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NEW AHP METHODS FOR HANDLING UNCERTAINTY WITHIN THE BELIEF FUNCTION THEORY

Amel Ennaceur

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Nahla BEN AMOR	Professeur, Université de Tunis	Président
Arnaud MARTIN	Professeur, Université de Rennes 1	Rapporteur
Allel HADJALI	Professeur, ENSMA	Rapporteur
Laurent DELAHOCHE	Maître de conférences-HDR, Université de Picardie	Examinateur
Gilles GONCALVES	Professeur, Université d'Artois	Invité
Zied Elouedi	Professeur, Université de Tunis	Directeur de thèse
Eric Lefevre	Professeur, Université d'Artois	Directeur de thèse

Ecole doctorale : Institut Supérieur de Gestion Tunis

Laboratoire/Unité de recherche : LARODEC/LGI2A

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Introduction

As society becomes more complex, people are faced with many situations in which they have to make a decision among different alternatives. However, the most preferable one is not always easily selected. Therefore, the need for decisions that balance conflicting criteria has grown. This is the main aim of researchers in the field of Multi-Criteria Decision Making (MCDM). Hence, the study of decision making has become a part of many of disciplines, including operations research, business, engineering, etc.

Several MCDM methods exist in the literature. These include two large families: the outranking techniques (Roy, 1968, 1990) and the value and utility-based approaches (Figueira et al., 2005; Triantaphyllou, 2000).

Outranking techniques such as ELECTRE (ELimination Et Choix Traduisant la REalité) (Roy, 1968, 1990) and PROMETHEE (Preference Ranking Organization METHod for Enrichment and Evaluation) (Vincke & Brans, 1985), are developed and which are based on the so-called partial comparability. On the other hand, the value and utility-based approaches mainly started by Keeney and Raiffa (1976) and then implemented in a number of methods (Figueira et al., 2005; Triantaphyllou, 2000): Multi-Attribute Value Theory (MAVT), Multi-Attribute Utility Theory (MAUT) and the Analytic Hierarchy Process (AHP).

In this Thesis, we will focus on one popular MCDM method, namely AHP (Saaty, 1977, 1980) since its simplicity and understandability. The advantages of this approach over other multi-criteria methods are its flexibility, intuitive appeal to the decision maker and its ability to check inconsistencies. While providing a useful mechanism for checking the consistency of the criteria and alternatives, AHP reduces bias in decision making.

The AHP method is one of the widely used MCDM methods. It effectively keeps both qualitative and quantitative data in decision making. Subsequently, due to its efficiency, there has been a growth of applications and mathematical development to this methodology. It has been extensively used in a wide variety of decision areas including those related to supplier selection problems (Chamodrakas et al., 2010; Kilincci & Onal, 2011), catering selection (Cebeci & Kahraman, 2002; Kahraman et al., 2004), resource allocation (Chamodrakas et al., 2010), economy, energy policy, health, conflict resolution, project selection, budget allocation, operations management, benchmarking, education, etc.

Motivation

In this dissertation, we have performed a detailed study of this approach. While doing so, we notice that the method has two main critical limitations. The first limitation is linked to the number of comparisons and the second one is related to pair-wise comparison procedure.

More precisely, these two shortcomings are highlighted as follows:

- Since in most cases, it is unrealistic to expect that the decision maker will have either complete information regarding all aspects of the decision making problem or full understanding of the problem, a degree of uncertainty will be associated with some or all of the pairwise comparisons. For instance, the decision maker can only express his judgment to those alternatives or criteria which he has a level of opinion towards. As result, using standard AHP, he can complete the pair-wise matrix with erroneous information.

  On the other hand, due to the exponentially increase of the number of pair-wise comparisons, the elicitation of preferences may be rather difficult when the number of alternatives and criteria is large. If the number of alternatives (criteria) in the hierarchy increases then, more comparisons are needed to be made.

- Though the main purpose of AHP is to capture the expert's knowledge, the standard method still cannot reflect the human thinking style. The method is often criticized for its use of an unbalanced scale of estimations and its inability to adequately handle the uncertainty and imprecision associated with the mapping of the decision maker's perception to a crisp number (Holder, 1995; Joaquin, 1990).

  As a result, there has been a serie of AHP related studies concerned with the question of what is the most appropriate set of scale values to be utilized. Indeed, studies such as Ma and Zheng (1991) and Donegan et al. (1992) have offered alternative sets of 9-unit scales, which they contest, are more appropriate. Other researchers have expanded the method by uncertain theories and group decision making such as Probabilistic AHP (Vargas, 1982; Escobar & Moreno-Jimenez, 2000; Manassero et al., 2004) and fuzzy AHP (Laarhoven & Pedrycz, 1983; Lootsma, 1997).

With regard to these proposed methods, we can frequently find limits. Firstly, in some cases, the decision maker might be unwilling to provide all comparisons necessary to construct full comparison matrices. In addition, these approaches deal only with numerical values to translate the expert preferences into quantitative information.

Consequently, the need to consider uncertainty within AHP method is proved.

Contribution

Here are the main ideas that we plan to explore in order to achieve the above expecting goals of our Thesis work. In the following, we will try to organize them into two main contributions:

- Firstly, our proposed AHP approach must be able to be efficient in terms of reducing the number of comparisons. Indeed, in many complex problems decision makers are able to compare only subsets of criteria and alternatives and cannot evaluate separate ones. For that, we will consider:

  - Groups of criteria. our method suggests to allow the expert to express his opinions on groups of criteria instead of single one. So, he chooses these subsets by assuming that criteria having the same importance are grouped together.
  - Subsets of alternatives. In order to properly model the decision maker knowledge, he needs only to identify and to express judgment to those alternatives which he has a level of opinion towards. Consequently, the ability of the expert to control the amount of information expressed on each criterion.

  Also, we have studied the dependency between alternatives and criteria. Our aim is to model the influences of the criteria on the evaluation of alternatives.

- Secondly, we will explore the effect of imperfection on our pair-wise comparison procedure. How we can properly model this imperfection is the basic problem of this dissertation. Actually, expert evaluation can be modeled quantitatively or qualitatively. As a result, two main approaches will be developed.
  - Our proposed solution avoids the standard pair-wise comparisons and proposes a new elicitation technique based on the belief function theory. The expert has then the ability to express his assessment freely. In other words, to quantify the subjective judgments with uncertainty, decision maker's response can be described by a belief distribution. For that, we are going to develop a new method, named Yes-No/AHP approach.
  - To express his assessments, the decision maker has to model his opinions qualitatively, based on knowledge and experience that he provides in response to a given question rather than direct quantitative information. He only selects the related linguistic variable using preference modeling. In this context, a new qualitative belief function method will be introduced. This model is able to generate quantitative mass distribution from qualitative assessments. This method will help us in the development of our new qualitative AHP method.

Thesis outline

Our Thesis is organized in the following six chapters partitioned into two parts.

Part I: Theoretical Aspects. The first part is composed of two chapters which are the following:

- **Chapter 1: Multi-Criteria Decision Making: An overview**. This Chapter gives the necessary background regarding the basic concepts of the MCDM with a special focus on the AHP method.
- **Chapter 2: AHP method under uncertainty**. This chapter reviews recent works in AHP method under uncertain theories. It involves the main tools and techniques used across the development of our new MCDM methods throughout this dissertation.

Part II: Contribution. The second part of this Thesis presents our contributions. Its purpose is to develop new MCDM methodologies under the belief function framework. To describe that, we can decompose this part into four chapters.

- **Chapter 3: Modeling dependency between alternatives and criteria**. This chapter presents a first MCDM method, named belief AHP. Its objective is to cover the limitations of the standard AHP by reducing the number of comparisons. The second part of the chapter is dedicated to the analysis of the influence of criteria in the evaluation of criteria. Then, we define a new MCDM called conditional belief AHP that models this dependency.
- **Chapter 4: A new ranking procedure by belief pair-wise comparisons**. This chapter proposes an extension of our previous methods; we call them Yes-No/AHP method and conditional Yes-No/AHP method. Our main aim is to introduce new elicitation technique under the belief function framework.
- **Chapter 5: Constructing belief functions from qualitative expert assessments**. In this chapter, we introduce a new qualitative model that is able to generate quantitative distributions from qualitative assessments.
- **Chapter 6: AHP method based on belief preference relations**. This chapter describes, in details, our new MCDM method which will be able to handle the problem of imperfection in the pair-wise comparison procedure. Our developed method uses the previous model (presented in Chapter 5) to properly model expert judgments.

Finally, a general conclusion gives a summary of the results achieved in this Thesis and presents possible future developments.

Two appendices complete this Thesis. The first appendix details simulation results. The second appendix gives an overview of sensitivity analysis.

PART I:

THEORETICAL ASPECTS

Part I presents the theoretical aspects of this Thesis. It provides the necessary background regarding the basic concepts of the Multi-Criteria Decision Making and more precisely the Analytic Hierarchy Process. Besides, it introduces the belief function theory as main technique adopted in this dissertation. In addition, some AHP approaches under uncertainty are also detailed.

Chapter 1

Multi-Criteria Decision Making: An overview

1.1 Introduction

Given the complexity of our life today, people have to make lots of decisions during their everyday life. Some decisions may be made considering a single criterion, but these are very limited to the simple and relatively unimportant ones. Therefore, the two terms "multi-criteria" and "decision-making" are nearly inseparable, especially when making complex decisions that require consideration of all the different aspects.

Multi-Criteria Decision Making (MCDM) is considered as one of the most well-known branches of decision making. It is a branch of a general class of Operations Research models which deal with decision problems under the presence of a number of decision criteria. This super class of models is very often called Multi-Attributes Decision Making (MADM). According to many authors (Zeleny, 1982), MCDM is divided into Multi-Objective Decision Making (MODM) and Multi-Attribute Decision Making (MADM) (Figueira et al., 2005).

MODM studies decision problems in which the decision space is continuous. A typical example is mathematical programming problems with multiple objective functions (Kuhn & Tucker, 1951). On the other hand, MADM (or namely MCDM) concentrates on problems with discrete decision spaces. In these problems, the set of alternatives has been predetermined (Zeleny, 1982).

Generally, the term MADM and MCDM are used to mean the same class of models.

In this work, we focus on what it is called MADM methods (namely also MCDM approaches), particularly the Analytic Hierarchy Process (AHP) (Saaty, 1977, 1980). As we will show, this approach is considered as one of the most known method, since it has been successfully applied to many practical problems (Zeleny, 1982).

In this Chapter, we firstly present an overview of MDCM: in Section 1.2, we briefly introduce some common concepts. Then, we expose in Section 1.3 several MCDM methods and we classify them according to the available data. In Section 1.4, we are interested especially in AHP method: we focus on its standard version where its procedure will be described. Then, an example will be detailed to illustrate this approach.

1.2 Multi-Criteria Decision Making

MCDM can be defined as a discipline which refers to making decisions in the presence of multiple, usually conflicting, criteria (Zeleny, 1982). For instance, consider buying a new car, some of the criteria to handle are cost, fuel consumption, safety, capacity and style. A decision maker wants to buy the cheapest car but also the most comfortable. After evaluating a list of possible cars against these conflictual criteria, a ranking can be obtained and the most appropriate choice can be selected.

Although MCDM methods may be widely diverse, many of them have some aspects in common (Roy, 1985). These are the notions of alternatives and criteria (or attributes, goals) as described next.

1.2.1 Basic concepts

Despite the fact that MCDM problems could be very different in context, they share the following common features. In this Section, we define the different concepts (Triantaphyllou, 2000; Figueira et al., 2005):

- **Decision maker:** actor for whom the decision-aid tools are developed and implemented.

- **Alternative:** usually alternatives represent the different choices of action available to the decision maker.
- **Criterion:** is also referred to as "goal" or "attribute". It represents the different dimensions from which the alternatives can be viewed. Criteria represent the different dimensions from which the alternatives can be viewed.

  Criteria can be both well defined and quantitatively measurable (price, size, etc.) or qualitatively but difficult to measure (appearance, satisfaction, etc.). It should be:

  - able to discriminate among the alternatives and to support the comparison of the performance of the alternatives,
  - complete to include all goals,
  - operational and meaningful,
  - non-redundant,
  - few in number,
  - usually conflict with one another,
  - hybrid nature: Criteria may have a different unit of measurement.
- **Weight:** Value that indicates the relative importance of one criterion in a particular decision process (denoted by ω). These weights are usually normalized ($\sum \omega = 1$).
- **Performance matrix:** Consider a MCDM problem with n criteria and m alternatives. Let $c_1, \ldots, c_n$ and $a_1, \ldots, a_m$ be the criteria and alternatives, respectively. A standard feature of MCDM methodology is the decision table as shown (see Table 1.1).

  In this table, each column belongs to a criterion and each row describes the performance of an alternative. The score v_{ij} describes the performance of alternative a_i against criterion c_j.

Table 1.1: A Typical decision matrix

	Criteria			
	c_1	c_2	$\ldots$	c_n
Alt.	w_1	w_2	$\ldots$	w_n
a_1	v_{11}	v_{12}	$\ldots$	v_{1n}
$\ldots$				
a_m	v_{m1}	v_{m2}	$\ldots$	v_{mn}

Example 1.1. *Let us consider the problem of buying a car. A decision maker has made a first selection for which he retains three cars ($A = \{$Renault, Ford, Peugeot$\}$). He has decided to evaluate each of them on four criteria, namely: Style, Reliability, Fuel and Price. Table 1.2 summarises how he evaluated the three cars on the four criteria.*

Table 1.2: Decision matrix

	Criteria			
	Style	Reliability	Fuel	Price
Alt.				
Renault	Small	Poor	Good	15000
Ford	Midsize	Good	Good	18000
Peugeot	Sport	Good	Poor	25000

Imagine that the decision maker expresses the following preferences: His preference goes for a sport car and he would like to pay as little as possible. Considering these observations, one can easily check that there exists no optimal car in A, one that would dominate all the other ones. As a consequence, a MCDM on this problem should reveal a compromise alternative.

1.2.2 Decision making process

Whether simple or complex, all decisions involve the same basic process (Belton & Stewart, 2003). For an alternative to be judged by a criterion, a scale of possible values must be defined for the criterion. A scale is defined by the direction and the magnitude of the values. In addition, different types of criteria may be used including measurable, ordinal, probabilistic or fuzzy criteria (Zeleny, 1982). To find the highest scoring alternative, the decision maker must evaluate all possible choices against each criterion. Then a prioritization method is applied to aggregate all judgments and create a ranking of the alternatives. Finally, the decision maker uses this information as a recommendation to select one of the alternatives according to his requirements and preferences.

The MCDM methodology is a process that includes five main steps beginning with the definition of the decision problem and ending with the selection of an alternative (Zeleny, 1982). The steps are shown as follows:

1. **Structure the decision problem:** A decision-making problem should start out by clearly

defining the problem, discerning the alternatives, identifying the actors, the objectives and any points in conflict, together with the constraints, the degree of uncertainty and the key issues. Even if it can be sometimes a long iterative process to come to such an agreement, it is a crucial and necessary point before proceeding to the next step.

2. **Establish alternatives:** Alternative identification means finding suitable alternatives to be modeled, evaluated and analyzed.

3. **Define criteria:** The criteria and the way to measure the alternatives for each criterion are defined. A weight for each criterion to reflect their relative importance to the decision is assigned.

4. **Select a decision making tool:** The selection of an appropriate tool is not an easy task and depends on the concrete decision problem, as well as on the objectives of the decision makers. Each alternative is judged against each criterion and the selected decision making tool can be applied to rank the alternatives or to choose a subset of the most promising alternatives.

5. **Recommendations:** recommendations are given to the decision maker based on the results from the previous step. The decision maker selects one of the alternatives.

Example 1.2. *In the next Figure 1.1, we see an example of the presented steps. First, the problem of choosing a car is defined. The main goal of this step is to analyze the problem: finding the real necessities of a car, discarding other types of vehicles, making a preliminary list of possible models and so on. The second step and after the initial analysis of the problem, three particular cars have been selected as the possible solution alternatives. On the third step, the evaluation criteria have been identified. Finally, on the last step, the decision process has been carried out and the best alternative has been selected.*

There exist several prioritization methods that aggregate the preferences in Step 4 and different methods may yield different results. Although studies have been carried out to compare different methods and to provide a framework for selecting the most appropriate one depending on the problem.

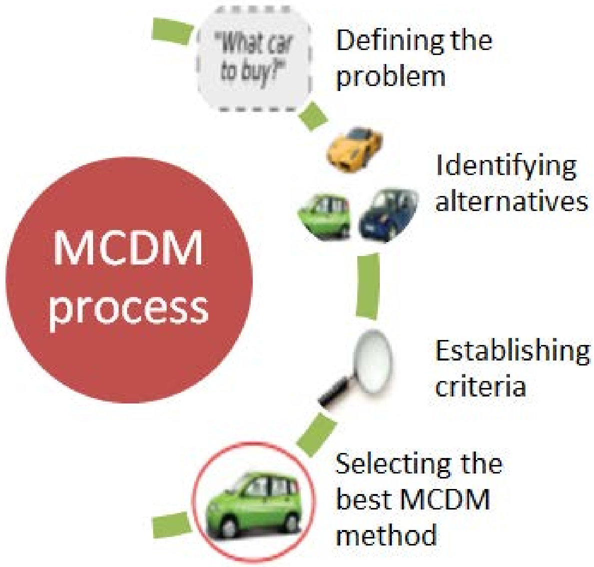

Figure 1.1: Decision making steps

1.3 Classification of MCDM methods

A wide collection of approaches is available to support individuals or groups in decision making but none outperforms all other methods. The selection of an appropriate method depends on the environment and is influenced by several factors such as available information, desired types of outcome or number of alternatives. In order to provide an overview of available MCDM methods, it is helpful to classify these methods.

Hajkowicz et al. (2000) classify MCDM methods under two major groupings namely continuous and discrete methods, based on the nature of the alternatives to be evaluated. Continuous methods aim to identify an optimal quantity, which can vary infinitely in a decision problem. Techniques such as linear programming, goal programming and aspiration-based models are considered continuous. Discrete MCDM methods can be defined as decision support techniques that have a finite number of alternatives, a set of objectives and criteria by which the alternatives are to be judged and a method of ranking alternatives, based on how well they satisfy the objectives and criteria. Discrete methods can be further subdivided into weighting methods and ranking methods. These categories can be further subdivided into qualitative, quantitative and mixed methods. Qualitative methods use only ordinal performance measures. Mixed qualitative and quantitative methods apply different decision rules based on the type of data available.

Quantitative methods require all data to be expressed in cardinal or ratio measurements. Our focus will be on the problems with a finite number of alternatives.

Value and utility-based approaches (Figueira et al., 2005; Triantaphyllou, 2000) use mathematical functions to assist decision makers to construct their preferences. Multi-Attribute Value Theory (MAVT), Multi-Attribute Utility Theory (MAUT) and the Analytic Hierarchy Process (AHP) are the most common approaches within this school. The Analytic Hierarchy Process (AHP), developed by Saaty (1977, 1980), uses the same paradigm as MAVT. However, the AHP is based on a different approach to estimate relative values of criteria (weights) and score alternatives over these criteria. The AHP is the source of several other variants, such as the geometric mean approach.

The French school uses outranking techniques such as ELECTRE (ELimination Et Choix Traduisant la REalité) (Roy, 1968, 1990) and PROMETHEE (Preference Ranking Organization METHod for Enrichment and Evaluation) (Vincke & Brans, 1985), which are based on the so-called partial comparability axiom (in contrast to the utility paradigm). Figure 1.2 presents the main MCDM families.

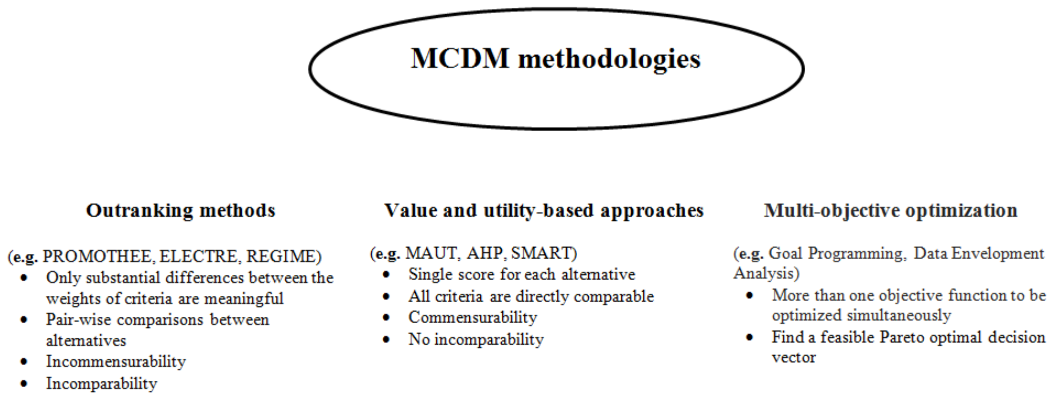

Figure 1.2: The main MCDM families

The choice of MCDM method depends not only on the criteria and the preferences of the decision maker, but also on the type of the problem. Hence, for all the methods applied, the analyst as well as the decision maker should acknowledge the prerequisites for its use, as well as the advantages and drawbacks the method has. Table 1.3 defines a comparative study of the main introduced approaches: AHP, multi-attribute utility theory and outranking methods.

Table 1.3: A comparative table of MCDM methods

Method	Important elements	Strengths	Weaknesses
Multi-Attribute Utility Theory	Expression of overall performance of an alternative in a single, nonmonetary number representing the utility of that alternative.	Easier to compare alternatives whose overall scores are expressed as single numbers.	Maximization of utility may not be important to decision makers.
	Criteria weights often obtained by directly surveying decision makers.	Choice of an alternative can be transparent if highest scoring alternative is chosen.	Criteria weights obtained through less rigorous stakeholder surveys may not accurately reflect decision makers.
Analytical Hierarchy Process	Criteria weights and scores are based on pair-wise comparisons of criteria and alternatives, respectively.	Surveying pair-wise comparisons is easy to implement.	The weights obtained from pair wise comparison are strongly criticized for not reflecting peoples true preferences. The number of pair-wise comparisons increases when the number of criteria or alternatives increases.
Outranking	One option outranks another if: It outperforms the other on enough criteria of sufficient importance (as reflected by the sum of criteria weights).	Does not require the reduction of all criteria to a single unit.	Does not always take into account whether over performance on one criterion can make up for under performance on another.
	It is not outperformed by the other in the sense of recording a significantly inferior performance on any one criterion	Explicit consideration of possibility that very poor performance on a single criterion may eliminate an alternative from consideration, even if that criterions performance is compensated for by very good performance on other criteria.	The algorithms used in outranking are often relatively complex and not well understood by decision makers.

From Table 1.3, there are many MDCM methods available in the literature. All these decision methodologies are differentiated by the way the objective and alternative weights are determined. In this work, we will discuss a widely used MCDM technique: the AHP approach, which is the main focus of our work.

1.4 Analytic Hierarchy Process as a MCDM method

The Analytic Hierarchy Process has been developed by Saaty (1977, 1980) and is one of the well-known and most widely used MCDM approaches.

The AHP has attracted the interest of many researchers because it provides a flexible and easily understood way to analyze and decompose the complex decision problem through breaking it into smaller and smaller parts. In addition, it is a MCDM methodology that allows subjective as well as objective factors to be considered in the evaluation process. The pertinent data are then derived by using a set of pair-wise comparisons. These comparisons are used to obtain the weights of importance of the decision criteria and the relative performance measures of the alternatives in terms of each individual decision criterion.

Indeed, that is the reason why AHP has successfully been applied to many practical problems (Saaty, 1990): from the simple problem of buying a car to the complex problems of economic planning, supplier selection (Kahraman et al., 2003), resource allocation (Chamodrakas et al., 2010), etc.

The AHP, as a compensatory method, assumes complete aggregation among criteria and develops a linear additive model. The weights and scores are achieved basically by pair-wise comparisons between all alternatives and criteria. The basic procedure to carry out the AHP methodology will be presented in the following subsections.

1.4.1 AHP hierarchy

Constructing the hierarchical structure is the most important step in AHP method. This step is based on findings indicating that when elaborating information, the human mind recognizes objects and concepts, and identifies relations existing between them. Because the human mind is not able to perceive simultaneously all factors affected by an action and their connections, it helps to break down complex systems into simple structures: this simplification is possible by means of a logical process which aims at the construction of suitable hierarchies. Therefore, the purpose of constructing the hierarchy is to evaluate the influence of the criteria on the alternatives to attain objectives.

The number of levels depends upon the complexity of the problem and the degree of detail in the problem. So, an AHP hierarchy has at least three levels: the main objective of the problem is represented at the top level of the hierarchy. Then, each level of the hierarchy contains criteria or sub-criteria that influence the decision. The last level of the structure contains the alternatives.

1.4.2 Pair-wise comparison

In AHP, once the hierarchy has been constructed, the decision maker starts with the prioritization procedure to determine the relative importance of the elements on each level of the hierarchy (criteria and alternatives). Elements of a problem on each level are paired (with respect to their common relative impacts on a property or criteria) and then compared.

To compare elements on each level of the hierarchy, AHP uses a quantitative comparison method that is based on pair-wise comparisons of the following type "How important is criterion c_i relative to criterion c_j ?". Questions of this type are used to establish the weights for criteria and similar questions are used to assess the performance scores for alternatives on the subjective (judgmental) criteria. "How important is alternative a_i when compared to alternative a_k with respect to a specific criterion c_j (in the level immediately higher)?".

The responses to the pair-wise comparison question use the following nine-point scale (Saaty scale). Table 1.4 expresses the intensity of preference for one element versus another.

In order to compute the weights for the different criteria, for example the AHP starts creating a pair-wise comparison matrix M ($n \times n$ matrix). Let c_{ij} denote the value obtained by comparing criterion c_i relative to criterion c_j. Of course, we set $c_{ii} = 1$. Furthermore, if we set $c_{ij} = k$, then we set $c_{ji} = \frac{1}{k}$. For example, if criterion c_i is absolutely more important than criterion c_j and is rated at 9, then c_j must be absolutely less important than c_i and is valued at $\frac{1}{9}$. The entries satisfy the following constraint:

$$c_{ij}.c_{ji} = 1 \tag{1.1}$$

Next, the comparison matrix is formed by repeating the process for each criterion.

Table 1.4: The Saaty Rating Scale

Intensity of importance	**Definition**	**Explanation**
1	Equal importance	Two factors contribute equally to the objective.
3	Somewhat more important	Experience and judgement slightly favour one over the other.
5	Much more important	Experience and judgement strongly favour one over the other.
7	Very much more important	Experience and judgement very strongly favour one over the other. Its importance is demonstrated in practice.
9	Absolutely more important.	The evidence favouring one over the other is of the highest possible validity.
2,4,6,8	Intermediate values	When compromise is needed.

1.4.3 Consistency ratio

In decision-making, it is important to know how good the consistency is. Consistency in this case means that the decision procedure is producing coherent judgments in specifying the pair-wise comparison of the criteria or alternatives.

However, perfect consistency rarely occurs in practice. In the AHP, the pair-wise comparisons, in a judgment matrix, are considered to be adequately consistent if the corresponding Consistency Ratio (CR) is less than 10%. The CR coefficient is calculated as follows.

First, the Consistency Index (CI) needs to be estimated. This is an index to assess how much the consistency of pair-wise comparison differs from perfect consistency. This is done by:

$$CI = (\lambda_{max} - n)/(n - 1) \tag{1.2}$$

where n is the matrix size and λ_{max} the maximum eigenvalue.

Then, AHP measures the overall consistency of judgment by means of Consistency Ratio CR. The CR index is obtained by dividing the computed CI index by a Random Index (RI):

$$CR = \frac{CI}{RI} \tag{1.3}$$

Table 1.5, derived from Saaty's book, shows the RI for matrices of order 1 through 10.

Table 1.5: Average random index (RI)

n	1	2	3	4	5	6	7	8	9	10
RI	0.0	0.0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

If $CR \leq 0.1$, the judgment matrix is acceptable otherwise it is considered inconsistent and the entries that are given by the decision maker have to be revised until a satisfactory consistency ratio is obtained.

1.4.4 Priority vectors and synthetic utility

After filling the pair-wise comparison matrices according to the 1-9 scale, the local priority weights are determined by using the eigenvalue method. The objective is then to find the weight of each element, or the score of each alternative by calculating the eigenvalue vector. Then, the global priorities is computed to obtain the final ranking of alternatives.

Hence, given a decision matrix the final priorities, denoted by A_i^{AHP}, of the alternatives in terms of all the criteria combined are determined according to the following formula:

$$A_i^{AHP} = \sum_{j=1}^{n} v_{ij}.w_j, \quad \text{for} \quad i = 1, \ldots, m \tag{1.4}$$

where v_{ij} describes the performance of alternative a_i against criterion c_j, and w_j indicates the relative importance of one criterion c_j.

The global priorities thus obtained are used for final ranking of the alternatives and selection of the best one.

1.5 Illustrative example

Let us treat a problem of purchasing a car. Suppose that this problem involves four criteria (Style, Fuel, Reliability and Price) and three alternatives (Peugeot, Renault and Ford) as shown in Figure 1.3. Here, both the criteria and the alternative weights should be calculated. Therefore, these two parts will be analyzed separately.

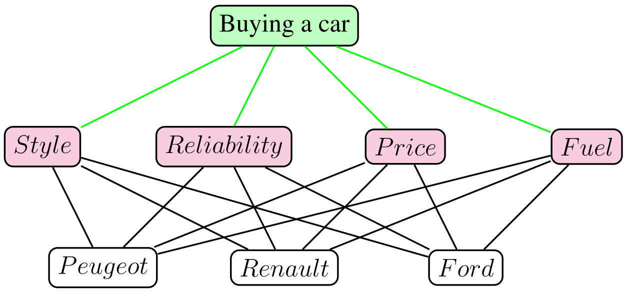

Figure 1.3: Hierarchy of car choice AHP model

1.5.1 Determining Weights of Criteria

The first step in AHP is to calculate the relative importance of the different criteria. We provide an initial matrix (see Table 1.6) for the pair-wise comparison criteria in which the principal diagonal contains entries of 1, as each factor is as important as itself. For instance, when Fuel criterion is compared to Style criterion then the decision maker has determined that Fuel is between to be classified "somewhat more important" than Style. Thus, the corresponding comparison assumes the value of 3. A similar interpretation is true for the rest of the entries.

Table 1.6: Pair-wise comparisons of criteria

Criteria	Style	Reliability	Fuel	Price
Style	1	$\frac{1}{2}$	3	5
Reliability	2	1	4	4
Fuel	$\frac{1}{3}$	$\frac{1}{4}$	1	3
Price	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	1

After completing the matrix, the Eigenvector method is applied to get the criteria weights. We start by computing the sum of the values in each column of the pair-wise comparison matrix as given in Table 1.7.

After that, we divide each element in the matrix by its column total to get a normalized pair-wise comparison matrix. Then, we compute the average of the elements in each row of the normalized matrix as shown in Table 1.8. Hence the relative weight of each criterion is given in Table 1.9.

At this level, the consistency index should be calculated. Perfect consistency rarely occurs in practice. Ratings should be consistent in two ways. First, ratings should be transitive. That

Table 1.7: Pair-wise comparison matrix

Criteria	Style	Reliability	Fuel	Price
Style	1	$\frac{1}{2}$	3	5
Reliability	2	1	4	4
Fuel	$\frac{1}{3}$	$\frac{1}{4}$	1	3
Price	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	1
Sum	$\frac{53}{15}$	2	$\frac{25}{3}$	13

Table 1.8: Normalized Pair-wise comparison matrix

Criteria	Style	Reliability	Fuel	Price
Style	$\frac{15}{53}$	$\frac{1}{4}$	$\frac{9}{25}$	$\frac{5}{13}$
Reliability	$\frac{30}{53}$	$\frac{1}{2}$	$\frac{12}{25}$	$\frac{4}{13}$
Fuel	$\frac{5}{53}$	$\frac{1}{8}$	$\frac{3}{25}$	$\frac{3}{13}$
Price	$\frac{3}{53}$	$\frac{1}{8}$	$\frac{1}{25}$	$\frac{1}{13}$

Table 1.9: Computing the criteria priority values

Criteria	Style	Reliability	Fuel	Price	**Priority**
Style	1	$\frac{1}{2}$	3	5	0.32
Reliability	2	1	4	4	0.47
Fuel	$\frac{1}{3}$	$\frac{1}{4}$	1	3	0.14
Price	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	1	0.07

means that if Reliability is better than Style and Style is better than Fuel, then Reliability must be better than Fuel. Second, ratings should be numerically consistent. For example, we know that “Reliability $=2$ Style” and “Reliability $=4$ Fuel” that means that “Style $=(4/2)$ Fuel”.

To calculate the consistency ratio we must solve:

$$Aw = \lambda_{max}.w$$

by solving:

$$det(\lambda I - A) = 0$$

We get $\lambda_{max} = 4.16272$. Then, the CI index is found by:

$$CI = (\lambda_{max} - n)/(n - 1) = 0.0542$$

The final step is to calculate the CR by using the table derived from Saaty's book (see Table 1.5):

$$CR = CI/RI = 0.0542/0.90 = 0.0602$$

where $RI = 0.90$ because the pair-wise comparison matrix is a matrix of order 4.

CR value is less than 0.1, so the evaluations are consistent. A similar procedure is repeated for the rest of matrix.

1.5.2 Determining priorities of alternatives with respect to criteria

After computing the importance of criteria, the same methodology is applied to find the respective values for alternatives. But now, the alternatives should be pair-wise compared with respect to each criterion particularly. That means, this analysis should be repeated for 4 more times for each criterion. For instance, if we compare alternatives regarding Style criterion, Table 1.10 is achieved.

Table 1.10: Comparison matrix for Style Criterion

Style	Peugeot	Ford	Renault
Peugeot	1	$\frac{1}{4}$	4
Ford	4	1	4
Renault	$\frac{1}{4}$	$\frac{1}{4}$	1

Then, we compare alternatives regarding Reliability and Fuel criteria and we obtain Tables 1.11 and 1.12.

Table 1.11: Comparison matrix for Reliability Criterion

Reliability	Peugeot	Ford	Renault
Peugeot	1	2	5
Ford	$\frac{1}{2}$	1	3
Renault	$\frac{1}{5}$	$\frac{1}{3}$	1

Table 1.12: Comparison matrix for Fuel Criterion

Fuel	Peugeot	Ford	Renault
Peugeot	1	2	5
Ford	$\frac{1}{2}$	1	3
Renault	$\frac{1}{5}$	$\frac{1}{3}$	1

Table 1.13: Comparison matrix for Price Criterion

Price	Quantity		**Priority**
Peugeot	25	$\frac{25}{58}$	0.43
Ford	18	$\frac{18}{58}$	0.31
Renault	15	$\frac{15}{58}$	0.26
sum	58		1

Finally, we compute the priority vector of Price criterion by normalizing the obtained data, since it's a quantitative criterion (see Table 1.13).

Similar to criterion calculation methodology, the Eigenvector method is computed to get the alternatives scores. Hence the relative priority matrix is given in Table 1.14.

Table 1.14: Alternatives priority matrix

	Style	Reliability	Fuel	Price
Peugeot	0.25	0.58	0.58	0.43
Ford	0.64	0.31	0.31	0.31
Renault	0.11	0.11	0.11	0.26

1.5.3 Synthetic utility

The next step is to calculate the global priorities to obtain the final ranking of alternatives and to select the best one. So, to determine these final scores, we will apply the Equation (1.4) by multiplying the criteria weights' by the ratings for the decision alternatives for each criterion and summing the respective products. The obtained results are shown in Table 1.15.

As a result, Peugeot will be preferred since it has the highest values.

Table 1.15: The global priority matrix

Criteria	Style	Reliability	Fuel	Price	**Priority**
	0.32	0.47	0.14	0.07	1
Peugeot	0.25	0.58	0.58	0.43	0.4639
Ford	0.64	0.31	0.31	0.31	0.4156
Renault	0.11	0.11	0.11	0.26	0.1205

1.6 Advantages and limits of AHP method

Triantaphyllou and Lin (1996) summarized the following advantages for AHP:

- It is the only known MCDM model that can measure the consistency in the decision maker judgments.
- It can also help decision makers to organize the critical aspects of a problem in a hierarchical structure, making the decision process easy to handle.
- Pair-wise comparisons in the AHP are often preferred by the decision makers, allowing them to derive weights of criteria and scores of alternatives from comparison matrices rather than quantify weights/scores directly.
- AHP can be combined with well-known operation research techniques to handle more difficult problems.
- AHP is easier to understand and can effectively handle both qualitative and quantitative data.

Despite wide applications of the AHP in a variety of domains, the method has been criticized from several viewpoints.

