Extended PCR Rules for Dynamic Frames

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***Abstract*—In most of classical fusion problems modeled from belief functions, the frame of discernment is considered as static. This means that the set of elements in the frame and the underlying integrity constraints of the frame are fixed forever and they do not change with time. In some applications, like in target tracking for example, the use of such invariant frame is not very appropriate because it can truly change with time. So it is necessary to adapt the Proportional Conflict Redistribution fusion rules (PCR5 and PCR6) for working with dynamical frames. In this paper, we propose an extension of PCR5 and PCR6 rules for working in a frame having some non-existential integrity constraints. Such constraints on the frame can arise in tracking applications by the destruction of targets for example. We show through very simple examples how these new rules can be used for the belief revision process.**

Keywords: Information fusion, DSmT, integrity constraints, belief functions.

I. Introduction

In most of classical fusion problems using belief functions, the frame of discernment $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ is considered static. This means that the set of elements in the frame (assumed to be non-empty and distinct) and the underlying integrity constraints of the frame[1] are fixed and they do not change with time. In some applications however, like in target tracking and battlefield surveillance for example, the use of such invariant frame is not very appropriate because it can truly change with time depending on the evolution of the events. So it is necessary to adapt the Proportional Conflict Redistribution fusion rules (PCR5 and PCR6) for working with dynamical frames. In this paper, we study in details how to work with PCR5 or PCR6 fusion rules in a dynamical frame subject to non-existential integrity constraint, when one or several elements of the frame disappear. This phenomena can occur in some applications, specially in defense and battlefield surveillance when foe targets (considered as element of the frame) can be shot and entirely destroyed and the initial belief one has on threat assessment must be revised according to the knowledge one has on this new fact obtained from intelligence services or observations systems. We show through very simple examples how this problem can be solved using PCR principle.

Example 1: Let's consider the set of three targets at a given time k to be $\Theta_k = \{\theta_1, \theta_2, \theta_3\}$ with $\theta_i \neq \emptyset$, $i = 1, 2, 3$ and assume that Θ_k satisfies Shafer's model (i.e. the targets are all distinct and exhaustive) and we work with normalized bba's. Suppose one has two basic belief assignments (bba) $m_1(.)$ and $m_2(.)$ defined with respect to the power-set of Θ_k given by two distinct sources of evidence to characterize their beliefs in the most threatening target. Let's assume that one receives at $k+1$ a new information confirming that one target, say target θ_3, has been destroyed. The problem one needs to solve is how to combine efficiently $m_1(.)$ and $m_2(.)$ taking into account this new non-existential integrity constraint $\theta_3 \equiv \emptyset$ in the new model of the frame to establish the most threatening and surviving targets belonging to $\Theta_{k+1} = \{\theta_1, \theta_2\}$.

The contribution of this paper is to propose a solution to such kind of belief revision problem involving dynamical frames including non-existential constraints on some of its elements. This paper is organized as follows. In section 1, we briefly recall the basis of DSmT (Dezert-Smarandache Theory) [4] and its main rule of combination (PCR5 and PCR6) for the fusion of bba's in a static frame. In section 2, we present an improvement/adaptation of PCR rules to work on frames with non-existential constraints (dynamical frames). In section 3, we apply our method on some examples. Conclusions are then given in section 4.

II. Basics of DSmT

The purpose of the development of Dezert-Smarandache Theory (DSmT) [4] is to overcome the limitations of Dempster-Shafer Theory (DST) [3] mainly by proposing new underlying models for the frames of discernment in order to fit better with the nature of real problems, and by proposing new efficient combination and conditioning rules. In DSmT framework, the elements θ_i, $i = 1, 2, \dots, n$ of a given frame Θ are not necessarily exclusive, and there is no restriction on θ_i but their exhaustivity. The hyper-power set D^Θ in DSmT, the hyper-power set is defined as the set of all composite propositions built from elements of Θ with operators $\cup$ and $\cap$. For instance, if $\Theta = \{\theta_1, \theta_2\}$, then $D^\Theta = \{\emptyset, \theta_1, \theta_2, \theta_1 \cap \theta_2, \theta_1 \cup \theta_2\}$. The hyper-power set D^Θ reduces to classical power-set 2^Θ as soon as we assume exclusivity between the elements of the frame (this is Shafer's model). A (generalized) basic belief assignment (bba for short) is defined as the mapping $m : D^\Theta \to [0, 1]$. The generalized belief and plausibility functions are defined in almost the same manner as in DST. More precisely, from a general frame Θ, we define a map $m(.) : D^\Theta \to [0, 1]$ associated to a given

[1] This is also called the model for Θ which can correspond to DSm free, DSm hybrid or Shafer's models in DSmT framework [4].

body of evidence $\mathcal{B}$ as

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{A \in D^{\Theta}} m(A) = 1 \tag{1}$$

The quantity $m(A)$ is called the *generalized* basic belief assignment/mass (or just "bba" for short) of A.
The *generalized* credibility and plausibility functions are defined in almost the same manner as within DST, i.e.

$$\text{Bel}(A) = \sum_{\substack{B \subseteq A \\ B \in D^{\Theta}}} m(B) \quad \text{and} \quad \text{Pl}(A) = \sum_{\substack{B \cap A \neq \emptyset \\ B \in D^{\Theta}}} m(B) \tag{2}$$

Two models[2] (the free model and hybrid model) in DSmT can be used to define the bba's to combine. In the free DSm model, the sources of evidence are combined without taking into account integrity constraints. When the free DSm model does not hold because the true nature of the fusion problem under consideration, we take into account some known integrity constraints[3] and define bba's to combine using the proper hybrid DSm model. Aside offering the possibility to work with different underlying models (not only Shafer's model as within DST), DSmT offers also new efficient combination rules based on proportional conflict redistribution (PCR rules no 5 and no 6) for combining highly conflicting sources of evidence. PCR5 transfers the conflicting mass only to the elements involved in the conflict and proportionally to their individual masses, so that the specificity of the information is entirely preserved in this fusion process. (see [4], Vol. 2 for full justification and examples): $m_{PCR5}(\emptyset) = 0$ and $\forall X \in D^{\Theta} \setminus \{\emptyset\}$

$$m_{PCR5}(X) = \sum_{\substack{X_1, X_2 \in D^{\Theta} \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2) +$$
$$\sum_{\substack{X_2 \in D^{\Theta} \\ X_2 \cap X = \emptyset}} [\frac{m_1(X)^2 m_2(X_2)}{m_1(X) + m_2(X_2)} + \frac{m_2(X)^2 m_1(X_2)}{m_2(X) + m_1(X_2)}] \tag{3}$$

