= \; & ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I) \\
= \; & ((x_1+x_2I)(a_1+a_2I))(a_1+a_2I) \\
\in \; & (N(S)L(S))(a_1+a_2I) \\
\subseteq \; & L(S)((a_1+a_2I)N(S)).
\end{aligned}$$

Thus $L(S) \cap (a_1+a_2I)(S) \subseteq L(S)(a_1+a_2I)(S)$. Let $(a_1+a_2I) \in L(S)$, thus $(a_1+a_2I) \in N(S)$ and since $N(S)$ is strongly regular so there exists an (x_1+x_2I) in $N(S)$ such that $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$ and $(a_1+a_2I)(x_1+x_2I) = (x_1+x_2I)(a_1+a_2I)$. Let $(y_1+y_2I) = ((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)$, then using left invertive law, we get

$$\begin{aligned}
& (y_1+y_2I) \\
= \; & ((x_1+x_2I)(a_1+a_2I))(x_1+x_2I) \\
= \; & ((a_1+a_2I)(x_1+x_2I))(x_1+x_2I) \\
= \; & (x_1+x_2I)^2(a_1+a_2I) \in N(S)L(S) \\
\subseteq \; & L(S).
\end{aligned}$$

Now using left invertive law and (1), we get

$$\begin{aligned}
& (y_1+y_2I)(a_1+a_2I) \\
= \; & [((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)](a_1+a_2I) \\
= \; & ((a_1+a_2I)(x_1+x_2I))((x_1+x_2I)(a_1+a_2I)) \\
= \; & ((x_1+x_2I)(a_1+a_2I))((a_1+a_2I)(x_1+x_2I)) \\
= \; & (a_1+a_2I)[((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)] \\
= \; & (a_1+a_2I)(y_1+y_2I).
\end{aligned}$$

Now using left invertive law and (1), we get

$$\begin{aligned}
& (a_1+a_2I) \\
= \; & ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I) \\
= \; & ((a_1+a_2I)(x_1+x_2I))[((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)] \\
= \; & ((a_1+a_2I)(x_1+x_2I))[((x_1+x_2I)(a_1+a_2I))(a_1+a_2I)] \\
= \; & ((a_1+a_2I)(x_1+x_2I))((a_1+a_2I)^2(x_1+x_2I)) \\
= \; & (a_1+a_2I)^2[((a_1+a_2I)(x_1+x_2I))(x_1+x_2I)] \\
= \; & ((a_1+a_2I)(a_1+a_2I))[((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)] \\
= \; & ((a_1+a_2I)(a_1+a_2I))(y_1+y_2I) \\
= \; & ((y_1+y_2I)(a_1+a_2I))(a_1+a_2I) \\
= \; & ((a_1+a_2I)(y_1+y_2I))(a_1+a_2I).
\end{aligned}$$

Therefore $L(S)$ is strongly regular.
$(ii) \Longrightarrow (i)$
Since $N(S)$ itself is a left ideal, therefore by assumption $N(S)$ is strongly regular. ■

Theorem 179 *For an Neutrosophic AG-groupoid $N(S)$ with left identity the following are equivalent,*
(i) $N(S)$ is strongly regular,
(ii) $(b_1+b_2I)N(S)\cap(a_1+a_2I)N(S) \subseteq (b_1+b_2I)N(S)(a_1+a_2I)N(S)$ and $(b_1+b_2I)N(S)$ is strongly regular AG-subgroupoid, where $(b_1+b_2I)N(S)$ is any bi ideal and $(a_1+a_2I)N(S)$ is any subset of $N(S)$.

Proof. $(i) \Longrightarrow (ii)$
Let $N(S)$ be a strongly regular Neutrosophic AG-groupoid with left identity. Let $(a_1+a_2I) \in (b_1+b_2I)N(S) \cap (a_1+a_2I)N(S)$, now since $N(S)$ is strongly regular so there exists some $(x_1+x_2I) \in N(S)$ such that $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$ and $(a_1+a_2I)(x_1+x_2I) =$

$(x_1+x_2I)(a_1+a_2I)$. Then using left invertive law and (1), we get

$$\begin{aligned}
& (a_1+a_2I) \\
= \; & ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I) \\
= \; & [\{((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)\}(x_1+x_2I)](a_1+a_2I) \\
= \; & [((x_1+x_2I)(a_1+a_2I))((a_1+a_2I)(x_1+x_2I))](a_1+a_2I) \\
= \; & [((a_1+a_2I)(x_1+x_2I))((x_1+x_2I)(a_1+a_2I))](a_1+a_2I) \\
= \; & [\{((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)\}(a_1+a_2I)](a_1+a_2I) \\
= \; & [\{((x_1+x_2I)\{((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)\}) \\
& (x_1+x_2I)\}(a_1+a_2I)](a_1+a_2I) \\
= \; & [\{\{((a_1+a_2I)(x_1+x_2I))((x_1+x_2I)(a_1+a_2I))\} \\
& (x_1+x_2I)\}(a_1+a_2I)](a_1+a_2I) \\
= \; & [\{\{(x_1+x_2I)((x_1+x_2I)(a_1+a_2I))\} \\
& ((a_1+a_2I)(x_1+x_2I))\}(a_1+a_2I)](a_1+a_2I) \\
= \; & [((a_1+a_2I)\{\{(x_1+x_2I)((x_1+x_2I)(a_1+a_2I))\} \\
& (x_1+x_2I)\})(a_1+a_2I)](a_1+a_2I) \\
= \; & [((a_1+a_2I)(t_1+t_2I))(a_1+a_2I)](a_1+a_2I) \\
\in \; & [((b_1+b_2I)(S)N(S))(b_1+b_2I)N(S)](a_1+a_2I)N(S) \\
\subseteq \; & ((b_1+b_2I)N(S))((a_1+a_2I)N(S))
\end{aligned}$$

where $(t_1+t_2I)=\{(x_1+x_2I)((x_1+x_2I)(a_1+a_2I))\}(x_1+x_2I)\}$

Thus $(b_1+b_2I)(S)\cap(a_1+a_2I)(S)\subseteq(b_1+b_2I)(S)(a_1+a_2I)(S)$. Let $(a_1+a_2I)\in(b_1+b_2I)(S)$, thus $(a_1+a_2I)\in N(S)$ and since $N(S)$ is strongly regular so there exists an (x_1+x_2I) in $N(S)$ such that $(a_1+a_2I)=((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$ and $(a_1+a_2I)(x_1+x_2I)=(x_1+x_2I)(a_1+a_2I)$. Let $(y_1+y_2I)=((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)$, then using left invertive law, (1), paramedial and medial law, we get

$$
\begin{aligned}
& (y_1+y_2I) \\
= \ & ((x_1+x_2I)(a_1+a_2I))(x_1+x_2I) \\
= \ & ((a_1+a_2I)(x_1+x_2I))(x_1+x_2I) \\
= \ & (x_1+x_2I)^2(a_1+a_2I) \\
= \ & (x_1+x_2I)^2[((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)] \\
= \ & (x_1+x_2I)^2[((x_1+x_2I)(a_1+a_2I))(a_1+a_2I)] \\
= \ & (x_1+x_2I)^2((a_1+a_2I)^2(x_1+x_2I)) \\
= \ & (a_1+a_2I)^2((x_1+x_2I)^2(x_1+x_2I)) \\
= \ & (a_1+a_2I)^2(t_1+t_2I) \\
= \ & ((a_1+a_2I)(a_1+a_2I))(t_1+t_2I) \\
= \ & ((t_1+t_2I)(a_1+a_2I))(a_1+a_2I) \\
= \ & [(t_1+t_2I)\{((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)\}](a_1+a_2I) \\
= \ & [(t_1+t_2I)\{((x_1+x_2I)(a_1+a_2I))(a_1+a_2I)\}](a_1+a_2I) \\
= \ & [(t_1+t_2I)((a_1+a_2I)^2(x_1+x_2I))](a_1+a_2I) \\
= \ & [(a_1+a_2I)^2((t_1+t_2I)(x_1+x_2I))](a_1+a_2I) \\
= \ & [((a_1+a_2I)(a_1+a_2I))((t_1+t_2I)(x_1+x_2I))](a_1+a_2I) \\
= \ & [((x_1+x_2I)(t_1+t_2I))((a_1+a_2I)(a_1+a_2I))](a_1+a_2I) \\
= \ & [(a_1+a_2I)\{((x_1+x_2I)(t_1+t_2I))(a_1+a_2I)\}](a_1+a_2I) \\
= \ & ((a_1+a_2I)(v_1+v_2I))(a_1+a_2I) \\
\in \ & ((b_1+b_2I)(N(S))N(S))(b_1+b_2I)N(S) \\
\subseteq \ & (b_1+b_2I)N(S).
\end{aligned}
$$

where $(t_1+t_2I) = ((x_1+x_2I)^2(x_1+x_2I))$ and $(v_1+v_2I) = ((x_1+x_2I)(t_1+t_2I))(a_1+a_2I)$

Now using left invertive law and (1), we get

$$
\begin{aligned}
& (y_1+y_2I)(a_1+a_2I) \\
= \ & [((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)](a_1+a_2I) \\
= \ & ((a_1+a_2I)(x_1+x_2I))((x_1+x_2I)(a_1+a_2I)) \\
= \ & ((x_1+x_2I)(a_1+a_2I))((a_1+a_2I)(x_1+x_2I)) \\
= \ & (a_1+a_2I)[((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)] \\
= \ & (a_1+a_2I)(y_1+y_2I).
\end{aligned}
$$

Now using left invertive law and (1), we get

$$
\begin{aligned}
& (a_1+a_2I) \\
= \ & ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I) \\
= \ & ((a_1+a_2I)(x_1+x_2I))[((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)] \\
= \ & ((a_1+a_2I)(x_1+x_2I))[((x_1+x_2I)(a_1+a_2I))(a_1+a_2I)] \\
= \ & ((a_1+a_2I)(x_1+x_2I))((a_1+a_2I)^2(x_1+x_2I)) \\
= \ & (a_1+a_2I)^2[((a_1+a_2I)(x_1+x_2I))(x_1+x_2I)] \\
= \ & ((a_1+a_2I)(a_1+a_2I))[((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)] \\
= \ & ((a_1+a_2I)(a_1+a_2I))(y_1+y_2I) \\
= \ & ((y_1+y_2I)(a_1+a_2I))(a_1+a_2I) \\
= \ & ((a_1+a_2I)(y_1+y_2I))(a_1+a_2I).
\end{aligned}
$$

Therefore $(b_1+b_2I)N(S)$ is strongly regular.

$(ii) \Longrightarrow (i)$

Since $N(S)$ itself is a bi ideal, therefore by assumption $N(S)$ is strongly regular. ■

Theorem 180 *For an Neutrosophic AG-groupoid $N(S)$ with left identity the following are equivalent,*

(i) $N(S)$ is strongly regular,

(ii) $Q(S)\cap(a_1+a_2I)N(S) \subseteq Q(S)(a_1+a_2I)N(S)$ and $Q(S)$ is strongly regular AG-subgroupoid, where $Q(S)$ is any Neutrosophic quasi ideal and $(a_1+a_2I)N(S)$ is any subset of $N(S)$.

Proof. $(i) \Longrightarrow (ii)$

Let $N(S)$ be a Neutrosophic strongly regular AG-groupoid with left identity. Let $(a_1+a_2I) \in Q(S)\cap(a_1+a_2I)(S)$, now since $N(S)$ is strongly regular so there exists some $(x_1+x_2I) \in N(S)$ such that $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$ and $(a_1+a_2I)(x_1+x_2I) = (x_1+x_2I)(a_1+a_2I)$.

Now using left invertive law and (1), we get

$$
\begin{aligned}
& (a_1+a_2I)(x_1+x_2I) \\
= \; & [((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)](x_1+x_2I) \\
= \; & [((x_1+x_2I)(a_1+a_2I))(a_1+a_2I)](x_1+x_2I) \\
= \; & ((a_1+a_2I)^2(x_1+x_2I))(x_1+x_2I) \\
= \; & (x_1+x_2I)^2(a_1+a_2I)^2 \\
= \; & (x_1+x_2I)^2((a_1+a_2I)(a_1+a_2I)) \\
= \; & (a_1+a_2I)((x_1+x_2I)^2(a_1+a_2I)) \\
\in \; & Q(S)N(S).(a_1+a_2I)(x_1+x_2I) \\
= \; & [((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)](x_1+x_2I) \\
= \; & ((x_1+x_2I)(a_1+a_2I))((a_1+a_2I)(x_1+x_2I)) \\
= \; & ((a_1+a_2I)(x_1+x_2I))((x_1+x_2I)(a_1+a_2I)) \\
= \; & [((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)](a_1+a_2I) \\
\in \; & N(S)Q(S).
\end{aligned}
$$

Thus $(a_1+a_2I)(x_1+x_2I) \in Q(S)N(S) \cap N(S)Q(S) \subseteq Q(S)$.

Also $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I) \in Q(S)(a_1+a_2I)(S)$. Let $(a_1+a_2I) \in Q(S)$, thus $(a_1+a_2I) \in N(S)$ and since $N(S)$ is strongly regular so there exists an (x_1+x_2I) in $N(S)$ such that $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$ and $(a_1+a_2I)(x_1+x_2I) = (x_1+x_2I)(a_1+a_2I)$. Let $(y_1+y_2I) = ((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)$, then using left invertive law, paramedial, medial law and (1), we get

$$
\begin{aligned}
& (y_1+y_2I) \\
= \; & ((x_1+x_2I)(a_1+a_2I))(x_1+x_2I) \\
= \; & ((a_1+a_2I)(x_1+x_2I))(x_1+x_2I) \\
= \; & (x_1+x_2I)^2(a_1+a_2I) \in N(S)Q(S),
\end{aligned}
$$

and

$$
\begin{aligned}
& (y_1+y_2I) \\
= \; & ((x_1+x_2I)(a_1+a_2I))(x_1+x_2I) \\
= \; & ((x_1+x_2I)(a_1+a_2I))((e+eI)(x_1+x_2I)) \\
= \; & ((x_1+x_2I)(e+eI))((a_1+a_2I)(x_1+x_2I)) \\
= \; & (a_1+a_2I)[((x_1+x_2I)(e+eI))(x_1+x_2I)] \\
\in \; & Q(S)N(S).
\end{aligned}
$$

Thus $(y_1+y_2I) \in Q(S)N(S) \cap N(S)Q(S) \subseteq Q(S)$. Now using left invertive law and (1), we get

$$\begin{aligned}
& (y_1+y_2I)(a_1+a_2I) \\
= & [((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)](a_1+a_2I) \\
= & ((a_1+a_2I)(x_1+x_2I))((x_1+x_2I)(a_1+a_2I)) \\
= & ((x_1+x_2I)(a_1+a_2I))((a_1+a_2I)(x_1+x_2I)) \\
= & (a_1+a_2I)[((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)] \\
= & (a_1+a_2I)(y_1+y_2I).
\end{aligned}$$

Now using left invertive law and (1), we get

$$\begin{aligned}
& (a_1+a_2I) \\
= & ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I) \\
= & ((a_1+a_2I)(x_1+x_2I))[((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)] \\
= & ((a_1+a_2I)(x_1+x_2I))[((x_1+x_2I)(a_1+a_2I))(a_1+a_2I)] \\
= & ((a_1+a_2I)(x_1+x_2I))((a_1+a_2I)^2(x_1+x_2I)) \\
= & (a_1+a_2I)^2[((a_1+a_2I)(x_1+x_2I))(x_1+x_2I)] \\
= & ((a_1+a_2I)(a_1+a_2I))[((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)] \\
= & ((a_1+a_2I)(a_1+a_2I))(y_1+y_2I) \\
= & ((y_1+y_2I)(a_1+a_2I))(a_1+a_2I) \\
= & ((a_1+a_2I)(y_1+y_2I))(a_1+a_2I).
\end{aligned}$$

Therefore $Q(S)$ is strongly regular.

$(ii) \Longrightarrow (i)$

Since $N(S)$ itself is a quasi ideal, therefore by assumption $N(S)$ is strongly regular. ■

Theorem 181 *Let $N(S)$ be a strongly regular Neutrosophic AG-groupoid with left identity. Then, for every $(a_1+a_2I) \in N(S)$, there exists $(y_1+y_2I) \in N(S)$ such that $(a_1+a_2I) = ((a_1+a_2I)(y_1+y_2I))(a_1+a_2I)$, $(y_1+y_2I) = ((y_1+y_2I)(a_1+a_2I))(y_1+y_2I)$ and $(a_1+a_2I)(y_1+y_2I) = (y_1+y_2I)(a_1+a_2I)$.*

Proof. Let $(a_1+a_2I) \in N(S)$, since $N(S)$ is strongly regular, there exists $(x_1+x_2I) \in N(S)$ such that $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$ and $(a_1+a_2I)(x_1+x_2I) = (x_1+x_2I)(a_1+a_2I)$. Now using (1), paramedial law and medial law, we get

$$
\begin{aligned}
& (a_1+a_2I) \\
= \;& ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I) \\
= \;& ((x_1+x_2I)(a_1+a_2I))(a_1+a_2I) \\
= \;& [(x_1+x_2I)\{((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)\}](a_1+a_2I) \\
= \;& [(x_1+x_2I)\{((a_1+a_2I)(x_1+x_2I))((e+eI)(a_1+a_2I))\}](a_1+a_2I) \\
= \;& [(x_1+x_2I)\{((a_1+a_2I)(e+eI))((x_1+x_2I)(a_1+a_2I))\}](a_1+a_2I) \\
= \;& [((a_1+a_2I)(e+eI))\{(x_1+x_2I)((x_1+x_2I)(a_1+a_2I))\}](a_1+a_2I) \\
= \;& [((a_1+a_2I)(e+eI))\{((e+eI)(x_1+x_2I))((a_1+a_2I)(x_1+x_2I))\}] \\
& (a_1+a_2I) \\
= \;& [((a_1+a_2I)(e+eI))\{((x_1+x_2I)(a_1+a_2I))((x_1+x_2I)(e+eI))\}] \\
& (a_1+a_2I) \\
= \;& [((x_1+x_2I)(a_1+a_2I))\{((a_1+a_2I)(e+eI))((x_1+x_2I)(e+eI))\}] \\
& (a_1+a_2I) \\
= \;& [((x_1+x_2I)(a_1+a_2I))\{((e+eI)(x_1+x_2I))((e+eI)(a_1+a_2I))\}] \\
& (a_1+a_2I) \\
= \;& [((x_1+x_2I)(a_1+a_2I))((x_1+x_2I)(a_1+a_2I))](a_1+a_2I) \\
= \;& [((a_1+a_2I)(x_1+x_2I))((a_1+a_2I)(x_1+x_2I))](a_1+a_2I) \\
= \;& [(a_1+a_2I)\{((a_1+a_2I)(x_1+x_2I))(x_1+x_2I)\}](a_1+a_2I) \\
= \;& [(a_1+a_2I)\{((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)\}](a_1+a_2I) \\
= \;& ((a_1+a_2I)(y_1+y_2I))(a_1+a_2I)
\end{aligned}
$$

where $(y_1+y_2I)=((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)$
Now using (1) and left invertive law, we get

$$
\begin{aligned}
& (y_1+y_2I) \\
= \;& ((x_1+x_2I)(a_1+a_2I))(x_1+x_2I) \\
= \;& [(x_1+x_2I)\{((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)\}](x_1+x_2I) \\
= \;& [((a_1+a_2I)(x_1+x_2I))((x_1+x_2I)(a_1+a_2I))](x_1+x_2I) \\
= \;& [\{((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)\}(a_1+a_2I)](x_1+x_2I) \\
= \;& ((y_1+y_2I)(a_1+a_2I))(x_1+x_2I) \\
= \;& [(y_1+y_2I)\{((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)\}](x_1+x_2I) \\
= \;& [(x_1+x_2I)\{((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)\}](y_1+y_2I) \\
= \;& [(x_1+x_2I)\{((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)\}](y_1+y_2I) \\
= \;& [((a_1+a_2I)(x_1+x_2I))((x_1+x_2I)(a_1+a_2I))](y_1+y_2I) \\
= \;& [\{((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)\}(a_1+a_2I)](y_1+y_2I) \\
= \;& ((y_1+y_2I)(a_1+a_2I))(y_1+y_2I).
\end{aligned}
$$

Now using (1) and left invertive law, we get

$$
\begin{aligned}
& (a_1+a_2I)(y_1+y_2I) \\
= \; & (a_1+a_2I)[((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)] \\
= \; & ((x_1+x_2I)(a_1+a_2I))((a_1+a_2I)(x_1+x_2I)) \\
= \; & ((a_1+a_2I)(x_1+x_2I))((x_1+x_2I)(a_1+a_2I)) \\
= \; & [((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)](a_1+a_2I) \\
= \; & (y_1+y_2I)(a_1+a_2I).
\end{aligned}
$$

■

Theorem 182 *For an Neutrosophic AG-groupoid $N(S)$ with left identity the following are equivalent,*

(i) $N(S)$ is strongly regular,

(ii) $N(S)$ is left regular, right regular and $(N(S)(a_1+a_2I))N(S)$ is a strongly regular AG-subgroupoid, of $N(S)$ for every $(a_1+a_2I) \in N(S)$.

(iii) For every $(a_1+a_2I) \in N(S)$, we have $(a_1+a_2I) \in (a_1+a_2I)N(S)$ and $(N(S)(a_1+a_2I))N(S)$ is a strongly regular AG-subgroupoid, of $N(S)$.

