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Smarandache curves in Euclidean 4- space E^4

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The purpose of this paper is to study Smarandache curves in the 4-dimensional Euclidean space E^4, and to obtain the Frenet–Serret and Bishop invariants for the Smarandache curves in E^4. The first, the second and the third curvatures of Smarandache curves are calculated. These values depending upon the first, the second and the third curvature of the given curve.

1. Introduction

It is well known that, if a curve is differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed, and these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curve. The set, whose elements are frame vectors and curvatures of a curve is called Frenet apparatus of the curves. In recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to Minkowski space [1–3] and Galilean space [4]. For instance in [1,2] the authors extended and studied Smarandache curves in Minkowski space-time. A regular curve in Euclidean space E^4, whose position vectors is composed by Frenet frame vectors on another regular curve is called Smarandache curve. Special Smarandache curves in three dimensional Euclidean space studied in [5].

The Bishop frame [6] or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an othonormal frame along a curve simply by parallel transporting each component of the frame in Euclidean 4-space. The parallel transport frame is based on the observation that while $T(s)$ for the given curve model is unique, we may chose any convenient arbitrary basis which consists of relatively parallel vector fields $\{M_1(s), M_2(s), M_3(s)\}$ of the frame, such that they are perpendicular to $T(s)$ at each point[7,8]. The parallel transport frame in four dimensional Euclidean space is studied in [9]. Smarandache curves were studied from deferent researchers in three dimensional Euclidean space [10–14]. Smarandache curves in 4-dimensional Galilean space are presented in [15]. In this paper we study Smarandache curves in 4-dimensional space according to the Frenet frame and parallel transport frame.

2. Preliminaries

Let $\alpha\colon R \to E^4$ be an arbitrary curve in the Euclidean space E^4. Let $\vec{a} = (a_1, a_2, a_3, a_4)$, $\vec{b} = (b_1, b_2, b_3, b_4)$ and $\vec{c} = (c_1, c_2, c_3, c_4)$ be three vectors in E^4, equipped with the standard inner product given by $< \vec{a}, \vec{b} >= a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$. The norm of a vector $a \in E^4$ is given by $\|\vec{a}\| = \sqrt{< \vec{a}, \vec{a} >}$. The curve α is said to be of a unit speed or parametrized by arc length function s if $< \alpha', \alpha' >= 1$. The vector product of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is defined by the determinant

$$\vec{a} \times \vec{b} \times \vec{c} = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_4 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

where $e_1 \times e_2 \times e_3 = e_4, e_2 \times e_3 \times e_4 = e_1, e_3 \times e_4 \times e_1 = e_2, e_4 \times e_1 \times e_2 = e_3, e_3 \times e_2 \times e_1 = e_4$.

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The vectors $t(s)$, $n(s)$, $b_1(s)$, $b_2(s)$ are the moving Frenet frame along the unit speed curve α. Then the Frenet formulas are given by

$$\begin{bmatrix} t' \\ n' \\ b_1' \\ b_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_2 \end{bmatrix}$$

t, n, b_1, and b_2 are called, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields of the curves. The functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ are called respectively, the first, the second and the third curvature of the curve α. The curve is called $W-curve$ if it has constant curvatures k_1, k_2 and k_3.

Let $\alpha = \alpha(t)$ be an arbitrary curve in E^4. The Frenet apparatus of the curve α can be calculated by the following equations.

$$t = \frac{\alpha'}{\|\alpha'\|}$$

$$n = \frac{\|\alpha'\|^2\alpha'' - <\alpha', \alpha''> \alpha'}{\left\| \|\alpha'\|^2\alpha'' - <\alpha', \alpha''> \alpha' \right\|}$$

$$b_1 = \eta b_2 \times t \times n$$

$$b_2 = \eta \frac{t \times n \times \alpha'''}{\|t \times n \times \alpha'''\|}$$

$$k_1 = \frac{\left\| \|\alpha'\|^2\alpha'' - <\alpha', \alpha''> \alpha' \right\|}{\|\alpha'\|^4}$$

$$k_2 = \frac{\|t \times n \times \alpha'''\| \|\alpha'\|}{\left\| \|\alpha'\|^2\alpha'' - <\alpha', \alpha''> \alpha' \right\|}$$

$$k_3 = \frac{<\alpha^{(IV)}, b_2>}{\|t \times n \times \alpha'''\| \|\alpha'\|}$$

where η is taken $\pm$ 1 such that determinant of matrix [t,n,b1,b2] is equal to one.

The Bishop frame or parallel transport frame is an alternative approach to define a moving frame that is well defined even when the curve has vanishing second derivative [9]. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame. The tangent vector and the convenient arbitrary basis for the remainder of the frame are used. The parallel transport equations in E^4 can be expressed as

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \\ M_3' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & 0 & 0 \\ -K_2 & 0 & 0 & 0 \\ -K_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

where K_1, K_2, and K_3 are the principal curvature functions according to parallel transport frame of the curve α. The set $\{T, M_1, M_2, M_3\}$ is called the parallel transport frame of α [6,14].

