Yang-Mills Field from Quaternion Space Geometry, and Its Klein-Gordon Representation

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Analysis of covariant derivatives of vectors in quaternion (Q-) spaces performed using Q-unit spinor-splitting technique and use of SL(2C)-invariance of quaternion multiplication reveals close connexion of Q-geometry objects and Yang-Mills (YM) field principle characteristics. In particular, it is shown that Q-connexion (with quaternion non-metricity) and related curvature of 4 dimensional (4D) space-times with 3D Q-space sections are formally equivalent to respectively YM-field potential and strength, traditionally emerging from the minimal action assumption. Plausible links between YM field equation and Klein-Gordon equation, in particular via its known isomorphism with Duffin-Kemmer equation, are also discussed.

1 Introduction

Traditionally YM field is treated as a gauge, "*auxiliary*", field involved to compensate local transformations of a 'main' (e.g. spinor) field to keep invariance of respective action functional. Anyway there are a number of works where YM-field features are found related to some geometric properties of space-times of different types, mainly in connexion with contemporary gravity theories.

Thus in paper [1] violation of $\mathrm{SO}(3,1)$-covariance in gauge gravitation theory caused by distinguishing time direction from normal space-like hyper-surfaces is regarded as spontaneous symmetry violation analogous to introduction of mass in YM theory. Paper [2] shows a generic approach to formulation of a physical field evolution based on description of differential manifold and its mapping onto "model" spaces defined by characteristic groups; the group choice leads to gravity or YM theory equations. Furthermore it can be shown [2b] that it is possible to describe altogether gravitation in a space with torsion, and electroweak interactions on 4D real spacetime C^2, so we have in usual spacetime with torsion a unified theory (modulo the non treatment of the strong forces).

Somewhat different approach is suggested in paper [3] where gauge potentials and tensions are related respectively to connexion and curvature of principle bundle, whose base and gauge group choice allows arriving either to YM or to gravitation theory. Paper [4] dealing with gravity in Riemann-Cartan space and Lagrangian quadratic in connexion and curvature shows possibility to interpret connexion as a mediator of YM interaction.

In paper [5] a unified theory of gravity and electroweak forces is built with Lagrangian as a scalar curvature of space-time with torsion; if trace and axial part of the torsion vanish the Lagrangian is shown to separate into Gilbert and YM parts. Regardless of somehow artificial character of used models, these observations nonetheless hint that there may exist a deep link between supposedly really physical object, YM field and pure math constructions. A surprising analogy between main characteristics of YM field and mathematical objects is found hidden within geometry induced by quaternion (Q-) numbers.

In this regard, the role played by Yang-Mills field cannot be overemphasized, in particular from the viewpoint of the Standard Model of elementary particles. While there are a number of attempts for describing the Standard Model of hadrons and leptons from the viewpoint of classical electromagnetic Maxwell equations [6, 7], nonetheless this question remains an open problem. An alternative route toward achieving this goal is by using quaternion number, as described in the present paper. In fact, in Ref. [7] a somewhat similar approach with ours has been described, i.e. the generalized Cauchy-Riemann equations contain 2-spinor and C-gauge structures, and their integrability conditions take the form of Maxwell and Yang-Mills equations.

It is long ago noticed that Q-math (algebra, calculus and related geometry) naturally comprise many features attributed to physical systems and laws. It is known that quaternions describe three "imaginary" Q-units as unit vectors directing axes of a Cartesian system of coordinates (it was initially developed to represent subsequent telescope motions in astronomical observation). Maxwell used the fact to write his equations in the most convenient Q-form. Decades later Fueter discovered a formidable coincidence: a pure math Cauchy-Riemann type condition endowing functions of Q-variable with analytical properties turned out to be identical in shape to vacuum equations of electrodynamics [9].

Later on other surprising Q-math — physics coincidences were found. Among them: "automatic" appearance of Pauli magnetic field-spin term with Bohr magneton as a coefficient when Hamiltonian for charged quantum mechanical particle was built with the help of Q-based metric [10]; possibility to endow "imaginary" vector Q-units with properties of not only stationary but movable triad of Cartan type and use it for a very simple description of Newtonian mechanics in rotating frame of reference [11]; discovery of inherited in Q-math variant of relativity theory permitting to describe motion of non-inertial frames [12]. Preliminary study shows that YM field components are also formally present in Q-math.

In Section 2 notion of Q-space is given in necessary detail. Section 3 discussed neat analogy between Q-geometric objects and YM field potential and strength. In Section 4 YM field and Klein-Gordon correspondence is discussed. Concluding remarks can be found in Section 5.

Part of our motivation for writing this paper was to explicate the hidden electromagnetic field origin of YM fields. It is known that the Standard Model of elementary particles lack systematic description for the mechanism of quark charges. (Let alone the question of whether quarks do exist or they are mere algebraic tools, as Heisenberg once puts forth: *If quarks exist, then we have redefined the word "exist"*.) On the other side, as described above, Maxwell described his theory in quaternionic language, therefore it seems natural to ask whether it is possible to find neat link between quaternion language and YM-fields, and by doing so provide one step toward describing mechanism behind quark charges.

Further experimental observation is of course recommended in order to verify or refute our propositions as described herein.

2 Quaternion spaces

Detailed description of Q-space is given in [13]; shortly but with necessary strictness its notion can be presented as following.