The first problem is that of rank reversal. This was indicated by Belton and Gear (1983). In many scenarios, the rankings of alternatives obtained by the AHP may change if a new alternative is added. Belton and Gear introduced one alternative, which was an exact copy of one of the alternatives and then re-evaluated the matrices. This amounted to adding one more column to the matrix with elements similar to those of the original entries in the column corresponding to the earlier alternative.

Secondly, the human preference model is uncertain and decision makers might be unable to assign exact numerical values to the comparison judgments. Although, the use of the discrete scale of $1-9$ for performing pair-wise comparative analysis has the advantage of simplicity, a decision maker may find it extremely difficult to express the strength of his preferences and to provide exact pair-wise comparison judgments.

Also, the number of pair-wise comparisons increases when the number of criteria or alternatives increases.

To handle these pitfalls, many extensions of this standard AHP method have been developed. A survey of these extensions will be provided in Chapter 2.

1.7 Conclusion

In the first part of this Chapter, we have briefly presented the fundamental concepts of the MCDM method. We have given an overview of the available MCDM methods. In the second part, we have presented with more details the basics of the AHP method, one of the most widely used approaches, with a detailed example.

Despite the advantages of AHP method, several researches are focusing on improving more and more the results provided by this approach, especially, in an environment where uncertainty may exist in the different levels relative to a decision making problem. Consequently, standard AHP method should be adapted to handle such imperfection. Therefore, several approaches are combined within uncertain theories such as probability theory, fuzzy set theory and belief function theory.

The next Chapter is devoted to the presentation of the main developed approaches, especially those developed under the belief function framework. This topic besides belief function theory dealing uncertainty will be at the core of some contributions of this Thesis.

Chapter 2

AHP method under uncertainty

2.1 Introduction

The Analytic Hierarchy Process (AHP) has emerged as a successful and practical Multicriteria Decision Making (MCDM) technique applied in a variety of areas. This approach gives good results in a context in which everything is known with certainty. However, the reality is connected to uncertainty and imprecision by nature. Hence, one of the main problems of the standard AHP is that it does not take into account uncertainty in the judgments since the matrices of judgments are deterministic. In real applications, the decision maker is always subject to uncertainty while expressing their judgments and do not like to be forced to give deterministic answers. Moreover, by eliminating uncertainty from the judgments it becomes impossible to evaluate its impact on the final decision's uncertainty. This limitation greatly reduces the confidence of the users on the final results of the AHP technique.

To overcome these difficulties and to extend the AHP on a more real elicitation procedure, several AHP methods are combined within uncertain theories. As a result, three different families of approaches for the problem are proposed: the fuzzy approach, the probabilistic approach and the belief approach.

In this Chapter, we will especially focus on both AHP method under fuzzy set theory and AHP method under the belief function framework, therefore some known fuzzy AHP will be detailed. Regarding the belief function framework, we will present Utkin method (Utkin &

Simanova, 2008) and Evidential Reasoning approach (ER-MCDM) (Dezert et al., 2010) and we will give more details about DS/AHP (Beynon et al., 2000) which constitutes one of the main focus of our work.

This Chapter is organized as follows: we start, in Section 2.2, by describing the main developed methods. Then, in Section 2.3, we introduce the belief functions that are used to represent knowledge under the belief function framework. Several basic operations are also detailed. In Section 2.4, we present some AHP approaches under the belief function theory.

2.2 AHP method under uncertainty

Probabilistic AHP (Vargas, 1982; Escobar & Moreno-Jimenez, 2000; Manassero et al., 2004), fuzzy AHP (Laarhoven & Pedrycz, 1983; Lootsma, 1997) are compact representations of uncertainty in different levels. Their success is due to their capacity of handling imperfection and solving more complex problems. In this Section, we briefly recall these approaches.

2.2.1 Brief refresher on Probabilistic AHP

The first work was introduced by Vargas (1982). He tries to demonstrate that if the judgments in a matrix are gamma distributed variables then the eigenvector follows a Dirichlet, or multinomial beta, distribution. However, there are two problems with this approach. First, the judgments in the pair-wise comparison matrix are reciprocals since the gamma distribution is not reciprocal. Therefore defining an element V_{ij}, of the matrix as a gamma random variable, we have that $V_{ji} = \frac{1}{V_{ij}}$, is not distributed as a gamma. Therefore, the order in which the decision maker gives his judgments modifies the probability distributions of the elements of the matrix and this goes against one of the main principle of the AHP methodology. Secondly, the elements of the principal diagonal are treated as random variables but by definition they must be equal to 1 (Manassero et al., 2004).

Escobar and Moreno-Jimenez (2000) have presented a method that solves both problems, demonstrating that if a judgment follows a reciprocal distribution then also its reciprocal is a random variable that follows the same kind of distribution. Arbel (1989) proposes a hybrid stochastic-interval AHP (SIAHP) approach to address uncertainty in group decision making by integrating interval judgment. The goal of this method is to process a matrix whose entries are intervals. These intervals may be assumed to be constraints in an optimization problem, or intervals characterized by some type of probability distribution. The optimization approach yields a vector of intervals, one for each of the components and the simulation approach yields a probability distribution, for the priorities.

2.2.2 Brief refresher on fuzzy AHP

Fuzzy set theory

The fuzzy set theory was introduced by Zadeh (1965). This theory, which was a generalization of classic set theory, allowed the membership functions to operate over the range of real numbers $[0, 1]$. The uncertainty can be represented by the fuzzy number. A triangular fuzzy number is a special fuzzy set $F = \{(x, \mu(x)), x \in \Re\}$, where x takes its values on the real line and $\mu(x)$ is a continuous mapping from $\Re$ to the closed interval $[0, 1]$.

A triangular fuzzy number is denoted by $\tilde{M} = (a, b, c)$, where $a \leq b \leq c$, can be described as:

$$\mu_{\tilde{M}}(x) = \begin{cases} 0 \ , \ x < a \\ \frac{x-a}{b-a} \ , \ a \leq x \leq b \\ \frac{c-x}{c-b} \ , \ b < x \leq c \\ 1 \ , \ x > c \end{cases} \tag{2.1}$$

in which the parameters a, b and c respectively denote the smallest possible value, the most promising value and the largest possible value that describe a fuzzy event.

Fuzzy AHP method

Fuzzy AHP (Laarhoven & Pedrycz, 1983; Chang, 1996) uses fuzzy set theory to express the uncertain comparison judgments as a fuzzy numbers. The main steps of fuzzy AHP are as follows:

1. Structuring decision hierarchy. Similar to standard AHP, the first step is to break down the complex decision making problem into a hierarchical structure.

2. Developing pair-wise fuzzy comparison matrices. Consider a prioritization problem at a level with n elements, where pair-wise comparison judgments are represented by fuzzy triangular numbers $\tilde{v_{ij}} = (\tilde{a_{ij}}, \tilde{b_{ij}}, \tilde{c_{ij}})$ as shown in Table 2.1. As in the standard AHP, each set of comparisons for a level requires $\frac{n(n-1)}{2}$ judgments.

Table 2.1: The fuzzy Saaty scale

Intensity of importance	**Definition**
$\tilde{1} = (1, 1, 3)$	Equal importance
$\tilde{3} = (1, 3, 5)$	Somewhat more important
$\tilde{5} = (3, 5, 7)$	Much more important
$\tilde{7} = (5, 7, 9)$	Very much more important
$\tilde{9} = (7, 9, 9)$	Absolutely more important
$\tilde{2}$,$\tilde{4}$,$\tilde{6}$,$\tilde{8}$	Intermediate values

3. Consistency check and deriving priorities. This step checks for consistency and extracts the priorities from the pair-wise comparison matrices.

4. Aggregation of priorities and ranking the alternatives. The final step aggregates local priorities obtained at different levels of the decision hierarchy into composite global priorities for the alternatives based on the weighted sum method.

The existing fuzzy AHP methods mainly differ on the employed fuzzy judgments in above-stated Step 2 or the developed fuzzy prioritization method in Step 3, or both. Laarhoven and Pedrycz (1983), Buckley (1985) and Lootsma (1997) used a triangular membership function and developed a fuzzy version of the logarithmic least squares method. Chang (1996) introduced an extent analysis method for the synthetic extent values of the pair-wise comparisons and applied a simple arithmetic mean algorithm to find fuzzy priorities from comparison matrices, whose elements are represented by triangular fuzzy numbers.

All these methods derive fuzzy priorities and, after aggregating, the final scores of the alternatives are also represented as fuzzy numbers or fuzzy sets. Due to the large number of multiplication and additional operations, the resulting fuzzy scores have wide supports and overlap over a large range (Wang et al., 2008; Javanbarg et al., 2012).

Wang et al. (2006) proposed a modified fuzzy logarithmic least squares method based on a constrained nonlinear optimization model. To reduce the computational requirement, Chang

(1996) proposed the extent analysis to derive the crisp weights from triangular membership functions and this has been applied to numerous real-life problems such as in implementing cleaner production in a manufacturing firm (Tseng et al., 2009) and prioritizing environmental issues in off-shore oil and gas operations (M. Yang et al., 2011). Also, fuzzy AHP extensions have been extensively applied to solve supplier selection problem (Junior et al., 2014).

Some drawbacks in using existing fuzzy AHP were also pointed out by Wang et al. (2006). Mikhailov (2003) argued that these fuzzy AHP variants require an additional defuzzification procedure to convert fuzzy weights to crisp weights and he proposed fuzzy preference programming technique to derive the crisp weights from the fuzzy pair-wise comparison judgment matrix.

Comments

Many AHP extensions have been developed and the difference between them mainly lies in the type of uncertainty related to the problem at hand and especially in the way of dealing with that uncertainty when making a decision. Therefore, probability and fuzzy set theories are widely used to represent such uncertainty but they fail to handle imprecise, uncertain and conflicting information. For that reason, we are interested in belief function theory which offers a natural and simple tool to handle imperfect information. It represents an appropriate framework for experts to express their partial beliefs in a much more flexible way.

The belief function theory, known also as the theory of evidence or Dempster-Shafer theory, is a general framework for reasoning with uncertainty (Smets & Kennes, 1994). It shows its efficiency in many real-world applications such classification (Elouedi et al., 2001; Trabelsi et al., 2011; Smets, 2001), image processing (Denoeux & Zouhal, 2001; Lefevre et al., 2000), multi-criteria decision making (Beynon et al., 2001; Frikha & Moalla, 2014; J. Yang et al., 2006), etc.

In the next Section, we will especially focus on the belief function theory and we will explain in details its different belief functions, which are used to represent knowledge. On the other hand, we will present more the main proposed approaches under this theory.

2.3 Belief function theory

The term Dempster-Shafer refers to the origin of the theory. In fact, in the 1960s, Dempster developed the basic ideas of a new mathematical theory of uncertainty that includes a kind of upper and lower probabilities (Dempster, 1967, 1968). Then, in 1970s, it was extended by Shafer (1976) to what is now known as belief function theory.

Several interpretations of this theory have been proposed (Smets, 1991) among them: the Dempster model (Dempster, 1967, 1968), the lower probability model (Walley, 1991), the theory of hints (Kohlas & Monney, 1995) and the Transferable Belief Model (TBM) (Smets & Kennes, 1994). In this Thesis, we deal with the interpretation of the belief function theory as explained by the TBM.

2.3.1 Basic concepts

Let us denote by Θ the finite non empty set including n elementary events (hypotheses) representing the solutions of a given problem. These events are assumed to be exhaustive and mutually exclusive. The set Θ, called the frame of discernment, is defined as:

$$\Theta = \{\theta_1, \theta_2, \dots, \theta_n\} \tag{2.2}$$

All the subsets of Θ belong to the power set of Θ, denoted by 2^Θ and every element of 2^Θ is called a proposition or an event:

$$2^\Theta = \{A/A \subseteq \Theta\} \tag{2.3}$$

In Shafer's model, Θ is assumed to be exhaustive (Shafer, 1976) which means that the solution to a given problem is unique and is necessarily included in this frame of discernment. However, in the TBM, Smets relaxed this condition, considering that it is sometimes difficult to list a priori all the possible hypotheses related to a given problem domain. He induced what he called the open-world assumption and the closed-world assumption (Smets, 1990, 1998).

Under the open-world assumption, Θ is not necessarily exhaustive. It means that we admit that the problem domain can include some unknown hypotheses that we did not mention into the frame of discernment, whereas under the closed-world assumption the frame of discernment is exhaustive.

Example 2.1. *Let us treat a problem of identification of childhood diseases. Some of the most common illnesses of childhood cause skin eruptions and are known as exanthems. The childhood exanthems include measles, rubella and fifth disease. All of these infections have the same symptoms.*

Suppose the frame of discernment Θ *related to this problem is defined as follows:*

$$\Theta = \{measles, rubella, fifth\ disease\}$$

Then, the power set of Θ *is:*

$$2^{\Theta} = \{\emptyset, \{measles\}, \{rubella\}, \{fifth\ disease\},$$
$$\{measles,\ rubella\}, \{measles,\ fifth\ disease\}, \{rubella,\ fifth\ disease\}, \Theta\}$$

Basic belief assignment

The basic belief assignment (bba), called initially by Shafer basic probability assignment (Shafer, 1976), assigns a belief in range $[0, 1]$ to every member of 2^{Θ} (bba can assign belief to any proposition in the frame and not only to the elementary ones) such that their sum is 1. That means, a function is called a basic belief assignment such that (Shafer, 1976):

$$\sum_{A \subseteq \Theta} m(A) = 1 \tag{2.4}$$

The value $m(A)$, named a basic belief mass (bbm), represents the portion of belief committed exactly to the event A and not for a particular subset of A. In this way, committing belief to a proposition A does not necessarily imply that the remaining belief is committed to $\bar{A}$. The mass $m(\Theta)$ quantifies the part of belief committed to the whole frame Θ. It represents the beliefs that are not assigned to the different subsets of Θ.

The subsets of Θ such that $m(A) > 0$ are called focal elements and $\mathcal{F}(m)$ is the set of all its elements relative to the bba m. The union of all focal elements is called core.

Similarly, $m(\emptyset)$ represents the part of belief allocated to the empty set. Shafer has initially imposed the condition $m(\emptyset) = 0$. This condition reflects the fact that no belief ought to be allocated to the empty set. Such bba is called a normalized basic belief assignment. However, this condition is relaxed in the TBM. The allocation of a positive mass to the empty set $(m(\emptyset) > 0)$ is interpreted as a consequence of the open-world assumption (Smets, 1990). A mass of belief is assigned to each possible subset.

The normalization process is defined as follows:

$$m(A) = \begin{cases} 0 \text{ if } A = \emptyset \\ K.m(A) \text{ otherwise} \end{cases} \tag{2.5}$$

where $K^{-1} = 1 - m(\emptyset)$. It is called the normalization factor.

Since the belief function theory models several types of imperfection, special bbas were defined. In particular, we have:

- **Vacuous bba**: is a normalized bba with Θ is its unique focal element (Shafer, 1976). So, its corresponding bba is defined as follows:

$$m(\Theta) = 1 \text{ and } m(A) = 0, \ \forall A \subset \Theta, \ A \neq \Theta \tag{2.6}$$

  Such basic belief assignment quantifies the state of total ignorance, in which there is no reason to belief in any proposition more than another and all the propositions are plausible.

- **Categorical bba**: the case where the bba has a unique focal element A (Mellouli, 1987):

$$m(A) = 1, \ \forall A \subset \Theta \tag{2.7}$$

- **Bayesian bba**: is a bba where all focal elements are singletons. It is a particular case of probabilities.

$$\text{if } m(A) > 0 \text{ then } |A| = 1, \text{ where } |A| \text{ stands for the cardinal of } A. \tag{2.8}$$

- **Certain bba**: is a bba having only one focal element which is a singleton.

$$m(A) = 1, \quad \forall A \in \Theta \tag{2.9}$$

- **Consonant bba**: is a bba in which all the focal elements $(A_1, A_2, \ldots, A_n)$ are nested, that is $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$ (Dubois et al., 2001).

Example 2.2. *Let us consider the example defined in Example 2.1. Table 2.2 shows an example of bbas expressed on the frame of discernment* Θ.

Table 2.2: Example of bbas expressed on the frame of discernment Θ

	certain	categorical	Bayesian	consonant	vacuous	any
$\emptyset$	0	0	0	0	0	0.1
{measles}	0	0	0.5	0.4	0	0.2
{rubella}	1	0	0.3	0	0	0.05
{fifth disease}	0	0	0.2	0	0	0.05
{measles,rubella}	0	0	0	0.1	0	0.3
{measles, fifth disease}	0	1	0	0	0	0.1
{rubella, fifth disease}	0	0	0	0	0	0.1
Θ	0	0	0	0.5	1	0.1

Belief function

Belief function or credibility function, denoted *bel*, corresponding to a specific bba m, assigns to every subset A of Θ the sum of the masses of belief committed exactly to every subset of A by m (Shafer, 1976).

Unlike the bbm $m(A)$ which measures the exact portion of belief assigned to the subset A, $bel(A)$ quantifies the total amount of belief assigned to the subsets implying A without implying $\bar{A}$. It is obtained by summing all the bbms given to the subsets of A. Since $m(\emptyset)$ supports not only A, but also $\bar{A}$, the empty set must be discarded from the sum.

The belief function *bel* is defined for $A \subseteq \Theta$ and $A \neq \emptyset$ as:

$$bel : 2^{\Theta} \rightarrow [0, 1]$$
$$bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B)$$

The belief function bel satisfies the following condition (Shafer, 1976):

$$bel(A_1 \cup \cdots \cup A_n) \geq \sum_i bel(A_i) - \sum_{i>j} bel(A_i \cap A_j) - \cdots - (-1)^n bel(A_i \cap A_n), \quad \forall A_1, \ldots, A_n \in 2^{\Theta} \tag{2.10}$$

Shafer assumed that $bel(\Theta) = 1$ (Shafer, 1976). This can be ignored in the TBM, under the open world assumption, requiring only that $bel(\Theta) < 1$.

Example 2.3. *Let us consider the same example defined in Example 2.1. Suppose a doctor expressing a piece of evidence concerning the diseases. The obtained bba is then defined as follows:*
$m(\{\textit{measles}\}) = 0.6$;
$m(\{\textit{measles, rubella}\}) = 0.2$;
$m(\Theta) = 0.2$.

The belief function bel *corresponding to the bba* m *is defined by:*
$bel(\{\textit{measles}\}) = 0.6$;
$bel(\{\textit{rubella}\}) = 0$;
$bel(\{\textit{fifth disease}\}) = 0$;
$bel(\{\textit{measles, rubella}\}) = 0.8$;
$bel(\{\textit{measles, fifth disease}\}) = 0.6$;
$bel(\{\textit{rubella, fifth disease}\}) = 0$;
$bel(\Theta) = 1$.

For example, 0.6 *represents the part of belief exactly committed to the hypothesis "the patient has measles".*

Plausibility function

The plausibility function pl, expresses the maximum amount of specific support that could be given to a proposition A in Θ. It measures the degree of belief committed to the propositions compatible with A. $pl(A)$ is then obtained by summing the bbms given to the subsets B such that $B \cap A \neq \emptyset$ (Shafer, 1976).

The plausibility function is defined by:

$$pl : 2^{\Theta} \rightarrow [0, 1]$$
$$pl(A) = \sum_{B \cap A \neq \emptyset} m(B)$$

There is a simple relationship between the belief function bel and the plausibility function pl associated with a mass function m:

$$pl(A) = bel(\Theta) - bel(\bar{A}) \quad \forall A \subseteq \Theta \tag{2.11}$$

where $\bar{A}$ denotes the complement of A.

Also,

$$pl(A) = bel(\Theta) - bel(\bar{A}) \quad \forall A \subseteq \Theta \tag{2.12}$$

Example 2.4. *Let us continue with Example 2.1. The plausibility function pl corresponding to the bba m is defined by:*
$pl(\{\textit{measles}\}) = 1$;
$pl(\{\textit{rubella}\}) = 0.4$;
$pl(\{\textit{fifth disease}\}) = 0.2$;
$pl(\{\textit{measles, rubella}\}) = 1$;
$pl(\{\textit{measles, fifth disease}\}) = 1$;
$pl(\{\textit{rubella, fifth disease}\}) = 0.4$;
$pl(\Theta) = 1$.

For example, 0.4 represents the maximum degree of belief that could be given to the proposition "rubella".

2.3.2 Basic tools

Combination rules

Let m_1 and m_2 be two bbas induced from two distinct information sources and defined on the same frame of discernment Θ. The combination of these bbas induces a bba on the same frame Θ and it can be either conjunctive or disjunctive.

The choice of one of these rules of combination for aggregating pieces of evidence may be guided by meta-belief concerning the reliability of the sources. In fact, if we know that both sources of information are fully reliable, then we combine them conjunctively. However, if we know that at least one of the two sources is reliable, then we combine them disjunctively (Smets, 1990, 1991).

1. **Conjunctive rule of combination:** When we know that both sources of information are fully reliable, the resulting bba is computed by the conjunctive rule of combination. Hence,

the induced bba quantifies the combined impact of the two pieces of evidence. It is defined as follows (Smets, 1990):

$$(m_1 \circledcap m_2)(A) = \sum_{B,C \subseteq \Theta, B \cap C = A} m_1(B) m_2(C), \quad \forall A \subseteq \Theta \tag{2.13}$$

The conjunctive rule is considered as the unnormalized Demspter rule of combination dealing with the closed world assumptions, defined as follows (Shafer, 1986):

$$(m_1 \oplus m_2)(A) = K. \sum_{B,C \subseteq \Theta, B \cap C = A} m_1(B) m_2(C) \tag{2.14}$$

where

$$K^{-1} = 1 - \sum_{B,C \subseteq \Theta, B \cap C = \emptyset} m_1(B) m_2(C) \tag{2.15}$$

and

$$(m_1 \oplus m_2)(\emptyset) = 0 \tag{2.16}$$

K is called the normalization factor.

The conjunctive rule of combination is characterized by the following properties:

- Commutativity:

$$m_1 \circledcap m_2 = m_2 \circledcap m_1 \tag{2.17}$$

- Associativity:

$$(m_1 \circledcap m_2) \circledcap m_3 = m_1 \circledcap (m_2 \circledcap m_3) \tag{2.18}$$

- Non-idempotency:

$$m \circledcap m \neq m \tag{2.19}$$

- Neutral element:
  The neutral element within the conjunctive rule of combination is the vacuous basic belief assignment representing the total ignorance:

$$m \circledcap m_0 = m \tag{2.20}$$

  where m_0 is a vacuous bba.

2. **Disjunctive rule of combination:** The dual of the conjunctive rule is the disjunctive rule of combination. We use it when we only know that at least one of the sources of information is reliable but we do not know which one is reliable (Smets, 1998). It is defined as:

$$(m_1 \circledcirc m_2)(A) = \sum_{B,C \subseteq \Theta, B \cup C = A} m_1(B) m_2(C) \tag{2.21}$$

   The disjunctive rule of combination (as the conjunctive rule of combination) is commutative and associative.

3. In addition to the conjunctive and disjunctive combination rule, a larger choice of combination rules has been recognized by many researchers involved in real-world applications (Lefevre et al., 2002; Yager, 1987): Yager's rule (Yager, 1987), the cautious rule (Denoeux, 2006), Inagaki's unified combination rule (Inagaki, 1991) and combination with adapted conflict (Lefevre & Elouedi, 2013), etc.

Example 2.5. *Let us consider two distinct doctors' evidences S_1 and S_2. The first evidence is expressed by a bba m_1 and defined as follows:*
$m_1(\{$*measles*$\}) = 0.4$;
$m_1(\{$*rubella*$\}) = 0.1$;
$m_1(\{$*measles, fifth disease*$\}) = 0.3$;
$m_1(\Theta) = 0.2$;

The second evidence is expressed by m_2 defined as follows:
$m_2(\{$*fifth disease*$\}) = 0.5$;
$m_2(\{$*measles, fifth disease*$\}) = 0.4$;
$m_2(\Theta) = 0.1$;

The bba corresponding to the conjunctive combination of both pieces of evidence is defined as follows:
$(m_1 \circledcap m_2)(\emptyset) = 0.2 + 0.05 + 0.04 = 0.29$;
$(m_1 \circledcap m_2)(\{$*measles*$\}) = 0.16 + 0.04 = 0.2$;
$(m_1 \circledcap m_2)(\{$*rubella*$\}) = 0.01$;
$(m_1 \circledcap m_2)(\{$*fifth disease*$\}) = 0.1 + 0.15 = 0.25$;
$(m_1 \circledcap m_2)(\{$*measles, fifth disease*$\}) = 0.08 + 0.12 + 0.02 = 0.22$;
$(m_1 \circledcap m_2)(\Theta) = 0.03$;

$m_1 \circledcap m_2$ represents the joint bba induced from the combination of m_1 and m_2 by using the conjunctive rule of combination.

Once the disjunctive rule of combination is applied, we get:
$(m_1 \circledcup m_2)(\{$*measles, fifth disease*$\}) = 0.2 + 0.1 + 0.16 + 0.08 = 0.54$*;*
$(m_1 \circledcup m_2)(\{$*rubella, fifth disease*$\}) = 0.05$*;*
$(m_1 \circledcup m_2)(\Theta) = 0.15 + 0.12 + 0.04 + 0.02 + 0.03 + 0.04 + 0.01 = 0.41$*;*

$m_1 \circledcup m_2$ *represents the joint bba induced from the combination of* m_1 *and* m_2 *by using the disjunctive rule of combination.*

Discounting

In the Transferable Belief Model, discounting allows to take in consideration the reliability of the information source that generates the bba m.

For $\alpha \in$ [0,1], let (1-α) be the degree of confidence ("reliability") that we assign to the source of information. It quantifies the strength of reliability given to the expert (Smets, 1992).

If the source is not fully reliable, the expert's opinions are represented as follows:

$$m^{\alpha}(A) = (1 - \alpha)m(A), \ \ \forall A \subset \Theta \tag{2.22}$$

$$m^{\alpha}(\Theta) = \alpha + (1 - \alpha)m(\Theta) \tag{2.23}$$

where α is the discounting factor.

Example 2.6. *The degree of reliability given to the expert is equal to* 0.7*. If we consider this bba defined as follows:*
$m(\{$*measles*$\}) = 0.4$*;*
$m(\{$*measles,fifth disease*$\}) = 0.3$*;*
$m(\Theta) = 0.3$*;*

So, we obtain after discounting this bba:
$m^{\alpha}(\{$*measles*$\}) = 0.4 \times 0.7 = 0.28$*;*
$m^{\alpha}(\{$*measles,fifth disease*$\}) = 0.3 \times 0.7 = 0.21$*;*
$m^{\alpha}(\Theta) = 0.3 + (0.3 \times 0.7) = 0.51$*;*

2.3.3 Decision making

It is necessary, when making a decision, to select the most likely hypothesis. Some solutions are developed to ensure the decision making within the belief function theory. One of the most used is the pignistic probability proposed within the TBM (Smets & Kennes, 1994).

The TBM is a model that aims at quantifying someone's degree of belief. It is based on two levels (Smets & Kennes, 1994):

- The credal level where beliefs are entertained and represented by belief functions.
- The pignistic level where beliefs are used to make decisions and represented by probability functions called the pignistic probabilities.

When a decision must be made, beliefs held at the credal level induce a probability measure at the pignistic measure denoted $BetP$ (Smets, 1998):

$$BetP(A) = \sum_{B \subseteq \Theta} \frac{|A \cap B|}{|B|} \frac{m(B)}{(1 - m(\emptyset))}, \ \ \forall A \in \Theta \tag{2.24}$$

It includes normalization and division of bbas assigned to focal elements by their cardinality.

Example 2.7. *Assume that at the credal level, beliefs are represented by the following bba:*
$m(\{$*measles*$\}) = 0.7$*;*
$m(\{$*measles,rubella*$\}) = 0.2$*;*
$m(\Theta) = 0.1$*;*

To select the most probably hypothesis, we have to compute the corresponding pignistic probabilities BetP to make the optimal decision, so we have:
$BetP(\{$*measles*$\}) = 0.84$*;*
$BetP(\{$*rubella*$\}) = 0.13$*;*
$BetP(\{$*fifth disease*$\}) = 0.03$*;*

We notice that the most probable disease is "measles". So, if we have to decide, we will choose this hypothesis.

2.3.4 Uncertainty Measures

In the case of the belief function framework, the bba is defined on an extension of the powerset: 2^Θ and not only on Θ. In the powerset, each element is not equivalent in terms of precision. Indeed, $\theta_i \subset \Theta$ ($i \in \{1, 2\}$) is more precise than $\theta_1 \cup \theta_2 \subseteq \Theta$.

In order to try to quantify this imprecision, different uncertainty measures have been defined (Pal et al., 1992, 1993). In this section we will focus on some of these measures proposed within the belief function framework. Therefore, two types of uncertainty can be expressed: nonspecificity or imprecision, and discord or strife.

Nonspecificity is connected with sizes (cardinalities) of relevant sets of alternatives while discord expresses conflicts among the various sets of alternatives. Composite measures, referred to as global or total measures of uncertainty, have also been proposed. They attempt to capture both nonspecificity and conflict.

Nonspecificity Measures

The nonspecificity measure is introduced by Dubois and Prade in order to measure the nonspecificity of a normal bba by a function N defined as (Pal et al., 1992, 1993):

$$N(m) = \sum_{A \in \mathcal{F}(m)} m(A) \log_2 |A| \tag{2.25}$$

The bba m is all the most imprecise (least informative) that $N(m)$ is large. The minimum ($N(m) = 0$) is obtained when m is a Bayesian bba (focal elements are singletons) and the maximum ($N(m) = \log_2 |A|$) is reached when m is a vacuous bba ($m(\Theta) = 1$).

Another measure of nonspecificity is defined as:

$$J(m) = 1 - \sum_{A \in \mathcal{F}(m)} \frac{m(A)}{|A|} \tag{2.26}$$

Conflict Measures

Conflict measures are a generalization of the Shannon's entropy and they were expressed as follows (Pal et al., 1992, 1993):

$$conflict(m) = -\sum_{A \in \mathcal{F}(m)} m(A) \log_2 f(A) \tag{2.27}$$

where f is, respectively, pl, bel or $BetP$ and the conflict measures are called, respectively, Dissonance (E), Confusion (C) and Discord (D).

Pal et al. (1993) proposed a different conflict measure, defined as:

$$I(m) = -\sum_{A \subseteq \Omega} \log_2 q(A) \tag{2.28}$$

Notice that this measure is not a generalization of the Shannon's entropy and it exists when $m(\Omega) > 0$.

Pal et al. (1992) proposed a measure of total conflict which is a generalization of Vajda's quadratic entropy, given by:

$$TC(m) = -\sum_{A,B \in \mathcal{F}(m)} m(A)m(B)CON(A,B) \tag{2.29}$$

where $CON(A,B) = 1 - \frac{|A \cap B|}{|A \cup B|}$ represents the conflict between propositions A and B.

Composite Measures

Different measures have been defined by Pal et al. (1992, 1993), such as: global uncertainty ($G1$), total uncertainty (T, H), and pignistic entropy (EP). These measures are defined, respectively, as:

$$EP(m) = -\sum_{\omega \in \Omega} BetP(\omega) \log_2 BetP(\omega) \tag{2.30}$$

$$G1(m) = E(m) + N(m) \tag{2.31}$$

$$T(m) = D(m) + N(m) \tag{2.32}$$

$$H(m) = \sum_{A \in \mathcal{F}(m)} m(A) \log_2 (\frac{|A|}{m(A)}) \tag{2.33}$$

The interesting feature of $H(m)$ is that it has a unique maximum.

2.3.5 Multi-variable operations

In the previous subsections, we have presented the basic concepts of the belief function theory. These mechanisms are based on the assumption that bbas are defined on the same frame of discernment. However, this constraint limits the practical applications. Let us consider in what follows, a first frame Θ and a second frame Ω.

Cylindrical extension and projection

- **Cylindrical extension:** allows to extend a set defined in low-dimensional domain into a higher-dimensional domain. Consider a subset $A \subseteq \Theta$, the cylindrical extension of A to $\Theta \times \Omega$ is denoted $A^{\uparrow \Theta \times \Omega}$. It is obtained as:

$$A^{\uparrow \Theta \times \Omega} = A \times \Omega. \tag{2.34}$$

Example 2.8. *Let us consider* $\Theta = \{\theta_1, \theta_2\}$*. We want to define* θ_1 *into the two-dimensional space* $\Theta \times \Omega$ *where* $\Omega = \{\omega_1, \omega_2\}$*. Its cylindrical extension is computed as:*

$$\theta_1^{\uparrow \Theta \times \Omega} = \{(\theta_1, \omega_1), (\theta_1, \omega_2)\}.$$

- **Projection:** is the opposite operation of cylindrical extension. It allows to reduce a set defined in a multi-dimensional domain to a set defined in a lower-dimensional domain. Let C be a subset of $\Theta \times \Omega$. Projecting C on Ω, denoted $C^{\downarrow \Omega}$, means dropping extra coordinates. It is obtained by:

$$C^{\downarrow \Omega} = \{\omega, \omega \in \Omega, C \cap \omega^{\uparrow \Omega \times \Theta} \neq \emptyset\}. \tag{2.35}$$

Example 2.9. *Let us consider* $\{(\theta_1, \omega_1), (\theta_2, \omega_1)\}$ *defined on* $\Theta \times \Omega$*. The projection of this set into* Ω *is equal to:* $\{(\theta_1, \omega_1), (\theta_2, \omega_1)\}^{\downarrow \Omega} = \omega_1$.

Vacuous Extension

This operation is useful, when the referential is changed by adding new variables. Thus, a marginal mass function m^{Θ} defined on Θ will be expressed in the frame $\Theta \times \Omega$ as follows (Smets, 1993):

$$m^{\Theta \uparrow \Theta \times \Omega}(C) = \begin{cases} m^{\Theta}(A) & \text{if } C = A \times \Omega, A \subseteq \Theta, \\ 0 \text{ otherwise.} \end{cases} \tag{2.36}$$

Example 2.10. *Given the following bba defined on* $\Theta = \{\theta_1, \theta_2, \theta_3\}$ *as:*

$m^{\Theta}(\{\theta_1\}) = 0.5$, $m^{\Theta}(\{\theta_2\}) = 0.2$ *and* $m^{\Theta}(\Theta) = 0.3$.

Let $\Omega = \{\omega_1, \omega_2\}$.

The bba defined on Θ *will be defined in a finer frame* $\Theta \times \Omega$ *using the vacuous extension as follows:*
$m^{\Theta\uparrow\Theta\times\Omega}(\{(\theta_1, \omega_1), (\theta_1, \omega_2)\}) = 0.5$
$m^{\Theta\uparrow\Theta\times\Omega}(\{(\theta_2, \omega_1), (\theta_2, \omega_2)\}) = 0.2$
$m^{\Theta\uparrow\Theta\times\Omega}(\Theta \times \Omega) = 0.3$

Marginalization

Given a mass distribution defined on the product space $\Theta \times \Omega$, marginalization corresponds to mapping over a subset of the product space by dropping the extra coordinates. The new belief defined on Θ is obtained by (Smets, 1993):

$$m^{\Theta\times\Omega\downarrow\Theta}(A) = \sum_{\{B\subseteq\Theta\times\Omega|B^{\downarrow\Theta}=A)\}} m^{\Theta\times\Omega}(B), \forall A \subseteq \Theta. \tag{2.37}$$

$B^{\downarrow\Theta}$ denotes the projection of B onto Θ.

Example 2.11. *Let us consider the bba defined on* $\Theta \times \Omega$*:*
$m^{\Theta\times\Omega}(\{(\theta_1, \omega_1), (\theta_1, \omega_2)\}) = 0.5$
$m^{\Theta\times\Omega}(\{(\theta_2, \omega_1), (\theta_2, \omega_2)\}) = 0.2$
$m^{\Theta\times\Omega}(\{(\theta_1, \omega_1)\}) = 0.3$

Marginalizing $m^{\Theta\times\Omega}$ *on the coarser frame* Θ, $m^{\Theta\times\Omega\downarrow\Theta}$ *will lead to the following distribution:*
$m^{\Theta\times\Omega\downarrow\Theta}(\{\theta_1\}) = 0.5 + 0.3 = 0.8$
$m^{\Theta\times\Omega\downarrow\Theta}(\{\theta_2\}) = 0.2$

Ballooning Extension

This operation is useful if an agent after conditioning realizes that the evidence he has considered as true was not and accordingly he would reconstruct the initial distribution. It can also be useful if beliefs are defined on a limited set and other alternatives were discovered afterwards. The agent should redistribute his beliefs to take them into account.

Let $m^{\Theta}[\omega]$ represents your beliefs on Θ conditionally on ω a subset of Ω. To get rid of conditioning, we have to compute its ballooning extension.

Conditional masses are transferred to C, the largest subset of $\Theta \times \Omega$ whose intersection with the vacuous extension of ω followed by a projection on Θ gives A: $(A \cap \omega^{\uparrow \Theta \times \Omega})^{\downarrow \Theta} = A$.