where all denominators in (3) are different from zero. If a denominator is zero, that fraction is discarded. The properties of PCR5 can be found in [2]. Extension of PCR5 for combining qualitative bba's can be found in [4], Vol. 2 & 3. All propositions/sets are in a canonical form. A variant of PCR5, called PCR6 has been proposed by Martin and Osswald in [4], Vol. 2, for combining $s > 2$ sources. The general formulas for PCR5 and PCR6 rules are given in [4], Vol. 2 also. PCR6 coincides with PCR5 when one combines two sources. The difference between PCR5 and PCR6 lies in the way the proportional conflict redistribution is done as soon as three or more sources are involved in the fusion. From the implementation point of view, PCR6 is much more simple to implement than PCR5. For convenience, very basic (not optimized) Matlab codes of PCR5 and PCR6 fusion rules can be found in [4], [5] and from the toolboxes repository on the web [7]. In DSmT framework, the classical pignistic transformation $BetP(.)$ is replaced by the more effective $DSmP(.)$ transformation to estimate the subjective probabilities of hypotheses for decision-making support once the combination of bba's has been done if compromise attitude is chosen. The max of credibility (pessimistic decision attitude) or max of plausibility (optimistic decision attitude) are also possible depending on the preference of decision maker. This topic is out of the scope of this paper and readers interested in decision-making based on DSmP must refer to [4], Vol.3 freely available on the web.

[2]Actually, Shafer's model, considering all elements of the frame as truly exclusive, can be viewed as a special case of hybrid model.

[3]but non-existential integrity constraints as shown in Example 2.

III. WORKING WITH NON-EXISTENTIAL CONSTRAINTS

In this section we show how this problem can be solved from the classical Shafer's approach and then we show how it can be solved with PCR rules to get more specific results.

A. *Shafer's approach*

Let's consider a finite and discrete frame $\Theta_k = \{\theta_1, \theta_2, \dots, \theta_n\}$ satisfying Shafer's model with all $\theta_i \neq \emptyset$ at a given time k, and two bba's $m_{1,k}(.)$ and $m_{2,k}(.)$ provided by two distinct sources of evidences. Each bba is defined in the power set 2^{Θ_k}. Let's assume now that at time $k+1$ extra knowledge is given about the non-existence of some elements of Θ_k. We denote such non-existential constraint as NE (the set of Non Existing elemnts). For example, if $\text{NE}_{k+1} = \{\theta_1\}$ means that actually $\theta_1 = \emptyset$, $\text{NE}_{k+1} = \{\theta_1, \theta_2\}$ means that both $\theta_1 = \emptyset$ and $\theta_2 = \emptyset$, and so on. The new frame of discernment we have to work with is then given by $\Theta_{k+1} = \Theta_k \setminus \text{NE}_{k+1}$. The question is how to combine at time $k+1$ the two original bba's $m_{1,k}(.)$ and $m_{2,k}(.)$ one had in taking into account our knowledge on the revised frame Θ_{k+1} obtained from Θ_k and NE_{k+1} ?

Dempster-Shafer Theory (DST) [3] offers a mathematical tool for answering to this question: Dempster-Shafer belief conditioning rule (DSCR) which consists in combining with Dempster-Shafer's rule the prior bba $m(.)$ with the conditioning bba $m_c(.)$ which is only focused on the conditioning event X, i.e. for which $m_c(X) = 1$. Mathematically, $m_{DS}(.|X)$ is then defined[4] by

$$m_{DS}(.|X) = [m \oplus m_c](.) \tag{4}$$

where $\oplus$ corresponds here to Dempster-Shafer's rule of combination and $m_c(X) = 1$.

For solving this fusion problem under non-existential integrity constraints, three methods are a priori possible based on DSCR:

• **The Fusion-Conditioning approach (FC)**: It consists to combine the sources at first and then apply Dempster-Shafer conditioning rule. This corresponds to the following formula:

$$m_{DS\text{-}FC}(.|\Theta_{k+1}) = [[m_{1,k} \oplus m_{2,k}] \oplus m_{c,k}](.) \tag{5}$$

where $\oplus$ corresponds here to Dempster-Shafer's rule of combination and $m_{c,k}(\Theta_{k+1}) = 1$. Note that $m_{c,k}(.)$ refers to the conditioning bba defined in 2^{Θ_k}.

[4]if $m(.)$ and $m_c(.)$ are not in total contradiction of course.

• **The Conditioning-Fusion approach (CF)**: It consists to apply the DS conditioning to the sources at first and then combine the conditioned bba's with Dempster-Shafer rule. This corresponds to the following formula:

$$m_{DS\text{-}CF}(.|\Theta_{k+1}) = [m_{1,k} \oplus m_{c,k}] \oplus [m_{2,k} \oplus m_{c,k}](.) \quad (6)$$

• **The Global Conditioning approach (GC)**: It consists to combine all the bba's altogether in a single step of fusion. This corresponds to the following formula:

$$m_{DS\text{-}GC}(.|\Theta_{k+1}) = [m_{1,k} \oplus m_{2,k} \oplus m_{c,k}](.) \quad (7)$$

Because of the commutativity and associativity of DS rule and since $[m_c \oplus m_c](.) = m_c(.)$ for any conditioning bba focused on only one specific element X, the three previous methods provide exactly the same results. This makes Shafer's approach very appealing since there is no ambiguity in the choice of the method to apply.

B. Example 1 (continued)

Let's take back the Example 1 and consider the two arbitrary prior bba's given in Table I.

bba's\focal elem.	θ_1	θ_2	θ_3	$\theta_1 \cup \theta_2$
Prior: $m_{1,k}(.)$	0.2	0.4	0.3	0.1
Prior: $m_{2,k}(.)$	0.3	0.1	0.4	0.2
Conditioning: $m_{c,k}(.)$	0	0	0	1
DS-FC: $m_{DS\text{-}FC}(.)$	0.4643	0.4643	0	0.0714
DS-CF: $m_{DS\text{-}CF}(.)$	0.4643	0.4643	0	0.0714
DS-GC: $m_{DS\text{-}GC}(.)$	0.4643	0.4643	0	0.0714

Table I
EXAMPLE 1: RESULTS WITH DS-BASED CONDITIONING.

Because in this example $\Theta_k = \{\theta_1, \theta_2, \theta_3\}$ and $\text{NE}_{k+1} = \{\theta_3\}$ then $\Theta_{k+1} = \{\theta_1, \theta_2\}$ (only targets θ_1 and θ_2 survive) and therefore the conditioning bba $m_{c,k}(.)$ is defined by $m_{c,k}(\theta_1 \cup \theta_2) = 1$. In applying DSCR, one gets with three methods the same following result: $m_{DS\text{-}GC}(.|\Theta_{k+1}) = m_{DS\text{-}CF}(.|\Theta_{k+1}) = m_{DS\text{-}FC}(.|\Theta_{k+1})$ as shown in the last three rows of Table I). This symmetrical result in θ_1 and θ_2 is very surprising since clearly the input bba's are asymmetrical in θ_1 and θ_2 and we don't see any intuitive nor rational justification to consider such DSCR-based behavior as efficient for applications.