Let U_N be a manifold, a geometric object consisting of points $M \in U_N$ each reciprocally and uniquely corresponding to a set of N numbers-coordinates $\{y^A\} : M \leftrightarrow \{y^A\}$, $(A=1,2\ldots N)$. Also let the sets of coordinates be transformed so that the map becomes a homeomorphism of a class C_k. It is known that U_N may be endowed with a proper tangent manifold T_N described by sets of orthogonal unite vectors $\mathbf{e}_{(A)}$ generating in T_N families of coordinate lines $M \to \{X^{(A)}\}$, indices in brackets being numbers of frames' vectors. Differentials of coordinates in U_N and T_N are tied as $dX^{(A)} = g^{(A)}_B dy^B$, with Lamé coefficients $g^{(A)}_B$, functions of y^A, so that $X^{(A)}$ are generally non-holonomic. Irrespectively of properties of U_N each its point may be attached to the origin of a frame, in particular presented by "imaginary" Q-units $\mathbf{q}_k$, this attachment accompanied by a rule tying values of coordinates of this point with the triad orientation $M \leftrightarrow \{y^A, \Phi_\xi\}$. All triads $\{\mathbf{q}_k\}$ so defined on U_N form a sort of "tangent" manifold $T(U,\mathbf{q})$, (really tangent only for the base U_3). Due to presence of frame vectors $\mathbf{q}_k(y)$ existence of metric and at least proper (quaternionic) connexion $\omega_{jkn} = -\omega_{jnk}$, $\partial_j \mathbf{q}_k = \omega_{jkn}\mathbf{q}_n$, is implied, hence one can tell of $T(U,\mathbf{q})$ as of a Q-tangent space on the base U_N. Coordinates x_k defined along triad vectors $\mathbf{q}_k$ in $T(U,\mathbf{q})$ are tied with non-holonomic coordinates $X^{(A)}$ in proper tangent space T_N by the transformation $dx_k \equiv h_{k(A)} dX^{(A)}$ with $h_{k(A)}$ being locally depending matrices (and generally not square) of relative $\mathbf{e}_{(A)} \leftrightarrow \mathbf{q}_k$ rotation. Consider a special case of unification $U \oplus T(U,\mathbf{q})$ with 3-dimensional base space $U = \mathbf{U}_3$. Moreover, let quaternion specificity of T_3 reflects property of the base itself, i.e. metric structure of $\mathbf{U}_3$ inevitably requires involvement of Q-triads to initiate Cartesian coordinates in its tangent space. Such 3-dimensional space generating sets of tangent quaternionic frames in each its point is named here "quaternion space" (or simply Q-space). Main distinguishing feature of a Q-space is non-symmetric form of its metric tensor* $\mathbf{g}_{kn} \equiv \mathbf{q}_k \mathbf{q}_n = -\delta_{kn} + \varepsilon_{knj}\mathbf{q}_j$ being in fact multiplication rule of "imaginary" Q-units. It is easy to understand that all tangent spaces constructed on arbitrary bases as designed above are Q-spaces themselves. In most general case a Q-space can be treated as a space of affine connexion $\Omega_{jkn} = \Gamma_{jkn} + Q_{jkn} + S_{jkn} + \omega_{jnk} + \sigma_{jkn}$ comprising respectively Riemann connexion Γ_{jkn}, Cartan contorsion Q_{jkn}, segmentary curvature (or ordinary non-metricity) S_{jkn}, Q-connexion ω_{jnk}, and Q-non-metricity σ_{jkn}; curvature tensor is given by standard expression $R_{knij} = \partial_i \Omega_{j\,kn} - \partial_j \Omega_{i\,kn} + \Omega_{i\,km}\Omega_{j\,mn} - \Omega_{j\,nm}\Omega_{i\,mk}$. Presence or vanishing of different parts of connexion or curvature results in multiple variants of Q-spaces classification [13]. Further on only Q-spaces with pure quaternionic characteristics (Q-connexion and Q-non-metricity) will be considered.

3 Yang-Mills field from Q-space geometry

Usually Yang-Mills field $A_{B\mu}$ is introduced as a gauge field in procedure of localized transformations of certain field, e.g. spinor field [14, 15]

$$\psi_a \to U(y^\beta)\,\psi_a\,. \tag{1}$$

If in the Lagrangian of the field partial derivative of ψ_a is changed to "covariant" one

$$\partial_\beta \to D_\beta \equiv \partial_\beta - gA_\beta\,, \tag{2}$$

*Latin indices are 3D, Greek indices are 4D; δ_{kn}, ε_{knj} are Kronecker and Levi-Civita symbols; summation convention is valid.

$$A_\beta \equiv i A_{C\,\beta} \mathbf{T}_C\,, \tag{3}$$

where g is a real constant (parameter of the model), $\mathbf{T}_C$ are traceless matrices (Lie-group generators) commuting as

$$[\mathbf{T}_B,\, \mathbf{T}_C] = i f_{BCD} \mathbf{T}_D \tag{4}$$

with structure constants f_{BCD}, then

$$D_\beta U \equiv (\partial_\beta - g A_\beta)\, U = 0\,, \tag{5}$$

and the Lagrangian keeps invariant under the transformations (1). The theory becomes "self consistent" if the gauge field terms are added to Lagrangian

$$L_{YM} \sim F^{\alpha\beta} F_{\alpha\beta}\,, \tag{6}$$

$$F_{\alpha\beta} \equiv F_{C\,\alpha\beta}\, \mathbf{T}_C\,. \tag{7}$$

The gauge field intensity $F^{\mu\nu}_B$ expressed through potentials $A_{B\mu}$ and structure constants as

$$F_{C\,\alpha\beta} = \partial_\alpha A_{C\,\beta} - \partial_\beta A_{C\,\alpha} + f_{CDE}\, A_{D\,\alpha} A_{E\,\beta}\,. \tag{8}$$

Vacuum equations of the gauge field

$$\partial_\alpha F^{\alpha\beta} + \left[A_\alpha, F^{\alpha\beta}\right] = 0 \tag{9}$$

are result of variation procedure of action built from Lagrangian (6).

