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QUASI SET TOPOLOGICAL VECTOR SUBSPACES

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PREFACE

In this book the authors introduce four types of topological vector subspaces. All topological vector subspaces are defined depending on a set. We define a quasi set topological vector subspace of a vector space depending on the subset S contained in the field F over which the vector space V is defined.

These quasi set topological vector subspaces defined over a subset can be of finite or infinite dimension. An interesting feature about these spaces is that there can be several quasi set topological vector subspaces of a given vector space. This property helps one to construct several spaces with varying basic sets.

Further we cannot define quasi set topological vector subspaces of all vector subspaces. We have given the number of quasi set topological vector subspaces in case of a vector space defined over a finite field.

It is still an open problem, "Will these quasi set topological vector spaces increase the number of finite topological spaces with n points, n a finite positive integer?".

Chapter one is introductory in nature and chapter two uses vector spaces to build quasi set topological vector subspaces. Not only we use vector spaces but we also use S-vector spaces, set vector spaces, semigroup vector spaces and group vector spaces to build set topological vector subspaces. These also give several finite set topological spaces. Such study is carried out in chapters three and four.

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Chapter One

INTRODUCTION

In this book the authors introduce the new notion of quasi set topological vector subspaces and New Set topological vector subspaces defined over the set S.

For the concept of vector spaces and Smarandache vector spaces please refer [15]. For the notion of set vector spaces please refer [16]. For the concept of topological spaces refer [1, 5].

Here S-quasi set topological vector subspaces are also defined which is quasi set topological vector subspaces defined over Smarandache rings (S-rings) [7].

Finally we in this book define the concept of New Set topological vector subspace (NS-topological vector subspace) of a set vector space V defined over the subset P of S where S is the set over which V is defined.

We enumerate the properties associated with them. These new topological vector subspaces are not like the usual topological spaces where are defined on the collection of sets and some topology is defined but the set topological vector subspaces depend highly on the set over which they are defined as well as the algebraic structure enjoyed by the set over which they are defined.

For instance if T is the quasi set topological vector subspace defined over the set P; then T depends on the vector space over it is defined as well as the set P $\subseteq$ F (F is the field over which V is defined). Likewise if M is a S-quasi set topological vector subspace of V defined over the set P $\subseteq$ R where R is a S-ring over which the S-vector space V is defined.

Finally W the New Set topological vector subspace S of V defined over the set L $\subseteq$ S where V is a set vector space defined over the set S.

Thus it is left as an open problem whether these three types of new topological vector subspaces are different from the already existing topological spaces. For these are dependent topological vector subspaces over the sets and the algebraic structures over which they are defined.

Chapter Two

QUASI SET TOPOLOGICAL VECTOR SUBSPACES

In this chapter we for the first time define set topological vector subspace using quasi set vector subspaces of a vector space. Here we develop and describe these structures.

DEFINITION 2.1: *Let V be a vector space defined over a field F. Let S $\subseteq$ V be a non empty subset of V and P $\subseteq$ F be a subset of the field F. If for all s $\in$ S and p $\in$ P, sp and ps $\in$ S then we define S to be a quasi set vector subspace of V defined over the subset P of F.*

We will first illustrate this by some examples.

Example 2.1: Let V = Q × Q × Q be a vector space defined over the field F = Q.

Consider S = {(3Z × 2Z × 5Z)} $\subseteq$ V and P = $Z^+ \cup \{0\}$ $\subseteq$ F be proper subset of V and F = Q respectively. S is a quasi set vector subspace of V defined over P $\subseteq$ V.

Example 2.2: Let

V = {collection of all 2 × 2 matrices with entries from Q} be a vector space defined over the field F = Q.

Let

$$S = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & d \\ c & 0 \end{bmatrix} \,\middle|\, a \in 3Z, b \in 5Z; c, d \in 7Z \right\} \subseteq V$$

be a subset of V and $T = 3Z^+ \cup \{0\} \subseteq F = Q$ be a subset of F. S is quasi set vector subspace of V defined over the set T of F.

Example 2.3: Let $V = Z_7 \times Z_7 \times Z_7 \times Z_7$ be a vector space defined over the field $F = Z_7$.

Let $S = \{(a, b, c, d) \mid a, b, c, d \in \{0, 1, 6\} \subseteq Z_7\} \subseteq V$ be a subset of V. $P = \{0, 1, 6\} \subseteq Z_7 = F$ be a subset of Z_7. S is a quasi set vector subspace of V defined over the set P of Z_7.

The following observations are interesting and important.

(1) For any given subset P of the field F; where V is the vector space defined over the field F we can have in general many number of quasi set vector subspaces of V defined over the set $P \subseteq F$.

(2) We can have any number of quasi set vector subspaces $S \subseteq V$ for varying subsets P of the field F.

(3) {0} is the trivial quasi set vector subspace of V defined over every proper subset P of the field F.

(4) V is also trivial (or not proper) quasi set vector subspace of V defined over every proper subset P of the field F.

Now we define the concept of substructures of a quasi set vector subspace of V defined over the subset P of a field F.

DEFINITION 2.2: *Let V be a vector space defined over the field F. S ⊆ V be a quasi set vector subspace of V defined over the subset P of F. If X ⊆ S is a proper subset such that X is a quasi set vector subspace of S over the set P of F; we define X to be a quasi subset vector subspace of S ⊆ V over the set P of F of type I. Suppose S ⊆ V is a quasi set vector subspace of V define over P and if T ⊆ P (T a proper subset of P) then we define S to be a quasi subset vector subspace of V over the subset T of P of type II.*

If S ⊆ V is a quasi subset vector subspace of V defined over the subset P of S and if W ⊆ S (W a proper subset of S) and T ⊆ P (T a proper subset of S), such that W is a quasi subset vector subspace of S ⊆ V defined over T ⊆ P, then we define W to be a quasi subset vector subspace of type I and type II, which we call as a Twin quasi set vector subspace of S ⊆ V define over T ⊆ P ⊆ F.

We will illustrate all these situations by some examples.

Example 2.4: Let V = Q × Q × Q × Q be a vector space defined over the field F = Q. Let S = {(3Z × 2Z × 5Z × 11Z)} ⊆ V be a quasi set vector subspace of V defined over the set P = (3Z ∪ 2Z) ⊆ Q = F.

Consider W = {(6Z × 10Z × 35Z × 44Z)} ⊆ S ⊆ V and T = {(6Z ∪ 16Z)} ⊆ P ⊆ Q. W is a Twin quasi set vector subspace of S over the set T of P.

THEOREM 2.1: *Let V be a vector space defined over a field F. If W ⊆ V is a Twin quasi set vector subspace defined over a set in F then W is both a type I quasi subset vector subspace and type II quasi subset vector subspace of V.*

The proof is direct from the definition.

Now we show however a type I or type II quasi subset vector subspace in general is not a Twin quasi set vector subspace of V defined over F (or used in the mutually exclusive sense).

This is described in the following.

THEOREM 2.2: *Let V be a vector space defined over a field F, V in general need not have a quasi set vector subspace defined over F.*

Proof: This is proved by the following example.

Let $V = Z_2 \times Z_2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ be a vector space over the field $Z_2 = F$. V has no quasi set vector subspace defined over a subset in F.

We call such vector spaces as strongly simple vector spaces.

THEOREM 2.3: *Let V be a vector space defined over the field F. Suppose $S \subseteq V$ is a quasi set vector subspace of V over $P \subseteq F$ of type I; S need not in general be a quasi set vector subspace of V over $P \subseteq F$ of type II.*

Proof: We prove this by a counter example.

Let $V = Z_3 \times Z_3 \times Z_3 \times Z_3 \times Z_3$ be a vector space defined over the field $Z_3 = F$. $S = \{(Z_3 \times Z_3 \times 0 \times 0 \times Z_3)\} \subseteq V$; be a quasi set vector subspace of V defined over the subset $P = \{0, 1\} \subseteq Z_3 = F$. $T = \{(Z_3 \times \{0\} \times \{0\} \times \{0\} \times Z_3)\} \subseteq S \subseteq V$; is a subset of S and T is a quasi set vector subspace of S of V over the subset $P = \{0, 1\} \subseteq Z_3 = F$ of type I. Clearly T is not a quasi set vector subspace of S of V over the subset $P = \{0, 1\} \subseteq Z_3$ of type II; hence the theorem.

Now we show a type II quasi set vector subspace in general is not a type I quasi set vector subspace.

THEOREM 2.4: *Let V be a vector space defined over a field F. Let $S \subseteq V$ be a quasi set vector subspace of V defined over the set $P \subseteq F$. S is a quasi set vector subspace of V defined over the* *subset $T \subseteq P \subseteq F$ of type II. S in general is not a type I quasi set vector subspace of V defined over T or P.*

Proof: The proof is by a counter example.

Consider $V = Z_5 \times Z_5$ a quasi set vector subspace defined over the field $Z_5 = F$. Let $S = \{\{0\} \times \{0, 5, 1\} \subseteq Z_5$ be a quasi set vector subspace of V over $T = \{0, 1, 5\} \subseteq F = Z_5$. We see S is a quasi set vector subspace of V defined over the set $P = \{0, 5\} \subseteq T \subseteq Z_5$ of type II but S can not have vector subspace of type I.

Hence the claim.

DEFINITION 2.3: *Let V be a vector space defined over the field F. Let T = {Collection of all subsets of S of V such that S is a quasi set vector subspace of V defined over a fixed subset P of F}.*

Clearly $\{0\} \in T$.

(1) If $T = \bigcup_{s \in T} S$ then T is also a quasi set vector subspace of V over P of F.

(2) {0} is trivially a quasi set vector subspace of V defined over the set P of F.

(3) Now if S_1 and $S_2 \in T$ then $S_1 \cap S_2$ also is in T.

(4) The union of any collection of sets in T is in T. So with T the given set of elements a topology T_q on T is a non empty collection of subsets of T called quasi set vector subspaces defined over P. The set T is topologised if a topology T_q is given on T associated with P. The topologised set T is called a quasi set topological vector subspace of V over the set P (or relative to P). The sets in T are called the quasi set vector subspaces relative to P of the topology T_q.

We will first illustrate this situation before we proceed to derive more properties. However the topology T_q is understood without explicitly mentioning it.

Example 2.5: Let V = $Z_3 \times Z_3 \times Z_3 \times Z_3$ be a vector space over the field Z_3 = F. Take W = {(1, 0, 0, 0), (2, 0, 0, 0), (2, 2, 2, 2), (1, 1, 1, 1), (2, 2, 1, 1), (1, 1, 2, 2), (1, 0, 0, 2), (2, 0, 0, 1)} $\subseteq$ V, a quasi set vector subspace of V over the set P = {0, 1} $\subseteq Z_3$.

We see infact every subset of V is a quasi set vector subspace of V over P = {0, 1} $\subseteq Z_3$. Let T be the collection of all quasi set vector subspaces of V over P = {0, 1} $\subseteq Z_3$. T is a quasi set topological vector subspace of V defined over P.

Example 2.6: Let M = $Z_3 \times Z_3$ be a vector space defined over the field F = Z_3. Consider T = {all subsets of M including M}; T is a collection of all quasi set vector subspaces of M defined over the set P = {0, 1} $\subseteq Z_3$. T is a quasi set topological of space of vector subspaces over P = {0, 1} $\subseteq Z_3$.

The basic quasi set of T are {(0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (1, 0), (2, 0), (2, 2)} $\subseteq$ T. Infact the number of elements in this quasi set topological vector subspace is finite.

Example 2.7: Let V = $Z_{11} \times Z_{11} \times Z_{11}$ be a vector space defined over the field Z_{11}. Consider T = {set of all subsets of V which are quasi set vector subspaces of V defined over the set P = {0, 1} $\subseteq Z_{11}$}. Clearly every set S in T contains (0, 0, 0) as an element. Further T is a quasi set topological vector subspace of V over P = {0, 1} $\subseteq Z_{11}$.

We see the basic set B of T contains pair {x, y} $\subseteq$ V such that x = (0, 0, 0) and x $\neq$ y $\in Z_{11} \times Z_{11} \times Z_{11}$. Thus B contains $11^3 - 1$ elements in it.

Inview of this we have the following theorem.

THEOREM 2.5: *Let V =* $\underbrace{Z_p \times Z_p \times \ldots \times Z_p}_{n\text{-times}}$ *be a vector space defined over the field* Z_p *= F. T = {Collection of all quasi set vector subspaces of V defined over the set P = {0, 1}* $\subseteq Z_p$ *= F}* *be the quasi set topological vector subspace defined over the set $P = \{0, 1\} \subseteq Z_p$.*

The basic set of T is $B = \{a, b / a = (0, 0, \ldots, 0), a \neq b \in V\}$ and the number of elements in B is $p^n - 1$.

Proof: Let V = $\underbrace{Z_p \times Z_p \times \ldots \times Z_p}_{\text{n-times}}$ be a vector space defined over the field Z_p. Let T = {All subsets of V which are quasi set topological vector subspaces of V over the set $P = \{0, 1\} \subseteq Z_p$}. Clearly T is a quasi set topological vector subspace of V over $P = \{0, 1\}$.

Now
$B = \{x = (0, 0, \ldots, 0), y = (a_1, \ldots, a_n) \mid a_i \in Z_p, 1 \leq i \leq n, x \neq y\}$
is the basic set of the quasi set topological vector subspace as every other element of T can be got as the union of elements from T and the intersection of any two elements of T or intersection of a finite number of elements of T is in T.

We can also give a lattice associated with the quasi set topological vector subspace T whether T is finite or infinite.

We give examples of infinite quasi set topological vector subspaces.

Example 2.8: Let $V = Q \times Q$ be a vector space over Q = F. T = {all quasi set vector subspaces of V defined over the set $P = \{0, 1\} \subseteq Q$}. T is an infinite quasi set topological vector subspace of V over the set $P = \{0, 1\} \subseteq Q$.

Infact every subset of V to be a quasi set vector subspace of V, must contain (0, 0). All subsets of V with (0, 0) as one of its elements is a quasi set vector subspace of V over $P = \{0, 1\} \subseteq Q$.

We see T is an infinite quasi set topological vector subspace of V over $P = \{0, 1\}$.

The basic set B is also infinite.
B = {(0, 0), (a, b) | (a, b) ≠ (0, 0) ∈ Q × Q} ⊆ T.

Example 2.9: Let

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix} \text{ where } a_i \in Q, 1 \leq i \leq 12 \right\}$$

be a vector space defined over the field Q. Let

$$T = \left\{ \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix} \right\} \right|$$

$$a_i \in Q, 1 \leq i \leq 12\}$$

denote the collection of all pairs. T generates a quasi set topological vector subspace of V over the set P = {0, 1} ⊆ Q.

With T as a basic set we get an infinite quasi set topological vector subspace defined over P of 3 × 4 matrices.

It is pertinent to mention here that we have a class of simple quasi set vector subspaces and on these vector subspaces we would not be in a position to define the concept of quasi set topological vector subspaces of finite or infinite basic set.

THEOREM 2.6: *Let V be a any vector space defined over the field Z_2 = {0, 1}. V is a simple quasi set vector space.*

Proof: Follows from the simple fact Z_2 = {0, 1} has no proper subset whose cardinality is two.

Example 2.10: Let

$$V = \left\{ \begin{pmatrix} a_1 & \ldots & a_5 \\ a_6 & \ldots & a_{10} \end{pmatrix} \middle| \; a_i \in Z_2 = \{0, 1\}, 1 \leq i \leq 10 \right\}$$

be a vector space defined over the field Z_2. V is a simple quasi set vector subspace defined over Z_2.

It is important and interesting to note that V has in general vector subspaces even if V is a simple quasi set vector subspace.

The claim follows from the following example.

Example 2.11: Let $V = Z_2 \times Z_2 \times Z_2$ be a vector space defined over the field Z_2. Take $W = Z_2 \times \{0\} \times Z_2 \subseteq V$; W is a vector subspace of V over Z_2.

Take $M = \{0\} \times Z_2 \times Z_2 \subseteq V$; M is also a vector subspace of V over Z_2; hence the claim.

Example 2.12: Let $V = R \times R \times R \times R$ be a vector space over the field Q (or R). T = {all subsets of V which contain {(0, 0, 0, 0)} as one of its elements}; T is a collection of quasi set vector subspaces of V over the set $P = \{0, 1\} \subseteq Q$ (or R).

Infact T is a quasi set topological vector subspace of V over the set $P = \{0, 1\} \subseteq Q$ (or R). T is an infinite quasi set topological vector subspace and the basic set of T is of infinite order.

Example 2.13: Let

$$V = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| \; a_i \in Z_3; 1 \leq i \leq 6 \right\}$$

be a vector space defined over the field Z_3.

$$T = \left\{ \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix}, \right.\right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 \\ a_4 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_6 \end{pmatrix}, \ldots, \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \right\} \Bigg|$$

$$a_i \in Z_3 \setminus \{0\},\ 1 \leq i \leq 6\}$$

be the collection of all matrices} is a quasi set vector subspace of V over the set $P = \{0, 1\} \subseteq Z_3$.

Infact T is a quasi set topological vector subspace of V over the set $P = \{0, 1\} \subseteq Z_3$.

We see the basic set of T is finite.

Example 2.14: Let

$$S = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \Bigg| \ a_i \in Z_3,\ 1 \leq i \leq 4 \right\}$$

be a vector space defined over Z_3.

Take

$$T = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

$$\left. \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \right\};$$

T is the quasi set topological vector subspace built over the set P = {0, 1} $\subseteq Z_3$ of finite dimension.

The basic set

$$\left\{\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, x\right\} \middle| \; x \neq (0) \in S\right\}$$

is the collection of pairs.

Can we have any other quasi set topological vector subspaces built using other subsets of Z_3?

The answer is yes.

For take

$$M = \left\{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix},\right.$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix},$$

$$\left.\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right\};$$

M is again a quasi set topological vector subspace defined over the set P = {1, 2} $\subseteq Z_3$.

We see

$$B = \left\{\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix}\right\} \middle| \; a, b, c, d \in Z_3\right\}$$

is the basic set of the quasi set topological vector subspace M defined over P = {1, 2}.

For

$$B = \left\{\left\{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\right\},\right.$$

$$\left.\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right\}, \ldots, \left\{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right\} \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right\}$$

is a basic set and the empty set is the least element $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$,

which is also the trivial quasi set vector subspace of V over $\{1, 2\} \subseteq Z_3$.

To every quasi set topological vector subspace T relative to the set $P \subseteq F$, we have a lattice associated with it we call this lattice as the Representative Quasi Set Topological Vector subspace lattice (RQTV-lattice) of T relative to P.

When T is finite we have a nice representation of them. In case T is infinite we have a lattice which is of infinite order. We can in all cases give the atoms of the lattice which is infact the basic set of T over P.

It is pertinent to keep on record that the T and the basic set (or the atoms of the RQTV-lattice) depends on the set P over which it is defined.

We will illustrate this situation by some examples.

Example 2.15: Let V = $Z_5 \times Z_5$ be a vector space defined over the field Z_5. Consider T = {all quasi set vector subspaces of V defined over the set P = {0, 1} $\subseteq Z_5$}. T is a quasi set topological vector subspace of V defined over the set P. The atoms of T relative to the RQTV - lattice whose least element is (0, 0) and the greatest element is V is as follows. Let A denote the atoms of L.

A = {{(0, 0), (1, 0)}, {(0, 0), (0, 1)}, {(0, 0), (2, 0)}, {(0, 0), (0, 2)}, {(0, 0), (3, 0)}, {(0, 0), (0, 3)}, {(0, 0), (4, 0)}, {(0, 0), (0, 4)}, …, {(0, 0), (4, 1)}, {(0, 0), (1, 4)}, {(0, 0), (4, 2)}, {(0, 0), (2, 4)} , …, {(0, 0), (4, 4)}}.

o(A) = 25 – 1 = $5^2 - 1$. With A as the basic set we can generate the quasi set topological vector subspace, T relative to the set P = {0, 1}.

Suppose we change the set P, do we get a new quasi set topological vector subspace? The answer in general is yes.

We may have different sets for which the T remains the same.

Take P_1 = {1, 4} $\subseteq Z_5$. We find the quasi set topological vector subspace relative to the set P_1 = {1, 4}. Let M denote the collection of all quasi set vector subspaces of V defined over the set P_1 = {1, 4}. To find the basic set of M or equivalently the atoms of the RQTV-lattice of M.

Let B denote the basic set or atoms of the RQTV - lattice of M. B = {(0, 0), {(1, 1), (4, 4)}, {(1, 0), (4, 0)}, {(0, 1), (0, 4)}, …, {(2, 4), (3, 1)}, {(1, 2), (4, 3)}, {(2, 2), (3, 3)}, {(1, 3), (4, 2)}, {(3, 1), (2, 4)},…}.

Clearly the number of elements in B is 13 and these 13 elements form the atoms of M relative to P_1 = {1, 4}.

We see the lattice of the quasi set topological vector subspace T over P = {0, 1} $\subseteq Z_5$ has 24 atoms and that of the lattice of quasi set topological vector subspace M over $P_1 = \{1, 4\} \subseteq Z_5$ has 13 atoms.

So both the quasi set topological vector subspaces T and M are different. Further quasi set topological vector subspaces defined over set P and P_1 respectively are distinct and as well as the lattices associated with them depend highly on the sets P and P_1 over which they are defined.

This is evident from the above examples.

Suppose we take $P_2 = \{2, 3\} \subseteq Z_5$ as the set over which the quasi set vector subspaces of V is defined. S = {Collection all quasi set vector subspace of V defined over the set $\{2, 3\} = P_2 \subseteq Z_5$} be the quasi set topological vector subspace of V defined over the set $P_2 = \{2, 3\}$.

The basic set of S is B = $\{A_1 = (0, 0), \{(2, 1), (4, 2), (1, 3), (3, 4)\} = A_2, \{(1, 2), (2, 4), (3, 1), (4, 3)\} = A_3, \{(1, 1), (2, 2), (3, 3), (4, 4)\} = A_4, A_5 = \{(1, 0), (2, 0), (3, 0), (4, 0)\}, A_6 = \{(0, 1), (0, 2), (0, 3), (0, 4)\}$ and $A_7 = \{(2, 3), (3, 2), (1, 4), (4, 1)\}\}$.

Now L the lattice associated with the quasi set topological vector subspace of V defined over the set S has the maximum element as V and the least element is the empty set ϕ. The atoms of the lattice are $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$. We see the associated lattice is a Boolean algebra of order 2^7.

Thus we see for any given vector space V over the field F we can have several quasi set topological vector subspaces of V depending on the subset P taken in F. The associated RQTV-lattice of these quasi set topological vector subspaces will be a Boolean algebra of finite or infinite order depending on the cardinality of the vector space V over F.

Recall a topological space X is said to satisfy the second axiom of countability if and only if its topology has a countable basis.

We further define basis and subbasis of quasi set topological vector subspaces defined on the subset P of a field F, where V is a vector space defined over the field F.

Let T be the collection of all quasi set vector subspaces of the vector space V defined over the set P $\subseteq$ F, F is the field over which V is defined. Let T be a quasi set topological vector subspace with topology T_q (we also denote T by T_q^P as T is defined over the set P $\subseteq$ F).

A basis of a topology in T is a subcollection B of T such that every quasi set vector subspace U of T is a union of some quasi set vector subspaces in B.

In other words for every quasi set vector subspace U in T and each quasi set vector subspace X in U there is a D in B such that X = D (or $\cup$ D) $\subseteq$ U. The quasi sets B will be called Basic quasi sets of vector subspaces of the quasi set topological vector subspace T. Subbasis of T can be defined in an analogous way.

We proceed onto give examples of quasi set topological vector subspaces T defined over a set P $\subseteq$ F of a vector space V defined over the field F which satisfy the second axiom of countability.

Example 2.16: Let V = $Z_{11} \times Z_{11} \times Z_{11}$ be a vector space defined over the field Z_{11} = F. Let T = {collection of all quasi set vector subspaces of V defined over the subset P = {0, 1} $\subseteq$ Z_{11}}. T is a quasi set topological vector subspace of V defined over (relative to) the subset P = {0, 1} $\subseteq$ F = Z_{11}. T satisfies second axiom of countability as it has a finite basis.

Example 2.17: Let V = Q × Q × Q × Q be a vector space defined over the field Q = F. T = {collection of all quasi set vector subspaces of V defined over the set P = {0, 1} $\subseteq$ Q = F}. T is a quasi set topological vector subspace of V defined over the set P = {0, 1} $\subseteq$ Q = F. T satisfies the second axiom of countability.

Now we have the following interesting theorem.

THEOREM 2.7: *Let V be a vector space defined over a field Z_p (p a prime number and number of elements in V is finite). Every quasi set topological vector subspace of V defined over the set P $\subseteq Z_p$ satisfies the second axiom of countability for every proper subset P $\subseteq Z_p$.*

The proof is direct and hence leave it as an exercise to the reader.

Corollary: If Z_p in theorem 2.7 is replaced by Z_q where $q = p^m$ and Z_q a field m > 1 then the above theorem is true for every subset P $\subseteq Z_q$.

Example 2.18: Let V = Q × Q × Q × Q × Q be a vector space of finite dimension defined over the field F = Q.

T = {collection of all quasi set vector subspaces of V over the set P = {0, 1} $\subseteq$ Q}; T is a quasi set topological vector subspace of V over the set P which satisfies the second axiom of countability. For take B = {{(0, 0, 0, 0, 0), x = (a, b, c, d, e)} | a, b, c, d, e $\in$ Q and x $\neq$ (0, 0, 0, 0, 0)} as a basis of T over the set P = {0, 1} $\subseteq$ Q. Hence the claim.

We define those quasi set topological vector subspaces defined over the set {0, 1} $\subseteq Z_p$ or F (F a field of characteristic zero and Z_p is the prime field of characteristic zero) of any vector space V, V defined over Z_p or F as the fundamental quasi set topological vector subspace of V defined over the set {0, 1} $\subseteq Z_p$ or F.

P = {0, 1} is also called in this book as the fundamental set in Z_p or F.

THEOREM 2.8: *Let V be a vector space defined over the field Q of finite dimension defined over Q = F. T = {collection of all quasi set vector subspaces of V defined over the fundamental set P = {0, 1} $\subseteq$ Q}; T is a fundamental quasi set topological vector* *subspace of V defined over the fundamental set P = {0, 1} ⊆ Q and this fundamental quasi set topological vector subspace satisfies the second axiom of countability.*

The proof is straight forward and hence is left as an exercise to the reader.

Example 2.19: Let V = {Q × Q} be a vector space defined over the field Q. Take S = {(0, –1) ⊆ Q} to be a subset of Q. Let T = {all quasi set vector subspaces of V over the set S ⊆ Q}; T is a quasi set topological vector subspace of V defined over (or relative) the set S = {0, –1} ⊆ Q.

We see the basic set of T assumes the following form B_T = {{(0, 0), x = (a, b), (–a, –b)} | a, b ∈ Q, x ≠ (0, 0)}. T also satisfies the second axiom of countability.

Example 2.20: Let V = $Z_7 \times Z_7$ be a vector space defined over the field Z_7 = F. Let S = {0, 6} ⊆ Z_7 be a proper subset of Z_7. T = {collection of all quasi set vector subspaces of V defined over the set S = {0, 6} ⊆ Z_7}. T is a quasi set topological vector subspace of V over the set S = {0, 6}.

Now B = {{(0, 0), (0, 1), (0, 6)}, {(0, 0), (1, 0), (6, 0)}, {(0, 0), (1, 6), (6, 1)}, {(0, 0), (6, 6), (1, 1)}, {(0, 0), (2, 0), (5, 0)}, {(0, 0), (0, 2), (0, 5)}, {(0, 0), (2, 2), (5, 5)}, {(0, 0), (2, 5), (5, 2)}, {(0, 0), (0, 3), (0, 4)}, {(0, 0), (3, 0), (4, 0)}, {(0, 0), (3, 3), (4, 4)}, {(0, 0), (3, 4), (4, 3)}, {(0, 0), (1, 2), (6, 5)}, {(0, 0), (2, 1), (5, 6)}, {(0, 0), (1, 3), (6, 4)}, {(0, 0), (4, 6), (3, 1)}, {(0, 0), (1, 4), (6, 3)}, {(0, 0), (4, 1), (3, 6)}, {(0, 0), (1, 5), (6, 2)}, {(0, 0), (5, 1), (2, 6)}, {(0, 0), (2, 3), (5, 4)}, {(0, 0), (3, 2), (4, 5)}, {(0, 0), (2, 4), (5, 3)}, {(0, 0), (4, 2), (3, 5)}} is the basic set of the quasi set topological vector subspace of V over S = {0, 6} ⊆ Z_7. o(B) = 24 = $(7^2 - 1) / 2$.

Example 2.21: Let V = $Z_5 \times Z_5$ be a vector space defined over the field F = Z_5. Let T = {all quasi set vector subspaces of V defined over the set S = {0, 4} ⊆ Z_5}; be the quasi set topological vector subspace of V over the set S = {0, 4} ⊆ Z_5.

The basic set of T = {{(0, 0), (1, 0), (4, 0)}, {(0, 0), (0, 1), (0, 4)}, {(0, 0), (1, 1), (4, 4)}, {(0, 0), (1, 4), (4, 1)}, {(0, 0), (2, 0), (3, 0)}, {(0, 0), (0, 2), (0, 3)}, {(0, 0), (3, 2), (2, 3)}, {(0, 0), (2, 2), (3, 3)}, {(1, 2), (0, 0), (4, 3)}, {(0, 0), (2, 1), (3, 4)}, {(1, 3), (4, 2), (0, 0)}, {(3, 1), (2, 4), (0, 0)}.

Clearly order of B is $(5^2 - 1) / 2 = 12$.

Thus the associated lattice of T is a Boolean algebra of order 2^{12} with {(0, 0)} as least element and V as the largest element.

In view of this we have the following theorem.

THEOREM 2.9: *Let $V = Z_p \times Z_p$ be a vector space defined over the field Z_p. $P = \{0, (p-1)\} \subseteq Z_p$ be a subset of Z_p. T = {all subsets of V which are quasi set vector subspaces of V defined over the set $P = \{0, p-1\} \subseteq Z_p$}; T is a quasi set topological vector subspace of V over the set $P = \{0, p-1\}$.*

(1) T has a finite basis B and $o(B) = \dfrac{(2^p - 1)}{2}$.

