On Neutrosophic Quadruple Algebraic Structures

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Abstract

In this paper we present the concept of neutrosophic quadruple algebraic structures. Specifically, we study neutrosophic quadruple rings and we present their elementary properties.
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1 Introduction

The concept of neutrosophic quadruple numbers was introduced by Florentin Smarandache [3]. It was shown in [3] how arithmetic operations of addition, subtraction, multiplication and scalar multiplication could be performed on the set of neutrosophic quadruple numbers. In this paper we studied neutrosophic sets of quadruple numbers together with binary operations of addition and multiplication and the resulting algebraic structures with their elementary properties are presented. Specifically, we studied neutrosophic quadruple rings and we presented their basic properties.

Definition 1.1. [3] A neutrosophic quadruple number is a number of the form (a, bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and $a, b, c, d \in \mathbb{R}$ or $\mathbb{C}$. The set NQ defined by

$$NQ = \{(a, bT, cI, dF) : a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C}\} \quad (1)$$

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is called a neutrosophic set of quadruple numbers. For a neutrosophic quadruple number (a, bT, cI, dF) representing any entity which may be a number, an idea, an object, etc, a is called the known part and (bT, cI, dF) is called the unknown part.

Definition 1.2. Let $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQ$. We define the following:

$$\begin{aligned} a + b &= (a_1 + b_1, (a_2 + b_2)T, (a_3 + b_3)I, (a_4 + b_4)F) && (2) \\ a - b &= (a_1 - b_1, (a_2 - b_2)T, (a_3 - b_3)I, (a_4 - b_4)F) && (3) \end{aligned}$$

Definition 1.3. Let $a = (a_1, a_2T, a_3I, a_4F) \in NQ$ and let α be any scalar which may be real or complex, the scalar product $\alpha.a$ is defined by

$$\alpha.a = \alpha.(a_1, a_2T, a_3I, a_4F) = (\alpha a_1, \alpha a_2T, \alpha a_3I, \alpha a_4F) \quad (4)$$

If $\alpha = 0$, then we have $0.a = (0, 0, 0, 0)$ and for any non-zero scalars m and n and $b = (b_1, b_2T, b_3I, b_4F)$, we have:

$$\begin{aligned} (m + n)a &= ma + na, \\ m(a + b) &= ma + mb, \\ mn(a) &= m(na), \\ -a &= (-a_1, -a_2T, -a_3I, -a_4F). \end{aligned}$$

Definition 1.4. [3] [Absorbance Law] Let X be a set endowed with a total order $x < y$, named " x prevailed by y" or "x less stronger than y" or "x less preferred than y". $x \leq y$ is considered as "x prevailed by or equal to y" or "x less stronger than or equal to y" or "x less preferred than or equal to y".

For any elements $x, y \in X$, with $x \leq y$, absorbance law is defined as

$$x.y = y.x = \text{absorb}(x, y) = \max\{x, y\} = y \quad (5)$$

which means that the bigger element absorbs the smaller element (the big fish eats the small fish). It is clear from (5) that

$$\begin{aligned} x.x &= x^2 = \text{absorb}(x, x) = \max\{x, x\} = x \quad \text{and} && (6) \\ x_1.x_2 \cdots x_n &= \max\{x_1, x_2, \cdots, x_n\}. && (7) \end{aligned}$$

Analogously, if $x > y$, we say that "x prevails to y" or "x is stronger than y" or "x is preferred to y". Also, if $x \geq y$, we say that "x prevails or is equal to y" or "x is stronger than or equal to y" or "x is preferred or equal to y".

Definition 1.5. Consider the set $\{T, I, F\}$. Suppose in an optimistic way we consider the prevalence order $T > I > F$. Then we have:

$$\begin{aligned} TI &= IT = \max\{T, I\} = T, && (8) \\ TF &= FT = \max\{T, F\} = T, && (9) \\ IF &= FI = \max\{I, F\} = I, && (10) \\ TT &= T^2 = T, && (11) \\ II &= I^2 = I, && (12) \\ FF &= F^2 = F. && (13) \end{aligned}$$

Analogously, suppose in a pessimistic way we consider the prevalence order $T < I < F$. Then we have:

$$\begin{aligned} TI &= IT = \max\{T, I\} = I, && (14) \\ TF &= FT = \max\{T, F\} = F, && (15) \\ IF &= FI = \max\{I, F\} = F, && (16) \\ TT &= T^2 = T, && (17) \\ II &= I^2 = I, && (18) \\ FF &= F^2 = F. && (19) \end{aligned}$$

Except otherwise stated, we will consider only the prevalence order $T < I < F$ in this paper.