The expressions of the principal curvatures are given as follows:

$$K_1 = k_1 \cos\theta \cos\psi,$$

$$K_2 = k_1(-\cos\phi \sin\psi + \sin\phi \sin\theta \cos\psi),$$

$$K_3 = k_1(\sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi) \ and$$

$$k_1 = \sqrt{K_1^2 + K_2^2 + K_3^2}, \ k_2 = -\psi' + \phi' \sin\theta, \ k_3 = \frac{\theta'}{\sin\psi},$$

$$\phi' \cos\theta + \theta' \cot\psi = 0$$

where $\theta' = \frac{k_3}{\sqrt{k_1^2 + k_2^2}}, \psi' = -(k_2 + k_3 \frac{\sqrt{k_3^2 - (\theta')^2}}{\sqrt{k_1^2 + k_2^2}}), \phi' = \frac{\sqrt{k_3^2 - (\theta')^2}}{\cos\theta}$

Note that k_1, k_2, k_3 are the principal curvature functions according to Frenet frame and K_1, K_2, K_3 are the principal curvature functions according to the parallel transport frame of the curve α.

3. Main results

3.1. tb_1 *Smarandache curves in* E^4 *according to the Frenet frame.*

In this subsection we define tb_1 Smarandache curves and obtain their Frenet apparatus.

Definition 1. A regular curve in E^4, whose position vector is obtained by Frenet frame vectors on another regular curve, is called Smarandache curve.

Definition 2. Let $\alpha = \alpha(s)$ be a unit-speed curve with constant and nonzero curvatures k_1, k_2, k_3 and $\{t, n, b_1, b_2\}$ be moving frame on it, tb_1 Smarandache curves are defined by $\beta(s_\beta) = \frac{1}{\sqrt{2}}(t(s) + b_1(s))$.

Theorem 1. *Let $\alpha(s)$ be a unit speed curve with constant non zero curvatures k_1, k_2, k_3 and $\beta(s_\beta)$ be tb_1 Smarandache curves in E^4 defined by the frame vectors of $\alpha(s)$. Then the Frenet apparatus of $\beta(\{t_\beta, n_\beta, b_{1_\beta}, b_{2_\beta}, k_{1_\beta}, k_{2_\beta}, k_{3_\beta}\})$ can be formed by Frenet apparatus of $\alpha(\{t, n, b_1, b_2, k_1, k_2, k_3\})$.*

Proof. Let $\beta = \beta(s_\beta)$ be tb_1 Smarandache curve of the curve α. Then

From Definition (2) we have $\beta(s_\beta) = \frac{1}{\sqrt{2}}(t(s) + b_1(s))$

By differentiating $\beta(s_\beta)$ with respect to s we obtain

$$\frac{d\beta(s_\beta)}{ds} = \frac{d\beta(s_\beta)}{ds_\beta}\frac{ds_\beta}{ds} = \frac{1}{\sqrt{2}}((k_1 - k_2)n + k_3 b_2)$$

The tangent vector of the curve β is given by

$$t_\beta = A_1 n + A_2 b_2 \tag{3.1}$$

where $\frac{ds_\beta}{ds} = \frac{\sqrt{(k_1-k_2)^2+k_3^2}}{\sqrt{2}}, A_1 = \frac{(k_1-k_2)}{\sqrt{(k_1-k_2)^2+k_3^2}}$ and $A_2 = \frac{k_3}{\sqrt{(k_1-k_2)^2+k_3^2}}$

Again differentiating the tangent vector of β with respect to s_β we can obtain β'' as follows

$$\beta'' = \frac{\sqrt{2}[-k_1(k_1 - k_2)t + (k_1 k_2 - k_2^2 - k_3^2)b_1]}{(k_1 - k_2)^2 + k_3^2}$$

The principal normal of the curve β is

$$n_\beta = A_3 t + A_4 b_1 \tag{3.2}$$

where $A_3 = \frac{-k_1(k_1-k_2)}{\sqrt{k_1^2(k_1-k_2)^2+(k_1k_2-k_2^2-k_3^2)^2}}$ and $A_4 = \frac{k_1k_2-k_2^2-k_3^2}{\sqrt{k_1^2(k_1-k_2)^2+(k_1k_2-k_2^2-k_3^2)^2}}$

$$\beta''' = A_5 n + A_6 b_2$$

where $A_5 = \frac{2(-k_1^2(k_1-k_2)-k_2(k_1k_2-k_2^2-k_3^2))}{((k_1-k_2)^2+k_3^2)^{\frac{3}{2}}}$ and $A_6 = \frac{k_3(k_1k_2-k_2^2-k_3^2)}{((k_1-k_2)^2+k_3^2)^{\frac{3}{2}}}$

The second binormal vector of the curve β is given easily as follows

$$b_{2_\beta} = \frac{(k_1k_2 - k_2^2 - k_3^2)t + k_1(k_1 - k_2)b_1}{\sqrt{(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2}} \tag{3.3}$$