(2) T satisfies second axiom of countability.

(3) The lattice L associated with T is a Boolean Algebra with the basic set B as atoms and {(0, 0)} is the least element and V is the largest element and $o(L) = 2^{(2^p - 1/2)} = 2^{o(B)}$.

The proof of the above theorem is straight forward and hence is left as an exercise to the reader.

Example 2.22: Let $V = Z_3 \times Z_3 \times Z_3$ be a vector space defined over the field $F= Z_3$. $P = \{0, 2\} \subseteq Z_3$. Let T = {all subsets of V which are quasi set vector subspaces of V defined over the set $\{0, 2\} \subseteq Z_3$}. T is a quasi set topological vector subspace of V defined over P = {0, 2}.

The basic set associated with T be B, B = {{(0, 0, 0), (0, 1, 0), (0, 2, 0)}, {(0, 0, 0), (0, 0, 1), (0, 0, 2)}, {(0, 0, 0), (1, 0, 0), (2, 0, 0)}, {(0, 0, 0), (1, 1, 0), (2, 2, 0)}, {(0, 0, 0), (1, 0, 1), (2, 0, 2)}, {(0, 0, 0), (0, 1, 1), (0, 2, 2)}, {(0, 0, 0), (1, 0, 2), (2, 0, 1)}, {(0, 0, 0), (0, 1, 2), (0, 2, 1)}, {(0, 0, 0), (1, 2, 0), (2, 1, 0)}, {(0, 0, 0), (1, 1, 1), (2, 2, 2)}, {(0, 0, 0), (1, 2, 1), (2, 1, 2)}, {(0, 0, 0), (2, 1, 1), (1, 2, 2)}, {(0, 0, 0), (1, 1, 2), (2, 2, 1)}}. The number of elements in B is $(3^3 - 1) / 2$.

Example 2.23: Let V = $Z_{11} \times Z_{11} \times Z_{11}$ be a vector space defined over the field F = Z_{11}. T = {set of all quasi set vector subspaces of V defined over the set P = {0, 10} $\subseteq Z_{11}$}. T is a quasi set topological vector subspace of V defined over the set P = {0, 10} $\subseteq Z_{11}$.

The basic set of T defined over P be B. B = {{(0,0,0), (1,0,0), (10,0,0)}, {(0,0,0), (0,1,0), (0,10,0)} …, {(0,0,0), (0,10,0), (10,1,10)}}. Clearly order of B is $(11^3 - 1)/2$.

We see the associated lattice of T is a Boolean algebra of order $2^{o(B)} = 2^{(11^3-1)/2}$.

In view of this we have the following theorem.

THEOREM 2.10: *Let V = $Z_p \times Z_p \times Z_p$; p a prime be a vector space defined over the field F = Z_p. Let P = {0, p–1} $\subseteq Z_p$ be a subset of Z_p. T = {all quasi set vector subspaces of V defined over the set P = {0, p–1} $\subseteq Z_p$}. T is a quasi set topological vector subspace of V defined over the set P = {0, p–1}. T is a second countable quasi set topological vector subspace. Let B be the basic set of T. Number of elements in B is p^3–1 / 2. Clearly the lattice associated with T is a Boolean algebra of order p^3–1/2.*

We can generalize this by the following theorem.

THEOREM 2.11: *Let V =* $\underbrace{Z_p \times Z_p \times \ldots \times Z_p}_{n\text{-times}}$ *be a vector space defined over the field F = Z_p. Let T = {all quasi set vector subspaces of V defined over the set {0, p–1} $\subseteq Z_p$} be the quasi set topological vector subspace of V over the set P = {0, p–1}.*

(1) T is quasi set topological vector subspace of V defined over the set P $\subseteq Z_p$ satisfies the second axiom of countability.

(2) The basic set B of T is of order $\frac{(p^n - 1)}{2}$.

(3) The lattice associated with T is a Boolean algebra of order $2^{\left(\frac{p^n-1}{2}\right)}$.

The proof of the two theorems is direct and can be easily proved.

Now we proceed onto define dual quasi set topological vector subspace of V over a set.

DEFINITION 2.4: *Let V be a vector space defined over the field Z_p. T = {collection of all quasi set vector subspaces of V defined over the set P = {0, p–1} $\subseteq Z_p$}. T is a quasi set topological vector subspace of V over P; T is defined as the fundamental dual quasi set topological vector subspace of V over P = {0, p–1} relative to the fundamental quasi set topological vector subspace of V defined over the set S = {0, 1} $\subseteq Z_p$.*

Example 2.24: Let V = $Z_{13} \times Z_{13}$ be a vector space defined over the field F = Z_{13}. Let T = {collection of all quasi set vector subspaces of V defined over the set P = {0, 1}}. T is a quasi set topological vector subspace of V over P.

Suppose B is the basic set of V then order of B is $13^2 – 1$.

Let S = {collection of all quasi set topological vector subspaces of V defined over the set $P_1 = \{0, 12\} \subseteq Z_{13}$}. S is a quasi set topological vector subspace of V defined over the set $P_1 = \{0, 12\}$.

Let B_1 be the basic set of S. Now order of B_1 is $\frac{(13^2 - 1)}{2}$.

S is the fundamental dual quasi set topological vector subspace of V defined over the set $P_1 = \{0, 12\}$ to the fundamental quasi set topological vector subspace T of V defined over the set $P = \{0, 1\} \subseteq Z_{13}$.

Now we have seen example of fundamental dual quasi set topological vector subspace of V and the fundamental quasi set topological vector subspace of V.

Now we have discussed and described the properties of quasi set topological vector subspaces.

Apart from quasi set fundamental dual and fundamental topological vector subspaces we have other than these more number of quasi set topological vector subspaces of V defined over subsets in Z_p.

Now for quasi set topological vector subspaces defined over Q or R. The fundamental dual quasi set topological vector subspaces are defined over the set $P_1 = \{0, -1\} \subseteq Q$ or R.

We illustrate this by an example.

Example 2.25: Let $V = Q \times Q$ be a vector space defined over the field Q. Let $P_1 = \{0, -1\} \subseteq Q$ be a proper subset of Q. T = {collection of all quasi set vector subspaces of V defined over the set $P_1 = \{0, -1\} \subseteq Q$}. T is a quasi set topological vector subspace of V defined over the set P_1.

Now the basic set B of T is as follows: B = {{(0, 0), (a, 0), (–a, 0)}, {(0, 0), (0, b), (0, –b)}, {(0, 0), (a, b), (–a, –b)} | a, b $\in$ Q \ {0}}. T is the dual quasi set topological vector subspace of

V defined over the set P_1 = {0, –1} ⊆ Q. Suppose S = {collection of all quasi set vector subspaces of V defined over the set P = {0, 1} ⊆ Q}; S is the fundamental quasi set topological vector subspace of V defined over the set P = {0, 1}.

Suppose B_1 is the basic set of S over the set P = {0, 1}. Now B_1 = {{(0, 0), (a, b)} | a, b ∈ Q, (a, b) ≠ (0, 0)} is the basic set. With the basic set as the atoms we can get an infinite Boolean algebra associated with S over P.

We see in case of the dual fundamental quasi set topological vector subspace the basic set B serves as the atom of the related infinite Boolean algebra. Clearly both B and B_1 are of different cardinality $o(B_1) > o(B)$.

Further we see both the fundamental dual quasi set topological vector subspaces as well as fundamental quasi set topological vector subspace over P_1 and P respectively satisfy the second axiom of countability.

Interested reader can study the above example by replacing Q by R.

Example 2.26: Let V = Q × Q × Q be the vector space defined over the field F = Q. Let P = {0, 1, –1} ⊆ Q = F be a set in V. T = {all quasi set vector subspaces of V defined over the set P = {0, 1, –1} ⊆ Q = F}. T is a quasi set topological vector subspace of V defined over the set P = {0, 1, –1}. The basic set B of T is given by B = {{(0, 0, 0), (a, b, c), –(a, b, c) = (–a, –b, –c)} | a, b, c ∈ Q and (a, b, c) ≠ (0, 0, 0)}. Clearly T satisfies the second axiom of countability.

Example 2.27: Let V = $Z_5 \times Z_5$ be the vector space defined over the field Z_5 = F. Let P = {0, 1} ⊆ Z_5. T = {all quasi set vector subspaces of V defined over the set P = {0, 1} ⊆ Z_5}. T is a quasi set topological vector subspace with the basic set B_T = {{(0, 0), (1, 0)}, {(0, 0), (0, 1)}, …, {(0, 0), (4, 4)}} and $o(B_T) = 5^2–1 = 24$.

Let S = {all quasi set vector subspaces of V defined over the set P_1 = {0, 4} $\subseteq$ Z_5}. S is a quasi set topological vector subspace with basic set B_S = {{(0, 0), (1, 0), (4, 0)}, {(0, 0), (0, 1), (0, 4)}, {(0, 0), (2, 0), (3, 0)}, {(0, 0), (0, 2), (0, 3)}, {(0, 0), (1, 1), (4, 4)}, {(0, 0), (2, 2), (3, 3)}, {(0, 0), (1, 2), (4, 3)}, {(0, 0), (2, 1), (3, 4)}, {(0, 0), (1, 3), (4, 2)}, {(0, 0), (3, 1), (2, 4)}, {(0, 0), (2, 3), (3, 2)}, {(0, 0), (1, 4), (4, 1)}} of S over the set P_1 = {0, 4} $\subseteq$ Z_5.

Let M = {all quasi set vector subspaces of V defined over the set P_2 = {1, 4}}. M is a topological space of quasi set vector subspaces of V over P_2. Let B_M be the basic set of M.

B_M = {(0, 0), {(1, 0), (4, 0)}, {(0, 1), (0, 4)}, {(0, 2), (0, 3)}, {(2, 0), (3, 0)} {(1, 2), (4, 3)}, {(2, 1), (3, 4)}, {(1, 3), (4, 2)}, {(3, 1), (2, 4)}, {(3, 2), (2, 3)}, {(1, 4), (4, 1)}, {(1, 1), (4, 4)}, {(2, 2), (3, 3)}} is the basic set of M.

Let N = {all quasi set vector subspaces of V defined over the set {0, 1, 4} $\subseteq$ Z_5}. N is a quasi set topological vector subspace of V defined over the set {0, 1, 4}.

The basic set B_N of N is as follows: B_N = {{(0, 0), (1, 0), (4, 0)}, {(0, 0), (0, 1), (0, 4)}, {(0, 0), (2, 0), (3, 0)}, {(0, 0), (0, 2), (0, 3)}, {(0, 0), (1, 1), (4, 4)}, {(0, 0), (2, 2), (3, 3)}, {(0, 0), (2, 3), (3, 2)}, {(0, 0), (1, 3), (4, 2)}, {(0, 0), (3, 1), (2, 4)}, {(0, 0), (1, 2), (4, 3)}, {(0, 0), (2, 1), (3, 4)} {(0, 0), (1, 4), (4, 1)}} is the basic set identical with the basic set B_S of S.

Inview of this we see we can have quasi set topological vector subspaces to be the same even for different subsets in the field over which the vector space is defined.

THEOREM 2.12: *Let V = $\underbrace{Z_p \times Z_p \times \ldots \times Z_p}_{n\text{-times}}$ be a vector space defined over the field Z_p. There exists atleast two quasi set topological vector subspaces of V which are identical (same) but defined over different subsets of Z_p.*

Proof: Let M = {collection of all set quasi vector subspaces of V defined over the set {0, p–1} ⊆ Z_p}; be the quasi set topological vector subspace of V defined over {0, p–1} ⊆ Z_p. Let P = {collection of all set quasi vector subspaces of V defined over the set {0, 1, p–1} ⊆ Z_p} be the quasi set topological vector subspace of V defined over the set {0, 1, p–1} ⊆ Z_p. M and P have the same basic sets, that is M and P are identical quasi set topological vector subspaces defined over the sets {0, p–1} and {0, 1, p–1} respectively.

Hence the claim.

Thus distinct sets need not pave way for different quasi set topological vector subspaces.

Example 2.28: Let V = $Z_5 \times Z_5$ be a vector space defined over Z_5. We have seen quasi set topological vector subspaces of V defined over the sets {0, 1}, {0, 4}, {4, 1} and {0, 1, 4}. Now we find on other subsets of Z_5 the quasi set topological vector subspaces defined over the set A = {0, 1, 2, 3} ⊆ Z_5. B = {collection of all quasi set vector subspaces defined over the set {0, 1, 2, 3} = A ⊆ Z_5}. B is quasi set topological vector subspace of V defined over the set A.

Suppose X is the basic set of B; then X = {{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)}, {(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)}, {(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)}, {(0, 0), (1, 2), (2, 4), (3, 1), (4, 3)}, {(0, 0), (2, 1), (4, 2), (1, 3), (3, 4)} {(0, 0), (2, 3), (3, 2), (4, 2), (2, 4)}}; we see the associated lattice of B is a Boolean algebra L and L is of order 2^6 with X as its atom set and {(0, 0)} is the least element L and V is the greatest element of L.

Let us consider a set W = {0, 3, 4} ⊆ Z_5. Suppose Y = {collection of all quasi set vector subspaces of V defined over the set W = {0, 3, 4} ⊆ Z_5}, Y is a quasi set topological vector subspace of V defined over the set W = {0, 3, 4} ⊆ Z_5.

The basic set of Y be D_Y = {{(0, 0), (1, 0), (3, 0), (4, 0), (2, 0)}, {(0, 0), (0, 1), (0, 3), (0, 2), (0, 4)}, {(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)}, {(0, 0), (1, 4), (3, 2), (4, 1), (2, 3)}, {((0, 0), (4, 3), (2, 4), (1, 2), (3, 1)}, {(0, 0), (2, 1), (1, 3), (4, 2), (3, 4)}}. D_Y is the basic set of the quasi set topological vector subspace of V defined over the set W = {0, 3, 4} $\subseteq Z_5$.

Let L = {0, 2, 3, 4} $\subseteq Z_5$ be a subset of Z_5. F = {collection of all quasi set vector subspaces of V defined over the set L = {0, 2, 3, 4} $\subseteq Z_5$}, be a quasi set topological vector subspace of V defined over the set L = {0, 2, 3, 4}.

The basic set of F over the set L is given by Z = {{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)}, {(0, 0), (0, 1), (0, 2), (0, 3), (4, 0)}, {(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)}, {(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)}, {(0, 0), (2, 1), (4, 2), (3, 4), (1, 3)}, {(0, 0), (2, 3), (4, 1), (3, 2), (1, 4)}}.

Clearly $o(Z) = 6 = \frac{(5^2 - 1)}{4}$.

The lattice associated with F is a Boolean algebra with {(0, 0)} as its least element and V as its greatest element. Further the order of the Boolean algebra is 2^6.

Let U = {1, 2, 3, 4} $\subseteq Z_5$ be a subset of Z_5. Suppose E = {collection of all quasi set vector subspaces of V defined over the set U = {1, 2, 3, 4} $\subseteq Z_5$} be the quasi set topological vector subspace defined over U.

The basic set of E over U is given by G = {{(1, 0), (2, 0), (3, 0), (4, 0)}, {(0, 1), (0, 2), (0, 3), (0, 4)}, {(0, 0)}, {(1, 1), (2, 2), (3, 3), (4, 4)}, {(1, 2), (2, 4), (3, 1), (4, 3)}, {(2, 1), (4, 2), (1, 3), (3, 4)}, {(1, 4), (4, 1), (2, 3), (3, 2)}}.

Clearly o(G) = 7 and for this lattice L associated with E we see ‘ϕ’ the empty set, is the least element and V is the largest element of the lattice L. Infact L is a Boolean algebra of order 2^7.

Thus we can using different subsets of the field get different topological quasi set vector subspaces defined over different subsets.

Now several questions are to be answered.

(i) If $S \subseteq F$ be a subset of a field and if $P \subseteq S \subseteq F$ and P a proper subset of S; does there exist any relation between the quasi set topological vector subspaces of V defined over S and that of over P.
To this end we first study some examples.

(ii) Characterize those sets P_i in F such that the quasi set topological vector subspaces of V defined over $P_i \subseteq F$ are isomorphic.

Example 2.29: Let $V = Z_7 \times Z_7 \times Z_7$ be a vector space defined over the field $F = Z_7$. Let S = {0, 1, 2, 3} and P = {0, 1, 3} be two subsets of Z_7. Clearly $P \subseteq S \subseteq Z_7$.

Let T = {all quasi set vector subspaces of V defined over P} be the quasi set topological vector subspace of V defined over P and let W = {all quasi set vector subspaces of V defined over S} be the quasi set topological vector subspaces of V over S.

We will denote the basic set of T by B_T and that of W by B_W respectively.

Now B_T = {{(0, 0), (1, 0), (3, 0), (2, 0), (6, 0), (4, 0), (5, 0)}, {(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6)}, {(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}}, {(0, 0), (1, 2), (3, 6), (2, 4), (6, 5), (4, 1), (5, 13)}, {(0, 0), (2, 1), (6, 3), (4, 2), (5, 6), (1, 4)}, {(0, 0), (1, 3), (3, 2), (2, 6), (6, 4), (4, 5), (5, 1)}, {(0, 0), (1, 5), (3, 1), (2, 3), (6, 2), (4, 6), (5, 4)}, {(0, 0), (1, 6), (3, 4), (2, 5), (6, 1), (4, 3), (5, 2)}}.

Clearly $o(B_T) = 8$.

Now we find B_W = {{(0, 0), (1, 0), (2, 0), (3, 0), (6, 0), (4, 0), (5, 0)}, {(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6)}, {(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}, {(0, 0), (1, 2), (2, 4), (4, 1), (3, 6), (6, 5), (5, 3)} {(0, 0), (2, 1), (4, 2), (1, 4), (3, 5), (5, 6), (6, 3)} {(0, 0), (1, 3), (2, 6), (4, 5), (3, 2), (6, 4), (5, 1)}, {(0, 0), (3, 1), (6, 2), (5, 4), (1, 5), (2, 3), (4, 6)}, {(0, 0), (1, 6), (2, 5), (4, 3), (3, 4), (6, 1), (5, 2)}}. Clearly $o(B_W) = 8$.

Thus though P = {0, 1, 3} ⊆ {0, 1, 2, 3} = S ⊆ Z_7 we see $B_W = B_T$.

Let B = {0, 1, 2, 3} and M = {0, 2} be subsets of Z_7. Clearly M ⊆ S. Now let

N = {all quasi set vector subspaces of V defined over M} be the topological space of quasi set vector subspaces defined over M ⊆ Z_7. Let B_N denote the basic set of N.

B_N = {{(0, 0), (1, 0), (2, 0), (4, 0)}, {(0, 0), (0, 1), (0, 2), (0, 4)}, {(0, 0), (3, 0), (6, 0), (5, 0)}, {(0, 0), (0, 3), (0, 6), (0, 5)}, {(0, 0), (1, 1), (2, 2), (4, 4)}, {(0, 0), (3, 3), (6, 6), (5, 5)}, {(0, 0), (1, 2), (2, 4), (4, 1)}, {(0, 0), (2, 1), (4, 2), (1, 4)}, {(0, 0), (1, 3), (2, 6), (4, 5)}, {(0, 0), (3, 1), (6, 2), (5, 4)}, {(0, 0), (1, 5), (2, 3), (4, 6)}, {(0, 0), (5, 1), (3, 2), (6, 4)}, {(0, 0), (1, 6), (2, 5), (4, 3)}, {(0, 0), (6, 1), (5, 2), (3, 4)}, {(0, 0), (3, 5), (6, 3), (5, 6)}, {(0, 0), (5, 3), (3, 6), (6, 5)}}.

Clearly $o(B_N) = 16$.

We see M ⊆ S but elements of B_N are subsets of the elements of B_W. This can be seen by observing B_N and B_W.

Example 2.30: Let V = $Z_{11} \times Z_{11}$ be a vector space defined over the field Z_{11}. Take P = {0, 6, 5} and P_1 = {0, 7, 4}, subsets of Z_{11}. To find the quasi set topological vector subspaces associated with (or over) P_1 and P respectively.

Let S = {all quasi set vector subspaces of V defined over the set P}; be the quasi set topological vector subspace of V defined over the set P. M = {all quasi set vector subspaces of V defined over the set P_1} be the quasi set topological vector subspace of V defined over the set P_1.

The basic set B_S of S is B_S = {{(0, 0), (1, 0), (6, 0), (5, 0), (8, 0), (4, 0), (9, 0), (7, 0), (2, 0), (10, 0), (3, 0)}, {(0, 0), (1, 1), (2, 2), …, (10, 10)}; {(0, 0), (1, 2), (5, 10), (3, 6), (4, 8), (9, 7), (6, 1), (8, 5), (7, 3), (2, 4), (10, 9)}, {(0, 0), (1, 3), (6, 7), (3, 9), (7, 10), (9, 5), (10, 8), (5, 4), (8, 2), (4, 1), (2, 6)}, {(0, 0), (1, 4) (6, 2), (3, 1), (7, 6), (9, 3), (10, 7), (5, 9), (8, 10), (4, 5), (2, 8), (1, 4), (6, 2)}, {(0, 0), (1, 5), (6, 8), (3, 4), (7, 2), (9, 1), (10, 6), (5, 3), (8, 7), (4, 9), (2, 10)}, {(0, 0), (5, 1), (8, 6), (4, 3), (2, 7), (1, 9), (6, 10), (3, 5), (7, 8), (9, 4), (10, 2)}, {(0, 0), (1, 6), (6, 3), (3, 7), (7, 9), (9, 10), (10, 5), (5, 8), (8, 4), (4, 2), (2, 1)}, {(0, 0), (1, 7), (6, 9), (3, 10), (7, 5), (9, 8), (10, 4), (5, 2), (8, 1), (4, 6), (2, 3)} {(0, 0), (1, 8), (6, 4), (3, 2), (7, 1), (9, 6), (10, 3), (5, 7), (8, 9), (4, 10), (2, 5)}, {(0, 0), (1, 9), (6, 10), (3, 5), (7, 8), (9, 4), (10, 2), (5, 1), (8, 6), (4, 3), (2, 7)}, {(0, 0), (1, 10), (6, 5), (3, 8), (7, 4), (9, 2), (10, 1), (5, 6), (8, 3), (4, 7), (2, 9)}}. $o(B_S) = 12$.

Now we consider B_M, the basic set of the quasi set topological vector subspace of M over P_1 = {0, 7, 4}.

B_M = {{(0, 0), (1, 0), (7, 0), (5, 0), (2, 0), (3, 0), (10, 0), (4, 0), (6, 0), (9, 0), (8, 0)}, {(0, 1), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (0, 10)}, {(0, 0), (1, 3), (7, 10), (5, 4), (2, 6), (3, 9), (10, 8), (4, 1), (6, 7), (9, 5), (8, 2)}, {(0, 0), (1, 4), (7, 6), (5, 9), (2, 8), (3, 1), (10, 7), (4, 5), (6, 2), (9, 3), (8, 10)}, {(0, 0), (1, 5), (7, 2), (5, 3), (2, 10), (3, 4), (10, 6), (4, 9), (6, 8), (9, 1), (8, 7)}, {(0, 0), (1, 6), (7, 9), (5, 8), (2, 1), (3, 7), (10, 5), (4, 2), (6, 3), (9, 10), (8, 4)}, {(0, 0), (1, 7), (7, 5), (5, 2), (2, 3), (3, 10), (10, 4), (4, 6), (6, 9), (9, 8), (8, 1)}, {(0, 0), (1, 8), (7, 1), (5, 7), (2, 3), (3, 2), (10, 3), (4, 10), (6, 4), (9, 6), (8, 9)}, {(0, 0), (1, 9), (7, 8), (5, 1), (2, 7), (3, 5), (10, 2), (4, 3), (6, 10), (9, 4), (8, 6)}, {(0, 0), (1, 10), (7, 4), (5, 6), (2, 9), (3, 8), (10, 1), (4, 7), (6, 5), (9, 2), (8, 3)}, {(0, 0), (1, 1), (2, 2), …, (10, 10)}, {(0, 0), (1, 2), (7, 3), (5, 10), (2, 4), (3, 6), (10, 9), (4, 8), (6, 1), (9, 7), (8, 5)}}.

$o(B_M) = 12$.

We see M and S are identical as topologies.

Take A = {0, 5, 10} and C = {0, 5} to be proper subsets of the field Z_{11}.

D = {all quasi set vector subspaces defined over the subset A = {0, 5, 10}} be the quasi set topological vector subspace of V over A and E = {all quasi set vector subspaces of V defined over the set C = {0, 5}}} be the quasi set topological vector subspace of V over C.

Let B_D and B_E be the basic sets of D and E respectively.

B_D = {{(0, 0), (1, 0), (5, 0), (3, 0), (4, 0), (9, 0), (10, 0), (8, 0), (7, 0), (2, 0), (6, 0)}, {(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (0, 10)}, {(0, 0), (1, 1), …, (10, 10)}, {(0, 0), (1, 2), (5, 10), (3, 6), (4, 8), (9, 7), (10, 9), (6, 1), (8, 5), (7, 3), (2, 4)}, …, {(1, 10), (5, 6), (3, 8), (4, 7), (9, 2), (10, 1), (6, 5), (8, 3), (7, 4), (2, 9)}}.

$o(B_D) = 12$.

B_E = {{(0, 0), (1, 0), (5, 0), (3, 0), (4, 0), (9, 0)}, {(0, 0), (2, 0), (10, 0), (6, 0), (8, 0), (7, 0)}, {(0, 0), (1, 1), (5, 5), (3, 3), (4, 4), (9, 9)}, {(0, 0), (2, 2), (10, 10), (6, 6), (8, 8), (7, 7)}, {(0, 0), (1, 2), (5, 10), (3, 6), (4, 8), (9, 7)}, {(0, 0), (2, 1), (10, 5), (6, 3), (8, 4), (7, 9)}, {(0, 0), (1, 3), (5, 4), (3, 9), (4, 1), (9, 5)}, {(0, 0), (3, 1), (4, 5), (9, 3), (1, 4), (5, 9)}, {(0, 0), (1, 5), (5, 3), (3, 4), (4, 9), (9, 1)}, {(0, 0), (5, 1), (3, 5), (4, 3), (9, 4), (1, 9)}, {(0, 0), (1, 6), (5, 8), (3, 7), (4, 2), (9, 10)}, {(0, 0), (6, 1), (8, 5), (7, 3), (4, 2), (9, 10)}, {(0, 0), (6, 1), (8, 5), (7, 3), (2, 4), (10, 9)}, {(0, 0), (1, 7), (5, 2), (3, 10), (4, 6), (9, 8)}, {(0, 0), (7, 1), (2, 5), (10, 3), (6, 4), (8, 9)}, {(0, 0), (1, 8), (5, 7), (3, 2), (4, 10), (9, 6)}, {(0, 0), (8, 1), (7, 5), (2, 3), (10, 4), (6, 9)}, {(0, 0), (1, 10), (5, 6), (3, 8), (4, 7), (9, 2)}, {(0, 0), (10, 1), (6, 5), (8, 3), (7, 4), (2, 9), {(0, 0), (2, 6), (10, 8), (6, 7), (8, 2), (7, 10)}, {(0, 0), (6, 2), (8, 10), (7, 6), (2, 8), (10, 7)}, {(0, 0), (2, 7), (10, 2), (6, 10), (8, 6), (7, 8)}, {(0, 0), (7, 2), (2, 10), (10, 6), (6, 8), (8, 7)}, {(0, 0), (2, 10), (10, 6), (6, 8), (8, 7), (7, 2)}, {(0, 0), (10, 2), (6, 10), (8, 6), (7, 8), (2, 7)}}.

Clearly $o(B_E) = 24$ and $C = \{0, 5\} \subseteq \{0, 5, 10\} = A \subseteq Z_{11}$. However the topologies are distinct. We see topology E has its related Boolean algebra to be of order 2^{24} where as the Boolean algebra of the topology D is of order 2^{12}.

Now we see examples of subbasic set of a basic set and the topologies generated by the subbasic sets. Suppose T is a quasi set topological vector subspace of V defined over a set $P \subseteq F$ (F a field over which V is defined).

Let B_T be the basic set of T. Let $S \subseteq B_T$ (S a proper subset of B_T). S will generate a topological vector subspace defined as the quasi set subtopological vector subspace of V defined over the set $P \subseteq F$.

We will illustrate the situation by some examples.

Example 2.31: Let $V = Z_5 \times Z_5$ be a vector space over Z_5. Let $P_1 = \{2, 0, 1\} \subseteq Z_5$ be a subset of Z_5. Let T = {collection of all quasi set vector subspaces of V defined over the set $P_1 \subseteq Z_5$}, be the quasi set topological vector subspace of V defined over the set P_1. Let B_T denote the basic set of T.

$B_T = \{v_1 = \{(0, 0), (1, 0), (2, 0), (4, 0), (3, 0)\}, v_2 = \{(0, 0), (0, 1), (0, 2), (0, 4), (0, 3)\}, v_3 = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}, v_4 = \{(0, 0), (1, 2), (2, 4), (4, 3), (3, 1)\}, \{(0, 0), (2, 1), (4, 2), (1, 3), (3, 4)\} = v_5, \{(0, 0), (1, 4), (2, 3), (4, 1), (3, 2)\} = v_6\}$.

$$o(B^T) = \frac{(5^2 - 1)}{4} = 6.$$

Thus the associated lattice is of order 2^6, that is a Boolean algebra. We see {(0)} is the least element so S = {(0, 0),v_i} will give a quasi set topological vector subspaces with only basic element v_i ($1 \leq i \leq 6$), of course which we call as indiscrete quasi set topological vector subspace (i fixed).

P = {(0, 0), v_i, v_j | $i \neq j$, $1 \leq i, j \leq 6$}, i and j fixed, be the basic set and P generates a topology T_P = {(0, 0), v_i, v_j, $v_i \cup v_j$} that is a quasi set topology with four elements.