Proof. $(i) \Longrightarrow (ii)$

Let $(a_1+a_2I) \in N(S)$, and $N(S)$ is strongly regular so there exists some $(x_1+x_2I) \in N(S)$ such that $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$ and $(a_1+a_2I)(x_1+x_2I) = (x_1+x_2I)(a_1+a_2I)$. Now left invertive law ,we get

$$
\begin{aligned}
& (a_1+a_2I) \\
= \; & ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I) \\
= \; & ((x_1+x_2I)(a_1+a_2I))(a_1+a_2I) \\
= \; & (a_1+a_2I)^2(x_1+x_2I).
\end{aligned}
$$

This implies that $N(S)$ is right regular. Now using medial law, (1) and paramedial law, we get

$$
\begin{aligned}
& (a_1 + a_2I) \\
= \;& ((a_1 + a_2I)(x_1 + x_2I))(a_1 + a_2I) \\
= \;& ((a_1 + a_2I)(x_1 + x_2I))[((a_1 + a_2I)(x_1 + x_2I))(a_1 + a_2I)] \\
= \;& [(a_1 + a_2I)((a_1 + a_2I)(x_1 + x_2I))]((x_1 + x_2I)(a_1 + a_2I)) \\
= \;& [(a_1 + a_2I)((x_1 + x_2I)(a_1 + a_2I))]((x_1 + x_2I)(a_1 + a_2I)) \\
= \;& [(x_1 + x_2I)((a_1 + a_2I)(a_1 + a_2I))]((x_1 + x_2I)(a_1 + a_2I)) \\
= \;& ((x_1 + x_2I)(a_1 + a_2I)^2)((x_1 + x_2I)(a_1 + a_2I)) \\
= \;& [(x_1 + x_2I)((a_1 + a_2I)(a_1 + a_2I))]((x_1 + x_2I)(a_1 + a_2I)) \\
= \;& [((e + eI)(x_1 + x_2I))((a_1 + a_2I)(a_1 + a_2I))]((x_1 + x_2I)(a_1 + a_2I)) \\
= \;& [((a_1 + a_2I)(a_1 + a_2I))((x_1 + x_2I)(e + eI))]((x_1 + x_2I)(a_1 + a_2I)) \\
= \;& [(a_1 + a_2I)^2((x_1 + x_2I)(e + eI))]((x_1 + x_2I)(a_1 + a_2I)) \\
= \;& [((x_1 + x_2I)(a_1 + a_2I))((x_1 + x_2I)(e + eI))](a_1 + a_2I)^2 \\
= \;& (u_1 + u_2I)(a_1 + a_2I)^2
\end{aligned}
$$

where $(u_1 + u_2I) = [((x_1 + x_2I)(a_1 + a_2I))((x_1 + x_2I)(e + eI))]$.

Let $(b_1 + b_2I) \in (N(S)(a_1 + a_2I))N(S) \subseteq N(S)$, thus $(b_1 + b_2I) \in N(S)$, and since $N(S)$ is strongly regular, so there exist $(x_1 + x_2I)_1 \in N(S)$, such that $(b_1 + b_2I) = ((b_1 + b_2I)(x_1 + x_2I)_1)(b_1 + b_2I)$ and $(x_1 + x_2I)_1 = ((x_1 + x_2I)_1(b_1 + b_2I))(x_1 + x_2I)_1$ and $(b_1 + b_2I)(x_1 + x_2I)_1 = (x_1 + x_2I)_1(b_1 + b_2I)$, since $(b_1 + b_2I) \in (N(S)(a_1 + a_2I))N(S) \Rightarrow (b_1 + b_2I) = (z(a_1 + a_2I))(t_1 + t_2I)$, for some $z, (t_1 + t_2I) \in N(S)$. Using paramedial, medial law, left invertive law and (1), we get

$$\begin{aligned}
& (x_1+x_2I)_1 \\
= & ((x_1+x_2I)_1(b_1+b_2I))(x_1+x_2I)_1 \\
= & ((x_1+x_2I)_1(b_1+b_2I))((e+eI)(x_1+x_2I)_1) \\
= & ((x_1+x_2I)_1(e+eI))((b_1+b_2I)(x_1+x_2I)_1) \\
= & (b_1+b_2I)[((x_1+x_2I)_1(e+eI))(x_1+x_2I)_1] \\
= & (b_1+b_2I)(u_1+u_2I) \\
= & [(z(a_1+a_2I))(t_1+t_2I)](u_1+u_2I) \\
= & ((u_1+u_2I)(t_1+t_2I))(z(a_1+a_2I)) \\
= & ((a_1+a_2I)z)((t_1+t_2I)(u_1+u_2I)) \\
= & [((t_1+t_2I)(u_1+u_2I))z](a_1+a_2I) \\
= & (v_1+v_2I)(a_1+a_2I) \\
= & (v_1+v_2I)((a_1+a_2I)^2(x_1+x_2I)) \\
= & (a_1+a_2I)^2((v_1+v_2I)(x_1+x_2I)) \\
= & ((a_1+a_2I)(a_1+a_2I))((v_1+v_2I)(x_1+x_2I)) \\
\in & (N(S)(a_1+a_2I))N(S)
\end{aligned}$$

where $(u_1+u_2I) = ((x_1+x_2I)_1(e+eI))(x_1+x_2I)_1$ and $(v_1+v_2I) = ((t_1+t_2I)(u_1+u_2I))z$.

This shows that $(N(S)(a_1+a_2I))N(S)$ is strongly regular.

$(ii) \Longrightarrow (iii)$

Let $(a_1+a_2I) \in N(S)$, and $N(S)$ is left regular so there exists some $(y_1+y_2I) \in N(S)$ such that $(a_1+a_2I) = (y_1+y_2I)(a_1+a_2I)^2$.

Now using (1), we get

$$\begin{aligned}
& (a_1+a_2I) \\
= & (y_1+y_2I)(a_1+a_2I)^2 \\
= & (y_1+y_2I)((a_1+a_2I)(a_1+a_2I)) \\
= & (a_1+a_2I)((y_1+y_2I)(a_1+a_2I)) \\
\in & (a_1+a_2I)(N(S)N(S)) \\
= & (a_1+a_2I)N(S).
\end{aligned}$$

$(iii) \Longrightarrow (i)$

Let $(a_1+a_2I) \in (a_1+a_2I)N(S)$ so there exists some $(t_1+t_2I) \in N(S)$ such that $(a_1+a_2I) = (a_1+a_2I)(t_1+t_2I)$, also $(a_1+a_2I) \in N(S)(a_1+a_2I)$ so there exists some $z \in N(S)$ such that $(a_1+a_2I) = z(a_1+a_2I)$.

Now

$$\begin{aligned}
& (a_1 + a_2I) \\
= \ & (a_1 + a_2I)(t_1 + t_2I) \\
= \ & (z(a_1 + a_2I))(t_1 + t_2I) \\
\in \ & (N(S)(a_1 + a_2I))N(S),
\end{aligned}$$

and as $(N(S)(a_1 + a_2I))N(S)$ strongly regular so there exists some $(x_1 + x_2I)$ in $N(S)$ such that $(a_1 + a_2I) = ((a_1 + a_2I)(x_1 + x_2I))(a_1 + a_2I)$ and $(a_1 + a_2I)(x_1 + x_2I) = (x_1 + x_2I)(a_1 + a_2I)$. So $N(S)$ is strongly regular. ■

Theorem 183 *For an Neutrosophic AG-groupoid $N(S)$ with left identity the following are equivalent,*
(i) $N(S)$ is strongly regular,
(ii) $(N(S)(a_1 + a_2I))N(S)$ is strongly regular and $N(S)$ is left duo.

Proof. $(i) \Longrightarrow (ii)$
Let $(a_1 + a_2I) \in (N(S)(a_1 + a_2I))N(S)$, so $(a_1 + a_2I) \in N(S)$ and since $N(S)$ is strongly regular so there exists some $(x_1 + x_2I) \in N(S)$ such that $(a_1 + a_2I) = ((a_1 + a_2I)(x_1 + x_2I))(a_1 + a_2I)$ and $(a_1 + a_2I)(x_1 + x_2I) = (x_1 + x_2I)(a_1 + a_2I)$. Let $(y_1 + y_2I) = ((x_1 + x_2I)(a_1 + a_2I))(x_1 + x_2I)$ for any $(y_1 + y_2I) \in N(S)$. Now using (1) and left invertive law ,we get

$$\begin{aligned}
& (y_1 + y_2I) \\
= \ & ((x_1 + x_2I)(a_1 + a_2I))(x_1 + x_2I) \\
= \ & [(x_1 + x_2I)\{((a_1 + a_2I)(x_1 + x_2I))(a_1 + a_2I)\}](x_1 + x_2I) \\
= \ & [((a_1 + a_2I)(x_1 + x_2I))((x_1 + x_2I)(a_1 + a_2I))](x_1 + x_2I) \\
= \ & [\{((x_1 + x_2I)(a_1 + a_2I))(x_1 + x_2I)\}(a_1 + a_2I)](x_1 + x_2I) \\
= \ & ((y_1 + y_2I)(a_1 + a_2I))(x_1 + x_2I) \\
\in \ & (N(S)(a_1 + a_2I))N(S).
\end{aligned}$$

Now using paramedial law,medial law and (1), we get

$$
\begin{aligned}
& (a_1+a_2I) \\
= & ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I) \\
= & ((x_1+x_2I)(a_1+a_2I))(a_1+a_2I) \\
= & [(x_1+x_2I)\{((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)\}](a_1+a_2I) \\
= & [(x_1+x_2I)\{((a_1+a_2I)(x_1+x_2I))((e+eI)(a_1+a_2I))\}](a_1+a_2I) \\
= & [(x_1+x_2I)\{((a_1+a_2I)(e+eI))((x_1+x_2I)(a_1+a_2I))\}](a_1+a_2I) \\
= & [((a_1+a_2I)(e+eI))\{(x_1+x_2I)((x_1+x_2I)(a_1+a_2I))\}](a_1+a_2I) \\
= & [((a_1+a_2I)(e+eI))\{((e+eI)(x_1+x_2I))((a_1+a_2I)(x_1+x_2I))\}](a_1+a_2I) \\
= & [((a_1+a_2I)(e+eI))\{((x_1+x_2I)(a_1+a_2I))((x_1+x_2I)(e+eI))\}](a_1+a_2I) \\
= & [((x_1+x_2I)(a_1+a_2I))\{((a_1+a_2I)(e+eI))((x_1+x_2I)(e+eI))\}](a_1+a_2I) \\
= & [((x_1+x_2I)(a_1+a_2I))\{((e+eI)(x_1+x_2I))((e+eI)(a_1+a_2I))\}](a_1+a_2I) \\
= & [((x_1+x_2I)(a_1+a_2I))((x_1+x_2I)(a_1+a_2I))](a_1+a_2I) \\
= & [((a_1+a_2I)(x_1+x_2I))((a_1+a_2I)(x_1+x_2I))](a_1+a_2I) \\
= & [(a_1+a_2I)\{((a_1+a_2I)(x_1+x_2I))(x_1+x_2I)\}](a_1+a_2I) \\
= & [(a_1+a_2I)\{((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)\}](a_1+a_2I) \\
= & ((a_1+a_2I)(y_1+y_2I))(a_1+a_2I),
\end{aligned}
$$

and using (1) and left invertive law, we get

$$
\begin{aligned}
& (a_1+a_2I)(y_1+y_2I) \\
= & (a_1+a_2I)[((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)] \\
= & ((x_1+x_2I)(a_1+a_2I))((a_1+a_2I)(x_1+x_2I)) \\
= & ((a_1+a_2I)(x_1+x_2I))((x_1+x_2I)(a_1+a_2I)) \\
= & [((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)](a_1+a_2I) \\
= & (y_1+y_2I)(a_1+a_2I).
\end{aligned}
$$

This shows that $(N(S)(a_1+a_2I))N(S)$ is strongly regular.

Let $L(S)$ be any left ideal in $N(S) \Rightarrow N(S)L(S) \subseteq L(S)$. Let $(a_1+a_2I) \in L(S), N(S) \in N(S)$. Since $N(S)$ is strongly regular, so there exists some $(x_1+x_2I) \in N(S)$, such that, $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$ and $(a_1+a_2I)(x_1+x_2I) = (x_1+x_2I)(a_1+a_2I)$. Now $(a_1+a_2I)N(S) \in$

$L(S)N(S)$

$$
\begin{aligned}
& (a_1+a_2I)N(S) \\
= \ & [((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)]N(S) \\
= \ & [((x_1+x_2I)(a_1+a_2I))(a_1+a_2I)]N(S) \\
= \ & ((a_1+a_2I)^2(x_1+x_2I))N(S) \\
= \ & (N(S)(x_1+x_2I))(a_1+a_2I)^2 \\
= \ & (N(S)(x_1+x_2I))((a_1+a_2I)(a_1+a_2I)) \\
\in \ & N(S)(N(S)L) \subseteq N(S)L \subseteq L.
\end{aligned}
$$

This shows that $L(S)$ is also right ideal and $N(S)$ is left duo.

$(ii) \Longrightarrow (i)$

Using medial and paramedial laws we get $(N(S)(a_1+a_2I))(N(S)N(S)) = (N(S)N(S))((a_1+a_2I)N(S)) = (N(S)(a_1+a_2I))N(S)$. Now since $N(S)$ is left duo, so $(a_1+a_2I)N(S) \subseteq N(S)(a_1+a_2I)$. Also we can show that $N(S)(a_1+a_2I) \subseteq (a_1+a_2I)N(S)$. Thus $N(S)(a_1+a_2I) = (a_1+a_2I)N(S)$. Now let $(a_1+a_2I) \in N(S)$, also $(a_1+a_2I) \in N(S)(a_1+a_2I) = (a_1+a_2I)N(S) \Rightarrow (a_1+a_2I) = (t_1+t_2I)(a_1+a_2I)$ and $(a_1+a_2I) = (a_1+a_2I)(v_1+v_2I)$ for some $(t_1+t_2I),(v_1+v_2I) \in N(S)$. Now

$$
\begin{aligned}
& (a_1+a_2I) \\
= \ & (a_1+a_2I)(v_1+v_2I) \\
= \ & ((t_1+t_2I)(a_1+a_2I))(v_1+v_2I) \\
\in \ & (N(S)(a_1+a_2I))N(S).
\end{aligned}
$$

As $(N(S)(a_1+a_2I))N(S)$ is strongly regular, so there exists some $(u_1+u_2I) \in (N(S)(a_1+a_2I))N(S)$, such that $(a_1+a_2I) = ((a_1+a_2I)(u_1+u_2I))(a_1+a_2I)$ and $(a_1+a_2I)(u_1+u_2I) = (u_1+u_2I)(a_1+a_2I)$. Hence $N(S)$ is regular. ■

Theorem 184 *For an Neutrosophic AG-groupoid $N(S)$ with left identity the following are equivalent,*

(i) $N(S)$ is strongly regular,

(ii) $N(S)(a_1+a_2I)$ is strongly regular for all (a_1+a_2I) in $N(S)$.

Proof. $(i) \Longrightarrow (ii)$

Let $(a_1+a_2I) \in N(S)(a_1+a_2I)$, so $(a_1+a_2I) \in N(S)$ and $N(S)$ is strongly regular so there exists some $(x_1+x_2I) \in N(S)$ such that $(a_1+a_2I) = ((a_1+a_2I)(x_1+x_2I))(a_1+a_2I)$ and $(a_1+a_2I)(x_1+x_2I) = (x_1+x_2I)(a_1+a_2I)$. Let $(y_1+y_2I) = ((x_1+x_2I)(a_1+a_2I))(x_1+x_2I)$ for some

$(y_1 + y_2I) \in N(S)$. Now using left invertive law we get

$$\begin{aligned}
& (y_1 + y_2I) \\
= \; & ((x_1 + x_2I)(a_1 + a_2I))(x_1 + x_2I) \\
= \; & ((a_1 + a_2I)(x_1 + x_2I))(x_1 + x_2I) \\
= \; & (x_1 + x_2I)^2(a_1 + a_2I) \\
\in \; & N(S)(a_1 + a_2I).
\end{aligned}$$

Now using paramedial law,medial law and (1), we get

$$\begin{aligned}
& (a_1 + a_2I) \\
= \; & ((a_1 + a_2I)(x_1 + x_2I))(a_1 + a_2I) \\
= \; & ((x_1 + x_2I)(a_1 + a_2I))(a_1 + a_2I) \\
= \; & [(x_1 + x_2I)\{((a_1 + a_2I)(x_1 + x_2I))(a_1 + a_2I)\}](a_1 + a_2I) \\
= \; & [(x_1 + x_2I)\{((a_1 + a_2I)(x_1 + x_2I))((e + eI)(a_1 + a_2I))\}](a_1 + a_2I) \\
= \; & [(x_1 + x_2I)\{((a_1 + a_2I)(e + eI))((x_1 + x_2I)(a_1 + a_2I))\}](a_1 + a_2I) \\
= \; & [((a_1 + a_2I)(e + eI))\{(x_1 + x_2I)((x_1 + x_2I)(a_1 + a_2I))\}](a_1 + a_2I) \\
= \; & [((a_1 + a_2I)(e + eI))\{((e + eI)(x_1 + x_2I))((a_1 + a_2I)(x_1 + x_2I))\}](a_1 + a_2I) \\
= \; & [((a_1 + a_2I)(e + eI))\{((x_1 + x_2I)(a_1 + a_2I))((x_1 + x_2I)(e + eI))\}](a_1 + a_2I) \\
= \; & [((x_1 + x_2I)(a_1 + a_2I))\{((a_1 + a_2I)(e + eI))((x_1 + x_2I)(e + eI))\}](a_1 + a_2I) \\
= \; & [((x_1 + x_2I)(a_1 + a_2I))\{((e + eI)(x_1 + x_2I))((e + eI)(a_1 + a_2I))\}](a_1 + a_2I) \\
= \; & [((x_1 + x_2I)(a_1 + a_2I))((x_1 + x_2I)(a_1 + a_2I))](a_1 + a_2I) \\
= \; & [((a_1 + a_2I)(x_1 + x_2I))((a_1 + a_2I)(x_1 + x_2I))](a_1 + a_2I) \\
= \; & [(a_1 + a_2I)\{((a_1 + a_2I)(x_1 + x_2I))(x_1 + x_2I)\}](a_1 + a_2I) \\
= \; & [(a_1 + a_2I)\{((x_1 + x_2I)(a_1 + a_2I))(x_1 + x_2I)\}](a_1 + a_2I) \\
= \; & ((a_1 + a_2I)(y_1 + y_2I))(a_1 + a_2I),
\end{aligned}$$

and using (1) and left invertive law, we get

$$\begin{aligned}
& (a_1 + a_2I)(y_1 + y_2I) \\
= \; & (a_1 + a_2I)[((x_1 + x_2I)(a_1 + a_2I))(x_1 + x_2I)] \\
= \; & ((x_1 + x_2I)(a_1 + a_2I))((a_1 + a_2I)(x_1 + x_2I)) \\
= \; & ((a_1 + a_2I)(x_1 + x_2I))((x_1 + x_2I)(a_1 + a_2I)) \\
= \; & [((x_1 + x_2I)(a_1 + a_2I))(x_1 + x_2I)](a_1 + a_2I) \\
= \; & (y_1 + y_2I)(a_1 + a_2I).
\end{aligned}$$

Which implies that $N(S)(a_1 + a_2I)$ is strongly regular.

$(ii) \Longrightarrow (i)$

Let $(a_1+a_2I) \in N(S)$, so $(a_1+a_2I) \in N(S)(a_1+a_2I)$ and $N(S)(a_1+a_2I)$ is strongly regular which implies $N(S)$ is strongly regular. ■

6 Neutrosophic Ideals in Semigroups

In this chapter we will introduce the concept of neutrosophic ideal, neutrosophic prime ideal, neutrosophic bi-ideal and neutrosophic quasi ideal of a neutrosophic semigroup. With counter example we have shown that the union and product of two neutrosophic quasi-ideals of a neutrosophic semigroup need not be a neutrosophic quasi-ideal of neutrosophic semigroup. We have also shown that every neutrosophic bi-ideal of a neutrosophic semigroup need not be a neutrosophic quasi-ideal of a neutrosophic semigroup. We have also characterized the regularity and intra-rgularity of a neutrosophic semigroup.

Algebra is one of the oldest branch of mathematics, defined as the manipulation of symbols. Basically the word “Algebra” is an Arabic word which means recombining the factors. In this regard the mathematicians such as al-Khwarizmi, Diophantus and Omar Khayyam make extra ordinary contributions. Mainly the history of Algebra is divided into two eras, firstly upto 19th century known as Elementary Algebra, secondly after 19th century known as abstract or modern Algebra. The Elementary Algebra have many applications in engineering, medicine and economics. While the abstract or modern algebra discus groups, vector spaces, rings topologies etc.

As from beginning to present, there are a lot of difficulties and complications in every branch of science which distress human life directly or indirectly. By applying communal technique these difficulties and complications cannot be handled. The uncertainty and disorderliness was lectured by Lotfi Zedah in 1965, hosting the concept of fuzzy set. A lot of theories have been developed to contended with uncertainty, haziness and ambiguity such as theory of probability, rough Set theory, fuzzy set theory. But all these theories were not sufficient to describe the situation of neutrality or indeterminacy. F. Samrandache perceived that the law of omitted middle is still sedentary, as in making a decision (making a decision, uncertain, not making), in sports games (defeating, wining, tie), and in voting system (yes, no, NA). To handle such type of situation he develop the idea of neutrosophic set . Neutrosophy is exploration of neutrosophic logic. When uncertainty and disorderedness is modeled, we will use fuzzy theory whereas we will use neutrosophic theory when there is indeterminacy or ambiguity involved. Using neutrosophic theory the "neutrosophic algebraic structure" is lately defined by Vasantha K Andasmy and Florentien Samarandache. Some of neutrosophic algebraic structure have been defined are, neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroup, neutrosophic bisemigroups, neutrosophic N-semigroups, neutrosophic loops, neutrosophic biloops, neutrosophic N-loops, neutrosophic groupiods and neutrosophic bigroupiods.