Group Lie, e.g. SU(2) generators in particular can be represented by "imaginary" quaternion units given by e.g. traceless 2×2-matrices in special representation (Pauli-type) $i\mathbf{T}_B \rightarrow \mathbf{q}_{\tilde{k}} = -i\sigma_k$ (σ_k are Pauli matrices),

Then the structure constants are Levi-Civita tensor components $f_{BCD} \rightarrow \varepsilon_{knm}$, and expressions for potential and intensity (strength) of the gauge field are written as:

$$A_\beta = g\,\frac{1}{2}\, A_{\tilde{k}\,\beta}\, \mathbf{q}_{\tilde{k}}\,, \tag{10}$$

$$F_{k\,\alpha\beta} = \partial_\alpha A_{k\,\beta} - \partial_\beta A_{k\,\alpha} + \varepsilon_{kmn}\, A_{m\,\alpha} A_{n\,\beta}\,. \tag{11}$$

It is worthnoting that this conventional method of introduction of a Yang-Mills field type essentially exploits *heuristic base* of theoretical physics, first of all the postulate of minimal action and formalism of Lagrangian functions construction. But since description of the field optionally uses quaternion units one can assume that some of the above relations are appropriate for Q-spaces theory and may have geometric analogues. To verify this assumption we will use an example of 4D space-time model with 3D spatial quaternion section.

Begin with the problem of 4D space-time with 3D spatial section in the form of Q-space containing only one geometric object: proper quaternion connexion. Q-covariant derivative of the basic (frame) vectors $\mathbf{q}_m$ identically vanish in this space:

$$\tilde{D}_\alpha\, \mathbf{q}_k \equiv (\delta_{mk}\, \partial_\alpha + \omega_{\alpha\, mk})\, \mathbf{q}_m = 0\,. \tag{12}$$

This equation is in fact equivalent to definition of the proper connexion $\omega_{\alpha\, mk}$. If a transformation of Q-units is given by spinor group (leaving quaternion multiplication rule invariant)

$$\mathbf{q}_k = U(y)\, \mathbf{q}_{\tilde{k}}\, U^{-1}(y) \tag{13}$$

($\mathbf{q}_{\tilde{k}}$ are constants here) then Eq. (12) yields

$$\partial_\alpha U\, \mathbf{q}_{\tilde{k}}\, U^{-1} + U\, \mathbf{q}_{\tilde{k}}\, \partial_\alpha U^{-1} = \omega_{\alpha kn}\, U\, \mathbf{q}_{\tilde{n}}\, U^{-1}. \tag{14}$$

But one can easily verify that each "imaginary" Q-unit $\mathbf{q}_{\tilde{k}}$ can be always represented in the form of tensor product of its eigen-functions (EF) $\psi_{(\tilde{k})}$, $\varphi_{(\tilde{k})}$ (no summation convention for indices in brackets):

$$\mathbf{q}_{\tilde{k}}\, \psi_{(\tilde{k})} = \pm i \psi_{(\tilde{k})}, \qquad \varphi_{(\tilde{k})} \mathbf{q}_{\tilde{k}} = \pm i \varphi_{(\tilde{k})} \tag{15}$$

having spinor structure (here only EF with positive parity (with sign +) are shown)

$$\mathbf{q}_{\tilde{k}} = i\,(2\psi_{(\tilde{k})} \varphi_{(\tilde{k})} - 1); \tag{16}$$

this means that left-hand-side (lhs) of Eq. (14) can be equivalently rewritten in the form

$$\begin{aligned} &\frac{1}{2}\,(\partial_\alpha U\, \mathbf{q}_{\tilde{k}}\, U^{-1} + U\, \mathbf{q}_{\tilde{k}}\, \partial_\alpha U^{-1}) = \\ &\quad = (\partial_\alpha U\, \psi_{(\tilde{k})})\, \varphi_{(\tilde{k})} U^{-1} + U\, \psi_{(\tilde{k})}\, (\varphi_{(\tilde{k})} \partial_\alpha U^{-1}) \end{aligned} \tag{17}$$

which strongly resembles use of Eq. (1) for transformations of spinor functions.

Here we for the first time underline a remarkable fact: *form-invariance of multiplication rule of Q-units under their spinor transformations gives expressions similar to those conventionally used to initiate introduction of gauge fields of Yang-Mills type.*

Now in order to determine mathematical analogues of these "physical fields", we will analyze in more details Eq. (14). Its multiplication (from the right) by combination $U\, \mathbf{q}_{\tilde{k}}$ with contraction by index $\tilde{k}$ leads to the expression

$$-3\, \partial_\alpha U\ + U\, \mathbf{q}_{\tilde{k}}\, \partial_\alpha U^{-1} U \mathbf{q}_{\tilde{k}} = \omega_{\alpha kn}\, U\, \mathbf{q}_{\tilde{n}}\, \mathbf{q}_{\tilde{k}}\,. \tag{18}$$