Let

B = {(0, 0), v_i, v_j, v_k | i, j and k are distinct; $1 \leq i, j, k \leq 6$} be a basic set for which the associated topology has 8 elements given by

T_B = {(0, 0), v_i, v_j, $v_j \cup v_i$, v_k, $v_k \cup v_i$, $v_j \cup v_k$, $v_i \cup v_j \cup v_k$}. Thus $v_i \cup v_j \cup v_k$ is the largest element of the associated Boolean algebra of order 2^3.

Let C = {(0, 0), v_i, v_j, v_k, v_i | v_i, v_j, v_k and v_l are distinct elements of B_T. $1 \leq i, j, k, l \leq 6$}. T_C the associated quasi set topology has, 2^4 elements and so on.

We call T_C, T_B, T_S, T_P as quasi set subtopologies of vector subspaces of the quasi set topological vector subspace of T.

Infact we have 62 distinct subtopologies for the quasi set topological vector subspaces of T over the same set P_1 = {0, 1, 2}.

Interested reader can construct such quasi set subtopological vector subspaces of any given quasi set topological vector subspace defined over a subset P of the field F. For all such subtopologies are defined only over P. S will generate a quasi set topological vector subspace defined as the quasi set subtopological vector subspace of V defined over the set $P \subseteq F$.

We will illustrate this situation by some examples.

Example 2.32: Let $V = Z_{11} \times Z_{11}$ be a vector space defined over field Z_{11}. Let $P = \{0, 2, 6, 8\} \subseteq Z_{11}$ be a proper subset of Z_{11}.

T = {all quasi set vector subspaces of V defined over the set P} be the quasi set topological vector subspace of V defined over the set P.

The basic set B_T of T is as follows: B_T = {{(0, 0), (1, 0), (2, 0), (6, 0), (8, 0), (4, 0), (3, 0), (9, 0), (5, 0), (10, 0), (7, 0)} = v_1, {(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (0, 10)} = v_2, v_3 = {(0, 0), (1, 1), (2, 2), …, (10, 10)}, v_4 = {(0, 0), (1, 2), (2, 4), (4, 8), (8, 5), (5, 10), (10, 9), (9, 7), (7, 3), (3, 6), (6, 1)}, v_5 = {(0, 0), (2, 1), (4, 2), (8, 4), (5, 8), (10, 5), (9, 10), (7, 9), (3, 7), (6, 3), (1, 6)}, v_6 = {(0, 0), (1, 3), (2, 6), (4, 1), (8, 2), (5, 4), (10, 8), (9, 5), (7, 10), (3, 9), (6, 7)}, v_7 = {(0, 0), (3, 1), (6, 2), (1, 4), (2, 8), (4, 5), (8, 10), (5, 9), (10, 7), (9, 3), (7, 6)}, v_8 = {(0, 0), (1, 5), (2, 10), (4, 9), (8, 7), (5, 3), (10, 6), (9, 1), (7, 2), (3, 4), (6, 8)}, v_9 = {(0, 0), (5, 1), (10, 2), (9, 4), (7, 8), (3, 5), (6, 10), (1, 9), (2, 7), (4, 3), (8, 6)}, v_{10} = {(0, 0), (1, 7), (2, 3), (4, 6), (8, 1), (5, 2), (10, 4), (9, 8), (7, 5), (3, 10), (6, 9)}, v_{11} = {(0, 0), (7, 1), (3, 2), (6, 4), (1, 8), (2, 5), (4, 10), (8, 9), (5, 7), (10, 3), (9, 6)}, v_{12} = {(0, 0), (1, 10), (2, 9), (4, 7), (8, 3), (5, 6), (10, 1), (6, 5), (5, 6), (3, 4), (4, 3)}.

Thus $B_T = \{v_1, v_2, \ldots, v_{12}\}$ and $o(B_T) = 12$. The lattice associated with the quasi set topological vector subspace of V defined over the set P is a Boolean algebra of order 2^{12}.

All subtopological quasi set vector subspaces of V defined over P will be a Boolean algebra of order 2^n; $1 \leq n \leq 11$.

Example 2.33: Let $V = Z_{19} \times Z_{19}$ be the vector space defined over the field Z_{19}. Let $P = \{0, 3, 6, 9, 12, 15, 18\} \subseteq Z_{19}$ be a subset of Z_{19}. T = {collection of all quasi set vector subspaces of V defined over $P \subseteq Z_{19}$}, be a quasi set topological vector subspace of V over P.

Let B_T be the basic set of T. B_T = {{(0, 0), (1, 0), (3, 0), (9, 0), (6, 0), (18, 0), (12, 0), (15, 0), (17, 0), (8, 0), (5, 0), (11, 0), (14, 0), (7, 0), (2, 0), (16, 0), (10, 0), (4, 0), (13, 0) and so on}} = {$v_1, v_2, \ldots, v_{20}$}; each v_i is of cardinality 19, $1 \leq i \leq 20$.

We see the associated lattice of T is a Boolean algebra of order 2^{20}. We get several quasi set subtopological vector subspaces of V defined over the set P.

Each of the subtopological quasi set subvector spaces give a lattice which is a Boolean algebra of order 2^n, $1 \leq n \leq 19$.

Now we define the concept of quasi subset subtopological vector subspace of V defined over a subset in F; V is a vector space defined over the field F.

DEFINITION 2.5: *Let V be a vector space defined over the field F. Let P $\subseteq$ F be a proper subset of F. T = {collection of all quasi set vector subspaces of V defined over the set P $\subseteq$ F} be a quasi set topological vector subspace of V defined over the set P.*

Let M $\subseteq$ P be a proper subset of M. If S = {collection of all quasi set vector subspaces of V defined over the set M $\subseteq$ P}, then we define S to be the collection of quasi subset vector subspaces of V defined over the set M $\subseteq$ P. Infact S is a quasi subset subtopological vector subspace of V defined over the subset M $\subseteq$ P.

We will illustrate this situation by some examples.

Example 2.34: Let V = $Z_7 \times Z_7$ be a vector space defined over the field F = Z_7. P = {0, 1, 6, 4} $\subseteq Z_7$ be a subset of Z_7. T = {all quasi set vector subspaces of V defined over the set P = {0, 1, 4, 6} $\subseteq Z_7$} be the quasi set topological vector subspace of V over the set P.

B_T^P = {{(0, 0), (1, 0), (6, 0), (4, 0), (5, 0), (2, 0), (3, 0)}, {(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6)}, {(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}, {(0, 0), (1, 2), (4, 1), (6, 5), (2, 4), (5, 3), (3, 6)}, {(0, 0), (2, 1), (1, 4), (5, 6), (4, 2), (3, 5), (6, 3)}, {(0, 0), (1, 3), (4, 5), (6, 4), (3, 2), (5, 1), (2, 6)}, {(0, 0), (3, 1), (5, 4), (4, 6), (2, 3), (1, 5), (6, 2)}, {(1, 6), (6, 1), (3, 4), (4, 3), (11, 5), {5, 11), (0, 0)}}.

$o(B_T^P) = 8$.

Let M = {0, 4} $\subseteq$ P = {0, 4, 6, 1} $\subseteq$ Z_7. Suppose S={all quasi set vector subspaces of V defined over the set M}; be the quasi set topological vector subspace of V defined over the set M = {0, 4} $\subseteq$ P.

Now B_S^M = {(0, 0), (1, 0), (4, 0), (2, 0)}, {(0, 0), (0, 1), (0, 4), (0, 2)}, {(0, 0), (1, 1), (4, 4), (2, 2)}, {(0, 0), (3, 0), (5, 0), (6, 0)}, {(0, 0), (0, 3), (0, 5), (0, 6)}, {(0, 0), (3, 3), (5, 5), (6, 6)}, {(0, 0), (1, 2), (4, 1), (2, 4)}, {(0, 0), (2, 1), (4, 1), (4, 2)}, {(0, 0), (1, 3), (4, 5), (2, 6)}, {(0, 0), (3, 1), (4, 5), (6, 2)}, {(0, 0), (1, 5), (4, 6), (2, 3)}, {(0, 0), (5, 1), (6, 4), (3, 2)}, {(0, 0), (1, 6), (4, 3), (2, 5)}, {(0, 0), (6, 1), (3, 4), (5, 2)}, {(0, 0), (3, 6), (5, 3), (6, 5)}, {(0, 0), (3, 6), (3, 5), (5, 6)}}.

$o(B_S^M) = 16$.

We see every set in B_S^M is a subset of a set in B_T^P. Thus we can say in general the larger the subset which is taken in the field Z_p the smaller is the cardinality of the basic set of the quasi set topological vector subspace and the smaller the subset taken in the field Z_p, the larger is the cardinality of the basic set of the quasi set topological vector subspace.

This is also seen from the above example. It may sometimes happen for both the subsets; the cardinality of the basic set is the same. This is the case for L = {0, 4, 6} $\subseteq$ {0, 6, 4, 1} = P. That is $o(B_A^L) = o(B_T^P)$ where A is the quasi set topological vector subspace of V defined over the subset L $\subseteq$ P $\subseteq$ Z_7.

Example 2.35: Let $Z_{13} \times Z_{13}$ be a vector space defined over the field Z_{13} = F. Take P = {0, 2, 5, 8, 10} $\subseteq Z_{13}$. Let T = {collection of all quasi set vector subspaces of V defined over the set P $\subseteq Z_{13}$} be the quasi set topological vector subspace of V defined over the set P $\subseteq Z_{13}$.

The basic set of T is denoted by

B_T^P = {{(0, 0), (1, 0), (2, 0), (5, 0), (8, 0), (10, 0), (4, 0), (3, 0), (6, 0), (12, 0), (11, 0), (9, 0), (7, 0)}, {(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 10), (0, 9), (0, 11), (0, 12)}, {(0, 0), (1, 1), (2, 2), …, (12, 12)}, {(0, 0), (1, 2), (2, 4), (4, 8), (8, 3), (3, 6), (6, 12), (12, 11), (11, 9), (9, 5), (5, 10), (10, 7), (7, 1)}, {(0, 0), (2, 1), (4, 2), (8, 4), (3, 8), (6, 3), (12, 6), (11, 12), (9, 11), (5, 9), (10, 5), (7, 10), (1, 7)}, {(0, 0), (1, 3), (2, 6), (4, 12), (8, 11), (3, 9), (6, 5), (12, 10), (11, 7), (9, 1), (5, 2), (10, 4), (7, 8)} {(0, 0), (3, 1), (6, 2), (12, 4), (11, 8), (9, 3), (5, 6), (10, 12), (7, 11), (1, 9), (2, 5), (4, 10), (8, 7)}, {(0, 0), (1, 4), (2, 8), (4, 3), (8, 6), (3, 12), (6, 11), (12, 9), (11, 5), (9, 10), (5, 7), (10, 1), (7, 2)} {(0, 0), (4, 1), (8, 2), (3, 4), (6, 8), (12, 3), (11, 6), (9, 12), (5, 11), (10, 9), (7, 5), (1, 10), (2, 7)}, {(0, 0), (1, 5), (2, 10), (4, 7), (8, 1), (3, 2), (6, 4), (12, 8), (11, 3), (9, 6), (5, 12), (10, 11), (7, 9)}, {(0, 0), (5, 1), (10, 2), (7, 4), (1, 8), (2, 3), (4, 6), (8, 12), (3, 11), (6, 9), (12, 5), (11, 10), (9, 7)}, {(0, 0), (1, 6), (2, 12), (4, 11), (8, 9), (3, 5), (6, 10), (12, 7), (11, 1), (9, 2), (5, 4), (10, 8), (7, 3)}, {(0, 0), (6, 1), (12, 2), (11, 4), (9, 8), (5, 3), (10, 6), (7, 12), (1, 11), (2, 9), (4, 5), (8, 10), (3, 7)}, {(0, 0), (1, 12), (2, 11), (4, 9), (8, 5), (3, 10), (6, 7), (12, 1), (11, 2), (9, 4), (5, 8), (10, 3), (7, 6)}}. o (B_T^P) = 14.

Now take M = {0, 8} $\subseteq$ P $\subseteq Z_{13}$. Let S = {all quasi set vector subspaces of V defined over the set M} be the quasi subset topological vector subspace of V defined over the set M.

B_S^M = {{(0, 0), (1, 0), (8, 0), (12, 0), (5, 0)}, {(0, 0), (0, 1), (0, 8), (0, 12), (0, 5)}, {(0, 0), (2, 0), (3, 0), (11, 0), (10, 0)}, {(0, 0), (0, 2), (0, 3), (0, 11), (0, 10)}, {(0, 0), (4, 0), (6, 0), (9, 0), (7, 0)}, {(0, 0), (0, 4), (0, 6), (0, 9), (0, 7)}, {(0, 0), (4, 4), (6, 6), (9, 9), (7, 7)}, {(0, 0), (2, 2), (3, 3), (11, 11), (10, 10)}, {(0, 0), (1, 1), (8, 8), (12, 12), (5, 5)}, {(0, 0), (1, 2), (8, 3), (12, 11), (5, 10)}, {(0, 0), (2, 1), (3, 8), (11, 12), (10, 5)}, {(0, 0), (1, 3), (8, 11), (12, 10), (5, 2)}, {(0, 0), (3, 1), (11, 8), (10, 12), (2, 5)}, {(0, 0), (1, 4), (8, 6), (12, 9), (5, 7)}, {(0, 0), (4, 1), (6, 8), (9, 12), (7, 5)}, {(0, 0), (1, 5), (8, 1), (12, 8), (5, 12)}, {(0, 0), (5, 1), (1, 8), (8, 12), (12, 5)}, {(0, 0), (1, 6), (8, 9), (12, 7), (5, 6)}, {(0, 0), (6, 1), (9, 8), (7, 12), (5, 6)}, {(0, 0), (1, 7), (8, 4), (12, 6), (5, 9)}, {(0, 0), (7, 1), (4, 8), (6, 12), (9, 5)}, {(0, 0), (1, 9), (8, 7), (12, 4), (5, 6)}, {(0, 0), (9, 1), (7, 8), (4, 12), (6, 5)}, {(0, 0), (1, 10), (8, 2), (12, 3), (5, 11)}, {(0, 0), (10, 0), (2, 8), (3, 12), (11, 5)}, {(0, 0), (1, 11), (8, 10), (12, 2), (5, 3)}, {(0, 0), (11, 1), (10, 8), (2, 12), (3, 5)}, {(0, 0), (1, 12), (8, 5), (12, 1), (5, 8)}, {(0, 0), (2, 3), (3, 11), (11, 10), (10, 2)}, {(0, 0), (3, 2), (11, 3), (10, 11), (2, 10)}, {(0, 0), (2, 4), (3, 6), (11, 9), (10, 7)}, {(0, 0), (4, 2), (6, 3), (9, 11), (7, 10)}, {(0, 0), (2, 6), (3, 9), (11, 7), (10, 4)}, {(0, 0), (6, 2), (9, 3), (7, 11), (4, 10)}, {(0, 0), (2, 7), (3, 4), (11, 6), (10, 9)}, {(0, 0), (7, 2), (4, 3), (6, 11), (9, 10)}, {(0, 0), (2, 9), (3, 7), (11, 4), (10, 6)}, {(0, 0), (9, 2), (7, 3), (4, 11), (6, 10)}, {(0, 0), (2, 11), (3, 10), (11, 2), (10, 3)}, {(0, 0), (6, 7), (9, 4), (7, 6), (4, 9)}, {(0, 0), (7, 9), (4, 7), (6, 4), (9, 6)}, {(0, 0), (9, 7), (4, 7), (4, 6), (6, 9)}} is of order 42 and the subtopological vector subspace of quasi subsets of M is of higher cardinality.

Now having seen the notion of quasi subset subtopological vector subspaces; we now proceed onto suggest some problems to the reader.

Problems:

1. Find some interesting properties enjoyed by quasi set vector subspaces of a vector space V defined over the set $P \subseteq F$; F is the field over which V is defined.

2. Find the number of quasi set vector subspaces of V; defined over the set $P = \{0, 2, 3, 4, 7\} \subseteq Z_{13}$, where $V = Z_{13} \times Z_{13}$ is defined over the field $F = Z_{13}$.

3. How many quasi set vector subspaces of V over the set $P = \{0, 10\} \subseteq Z_{11}$ exists? ($V = Z_{11} \times Z_{11} \times Z_{11}$ vector space defined over the field Z_{11}).

4. How many quasi set vector subspaces can be constructed using different subsets of the field $F = Z_5$? ($V = Z_5 \times Z_5 \times Z_5 \times Z_5$ is a vector space defined over the field F).

5. Let $V = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| a_i \in Z_{11}, 1 \leq i \leq 6\right\}$ be a vector space defined over the field Z_{11}.
Take $P = \{0, 1, 3, 7\} \subseteq Z_{11}$. T = {collection of all quasi set vector subspaces of V defined over the set $P = \{0, 1, 3, 7\} \subseteq Z_{11}$}.

(i) Is T a quasi set topological vector subspace defined over the set P?
(ii) Find the basic set of T.
(iii) Find the lattice L associated with T.
(iv) Is L a Boolean algebra?
(v) How many quasi set subtopological vector subspaces of T exists over P?
(vi) How many quasi subset subtopological vector subspaces of T exist over P?

6. Let $V = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in Z_{19}, 1 \leq i \leq 9\right\}$ be a vector space over the field Z_{19}. Study (i) to (vi) mentioned in case of V in problem 5 by taking $P = \{6, 1, 3, 2\ 17\} \subseteq Z_{19}$.

7. Let $V = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in Q, 1 \leq i \leq 5\}$ be a vector space defined over the field Q. For $P = \{0, -1, 1\} \subseteq Q$;

   (i) Find the quasi set topological vector subspace T defined over the set P.
   (ii) Does T satisfy second countability axiom?
   (iii) If L is the associated lattice with minimum (least) element as $\{(0, 0, 0, 0, 0)\}$ and maximum element as V; will atoms of L be the basic set of T?

8. Study problem (7) in case of

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{19} & a_{20} \end{bmatrix} \middle| \, a_i \in Q, 1 \leq i \leq 20 \right\} \text{ and the set}$$

   $P = \{0, -1\} \subseteq Q$.

9. Let $V = \{R \times R \times R \times R\}$ be a vector space defined over the field Q.

   (i) For $P = \{0, -1, 1\} \subseteq Q$ find the quasi set topological vector subspace T of V defined over P.
   (ii) Does T satisfy the second axiom of countability?

10. Let $V = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix}\right| a_i \in R, 1 \leq i \leq 10\right\}$ be a vector space defined over the field R. For $P = \{0, -1\} \subseteq R$. Will the quasi set topological vector subspace T of V defined over the set P be second countable? Justify your claim.

11. Let $V = Z_{23} \times Z_{23}$ be a vector space defined over the field Z_{23}. Let $P = \{1, 22\} \subseteq Z_{23}$ be a proper subset of Z_{23}.

    (i) Find the quasi set vector subspaces of V associated with P.
    (ii) Will this collection be a quasi set vector subspaces defined over the set P be a quasi topological vector subspace T?
    (iii) Find the basic set of T.
    (iv) Does T satisfy second and first axiom of countability?
    (v) Is the quasi set topological vector subspace T pseudo simple?
    (we say T is pseudo simple if T has no proper quasi subset subtopological vector subspaces. If $P \subseteq F$ (F a field) and $o(P) = 2$; then T is pseudo simple).

12. Find some interesting features related with pseudo simple quasi set topological vector subspaces of V defined over a set $P \subseteq F$ (F a field over which the vector space V is defined).

13. Find some nice applications of quasi set vector subspaces of a vector space V defined over the subset P of a field F.

14. What are the special features enjoyed by the quasi set topological vector subspaces of a vector space V defined over a subset P of a field F?

15. Does there exist a quasi set topological vector subspace of a vector space V defined over a set P which does not satisfy the second axiom countability?

16. Does there exist a quasi set topological vector subspace defined over a set P which does not satisfy the first axiom of countability?

17. Can one say all quasi set topological vector subspaces of a vector space V = $\underbrace{Z_p \times Z_p \times \ldots \times Z_p}_{n\text{-times}}$ defined over a set P ⊆ Z_p always has its associated lattice to be a finite Boolean algebra?

18. Let V = $\left\{ \left[a_{ij}\right]_{n \times m} \,\middle|\, a_{ij} \in Z_{43}; 1 \leq i \leq n \text{ and } 1 \leq j \leq m \right\}$

    be a vector space defined over the field Z_{43}.
    Let P = {0, 2, 4, 6, 8, …, 42} ⊆ Z_{43}.

    (i) Find quasi set vector subspaces of V defined over the set P.
    (ii) If T is the quasi set topological vector subspace of V defined over P; find the basic set B^T of T.
    (iii) Is T ≅ M where M is a quasi set topological vector subspace of S = $\underbrace{Z_{43} \times \ldots \times Z_{43}}_{mn\text{-times}}$ defined over the set P = {0, 2, …, 42} ⊆ Z_{43}?
    (iv) Find lattices L_1 and L_2 associated with T and M respectively.
    (v) Will L_1 be lattice isomorphic with the lattice L_2?

19. Does it imply isomorphic quasi set topological vector subspaces must have isomorphic Boolean algebras?

20. Can we have isomorphic quasi set topological vector subspaces of V which are defined over different subsets of the field?

21. Can we have isomorphic quasi set topological vector subspaces of different vector spaces V_1 and V_2; $V_1 \neq V_2$ defined over different subsets $P_1 \subseteq F_1$ and $P_2 \subseteq F_2$?
(F$_i$ is the field over which V_i is defined i = 1, 2).

22. Give any other interesting property about quasi set topological vector subspaces of a vector space defined over a subset of a field.

23. Let V = $Z_5[x]$ be a vector space defined over the field Z_5. Let P = {0, 1} $\subseteq Z_5$.

    (i) Can we have a quasi set topological vector subspace T of V defined over the set P?
    (ii) Will T be second countable?
    (iii) Can T have a countable basic set B_T?

24. Let $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in Z_{47}, 1 \leq i \leq 5 \right\}$ be a vector space defined over the field Z_{47}. P = {0, 3, 7, 5, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43} $\subseteq Z_{47}$.

    (i) Find at least two different quasi set vector subspaces of V defined over the set P.

(ii) Find the quasi set topological vector subspaces of T of V defined over P.
(iii) Find the lattice associated with T.

25. Let V = C × C × C be a complex vector space defined over the field C. Let P = {–1, 1, i, 0} ⊆ C.

(i) Find all the quasi set vector subspaces of V defined over P.
(ii) Find the quasi set topological vector subspace T of V defined over P.
(iii) Find the basic set of T.
(iv) Does T satisfy the axiom of first and second countability?
(v) Prove T is not pseudo simple.

26. Let V = $C(Z_7) \times C(Z_7)$ be a vector space defined over the field of complex modulo integers; $C(Z_7)$. Let P = {0, 1, i_F} ⊆ $C(Z_7)$.

(i) Find two distinct quasi set vector subspaces of V defined over the set P.
(ii) Let T be the quasi set topological vector subspace defined over the set P. Is T second countable?
(iii) Find B_T the basic set of T.
(iv) Prove T is not pseudo simple.
(v) If V is defined over Z_7 and P = {0, 1, 6} ⊆ Z_7; study the problems (i) to (iv).

27. Let V = $C(Z_{13}) \times C(Z_{13}) \times C(Z_{13})$ be a vector space over the field Z_{13}. Take P = {0, 4, 5, 10, 11} ⊆ Z_{13}.

(i) Find all the quasi set vector subspaces of V defined over the set P.
(ii) Find the quasi set topological vector subspaces, T of V defined over the set P.

(iii) Can T have isomorphic quasi set subtopological vector subspaces of V defined over P?

(iv) Prove T is not pseudo simple.

28. Let V = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix}\right|$ $a_i \in C(Z_{17})$; $1 \leq i \leq 7\}$ be a complex modulo integer vector space defined over the field Z_{17}. Let P = {0, 4, 16} $\subseteq Z_{17}$ be a set.

(i) Find atleast 3 distinct quasi set vector subspaces of V defined over P $\subseteq Z_{17}$.

(ii) Find the quasi set topological vector subspace T of V defined over the set P $\subseteq Z_{17}$.

(iii) Is T pseudo simple?

(iv) Find B_T.

(v) Compare V with W; where

W = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix}\right|$ $a_i \in Z_{17}$; $1 \leq i \leq 7\}$ is a vector space over Z_{17} for problems (i) to (iv) of V.

29. Let W = $\underbrace{C(Z_{19}) \times C(Z_{19}) \times \ldots \times C(Z_{19})}_{10\text{-times}}$ be a vector space defined over the complex modulo integer vector space over the complex modulo integer field $C(Z_{19})$.

Let $P_1 = \{3 + i_F, 0, 3i_F+1, i_F, 18i_F, 1\} \subseteq C(Z_{19})$. Study problems (i) to (iv) described in problem 28 for this W and P_1.

30. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \end{bmatrix} \middle| a_i \in C(Z_{23}); 1 \leq i \leq 12 \right\}$

be a vector space defined over the complex modulo integer field $C(Z_{23})$. $P = \{3i_F, 1, i_F, 8i_F\} \subseteq C(Z_{23})$.

(i) Find at least four quasi set vector subspaces of V defined over P.
(ii) Find the quasi set complex modulo integer topological vector subspaces T of V defined over P.
(iii) Is T second countable?

31. Find any other interesting properties enjoyed by quasi set complex modulo integer topological vector subspaces defined over the field $C(Z_p)$. $p \neq r^2 + n^2$ $(1 \leq r, n < p)$.

32. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in C(Z_{43}), \ 1 \leq i \leq 9 \right\}$

be the complex modulo integer vector space defined over the complex modulo integer field $C(Z_{43})$. $P = \{Z_{43}\} \subseteq C(Z_{43})$.

(i) Find quasi set complex modulo integer vector subspaces of V defined over the set P.
(ii) Is the quasi set topological vector subspace T of V defined over P second countable?
(iii) Find B_T.

33. Let $V = \{a + bg \mid a, b \in C(Z_{17}); g = 5 \in Z_{25}\}$ be a vector space defined over the field Z_{17}.
$P = \{0, 2, 8, 16\} \subseteq Z_{17}$.

(i) Find all quasi set vector subspaces of V defined over $P \subseteq Z_{17}$.
(ii) Find the quasi set dual number topological vector subspace T of V over P.
(iii) Find B_T the basic set of T.

34. Let $M = \{P \times P \times P \times P \mid P = \{a + bg \mid a, b \in C(Z_5), g = 3 \in Z_9\}$ be a dual number vector space defined over the field Z_5.
Let $P = \{a + bg \mid a, b \in \{0, 2, 4i_F\} \subseteq C(Z_5)$.

(i) Find quasi set dual complex number vector subspaces of M defined over P.
(ii) Find the quasi set dual number subtopological vector subspace T of M defined over the set P.
(iii) Is B_T finite?

35. Let $W = \{C(\langle Z_{11} \cup I\rangle) \times C(\langle Z_{11} \cup I\rangle) \times C(\langle Z_{11} \cup I\rangle)\}$ be a quasi set complex modulo integer neutrosophic vector space defined over the field Z_{11}.
$P = \{0, 2, 7\} \subseteq Z_{11}$.

(i) Find quasi set neutrosophic complex modulo integer vector subspaces of W defined over the set P.
(ii) Find the quasi set neutrosophic complex modulo integer topological vector subspaces of W defined over the set $P \subseteq Z_{11}$.
(iii) Find the basic set of T.
(iv) Is T first and second countable?

36. Let B = ⟨Q ∪ I⟩ × ⟨Q ∪ I⟩ × ⟨Q ∪ I⟩ be the neutrosophic vector space defined over the field Q. Let P = {0, 1, –1} ⊆ Q.

    (i) Find the quasi set vector subspaces of B defined over P.
    (ii) Find the quasi set neutrosophic topological vector subspace of T of B defined over P.
    (iii) Find the basic set of T.
    (iv) Is T second countable?
    (v) Is T a pseudo simple space?

37. Let V = $\{(a_1, a_2) \mid a_i = (x_1^i + x_2^i i_F + x_3^i i_F I + x_4^i I) + (y_1^i + y_2^i i_F + y_3^i i_F I + y_4^i I)g \mid 1 \leq i \leq 2,\ g = 10 \in Z_{20};$ $x_k, y_j \in Z_{47}; 1 \leq k, j \leq 4\}$ be a neutrosophic complex modulo integer dual number vector space defined over the field Z_{47}. P = {0, 1, 10, 20, 30, 40} ⊆ Z_{47}.

    (i) Find quasi set neutrosophic dual number complex modulo integer vector subspaces of V defined over P ⊆ Z_{47}.
    (ii) Let T be the quasi set neutrosophic dual number complex modulo integer topological vector subspace of V defined over the set P.

        (a) Find the basic set B_T of T.
        (b) Is T second countable?
        (c) Find the lattice associated with T.
        (d) Is T pseudo simple?

38. Let V = $\{(a_1, a_2, a_3) \mid a_i = x_i + y_i g$ where $x_i, y_i \in Z_{23}$ and g = I the neutrosophic number such that $g^2 = g = I$, $1 \leq i \leq 3\}$ be the vector space defined over the field Z_{23} of special dual like numbers.

(i) Find atleast 3 distinct quasi set vector subspaces of V defined over P = {0, 7, 11} $\subseteq Z_{23}$.

(ii) Find the quasi set special dual like number topological T vector subspace of V defined over P.

(iii) Find the basic set B_T of T.

(iv) Is T second countable?

(v) Find quasi set special dual like number subtopological vector subspaces of T defined over P.

(vi) Find quasi subset special dual like number subtopological vector subspaces of T defined over P.

(vii) Find two isomorphic quasi subset special dual like number subtopological vector subspaces which are isomorphic.

39. Let V = $\{(a_1, a_2, a_3, a_4) \mid a_i = x_1^i + x_2^i I + x_3^i i_F + x_4^i i_F I + (y_1^i + y_2^i I + y_3^i i_F + y_4^i i_F I)g$ where $g = 3 \in Z_6$; $x_j^i, y_j^i \in Z_{19}$; $1 \le i \le 4$; $1 \le j \le 4\}$ be a vector space of special dual like numbers of finite complex neutrosophic modulo integers defined over the complex modulo integer field $C(Z_{19})$. Let P = $\{0, i_F, 3i_F + 5, 8+7i_F\} \subseteq C(Z_{19})$.