The "neutrosophic" comes from Greek word Sophia which means study of non-combatant speculation. Thus neutrosophic is the study of Essence, Provenance and Breadth of non-interference as well as there synergy with different intellectual continuous. One first present the alteration from fuzzy set to neutrosophic set. Then one introduce the neutrosophic constituent T, I, F, which represent the existence /membership, indeterminacy and non-membership values respectively, where $]0^-,1^+[$ is the non-standard unit interval. So one introduce the neutrosophic set. One define the neutrosophic set operation such as complement, intersection union, addition/difference, Cartesian product and inclusion.

The algebraic structure $(S, *)$ is called semigroup. Where S is a non-empty set and $*$ is closed and associative binary operation. i.e. $ab \in S$, and,

$$a * (b * c) = (a * b) * c$$

$\forall\, a$, b, $c \in S$. For a semigroup $(S, *)$ it is not vital that the identity element or inverses of all element also prevail in S. The set of natural numbers $\mathbb{N}$ and that of real numbers $\mathbb{R}$ are semigroups withe the binary operation usual addition and multiplication.

The algebraic structure $(N(S), *)$ is called neutrosophic semigroup . Where $N(S)$ is a non-empty neutrosophic set and $*$ is closed and associative binary operation i.e $(a + aI) * (b + bI) \in N(S)$, and,

$$\begin{aligned} & (a + aI) * [(b + bI) * (c + cI)] \\ = \; & [(a + aI) * (b + bI)] * (c + cI) \forall (a + aI), (b + bI), (c + cI) \in N(S). \end{aligned}$$

For a neutrosophic semigroup $(N(S), *)$ it is not vital that the identity element or inverses of all elements also prevail in $(N(S), *)$ Onward, in case of semigroup, we will write S instead of $(S, *)$ and operation on it as ab instead of $a*b$, $a(bc)$ instead of $a*(b*c)$ and so on. Similarly for neutrosophic semigroup we will use the notation $N(S)$ instead of $(N(S), *)$ and the corresponding operation as $(a + aI)(b + bI)$ instead of $(a + aI) * (b + bI)$, $(a+aI)[(b+bI)(c+cI)]$ instead of $(a+aI)*[(b+bI)*(c+cI)]$ and so on. The operation usually refer to multiplication on S or $N(S)$ respectively. The neutrosophic set of natural numbers $N(\mathbb{N})$ and that of real numbers $N(\mathbb{R})$ are neutrosophic semigroups w.r.t usual addition and multiplication. A neutrosophic semigroup $N(S)$ may or may not commutative. If

$$(a + aI)(b + bI) = (b + bI)(a + aI) \;\forall\; (a + aI),\, (b + bI) \in N(S).$$

Then $N(S)$ is called neutrosophic commutative semigroup.

Definition 185 *A non-empty neutrosophic subset $N(Q)$ of a neutrosophic semigroup $N(S)$ is called a neutrosophic quasi-ideal if*

$$N(Q)N(S) \cap N(S)N(Q) \subseteq N(Q).$$

Proposition 186 *Every neutrosophic quasi-ideal of a neutrosophic semigroup $N(S)$ is a neutrosophic subsemigroup of $N(S)$.*

Proof. Let $N(Q)$ be a neutrosophic quasi-ideal of a neutrosophic semigroup $N(S)$. Then

$$N(Q)N(S) \cap N(S)N(Q) \subseteq N(Q).$$

Also

$$\begin{aligned}[N(Q)]^2 &= N(Q)N(Q) \\ &\subseteq N(Q)N(S).\end{aligned}$$

And

$$\begin{aligned}[N(Q)]^2 &= N(Q)N(Q) \\ &\subseteq N(S)N(Q).\end{aligned}$$

This implies that

$$\begin{aligned}[N(Q)]^2 &\subseteq N(Q)N(S) \cap N(S)N(Q) \\ &\subseteq N(Q).\end{aligned}$$

Thus $N(Q)$ is a neutrosophic subsemigroup of $N(S)$. ■

Proposition 187 *A neutrosophic semigroup $N(S)$ without zero is a neutosophic group if and only if $N(S)$ has no proper neutrosophic quasi-ideal.*

Proof. Let

$$(x + yI) \in N(Q).$$

And $N(Q)$ be a neutrosophic quasi-ideal of a neutrosophic group $N(S)$.Then

$$\begin{aligned}N(S) &= N(S)(x + yI) \\ &= (x + yI)\,N(S) \\ &= N(S)(x + yI) \cap (x + yI)N(S) \\ &\subseteq N(S)N(Q) \cap N(Q)N(S) \\ &\subseteq N(Q).\end{aligned}$$

Hence

$$N(S) = N(Q).$$

Conversely let the neutrosophic semigroup $N(S)$ has no neutrosophic proper quasi-ideal. Then for any

$$(x + yI) \in N(S)$$

the product $N(S)(x+yI)$ and $(x+yI)N(S)$ are left and right neutrosophic quasi-ideals of$N(S)$, respectively. Thus

$$\begin{aligned} N(S) &= N(S)(x+yI) \\ &= (x+yI)N(S). \end{aligned}$$

i.e $N(S)$ is a neutrosophic group. ■

Proposition 188 *Let $N(S)$ be a neutrosophic semigroup with zero such that*

$$[N(S)]^2 \neq 0.$$

Then $N(S)$ is a group with zero if and only if it contains no proper neutrosophic quasi-ideal.

Proof. *Consider $N(S)$ is a neutrosophic group with zero. Let $N(Q)$ be a neutrosophic quasi-ideal of $N(S)$ and $(x+yI)$ be a non zero element of $N(Q)$. Then*

$$\begin{aligned} N(S) &= N(S)(x+yI) \\ &= (x+yI)N(S) \\ &= N(S)(x+yI) \cap (x+yI)N(S) \\ &\subseteq N(S)N(Q) \cap N(Q)N(S) \\ &\subseteq N(Q). \end{aligned}$$

Conversely, assume that the neutrosophic semigroup $N(S)$ with zero contains no neutrosophic proper quasi-ideals. Let $(a+aI) \cup N(S)(a+aI)$ be the neutrosophic principal left ideal of $N(S)$ generated by

$$(0+0I) \neq (a+aI) \in N(S).$$

Then

$$(a+aI) \cup N(S)(a+aI) = N(S).$$

Thus

$$\begin{aligned} N(S)[(a+aI) \cup N(S)(a+aI)] &= N(S)N(S) \\ &= [N(S)]^2 \\ &\neq (0+0I). \end{aligned}$$

As $N(S)(a+aI)$ is a non zero neutrosophic ideal of $N(S)$, therefore

$$N(S)(a+aI) = N(S).$$

Similarly we obtain

$$(a+aI)N(S) = N(S)$$

for every non zero element $(a+aI)$ of $N(S)$. This implies that $N(S)$ is a neutrosophic group. ■

Proposition 189 *The intersection of a neutrosophic right ideal $N(R)$ and left ideal $N(L)$ of a neutrosophic semigroup $N(S)$ is a neutrosophic quasi ideal of $N(S)$.*

Proof. Since

$$N(R)N(L) \subseteq N(R) \cap N(L).$$

Thus $N(R) \cap N(L)$ is non-empty. Further,

$$\begin{aligned}[N(R) \cap N(L)]N(S) \cap N(S)[N(R) \cap N(L)] &\subseteq N(R)N(S) \cap N(S)N(L) \\ &\subseteq N(R) \cap N(L).\end{aligned}$$

Hence $N(R) \cap N(L)$ is a neutosophic quasi ideal of $N(S)$. ■

Proposition 190 *The intersection of a neutrosophic quasi-ideal $N(Q)$ and a neutrosophic subsemigroup $N(T)$ of a neutrosophic semigroup$N(S)$ is either empty or a neutrosophic quasi-ideal of $N(T)$.*

Proof. If $N(T) \cap N(Q)$ is non-empty then, $N(T) \cap N(Q)$ is a neutrosophic subset of $N(T)$ such that

$$\begin{aligned}[N(T) \cap N(Q)]N(T) \cap N(T)[N(T) \cap N(Q)] &\subseteq N(T)N(T) \cap N(T)N(T) \\ &\subseteq [N(T)]^2 \\ &\subseteq N(T).\end{aligned}$$

And

$$\begin{aligned}[N(T) \cap N(Q)]N(T) \cap N(T)[N(T) \cap N(Q)] &\subseteq N(Q)N(S) \cap N(Q)N(S) \\ &\subseteq N(Q).\end{aligned}$$

Thus

$$[N(T) \cap N(Q)]N(T) \cap N(T)[N(T) \cap N(Q)] \subseteq N(T) \cap N(Q).$$

Hence $N(T) \cap N(Q)$ is a neutrosophic quasi-ideal of $N(T)$. ■

Proposition 191 *Let $(e+eI)$ be an idempotent element of a neutrosophic semigroup $N(S)$. $N(R)$ and , $N(L)$ be neutrosophic right and left ideals of $N(S)$ respectively. Then $N(R)(e+eI)$ and $(e+eI)N(L)$ are neutrosophic quasi-ideals of $N(S)$ such that*

$$N(R)(e+eI) = N(R) \cap N(S)(e+eI).$$

And

$$(e+eI)N(L) = (e+eI)N(S) \cap N(L).$$

Proof. *If $N(R)$ and $N(L)$ are neutrosophic right and left ideals and $N(X)$ be a non-empty neutrosophic subset of a neutrosophic semigroup $N(S)$.*

Then $N(X)N(R)$ and $N(L)N(X)$ are neutrosophic right and left ideals of $N(S)$ and

$$N(R)N(L) \subseteq N(R) \cap N(L).$$

Thus by this result $N(S)(e+eI)$ and $(e+eI)N(S)$ are neutrosophic left and right ideal of $N(S)$. Now we show that

$$N(R)(e+eI) = N(R) \cap N(S)(e+eI).$$

And

$$(e+eI)N(L) = (e+eI)N(S) \cap N(L).$$

Since $N(R)$ is a neutrosophic right ideal so,

$$N(R)(e+eI) \subseteq N(R) \cap N(S)(e+eI).$$

Let $x \in N(R) \cap N(S)(e+eI)$. Then

$$\begin{aligned} x &= (r+rI) \\ &= (s+sI)(e+eI). \end{aligned}$$

Where $(r+rI) \in N(R)$ and $(s+sI) \in N(S)$. This implies that

$$\begin{aligned} x &= (s+sI)(e+eI) \\ &= (s+sI)(e+eI)(e+eI) \\ &= (r+rI)(e+eI) \\ &\in N(R)(e+eI). \end{aligned}$$

Thus

$$N(R) \cap N(S)(e+eI) \subseteq N(R)(e+eI).$$

Therefore

$$N(R)(e+eI) = N(R) \cap N(S)(e+eI).$$

Hence by proposition 4,

$$N(R)(e+eI) = N(R) \cap N(S)(e+eI).$$

is a neutrosophic quasi-ideal of $N(S)$.

Similarly we can show that,

$$(e+eI)N(L) = (e+eI)N(S) \cap N(L).$$

is a neutrosophic quasi-ideal of $N(S)$. ■

Proposition 192 *Every neutrosophic quasi ideal $N(Q)$ of a neutrosophic semigroup $N(S)$ is the intersection of the neutrosophic principle left ideal $N(Q) \cup N(S)N(Q)$ and neutrosophic principle right ideal $N(Q) \cup N(S)N(Q)$ of $N(S)$.*

Proof. $N(Q) \cup N(S)N(Q)$ and $N(Q) \cup N(Q)N(S)$ are neutrosophic principle left and right ideals of $N(S)$ generated by $N(Q)$. Also

$$N(Q) \subseteq [N(Q) \cup N(S)N(Q)] \cap [N(Q) \cup N(Q)N(S)].$$

Conversely

$$\begin{aligned} & [N(Q) \cup N(S)N(Q)] \cap [N(Q) \cup N(Q)N(S)] \\ = \; & N(Q) \cup [N(S)N(Q) \cap N(Q)N(S)]. \end{aligned}$$

Since $N(Q)$ is neutrosophic quasi-ideal, therefore

$$N(S)N(Q) \cap N(Q)N(S) \subseteq N(Q).$$

Hence

$$\begin{aligned} & [N(Q) \cup N(S)N(Q)] \cap [N(Q) \cup N(Q)N(S)] \\ \subseteq \; & N(Q). \end{aligned}$$

■

Corollary 193 *A non-empty neutrosophic subset of a neutrosophic semigroup $N(S)$ is a neutrosophic quasi-ideal of $N(S)$ iff it is the intersection of a neutrosophic left and a right ideal of $N(S)$.*

Proposition 194 *If $N(Q)$ is a neutrosophic proper quasi-ideal of a neutrosophic semigroup $N(S)$ such that $N(Q)$ is one-sided neutrosophic ideal of $N(S)$, then $N(S)N(Q)$ $(N(Q)N(S))$ is a neutrosophic left (right) ideal of $N(S)$.*
Proof. *Assume that $N(S)N(Q)$ is not a neutrosophic proper left ideal of the neutrosophic semigroup $N(S)$, then*

$$N(S)N(Q) = 0$$

or

$$N(S)N(Q) = N(S).$$

The case

$$N(S)N(Q) = 0$$

is impossible. In fact

$$\begin{aligned} N(S)N(Q) &= 0 \\ &\subseteq N(Q) \end{aligned}$$

contradicts the assumption that $N(Q)$ is not a neutrosophic proper left ideal of the neutrosophic semigroup $N(S)$. On the other hand if

$$N(S)N(Q) = N(S)$$

then

$$N(Q) \subseteq N(S)N(Q),$$

thus by previous proposition 7,

$$\begin{aligned} N(Q) &= [N(Q) \cup N(S)N(Q)] \cap [N(Q) \cup N(Q)N(S)] \\ &= N(S) \cap [N(Q) \cup N(Q)N(S)] \\ &= N(Q) \cup N(Q)N(S). \end{aligned}$$

This implies that

$$N(Q)N(S) \subseteq N(Q)$$

which is a contradiction to the condition that $N(Q)$ is not a neutrosophic right ideal of $N(S)$. So $N(S)N(Q)$ must be a neutrosophic proper left ideal of the neutrosophic semigroup $N(S)$. Similarly we can show that $N(Q)N(S)$ is a neutrosophic proper right ideal of the neutrosophic semigroup $N(S)$. ■

Proposition 195 *The intersection of any set of a neutrosophic quasi-ideal of a neutrosophic semigroup $N(S)$ is either empty or a neutrosophic quasi-ideal of $N(S)$.*

Proof. Let $\{N(Q_\lambda) : \lambda \in \Lambda\}$ be a set of neutrosophic quasi-ideals of a neutrosophic semigroup $N(S)$. If $\underset{\lambda\in\Lambda}{\cap} N(Q_\lambda)$ is non-empty, then, for every $(\mu \in \Lambda)$,

$$\begin{aligned} N(D) &= N(S)[\underset{\lambda\in\Lambda}{\cap} N(Q_\lambda)] \cap [\underset{\lambda\in\Lambda}{\cap} N(Q_\lambda)]N(S) \\ &\subseteq N(S)N(Q_\mu) \cap N(Q_\mu)N(S) \\ &\subseteq N(Q_\mu). \end{aligned}$$

Hence

$$N(D) \subseteq \underset{\lambda\in\Lambda}{\cap} N(Q_\lambda)$$

i.e. $\underset{\lambda\in\Lambda}{\cap} N(Q_\lambda)$ is a neutrosophic quasi ideal of $N(S)$. ■

Remark 196 *The union and product of two neutrosophic quasi-ideals of a neutrosophic semigroup $N(S)$ need not be a neutrosophic quasi-ideal of $N(S)$. Let see examples.*

Example 197 *Let*

$$N(S) = \left\{ \begin{array}{c} \begin{bmatrix} l+lI & m+mI \\ p+pI & q+qI \end{bmatrix} : \\ l+lI,\, m+mI,\, p+pI\ ,\, q+qI \in N(\mathbb{Z}^+) \end{array} \right\},$$

then $N(S)$ is a neutrosophic semigroup under the usual multiplication of matrices.

Let

$$N(Q_1) = \left\{ \begin{bmatrix} l+lI & 0+0I \\ 0+0I & 0+0I \end{bmatrix} : l+lI \in N\left(\mathbb{Z}^+\right) \right\},$$

and

$$N(Q_2) = \left\{ \begin{bmatrix} 0+0I & 0+0I \\ 0+0I & q+qI \end{bmatrix} : q+qI \in N\left(\mathbb{Z}^+\right) \right\},$$

then clearly both $N(Q_1)$ and $N(Q_2)$ are neutrosophic quasi ideal of $N(S)$.

Let

$$N(Q) = N(Q_1) \cup N(Q_2),$$

then $N(Q)$ is not a neutrosophic quasi ideal of $N(S)$ because,

$$\begin{aligned} & \begin{bmatrix} 0+0I & 1+1I \\ 0+0I & 0+0I \end{bmatrix} \begin{bmatrix} 0+0I & 0+0I \\ 0+0I & a+aI \end{bmatrix} \\ = & \begin{bmatrix} 0+0I & a+aI \\ 0+0I & 0+0I \end{bmatrix} \\ = & \begin{bmatrix} a+aI & 0+0I \\ 0+0I & 0+0I \end{bmatrix} \begin{bmatrix} 0+0I & 1+1I \\ 0+0I & 0+0I \end{bmatrix} \\ \in & \ N(S)N(Q) \cap N(Q)N(S). \end{aligned}$$

But does not belong to $N(Q)$. So the union of neutrosophic quasi ideals need not be a neutrosophic quasi ideal.

Example 198 *Let*

$$N(S) = \left\{ \begin{bmatrix} l+lI & 0+0I \\ p+pI & 1+1I \end{bmatrix} : l+lI,\, p+pI \ \in N\left(\mathbb{R}^+\right) \right\},$$

then $N(S)$ is a neutrosophic semigroup under the usual multiplication of matrices.

Let

$$N(R) = \left\{ \begin{array}{c} \begin{bmatrix} l+lI & 0+0I \\ p+pI & 1+1I \end{bmatrix} : l+lI,\, p+pI \ \in N\left(\mathbb{R}^+\right), \\ 0+0I < l+lI < p+pI \end{array} \right\},$$

and

$$N(L) = \left\{ \begin{array}{c} \begin{bmatrix} p+pI & 0+0I \\ q+qI & 1+1I \end{bmatrix} : p+pI,\, q+qI \ \in N\left(\mathbb{R}^+\right), \\ 0+0I < p+pI \text{ and } 5+5I < q+qI \end{array} \right\},$$

then $N(R)$ being neutrosophic right ideal and $N(L)$ being neutrosophic left ideals are quasi ideal of $N(S)$. Then the product $N(R)N(L)$ is a neutrosophic bi-ideal of $N(S)$ but by p224 it is not a neutrosophic quasi ideal *of* $N(S)$*. Indeed*

$$\begin{aligned}
& \begin{bmatrix} 5+5I & 0+0I \\ 10+10I & 1+1I \end{bmatrix} \\
= & \begin{bmatrix} 10+10I & 0+0I \\ 6+6I & 1+1I \end{bmatrix} \begin{bmatrix} \frac{1}{2}+\frac{1}{2}I & 0+0I \\ 1+1I & 1+1I \end{bmatrix} \begin{bmatrix} 1+1I & 0+0I \\ 6+6I & 1+1I \end{bmatrix} \\
= & \begin{bmatrix} 2+2I & 0+0I \\ 3+3I & 1+1I \end{bmatrix} \begin{bmatrix} 10+10I & 0+0I \\ 6+6I & 1+1I \end{bmatrix} \begin{bmatrix} \frac{1}{4}+\frac{1}{4}I & 0+0I \\ 1+1I & 1+1I \end{bmatrix} \\
\in & N(S)[N(R)N(L)] \cap [N(R)N(L)]N(S).
\end{aligned}$$

But do not contain in $N(R)N(L)$*. Hence,*

$$N(S)[N(R)N(L)] \cap [N(R)N(L)]N(S) \nsubseteq N(R)N(L).$$

Definition 199 *A neutrosophic subsemigroup* $N(B)$ *of a neutrosophic semigroup* $N(S)$ *is said to be bi-ideal if*

$$N(B)N(S)N(B) \subseteq N(B).$$

Proposition 200 *Let* $N(A)$ *be a neutrosophic ideal of a neutrosophic semigroup* $N(S)$ *and* $N(Q)$ *be neutrosophic quasi-ideal of* $N(A)$*, then* $N(Q)$ *is a bi-ideal of* $N(S)$*.*

Proof. Since $N(Q)$ is a neutrosophic quasi-ideal of $N(A)$, therefore by proposition 1, $N(Q)$ is a neutrosophic subsemigroup of $N(S)$. Also since

$$N(Q) \subseteq N(A),$$

so we have

$$\begin{aligned}
N(Q)N(S)N(Q) &\subseteq N(Q)N(S)N(A) \cap N(A)N(S)N(Q) \\
&\subseteq N(Q)N(A) \cap N(A)N(Q) \\
&\subseteq N(Q)
\end{aligned}$$

Hence $N(Q)$ is a bi-ideal. ■

Corollary 201 *Every neutrosophic quasi ideal of a neutrosophic semigroup* $N(S)$ *is a neutrosophic bi-ideal of* $N(S)$*.*

Proof. Let $N(Q)$ be a neutrosophic quasi ideal of a neutrosophic semigroup $N(S)$. Then by proposition 1, $N(Q)$ is a neutrosophic subsemigroup of $N(S)$. Also

$$\begin{aligned}
N(Q)N(S)N(Q) &\subseteq N(Q)N(S) \cap N(S)N(Q) \\
&\subseteq N(Q).
\end{aligned}$$

Thus $N(Q)$ is a neutrosophic bi-ideal of $N(S)$. ■

Proposition 202 *The product of two neutrosophic quasi-ideals $N(Q_1)$ and $N(Q_2)$ of a neutrosophic semigroup $N(S)$ is a neutrosophic bi-ideal of $N(S)$.*

Proof. *Let $N(Q_1)$ and $N(Q_2)$ be neutrosophic quasi-ideal of $N(S)$. Since every neutrosophic quasi-ideal of a neutrosophic semigroup $N(S)$ is a neutrosophic bi-ideal of $N(S)$, so we can write*

$$N(Q_2)N(S)N(Q_2) \subseteq N(Q_2),$$

therefore

$$\begin{aligned} N(Q_1)N(Q_2)N(Q_1)N(Q_2) &\subseteq N(Q_1)N(Q_2)N(S)N(Q_2) \\ &\subseteq N(Q_1)N(Q_2). \end{aligned}$$

i.e $N(Q_1)N(Q_2)$ is a neutrosophic subsemigroup of $N(S)$. Also

$$\begin{aligned} N(Q_1)N(Q_2)N(S)N(Q_1)N(Q_2) &\subseteq N(Q_1)N(Q_2)N(S)N(Q_2) \\ &\subseteq N(Q_1)N(Q_2). \end{aligned}$$

Thus $N(Q_1)N(Q_2)$ is a neutrosophic bi-ideal of $N(S)$. ■

Remark 203 *Every neutrosophic bi-ideal of a neutrosophic semigroup $N(S)$ need not be a neutrosophic quasi-ideal of $N(S)$. Consider the examples.*

Example 204 *Let*

$$N(S) = \left\{ \begin{bmatrix} a+aI & 0+0I \\ b+bI & 1+1I \end{bmatrix} \ : \ a+aI,\, b+bI \in N\left(\mathbb{R}^+\right) \right\},$$

then $N(S)$ is a neutrosophic semi group under usual matrix multiplication.