Thus, $C = (A \times \omega \cup \Theta \times \bar{\omega})$ where $\bar{\omega}$ stands for the complement of ω.

Accordingly, the ballooning extension is defined as:

$$m^{\Theta}[\omega]^{\Uparrow \Theta \times \Omega}(A \times \omega \cup \Theta \times \bar{\omega}) = m^{\Theta}[\omega](A), \forall A \subseteq \Theta. \tag{2.38}$$

Example 2.12. *Let us consider* $\Theta = \{\theta_1, \theta_2, \theta_3\}$, $\Omega = \{\omega_1, \omega_2\}$ *and the conditional bba* $m^{\Theta}[\omega_1](\theta_1) = 0.6$. *Its corresponding basic belief mass on* $\Theta \times \Omega$ *is obtained by taking into consideration* $\{(\theta_1, \omega_1)\}$ *and all the instances of* Θ *for the complement of* ω_1.

Hence, $m^{\Theta}[\omega_1]^{\Uparrow \Theta \times \Omega}(\{(\theta_1, \omega_1), (\theta_1, \omega_2), (\theta_2, \omega_2), (\theta_3, \omega_2)\}) = m^{\Theta}[\omega_1](\theta_1)$.

Refinement and coarsening

Sometimes, beliefs are induced by information sources with different but compatible frames of discernment. The coarsening and refinement operations allow to establish relationships between these different frames in order to express beliefs on anyone of them.

Let Ω and Θ be two finite sets. The idea behind the refinement consists in obtaining one frame of discernment Ω from the set Θ by splitting some or all of its events (Shafer, 1976).

Conversely, the coarsening consists in forming a frame Θ by grouping together the events of the frame of discernment Ω.

Let us define a mapping ρ: $2^{\Theta} \rightarrow 2^{\Omega}$ such that (Shafer, 1976):

$$\rho(\{\theta\}) \neq \emptyset \;\; \forall \theta \in \Theta \tag{2.39}$$

$$\rho(\{\theta\}) \cap \rho(\{\theta^{'}\}) = \emptyset \;\; \text{if } \theta \neq \theta^{'} \tag{2.40}$$

$$\bigcup_{\theta \in \Theta} \rho(\{\theta\}) = \Omega \tag{2.41}$$

So, given a disjoint partition $\rho(\{\theta\})$ one may set (Shafer, 1976):

$$\rho(A) = \bigcup_{\theta \in A} \rho(\{\theta\}) \tag{2.42}$$

For each $A \in \Theta$, $\rho(A)$ consists of all the possibilities in Ω by splitting the elements of A (Shafer, 1976).

The mapping ρ: $2^{\Theta} \to 2^{\Omega}$ is called a refining, Ω is a refinement of Θ and Θ is the coarsening of Ω.

Example 2.13. *Let us continue with the same problem domain:*

$$\Theta = \{\mathit{measles}, \mathit{rubella}, \mathit{fifth\ disease}\}$$

A possible refinement of the frame of discernment Θ is:

$$\Omega = \{\mathit{childhood\_measles}, \mathit{congenital\_measles}, \mathit{childhood\_rubella}, \mathit{congenital\_rubella}, \\ \mathit{childhood\_fifth\ disease}, \mathit{congenital\_fifth\ disease}\}$$

where
$\rho(\{\mathit{measles}\}) = \{\mathit{childhood\_measles}, \mathit{congenital\_measles}\}$
$\rho(\{\mathit{rubella}\}) = \{\mathit{childhood\_rubella}, \mathit{congenital\_rubella}\}$
$\rho(\{\mathit{fifth\ disease}\}) = \{\mathit{childhood\_fifth\ disease}, \mathit{congenital\_fifth\ disease}\}$

Inversely, Θ is considered as the coarsening of Ω.

2.3.6 MCDM under the belief function theory

Under this framework, many MCDM method are proposed by various authors (Frikha & Moalla, 2014; J. Yang et al., 2006; Fan & Nguyen, 2011), etc. In this Section, we present MCDM method under the belief function framework. Then, we will give more details about belief AHP extensions.

An interesting method has been introduced by J. Yang (1994) is the evidential reasoning (ER) approach. This method is suitable for representation and quantification of subjective judgments with uncertainty. To solve a hybrid MCDM problem, the first step is to evaluate and quantify the state of a qualitative criterion at each alternative by defining a few evaluation grades for the criterion, which are quantified using a certain scale. The state of the criterion at an alternative may be evaluated to one of the grades. The scale of the confirmed grade may then be used as a numerical value for measuring the state of the criterion at the alternative.

Boujelben et al. (2011) propose a method under the belief function framework inspired by ELECTRE I. Under this approach, the authors consider MCDM problems where the alternatives are evaluated on a set of ordinal criteria. The evaluation of each alternative with respect to each criterion may be uncertain and imprecise and is provided by one or several experts. Therefore, they model this evaluation as a basic belief assignment (bba). In order to compare the different pairs of alternatives according to each criterion, they introduce the concept of first belief dominance.

Tacnet and Dezert (2011) developed a new methodology called Cautious Ordered Weighted Averaging with Evidential Reasoning (COWA-ER). COWA-ER is proposed for decision making under uncertainty to take into account imperfect evaluations of the alternatives and unknown beliefs about groups of the possible states of the world (scenarii). COWA-ER mixes cautiously the principle of Yager's Ordered Weighted Averaging (OWA) approach (Yager, 1988) with the efficient fusion of belief functions proposed in Dezert-Smarandache Theory (DSmT) (Smarandache & Dezert, 2004).

2.4 AHP method under the belief function framework

Many AHP extensions have been developed under the belief function framework. In particular, Beynon et al. have proposed a method called the DS/AHP method (Beynon et al., 2000; Beynon, 2002b) comparing not only single alternatives, but also groups of alternatives. Besides, several works have been proposed by Utkin (2009), (Utkin & Simanova, 2008). The main feature of his approach is that it allows the expert to deal with comparisons of arbitrary subsets of alternatives and criteria (Utkin & Simanova, 2012). Additionally, Dezert et al. (2010) have developed the DSmT/AHP which is based on the Dezert-Smarandache theory (Smarandache & Dezert, 2004). This method aimed at performing a similar purpose as DS/AHP that is to compare groups of alternatives. This method has been extended by Schubert (2014). In the following subsections, we will present the main developed methods.

2.4.1 DS/AHP approach

Though the popularity and efficiency of the AHP approach, this method is often criticized, because it cannot be applied in an uncertain and imprecise context. In fact, in some cases, the decision maker cannot make pair-wise comparisons between all the alternatives.

To solve this problem, Beynon et al. (2000) propose to extend the AHP on a more real elicitation procedure. Beynon et al. have proposed a method called the DS/AHP method, to compare not only a single alternatives but also groups of alternatives between each other.

Within DS/AHP method, for each criterion, there are certain groups of decision alternatives, including Θ, about which the decision maker can express some degree of favorable knowledge.

Through comparing a group of alternative to Θ, the decision maker will express some degree of favorable knowledge on each of these groups of alternative. This differs from the AHP method that makes pair-wise comparisons between individual decision alternatives (Beynon et al., 2000, 2001), here each group of alternatives identified is compared to all possible alternatives in the frame of discernment.

After identifying the candidate sets of criteria, what is left is setting priorities of the sets of alternatives. At this point, classical pair-wise comparisons of the elements are made to obtain these priorities.

In DS/AHP method, the Saaty's scale was modified for simplicity, it is reduced to 5 unit scale (see Table 2.3).

Table 2.3: Knowledge/favourable scale

Opinion/Knowledgeable	Numerical rating
Extremely favourable	6
Strongly to extremely	5
Strongly favourable	4
Moderately to strongly	3
Moderately favourable	2

To calculate the priority vector, the weight of criteria must be incorporated in the pair-wise comparison matrix. This is done by multiplying the elements in the last column (except the last entry in that column) by the respective importance value for that criterion. If p is the weight of the criterion j and v_{ij} is the favorability opinion for a particular group of alternatives, then the resultant value is $p \times v_{ij}$ (the resultant change in the bottom row of the matrix is similarly $(1/(p \times v_{ij}))$.

After computing the priority vector, the priority values in each column sum to one. As a result, Beynon et al. (2001) consider that these priorities values are a basic belief assignment.

Then, the obtained priorities vectors are combined using the Dempster rule of combination to integrate them into a single bba. In fact, Beynon et al. (2001) assume that criteria are independent pieces of evidence, offering information on the decision maker's knowledge towards the favorability of the identified groups of alternatives, hence the associated bbas are independent.

Finally, to choose the best alternative, the presented approach uses the belief and plausibility functions.

Example 2.14. *To describe the DS/AHP method, we will use a simple example (Beynon, 2002b). Indeed, the decision involves buying a new car, from a choice of three well-known types of car (A, B and C). However, there are four criteria: Price, Fuel economy, Comfort and Style that are believed influence the choice of car. Hence, the overall objective is to decide which is the best car to buy.*

First, we establish the hierarchy frame for purchasing a car, as shown in Figure 2.1. So, for each criterion, there are certain groups of decision alternatives, including Θ.

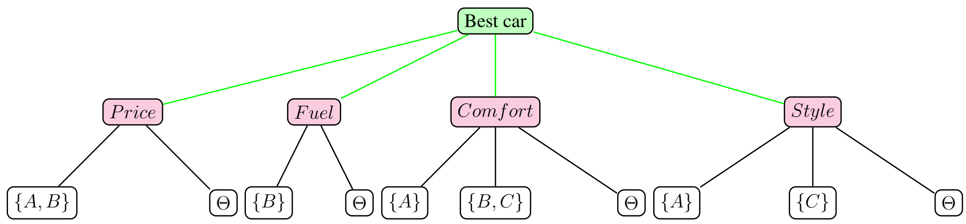

Figure 2.1: Hierarchy of modified car choice model

To calculate the weight of criteria, the standard AHP method is applied to get the priority vector (see Table 2.4).

Also, the same process is used to get the alternatives priorities. For example, the pair-wise comparison for the Comfort criterion is shown in Table 2.5.

The zero's which appears in the knowledge matrix indicates no attempt to assert knowledge

Table 2.4: Criteria priority values

Criterion	Price	Fuel	Comfort	Style
Priority	0.3982	0.0851	0.2159	0.2988

Table 2.5: Initial pair-wise matrix for Comfort criterion

Comfort	$\{A\}$	$\{B,C\}$	$\{A,B,C\}$
$\{A\}$	1	0	$4p$
$\{B,C\}$	0	1	$6p$
$\{A,B,C\}$	$1/4p$	$1/6p$	1

between groups of decision alternatives.

Next, to calculate the priority vector, we multiply each element of the pair-wise matrices (except the last entry in that column) by the respective importance value for that criterion.

Let us continue with the Comfort criterion, which had an importance value $p = 0.2159$*, we obtain the Table 2.6.*

Table 2.6: Pair-wise matrix for Comfort criterion after influence of its priority rating

Comfort	$\{A\}$	$\{B,C\}$	$\{A,B,C\}$
$\{A\}$	1	0	0.8714
$\{B,C\}$	0	1	1.3072
$\{A,B,C\}$	1.1475	0.7650	1

For the other three criteria, Table 2.7 summarizes the obtained comparisons.

In these knowledge matrices, the eigenvector method is again used to calculate the priority values (see Table 2.8). That is, the normalized elements of the eigenvector associated with the largest eigenvalue from the matrix.

The priority values in each column sum to one. These are directly defined as basic belief assignments. So, for Comfort criterion, we note m_c *as the associated bba:*

$$m_c(\{A\}) = 0.2417,\ m_c(\{B\}) = 0.3625,\ m_c(\Theta) = 0.3958$$

We can go through a similar process with Price, Fuel and Style. These bbas are independent

Table 2.7: Pair-wise matrix for Price, Fuel and Style

Price	$\{A, B\}$	Θ	
$\{A, B\}$	1	$6p$	
Θ	$1/6p$	1	
Fuel	$\{B\}$	Θ	
$\{B\}$	1	$3p$	
Θ	$1/3p$	1	
Style	$\{A\}$	$\{C\}$	Θ
$\{A\}$	1	0	$5p$
$\{C\}$	0	1	$2p$
Θ	$1/5p$	$1/2p$	1

Table 2.8: Priority values

Price	Priority	Fuel	Priority	Comfort	Priority	Style	Priority
$\{A, B\}$	0.7049	$\{B\}$	0.2034	$\{A\}$	0.2417	$\{A\}$	0.4261
Θ	0.2951	Θ	0.7966	$\{B, C\}$	0.3625	$\{C\}$	0.1705
				Θ	0.3958	Θ	0.4034

pieces of evidence. Hence, the associated bba are independent. So, we use the Dempster rule of combination to get a single bba and the resulting bba is shown in Table 2.9.

Table 2.9: The bba m_{car} after combining all evidence

DA	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{B, C\}$	Θ
m_{car}	0.4309	0.2312	0.0511	0.1650	0.0527	0.0691

Now, we can calculate the belief and plausibility functions and we obtain the following Table 2.10

If we consider the focal element $\{A\}$, then there is a small amount of evidence in favour as well as against the hypothesis A being the best car. For $\{C\}$ the table shows strong evidence against being the best choice. Interestingly the set $\{A, B\}$ shows strong evidence in favour of including within its elements the best choice of car.

Table 2.10: Belief and plausibility values for subsets of cars

Cars	Bel	Pl
$\{A\}$	0.4309	0.6650
$\{B\}$	0.2312	0.5180
$\{C\}$	0.0511	0.1729
$\{A, B\}$	0.8271	0.9489
$\{A, C\}$	0.4820	0.7688
$\{B, C\}$	0.3350	0.5691

2.4.2 Utkin method

Utkin and Simanova (2008) make an extension of the DS/AHP method. The proposed approach takes into account the fact that a MCDM, problem might have several levels of criteria. Moreover, it is assumed that expert judgments concerning the criteria are imprecise and incomplete. The proposed extension also uses groups of experts for comparing decision alternatives and criteria.

The proposed approach is summarized in the following steps (Utkin, 2009):

- We suppose that there is a set of alternatives consisting of m and a set of criteria consisting of n. An expert chooses some subset of alternatives and compares this subset with another subset of alternatives with respect to a certain criterion. In the same way, the decision maker chooses some subset of criteria from the set and compares this subset with another group of criteria.
- The extended matrix of pair-wise comparisons of alternatives in this case has $2^m - 1$ columns and rows (the empty set is not considered here). Similarly, the extended matrix of comparisons of criteria has $2^n - 1$ columns and $2^n - 1$ rows.
- It is supposed that expert compares only subsets of alternatives and criteria, but they do not provide preference values or weights. At that, if an expert supplies the comparison assessment: a_i is preferred to a_j, then the value 1 is added to the corresponding cell in the comparison matrix (i-th row and j-th column). In this case, the preference value a_{ij} can be regarded as the number of experts chosen this comparison assessment.
- For every pair-wise comparison in the extended comparison matrix, we define its bba as follows. If N is the number of expert providing the judgments, then we can compute the

bba by:

$$m(\{(a_i, a_j)\}) = \frac{a_{ij}}{N} \tag{2.43}$$

Example 2.15. *Let us study a decision problem where the decision maker has to choose which one of three alternatives evaluated based on two criteria c_1 and c_2. Suppose that 15 experts provide preferences concerning criteria as shown in Table 2.11.*

Table 2.11: Expert preferences related to criteria

	$\{c_1\}$	$\{c_2\}$	$\{c_1, c_2\}$
	D_1	D_2	D_3
c_k	6	4	5
$m(D_k)$	$\frac{6}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

By introducing the special algebra of sets of preferences, a combination procedure is defined to aggregate the references for alternatives and for criteria. At this level, the DS/AHP method is applied. This method is used to identify the most favorable subsets from the available set of criteria or alternatives. Then, the second task is to use the belief Bel and plausibility Pl functions of preferences as a lower and upper probabilities. Therefore, Utkin et al. solve this problem as a linear programming problems.

2.4.3 ER-MCDM method

This framework called ER-MCDM mixes the AHP and uncertainty theories including fuzzy sets, possibility and belief function theories (Tacnet et al., 2010, 2011). The principle of the ER-MCDM methodology is to use AHP to analyze the decision problem and to replace the aggregation step by two successive fusion processes. Its main objective is to take into account both information imperfection, source reliability and conflict.

ER-MCDM is described as follows:

- Analyzing the decision problem through a hierarchical structure.
- Defining the evaluation classes for decision through a common frame of discernment.
- Evaluating the qualitative or quantitative criteria.

- Mapping the evaluations of criteria into the common frame of discernment for decision.
- Fusing the mapped evaluations of criteria to get a basic belief assignment related to the evaluations classes of decision (frame of discernment for decision).

The decision consists in choosing a sensitivity level (no, low, medium, high sensitivity). Those levels are the decision hypotheses that are used to define the common frame of discernment for information fusion.

These steps are independent one from each other. Therefore, imprecise and uncertain evaluations of quantitative or qualitative criteria can be done by the sources (experts) and re-used with different mapping models. Uncertainty theories are then used:

- to represent different kinds of information imperfection (unconsistency, imprecision, uncompletness and uncertainty).
- to evaluate the criteria (possibility theory and belief function theory).
- to map the evaluations into the frame of discernment (fuzzy sets theory).
- to consider the multiple information sources and produce a decision (belief function theory).

Example 2.16. *The ER-MCDM methodology uses the conceptual framework of AHP both to analyze the decision problem (representing it through a weighted hierarchy of criteria and sub-criteria) and to identify preferences between criteria. However, it introduces several new features as described below on Figure 2.2.*

Quantitative criteria, such as the number of occupants, are evaluated through possibility distributions representing both imprecision and uncertainty. The expert provides evaluations as intervals with confidence level. The question is then to transform a number of occupants into a level of sensitivity keeping information about uncertainty. Possibility distribution is derived into bba. We use a mapping process that project the bbas expressed on intervals on bbas expressed on the frame of discernment of decision (low-LS, medium-MS and high-HS sensitivity levels). After this evaluation step, we get, for each criterion, bbas related to the same frame of discernment.

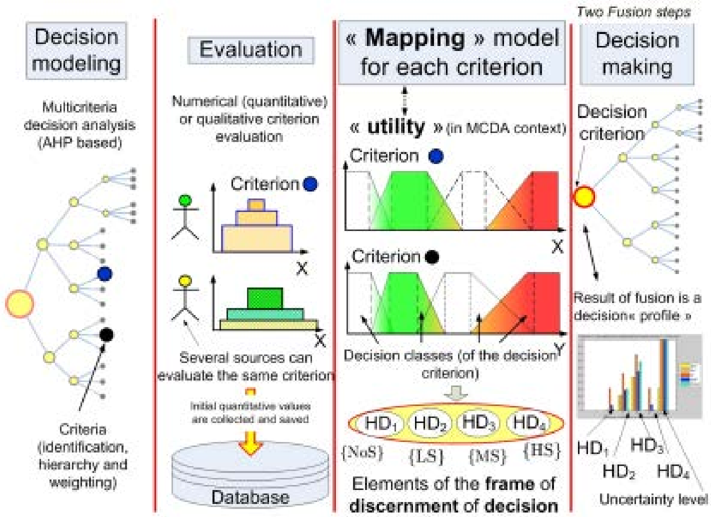

Figure 2.2: Decision problem related to avalanche risk zoning (Tacnet et al., 2010)

2.4.4 TIN-DS/AHP method

Du et al. (2014) propose to fuse the Three-point Interval Number (TIN) method (Zhu et al., 2007) and the DS/AHP approach. This method introduces a new elicitation procedure based on TIN matrix.

TIN knowledge matrix is used to describe cognitive inference of experts group, for one thing, it could overcome the defect which is lost easily in the process of decision information gathering, for another thing, it could also prevent the decision invalid problems which are caused by dispersive opinion of experts. Then, decision information could not be only derived from relatively judgment between group, but also could be derived from relatively judgement between group and recognition framework, which is benefit for experts with their own structure of knowledge to choose the most appropriate way to express deduction information and also benefit for mutual corroboration and conflict revising between decision information.

Example 2.17. *Let us consider the following knowledge matrices, our main objective is to generate their corresponding bbas.*

The first step is to compute their corresponding the element $\tilde{v}_j^{nn'}$ *of TIN matrices which include minimum* $\overleftarrow{v_j^{nn'}}$*, center* $\overline{v_j^{nn'}}$ *and* $v_j^{nn'}$ *maximum elements. So generating a random* $1-9$ *scale value in the interval* $[\tilde{v}_j^{nn'}-1, \tilde{v}_j^{nn'}+1]$ *treat it as* $\overline{v_j^{nn'}}$*.* $[v_j^{nn'}-1, \overline{v_j^{nn'}}]$ *and* $[\overline{v_j^{nn'}}, \overline{v_j^{nn'}}+1]$*,*

$$V_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 1/5 & 1/6 & 1/4 & 1/3 & 1 \end{bmatrix},$$
$$V_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 1/5 & 1/4 & 1/3 & 1/2 & 1 \end{bmatrix},$$
$$V_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 1/6 & 1/4 & 1/2 & 1/3 & 1 \end{bmatrix}.$$

treat them as $\overleftarrow{v_j^{nn'}}$ *and* $v_j^{nn'}$.

The TIN knowledge matrices generated by above process is as shown in Figure 2.3.

$$\left.\begin{array}{l}
\tilde{V}_1 = \begin{bmatrix} [1/1,1/1,1/1] & [1/1,2/1,4/1] & [\quad] & [\quad] & [4/1,5/1,7/1] \\ [\quad] & [1/1,1/1,1/1] & [1/1,3/2,3/1] & [\quad] & [4/1,6/1,7/1] \\ [2/7,2/3,1/1] & [\quad] & [1/1,1/1,1/1] & [\quad] & [2/1,4/1,5/1] \\ [\quad] & [1/5,1/3,1/2] & [\quad] & [1/1,1/1,1/1] & [2/1,3/1,4/1] \\ [\quad] & [\quad] & [\quad] & [2/7,1/2,4/5] & [1/1,1/1,1/1] \end{bmatrix} \\
\tilde{V}_2 = \begin{bmatrix} [1/1,1/1,1/1] & [\quad] & [1/1,3/2,3/1] & [\quad] & [3/1,5/1,6/1] \\ [\quad] & [1/1,1/1,1/1] & [\quad] & [1/1,3/2,2/1] & [6/5,4/1,9/2] \\ [1/2,2/3,1/1] & [\quad] & [1/1,1/1,1/1] & [\quad] & [2/1,3/1,4/1] \\ [\quad] & [2/5,1/2,4/5] & [\quad] & [1/1,1/1,1/1] & [1/1,2/1,3/1] \\ [\quad] & [1/3,1/2,3/5] & [\quad] & [\quad] & [1/1,1/1,1/1] \end{bmatrix} \\
\tilde{V}_3 = \begin{bmatrix} [1/1,1/1,1/1] & [\quad] & [\quad] & [2/1,3/1,7/2] & [4/1,6/1,7/1] \\ [1/3,5/8,1/1] & [1/1,1/1,1/1] & [\quad] & [\quad] & [5/2,4/1,5/1] \\ [1/5,3/8,4/9] & [\quad] & [1/1,1/1,1/1] & [\quad] & [3/2,2/1,3/1] \\ [\quad] & [2/7,1/2,1/1] & [\quad] & [1/1,1/1,1/1] & [2/1,3/1,7/2] \\ [1/7,1/6,1/5] & [\quad] & [\quad] & [\quad] & [1/1,1/1,1/1] \end{bmatrix}
\end{array}\right\rangle$$

Figure 2.3: The corresponding TIN knowledge matrices

Then, according to TIN-DS/AHP, the corresponding bbas are solved and the decision is make using pignistic probabilities.

2.5 Conclusion

In this Chapter, we have presented an overview of the main developed AHP approaches under uncertain theories. In the second part, we have presented the basic concepts of belief function theory as understood in the Transferable Belief Model. This presentation shows that the belief function theory provides a convenient tool to handle uncertainty in decision problems, especially within Multi-Criteria Decision Making methods. For this reason, this theory will be used as a tool to formalize different forms of uncertainty under the AHP model.

As we have described, AHP method has some limits. Therefore, in the next Chapter, we will present belief AHP method as a possible solution to deal with uncertainty using the belief function theory framework. An improvement of this latter will also be proposed via new elicitation techniques.

PART II:

CONTRIBUTIONS

Part II presents the contributions of this Thesis. This part focuses on presenting our newly developed approaches solving some of the AHP mentioned limitations. The two first Chapters present new approaches based on a quantitative scale under the belief function framework. Secondly, the next two Chapters detail solutions dealing with qualitative elicitation technique. The proposed latter methods are based on a new qualitative reasoning approach that derive crisp priorities from qualitative judgments.

Chapter 3

Modeling dependency between alternatives and criteria

3.1 Introduction

Uncertainty is a source of complexity in decision making. There are various forms that may arise in multi-criteria decision making (MCDM), in particular the Analytic Hierarchy Process (AHP), from impression to lack of knowledge or ignorance. As described in the previous Chapter, at one level there is an uncertainty about alternatives that appear in the identification of the candidate alternatives. At another level, there is an uncertainty about the ability of the selected criteria to adequately represent the objective that the decision maker tries to achieve. However, imperfection may also be in the evaluation process. So, variability in all these factors has the potential to affect the ranking of alternatives of a MCDM problem.

Our objectives through this research are to handle the different types of uncertainty in order to solve more complex MCDM problems. Hence, in this Chapter, we will introduce a new approach, which is able to model dependency between alternatives and criteria.

Since the main purpose is to rank alternatives, the proposed methodology is constructed upon the basis of the AHP method. As mentioned previously, AHP is a pair-wise comparison technique, which can model a complex problem in a unidirectional hierarchical structure assuming that there are no interdependencies between or within the levels. It assumes that there are no relationships between the alternatives and criteria. However, at the alternative level, when modeling his assessment, the decision maker expresses his evaluation regarding each criterion. In other words, the alternative preferences depend on each criterion, which has an effect on the chosen alternatives.

Therefore, a new MCDM method has been developed as part of this research. The main aim of the proposed approach is to find a way to model the relationship between alternatives and criteria. While the methodology handles the existence of dependency between criteria and alternatives, it can also incorporate the belief function theory in order to correctly reflect the decision behavior of the expert. First of all, our method, called belief AHP, deals with uncertainty into two levels namely the criterion and alternative. So, the decision maker may express his preferences in groups of criteria and alternatives instead of a single one. In other ways and as an extension of the belief AHP approach, our second solution is introduced to represent the influences of the criteria on the evaluation of alternatives.

The rest of this Chapter is organized as follows: in Section 3.2, we introduce belief AHP, a MCDM approach that combines the belief function theory with the AHP method. Section 3.3 investigates the relationship between alternatives and criteria and presents our new belief AHP method where beliefs are expressed in terms of conditional mass distributions. In the last Section (Section 3.4), we present some experimental studies.

3.2 Belief AHP method

3.2.1 Introduction

This Section is dedicated to the presentation of the basic steps needed to ensure the ranking of alternatives in an uncertain environment. In this context, we introduce our belief AHP approach which is a combination of AHP method and the belief function theory. A first work has been introduced in Ennaceur et al. (2011). Its main idea is to compare groups of criteria and alternatives. Then, it represents the weights of criteria by a reliability measure and the alternatives priorities using basic belief assignments (bba).

This Section illustrates belief AHP method and investigates its fundamentals steps. Examples will be given to illustrate the approach.

3.2.2 Computational procedure

As with standard AHP method, building a belief AHP falls to the definition of its fundamental steps seen in the previous Chapter, namely, hierarchical model, pair-wise comparisons, local priorities and global priorities. The specific steps of the method are illustrated in the following subsection (Ennaceur et al., 2011).

A MCDM problem is defined as a set of alternatives $\Theta = \{a_1, \ldots, a_m\}$, a set of criteria $\Omega = \{c_1, \ldots, c_n\}$, a qualitative or quantitative assessment representing the performance of each alternative with respect to each criterion leading to the determination of a decision matrix for the alternatives and a weighting vector representing the relative importance of the evaluation criteria with respect to the overall objective of the problem. The general decision making paradigm based on the proposed method is shown in Figure 3.1.

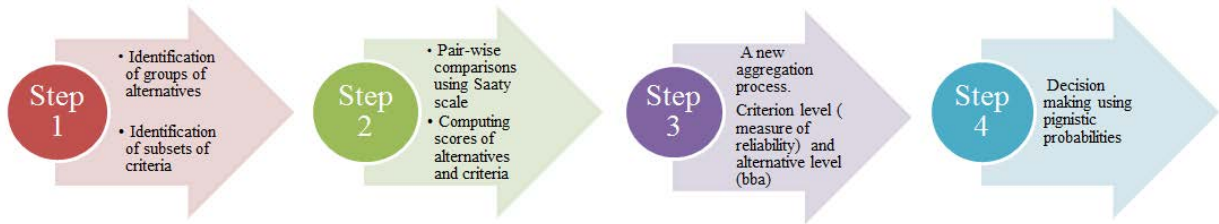

Figure 3.1: The general decision-making paradigm based on the belief AHP

Identification of the candidate criteria and alternatives

As shown in Figure 3.1, the ranking procedure starts at the determination of the criteria importance and alternative performance. Let Ω be a set of criteria, we denote the set of all subsets of Ω by 2^Ω and let C_k be the short notation of a subset of Ω (here k is the order number of the corresponding subset of Ω). Besides, in many complex problems, decision makers are able to compare only subsets of criteria and cannot evaluate separate ones. For example, for the Fuel, we may get two groups: the first one containing both natural gas and gasoline criteria and the second one with only singleton criterion which is diesel.

To solve this problem, that means to reduce the number of criteria which decreases the number of comparisons, this method suggests to allow the expert to express his opinions on groups of criteria instead of single one. So, the expert chooses these subsets by assuming that criteria having the same degree of preference are grouped together. For instance, if an expert identifies a group of criteria, then we could suppose that all of them have the same importance. Thus, these groups can be defined as:

$$\forall\, k, j | C_k, C_j \in 2^{\Omega}, C_k \cap C_j = \emptyset \text{ and } \bigcup_j C_j = \Omega \quad (\text{with } C_j \text{ exclusive}). \tag{3.1}$$

Since we are not performing pair-wise comparisons of criteria but relating to groups of criteria, these sets should not consider a criterion in common. If one criterion is included in two groups, then each group will give a different level of favorability and the expert judgment could be inconsistent.

Like the criterion level, the proposed model denotes the set of all subsets of alternatives Θ by 2^{Θ} and let A_k be the short notation of a subset of A, i.e., $A_k \subseteq \Theta$. At this level, this method suggests to not necessarily consider all of them but just to choose groups of those alternatives. One of the possible solutions of this task is to use the DS/AHP method (Beynon et al., 2000). Besides, the proposed approach applies the same hypothesis assumed in DS/AHP to identify the subsets of alternatives (see Chapter 2). The decision maker compares not only a single one but also sets of alternatives between each other.

Example 3.1. *To describe this approach, we consider the problem of purchasing a car. Suppose that this problem involves four criteria:* $\Omega = \{$*Price* (c_1), *Style* (c_2), *Fuel* (c_3), *Reliability* $(c_4)\}$ *and three selected alternatives:* $\Theta = \{$*Peugeot* (p), *Renault* (r), *Ford* $(f)\}$.

The expert can identify the following subsets $\{c_1\}$, $\{c_4\}$ *and* $\{c_2, c_3\}$. *He assumed that the two criteria* c_2 *and* c_3 *have the same degree of importance. Therefore, they are grouped together.*

Similarly to the criterion level, the judgments between decision alternatives over different criteria are dealt within an identical manner. For example, to evaluate the alternatives according to the criterion c_1, *the expert is asked to evaluate the following subsets of alternatives:* $\{p\}$ *and* $\{p, r, f\}$.

The belief AHP hierarchy is then constructed as shown in Figure 3.2.

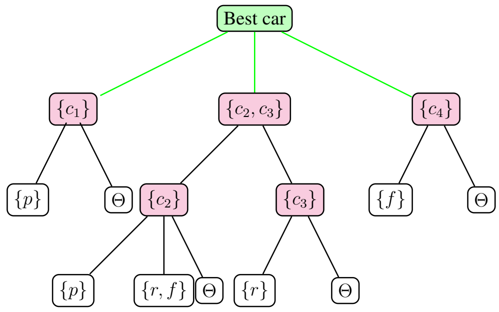

Figure 3.2: The belied AHP Hierarchy of car choice model

Pair-wise comparisons and preference elicitation

Once the sets of criteria and alternatives are defined, the expert tries to specify his assessments. In this study, the proposed model has adopted the Saaty's scale to evaluate the importance of pairs of grouped elements in terms of their contribution (see Step 2 in Figure 3.1). Thus, the priority vectors are then generated using the eigenvector method and the standard consistency index is chosen in order to ensure that belief AHP's pair-wise comparison matrices are consistent.

Example 3.2. *If we consider the previous example where: $\{c_1\}$, $\{c_4\}$ and $\{c_2, c_3\}$ are the groups of criteria which belong to the power set of 2^{Ω}. The decision maker can define his preferences by using the pair-wise comparison of the following type "How important is criterion $\{c_1\}$ relative to criterion $\{c_4\}$?". To respond to this question, the expert uses Saaty's scale expressing the intensity of the preference for one criterion versus another (see Table 3.1).*

Table 3.1: The weights assigned to the criteria according to the expert's opinion

Criteria	$\{c_1\}$	$\{c_4\}$	$\{c_2, c_3\}$	Priority
$\{c_1\}$	1	2	6	0.58
$\{c_4\}$	$\frac{1}{2}$	1	4	0.32
$\{c_2, c_3\}$	$\frac{1}{6}$	$\frac{1}{4}$	1	0.1

At the alternative level, a sample matrix for the Price criterion, for example, being shown in Table 3.2.

Table 3.2: Comparison matrix for Price criterion

c_1	$\{p\}$	$\{p,r,f\}$	Priority
$\{p\}$	1	9	0.806
$\{p,r,f\}$	$\frac{1}{9}$	1	0.194

Aggregation process: Updating the alternatives priorities'

The next step is to update the alternatives priorities with respect to the criterion weight (Step 3 Figure 3.1). On the one hand, we have priorities concerning criteria and groups of criteria instead of single ones. On the other hand, the sets of alternatives are compared pair-wise with respect to a specific single criterion. Within this structure of alternatives and criteria, the belief AHP method cannot use the strategy used by the standard method which aggregates all local priorities from the decision table by a simple weighted sum.

At the decision alternative level, uncertainty on the decision maker preferences over the set of alternatives is represented by a basic belief assignment (bba) defined on the set of possible alternatives. In fact, within this framework, we have $A_k \subseteq 2^\Theta$ and we have the priority values of each A_k in each comparison matrix representing the opinions-beliefs of the expert about his preferences. So, we assume that the set of alternatives is the frame of discernment and we notice that the priority vector sums to one which can be considered as a bba ($m(A_k)$) which represents its power set.

Given a pair-wise comparison matrix which compares the sets of alternatives according to a specific criterion. For each set of alternatives $A_k \subseteq 2^\Theta$ and A_k belongs to this pair-wise matrix, we get:

$$m(A_k) = v_k \tag{3.2}$$

where v_k is the eigenvalue of the k^{th} sets of alternatives.

The next step is to update the obtained bba with the importance of their respective criteria to measure their contribution. In AHP, the fusion is done from the product of the bbas matrix with the weighting vector of criteria. Such AHP fusion does not actually process efficiently the conflicting information between the sources.

To palliate these problems, we propose a new solution that regard each priority value of a specific set of criteria as a measure of reliability. In fact, this factor is used to update experts' beliefs (bba) by taking into account the important of each set of criteria. The idea is then to measure most heavily the bba evaluated according to the most important criteria and conversely for the less important ones.

If we have C_k (as defined above) as a subset of criteria, then we get β_k its corresponding measure of reliability:

$$\beta_k = \frac{\omega_k}{\omega_{max}} \tag{3.3}$$

where w_k is the eigenvalue of the k^{th} sets of criteria and ω_{max} is the maximum eigenvalue such as:

$$\omega_{max} = max_{k \in [1:l]} \omega_k \tag{3.4}$$

where l is the number of subsets of criteria.

Consequently, two cases will be presented. First, if the reliability factor represents a single criterion, then the corresponding bba will be directly discounted.

If C_k is a singleton criterion, then we apply the discounting rule and we get:

$$m^{\alpha_k}_{C_k}(A_j) = \beta_k.m_{C_k}(A_j),\ \forall A_j \subset \Theta \tag{3.5}$$

$$m^{\alpha_k}_{C_k}(\Theta) = (1 - \beta_k) + \beta_k.m_{C_k}(\Theta) \tag{3.6}$$

where A_j a subset of alternatives that are evaluated with respect to the criterion C_k, $m_{C_k}(A_k)$ the relative bbm for the subset A_k, β_k its corresponding measure of reliability and we denote $\alpha_k = 1 - \beta_k$.