Study problems (i) to (vii) given in problem 38.

40. Let V = $\left\{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| a_i = x_i + y_i g \text{ where } x_i, y_i \in Z_{53}, 1 \le i \le 4; g = 6 \in Z_{30}\right\}$ be a special dual like number vector space defined over the field Z_{53}.

Study problems (i) to (vii) given in problem 38.

41. Let V = {(a_1, a_2, a_3, …, a_7) | a_i = $x_1^i + x_2^i g_1 + x_3^i g_2$ with x_j^i ∈ Z_{17}; 1 ≤ i ≤ 3; 1 ≤ j ≤ 7 and g_1 = 3 ∈ Z_6 and g_2 = 4 ∈ Z_8} be a vector space of mixed dual numbers defined over the field Z_{17}.
P = {0, 2, 6, 12, 15} ⊆ Z_{17}.

(i) Find at least 3 quasi set mixed dual number vector subspaces of V defined over the set P ⊆ Z_{17}.

(ii) Let T be the mixed dual number quasi set topological vector subspace of V defined over the set P ⊆ Z_{17}.

(a) Find the basic set B_T of T.
(b) Find at least three quasi set subtopological vector subspaces of T over P.
(c) Find mixed dual number quasi subset subtopological vector subspaces of T defined over S ⊆ P.
(d) Prove T is not pseudo simple.

42. Let V = $\left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix}\right|\right.$ a_i = $x_1^i + x_2^i g$ where g = (–1, –1, –1, –I, –1, 0) x_j^i ∈ Z_{61}; 1 ≤ i ≤ 12, 1 ≤ j ≤ 2, with g^2 = –g = – (1, 1, 1, I, 1, 0)} be a vector space of special quasi dual numbers defined over the field Z_{61}. Let P = {0, 10, 20, 30, 40, 50, 60} ⊆ Z_{61}.

(i) Find three quasi set vector subspaces of V defined over P.

(ii) Find the quasi set topological vector subspace T of V defined over P.
(iii) What is the basic set of T?

43. Let V = $\{a + bg_1 + cg_2 + dg_3 \mid a, b, c, d \in Z_{11};\ g_1 = 6,\ g_2 = 3,\ g_3 = 4\}$ be a special mixed dual number vector space defined over Z_{11}.
Let P = $\{0, 1, 4, 5\} \subseteq Z_{11}$.

(i) Find the quasi set vector subspace of V defined over P.
(ii) Find the quasi set special mixed dual number topological vector subspace T of V defined over P.
(iii) Find the basic set of T.
(iv) Prove T is not pseudo simple?
(v) Find quasi set special mixed dual number subtopological vector subspaces of T defined over P.

44. Let V = $\{\langle Z_{19} \cup I\rangle \times \langle Z_{19} \cup I\rangle\}$ be a neutrosophic vector space over the field Z_{19}.
Let P = $\{0, -1, 1\} \subseteq Z_{19}$.

(i) Find non isomorphic quasi set vector subspaces of V defined over P.
(ii) Find the neutrosophic quasi set topological vector subspace T of V defined over P.

(a) Is T pseudo simple?
(b) Find B_T.
(c) Can T have quasi set subtopological vector subspaces?

45. Let V = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in \langle Z_7 \cup I \rangle; \right.$

$1 \leq i \leq 16\}$ be a neutrosophic vector space defined over the field Z_7. Take P = {0, 1, 2, 5} $\subseteq Z_7$.

(i) Find six distinct non isomorphic quasi set neutrosophic vector subspaces of V defined over P.

(ii) Find the quasi set neutrosophic topological vector subspaces T of V defined over P.

(a) If P_1 = {0, 1} $\subseteq$ P find the corresponding quasi set topological vector subspace S defined over P_1. Is S $\cong$ T?

(b) If P_2 = {2, 1} $\subseteq$ P find the quasi set neutrosophic topological vector subspace W of V defined over P_2. Is W $\cong$ T? Is W $\cong$ S?

(c) Is W pseudo simple?

(d) Is T pseudo simple?

(e) Can S be pseudo simple?

(iii) Find the corresponding lattices of W, S and T and compare them.

Chapter Three

S-QUASI SET TOPOLOGICAL VECTOR SUBSPACES

In this chapter we for the first time introduce the notion of both Smarandache quasi set vector subspaces of a Smarandache vector space defined over the set P $\subseteq$ R, R a S-ring and Smarandache quasi set topological vector subspace defined over a set P (quasi set Smarandache topological vector subspace defined over P).

We illustrate, define and describe these structures in this chapter. For the concept about Smarandache vector spaces, Smarandache rings and their properties please refer [S-ring, S-linear alg. books].

DEFINITION 3.1: *Let V be a Smarandache vector space (S-vector space) defined over the S-ring R. Let P $\subseteq$ R be a proper subset of R. Let M $\subseteq$ V be a proper subset of V. If for all m $\in$ M and p $\in$ P; mp and pm $\in$ M then we define M to be a Smarandache quasi set vector subspace (S-quasi set vector subspace) of V defined over the set P $\subseteq$ R.*

We will first illustrate this situation by some simple examples.

Example 3.1: Let $V = Z_{10} \times Z_{10} \times Z_{10}$ be a Smarandache vector space defined over the S-ring Z_{10}.

Take $M = \{Z_{10} \times \{0\} \times \{0\}, \{0\} \times Z_{10} \times \{0\}\} \subseteq V$ and $P = \{0, 7, 3, 5\} \subseteq Z_{10}$. M is a Smarandache quasi set vector subspace of V defined over the set $P \subseteq Z_{10}$.

Example 3.2: Let $V = Z_{12} \times Z_{12} \times Z_{12}$ be a Smarandache vector space defined over the S-ring, $Z_{12} = R$. Take $P = \{5, 0, 1\} \subseteq Z_{12}$. $S_1 = \{(0, 0), (2, 1), (10, 5)\} \subseteq V$ is a S-quasi set vector subspace of V defined over the set P.

Take $S_2 = \{(0, 0), (1, 1), (5, 5)\} \subseteq V$, S_2 is also a S-quasi set vector subspace of V defined over the set P.

Take $S_3 = \{(0, 0), (2, 2), (10, 10), (3, 5), (3, 1), (6, 6)\}$, S_3 is also a S-quasi set vector subspace of V defined over the set P.

We see for a given set $P \subseteq Z_{12}$ we can have several S-quasi set vector subspaces of V defined over the set P.

Also we will show a set $S \subseteq V$ can be S-quasi set vector subspace of V defined over more than one subset of Z_{12}.

Example 3.3: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} \middle| \, a_i \in Z_{15}, 1 \leq i \leq 6 \right\}$$

be a S-vector space defined over the S-ring Z_{15}.

Let $P = \{7, 1, 5, 0\} \subseteq Z_{15}$.

Let

$$M_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 13 \\ 13 \\ 13 \\ 13 \\ 13 \\ 13 \end{bmatrix} \right\} \subseteq V$$

be S-quasi set vector subspace of V defined over the set $P \subseteq Z_{15}$.

Let

$$M_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 10 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 14 \\ 0 \\ 6 \\ 13 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 0 \\ 12 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 13 \\ 11 \\ 0 \\ 9 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 4 \\ 0 \end{bmatrix} \right\} \subseteq V$$

be a S-quasi set vector subspace of V defined over the set $P \subseteq Z_{15}$.

Take

$$M_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 14 \\ 6 \\ 0 \\ 12 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 8 \\ 7 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 11 \\ 4 \\ 0 \\ 3 \end{bmatrix} \right\} \subseteq V,$$

M_3 is a S-quasi set vector subspace of V over the set $P \subseteq Z_{15}$.

We can have many more S-quasi set vector subspaces defined over P. Take L = {0, 5} $\subseteq$ P $\subseteq$ Z_{15}. We find the S-quasi set vector subspaces of V defined over the set L.

$$S_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 10 \\ 10 \\ 0 \\ 0 \\ 10 \\ 10 \end{bmatrix} \right\} \subseteq V$$

is a S-quasi set vector subspace of V defined over the set L $\subseteq$ Z_{15}.

$$S_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq V$$

is a S-quasi set vector subspace of V defined over the set L.

We can have many more S-quasi set vector subspaces of V defined over the set L.

Example 3.4: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in Z_{21}; 1 \le i \le 12 \right\}$$

be a S-vector space defined over the S-ring Z_{21}.

Let P = {0, 2, 4, 5, 10} $\subseteq Z_{21}$ be a subset of Z_{21}.

Let

$$M = \left\{\begin{bmatrix}1&2&0&0\\0&1&2&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}2&4&0&0\\0&2&4&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\right.$$

$$\begin{bmatrix}4&8&0&0\\0&4&8&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}8&16&0&0\\0&8&16&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$\begin{bmatrix}16&11&0&0\\0&16&11&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}11&1&0&0\\0&11&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}5&10&0&0\\0&5&10&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$\left.\begin{bmatrix}19&17&0&0\\0&19&17&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}17&13&0&0\\0&17&13&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}13&5&0&0\\0&13&5&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\right\} \subseteq V$$

be a S-quasi set vector subspace of V defined over the set P.

Let

$$N = \left\{\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}4&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}8&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\right.$$

$$\begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 11 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 19 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 13 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left.\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} \subseteq V$$

be a S-quasi set vector subspace of V defined over the set $P \subseteq Z_7$.

Now we proceed onto define the notion of S-quasi subset vector subspace of V defined over a set $P \subseteq R$. R is a S-ring over which the S-vector space V is defined.

DEFINITION 3.2: *Let V be a S-vector space defined over the S-ring R. Let $P \subseteq R$ be a proper subset of R. M be the S-quasi set vector subspace of V defined over the set P. Let $S \subseteq P \subseteq R$ (S a proper subset of P). If $N \underset{\neq}{\subset} M$ is a quasi set S-vector subspace of V defined over S then we call N to be a quasi subset S-vector subspace of V defined over the subset S of the set P (N is only a proper subset of M). If N happens to be equal to M then we call the subset $S \subseteq P$ to be invariant subset relative to the S-quasi set vector subspace M of V.*

We will illustrate these situations by some examples.

Example 3.5: Let $V = Z_{14} \times Z_{14}$ be a S-vector space defined over the S-ring Z_{14}. Let $P = \{0, 2, 1, 4, 7, 3\} \subseteq Z_{14}$. $M = \{(0, 0), (1, 0), (2, 0), (4, 0), (8, 0), (3, 0), (7, 0), (6, 0), (9, 0), (13, 0), (12, 0), (10, 0), (11, 0), (5, 0)\}$ is a S-quasi set vector subspace of V over the set $P \subseteq Z_{14}$.

Consider $L = \{0, 2, 4, 7, 3\} \subseteq P \subseteq Z_{14}$; we see M is also a S-quasi set vector subspace of V defined over the set $L \subseteq P \subseteq Z_{14}$. Take $B = \{0, 2, 7, 3\} \subseteq P$, M is also a S-quasi set vector subspace of V defined over B. Thus the subset B and L are the invariant sets of P for the S-quasi set vector subspace M of V defined over P.

Now consider the subset $A = \{0, 2, 4, 1\} \subseteq P$. Then consider the set $N = \{(0, 0), (1, 0), (2, 0), (4, 0), (8, 0)\} \subseteq M$; N is a S-quasi set vector subspace of V defined over the set A. N is also the S-quasi subset vector subspace of a S-quasi set vector subspace M of V defined over $A \subseteq P$.

Example 3.6: Let $V = Z_6 \times Z_6 \times Z_6$ be a S-vector space defined over the S-ring Z_6. Let $P = \{1, 5, 0\} \subseteq Z_6$ be a subset of Z_6. $M_1 = \{(0, 0), (1, 0), (5, 0)\} \subseteq V$ is a S-quasi set vector subspace of V defined over the set P.

$M_2 = \{(0, 0), (1, 2), (5, 4)\} \subseteq V$ is a S-quasi set vector subspace of V defined over P.

Let $B = \{0, 5\} \subseteq V$. We see M_1 and M_2 are S-quasi set vector subspaces of V defined over B. That is B is an invariant set of these S-quasi set vector subspaces.

Take $A = \{0, 1\} \subseteq P$. Take $N_1 = \{(0, 0), (1, 0)\} \subseteq M_1$ and $N_2 = \{(0, 0), (1, 2)\} \subseteq M_2$, N_1 and N_2 are S-quasi subset vector subspaces of M_1 and M_2 respectively defined over the subset $A \subseteq P$.

Now we proceed onto define quasi set Smarandache topological vector subspace of V defined over a set $P \subseteq R$

(quasi set S-topological vector subspace of V defined over the set P ⊆ R) or Smarandache quasi set topological vector subspace of V defined over the set P ⊆ R (S-quasi set topological vector subspace of V defined over P ⊆ R).

DEFINITION 3.3: *Let V be a S-vector space over the S-ring R. Let P ⊆ R be a proper subset of R. T = {collection of all S-quasi set vector subspaces of V defined over the set P ⊆ R}.*

We see T is non empty.

(1) The empty set in T is a S-quasi set vector subspace or the zero set is in T which is a S-quasi set vector subspace of V and is in T (we assume empty set is in T if T has no zero set).

(2) The set V is itself in T and V is again a S-quasi set vector subspace of V defined over P.

(3) Union of any number of S-quasi set vector subspaces defined over P in T is again in T.

(4) Similarly intersection of any two S-quasi sets of vector subspaces is in T.

Thus T is defined as the Smarandache quasi set topological vector subspace (S-quasi set topological vector subspace) of V defined over the set P.

We give examples of this.

Example 3.7: Let V = $Z_6 \times Z_6$ be a S-vector space defined over the S-ring Z_6. Let P = {0, 5, 3} ⊆ Z_6 be a proper subset of Z_6.

Let T = {collection of all S-quasi set vector subspaces of V defined over the set P}. T is a S-quasi set topological vector subspace of V defined over the set P ⊆ Z_6.

Example 3.8: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| \; a_i \in Z_{35}, 1 \leq i \leq 12 \right\}$$

be a S-vector space defined over the S-ring Z_{35}.

Choose P = {0, 2, 3, 5, 123, 16, 28, 31} $\subseteq Z_{35}$.

Let T = {collection of all S-quasi set vector subspaces of V defined over the set P $\subseteq Z_{35}$} be the quasi set Smarandache topological vector subspace of V defined over the set P $\subseteq Z_{35}$.

Example 3.9: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \middle| \; a_i \in Z_{39}, 1 \leq i \leq 8 \right\}$$

be a S-vector space defined over the S-ring Z_{39}.

Let P = {0, 8, 16, 9, 25, 33} $\subseteq Z_{39}$. T = {all S-quasi set vector subspaces of V defined over the set P $\subseteq Z_{39}$} be a S-quasi set topological vector subspace of V defined over the set P $\subseteq Z_{39}$.

Example 3.10: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in \langle Q \cup I \rangle; \; 1 \leq i \leq 16 \right\}$$

be a S-vector space defined over the S-ring, $\langle Q \cup I \rangle$.

Let P = {–I, 1, I, 1, 0} $\subseteq$ $\langle Q \cup I\rangle$}. T = {all S-quasi set vector subspaces of V defined over the set P $\subseteq$ $\langle Q \cup I\rangle$} be the S-quasi set neutrosophic topological vector subspace of V defined over the set P.

Example 3.11: Let V = $\langle Z_5 \cup I\rangle \times \langle Z_5 \cup I\rangle$ be a S-vector space defined over the S-ring, $\langle Z_5 \cup I\rangle$. Take P = {0, I, 2I, 1} $\subseteq$ $\langle Z_5 \cup I\rangle$. T = {all S-quasi set vector subspaces of V defined over the set P}; be the S-quasi set topological vector subspace of V defined over the set P.

Example 3.12: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{13} & a_{14} \end{bmatrix} \middle| \; a_i \in \langle Z_{35} \cup I\rangle;\ 1 \le i \le 14 \right\}$$

be a S-vector space defined over the S-ring $\langle Z_{35} \cup I\rangle$.

P = {0, I, 3I+4, 8+5I, 7I, 11+23I, 31I+17} $\subseteq$ $\langle Z_{35} \cup I\rangle$. T = {collection of all S-quasi set vector subspaces of V defined over the set P}; be the S-quasi set topological vector subspace of V defined over P $\subseteq$ $\langle Z_{35} \cup I\rangle$.

As in case of usual topological spaces we define the basic set. It is pertinent to mention here that the basic set is also the set which generates T. Further we will call the basic set also as the fundamental set associated with this topological space or as the Smarandache basic set of the S-quasi set topological vector subspace defined over the set P.

We will give examples of basic sets of the S-quasi set topological vector subspaces.

Example 3.13: Let V = $\langle Z_{18} \cup I\rangle \cup \langle Z_{18} \cup I\rangle \times \langle Z_{18} \cup I\rangle$ be a S-vector space defined over the S-ring, $\langle Z_{18} \cup I\rangle$. Let P = {0, 5I, 7} ⊆ ⟨$Z_{18}$ ∪ I⟩. T = {collection of all S-quasi set vector subspaces of V defined over the set P} be the quasi set S-topological vector subspace of V defined over the set P of ⟨Z_{18} ∪ I⟩.

Let B_T^S denote the Smarandache basic set of the S-topological quasi set vector subspace defined over P.

B_T^S = {{(0, 0, 0), (1, 0, 0), (5I, 0, 0), (7, 0, 0), (7I, 0, 0), (13, 0, 0), (13I, 0, 0), (17I, 0, 0), (I, 0, 0), …} and so on}.

Example 3.14: Let V = $Z_{10} \times Z_{10}$ be a S-vector space defined over the S-ring Z_{10}. Let P = {0, 3, 2, 9} ⊆ Z_{10}. Let T = {all S-quasi set vector subspaces of V defined over the set P}; be the S-quasi set topological vector subspace of V defined over P.

The Smarandache basic set B_T^S = {{(0, 0), (1, 0), (2, 0), (3, 0), (9, 0), (6, 0), (7, 0), (4, 0), (8, 0)}, {(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 6), (0, 7), (0, 8), (0, 9)}, {(0, 0), (1, 2), (2, 4), (3, 6), (9, 8), (6, 2), (8, 6), (7, 4), (4, 8)}, {(0, 0), (2, 1), (4, 2), (6, 3), (8, 9), (2, 6), (6, 8), (4, 7), (8, 4)}, {(0, 0), (1, 3), (2, 6), (3, 9), (9, 7), (8, 4), (6, 8), (7, 1), (4, 2)}, {(0, 0), (1, 7), (3, 1), (6, 2), (9, 3), (7, 9), (4, 8), (8, 6), (2, 4)}, {(0, 0), (1, 4), (2, 8), (3, 2), (9, 6), (8, 2), (7, 8), (6, 4), (4, 6)}, {(0, 0), (4, 1), (8, 2), (2, 8), (2, 3), (6, 9), (8, 7), (4, 6), (6, 4)}, {(0, 0), (1, 5), (2, 0), (3, 5), (7, 5), (9, 5), (6, 0), (4, 0), (8, 0)} {(0, 0), (5, 1), (0, 2), (5, 3), (5, 7), (5, 9), (0, 6), (0, 4), (0, 8)}, {(0, 0), (5, 5)}, {(0, 0), (1, 6), (2, 2), (3, 8), (9, 4), (4, 4), (8, 8), (7, 2), (6, 6)}, {(0, 0), (6 1), (2, 2), (4, 4), (6, 6), (8, 8), (4, 9), (8, 3), (2, 7)} {(0, 0), (1, 8), (2, 6), (4, 2), (3, 4), (9, 2), (8, 4), (6, 8), (7, 6)}, {(0, 0), (8, 1), (6, 2), (2, 4), (4, 3), (2, 9), (4, 8), (8, 6), (6, 7)}, {(0, 0), (1, 9), (2, 8), (4, 6), (3, 7), (9, 1), (6, 4), (8, 2), (7, 3)} and so on}.

We see the elements of the basic set are not disjoint. They have common terms.

This is the marked difference between the S-vector spaces and vector spaces using which the S-quasi set topology and quasi set topology are built.

Example 3.15: Let V = $Z_6 \times Z_6$ be a S-vector space defined over Z_6, the S-ring. Let P = {0, 2, 3} $\subseteq Z_6$ be the subset.

T = {collection of all S-quasi set vector subspaces of V defined over the set P}; be the S-quasi set topological vector subspace of V defined over P. The S-basic set of T is given by B_T^S;

B_T^S = {{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)}, {(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)}, {(0, 0), (2, 5), (4, 4), (0, 3), (2, 2)}, {(0, 0), (5, 2), (3, 0), (4, 4), (2, 2)}, {(0, 0), (3, 5), (3, 3), (0, 4), (0, 2)}, {(0, 0), (5, 3), (3, 3), (4, 0), (2, 0)}, {(0, 0), (4, 5), (2, 4), (0, 3), (4, 2)}, {(0, 0), (5, 4), (4, 2), (3, 0), (2, 4)}, {(0, 0), (3, 4), (0, 4), (3, 0), (0, 2)}, {(0, 0), (4, 3), (2, 0), (0, 3), (4, 0)} {(2, 3), (4, 0), (0, 3), (2, 0), (0, 0)}, {(3, 2), (0, 4), (3, 0), (0, 2), (0, 0)}, {(0, 0), (3, 4), (0, 2), (0, 4), (3, 0)}, {(0, 0), (4, 3), (0, 3), (2, 0), (4, 0)}, {(0, 0), (5, 5), (4, 4), (2, 2), (3, 3)}; we see the order of the S-basic set, o(B_T^S) = 2^4.

This is the Smarandache basic set associated with T whose intersection is {(0, 0)}.

They serve as the atom to the lattice of the S-quasi set topological vector subspace of V defined over the set P = {0, 2, 3}. Now take S = {0, 1} $\subseteq Z_6$. Then M = {collection of all S-quasi set vector subspaces of V defined over the set S = {0, 1}}.

The S-basic set of M of the S-topological quasi set vector subspace; B_M^S = {{(0, 0), (1, 0)}, {(0, 0), (0, 2)}, …, {(0, 0), (4, 5)}, {(0, 0), (5, 4)}, {(0, 0), (5, 5)}}.

We see o(B_S^S) = 35 = $6^2 - 1$. Further each of the sets have only (0, 0) to be the common element which will be the least element of the lattice associated with the S-topological quasi set vector subspace of V defined over the set S = {0, 1}.

Thus depending on the subset we choose in the S-ring, the basic set will be over lapping or disjoint. Let A = {0, 1, 5} $\subseteq$ Z_6. Let X = {collection of all S-quasi set vector subspaces of V defined over the set A = {0, 1, 5}} be the S-quasi set topological vector subspace of V defined over the set A.

Let B_X^S be the S-basic set of X.

B_X^S = {{(0, 0), (1, 0), (5, 0)}, {(0, 0), (2, 0), (4, 0)}, {(0, 0), (3, 0)}, {(0, 0), (0, 1), (0, 5)}, {(0, 0), (0, 3)}, {(0, 0), (0, 4), (0, 2)}, {(0, 0), (1, 1), (5, 5)}, {(0, 0), (2, 2), (4, 4)}, {(3, 3), (0, 0)}, {(0, 0), (1, 2), (5, 4)}, {(0, 0), (2, 1), (4, 5)}, {(0, 0), (1, 3), (5, 3)}, {(0, 0), (3, 1), (3, 5)}, {(0, 0), (1, 4), (5, 2)}, {(0, 0), (4, 1), (2, 5)}, {(0, 0), (2, 3), (4, 3)} {(0, 0), (3, 2), (3, 4)}, {(0, 0), (2, 4), (4, 2)}, {(0, 0), (1, 5), (5, 1)}}

o(B_X^S) = 19. {(0, 0)} is the least element and V is the maximum element of X.

The lattice associated with the quasi set S-topological vector subspace of V defined over A is a Boolean algebra of order 2^{19}.

Let C = {0, 1, 5, 3} $\subseteq$ Z_6 be the set for which we construct the quasi set S-topological vector subspace of V defined over C.

Let N = {collection of all Smarandache quasi vector subspaces of V defined over the set C $\subseteq$ Z_6} be the S quasi set topological vector subspace of V defined over the set C.

The S-basic set of N denoted by

B_N^S = {(0, 0), (1, 0), (3, 0), (5, 0)}, {(0, 0), (0, 1), (0, 3), (0, 5)}, {(2, 0), (0, 0), (4, 0)}, {(0, 0), (0, 2), (0, 4)}, {(1, 1), (0, 0), (3, 3), (5, 5)} {(2, 2), (0, 0), (4, 4)}, {(0, 0), (1, 2), (3, 0), (5, 4)}, {(0, 0), (2, 1), (0, 3), (4, 5)}, {(0, 0), (1, 3), (3, 3), (5, 3)}, {(0, 0), (3, 1), (3, 3), (5, 3)}, {(0, 0), (3, 1), (3, 3), (3, 5)}, {(0, 0), (1, 4), (3, 0), (5, 2)}, {(0, 0), (4, 1), (0, 3), (2, 5)}, {(0, 0), (1, 5), (5, 1), (3, 3)}, {(0, 0), (2, 3), (0, 3)}, {(0, 0), (3, 2), (3, 4), (3, 0)}, {(0, 0), (2, 4), (4, 2)}}. o(B_N^S)=16.

However for the associated lattice of the S-topological space of N we take a Boolean algebra with least element zero, greatest element is V and atoms are the 16 elements of B_N^s.

Suppose Y = {0, 1, 5, 4} $\subseteq Z_6$. Let Z = {all Smarandache quasi set vector subspaces of V defined over the subset Y of Z_6} be the quasi set S-topological vector subspace of V defined over Y.

The S-basic set of Z is B_Z^S = {(0, 0), (1, 0), (5, 0), (2, 0), (4, 0)}, {(0, 0), (0, 1), (0, 5), (0, 2), (0, 4)}, {(0, 0), (1, 1), (5, 5), (2, 2), (4, 4)}, {(0, 3), (0, 0)}, {(0, 0), (3, 0)}, {(1, 2), (0, 0), (5, 4), (4, 2), (2, 4)}, {(2, 1), (0, 0), (4, 5), (4, 2), (2, 4)}, {(1, 3), (0, 0), (5, 3), (4, 0), (2, 0)}, {(3, 1), (0, 0), (0, 4), (3, 5)}, {(1, 4), (0, 0), (5, 2), (4, 4), (2, 2)}, {(0, 0), (4, 1), (2, 5), (4, 4), (2, 2)}, {(1, 5), (0, 0), (5, 1), (4, 2), (2, 4)}, {(0, 0), (3, 3)}, {(0, 0), (2, 3), (4, 3), (2, 0), (4, 0)}, {(0, 0), (3, 4), (3, 2), (0, 4), (0, 2)}}.

o(B_Z^S) = 15. Thus the associated lattice of the quasi set S-topological vector subspace Z defined over Y is a Boolean algebra of order 2^{15} with the elements of B_Z^S as its atoms and (0, 0) is the least element of the Boolean algebra and V is the largest element of Z.

Example 3.16: Let V = $\langle Z_3 \cup I\rangle \times \langle Z_3 \cup I\rangle$ be a neutrosophic S-vector space defined over the S-ring, $\langle Z_3 \cup I\rangle$.

Let P = {0, 1, 2, I} $\subseteq \langle Z_3 \cup I\rangle$ be a subset of $\langle Z_3 \cup I\rangle$. T = {collection of all S-quasi set vector subspaces of V defined over the set P $\subset \langle Z_3 \cup I\rangle$}, be the S-quasi set topological vector subspace of V defined over the set P. Let B_T^S be the S-basic set of T.

B_T^S = {(0, 0), (1, 0), (I, 0), (2, 0), (2I, 0)}, {(0, 0), (0, 1), (0, I), (0, 2), (0, 2I)}, {(0, 0), (1, 1), (I, I), (2, 2), (2I, 2I)}, {(0, 0), (1, 2), (I, 2I), (2, 1), (2I, I)}, {(0, 0), (1+I, 0), (2+2I, 0), (2I, 0)} {(0, 0), (0, 1+I), (0, 2+2I), (0, 2I), (0, 2+I), (0, 1+2I)}, {(0, 0), (I, 1+I), (2I, 2+2I), (I, 2I), (2I, I)}, {(0, 0), (1+I, I), (2+2I, 2I), (2I, I), (I, 0), (I, 2I)}, {(0, 0), (1+2I, I), (0, I), (I, 2I), (2I, I)} and so on}.

We using elements of B_T^S as atoms get a lattice associated with T.

Example 3.17: Let V = {$a_1 + a_2g_1 + a_3g_2$ | $a_i \in Z_{12}$, $1 \leq i \leq 3$, $g_1 = 3 \in Z_6$ and $g_2 = 4 \in Z_6$} be the higher dimensional dual like numbers S-vector space defined over the S-ring Z_{12}.

Take P = {0, 3, 5, 7} $\subseteq Z_{12}$. Let M = {collection of all Smarandache quasi set vector subspaces of V defined over the set P}, be the quasi set S-topological vector subspace defined over the set P.

The S-basic set B_M^S = {{0, 1, 3, 5, 7, 9, 11}, {0, 2, 6, 10}, {0, 4, 8}, {0, $1+g_1$, $3(1+g_1)$, $5(1+g_1)$, $7(1+g_1)$, $9(1+g_1)$, $11(1+g_1)$}, {0, $2(1+g_1)$, $6(1+g_1)$, $10(1+g_1)$}, {0, $4(1+g_1)$, $8(1+g_1)$, {0, $3g_1$, g_1, $5g_1$, $7g_1$, $9g_1$, $11g_1$}, …}.

We see the sets of B_M^S do not have the same cardinality. Let P_1 = {0, 1} $\subseteq Z_{12}$. W = {all S-quasi set vector subspaces of V defined over the set P_1 = {0, 1} $\subseteq Z_{12}$}, be the quasi set S-topological vector subspace defined over the set P_1. The S-basic set of W is

B_M^S = {{0, 1}, {0, 2}, {0, 3}, …, {0, $1+g_1 + g_2$}, …}.

The o(B_M^S) = $12^3 - 1$.