Let

$$N(R) = \left\{ \begin{array}{c} \begin{bmatrix} a+aI & 0+0I \\ b+bI & 1+1I \end{bmatrix} : l+lI,\, p+pI \ \in N\left(\mathbb{R}^+\right), \\ 0+0I < l+lI < p+pI. \end{array} \right\},$$

and

$$N(L) = \left\{ \begin{array}{c} \begin{bmatrix} p+pI & 0+0I \\ q+qI & 1+1I \end{bmatrix} : p+pI,\, q+qI \ \in N\left(\mathbb{R}^+\right), \\ 0+0I < p+pI \text{ and } 5+5I < q+qI. \end{array} \right\},$$

then $N(R)$ being neutrosophic right ideal and $N(L)$ being neutrosophic left ideals are quasi ideal of $N(S)$. Then the product $N(R)N(L)$ is a neutrosophic bi-ideal of $N(S)$ but by proposition 11 it is not a neutrosophic *quasi ideal of $N(S)$. Indeed*

$$\begin{aligned}
&\begin{bmatrix} 5+5I & 0+0I \\ 10+10I & 1+1I \end{bmatrix} \\
= &\begin{bmatrix} 10+10I & 0+0I \\ 6+6I & 1+1I \end{bmatrix} \begin{bmatrix} \frac{1}{2}+\frac{1}{2}I & 0+0I \\ 1+1I & 1+1I \end{bmatrix} \begin{bmatrix} 1+1I & 0+0I \\ 6+6I & 1+1I \end{bmatrix} \\
= &\begin{bmatrix} 2+2I & 0+0I \\ 3+3I & 1+1I \end{bmatrix} \begin{bmatrix} 10+10I & 0+0I \\ 6+6I & 1+1I \end{bmatrix} \begin{bmatrix} \frac{1}{4}+\frac{1}{4}I & 0+0I \\ 1+1I & 1+1I \end{bmatrix} \\
\in \ & N(S)[N(R)N(I)] \cap [N(R)N(I)]N(S).
\end{aligned}$$

But do not contain in $N(R)N(L)$. Hence,

$$N(S)[N(R)N(L)] \cap [N(R)N(L)]N(S) \nsubseteq N(R)N(L).$$

Proposition 205 *The intersection of any set of neutrosophic bi-ideals of a neutrosophic semigroup $N(S)$ is either empty or a neutrosophic bi-ideal of $N(S)$.*

Proof. Let $\{N(B_\lambda) : \lambda \in \Lambda\}$ be a family of neutrosophic bi-ideals of a neutrosophic semigroup $N(S)$. If $\underset{\lambda\in\Lambda}{\cap} N(B)$ is non-empty then,

$$\left(\underset{\lambda\in\Lambda}{\cap} N(B_\lambda)\right)\left(\underset{\lambda\in\Lambda}{\cap} N(B_\lambda)\right) \subseteq N(B_\mu)N(B_\mu) \subseteq N(B_\mu).$$

for every $\mu \in \Lambda$ and this implies that

$$\left(\underset{\lambda\in\Lambda}{\cap} N(B_\lambda)\right)\left(\underset{\lambda\in\Lambda}{\cap} N(B_\lambda)\right) \subseteq \underset{\lambda\in\Lambda}{\cap} N(B_\lambda).$$

i.e $\underset{\lambda\in\Lambda}{\cap} N(B_\lambda)$ is a subsemigroup of $N(S)$. Also

$$\left(\underset{\lambda\in\Lambda}{\cap} N(B_\lambda)\right) N(S) \left(\underset{\lambda\in\Lambda}{\cap} N(B_\lambda)\right) \subseteq N(B_\mu)N(S)N(B_\mu) \subseteq N(B_\mu).$$

for every $\mu \in \Lambda$ This implies that

$$\left(\underset{\lambda\in\Lambda}{\cap} N(B_\lambda)\right) N(S) \left(\underset{\lambda\in\Lambda}{\cap} N(B_\lambda)\right) \subseteq \underset{\lambda\in\Lambda}{\cap} N(B_\lambda).$$

Thus $\underset{\lambda\in\Lambda}{\cap} N(B_\lambda)$ is a neutrosophic bi-ideal of $N(S)$.

The intersection of any set of neutrosophic bi-ideals of a neutrosophic semigroup $N(S)$ with zero is a neutrosophic bi-ideal of $N(S)$. ■

Corollary 206 *The intersection of a neutrosophic bi-ideal $N(B)$ of a neutrosophic semigroup $N(S)$ and a neutrosophic subsemigroup $N(T)$ of $N(S)$ is a neutrosophic bi-ideal of $N(T)$.*

Proof. Obviously $N(B)\cap N(T)$ is a neutrosophic semigroup of $N(S)$. Since

$$N(B)\cap N(T)\subseteq N(T),$$

we have

$$\begin{aligned}[N(B)\cap N(T)]N(T)[N(B)\cap N(T)] &\subseteq N(T)\cap N(T)\cap N(T)\\ &\subseteq N(T).\end{aligned}$$

And

$$\begin{aligned}[N(B)\cap N(T)]N(T)[N(B)\cap N(T)] &\subseteq N(B)\cap N(T)\cap N(B)\\ &\subseteq N(B)\cap N(S)\cap N(B)\\ &\subseteq N(B).\end{aligned}$$

Hence

$$[N(B)\cap N(T)]N(T)[N(B)\cap N(T)]\subseteq N(B)\cap N(T).$$

Thus $N(B)\cap N(T)$ is a neutrosophic bi-ideal of $N(T)$. ■

Proposition 207 *Let $N(T)$ be an arbitrary neutrosophic subset and $N(B)$ be a neutrosophic bi-ideal of a neutrosophic semigroup $N(S)$. Then the product $N(B)N(T)$ and $N(T)N(B)$ are both neutrosophic bi-ideals of $N(S)$.*

Proof. Since

$$N(T)N(S)\subseteq N(S),$$

and

$$N(B)N(S)N(B)\subseteq N(B),$$

so we have

$$\begin{aligned}[N(B)N(T)]N(S)[N(B)N(T)] &\subseteq N(B)[N(T)N(S)][N(B)N(T)]\\ &\subseteq [N(B)N(S)][N(B)N(T)]\\ &= [N(B)N(S)N(B)]N(T)\\ &\subseteq N(B)N(T).\end{aligned}$$

And

$$\begin{aligned}[N(B)N(T)][N(B)N(T)] &= [N(B)N(T)N(B)]N(T)\\ &\subseteq [N(B)N(S)N(B)]N(T)\\ &\subseteq N(B)N(T).\end{aligned}$$

This implies that $N(B)N(T)$ is a neutrosophic bi-ideal of $N(S)$. Similarly we can show that $N(T)N(B)$ is is a neutrosophic bi-ideal of $N(S)$. ■

Proposition 208 *Let $N(B)$ be an arbitrary neutrosophic bi-ideal of a neutrosophic semigroup $N(S)$ and $N(C)$ be a neutrosophic bi-ideal of the neutrosophic semigroup $N(B)$ such that*

$$[N(C)]^2 = N(C).$$

Then $N(C)$ is a neutrosophic bi-ideal of $N(S)$.

Proof. Since

$$N(B)N(S)N(B) \subseteq N(B),$$

and

$$N(C)N(B)N(C) \subseteq N(C),$$

so we have

$$\begin{aligned} N(C)N(S)N(C) &= [N(C)]^2 N(S)[N(C)]^2 \\ &= N(C)[N(C)N(S)N(C)]N(C) \\ &\subseteq N(C)[N(B)N(S)N(B)]N(C) \\ &\subseteq N(B)N(C)N(B) \\ &\subseteq N(C)N(B)N(C) \\ &\subseteq N(C). \end{aligned}$$

Thus $N(C)$ is a neutrosophic bi-ideal of $N(S)$. ■

6.1 Some Characterizations of neutrosophic Regular Semigroups

Theorem 209 *For a neutrosophic semigroup $N(S)$ the following conditions are equivalent:*

(i) $N(S)$ is regular.
(ii) $N(R)N(L) = N(R) \cap N(L)$ where $N(R)$ and $N(L)$ is any neutrosophic right and left ideals of $N(S)$.
(iii)
(a) $[N(R)]^2 = N(R)$
(b) $[N(L)]^2 = N(L)$
(c) $N(R)N(L)$ is a neutrosophic quasi-ideal of $N(S)$. Where $N(R)$ and $N(L)$ are any neutrosophic right and left ideals of $N(S)$ respectively.
(iv) The set of all quasi-ideals of $N(S)$ is a neutrosophic regular semigroup.
(v) $N(Q)N(S)N(Q) = N(Q)$ where $N(Q)$ is any neutrosophic quasi-ideal of $N(S)$.
Proof. $(i) \Longrightarrow (ii)$

Let $N(R)$ and $N(L)$ be the neutrosophic right and left ideals of $N(S)$ then,

$$\begin{aligned} N(R)N(L) &\subseteq N(R)N(S) \\ &\subseteq N(R). \end{aligned}$$

And

$$\begin{aligned} N(R)N(L) &\subseteq N(S)N(L) \\ &\subseteq N(L). \end{aligned}$$

This implies that

$$N(R)N(L) \subseteq N(R) \cap N(L).$$

Again let

$$(a + aI) \in N(R) \cap N(L),$$

since $N(S)$ is neutrosophic regular so there exist some

$$(x + xI) \in N(S),$$

such that

$$(a + aI) = (a + aI)(x + xI)(a + aI),$$

since

$$(a + aI) \in N(R),$$

and

$$(x + xI)(a + aI) \in N(L),$$

thus

$$\begin{aligned} (a + aI) &= (a + aI)(x + xI)(a + aI) \\ &\in N(R)N(L). \end{aligned}$$

This implies that

$$N(R) \cap N(L) \subseteq N(R)N(L).$$

Hence

$$N(R)N(L) = N(R) \cap N(L).$$

$(ii) \Longrightarrow (iii)$

Since the intersection of a neutrosophic right ideal $N(R)$ and left ideal $N(L)$ of a neutrosophic semigroup $N(S)$ is a neutrosophic quasi-ideal of $N(S)$. Also by (ii),

$$N(R)N(L) = N(R) \cap N(L),$$

so $N(R)N(L)$ is a neutrosophic quasi-ideal. The two sided neutrosophic ideal of $N(S)$ generated by $N(R)$ is $N(R) \cup N(S)N(R)$ so from (ii) it follows that

$$\begin{aligned} N(R) &= N(R) \cap [N(R) \cup N(S)N(R)] \\ &= N(R)[N(R) \cup N(S)N(R)] \\ &= [N(R)]^2 \cup N(R)N(S)N(R) \\ &= [N(R)]^2 \cup [N(R)N(S)]N(R) \\ &= [N(R)]^2. \end{aligned}$$

Thus

$$[N(R)]^2 = N(R).$$

Similarly we can prove that

$$[N(L)]^2 = N(L).$$

$(iii) \Longrightarrow (iv)$

If $N(Q)$ is a neutrosophic quasi-ideal of $N(S)$ then $N(Q) \cup N(S)N(Q)$ is a neutrosophic left ideal of $N(S)$ generated by $N(Q)$. By $(iii)(b)$ we have

$$\begin{aligned} N(Q) &\subseteq N(Q) \cup N(S)N(Q) \qquad (6.1) \\ &= [N(Q) \cup N(S)N(Q)]^2 \\ &= [N(Q) \cup N(S)N(Q)][N(Q) \cup N(S)N(Q)] \\ &= [N(Q)]^2 \cup [N(S)N(Q)]^2 \cup N(S)[N(Q)]^2 \cup N(Q)N(S)N(Q) \\ &\subseteq N(S)N(Q). \end{aligned}$$

Similarly

$$N(Q) \subseteq N(Q)N(S).$$

These relation and definition of $N(Q)$ implies that

$$\begin{aligned} N(Q) &\subseteq N(S)N(Q) \cap N(Q)N(S) \\ &\subseteq N(Q) \end{aligned}$$

i.e

$$N(Q) = N(S)N(Q) \cap N(Q)N(S) \qquad ((i))$$

(I) and $(iii)(c)$ together implies that

$$N(R)N(L) = N(S)N(R)N(L) \cap N(R)N(L)N(S),$$

for every neutrosophic right ideal $N(R)$ and left ideal $N(L)$ of $N(S)$ respectively. Furthermore from $(iii)(a)$ and (b) we get

$$N(S)N(Q_1)N(Q_2) = [N(S)N(Q_1)N(Q_2)][N(S)N(S)N(Q_1)N(Q_2)].$$

And

$$N(Q_1)N(Q_2)N(S) = [N(Q_1)N(Q_2)N(S)N(S)][N(Q_1)N(Q_2)N(S)].$$

From above relation and (ii) we obtained

$$\begin{aligned}
& [N(Q_1)N(Q_2)]N(S) \cap N(S)[N(Q_1)N(Q_2)] \\
= \; & [N(Q_1)N(Q_2)N(S)N(S)][N(Q_1)N(Q_2)N(S)] \cap \\
& [N(S)N(Q_1)N(Q_2)][N(S)N(S)N(Q_1)N(Q_2)] \\
= \; & [N(Q_1)N(Q_2)N(S)][N(S)N(Q_1)N(Q_2)]N(S) \cap \\
& N(S)[N(Q_1)N(Q_2)N(S)][N(S)N(Q_1)N(Q_2)] \\
= \; & [N(Q_1)N(Q_2)N(S)][N(S)N(Q_1)N(Q_2)] \\
\subseteq \; & N(Q_1)[N(Q_2)N(S)N(Q_2)] \subseteq N(Q_1)N(Q_2).
\end{aligned}$$

i.e $N(Q_1)N(Q_2)$ is a quasi-ideal of $N(S)$.
$(iv) \Longrightarrow (v)$
Let $N(Q)$ be a neutrosophic quasi ideal of $N(S)$. Then by (iv) there exist a neutrosophic quasi ideal $N(X)$ of $N(S)$ such that

$$\begin{aligned}
N(Q) &= N(Q)N(X)N(Q) \\
&\subseteq N(Q)N(S)N(Q) \\
&\subseteq N(Q)N(S) \cap N(S)N(Q) \\
&\subseteq N(Q).
\end{aligned}$$

i.e $N(Q) = N(Q)N(S)N(Q)$
$(v) \Longrightarrow (i)$
The intersection $(a+aI)_l \cap (a+aI)_r$ of neutrosophic principal right ideal $(a+aI)_r$ and left ideal $(a+aI)_l$ of $N(S)$ generated by the element $(a+aI)$ of $N(S)$ is a neutrosophic quasi -ideal of $N(S)$. So by (v) we have

$$\begin{aligned}
(a+aI)_l \cap (a+aI)_r &= [(a+aI)_l \cap (a+aI)_r]N(S)[(a+aI)_l \cap (a+aI)_r] \\
&\subseteq (a+aI)_r N(S)(a+aI)_l.
\end{aligned}$$

Since $(a+aI) \in (a+aI)_l \cap (a+aI)_r$ so

$$\begin{aligned}
(a+aI) &\in (a+aI)_r N(S)(a+aI)_l \\
&= (a+aI)N(S)(a+aI)_l \\
&= (a+aI)N(S)(a+aI).
\end{aligned}$$

i.e any element of $N(S)$ is neutrosophic regular. Hence $N(S)$ is neutrosophic regular. ■

Corollary 210 *Let $N(S)$ be a neutrosophic regular semigroup. Then the following conditions are equivalent*:

(*i*)
$$N(Q) = N(R) \cap N(L) = N(R)N(L),$$
for all quasi-ideal $N(Q)$ of $N(S)$,where $N(R)$ and $N(L)$ are neutrosophic right and left ideal of $N(A)$ generated by $N(Q)$.
(*ii*)
$$[N(Q)]^2 = [N(Q)]^3,$$
for neutrosophic quasi-ideal $N(Q)$ of $N(S)$.
(*iii*) Every neutrosophic bi-ideal of $N(S)$ is a quasi-ideal of $N(S)$.
(*iv*) Every neutrosophic bi-ideal of a neutrosophic two sided ideal of $N(S)$ is a neutrosophic quasi-ideal of $N(S)$.
Proof. It is easy to see that (*i*) is hold. And
$$[N(Q)]^3 \subseteq [N(Q)]^2,$$
is obvious. Since $N(Q)$ is a neutrosophic quasi ideal so $[N(Q)]^2$, thus there exist a neutrosophic quasi-ideal $N(X)$ of $N(S)$ such that
$$[N(Q)]^2 = [N(Q)]^2 N(X)[N(Q)]^2.$$
Hence
$$\begin{aligned} [N(Q)]^2 &= [N(Q)]^2 N(X)[N(Q)]^2 \\ &\subseteq [N(Q)]^2 N(S)[N(Q)]^2 \\ &= N(Q) \cdot N(Q)N(S)N(Q) \cdot N(Q) \\ &\subseteq [N(Q)]^3. \end{aligned}$$
i.e
$$[N(Q)]^2 = [N(Q)]^3.$$
Let $N(B)$ be a neutrosophic bi-ideal of $N(S)$. Then $N(S)N(B)$ and $N(B)N(S)$ are neutrosophic left and right ideal of $N(S)$. Now we obtained
$$\begin{aligned} N(B)N(S) \cap N(S)N(B) &= N(B)N(S) \cdot N(S)N(B) \\ &= N(B)N(S)N(B) \\ &\subseteq N(B). \end{aligned}$$
Thus $N(B)$ is a quasi ideal of $N(S)$.
Let $N(A)$ denote a neutrosophic two sided ideal of $N(S)$ and $N(B)$ be a neutrosophic bi-ideal of $N(A)$. Since every two sided neutrosophic ideal of a neutrosophic semigroup $N(S)$ is a neutrosophic regular subsemigroup of $N(S)$. And together with (*iii*) it implies that, $N(B)$ is a neutrosophic quasi-ideal of $N(A)$ and $N(B)$ is a neutrosophic bi-ideal of $N(S)$. Again by (*iii*), $N(B)$ is a neutrosophic quasi-ideal of $N(S)$ ■

Theorem 211 *For a neutrosophic semigroup $N(S)$ the following are equivalent*:

Theorem 212 *(i) $N(S)$ is regular.*

(ii) For all neutrosophic ideals $N(I)$ and bi-ideals $N(B)$ of $N(S)$,

$$N(I) \cap N(B) = N(B)N(I)N(B).$$

(iii) For all neutrosophic ideal $N(I)$ and neutrosophic quasi-ideals $N(Q)$ of $N(S)$,

$$N(I)N(Q) = N(Q)N(I)N(Q).$$

Proof. $(i) \Longrightarrow (ii)$

Suppose $N(I)$ and $N(B)$ are neutrosophic ideal and bi-ideal of a neutrosophic semigroup $N(S)$ respectively, then

$$N(B)N(I)N(B) \subseteq N(I),$$

and

$$\begin{aligned} N(B)N(I)N(B) &\subseteq N(B)N(S)N(B) \\ &\subseteq N(B), \end{aligned}$$

so

$$N(B)N(I)N(B) \subseteq N(I) \cap N(B).$$

Let

$$x + yI \in N(I) \cap N(B),$$

then

$$x + yI = (x + yI)(a + bI)(x + yI),$$

for some $(a + bI) \in N(S)$.