Second, if this factor represents a group of criteria, their corresponding bbas must be combined using the conjunctive rule (see Equation 2.13), then it will be discounted by the measure of reliability relative to this group of criteria.

Let C_k a subset of criteria and $c_i \in C_k$, then we apply the conjunctive rule of combination to obtain m_{C_k}:

$$m_{C_k} = \textcircled{\cap} m_{c_i},\ \ i = \{1, \ldots, h\} \tag{3.7}$$

where h is the number of elements of a specific group of criteria C_k and c_i is a singleton criterion.

Finally, these obtained bbas (m_{C_k}) will be discounted by their corresponding measure of reliability β_k. We apply the same idea used in Equations 3.5 and 3.6, to get $m^{\alpha_k}_{C_k}$.

After updating the priorities of the alternatives sets with respect to their set of criteria, the overall bba must be computed. The conjunctive rule of combination is used as an aggregation operator in the belief function framework combining between two or several bbas:

$$m_{final} = \textcircled{\cap} m_{C_k}^{\alpha_k}, \quad k = \{1, \ldots, l\} \tag{3.8}$$

where l is the number of subsets of criteria.

The aggregation process is summarized in Figure 3.3.

Figure 3.3: A new aggregation process

Example 3.3. *Let us continue with the previous example. From Table 3.2, we can notice that the priority value is sum to one. So, we suppose that these priorities are the bbas. We denote these bbas by m_{c_1} and we get:*

$$m_{c_1}(\{p\}) = 0.806 \text{ and } m_{c_1}(\{p, r, f\}) = 0.194.$$

Then, we can go through a similar process with Reliability, Fuel and Style. We get the following information shown on Table 3.3.

Table 3.3: Priorities values

c_1	m_{c_1}	c_2	m_{c_2}	c_3	m_{c_3}	c_4	m_{c_4}
$\{p\}$	0.806	$\{p\}$	0.4	$\{r\}$	0.889	$\{f\}$	0.606
$\{p, r, f\}$	0.194	$\{r, f\}$	0.405	$\{p, r, f\}$	0.111	$\{p, r, f\}$	0.394
		$\{p, r, f\}$	0.195				

Then, these bbas must be combined with their criteria. Firstly, this step concerns the groups of criteria, that is the $\{c_2, c_3\}$ subset. Our aim is to update the priority alternatives relative to the Style and Fuel criteria. Therefore, by using Equation 3.7, belief AHP combines the bba relative to the Fuel and Style criteria:

$$m_{c_2,c_3} = m_{c_2} \textcircled{\cap} m_{c_3}$$

Then, these obtained bbas are discounted by the measure of reliability $\beta_{c_2,c_3} = 0.17$ *(see Table 3.4) using Equations 3.5 and 3.6.*

Table 3.4: The measure of reliability

Criteria	$\{c_1\}$	$\{c_4\}$	$\{c_2, c_3\}$
β_i	1	0.55	0.17

After that, this step concerns the single criterion $\{c_1\}$ *and* $\{c_4\}$*, the relative bba are directly discounting using Equation 3.5 and 3.6, where the reliability measure* $\beta_{c_1} = 1$ *and* $\beta_{c_4} = 0.55$.

Finally, the conjunctive rule of combination is applied (see Equation 3.8), this leads us to get a single bba denoted by m_{car} *(see Table 3.5).*

Table 3.5: The bba m_{car} after combining all evidence

	$\emptyset$	$\{p\}$	$\{f\}$	$\{r\}$	$\{f, r\}$	Θ
m_{car}	0.37204	0.45301	0.0544	0.011728	0.00098945	0.10783

Decision making

To this end, the final step is to choose the best alternative. In this context, the pignistic transformation is used. The decision maker will choose the alternative which has the highest value of pignistic probabilities as shown in Step 4 in Figure 3.1.

Example 3.4. *The final step is then to choose the best alternatives; After computing the pignistic probabilities, we get:*

$$BetP_{car}(\{p\}) = 0.77864,\ BetP_{car}(\{f\}) = 0.14465 \text{ and } BetP_{car}(\{r\}) = 0.076701.$$

The score of each alternative is depicted in Figure 3.4, which shows that Peugeot has higher priority than the other alternatives. As a result, Peugeot will be preferred since it has the highest values.

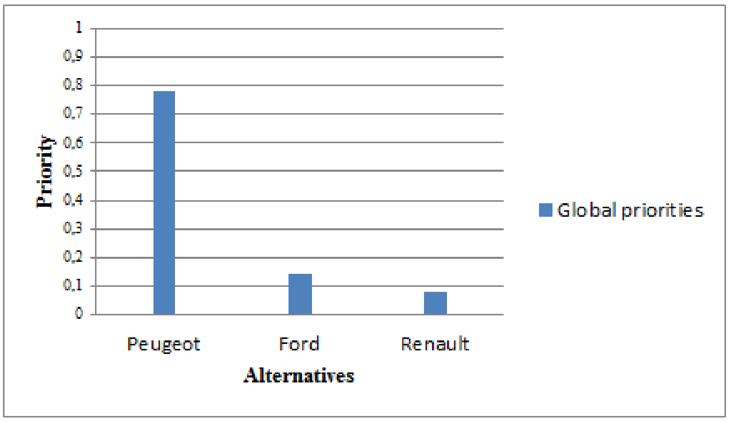

Figure 3.4: Ranking of alternatives using Belief AHP

Comments

We have shown, in the previous Section, that the belief AHP approach combines the AHP method and the belief function theory. On the one hand, the AHP allows to build bbas from decision maker preferences which are established with respect to several criteria and to compute the criteria weights. On the other hand, the belief function theory allows aggregating efficiently the (possibly highly conflicting) bbas based on each criterion.

The belief AHP approach provides for simplification of a complex multi-criteria decision making problem. A major advantage of this methodology is that it does not only compare pairs of criteria and alternatives but also enables us to evaluate groups of alternatives and subsets of criteria. Hence, by using belief AHP, we have reduced the number of comparisons. So, if we consider a two layer decision making model with n criteria and m alternatives then for the standard AHP method there will be $\frac{n(n-1)}{2}$ criterion comparisons and there would be $n \times \frac{m(m-1)}{2}$ alternative comparisons.

When we apply belief AHP method, in the worst case we can consider there to be n judgments made at the criterion level. Our methodology compares then singleton instead of groups of criteria. So, the number of comparison is the same as standard AHP ($\frac{n(n-1)}{2}$).

At the alternative level, we need to made m judgments regarding each criterion. The judgments can be thought of as deciding for each criterion whether to include it in a particular preference group of alternatives or to not include it. In this case, we are performing the same number of comparisons as DS/AHP method.

However, in the evaluation process, AHP assumes linear independence of criteria and alternatives. It does not properly represent the evaluation of each alternative regarding each criterion. For instance, in a problem of buying a car, the expert might consider that Peugeot is evaluated to be more important than Renault regarding Price criterion, but Renault is more important than Peugeot with respect to Style criterion. As we can see, the alternative priorities are dependent on each specific criterion. Hence, an extension of belief AHP method is proposed in order to model the dependency relation between alternatives and criteria.

3.3 A new belief AHP extension

In this Section, we develop a new AHP extension, called conditional belief AHP. Our approach tries to model a new relationship between alternatives and criteria.

3.3.1 Modeling dependency between alternatives and criteria

One of basic assumptions of AHP technique is that all the elements in the same hierarchy are totally independent. However, this assumption is hard to be satisfied due to ambiguousness and complexity of questions. In addition to this problem, the evaluating elements include dependent properties.

Under the AHP approach, criteria are assumed independent of the alternatives. However, paired comparisons imply dependence of a different kind. In fact, the importance assigned to an alternative depends on the evaluated criterion. This dependency is not according to structure, because we usually try to respect AHP axioms, but according to function.

In reality, when representing his assessments using the pair-wise comparison technique, the decision maker expresses his preferences regarding the upper level. For instance, the alternative priorities are usually dependent on each specific criterion. If the dependency among the elements exists, it can distort the result significantly. So, to raise reliance of AHP, it is needed to consider the effect of the degree of dependency between elements. In this study, we investigated the dependency and reflected that on belief AHP method.

Therefore, in the following Section, we propose a new AHP extension (Ennaceur et al., 2012b). Close to belief AHP, this model is called conditional belief AHP. Our aim through this work is then to represent uncertainty and to more imitate the expert reasoning, since he tries to express his preferences over the sets of alternatives regarding each criterion and not regardless of the criteria. Consequently, we model the influences of the criteria on the evaluation of alternatives. In making a decision, we need to allow for both and not to take the simple way out by always assuming independency.

3.3.2 A new aggregation process using the belief function framework

In this part of this Chapter, we develop our second MCDM model named conditional belief AHP. The difference between the two proposed methods is based on the aggregation procedure, as presented previously in Figure 3.1 (Step 3). Figure 3.5 summarizes the new computational procedure.

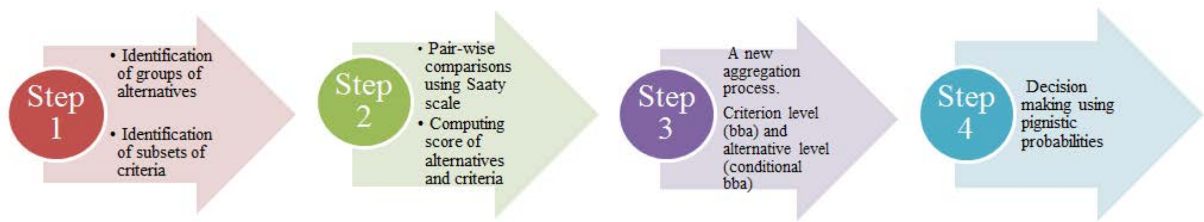

Figure 3.5: The general decision-making paradigm based on the conditional belief AHP

By introducing this approach, we try to model the relationship between alternatives and criteria. The developed method has the same steps as belief AHP. Nevertheless, we suggest a new aggregation procedure as shown in Figure 3.6.

After building the belief AHP hierarchy and completing the different pair-wise comparison matrices, we will be interested in this stage by showing how to combine the alternatives priorities with the importance of their corresponding criteria.

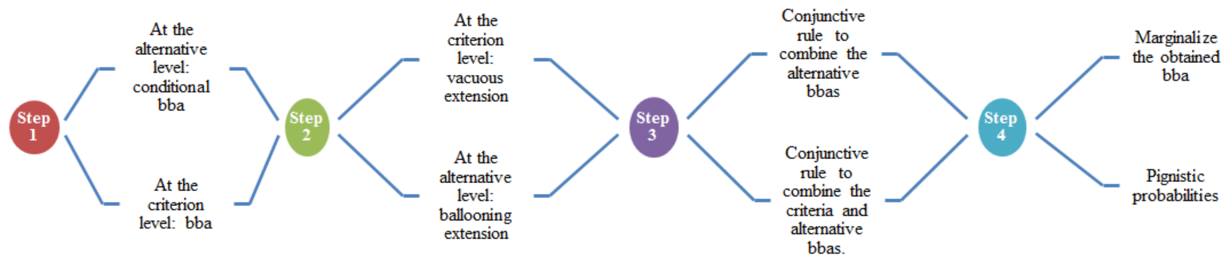

Figure 3.6: The new aggregation procedure under the conditional belief AHP method

Step 1: Representation of uncertainty

Through our approach, the uncertainty is introduced in the decision maker's preferences. Besides, we propose to represent the imperfection over the sets of criteria. Within our framework, we have $C_i \subseteq 2^{\Omega}$ and we have the priority values of each C_i representing the opinions-beliefs of the expert about his preferences. We also notice that this priority vector sums to one which can be regarded as a bba. As a result, this bba can be denoted by m^{Ω}.

Furthermore, we propose to represent the uncertainty at the level of alternatives. In fact, the expert tries to express his preferences regarding the sets of alternatives in relation with each criterion. Accordingly, and to more imitate the expert's reasoning, we indicate that to define the influences of the criteria on the evaluation of alternatives, we might use a conditional belief.

Given a pair-wise comparison matrix which compares the sets of alternatives according to a specific criterion, a conditional bba can be represented by:

$$m^{\Theta}[c_j](A_k) = v_k, \quad \forall A_k \subseteq 2^{\Theta} \text{ and } c_j \in \Omega \ . \tag{3.9}$$

where A_k represents a subset of 2^{Θ}, v_k is the eigen value of the k^{th} sets of alternatives regarding the criterion c_j. $m^{\Theta}[c_j](A_k)$ means that we know the belief about A_k regarding c_j.

Example 3.5. *Let us consider the same problem of purchasing a car (Example 3.1). We have four criteria:* $\Omega = \{$*Price* (c_1), *Style* (c_2), *Fuel* (c_3), *Reliability* $(c_4)\}$ *and three selected alternatives:* $\Theta = \{$*Peugeot* (p), *Renault* (r),
Ford $(f)\}$. *The first step is to identify the subsets of criteria and the groups of alternatives as shown in the previous example. Then, we have to fill all the pair-wise comparison matrices and we get the previous defined matrices (see Tables 3.1 and 3.3).*

As indicated above, the criterion weights are expressed by a basic belief assessment (bba) *and we propose to model the alternative score by means of conditional bba. For instance, the alternative $\{p\}$ given c_1 can be represented by $m^{\Theta}[c_1](\{p\}) = 0.806$, which means that we know the belief about $\{p\}$ regarding the criterion c_1.*

Step 2: Standardization of the frame of discernment

As indicated above, our objective through the second step is to combine the obtained conditional belief with the importance of their respective criteria to measure their contribution. In this context, our major problem here is that we have priorities concerning criteria and groups of criteria that are defined on the frame of discernment Ω, whereas the sets of decision alternatives are generally defined on another frame Θ.

In order to solve this problem, we propose to standardize our frame of discernment. First, at the criterion level, our objective is then to redefine the bba that represents criteria weights. Indeed, we propose to extend this bba m^{Ω} from Ω to $\Theta \times \Omega$:

$$m^{\Omega \uparrow \Theta \times \Omega}(B) = m^{\Omega}(C_i) \quad B = \Theta \times C_i, C_i \subseteq \Omega \ . \tag{3.10}$$

Example 3.6. *Let us continue with the previous example. According to this approach, the second step is to standardize the criterion and the alternative frames of discernment. For the criterion level, we suggest to apply the extension procedure. Hence, Equation 3.10 is used and the resulting bbas is summarized in Table 3.6.*

Table 3.6: Vacuous extension of bba

$m^{\Omega}(.)$	$m^{\Omega \uparrow \Theta \times \Omega}$	bbm
$m^{\Omega}(\{c_1\})$	$\{(p, c_1), (r, c_1), (f, c_1)\}$	0.58
$m^{\Omega}(\{c_4\})$	$\{(p, c_4), (r, c_4), (f, c_4)\}$	0.32
$m^{\Omega}(\{c_2, c_3\})$	$\{(p, c_2), (r, c_2), (f, c_2), (p, c_3), (r, c_3), (f, c_3)\}$	0.1

Secondly, at the alternative level, the idea was to use the deconditionalization process in order to transform the conditional belief into a new belief function. In this case, the ballooning extension (see Equation 2.38) technique is applied:

$$m^{\Theta}[c_j]^{\Uparrow \Theta \times \Omega}(A_k \times c_j \cup \Theta \times \bar{c_j}) = m^{\Theta}[c_j](A_k), \forall A_k \subseteq \Theta \ . \tag{3.11}$$

Example 3.7. *Using the same pair-wise comparison matrices presented in Table 3.3, an illustrative example of the above mentioned calculations is given in what follows. Our method suggests* *to transform the conditional belief into joint distribution by applying the ballooning extension (see Equation 3.11). For instance, we present the priority matrix that evaluates the candidate subsets of alternatives regarding the criterion c_1.*

*Let us consider, $m^\Theta[c_1](\{p\})$ its corresponding basic belief mass on $\Theta \times \Omega$ is obtained by $\{(p, c_1)\}$ and all the instances of Θ $\{(p, c_1), (p, c_2), (p, c_3), (p, c_4)\}$ for the complement of c_1 $\{(r, c_2), (r, c_3), (r, c_4), (f, c_2),$
$(f, c_3), (f, c_4)\}$. The following Table 3.7 is calculated.*

Table 3.7: Ballooning extension of conditional bba

$m^\Theta[c_1](.)$	$m^\Theta[c_j]^{\Uparrow\Theta\times\Omega}$	bbm
$m^\Theta[c_1](\{p\})$	$\{(p, c_1), (p, c_2), (p, c_3), (p, c_4), (r, c_2), (r, c_3), (r, c_4), (f, c_2), (f, c_3), (f, c_4)\}$	0.806
$m^\Theta[c_1](\{p, r, f\})$	$\{(p, c_1), (p, c_2), (p, c_3), (p, c_4), (r, c_1), (r, c_2), (r, c_3), (r, c_4), (f, c_1), (f, c_2), (f, c_3), (f, c_4)\}$	0.194

Then, the same process is repeated for the rest of alternatives regarding each criterion.

Step 3: Combination rule

Once the frame of discernment $\Theta \times \Omega$ is formalized, our approach proposes to combine the alternative priorities as shown in Figure 3.6. In fact, we assume that each pair-wise comparison matrix is considered as a distinct source of evidence, which provides opinions towards the preferences of particular decision alternatives. Then, based on the belief function framework, we can apply the conjunctive rule of combination. The obtained bba represents the belief in groups of alternatives based on the combined evidence from the decision matrices.

Finally, we might combine the obtained bbas with the importance of their respective criteria to measure their contribution. That is, we will apply the conjunctive rule of combination and we get:

$$m^{\Theta\times\Omega} = \left[\textcircled{\cap}_{j=1,\ldots,m} m^\Theta[c_j]^{\Uparrow\Theta\times\Omega}\right] \textcircled{\cap} m^{\Omega\uparrow\Theta\times\Omega} \quad . \tag{3.12}$$

So, we obtain $m^{\Theta\times\Omega}$ reflecting the importance of alternatives to the given criteria.

Example 3.8. *To illustrate the described operations, we continue with the same example given* *in Example 3.7. The obtained bbas $m^{\Omega\uparrow\Theta\times\Omega}$ and $m^{\Theta}[c_j]^{\Uparrow\Theta\times\Omega}$ can be directly combined using the conjunctive rule of combination to get Table 3.8.*

Table 3.8: The obtained bba: $m^{\Theta\times\Omega}$

$m^{\Theta\times\Omega}$	bbm
$\{(p,c_1),(f,c_1),(r,c_1)\}$	0.362
$\{(p,c_1)\}$	0.315
$\{(p,c_4),(f,c_4),(r,c_4)\}$	0.1302
$\{(f,c_1)\}$	0.0064
$\{(p,c_2),(f,c_2),(r,c_2),(p,c_3),(f,c_3),(r,c_3)\}$	0.008
$\{(r,c_2),(r,c_3),(f,c_2),(p,c_2)\}$	0.0664
$\emptyset$	0.112

Step 4: Decision making

To this end and after combining the resulting ballooning extension, a decision under uncertainty must be defined. In the sequel, the pignistic transformation is used. However, our obtained beliefs are defined on the product space $\Theta \times \Omega$. To solve this problem, we propose to marginalize this bba on Θ (frame of alternatives) by transferring each mass $m^{\Theta\times\Omega}$ to its projection on Θ (see Equation 2.35):

$$m^{\Theta\times\Omega\downarrow\Theta}(A_j) = \sum_{\{B\subseteq\Theta\times\Omega|Proj(B\downarrow\Theta)=A_j)\}} m^{\Theta\times\Omega}(B), \forall A_j \subseteq \Theta \ . \tag{3.13}$$

Finally, we can compute the pignistic probabilities to choose the best alternatives:

$$BetP(a_j) = \sum_{A_i\subseteq\Theta} \frac{|a_j \cap A_i|}{|A_i|} \frac{m^{\Theta\times\Omega\downarrow\Theta}(A_i)}{(1 - m^{\Theta\times\Omega\downarrow\Theta}(\emptyset))}, \forall a_j \in \Theta \ . \tag{3.14}$$

Example 3.9. *To choose the best alternatives, we need to define our beliefs over the frame of alternatives. As a result, the obtained bba is marginalized on Θ using Equation 3.13 (see Table 3.9).*

We can now calculate the overall performance for each alternative and determine its corresponding ranking by computing the pignistic probabilities as shown in Figure 3.7 (see Table 3.10).

Table 3.9: The obtained bba: $m^{\Theta\times\Omega\downarrow\Theta}$

$m^{\Theta\times\Omega\downarrow\Theta}$	bbm
$m^{\Theta\times\Omega\downarrow\Theta}(\{p,r,f\})$	0.5666
$m^{\Theta\times\Omega\downarrow\Theta}(\{p\})$	0.315
$m^{\Theta\times\Omega\downarrow\Theta}(\{f\})$	0.0064
$m^{\Theta\times\Omega\downarrow\Theta}(\emptyset)$	0.112

Table 3.10: The Final ranking of alternative

Alternatives	$\{p\}$	$\{r\}$	$\{f\}$
$BetP$	0.567	0.220	0.213

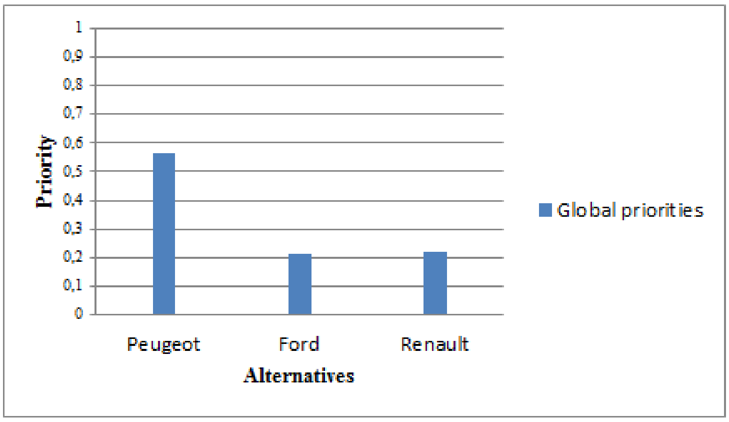

Figure 3.7: Ranking of alternatives using conditional belief AHP

3.4 Computational experiments

In this Section, the performances of the proposed methods, belief AHP and conditional belief AHP, are compared. Firstly, the evaluation algorithm used in the comparisons are described. Secondly, the methods are tested on randomly generated matrices. Then, we discuss the obtained result. Lastly, proposed methods are compared based on an example that is commonly used in the literature.

3.4.1 Evaluation algorithm

Despite the possible differences between the obtained results of two MCDM methods, we cannot conclude the superiority of one over another. Unless we have a solid basis, we compare the ranking results against the closeness of the rankings of each method to the actual rankings.

To do so, we need to compare each set of rankings provided by AHP and the proposed methods with a ranking that has already been produced by an alternative, yet reliable ranking method. This alternative ranking will be considered as a basis, or actual ranking of the alternatives and will be used to measure the closeness of the rankings provided by AHP, belief AHP and conditional belief AHP to reality. These evaluations are provided in the following Sections.

A method which is accurate in MCDM problems should also be accurate in single dimensional problems. Therefore, we use the Weighted Sum Model (WSM) method since in single-dimensional environment, it yields the most reasonable results. Hence, Triantaphyllou and Mann (1989); Triantaphyllou and Lin (1996) compare the obtained results using WSM to those obtained by other MCDM method. This evaluation criterion has been applied in order to evaluate crisp and fuzzy MCDM methods (Triantaphyllou & Mann, 1989; Triantaphyllou & Lin, 1996).

Besides, WSM is the simplest and still the widest used MCDM method. In this method, each criterion is given a weight, and the sum of all weights must be 1. Each alternative is assessed with regard to every criterion. The overall or composite performance score of an alternative is given by the equation:

$$P_i = \sum_j v_{ij} * \omega_j \tag{3.15}$$

where P_i is the priority of each alternative, ω_j is the weight of each criterion and v_{ij} is the score of each alternative regarding each criterion.

In order to overcome these issues, in the next Section, we test a simulation algorithm that compares the ranking results of belief AHP and conditional belief AHP methods under different scenarios. Regarding the two proposed approaches, we will try to make some comparisons not to decide which one is the best one but just to make some conclusions.

3.4.2 Simulation algorithm

To generate reliable data for a numerical analysis in AHP, simulation has been extensively used in prior research (Triantaphyllou & Lin, 1996; Triantaphyllou & Mann, 1989). The simulation algorithm is summarized in Figure 3.8.

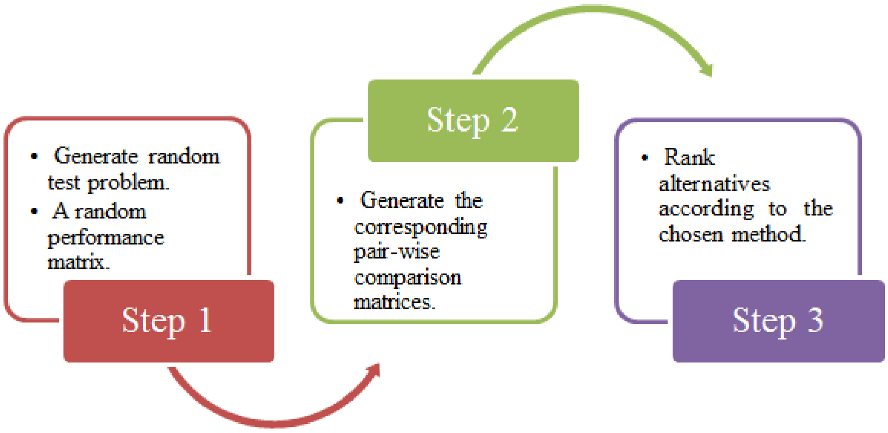

Figure 3.8: Simulation algorithm steps

The experiment is based on the following steps:

1. We generate a random matrix for the decision performance and another one to represent the weight of each decision criteria. Based on these two matrices, the overall scores and ranks of the decision alternatives are calculated. These steps are usual steps in the WSM method.

2. From the performance matrix, we generated pair-wise comparison matrices of different alternatives that are compared to each criterion.

3. We apply the suggested method to compute the overall priorities and to rank alternatives.

4. We compare the obtained result with the ranking of the WSM method.

Example 3.10. *Let's demonstrate the evaluation procedure using the same example used in Triantaphyllou and Mann (1989). We consider* 3 *alternatives* a_1*,* a_2 *and* a_3 *and three criteria*

c_1, c_2 *and* c_3. *The decision making problem is described using the matrix presented in Table 3.11.*

Table 3.11: Decision matrix

	Criteria		
	c_1	c_2	c_3
Alt.	$\frac{8}{13}$	$\frac{2}{13}$	$\frac{3}{13}$
a_1	1	9	9
a_2	5	2	2
a_3	1	5	9

This example has been solved using the WSM and AHP in (Triantaphyllou & Mann, 1989). Applying the WSM, it can be shown that the alternative a_1 *is the best one. However, AHP turns out that the alternative* a_2 *is the best one. Obviously, this contradicts with the conclusion derived using the WSM.*

Now, let us model this example using belief AHP. If the decision maker knew the actual data shown in the original crisp decision matrix, then the matrix of the actual pair-wise comparisons, when the three alternatives are compared regarding each criterion, will be illustrated in Table 3.12.

Next, and after using the eigenvector method to compute the alternative priorities, we use the belief AHP method to aggregate the obtained priorities. We obtain the following order: $a_1 > a_3 > a_2$. *Obviously, this is in contradiction with the results derived when the AHP method was applied at the beginning of this illustrative example. However, we have obtained encouraging results since it can be observed that the ranking order of the alternatives as derived by the WSM and the belief AHP is the same.*

Table 3.12: The preference relations matrices

c_1			
	a_1	a_2	a_3
a_1	1	$1/5$	1
a_2	5	1	5
a_3	1	$1/5$	1

c_2			
	a_1	a_2	a_3
a_1	1	$9/2$	$9/5$
a_2	$2/9$	1	$2/5$
a_3	$5/9$	$5/2$	1

c_3			
	a_1	a_2	a_3
a_1	1	$9/2$	$9/9$
a_2	$2/9$	1	$2/9$
a_3	$9/9$	$9/2$	1

In order to gain a deeper understanding, a computational study was undertaken. The data were random numbers from the interval $[1, 9]$ (in order to be compatible with the numerical properties of the Saaty scale). In these test problems, the number of alternatives was equal to the following values: $3, 4, 5, 7, 9$, and 10. Similarly, the number of criteria was equal to $3, 4, 5, 6, 7, 8, 9$, and 10. Psychological experiments have shown that individuals cannot simultaneously compare more than seven objects (plus or minus two) (Miller, 1956). Therefore, we choose that the number of criteria and alternatives in the analysis should not exceed 10. Thus, a total of (8×8) different cases were examined with 100 replications (in order to derive statistically significant results) per each case. Each random problem was solved using the original and belief AHP methods. The test problems were treated as the previous illustrative example. Any ranking irregularity was recorded.

3.4.3 Simulation results

In this Section, we present the results of our simulation study. Note that the purpose of the simulation algorithm was to compare the performance of AHP with belief AHP and conditional belief AHP methods in terms of score and rank of decision alternatives for different number of criteria and alternatives.

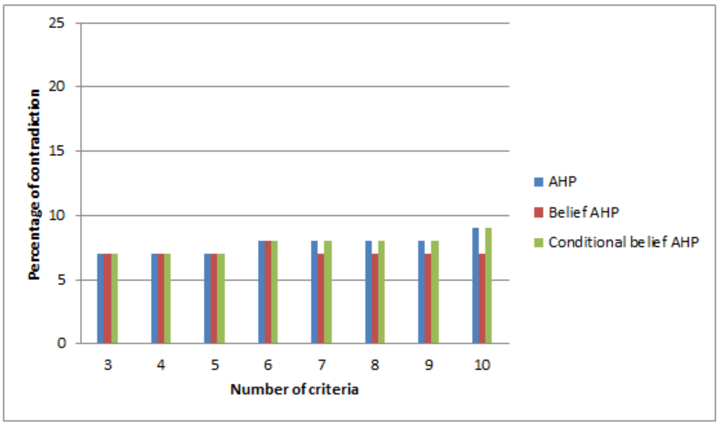

Figure 3.9: Percentage of contradiction (%) based on 3 alternatives

Figures 3.9, 3.10 and 3.11 refer to the use of the standard, belief and conditional belief AHP methods. Different curves correspond to problems with different numbers of alternatives. As it can be seen from these figures, problems with few alternatives had smaller percentage of contradiction. Also, these figures show that the three versions of the AHP have nearly the same

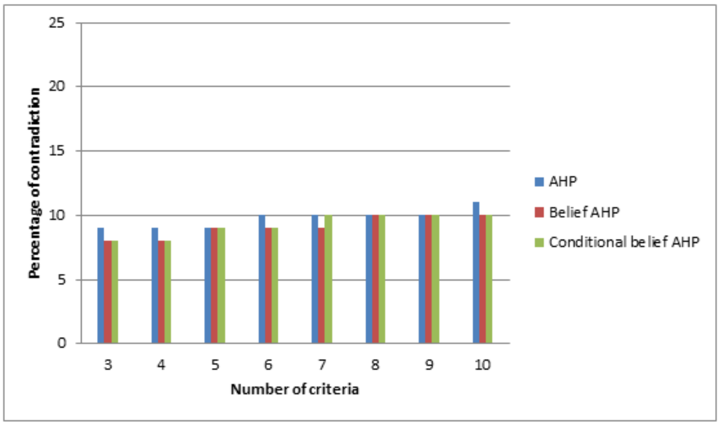

Figure 3.10: Percentage of contradiction (%) based on 6 alternatives

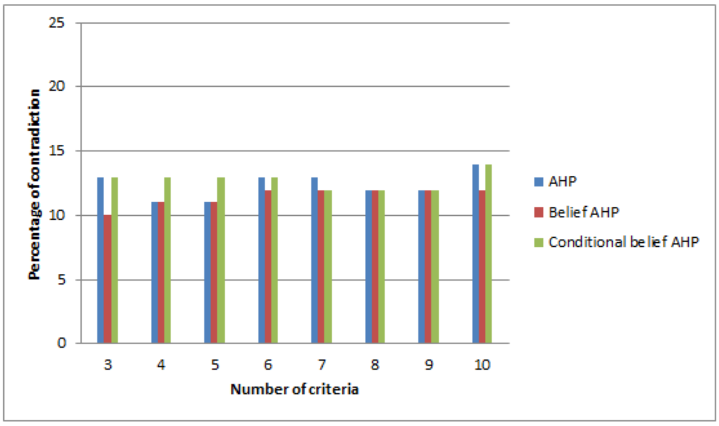

Figure 3.11: Percentage of contradiction (%) based on 9 alternatives

percentage of contradiction.

From Figure 3.9, we can notice that belief AHP method outperforms standard AHP, when the number of alternatives is 3. For example, when applying our method with 7 criteria, the percentage of contradiction is set to 7%. Nevertheless, when applying the standard AHP the percentage is 8%. We can conclude that when the number of criteria increases (greater than 10) the percentage of contradiction increases. The previous remark is confirmed by the results appearing in Figures 3.10 and 3.11. This can be justified by the use of the combination technique. The more we have criteria, the more we use the conjunctive rule, which can increase the conflict between the combined bbas. Conditional belief AHP method is also capable of producing relatively the same percentage of contradiction as standard AHP.

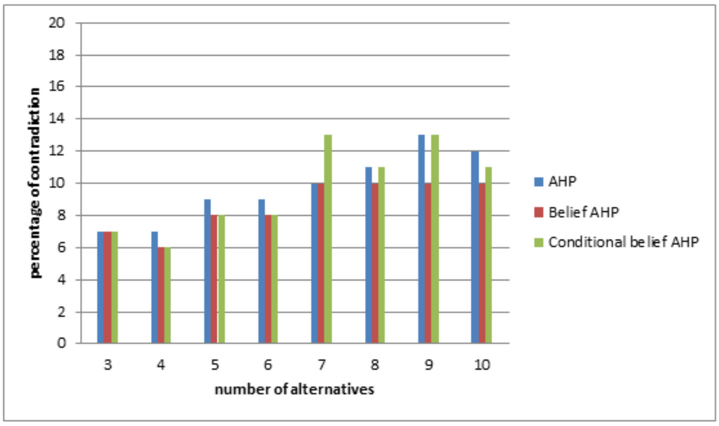

Figure 3.12: Percentage of contradiction (%) based on 3 criteria

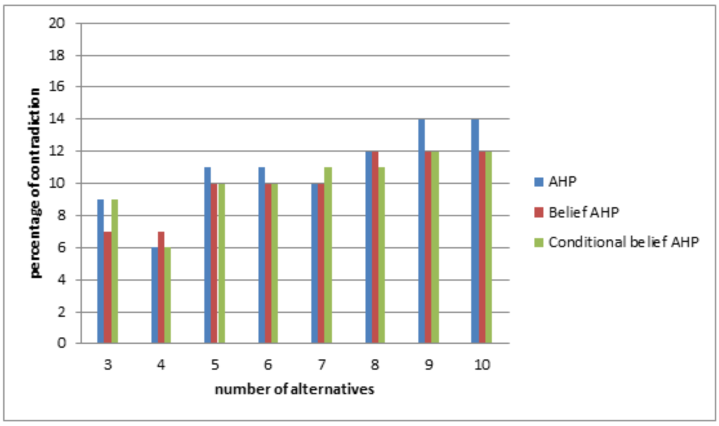

Figure 3.13: Percentage of contradiction (%) based on 10 criteria

On the other hand, Figures 3.12 and 3.13 depict contradictions in the ranking of alternatives for respectively 3 and 10 criteria. For the first case, the number of alternatives plays a decisive role; if the number of alternatives increase then the contradiction increases too. For instance, for problems with 10 criteria, the percentage of contradiction is almost 10%. As before, there is no much difference between the results (more details can be found in Appendix A).

In all these results, problems with less alternatives yielded fewer ranking contradictions than problems with more alternatives. As it can be seen, the results derived from the computational experiments lead to some interesting observations. Our new approaches perform much better than traditional AHP. Despite of these satisfactory results, the methods have some limits, mainly the use of Saaty scale. Trying to focus on this limitation, we have developed a new extension where its details are presented in the next Chapter.

3.4.4 Comparison with DS/AHP method

In this Section, we compare the performance of our proposed belief AHP methods to DS/AHP approach. As we are focusing on the percentage of contradiction, the comparison made is in terms of the number of erroneous rank.