Suppose P_2 = {0, 1, 11} $\subseteq Z_{12}$ and X = {collection of all S-quasi set topological vector subspaces of V defined over the set

P_2} be the S-quasi set topological vector subspace of V defined over the set P_2.

If B_X^S is the S-basic set of X then

B_X^S={{0, 1, 11}, {0, 2, 10}, …, {0, 9+8g_1 + 11g_2, 3+7g_1 + g_2}}
and

$o(B_X^S) = (12^3 - 1) / 2$.

Example 3.18: Let

$V = \{a + bg \mid a, b \in Z_6 \text{ and } g = 8, g^2 = -g = -8 = 4, g \in Z_{12}\}$ be the S-special quasi dual number vector space defined over the S-ring, Z_6. Let P = {0, 1} $\subseteq Z_6$ and

T = {all S-quasi set vector subspaces of V defined over set P} be the S-quasi set topological vector subspace of V over the S-ring, Z_6. Let B_T^S be the S-basic set of T, B_T^S = {(0, 1), (0, 2), …, (0, 5), (1, 0), (2, 0), …, (5, 0), (0, g), (0, 2g), …, (0, 5g), (g, 0), (2g, 0), …, (5g, 0), (1 + g, 0), …, (5 + 5g, 0), …, (0, 5+5g)}.

Clearly $o(B_T^S) = o(V) - 1$.

If we take instead of P = {0, 1} say P_1 = {0, 1, 5} then B = {collection of all S-quasi set vector subspaces of V defined over the set P_1 = {0, 1, 5}} is the S-quasi set topological vector subspace of V defined over P.

The S-basic set of B is
B_B^S = {{0, 1, 5}, {0, 2, 4}, …, {0, 5 + 4g, 1+2g}} with $o(B_B^S) = (o(V) - 1) / 2$.

Now having seen examples of S-basic sets of a S-topological quasi set vector subspaces we now proceed onto define substructures and give examples of them.

DEFINITION 3.4: *Let V be a S-vector space defined over the S-ring R. Let P $\subseteq$ R (P a proper subset of R). T be the S-quasi set topological vector subspace of V defined over the set P; P $\subseteq$ R. Let B_T^S be the S-basic set of T. Every subset M $\subseteq$ B_T^S generates a S-quasi set topological vector subspace of V over P defined as a quasi set Smarandache subtopological vector subspace of V defined over the set P.*

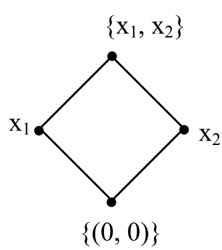
We will first illustrate this situation by some examples.

Example 3.19: Let V = $Z_{10} \times Z_{10}$ be a S-vector space defined over the S-ring R = Z_{10}. Let P = {0, 3, 1, 8} $\subseteq Z_{10}$ be a proper subset of Z_{10}. T = {all S-quasi set vector subspaces of V defined over the set P} be the S-quasi set topological vector subspace of V over P.

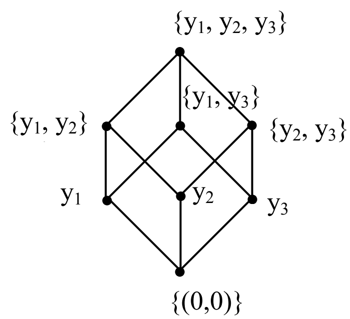
Let B_T^S = {{(0, 0), (1, 0), (3, 0), (8, 0), (4, 0), (7, 0), (2, 0), (9, 0), (6, 0)}, {(0, 0), (0, 1), (9, 0), (7, 0), (0, 3), (0, 4), (0, 8), (0, 6), (0, 2)}, {(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (9, 9), (7, 7), (6, 6), (8, 8)}, {(0, 0), (5, 0)}, {(0, 0), (0, 5)}, {(0, 0), (5, 5)}, {(1, 2), (0, 0), (3, 6), (9, 8), (8, 6), (4, 8), (6, 2), (7, 4), (2, 4)}, {(2, 1), (0, 0), (6, 3), (8, 9), (6, 8), (8, 4), (2, 6), (4, 7), (4, 2)} …} be the S-basic set.

Let {x_1 = {(0, 0), (0, 5)}, x_2 = {(0, 0), (1, 0), (3, 0), (4, 0), (8, 0), (7, 0), (9, 0), (6, 0)}} $\subseteq B_T^S$. $\langle x_1, x_2\rangle$ generates a S-quasi set subtopological vector subspace, W defined over the set P of T.

The lattice associated with W is as follows:



Thus a Boolean algebra of order four. Let $\{y_1 = \{(0, 0), (5, 5)\}, y_2 = \{(0, 0), (5, 0)\}, y_3 = \{(0, 0), (0, 5)\}\} \subseteq B_T^S$; the quasi set S-topological vector subspace generated by (y_1, y_2, y_3) be A; A has its associated lattice which is a Boolean algebra of order 8, given by the following diagram.



Let us take $v_1 = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (6, 0), (8, 0), (7, 0), (9, 0)\}$, $v_2 = \{(0, 0), (0, 5)\}$, $v_3 = \{(0, 0), (5, 0)\}$, $v_4 = \{(0, 0), (5, 5)\}$ and $v_5 = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 6), (0, 7), (0, 8), (0, 9)\}$.

Let B be the S-quasi set subtopological vector subspace generated by the set $\{v_1, v_2, v_3, v_4, v_5\} \subseteq B_T^S$. The lattice associated with B is a Boolean algebra of order 2^5 with $\{(0, 0)\}$ as its least element and $\{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0), (8, 0), (9, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (5, 5)\}$ as the largest element.

Associated with B we have 2^5, S-quasi set vector subspaces including $\{(0, 0)\}$ of V defined over P.

In this way we can find several S-quasi set subtopological vector subspaces defined over P for a given S-quasi set topological vector subspace of T defined over P.

Example 3.20: Let $V = Z_{15} \times Z_{15} \times Z_{15} \times Z_{15}$ be a S-vector space defined over the S-ring Z_{15}. Take $P = \{0, 1\} \subseteq Z_{15}$. Let

T = {collection of all S-quasi set vector subspaces of V defined over the set P} be the S-quasi set topological vector subspace of V defined over the set P.

The S-basic set of T defined over the set P is B_T^S = {{(0, 0, 0, 0), (1, 0, 0, 0)}, {(0, 0, 0, 0), (0, 1, 0, 0)}, {(0, 0, 0, 0), (0, 0, 0, 1)}, {(0, 0, 0, 0), (0, 0, 1, 0)}, {(0, 0, 0, 0), (1, 1, 0, 0)} … {(0, 0, 0, 0), (14, 14, 14, 14)}}. Clearly o(B_T^S) = $15^4 - 1$.

We can take any desired number of elements from B_T^S and generate a S-quasi set subtopological vector subspace of T defined over the set P. Let B = {{(0, 0, 0, 0), (1, 0, 0, 0)}, {(0, 0, 0, 0), (0, 8, 9, 0)}, {(0, 0, 0, 0), (0, 0, 0, 11)}, {(0, 0, 0, 0), (5, 2, 4, 3)}} $\subseteq B_T^S$.

Now B generates a S-quasi set subtopological vector subspaces B_1 of T defined over the set P. The lattice associated with B is a Boolean algebra of order 2^4.

Let D = {{(0, 0, 0, 0), (1, 2, 3, 4)} = s_1, s_2 = {(0, 0, 0, 0), (5, 6, 7, 8)}, s_3 = {(0, 0, 0, 0), (7, 0, 7, 0)}, s_4 = {(0, 0, 0, 0), (1, 0, 5, 8)}, s_5 = {(0, 0, 0, 0), (1, 9, 2, 3)}} $\subseteq B_T^S$; D generates a S-quasi set subtopological vector subspace of order 2^5. The lattice associated with D is a Boolean algebra of order 2^5.

Example 3.21: Let V = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid a_i \in Z_{35}, 1 \leq i \leq 4, g_1 = 6, g_2 = 4, g_3 = 3 \in Z_{12}\}$ be the S-vector space defined over the S-ring Z_{35}.

Now we give the definition of S-quasi subset subtopological vector subspace T of a S-quasi set topological vector subspace of T defined over a subset A $\subseteq$ P; where T is defined over P.

DEFINITION 3.5: *Let V be a S-vector space over the S-ring R. Let P $\subseteq$ R; T be the S-quasi set topological vector subspace of V defined over the set P. Let S $\subseteq$ P (S a proper subset of P). M = {all S-quasi set vector subspaces of V defined over the set*

$S \subseteq P$}, be the S-quasi set topological vector subspace of V defined over the set S; we define M to be a Smarandache quasi subset subtopological vector subspace (S-quasi subset subtopological vector subspace) of T defined over the subset $S \subseteq P$.

We will illustrate this situation by some examples.

Example 3.22: Let $V = Z_{12} \times Z_{12}$ be a S-vector space defined over the S-ring Z_{12}. Let P = {0, 1, 11, 5} $\subseteq Z_{12}$. T be the S-quasi set topological vector subspace of V defined over the set P. Let X = {0, 1} $\subseteq P \subseteq Z_{12}$ be a subset of the set P. S be the S-quasi set topological vector subspace of V defined over the set X.

S is the S-quasi subset subtopological vector subspace of the S-quasi set topological vector subspace T defined over the set $X \subseteq P$.

The S-basic set of S, B_S^S = {{(0, 0), (1, 0)}, {(0, 0), (0, 1)}, …, {(0, 0), (0, 2)}, {(0, 0), (2, 0)}, …, {(0, 0), (11, 11)}}.

Now the S-basic set of T. B_T^S = {{(0, 0), (1, 0), (5, 0), (11, 0), (7, 0)}, {(0, 0), (0, 1), (0, 5), (11, 0), (0, 7)}, {(0, 0), (2, 0), (10, 0)}, {(0, 0), (0, 2), (0, 10)}, {(0, 0), (3, 0), (9, 0)}, {(0, 0), (0, 3), (0, 9)}, …, {(0, 0), (10, 11), (2, 1), (2, 7)}}.

We see $o(B_S^S) > o(B_T^S)$ and however S is a S-quasi subset subtopological vector subspace of T defined over the subset $X \subseteq P$.

Example 3.23: Let $V = Z_{10} \times Z_{10}$ be a S-vector space defined over the S-ring Z_{10}. P = {0, 5, 1, 9} $\subseteq Z_{10}$.

T = {all S-quasi set vector subspaces of V defined over the set P}, be the S-quasi set topological vector subspace of V defined over P. The S-basic set of T;

B_T^S = {{(0, 0), (1, 0), (5, 0), (9, 0)}, {(0, 0), (0, 1), (0, 5), (0, 9)}, {(0, 0), (1, 1), (5, 5), (9, 9)}, {(0, 0), (0, 2), (0, 8)}, {(0, 0), (2, 0), (8, 0)}, …, {(0, 0), (8, 9), (0, 5), (2, 1)}, {(0, 0), (9, 8), (5, 0), (1, 2)}}.

Now take M = {0, 1, 5} ⊆ P. Let W = {collection of all S-quasi set vector subspaces of V defined over the set M}, W is the S-quasi subset, subtopological vector subspace of T defined over the subset M of P. Let B_W^S be the basic set of W.

B_W^S = {{(0, 0), (1, 0), (5, 0)}, {(0, 0), (0, 1), (0, 5)}, {(0, 0), (2, 0)}, {(0, 0), (0, 2)}, {(0, 0), (1, 2), (5, 0)}, {(0, 0), (2, 1), (0, 5)}, …, {(0, 0), (8, 9), (0, 5)}, {(0, 0), (9, 8), (5, 0)}}.

We see o(B_W^S) > o(B_T^S).

Now having seen examples of substructures we proceed onto suggest some problems for the reader.

We wish to study if Z_n is the S-ring if the set P contains all primes p < n and p/n then does the corresponding S-topology has special properties.

Problems:

1. Find some interesting properties associated with S-quasi set vector subspaces of V defined over the subset P of the S-ring R.

2. Let V = $Z_{35} \times Z_{35} \times Z_{35} \times Z_{35}$ be a S-vector space defined over the S-ring.
For the set P = {0, 2, 7, 11, 31, 29} ⊆ Z_{35} find the number of S-quasi set vector subspaces of V defined over the set P ⊆ Z_{35}.

3. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| a_i \in Z_{12}; 1 \le i \le 6\right\}$ be a S-vector space defined over the S-ring Z_{12} = R. For the subsets $P_1 = \{0, 1\} \subseteq Z_{12}$, $P_2 = \{0, 1, 11\} \subseteq Z_{12}$, $P_3 = \{0, 2\} \subseteq Z_{12}$, $P_4 = \{0, 3, 5\} \subseteq Z_{12}$ and $P_5 = \{0, 7, 5, 3, 2\} \subseteq Z_{12}$ find the corresponding S-quasi set vector subspaces of V.

4. Let V = $Z_{26} \times Z_{26}$ be a S-vector space defined over the S-ring; Z_{26}. Let $P_1 = \{0, 13\} \subseteq Z_{26}$, $P_2 = \{1, 13\} \subseteq Z_{26}$, $P_3 = \{1, 25\} \subseteq Z_{26}$ and $P_4 = \{1, 2, 3, 5, 7, 11, 13, 17, 19, 21, 23\} \subseteq Z_{26}$ be subsets of Z_{26}.

   Find the number of S-quasi set vector subspaces of V associated with each of these subsets.

5. Is Z_{25} a S-ring?

6. Can Z_{p^2} be a S-ring?

7. Can Z_{p^n} be a S-ring, p any prime? ($n \ge 2$).

8. Find some interesting features enjoyed by the S-quasi set topological vector subspace of V defined over the set $P \subseteq R$, R a S-ring defined over which the S-vector space is defined.

9. Find the difference between the S-quasi set topological vector subspace of V and quasi set topological vector subspace of W where V is a S-vector space and W is a vector space defined over a S-ring and a field respectively.

10. Let V = $\langle Q \cup I\rangle \times \langle Q \cup I\rangle$ be a S-vector space of neutrosophic rationals defined over the S-ring.

(i) Find the S-quasi set vector subspace of V defined over the set P = {0, 1, I, –1, –I} $\subseteq$ $\langle Q \cup I\rangle$.
(ii) Find the S-quasi set topological vector subspace T of V defined over the set P.
(iii) Find the S-basic set of T.
(iv) Is T a second countable S-topological space?
(v) Let P_1 = {0, I} $\subseteq$ $\langle Q \cup I\rangle$; find the S-quasi set topological vector subspace of V over P_1.

11. Let V = $Z_{18} \times Z_{18} \times Z_{18}$ be a S-vector space defined over the S-ring Z_{18}.

(i) Find three S-quasi set vector subspaces of V defined over the set P = {0, 1, 17}.
(ii) Find S-quasi set subtopological vector subspaces of T of V defined over P.
(iii) What is the order of S-basic set of T?
(iv) If P is replaced by P_1 = {0, 17} will those two S-topological spaces be isomorphic?

12. Let V = $Z_{12} \times Z_{12} \times Z_{12} \times Z_{12}$ be a S-vector space defined over the S-ring Z_{12}.
Let P = {0, 5, 7, 11} $\subseteq$ Z_{12}.

(i) Find how many S-quasi set vector subspaces can be defined on P?
(ii) Find the S-quasi set topological vector subspace T of V defined over the set P $\subseteq$ Z_{12}.
(iii) Find the S-basic set of T.
(iv) Does T contain S-quasi set subtopological vector subspace of V defined over P? (find atleast 3 such spaces).
(v) Can T contain S-quasi subset subtopological vector subspaces defined over proper subsets of P?
(vi) Find the lattice associated with T.

13. Let $V = \langle Z_{19} \cup I\rangle \times \langle Z_{19} \cup I\rangle \times \langle Z_{19} \cup I\rangle$ be a S-neutrosophic vector space defined over the S-ring $\langle Z_{19} \cup I\rangle$.

(i) For $P = \{0, I\} \subseteq \langle Z_{19} \cup I\rangle$; find the S-quasi set neutrosophic topological vector subspace T of V defined over P.
(ii) Is T pseudo simple?
(iii) Find S-quasi set neutrosophic subtopological vector subspace of T defined over P.
(iv) Let $M = \{0, 3, 3I, 5, 5I, 7, 7I, 11, 11I, 13, 13I, 17, 17I\} \subseteq \langle Z_{19} \cup I\rangle$. Find a S-quasi set topological vector subspace A of V defined over the set M.
(v) Does M enjoy any other special properties?
(vi) Prove M is not pseudo simple.
(vii) Find 3 distinct S-quasi subset subtopological vector subspaces of A defined over some three distinct subsets of M.
(viii) Find three distinct S-quasi set subtopological vector subspaces of A defined over three subsets of M.
(ix) Find B_A^S and the associated lattice with A.

14. Let $V = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i = x_i + y_i g$ with $x_i, y_i \in Z_{12};\ 1 \leq i \leq 5,\ g = 4 \in Z_{16}\}$ be a dual number S-vector space defined over the S-ring Z_{12}.

(i) Let $P = \{0, 2, 6, 4, 8\} \subseteq Z_{12}$. Find the S-quasi set topological vector subspace T of V defined over $P \subseteq Z_{12}$.
(ii) Find B_T^S.
(iii) Let $P_1 = \{0, 1, 3, 5, 7, 9\} \subseteq Z_{12}$. Find the S-quasi set dual number topological vector subspace M of V defined over P_1.
(iv) Find B_M^S.
(v) Compare the S-topological spaces T and M.

15. Let $V = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{pmatrix} \middle| \; a_i = x_i + y_ig_1 + z_ig_2 + s_ig_3 \text{ with} \right.$

$x_i, y_i, z_i, s_i \in Z_{15}$; $1 \leq i \leq 6$ and $g_1 = 6$, $g_2 = 8$ and $g_3 = 9 \in Z_{12}\}$ be a S quasi set vector space defined over the S-ring Z_{15}.

(i) For the set $P_1 = \{0, 1, 14\}$ find the S-quasi set topological vector space T_1 of V defined over P_1.

(ii) For $P_2 = \{0, 3, 5\} \subseteq Z_{15}$, find the S-quasi set topological vector subspace T_2 of V over P_2.

(iii) For $P_3 = \{0, 2, 7, 11, 13\} \subseteq Z_{15}$, find the S-quasi set topological vector subspace T_3 of V over P_3.

(iv) Let $P_4 = \{0, 6, 9, 10, 12\} \subseteq Z_{15}$, find the S-quasi set topological vector subspace T_4 of V over P_4.

(v) Compare all the four topological spaces T_1, T_2, T_3 and T_4.

(vi) Compare the S-basic sets of T_1, T_2, T_3 and T_4.

16. Let $V = \langle Z_{17} \cup I\rangle \cup \langle Z_{17} \cup I\rangle$ be a S-vector space over the S-ring, $R = \langle Z_{17} \cup I\rangle$.

(i) For $P_1 = Z_{17} \subseteq \langle Z_{17} \cup I\rangle$ find the S-quasi set topological vector subspace T_1 of V defined over P_1.

(ii) Let $P_2 = \{0, 1\} \subseteq \langle Z_{17} \cup I\rangle$ be a subset of R; find the S-quasi set topological vector subspace of V; T_2 defined over P_2.

(iii) Let $P_3 = \{0, I\} \subseteq \langle Z_{17} \cup I\rangle$ be a subset of R; find the S-quasi set topological vector subspace T_3 of V defined over P_3.

(iv) Compare the 3 spaces T_1, T_2 and T_3.

(v) Find S-quasi subset topological vector subspace of T_1 defined over P_1.

(vi) Prove T_2 and T_3 are pseudo simple!

(vii) Find S-quasi set subtopological vector subspaces of T_1, T_2 and T_3 defined over P_1, P_2 and P_3 respectively.

17. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \end{bmatrix} \middle| a_i \in Z_{26}; (g_1, g_2) = \right.$

$\{x_i + y_i g_1 + z_1 g_2 \mid x_i, y_i, z_i \in Z_{26}, g_1 = 4, g_2 = 6 \in Z_{12}\}, 1 \leq i \leq 14\}$ be a S-vector space defined over the S-ring Z_{26}. Let $P_1 = \{0, 13\} \subseteq Z_{26}$. T be the S-quasi set topological vector subspace of V over the set P_1 and $P_2 = \{0, 1, 25\} \subseteq Z_{26}$.

M be the S-quasi set topological vector subspace of V defined over the set P_2. $P_3 = \{0, 3, 5, 7, 11, 13, 17, 23\} \subseteq Z_{26}$. Let W be the S-quasi set topological vector subspace of V over P_3.

(i) Prove T is not pseudo simple.

(ii) Prove M and W are not pseudo simple.

(iii) Find S-quasi set subtopological vector subspaces of T, M and W.

(iv) Find S-quasi subset subtopological vector subspaces of M and W.

18. Does there exist a S-quasi set topological vector subspace which is not second countable?

19. Does there exist a S-quasi set vector subspace which is both first and second countable?

20. Give an example of a pseudo S-quasi set topological vector subspace of infinite order.

21. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in \langle Q \cup I\rangle, 1 \leq i \leq 4 \right\}$ be a S-quasi set vector space defined over the S-ring $R = \langle Q \cup I\rangle$.

    Let $P = \{0, 1\} \subseteq R$. Is T the S-quasi set topological vector subspace of V defined over P, pseudo simple.

22. Let $V = \{(a_1, a_2, \ldots, a_{10}) \mid a_i \in \langle R \cup I\rangle; 1 \leq i \leq 10\}$ be a S-vector space defined over the S-ring $\langle R \cup I\rangle$.

    (i) Let T be a S-quasi set topological vector subspace of V defined over the set $P = \{0, 1\} \subseteq \langle R \cup I\rangle$.

    (a) Is T second countable?
    (b) Is T first countable?
    (c) Is T pseudo simple?
    (d) Give two S-quasi set topological vector subspaces of V defined over the set $P = \{0, 1\}$.

23. Let

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \end{pmatrix} \middle| \; a_i \in \langle R \cup I\rangle, 1 \leq i \leq 12 \right\}$$

be a S-vector space defined over the S-ring $\langle R \cup I\rangle$.

(i) Let $P = \{\sqrt{2}, \sqrt{3}, 0, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13}\} \subseteq \langle R \cup I\rangle$; T be a S-quasi set topological vector subspace of M over P.
(ii) Is T first countable?
(iii) Is T second countable?
(iv) Find S-quasi set subtopological vector subspaces of T defined over P.

24. Let P = $\{Z_{28} \times Z_{28} \times Z_{28}\}$ be a S-vector space defined over the S-ring Z_{28}.

(i) Find the total number of S-quasi topological vector subspaces of P.
(ii) How many of them are pseudo simple?
(iii) Does there exist atleast 27 pseudo simple S-quasi set topological vector subspaces?

25. Let V = $\langle Z_{10} \cup I\rangle \times \langle Z_{10} \cup I\rangle \times \langle Z_{10} \cup I\rangle$ be a S-vector space defined over the S-ring Z_{10}.

(i) How many S-quasi set topological vector subspaces can be constructed using V?
(ii) How many are pseudo simple?
(iii) Will all the S-quasi set topological vector subspaces of V defined over subsets of Z_{10} be second and first countable?

26. Does S-quasi set neutrosophic topological vector subspace of a S-neutrosophic vector space defined over a S-ring R enjoy any striking and special properties?

27. Let V be a S-dual number vector space defined over a S-ring. Does the S-quasi set dual number topological vector subspaces of V enjoy any special features?

28. If dual numbers in problem (27) is replaced by special dual like numbers will those S-quasi set special dual like number topological spaces enjoy any special properties?

29. Study the same question in (28) when special dual like numbers are replaced by special quasi dual numbers.

30. Can every S-quasi set neutrosophic topological vector subspace be realized as the S-quasi set special dual like number topological vector subspace? (Justify your claim).

31. Show a S-quasi set special dual like number topological vector subspace in general is not a S-quasi special dual like number topological vector subspace.

32. Every S-quasi set topological vector subspace defined over a set P of cardinality two is always pseudo simple.

33. Can every S-quasi set topological vector subspace defined over a set P of cardinality greater than two always have a S-quasi subset subtopological vector subspace?

34. Compare the S-quasi set topological vector subspaces and set topological vector subspaces.

35. Let V be any S-vector space defined over the S-ring R.

    (i) Characterize those S-quasi subset topological vector subspaces whose associated lattice is not a Boolean lattice.

36. Suppose T is a S-quasi set topological vector subspace of V defined over a set P.

    Will the associated lattice of T be a Boolean algebra?

37. Let V be a S-vector space $\langle R \cup I \rangle [x]$ defined over the S-ring, $\langle R \cup I \rangle$.

(i) Can we have S-quasi set topological vector subspace of V which is finite?

(ii) Can we have S-quasi set topological vector subspace of V which is not second or first countable?

(iii) Can V have S-quasi set topological vector subspace which is both first and second countable?

38. Let W = $\langle Z_n \cup I\rangle$ [x] be a S-vector space defined over the S-ring, $\langle Z_n \cup I\rangle$.

Study the problems (i) to (iii) given in problem 37 in case of this W.

Chapter Four

NEW SET TOPOLOGICAL VECTOR SPACES

In this chapter we for the first time introduce the notion of New Set topological vector subspaces defined over the set. For more information about set vector spaces refer [17].

DEFINITION 4.1: *Let V be a set vector space defined over the set S. Let $P \subseteq S$. T = {collection of all subset vector subspaces of V defined over the set P} (P is a proper subset of S).*

T is given a topology with respect to P and it is easily verified T is a topological space and we define T to be the New Set topological vector subspace of V with respect to P and they are abbreviated as NS-topological vector subspace of V defined over $P \subseteq S$.

We will illustrate this situation by some examples.

Example 4.1: Let

V = {0, 2, 4, 6, 8, 10, …, 2n, …, 5, 15, 25, 35, …, ∞} be a set vector space over the set S = {0, 1, 3, 7, 11, 13, 9, 17}.

Let T = {collection of all subset vector subspaces of V defined over the set P = {0, 1} ⊆ S}. T is a NS-topological vector subspace of V defined over P.

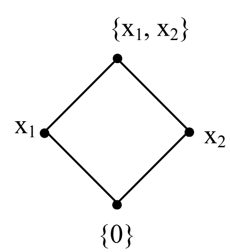
The basic set of T defined as the new basic set or NB-set of T is given by NB_{T_1} or $B^N_{T_1}$ = {(0, 0), (0, 2), (0, 4), …, (0, 2n), …, (0, 5), (0, 15), …, (0, 2m), …} ⊆ T.

Clearly the lattice associated with T is a Boolean algebra of infinite order.

Example 4.2: Let

V = {1, 0, 2, 4, 6, 8, 12, 14, 16, 18, 5, 10, 15, 7, 14} ⊆ Z_{20}. S = {0, 1, 5, 3, 6, 10, 4, 8} ⊆ Z_{20}. V is a set vector space defined over the set S. We see if s ∈ S and v ∈ V, s.v ≡ t (mod 20) ∈ V. For P = {0, 1, 3, 5} ⊆ S. T = {collection of all subset vector subspaces of V defined over the set P} be the NS-topological vector subspace of V over the set P. The NS-basic set B^N_T = {x_1 = (0, 2, 6, 10, 18, 14), x_2 = (0, 4, 12, 16, 8) and so on}.

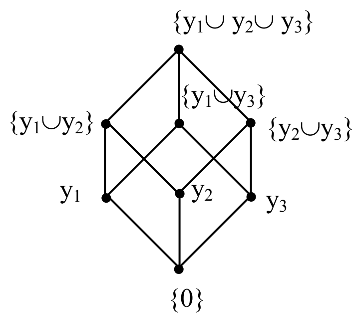
The lattice associated with x_1 and x_2 is as follows.



It is important to note that {$x_1 \cup x_2$} ≠ V, infact $x_1 \cup x_2$ ⊆ V. x_1, x_2 generates a NS-subset subtopological vector subspace of V over the set P. Take P_1 = {0, 2} ⊆ S. Let M = {collection of all subset vector subspaces of V defined over the set P_1 ⊆ S} to be the NS-topological vector subspace of V over the set P_1 ⊆ S. Consider the NS-basic set of M.

$B_M^N = \{(0, 1, 2, 4, 8, 16, 12) = y_1, y_2 = (0, 5, 10), y_3 = (0, 7, 14, 8, 16, 12, 4), y_4 = \{0, 6, 12, 4, 8, 16\}, y_5 = \{0, 15, 10\}$ and $y_6 = \{0, 18, 16, 12, 4\}\}$.

The lattice associated with $\{y_1, y_2, y_3)$ of B_M^N is as follows:



Here also $y_1 \cup y_2 \cup y_3 \neq V$.

Example 4.3: Let V = {1, 0, 10, 20, 40, 5, 15, 25, 30, 35, 45, 2, 4, 6, 8, 12, 14, 16, 18, 22, 24, 26, 32, 34, 36, 38, 42, 44, 46, 48} be a set vector space defined over the set S = {0, 1, 2, 10, 5, 8, 44} $\subseteq Z_{50}$.

Take P = {0, 1, 5, 8} $\subseteq Z_{50}$. Let T = {collection of all subset vector subspaces of V defined over the set P}; be the New Set topological vector subspace of V over the set P.

Consider the new basic set B_T^N = {(0, 1, 5, 8, 25, 40, 14, 20, 12, 46, 30, 18, 10, 44, 2, 16, 28, 24, 42, 36, 38, 6, 4, 32, 48}, (0, 15, 20, 25, 30, 40, 10), (0, 35, 40, 20, 10, 30, 25), (0, 45, 25, 10, 30, 40, 20)}.

Clearly o(B_T^N) = 4.

We see the associated lattice of T is a Boolean algebra of order 2^4.

Example 4.4: Let

V = {0, 3, 6, 9, …, 3n, …, 7, 14, 21, …, 7n …} be a set vector space over the set S = $\{Z^+ \cup \{0\}\}$. Take P = $\{2Z^+ \cup \{0\}\} \subseteq S$.

Let T = {collection of all subset vector subspaces of V over P}. B_T^N has only two sets and the lattice associated with it is a Boolean algebra of order four.