$$\begin{aligned} x + yI &= (x + yI)(a + bI)(x + yI) \\ &= (x + yI)(a + bI)(x + yI)(a + bI)(x + yI) \\ &= (x + yI)[(a + bI)(x + yI)(a + bI)](x + yI) \\ &\in N(B)N(I)NB. \end{aligned}$$

Thus

$$N(I) \cap N(B) \subseteq N(B)N(I)N(B).$$

Hence

$$N(I) \cap N(B) = N(B)N(I)N(B).$$

$(ii) \Longrightarrow (iii)$

Let $N(I)$ be a neutrosophic ideal of $N(S)$ and $N(Q)$ be a neutrosophic quasi-ideal of $N(S)$. Since every neutrosophic quasi ideal is a neutrosophic bi-ideal therefore by (ii),

$$N(I) \cap N(Q) = N(Q)N(I)N(Q).$$

$(iii) \Longrightarrow (i)$

Let $N(Q)$ be a neutrosophic quasi ideal of $N(S)$. As $N(S)$ is a neutrosophic ideal of $N(S)$ therefore by (iii),

$$N(S) \cap N(Q) = N(Q)N(S)N(Q).$$

i.e

$$N(Q) = N(Q)N(S)N(Q).$$

Thus by theorem 1, $N(S)$ is a neutrosophic regular group. ■

6.2 Some Characterizations of neutrosophic Regular and Intra-regular Semigroups

Theorem 213 *For a neutrosophic semigroup $N(S)$ the following conditions are equivalent*:

(i) *$N(S)$ is both neutrosophic regular and intra-regular.*

(ii) *For any neutrosophic bi-ideal $N(B)$ of $N(S)$,*

$$[N(B)]^2 = N(B).$$

(iii) *For any neutrosophic quasi-ideal $N(Q)$ of $N(S)$,*

$$[NQ)]^2 = N(Q).$$

Proof. $(i) \Longrightarrow (ii)$

Let $N(B)$ be a neutrosophic bi-ideal of $N(S)$ then by theorem 1, and corollary 4,

$$N(B) = N(B)N(S)N(B),$$

therefore

$$\begin{aligned}
[N(B)]^2 &= N(B)N(B) \\
&= N(B)N(B)N(S)N(B) \\
&\subseteq N(B)N(S)N(B).
\end{aligned}$$

Since $N(S)$ contains multiplicative identity and

$$N(B)N(S)N(B) \subseteq N(B),$$

so we get

$$[N(B)]^2 \subseteq N(B).$$

Let $(b + bI) \in N(B)$, as $N(S)$ is regular so that

$$(b + bI) = (b + bI)(v + vI)(b + bI),$$

for some $(v+vI) \in N(S)$, Also since $N(S)$ is neutrosophic intra-regular therefore

$$(b+bI) = (x+xI)(b+bI)^2(y+yI),$$

for some $(x+xI)$, $(y+yI) \in N(S)$ Thus

$$\begin{aligned}
& (b+bI) \\
= \;& (b+bI)(v+vI)(b+bI) \\
= \;& (b+bI)(v+vI)(b+bI)(v+vI)(b+bI) \\
= \;& (b+bI)(v+vI)(b+bI)(v+vI)(b+bI)(v+vI)(b+bI) \\
= \;& (b+bI)(v+vI)(b+bI)(v+vI)(x+xI)(b+bI)^2(y+yI)(v+vI)(b+bI) \\
= \;& [(b+bI)(x+xI)^{/}(b+bI)][(b+bI)(y+yI)^{/}(b+bI)] \\
\in \;& N(B)N(S)N(B) \subseteq N(B)N(B) = [N(B)]^2.
\end{aligned}$$

Thus $(b+bI) \in [N(B)]^2$ therefore

$$N(B) \subseteq [N(B)]^2.$$

Hence

$$[N(B)]^2 = N(B).$$

$(ii) \Longrightarrow (iii)$

Let $N(R)$ be a neutrosophic right ideal and $N(L)$ be a neutrosophic left ideal of $N(S)$. Then $N(R) \cap N(L)$ is a neutrosophic quasi ideal of $N(S)$. Thus by (iii)

$$\begin{aligned}
N(R)N(L) &= [N(R)N(L)]^2 \\
&= [N(R) \cap N(L)][N(R) \cap N(L)] \\
&\subseteq N(R)N(L).
\end{aligned}$$

But

$$N(R)N(L) \subseteq N(R) \cap N(L),$$

therefore,

$$N(R) \cap N(L) = N(R)N(L).$$

Thus $N(S)$ is neutrosophic regular. Also

$$\begin{aligned}
N(R) \cap N(L) &= [N(R) \cap N(L)]^2 \\
&= [N(R) \cap N(L)][N(R) \cap N(L)] \\
&\subseteq N(R)N(L)
\end{aligned}$$

And for a neutrosophic semigroup $N(S)$,

$$N(R)N(L) \subseteq N(R) \cap N(L)$$

iff $N(S)$ is neutrosophic intra-regular. Hence $N(S)$ is neutrosophic intra-regular. ■

Theorem 214 *The following conditions are equivalent for a neutrosophic semigroup $N(S)$ with multiplicative identity $(1+1I)$.*

(i) $N(S)$ is both neutrosophic regular and intra-regular.
(ii) For every neutrosophic bi-ideal $N(B)$ and $N(B)$ of $N(S)$,

$$N(B_1) \cap N(B_2) = [N(B_1)N(B_2)] \cap [N(B_2)N(B_1)].$$

(iii) For every neutrosophic bi-ideal $N(B)$ and quasi-ideal $N(Q)$ of $N(S)$,

$$N(B) \cap N(Q) = [N(B)N(Q)][\cap N(Q)N(B)].$$

(iv) For every neutrosophic quasi-ideal $N(Q_1)$ and $N(Q_2)$ of $N(S)$,

$$N(Q_1) \cap N(Q_2) = [N(Q_1)N(Q_2)] \cap [N(Q_2)N(Q_1)].$$

Proof. $(i) \Longrightarrow (ii)$
Let $N(B_1)$ and $N(B_2)$ be neutrosophic bi-ideal of $N(S)$. Then $N(B_1) \cap N(B_2)$ is a neutrosophic bi-ideal of $N(S)$. Then

$$\begin{aligned} N(B_1) \cap N(B_2) &= [N(B_1) \cap N(B_2)]^2 \\ &= [N(B_1) \cap N(B_2)][N(B_1) \cap N(B_2)] \\ &\subseteq N(B_1)N(B_2). \end{aligned}$$

Similarly
$$N(B_1) \cap N(B_2) \subseteq N(B_2)N(B_1).$$
Thus
$$N(B_1) \cap N(B_2) \subseteq [N(B_1)N(B_2)] \cap [N(B_2)N(B_1)].$$
Again Since $[N(B_1)N(B_2)] \cap [N(B_2)N(B_1)]$ is a neutrosophic bi-ideal

$$\begin{aligned} & [N(B_1)N(B_2)] \cap [N(B_2)N(B_1)] \\ = \; & [N(B_1)N(B_2) \cap N(B_2)N(B_1)]^2 \\ = \; & [N(B_1)N(B_2) \cap N(B_2)N(B_1)][N(B_1)N(B_2) \cap N(B_2)N(B_1)] \\ = \; & N(B_1)N(B_2)[N(B_1)N(B_2) \cap N(B_1)N(B_2) \cdot N(B_2)N(B_1) \cap N(B_1)N(B_2) \\ & \cdot N(B_1)N(B_2) \cap N(B_2)N(B_1)]N(B_2)N(B_1) \\ \subseteq \; & N(B_1) \cap N(B_2). \end{aligned}$$

Hence
$$N(B_1) \cap N(B_2) = N(B_1)N(B_2) \cap N(B_2)N(B_1).$$
$(ii) \Longrightarrow (iii)$
Since every neutrosophic quasi ideal is a neutrosophic bi-ideal therefore by (ii),
$$N(B) \cap N(Q) = [N(B)N(Q)] \cap [N(Q)N(B)].$$
$(iii) \Longrightarrow (iv)$

Since every neutrosophic quasi-ideal is a neutrosophic bi-ideal therefore by (iii),

$$N(Q_1) \cap N(Q_2) = [N(Q_1)N(Q_2)] \cap [N(Q_2)N(Q_1)].$$

$(iv) \Longrightarrow (i)$

Let $N(R)$ and $N(L)$ be a neutrosophic right and left ideal of $N(S)$ respectively. Then by (iv),

$$N(R) \cap N(L) = [N(R)N(L)] \cap [N(L)N(R)].$$

So

$$N(R) \cap N(L) \subseteq N(R)N(L).$$

But

$$N(R)N(L) \subseteq N(R) \cap N(L).$$

Thus

$$N(R) \cap N(L) = N(R)N(L).$$

Which implies that $N(S)$ is a neutrosophic regular semigroup. Also

$$N(R) \cap N(L) \subseteq N(L)N(R).$$

This implies that $N(S)$ is an neutrosophic intra-regular semigroup. ■

6.3 Neutrosophic Prime Ideals

Definition 215 *A neutrosophic bi-ideal $N(B)$ of a neutrosophic semigroup $N(S)$ is said to be neutrosophic prime bi-ideal if*

$$N(B_1)N(B_2) \subseteq N(B),$$

implies either

$$N(B_1) \subseteq N(B),$$

or

$$N(B_2) \subseteq N(B),$$

for every neutrosophic bi-ideal $N(B_1)$ and $N(B_2)$ of $N(S)$.

Definition 216 *A neutrosophic bi-ideal $N(B)$ of a neutrosophic semigroup $N(S)$ is said to be neutrosophic strongly irreducible bi-ideal*

$$N(B_1) \cap N(B_2) \subseteq N(B)$$

implies

$$\text{either } N(B_1) \subseteq N(B) \text{ or } N(B_2) \subseteq N(B).$$

Definition 217 *A neutrosophic bi-ideal of a neutrosophic semigroup $N(S)$ is called neutrosophic semiprime bi-ideal if*

$$[N(B_1)]^2 \subseteq N(B),$$

implies

$$N(B_1) \subseteq N(B),$$

for every neutrosophic bi-ideal $N(B_1)$ of $N(S)$.

Remark 218 (i) *every neutrosophic strongly prime bi-ideal is neutrosophic prime bi-ideal.*

(ii) Every neutrosophic prime bi-ideal is neutrosophic semi prime bi-ideal.

(iii) If the set of neutrosophic bi-ideals of a neutrosophic semigroups $N(S)$ is totally ordered under inclusion then these conditions are coincide.

Proof. Let $N(B)$ be a neutrosophic bi-ideal of $N(S)$ and

$$N(B_1)N(B_2) \cap N(B_2)N(B_1) \subseteq N(B),$$

for $N(B_1)$,$N(B_2)$ of $N(S)$. Since the set of neutrosophic bi-ideals of $N(S)$ is totally ordered under inclusion therefore either

$$N(B_1)N(B_2) \subseteq N(B_2)N(B_1),$$

or

$$N(B_2)N(B_1) \subseteq N(B_1)N(B_2).$$

Suppose

$$N(B_1)N(B_2) \subseteq N(B_2)N(B_1),$$

then,

$$N(B_1)N(B_2) \cap N(B_2)N(B_1) \subseteq N(B_1)N(B_2)$$

so

$$N(B_1)N(B_2) \subseteq N(B).$$

Since $N(B)$ is a neutrosophic prime bi-ideal therefore either

$$N(B_1) \subseteq N(B) \text{ or } N(B_2) \subseteq N(B).$$

i.e $N(B)$ is strongly neutrosophic bi-ideal. Similarly if $N(B)$ is neutrosophic semiprime bi-ideal of $N(S)$ and

$$N(B_1)N(B_2) \subseteq N(B),$$

where $N(B_1)$, $N(B_2)$ are neutrosophic bi-ideal of $N(S)$. Then either

$$N(B_1) \subseteq N(B_2) \text{ or } N(B_2) \subseteq N(B_1).$$

Suppose

$$N(B_1) \subseteq N(B_2),$$

then

$$[N(B_1)]^2 \subseteq N(B_1)N(B_2) \subseteq N(B).$$

Since $N(B)$ is a neutrosophic semiprime bi-ideal, therefore

$$N(B_1) \subseteq N(B),$$

i.e $N(B)$ is a neutrosophic prime bi-ideal of $N(S)$. ■

Proposition 219 *The intersection of neutrosophic prime bi-ideals of a neutrosophic semigroup $N(S)$ is a neutrosophic semiprime bi-ideal of $N(S)$.*

Proof. Let $\{N(B_\alpha) : \alpha \in I\}$ be a collection of neutrosophic prime bi-ideal of $N(S)$. Then $\underset{\alpha \in I}{\cap} N(B_\alpha)$ is a neutrosophic prime bi-ideal of $N(S)$. If $N(B)$ is a neutrosophic bi-ideal of $N(S)$ such that

$$[N(B)]^2 \subseteq \underset{\alpha \in I}{\cap} N(B_\alpha),$$

then

$$[N(B)]^2 \subseteq N(B_\alpha)$$

for all $\alpha \in I$. Thus

$$N(B) \subseteq \underset{\alpha \in I}{\cap} N(B_\alpha).$$

Hence $\underset{\alpha \in I}{\cap} N(B_\alpha)$ is a neutrosophic semiprime bi-ideal. ■

Definition 220 *A neutrosophic bi-ideal $N(B)$ of a neutrosophic semigroup $N(S)$ is said to be a neutrosophic irreducible bi-ideal if*

$$N(B_1) \cap N(B_2) = N(B),$$

implies either

$$N(B_1) = N(B) \text{ or } N(B_2) = N(B).$$

Definition 221 *A neutrosophic bi-ideal $N(B)$ of a neutrosophic semigroup $N(S)$ is said to be a neutrosophic irreducible bi-ideal if*

$$N(B_1) \cap N(B_2) \subseteq N(B),$$

implies either

$$N(B_1) \subseteq N(B) \text{ or } N(B_2) \subseteq N(B).$$

Proposition 222 *Every neutrosophic strongly irreducible, semiprime bi-ideal of a neutrosophic semigroup $N(S)$ is a neutrosophic strongly prime bi-ideal of $N(S)$.*

Proof. Let $N(B)$ be a neutrosophic strongly irreducible, semiprime bi-ideal of a neutrosophic semigroup $N(S)$. Let $N(B_1)$ and $N(B_2)$ be any two neutrosophic bi-ideal of $N(S)$ such that

$$N(B_1)N(B_2) \cap N(B_2)N(B_1) \subseteq N(B).$$

As

$$[N(B_1) \cap N(B_2)]^2 \subseteq N(B_1)N(B_2),$$

also

$$[N(B_1) \cap N(B_2)]^2 \subseteq N(B_2)N(B_1).$$

Thus

$$\begin{array}{rcl} [N(B_1) \cap N(B_2)]^2 & \subseteq & N(B_1)N(B_2) \cap N(B_2)N(B_1) \\ & \subseteq & N(B). \end{array}$$

Since $N(B)$ is a neutrosophic semiprime bi-ideal, therefore

$$N(B_1) \cap N(B_2) \subseteq N(B).$$

As $N(B)$ is a neutrosophic strongly irreducible prime bi-ideal of $N(S)$, so either

$$N(B_1) \subseteq N(B) \text{ or } N(B_2) \subseteq N(B).$$

Thus $N(B)$ is a neutrosophic strongly prime bi-ideal of $N(S)$. ■

Example 223 *Consider the neutrosophic semigroup*
$N(S) = \{0+0I,\ 0+aI,\ 0+bI,\ a+0I,\ a+aI,\ a+bI,\ b+0I,\ b+aI,\ b+bI\}$

$\cdot$	$0+0I$	$0+aI$	$0+bI$	$a+0I$	$a+aI$	$a+bI$	$b+0I$	$b+aI$	$b+bI$
$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$0+aI$	$0+0I$	$0+aI$	$0+aI$	$0+aI$	$0+aI$	$0+aI$	$0+aI$	$0+aI$	$0+aI$
$0+bI$	$0+0I$	$0+bI$	$0+bI$	$0+bI$	$0+bI$	$0+bI$	$0+bI$	$0+bI$	$0+bI$
$a+0I$	$0+0I$	$0+0I$	$0+0I$	$a+0I$	$a+0I$	$a+0I$	$a+0I$	$a+0I$	$a+0I$
$a+aI$	$0+0I$	$0+aI$	$0+aI$	$a+0I$	$a+aI$	$a+aI$	$a+0I$	$a+aI$	$a+aI$
$a+bI$	$0+0I$	$0+bI$	$0+bI$	$a+0I$	$a+bI$	$a+bI$	$a+0I$	$a+bI$	$a+bI$
$b+0I$	$0+0I$	$0+0I$	$0+0I$	$b+0I$	$b+0I$	$b+0I$	$b+0I$	$b+0I$	$b+0I$
$b+aI$	$0+0I$	$0+aI$	$0+aI$	$b+0I$	$b+aI$	$b+aI$	$b+0I$	$b+aI$	$b+aI$
$b+bI$	$0+0I$	$0+bI$	$0+bI$	$b+0I$	$b+bI$	$b+bI$	$b+0I$	$b+bI$	$b+bI$

Then $N(S)$ is neutrosophic regular and intra-regular.
The neutrosophic right ideals of $N(S)$ are :
$\{0+0I\}$, $\{0+0I,\, a+aI\}$, $\{0+0I,\, b+bI\}$, $\{0+0I,\, a+aI,\, b+bI\}$

The neutrosophic left ideals of $N(S)$ are :
$\{0+0I\}$, $\{0+0I,\, a+aI,\, b+bI\}$

The neutrosophic ideals of $N(S)$ are :
$\{0+0I\}$, $\{0+0I,\, a+aI,\, b+bI\}$

The neutrosophic bi-ideals of $N(S)$ are :
$\{0+0I\}$, $\{0+0I,\, a+aI\}$, $\{0+0I,\, b+bI\}$, $\{0+0I,\, a+aI,\, b+bI\}$

All these neutrosophic bi-ideals are neutrosophic prime and hence semi prime. The neutrosophic prime bi-ideal $\{0+0I\}$ is not neutrosophic strongly prime bi-ideal because

$$\begin{aligned} & \{0+0I,\, a+aI\}\{0+0I,\, b+bI\} \cap \{0+0I,\, b+bI\}\{0+0I,\, a+aI\} \\ = & \{0+0I,\, a+aI\} \cap \{0+0I,\, b+bI\} \\ = & \{0+0I\} \subseteq \{0+0I\}. \end{aligned}$$

but neither $\{0+0I,\, a+aI\}$ nor $\{0+0I,\, b+bI\}$ contained in $\{0+0I\}$. Also $\{0+0I\}$ is not a neutrosophic strongly irreducible bi-ideal because

$$\begin{aligned} \{0+0I,\, a+aI\} \cap \{0+0I,\, b+bI\} &= \{0+0I\} \\ &\subseteq \{0+0I\}. \end{aligned}$$

but neither $\{0+0I,\, a+aI\}$ nor $\{0+0I,\, b+bI\}$ contained in $\{0+0I\}$. Which shows that a neutrosophic prime bi-ideal need not be a neutrosophic strongly irreducible or irreducible.