This matrix equation can be simplified with the help of the always possible development of transformation matrices

$$U \equiv a\ + b_k\, \mathbf{q}_{\tilde{k}}\,, \qquad U^{-1} = a\ - b_k\, \mathbf{q}_{\tilde{k}}\,, \tag{19}$$

$$U U^{-1} = a^2\ + b_k\, b_k = 1\,, \tag{20}$$

where a, b_k are real scalar and 3D-vector functions, $\mathbf{q}_{\tilde{k}}$ are Q-units in special (Pauli-type) representation. Using Eqs. (19), the second term in lhs of Eq. (18) after some algebra is reduced to remarkably simple expression

$$\begin{aligned} &U\, \mathbf{q}_{\tilde{k}}\, \partial_\alpha U^{-1} U \mathbf{q}_{\tilde{k}} = \\ &= \ (a + b_n \mathbf{q}_{\tilde{n}})\, \mathbf{q}_{\tilde{k}} (\partial_\alpha a - \partial_\alpha b_m \mathbf{q}_{\tilde{m}})\, (a + b_l\, \mathbf{q}_{\tilde{l}})\, \mathbf{q}_{\tilde{k}} = \\ &= \ \partial_\alpha (a + b_n \mathbf{q}_{\tilde{n}}) = -\,\partial_\alpha U \end{aligned} \tag{21}$$

so that altogether lhs of Eq. (18) comprises $-4\,\partial_\alpha U$ while right-hand-side (rhs) is

$$\omega_{\alpha kn} U\, \mathbf{q}_{\tilde{n}} \mathbf{q}_{\tilde{k}} = -\varepsilon_{knm}\omega_{\alpha kn} U\, \mathbf{q}_{\tilde{m}}\,; \tag{22}$$

then Eq. (18) yields

$$\partial_\alpha U - \frac{1}{4}\,\varepsilon_{knm}\omega_{\alpha kn} U\, \mathbf{q}_{\tilde{m}} = 0\,. \tag{23}$$

If now one makes the following notations

$$A_{k\,\alpha} \equiv \frac{1}{2}\,\varepsilon_{knm}\omega_{\alpha kn}\,, \tag{24}$$

$$A_{\alpha} \equiv \frac{1}{2}\,A_n \mathbf{q}_{\tilde{n}}\,, \tag{25}$$

then notation (25) exactly coincides with the definition (10) (provided $g = 1$), and Eq. (23) turns out equivalent to Eq. (5)

$$U\overleftarrow{D}_\alpha \equiv U(\overleftarrow{\partial}_\alpha - A_\alpha) = 0\,. \tag{26}$$

Expression for "covariant derivative" of inverse matrix follows from the identity:

$$\partial_\alpha U\, U^{-1} = -U\partial_\alpha U^{-1}. \tag{27}$$

Using Eq. (23) one easily computes

$$-\partial_\alpha U^{-1} - \frac{1}{4}\,\varepsilon_{knm}\omega_{\alpha kn}\, \mathbf{q}_{\tilde{m}} U^{-1} = 0 \tag{28}$$

or

$$D_\alpha U^{-1} \equiv (\partial_\alpha + A_\alpha)\, U^{-1} = 0\,. \tag{29}$$

Direction of action of the derivative operator is not essential here, since the substitution $U^{-1} \to U$ и $U \to U^{-1}$ is always possible, and then Eq. (29) exactly coincides with Eq. (5).

Now let us summarize first results. We have a remarkable fact: form-invariance of Q-multiplication has as a corollary "covariant constancy" of matrices of spinor transformations of vector Q-units; moreover one notes that proper Q-connexion (contracted in skew indices by Levi-Civita tensor) plays the role of "gauge potential" of some Yang-Mills-type field. By the way the Q-connexion is easily expressed from Eq. (24)

$$\omega_{\alpha kn} = \varepsilon_{mkn} A_{m\,\alpha}\,. \tag{30}$$

Using Eq. (25) one finds expression for the gauge field intensity (11) (contracted by Levi-Civita tensor for convenience) through Q-connexion

$$\begin{aligned} &\varepsilon_{kmn}F_{k\,\alpha\beta} = \\ &= \varepsilon_{kmn}(\partial_\alpha A_{k\,\beta} - \partial_\beta A_{k\,\alpha}) + \varepsilon_{kmn}\varepsilon_{mlj}A_{l\,\alpha}A_{j\,\beta} = \\ &= \partial_\alpha\omega_{\beta\,mn} - \partial_\beta\omega_{\alpha\,mn} + A_{m\,\alpha}A_{n\,\beta} - A_{m\beta}A_{n\alpha}\,. \end{aligned} \tag{31}$$

If identically vanishing sum

$$-\delta_{mn}A_{j\,\alpha}A_{j\,\beta} + \delta_{mn}A_{j\beta}A_{j\alpha} = 0 \tag{32}$$

is added to rhs of (31) then all quadratic terms in the right hand side can be given in the form

$$\begin{aligned} &A_{m\,\alpha}A_{n\,\beta} - A_{m\beta}A_{n\,\alpha} - \delta_{mn}A_{j\,\alpha}A_{j\,\beta} + \delta_{mn}A_{j\beta}A_{j\alpha} = \\ &= (\delta_{mp}\delta_{qn} - \delta_{mn}\delta_{qp})(A_{p\,\alpha}A_{q\,\beta} - A_{p\beta}A_{q\,\alpha}) = \\ &= \varepsilon_{kmq}\varepsilon_{kpn}(A_{p\,\alpha}A_{q\,\beta} - A_{p\beta}A_{q\,\alpha}) = \\ &= -\omega_{\alpha\,kn}\,\omega_{\beta\,km} + \omega_{\beta\,kn}A_{\alpha\,km}\,. \end{aligned}$$

Substitution of the last expression into Eq. (31) accompanied with new notation

$$R_{mn\,\alpha\beta} \equiv \varepsilon_{kmn}F_{k\,\alpha\beta} \tag{33}$$

leads to well-known formula:

$$\begin{aligned} R_{mn\,\alpha\beta} = \partial_\alpha\omega_{\beta\,mn} - \partial_\beta\omega_{\alpha\,mn} + \\ + \omega_{\alpha\,nk}\,\omega_{\beta\,km} - \omega_{\beta\,nk}\,\omega_{\alpha\,km}\,. \end{aligned} \tag{34}$$

This is nothing else but curvature tensor of Q-space built out of proper Q-connexion components (in their turn being functions of 4D coordinates). By other words, Yang-Mills field strength is mathematically (geometrically) identical to quaternion space curvature tensor. But in the considered case of Q-space comprising only proper Q-connexion, all components of the curvature tensor are identically zero. So Yang-Mills field in this case has potential but no intensity.