Example 4.5: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, (b_1, b_2, b_3, b_4, b_5), \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{10} \end{bmatrix}, \begin{bmatrix} d_1 & d_2 & \ldots & d_7 \\ d_8 & d_9 & \ldots & d_{14} \end{bmatrix} \right|$$

$a_i \in 3Z$, $b_k \in 5Z$, $c \in 2Z$ and $d_m \in 7Z$; $1 \leq i \leq 4$, $1 \leq k \leq 5$,

$1 \leq j \leq 10$ and $1 \leq m \leq 14\}$

be a set vector space defined over the set S = Z.

Take P = $\{Z^+ \cup \{0\}\} \subseteq S$.

T = {Collection of all subset vector subspaces of V over P}; is NS-topological vector subspace of V defined over the set P.

Now

$$B_T^N = \left\{ \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix} \right\}, \right.$$

$\{(0, 0, 0, 0, 0), (b_1, b_2, b_3, b_4, b_5),$

$$(-b_1, -b_2, -b_3, -b_4, -b_5)\}, \left\{\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}c_1\\c_2\\\vdots\\c_{10}\end{bmatrix}, \begin{bmatrix}-c_1\\-c_2\\\vdots\\-c_{10}\end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix}0 & 0 & \ldots & 0\\0 & 0 & \ldots & 0\end{bmatrix}, \begin{bmatrix}d_1 & d_2 & \ldots & d_7\\d_8 & d_9 & \ldots & d_{14}\end{bmatrix}, \begin{bmatrix}-d_1 & -d_2 & \ldots & -d_7\\-d_8 & -d_9 & \ldots & -d_{14}\end{bmatrix}\right\}\right\}.$$

Clearly $o(B_T^N) = 4$.

Thus the associated lattice of T is a Boolean algebra of order 2^4.

Example 4.6: Let M = $\{3Z \times 3Z, 5Z \times 5Z, 7Z \times 7Z \times 7Z\}$ be a set vector space over defined the set S = $\{0, \pm 1, \pm 3, \pm 5, \pm 7\}$. Take P = $\{0, \pm 1, \pm 3\} \subseteq$ S, W = {collection of all subset vector subspaces of V defined over P}, be the NS-topological vector subspace of M defined over the set P.

Now we proceed onto define the notion of NS-subtopological vector subspaces defined over the set P of a NS-topological vector subspace over the set P.

DEFINITION 4.2: *Let V be a set vector space defined over the set S. P $\subseteq$ S (P a proper subset of S). T = {Collection of all subset vector subspaces of V defined over the set P} be the NS-topological vector subspace of V over the set P.*

Let W $\subseteq$ T (W a proper subset of T), where W = {collection of subset vector subspaces of V defined of the set P}; we define W as the NS- subtopological vector subspace of T defined over the set P.

We will illustrate this situation by some examples.

Example 4.7: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}, (a_1, a_2, \dots, a_{12}) \,\middle|\, a_i \in Z_{10}, 1 \leq i \leq 20 \right\}$$

be a set vector space over defined the set S = {0, 2, 4, 1, 5} ⊆ Z_{10}.

Let P = {0, 1, 5, 4} ⊆ S and T = {collection of all subset vector subspaces of V defined over the set P} be the NS-topological vector subspace of V over P.

Let

$$W = \left\{ (a_1, a_2, \dots, a_{10}), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}, \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \,\middle|\, a_i \in Z_{10}; 1 \leq i \leq 5 \right\}$$

where these elements in W are subset vector subspaces of V defined over the set P contained in T}; W ⊆ T is the NS-subtopological vector subspace of T over P. T has several NS-subtopological vector subspaces over P.

Example 4.8: Let

$$V = \{Z_6 \times Z_6, \left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right| a_i \in Z_6, 1 \leq i \leq 6\}$$

be a set vector space defined over the set S = {0, 2, 4} ⊆ Z_6. T = {Collection of all subset vector subspaces of V defined over the set P} be the NS-topological vector subspace of V over P.

B_T^N, the new basic set of T is {{(0, 0), (0, 1), (0, 2), (0, 4)},

$$\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}\right\}, \{(0, 0), (1, 0), (2, 0), (4, 0)\},$$

$$\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}\right\}, \{(1, 1), (0, 0), (2, 2), (4, 4)\},$$

{(0, 0), (3, 3)}, {(3, 0), (0, 0)}, {(0, 0), (0, 3)},

{(0, 0), (5, 0), (4, 0), (2, 0)} …}.

Consider

$$M = \{\{(0, 0), (1, 0), (2, 0), (4, 0)\}, \left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}\right\}, \{(0, 0), (0, 1), (0, 2), (0, 4)\}, \left\{\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\},$$

{(1, 4), (0, 0), (2, 2), (4, 2)}}.

M generates a NS-topological vector subspace of T over P. Thus ⟨M⟩ is a NS-subtopological vector subspace of T over P.

Consider

$$N = \{\{(0, 0), (1, 5), (2, 4), (4, 2)\}, \{(0, 0), (5, 1), (4, 2), (2, 4)\}, \{(0, 0), (3, 1), (0, 2), (0, 4)\}, \{(0, 0), (1, 3), (2, 0), (4, 0)\},$$

$$\left\{\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}2\\2\\4\end{bmatrix},\begin{bmatrix}4\\4\\2\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}3\\0\\5\end{bmatrix},\begin{bmatrix}0\\0\\4\end{bmatrix},\begin{bmatrix}0\\0\\2\end{bmatrix}\right\}\}.$$

Now N generates again a NS-subtopological vector subspace of T over P.

Example 4.9: Let

$$V = \left\{Z_5 \times Z_5 \times Z_5 \times Z_5 \times Z_5, \begin{bmatrix}a\\b\end{bmatrix} \middle| \, a, b \in Z_5\right\}$$

be a set vector space defined over the set S = {0, 1, 2, 3} $\subseteq Z_5$. Take M = {collection of all subset vector subspaces of V over the set P = {0, 2, 3} ⊆ S}. M is a NS-topological vector subspace of V over the set P.

The new basic set of T denoted by

$$B_S^N = \{\{(0, 0, 0), (1, 0, 0), (2, 0, 0), (3, 0, 0), (4, 0, 0)\},$$

$$\{(0, 0, 0), (0, 1, 0), (0, 2, 0),$$

$$(0, 3, 0), (0, 4, 0)\}, \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}3\\0\end{bmatrix},\begin{bmatrix}4\\0\end{bmatrix}\right\}, \{(0, 0), (2, 0),$$

$$(4, 0), (1, 0), (3, 0)\} \ldots\}.$$

Let W = $\left\{\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}3\\0\end{bmatrix},\begin{bmatrix}4\\0\end{bmatrix}\right\}\right.$, {(0, 0), (2, 0), (4, 0), (1, 0), (3, 0)}, {(0, 0), (1, 2), (2, 4), (3, 1), (4, 3)}, {(0, 0, 0), (1, 0, 0), (2, 0, 0), (3, 0, 0), (4, 0, 0)}}; generate the NS-subtopological vector subspace of T over P. ⟨W⟩ ⊆ T; ⟨W⟩ is a NS-subtopological vector subspace of T over P.

Example 4.10: Let

$$V = \left\{Z_4 \times Z_4, \begin{bmatrix}a_1 & a_2\\a_3 & a_4\\a_5 & a_6\\a_7 & a_8\end{bmatrix}, \begin{bmatrix}a_1 & a_2 & \ldots & a_{10}\\a_{11} & a_{12} & \ldots & a_{20}\end{bmatrix} \middle| a_i \in Z_4, 1 \leq i \leq 20\right\}$$

be a set vector space over the set S = {0, 1, 3}.

Let P = {0, 3} ⊆ S;
T = {all subset vector subspaces of V over the set P}, be the NS-topological vector subspace of V over the set P. The new basic set of T is as follows:

B_T^N = {{(0, 0), (1, 0), (3, 0)}, {(0, 0), (2, 0)}, {(0, 0), (1, 1), (3, 3)}, {(0, 0), (2, 2)}, {(1, 2), (0, 0), (3, 2)}, {(2, 1), (0, 0), (2, 3)}, {(0, 0), (0, 1), (0, 3)}, {(0, 0), (0, 2)},

$$\{(0, 0), (1, 3), (3, 1)\}, \left\{\begin{bmatrix}0&0\\0&0\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&3\\0&0\\0&0\\0&0\end{bmatrix}\right\}, \ldots,$$

$$\left\{\begin{pmatrix}0&0&\ldots&0\\0&0&\ldots&0\end{pmatrix},\begin{pmatrix}1&0&\ldots&0\\0&0&\ldots&0\end{pmatrix},\begin{pmatrix}3&0&\ldots&0\\0&0&\ldots&0\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}\right\}, \ldots\}.$$

Consider the subset

$$L = \{\{(0, 0), (1, 0), (3, 0)\}, \{(0, 0), (0, 1), (0, 3)\},$$

$$\left\{\begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & 1 & \ldots & 1 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 & 3 & 0 & \ldots & 0 \\ 0 & 0 & 3 & 0 & 3 & \ldots & 3 \end{pmatrix}\right\}\} \subseteq B_T^N.$$

L generates a NS-subset subtopological set vector subspace of T over the set $P_1 = \{0, 3\} \subseteq S$.

We can in this way get many NS-subtopological set vector subspaces of T, by varying the subsets of P where T is the NS-set topological vector subspace defined over P.

If a NS-topological set vector subspace, T does not contain NS-subtopological vector subspaces then we define T to be simple. If T does not contain new subset subtopological vector subspaces then we define T to be pseudo simple.

We will give examples of them.

Example 4.11: Let

$$V = \left\{(a, b), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_3 & a_3 \end{pmatrix} \middle| \; a_i, a, b \in Z_{12}, 1 \leq i \leq 4\right\}$$

be a set vector space defined over the set P = {0, 1}.

We see using V cannot define any NS-topological vector subspaces over any subset of P as P cannot have a proper subset of order two.

Thus it is in the first place very important to note all set vector spaces do not pave way to built NS-topological vector subspaces. We call such set vector subspaces as topologically orthodox set vector spaces.

We will first give examples of them and then characterize them.

Example 4.12: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{bmatrix}, (a_1, a_2, \ldots, a_{11}) \right| a_i \in Z_{10};\ 1 \leq i \leq 20\}$$

be a set vector space defined over the set P = {0, 2}. V is a topologically orthodox set vector space defined over P.

Example 4.13: Let

$$M = \left\{Q \times Q, \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right| a_i \in Q;\ 1 \leq i \leq 4\}$$

be a set vector space over the set P = {1, –2}. M is a topologically orthodox set vector space defined over P.

Inview of this we have the following theorem.

THEOREM 4.1: *Let V be a any set vector space defined over a set P of cardinality two. Then V is an topologically orthodox set vector space.*

Proof : Follows from the fact the order of P is two so P cannot have proper subsets of order two.

THEOREM 4.2: *Let V be any topologically orthodox set vector space defined over a set P of cardinality two.*

V cannot have even any simple NS-topological vector subspace associated with it.

Proof: Follows from the fact that on V no NS-topological set vector subspace can be defined as V is topologically orthodox set vector space.

Example 4.14: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \end{bmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| \; a_i \in Z_{15}; 1 \leq i \leq 10 \right\}$$

be a set vector space defined over the set P = {0, 1, 5} ⊆ Z_{15}. Let T = {Collection of all set vector subspaces of V defined over the set S = {0, 1} ⊆ P ⊆ Z_{15}} be a NS-topological vector subspace of V over the set S. T is pseudo simple NS-subtopological vector subspace of V over S = {0, 1}.

However we say T is pseudo simple if we cannot find a NS-subtopological set vector subspace of T over the subset M ⊆ S; that S has no proper subset of order two.

In view of this we have the following theorem.

THEOREM 4.3: *Let V be a set vector space defined over the set P. Suppose T is a NS-set topological vector subspace of V over the set S = {a, b} ⊆ P, then T is pseudo simple and in general not simple.*

Proof: Pseudo simplicity of T is direct from the fact that S has only two elements so S cannot have a proper subset with two elements.

However even if T has atleast two elements in the new basic set B_T^N we can take one element and generate the two element set topology which will be NS-subtopological vector subspace of T. Hence the theorem.

We will describe this by examples.

Example 4.15: Let

$$V = \left\{Z_{14} \times Z_{14}, \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \end{bmatrix} \middle| \; a, b, c, a_i \in Z_{14}, \; 1 \leq i \leq 16\right\}$$

be a set vector space defined over the set S = {0, 1, 2, 3} $\subseteq Z_{14}$. Consider T = {all set vector subspaces of V defined over the set P = {0, 1} $\subseteq$ S}, this T is a NS-topological vector subspace of V over the set P = {0, 1}.

The new basic set of T;

$$B_T^N = \{\{(0, 0), (1, 0)\}, \{(0, 0), (0, 1)\}, \{(0, 0), (1, 1)\},$$

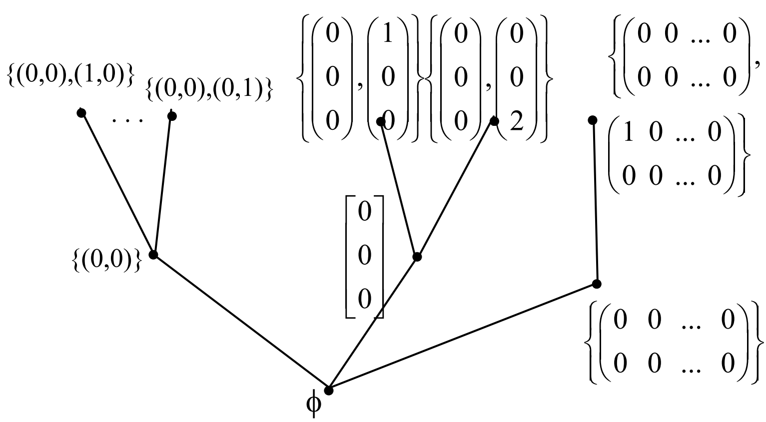
$$\{(0, 0), (2, 2)\}, \ldots, \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}, \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}, \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right\}, \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right\}, \ldots, \left\{\begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}\right\}, \ldots,$$

$$\left\{\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 2 & 0 & 1 & 13 & 0 & 4 \end{pmatrix}\right\}.$$

We see any element in B_T^N will generate a NS-subtopological set vector subspace of T. The least element of the associated lattice of T is 'ϕ' the empty set since B_T^N is the new basic set we see even intersection of {(0, 0), (1, 0)} ∩ {(0, 0), (0, 1)} is (0, 0) and so on. Thus the lattice is not a Boolean algebra but the atoms are not defined.



One need to study such lattices.

However we can get many number of NS-subtopological vector subspaces over P. T is NS-pseudo simple and V is not a topologically orthodox set vector space over the set S.

Example 4.16: Let

$$V = \{Z_4 \times Z_4 \times Z_4, \begin{bmatrix} a \\ b \end{bmatrix} \Bigg| \; a, b \in Z_4\}$$

be a set vector space over the set S = {0, 1, 2} $\subseteq Z_4$. Let T = {Collection of all subset vector subspaces of $Z_4 \times Z_4$ defined over the set P = {0, 2} $\subseteq$ S} be a NS-subtopological set vector subspace of V over the set P $\subseteq$ S. The new basic set of T is as follows:

B_T^N = {{(0, 0), (1, 0), (2, 0)}, {(0, 0), (0, 1), (0, 2)}, {(0, 0), (1, 1), (2, 2)}, {(0, 0), (3, 0), (2, 0)}, {(0, 0), (0, 3), (0, 2)}, {(0, 0), (3, 3), (2, 2)}, {(0, 0), (1, 2), (2, 0)}, {(0, 0), (2, 1), (0, 2)}, {(0, 0), (1, 3), (2, 2)}, {(0, 0), (3, 1), (2, 2)}, {(0, 0), (2, 3), (0, 2)}, {(0, 0), (3, 2), (2, 0)}} .

We see o(B_T^N) = 12 and T has 2^{12} elements in it. Further the lattice L associated with T is a Boolean algebra of order 2^{12}. These 12 elements of B_T^N serve as atoms of L. The least element of the lattice L is {(0, 0)} and the largest element is $Z_4 \times Z_4$.

Now we give the NS-topological set vector subspace for which V is the largest element and empty set is the least element. We work only with the same set P = {0, 2}.

Let

M = {set of all subset vector subspaces of V over the set P} be the NS-topological set vector subspace of V defined over the set P = {0, 2}. The new basic set of M denoted by B_M^N and

$$B_M^N = \{ B_T^N, \{0, 1, 2\}, \{0, 3, 2\}, \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\},$$

$$\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}3\\0\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\3\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}3\\2\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}3\\3\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix}\right\}\Bigg\}.$$

We see o(B_M^N) = 26. Thus the lattice L associated with M is of order 2^{26} and empty set as the least element and V as the largest element of M. Clearly T $\subseteq$ M is a NS-subtopological set vector subspace of V over P = {0, 2}.

Now if L = {Collection of all set vector subspaces of Z_4 over the set P = {0, 2}}, L is a NS- subtopological set vector subspace of M defined over the set P. That is L $\subseteq$ M. B_L^N = {{0, 1, 2}, {0, 3, 2}}.

Now S = {collection of all set vector subspaces of $\begin{bmatrix}a\\b\end{bmatrix}$ with a, b $\in$ Z_4 over the set P}; is the NS-subtopological subvector subspace of M over the set P = {0, 2}. That is S $\subseteq$ M.

We have given three NS-subtopological set vector subspaces of V defined over the set P = {0, 2}. However we have several other NS-subtopological set vector subspaces of M.

Suppose W is generated by the set

{{(0, 0), (1, 0), (2, 0)}, {(0, 0), (1, 3), (2, 2)}, {0, 1, 2},

$$\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}3\\0\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix}\right\}\} \subseteq B_M^N.$$

W is a NS-subtopological set vector subspace of M with 2^5 elements and ϕ is the least element of W and

$$\{0, 1, 2, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix}, (1, 0), (0, 0), (2, 0), (1, 3), (2, 2)\}$$

is the greatest element of W.

Consider the NS-subtopological subvector subspace B of M over P where B is generated by the set

$$\{0, 2, 3, \begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix},\begin{bmatrix}2\\2\end{bmatrix}, (0, 0), (1, 2), (2, 0), (3,1),(2,2)\}.$$

Clearly order of B is 2^{13}.

Now we proceed onto define the new notion of semigroup topological vector subspace of a semigroup vector space over a semigroup defined over a set $P \subseteq S$.

Here we describe some properties associated with it.

DEFINITION 4.3: *Let V be a semigroup vector space defined over a semigroup S. Let $W \subseteq V$; if W is a set semigroup vector subspace of V defined over the subset $P \subseteq S$; that is if wp, pw $\in$ W for all $w \in W$ and $p \in P$.*

We will first illustrate this by some simple examples.

Example 4.17: Let V = $\{5Z_{15} \times 3Z_{15}\}$ be a semigroup vector space defined over the semigroup Z_{15} under product.

Take W = $\{5Z_{15} \times \{0\}\} \subseteq V$. W is a set semigroup vector subspace of V over the set $\{0, 3, 5, 10\} \subseteq Z_{15}$. Take M = {(5, 0), (10, 0), (0, 3), (0, 9), (0, 12)} $\subseteq$ V. M is a set semigroup vector subspace of V over the set $\{0, 3, 5\} \subseteq Z_{15}$.

Example 4.18: Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in 2Z_{20} \cup 5Z_{20} \right\}$$

be a semigroup vector space defined over the semigroup $Z_{20} = S$.

$$\text{Let } W = \left\{ \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}, \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} \middle| \; a, b, c, d \in 4Z_{20} \cup Z_{20} \right\} \subseteq V$$

be a set semigroup vector subspace of V defined over the set $P = \{0, 4, 8, 10, 16\} \subseteq Z_{20}$.

Example 4.19: Let

$V = \{(a, b, c, d) \mid a, b, c, d \in 3Z^+ \cup 5Z^+ \cup 19Z^+ \cup \{0\}\}$

be a semigroup vector space defined over the semigroup $S = Z^+ \cup \{0\}$.

Consider $M = \{(a, b, c, d) \mid a, b, c, d \in 38Z^+ \cup 10Z^+\} \subseteq V$; M is a set semigroup vector subspace of V defined over the set $S = 5Z^+ \cup \{0\} \cup 2Z^+ \cup \{57Z^+\}$.

Example 4.20: Let

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \; a, b, c, d \in 8Z_{40} \cup 5Z_{40} \right\}$$

be a semigroup vector space defined over the semigroup Z_{40} under product. Take

$$N = \left\{ \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix} \middle| \; a, b \in 16Z_{40} \cup 10Z_{40} \right\} \subseteq V;$$

N is a set semigroup vector subspace of V defined over the set $S = \{16, 10, 0, 20, 4\} \subseteq Z_{40}$.

Example 4.21: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \end{bmatrix} \middle| \; a_i \in 5Z_{100} \cup 4Z_{100}; 1 \leq i \leq 10 \right\}$$

be a semigroup vector space defined over the semigroup Z_{100}.

Consider

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & a_6 & 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in 10Z_{100} \cup 16Z_{100}; 1 \leq i \leq 6 \right\}$$

$\subseteq V$, $V \in M$ as V is a trivial set semigroup vector subspace of V over the set P.

If $0 \in P$; $\{0\}$ is the least element in M. If $0 \notin P$, the empty set ϕ in M is the least element.

We see union of elements in M is in M. Also finite intersection of elements in M are in M.

Thus a topology can be defined on M and this topology is defined as the semigroup topological set vector subspace of V over the set $P \subseteq S$.

The following observations are interesting.

(i) The semigroup topological set vector subspace depends on the set over which it is defined.
(ii) There exist several semigroup topological set vector subspaces depending on the number of subsets in the semigroup over which it is defined.

We will illustrate this situation by some examples.

Example 4.22: Let V = {(a_1, a_2) | a_1, a_2 $\in$ {0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 22, 5, 30, 20, 25, 35, 15, 24, 26, 28, 32, 34, 38, 36} $\subseteq Z_{40}$} be a semigroup vector space over the semigroup Z_{40}.

T = {Collection of all semigroup set vector subspaces of V defined over the set S = {0, 2, 8, 32, 5, 15, 20, 35} $\subseteq Z_{40}$}. T is the semigroup topological set vector subspace of V defined over the set S.

Now if we try to find the semigroup basic set of T and denote it by S(B_T). S(B_T) = {{(0, 0), (1, 0), (2, 0), (8, 0), (5, 0), (32, 0), (15, 0), (20, 0), (35, 0)}, {(0, 0), (0, 1), …, (0, 35)}, {(0, 0), (1, 1), …, (35, 35)}, …, {(0, 0), (39, 38), …}}.

We see the least element is (0, 0) and the largest element is V.

Example 4.23: Let V = {(a, b) | a, b $\in$ {0, 2, 4, 3} $\subseteq Z_6$} be the semigroup vector space defined over the semigroup Z_6.

Let T = {Collection of all set semigroup vector subspaces of V over the set P = {0, 2, 1} $\subseteq Z_6$} be the semigroup topological set vector subspace of V defined over the set P = {0, 1, 2} $\subseteq Z_6$.

S(B_T) of T is as follows:

S(B_T) = {{(0, 0), (2, 0), (4, 0)}, {(0, 0), (0, 2), (0, 4)}, {(0, 0), (2, 2), (4, 4)}, {(0, 0), (3, 0)}, {(0, 0), (0, 3)}}.

Clearly the lattice of T is of order 2^5. This is a Boolean algebra of order 2^5. Clearly T is of order 2^5. The least element is (0, 0) and the largest element is V.

Example 4.24: Let

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \; a, b, c \in \{0, 2, 5, 4, 6, 8\} \subseteq Z_{10} \right\}$$

be a semigroup vector space over the semigroup Z_{10}.

Let T = {Collection of all set semigroup vector subspaces of V over the set P = {0, 2, 1, 5} $\subseteq Z_{10}$} be the semigroup topological set vector subspace of V over the set P.

The new basic set T is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\},$$

$$\left\{ \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

$$\left\{ \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\} \ldots \Bigg\} \text{ is the } S(B_T) \text{ of T.}$$

Example 4.25: Let V = {3Z × 5Z} be the semigroup vector space defined over the semigroup S = Z under product. Let T = {Collection of all set semigroup vector subspaces of V defined over the set $2Z^+ \cup \{0\} \subseteq Z$} be the semigroup topological vector subspace of V over the set $2Z^+ \cup \{0\}$ under $\times$. The semigroup basic set $S(B_T)$ is of infinite order. We can choose $L = 3Z^+ \cup \{0\}$ also to be a set over which semigroup topological set vector subspace can be defined.

Example 4.26: Let $V = 3Z_{24} \times 4Z_{24}$ be a semigroup vector space defined over the semigroup Z_{24}.

T = {Collection of all set semigroup vector subspaces of V defined over the set $P = \{0, 3, 4, 6, 8, 10, 15, 21\} \subseteq Z_{24}$} be semigroup topological set vector subspace of V defined over P. The semigroup basic set of T is $S(B_T) = \{\{(0, 0), (3, 0), (12, 0), (9, 0), (18, 0), (6, 0), (21, 0), (15, 0)\}, \{(0, 0), (4, 0), (12, 0), (16, 0), (8, 0), (16, 0)\} \ldots\}$.

Now we proceed onto define substructures on them.

DEFINITION 4.4: *Let V be a semigroup vector space defined over the semigroup S. Let $W \subseteq V$ be a set semigroup vector subspace of V defined over the set $P \subseteq S$. Let $M \subseteq W$; if M is itself a set semigroup vector subspace of V defined over the set $P \subseteq S$; we define M to be a set semigroup vector subspace of W defined over the same set. Let $L \subseteq P$ where L is a subset of P if $T \subseteq V$ and if T is a set semigroup vector subspace of W defined over the subset $L \subseteq P$ we define T to be a set semigroup vector subspace of W defined over the set L of P.*

We will illustrate this situation by some examples.

Example 4.27: Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in 2Z_{18} \cup 3Z_{18} \right\}$$

be a semigroup vector space defined over the semigroup $S = Z_{18}$.

$$P = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in 4Z_{18} \cup 6Z_{18} \right\} \subseteq V$$

is a semigroup vector subspace defined over the set {0, 6, 4} ⊆ Z_{18}. Now T = {Collection of all set semigroup vector subspaces of V defined over the set {0, 4} ⊆ Z_{18}} is the set semigroup topological vector subspace of V defined over the set {0, 4}.

Example 4.28: Let V = {$5Z_{15} \times 3Z_{15}$} be a semigroup vector space defined over the semigroup Z_{15}.

Consider the set P = {0, 1} ⊆ Z_{15}. We see T = {Collection of all set semigroup vector subspaces of V defined over the set P = {0, 1}} is the set semigroup topological vector subspace of V over the set P.

The semigroup basic set $S(B_T)$ = {{(0, 0), (5, 0)}, {(0, 0), (0, 3)}, {(0, 0), (10, 0)}, {(0, 0), (0, 6)}, {(0, 0), (0, 9)}, {(0, 0), (0, 12)}, {(0, 0), (5, 3)}, {(0, 0), (5, 6)}, {(0, 0), (5, 9)}, {(0, 0), (5, 12)}, {(0, 0), (10, 3)}, {(0, 0), (10, 6)}, {(0, 0), (10, 9)}, {(0, 0), (10, 12)}}.

$o(S(B_T)) = 14$. We see $o(V) = 15$ and $o(T) = 2^{14}$.

So the topological space has 2^{14} elements with {(0, 0)} as the least element and V as the largest element.

Example 4.29: Let V = {$2Z_{10} \times 5Z_{10}$} be a semigroup vector space defined over the semigroup S = Z_{10}. Let P_1 = {0, 1} ⊆ Z_{10}. Suppose M = {Collection of all set semigroup vector subspaces of V defined over the set P_1}; M is a set semigroup topological vector subspace of V defined over the set P_1.

The semigroup basic set of M is $S(B_M)$ = {{(0, 0), (2, 0)}, {(0, 0), (4, 0)}, {(0, 0), (6, 0)}, {(0, 0), (8, 0)}, {(0, 0), (0, 5)}, {(0, 0), (2, 5)}, {(0, 0), (4, 5)}, {(0, 0), (6, 5)}, {(0, 0), (8, 5)}}.

Clearly $o(S(B_M)) = 9$ and the number of elements in M is 2^9. M is a set semigroup topological vector subspace with (0, 0) as the least element and V as its largest element.

Example 4.30: Let $V = \{2Z_{26} \times 13Z_{26}\}$ be a semigroup vector space defined over the semigroup $S = Z_{26}$.

Take $P = \{0, 1\} \subseteq Z_{26}$. D = {Collection of all set semigroup vector subspaces of V defined over the set P = {0, 1}} is a set semigroup topological vector subspace of V over P.

Clearly $oS(B_D) = 2^{25}$.

Inview of all these examples we first make a definition.

DEFINITION 4.5: *Let $V = \{p_1Z_n \times p_2Z_n \times \ldots \times p_t Z_n \mid n = p_1 p_2 \ldots p_t$ where p_i's are distinct t primes $1 \leq i \leq t\}$ be a semigroup vector space defined over the semigroup $S = Z_n$.*

Let P = {0, 1} and M = {Collection of all set semigroup vector subspaces of V defined over the set P = {0, 1}} be the set semigroup topological vector subspace of V defined over the set P = {0, 1}. We define M to be the fundamental set semigroup topological vector subspace of V defined over the set P = {0, 1}.

We give examples and derive some associated properties with them.

Example 4.31: Let $V = \{3Z_{30} \times 2Z_{30} \times 5Z_{30}\}$ be the semigroup vector space defined over the semigroup $S = Z_{30}$. $P = \{0, 1\} \subseteq Z_{30}$. Let A = {Collection of all set semigroup vector subspaces of V defined over the set P} be the set semigroup topological vector subspace of V defined over P. Clearly A is the fundamental set semigroup topological vector subspace of V defined over P.