Definition 1.6. Let $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQ$. Then

$$\begin{aligned} a.b &= (a_1, a_2T, a_3I, a_4F).(b_1, b_2T, b_3I, b_4F) \\ &= (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2)T, (a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3)I, \\ &\quad (a_1b_4 + a_2b_4, a_3b_4 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)F). \qquad (20) \end{aligned}$$

2 Main Results

All neutrosophic quadruple numbers to be considered in this section will be real neutrosophic quadruple numbers i.e $a, b, c, d \in \mathbb{R}$ for any neutrosophic quadruple number $(a, bT, cI, dF) \in NQ$.

Theorem 2.1. *$(NQ, +)$ is an abelian group.*

Proof. Suppose that $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, c = (c_1, c_2T, c_3I, c_4F \in NQ$ are arbitrary. It can easily be shown that $a + b = b + a$. $a + (b + c) = (a + b) + c$. $a+(0, 0, 0, 0) = (0, 0, 0, 0) = a$ and $a+(-a) = -a+a = (0, 0, 0, 0)$. Thus, $0 = (0, 0, 0, 0)$ is the additive identity element in $(NQ, +)$ and for any $a \in NQ$, $-a$ is the additive inverse. Hence, $(NQ, +)$ is an abelian group. □

Theorem 2.2. *$(NQ, .)$ is a commutative monoid.*

Proof. Let $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, c = (c_1, c_2T, c_3I, c_4F$ be arbitrary elements in NQ. It can easily be shown that $ab = ba$. $a(bc) = (ab)c$. $a.(1, 0, 0, 0) = a$. Thus, $e = (1, 0, 0, 0)$ is the multiplicative identity element in $(NQ, .)$. Hence, $(NQ, .)$ is a commutative monoid. □

Theorem 2.3. *$(NQ, .)$ is not a group.*

Proof. Let $x = (a, bT, cI, dF)$ be any arbitrary element in NQ. Since we cannot find any element $y = (p, qT, rI, sF) \in NQ$ such that $xy = yx = e = (1, 0, 0, 0)$, it follows that x^{-1} does not exist in NQ for any given $a, b, c, d \in \mathbb{R}$ and consequently, $(NQ, .)$ cannot be a group. □

Example 1. Let $X = \{(a, bT, cI, dF) : a, b, c, d \in \mathbb{Z}_n\}$. Then $(X, +)$ is an abelian group.

Example 2. Let

$$M_{2\times 2} = \left\{ \begin{bmatrix} (a, bT, cI, dF) & (e, fT, gI, hF) \\ (i, jT, kI, lF) & (m, nT, pI, qF) \end{bmatrix} : a, b, c, d, e, f, g, h, i, j, k, l, m, n, p, q \in \mathbb{R} \right\}.$$

Then $(M_{2\times 2}, .)$ is a non-commutative monoid.

Theorem 2.4. $(NQ, +, .)$ *is a commutative ring.*

Proof. It is clear that $(NQ, +)$ is an abelian group and $(NQ, .)$ is a semigroup. To complete the proof, suppose that $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, c = (c_1, c_2T, c_3I, c_4F) \in NQ$ are arbitrary. It can easily be shown that $a(b + c) = ab + ac, (b + c)a = ba + ca$ and $ab = ba$. Hence, $(NQ, +, .)$ is a commutative ring. □

From now on, the ring $(NQ, +, .)$ will be called neutrosophic quadruple ring and it will be denoted by NQR. The zero element of NQR will be denoted by $(0, 0, 0, 0)$ and the unity of NQR will be denoted by $(1, 0, 0, 0)$.