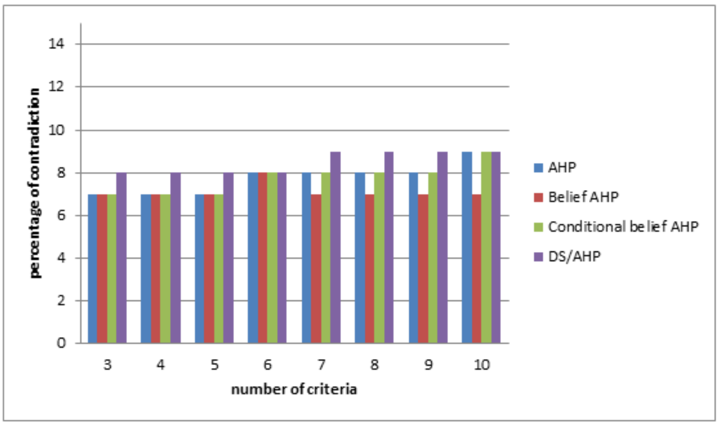

Figure 3.14: Comparison of AHP extensions according to the percentage of contradiction (based on 3 alternatives)

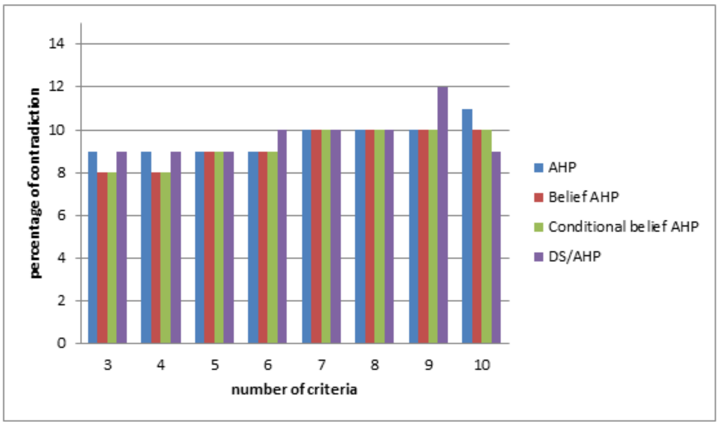

Figure 3.15: Comparison of AHP extensions according to the percentage of contradiction (based on 6 alternatives)

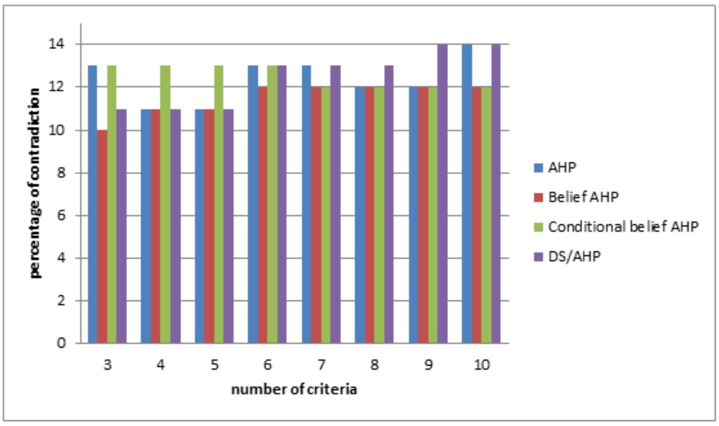

Figure 3.16: Comparison of AHP extensions according to the percentage of contradiction (based on 9 alternatives)

Figure 3.14, 3.15 and 3.16, also, show that our belief AHP approaches outperform the DS/AHP in terms of the overall percentage of contradiction.

3.4.5 Catering selection problem

The presented methods are illustrated with an example problem taken from literature: Catering selection problem. This latter is a MCDM problem which includes both qualitative and quantitative factors. In order to select the best catering firms, it is necessary to make a tradeoff between these tangible and intangible factors some of which may be conflict. The objective of catering selection is to identify firms with the highest potential for meeting customers' needs consistently and at an acceptable cost.

Some catering firms were compared using four attributes and fuzzy AHP (Cebeci & Kahraman, 2002). Creed (2001) discusses the results of a survey on how consumers perceive the acceptability of the prepared meals according to age group, social class, gender and frequency of eating out and the potential for extending the use of prepared meals to those who could benefit from. Pi and Low (2006) proposed a supplier evaluation and selection system via Taguchi loss function and AHP. Sevkli et al. (2007) used data envelopment AHP to select the best supplier.

In this Section, we apply the same problem proposed by Kahraman et al. (2004). Thus, we consider a "problem of catering firms in Turkey": a case study of the application of the fuzzy AHP to the selection of best catering firms.

Catering companies in Turkey have to be very competitive. Their customers change frequently their catering suppliers because it is easy to replace them when a complaint or non-conformity happens as there are many competing companies in the sector. Three catering firms, Durusu (D), Mertol (M) and, Afiyetle (A) were compared to select the best one. A questionnaire was given to the customers of these catering firms.

The candidate criteria determined by the questionnaire were Hygiene (H), Quality of meal (ingredients) (QM) and Quality of service (QS).

Figure 3.17 summarizes our decision making problem.

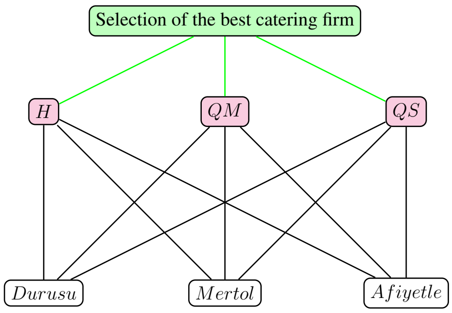

Figure 3.17: Hierarchy of catering firm selection problem

The hierarchical structure is simplified by the belief and conditional belief methods, and the sub-criteria in total are removed from decision analysis. In this situation, it is no wonder that the extent analysis method makes a wrong decision, and it is concentrated on the three main criteria. The removal of the sub-criteria from decision analysis will not affect the final result.

This problem has been solved using belief and conditional belief AHP methods. We have obtained the results shown in Table 3.13. The two models recommended “Durusu” as the best alternative since it has the highest values.

For the sake of comparison, we compare our results to those obtained using fuzzy AHP (Kahraman et al., 2004). We have priority of $D = 0.21$, priority of $A = 0.69$ and priority of $M = 0.10$. Using our approach we have noticed that the alternatives M and A have nearly the same importance compare to Fuzzy AHP results. In addition, in both models, the alternative M is the worst one. Furthermore, the obtained results have been also confirmed by Wang et al. (2006). They have proved that alternative M is the worst one.

The next stage is to check the stability of our models. Applying sensitivity analysis to such

Table 3.13: Decision matrix

		Criteria	
	H	QM	QS
	0.65	0.3	0.05
Alt.		**Ranking**	
	Belief AHP		conditional belief AHP
Durusu	0.13		0.23
Mertol	0.11		0.18
Afiyetle	0.76		0.59

decision making processes is essential to ensure the consistency of final decision. Through sensitivity analysis, different what-if scenarios, which are helpful to observe the impact of changing on criteria to final alternative rank, can be visualized (see Appendix B). Sensitivity analysis as shown in Figures 3.18 and 3.19 let evaluator observe how final evaluation is likely to change. It also helps in measuring how many changes made by certain extent of deviations in weights of criteria.

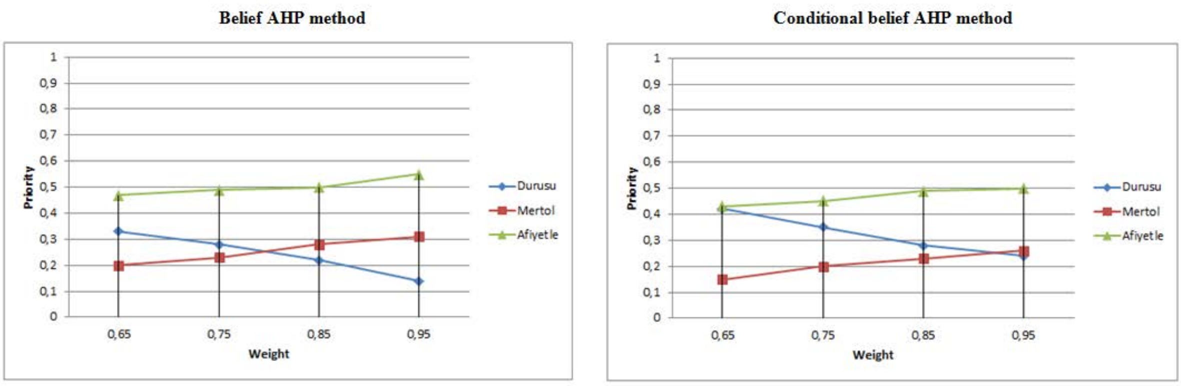

Figure 3.18: Sensitivity analysis of Hygiene criterion

In this case, simulation of sensitivity analysis is carried out by making gradual changes on the value of each criterion and then observing the rank order induced by such changes. It is revealed that by shifting the value of each criterion lowering up (H), it did not effect the first rank.

We start by considering the Hygiene criterion. By increasing the share of this criterion to an extreme of 95% of the main goal, it has been noticed that the model is still in favor of Afiyetle with a score of 0.68 followed by Durusu and lastly Mertol. The same conclusion can be drawn

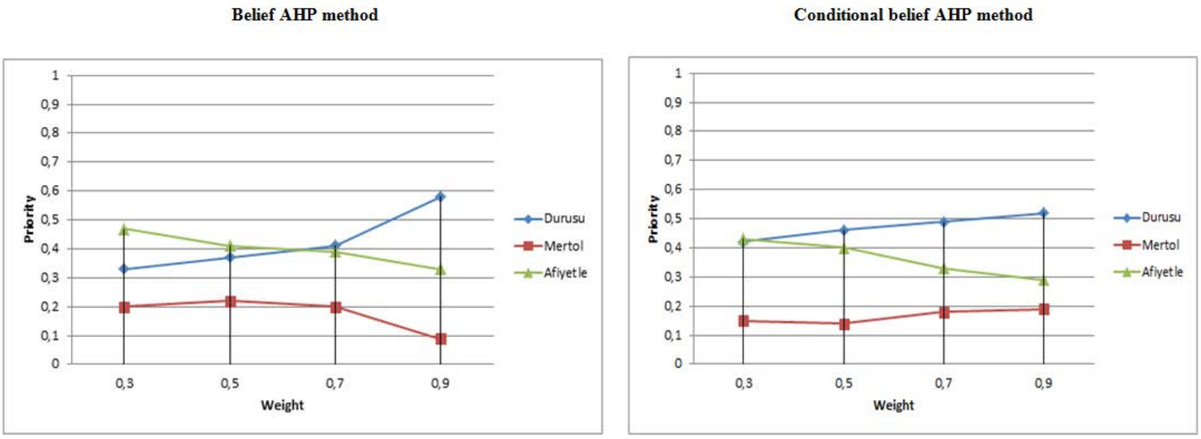

Figure 3.19: Sensitivity analysis of Quality of meal criterion

for the Quality of meal criterion, where Afiyetle remains the best choice. Rank reversal occurs only to the first and second ranks (Afiyetle and Durusu respectively) when Quality of meal is increased to 0.9.

The results show that Afiyetle is always in the lead with a persistent score. The sensitivity analysis presented here demonstrates how consistent the decision is. The choice of Afiyetle remains the same even with significant changes in the criteria weights, which can be justified by the consistent judgments of the pair-wise comparisons.

We ignored the criterion “Quality of service” because it is the least important one, and according to the expert, sensitivity results will be erroneous.

To conclude, the belief AHP and conditional belief AHP methods are used to solve real application problem and they have proved a good result. They have identified the best alternative and even with significant changes in the criteria weights the methods have maintained the best alternative.

3.5 Conclusion

In this Chapter, we have formulated two MCDM methods in an environment characterized by the imperfection of the information. The first one, belief AHP, deals with the groups of criteria and alternatives. Then, an extension of this approach, named conditional belief AHP, is introduced. It takes into account the dependency relationships between criteria and alternatives.

The belief AHP approaches proved to be a convenient method in tackling practical MCDM problems. They were used to synthesize the opinions of the decision maker to reduce the number of comparisons. They show the advantage of being able to imitate human thinking and aid in solving the research problem through modeling dependency between alternatives and criteria. The two proposed methods have been illustrated by a simple example. Then, we have compared them through random problems. Finally, we have used them to solve a problem from literature.

Nevertheless, these methods assume that a decision maker can provide precise point estimates of his preferences. Hence, Saaty scale is not suitable for scenarios when there is high uncertainty in decision makers judgment.

To overcome this limitation, belief AHP approaches will serve as a base to the proposed methods that will be introduced in the following Chapter. A new preference elicitation technique, which incorporates belief distributions to include uncertainty in the judgments, will be suggested.

Chapter 4

A new ranking procedure by belief pair-wise comparisons

4.1 Introduction

Despite its popularity and efficiency, the AHP method is often criticized for its use of an unbalanced scale of estimations and its inability to adequately handle the uncertainty associated with the mapping of the decision maker's perception to a crisp number. Besides, in MCDM process, an expert may be uncertain about his level of preference due to incomplete information or knowledge, inherent complexity and uncertainty within the decision environment.

Therefore, our main aim through this research is to extend the pair-wise comparison process on a more flexible method that integrates additional uncertain and/or imprecise knowledge. A natural way to cope with uncertain judgments is to express the comparison ratios as a belief function, which incorporates the imperfection of the human thinking. Thus, a new procedure is employed to derive crisp priorities from belief distributions based on a new set of choices.

In this Chapter, we give an insight into our proposed methods. We first present, in Section 4.2, some motivations to develop the belief pair-wise comparison. Next, Section 4.3 details our new belief pair-wise comparisons along with examples to illustrate them. Then, we introduce, in Section 4.4, our proposed MCDM methodology based on belief pair-wise comparison. Section 4.5 models dependency between alternatives and criteria under Yes-No/AHP framework. In the last Section (Section 4.6), we present some experimental studies.

4.2 Limits of Saaty scale

In MCDM, the pair-wise comparisons represent a useful starting point for determining a ranking on a set of alternatives or criteria. It aims at quantifying relative priorities for a given set of alternatives as well as the set of criteria, on a ratio scale, based on the judgment of the decision maker.

The scale used in the AHP for pair-wise comparisons is the 1 to 9 ratio scale (Saaty, 1977, 1980). We note that the inverse relationships are simply the multiplicative inverses of the values from within the 1 to 9 scale, hence the actual scale used in the AHP is $1/9$ to 1 to 9.

Using this approach, the decision maker has to express his opinion about the value of one single pair-wise comparison at a time. Usually, he has to choose his answer among discrete choices. Each choice is a linguistic phrase. Some examples of such linguistic phrases are: "A is more important than B", or "A is of the same importance as B", or "A is a little more important than B", and so on. He uses Saaty's scale (see Table 1.4) to map the labels which indicate the decision maker view to a numeric value.

However, as shown in Joaquin (1990) and Holder (1995), this scale was criticized since the user cannot be consistent. Sometimes, the decision maker may well want to say that A is twice as important as B, A is 3 times as important as C and B is 1.5 times as important as C, yet he is constrained to make the last judgment 1 or 2. In addition, the decision maker might find difficult to distinguish among them and tell for example whether one alternative is 6 or 7 times more important than another. Furthermore, the AHP method cannot cope with the fact that alternative A is 25 times more important than alternative C. Expert would not be able to efficiently express any kind of preference degree between the available alternatives and criteria. As a result, the scale is further incomplete and unnecessarily restricting because of the arbitrary cut-off at 9 for the maximum allowable ratio of weights.

All these criticisms have been discussed in the literature and some solutions for them have been developed. To take judgmental uncertainty into account, alternative methods such as applications of the fuzzy theory are developed for AHP (Laarhoven & Pedrycz, 1983; Lootsma, 1997). Different scaling methods have also been provided (Beynon, 2002a).

Consequently, our problem through this work is: how to quantify the linguistic choices selected by the decision maker during the evaluation of the pair-wise comparisons under the belief function framework? Is it necessary to decompose even more the different levels of the Saaty's scale?

4.3 A pair-wise comparison process: A new elicitation technique

Since using belief distribution as an element of the pair-wise comparison matrix is more expressive than using crisp number, we expect that belief approach allows for a more accurate description of the decision making process. Rather than forcing the expert to provide exact representations of imprecise perceptions, we suggest using an imprecise representation instead.

In this Section, we suggest to modify the structure of the Saaty scale by adopting a new set of choices. We present the definitions and notations for the pair-wise comparison matrix. Then, the inconsistency identification method is introduced.

4.3.1 Pair-wise comparison matrix with belief distributions

Under this approach, a new elicitation procedure is introduced. Thus to model his assessments, the decision maker has to express his opinions qualitatively. He indicates whether a criterion (or alternative) was more or less important to its partner. Therefore, we suggest a new set of choices. For instance, to respond to the following question: "Is the subset of criteria C_j important?", the expert only selects the related linguistic variable. He indicated whether a criterion was more or less important by "yes" or "no". In other terms:

$$\Omega_{C_j} = \{yes, no\} \tag{4.1}$$

where Ω_{C_j} is the set of possible choices.

Moreover, we accept that the expert may define uncertain or even unknown assessments. Indeed, we assume that each subset of criteria is described by a basic belief assignment defined on the possible responses ($m^{\Omega_{C_j}}$). In other words, to quantify the subjective judgments with uncertainty, a preference degree may be assigned to each decision maker's response. For instance, in a problem of purchasing a car, the following type of uncertain subjective judgments was frequently used: "the Price criterion is evaluated to be more important than Style with a confidence degree of 0.8". In fact, the decision maker responds to the question "is Price criterion important regarding the Style criterion?". Thus, the answer was: Price criterion is more preferable than Style criterion and 0.8 is referred to the degree of belief. Finally, the same process is repeated for each pair of elements.

To model the pair-wise comparison matrix, some priorities must be respected. We consider X, the pair-wise comparison matrix, is an $k \times k$ matrix in which k is the number of groups of criteria being compared. It has the following characteristics:

1. The first step is to model the pair-wise comparison matrix. Let d_{ij} is the entry from the i^{th} column of pair-wise comparison matrix (d_{ij} represents the different bbms of the identified bba).

$$\text{If } m_j^{\Omega_{C_i}}(.) = d_{ij}, \text{ then } \overline{m}_i^{\Omega_{C_j}}(.) = m_j^{\Omega_{C_i}}(.) = d_{ij} \tag{4.2}$$

   where $m_j^{\Omega_{C_i}}$ represents the importance of C_i with respect to the subset of criteria C_j (for simplicity we denote the subset of criteria by j instead of C_j), $i \neq j$ and $\overline{m}$ is the negation of m.

   The negation (or complement) $\overline{m}$ of a bba m is defined as the bba verifying (Dubois & Prade, 1986):

$$\overline{m}(A) = m(\overline{A}), \forall A \subset \Omega \tag{4.3}$$

   As regarding the previous example, if "the Price criterion (C) is evaluated to be more important than Style criterion (S) with a confidence degree of 0.8", that is $m_S^{\Omega_C}(\{yes\}) = 0.8$, then we can say that "the Style criterion is evaluated to be less important than Price criterion with a confidence degree of 0.8": $m_C^{\Omega_S}(\{no\}) = 0.8$.

2. Like the traditional AHP method, where the principal diagonal contains entries of 1, we set:

$$m_i^{\Omega_{C_i}}(\Omega_{C_i}) = d_{ii} = 1 \tag{4.4}$$

Once the pair-wise comparison matrix is complete, our objective is then to obtain the priority of each subset of criteria. In fact, within the belief comparison matrix, our problem is "what is the appropriate function to use in order to obtain a single representation value of these different bbas to get the priority vector". The idea is to combine the obtained bbas using the conjunctive rule of combination. Indeed, this function is chosen since we can regard each subset of criteria as a distinct source of information which provides distinct pieces of evidence.

To better understand, we consider X, as defined above, the pair-wise comparison matrix. For each row of the matrix, we apply the conjunctive rule. That means, for each subset of criteria $(i = \{1, \ldots, k\})$, we will get the following bba:

$$m^{\Omega_{C_i}} = \textcircled{\cap} m_j^{\Omega_{C_i}}, \text{ where } j = \{1, \ldots, k\} \tag{4.5}$$

The new belief pair-wise comparison technique is summarized in Figure 4.1.

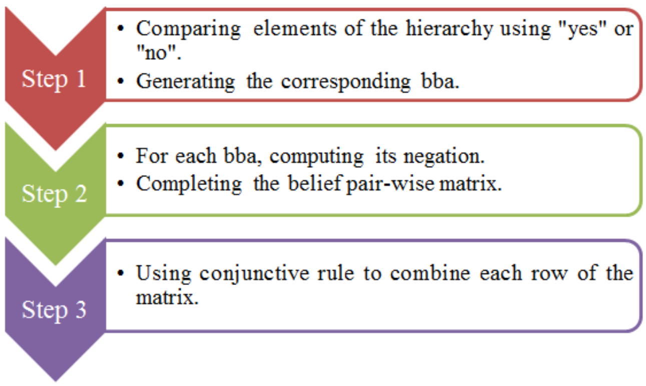

Figure 4.1: Belief pair-wise comparison technique

Example 4.1. *To describe this approach, we consider the problem of "purchasing a car". Suppose that this problem involves four criteria:* $\Omega = \{$*Price* (c_1), *Style* (c_2), *Fuel* (c_3), *Reliability* $(c_4)\}$ *and three selected alternatives:* $\Theta = \{$*Peugeot* (p), *Renault* (r), *Ford* $(f)\}$.

After identifying the subsets of criteria and alternatives, the pair-wise comparison matrices should be constructed.

From Table 4.1, the expert may say that $\{c_1\}$ *is evaluated to be more important than* $\{c_4\}$ *with a confidence degree of* 0.4. *That means,* 0.4 *of beliefs are exactly committed to the criterion* $\{c_1\}$ *is more important than* $\{c_4\}$, *whereas* 0.6 *is assigned to the whole frame of discernment (ignorance).*

Table 4.1: The weights preferences assigned to the criteria according to the expert's opinion

	$\{c_1\}$	$\{c_4\}$	$\Omega_1 = \{c_2, c_3\}$
$\{c_1\}$	$m^{\Omega_{\{c_1\}}}_{\{c_1\}}(\Omega_{\{c_1\}}) = 1$	$m^{\Omega_{\{c_1\}}}_{\{c_4\}}(\{yes\}) = 0.4$ $m^{\Omega_{\{c_1\}}}_{\{c_4\}}(\Omega_{\{c_1\}}) = 0.6$	$m^{\Omega_{\{c_1\}}}_{\Omega_1}(\{yes\}) = 0.9$ $m^{\Omega_{\{c_1\}}}_{\Omega_1}(\Omega_{\{c_1\}}) = 0.1$
$\{c_4\}$	$m^{\Omega_{\{c_4\}}}_{\{c_1\}}(\{no\}) = 0.4$ $m^{\Omega_{\{c_4\}}}_{\{c_1\}}(\Omega_{\{c_4\}}) = 0.6$	$m^{\Omega_{\{c_4\}}}_{\{c_4\}}(\Omega_{\{c_4\}}) = 1$	$m^{\Omega\{c_4\}}_{\Omega_1}(\{no\}) = 0.3$ $m^{\Omega_{\{c_4\}}}_{\Omega_1}(\Omega_{\{c_4\}}) = 0.7$
$\Omega_1 = \{c_2, c_3\}$	$m^{\Omega_{\Omega_1}}_{\{c_1\}}(\{no\}) = 0.9$ $m^{\Omega_{\Omega_1}}_{\{c_1\}}(\Omega_{\Omega_1}) = 0.1$	$m^{\Omega_{\Omega_1}}_{\{c_4\}}(\{yes\}) = 0.3$ $m^{\Omega_{\Omega_1}}_{\{c_4\}}(\Omega_{\Omega_1}) = 0.7$	$m^{\Omega_{\Omega_1}}_{\Omega_1}(\Omega_{\Omega_1}) = 1$

Then, the next step consists in combining the bbas corresponding to each criterion using Equation 4.5. The obtained bbas are reported in Table 4.2.

Table 4.2: Belief pair-wise matrix: Partial combination

	$\{c_1\}$	$\{c_4\}$	$\Omega_1 = \{c_2, c_3\}$
Weight	$m^{\Omega_{\{c_1\}}}(\{yes\}) = 0.94$ $m^{\Omega_{\{c_1\}}}(\Omega_{\{c_1\}}) = 0.06$	$m^{\Omega_{\{c_4\}}}(\{no\}) = 0.58$ $m^{\Omega_{\{c_4\}}}(\Omega_{\{c_4\}}) = 0.42$	$m^{\Omega_{\Omega_1}}(\{yes\}) = 0.03$ $m^{\Omega_{\Omega_1}}(\{no\}) = 0.63$ $m^{\Omega_{\Omega_1}}(\emptyset) = 0.27$ $m^{\Omega_{\Omega_1}}(\Omega_{\Omega_1}) = 0.07$

4.3.2 Special cases

Within the belief function framework, two extreme cases such as the total ignorance and the total knowledge can be easily expressed.

1. When the preferences of the decision maker are perfectly known, he is able to use our proposed model. Our solution has proved encouraging results when handling certain case.
2. When the expert is not able to give any information, this case is referred to total ignorance and our technique will be the appropriate solution.

4.3.3 A new consistency index

Rather than the standard consistency index, we propose here to define the conflict in expert judgment through a distance between the defined bbas. Therefore, if the opinions are far from each other, we consider that they are in conflict.

Under the belief function framework, when information $c_i \in \Omega$ is provided as a single set is consistent if and only if $m(c_i) \neq 0$.

A consistency degree has values included in the unit interval $[0, 1]$. Since only two situations can occur in the case of single sets, we can define the consistency degree as the function ϕ:

$$\phi(c_i) = \begin{cases} 1 \text{ , if } c_i \neq \emptyset \\ 0 \text{ , if } c_i = \emptyset \end{cases} \tag{4.6}$$

Moreover, to define a consistency degree on mass assignments, we start by introducing totally consistent and totally inconsistent information in terms of such mass assignments. A mass assignment modeling the empty set is the empty mass assignment (i.e., $m(\emptyset) = 1$). It is natural to associate it with a totally inconsistent information state. However, totally consistent information can be extended by:

- Logical consistency: A mass assignment m is logically consistent if and only if $\cap_{c_i \in \mathcal{F}} \neq \emptyset$

Let $m_1, ..., m_N$ be N mass assignments and $m_{\textcircled{\cap}}$ be a conjunctive combination of these masses. Then, we define a conflict measure $k(.)$:

$$k(c_i, c_j) = 1 - \phi(c_i \cap c_j) \begin{cases} 1 \text{ , if } c_i \cap c_j = \emptyset \\ 0 \text{ , if } c_i \cap c_j \neq \emptyset \end{cases} \tag{4.7}$$

where $c_i, c_j \subseteq \mathcal{F}(m_k) \ \forall k = \{1, \ldots, N\}$.

4.4 Yes-No/AHP: A new aggregation process using the "yes" or "no" framework

In this Section, we develop our Yes-No/AHP method based on belief pair-wise comparison. Our approach has the same skeleton as belief AHP method. The difference between the two models lies in the preference elicitation technique. Figure 4.2 summarizes the computational procedure.

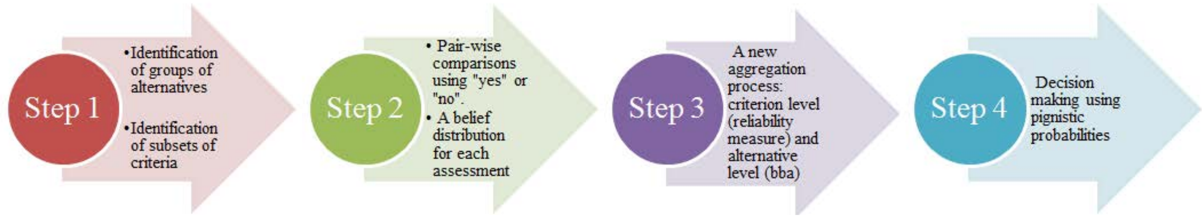

Figure 4.2: The general decision-making paradigm based on Yes-No/AHP

In what follows, we expose the different construction steps of Yes-No/AHP.

4.4.1 Identification of the candidate criteria and alternatives

Our set of criteria is given by $\Omega = \{c_1, \ldots, c_n\}$ where Ω is the frame of discernment involving all the possible criteria related to the MCDM problem. Denote the set of all subsets of criteria by 2^Ω and let C_k be a subset of Ω. We assume that there is a set of all the possible alternatives $\Theta = \{a_1, \ldots, a_m\}$ consisting of m elements. Denote the set of all subsets of Θ by 2^Θ and let A_k be a subset of alternatives.

To identify the candidate groups of criteria and alternatives, we have to apply the same hypotheses assumed in belief AHP (see Chapter 3).

4.4.2 Computing the weight of considered criteria

After identifying the subsets of criteria, the decision maker has to express his preferences in order to complete the belief pair-wise comparisons matrix as shown in Section 4.3.

Then, the obtained bba $m^{\Omega_{c_i}}$ is transformed into pignistic probabilities, denoted by $BetP^{\Omega_{c_i}}$ using the pignistic transformation.

Example 4.2. *Let us continue with Example 4.1. After computing the pair-wise comparison matrix, the resulting bbas are transformed into pignistic probabilities (see Table 4.3).*

Table 4.3: Belief pair-wise matrix: pignistic probabilities

	$\{c_1\}$	$\{c_4\}$	$\Omega_1 = \{c_2, c_3\}$
BetP	$BetP^{\Omega_{\{c_1\}}}(\{yes\}) = 0.97$	$BetP^{\Omega_{\{c_4\}}}(\{yes\}) = 0.21$	$BetP^{\Omega_{\Omega_1}}(\{yes\}) = 0.089$
	$BetP^{\Omega_{\{c_1\}}}(\{no\}) = 0.03$	$BetP^{\Omega_{\{c_4\}}}(\{no\}) = 0.79$	$BetP^{\Omega_{\Omega_1}}(\{no\}) = 0.911$

4.4.3 Computing the alternatives priorities

Following the same reasoning, belief pair-wise comparison matrices evaluate each group of alternatives regarding each criterion. By applying Equation 4.5 for each obtained pair-wise comparison matrix, we get a bba which quantifies the degree of belief assigned by the expert to each subset of alternatives. However, our purpose is to combine these obtained bbas to get a single belief function. The problem here is that each subset of alternatives has its own frame of discernment. For instance, if we say that alternative A is preferred to B and alternative C is preferred to B, this does not means that alternative A is indifferent to C. Consequently, each obtained bba is defined on a different frame of discernment.

The idea to use is then to standardize all the frames of discernment. Obviously, we propose to use the concept of refinement operations (Shafer, 1976), which allows us to establish relationships between different frames of discernment in order to express beliefs on anyone of them. The objective consists then in obtaining one frame of discernment Θ from the set Θ_{A_k} by splitting some or all of its events.

As mentioned above, each bba $m^{\Theta_{A_k}}$ represents the belief over all possible answers (yes or no). However, at this stage, we want to know which alternative is the best (Θ). Θ_{A_k} is considered as a coarsening of Θ, we get the following relation:

$$m^{\Theta_{A_k}\uparrow\Theta}(\rho_k(\omega)) = m^{\Theta_{A_k}}(\omega) \qquad \forall\omega \subseteq \Theta_{A_k} \tag{4.8}$$

where the mapping ρ_k from Θ_{A_k} to Θ is a refinement, $\rho_k(\{yes\}) = \{A_k\}$ and $\rho_k(\{no\}) = \overline{\{A_k\}}$.

Once we have standardized our frame of discernment Θ, we can now combine the resulting bbas using the conjunctive rule in order to obtain a belief function reflecting the importance of alternatives to a given criterion:

$$m_{c_k} = \textcircled{\scriptsize$\cap$} m^{\Theta_{A_i}\uparrow\Theta} \qquad \text{where } i = \{1, \ldots, l\} \tag{4.9}$$

where l is the number of subsets of alternatives.

Finally, we obtain m_{c_k} representing the opinions-beliefs of the expert about his preferences regarding the set of alternatives. At this step, the consistency of the obtained bba must be checked using our new index.

Example 4.3. *Like the criterion level, the judgments between decision alternatives over different criteria are dealt with in an identical manner. For example, to evaluate the alternatives according to the criterion c_1, the decision maker is required to evaluate the following subsets of alternatives: $\{p\}$ and $\{r, f\}$. We get the matrix presented in Table 4.4.*

Table 4.4: Belief pair-wise matrix regarding c_1 criterion

c_1	$\{p\}$	$\{r, f\}$
$\{p\}$	$m_{\{p\}}^{\Theta_{\{p\}}}(\Theta_{\{p\}}) = 1$	$m_{\{r,f\}}^{\Theta_{\{p\}}}(\{yes\}) = 0.95$ $m_{\{r,f\}}^{\Theta_{\{p\}}}(\Theta_{\{p\}}) = 0.05$
$\{r, f\}$	$m_{\{p\}}^{\Theta_{\{r,f\}}}(\{no\}) = 0.95$ $m_{\{p\}}^{\Theta_{\{r,f\}}}(\Theta_{\{r,f\}}) = 0.05$	$m_{\{r,f\}}^{\Theta_{\{r,f\}}}(\Theta_{\{r,f\}}) = 1$

As in the criterion level, for the subset of alternatives $\{p\}$, we use Equation 4.5 in order to combine the obtained bbas:

$$m^{\Theta_{\{p\}}} = m_{\{p\}}^{\Theta_{\{p\}}} \circledcap m_{\{r,f\}}^{\Theta_{\{p\}}}$$

We get the following bba:

$$m^{\Theta_{\{p\}}}(\{yes\}) = 0.95 \textit{ and } m^{\Theta_{\{p\}}}(\{\Theta_{\{p\}}\}) = 0.05.$$

The obtained bba is totally consistent since $\phi(\emptyset) = 0$. Then, a similar process is repeated for the rest of alternatives and results are shown in Table 4.5.

Table 4.5: Belief pair-wise matrix regarding c_1 criterion

c_1	$\{p\}$	$\{r, f\}$
bba	$m^{\Theta_{\{p\}}}(\{yes\}) = 0.95$ $m^{\Theta_{\{p\}}}(\Theta_{\{p\}}) = 0.05$	$m^{\Theta_{\{r,f\}}}(\{no\}) = 0.95$ $m^{\Theta_{\{r,f\}}}(\Theta_{\{r,f\}}) = 0.05$

Subsequently, we proceed now with the standardization of our frame of discernment. We get the following bba:

$$m^{\Theta_{\{p\}}\uparrow\Theta}(\{p\}) = m^{\Theta_{\{p\}}}(\{yes\})$$

$$m^{\Theta_{\{p\}}\uparrow\Theta}(\overline{\{p\}}) = m^{\Theta_{\{p\}}}(\{no\})$$

$$m^{\Theta_{\{p\}}\uparrow\Theta}(\Theta) = m^{\Theta_{\{p\}}}(\Theta_{\{p\}})$$

By applying Equation 4.8, we get Table 4.6.

Table 4.6: Belief pair-wise matrix regarding c_1 criterion

c_1	$\{p\}$	$\{r, f\}$
$\{bbm\}$	$m^{\Theta_{\{p\}}\uparrow\Theta}(\{p\}) = 0.95$ $m^{\Theta_{\{p\}}\uparrow\Theta}(\Theta) = 0.05$	$m^{\Theta_{\{r,f\}}\uparrow\Theta}(\{p\}) = 0.95$ $m^{\Theta_{\{r,f\}}\uparrow\Theta}(\Theta) = 0.05$

Finally, the obtained bbas can be directly combined using the conjunctive rule of combination. For simplicity, we denote $m^{\Theta_{\{p\}}\uparrow\Theta}$ by m^{Θ}, we get:

$$m^{\Theta}(\{p\}) = 0.9975 \textit{ and } m^{\Theta}(\{\Theta\}) = 0.0025.$$

Then, as shown in the previous step, the computation procedure is repeated for the rest of comparison matrices to get Table 4.7. Our obtained bbas are consistent since $\phi(\emptyset) < 1$.

Table 4.7: The bbas m_{c_2}, m_{c_3} and m_{c_4}

	$\emptyset$	$\{p\}$	$\{r, f\}$	Θ
m_{c_2}	0.541	0.128	0.133	0.198
	$\{r\}$	Θ		
m_{c_3}	0.905	0.095		
	$\{f\}$	Θ		
m_{c_4}	0.61	0.39		

4.4.4 Updating the alternatives priorities

As shown in the previous Chapter, we apply the belief AHP aggregation procedure. If we have C_k a subset of criteria, then we get β_k its corresponding measure of reliability:

$$BetP^{\Omega_{C_k}}(\{yes\}) = \beta_k \text{ and } BetP^{\Omega_{C_k}}(\{no\}) = 1 - \beta_k \quad (4.10)$$

Two cases will be presented. First, if the reliability factor represents a single criterion c_k, then the corresponding bba will be directly discounted:

$$m_{c_k}^{\alpha_k}(A_j) = \beta_k.m_{c_k}(A_j),\ \forall A_j \subset \Theta \tag{4.11}$$

$$m_{c_k}^{\alpha_k}(\Theta) = (1-\beta_k) + \beta_k.m_{c_k}(\Theta) \tag{4.12}$$

where $m_{c_k}(A_j)$ the relative bba for the subset A_j (obtained in the previous step), β_k is a measure of reliability and we denote $\alpha_k = 1 - \beta_k$.