• **Direct approach**: Note that this result can be also simply obtained in a direct manner using DS rule for combining $m_{1,k}(.)$ with $m_{2,k}(.)$ and in taking into account the constraint $\theta_3 = \emptyset$ in the DS formula. In this example 1, one gets:

$$\begin{aligned} m_{12}(\theta_1) &= m_{1,k}(\theta_1)m_{2,k}(\theta_1) + m_{1,k}(\theta_1)m_{2,k}(\theta_1 \cup \theta_2) \\ &\quad + m_{2,k}(\theta_1)m_{1,k}(\theta_1 \cup \theta_2) = 0.13 \\ m_{12}(\theta_2) &= m_{1,k}(\theta_2)m_{2,k}(\theta_2) + m_{1,k}(\theta_2)m_{2,k}(\theta_1 \cup \theta_2) \\ &\quad + m_{2,k}(\theta_2)m_{1,k}(\theta_1 \cup \theta_2) = 0.13 \\ m_{12}(\theta_1 \cup \theta_2) &= m_{1,k}(\theta_1 \cup \theta_2)m_{1,k}(\theta_1 \cup \theta_2) = 0.02 \end{aligned}$$

For θ_3, one has $m_{12}(\theta_3) = m_{1,k}(\theta_3)m_{1,k}(\theta_3) = 0.12$. Since actually $\theta_3 = \emptyset$, then $m_{12}(\theta_3 = \emptyset) = 0.12$ must be added to mass already committed to the empty set coming from other possible conflicting conjunctions so that finally one will get the total conflicting mass $m_{12}(\emptyset) = 0.72$. After normalization step, we finally get

$$\begin{aligned} m_{DS}(\theta_1) &= \frac{m_{12}(\theta_1)}{1 - m_{12}(\emptyset)} = 0.13/0.28 = 0.4643 \\ m_{DS}(\theta_2) &= \frac{m_{12}(\theta_2)}{1 - m_{12}(\emptyset)} = 0.13/0.28 = 0.4643 \\ m_{DS}(\theta_1 \cup \theta_2) &= \frac{m_{12}(\theta_1 \cup \theta_2)}{1 - m_{12}(\emptyset)} = 0.02/0.28 = 0.0714 \end{aligned}$$

• *Advantages of DS approach*: The main interest of this DSCR-based methods lies in the fact that DSCR can be interpreted as a generalization of Bayesian conditioning and that the conditioning and the DS fusion commute, so that the three methods FC, CF or GC based all on DSCR coincide.

• *Drawbacks of DS approach*: Although attractive, DSCR approach cannot however circumvent the problem inherent to DS rule itself when the sources to combine are highly conflicting or are in worst case in total conflict. Even if the sources are not too conflicting, DSCR can yield to questionable results as pointed out in Example 1 (i.e. symmetrical results based on asymmetrical inputs) – see Table I.

C. Example 2

This example is an extension of Zadeh's example including non-existential constraint. Let's take $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ satisfying Shafer's model and the following prior bba's given in Table II, and let's assume at time $k+1$ that we learn $\theta_4 = \emptyset$, so that $\Theta_{k+1} = \{\theta_1, \theta_2, \theta_3\}$. Applying all previous methods, provide same counter-intuitive result $m_{DS}(\theta_3) = 1$ as in classical Zadeh's example.

bba's\focal elem.	θ_1	θ_2	θ_3	θ_4	$\theta_1 \cup \theta_2 \cup \theta_3$
Prior: $m_{1,k}(.)$	0.98	0	0.01	0.01	0
Prior: $m_{2,k}(.)$	0	0.98	0.01	0.01	0
Conditioning: $m_{c,k}(.)$	0	0	0	0	1
DS-FC: $m_{DS\text{-}FC}(.)$	0	0	1	0	0
DS-CF: $m_{DS\text{-}CF}(.)$	0	0	1	0	0
DS-GC: $m_{DS\text{-}GC}(.)$	0	0	1	0	0

Table II
EXAMPLE 2–A: RESULTS WITH DS-BASED CONDITIONING.

This example can be generalized as in Table III where all bba's are normalized and the non-existential constraint is $A_4 \cup \ldots \cup A_n = \emptyset$. The result of DSCR approach is given in the right column of Table III.
for $n \geq 1$, where ϵ_1, ϵ_2, and δ_{ij} are very tiny positive numbers in [0,1], a_1 and a_2 are positive numbers closer to 1, but smaller than 1, and the sum on each column is 1; all intersections $A_i \cap A_j$ are empty, where A_i can be singletons or unions of singletons. So, this is a Bayesian and non-Bayesian example.

D. Example 3

Here we give two very simple classes of examples with Bayesian or non-Bayesian bba's where DSCR cannot be applied to solve the problem. We assume Shafer's model for

Focal elem.\ bba's	$m_{1,k}(.)$	$m_{2,k}(.)$	$m_{c,k}(.)$	$m_{DS}(.)$
A_1	a_1	0	0	0
A_2	0	a_2	0	0
A_3	ϵ_1	ϵ_2	0	1
A_4	δ_{11}	δ_{21}	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_n	δ_{1n}	δ_{2n}	0	0
$A_1 \cup A_2 \cup A_3$	0	0	1	0

Table III
GENERALIZATION OF EXAMPLE 2–A.

the frames. In example 3–A, the non-existential constraint is $\theta_1 = \emptyset$ and the parameters a and b belong to $[0, 1]$.

bba's\focal elem.	θ_1	θ_2	θ_3	$\theta_2 \cup \theta_3$
Prior: $m_{1,k}(.)$	a	0	1-a	0
Prior: $m_{2,k}(.)$	b	1-b	0	0
Conditioning: $m_{c,k}(.)$	0	0	0	1

Table IV
BBA'S FOR EXAMPLE 3–A. (BAYESIAN CASE WITH $\theta_1 = \emptyset$)

Example 3–A gives 0/0 when using Dempster-Shafer's conditioning rule.

In example 3–B, we consider non-Bayesian bba's. The parameters a and b belong to $[0; 1]$. The non-existential constraint is $\theta_1 = \theta_2 = \emptyset$.

bba's\focal elem.	$\theta_1 \cup \theta_2$	θ_3	θ_4
Prior: $m_{1,k}(.)$	a	0	1-a
Prior: $m_{2,k}(.)$	b	1-b	0
Conditioning: $m_{c,k}(.)$	0	0	1

Table V
BBA'S FOR EXAMPLE 3–B. (NON-BAYESIAN CASE WITH $\theta_1 \cup \theta_2 = \emptyset$)

An infinity of Bayesian or Non Bayesian classes with total conflicting sources can be constructed where DSCR rule cannot be applied.