Example 3. (i) Let X be as defined in EXAMPLE 1. Then $(X, +, .)$ is a commutative neutrosophic quadruple ring called a neutrosophic quadruple ring of integers modulo n.

It should be noted that $NQR(\mathbb{Z}_n)$ has 4^n elements and for $NQR(\mathbb{Z}_2)$ we have

$$\begin{aligned} NQR(\mathbb{Z}_2) = \{&(0, 0, 0, 0), (1, 0, 0, 0), (0, T, 0, 0), (0, 0, I, 0), (0, 0, 0, F), (0, T, I, F), (0, 0, I, F), \\ &(0, T, I, 0), (0, T, 0, F), (1, T, 0, 0), (1, 0, I, 0), (1, 0, 0, F), (1, T, 0, F), (1, 0, I, F), \\ &(1, T, I, 0), (1, T, I, F)\} \end{aligned}$$

(ii) Let $M_{2\times 2}$ be as defined in EXAMPLE 2. Then $(M_{2\times 2}, +, .)$ is a non-commutative neutrosophic quadruple ring.

Definition 2.5. Let NQR be a neutrosophic quadruple ring.

(i) An element $a \in NQR$ is called idempotent if $a^2 = a$.

(ii) An element $a \in NQR$ is called nilpotent if there exists $n \in \mathbb{Z}^+$ such that $a^n = 0$.

Example 4. (i) In $NQR(\mathbb{Z}_2)$, $(1, T, I, F)$ and $(1, T, I, 0)$ are idempotent elements.

(i) In $NQR(\mathbb{Z}_4)$, $(2, 2T, 2I, 2F)$ is a nilpotent element.

Definition 2.6. Let NQR be a neutrosophic quadruple ring. NQR is called a neutrosophic quadruple integral domain if for $x, y \in NQR$, $xy = 0$ implies that $x = 0$ or $y = 0$.

Example 5. $NQR(\mathbb{Z})$ the neutrosophic quadruple ring of integers is a neutrosophic quadruple integral domain.

Definition 2.7. Let NQR be a neutrosophic quadruple ring. An element $x \in NQR$ is called a zero divisor if there exists a nonzero element $y \in NQR$ such that $xy = 0$. For example in $NQR(\mathbb{Z}_2)$, $(0, 0, I, F)$ and $(0, T, I, 0)$ are zero divisors even though $\mathbb{Z}_2$ has no zero divisors. This is one of the distinct features that characterize neutrosophic quadruple rings.

Definition 2.8. Let NQR be a neutrosophic quadruple ring and let NQS be a nonempty subset of NQR. Then NQS is called a neutrosophic quadruple subring of NQR if $(NQS, +, .)$ is itself a neutrosophic quadruple ring. For example, $NQR(n\mathbb{Z})$ is a neutrosophic quadruple subring of $NQR(\mathbb{Z})$ for $n = 1, 2, 3, \cdots$.

Theorem 2.9. *Let NQS be a nonempty subset of a neutrosophic quadruple ring NQR. Then NQS is a neutrosophic quadruple subring if and only if for all $x, y \in NQS$, the following conditions hold:*

(i) $x - y \in NQS$ *and*

(ii) $xy \in NQS$.

Proof. Same as the classical case and so omitted. □

Definition 2.10. Let NQR be a neutrosophic quadruple ring. Then the set

$$Z(NQR) = \{x \in NQR : xy = yx \ \ \forall \ y \in NQR\}$$

is called the centre of NQR.

Theorem 2.11. *Let NQR be a neutrosophic quadruple ring. Then $Z(NQR)$ is a neutrosophic quadruple subring of NQR.*

Proof. Same as the classical case and so omitted. □

Theorem 2.12. *Let NQR be a neutrosophic quadruple ring and let NQS_j be famlies of neutrosophic quadruple subrings of NQR. Then $\bigcap_{j=1}^{n} NQS_j$ is a neutrosophic quadruple subring of NQR.*

Definition 2.13. Let NQR be a neutrosophic quadruple ring. If there exists a positive integer n such that $nx = 0$ for each $x \in NQR$, then the smallest such positive integer is called the characteristic of NQR. If no such positive integer exists, then NQR is said to have characteristic zero. For example, $NQR(\mathbb{Z})$ has characteristic zero and $NQR(\mathbb{Z}_n)$ has characteristic n.