Second, if this factor represents a group of criteria, its corresponding bbas must be combined, then it will be discounted by their corresponding measure of reliability:

$$m_{C_k} = \textcircled{\cap} m_{c_i},\ \ i = \{1, \ldots, h\} \tag{4.13}$$

where h is the number of items of a specific group of criteria C_k.

Finally, these obtained bbas will be discounted by their corresponding measure of reliability. We apply the same process used in Equations 4.11 and 4.12 to get $m_{C_k}^{\alpha_k}$.

Example 4.4. *As shown previously, after computing the belief functions for each set of alternatives with respect to each criterion, we must update the obtained bba with their corresponding measure of reliability.*

Firstly, this step concerns the groups of criteria, that is the subset $\{c_2, c_3\}$. Therefore, by using Equation 3.7, we propose to combine the bbas relative to the c_2 and c_3 criteria ($m_{c_2,c_3} = m_{c_2} \textcircled{\cap} m_{c_3}$).

Then, these obtained bbas are discounted by the measure of reliability $\beta_{c_2,c_3} = 0.089$ (obtained from Table 4.3 and using Equation 4.13), so we use Equations 4.11 and 4.12 to get Table 4.8.

Table 4.8: The bbas m_{c_2,c_3} and $m_{c_2,c_3}^{\alpha_{c_2,c_3}}$

	$\emptyset$	$\{p\}$	$\{r\}$	$\{r,f\}$	Θ
m_{c_2,c_3}	0.6518	0.0121	0.2994	0.0126	0.0241
$m_{c_2,c_3}^{\alpha_{c_2,c_3}}$	0.058	0.001	0.0266	0.0011	0.9133

The next step concerns the criteria $\{c_1\}$ and $\{c_4\}$. The relative bba are directly discounting using Equations 4.11 and 4.12, where the reliability measure $\beta_{c_1} = 0.97$ and $\beta_{c_4} = 0.21$ (see Table 4.9).

Table 4.9: The bbas $m_{c_1}^{\alpha_{c_1}}$ and $m_{c_4}^{\alpha_{c_4}}$ after discounting

	$\{p\}$	Θ
$m_{c_1}^{\alpha_{c_1}}$	0.9675	0.0325
	$\{f\}$	Θ
$m_{c_4}^{\alpha_{c_4}}$	0.1281	0.8719

4.4.5 Decision making

To this end, and after updating the priorities of the alternative sets with respect to their set of criteria, we must combine the overall bba in order to help the expert to make a decision:

$$m_{final} = \textcircled{\cap} m_{C_k}^{\alpha_k}, \ \ k = \{1, \ldots, l\} \tag{4.14}$$

where l is the number of subsets of criteria.

In this context, we choose to transform the final bba into pignistic probabilities. The decision maker will choose the alternative which has the highest value.

Example 4.5. *The final step is then to choose the best alternatives. First, we apply the conjunctive rule of combination (see Equation 3.8), that leads us to get a single bba denoted by m_{final}. Then, we transform the obtained bba to a pignistic probability, and we get this final result (see Table 4.11).*

Table 4.10: The overall bba

	$\emptyset$	$\{p\}$	$\{f\}$	$\{r\}$	$\{r, f\}$	Θ
m_{car}	0.19823	0.7713	0.0038	0.00075	0.00004	0.02588

Table 4.11: The final result using the Yes-No/AHP approach

Alternatives	p	r	f
BetP	0.97275	0.011719	0.015531

The alternatives are now ranked according to their pignistic probabilities, as follows: $\{p\}$, $\{f\}$ and $\{r\}$.

Figure 4.3 presents the ratio of each alternative, where Peugeot is evidently the most important alternative in the presented case study with a total aggregate priority of 0.97275. *Conversely, Renault is shown to be the least important carrying a priority of* 0.011719.

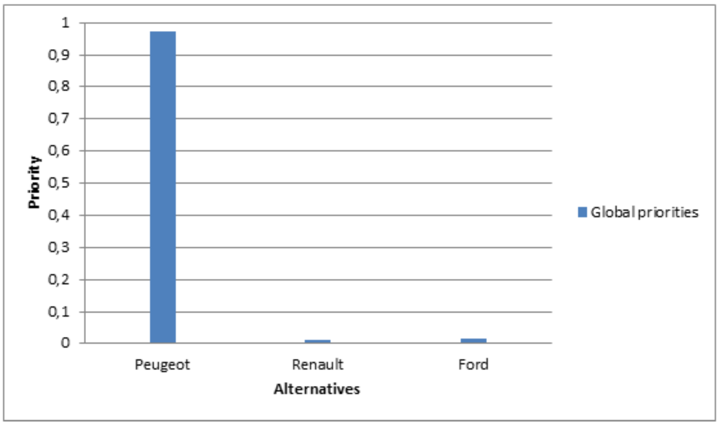

Figure 4.3: The resulting contribution of Yes-No/AHP

4.5 Handling dependency between alternatives and criteria under Yes-No/AHP method

As we have presented previously in Chapter 3, we need to model the relationship between alternatives and criteria. Therefore, in this Section, our goal is to model dependency between the two levels under the Yes-No/AHP model. In other words, our objective is to extend the conditional belief AHP by using belief pair-wise comparisons instead of Saaty scale. Figure 4.4 summarizes the computational procedure.

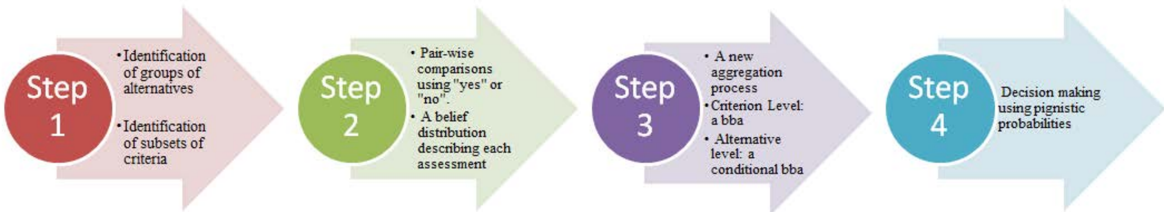

Figure 4.4: The general decision-making paradigm based on conditional Yes-No/AHP

According Figure 4.4, the major difference between conditional Yes-No/AHP and conditional belief AHP is at Step 2. To use the conditional Yes-No/AHP, the following stages must be respected (Ennaceur et al., 2014a):

1. Identification of the sets of criteria and groups of alternatives.

2. The criterion level is described by the obtained bba using belief pair-wise matrix.

3. At the alternative level, each belief pair-wise comparison matrix is described by a conditional bba.

4. Standardizing the frame of discernment. At the criterion level, the vacuous extension is used. At the alternative level, we apply the ballooning extension technique.

5. The conjunctive rule is defined to combine the overall bbas.

6. Decision making using pignistic transformation.

4.6 Experimental analysis

To test the validity of our developed methods, our experiments are performed on the same random data sets presented in Chapter 3.

4.6.1 Simulation algorithm

In order to compare AHP, Yes-No/AHP and conditional Yes-No/AHP, we have to generate reliable data. To do so, the following steps may be applied:

1. We generate a random matrix for the decision performance as well as a random matrix to represent the weight of each decision criteria. Based on these two matrices, the overall scores and ranks of the decision alternatives are calculated. These are the usual steps in the WSM method.

2. From the performance matrix, we generated pair-wise comparison matrices of different alternatives that are compared to each criterion.

3. Each pair-wise matrix is transformed into belief pair-wise comparison. Indeed, the resulting bbas has only one focal element since we are in a certain context:

   - If the actual preference value regarding the alternative a with bbm, $m(\{yes\}) = 1$ (If $1 < p \leq 9$).
   - If $p = 1$ then $m(\Theta) = 1$
   - If $\frac{1}{9} \leq p \leq \frac{1}{2}$ then $m(\{no\}) = 1$

4. We apply the suggested method to compute the overall priorities and rank alternatives.

5. We compare the obtained result with the ranking of the WSM method.

Example 4.6. *Let us continue with the same example 3.10 introduced in Chapter 3. We consider* 3 *alternatives* a_1*,* a_2 *and* a_3 *and three criteria* c_1*,* c_2 *and* c_3*. The decision making problem is described using the matrix presented in Table 4.12.*

Table 4.12: Decision matrix

	Criteria		
Alt.	c_1	c_2	c_3
	$\frac{8}{13}$	$\frac{2}{13}$	$\frac{3}{13}$
a_1	1	9	9
a_2	5	2	2
a_3	1	5	9

This example has been solved using the WSM and AHP in (Triantaphyllou & Mann, 1989). Applying the WSM, it can be shown that the alternative a_1 *is the best one and AHP turns out that the alternative* a_2 *is the best one. Obviously, this is in contradiction with the conclusion derived using the WSM.*

Now, let us model this example using Yes-No/AHP. If the decision maker knew the actual data shown in the original crisp decision matrix, then the pair-wise comparisons matrices will be illustrated in Table 4.13.

Next, and after modeling pair-wise comparison matrices using the Saaty scale, we transform them into bba. We obtain the results shown in Table 4.14.

We use the Yes-No/AHP method to aggregate the obtained priorities. We obtain the following order: $a_1 > a_2 > a_3$*. Obviously, this is in contradiction with the results derived when the AHP*

Table 4.13: The preference relations matrices

c_1			
	a_1	a_2	a_3
a_1	$-$	$1/5$	1
a_2	$-$	$-$	5
a_3	$-$	$-$	$-$

c_2			
	a_1	a_2	a_3
a_1	$-$	$9/2$	$9/5$
a_2	$-$	$-$	$2/5$
a_3	$-$	$-$	$-$

c_3			
	a_1	a_2	a_3
a_1	$-$	$9/2$	$9/9$
a_2	$-$	$-$	$2/9$
a_3	$-$	$-$	$-$

Table 4.14: Belief pair-wise comparison matrices

c_1	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$
$\{a_1\}$	$m^{\Omega_{\{a_1\}}}_{\{a_1\}}(\Omega_{\{a_1\}}) = 1$	$m^{\Omega_{\{a_1\}}}_{\{a_2\}}(\{no\}) = 1$	$m^{\Omega_{\{a_1\}}}_{\{a_3\}}(\Omega_{\{a_1\}}) = 1$
$\{a_2\}$	$m^{\Omega_{\{a_2\}}}_{\{a_1\}}(\{yes\}) = 1$	$m^{\Omega_{\{a_2\}}}_{\{a_2\}}(\Omega_{\{a_2\}}) = 1$	$m^{\Omega\{a_2\}}_{\{a_3\}}(\{yes\}) = 1$
$\{a_3\}$	$m^{\Omega_{\{a_3\}}}_{\{a_1\}}(\Omega_{\{a_3\}}) = 1$	$m^{\Omega_{\{a_3\}}}_{\{a_2\}}(\{no\}) = 1$	$m^{\Omega_{\{a_3\}}}_{\{a_3\}}(\Omega_{\{a_3\}}) = 1$

c_2	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$
$\{a_1\}$	$m^{\Omega_{\{a_1\}}}_{\{a_1\}}(\Omega_{\{a_1\}}) = 1$	$m^{\Omega_{\{a_1\}}}_{\{a_2\}}(\{yes\}) = 1$	$m^{\Omega_{\{a_1\}}}_{\{a_3\}}(\{yes\}) = 1$
$\{a_2\}$	$m^{\Omega_{\{a_2\}}}_{\{a_1\}}(\{no\}) = 1$	$m^{\Omega_{\{a_2\}}}_{\{a_2\}}(\Omega_{\{a_2\}}) = 1$	$m^{\Omega\{a_2\}}_{\{a_3\}}(\{no\}) = 1$
$\{a_3\}$	$m^{\Omega_{\{a_3\}}}_{\{a_1\}}(\{no\}) = 1$	$m^{\Omega_{\{a_3\}}}_{\{a_2\}}(\{yes\}) = 1$	$m^{\Omega_{\{a_3\}}}_{\{a_3\}}(\Omega_{\{a_3\}}) = 1$

c_3	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$
$\{a_1\}$	$m^{\Omega_{\{a_1\}}}_{\{a_1\}}(\Omega_{\{a_1\}}) = 1$	$m^{\Omega_{\{a_1\}}}_{\{a_2\}}(\{yes\}) = 1$	$m^{\Omega_{\{a_1\}}}_{\{a_3\}}(\Omega_{\{a_1\}}) = 1$
$\{a_2\}$	$m^{\Omega_{\{a_2\}}}_{\{a_1\}}(\{no\}) = 1$	$m^{\Omega_{\{a_2\}}}_{\{a_2\}}(\Omega_{\{a_2\}}) = 1$	$m^{\Omega\{a_2\}}_{\{a_3\}}(\{no\}) = 1$
$\{a_3\}$	$m^{\Omega_{\{a_3\}}}_{\{a_1\}}(\Omega_{\{a_3\}}) = 1$	$m^{\Omega_{\{a_3\}}}_{\{a_2\}}(\{yes\}) = 1$	$m^{\Omega_{\{a_3\}}}_{\{a_3\}}(\Omega_{\{a_3\}}) = 1$

method was applied at the beginning of this illustrative example. However, we have identified the same best alternative as WSM.

By applying the conditional Yes-No/AHP approach, we get the following ranking: $a_1 > a_2 > a_3$. *Our proposed approach has also identified alternative* a_1 *as the best.*

4.6.2 Simulation results

One of the difficulties in comparing many MCDM methods is that there is rarely any way to check the accuracy of the methods. At the same time, it must be understood that these methods have different relative advantages in dealing with different types of data such as ordinal and uncertain data. The intent is not to establish the dominance of any one method, but rather to compare the relative accuracy in the case of known outcome.

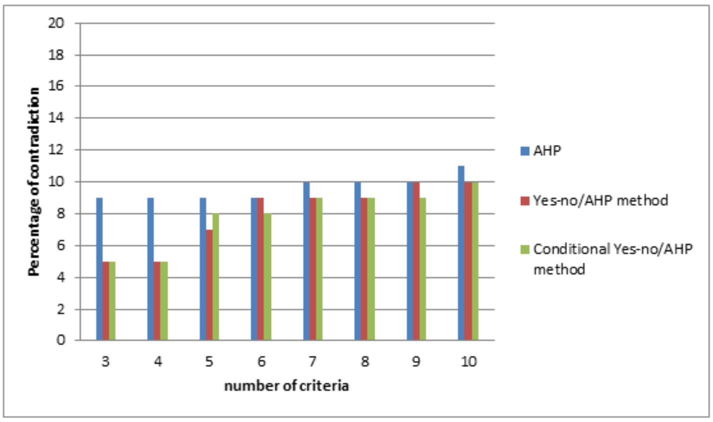

Figure 4.5: Percentage of contradiction (%) based on 5 alternatives

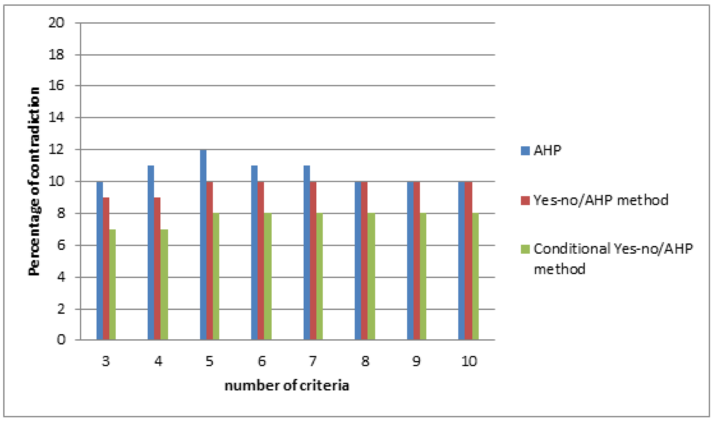

Figure 4.6: Percentage of contradiction (%) based on 7 alternatives

Previously with belief AHP methods, we have introduced our validation method. Comparing the results in terms of number of alternatives and criteria, in Figure 4.5, we can notice that in most cases conditional Yes-No/AHP is able to produce a smaller percentage of contradiction. For instance, when applying Yes-No/AHP with 5 alternatives, the percentage of contradiction is under 8%.

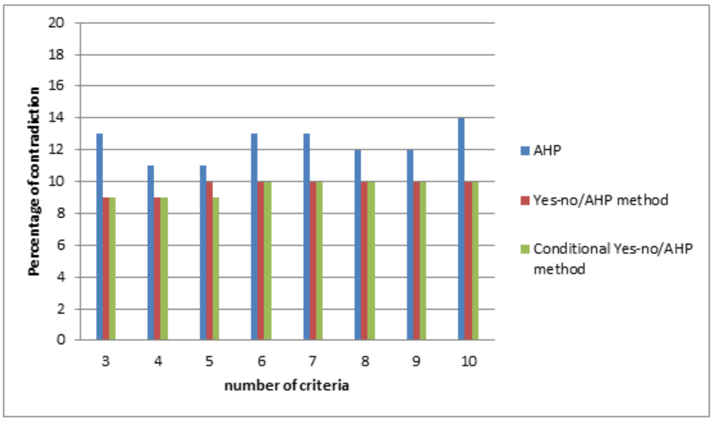

Figure 4.7: Percentage of contradiction (%) based on 9 alternatives

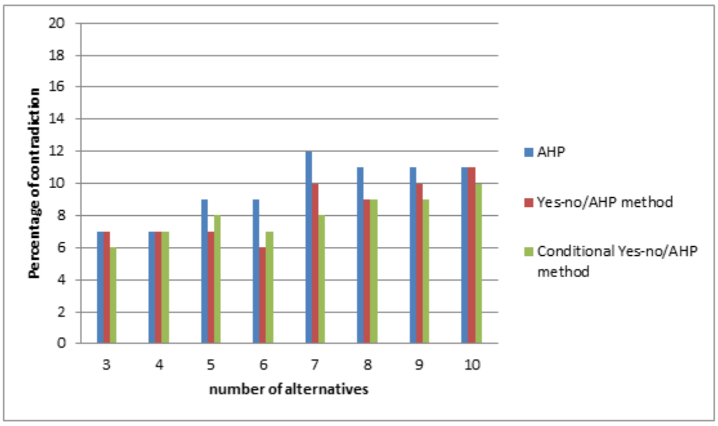

Figure 4.8: Percentage of contradiction (%) based on 5 criteria

We also notice, from Figures 4.6 and 4.7, that there is no significant difference in the percentage of contradiction between Yes-No/AHP and conditional Yes-No/AHP methods. In some other cases, Yes-No/AHP outperforms conditional Yes-No/AHP and vice-versa. Most importantly, when comparing the overall results, we can remark that problems with less alternatives yielded fewer ranking contradictions. As it can be seen, the number of criteria did not play a prime role. For instance, from Figure 4.8, our methods are compared regarding 5 criteria and different alternatives the percentage of contradiction is under 10%.

Importantly, both approaches perform equally. In some others, Yes-No/AHP outperforms conditional Yes-No/AHP. When comparing the methods regarding 5 criteria and 6 alternatives the percentage of contradiction is set to 6%. However, the percentage of contradiction of conditional

Yes-No/AHP is set to 7%. On the other hand, the percentage of contradiction of conditional Yes-No/AHP approach is lower than the percentage of Yes-No/AHP method (more details can be found in Appendix A).

Remarkably, Yes-No/AHP methods use only the minimum information to model the decision maker preferences. However, more uncertainty can be handled within the scale of preferences. This specific point was tackled by our next proposed qualitative AHP method.

4.6.3 Catering selection problem

Let us remind that the main objective of this section is to validate our findings by a real-world example of catering selection. This practical example does not only validate our results as a new MCDM method, but also ensures the consistency of final decision.

A sensitivity analysis is then held to show the effect of altering different parameters of the model on the choice of the right firm. First, the current values of the models are presented in Table 4.15 according to the pair-wise comparison that has been carried out by the experts in the construction fields. Obviously, the results are in favor of Afiyetle. Now that the best firm has been identified, how would the model respond to any changes in the weights of the listed criteria?

Table 4.15: Decision matrix

	Yes-No/AHP	conditional Yes-No/AHP
Durusu	0.33	0.42
Mertol	0.20	0.15
Afiyetle	0.47	0.43

The next stage is to check the stability of our models. Sensitivity analysis as shown in Figures 4.9 and 4.10.

The sensitivity analysis of the Hygiene criterion still demonstrates Afiyetle as the best scorer; however, as more weight is assigned, Mertol will tend to advance in rank. Similar analysis is held for Quality of meal criterion. The results show that Afiyetle is always in the lead with a persistent score. Nevertheless, we can remark that rank reversal occurs only in the first and second ranks (Afiyetle and Durusu respectively) when the Quality of meal is increased to 0.7 under the first model. Overall, based on sensitivity analysis, it can be concluded that the alternative Mertol is always presented as the worst one.

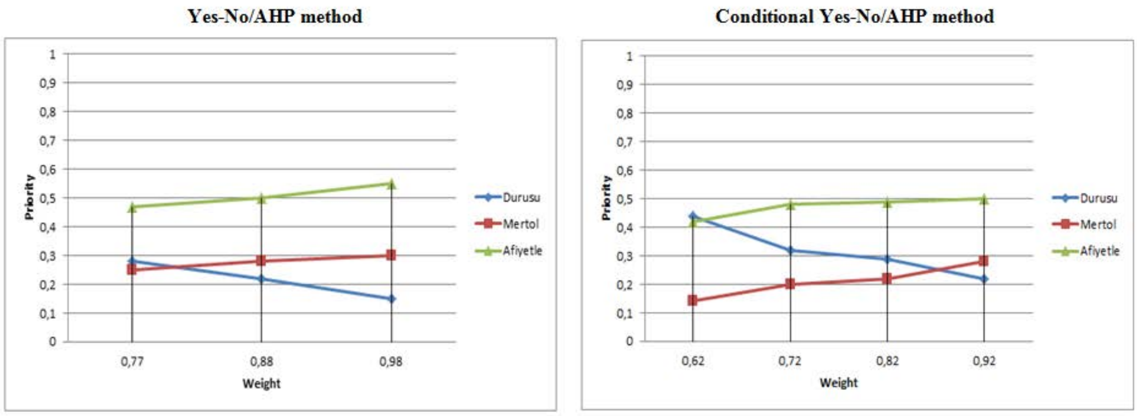

Figure 4.9: Sensitivity analysis of Hygiene criterion

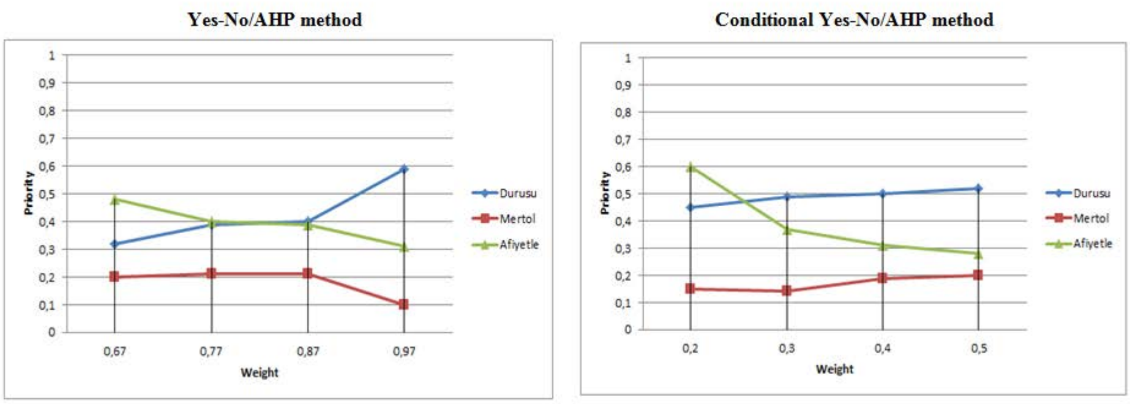

Figure 4.10: Sensitivity analysis of Quality of meal criterion

Furthermore, we notice that for both methods there is no significant difference. The two approaches maintain Afiyetle as the best alternative. This can proof that our method can perfectly model the expert judgment since the choice of Afiyetle remains the same even with significant changes in the criteria weights, which can be justified by the consistent judgments of the pair-wise comparisons.

4.7 Conclusion

In this Chapter, belief pair-wise comparison matrix was developed. We have exposed what kind of uncertainty is handled by this technique as well as its different objectives. Then, new AHP extensions were defined based on belief pair-wise procedure. We have explained their main steps and evaluated their performance using generated random data.

Despite the noticed advantages of our proposed Yes-No/AHP methods, there are some limitations. In some cases, the decision maker cannot estimate his assessment with a numerical value. This prompted our research into the use of qualitative belief function method in order to generate quantitative information from qualitative assessments. This will be dealt with in the next Chapter.

Chapter 5

Constructing belief functions from qualitative expert assessments

5.1 Introduction

For many years, qualitative methods for reasoning under uncertainty have increasingly attracted people (Parsons, 1994; Wong & Lingras, 1994). Their aim is to propose solutions for processing qualitative information which efficiently take into account information provided by human sources. The aim of qualitative reasoning methods is to help in decision-making for situations in which the precise numerical methods are not appropriate (Parsons & Hunter, 1998). Several formalisms for qualitative reasoning have been proposed by Ben Yaghlane et al. (2006), Bryson and Mobolurin (1999), Parsons (1994), and Wong and Lingras (1994).

However, modeling the decision maker's preferences is not an easy task because he usually prefers to express his opinions in natural language based on knowledge and experience that he provides rather than direct quantitative information. In other words, solving a problem dealing with the expert's preferences is usually characterized by a high degree of uncertainty. Besides, in some cases, the decision maker may be unable to express his opinions due to his lack of knowledge. He is then forced to provide incomplete or even erroneous information. Obviously, ignoring this difficulty in eliciting the expert's preference is not a good practice. Therefore, preferences need to be implemented in an assessment, which reflects as accurately as possible the human mind.

To tackle the problem, a numerical representation under the belief function framework is introduced.

In order to do this, this Chapter is set out as follows. First, in Section 5.2, we give some motivations. We recall in Section 5.3 what can be called preference modeling which serves as a “benchmark” for most of the works in the area. Then, we provide a brief description of some existing qualitative reasoning methods (Section 5.4). Next, in Section 5.5, our suggested solutions are described and examples are presented to illustrate our methods.

5.2 Motivations

When solving problems dealing with belief function theory, experts are usually required to provide precise numerical value, for determining the portion of belief committed exactly to an event in a particular domain. However, when handling such a situation, the main difficulty is how to quantify these numeric values. Therefore, linguistic assessments could be used instead. The expert is then asked to express his opinions qualitatively based on knowledge and experience that he provides in response to a given question rather than direct quantitative information.

Besides, in preference modeling, the expert may express preferences towards a pair of alternatives in distinct ways: he either has a strict preference towards one alternative, or is indifferent to both alternatives. These two interpretations are possible because we made the assumption based on complete and certain information.

However, these two preferences do not apply to all possible situations that a decision maker may faced. Consider now the situations in which the expert has symmetrically lack and excess of information in the sense that he has contradictory inputs. He is then forced to provide incomplete or even erroneous information.

Consequently, new situations are introduced, such as incompleteness and incomparability. The intuition is that the expert cannot compare apple and cheese because they are too different. For instance, he may consider that alternatives may be incomparable because the expert does not wish very dissimilar alternatives to be compared. Incompleteness, on the other hand, simply represents an absence of knowledge about the preference of certain pairs of alternatives. It arises when we have not fully elicited an expert’s preferences or when the expert does not have the full information. Moreover, we suggest to include the weak preference relation (Roy, 1987), that separates the preference area from the indifference area. A possible interpretation is an hesitation between strict preference and indifference.

To deal with such situations, a more realistic solution is proposed, that is able to efficiently imitate the expert's reasoning using belief function theory (Ennaceur et al., 2012a, 2013a, 2014b). Our main aim is then to elaborate on how may be incomparability, incompleteness and weak preference be represented in the belief function framework. In the next Section, we will start by introducing some classical binary relations.

5.3 Binary Relations and Preference Modeling

The literature on preference modeling is vast. This can first be explained by the fact that the question of modeling preferences occurs in several disciplines, such as the following:

- in Economics, where we try to model the preferences of a rational consumer (Debreu, 1959).
- in Psychology, in which the study of preference judgments collected in experiments is quite common (Kahneman & Tversky, 1979).
- in Political Sciences, in which the question of defining a collective preference on the basis of the opinion of several voters is central (Sen, 1986).
- in Operations Research, in which optimizing an objective function implies the definition of a direction of preference (Roy, 1985).
- in Artificial Intelligence, in which the creation of autonomous agents able to make decisions involves the modeling of their vision of what is desirable and what is less so (Doyle & Wellman, 1992).

A binary relation T on a set A is a subset of the Cartesian product $A \times A$, i.e. a set of ordered pairs (a, b) of elements of A. If the ordered pair (a, b) belongs to the set T, we will often write aTb. In the opposite case, we write $a\bar{T}b$. Except when explicitly mentioned otherwise, we will suppose in all what follows that the set A is finite (Roubens & Vincke, 1985; Bouyssou & Vincke, 2009).

Consider an ordered pair (a, b) of objects. In the classical theory, it is supposed that the answer to the question: is a least as good as b? can only have two possible answers: yes or no.

Asking such a question for all possible ordered pairs of objects defines a binary relation on T in the following way:

aTb iff the answer to the question "is a at least as good as b?" "is yes".

It is easy to see that considering a pair (a, b) of objects, four and only four mutually exclusive cases arise:

- $aTb \wedge bTa$ denoted by $a \sim b$, which reads "a is indifferent to b",
- Not $aTb \wedge$ Not bTa denoted by $a?b$, which reads "a is incomparable to b",
- $aTb \wedge$ Not bTa denoted by $a \succ b$, which reads "a is strictly preferred to b",
- Not $aTb \wedge bTa$ denoted by $b \succ a$, which reads "b is strictly preferred to a".

In some cases, we may consider answers other than yes or no to the question "is a at least as good as b?", such as the following:

- answers such as I do not know,
- answers including information on the intensity of the preference,
- answers including information on the credibility of the proposition.

Admitting such answers imply using a language that is richer than that of binary relations, e.g.:

- the language of fuzzy relations, each assertion of the type aSb having a degree of credibility (Doignon et al., 1986),
- languages tolerating hesitation (Perny & Roy, 1992),
- languages using the idea of intensity of preference (Doignon, 1987; Costa & Vansnick, 1994), an assertion such that aSb and $b\bar{S}a$ being further qualified (weak, strong or extreme preference, for instance).

Therefore, in this Chapter, we introduce a new elicitation method under the belief function framework, based on preference relations. We start by describing some existing approaches. Then, we will present our proposed solution.

5.4 Overview of qualitative reasoning methods

The problem of qualitatively eliciting experts' opinions and generating basic belief assignments have been addressed by many researchers: Wong and Lingras (1994), Parsons (1994), Bryson and Mobolurin (1999) and Ben Yaghlane et al. (2006). In this Section, we provide an overview of some existing approaches.

5.4.1 Wong and Lingras' Method

Wong and Lingras (1994) proposed a method for generating quantitative belief functions from qualitative preference assessments. So, given a pair of propositions, experts may express which of the propositions is more likely to be true. Thus, they defined two binary relations: preference ($\succ$) and indifference ($\sim$) defined on 2^{Θ} such as:

$$a \succ b \text{ is equivalent to } bel(a) > bel(b) \tag{5.1}$$

$$a \sim b \text{ is equivalent to } bel(a) = bel(b) \tag{5.2}$$

where $a, b \in 2^{\Theta}$.

This approach is based on two steps. The first one consists in considering that all the propositions that appear in the preference relations are potential focal elements. However, some propositions are eliminated according to the following condition: if $a \sim b$ for some $a \subset b$, then a is not a focal element.

After that, the basic belief assignment is generated using the two presented Equations 5.1 and 5.2. This formulation has multiple belief functions that are consistent with the input qualitative information, and so their procedure only generates one of them.

It should be noted that Wong and Lingras' approach does not address the issue of inconsistency in the pair-wise comparisons. For example, the expert could specify the apparently inconsistent preference relationships: $bel(a) > bel(b)$, $bel(b) > bel(c)$ and $bel(c) > bel(a)$.

5.4.2 Parsons Method

Parsons (1994) suggests a qualitative approach in which the mass of each proposition is expressed vaguely. He defines a qualitative mass function $[m(A)]$ that corresponds to $m(A)$.

$$\text{if } m(A) > 0, \text{ then } [m(A)] = + \tag{5.3}$$

$$\text{if } m(A) = 0, \text{ then } [m(A)] = 0$$

where $+$ is any mass that is known to be non-zero ($0 < + < 1$).

He demonstrates how these qualitative mass functions can be combined based on the Dempster combination rule. Parsons (1994) further suggests that linguistic quantifiers can be used to aid the transformation from qualitative representation of belief to numeric representation of belief. He suggested the linguistic qualifiers None, Little, About Half, Much, All where:

$$\text{None} \leq \text{Little} \leq \text{About Half} \leq \text{Much} \leq \text{All}.$$

with each linguistic qualifier corresponding to a subinterval of the unit interval $[0, 1]$.

Parsons also suggests that bba could be represented numerically using numeric intervals. He notes that this approach gives more precise results than the linguistic approach because intervals do not have to be rounded to the nearest numeric interval. With this approach it becomes necessary to do interval ranking, which could be done using fuzzy arithmetic.

5.4.3 Bryson Method

Qualitative Discrimination Process (QDP), a model for generating belief functions from qualitative preferences, was presented by Bryson and Mobolurin (1999).

This method is based on the following steps. First, using this QDP approach, each proposition is assigned into a broad category bucket, then to a corresponding intermediate bucket and finally to a corresponding narrow category bucket. The qualitative scoring is done using a table where each broad category is a linguistic quantifier in the sense of Parsons (1994). He considers that linguistic quantifiers could provide a useful approach to representing beliefs vaguely. Hence bba should be represented using numeric intervals.

Then, in Step 2, the previous broad category is used to identify and remove non focal propositions. For each superset proposition, determine if the expert is indifferent in his strength of belief, in the truthfulness of the given proposition and any of its subset propositions in the same or lower narrow category bucket.

Step 3 is called “imprecise pair-wise comparisons” because the expert is required to provide numeric intervals to express his beliefs on the relative truthfulness of the propositions.

In Step 4, the consistency of the belief information provided by the expert is checked. Then, the belief function is generated in Step 5 by providing a bba interval for each focal element.

Finally, in Step 6, the expert examines the generated belief functions and stops the QDP if it is acceptable, otherwise the process is repeated.

5.4.4 Ben Yaghlane et al.’s Method

Ben Yaghlane et al. (2006) proposed a method for generating optimized belief functions from qualitative preferences.

So given two alternatives, an expert can usually express which of the propositions is more likely to be true, thus he used two binary preference relations: the preference and the indifference relations.

The objective of this method is then to convert these preferences into constraints of an optimization problem whose resolution, according to some uncertainty measures (UM) (nonspecificity measures, conflict measures, composite measures (see Section 2.3.4)), allows the generation of the least informative or the most uncertain belief functions defined as follows:

$$a \succ b \Leftrightarrow bel(a) - bel(b) \geq \varepsilon \tag{5.4}$$

$$a \sim b \Leftrightarrow |bel(a) - bel(b)| \leq \varepsilon \tag{5.5}$$

where ε is considered to be the smallest gap that the expert may discern between the degrees of belief in two propositions a and b. Note that ε is a constant specified by the expert before beginning the optimization process.

Ben Yaghlane et al. (2006) propose a method which requires that propositions are represented in terms of focal elements, they assume that Θ (where Θ is the frame of discernment) should always be considered as a potential focal element. Then, a mono-objective technique was used to solve such constrained optimization problem:

$$\begin{array}{c} Max_m UM(m) \\ s.t. \\ bel(a) - bel(b) \geq \varepsilon \quad \forall (a,b) \text{ for which } a \succ b \\ bel(a) - bel(b) \geq -\varepsilon \quad \forall (a,b) \text{ for which } a \sim b \\ bel(a) - bel(b) \leq \varepsilon \quad \forall (a,b) \text{ for which } a \sim b \\ \sum\limits_{a \in \mathcal{F}(m)} m(a) = 1, m(a) \geq 0, \forall a \subseteq \Theta; m(\emptyset) = 0 \end{array} \tag{5.6}$$

where the first, second and third constraints are derived from Equations 5.4 and 5.5, representing the quantitative constraints that correspond to the qualitative preference relations.