Proposition 224 *Let $N(B)$ be a neutrosophic bi-ideal of a neutrosophic semigroup $N(S)$ and $(x+xI) \in N(S)$ such that $(x+xI) \notin N(B)$. Then there exist a neutrosophic irreducible bi-ideal $N(I)$ of $N(S)$ such that $N(B) \subseteq N(I)$ and $(x+xI) \notin N(I)$.*

Proof. Let $N(D)$ be the set of all neutrosophic bi-ideal of $N(S)$ such that $N(B) \in N(D)$ and $(x+xI) \notin N(D)$.Then

$$N(D) \neq 0,$$

because $N(B) \in N(D)$. The collection $N(D)$ is totally ordered set under inclusion. As every totally ordered subset of $N(D)$ is bounded above, thus by Zorn's Lemma there exist a maximal element $(d+dI) \in N(D)$. Let $N(E)$ and $N(F)$ be two neutrosophic bi-ideal of $N(S)$ such that

$$N(I) = N(C) \cap N(D).$$

If both $N(C)$ and $N(D)$ properly contains $N(I)$ then $(x+xI) \in C$ and $(x+xI) \in D$. Hence

$$(x+xI) \in N(C) \cap N(D) = N(I),$$

this contradiction the fact that $(x+xI) \notin N(I)$. Hence

$$N(I) = N(C) \text{ or } N(I) = N(D).$$

■

Theorem 225 *Let $N(S)$ be a neutrosophic semigroup, then the following conditions are equivalent:*

(i) $N(S)$ is neutrosophic regular and intra regular.
(ii) For every neutrosophic bi-ideal $N(B)$ of $N(S)$,

$$[N(B)]^2 = N(B)$$

(iii) For every neutrosophic bi-ideal $N(B_1)$ and $N(B_2)$,

$$N(B_1) \cap N(B_2) = [N(B_1)N(B_2)] \cap [N(B_2)N(B_1)]$$

(iv) Every neutrosophic bi-ideal of $N(S)$ is neutrosophic semiprime.
(v) every neutrosophic bi-ideal of $N(S)$ is the intersection of irreducible neutrosophic semiprime bi-ideal of $N(S)$ which lies in it.
Proof. $(i) \Longrightarrow (ii)$
Let $N(B)$ be a neutrosophic bi-ideal of $N(S)$. As $N(B)$ is a neutrosophic subsemi group of $N(S)$, so $[N(B)]^2 \subseteq N(B)$.
Let $(b+bI) \in N(B)$, since $N(S)$ is neutrosophic regular and intra-regular semigroup so there exist some $(x+xI)$, $(y+yI)$, $(z+zI) \in N(S)$ such that

$$(b+bI) = (b+bI)(x+xI)(b+bI),$$

and

$$(b+bI) = (y+yI)(b+bI)^2(z+zI),$$

respectively. Now

$$\begin{aligned}
(b+bI) &= (b+bI)(x+xI)(b+bI) \\
&= (b+bI)(x+xI)(b+bI)(x+xI)(b+bI) \\
&= (b+bI)(x+xI)(y+yI)(b+bI)^2(z+zI)(x+xI)(b+bI) \\
&= [(b+bI)[(x+xI)(y+yI)](b+bI)][(b+bI)[(z+zI)(x+xI)](b+bI)] \\
&\in [N(B)N(S)N(B)][N(B)N(S)N(B)] \\
&\subseteq N(B)N(B) = [N(B)]^2.
\end{aligned}$$

i.e

$$N(B) \subseteq [N(B)]^2.$$

Hence
$$[N(B)]^2 = N(B).$$

$(ii) \Longrightarrow (iii)$

Let $N(B_1)$ and $N(B_2)$ be any neutrosophic bi-ideals of $N(S)$, then by (ii)

$$\begin{aligned} N(B_1) \cap N(B_2) &= [N(B_1) \cap N(B_2)]^2 \\ &= [N(B_1) \cap N(B_2)][N(B_1) \cap N(B_2)] \\ &\subseteq N(B_1)N(B_2). \end{aligned}$$

Similarly
$$N(B_1) \cap N(B_2) \subseteq N(B_2)N(B_1).$$

Thus
$$N(B_1) \cap N(B_2) = [N(B_1)N(B_2)] \cap [N(B_2)N(B_1)].$$

Since the product and intersection of neutrosophic bi-ideals is also a neutrosophic bi-ideal, so $N(B_1)N(B_2)$ and $N(B_1) \cap N(B_2)$ are neutrosophic bi-ideals. Thus

$$\begin{aligned} & N(B_1)N(B_2) \cap N(B_2)N(B_1) \\ = \; & [N(B_1)N(B_2) \cap N(B_2)N(B_1)][N(B_1)N(B_2) \cap N(B_2)N(B_1)] \\ \subseteq \; & [N(B_1)N(B_2)][N(B_2)N(B_1)] \\ \subseteq \; & N(B_1)N(S)N(B_1) \subseteq N(B_1). \end{aligned}$$

Similarly
$$N(B_1)N(B_2) \cap N(B_2)N(B_1) \subseteq N(B_1).$$

Thus
$$N(B_1)N(B_2) \cap N(B_2)N(B_1) \subseteq N(B_1) \cap N(B_2).$$

Hence (I) and (II) gives us,

$$N(B_1) \cap N(B_2) = N(B_1)N(B_2) \cap N(B_2)N(B_1).$$

$(iii) \Longrightarrow (iv)$

Let $N(B_1)$ and $N(B_2)$ be the neutrosophic bi-ideals of $N(S)$ such that

$$[N(B)]^2 \subseteq N(B).$$

By (iii)

$$\begin{aligned} N(B_1) &= N(B_1) \cap N(B_1) \\ &= N(B_1)N(B_1) \cap N(B_1)N(B_1) \\ &= [N(B_1)]^2. \end{aligned}$$

Thus
$$N(B_1) \subseteq N(B).$$

Hence every neutrosophic bi-ideal of $N(S)$ is neutrosophic semiprime.

$(iv) \Longrightarrow (v)$

Let $N(B)$ be a neutrosophic bi-ideal of $N(S)$, then $N(B)$ is contained in the intersection of all irreducible neutrosophic bi-ideal of $N(S)$ which contain $N(B)$. So by previous proposition, if $(b + bI) \notin N(B)$ then there exist irreducible neutrosophic bi-ideal $N(I)$ of $N(S)$ such that

$$N(B) \subseteq N(I),$$

and $(b + bI) \notin N(I)$. Thus $N(B)$ is the intersection of all irreducible neutrosophic bi-ideal which contain it. By (iv) every neutrosophic bi-ideal is neutrosophic semiprime, so every neutrosophic bi-ideal is the intersection of irreducible neutrosophic semiprime bi-ideal of $N(S)$ which contain in it.

$(v) \Longrightarrow (ii)$

Let $N(B)$ be a neutrosophic bi-ideal of $N(S)$. If

$$[N(B)]^2 = N(S),$$

then

$$[N(B)]^2 = N(B).$$

i.e $N(B)$ is idempotent. If

$$[N(B)]^2 \neq N(S),$$

then $[N(B)]^2$ is a proper bi-ideal of $N(S)$, thus by (v)

$[N(B)]^2 = \underset{\alpha}{\cap}\{N(B_\alpha) : N(B_\alpha)$ is neutrosophic irreducible semiprime bi-ideal$\}$

This implies

$$N(B) \subseteq N(B_\alpha),$$

for all α. Since each $N(B_\alpha)$ is a neutrosophic semi prime bi-ideal therefore

$$N(B) \subseteq N(B_\alpha),$$

for all α so

$$N(B) \subseteq \underset{\alpha}{\cap} N(B_\alpha) = [N(B)]^2.$$

Hence every bi-ideal in $N(S)$ is neutrosophic idempotent.

$(ii) \Longrightarrow (i)$

Let $N(R)$ and $N(L)$ be the neutrosophic right and left ideals of $N(S)$. Then $N(R) \cap N(L)$ is a neutrosophic quasi ideal and so neutrosophic bi-ideal of $N(S)$. By (ii)

$$\begin{aligned} N(R) \cap N(L) &= [N(R) \cap N(L)][N(R) \cap N(L)] \\ &\subseteq N(R)N(L). \end{aligned}$$

Also

$$N(R)N(L) \subseteq N(R),$$

and

$$N(R)N(L) \subseteq N(L).$$

Thus

$$N(R)N(L) \subseteq N(R) \cap N(L).$$

So

$$N(R) \cap N(L) = N(R)N(L).$$

Hence by theorem 1, $N(S)$ is neutrosophic regular.

Similarly

$$N(R) \cap N(L) = [N(R) \cap N(L)][N(R) \cap N(L)].$$

And

$$N(R) \cap N(L) = [N(R) \cap N(L)][N(R) \cap N(L)]$$

iff $N(S)$ is neutrosophic intra-regular. Hence $N(S)$ is neutrosophic intra-regular. ■

Proposition 226 *Let $N(B)$ be a neutrosophic bi-ideal of a neutrosophic regular and intra-regular semigroup of $N(S)$, then the following conditions are equivalent:*

(i) $N(B)$ is neutrosophic strongly irreducible.
(ii) $N(B)$ is neutrosophic strongly prime.

Proof. $(i) \Longrightarrow (ii)$

Let $N(B_1)$ and $N(B_2)$ be two neutrosophic bi-ideals of $N(S)$ such that,

$$N(B_1)N(B_2) \cap N(B_2)N(B_1) \subseteq N(B).$$

By theorem 5,

$$N(B_1) \cap N(B_2) = N(B_1)N(B_2) \cap N(B_2)N(B_1),$$

thus

$$N(B_1) \cap N(B_2) \subseteq N(B).$$

By (i)

$$\text{either } N(B_1) \subseteq N(B) \text{ or } N(B_2) \subseteq N(B).$$

Hence $N(B)$ is neutrosophic strongly prime bi-ideal.

$(ii) \Longrightarrow (i)$

Let $N(B_1)$ and $N(B_2)$ be two neutrosophic bi-ideals of $N(S)$ such that,

$$N(B_1) \cap N(B_2) \subseteq N(B).$$

By definition,

$$N(B_1) \cap N(B_2) \subseteq N(B_1)N(B_2) \cap N(B_2)N(B_1),$$

thus

$$N(B_1)N(B_2) \cap N(B_2)N(B_1) \subseteq N(B).$$

Thus by (ii),

$$\text{either } N(B_1) \subseteq N(B) \text{ or } N(B_2) \subseteq N(B).$$

Hence $N(B)$ is neutrosophic strongly irreducible bi-ideal of $N(S)$. ■

Theorem 227 *Every neutrosophic bi-ideal of a neutrosophic semigroup $N(S)$ is neutrosophic strongly prime iff $N(S)$ is neutrosophic regular, intra-regular and the set of neutrosophic bi-ideals of $N(S)$ is totally ordered under inclusion*

Proof. Let every neutrosophic bi-ideal of $N(S)$ is strongly prime then each neutrosophic bi-ideal of $N(S)$ is neutrosophic semiprime, thus by theorem 5, $N(S)$ is neutrosophic regular and intra-regular. Now consider $N(B_1)$ and $N(B_2)$ be any two neutrosophic bi-ideal of $N(S)$, so by theorem 5,

$$N(B_1) \cap N(B_2) = N(B_1)N(B_2) \cap N(B_2)N(B_1).$$

As every neutrosophic bi-ideal is strongly neutrosophic prime, so $N(B_1) \cap N(B_2)$ is strongly neutrosophic prime. Thus

$$\text{either } N(B_1) \subseteq N(B_1) \cap N(B_2) \text{ or } N(B_2) \subseteq N(B_1) \cap N(B_2).$$

If

$$N(B_1) \subseteq N(B_1) \cap N(B_2),$$

then

$$N(B_1) \subseteq N(B_2).$$

And if

$$N(B_2) \subseteq N(B_1) \cap N(B_1),$$

then

$$N(B_2) \subseteq N(B_1).$$

Conversely suppose that $N(S)$ is neutrosophic regular, intra-regular and the set of neutrosophic bi-ideals of $N(S)$ is totally ordered under inclusion. Now let $N(B)$ be an arbitrary neutrosophic bi-ideal of $N(S)$ and $N(B_1)$,and $N(B_2)$ be neutrosophic bi-ideals of $N(S)$ such that

$$N(B_1)N(B_2) \cap N(B_2)N(B_1) \subseteq N(B).$$

As $N(S)$ is neutrosophic regular and intra-regular, thus by theorem 5,

$$N(B_1) \cap N(B_2) = N(B_1)N(B_2) \cap N(B_2)N(B_1).$$

Also

$$N(B_1)N(B_2) \cap N(B_2)N(B_1) \subseteq N(B),$$

implies that

$$N(B_1) \cap N(B_2) \subseteq N(B),$$

since the set of neutrosophic bi-ideal of $N(S)$ is totally ordered under inclusion , so

$$\text{either } N(B_1) \subseteq N(B_2) \text{ or } N(B_2) \subseteq N(B_1),$$

i.e.

$$N(B_1) \cap N(B_2) = N(B_1) \text{ or } N(B_1) \cap N(B_2) = N(B_2).$$

Thus

$$\text{either } N(B_1) \subseteq N(B) \text{ or } N(B_2) \subseteq N(B).$$

Hence $N(B)$ is strongly neutrosophic prime. ■

Theorem 228 *Let the set of neutrosophic bi-ideals of a neutrosophic semigroup $N(S)$ is totally ordered, then $N(S)$ is both neutrosophic regular and intra-regular iff every neutrosophic bi-ideal of $N(S)$ is neutrosophic prime.*

Proof. Let $N(S)$ be both neutrosophic regular and intra-regular. Suppose $N(B)$ be any neutrosophic bi-ideal of $N(S)$ and $N(B_1)$, $N(B_2)$ are bi-ideals of $N(S)$ such that

$$N(B_1)N(B_2) \subseteq N(B).$$

Since the set of neutrosophic bi-ideals of $N(S)$ is totally ordered therefore

$$\text{either } N(B_1) \subseteq N(B_2) \text{ or } N(B_2) \subseteq N(B_1).$$

If

$$N(B_1) \subseteq N(B_2),$$

then,

$$[N(B_1)]^2 \subseteq N(B_1)N(B_2) \subseteq N(B).$$

By theorem 5, $N(B)$ neutrosophic semiprime, so

$$N(B_1) \subseteq N(B).$$

$N(B)$ is a neutrosophic semiprime bi-ideal of $N(S)$.

Conversely suppose that every neutrosophic bi-ideal of $N(S)$ is neutrosophic prime. Since the set of neutrosophic bi-ideal of $N(S)$ is totally ordered, so by remark 3(iii), the conditions of neutrosophic prime and strongly prime coincides. Hence by theorem 6,,$N(S)$ is neutrosophic regular and intra-regular. ■

Theorem 229 *Let $N(S)$ be a neutrosophic semigroup, then the following conditions are equivalent:*

(i) The set of neutrosophic bi-ideals of $N(S)$ is totally ordered under inclusion.

(ii) Every neutrosophic bi-ideal of $N(S)$ is strongly neutrosophic irreducible.

(iii) Every neutrosophic bi-ideal of $N(S)$ is neutrosophic irreducible.

Proof. $(i) \Longrightarrow (ii)$

Suppose $N(B)$ be a neutrosophic bi-ideal of $N(S)$ and $N(B_1)$, $N(B_2)$ be two bi-ideals of $N(S)$ such that

$$N(B_1) \cap N(B_2) \subseteq N(B),$$

since the set of bi-ideals of $N(S)$ is totally ordered therefore,

$$\text{either } N(B_1) \subseteq N(B_2) \text{ or } N(B_2) \subseteq N(B_1).$$

Thus

$$\text{either } N(B_1) \cap N(B_2) = N(B_1) \text{ or } N(B_1) \cap N(B_2) = N(B_2).$$

Hence

$$N(B_1) \cap N(B_2) = N(B),$$

which further implies that,

$$\text{either } N(B_1) \subseteq N(B) \text{ or } N(B_2) \subseteq N(B).$$

Consequently $N(B)$ is neutrosophic strongly irreducible bi-ideal of $N(S)$.

$(ii) \Longrightarrow (iii)$

Suppose $N(B)$ be a neutrosophic bi-ideal of $N(S)$ and $N(B_1)$, $N(B_2)$ be two bi-ideals of $N(S)$ such that

$$N(B_1) \cap N(B_2) = N(B),$$

thus,

$$N(B) \subseteq N(B_1) and N(B) \subseteq N(B_2).$$

And by (ii)

$$\text{either } N(B_1) \subseteq N(B) \text{ or } N(B_2) \subseteq N(B).$$

So,

$$\text{either } N(B_1) = N(B) \text{ or } N(B_2) = N(B).$$

Hence $N(B)$ is an irreducible neutrosophic bi-ideal.

$(iii) \Longrightarrow (i)$

Suppose $N(B_1)$ and $N(B_2)$ be any two neutrosophic bi-ideals of $N(S)$, then $N(B_1) \cap N(B_2)$ is a neutrosophic bi-ideal of $N(S)$. Also

$$N(B_1) \cap N(B_2) = N(B_1) \cap N(B_2),$$

so by (iii),

$$N(B_1) = N(B_1) \cap N(B_2) \text{ or } N(B_2) = N(B_1) \cap N(B_2)$$

i.e.

$$N(B_1) \subseteq N(B_2) \text{ or } N(B_2) \subseteq N(B_1).$$

Hence the set of bi-ideals of $N(S)$ is totally ordered. ■

Example 230 *Consider the neutrosophic semigroup*

$N(S) = \{0 + 0I,\ 0 + 1I,\ 0 + aI,\ 0 + bI,\ 1 + 0I,\ 1 + 1I,\ 1 + aI,\ 1 + bI,$
$a + 0I,\ a + 1I,\ a + aI,\ a + bI,\ b + 0I,\ b + 1I,\ b + aI,\ b + bI\}$

$\cdot$	$0+0I$	$0+eI$	$0+aI$	$0+bI$	$e+0I$	$e+eI$	$e+aI$	$e+bI$	$a+0I$	$a+eI$	$a+aI$	$a+bI$	$b+0I$	$b+eI$	$b+aI$	$b+bI$
$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$0+eI$	$0+0I$	$0+eI$	$0+aI$	$0+0I$	$0+0I$	$0+eI$	$0+aI$	$0+0I$	$0+0I$	$0+eI$	$0+aI$	$0+0I$	$0+0I$	$0+eI$	$0+aI$	$0+0I$
$0+aI$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$0+bI$	$0+0I$	$0+bI$	$0+0I$	$0+0I$	$0+0I$	$0+bI$	$0+0I$	$0+0I$	$0+0I$	$0+bI$	$0+0I$	$0+0I$	$0+0I$	$0+bI$	$0+0I$	$0+0I$
$e+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$e+0I$	$e+0I$	$e+0I$	$e+0I$	$a+0I$	$a+0I$	$a+0I$	$a+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$e+eI$	$0+0I$	$0+eI$	$0+aI$	$0+0I$	$e+0I$	$e+eI$	$e+aI$	$e+0I$	$a+0I$	$a+eI$	$a+aI$	$a+0I$	$0+0I$	$0+eI$	$0+aI$	$0+0I$
$e+aI$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$e+0I$	$e+0I$	$e+aI$	$e+0I$	$a+0I$	$a+0I$	$a+aI$	$a+0I$	$0+0I$	$0+0I$	$0+aI$	$0+0I$
$e+bI$	$0+0I$	$0+bI$	$0+0I$	$0+0I$	$e+0I$	$e+bI$	$e+0I$	$e+bI$	$a+0I$	$a+bI$	$a+0I$	$a+bI$	$0+0I$	$0+bI$	$0+0I$	$0+bI$
$a+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$a+0I$	$a+0I$	$a+0I$	$a+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$a+eI$	$0+0I$	$0+0I$	$0+aI$	$0+0I$	$a+0I$	$a+eI$	$a+aI$	$a+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$a+aI$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$a+bI$	$0+0I$	$0+bI$	$0+0I$	$0+bI$	$0+0I$	$0+bI$	$0+0I$	$0+bI$	$0+0I$	$0+bI$	$0+0I$	$0+bI$	$0+0I$	$0+bI$	$0+0I$	$0+bI$
$b+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$b+eI$	$0+0I$	$0+eI$	$0+aI$	$0+0I$	$b+0I$	$b+eI$	$b+aI$	$b+0I$	$0+0I$	$0+eI$	$0+aI$	$0+0I$	$0+0I$	$0+eI$	$0+aI$	$0+0I$
$b+aI$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$b+0I$	$b+0I$	$b+0I$	$b+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$	$0+0I$
$b+bI$	$0+0I$	$0+bI$	$0+0I$	$0+0I$	$b+0I$	$b+bI$	$b+0I$	$b+0I$	$0+0I$	$0+bI$	$0+0I$	$0+0I$	$0+0I$	$0+bI$	$0+0I$	$0+0I$

Then N(S) is both regular and intra-regular.

The neutrosophic right ideal of N(S) are :

$\{0 + 0I\}$, $\{0 + 0I,\ a + aI\}$, $\{0 + 0I,\ b + bI\}$, $\{0 + 0I,\ a + aI,\ b + bI\}$,
$\{0 + 0I,\ e + eI,\ a + aI\}$, $\{0 + 0I,\ e + eI,\ a + aI,\ b + bI\}$

The neutrosophic left ideal of N(S) are :

$\{0 + 0I\}$, $\{0 + 0I,\ a + aI\}$, $\{0 + 0I,\ b + bI\}$, $\{0 + 0I,\ a + aI,\ b + bI\}$,
$\{0 + 0I,\ e + eI,\ a + aI\}$, $\{0 + 0I,\ e + eI,\ a + aI,\ b + bI\}$

The neutrosophic bi-ideals of N(S) are :

$\{0+0I\}$, $\{0+0I,\ a+aI\}$, $\{0+0I,\ b+bI\}$, $\{0+0I,\ e+eI\}$, $\{0+0I,\ a+aI,\ b+bI\}$, $\{0+0I,\ a+aI,\ e+eI\}$, $\{0+0I,\ e+eI,\ b+bI\}$, $\{0+0I,\ e+eI,\ a+aI,\ b+bI\}$.