Definition 2.14. Let NQJ be a nonempty subset of a neutrosophic quadruple ring NQR. NQJ is called a neutrosophic quadruple ideal of NQR if for all $x, y \in NQJ, r \in NQR$, the following conditions hold:

(i) $x - y \in NQJ$.

(ii) $xr \in NQJ$ and $rx \in NQJ$.

Example 6. (i) $NQR(3\mathbb{Z})$ is a neutrosophic quadruple ideal of $NQR(\mathbb{Z})$.

(ii) Let $NQJ = \{(0,0,0,0),(2,0,0,0),(0,2T,2I,2F),(2,2T,2I,2F)\}$ be a subset of $NQR(\mathbb{Z}_4)$. Then NQJ is a neutrosophic quadruple ideal.

Theorem 2.15. *Let NQJ and NQS be neutrosophic quadruple ideals of NQR and let $\{NQJ_j\}_{j=1}^{n}$ be a family of neutrosophic quadruple ideals of NQR. Then:*

(i) $NQJ + NQJ = NQJ$.

(ii) $x + NQJ = NQJ$ *for all* $x \in NQJ$.

(iii) $\bigcap_{j=1}^{n} NQJ_j$ *is a neutrosophic quadruple ideal of* NQR.

(iv) $NQJ + NQS$ *is a neutrosophic quadruple ideal of* NQR.

Definition 2.16. Let NQJ be a neutrosophic quadruple ideal of NQR. The set

$$NQR/NQJ = \{x + NQJ : x \in NQR\}$$

is called a neutrosophic quadruple quotient ring.

If $x + NQJ$ and $y + NQJ$ are two arbitrary elements of NQR/NQJ and if $\oplus$ and $\odot$ are two binary operations on NQR/NQJ defined by:

$$\begin{aligned}(x + NQJ) \oplus (y + NQJ) &= (x + y) + NQJ,\\ (x + NQJ) \odot (y + NQJ) &= (xy) + NQJ,\end{aligned}$$

it can be shown that $\oplus$ and $\odot$ are well defined and that $(NQR/NQJ, \oplus, \odot)$ is a neutrosophic quadruple ring.

Example 7. Consider the neutrosophic quadruple ring $NQR(\mathbb{Z})$ and its neutrosophic quadruple ideal $NQR(2\mathbb{Z})$. Then

$$\begin{aligned}NQR(\mathbb{Z})/NQR(2\mathbb{Z}) = \{&NQR(2\mathbb{Z}), (1,0,0,0) + NQR(2\mathbb{Z}), (0,T,0,0) + NQR(2\mathbb{Z}), (0,0,I,0) + NQR(2\mathbb{Z}),\\ &(0,0,0,F) + NQR(2\mathbb{Z}), (0,T,I,F) + NQR(2\mathbb{Z}), (0,0,I,F) + NQR(2\mathbb{Z}),\\ &(0,T,I,0) + NQR(2\mathbb{Z}), (0,T,0,F) + NQR(2\mathbb{Z}), (1,T,0,0) + NQR(2\mathbb{Z}),\\ &(1,0,I,0) + NQR(2\mathbb{Z}), (1,0,0,F) + NQR(2\mathbb{Z}), (1,T,0,F) + NQR(2\mathbb{Z}),\\ &(1,0,I,F) + NQR(2\mathbb{Z}), (1,T,I,0) + NQR(2\mathbb{Z}), (1,T,I,F) + NQR(2\mathbb{Z})\}\end{aligned}$$

which is clearly a neutrosophic quadruple ring.

Definition 2.17. Let NQR and NQS be two neutrosophic quadruple rings and let $\phi : NQR \to NQS$ be a mapping defined for all $x, y \in NQR$ as follows:

(i) $\phi(x + y) = \phi(x) + \phi(y)$.

(ii) $\phi(xy) = \phi(x)\phi(y)$.

(iii) $\phi(T) = T$, $\phi(I) = I$ and $\phi(F) = F$.