The picture absolutely changes for the case of quaternion space with Q-connexion containing a proper part $\omega_{\beta\,kn}$ and also Q-non-metricity $\sigma_{\beta\,kn}$

$$\Omega_{\beta\,kn}(y^\alpha) = \omega_{\beta\,kn} + \sigma_{\beta\,kn} \tag{35}$$

so that Q-covariant derivative of a unite Q-vector with connexion (35) does not vanish, its result is namely the Q-non-metricity

$$\hat{D}_\alpha \mathbf{q}_k \equiv (\delta_{mk}\partial_\alpha + \Omega_{\alpha\,mk})\,\mathbf{q}_m = \sigma_{\alpha\,mk}\,\mathbf{q}_k\,. \tag{36}$$

For this case "covariant derivatives" of transformation spinor matrices may be defined analogously to previous case definitions (26) and (29)

$$U\hat{\overleftarrow{D}}_\alpha \equiv \hat{U}(\overleftarrow{\partial}_\alpha - \hat{A}_\alpha), \quad \hat{D}_\alpha U^{-1} \equiv (\partial_\alpha + \hat{A}_\alpha)U\,. \tag{37}$$

But here the "gauge field" is built from Q-connexion (35)

$$\hat{A}_{k\,\alpha} \equiv \frac{1}{2}\,\varepsilon_{knm}\Omega_{\alpha kn}, \qquad \hat{A}_{\alpha} \equiv \frac{1}{2}\,\hat{A}_n \mathbf{q}_{\tilde{n}}\,. \tag{38}$$

It is not difficult to verify whether the definitions (37) are consistent with non-metricity condition (36). Action of the "covariant derivatives" (37) onto a spinor-transformed unite Q-vector

$$\begin{aligned} &\hat{D}_\alpha\,\mathbf{q}_k \to (\hat{D}_\alpha U)\,\mathbf{q}_{\tilde{k}}\,\partial_\alpha U^{-1} + U\,\mathbf{q}_{\tilde{k}}\,(\hat{D}_\alpha U^{-1}) = \\ &= \left(U\overleftarrow{D}_\alpha - \frac{1}{4}\,\varepsilon_{jnm}\Omega_{\alpha\,nm}U\mathbf{q}_{\tilde{j}}\,\mathbf{q}_{\tilde{k}}\right)U^{-1} + \\ &+ U\,\mathbf{q}_{\tilde{k}}\left(D_\alpha U^{-1} + \frac{1}{4}\,\varepsilon_{jnm}\Omega_{\alpha\,nm}\,\mathbf{q}_{\tilde{j}}\,U^{-1}\right) \end{aligned}$$

together with Eqs. (26) and (29) demand:

$$U\overleftarrow{D}_{\alpha} = D_{\alpha} U^{-1} = 0 \tag{39}$$

leads to the expected results

$$\hat{D}_{\alpha}\mathbf{q}_k \to \frac{1}{2}\,\varepsilon_{jnm}\sigma_{\alpha\,nm} U \varepsilon_{jkl}\,\mathbf{q}_{\tilde{l}}\,U^{-1} = \\ = \sigma_{\alpha\,kl}\,U\,\mathbf{q}_{\tilde{l}}\,U^{-1} = \sigma_{\alpha\,kl}\,\mathbf{q}_l$$

i.e. "gauge covariant" derivative of any Q-unit results in Q-non-metricity in full accordance with Eq. (36).

Now find curvature tensor components in this Q-space; it is more convenient to calculate them using differential forms. Given Q-connexion 1-form

$$\Omega_{\,kn} = \Omega_{\beta\,kn} dy^{\beta} \tag{40}$$

from the second equation of structure

$$\frac{1}{2}\,\hat{R}_{kn\alpha\beta}\,dy^{\alpha}\wedge dy^{\beta} = d\Omega_{kn} + \Omega_{km}\wedge\Omega_{mn} \tag{41}$$

one gets the curvature tensor component

$$\hat{R}_{kn\alpha\beta} = \partial_{\alpha}\Omega_{\beta\,kn} - \partial_{\beta}\Omega_{\alpha\,kn} + \\ + \Omega_{\alpha\,km}\Omega_{\beta\,mn} - \Omega_{\alpha\,nm}\Omega_{\beta\,mk} \tag{42}$$

quite analogously to Eq. (34). Skew-symmetry in 3D indices allows representing the curvature part of 3D Q-section as 3D axial vector

$$\hat{F}_{m\,\alpha\beta} \equiv \frac{1}{2}\,\varepsilon_{knm}\hat{R}_{kn\alpha\beta} \tag{43}$$

and using Eq. (38) one readily rewrites definition (43) in the form

$$\hat{F}_{m\,\alpha\beta} = \partial_{\alpha}\hat{A}_{m\,\beta} - \partial_{\beta}\hat{A}_{m\,\alpha} + \varepsilon_{knm}\hat{A}_{k\,\alpha}\hat{A}_{n\,\beta} \tag{44}$$

which exactly coincides with conventional definition (11). QED.