Furthermore, the proposed method addresses the problem of inconsistency. In fact, if the preference relations are consistent, then the optimization problem is feasible. Otherwise no solutions will be found. Thus, the expert may be guided to reformulate his preferences.

An extension of the proposed solution is also presented. In fact, they suggested to use the goal programming, a multiobjective method, in order to take into account several objectives simultaneously in the formulation of the problem. So, the idea behind the use of this method is to be able to integrate additional information about the belief functions to be generated.

It should be noted that this method does not address the issue of incomparability in the pairwise comparisons. In fact, this proposed method treats incomparability as incompleteness. However, we believe that this interpretation is not appropriate. If an expert is unable to compare two alternatives then this situation should be reflected in the preference relation not as an incomplete situation, but with an entry for that particular pair of alternatives. Moreover, we suggest to include the weak preference relation, that separates the preference area from the indifference one. A possible interpretation is hesitation between strict preference and indifference. So, in the following Section, we present our method which deals with these problems.

5.5 A new elicitation method using qualitative assessments

As presented above, efficiently representing the expert's preferences is a crucial task in elaborating the necessary data for a considered problem. Therefore, we propose a realistic solution that is able to efficiently imitate the expert's reasoning.

Our first step is then to elaborate on how may incomparability and incompleteness be represented in qualitative belief functions. The solution we suggest is then a qualitative model for constructing belief functions from elicited experts' opinions when dealing with qualitative preference relations based only on strict preference and indifference relations. The second stage is to introduce the weak preference relation, that separates the preference area from the indifference one by inserting an intermediate zone.

5.5.1 Preference articulation: Incompleteness

Incomparability and incompleteness represent very different concepts. In this subsection, we try to differentiate incomplete preferences from incomparable ones. This situation is illustrated by complete ignorance, missing information, lack of knowledge or an ongoing preference elicitation process. Incompleteness represents then simply an absence of knowledge of the relationship between these pairs of alternatives.

Given such considerations, it may perhaps be useful at times to take incomplete order as the primitive of analysis. Besides, the expert is freely allowed to assign this belief to any pairs of alternatives. In other words, a partial order allows some relations between pairs of alternatives to be unknown.

Example 5.1. *Given three alternatives $\Theta = \{a, b, c\}$, an incomplete order can be for example: $(a \succ c, b \sim c)$ or $(c \succ a, a \sim b)$, where some relations between pairs of alternatives are unknown.*

5.5.2 Preference articulation: Incomparability

A missing value in a linguistic preference relation is not always equivalent to a lack of preference of one alternative rather than another. A missing value can also be the result of an expert's inability to compare one alternative to another because they are too different. In such cases, the expert may not put his opinion about certain aspects of the problem; he would not be able to efficiently express his preference towards two or more of the available alternative. As a result, he may find some of them to be incomparable and thus has an incomplete preference ordering, i.e., he neither prefers one alternative over the other nor finds them equally as good. Therefore, it would be of great importance to provide the expert with tools that would him allow to efficiently model his preferences.

In order to model this situation, we first consider how to represent the incomparability relation $(?)$. Our problem here is that incomparability is expressed entirely in terms of negations:

$$a?b \text{ iff } \neg(a \succ b) \wedge \neg(b \succ a) \tag{5.7}$$

By definition, a couple of alternatives (a, b) belongs to the incomparability relation if and only if the expert is unable to compare a and b. Furthermore, it is hard to see what kind of behavior could correspond to Equation 5.7. If neither a nor b is chosen, the expert may not be able to tell which alternative is better, since not $a \succ b$, not $b \succ a$ and not $a \sim b$. In other terms, we apply incomparability when the preference profiles of two alternatives are severely conflicting.

The question now is how to formalize this situation in the belief function framework. In order to build this new preference relation, we may accept that there exist positive reasons which support the relation $\neg(a \succ b)$ and also there exists sufficient negative information to establish the relation $(a \succ b)$. These two assumptions can properly model the contradictory information. Besides, we can surely establish that "a is preferred to b" as there are not sufficient reasons supporting the opposite, and there are sufficient information against it, while we can also surely establish that "b is preferred to a" for the same reasons. Therefore, a and b are in a conflicting position.

On the other hand, and based on the belief function framework and as defined by Boujelben et al. (2011), the incomparability situation appears between two alternatives when their evaluations given by basic belief assessments differ significantly.

Consider two alternatives a and b, as proved in Wong and Lingras (1994); the belief function exists since the preference relation $\succ$ satisfies the following axioms:

1. Asymmetry: $a \succ b \Rightarrow \neg(b \succ a)$
2. Negative Transitivity: $\neg(a \succ b) \wedge \neg(b \succ c) \Rightarrow \neg(a \succ c)$

3. Dominance: $\forall\, a, b \in 2^{\Theta}, a \supseteq b \Rightarrow a \succ b$ or $a \sim b$

4. Partial monotonicity: $\forall\, a, b, c \in 2^{\Theta},$ if $a \supset b$ and $a \cap c = \emptyset \Rightarrow (a \cup c) \succ (b \cup c)$

According to Wong and Wong and Lingras (1994), functions other than the belief ones may exist, which are also compatible with a preference relation such that for every $a, b \in 2^{\Theta}$:

$$a \succ b \Leftrightarrow f(a) \geq f(b) \tag{5.8}$$

if and only if the relation $\succ$ satisfies the previous axioms.

Similarly to this idea, we can prove that the plausibility function also exists since the preference relation $\succ$ satisfies the previous axioms. Besides, we can conclude that there exists a plausibility function pl: $2^{\Theta} \rightarrow [0, 1]$ such as:

$$a \succ b \Leftrightarrow pl(a) \geq pl(b) \tag{5.9}$$

To summarize, we can get the following relations:

$$a \succ b \Leftrightarrow bel(a) \geq bel(b) \;\text{ and }\; pl(a) \geq pl(b) \tag{5.10}$$

As we have defined previously, the incomparability situation appears between two alternatives when their preference profiles are severely conflicting. That is when their evaluations given by basic belief assessments differ significantly. We can then intuitively conclude from Equation 5.10 that, if a is incomparable with b, then:

$$pl(a) \leq pl(b) \wedge bel(a) \geq bel(b) \tag{5.11}$$

The first part of Equation 5.11 supports the assumption "a is preferred to b"; however, the second one supports the opposite claim. Also, the second part of the equation supports the assumption "b is preferred to a" and the first part affirms the opposite assumption.

Consequently, our purpose is then to prove the existence of the previous Equation 5.11 in order to correctly represent the bba relative to the incomparability relation.

Demonstration

$$a?b \Leftrightarrow \neg(pl(a) \geq pl(b)) \wedge \neg(bel(b) \geq bel(a)) \tag{5.12}$$

Then:

$$pl(b) \geq pl(a) \wedge bel(a) \geq bel(b) \tag{5.13}$$

$$\Leftrightarrow pl(a) \leq pl(b) \wedge bel(a) \geq bel(b) \tag{5.14}$$

As a result, we obtain the relation presented in Equation 5.11.

We propose the following proof to justify the previous relation:

$$pl(a) \leq pl(b) \Leftrightarrow bel(\Theta) - bel(\overline{a}) \leq bel(\Theta) - bel(\bar{b})$$

$$\Leftrightarrow -bel(\overline{a}) \leq -bel(\bar{b})$$

$$\Leftrightarrow bel(\overline{a}) \geq bel(\bar{b})$$

However, we replace in Equation 5.14 $pl(a) \leq pl(b)$ by $bel(\overline{a}) \geq bel(\bar{b})$. We obtain:

$$bel(\overline{a}) \geq bel(\bar{b}) \wedge bel(a) \geq bel(b) \tag{5.15}$$

Then:

$$[bel(\overline{a}) > bel(\bar{b}) \vee bel(\overline{a}) = bel(\bar{b})] \wedge [bel(a) > bel(b) \vee bel(a) = bel(b)] \tag{5.16}$$

Moreover:

$$[\overline{a} \succ \bar{b} \vee \overline{a} \sim \bar{b}] \wedge [a \succ b \vee a \sim b] \tag{5.17}$$

Also, we have:

$$a \sim b \Leftrightarrow a \succ b \wedge b \succ a \tag{5.18}$$

As a result:

$$[\overline{a} \succ \bar{b} \vee (\overline{a} \succ \bar{b} \wedge \bar{b} \succ \overline{a})] \wedge [a \succ b \vee (a \succ b \wedge b \succ a)] \tag{5.19}$$

Then, we apply the absorption law: $A \vee A \wedge B = A$

$$\Rightarrow [\overline{a} \succ \bar{b}] \wedge [a \succ b] \tag{5.20}$$

Besides, we have:

$$\neg(a \succ b) \wedge a \succ b \tag{5.21}$$

As a result:

$$\neg(a \succ b) \wedge \neg(b \succ a) \tag{5.22}$$

Finally, we obtain the incomparability relation.

As a conclusion, such representation of incomparability (see Equation 5.10) enables us to correctly express the conflicting information produced by the alternative a and the alternative b. In fact, the first part of the Equation 5.11 "$bel(a) \geq bel(b)$" implies that a is preferred to b. Then, the plausibility function is used since it expresses the maximum amount of specific support that could be given to a proposition a. However, when we define the second part of the Equation 5.11, we propose to assume that $pl(a) \leq pl(b)$ which means that b is preferred to a. This contradicts with the first assumption and can properly express the conflicting information produced by a and b. Using this relation some new focal elements may appear two express the conflicting relation.

Computational procedure

Now and after modeling the incompleteness and the incomparability preferences, we propose to extend Ben Yaghlane et al. method (Ben Yaghlane et al., 2006). We transform these preference relations into constraints. We get:

$$\begin{gathered}
Max_m UM(m) \\
s.t. \\
bel(a) - bel(b) \geq \varepsilon \quad \forall (a,b) \text{ for which } \; a \succ b \\
bel(a) - bel(b) \leq \varepsilon \quad \forall (a,b) \text{ for which } \; a \sim b \\
bel(a) - bel(b) \geq -\varepsilon \quad \forall (a,b) \text{ for which } \; a \sim b \\
bel(a) \geq bel(b) \quad \forall (a,b) \text{ for which } \; a?b \\
pl(a) \leq pl(b) \quad \forall (a,b) \text{ for which } \; a?b \\
\sum_{a \in \mathcal{F}(m)} m(a) = 1; m(a) \geq 0; \forall a \subseteq \Theta; m(\emptyset) = 0
\end{gathered} \tag{5.23}$$

where the first, second and third constraints of the model are derived from the preference and indifference relations. The fourth and fifth constraints correspond to the incomparability relation. ε is a constant speci.fied by the expert before beginning the optimization process.

A crucial step is needed before beginning the task of generating belief functions; it consists in the identification of the candidate focal elements. Thus, as applied in the existing approaches, we may initially assume that prepositions which may appear in the preference relationships are considered as focal elements. Then, other focal elements could appear or be eliminated. The next phase of our procedure consists in establishing the local preference relations between each pair of two alternatives. Finally, these obtained relations are transformed into constraints to obtain the quantitative belief function.

Example 5.2. *Let us consider a problem of eliciting the weight of the candidate criteria. The problem involves six criteria:* $\Theta = \{a, b, c, d, e, f\}$*. The focal elements are:* $F1 = \{a\}$*,* $F2 = \{a, b, c\}$*,* $F3 = \{b, e\}$*,* $F4 = \{e, f\}$ *and* $F5 = \{a, e, d\}$*.*

Next, the expert's opinions should be elicited. For this purpose, an interview with the expert is carried out in order to model his preferences. Consequently, he has validated the following relations:

$$F2 \succ F1,\ F1?F3,\ F4 \succ F1$$
$$F5 \succ F1,\ F3 \sim F2,\ F5 \succ F4,$$

After eliciting the expert's preferences, the following step is to identify the candidate focal elements. So, we get:

$$\mathcal{F}(m) = \{F1, F2, F3, F4, F5, \Theta\}$$

Next, these obtained relations are transformed into optimization problem according to our proposed method. We assume that $\varepsilon = 0.01$ *and the uncertainty measures is* H *(as defined in Chapter 2). The following step is then to transform the obtained relations into constraints. We obtain:*

1. $F2 \succ F1 \Leftrightarrow bel(F2) - bel(F1) \geq 0.01$
2. $F1?F3 \Leftrightarrow pl(F3) \leq pl(F1)$

Table 5.1: The obtained bba using our proposed model

Criteria	$\{a\}$	$\{a,b,c\}$	$\{b,e\}$	$\{e,f\}$	$\{a,e,d\}$	$\{b,c,d,e,f\}$	$\{a,c,d,f\}$	Θ
bba	0.039	0.061	0.09	0.077	0.077	0.006	0.134	0.516
bel	0.039	0.1	0.09	0.077	0.116	0.173	0.173	1
pl	0.827	0.923	0.827	0.9	1	0.961	0.91	1

3. $F1?F3 \Leftrightarrow bel(F3) \geq bel(F1)$

4. $F4 \succ F1 \Leftrightarrow bel(F4) - bel(F1) \geq 0.01$

5. $F5 \succ F1 \Leftrightarrow bel(F5) - bel(F1) \geq 0.01$

6. $F3 \sim F2 \Leftrightarrow bel(F3) - bel(F2) \leq 0.01$

7. $F3 \sim F2 \Leftrightarrow bel(F3) - bel(F2) \geq -0.01$

8. $F5 \succ F4 \Leftrightarrow bel(F5) - bel(F4) \geq 0.01$

Then, we obtain the following optimization problem example:

$$\begin{gathered}
Max_m H(m) = m(F1) * log_2(1/m(F1)) + m(F2) * log_2(3/m(F2)) \\
+m(F3) * log_2(2/m(F3)) + m(F4) * log_2(2/m(F4)) \\
+m(F5) * log_2(3/m(F5)) + m(\Theta) * log_2(6/m(\Theta)); \\
s.t. \\
bel(F2) - bel(F1) \geq 0.01 \\
pl(F3) \leq pl(F1) \\
bel(F3) \geq bel(F1) \\
bel(F4) - bel(F1) \geq 0.01 \\
bel(F5) - bel(F1) \geq 0.01 \\
bel(F3) - bel(F2) \leq 0.01 \\
bel(F3) - bel(F2) \geq -0.01 \\
bel(F5) - bel(F4) \geq 0.01 \\
\sum_{Fi \in \mathcal{F}(m)} m(Fi) = 1, m(Fi) \geq 0, \forall Fi \subseteq \Theta; m(\emptyset) = 0,
\end{gathered} \tag{5.24}$$

After that, we obtain the following results presented in Table 5.1.

Table 5.2: The obtained bba using Ben Yaghlane et al. method

Criteria	$\{a\}$	$\{a,b,c\}$	$\{b,e\}$	$\{e,f\}$	$\{a,e,d\}$	Θ
bba	0.063	0.096	0.149	0.126	0.126	0.314
bel	0.063	0.159	0.149	0.126	0.189	1

Table 5.1 gives the results of all ordered couples on the basis of their preference relation. Besides, we are interested in obtaining their corresponding quantitative bba.

Now, we propose to apply Ben Yaghlane et al. method. By using this model, we assume that the incomparability and the incompleteness are modeled in the same way. In other words, the relation $F1?F3$ will be eliminated, and we get the following Table 5.2.

In absence of incomparabilities, we note the couple of alternatives $\{b,c,d,e,f\}$ and $\{a,c,d,f\}$ do not appear because the incomparability relation has been assigned to other relation: the incompleteness. Observing the two obtaining results, it is possible to see that in spite of the use of two different models, we get almost the same partial order.

5.5.3 Preference articulation: Weak preference relation

Let a and b be two alternatives. Besides, crisp binary relations are based on two basic relations strict preference $\succ$ and indifference $\sim$ (Roubens & Vincke, 1985) as defined previously.

However, in this case, we want to answer to the question "Is the alternative a at least as good as the alternative b?". We can have then the following answers:

- Either yes or no. The decision maker answers to the previous question by "yes" or "no".
- I don't know: The decision maker can also express his ignorance.
- Answers including the intensity of preference: for example, "a is strongly - weakly, moderately preferred to b".

For these reasons, we will assume that the comparison of a and b gives a choice between two other possible cases:

- a is weakly preferred to b ($(a, b) \in Q$) iff ($a \succeq b$), means that the decision maker thinks that a is at least as good as b;
- the relation between a and b is unknown;

From this relation $\succeq$, we can derive two other important relations on A:

1. Strict preference relation, $\succ$, defined by:

$$a \succ b \Leftrightarrow a \succeq b \wedge \text{ not } (b \succeq a)$$

2. Strict Indifference relation, $\sim$, defined by:

$$a \sim b \Leftrightarrow a \succeq b \wedge b \succeq a$$

Under the previous approach (Ben Yaghlane et al., 2006), in general, when comparing two alternatives a and b, the expert uses two binary relations, the preference and indifference relations, no matter how large the difference is.

In real-life problems, however, a small positive difference of scores is not always a justification for a preference. A classical attitude is to assess discrimination thresholds to distinguish between significant and insignificant differences of scores. Therefore, the indifference threshold ε was introduced (Ben Yaghlane et al., 2006). If the performances of two alternatives differ by less than ε, there is an indifference relation and not a preference one.

However, this model presents some drawbacks (Fodor & Roubens, 1993). Suppose two alternatives a and b are such that:

$$bel(a) - bel(b) = \varepsilon - \frac{\mu}{2} \tag{5.25}$$

where μ is a positive quantity very small compared to ε.

If a slightly superior score (μ) was attached to a, we would obtain:

$$bel(a) - bel(b) = \varepsilon + \frac{\mu}{2} \tag{5.26}$$

transforming the previous indifference ($a \sim b$) into strict preference ($a \succ b$).

We may overcome these difficulties by separating the preference and the indifference relations by inserting an intermediate zone called weak preference relation (Perny & Roy, 1992). A possible interpretation is an hesitation between strict preference and indifference.

Formally, one may consider a strict preference threshold γ to distinguish between strict preference and weak preference. This strict preference threshold is a value such as if the performances of a and b differ by at least γ, then we are in a situation when one alternative is strongly preferred to the other. This is illustrated as follows:

$$a \succ b \Leftrightarrow bel(a) - bel(b) \geq \gamma \tag{5.27}$$

$$a \succeq b \Leftrightarrow 0 \leq bel(a) - bel(b) \leq \gamma \tag{5.28}$$

However, when comparing two alternatives, we might want to use both the indifference and the strict preference thresholds, where $\gamma \geq \varepsilon$:

$$a \succ b \Leftrightarrow bel(a) - bel(b) \geq \gamma \tag{5.29}$$

$$a \succeq b \Leftrightarrow \varepsilon \leq bel(a) - bel(b) \leq \gamma \tag{5.30}$$

$$a \sim b \Leftrightarrow |bel(a) - bel(b)| \leq \varepsilon \tag{5.31}$$

Nevertheless, there exist different ways for choosing the preference and indifference threshold. For instance, Perny and Roy (1992) believe that the fixing of thresholds involves not only the estimation of error in a physical sense, but also a significant subjective input by the decision-maker himself. They assume that these two thresholds can be constant values or can take the linear form. Besides, in other works (Perny & Roy, 1992), γ and ε are derived from mathematical equations.

In this work, we assume that the thresholds γ and ε can be constant values. We interpret the indifference threshold as the minimum margin of uncertainty associated with a given alternative and the preference threshold as the maximum margin of error associated with the alternative in question.

Computational Procedure

Now, we transform these preference relations into constraints as presented in previous Section. We get:

$$
\begin{aligned}
& Max_m UM(m) \\
& s.t. \\
& bel(a) - bel(b) \geq \gamma \quad \forall (a,b) \text{ for which } a \succ b \\
& bel(a) - bel(b) \leq \gamma \quad \forall (a,b) \text{ for which } a \succeq b \\
& bel(a) - bel(b) \geq \varepsilon \quad \forall (a,b) \text{ for which } a \succeq b \\
& bel(a) - bel(b) \leq \varepsilon \quad \forall (a,b) \text{ for which } a \sim b \\
& bel(a) - bel(b) \geq -\varepsilon \quad \forall (a,b) \text{ for which } a \sim b \\
& \sum_{a \in \mathcal{F}(m)} m(a) = 1; m(a) \geq 0; \forall a \subseteq \Theta; m(\emptyset) = 0
\end{aligned}
\tag{5.32}
$$

where the first constraint of the model is derived from the preference relation. The second and third constraints model the weak preference relation. The fourth and fifth constraints correspond to the indifference relation.

ε and γ are constants specified by the expert before beginning the optimization process.

The choice of thresholds intimately affects whether a particular binary relationship holds. While the choice of appropriate thresholds is not easy, in most realistic decision making situations there are good reasons for choosing non-zero values for ε and γ.

Figure 5.1 summarizes the obtained transformation. These thresholds define five different intervals in the domain of preference of two alternatives.

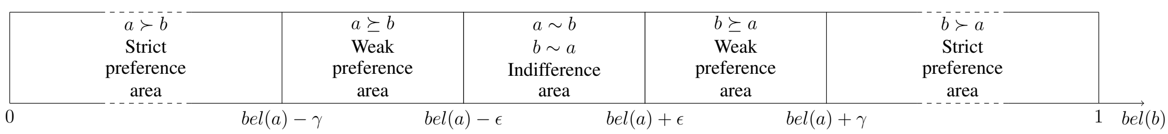

Figure 5.1: Belief relations built from thresholds and crisp scores.

Example 5.3. *Let us consider the problem of eliciting the weight of the candidate alternatives. The problem involves five alternatives:*

$$\Theta = \{a, b, c, d, e\}.$$

The focal elements are:

$$F1 = \{a\},\ F2 = \{a, b, c\},\ F3 = \{b, d\},\ F4 = \{e\},\ F5 = \{a, e\}.$$

Next, the expert's opinions should be elicited. For this purpose, an interview with the expert is carried out in order to model his preferences. Consequently, he has validated the following relations:

$$F2 \succ F1,\ F1 \succeq F3,\ F4 \sim F1,\ F5 \succ F1,\ F5 \succ F4.$$

After eliciting his preferences, the next step is to transform the obtained relations into optimization problem according to our proposed method.

We assume that $\varepsilon = 0.01$, $\gamma = 0.02$ *and the uncertainty measures is* H.

The following step is then to transform the obtained relations into constraints. We obtain:

1. $F2 \succ F1 \Leftrightarrow bel(F2) - bel(F1) \geq 0.02$
2. $F1 \succeq F3 \Leftrightarrow bel(F1) - bel(F3) \leq 0.02$
3. $F1 \succeq F3 \Leftrightarrow bel(F1) - bel(F3) \geq 0.01$
4. $F4 \sim F1 \Leftrightarrow bel(F4) - bel(F1) \leq 0.01$
5. $F4 \sim F1 \Leftrightarrow bel(F4) - bel(F1) \geq -0.01$
6. $F5 \succ F1 \Leftrightarrow bel(F5) - bel(F1) \geq 0.02$
7. $F5 \succ F4 \Leftrightarrow bel(F5) - bel(F4) \geq 0.02$

Table 5.3: The obtained bba using the presented model

Criteria	$\{a\}$	$\{a,b,c\}$	$\{e\}$	$\{b,d\}$	$\{a,e\}$	Θ
m	0.092	0.203	0.082	0.082	0.203	0.338
bel	0.092	0.295	0.082	0.082	0.377	1

Then, we obtain the following optimization problem example:

$$\begin{aligned}
Max_m H(m) = Max(m(F1) * log_2(1/m(F1)) + m(F2) * log_2(3/m(F2)) \\
+m(F3) * log_2(2/m(F3)) + m(F4) * log_2(1/m(F4)) \\
+m(F5) * log_2(2/m(F5)) + m(\Theta) * log_2(5/m(\Theta))); \\
s.t. \\
bel(F2) - bel(F1) \geq 0.02 \\
bel(F1) - bel(F3) \leq 0.02 \\
bel(F1) - bel(F3) \geq 0.01 \\
bel(F4) - bel(F1) \leq 0.01 \\
bel(F4) - bel(F1) \geq -0.01 \\
bel(F5) - bel(F1) \geq 0.02 \\
bel(F5) - bel(F4) \geq 0.02 \\
\sum_{Fi \in \mathcal{F}(m)} m(Fi) = 1, m(Fi) \geq 0, \forall Fi \subseteq \Theta; m(\emptyset) = 0,
\end{aligned} \tag{5.33}$$

Finally, the obtained results are represented in Table 5.3.

Table 5.3 gives the results of all ordered couples on the basis of their preference relation. Besides, we show that a new subset Θ which express the part of ignorance, is introduced.

Indeed, using our model the expert expresses his assessments freely. By applying our presented solution, it is easy to see that our method aggregates all the elicited data.

Here, in the present example, the expert expressed his assessments only in some pairs of alternatives. Thus, a quantitative information is constructed from these incomplete and even uncertain preference relations.

We are then able to represent all the expert knowledge and to transform this information into quantitative data. We have obtained encouraging results since we have the same ranking of alternatives as expressed by the expert.

5.5.4 A global model in the presence of different relations

After modeling the different preference relation, we propose to introduce a global model. This method is concerned with preference models including: strict preference, weak preference, indifference, incompleteness and incomparability:

$$\begin{array}{c} Max_m UM(m) \\ s.t. \\ bel(a) - bel(b) \geq \gamma \quad \forall (a,b) \text{ for which } a \succ b \\ bel(a) - bel(b) \leq \gamma \quad \forall (a,b) \text{ for which } a \succeq b \\ bel(a) - bel(b) \geq \varepsilon \quad \forall (a,b) \text{ for which } a \succeq b \\ bel(a) - bel(b) \leq \varepsilon \quad \forall (a,b) \text{ for which } a \sim b \\ bel(a) - bel(b) \geq -\varepsilon \quad \forall (a,b) \text{ for which } a \sim b \\ bel(a) \geq bel(b) \quad \forall (a,b) \text{ for which } a?b \\ pl(a) \leq pl(b) \quad \forall (a,b) \text{ for which } a?b \\ \sum\limits_{a \in \mathcal{F}(m)} m(a) = 1; m(a) \geq 0; \forall a \subseteq \Theta; m(\emptyset) = 0 \end{array} \tag{5.34}$$

5.6 Conclusion

In this Chapter, a new model for constructing belief functions, that takes into account different levels of uncertainties, from elicited expert opinions has been defined. The originality of our model is then to provide additional interpretation values to the existing methods.

In the following Chapter, we will focus on applying our proposed model in the multi-criteria decision making field, which can be interesting in eliciting experts' judgments.

Chapter 6

AHP method based on belief preference relations

6.1 Introduction

Within the framework of AHP problems, a decision maker often needs to express his assessments using crisp number. However, an expert may be uncertain about his level of preference due to incomplete information or knowledge, inherent complexity and uncertainty within the decision environment. Thus, in such cases, the decision maker cannot estimate his assessments with a numerical value.

To overcome these difficulties, several AHP extensions were developed. However, these approaches deal only with numerical numbers to translate the expert preferences into quantitative information.

To solve the problems presented above, and to facilitate the pair-wise comparison process, a new MCDM method under uncertainty that eliminates some of the drawbacks of the existing prioritization methods, is proposed. A natural way to cope with uncertain judgments is to express the comparison ratios as a belief function, which incorporates the imperfection of the human thinking. Indeed, preferential assessments are used in order to express the decision maker's subjective assessments instead of using numerical values. Within our method, the expert does not require to complete all the comparison matrices; he can then derive priorities from incomplete set of judgments. Therefore, a new procedure is employed to derive crisp priorities from qualitative judgments corresponding to each level.

In this Chapter, we give an insight into our proposed methods. We first give some motivations to develop the qualitative AHP method for handling uncertainty. Then, we detail the proposed methods and we give some examples to illustrate them.

6.2 On the extraction of scores from qualitative pair-wise comparison matrices

Pair-wise comparisons aims at quantifying relative priorities for a given set of alternatives as well as the set of criteria, on a ratio scale, based on the judgment of the decision maker. Using this approach, the decision maker has to express his opinion about the value of one single pair-wise comparison at a time. Usually, he has to choose his answer among discrete choices. Each choice is a linguistic phrase. Some examples of such linguistic phrases are: "A is more important than B", or "A is of the same importance as B", or "A is a little more important than B", and so on. Then, the decision maker uses the Saaty scale (see Table 1.4) to map the labels which indicate the decision maker view to a numeric value.

However, as shown in Chapter 4, this scale was criticized. Moreover, using a quantitative scale, the expert cannot be consistent due to its inability to adequately handle the uncertainty and imprecision associated with the mapping of the decision maker's perception. For instance, he cannot estimate his degree of belief with a numerical value. Therefore, a new elicitation procedure is introduced. In the proposed methodology, the expert is allowed to use preference relations only. Thus to express his judgments, the decision maker has to express his opinions qualitatively, based on knowledge and experience that he provides in response to a given question rather than direct quantitative information. He only selects the related linguistic variable using preference modeling.

For instance, to determine the criteria weights, the preference relation matrix is obtained (see Table 6.1).

In this table, P_{ij} is a preference relations. It may be:

1. a strict preference relation $\succ$ iff $(c_i \succ c_j) \wedge \neg(c_j \succ c_i)$

Table 6.1: Preference relation matrix

	c_1	c_2	$\dots$	c_m
c_1	-	P_{12}	$\dots$	P_{1m}
c_2	-	-	$\dots$	P_{2m}
$\dots$	-	-	-	$\dots$
c_m	-	-	-	-

2. an indifference relation $\sim$ iff $(c_i \succ c_j) \wedge (c_j \succ c_i)$
3. a weak preference relation $\succeq$ ($(a, b) \in Q$) iff $(a \succeq b)$,
4. an unknown relation.

To complete the pair-wise matrix, the expert does not need to quantify his assessments and to fill all the pair-wise comparisons matrix. He is able to express his preferences freely. The expert has to complete only the matrix without quantifying the diagonal and the reciprocal matrix.

Example 6.1. *To describe this approach, we consider the same problem of purchasing a car (Example 3.1). As we have already presented, this problem involves four criteria: $\Omega = \{$Price (c_1), Style (c_2), Fuel (c_3), Reliability $(c_4)\}$ and three selected alternatives: $\Theta = \{$Peugeot (p), Renault (r), Ford $(f)\}$. The expert has identified the following subsets of criteria $\{c_1\}$, $\{c_4\}$ and $\{c_2, c_3\}$.*

Along with our qualitative pair-wise comparison process, the preference modeling defined in Table 6.2 was obtained.

Table 6.2: Preference relation matrix for criterion level

Criteria	$\{c_1\}$	$\{c_4\}$	$\{c_2, c_3\}$
$\{c_1\}$	-	$\succ$	$\succ$
$\{c_4\}$	-	-	$\succ$
$\{c_2, c_3\}$	-	-	-

From this table, we conclude that the expert may say that $\{c_1\}$ is evaluated to be more important than $\{c_4\}$ and $\{c_1\}$ is evaluated to be more preferred than $\{c_2, c_3\}$.

Once the pair-wise comparison matrix is complete, our objective is then to obtain the priority of each subset of element. In fact, within our model, we propose to transform these preference relations into numerical values using the belief function theory. Besides, we try to closely imitate the expert's reasoning without adding any additional information. Therefore, we suggest to apply Ennaceur et al. (2013b, 2013c) model that converts the preference relations into constraints of an optimization problem whose resolution, according to some uncertainty measures, allows the generation of the least informative or the most uncertain belief functions. It can then be determined by the resolution of an optimization problem.

As already presented in Chapter 5, each preference relation is transformed into constraints of an optimization problem. For instance, if we use the preference relation matrix relative to the criterion level we get:

$$\begin{gathered}
Max_m H(m) = m(\{c_1\}) * log_2(|c_1|/m(\{c_1\})) + m(\{c_2\}) log_2(|c_2|/m(\{c_2\})) \\
+ ... + m(\{c_m\}) * log_2(|c_m|/m(\{c_m\})) + m(\Omega) * log_2(|\Omega|/m(\Omega)); \\
s.t. \\
bel(\{c_1\}) - bel(\{c_2\}) \geq \gamma \quad \forall (c_1, c_2) \text{ for which } c_1 \succ c_2 \\
bel(\{c_1\}) - bel(\{c_2\}) \leq \gamma \quad \forall (c_1, c_2) \text{ for which } c_1 \succeq c_2 \\
bel(\{c_1\}) - bel(\{c_2\}) \geq \varepsilon \quad \forall (c_1, c_2) \text{ for which } c_1 \succeq c_2 \\
bel(\{c_1\}) - bel(\{c_2\}) \leq \varepsilon \quad \forall (c_1, c_2) \text{ for which } c_1 \sim c_2 \\
bel(\{c_1\}) - bel(\{c_2\}) \geq -\varepsilon \quad \forall (c_1, c_2) \text{ for which } c_1 \sim c_2 \\
... \\
\sum_{c_i \in \mathcal{F}(m)} m(c_i) = 1, m(A) \geq 0, \forall A \subseteq \Omega; m(\emptyset) = 0.
\end{gathered} \tag{6.1}$$

H, a measure of uncertainty, is used since it takes into account the non-specificity and quantifies the conflict presented in the body of evidence (measure of total uncertainty).

After solving this optimization problem, each element is described by a basic belief assignment (bba) defined on the possible responses.

Furthermore, the proposed method addresses the problem of inconsistency. In fact, if the preference relations are consistent, then the optimization problem is feasible. Otherwise, no solution will be found.

6.3 AHP method with belief preference relations

To handle uncertainty in the AHP method, we have first developed belief AHP method (Ennaceur et al., 2011), that deals with groups of criteria and alternatives and uses Saaty scale to model expert judgments. Then, we have studied the limit of this $1-9$ scale and we have shown that using Saaty scale is not the most appropriate solution in some cases. This is because this scale cannot perfectly model the expert assessments in an uncertain environment. Therefore, we have proposed a new AHP extension based on new set of choices, Yes-No/AHP approach.

On the other hand, we have shown in (Ennaceur et al., 2012a, 2013a, 2014b) that the expert has some difficulties in expressing his preferences using quantitative information. To solve this issue, we have proposed new elicitation technique based on preference relations. In these works, we have shown that qualitative scale can perfectly model expert preferences.

Taking all these facts into consideration, we try to develop an overall MCDM method that identifies groups of criteria and alternatives and compares then using belief preference relations.

Let us remind that in the third Chapter of this dissertation, we have presented two AHP extensions; namely belief AHP and conditional belief AHP. So to check the validity of our new elicitation technique, we will use our presented belief pair-wise comparisons with the aggregation procedure adopted by these previously proposed methods. This leads us to develop two AHP versions which we called qualitative AHP and conditional qualitative AHP. Details of these algorithms are given in the following Sections.

6.3.1 Qualitative AHP method

This Section is dedicated to the presentation of our new MCDM, called qualitative AHP method (Ennaceur et al., 2013b, 2013c). Indeed, our model has the same features as belief and Yes-No/AHP methods; however, it introduces a new pair-wise comparison technique. Nevertheless, we suggest to extend the belief AHP method under qualitative scale. A new aggregation procedure is then presented, as shown in Figure 6.1.

To present the qualitative AHP method, we introduce its different construction steps, described as follows.

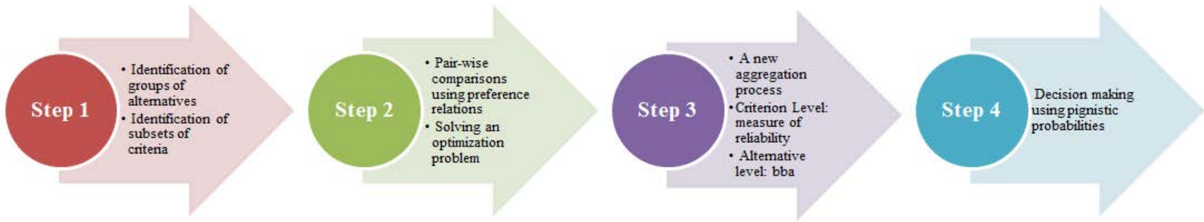

Figure 6.1: The general decision-making paradigm based on qualitative AHP

6.3.2 Identification of the candidate criteria and alternatives

As previously stated, our set of criteria is given by $\Omega = \{c_1, \ldots, c_n\}$ and we assume that there is a set of alternatives $\Theta = \{a_1, \ldots, a_m\}$ consisting of m elements. As explained previously in Chapter 3, our method suggests to allow the expert to express his opinions on groups of criteria and subsets of alternatives instead of single ones. So, he chooses these groups by assuming that criteria/alternatives having the same degree of preference are grouped together.

6.3.3 Computing the weight of considered criteria

After identifying the subsets of criteria, the decision maker has to express his preferences in order to complete the qualitative pair-wise comparison matrix as shown in Section 6.2.