E. DSmT approach

Since the PCR5 or PCR6 circumvent the problem of DS rule for combining potentially highly conflicting sources of evidence, it is natural to try at first to use the same methodology for solving the problem just in replacing the DS fusion operator $\oplus$ by PCR5 (or PCR6) fusion operators. This is called PCR5CR (PCR5-based conditioning rule) or PCR6CR if one prefers to use PCR6. Unfortunately, the solution based on these PCR rules is not so simple because PCR rules are not associative and thus the result one gets highly depends on the conditioning method we adopt: FC, CF or Global. Moreover, the direct approach based on classical/original PCR5 rule under non-existential constraint cannot be applied as it will be shown from Example 1. That's why we propose a new solution to solve this important problem in the sequel.

Example 1 (continued): Let's take back example 1 and examine the results given by PCR5-FC, PCR5-CF and PCR5-GC methods[5]. The results are given in Table VI.

bba's\focal elem.	θ_1	θ_2	θ_3	$\theta_1 \cup \theta_2$
Prior: $m_{1,k}(.)$	0.2	0.4	0.3	0.1
Prior: $m_{2,k}(.)$	0.3	0.1	0.4	0.2
Conditioning: $m_{c,k}(.)$	0	0	0	1
PCR5-FC: $m_{PCR5\text{-}FC}(.)$	0.2664	0.2927	0.3320	0.1089
PCR5-CF: $m_{PCR5\text{-}CF}(.)$	0.3526	0.3822	0.0470	0.2182
PCR5-GC: $m_{PCR5\text{-}GC}(.)$	0.1811	0.1975	0.1597	0.4617

Table VI
EXAMPLE 1: RESULTS WITH PCR5-BASED CONDITIONING.

From Table VI, one sees clearly that the original PCR5 rule used for solving this example generates different results depending the method (PCR5-FC, PCR5-CF or PCR5-GC) which is not very satisfactory, and that all methods commit a positive mass to $\theta_3 = \emptyset$ which is not acceptable since we assume to work within Shafer's model in this example.

Direct approach: If we now use a direct PCR5-based approach for trying to solve the problem, we need to replace θ_3 by $\emptyset$ in the bba's inputs and apply the PCR5$_\emptyset$ fusion rule proposed in [5]. $PCR5_\emptyset$ fusion formula is same as $PCR5$ fusion formula (3) except that $X \in D^\Theta$ where D^Θ includes the empty set as well. In clear, PCR5$_\emptyset$ fusion rule allows $\emptyset$ as focal element (as in Smets' TBM). If we apply this PCR5$_\emptyset$ direct fusion, one will get results in Table VII consistent with the result of the last row of Table VI which is normal.

bba's\focal elem.	θ_1	θ_2	$\emptyset$	$\theta_1 \cup \theta_2$
Prior: $m_{1,k}(.)$	0.2	0.4	0.3	0.1
Prior: $m_{2,k}(.)$	0.3	0.1	0.4	0.2
Conditioning: $m_{c,k}(.)$	0	0	0	1
$m_{PCR5_\emptyset\text{-}Direct}(.)$	0.1811	0.1975	0.1597	0.4617

Table VII
BBA'S FOR EXAMPLE 1 AND PCR5$_\emptyset$-DIRECT RESULTS.

Example 2 (continued): Let's take back example 2 and examine the results given by PCR5-FC, PCR5-CF and PCR5-GC methods. The results are given in Table VIII (rounded when possible at the fourth decimal).

bba's\focal elem.	θ_1	θ_2	θ_3	$\theta_4 \equiv \emptyset$	$\theta_1 \cup \theta_2 \cup \theta_3$
Prior: $m_{1,k}(.)$	0.98	0	0.01	0.01	0
Prior: $m_{2,k}(.)$	0	0.98	0.01	0.01	0
Conditioning: $m_{c,k}(.)$	0	0	0	0	1
$m_{PCR5\text{-}FC}(.)$	0.49960202	0.49960202	0.00039798	0.00000016	0.00039782
$m_{PCR5\text{-}CF}(.)$	0.49970100	0.49970100	0.00049796	0.00000007	0.00009997
$m_{PCR5\text{-}GC}(.)$	0.32762253	0.32762253	0.00020045	0.00010047	0.34445402

Table VIII
EXAMPLE 2–A: RESULTS WITH PCR5-FC, PCR5-CF & PCR5-GC.

Example 3 (continued): Let's take back example 3–A with $a = b = 0.9$ and $1 - a = 1 - b = 0.1$. The results given by PCR5-FC, PCR5-CF and PCR5-GC are given in Table IX.

[5]i.e. FC, CF and GC approaches when using PCR5 rule of combination instead of DS rule.

bba's\focal elem.	$\theta_1 \equiv \emptyset$	θ_2	θ_3	$\theta_2 \cup \theta_3$
Prior: $m_{1,k}(.)$	0.9	0	0.1	0
Prior: $m_{2,k}(.)$	0.9	0.1	0	0
Conditioning: $m_{c,k}(.)$	0	0	0	1
$m_{PCR5\text{-}FC}(.)$	0.4791	0.0140	0.0140	0.4929
$m_{PCR5\text{-}CF}(.)$	0.4421	0.0605	0.0605	0.4369
$m_{PCR5\text{-}GC}(.)$	0.4435	0.0053	0.0053	0.5458

Table IX
EXAMPLE 3–A: RESULTS WITH PCR5-FC, PCR5-CF & PCR5-GC.

In summary, one has shown from very simple examples that original PCR5-based approaches cannot be used directly to solve the problem because they generate a non normalized bba (i.e. a bba with a positive value committed to $\emptyset$) and moreover the result depends the choice of the methods because of non associativity of PCR5 (or PCR6 as well). It is worth to note however that the results provided by the PCR5-based approaches commit different masses on non-empty focal lements contrariwise to DS-based approaches. In the next section we present new approaches for trying to solve the problem.

IV. EXTENDED PCR RULES

In this section we propose several ways to deal with the fusion of sources under non-existential integrity constraints since original PCR5 (or PCR6) cannot be applied directly. This is the main reason why new solutions have to be found and this is the main contribution of this paper.