(iv) $\phi(1,0,0,0) = (1,0,0,0)$.

Then ϕ is called a neutrosophic quadruple homomorphism. Neutrosophic quadruple monomorphism, endomorphism, isomorphism, and other morphisms can be defined in the usual way.

Definition 2.18. Let $\phi : NQR \to NQS$ be a neutrosophic quadruple ring homomorphism.

(i) The image of ϕ denoted by $Im\phi$ is defined by the set

$$Im\phi = \{y \in NQS : y = \phi(x), \text{for some } x \in NQR\}.$$

(ii) The kernel of ϕ denoted by $Ker\phi$ is defined by the set

$$Ker\phi = \{x \in NQR : \phi(x) = (0, 0, 0, 0)\}.$$

Theorem 2.19. *Let $\phi : NQR \to NQS$ be a neutrosophic quadruple ring homomorphism. Then:*

(i) $Im\phi$ is a neutrosophic quadruple subring of NQS.

(ii) $Ker\phi$ is not a neutrosophic quadruple ideal of NQR.

Proof. (i) Clear.
(ii) Since T, I, F cannot have image $(0, 0, 0, 0)$ under ϕ, it follows that the elements $(0, T, 0, 0), (0, 0, I, 0), (0, 0, 0, F)$ cannot be in the $Ker\phi$. Hence, $Ker\phi$ cannot be a neutrosophic quadruple ideal of NQR. □

Example 8. Consider the projection map $\phi : NQR(\mathbb{Z}_2) \times NQR(\mathbb{Z}_2) \to NQR(\mathbb{Z}_2)$ defined by $\phi(x, y) = x$ for all $x, y \in NQR(\mathbb{Z}_2)$. It is clear that ϕ is a neutrosophic quadruple homomorphism and its kernel is given as

$$\begin{aligned} Ker\phi = \{&\{((0,0,0,0),(0,0,0,0)),((0,0,0,0),(1,0,0,0)),((0,0,0,0),(0,T,0,0)),((0,0,0,0),(0,0,I,0)),\\ &((0,0,0,0),(0,0,0,F)),((0,0,0,0),(0,T,I,F)),((0,0,0,0),(0,0,I,F)),((0,0,0,0),(0,T,I,0)),\\ &((0,0,0,0),(0,T,0,F)),((0,0,0,0),(1,T,0,0)),((0,0,0,0),(1,0,I,0)),((0,0,0,0),(1,0,0,F)),\\ &((0,0,0,0),(1,T,0,F)),((0,0,0,0),(1,0,I,F)),((0,0,0,0),(1,T,I,0)),((0,0,0,0),(1,T,I,F))\}. \end{aligned}$$

Theorem 2.20. *Let $\phi : NQR(\mathbb{Z}) \to NQR(\mathbb{Z})/NQR(n\mathbb{Z})$ be a mapping defined by $\phi(x) = x + NQR(n\mathbb{Z})$ for all $x \in NQR(\mathbb{Z})$ and $n = 1, 2, 3, \ldots$. Then ϕ is not a neutrosophic quadruple ring homomorphism.*

References

[1] A.A.A. Agboola, *On Refined Neutrosophic Algebraic StructuresI*, Neutrosophic Sets and Systems **10** (2015), 99-101.

[2] F. Smarandache, Neutrosophy/Neutrosophic Probability, Set, and Logic, American Research Press, Rehoboth, USA, 1998
http://fs.gallup.unm.edu/eBook-otherformats.htm

[3] F. Smarandache, *Neutrosophic Quadruple Numbers, Refined Neutrosophic Quadruple Numbers, Absorbance Law, and the Multiplication of Neutrosophic Quadruple Numbers*, Neutrosophic Sets and Systems **10** (2015), 96-98.

[4] F. Smarandache, *(t,i,f) - Neutrosophic Structures and I-Neutrosophic Structures*, Neutrosophic Sets and Systems **8** (2015), 3-10.

[5] F. Smarandache,*n-Valued Refined Neutrosophic Logic and Its Applications in Physics*, Progress in Physics **4** (2013), 143-146.