The semigroup basic set of A is $S(B_A)$ = {{(0, 0, 0), (3, 0, 0)} {(0, 0, 0), (6, 0, 0)}, …, {(0, 0, 0), (27, 0, 0)}, {(0, 0, 0), (0, 2, 0)}, …, {(0, 0, 0), (0, 28, 0)}, {(0, 0, 0), (0, 0, 5)}, …, {(0, 0, 0), (0, 0, 25)}, {(0, 0, 0), (3, 2, 0)}, …, {(0, 0, 0), (3, 2, 25)} , …, {(0, 0, 0), (3, 4, 0)}, {(0, 0, 0), (3, 4, 5)}, …, {(0, 0, 0), (3, 4, 25)}, …, {(0, 0, 0), (27, 28, 25)}}.

$o(S(B_A)) = 9 + 14 + 5 + 9 \times 14 + 9 \times 5 + 14 \times 5 + 9 \times 14 \times 5$

$= 899.$

We see this can be generalized into a theorem.

THEOREM 4.4: *Let $V = \{p_1Z_n \times p_2Z_n \times \ldots \times p_tZ_n \mid t < n, p_i$'s are distinct primes $p_i / n;\ 1 \leq i \leq t\}$ be a semigroup vector space defined over the semigroup Z_n. If T is the set semigroup topological vector subspace of V over $P = \{0, 1\}$ that is the fundamental set semigroup topological vector subspace of V, then the number of elements in T is $o(V) - 1$.*

Proof is direct hence left as an exercise to the reader.

Example 4.32: Let $V = \{2Z_{70} \times 5Z_{70} \times 7Z_{70}\}$ be a semigroup vector subspace of V defined over the semigroup Z_{70}.

T = {collection of all set semigroup vector subspaces of V defined over the set $P = \{0, 1\} \subseteq Z_{70}\}$ be the set semigroup topological vector subspaces of V defined over P.

$o(SB_T) = \{o(V) - 1\}$.

We define now dual of the fundamental semigroup topological space after giving a few examples.

Example 4.33: Let $V = \{3Z_6 \times 2Z_6\}$ be a semigroup vector space defined over the semigroup Z_6. Let $P = \{0, 5\} \subseteq Z_6$.

T = {Collection of all set semigroup vector subspace of V defined over the set $P = \{0, 5\} \subseteq Z_6\}$ be the set semigroup topological vector subspace of V defined over P = {0, 5}. The semigroup basic set associated with T is SB_T = {{(0, 0), (3, 0)}, {(0, 0), (0, 2), (0, 4)}, {(0, 0), (3, 2), (3, 4)} and $o(SB_T) = 3$.

Example 4.34: Let V = {$2Z_{14} \times 7Z_{14}$} be a semigroup vector space defined over the semigroup Z_{14}. Let P = {0, 13} $\subseteq Z_{14}$ and W = {Collection of all set semigroup vector subspaces of V over the set P = {0, 13}} be the set semigroup topological vector subspaces of V defined over the set P = {0, 13}. The semigroup basic set of T is SB_T = {{(0, 0), (2, 0), (12, 0)}, {(0, 0), (4, 0), (10, 0)}, {(0, 0), (6, 0), (8, 0)}, {(0, 0), (0, 7)}, {(0, 0), (2, 7), (12, 7)}, {(0, 0), (4, 7), (10, 7)}, {(0, 0), (6, 7), (8, 7)}}.

$o(SB_T) = 7 = o(V)/2$.

Example 4.35: Let V = {$3Z_{15} \times 5Z_{15}$} be a semigroup vector space over the semigroup S = Z_{15}. Take P = {0, 14} be a proper subset of S = Z_{15}.

Let T = {Collection of all set semigroup vector subspaces of V defined over the set P = {0, 14}} be the set semigroup topological vector subspace of V defined over P.

The semigroup basic set of T is SB_T = {{(0, 0), (3, 0), (12, 0)}, {(0, 0), (6, 0), (9, 0)}, {(0, 0), (0, 5), (0, 10)}, {(0, 0), (3, 5), (12, 10)}, {(0, 0), (3, 10), (12, 5)}, {(0, 0), (6, 5), (9, 10)}, {(0, 0), (9, 10), (6, 5)}} $o(SB_T) = (15–1)/2 = 7$.

$o(T) = 2^7$.

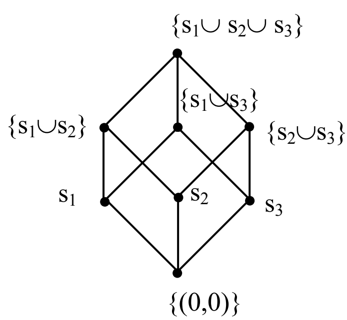
DEFINITION 4.6: *Let V = {$p_1Z_n \times p_2Z_n \times \ldots \times p_tZ_n \mid n = p_1p_2 \ldots p_t$; $t < n$; p_i's are distinct primes, $1 \leq i \leq t$} be a semigroup vector space defined over the semigroup. Let P = {0, n–1} $\subseteq Z_n$. T = {Collection of all set semigroup vector subspaces of V defined over the set P} be the set semigroup topological vector subspace of V defined over the set P. We define T to be the dual fundamental set semigroup topological vector subspace of V defined over the dual set P = {0, n–1} (P = (0, n–1) is defined as dual set of {0, 1} $\subseteq Z_n$).*

Example 4.36: Let M = {$3Z_{42} \times 2Z_{42} \times 7Z_{42}$} be a semigroup vector space over the semigroup Z_{42}. Let P = {0, 41}. T = {Collection of all set semigroup vector subspaces of M over the set P} be the semigroup topological set vector subspace of V over P.

Now the semigroup basic set SB_T = {{(0, 0, 0), (3, 0, 0), (39, 0, 0)}, {(0, 0, 0), (6, 0, 0), (36, 0, 0)}, {(0, 0, 0), (9, 0, 0), (33, 0, 0)}, {(0, 0, 0), (12, 0, 0), (30, 0, 0)}, {(0, 0, 0), (15, 0, 0), (27, 0, 0)}, {(18, 0, 0), (0, 0, 0), (24, 0, 0)}, {(0, 0, 0), (21, 0, 0)}, {(0, 0, 0), (0, 2, 0), (0, 40, 0)}, {(0, 0, 0), (0, 4, 0), (0, 38, 0)}, {(0, 0, 0), (0, 6, 0), (0, 36, 0)}, {(0, 0, 0), (0, 8, 0), (0, 34, 0)}, {(0, 0, 0), (0, 10, 0), (0, 32, 0)}, {(0, 0, 0), (0, 12, 0), (0, 30, 0)}, {(0, 0, 0), (0, 14, 0), (0, 28, 0)}, {(0, 0, 0), (0, 16, 0), (0, 26, 0)}, {(0, 0, 0), (0, 18, 0), (0, 24, 0)}, {(0, 20, 0), (0, 0, 0), (0, 22, 0)}, {(0, 0, 0), (0, 0, 7), (0, 0, 35)}, {(0, 0, 0), (0, 0, 14), (0, 0, 28)}, {(0, 0, 0), (0, 0, 21), (39, 40, 0)}, {(0, 0, 0), (3, 4, 0), (39, 38, 0), …, {(0, 0, 0), (21, 20, 0), (21, 22, 0)}, …, {(0, 0, 0), (21, 20, 21), (21, 22, 21)}}.

Example 4.37: Let V = {$2Z_{14} \times 7Z_{14}$} be a semigroup vector space defined over the semigroup S = Z_{14}. Let P = {0, 1, 3, 5, 11, 13} $\subseteq$ S. T = {Collection of all set semigroup vector subspaces of V defined over the set P}; be the set semigroup topological vector subspace of V defined over P. The semigroup basic set of T; SB_T = {{(0, 0), (2, 0), (6, 0), (4, 0), (12, 0), (8, 0), (10, 0)}, {(0, 0), (0, 7)}, {(0, 0), (2, 7), (4, 7), (8, 7), (6, 7), (10, 7), (12, 7)}} = {s_1, s_2, s_3}.

o(SB_T) = 3. The lattice L associated with T is as follows:



L is a Boolean algebra of order 2^3 with {(0, 0)} as the least element and V as its largest element. However o(V) = 14.

Let us take P_1 = {0, 1, 13} ⊆ P. Suppose M = {collection of all set semigroup vector subspace of V defined over the set P_1}, M, is the set semigroup topological vector subspace of V over P_1.

The semigroup basic set of M is SB_M = {{(0, 0), (2, 0), (12, 0)}, {(0, 0), (4, 0), (10, 0)}, {(0, 0), (6, 0), (8, 0)}, {(0, 0), (0, 7)}, {(0, 0), (2, 7), (12, 7)}, {(0, 0), (4, 7), (10, 7)}, {(0, 0), (6, 7), (8, 7)}, {(0, 0), (7, 7)}} and o(SB_M) = 8. The least element of M is {(0, 0)} and the greatest element is V.

The lattice associated with M is a Boolean algebra of order 2^8.

Let P_2 = {0, 1} ⊆ P and N = {Collection of all set semigroup vector subspaces of V defined over the set P_2 = {0, 1} ⊆ P} be the set semigroup topological vector subspace of V over P_2.

The semigroup basic set of N; SB_N = {{(0, 0), (2, 0)}, {(0, 0), (4, 0)}, {(0, 0), (6, 0)}, {(0, 0), (8, 0)}, {(0, 0), (10, 0)}, {(0, 0), (12, 0)}, {(0, 0), (0, 7)}, {(0, 0), (2, 7)}, {(0, 0), (4, 7)} {(0, 0), (6, 7)}, {(0, 0), (8, 7)}, {(0, 0), (10, 7)}, {(0, 0), (12, 7)}}.

o(SB_N) = 13.

The associated lattice of N is a Boolean algebra of order 2^{13}.

We see N and M are subset semigroup topological vector subspace of T over P_1 and P_1 respectively.

Example 4.38: Let V = $4Z_{20} \times 10Z_{20}$ be a semigroup vector space defined over the semigroup Z_{20}. Let P_1 = {0, 1, 3, 7, 11, 13, 17, 19} ⊆ Z_{20} and T_1 = {Collection of all set semigroup vector subspaces of V defined over the set P_1} be the set semigroup topological vector subspace of V over P_1. The semigroup basic set of T_1 be SB_{T_1} = {{(0, 0), (4, 0), (12, 0), (16, 0), (8, 0)}, {(0, 0), (0, 10)}, {(0, 0), (4, 10), (8, 10), (12, 10), (16, 10)}} = (v_1, v_2, v_3).

$o(SB_T) = 3$ and the lattice associated with T_1 is a Boolean algebra of order 2^3. Let us take $P_2 = \{0, 1\} \subseteq P_1$. T_2 = {Collection of all set semigroup vector subspace of V defined over the set P_2} be the set semigroup topological vector subspace of V over P_2. The semigroup basic set of T_2 be

SB_{T_2} = {{(0, 0), (4, 0)}, {(0, 0), (8, 0)}, {(0, 0), (12, 0)}, {(0, 0), (16, 0)}, {(0, 0), (0, 10)}, {(0, 0), (4, 10)}, {(0, 0), (8, 10)}, {(0, 0), (12, 10)}, {(0, 0), (16, 10)}}.

$o(SB_T) = 9$.

Take $P_3 = \{0, 19\} \subseteq P_1$; let T_3 = {Collection of all set semigroup vector subspaces of V over the set P_3} be the set semigroup topological vector subspace of V over P_3. The semigroup basic set of T_3 is SB_{T_3} = {{(0, 0), (4, 0), (16, 0)}, {(0, 0), (8, 0), (12, 0)} {(0, 0), (0, 10)}} and $o(SB_{T_3}) = 3$. T_3 is also a subset semigroup subtopological vector subspaces of T_1 defined over the subset $P_3 \subseteq P_1$.

Example 4.39: Let $V = \{3Z_{210} \times 2Z_{210} \times 7Z_{210} \times 5Z_{210}\}$ be a semigroup vector space defined over the semigroup Z_{210}. Let $P_1 = \{0, 1\} \subseteq Z_{210}$; T_1 = {Collection of all set semigroup vector subspaces of V defined over the set P_1} be the set semigroup topological vector subspace of V over P_1.

The semigroup basic set of T_1 is SB_{T_1} = {{(0, 0, 0, 0), (3, 0, 0, 0)}, {(0, 0, 0, 0), (6, 0, 0, 0)}, …, {(0, 0, 0, 0), (207, 0, 0, 0)}, …, {(0, 0, 0, 0), (207, 208, 203, 205)}}.

$o(SB_{T_1}) = o(V) - 1$.

Take $P_2 = \{0, 1, 209\} \subseteq Z_{210}$. Now T_2 = {Collection of all set semigroup vector subspaces of V defined over the set P_2} is the semigroup topological set vector subspace of V defined over the set P_2.

The semigroup basic set of T_2 is SB_{T_2} = {{(0, 0, 0, 0), (3, 0, 0, 0), (207, 0, 0, 0)}, {(0, 0, 0, 0), (6, 0, 0, 0), (204, 0, 0, 0)}, {(0, 0, 0, 0), (9, 0, 0, 0), (201, 0, 0, 0)}, …, {(0, 0, 0, 0), (3, 2, 7, 5), (207, 208, 203, 205)}}.

Example 4.40: Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, (a,b,c) \;\middle|\; a, b, c, d \in 3Z_{18} \right\}$$

be a special semigroup vector space defined over the semigroup $S = Z_{18}$.

Take $P = \{0, 5, 7, 11, 13, 17\} \subseteq Z_{18}$; T = {Collection of all semigroup vector subspaces of V defined over the set P} be the special set semigroup topological vector subspace of V defined over the set P. The special semigroup basic set of T is denoted by

$$SB_T = \left\{ \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 15 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 12 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \right.$$

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \ldots,$$

{(0, 0, 0), (3, 0, 0), (15, 0, 0)}, …, {(0, 0, 0), (3, 3, 3), (15, 15, 15)}, {(0, 0, 0), (6, 6, 6), (12, 12, 12)}, {(0, 0, 0), (9, 9, 9)}, {(0, 0, 0), (3, 6, 9), (15, 12, 9)}}.

Example 4.41: Let $V = \{3Z_{48}, 4Z_{48}, Z_{48} \times 6Z_{48}\}$ be a special semigroup vector space defined over the semigroup Z_{48}.

Let P = {0, 1, 47} and T = {Collection of all set special semigroup vector subspaces of V defined over the set P} be the special set semigroup topological vector subspace of V defined over the set P.

Suppose the special semigroup basic set of T be SB_T and if P_1 = {0, 47} be a subset of Z_{48} with T_1 = {Collection of all set special semigroup vector subspaces of V defined over the set P_1} as the special set semigroup topological vector subspace of V defined over the set P_1.

Let the special semigroup basic set of T_1 be SB_{T_1} then $o(SB_{T_1}) = o(SB_T)$ and they are the same in structure.

However if $P_2 = \{0, 1\} \subseteq Z_{48}$ and if T_2 = {Collection of all set special semigroup vector subspaces of V defined over the set P_2} is the set special semigroup topological vector subspace of V defined over P_2. Let SB_{T_2} be the special set semigroup basic set. Clearly $SB_{T_2} \neq SB_{T_1}$.

If we take $P_3 = \{0, 1, 5, 7, 11, 13, 19, 23, 29, 31, 37, 41, 43\} \subseteq Z_{48}$ the associated set special semigroup topological vector subspace would be distinctly different from T_1, T_2 and T.

Example 4.42: Let

$$V = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, 2Z_{10} \times 7Z_{10} \right| a_j \in Z_{70}, 1 \leq j \leq 10\}$$

be the special semigroup vector space defined over the semigroup S = Z_{70}. Let $P = \{0, 1, 5, 18\} \subseteq Z_{70}$ and T = {Collection of all set special semigroup vector subspaces of

V over the set P} be the set special semigroup topological vector subspace of V defined over the set P.

Example 4.43: Let

$$V = \{Z \times Z, \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \middle| \; a, b, c, d, e, f \in 3Z \cup 5Z\}$$

be a special semigroup vector space defined over the semigroup $S = Z^+ \cup \{0\}$.

Let P = {0, 1, 5} ⊆ S and T = {Collection of all set special semigroup vector subspaces of V defined over the set P} be the set special semigroup topological vector subspace of V defined over the set P. The cardinality of the semigroup basic set of T is infinite.

Now we proceed onto define, describe and develop the concept of set group vector subspaces of a group vector space defined over the group G and the notion of set group topological vector subspaces of G defined over the set S ⊆ G.

DEFINITION 4.7: *Let V be a group vector space defined over the group G. Take P a proper subset of G and W ⊆ V (W also a proper subset of V). If for all p ∈ P and g ∈ G; pg and gp ∈ P then we define W to be a set group vector subspace of V defined over the set P ⊆ G.*

We will first illustrate these situations by some examples.

Example 4.44: Let V = {3Z × 5Z × 7Z × 11Z} be a group vector space defined over the group G = Z. Let B = {9Z × {0} × 14Z × {0} ⊆ V and P = 2Z ∪ 5Z ∪ 11Z ⊆ Z be a subset of Z.

We see B is set group vector subspace of V defined over the set P.

Example 4.45: Let V = $\{\{Z_7 \setminus \{0\} \times Z_7 \setminus \{0\}\}\}$ be a group vector space defined over the group G = $Z_7 \setminus \{0\}$. Take M = {(2, 2), (5, 5), (6, 6), (3, 1), (4, 6), (2, 1), (5, 6), (1, 1)} $\subseteq$ V, a proper subset of V. Let P = {1, 6} $\subseteq Z_7 \setminus \{0\}$. M is a set group vector subspace of V defined over the set P.

Consider N = {(3, 3), (4, 4), (3, 4), (4, 3)} $\subseteq$ V; N is also a set group vector subspace of V defined over the set P.

Example 4.46: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \end{bmatrix} \begin{bmatrix} a_1 & a_{11} \\ a_2 & a_{12} \\ \vdots & \vdots \\ a_{10} & a_{20} \end{bmatrix}, (a_1, a_2, a_3) \right| a_i \in Z_{19} \setminus \{0\};\ 1 \leq i \leq 20\}$$

be a group vector space defined over the group G = $Z_{19} \setminus \{0\}$.

Take P = {9, 2, 1, 18} $\subseteq Z_{19} \setminus \{0\}$ = G and

$$M = \left\{ \begin{bmatrix} a & a & a & \ldots & a \\ a & a & a & \ldots & a \end{bmatrix}, (a_1, a_2, a_3) \right| a, a_i \in Z_{19} \setminus \{0\} = G\} \subseteq V;$$

M is a set group vector subspace of V defined over the set P.

M is still a set group vector subspace of V defined over the set P_1 = {1, 2} $\subseteq$ P.

Example 4.47: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, (a_1, a_2, \ldots, a_{10}), \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \right| a_i \in Z_{23} \setminus \{0\};\ 1 \leq i \leq 9\}$$

be a group vector space defined over the group $G = Z_{23} \setminus \{0\}$.

Take $P = \{0, 5, 3, 7, 11, 13, 17, 19\} \subseteq G$ and

$$M = \left\{ (a_1, a_2, \ldots, a_{10}), \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| \, a_i \in G;\ 1 \leq i \leq 10 \right\} \subseteq V;$$

M is a set group vector subspace of V defined over the set P.

Example 4.48: Let

$$V = \left\{ 6Z_{18} \times Z_{18}, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, a, b, c \in 2Z_{18} \right\}$$

be a group vector space defined over the group $(Z_{18}, +)$.

Let

$$M = \left\{ \{0\} \times 3Z_{18}, \begin{bmatrix} a \\ a \\ a \end{bmatrix} \middle| \, a \in 2Z_{18} \right\} \subseteq V;$$

M is a set group vector subspace of V defined over the set $P = \{0, 1, 17\} \subseteq Z_{18}$.

Example 4.49: Let

$$V = \left\{ 3Z_{24} \times 2Z_{24}, \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \middle| \, a \in Z_{24} \right\}$$

be a group vector space defined over the group $G = Z_{24}$ under addition.

Let

$$M = \{6Z_{24} \times 4Z_{24}, \left.\begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}\right| a \in 3Z_{24}\} \subseteq V$$

be a set group vector subspace of V defined over the set P = {0, 3, 2, 1} ⊆ G.

Now we can define two substructures on set group vector subspaces of a group vector space defined over a set.

DEFINITION 4.8: *Let V be a group vector space defined over the group G. Let M ⊆ V be a set group vector subspace of V defined over the set P ⊆ G. Suppose N ⊆ M and N is a set group vector subspace of V defined over the set P ⊆ G then we define N to be the set group strong vector subspace of M defined over P. If M has no such set group strong vector subspace then we define M to be simple over P.*

Example 4.50: Let

$$V = \{3Z_{30} \times 5Z_{30}, (a_1, a_2, a_3, a_4, a_5), \left.\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}\right|$$

$$a_i \in 2Z_{30}, b_j \in 10Z_{30}, 1 \leq i \leq 5 \text{ and } 1 \leq j \leq 4\}$$

be a group vector space defined over the group G = Z_{30} under addition.

Take

$$P = \{0, 1, 15, 5\} \subseteq Z_{30} \text{ and } M = \{(a, a, a, a, a) \begin{bmatrix} b \\ b \\ b \\ b \end{bmatrix} \Bigg|$$

$$a \in 2Z_{30} \text{ and } b \in 10Z_{30}\} \subseteq V$$

be the set group vector subspace of V defined over the set P.

Take N = {(a, a, a, a, a) | a ∈ $2Z_3$} ⊆ M, N is a set group strong vector subspace of M defined over P.

Take

$$L = \left\{ \begin{bmatrix} b \\ b \\ b \\ b \end{bmatrix} \Bigg| \; b \in 10Z_{30} \right\} \subseteq M;$$

L is also a set group strong vector subspace of M defined over P. Thus M is not simple over P.

Example 4.51: Let V = {$3Z_6 \times 2Z_6$} be a group vector space defined over the group Z_6 under addition. Now V = {(0, 0), (3, 0), (0, 2), (0, 4), (3, 2), (3, 4)}. Let P = {0, 1}. M = {(0, 0), (3, 0)} ⊆ V is a set group vector subspace of V defined over the set P.

Clearly M is simple as M can have only {(0, 0)} to be a set group strong vector subspace which is obviously trivial. Another set group strong vector subspace being M itself.

Example 4.52: Let V = {$3Z_9 \times Z_9$} be a group vector space defined over the group Z_9 under addition. Take M = {$3Z_9 \times \{0\}$} ⊆ V to be a set group vector subspace of V defined over the set P = {0, 1, 8} ⊆ Z_9.

M = {(0, 0), (3, 0), (6, 0)} ⊆ V. M is a simple set group vector subspace of V defined over the set P.

It is important and interesting to make the following observation. A simple set group vector subspace defined over a set P need not continue to be simple over some other set P_1.

This is explained by the following example.

Example 4.53: Let V = {$5Z_{25} \times Z_{25}$} be a group vector space defined over the group Z_{25} under addition. Let M = {(0, 0), (5, 0), (20, 0)} ⊆ V; be the set group vector subspace of V defined over the set P = {0, 1, 24}.

Now M = {(5, 0), (0, 0), (28, 0)} is a simple group vector subspace defined over P.

However take the set P_1 = {0, 5, 1} instead of P; M = {(0, 0), (5, 0), (20, 0)} is not simple for N = {(0, 0), (5, 0)} ⊆ M is a set group strong vector subspace of M defined over P_1.

Thus the notion of simple is a relative concept depends on the set chosen from the group G over which the group vector space is defined.

Now we proceed onto define another type of substructure.

DEFINITION 4.9: *Let V be a group vector space defined over the group G. Let M ⊆ V be a set group vector subspace of V defined over the set P ⊆ G. Let S ⊆ P where S is a proper subset of P if N ⊆ M is such that N is a set vector subspace of V defined over the set S ⊆ P then we define N to be a subset group vector subspace of M defined over the subset S ⊆ P. If M has no subset group vector subspace of V defined over any subset in P then we define M to be a pseudo simple set group vector subspace of V defined over the set P.*

We will illustrate this situation by some examples.

Example 4.54: Let V = $\{2Z_{10} \times 5Z_{10}\}$ be a group vector space defined over the group G = Z_{10} under addition. Let M = $\{2Z_{10} \times \{0\}\} \subseteq V$ be a set group vector subspace of V defined over the set P = {0, 1, 5} ⊆ G. Take N = {(0, 0), (4, 0), (8, 0)} ⊆ M. N is a subset vector subspace of M defined over the subset P_1 = {0, 5} ⊆ P. So M is not pseudo simple.

Example 4.55: Let V = $\{2Z_{12} \times 3Z_{12}\}$ be a group vector space defined over the group G = Z_{12} under addition modulo 12.

Let M = $\{\{0\} \times 3Z_{12}\} \subseteq V$ be a group vector subspace of V defined over the set P = {0, 1}. M is pseudo simple.

Example 4.56: Let V = $\{4Z_{20} \times 5Z_{20}\}$ be a group vector space defined over the group Z_{20} under addition modulo 20.

Let M = $\{8Z_{20} \times 10Z_{20}\} \subseteq V$ be a set group vector subspace of V defined over the set P = {0, 1} ⊆ Z_{20}. M is pseudo simple but is not simple for N = $\{8Z_{20} \times \{0\}\} \subseteq M$ is a set group strong vector subspace of M defined over the set P = {0, 1}.

In view of these we have the following theorems.

THEOREM 4.5: *Let V be a group vector space defined over the group G. Let M ⊆ V be a set group vector subspace of V defined over the set P = {a, b} ⊆ G. M is a pseudo simple set group vector subspace of V defined over P.*

The proof follows from the fact the cardinality of P is two and so P cannot have a proper subset of cardinality two.

THEOREM 4.6: *Let V be a group vector space defined over the group G. Let M be a pseudo simple set group vector subspace of V defined over a set P ⊆ G. M in general need not be simple.*

Proof: Follows from the following example. Take V = $\{3Z_{30} \times 5Z_{30}\}$ to be a group vector space defined over the group Z_{30} under addition.

Let $P = \{0, 1\} \subseteq Z_{30}$ and $M = \{6Z_{30} \times 10Z_{30}\} \subseteq V$ be the set group vector subspace of V defined over the set $P = \{0, 1\}$. M is a pseudo simple set group vector subspace of V defined over $P = \{0, 1\}$. However M is not simple for take $N = \{6Z_{30} \times \{0\}\} \subseteq M \subseteq V$. N is a set group strong vector subspace of V defined over the set $P = \{0, 1\}$. So M is not a simple set group vector subspace of V.

We now proceed onto define set group topological vector subspace associated with group vector space.

DEFINITION 4.10: *Let V be a group vector space defined over a group G. $P \subseteq G$ (P a proper subset of G). Let T = {Collection of all set group vector subspaces of V defined over the set P}. We can define topology as in case of set semigroup vector spaces. We define T as a set group topological vector subspace of V defined over the set P.*

It is important to note that the set group topological vector subspace is dependent on the set P in general. At times T is the same for more than one set.

We first illustrate this situation by some examples.

Example 4.57: Let $V = 3Z_6 \times 2Z_6$ be a group vector space defined over the group Z_6 under addition. $P = \{0, 1\} \subseteq Z_6$.

T = {Collection of all set group vector subspaces of V over the set P = {0, 1}} = $\{v_1 = \{(0, 0), (0, 2)\}$, $v_2 = \{(0, 0), (0, 4)\}$, $v_3 = \{(0, 0), (3, 0)\}$, $v_4 = \{(0, 0), (3, 2)\}$, $v_5 = \{(0, 0), (3, 4)\}$, $v_1 \cup v_2$, $v_1 \cup v_3$, $v_1 \cup v_4$, $v_1 \cup v_5$, $v_2 \cup v_3$, $v_2 \cup v_4$, $v_2 \cup v_5$, $v_3 \cup v_4$, $v_3 \cup v_5$, $v_4 \cup v_5$, $v_1 \cup v_2 \cup v_3$, $v_1 \cup v_2 \cup v_4$, $v_1 \cup v_2 \cup v_5$, $v_1 \cup v_3 \cup v_4$, $v_1 \cup v_3 \cup v_5$, $v_1 \cup v_4 \cup v_5$, $v_2 \cup v_3 \cup v_4$, $v_2 \cup v_3 \cup v_5$, $v_2 \cup v_4 \cup v_5$, $v_3 \cup v_4 \cup v_5$, $v_1 \cup v_2 \cup v_3\ v_5$, $v_1 \cup v_2 \cup v_3 \cup v_4$, $v_1 \cup v_2 \cup v_4 \cup v_5$, $v_2 \cup v_3 \cup v_4 \cup v_5$, $v_1 \cup v_3 \cup v_4 \cup v_5$, $\{(0, 0)\}$, $v_1 \cup v_2 \cup v_3 \cup v_4 \cup v_5 = V\}$ is the set group topological vector subspace of V over the set P = {0, 1}. o(T) = 32.

Example 4.58: Let V = {3Z × 5Z × 11Z × 19Z} be a group vector space defined over the group Z under addition. P = {0, 1, –1} ⊆ Z be a subset of Z. T = {Collection of all set group vector subspaces of V defined over the set P = {0, 1, –1}}. T is set group topological vector subspace of V defined over P.

We can as in case of set semigroup vector subspaces defined over a set define the group basic set of a set group topological vector subspace.

We will illustrate this situation by some examples.

Example 4.59: Let V = $\{7Z_{42} \times 3Z_{42} \times 2Z_{42}\}$ be a group vector space defined over the group Z_{42} under addition.

Let P = {0, 1} ⊆ Z_{42} be a set in Z_{42}. Let T = {Collection of all set group vector subspaces of V defined over the set P} be the set group topological vector subspace of V defined over P.

The group basic set GB_T = {{(0, 0, 0), (7, 0, 0)}, {(0, 0, 0), (14, 0, 0)}, {(0, 0, 0), (21, 0, 0)}, {(0, 0, 0), (28, 0, 0)}, {(0, 0, 0), (35, 0, 0)}, {(0, 0, 0), (0, 3, 0)}, …, {(0, 0, 0), (0, 39, 0)}, {(0, 0, 0), (0, 0, 2)}, …, {(0, 0, 0), (0, 0, 40)}, …, {(0, 0, 0), (35, 39, 40)}}.

Infact $o(GB_T)$ is finite and the associated lattice L of T is a Boolean algebra of order $2^{o(G(B_T)}$ and $o(GB_T) = o(V) – 1$.