A. Simple solution based on normalization

A simple solution would consist to use original PCR5 or direct PCR5$_\emptyset$ rules with a normalization final step (not included in original formulas) consisting in dividing all the mass of non-empty focal elements by $(1-m(\emptyset))$. This method can be applied only when $m(\emptyset) < 1$ of course. In example 1, one will get results given in Table X.

bba's \ focal elem.	θ_1	θ_2	θ_3	$\theta_1 \cup \theta_2$
Prior: $m_{1,k}(.)$	0.2	0.4	0.3	0.1
Prior: $m_{2,k}(.)$	0.3	0.1	0.4	0.2
Conditioning: $m_{c,k}(.)$	0	0	0	1
Normalized PCR5-FC bba	0.3988	0.4382	0	0.1630
Normalized PCR5-CF bba	0.3700	0.4010	0	0.2290
Normalized PCR5-GC bba	0.2155	0.2350	0	0.5495
Normalized PCR5$_\emptyset$ bba	0.2155	0.2350	0	0.5495
Normalized PCR6-GC bba	0.2133	0.2326	0	0.5541
Normalized PCR6$_\emptyset$ bba	0.2133	0.2326	0	0.5541

Table X
BBA'S FOR EXAMPLE 1 AND PCR5CR-BASED RESULTS AFTER NORMALIZATION.

Note that another result can be obtained from PCR5 and CF approach if one first normalizes the bba's $m_1^{PCR5}(.|\theta_1 \cup \theta_2) = m_{1,k} \oplus m_{c,k}(.)$, and $m_2^{PCR5}(.|\theta_1 \cup \theta_2) = m_{2,k} \oplus m_{c,k}(.)$ and then if we apply original PCR5 formula to combine them. We denote this method as PCR5-CnF (n standing for the position where the normalization step is done). In this case, one will get: $m_{PCR5\text{-}CnF}(\theta_1) = 0.391$, $m_{PCR5\text{-}CnF}(\theta_2) = 0.414$ and $m_{PCR5\text{-}CnF}(\theta_1 \cup \theta_2) = 0.195$ which is still different from previous results.

As one sees, all methods including a normalization step provide now different results and all agree that θ_2 corresponds to the hypothesis that has highest belief or plausibility. There is no ambiguity in the choice between θ_1 and θ_2 contrariwise to DS approach. The least uncertainty level is obtained with $PCR5 - FCn$ approach in this example.

B. A more efficient solution

Here we propose another way to solve the problem using new extended PCR5 fusion formulas denoted $PCR5a$, $PCR5b$ and $PCR5c$.

- The PCR5a fusion rule: $m_{PCR5a}(\emptyset) = 0$ and $\forall A \in G^\Theta \setminus \boldsymbol{\emptyset}$

$$m_{PCR5a}(A) = m_{12}(A) + \sum_{\substack{X \in G^\Theta \setminus \boldsymbol{\emptyset} \\ X \cap A = \emptyset}} [\frac{m_1(A)^2 m_2(X)}{m_1(A)+m_2(X)} + \frac{m_2(A)^2 m_1(X)}{m_2(A)+m_1(X)}] + \sum_{X \in \boldsymbol{\emptyset}} [m_1(A)m_2(X) + m_2(A)m_1(X)] + m_{12}(A) \cdot \frac{\sum_{X,Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y)}{\sum_{Z \in G^\Theta \setminus \boldsymbol{\emptyset}} m_{12}(Z)} \quad (8)$$

In PCR5a rule, one transfers the remaining conflicting masses proportionally with respect to the non-null masses resulted from the conjunctive rule.

- The PCR5b fusion rule: $m_{PCR5b}(\emptyset) = 0$ and$\forall A \in G^\Theta \setminus \boldsymbol{\emptyset}$

$$m_{PCR5b}(A) = m_{12}(A) + \sum_{\substack{X \in G^\Theta \setminus \boldsymbol{\emptyset} \\ X \cap A = \emptyset}} [\frac{m_1(A)^2 m_2(X)}{m_1(A)+m_2(X)} + \frac{m_2(A)^2 m_1(X)}{m_2(A)+m_1(X)}] + \sum_{X \in \boldsymbol{\emptyset}} [m_1(A)m_2(X) + m_2(A)m_1(X)] + \frac{\sum_{X,Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y)}{Card(\{Z | Z \in G^\Theta \setminus \boldsymbol{\emptyset}, m_{12}(Z) \neq 0\})} \quad (9)$$

In PCR5b rule, one uniformly transfers the remaining conflicting masses to all non-null masses resulted from the conjunctive rule.

- The PCR5c fusion rule: $m_{PCR5c}(\emptyset) = 0$ and$\forall A \in G^\Theta \setminus \boldsymbol{\emptyset}$

$$m_{PCR5c}(A) = m_{12}(A) + \sum_{\substack{X \in G^\Theta \setminus \boldsymbol{\emptyset} \\ X \cap A = \emptyset}} [\frac{m_1(A)^2 m_2(X)}{m_1(A)+m_2(X)} + \frac{m_2(A)^2 m_1(X)}{m_2(A)+m_1(X)}] + \sum_{X \in \boldsymbol{\emptyset}} [m_1(A)m_2(X) + m_2(A)m_1(X)] + \sum_{X,Y \in \boldsymbol{\emptyset}, A = I_t} m_1(X)m_2(Y) \quad (10)$$

In PCR5c rule, one transfers all remaining conflicting masses to the total ignorance I_t.

For PCR5a–PCR5c formulas (8)–(10) (and the next DSmHa–DSmHc, DSa–DSc formulas too) if a denominator is equal to zero, then its respective fraction is discarded, and $\sum_{X,Y\in\boldsymbol{\emptyset}} m_1(X)m_2(Y)$ is transferred to the total ignorance. In this case all three PCR5a–c coincide. Similarly, all DSmHa–c coincide, and all DSa–c coincide as well. In the above formulas, $m_{12}(A)$ is the mass obtained by the classical conjunctive consensus obtained by

$$m_{12}(A) = \sum_{\substack{X_1,X_2\in G^\Theta\\ X_1\cap X_2=A}} m_1(X_1)m_2(X_2) \tag{11}$$

G^Θ is the fusion space (power-set, hyper-power set or super-power set) depending on the underlying model chosen for the frame Θ and $\boldsymbol{\emptyset}$ is the set of all empty sets that occur in the fusion due to the integrity constraints.

Remarks:

1) If no constraint occurs (i.e. no focal element becoming empty), then all PCR5a–PCR5c formulas coincide with classical PCR5 fusion rule. All these extended PCR5 rules can be extended for combining $N > 2$ sources of evidences.
2) If all information about $m_1(.)$ and $m_2(.)$ and constraints (the sets which become empty in the fusion space) come simultaneously, we can use any of these three formulas.
3) PCR5a formula is the best. PCR5a and PCR5b formulas keep the specificity resulted after applying the conjunctive rule. PCR5c rule is less specific (and not recommended).
4) These formulas can be modified easily into PCR6a–PCR6c formulas by applying PCR6 redistribution principle to $m_1(.)$ and $m_2(.)$ and transferring the remaining mass committed to empty set as in PCR5a–PCR5c formulas.
5) In the case when the information comes sequentially, we combine it in that order.