Then, the obtained bba m^Ω is transformed into pignistic probabilities, denoted by $BetP^\Omega$ using the pignistic transformation (Smets, 1998):

$$BetP^\Omega(c_i) = \sum_{c_j \subseteq \Theta} \frac{|c_i \cap c_j|}{|c_j|} \frac{m(c_j)}{(1 - m(\emptyset))}, \quad \forall c_i, c_i \subset \Omega \tag{6.2}$$

Example 6.2. *Let us continue with the Example 6.1. After deriving the weights of criteria from the optimization problem (see Table 6.3), the obtained bba is transformed into pignistic probabilities.*

Table 6.3: The weights assigned to the subset of criteria

Subsets of criteria	$\{c_1\}$	$\{c_4\}$	$\{c_2, c_3\}$	Ω
m^Ω	0.228	0.218	0.208	0.346
Criteria	$\{c_1\}$	$\{c_2\}$	$\{c_3\}$	$\{c_4\}$
$BetP^\Omega$	0.3145	0.3045	0.1905	0.1905

6.3.4 Computing and updating the alternatives priorities

Following the same reasoning, qualitative pair-wise comparison matrices are constructed to evaluate each subset of alternatives regarding each criterion c_i. The obtained bba is denoted by m^{c_i}.

As shown in Chapter 3, we apply the belief AHP aggregation procedure. Firstly, we propose to regard each pignistic probabilities of a specific subset of criteria as a measure of reliability.

If we have C_i as a subset of criteria, then we get β_i its corresponding measure of reliability:

$$\beta_i = \frac{BetP^{\Omega}(c_i)}{max_k BetP^{\Omega}(c_k)} \tag{6.3}$$

Secondly, as the reliability factor represents a single criterion c_k, the corresponding bba will be directly discounted:

$$m^{\alpha_k}_{c_k}(A_j) = \beta_k.m_{c_k}(A_j),\ \forall A_j \subset \Theta \tag{6.4}$$

$$m^{\alpha_k}_{c_k}(\Theta) = (1 - \beta_k) + \beta_k.m_{c_k}(\Theta) \tag{6.5}$$

where $m_{c_k}(A_j)$ the relative bba for the subset A_j (obtained in the previous step), β_k its corresponding measure of reliability and, we denote $\alpha_k = 1 - \beta_k$.

Example 6.3. *As shown previously, after modeling the qualitative pair-wise comparison matrix for each sets of alternatives regarding each criterion, we obtain Table 6.4.*

Table 6.4: Priorities values

c_1	$m^{c_1}(.)$	c_2	$m^{c_2}(.)$	c_3	$m^{c_3}(.)$	c_4	$m^{c_4}(.)$
$\{p\}$	0.505	$\{p\}$	0.319	$\{r\}$	0.505	$\{f\}$	0.505
$\{p,r,f\}$	0.495	$\{r,f\}$	0.535	$\{p,r,f\}$	0.495	$\{p,r,f\}$	0.495
		$\{p,r,f\}$	0.146				

The next stage is to compute the measure of reliability. Table 6.5 presents the corresponding measure.

Table 6.5: The measure of reliability assigned to the subset of criteria

Criteria	$\{c_1\}$	$\{c_2\}$	$\{c_3\}$	$\{c_4\}$
β_k	1	0.96	0.6	0.6

After computing the belief functions for each set of alternatives with respect to each criterion, we need to combine the weight of criteria and the alternatives priorities. Table 6.6 defines the discounted bbas.

Table 6.6: The discounted bbas

c_1	$m_{c_1}^{\alpha_1}(.)$	c_2	$m_{c_2}^{\alpha_2}(.)$	c_3	$m_{c_3}^{\alpha_3}(.)$	c_4	$m_{c_4}^{\alpha_k}(.)$
$\{p\}$	0.505	$\{p\}$	0.1914	$\{r\}$	0.303	$\{f\}$	0.4848
$\{p, r, f\}$	0.495	$\{r, f\}$	0.321	$\{p, r, f\}$	0.697	$\{p, r, f\}$	0.5152
		$\{p, r, f\}$	0.4876				

6.3.5 Decision making

To this end and after updating the relative bba, a decision under uncertainty must be defined. In the sequel, the pignistic probabilities are used. Therefore, our obtained beliefs must be combined. To do so, we propose to apply the conjunctive rule:

$$m_{final} = \textcircled{\cap} m_{c_k}^{\alpha_k}, \quad k = \{1, \ldots, n\} \tag{6.6}$$

Now, we can compute the pignistic probabilities to choose the best alternatives.

Example 6.4. *Let us continue with the previous example and remind that our objective is to rank alternative in an uncertain environment. Hence, our first stage is to combine the overall bbas. The obtained result is illustrated in Table 6.7.*

Table 6.7: The overall bba

	$\emptyset$	$\{p\}$	$\{r\}$	$\{r, f\}$	Θ
m_{final}	0.4072	0.3866	0.06248	0.0570	0.0867

Finally, the ranking of alternative is obtained according to the pignistic transformation (see Table 6.8).

Table 6.8: The final result using the Yes-No/AHP approach

Alternatives	p	r	f
BetP	0.70088	0.20226	0.096861

6.4 Introducing dependency under qualitative AHP method: Conditional qualitative AHP

As we have presented previously in Chapter 3, we need to model the relationship between alternatives and criteria. Thus, in this Section, our main objective is to handle dependency between the two levels under the qualitative AHP methodology. Besides, we propose the latter by using preference relation in order to translate the expert's assessments. Figure 6.2 summarizes the computational procedure.

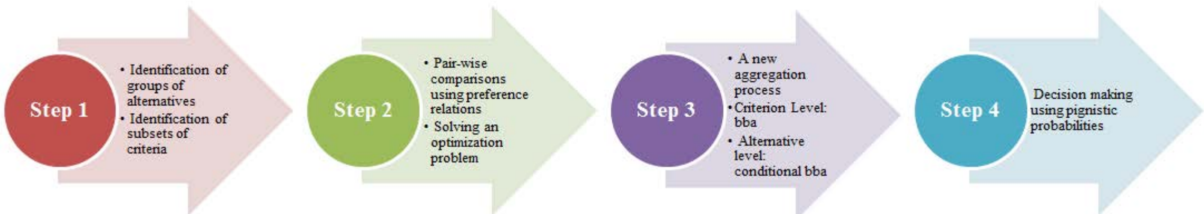

Figure 6.2: The general decision-making paradigm based on conditional qualitative AHP

From Figure 6.2, the major difference between conditional qualitative AHP and other AHP extensions is in Step 2. So, our solution proposes to use preferential judgments instead of Saaty scale. Therefore, the following stages must be respected (Ennaceur et al., 2014c):

1. Identification of the sets of criteria and groups of alternatives.
2. The criterion level is described by the obtained bba using preference relations.
3. At the alternative level, each pair-wise comparison matrix is constructed using preference relations and the obtained bba is described by conditional beliefs.
4. Standardizing the frame of discernment. At the criterion level, the vacuous extension is used. At the alternative level, we apply the ballooning extension technique.
5. The conjunctive rule is defined to combine the overall bbas.
6. Decision making using pignistic transformation.

6.5 Experimental analysis

To test the validity of our developed methods, our experiments are performed on the same random data sets presented in Chapter 3.

6.5.1 Simulation algorithm

Despite the possible differences between the obtained results through three MCDM methods, we cannot conclude the superiority of one over another. Unless we have a solid basis, we compare the ranking results against the closeness of the rankings of each method to the actual rankings. To do so, we need to compare each set of rankings provided by AHP and the proposed methods with a ranking that has been already produced by an alternative, yet reliable ranking method. This alternative ranking will be considered as a basis, or actual ranking of the alternatives and will be used to measure the closeness of the rankings provided by AHP, qualitative AHP and conditional qualitative AHP to reality.

In order to overcome these issues, in the next Section, we test a simulation algorithm that compares the ranking results of the three methods under different scenarios.

To generate reliable data for a numerical analysis, we have applied the following algorithm:

1. We generate a random matrix for the decision performance as well as a random matrix to represent the weight of each decision criteria. Based on these two matrices, the overall scores and ranks of the decision alternatives are calculated. These are usual steps in the WSM method.
2. From the performance matrix, we generated pair-wise comparison matrices of different alternatives that are compared to each criterion.
3. Each pair-wise comparison matrix is transformed into qualitative assessment using preference relations. The qualitative matrix of the actual pair-wise comparisons would be obtained as follows:
    - "Equal importance" is equivalent to indifference relation.
    - "Somewhat more important" and "Much more important" are equivalent to weak preference relation.

- "Very much more important" and "Absolutely more important" are equivalent to strong preference relation.

4. We apply the suggested method to compute the overall priorities and to rank alternatives.
5. We compare the obtained result with the ranking of the WSM method.

Let's demonstrate the evaluation procedure using the same example 3.10 introduced in Chapeter 3. We consider 3 alternatives a_1, a_2 and a_3 and three criteria c_1, c_2 and c_3. The decision making problem is described using the matrix presented in Table 6.9.

Table 6.9: Decision matrix

	Criteria		
	c_1	c_2	c_3
	$\frac{8}{13}$	$\frac{2}{13}$	$\frac{3}{13}$
a_1	1	9	9
a_2	5	2	2
a_3	1	5	9

This example has been solved using the WSM and AHP in (Triantaphyllou & Mann, 1989). While applying the WSM, it can be shown that the alternative a_1 is the best one. However, AHP turns out that the alternative a_2 is the best. Obviously, this is in contradiction with the conclusion derived using the WSM.

Now, let us model this example using qualitative AHP. As a result, the pair-wise comparison matrix of the three alternatives in terms of each criterion is illustrated in Table 6.10.

Next, an optimization model is used to transform preference relation into constraints and to generate quantitative information from these qualitative assessments. Finally, we obtain the

Table 6.10: The preference relations matrices

c_1			
	a_1	a_2	a_3
a_1	-	$\prec$	$\sim$
a_2	-	-	$\succ$
a_3	-	-	-

c_2			
	a_1	a_2	a_3
a_1	-	$\succeq$	$\sim$
a_2	-	-	$\sim$
a_3	-	-	-

c_3			
	a_1	a_2	a_3
a_1	-	$\succeq$	$\sim$
a_2	-	-	$\sim$
a_3	-	-	-

following order: $a_1 \succ a_3 \succ a_2$. Obviously, this is in contradiction with the results derived when the AHP method was applied at the beginning of this illustrative example. However, we have obtained encouraging results since it can be observed that the ranking order of the alternatives as derived by the WSM and the qualitative AHP is the same.

In order to gain a deeper understanding, a computational study was undertaken. The data were random numbers from the interval $[1, 9]$. In these test problems, the number of alternatives was equal to the following values: $3, 4, 5, 6, 7, 8, 9$ and 10. Similarly, the number of criteria was equal to $3, 4, 5, 6, 7, 8, 9$ and 10. Test problems were treated as the previous illustrative example. Any ranking irregularity was recorded.

6.5.2 Simulation results

Let us reiterate the immediate goal of this Section: We wish to check the accuracy of our proposed models by comparing them to WSM method.

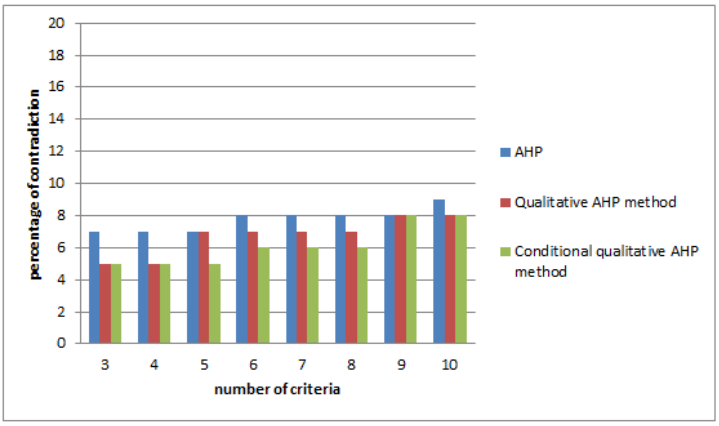

Figure 6.3: Percentage of contradiction (%) based on 3 alternatives

From Figures 6.3 and 6.4, our qualitative AHP achieves a satisfactory percentage of contradiction when it is compared regarding a few numbers of alternatives and criteria. Indeed, we have to point out that our two methods, qualitative AHP and conditional qualitative AHP, give the lowest percentage of contradiction in almost the cases in comparison to the other presented methods (see Chapters 3 and 4). This can be explained by the appropriate use of the preference relations to model expert assessments.

The qualitative AHP methods perform much better than standard AHP on the whole. For in-

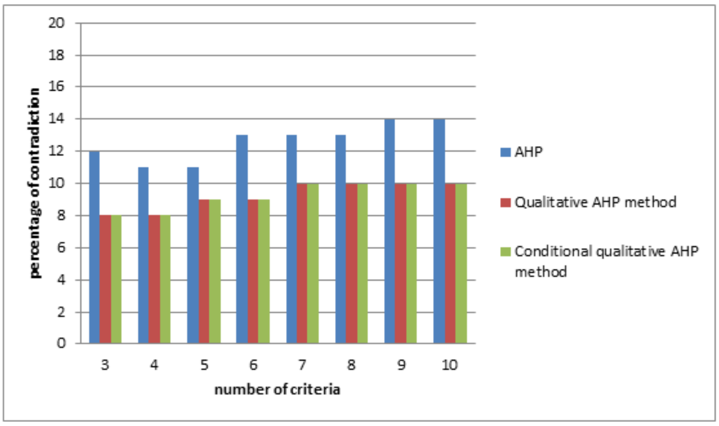

Figure 6.4: Percentage of contradiction (%) based on 10 alternatives

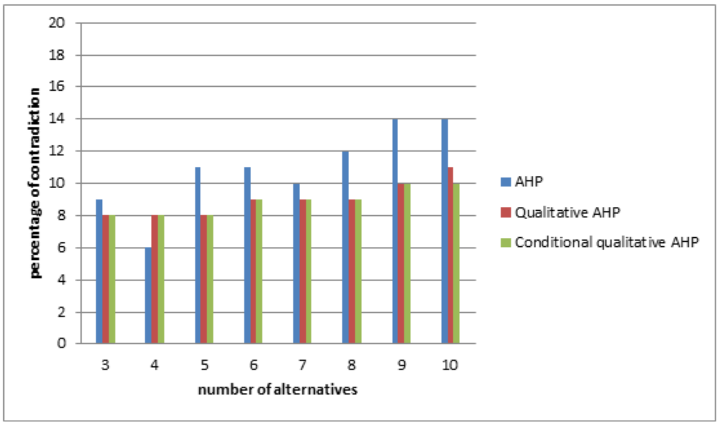

Figure 6.5: Percentage of contradiction (%) based on 10 criteria

stance, applying qualitative AHP, conditional qualitative AHP and standard AHP to 3 alternatives and 4 criteria the percentage of contradiction is set 5%, 5%, and 7%, respectively.

From the experimental results in Figure 6.5, we obtain a similar observations as before. This experiment further validates the satisfactory results obtained by our method in most cases in terms of percentage of contradiction. Although, the number of alternative and/or criteria increases, the qualitative AHP methods give the best results. More results can be found in Appendix A.

In summary, these results show that qualitative AHP methods display good performance. This is explained by the fact that our models use a more convenient elicitation technique to model the preference assessments.

Yet, it is important to mention that the main conclusion from the works presented in Ennaceur et al. (2011, 2012b, 2014a), detailed in Chapter 3 and Chapter 4, and Ennaceur et al. (2013b, 2013c, 2014c) are, also, described in this Chapter, is the superiority of qualitative AHP approaches comparing to the studied methods. This is because the method is able to model expert preferences adequately using only belief preference relations. Consequently, in the following Section, we will check the consistency of the presented methodologies.

6.5.3 Catering selection problem

Figures 6.6 and 6.7 show a sensitivity analysis between the qualitative AHP method and conditional qualitative AHP version. Focusing on the consistency of the methods, Table 6.11 shows the obtained ranking.

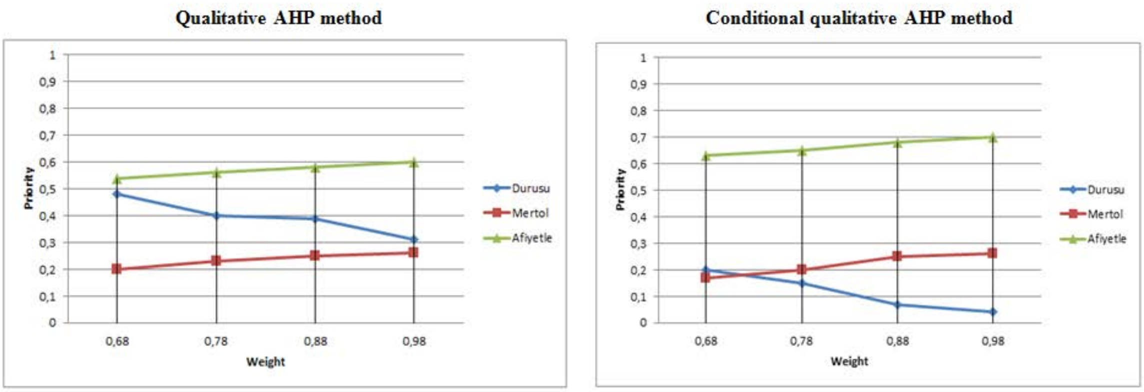

Figure 6.6: Sensitivity analysis of Hygiene criterion

Table 6.11: Decision matrix

Alt.	**Ranking**	
	Qualitative AHP	Condtional qualitative AHP
Durusu	0.34	0.23
Mertol	0.29	0.18
Afiyetle	0.37	0.59

Firstly, consider the Hygiene criterion. By increasing the share of this latter, it has been noticed that the model is still in favor of Afiyetle with a score followed by Durusu and lastly

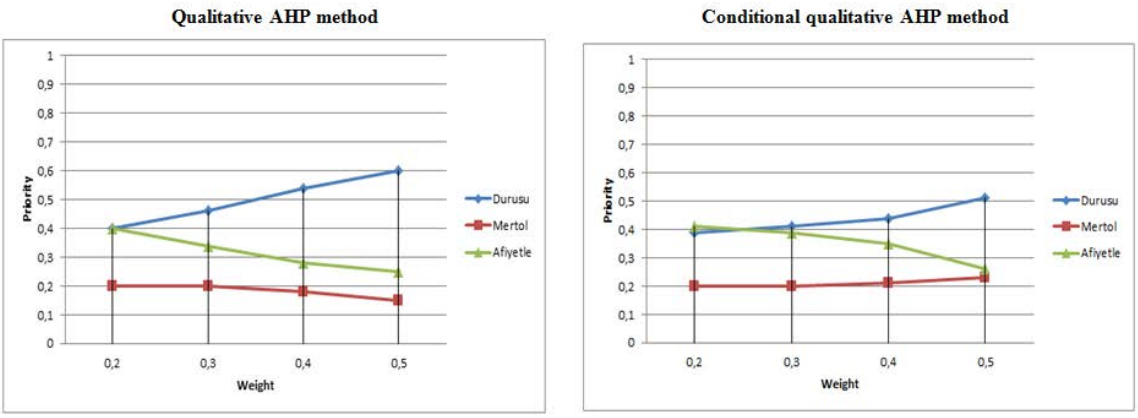

Figure 6.7: Sensitivity analysis of Quality of meal criterion

Mertol. The same conclusion can be drawn for the Quality of meal criterion, where Afiyetle remains as the best choice. Rank reversal occurs only to the first and second ranks (Afiyetle and Durusu respectively) when Quality of meal is increased.

Secondly, the same conclusion can be drawn for the Quality of meal criterion, where Afiyetle remains the best choice.

The results show that Afiyetle is always in the lead with a persistent score. The sensitivity analysis presented here demonstrates how consistent the decision is.

To conclude, the qualitative AHP and conditional qualitative AHP methods are used to solve real application problem and they have proved a good result. They have identified the best alternative and even with significant changes in the criteria weights the methods have maintained the best alternative.

6.6 Conclusion

In this Chapter, we have formulated qualitative MCDM methods in an environment characterized by imperfection. Our approaches deal with qualitative reasoning to model the uncertainty related to expert's assessment. The advantage of these newly proposed models are their ability to represent the decision maker's preferences without using numerical values. The expert is then allowed to freely express his judgments using belief preferences relations.

Conclusion

This PhD Thesis was devoted to the modelling of uncertainty under the AHP method. Uncertainty is formalized using the belief function framework.

In real-life decision making situation, the decision maker may encounter several difficulties when expressing his own level of preferences between alternatives or also criteria. However, standard version of AHP method is not adapted to ensure its role in such environment. Thus, the need of the development of appropriate approach to this kind of environment is vital.

For that reason, as a first contribution, we have revised the theoretical aspects of AHP method and we have proposed a new belief AHP method. When building the AHP hierarchy, our new approach uses subsets of criteria and alternatives instead of singletons, in order to reduce the number of comparisons. So, the expert has to express his knowledge to alternatives and criteria which he has a level of opinion towards.

Another main contribution of this PhD Thesis is to model the dependency between the alternatives and criteria. Consequently, we have been focused on representing the influences of the criteria on the evaluation of alternatives. In this latter, uncertainty is given in terms of conditional mass distributions. While making a decision, we need to allow for both and not to take the simple way out by always assuming independency.

The last part of the work provides some criticisms according to the comparison procedure. In fact, we have extended the belief AHP approach on a more flexible method by introducing uncertainty in the pair-wise comparison matrix. Thus, to evaluate the responses of the pair-wise comparison question, the decision maker expresses his judgment with some degrees of uncertainty.

Focusing on this aspect, we have proposed two different AHP extensions: Yes-No/AHP and qualitative AHP. The first expert judgment elicitation method is based on mass distributions, while the second one is based on preference relations. These introduced methods could produce significantly better results than standard AHP. Besides, using the first model, the uncertainty may appear in pair-wise comparisons values and is represented by bbas notion. This approach allows the expert to model his ignorance and to represent his imperfection. Despite of the advantages of the developed Yes-No/AHP method, the decision maker needs to quantify his degree of preferences. To handle this limitation, qualitative AHP approach has been developed. This methodology uses preference relations to model expert judgments. As a result, this qualitative AHP method is advantageous over the other proposed techniques.

While developing our qualitative AHP method, we have noticed that we need to introduce a new model that generates quantitative information from qualitative assessment. For this reason, we have proposed a new expert judgment elicitation method under the belief function framework that produces quantitative mass distributions from preference relations. The originality of our model is its ability to model different preference levels and to generate quantitative information from qualitative one only.

Finally, the performances of the proposed methods and the other approaches are compared according to the defined performance measure. In the computational experiments, the proposed Yes-No/AHP and qualitative AHP methods showed the best performances in all of the randomly generated matrices. When these two methods are compared, the second variation showed a better performance as the dimension of the matrices increases. Also, we have checked the robustness and the stability of the developed methods for maintaining the best solution. Applying sensitivity analysis to catering selection problem is essential to ensure the consistency of final decision.

In summary, the whole work is presented by Figure 6.8 which shows the main characteristics of the proposed methods.

Nevertheless, there exists a certain amount of problems which cannot be solved by standard AHP framework. With this work, we have been able to show that the way of solving a decision problem depends on the available information, on the objectives, on the possible and potential interactions with the decision maker and on the type of process which is eligible.

Some interesting future works have to be mentioned. First, in some situations, the decision maker may be a group or an organization. An application of the proposed method to the group decision making situations can then be done. We can, also, be interested in studying more com-

	Without dependency	With dependency
Standard AHP	Saaty scale	
Belief AHP	Saaty scale	Saaty scale
	A new aggregation process: discounting technique	A new aggregation process: standardization process
	Decision making using pignistic probabilities	Decision making using pignistic probabilities
Yes-No/AHP	A new set of choices: Yes or no	A new set of choices: Yes or no
	A belief distribution to describe each assessment	A belief distribution to describe each assessment
	A new aggregation process: discounting technique	A new aggregation process: standardization process
	Decision making using pignistic probabilities	Decision making using pignistic probabilities
Qualitative AHP	Preference relations to model each judgment	Preference relations to model each judgment
	A new aggregation process: discounting technique	A new aggregation process: standardization process
	Decision making using pignistic probabilities	Decision making using pignistic probabilities

Figure 6.8: A Summary of the whole work

plex hierarchical problems. That is the hierarchical structure is characterized by more than three levels: the overall objective, criteria, sub-criteria and alternatives. In such case, we propose to handle uncertainty in three levels instead of two.

From application point of view and in parallel to further theoretical elaboration, it would be interesting to apply our developed methods in other particular domain. They may be dedicated to situations where the decision is mainly based on expert experiences and knowledge. The developed methods in our PhD research can be exploited to cope with such situations. One interesting application of one of our previously developed methods is studying consumer behavior. Indeed, we could apply the qualitative AHP method since it models several types of judgments that can be strong preference, weak preference, indifference, incomparability or incompleteness. Note that our approach can also be used in other fields (e.g., supplier selection).

Moreover, it is interesting to solve other MCDM problem by an integrated approach. This latter combines one of our previously developed methods and an outranking method such as ELECTRE, PROMETHEE. In this methodology, the criteria weights will be generated by one of our proposed approaches and an outranking method under the belief function framework will be employed to assess different alternatives under uncertainty.

Another line of research could be the integration of machine learning methods in decision analysis. The decision process has two major aims: first, to explain decisions and second, to give recommendations: how to make a decision under specific circumstances. These are very similar to the goals of the machine learning approach, where the first step is to establish a model from previous experience which is then applied to predict future situations. This parallelism leads to the application of these methods in decision making. Therefore, it will be useful to study how these two fields can interact and how they can handle uncertainty in their parameters.

For instance, in a problem of credit scoring, one of its most important parts is determining the class of customers to run a classification algorithm. The purpose of this research will be the allocation of the labels of credit customers using one of the developed AHP methods. Here, in the first step, each customer will be labeled by an AHP extension and then a classification algorithm will be applied. In this way, via this method the acquired results of data mining algorithms could be improved.

Appendix A

Experimental results

A.1 Introduction

In this Appendix, all the experimental results will be given. To test the validity of our proposed models, our experiments are performed on the same generated data as described in the previous Chapters. We will show that some methods are better than others in some cases even though none of the methods is perfectly accurate.

A.2 Experimental analysis

In this Section, we present the results of our simulation study using AHP, belief AHP, conditional belief AHP, Yes-No/AHP, conditional Yes-No/AHP, qualitative AHP and conditional qualitative AHP methods.

Table A.1: Percentage of contradiction (%) between the WSM and the AHP

	Number of alternatives							
Number of criteria	3	4	5	6	7	8	9	10
3	7	7	9	9	10	11	13	12
4	7	7	9	9	11	11	11	11
5	7	7	9	9	12	11	11	11
6	8	6	9	10	11	12	13	13
7	8	6	10	10	11	12	13	13
8	8	6	10	10	10	1	12	13
9	8	6	10	10	10	12	12	14
10	9	6	11	11	10	12	14	14

Table A.2: Percentage of contradiction (%) between the WSM and the belief AHP

	Number of alternatives							
Number of criteria	3	4	5	6	7	8	9	10
3	7	6	8	8	10	10	10	10
4	7	5	8	8	11	11	11	10
5	7	6	9	9	11	11	11	11
6	8	6	9	9	12	11	12	12
7	7	7	10	9	11	11	12	12
8	7	7	10	10	10	12	12	12
9	7	7	10	10	10	12	12	12
10	7	7	10	10	10	12	12	12

Table A.3: Percentage of contradiction (%) between the WSM and the conditional belief AHP

	Number of alternatives							
Number of criteria	3	4	5	6	7	8	9	10
3	7	6	8	8	13	11	13	11
4	7	6	8	8	12	11	13	11
5	7	6	9	9	11	11	13	12
6	8	6	9	9	11	11	13	12
7	8	6	10	10	11	11	12	12
8	8	6	10	10	11	11	12	12
9	8	6	10	10	11	11	12	12
10	9	6	10	10	11	11	12	12

Table A.4: Percentage of contradiction (%) between the WSM and the Yes-No/AHP

	Number of alternatives							
Number of criteria	3	4	5	6	7	8	9	10
3	5	5	5	6	9	9	9	10
4	5	5	5	6	9	9	9	11
5	7	7	7	6	10	9	10	11
6	7	7	9	6	10	9	10	12
7	8	8	9	6	10	10	10	12
8	8	8	9	10	10	10	10	12
9	8	8	10	10	10	10	10	12
10	8	8	10	10	10	10	10	12

Table A.5: Percentage of contradiction (%) between the WSM and the conditional Yes-No/AHP

	Number of alternatives							
Number of criteria	3	4	5	6	7	8	9	10
3	5	5	5	7	7	7	9	10
4	5	5	5	7	7	7	9	10
5	6	7	8	7	8	9	9	10
6	6	7	8	8	8	9	10	11
7	6	8	9	8	8	10	10	11
8	7	8	9	8	8	10	10	11
9	7	8	9	10	8	10	10	11
10	7	8	10	10	8	10	10	11

Table A.6: Percentage of contradiction (%) between the WSM and the qualitative AHP

	Number of alternatives							
Number of criteria	3	4	5	6	7	8	9	10
3	5	7	7	7	7	7	8	8
4	5	7	7	7	7	7	10	8
5	7	8	8	8	9	9	10	9
6	7	8	8	8	8	10	10	9
7	7	8	8	7	7	10	10	10
8	7	8	8	8	8	8	10	10
9	8	9	9	10	10	10	10	10
10	8	8	8	9	9	9	10	11

Table A.7: Percentage of contradiction (%) between the WSM and the conditional qualitative AHP

	Number of alternatives							
Number of criteria	3	4	5	6	7	8	9	10
3	5	7	7	7	7	7	8	8
4	5	7	7	7	7	7	10	8
5	5	8	8	8	9	9	10	9
6	6	8	8	9	9	10	10	9
7	6	8	8	9	9	9	10	10
8	6	8	8	9	9	9	10	10
9	8	9	9	10	10	10	10	10
10	8	8	8	9	9	9	10	10

A.3 Conclusion

In this Appendix, we have elucidated the different results obtained using the different developed approaches.

Appendix B

Sensitivity analysis

B.1 Introduction

The objective of this Appendix is to briefly review typical sensitivity analysis methods and to recommend the selection of method to study the case of catering selection problem. Sensitivity analysis methods may be broadly classified as mathematical methods, statistical (or probabilistic) methods, and graphical methods. This classification helps in understanding applicability of sensitivity analysis methods for different types of models, and in selecting appropriate methods according to their usefulness to a decision maker.

Mathematical methods are useful for deterministic and probabilistic models. Statistical methods are generally used for probabilistic models. Graphical methods can be used for any kind of model (Frey & Patil, 2002).

In Section B.2, the basics of mathematical methods will be given. In Section B.3, the main notions of the statistical methods will be introduced while in Section B.4, the fundamental notions graphical methods will be highlighted.

B.2 Mathematical methods

This group of sensitivity analysis methods uses mathematical models when it is possible to express the relationship between the input data and the problem solution. Mathematical models have better performance and are the most efficient as they do not require iteration. Moreover, their results are much more accurate when using verified formulas. Many works have been presented such as: (Frey & Patil, 1990), (Huang, 1990) and (Erkut & Tarimcilar, 1991).

Several authors have developed mathematical models for sensitivity analysis in AHP. Frey and Patil (1990) studied how changes throughout the whole domain in the weights of criteria may affect the ranking of alternatives and proposed a sensitivity coefficient representing the possibility of rank reversal from these changes.

Huang (1990) found an inconsistency in Masudas coefficient as a large value of the coefficient may not necessarily mean that a rank reversal will occur, and vice versa, a low value of the coefficient may produce a rank reversal. To overcome this, Huang proposed a new sensitivity coefficient and demonstrated it reveals the sensitivity of an AHP model with more accuracy than Masuda's coefficient.

Erkut and Tarimcilar (1991) presented a method to analyze and visualize in the weight space the ranking of alternatives with respect to all possible combinations of weights for the first level of criteria. The method works by partitioning the weight space into n subsets, where n is the number of criteria. However, there are limitations in their method. It does not provide a method for performing sensitivity analysis when the problem has more than one level of criteria. Moreover, if more than three criteria are simultaneously considered, then visual representation is not possible and the weight space is no longer a triangle, it becomes a convex polyhedron.

B.3 Statistical methods

Simulation methods replace judgments in the pair-wise comparison matrix with values from probability distributions and perform a number of simulations to calculate the expected ranking of alternatives. As probabilistic input is used, the problem is no longer deterministic. This approach allows for changing more than one parameter at a time.

Statistical methods are widely used for probabilistic models as these methods can evaluate the effect of variance in the inputs on the output. A probabilistic model is itself deterministic in nature, but inputs are assigned distributions (Frey & Patil, 2002).

Butler et al. (1997) proposed a method using Monte-Carlo simulations that allows random change of all weights simultaneously to explore the effect on the ranking. They presented three types of simulations:

- Random weights: All criteria weights are generated completely at random in order to discover how the ranking of the alternatives changes under any conditions.
- Random weights preserving rank order: If the ranking of criteria weights are of importance and have to be preserved, then the procedure is similar to the previous one with the difference that the random weights are ranked according to the original criteria ranking and then are used to calculate the solution.
- Random weights from a response distribution: This type of simulation considers the current weights as means of probability distributions and the random weights for the simulation are generated from these distributions.

B.4 Graphical methods

This approach involves changing the weight values and calculating the new solution. The method, also known as One-at-a-time (OAT), works by incrementally changing one parameter at a time, calculating the new solution and graphically presenting how the global ranking of alternatives changes. This is the most popular in the literature where AHP is used to solve problems.

Graphical methods give representation of sensitivity in the form of graphs, charts, or surfaces. Generally, they are used to give a visual indication of how an output is affected by variation in inputs. Also, they can be used as a screening method before further analysis of a model or to represent complex dependencies between inputs and outputs.

Graphical methods can be used to complement the results of mathematical and statistical methods for better interpretation.

B.5 Conclusion

This Appendix has presented a discussion of sensitivity analysis was presented and its different types were covered.

Sensitivity analysis methods may be grouped into three main categories: graphical methods, probabilistic simulations and mathematical models. The first method is the most popular because of its simplicity and easy implementation. Therefore, in this work, we have use it as a sensitivity analysis technique. However, the other two groups of methods may provide more insights on a decision problem as they allow simultaneous analysis on more than one decision element.

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Résumé:

L'aide à la décision multicritères regroupe des méthodes permettant de choisir la meilleure solution en fonction des différents critères et compte tenu des préférences des experts. Toutefois, ces préférences sont parfois exprimées de manière imparfaite. La théorie des fonctions de croyance modélise de manière souple les connaissances et fournit des outils mathématiques pour gérer les différents types d'imperfection. Ainsi dans cette thèse, nous nous intéressons à la prise de décision multicritères dans un cadre incertain en étendant la méthode d'Analyse Hiérarchique des Procédés (AHP) à la théorie des fonctions de croyance. Après avoir présenté les fondements théoriques de la méthode AHP, nous avons proposé une approche qui permet de réduire le nombre de comparaisons par paires en jugeant des sous-ensembles de critères et d'alternatives. En outre, nous avons examiné la dépendance entre les critères et les alternatives. Dans ce cas, l'incertitude au niveau des évaluations est donnée sous forme de masses conditionnelles. Une autre partie de nos travaux répond aux critiques concernant la procédure de comparaison. Pour cela, nous avons proposé deux approches. La première technique d'élicitation des jugements de l'expert est fondée sur des distributions de masses, alors que la seconde s'appuie sur des relations de préférence. Dans ce cadre, nous avons introduit un modèle qui permet de générer des distributions de masse quantitatives à partir des relations de préférence. Ainsi, nous avons développé une méthode multicritères qui permet d'imiter le raisonnement humain. Cette méthode produit des résultats meilleurs et plus robustes que les approches de la littérature.

Mots clés : Décision multicritères, théorie des fonctions de croyance, AHP, comparaison par pair

Abstract:

Multi-criteria decision making is the study of identifying and choosing alternatives to find the best solution based on different criteria and considering the decision makers' expectations. However, the expert assessments are sometimes expressed imperfectly. Belief function theory can then provide more flexible and reliable tools to manage different types of imperfection. Thus, in this thesis, we are interested in multi-criteria decision making in an uncertain framework by extending the Analytic Hierarchy Process (AHP) method to the belief function framework. After presenting the theoretical foundations of the AHP method, we proposed an approach that reduces the number of pair-wise comparisons by judging subsets of criteria and alternatives. In addition, we examined the dependence between the criteria and alternatives. In this case, the uncertainty is given in terms of conditional mass distributions. Another part of the work provides critical concerning the pair-wise comparison process. For this purpose, we proposed two approaches. The first expert judgment elicitation method is based on mass distributions, while the second one is based on preference relations. In this context, we have introduced a model that is able to generate quantitative mass distributions from preference relations. Thus, we have developed a multi-criteria decision making method that imitates human reasoning. This method gives better and more robust results than existing approaches.

Keywords : multi-criteria decision making, belief function theory, AHP, pair-wise comparison