$\{0+0I,\ e+eI\}$ is not a neutrosophic symmetric bi-ideal, because,

$$\begin{aligned}(e+eI)(a+aI)(e+eI) &= (a+aI)(e+eI)\\ &= (0+0I)\\ &\in \{0+0I,\ e+eI\}.\end{aligned}$$

but

$$\begin{aligned}(e+eI)(e+eI)(a+aI) &= (e+eI)(a+aI)\\ &= (a+aI)\\ &\notin \{0+0I,\ e+eI\}.\end{aligned}$$

Proposition 231 *Every neutrosophic symmetric bi-ideal $N(B)$ of a neutrosophic regular semigroup $N(S)$ is a neutrosophic right ideal.*

Proof. Suppose $N(B)$ be a neutrosophic symmetric bi-ideal of $N(S)$ and $(b+bI)\in N(B)$, then there exist $(s+sI)\in N(S)$, such that

$$(b+bI)=(b+bI)(s+sI)(b+bI).$$

Now for all $(s^{'}+s^{'}I)\in N(S)$,

$$\begin{aligned}(b+bI)[(s^{'}+s^{'}I)(s+sI)](b+bI) &\in N(B)N(S)N(B)\\ &\subseteq N(B).\end{aligned}$$

Since $N(B)$ is a neutrosophic symmetric bi-ideal, therefore $(b+bI)(s+sI)(b+bI)(s^{'}+s^{'}I)\in N(B)$. But

$$(b+bI)(s^{'}+s^{'}I)=(b+bI)(s+sI)(b+bI)(s^{'}+s^{'}I),$$

thus $(b+bI)(s^{'}+s^{'}I)\in N(B)$. Hence

$$N(B)N(S)\subseteq N(B),$$

i.e $N(B)$ is a neutrosophic right ideal ■

Corollary 232 *Let $N(S)$ be a neutrosophic regular semigroup with multiplicative identity, then every neutrosophic symmetric bi-ideal of $N(S)$ is a neutrosophic ideal of $N(S)$.*

Proof. Let $N(B)$ be a neutrosophic symmetric bi-ideal of a neutrosophic regular semigroup $N(S)$. Since every neutrosophic symmetric bi-ideal $N(B)$ of a neutrosophic regular group $N(S)$ is neutrosophic right ideal, so $N(B)$

is neutrosophic right ideal of $N(S)$. Now let $(b+bI) \in N(B)$ and $(s+sI) \in N(S)$, then

$$\begin{aligned}(b+bI)(s+sI) &= (e+eI)(b+bI)(s+sI) \\ &= (e+eI)(s+sI)(b+bI) \\ &= (s+sI)(b+bI).\end{aligned}$$

Similarly

$$\begin{aligned}(s+sI)(b+bI) &= (e+eI)(s+sI)(b+bI) \\ &= (e+eI)(b+bI)(s+sI) \\ &= (b+bI)(s+sI).\end{aligned}$$

Thus

$$N(B)N(S) = N(S)N(B),$$

so

$$N(S)N(B) \subseteq N(B),$$

i.e $N(B)$ is a neutrosophic left ideal of $N(S)$. Hence $N(B)$ is a neutrosophic two-sided ideal, i.e $N(B)$ is a neutrosophic ideal of $N(S)$. ■

Definition 233 *A neutrosophic bi-ideal $N(B)$ of a neutrosophic semigroup $N(S)$ is said to be neutrosophic completely prime if for any $a+aI$, $b+bI \in N(S)$, $(a+aI)(b+bI) \in N(B)$ implies either $(a+aI) \in N(B)$ or $(b+bI) \in N(B)$.*

Lemma 234 *A completely neutrosophic prime bi-ideal of $N(S)$ is neutrosophic prime bi-ideal of $N(S)$.*

Proof. Let $N(B)$ be a neutrosophic completely prime bi-ideal of $N(S)$. Let $N(B_1)$ and $N(B_2)$ be neutrosophic bi-ideal of $N(S)$ such that

$$N(B_1)N(B_2) \subseteq N(B).$$

Let

$$N(B_1) \subsetneq N(B),$$

so there exist $(a+aI) \in N(B_1)$ such that $(a+aI) \notin N(B)$. Now for each $(b+bI) \in N(B_2)$,

$$\begin{aligned}(a+aI)(b+bI) &\in N(B_1)N(B_2) \\ &\subseteq N(B).\end{aligned}$$

Since $N(B)$ is neutrosophic completely prime so either $(a+aI) \in N(B)$ or $(b+bI) \in N(B)$. But $(a+aI) \notin N(B)$ thus $(b+bI) \in N(B)$ which further implies that

$$N(B_2) \subseteq N(B).$$

Hence $N(B)$ is a neutrosophic prime bi-ideal of $N(S)$. ■

Theorem 235 *Let N(S) be a neutrosophic regular semigroup then,*

(i) The intersection of neutrosophic symmetric bi-ideals of $N(S)$ is a symmetric bi-ideal of $N(S)$.

(ii) The union of neutrosophic symmetric bi-ideals of $N(S)$ is a symmetric bi-ideal of $N(S)$.

Proof. (i) Let $\{N(B_\alpha) : \alpha \in I\}$ be a collection of neutrosophic symmetric bi-ideals of $N(S)$. Then $\underset{\alpha\in I}{\cap} N(B_\alpha)$ is a neutrosophic bi-ideal of $N(S)$. Let $(x+xI)(y+yI)(z+zI) \in \underset{\alpha\in I}{\cap} N(B_\alpha)$ then $(x+xI)(y+yI)(z+zI) \in N(B_\alpha)$ for any $\alpha \in I$. Since $N(B_\alpha)$ are neutrosophic symmetric bi-ideal of $N(S)$, so $(x + xI)(y + yI)(z + zI) \in N(B_\alpha)$ for any $\alpha \in I$. Hence $(x + xI)(y + yI)(z + zI) \in \underset{\alpha\in I}{\cap} N(B_\alpha)$ i.e. $\underset{\alpha\in I}{\cap} N(B_\alpha)$ is neutrosophic symmetric bi-ideal of $N(S)$.

(ii) Let $\{N(B_\alpha) : \alpha \in I\}$ be a collection of neutrosophic symmetric bi-ideals of $N(S)$. Then by p335 each $N(B_\alpha)$ is a neutrosophic right ideal of $N(S)$, so $\underset{\alpha\in I}{\cup} N(B_\alpha)$ is a neutrosophic right ideal of $N(S)$. Thus $\underset{\alpha\in I}{\cup} N(B_\alpha)$ is a neutrosophic bi-ideal of $N(S)$. If $(x+xI)(y+yI)(z+zI) \in \underset{\alpha\in I}{\cup} N(B_\alpha)$, then there exist some $\alpha \in I$ such that $(x + xI)(y + yI)(z + zI) \in N(B_\alpha)$. Since $N(B_\alpha)$ is neutrosophic symmetric bi-ideal of $N(S)$, so $(x + xI)(y + yI)(z + zI) \in N(B_\alpha)$. Thus $(x + xI)(y + yI)(z + zI) \in \underset{\alpha\in I}{\cup} N(B_\alpha)$. i.e. $\underset{\alpha\in I}{\cup} N(B_\alpha)$ is neutrosophic symmetric bi-ideal of $N(S)$. ■

7

Neutrosophic Left Almost Rings

In this chapter we introduced neutrosophic ideals in neutrosophic left almost rings. Further we characterized these ideals.

7.1 Neutrosophic Ideals

Definition 236 *A groupoid $(N(S), *)$ is called a Neutrosophic left almost semigroup; abbreviated as an LA-semigroup; if it satisfies left invertive law, that is $\{(x+xI)*(y+yI)\}*(z+zI) = \{(z+zI)*(y+yI)\}*(x+xI)$ for all $(x+xI)$, $(y+yI)$, $(z+zI) \in N(S)$.*

To understand the above concept we give an example. The following example has been taken from the paper[8].

Example 237 *Let $N(Z)$ denote the set of integers. Let the binary operation "$*$" in $N(Z)$ is defined in the following manner: $(l+lI)*(m+mI) = (m+mI)-(l+lI)$ for all $l+lI$, $m+mI \in z$ where "$-$" denotes the ordinary subtraction. Then $(N(Z), *)$ is an neutrosophic LA-semigroup. Let us present some properties of LA-semigroups which have been taken from [2] and will be used later.*

Lemma 238 *Let $(N(S), *)$ be an Neutrosophic LA-semigroup. Then the following law holds. for all $(x+xI)$, $(y+yI)$, $(z+zI)$, $w+wI \in N(S)$. $\{(x+xI)*(y+yI)\}*\{(z+zI)*(w+wI)\} = \{(x=xI)*(z+zI)\}*\{(y+yI)*(w+wI)\}$. The above law is called medial law.*

Lemma 239 *Let $(N(S), *)$ be an Neutrosophic LA-semigroup with left identity $(e+eI)$. Then the following law holds for all $(x+xI)$, $(y+yI)$, $(z+zI)$, $(w+wI) \in N(S)$.*

$$\{(x+xI)*(y+yI)\}*\{(z+zI)*(w+wI)\} = \{(w+wI)*(y+yI)\}*\{(z+zI)*(x+xI)\}$$

Lemma 240 *if $(N(S), *)$ is an neutrosophic LA-semigroup with left identity $(e+eI)$ then, for all $x+xI$, $y+yI$, $z+zI \in N(S)$.*

$$(x+xI)*\{(y+yI)*(z+zI)\} = (y+yI)*\{(x+xI)*(z+zI)\}$$

Definition 241 *A groupoid $N(G)$ with the binary operation "$*$" is said to be an neutrosophic LA-group if the following conditions are satisfied:*

(i) There exists an element $(e+eI) \in N(G)$ such that $(e+eI)*(a+aI) = (a+aI)$ for all $(a+aI) \in N(G)$,

(ii) For $(a+aI) \in N(G)$ there exists $(a+aI)' \in N(G)$ such that $(a+aI)' * (a+aI) = (a+aI) * (a+aI)' = (e+eI)$, i.e. left inverse of each element of $N(G)$ exists in $N(G)$,

(iii) Left invertive law holds in $N(G)$.

Definition 242 *Let $N(G)$ be an Neutrosophic LA-Semigroup and let $\phi \neq N(H) \subseteq N(G)$. Then $N(H)$ is called an Neutrosophic LA-subgroup of $N(G)$ if $N(H)$ itself is an Neutrosophic LA-group under the same binary operation as defined in $N(G)$. If $N(H)$ is an LA-subgroup of $N(G)$, then we write $N(H) \leq N(G)$.*

Theorem 243 *Let $N(G)$ be an Neutrosophic LA-group and let $\phi = N(H) \subseteq N(G)$. Then $N(H)$ is an Neutrosophic LA-subgroup of $N(G)$ if and only if $(a+aI)(b+bI)\prime \in N(H)$ for all $(a+aI)$, $(b+bI) \in N(H)$.*

Theorem 244 *Intersection of any family of Neutrosophic LA-subgroup of an Neutrosophic LA-group is again an LA-subgroup.*

Definition 245 *A Neutrosophic left almost ring is a non-empty set $N(R)$ together with two binary operations "+" and "." satisfying the following:*

(i) $(N(R), +)$ is an Neutrosophic LA-group,

(ii) $(N(R), \cdot)$ is an Neutrosophic LA-semigroup,

(iii) Both left and right distributive laws hold. That is for all $(l+lI)$, $(m+mI)$, $(n+nI) \in N(R)$

$$\begin{aligned} (l+lI)\cdot\{(m+mI)+(n+nI)\} &= (l+lI)\cdot(m+mI)+(l+lI)\cdot(n+nI) \text{ and} \\ \{(l+lI)+(m+mI)\}\cdot(n+nI) &= (l+lI)\cdot(n+nI)+(m+mI)\cdot(n+nI). \end{aligned}$$

Example 246 *Let $(N(R), +, \cdot)$ be a commutative ring, then we can always get an Neutrosophic LA-ring $(N(R), +, .)$ by defining for $(m+mI)$, $(n+nI) \in N(R)$, $(m+mI)+(n+nI) = (n+nI)-(m+mI)$ and $(m+mI)\cdot(n+nI)$ is the same as in the ring $(N(R), +, \cdot)$.*

Definition 247 *Let $(N(R), +, \cdot)$ be an Neutrosophic LA-ring. if $N(B)$ is non empty subset of $N(R)$ and $N(B)$ is itself an LA-ring under the same binary operation as defined in $N(R)$, then $N(B)$ is called an Neutrosophic LA-ring of $N(R)$.*

Let us describe some properties which have been taken from [9]. The following result gives us equivalent conditions for Neutrosophic LA-subring.

Lemma 248 *if $N(B)$ is non empty subset of an LA-ring $(N(R), +, \cdot)$, then $N(B)$ is an LA-subring of $N(R)$ if and only if $(a+aI)-(b+bI)$, $(a+aI)\cdot(b+bI) \in N(B)$ for all $(a+aI)$, $(b+bI) \in N(B)$.*

Theorem 249 *The intersection of any family of Neutrosophic LA-subring of an Neutrosophic LA-ring $N(R)$ is again an Neutrosophic LA-subring.*

It follows from the above theorem that if $N(A)$ and $N(B)$ are two Neutrosophic LA-ring $N(R)$, then the intersection of $N(A)$ and $N(B)$ is again an Neutrosophic LA-subring of $N(R)$. we are now going to define the second substructure of an Neutrosophic LA-ring $N(R)$ which is called an ideal. The following definition has been taken from the source [9].

Definition 250 *Let $(N(R), +, \cdot)$ be an Neutrosophic LA-ring and $N(I)$ an Neutrosophic LA-subring of $N(R)$. Then $N(I)$ is said to be a left ideal of $N(R)$ if $N(R)N(I) \subseteq N(I)$ and $N(I)$ is called a right ideal of $N(R)$ if $N(I)N(R) \subseteq N(I).N(I)$ is said to be a two sided ideal or simply an ideal of $N(R)$ if it is both left and right ideal of $N(R)$. Let us present some properties which have been taken from [1].*

Theorem 251 *Let $(N(R), +, \cdot)$ be an Neutrosophic LA-ring with left identity $(e + eI)$, then every right ideal is a left ideal.*

Theorem 252 *Intersection of two left(right) ideals of an Neutrosophic LA-ring is again a left(right) ideal.*

Corollary 253 *The intersection of any family of left(right) ideals of an Neutrosophic LA-ring is a left(right) ideal.*

We are now going to define sum of two ideals of an Neutrosophic LA-ring. The following definition has been taken from [1].

Definition 254 *Let $(N(R), +, \cdot)$ be an Neutrosophic LA-ring. Let $N(I)$, $N(J)$ be ideals of $N(R)$. The sum of $N(I)$ and $N(J)$ is defined as:*

$$N(I) + N(J) = \{(i + iI) + (j + jI) : i + iI \in N(I) \text{ and } j + jI \in N(J)\}.$$

It follows from the above definition that $N(I) + N(J) \subseteq N(R)$. Let us describe some properties, which have been taken from [1].

Theorem 255 *Let $(N(R), +, \cdot)$ be an Neutrosophic LA-ring. Then sum of two left(right) ideals of $N(R)$ is again a left(right) ideals of $N(R)$.*

Corollary 256 *The sum of one left and one right ideal of an Neutrosophic LA-ring with left identity $(e+eI)$ is a left ideal. We are now going to define product of two ideals of an Neutrosophic LA-ring. The definition has been taken from [1].*

Definition 257 *Let $(N(R), +, \cdot)$ be an Neutrosophic LA-ring and $N(I)$ and $N(J)$ be two ideals of $N(R)$. Then the product of $N(I)$ and $N(J)$ is* *denoted by $N(I)N(J)$ and is defined as*

$$
\begin{aligned}
N(I)N(J) &= \left\{ \sum_{i=1}^{n} (r+rI)_i (s+sI)_i : (r+rI)_i \in N(I) \& (s+sI)_i \in N(j) \right\} \\
&= \{ (...((((r+rI)_1(s+sI)_1 + (r+rI)_2(s+sI)_2) + \\
&\quad (r+rI)_3(s+sI)_3) + ... + (r+rI)_{n-1}(s+sI)_{n-1}) + (r+rI)_n(s+sI)_n)
\end{aligned}
$$

and $(r+rI)_i \in N(I)$, $(s+sI)_i \in N(J)\}$. Let us describe some properties.

Theorem 258 *Let $(N(R), +, \cdot)$ be an neutrosophic LA-ring with left identity $(e+eI)$. Then the product of two left(right) ideal is again a left(right) ideal of $N(R)$.*

The following result is a direct consequence of the above theorem.

Corollary 259 *if $N(I)$ is a right ideal of an LA-ring $N(R)$ with left identity $(e+eI)$ then $[N(I)]^2$ is an ideal of $N(R)$.*

7.2 Quasi and Bi ideals

Corresponding to quasi and bi ideals of rings in this section we define quasi and bi ideals in Neutrosophic LA-rings. we give some properties of quasi and bi ideals. we show that under some given condition every quasi-ideal is a bi-ideal.

Definition 260 *Let $(N(R), +, \cdot)$ be an Neutrosophic LA-ring. A non empty subset $N(Q)$ of $N(R)$ is said to be a quasi-ideal of $N(R)$, if $(N(Q), +)$ is an Neutrosophic LA-subgroup of $(N(R), +)$ such that*

$$N(R)N(Q) \cap N(Q)N(R) \subseteq N(Q).$$

It is clear that every one-sided ideal of an Neutrosophic LA-ring $(N(R), +, \cdot)$ is a quasi-ideal of N(R).To understand quasi ideals we give an example.

Proposition 261 *Each quasi-ideal of an Neutrosophic LA-ring $(N(R), +, \cdot)$ is an Neutrosophic LA-subring of $(N(R), +, \cdot)$.*

Proof. Let $N(Q)$ be a quasi-ideal of an Neutrosophic LA-ring $(N(R), +, \cdot)$, then by definition $(N(Q), +)$ is an Neutrosophic LA-subgroup of $(N(R), +)$. Now

$$
\begin{aligned}
[N(Q)]^2 &= N(Q)N(Q) \subseteq N(R)N(Q), \text{ i.e. } [N(Q)]^2 \subseteq N(R)N(Q) \text{ And} \\
[N(Q)]^2 &= N(Q)N(Q) \subseteq N(R)N(Q), \text{ i.e. } [N(Q)]^2 \subseteq N(R)N(Q) \text{ So} \\
[N(Q)]^2 &\subseteq N(R)N(Q) \cap N(Q)N(R) \subseteq N(Q), \text{ i.e. } [N(Q)]^2 \subseteq N(Q).
\end{aligned}
$$

Thus N(Q) is an Neutrosophic LA-subring of $(N(R), +, \cdot)$. ■

Proposition 262 *The intersection of any family of quasi-ideals of an Neutrosophic LA-ring* $(N(R), +, \cdot)$ *is a quasi-ideal of* $(N(R), +, .)$.

Proof. Let $\{N(Q) : i \in \Omega\}$ be a family of quasi-ideals of an Neutrosophic LA-ring $(N(R), +, \cdot)$. Then clearly $\underset{i\in\Omega}{\cap} N(Q)_i$. is an Neutrosophic LA-subgroup of $(N(R), +)$. Now

$$\begin{aligned} & N(R)\left\{\underset{i\in\Omega}{\cap} N(Q)_i\right\} \cap \left\{\underset{i\in\Omega}{\cap} N(Q)_i\right\} N(R) \\ \subseteq \quad & N(R)N(Q)_i \cap N(Q)_i N(R) \subseteq N(Q)_i \text{ for all } i \in \Omega \end{aligned}$$

This gives $N(R)\left\{\underset{i\in\Omega}{\cap} N(Q)_i\right\} \cap \left\{\underset{i\in\Omega}{\cap} N(Q)_i\right\} N(R) \subseteq \underset{i\in\Omega}{\cap} N(Q)_i$.

Thus $\underset{i\in\Omega}{\cap} N(Q)_i$ is a quasi-ideal of $(N(R), +, \cdot)$. ■

We are now going to state and prove a result which is based on the above theorem.

Corollary 263 *The intersection of a right ideal* $N(I)$ *and a left ideal* $N(J)$ *of an Neutrosophic LA-ring* $(N(R), +, \cdot)$ *is a quasi ideal of* $N(R)$.

Proof. The right ideal $N(I)$ and the left ideal $N(J)$ of the Neutrosophic LA-ring $(N(R), +, \cdot)$ being one-sided ideals are quasi-ideal of $(N(R), +, \cdot)$. Thus by the above proposition $N(I) \cap N(J)$ is a quasi ideal of N(R). ■

We are now going to define bi ideals in Neutrosophic LA-ring.

Definition 264 *Let* $(N(R), +, \cdot)$ *be an Neutrosophic LA-ring and* $N(B)$ *an Neutrosophic LA-subring* $(N(R), +, \cdot)$ *of* $N(R)$*, then* $N(B)$ *is called a bi-ideal of* $N(R)$ *if* $\{N(B)N(R)\}N(B) \subseteq N(B)$.

It is easy to see that every one sided ideal is a bi-ideal. Let us state and prove some properties of bi ideals.

Theorem 265 *Let* $(N(R), +, \cdot)$ *be an Neutrosophic LA-ring with left identity* $(e + eI)$ *such that* $\{(x + xI)(e + eI)\}N(R) = (x + xI)N(R)$ *for all* $(x + xI) \in N(R)$ *then every quasi-ideal of* $N(R)$ *is a bi-ideal of* $N(R)$.

Proof. Let $N(Q)$ be a quasi ideal of $N(R)$. Then $N(Q)$ is an Neutrosophic LA-subring of $N(R)$. Now by medial law and by (1) i.e $\{(x + xI)(e + eI)\}N(R) = (x + xI)N(R)$

$$\begin{aligned} \{N(Q)N(R)\}N(Q) \quad & \subseteq \quad N(R)N(Q) \text{ and } \{(N(Q)N(R)\}N(Q) \\ & \subseteq \quad \{N(Q)N(R)\}N(R) \\ & = \quad \{N(Q)N(R)\}\{(e + eI)N(R) \\ & = \quad \{N(Q)(e + eI)\}\{N(R)N(R)\} \\ & = \quad \{N(Q)(e + eI)\}N(R) = N(Q)N(R). \end{aligned}$$

Hence it follows that

$$\{N(Q)N(R)\}N(Q) \subseteq N(Q)N(R) \cap N(R)N(Q).$$

Hence $\{N(Q)N(R)\}N(Q) \subseteq N(Q)$. ■

Theorem 266 *The intersection of any family of bi ideals of an Neutrosophic LA-ring $(N(R), +, \cdot)$ is a bi ideal of $N(R)$.*

Proof. Let $\{N(B)_i : i \in \Omega\}$ be a family of bi ideals of the Neutrosophic LA-ring $N(R)$. Then$N(B) = \cap_{i\in\Omega} N(B)_i$ being the intersection of Neutrosophic LA-subrings of $N(R)$ is also an Neutrosophic LA-subrings of $N(R)$. Now

$$\begin{aligned}
\{N(B)N(R)\}N(B) &\subseteq \{N(B)_i N(R)\}N(B)_i \subseteq N(B)_i \text{ for all } i \in \Omega \\
\text{Thus } \{N(B)N(R)\}N(B) &\subseteq N(B)_i \text{ for all } i \in \Omega \\
\{N(B)N(R)\}N(B) &\subseteq \cap_{i\in\Omega} N(B)_i = N(B).
\end{aligned}$$

■

It follows from the above result that intersection of a right ideal $N(I)$ and a left ideal $N(J)$ of an Neutrosophic LA-ring $N(R)$ is a bi ideal of $N(R)$.