PCR5a is better than PCR5b and PCR5c because PCR5a is more specific than both of them. Its bigger specificity is due to the fact that all masses of degenerated intersections $m_{12}(A \cap B)$, where $A = B = \emptyset$, are redistributed proportionally to all non-empty elements resulted from the conjunctive rule. While PCR5c redistributes this whole degenerated mass to the total ignorance (hence the lowest specificity among this group of three related formulas), and PCR5b uniformly splits this whole degenerated mass to all non-empty elements (but this means that PCR5b gives the same amount to each non-empty element, while PCRa gives more generated mass to the elements which have a bigger mass from the conjunctive rule).

Except Smets' fusion rule in TBM, we can adapt many fusion rules which are based on the conjunctive rule, including PCR6 too of course. We can adapt in three ways, corresponding to the previous PCR5a–PCR5c improved rules, replacing only the PCR5 first summation in all three formulas with DSmH summation S_2 [4], Vol.1. For example, the DSmHa, DSmHa and DSmHc extended rules are given by:

- DSmHa fusion rule: $m_{DSmHa}(\emptyset) = 0$ and $\forall A \in G^\Theta \setminus \boldsymbol{\emptyset}$

$$\begin{aligned} m_{DSmHa}(A) = m_{12}(A) + &\sum_{\substack{X\in G^\Theta\setminus\boldsymbol{\emptyset}\\ X\cap Y=\emptyset\\ X\cup Y=A}} m_1(X)m_2(Y)\\ &+ \sum_{X\in\boldsymbol{\emptyset}} [m_1(A)m_2(X) + m_2(A)m_1(X)]\\ &+ m_{12}(A)\cdot \frac{\sum_{X,Y\in\boldsymbol{\emptyset}} m_1(X)m_2(Y)}{\sum_{Z\in G^\Theta\setminus\boldsymbol{\emptyset}} m_{12}(Z)} \end{aligned} \tag{12}$$

- DSmHb fusion rule: $m_{DSmHb}(\emptyset) = 0$ and $\forall A \in G^\Theta \setminus \boldsymbol{\emptyset}$

$$\begin{aligned} m_{DSmHb}(A) = m_{12}(A) + &\sum_{\substack{X\in G^\Theta\setminus\boldsymbol{\emptyset}\\ X\cap Y=\emptyset\\ X\cup Y=A}} m_1(X)m_2(Y)\\ &+ \sum_{X\in\boldsymbol{\emptyset}} [m_1(A)m_2(X) + m_2(A)m_1(X)]\\ &+ \frac{\sum_{X,Y\in\boldsymbol{\emptyset}} m_1(X)m_2(Y)}{Card(\{Z|Z\in G^\Theta\setminus\boldsymbol{\emptyset}, m_{12}(Z)\neq 0\})} \end{aligned} \tag{13}$$

- DSmHc fusion rule: $m_{DSmHc}(\emptyset) = 0$ and $\forall A \in G^\Theta \setminus \boldsymbol{\emptyset}$

$$\begin{aligned} m_{DSmHc}(A) = m_{12}(A) + &\sum_{\substack{X\in G^\Theta\setminus\boldsymbol{\emptyset}\\ X\cap Y=\emptyset\\ X\cup Y=A}} m_1(X)m_2(Y)\\ &+ \sum_{X\in\boldsymbol{\emptyset}} [m_1(A)m_2(X) + m_2(A)m_1(X)]\\ &+ \sum_{X,Y\in\boldsymbol{\emptyset}, A=I_t} m_1(X)m_2(Y) \end{aligned} \tag{14}$$

DSmH classic rule [4] (Vol.1) redistributes the whole conflicting mass of the form $m_{12}(A \cap B)$, with $A = B = \emptyset$, resulted from the conjunctive rule, to the total ignorance; DSmH classic is equivalent (gives the same result) as DSmHc. But DSmHa and DSmHb are more specific than DSmHc (=DSmH classic) from exactly the same reason as explained before regarding the more specificity of PCR5a with respect to PCR5 and PCR5b. DSmHa is the most specific among all three DSmHa–DSmHc. That's why we need DSmHa. In addition, in the three formulas of DSmHa–DSmHc we can condensed the first two summations $(m_{12}(A) + \sum \ldots + \sum \ldots + \ldots)$ into one summation only, i.e under the first summation we can write $X, Y \in G^\Theta$ (so X, Y can be empty as well) and the second summation disappears (it is absorbed by the first).

Similarly for Dempster-Shafer's extended rule in the DSm way, we replace in all first three formulas the first PCR5 summation by

$$m_{12}(A)\cdot \frac{\sum_{\substack{X,Y\in\boldsymbol{\emptyset}\\ X\cap Y=\emptyset}} m_1(X)m_2(Y)}{\sum_{Z\in G^\Theta\setminus\boldsymbol{\emptyset}} m_{12}(Z)}$$

• DSa fusion rule: $m_{DSa}(\emptyset) = 0$ and $\forall A \in G^\Theta \setminus \boldsymbol{\emptyset}$

$$m_{DSa}(A) = m_{12}(A) + m_{12}(A) \cdot \frac{\sum_{\substack{X,Y \in \boldsymbol{\emptyset} \\ X \cap Y = \emptyset}} m_1(X)m_2(Y)}{\sum_{Z \in G^\Theta \setminus \boldsymbol{\emptyset}} m_{12}(Z)} + \sum_{X \in \boldsymbol{\emptyset}} [m_1(A)m_2(X) + m_2(A)m_1(X)] + m_{12}(A) \cdot \frac{\sum_{X,Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y)}{\sum_{Z \in G^\Theta \setminus \boldsymbol{\emptyset}} m_{12}(Z)} \quad (15)$$

• DSb fusion rule: $m_{DSb}(\emptyset) = 0$ and $\forall A \in G^\Theta \setminus \boldsymbol{\emptyset}$

$$m_{DSb}(A) = m_{12}(A) + m_{12}(A) \cdot \frac{\sum_{\substack{X,Y \in \boldsymbol{\emptyset} \\ X \cap Y = \emptyset}} m_1(X)m_2(Y)}{\sum_{Z \in G^\Theta \setminus \boldsymbol{\emptyset}} m_{12}(Z)} + \sum_{X \in \boldsymbol{\emptyset}} [m_1(A)m_2(X) + m_2(A)m_1(X)] + \frac{\sum_{X,Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y)}{Card(\{Z | Z \in G^\Theta \setminus \boldsymbol{\emptyset}, m_{12}(Z) \neq 0\})} \quad (16)$$

• DSc fusion rule: $m_{DSc}(\emptyset) = 0$ and $\forall A \in G^\Theta \setminus \boldsymbol{\emptyset}$

$$m_{DSc}(A) = m_{12}(A) + m_{12}(A) \cdot \frac{\sum_{\substack{X,Y \in \boldsymbol{\emptyset} \\ X \cap Y = \emptyset}} m_1(X)m_2(Y)}{\sum_{Z \in G^\Theta \setminus \boldsymbol{\emptyset}} m_{12}(Z)} + \sum_{X \in \boldsymbol{\emptyset}} [m_1(A)m_2(X) + m_2(A)m_1(X)] + \sum_{X,Y \in \boldsymbol{\emptyset}, A = I_t} m_1(X)m_2(Y) \quad (17)$$

Note that all these extended fusion rules are however not associative and therefore if one has several sources available at a given time to combine, the combination must be applied with all sources together to get optimal fusion result.