Example 4.60: Let V = $\{2Z_{14} \times 7Z_{14}\}$ be a group vector space defined over the group Z_{14} under addition. Let P = {0, 1, 13} ⊆ Z_{14} and T = {Collection of all set group vector subspaces of V defined over the set P = {0, 1, 13}} be the set group topological vector subspace of V defined over the set P = {0, 1, 13}.

The group basic set of T is GB_T = {{(0, 0), (2, 0), (12, 0)}, {(0, 0), (4, 0), (10, 0)}, {(0, 0), (6, 0), (8, 0)}, {(0, 0), (0, 7)}, {(0, 0), (2, 7), (12, 7)}, {(0, 0), (4, 7), (10, 7)}, {(0, 0), (6, 7), (8, 7)}}.

$o(GB_T) = 7$ and if L is the lattice with T and order of L is 2^7. Infact L is a Boolean algebra of order and $o(T) = 2^7$.

Example 4.61: Let $V = \{3Z_{42} \times 7Z_{42} \times 2Z_{42}\}$ be a group vector space defined over the group $G = Z_{42}$ under addition. Let $P_1 = \{0, 1\} \subseteq Z_{42}$ and T_1 = {Collection of all set group vector subspaces of V defined over the set P_1} is the set group topological vector subspace of V defined over the set $P_1 = \{0, 1\}$.

The group basic set of T denoted by GB_{T_1} = {{(0, 0, 0), (3, 0, 0)}, {(0, 0, 0), (6, 0, 0), …, {(0, 0, 0), (39, 0, 0)}, {(0, 0, 0), (0, 7, 0)}, {(0, 0, 0), (0, 14, 0)}, …, {(0, 0, 0), (0, 35, 0)}, {(0, 0, 0), (0, 0, 2)}, …, {(0, 0, 0), (0, 0, 42)}, {(0, 0, 0), (3, 7, 0)}, …, {(0, 0, 0), (39, 35, 40)}}. Clearly $o(GB_{T_1}) = o(V) – 1$.

Take $P_1 = \{0, 1, 41\} \subseteq Z_{42}$ and T_2 = {Collection of all set group vector subspaces of V defined over the set $P_2 = \{0, 1, 41\}$} is the set group topological vector subspace of V defined over P_2.

The group basic set of T_2 denoted by GB_{T_2} = {{(0, 0, 0), (3, 0, 0), (39, 0, 0)}, {(0, 0, 0), (6, 0, 0), (36, 0, 0)}, {(0, 0, 0), (9, 0, 0), (33, 0, 0)}, …, {(0, 0, 0), (0, 7, 0), (0, 35, 0)}, …, {(0, 0, 0), (0, 0, 2), (0, 0, 40)}, …, {(0, 0, 0), (3, 7, 2), (39, 35, 40)}}, we see $o(GB_{T_1}) > o(GB_{T_2})$.

Example 4.62: Let $V = \{2Z_{210} \times 3Z_{210} \times 5Z_{210} \times 7Z_{210}\}$ be a group vector space defined over the group $G = Z_{210}$ under addition.

Let $P = \{0, 1, 2, 3, 5, 7\} \subseteq Z_{210}$. T = {Collection of all set group vector subspaces of V defined over the set P} be the set group topological vector subspace of V defined over the set P.

The group basic set of T is GB_T = {{(0, 0, 0, 0), (2, 0, 0, 0), (4, 0, 0, 0), (8, 0, 0, 0), (16, 0, 0, 0), (32, 0, 0, 0), …, (208, 0, 0, 0), (6, 0, 0, 0), (10, 0, 0, 0), (14, 0, 0, 0)}, {(0, 0, 0, 0), (0, 3, 0, 0), (0, 6, 0, 0), (0, 15, 0, 0), …, (0, 207, 0, 0)}, {(0, 0, 0, 0), (0, 0, 5, 0), (0, 0, 10, 0), (0, 0, 15, 0), …, (0, 0, 205, 0),}, {(0, 0, 0, 0), (0, 0, 0, 7), (0, 0, 0, 14), (0, 0, 0, 21), (0, 0, 0, 28), (0, 0, 0, 35), (0, 0, 0, 42), (0, 0, 0, 49), …, (0, 0, 0, 203)}, …, {(0, 0, 0, 0), (2, 3, 5, 7), …, (208, 207, 205, 203)}}.

Example 4.63: Let V = {$2Z_{18} \times 3Z_{18}$} be a group vector space defined over the group G = Z_{18}. Take P = {0, 1, 3} $\subseteq Z_{18}$ and let T = {collection of all set group vector subspaces of V defined over the set P} be the set group topological vector subspace of V defined over P.

The group basic set T; GB_T = {(0, 0), (2, 0), (6, 0)}, {(0, 0), (4, 0), (12, 0)}, {(0, 0), (8, 0), (6, 0)}, {(0, 0), (10, 0), (12, 0)}, {(0, 0), (14, 0), (6, 0)}, {(0, 0), (16, 0), (12, 0)}, {(0, 0), (0, 3), (0, 9)}, {(0, 0), (0, 6)}, {(0, 0), (0, 12)} {(0, 0), (0, 15), (0, 9)}, {(0, 0), (2, 6), (6, 0)}, {(0, 0), (2, 9), (6, 9), (0, 9)}, {(0, 0), (2, 12), (6, 0)}, {(0, 0), (2, 15), (6, 9), (0, 9)}, {(0, 0), (4, 3), (12, 9), (0, 9)}, {(0, 0), (4, 6), (12, 0)}, {(0, 0), (6, 6)}, {(0, 0), (8, 6), (6, 0)}, {(0, 0), (10, 6), (12, 0)}, {(0, 0), (12, 6)}, {(0, 0), (14, 6), (6, 0)}, {(0, 0), (16, 6), (12, 0)}, …, {(0, 0), (16, 3), (12, 9), (0, 9)}, {(0, 0), (16, 15), (12, 9), (0, 9)}}.

We see elements in GB_T are such that they have non empty intersection in many cases. Thus depending on the choice of the set P the elements of the group basic set GB_T happens to be distinct or overlapping.

Now we proceed onto define substructures of set group topological vector subspace of a group vector space.

DEFINITION 4.11: *Let V be a group vector space defined over the group G and P $\subseteq$ G (P a proper subset of G). T = {Collection of all set group vector subspaces of V defined over the set P} be the set group topological vector subspace of V defined over P. If S $\subseteq$ T (S a proper subset of T} is a set group topological vector subspace of V defined over P, then we define S to be a set group subtopological vector subspace of T* *defined over P. If T has no set group subtopological vector subspace then we define T to be simple.*

We will give examples of this situation.

Example 4.64: Let V = {$3Z_{15} \times 5Z_{15}$} be a group vector space defined over the group G = Z_{15} under addition. Let P = {0, 1, 14} $\subseteq Z_{15}$ and T = {Collection of all set group vector subspaces of V defined over the set P} be the set group topological vector subspace of V defined over P.

The group basic set of T, GB_T = {{(0, 0), (3, 0), (12, 0)}, {(0, 0), (6, 0), (9, 0)}, {(0, 0), (6, 0), (9, 0)}, {(0, 0), (0, 5), (0, 10)}, {(0, 0), (3, 5), (12, 10)}, {(0, 0), (6, 5), (9, 10)}}.

Consider the set group topological vector subspace generated by S = ⟨{{(0, 0), (6, 0), (9, 0)}, {(0, 0), (6, 5), (9, 10)}}⟩ $\subseteq GB_T$. S $\subseteq$ T and S is the set group subtopological vector subspace of V defined over P.

Infact T has several set group subtopological vector subspaces.

Example 4.65: Let V = {$2Z_{10}$} be a group vector space defined over additive group Z_{10} = G. Let P = {0, 1, 3} $\subseteq Z_{10}$ and T = {Collection of all set group vector subspaces of V defined over the set P = (0, 1, 3)} be the set group topological vector subspace of V defined over P.

The group basic set GB_T of T is {0, 2, 6, 8, 4} = V. Thus for this P, GB_T is a singleton set V and so T is simple.

Example 4.66: Let V = {$2Z_{62}$} be a group vector space defined over the group G = Z_{62} under addition. Let P = {0, 1, 3} $\subseteq Z_{62}$ and T = {Collection of all set group vector subspaces of V defined over the set P} be the set group topological vector subspace of V defined over P. Let GB_T be the group basic set of T; then GB_T = {{0, 2, 6, 18, 54, 38, 52, 32, …}. We see T is not simple.

We leave it as an open problem.

Problem: Let V = $\{2Z_{2p}\}$ be a group vector space defined over the group G (p a prime). Let P = $\{0, p_1, 1 / p_1$ a prime different from p and 2$\}$. T = {Collection of all set group vector subspaces of V defined over P} be the set group topological vector subspace defined over P.

Will T be simple? Find those p_1 in Z_{2p} for which T is simple.

Example 4.67: Let V = $\{2Z_{22}\}$ be a group vector space defined over the additive group G = Z_{2p} (p = 11). Let $P_1 = \{0, 1, 3\}$ and T_1 = {Collection of all set group vector subspaces of V defined over the set P_1} be the set group topological vector subspace of V defined over the set P_1.

The group basic set of T_1 be $GB_{T_1} = \{\{0, 2, 6, 18, 10, 8\}, \{0, 4, 12, 14, 20, 16\}\}$. $o(GB_{T_1}) = 2$ so T_1 is not simple.

Take $P_2 = \{0, 1, 5\} \subseteq Z_{22}$ and let T_2 = {Collection of all set group vector subspaces of V defined over the set P_2} be the set group topological vector subspace of V defined over the set P_2.

The group basic set of T_2 be GB_{T_2} ; $GB_{T_2} = \{\{0, 2, 10, 6, 8, 18\}, \{0, 4, 20, 12, 16, 14\}\}$. $o(GB_{T_2}) = 2$ so T_2 is not simple.

Consider $P_3 = \{0, 7, 1\} \subseteq Z_{22}$ and T_3 = {Collection of all set group vector subspaces of V defined over the set P_3} be the set group topological vector subspace of V defined over the set P_3. The group basic set of T_3 be $GB_{T_3} = \{\{2, 0, 14, 10, 4, 6, 20, 8, 12, 18, 16\}\}$; $o(GB_{T_3}) = 1$ so T_3 is a simple topological space.

Consider $P_4 = \{0, 1,11\} \subseteq Z_{22}$ and T_4 = {Collection of all set group vector subspaces of V defined over the set P_4} be the set group topological vector subspace of V defined over P_4. The group basic set of T_4 be GB_{T_4} = {{0, 2}, {0, 4}, {0, 6}, {0, 8}, {0, 10}, {0, 12}, {0, 14}, {0, 16}, {0, 18}, {0, 20}} and $o(GB_{T_4})$ = 10 so T_4 is not simple. Consider P_5 = {0, 13, 1} $\subseteq$ Z_{22}.

Let T_5 = {Collection of all set group vector subspaces of V defined over the set P_5} be the set group topological vector subspace of V defined over P_5. Suppose the group basic set GB_{T_5} = {{0, 2, 4, 8, 16, 10, 20, 18, 14, 6, 12}}; $o(GB_{T_5})$ = 1 so T_5 is simple. Let P_6 = {0, 17, 1} $\subseteq Z_{22}$ and let T_6 = {Collection of all set group vector subspaces of V defined over the set P_6} be the set group topological vector subspace of V defined over P_6.

The group basic set of T_6 is GB_{T_6} = {{0, 2, 12, 6, 14, 18, 20, 10, 16, 8, 4}} and $o(GB_{T_6})$ = 1 so T_6 is a simple topological space.

Let P_7 = {0, 1, 19} $\subseteq Z_{22}$ and T_7 = {Collection of all set group vector subspaces of V defined over the set P_7} be the set group topological vector subspace of V defined over P_7.

The group basic set of T_7 be GB_{T_7} = {0, 2, 16, 18, 12, 8, 20, 6, 4, 10, 14} and $o(GB_{T_7})$=1 so T_7 is a simple topological space.

Thus it is yet another interesting open problem.

Problem: Let V = {$2Z_{2p}$ / p is a odd prime} be a group vector space defined over the group G = Z_{2p}. For which of the subsets P in G the related / associated set group topological vector subspace T_P of V defined over the set P $\subseteq Z_{2p}$ is simple.

Characterize those prime numbers q $\in Z_{2p}$ which give way to simple set group topological vector subspaces.

Example 4.68: Let V = {$2Z_{34}$} be a group vector space defined over the group G = Z_{34}.

Let P_1 = {0, 1, 3} $\subseteq$ Z_{34} and T_1 = {Collection of all set group vector subspaces of V defined over P_1} be the set group topological vector subspace of V over P_1. The group basic set of T_1 is GB_{T_1} = {{0, 2, 6, 18, 20, 26, 10, 30, 22, 32, 28, 16, 14, 8, 24, 4, 12}}. We see o(GB_{T_1}) = 1 so T_1 is simple.

Take P_2 = {0, 5, 1} $\subseteq$ Z_{34}, T_2 = {collection of all set group topological vector subspaces of V defined over the set P_2} be the set group topological vector subspace of V defined over P_2. The group basic set of T_2 is GB_{T_2} = {{0, 2, 10, 16, 12, 28, 4, 20, 32, 24, 26, 18, 22, 8, 6, 30, 14}}. o(GB_{T_2}) = 2 so T_2 is not simple.

Let P_3 = {0, 7, 1} $\subseteq$ Z_{34} and T_3 = {Collection of all set group vector subspaces of V defined over the set P_3} be the set group topological vector subspace of V defined over the set P_3. The group basic set of T_3 be GB_{T_3} = {{0, 2, 14, 30, 6, 8, 22, 18, 24, 32, 20, 4, 28, 26, 12, 16, 10} and o(GB_{T_3}) = 1.

Now we proceed onto define substructures in the set group topological vector subspaces.

DEFINITION 4.12: *Let V be a group vector space defined over the group G. P $\subseteq$ G be the subset G. T = {collection of all set group vector subspaces of V defined over the set P} be the set group topological vector subspace of V defined over P. Let P_1 $\subseteq$ P (P_1 a proper subset of P). If M $\subseteq$ T; (M a proper subset of T) is a set group topological vector subspace of V defined over the subset P_1 of P, then we define M to the subset group subtopological vector subspace of T defined over the subset P_1 of P. If T has no subtopological vector subspace we say T is pseudo simple.*

We will illustrate this situatin by some examples.

Example 4.69: Let V = $\{2Z_{22}\}$ be a group vector space defined over the group G = Z_{22}. Let P = {0, 5, 1} be a subset of Z_{22} and T_P be the set group topological vector subspace of V defined over the set P. The group basic set of T_P is

GB_{T_P} = {{0, 2, 10, 6, 8, 18}, {0, 4, 20, 12, 16, 14}}. We see by taking P_1 = {0, 5} $\subseteq$ P, let T_{P_1} be the subset group subtopological vector subspace of V defined over P_1. {0} $\in T_{P_1}$. Consider P_2 = {0, 1, 7} $\subseteq Z_{22}$ and T_{P_2} = {collection of all set group topological vector subspaces of V defined over the set P_2} be set group topological vector subspace of V defined over the set P_2.

Now the group basic set of T_{P_2} is $GB_{T_{P_2}}$ = {0, 2, 14, 10, 4, 6, 20, 8, 12, 18, 16}. $o(GB_{T_{P_2}})$ = 1. So T_{P_2} has no subset subtopological spaces though P_2 has subsets.

Based on this we have the following theorem.

THEOREM 4.7: *Let V be a group vector space defined over a group G. Let P $\subseteq$ G (P a set with cardinality greater than two) and T_P the set group topological vector subspace of V defined over P. T_P can be pseudo simple. That is even if o(P) > 2 still the set group topological vector subspace may be pseudo simple as well as simple.*

Examples given earlier are evidence of this claim.

DEFINITION 4.13: *Let V be a group vector space defined over a group G. P $\subseteq$ G and T_P the set group topological vector subspace of V defined over the set P. If T_P is both simple and pseudo simple we then call T_P to be a super simple set group topological vector subspace of V defined over the set P.*

We will give some examples of this situation.

Example 4.70: Let V = {$2Z_{14}$} be a group vector space defined over the group G = Z_{14}. Let P = {0, 3, 1} $\subseteq$ Z_{14}. T_P = {Collection of all set group vector subspaces of V defined over the set P} be the set group topological vector subspace of V defined over P. GB_{T_P} = {{0, 2, 6, 4, 12, 8, 10}} is the group basic set of T_P.

We see o(GB_{T_P}) = 1 so T_P is both simple and pseudo simple so super simple.

We suggest the following problem.

Problem: Let V be a group vector space defined over a group G. P $\subseteq$ G and T_P the set group topological vector subspace of V defined over P.

(i) Find conditions for T_P to be simple.
(ii) Find conditions for T_P to be pseudo simple.
(iii) Find conditions for T_P to be super simple.

Example 4.71: Let V = {$3Z_{15}$} be a group vector space defined over the group G = Z_{15}. P = {0, 2, 1} $\subseteq$ Z_{15} and T_P be the set group topological vector subspace of V defined over P. The group basic set GB_{T_P} = {{0, 3, 6, 12, 9}}. So T_P is simple, pseudo simple and super simple.

Example 4.72: Let V = {$3Z_{21}$} be a group vector space defined over the group Z_{21}. P = {0, 2, 1} $\subseteq$ Z_{21}. T_P be the set group topological vector subspace of V defined over P. The group basic set of T_P is GB_{T_P} = {{0, 6, 3, 12}, {0, 9, 18, 15}}. o(GB_{T_P}) = 2.

T_P is not simple. T_P is not pseudo simple. But if we replace P by P_1 = {0, 1} then T_{P_1} is not simple.

However $GB_{T_P} \neq GB_{T_{P_1}}$.

$GB_{T_{P_1}}$ = {{0, 3}, {0, 6}, {0, 9}, {0, 12}, {0, 15}, {0, 18}}.
The associated lattice of T_{P_1} is a Boolean algebra of order 26 with {0} as the least element and V as the largest element.

Now we proceed onto suggest a few problems.

Problems:

1. Find some special properties enjoyed by NS-topological vector subspaces defined over a set P.

2. Let V = {0, 5, 10, …, 5n, …, 2, 4, 6, …, 2n} be a set vector space defined over the set N = {0, 5, 2, 18, 25, 48}. Let P_1 = {0, 5, 25} $\subseteq$ N.

   (i) Find the NS-topological vector subspace T_1 of V defined over P_1.
   (ii) Let P_2 = {0, 2, 18} $\subseteq$ N; find the NS-topological vector subspace T_2 of V defined over P_2.
   (iii) Compare T_1 and T_2.

3. Let V = {0, 2, 6, 4, 8, 10, 12, 14} $\subseteq Z_{16}$ be a set vector space defined over the set S = {0, 5, 10, 2, 9, 3}.

   (i) Find the number of NS-topological vector subspace of V defined over subsets of S.
   (ii) Let P_1 = {0, 5, 10} $\subseteq$ S; find the NS-topological vector subspace T_1 of V defined over P_1.

      (a) Find $B^N_{T_1}$.
      (b) Find the lattice associated with T_1.

   (iii) If P_2 = {0, 10} $\subseteq$ S; find the NS-topological vector space T_2 defined over P_2 and its new basic set B^N_T.
   (iv) Compare T_1 and T_2.

4. Let $V = \left\{Z_8 \times Z_8, \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \text{ a, b, c, d, e, } a_i \in Z_8,\right.$

   $1 \leq i \leq 4\}$ be a set vector space defined over the set $S = \{0, 1, 3, 4, 5, 6\} \subseteq Z_8$. Take $P = \{0, 1, 4\} \subseteq S \subseteq Z_8$. Let T = {Collection of all subset vector subspaces of V defined over the set P} be the NS-topological vector subspace of V defined over P.

   (i) Find B_T^N and $o(B_T^N)$.
   (ii) Can T have NS-subtopological vector subspaces?
   (iii) Find how many NS-topological vector subspaces of V defined over subsets of S can be constructed.

5. Obtain some interesting properties enjoyed by NS-topological vector subspace defined over a set.

6. Characterize those pseudo simple NS-topological vector subspaces defined over a set P.

7. Can we have a pseudo simple NS-topological vector subspace defined over a set P of cardinality equal to 5?

8. Can the associated lattice of a NS-topological vector subspace defined over a set P be a modular lattice?

9. Will all lattices of a NS-topological vector subspace be a Boolean algebra?

10. Obtain some special features of set semigroup vector subspaces of a semigroup vector space defined over a semigroup.

11. Give examples of set semigroup vector subspaces of a semigroup vector space V defined over a semigroup S.

12. Let V = $\{3Z_{15} \times 5Z_{15}\}$ be a semigroup vector space defined over the semigroup S = Z_{15}. Let P = {2, 5, 0, 8, 6, 9, 11, 7} $\subseteq Z_{15}$. Find all set semigroup vector subspaces of V defined over the set P.

    How many set semigroup vector subspaces can be defined on the set P?

13. Obtain some special properties enjoyed by semigroup topological set vector subspaces defined over a set in the semigroup.

14. Let V = $\{3Z_{420} \times 4Z_{420} \times 7Z_{420}\}$ be a semigroup vector space defined over the semigroup S = Z_{420}.

    (i) Let P = {0, 2, 3, 5, 7, 11, 13} $\subseteq$ S; Find how many set semigroup vector subspaces of V can be defined on P?
    (ii) If P_1 = {0, 11} $\subseteq$ P; then will the set semigroup vector subspace of V defined over P_1 be a substructure of every set semigroup vector subspace defined over P?
    (iii) Find the set semigroup topological vector subspaces T_1 and T defined over the set P_1 and P respectively.
    (iv) Find the semigroup basic sets of both the set semigroup topological vector subspaces T_1 and T.

15. Let V be a special semigroup vector space defined over the semigroup S.

    $$V = \left\{Z_{12} \times Z_{12}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}, (a_1, a_2, \ldots, a_8) \,\middle|\, a, b, c, d \in 2Z_{12}\right.$$

    and $a_j \in 3Z_{12}$; $1 \leq j \leq 8$} be a special semigroup vector space defined over the semigroup Z_{12} and multiplication modulo 12.

(i) Find the number of set semigroup vector subspaces of V defined over the set P = {0, 1, 8} $\subseteq Z_{12}$.
(ii) Find pseudo simple set semigroup vector subspaces of V defined over the set N $\subseteq Z_{12}$.
(iii) Find simple set semigroup vector subspaces of V defined over a set T $\subseteq Z_{12}$.
(iv) Find the corresponding set semigroup topological vector subspaces defined over the sets P, N and T.

16. Compare quasi set topological vector subspaces defined over a set with set semigroup topological vector subspace defined over a set.

17. Does these exist set semigroup topological vector subspace of V defined over a set P $\subseteq$ S, V the semigroup vector space defined over the semigroup S which is both simple and pseudo simple?

18. Give an example of a set semigroup topological vector subspace which is simple.

19. Give an example of a set semigroup topological vector subspace which is pseudo simple but not simple.

20. Does there exist a set semigroup topological vector subspace which is both simple and pseudo simple?

21. Find all the set semigroup topological vector subspaces of the semigroup vector space V = $\{2Z_{194}\}$ defined over the multiplication semigroup S = Z_{194}.

22. Let V = $\{3Z_{930} \times 2Z_{930} \times 5Z_{930}\}$ be the semigroup vector space defined over the semigroup Z_{930}.

(i) How many set semigroup vector subspaces be defined using the set P = {0, 1, 7, 11, 13, 17, 19, 23, 29} $\subseteq Z_{930}$?
(ii) How many set semigroup topological vector subspace of V are simple?

(iii) How many set semigroup topological vector subspaces of V are pseudo simple?
(iv) Can one say there exists atleast 929 pseudo simple set semigroup topological vector subspaces?
(v) Does there exists set semigroup topological vector subspace of V which are both simple and pseudo simple?
(vi) Give at least five distinct set semigroup topological vector subspaces of V which are simple but not pseudo simple.
(vii) Find the lattices associated with them (given by (vi)).

23. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \middle| a_i \in 3Z_{30} \cup 2Z_{30} \right\}$

be a semigroup vector space defined over the semigroup $S = Z_{30}$.

(i) Find atleast three pseudo simple set topological semigroup vector subspaces of V.
(ii) Give atleast three set semigroup topological vector subspaces of V which are not simple.
(iii) Give an example of a set semigroup topological vector subspace which is not simple and not pseudo simple.

24. Does there exist semigroup vector space V defined over a semigroup S using which we can have one and only one set semigroup topological vector subspace?

25. Study the special features enjoyed by fundamental set semigroup topological vector subspaces.

26. Characterize those dual fundamental set semigroup topological vector subspaces of V, V a semigroup vector space defined over a semigroup.

27. Enumerate some special properties enjoyed by group vector spaces V defined over a group G.

28. Can we always define a set group vector subspace of a group vector space V defined over G?

29. Give some interesting features enjoyed by set group vector subspaces of a group vector space defined over a set.

30. Obtain some special features enjoyed by set group topological vector subspaces which are simple.

31. How does the associated lattice of a simple set group topological vector subspace look like?

32. Can one say any thing about the order of lattice associated with simple set group topological vector subspaces defined over a set?

33. Find all set group topological vector subspaces of the group set vector space V = $\{2Z_{214}\}$ defined over the group G = Z_{214}.

34. Characterize those set group topological vector subspaces which are super simple.

35. Let V = $\{3Z_{291}\}$ be a group vector space defined over the group G = Z_{291} under addition.

    (i) Find at least 2 pseudo simple set group topological vector subspaces of V defined over subsets in G.
    (ii) Find atleast one super simple set group topological vector subspace of V defined over the set P in G.
    (iii) Let P = {0, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43} $\subseteq$ G be a proper subset of G. T_P be the set group topological vector subspace of V defined over P.

(a) Find order of GB_{T_P}.
(b) Is T_P simple?
(c) Is T_P pseudo simple?
(d) Find the lattice associated with T_P.
(e) Is T_{P_1} with $P_1 = \{0, 1, 2\}$ simple or pseudo simple?

36. Let V = $\{2Z \times 3Z \times 5Z\}$ be a group vector space defined over the group Z under addition.

(i) Take $P = \{0, 1\} \subseteq Z$ and find T_P the set group topological vector subspace of V over P.

(a) Is T_P pseudo simple?
(b) Is T_P simple?
(c) Find $o(GB_{T_P})$.
(d) Is T_P the fundamental dual set group vector subspace of V over P?

(ii) Take $P_1 = \{0, 1, -1\} \subseteq Z$.
Let T_{P_1} be the set group topological vector subspace of V defined over the set P_1.

(a) Is T_{P_1} simple?
(b) Is T_{P_1} pseudo simple?
(c) If P_1 is replaced by $P_2 = \{0, -1\} \subseteq P$ is $T_{P_1} \cong T_{P_2}$?
(d) Can any of the set group topological vector subspaces of V defined over any set P′ in Z yield a finite topological space?

(iii) Does every set group topological vector subspace of V defined over every subset P of Z satisfy the second and first axiom of countability?

37. Let V = {B × B × B × B | B = Z_{41} \ {0}} be a group vector space defined over the group B under product.

    (i) Let P = {1, 3} ⊆ B find the special features enjoyed by the set group topological vector subspace T_P of V over P.
    (ii) If P_1 = {6, 8, 24} ⊆ B study T_{P_1} the set group topological vector subspace of V over P_1.

        (a) Is T_{P_1} simple?
        (b) Can T_{P_1} be pseudo simple?
        (c) Find $o(GB_{T_{P_1}})$.

38. Does there exist a group vector space V such that every set group topological vector subspace built using V is pseudo simple?

39. Does there exist a group vector space V such that every set group topological vector subspace built using V is simple?

40. Can there exist a group vector space V such that V has no pseudo simple set group topological vector subspace?

41. Compare the set semigroup topological vector subspaces with set group topological vector subspaces.

42. Will every group vector space V yield for the construction of a super simple set group topological vector subspace?

43. Let V = $\left\{\left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right| a_i \in 2Z_{82}; 1 \le i \le 4\right\}$ be a group vector space defined over the group Z_{82}.

(i) Can V have pseudo simple set group topological vector subspaces?
(ii) Can V have simple set group topological vector subspaces?
(iii) If P = {0, 41} find the set group topological vector subspace of V defined over the P.

44. Let V = $\{Q^+ \times Q^+ \cup Q^+\}$ be a group vector space defined over the group G = Q^+ under product.

(i) For P = {1, 2} $\subseteq$ G find the set group topological vector subspace T_P of V defined over P.
(ii) Is T_P second countable?
(iii) If Q^+ is replaced by R^+ will T_P be first countable and second countable?

45. Does there exist a set group topological vector subspace T_P of V, (V is a group vector space defined over a group G) defined over P $\subseteq$ G which is not second countable?

46. Is every set group topological vector subspace T_P of V ($o(V) < \infty$) second countable and first countable?

47. Let V = $\{a + bg \mid a, b \in Z_{40}, g = 6 \in Z_{12}\}$ be a group vector space of dual numbers defined over the group G = Z_{40} under addition.

(i) Find pseudo simple set group topological vector subspace of dual numbers.
(ii) Is every set group topological vector subspace T_P associated with V first and second countable?
(iii) Using this V can we built super simple set group topological vector subspaces?

48. Let V = {a + bg + cg_1 | a, b, c ∈ Z_{17}, g = 4 and g_1 = 6 ∈ Z_{12}} be a group vector space of mixed dual numbers defined over the group G = Z_{17} \ {0} under product.

(i) For P = {1, 16} ⊆ Z_{17} \ {0} let T_P be the set group topological vector subspace over P.

(a) Find o(GB_{T_P}).
(b) Is T_P simple?
(c) Prove T_P is pseudo simple.
(d) Is T_P second countable?

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QUASI SET TOPOLOGICAL VECTOR SUBSPACES

VASANTHA KANDASAMY FLORENTIN SMARANDACHE

In this book, the authors introduce the notion of quasi set topological vector subspaces. The advantage of such study is that given a vector space we can have only one topological space associated with the collection of all subspaces. However, we can have several quasi set topological vector subspaces of a given vector space. Further, we have defined topological spaces for set vector spaces, semigroup vector spaces and group vector spaces.

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