4 Klein-Gordon representation of Yang-Mills field

In the meantime, it is perhaps more interesting to note here that such a neat linkage between Yang-Mills field and quaternion numbers is already known, in particular using Klein-Gordon representation [16]. In turn, this neat correspondence between Yang-Mills field and Klein-Gordon representation can be expected, because both can be described in terms of $SU(2)$ theory [17]. In this regards, quaternion decomposition of $SU(2)$ Yang-Mills field has been discussed in [17], albeit it implies a different metric from what is described herein:

$$ds^2 = d\alpha_1^2 + \sin^2\alpha_1\,d\beta_1^2 + d\alpha_2^2 + \sin^2\alpha_2\,d\beta_2^2\,. \tag{45}$$

However, the $O(3)$ non-linear sigma model appearing in the decomposition [17] looks quite similar (or related) to the Quaternion relativity theory (as described in the Introduction, there could be neat link between Q-relativity and $SO(3,1)$).

Furthermore, sometime ago it has been shown that four-dimensional coordinates may be combined into a quaternion, and this could be useful in describing supersymmetric extension of Yang-Mills field [18]. This plausible neat link between Klein-Gordon equation, Duffin-Kemmer equation and Yang-Mills field via quaternion number may be found useful, because both Duffin-Kemmer equation and Yang-Mills field play some kind of significant role in description of standard model of particles [16].

In this regards, it has been argued recently that one can derive standard model using Klein-Gordon equation, in particular using Yukawa method, without having to introduce a Higgs mass [19, 20]. Considering a notorious fact that Higgs particle has not been observed despite more than three decades of extensive experiments, it seems to suggest that an alternative route to standard model of particles using (quaternion) Klein-Gordon deserves further consideration.

In this section we will discuss a number of approaches by different authors to describe the (quaternion) extension of Klein-Gordon equation and its implications. First we will review quaternion quantum mechanics of Adler. And then we discuss how Klein-Gordon equation leads to hypothetical imaginary mass. Thereafter we discuss an alternative route for quaternionic modification of Klein-Gordon equation, and implications to meson physics.

4.1 Quaternion Quantum Mechanics

Adler's method of quaternionizing Quantum Mechanics grew out of his interest in the Harari-Shupe's rishon model for composite quarks and leptons [21]. In a preceding paper [22] he describes that in quaternionic quantum mechanics (QQM), the Dirac transition amplitudes are quaternion valued, i.e. they have the form

$$q = r_0 + r_1 i + r_2 j + r_3 k \tag{46}$$

where r_0, r_1, r_2, r_3 are real numbers, and i, j, k are quaternion imaginary units obeying

$$\begin{aligned} i^2 = j^2 = k^2 = -1, &\quad ij = -ji = k, \\ jk = -kj = i, &\quad ki = -ik = j\,. \end{aligned} \tag{47}$$

Using this QQM method, he described composite fermion states identified with the quaternion real components [23].

4.2 Hypothetical imaginary mass problem in Klein-Gordon equation

It is argued that dynamical origin of Higgs mass implies that the mass of W must always be pure imaginary [19, 20]. Therefore one may conclude that a real description for (composite) quarks and leptons shall avoid this problem, i.e. by not including the problematic Higgs mass.

Nonetheless, in this section we can reveal that perhaps the problem of imaginary mass in Klein-Gordon equation is not completely avoidable. First we will describe an elementary derivation of Klein-Gordon from electromagnetic wave equation, and then by using Bakhoum's assertion of total energy we derive alternative expression of Klein-Gordon implying the imaginary mass.

We can start with 1D-classical wave equation as derived from Maxwell equations [24, p.4]:

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0\,. \tag{48}$$

This equation has plane wave solutions:

$$E(x,t) = E_0 e^{i(kx-\omega t)} \tag{49}$$

which yields the relativistic total energy:

$$\varepsilon^2 = p^2c^2 + m^2c^4. \tag{50}$$

Therefore we can rewrite (48) for non-zero mass particles as follows [24]:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\right)\Psi e^{\frac{i}{\hbar}(px-Et)} = 0\,. \tag{51}$$

Rearranging this equation (51) we get the Klein-Gordon equation for a free particle in 3-dimensional condition:

$$\left(\nabla - \frac{m^2c^2}{\hbar^2}\right)\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}\,. \tag{52}$$

It seems worthnoting here that it is more proper to use total energy definition according to Noether's theorem in lieu of standard definition of relativistic total energy. According to Noether's theorem [25], the total energy of the system corresponding to the time translation invariance is given by:

$$E = mc^2 + \frac{cw}{2}\int_0^\infty \left(\gamma^2\, 4\pi r^2\, dr\right) = k\mu c^2 \tag{53}$$

where k is *dimensionless* function. It could be shown, that for low-energy state the total energy could be far less than $E = mc^2$. Interestingly Bakhoum [25] has also argued in favor of using $E = mv^2$ for expression of total energy, which expression could be traced back to Leibniz. Therefore it seems possible to argue that expression $E = mv^2$ is more generalized than the standard expression of special relativity, in particular because the total energy now depends on actual velocity [25].

From this new expression, it is possible to rederive Klein-Gordon equation. We start with Bakhoum's assertion that it is more appropriate to use $E = mv^2$, instead of more convenient form $E = mc^2$. This assertion would imply [25]:

$$H^2 = p^2c^2 - m_0^2c^2v^2. \tag{54}$$

A bit remark concerning Bakhoum's expression, it does not mean to imply or to interpret $E = mv^2$as an assertion that it implies zero energy for a rest mass. Actually the problem comes from "mixed" interpretation of what we mean with "velocity". In original Einstein's paper (1905) it is defined as "kinetic velocity", which can be measured when standard "steel rod" has velocity approximates the speed of light. But in quantum mechanics, we are accustomed to make use it deliberately to express "photon speed" $= c$. Therefore, in special relativity 1905 paper, it should be better to interpret it as "speed of free electron", which approximates c. For hydrogen atom with 1 electron, the electron occupies the first excitation (quantum number $n = 1$), which implies that their speed also approximate c, which then it is quite safe to assume $E \sim mc^2$. But for atoms with large number of electrons occupying large quantum numbers, as Bakhoum showed that electron speed could be far less than c, therefore it will be more exact to use $E = mv^2$, where here v should be defined as "average electron speed" [25].