C. Example 1 (continued)

Let's examine in details the results obtained on Example 1 with all these extended fusion formulas. Because $\theta_3 = \emptyset$ and Shafer's model is assumed for Θ_k, the set of elements becoming empty is $\boldsymbol{\emptyset} = \{\theta_1 \cap \theta_2, \theta_1 \cap \theta_3, \theta_2 \cap \theta_3, (\theta_1 \cup \theta_2) \cap \theta_3, \theta_3\}$ and one has: $m_{12}(\theta_1 \cap \theta_2) = 0.14$, $m_{12}(\theta_1 \cap \theta_3) = 0.17$, $m_{12}(\theta_2 \cap \theta_3) = 0.19$, $m_{12}((\theta_1 \cup \theta_2) \cap \theta_3) = 0.10$, $m_{12}(\theta_3) = 0.12$. $m_{12}(\theta_1 \cap \theta_2 \in \boldsymbol{\emptyset}) = 0.14$ is redistributed back to θ_1 and θ_2 using PCR5 principle:

$$\frac{x_{1\theta_1}}{0.2} = \frac{y_{1\theta_2}}{0.1} = \frac{0.02}{0.3} = \frac{0.2}{3}, \qquad \frac{x_{2\theta_1}}{0.3} = \frac{y_{2\theta_2}}{0.4} = \frac{0.12}{0.7} = \frac{1.2}{7}$$

$$x_{1\theta_1} = 0.2\frac{0.2}{3} \approx 0.013 \qquad x_{2\theta_1} = 0.3\frac{1.2}{7} \approx 0.051$$

$$y_{1\theta_2} = 0.1\frac{0.2}{3} \approx 0.007 \qquad y_{2\theta_2} = 0.4\frac{1.2}{7} \approx 0.069$$

$m_{12}(\theta_1 \cap \theta_3 \in \boldsymbol{\emptyset}) = 0.17$ is all redistributed back to θ_1 since $\theta_3 = \emptyset$ (non-existential constraint). $m_{12}(\theta_2 \cap \theta_3 \in \boldsymbol{\emptyset}) = 0.19$ is all redistributed back to θ_2 since $\theta_3 = \emptyset$ (non-existential constraint). $m_{12}((\theta_1 \cup \theta_2) \cap \theta_3) \in \boldsymbol{\emptyset}) = 0.10$ is all redistributed back to $\theta_1 \cup \theta_2$ since $\theta_3 = \emptyset$ (non-existential constraint). While $m_{12}(\theta_3 = \emptyset) = 0.12$ is redistributed differently in each PCR5a, PCR5b and PCR5c formulas:

1) In PCR5a:
$$\frac{x_{\theta_1}}{0.13} = \frac{y_{\theta_2}}{0.13} = \frac{z_{\theta_1 \cup \theta_2}}{0.02} = \frac{0.12}{0.28} = \frac{3}{7}$$
whence $x_{\theta_1} = y_{\theta_2} = 0.13 \cdot 3/7 \approx 0.056$ and $z_{\theta_1 \cup \theta_2} = 0.02 \cdot 3/7 \approx 0.008$.
2) In PCR5b: $x_{\theta_1} = y_{\theta_2} = z_{\theta_1 \cup \theta_2} = 0.12/3 = 0.04$.
3) In PCR5c: $z_{\theta_1 \cup \theta_2} = 0.12$.

Finally, one then gets results shown in the Table XI. From these results, one sees that PCR5a rules provides the most specific result since the mass committed to the uncertainty is lowest with respect to what we get with PCR5b, PCR5c and other PCR5-based normalized conditioning rules given in the Table X. PCR5b is also a bit better (more specific) than PCR5-based normalized conditioning rules also. As we see and as expected from the theory PCR5c is less specific than PCR5a and PCR5b. If we use DSmHa-DSmHc fusion rules on this example, $m_{12}(\theta_1 \cap \theta_2 \in \boldsymbol{\emptyset}) = 0.14$ is all redistributed back to $\theta_1 \cup \theta_2$ using DSmH principle [4], Vol.1. The other conflicting masses are redistributed respectively in the same way in PCR5a–PCR5c rules. The same example for Dempster-Shafer's rule extended in DSm style: $m_{12}(\theta_1 \cap \theta_2 \in \boldsymbol{\emptyset}) = 0.14$ is all redistributed back to θ_1, θ_2, and $\theta_1 \cup \theta_2$ since they are non-empty proportionally with respect to their conjunctive rule masses 0.13, 0.13 and respectively 0.02:

$$\frac{x_{\theta_1}}{0.13} = \frac{y_{\theta_2}}{0.13} = \frac{z_{\theta_1 \cup \theta_2}}{0.02} = \frac{0.14}{0.28} = 0.5$$

whence $x_{\theta_1} = y_{\theta_2} = 0.13(0.5) = 0.065$ and $z_{\theta_1 \cup \theta_2} = 0.02(0.5) = 0.010$. The other conflicting masses are redistributed respectively in the same way as in PCR5a–PCR5c rules. The results obtained with DSmHa–DSmHc and DSa–DSc rules are given in Table XI. In this example, one sees that PCR5a is the most specific rule and in all cases, the rational decision to take will be θ_2 without ambiguity contrariwise to DSCR approach.

bba's \ focal elem.	θ_1	θ_2	$\theta_3 \equiv \emptyset$	$\theta_1 \cup \theta_2$
Prior: $m_{1,k}(.)$	0.2	0.4	0.3	0.1
Prior: $m_{2,k}(.)$	0.3	0.1	0.4	0.2
m_{PCR5a}	**0.420**	**0.452**	0	**0.128**
m_{PCR5b}	0.404	0.436	0	0.160
m_{PCR5c}	0.364	0.396	0	0.240
m_{DSa}	0.421	0.441	0	0.138
m_{DSb}	0.405	0.425	0	0.170
m_{DSc}	0.365	0.385	0	0.250
m_{DSmHa}	0.356	0.376	0	0.268
m_{DSmHb}	0.340	0.360	0	0.300
m_{DSmHc}	0.300	0.320	0	0.380

Table XI
EXAMPLE 1: PCR5A–C & DSA–C & DSMHA–C RESULTS.