7.3 Regular and Intra-regular Neutrosophic LA-ring

corresponding to regular and intra-regular rings in this section we define regular and intra-regular Neutrosophic LA-rings. firstly we are going to define regular LA-rings.

Definition 267 *Let $(N(R), +, \cdot)$ be an LA-ring and let c be an element of $N(R)$,then c is called a regular element of $N(R)$, if and only if $\{(c + cI)(u + uI)\} = (c + cI)$ for some $(u + uI) \in N(R)$.*

If every element of an LA-ring $N(R)$ is regular, then the Neutrosophic LA-ring $N(R)$ is called regular. Let us describe some properties.

Theorem 268 *$(N(R), +, \cdot)$ be a regular Neutrosophic La-ring with left identity $e+eI$ then $N(I) \cap N(B) = \{(N(B)N(I)\}N(B)$ for every ideal $N(I)$ of $N(R)$ and and every bi-ideal $N(B)$ of $N(R)$.*

Proof. Given that $(N(R), +, \cdot)$ is a regular neutrosophic LA-ring with left identity $(e + eI)$ and an ideal of $N(R)$ and $N(B)$ every ideal $N(I)$ of $N(R)$

and every bi-ideal $N(B)$ of $N(R)$. Now

$$\begin{aligned}
\{N(B)N(I)\}N(B) &\subseteq N(I)N(B) \subseteq N(I)\\
N(B)N(I) &\subseteq N(I) \text{ and } N(I)N(B) \subseteq N(I)\\
\{N(B)N(I)\}N(B) &\subseteq \{N(B)N(R)\}N(B) \subseteq N(B).\\
\text{Thus,}\{N(B)N(I)\}N(B) &\subseteq N(I) \cap N(B).
\end{aligned}$$

Conversely we get
$(a+aI)\{(b+bI)(c+cI)\} = (b+bI)\{(a+aI)(c+cI)\}$let

$$\begin{aligned}
(u+uI) &\in N(I) \cap N(B), \text{ Then}\\
(u+uI) &= \{(u+uI)(v+vI)\}(u+uI) \text{ for some } v+vI. \text{ Now}\\
(u+uI) &= \{(u+uI)(v+vI)\}(u+uI)\\
&= [[\{(u+uI)(v+vI)\}(u+uI)](v+vI)](u+uI)\\
&= [\{(v+vI)(u+uI)\}\{(u+uI)(v+vI)\}](u+uI)\\
&= [(u+uI)[\{(v+vI)(u+uI)\}(v+vI)](u+uI)\\
&\in \{N(B)N(I)\}N(B). \text{ Thus,}\\
N(I) \cap N(B) &\subseteq \{N(B)N(I)\}N(B) \text{ and so}\\
N(I) \cap N(B) &= \{N(B)N(I)\}N(B).
\end{aligned}$$

■

Theorem 269 *Let $(N(R), +, \cdot)$ be a regular Neutrosophic LA-ring. Then $N(I)N(J) = N(I) \cap N(J)$ for every right ideal $N(I)$ and left Ideal $N(J)$ of $N(R)$.*

Proof. Given that $N(I)$ is a right ideal and $N(J)$ a left ideal of $(N(R), +, \cdot)$.Then obviously $N(I)N(J) \subseteq N(I) \cap N(J)$. Now let $(u+uI) \in N(I)N(J)$, then $(u+uI) \in N(I)$ and $(u+uI) \in N(J)$. As $N(R)$ is regular, so there exists $(v+vI) \in N(R)$ such that $(u+uI) = \{(u+uI)(v+vI)\}(u+uI)$ and $\{(u+uI)(v+vI)\}(u+uI) \in N(I)N(J)$. It follows that $N(I) \cap N(J) \subseteq N(I)N(J)$. This complete the proof. ■

We are now going to define intra-regular Neutrosophic LA-rings.

Definition 270 *Let $(N(R), +, .)$ be an Neutrosophic LA-rings,then an element $(a+aI)$ of $N(R)$ is called an intra-regular if there exists $(u+uI)$, $(v+vI) \in N(R)$ such that $(a+aI) = \{(u+uI)(a+aI)^2\}(v+vI)$. If every element of $N(R)$ is intra regular,then the Neutrosophic LA-ring $N(R)$ is called intra-regular. To understand the above definition we give an example.*

Let us state and prove some properties of intra-regular Neutrosophic LA-ring.

Theorem 271 *If $(N(R), +, \cdot)$ is an intra-regular Neutrosophic LA-ring with left identity $(e+eI)$, then $\{N(B)N(R)\}N(B) = N(B) \cap N(R)$, for every bi ideal $N(B)$ of $(N(R), +, \cdot)$.*

Proof. Let $(N(R), +, \cdot)$ be an intra-regular Neutrosophic LA-ring with left identity $(e + eI)$ then

$$\begin{aligned} \{N(B)N(R)\}N(B) &\subseteq N(R)N(B) \subseteq N(R) \text{ and} \\ \{N(B)N(R)\}N(B) &\subseteq N(B). \end{aligned}$$

Because $N(B)$ is a bi ideal of $N(R)$. Therefore

$$\{N(B)N(R)\}N(B) \subseteq N(B) \cap N(R).$$

Now $(a+aI)\{(b+bI)(c+cI)\} = (b+bI)\{(a+aI)(c+cI)\}$ by left invertive law by medial law and by paramedial law i.e let $(a+aI) \in N(B) \cap N(R) \Longrightarrow (a+aI) \in N(B)$ and $(a+aI) \in N(R)$.since is an intra-regular Neutrosophic LA-ring $(N(R), +, \cdot)$,so there exists $(u + uI), (v + vI) \in N(R)$ such that

$$\begin{aligned} (a + aI) &= \{(u + uI)(a + aI)^2\}(v + vI) \\ &= [(u + uI)\{(a + aI)(a + aI)\}](v + vI) \\ &= [(a + aI)[\{(u + uI)(a + aI)\}](v + vI) \\ &= [(v + vI)\{(u + uI)(a + aI)\}(a + aI) \\ &= (v + vI)(u + uI)[\{(u + uI)(a + aI)^2\}(v + vI)] \\ &= [(v + vI)[\{(u + uI)(a + aI)^2\}\{(u + uI)(v + vI)\}]](a + aI) \\ &= [\{(u + uI)(a + aI)^2\}[(v + vI)\{(u + uI)(v + vI)\}]] \\ &= [\{(u + uI)(v + vI)\}[(a + aI)^2\{(u + uI)(v + vI)\}]](a + aI) \\ &= [(a + aI)^2[\{(u + uI)(v + vI)\}\{(u + uI)(v + vI)\}]](a + aI) \\ &= [\{(a + aI)(a + aI)\}\{(u + uI)^2(v + vI)^2\}](a + aI) \\ &= [\{(a + aI)(u + uI)^2\}\{(a + aI)(v + vI)^2\}](a + aI) \\ &= [\{(v + vI)^2(u + uI)^2\}\{(a + aI)(a + aI)\}](a + aI) \\ &= [(a + aI)[\{(v + vI)^2(u + uI)^2\}(a + aI)]](a + aI) \\ &\in \{N(B)N(R)\}N(B). \text{ Thus,} \\ N(B) \cap N(R) &\subseteq \{N(B)N(R)\}N(B). \text{ Therefore,} \\ \{N(B)N(R)\}N(B) &= N(B) \cap N(R) \end{aligned}$$

■

we are now going to state a result which is based on the above theorem.

Corollary 272 *If $(N(R), +, \cdot)$ is an intra-regular Neutrosophic LA-ring with left identity $(e + eI)$, then $\{N(B)N(R)\}N(B) = N(B)$ for every bi ideal $N(B)$ of $(N(R), +, \cdot)$.*

The following result gives us equivalent conditions for bi ideals in intra-regular Neutrosophic LA-rings.

Theorem 273 *If $(N(R), +, \cdot)$ is an intra-regular Neutrosophic LA-ring with left identity $(e+eI)$ and $N(B)$ a non empty subset of $N(R)$, then the following conditions are equivalent*

(i) $N(B)$ is a bi-ideal of $N(R)$.
(ii) $\{N(B)N(R)\}N(B) = N(B)$ and $[N(B)]^2 = N(B)$

Proof. $(1) \Rightarrow (2)$

Let $N(B)$ be the bi ideal of the intra-regular Neutrosophic LA-ring $N(R)$ with left identity $(e+eI)$ then $\{(N(B)N(R)\}N(B) \subseteq N(B)$. Now $(a+aI)\{(b+bI)(c+cI)\} = (b+bI)\{(a+aI)(c+cI)\}$ by left invertive law and by medial law. Let $(b+bI) \in N(B)$, then since $N(R)$ is an intra-regular, so there exists $(u+uI), (v+vI) \in N(R)$ such that

$$\begin{aligned}
b+bI &= \{(u+uI)(b+bI)^2\}(v+vI) \\
&= [(u+uI)\{(b+bI)(b+bI)\}](v+vI) \\
&= [(b+bI)\{(u+uI)(b+bI)\}](v+vI) \\
&= [(v+vI)\{(u+uI)(b+bI)\}](b+bI) \\
&= [(v+vI)[(u+uI)\{(u+uI)(b+bI)^2\}](v+vI) \\
&= [(v+vI)[\{(u+uI)(b+bI)^2\}\{(u+uI)(v+vI)\}]](b+bI) \\
&= [\{(u+uI)(b+bI)^2\}[(v+vI)\{(u+uI)(v+vI)\}]](b+bI) \\
&= [[(u+uI)\{(b+bI)(b+bI)\}][(v+vI)\{(u+uI)(v+vI)\}]](b+bI) \\
&= [[(b+bI)[(u+uI)(b+bI)\}][(v+vI)\{(u+uI)(v+vI)\}]](b+b) \\
&= [\{(b+bI)(v+vI)\}\{(u+uI)(b+bI)\}\{(u+uI)(v+vI)\}](b+bI) \\
&= [\{(u+uI)(b+bI)\}\{(b+bI)(u+uI)\}(v+vI)^2](b+bI) \\
&= [\{(b+bI)(u+uI)\}\{(u+uI)(b+bI)\}(v+vI)^2](b+bI) \\
&= [\{(b+bI)(u+uI)\}\{(v+vI)^2(b+bI)\}(u+uI)](b+bI) \\
&= \{(v+vI)^2(b+bI)\}\{(b+bI)(u+uI)\}(u+uI)](b+bI) \\
&= [(v+vI)^2\{(b+bI)(u+uI)\}\{(b+bI)(u+uI)\}](b+bI) \\
&= [(b+bI)[(v+vI)^2\{(b+bI)(u+uI)\}(u+uI)](b+bI) \\
&\in \{N(B)N(R)\}N(B). \text{ This implies that} \\
N(B) &\subseteq \{N(B)N(R)\}N(B). \text{ Thus} \\
\{N(B)N(R)\}N(B) &= N(B).
\end{aligned}$$

Now $(a+aI)\{(b+bI)(c+cI)\} = (b+bI)\{(a+aI)(c+cI)\}$ by left invertive law by paramedial law we have to show that $[N(B)]^2 = N(B)$. Obviously $[N(B)]^2 \subseteq N(B)$. Now let $(b+bI) \in N(B)$, then there exists $(u+uI)$,

$(v+vI)\in N(R)$ such that

$$\begin{aligned}
& (b+bI) \\
= \;& \{(u+uI)(b+bI)^2\}(v+vI) \\
= \;& [(u+uI)\{(b+bI)(b+bI)\}](v+vI) \\
= \;& [(b+bI)\{(u+uI)(b+bI)\}](v+vI) \\
= \;& [(v+vI)\{(u+uI)(b+bI)\}](b+bI) \\
= \;& [(v+vI)[(u+uI)\{(u+uI)(b+bI)^2\}(v+vI)]](b+bI) \\
= \;& [(v+vI)[\{(u+uI)(b+bI)^2\}\{(u+uI)(v+vI)\}]](b+bI) \\
= \;& [\{(u+uI)(b+bI)^2\}[(v+vI)\{(u+uI)(v+vI)\}]](b+bI) \\
= \;& [(u+uI)\{(b+bI)(b+bI)\}][(v+vI)\{(u+uI)(v+vI)\}](b+bI) \\
= \;& [[(b+bI)\{(u+uI)(b+bI)\}][(v+vI)\{(u+uI)(v+vI)\}]](b+bI) \\
= \;& [[(v+vI)\{(u+uI)(v+vI)\}][\{(u+uI)(b+bI)\}(b+bI)]](b+bI) \\
= \;& [[(b+bI)\{(u+uI)(v+vI)\}][\{(u+uI)(v+vI)\}(b+bI)]](b+bI) \\
= \;& [[\{(b+bI)(u+uI)\}\{(u+uI)(v+vI)\}(v+vI)](b+bI)](b+bI) \\
= \;& [[\{(b+bI)(u+uI)\}[\{(v+vI)(v+vI)\}(u+uI)][(b+bI)](b+bI) \\
= \;& [\{(b+bI)(u+uI)\}\{(v+vI)^2(u+uI)\}(b+bI)](b+bI) \\
= \;& [[\{(b+bI)(v+vI)^2\}(u+uI)^2](b+bI)](b+bI) \\
= \;& [\{(u+uI)^2(v+vI)^2\}(b+bI)](b+bI)](b+bI) \\
= \;& [\{(u+uI)^2(v+vI)^2\}[(u+uI)\{(b+bI)(b+bI)\}](v+vI)](b+bI)](b+bI) \\
= \;& [\{(u+uI)^2(v+vI)^2\}[(b+bI)\{(u+uI)(b+bI)\}(v+vI)](b+bI)](b+bI) \\
= \;& [[(u+uI)^2(b+bI)\{(u+uI)(b+bI)\}][\{(v+vI)^2(v+vI)\}(b+bI)](b+bI) \\
= \;& [[[(b+bI)\{(u+uI)(u+uI)\}\{(u+uI)(b+bI)\}](v+vI)^3](b+bI)](b+bI) \\
= \;& [[(b+bI)\{(b+bI)(u+uI)\}\{(u+uI)(u+uI)\}(v+vI)^3](b+bI)](b+bI) \\
= \;& [[\{(b+bI)(u+uI)\}\{(b+bI)(u+uI)^2\}(v+vI)^3](b+bI)](b+bI) \\
= \;& [[\{(b+bI)(b+bI)\}\{(u+uI)(u+uI)^2\}(v+vI)^3](b+bI)](b+bI) \\
= \;& [[\{(v+vI)^3(u+uI)^3\}\{(b+bI)(b+bI)\}](b+bI)](b+bI) \\
= \;& [[(b+bI)\{(v+vI)^3(u+uI)^3\}(b+bI)](b+bI)](b+bI) \\
\in \;& [\{N(B)N(R)\}N(B)]N(B)\subseteq N(B)N(B)=[N(B)]^2
\end{aligned}$$

Thus $N(B) \subseteq [N(B)]^2$ and so $N(B)=[N(B)]^2$

$(ii)\Longrightarrow(i)$

$\{N(B)N(R)\}N(B)=N(B)$ and $N(B)^2=N(B)\Longrightarrow\{N(B)N(R)\}N(B)\subseteq N(B)$ and $N(B)\subseteq N(B)^2\Longrightarrow N(B)$ is a bi ideal of $(N(R),+,\cdot)$. ■

The following result gives us equivalent condition for quasi ideals in intra-regular Neutrosophic LA-rings with left identity $(e+eI)$.

Theorem 274 *Let $(N(R),+,\cdot)$ be an intra-regular Neutrosophic LA-ring with left identity $(e+eI)$ and $N(Q)$ a non empty subset of $N(R)$, then the* *following condition are equivalent:*

$(i)N(Q)$ is a quasi ideal of $N(R)$,
(ii) $N(R)N(Q) \cap N(Q)N(R) = N(Q)$.

Proof. $(i) \Longrightarrow (ii)$

Let $N(Q)$ be the quasi ideal of the intra regular Neutrosophic LA-ring $N(R)$ with left identity $(e+eI)$. Then $N(R)N(Q) \cap N(Q)N(R) \subseteq N(Q)$. Now $(a+aI)\{(b+bI)(c+cI)\} = (b+bI)\{(a+aI)(c+cI)\}$ by medial law by paramedial law, let $(a+aI) \in N(Q)$, then since $(N(R), +, \cdot)$ is an intra regular neutrosophic LA-ring, so there exists $(u+uI)$, $(v+vI) \in N(R)$ such that $(a+aI) = \{(u+uI)(a+aI)^2\}$. Now for $(m+mI) \in N(R)$, $(m+mI)(a+aI) \in N(R)N(Q)$. Further

$$\begin{aligned}
(m+mI)(a+aI) &= (m+mI)[\{(u+uI)(a+aI)^2\}(v+vI)] \\
&= \{(u+uI)(a+aI)^2\}\{(m+mI)(v+vI)\} \\
&= [(u+uI)\{(a+aI)(a+aI)\}]\{(m+mI)(v+vI)\} \\
&= [(a+aI)\{(u+uI)(a+aI)\}]\{(m+mI)(v+vI)\} \\
&= \{(a+aI)(m+mI)\}[\{(u+uI)(a+aI)\}(v+vI)] \\
&= \{(u+uI)(a+aI)\}[(a+aI)\{(m+mI)(v+vI)\}] \\
&= [(u+uI)\{(a+aI)(m+mI)\}]\{(a+aI)(v+vI)\} \\
&= [(v+vI)\{(a+aI)(m+mI)\}]\{(a+aI)(u+uI)\} \\
&= (a+aI)[[(v+vI)\{(a+aI)(m+mI)\}](u+uI)] \\
&\in N(Q)N(R).
\end{aligned}$$

Now by left invertive law $(a+aI)\{(b+bI)(c+cI)\} = (b+bI)\{(a+aI)(c+cI)\}$ by medial law and by paramedial law $m+mI \in N(R)$,$(a+aI)(m+mI) \in N(Q)N(R)$. Further

$$\begin{aligned}
(a+aI)(m+mI) &= [\{(u+uI)(a+aI)^2\}(v+vI)](m+mI) \\
&= \{(m+mI)(v+vI)\}\{(u+uI)(a+aI)^2\} \\
&= \{(m+mI)(v+vI)\}[(u+uI)\{(a+aI)(a+aI)\}] \\
&= (u+uI)[\{(m+mI)(v+vI)\}\{(a+aI)(a+aI)\}] \\
&= (u+uI)[\{(m+mI)(a+aI)\}\{(v+vI)(a+aI)\}] \\
&= (u+uI)[\{(a+aI)(a+aI)\}\{(v+vI)(m+mI)\}] \\
&= \{(a+aI)(a+aI)\}[(u+uI)\{(v+vI)(m+mI)\}] \\
&= [[(u+uI)\{(v+vI)(m+mI)\}](a+aI)](a+aI) \\
&\in N(R)N(Q).
\end{aligned}$$

Hence, $N(R)N(Q) = N(Q)N(R)$.Now by property (3) i.e. $(a + aI)\{(b + bI)(c + cI)\} = (b + bI)\{(a + aI)(c + cI)\}$ by left invertive law

$$\begin{aligned}
(a + aI) &= [(u + uI)(a + aI)^2](v + vI) \\
&= [(u + uI)\{(a + aI)(a + aI)\}](v + vI) \\
&= [(a + aI)\{(u + uI)(a + aI)\}](v + vI) \\
&= [(v + vI)\{(u + uI)(a + aI)\}](a + aI) \\
&\in N(R)N(Q).
\end{aligned}$$

Now as $N(R)N(Q) = N(Q)N(R)$, so it follows that $(a+aI) \in N(Q)N(R)$. Thus $(a + aI) \in N(R)N(Q) \cap N(Q)N(R) \implies N(Q) \subseteq N(R)N(Q) \cap N(Q)N(R)$. Thus $N(R)N(Q) \cap N(Q)N(R) = N(Q)$.

$(ii) \implies (i)$

$N(R)N(Q) \cap N(Q)N(R) = N(Q)$ this implies that $N(R)N(Q) \cap N(Q)N(R) \subseteq N(Q)$ which further implies that $N(Q)$ is a quasi ideal of $N(R)$. ■

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This book consists of seven chapters. In chapter one we introduced neutrosophic ideals (bi, quasi, interior, (m,n) ideals) and discussed the properties of these ideals. Moreover, we characterized regular and intra-regular AG-groupoids using these ideals.

In chapter two we introduced neutrosophic minimal ideals in AG-groupoids and discussed several properties.

In chapter three, we introduced different neutrosophic regularities of AG-groupoids. Further we discussed several condition where these classes are equivalent.

In chapter four, we introduced neutrosophic M-systems and neutrosophic p-systems in non-associative algebraic structure and discussed their relations with neutrosophic ideals.

In chapter five, we introduced neutrosophic strongly regular AG-groupoids and characterized this structure using neutrosophic ideals.

In chapter six, we introduced the concept of neutrosophic ideal, neutrosophic prime ideal, neutrosophic bi-ideal and neutrosophic quasi ideal of a neutrosophic semigroup. With counter example we have shown that the union and product of two neutrosophic quasi-ideals of a neutrosophic semigroup need not be a neutrosophic quasi-ideal of neutrosophic semigroup. We have also shown that every neutrosophic bi-ideal of a neutrosophic semigroup need not be a neutrosophic quasi-ideal of a neutrosophic semigroup. We have also characterized the regularity and intra-regularity of a neutrosophic semigroup.

In chapter seven, we introduced neutrosophic left almost rings and discussed several properties using their neutrosophic ideals.

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