In the first approximation of relativistic wave equation, we could derive Klein-Gordon-type relativistic equation from equation (54), as follows. By introducing a new parameter:

$$\zeta = i\,\frac{v}{c}\,, \tag{55}$$

then we can use equation (55) in the known procedure to derive Klein-Gordon equation:

$$E^2 = p^2c^2 + \zeta^2 m_0^2 c^4, \tag{56}$$

where $E = mv^2$. By using known substitution:

$$E = i\hbar\,\frac{\partial}{\partial t}\,, \qquad p = \frac{\hbar}{i}\,\nabla\,, \tag{57}$$

and dividing by $(\hbar c)^2$, we get Klein-Gordon-type relativistic equation [25]:

$$-c^{-2}\,\frac{\partial\Psi}{\partial t} + \nabla^2\Psi = k_0^{'2}\,\Psi\,, \tag{58}$$

where

$$k_0^{'} = \frac{\zeta m_0 c}{\hbar}\,. \tag{59}$$

Therefore we can conclude that imaginary mass term appears in the definition of coefficient $k_0^{'}$ of this new Klein-Gordon equation.

4.3 Modified Klein-Gordon equation and meson observation

As described before, quaternionic Klein-Gordon equation has neat link with Yang-Mills field. Therefore it seems worth to discuss here how to quaternionize Klein-Gordon equation. It can be shown that the resulting modified Klein-Gordon equation also exhibits imaginary mass term.

Equation (52) is normally rewritten in simpler form (by asserting $c = 1$):

$$\left(\nabla - \frac{\partial^2}{\partial t^2}\right)\Psi = \frac{m^2}{\hbar^2}\,. \tag{60}$$

Interestingly, one can write the Nabla-operator above in quaternionic form, as follows:

A. Define quaternion-Nabla-operator as analog to quaternion number definition above (46), as follows [25]:

$$\nabla^q = -i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\,, \tag{61}$$

where e_1, e_2, e_3 are quaternion imaginary units. Note that equation (61) has included partial time-differentiation.

B. Its quaternion conjugate is defined as follows:

$$\bar{\nabla}^q = -i\frac{\partial}{\partial t} - e_1\frac{\partial}{\partial x} - e_2\frac{\partial}{\partial y} - e_3\frac{\partial}{\partial z}\,. \tag{62}$$

C. Quaternion multiplication rule yields:

$$\nabla^q\,\bar{\nabla}^q = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{\partial^2}{\partial^2 z}\,. \tag{63}$$

D. Then equation (63) permits us to rewrite equation (60) in quaternionic form as follows:

$$\nabla^q\bar{\nabla}^q\Psi = \frac{m^2}{\hbar^2}\,. \tag{64}$$

Alternatively, one used to assign standard value $c=1$ and also $\hbar=1$, therefore equation (60) may be written as:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\varphi(x,t) = 0\,, \tag{65}$$

where the first two terms are often written in the form of square Nabla operator. One simplest version of this equation [26]:

$$-\left(\frac{\partial S_0}{\partial t}\right)^2 + m^2 = 0 \tag{66}$$

yields the known solution [26]:

$$S_0 = \pm mt + \text{constant}\,. \tag{67}$$

The equation (66) yields wave equation which describes a particle at rest with positive energy (lower sign) or with negative energy (upper sign). Radial solution of equation (66) yields Yukawa potential which predicts meson as observables.

It is interesting to note here, however, that numerical 1-D solution of equation (65), (66) and (67) each yields slightly different result, as follows. (All numerical computation was performed using Mathematica [28].)

- For equation (65) we get:

(−D[#,x,x]+mˆ2+D[#,t,t])&[y[x,t]]==

$$m^2 + y^{(0,2)}[x,t] - y^{(2,0)}[x,t] = 0$$

DSolve[%,y[x,t],{x,t}]

$$\left\{\left\{y[x,t] \to \frac{m^2x^2}{2} + C[1][t-x] + C[2][t+x]\right\}\right\}$$

- For equation (66) we get:

(mˆ2−D[#,t,t])&[y[x,t]]==

$$m^2 + y^{(0,2)}[x,t] = 0$$

DSolve[%,y[x,t],{x,t}]

$$\left\{\left\{y[x,t] \to \frac{m^2t^2}{2} + C[1][x] + tC[2][x]\right\}\right\}$$

One may note that this numerical solution is in quadratic form $\frac{m^2t^2}{2}$ + constant, therefore it is rather different from equation (67) in [26].

In the context of possible supersymetrization of Klein-Gordon equation (and also PT-symmetric extension of Klein-Gordon equation [27, 29]), one can make use biquaternion number instead of quaternion number in order to generalize further the differential operator in equation (61):

E. Define a new "diamond operator" to extend quaternion-Nabla-operator to its biquaternion counterpart, according to the study [25]:

$$\begin{aligned}\diamondsuit = \nabla^q + i\,\nabla^q = \left(-i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\right) + \\ + i\left(-i\frac{\partial}{\partial T} + e_1\frac{\partial}{\partial X} + e_2\frac{\partial}{\partial Y} + e_3\frac{\partial}{\partial Z}\right),\end{aligned} \tag{68}$$

where e_1, e_2, e_3 are quaternion imaginary units. Its conjugate can be defined in the same way as before.