V. EXAMPLES

Here we present the solution of Examples 2–A, 3A–3B obtained with our new extended PCR5a–PCR5c rules of combination for solving the fusion of bba's under non-existential constraints in degenerate cases.

A. Example 2 (continued)

Let's consider the Example 2–A and apply PCR5a–PCR5c formulas. Using the PCR5 principle, $m_{12}(\theta_1 \cap \theta_2) = 0.98 \cdot 0.98 = 0.9604$ is redistributed back to θ_1 and θ_2 with the same proportions $x_{\theta_1} = x_{\theta_2} = 0.4802$; $m_{12}(\theta_1 \cap \theta_3) = 0.98 \cdot 0.01 = 0.0098$ is redistributed to θ_1 and θ_3 with $x_{\theta_1} = 0.00970101$ and $x_{\theta_3} = 0.00009899$; $m_{12}(\theta_2 \cap \theta_3) = 0.0098$ is redistributed to θ_2 and θ_3 with $x_{\theta_2} = 0.00970101$ and $x_{\theta_3} = 0.00009899$; $m_{12}(\theta_1 \cap \theta_4) = 0.0098$ is transferred to θ_1 only since $\theta_4 \equiv \emptyset$; $m_{12}(\theta_2 \cap \theta_4) = 0.0098$ is transferred to θ_2 only since $\theta_4 \equiv \emptyset$; $m_{12}(\theta_3 \cap \theta_4) = 0.0002$ is transferred to θ_3 only since $\theta_4 \equiv \emptyset$; Since only $m_{12}(\theta_3) \neq 0$ with $\theta_3 \neq \emptyset$ the mass $m_{12}(\theta_4) = 0.0001$ is transferred to θ_3 in both PCR5a and PCR5b formulas. But in PCR5c rule, $m_{12}(\theta_4)$ is transferred to the total ignorance $I_t = \theta_1 \cup \theta_2 \cup \theta_3$. The final results obtained with PCR5a, PCR5b (same as with PCR5a for this example) and PCR5c are given in Table XII below.

focal el.\bba's	$m_{1,k}$	$m_{2,k}$	$m_{PCR5a,b}(.)$	$m_{PCR5c}(.)$
θ_1	0.98	0	0.49970101	0.49970101
θ_2	0	0.98	0.49970101	0.49970101
θ_3	0.01	0.01	$5.9798 \cdot 10^{-4}$	$4.9798 \cdot 10^{-4}$
$\theta_4 \equiv \emptyset$	0.01	0.01	0	0
$\theta_1 \cup \theta_2 \cup \theta_3$	0	0	0	0.0001

Table XII
EXAMPLE 2–A: RESULTS WITH PCR5A–C

B. Example 3 (continued)

In Example 3–A, θ_1 becomes empty and therefore: $m_{12}(\theta_1 \cap \theta_2) = a(1-b)$ goes to θ_2, $m_{12}(\theta_1 \cap \theta_3) = b(1-c)$ goes to θ_3 and $m_{12}(\theta_2 \cap \theta_3) = 1 - a - b + ab$ is split between θ_2 and θ_3 proportionally to $1-b$ and $1-a$ respectively:

$$\frac{x_{\theta_2}}{1-b} = \frac{x_{\theta_3}}{1-a} = \frac{1-a-b+ab}{2-a-b}$$

Therefore, one gets finally

$$x_{\theta_2} = \frac{1-a-2b+ab+b^2-ab^2}{2-a-b}$$

$$x_{\theta_3} = \frac{1-2a-b+2ab+a^2-a^2b}{2-a-b}$$

Since $\theta_1 = \emptyset$, $m_{12}(\theta_1) = ab$ is redistributed to $\theta_2 \cup \theta_3$ in PCR5a–PCR5c formulas because all $m_{12}(X) = 0$ for $X \neq \emptyset$. The final results are given in Table XIII depending on the values of parameters a and b

Cases	$a \neq 1, b \neq 1$	$a = b = 1$
focal elem. \ bba's	$m_{PCR5a,b,c}(.)$	$m_{PCR5a,b,c}(.)$
θ_1	0	0
θ_2	$a(1-b) + \frac{(1-b)(1-a-b+ab}{2-a-b}$	0
θ_3	$b(1-a) + \frac{(1-a)(1-a-b+ab}{2-a-b}$	0
$\theta_2 \cup \theta_3$	ab	1

Table XIII
EXAMPLE 3–A: RESULTS WITH PCR5A–PCR5C

Extended PCR5 rules for Example 3–B give same results as for Example 3–A, where we replace θ_1 by $\theta_1 \cup \theta_2$, θ_2 by θ_3, and θ_3 by θ_4, and $\theta_2 \cup \theta_3$ by $\theta_3 \cup \theta_4$. If we take by example, $a = b = 0.9$ and $1 - a = 1 - b = 0.1$ in examples 3–A and 3–B then we will finally obtain for Examples 3–A & 3–B:

bba's\focal elem.	θ_1	θ_2	θ_3	$\theta_2 \cup \theta_3$
$m_{PCR5a-c}(.)$	0	0.095	0.095	0.810
$m_{DSmHa-c}(.)$	0	0.090	0.090	0.820
$m_{DSa-c}(.)$	0	0.090	0.090	0.820

Table XIV
EXAMPLE 3–A: RESULTS WITH $a = b = 0.9$ AND $1 - a = 1 - b = 0.1$

bba's\focal elem.	$\theta_1 \cup \theta_2$	θ_3	θ_4	$\theta_3 \cup \theta_4$
$m_{PCR5a-c}(.)$	0	0.095	0.095	0.810
$m_{DSmHa-c}(.)$	0	0.090	0.090	0.820
$m_{DSa-c}(.)$	0	0.090	0.090	0.820

Table XV
EXAMPLE 3–B: RESULTS WITH $a = b = 0.9$ AND $1 - a = 1 - b = 0.1$

Dempster-Shafer's rule cannot be applied in these examples since it gives 0/0.

VI. CONCLUSIONS

In this paper we extend the classical PCR5 and DSmH combination fusion rules to two ensembles of new fusion rule formulas, PCR5a–PCR5c and respectively DSmHa–DSmHc, in order to be able to take into consideration the non-existence constraints (i.e. when some sets become empty) that may occur during a dynamic fusion. Further, we show that the same DSmT extension procedure applied to PCR5 and DSmH can be applied to Dempster's rule and other rules as well. We provide several examples with these PCR5a–PCR5c and DSmHa–DSmHc rules, and also with Dempster-Shafer conditioning rule (DSCR). We have presented some classes of counter-examples to DSCR. If we have two sources, what to do first Fusion and then Conditioning, or Conditioning and then Fusion? A simple answer would be to do them in the order we receive the information. But in the case we receive all of them simultaneously, it is better to use these new extended rules depending on the specificity quality we want to get, PCR5a being the most specific rule.

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