To generalize Klein-Gordon equation, one can generalize its differential operator to become:

$$\left[\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) + i\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\right]\varphi(x,t) = -m^2\varphi(x,t), \tag{69}$$

or by using our definition in (68), one can rewrite equation (69) in compact form:

$$\left(\diamondsuit\bar{\diamondsuit} + m^2\right)\varphi(x,t) = 0, \tag{70}$$

and in lieu of equation (66), now we get:

$$\left[\left(\frac{\partial S_0}{\partial t}\right)^2 + i\left(\frac{\partial S_0}{\partial t}\right)^2\right] = m^2. \tag{71}$$

Numerical solutions for these equations were obtained in similar way with the previous equations:

- For equation (70) we get:

(−D[#,x,x]+D[#,t,t]−I*D[#,x,x]+I*D[#,t,t]+mˆ2)
&[y[x,t]]==

$$m^2 + (1+i)\,y^{(0,2)}[x,t] - (1+i)\,y^{(2,0)}[x,t] = 0$$

DSolve[%,y[x,t],{x,t}

$$\left\{\left\{y[x,t] \to \left(\frac{1}{4} - \frac{i}{4}\right)m^2\,x^2 + C[1][t-x] + C[2][t+x]\right\}\right\}$$

- For equation (71) we get:

$$(-\text{m}\hat{\ }2\text{+D[\#,t,t]+I*D[\#,t,t])\&[y[x,t]]==}$$

$$m^2 + (1+i)\, y^{(0,2)}[x,t] = 0$$

$$\text{DSolve[\%,y[x,t],\{x,t\}]}$$

$$\left\{\left\{y[x,t] \to \left(\frac{1}{4} - \frac{i}{4}\right) m^2 x^2 + C[1][x] + t\,C[2][x]\right\}\right\}$$

Therefore, we may conclude that introducing biquaternion differential operator (in terms of "diamond operator") yield quite different solutions compared to known standard solution of Klein-Gordon equation [26]:

$$y(x,t) = \left(\frac{1}{4} - \frac{i}{4}\right) m^2 t^2 + \text{constant}\,. \tag{72}$$

In other word: we can infer hat $t = \pm\, \frac{1}{m} \sqrt{y/\left(\frac{1}{4} - \frac{i}{4}\right)}$, therefore it is likely that there is imaginary part of time dimension, which supports a basic hypothesis of the aforementioned BQ-metric in Q-relativity.

Since the potential corresponding to this biquaternionic KGE is neither Coulomb, Yukawa, nor Hulthen potential, then one can expect to observe a new type of matter, which may be called "*supersymmetric-meson*". If this new type of particles can be observed in near future, then it can be regarded as early verification of the new hypothesis of PT-symmetric QM and CT-symmetric QM as considered in some recent reports [27, 29]. In our opinion, its presence may be expected in particular in the process of breaking of Coulomb barrier in low energy schemes.

Nonetheless, further observation is recommended in order to support or refute this proposition.

5 Concluding remarks

If 4D space-time has for its 3D spatial section a Q-space with Q-connexion $\Omega_{\beta\, kn}$ containing Q-non-metricity $\sigma_{\beta\, kn}$, then the Q-connexion, geometric object, is algebraically identical to Yang-Mills potential

$$\hat{A}_{k\alpha} \equiv \frac{1}{2}\,\varepsilon_{knm}\Omega_{\alpha kn}\,,$$

while respective curvature tensor $\hat{R}_{kn\alpha\beta}$, also a geometric object, is algebraically identical to Yang-Mills "physical field" strength

$$\hat{F}_{m\alpha\beta} \equiv \frac{1}{2}\,\varepsilon_{knm}\hat{R}_{kn\alpha\beta}\,.$$

Thus Yang-Mills gauge field Lagrangian

$$L_{YM} \sim \hat{F}^{\alpha\beta}_{k}\hat{F}_{k\alpha\beta} = \frac{1}{4}\,\varepsilon_{kmn}\varepsilon_{kjl}\,\hat{R}^{\alpha\beta}_{mn}\hat{R}_{jl\alpha\beta} = \frac{1}{2}\,\hat{R}^{\alpha\beta}_{mn}\hat{R}_{mn\alpha\beta}$$

can be geometrically interpreted as a Lagrangian of "nonlinear" or "quadratic" gravitational theory, since it contains quadratic invariant of curvature Riemann-type tensor contracted by all indices. Hence Yang-Mills theory can be regarded as a theory of pure geometric objects: Q-connexion and Q-curvature with Lagrangian quadratic in curvature (as: Einstein's theory of gravitation is a theory of geometrical objects: Christoffel symbols and Riemann tensor, but with linear Lagrangian made of scalar curvature).

Presence of Q-non-metricity is essential. If Q-non-metricity vanishes, the Yang-Mills potential may still exist, then it includes only proper Q-connexion (in particular, components of Q-connexion physically manifest themselves as "forces of inertia" acting onto non-inertially moving observer); but in this case all Yang-Mills intensity components, being in fact components of curvature tensor, identically are equal to zero.

The above analysis of Yang-Mills field from Quaternion Space geometry may be found useful in particular if we consider its plausible neat link with Klein-Gordon equation and Duffin-Kemmer equation. We discuss in particular a biquaternionic-modification of Klein-Gordon equation. Since the potential corresponding to this biquaternionic KGE is neither Coulomb, Yukawa, nor Hulthen potential, then one can expect to observe a new type of matter. Further observation is recommended in order to support or refute this proposition.

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