The first binormal vector of the curve β is

$$b_{1_\beta} = \frac{-k_3 n + (k_1 - k_2)b_2}{\sqrt{k_3^2 + (k_1 - k_2)^2}} \tag{3.4}$$

The first, second and third curvature of the of the curve β are

$$k_{1_\beta} = \frac{2[(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2]}{(k_3^2 + (k_1 - k_2)^2)^2} \tag{3.5}$$

$$k_{2_\beta} = \frac{\sqrt{2}k_3[k_1(k_1k_2 - k_2^2 - k_3^2) + k_1^2(k_1 - k_2)]}{(k_3^2 + (k_1 - k_2)^2)\sqrt{(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2}} \tag{3.6}$$

$$k_{3_\beta} = \frac{\sqrt{2}(-k_1A_4A_5 - k_2A_3A_5 + k_3A_3A_6)}{\sqrt{k_3^2 + (k_1 - k_2)^2}\sqrt{(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2}} \tag{3.7}$$

This completes the proof. □

3.2. TM_1 *Smarandache curves in* E^4 *according to the parallel transport frame.*

In this subsection we define TM_1 Smarandache curves and obtain their parallel transport frame and the principal curvatures.

Definition 3. Let $\alpha = \alpha(s)$ be a unit speed curve in E^4 and $\{T_\alpha, M_{1_\alpha}, M_{2_\alpha}, M_{3_\alpha}\}$ be its moving parallel transport frame. TM_1 Smarandache curves is defined by $\beta(s_\beta) = \frac{1}{\sqrt{2}}(T_\alpha + M_{1_\alpha})$.

Theorem 2. *Let $\alpha = \alpha(s)$ be a unit speed curve with constant principal curvatures $K_{1_\alpha}, K_{2_\alpha}, K_{3_\alpha}$ and $\beta(s_\beta)$ be TM_1 Smarandache curves in E^4 defined by the parallel transport frame vectors of $\alpha = \alpha(s)$. Then the parallel transport frame of β can be formed by the parallel transport frame of α and the principle curvatures of β $(K_{1_\beta}, K_{2_\beta}, K_{3_\beta})$can be obtained by the principal curvatures of α.*

Proof. To investigate the parallel transport frame of TM_1 Smarandache curve according to $\alpha(s)$ differentiating $\beta(s_\beta) = \frac{1}{\sqrt{2}}(T_\alpha + M_{1_\alpha})$ with respect to s

$$T_\beta \frac{ds_\beta}{ds} = \frac{1}{\sqrt{2}}(-K_{1_\alpha}T_\alpha + K_{1_\alpha}M_{1_\alpha} + K_{2_\alpha}M_{2_\alpha} + K_{3_\alpha}M_{3_\alpha})$$

The tangent vector of the curve β can be written as follows

$$T_\beta = \frac{-K_{1_\alpha}T_\alpha + K_{1_\alpha}M_{1_\alpha} + K_{2_\alpha}M_{2_\alpha} + K_{3_\alpha}M_{3_\alpha}}{\sqrt{2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2}} \tag{3.8}$$

where $\frac{ds_\beta}{ds} = \frac{1}{\sqrt{2}}\sqrt{2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2}$

Differentiating (3.8) with respect to s

$$T'_\beta = \frac{dT_\beta}{ds_\beta} = \lambda_0 T_\alpha + \lambda_1 M_{1_\alpha} + \lambda_2 M_{2_\alpha} + \lambda_3 M_{3_\alpha}$$

where $\lambda_0 = \frac{-\sqrt{2}(K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2)}{(2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2)}$, $\lambda_1 = \frac{-\sqrt{2}K_{1_\alpha}^2}{(2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2)}$

$\lambda_2 = \frac{-\sqrt{2}K_{1_\alpha}K_{2_\alpha}}{(2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2)}$, $\lambda_3 = \frac{-\sqrt{2}K_{1_\alpha}K_{3_\alpha}}{(2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2)}$

The first curvature of the curve β according to Frenet frame is

$$k_{1_\beta} = \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2} = \frac{\sqrt{2}\sqrt{K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2}}{\sqrt{2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2}} \tag{3.9}$$

The principal normal of the curve β is given by the following formula

$$n_\beta = \frac{\lambda_0 T_\alpha + \lambda_1 M_{1_\alpha} + \lambda_2 M_{2_\alpha} + \lambda_3 M_{3_\alpha}}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \tag{3.10}$$

The third derivative of the curve β reads

$$\beta''' = (\lambda_0 T'_\alpha + \lambda_1 M'_{1_\alpha} + \lambda_2 M'_{2_\alpha} + \lambda_3 M'_{3_\alpha})\frac{\sqrt{2}}{\sqrt{2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2}}$$

$$\beta''' = \frac{\sqrt{2}[(-\lambda_1 K_{1_\alpha} - \lambda_2 K_{2_\alpha} - \lambda_3 K_{3_\alpha})T_\alpha + \lambda_0 K_{1_\alpha}M_{1_\alpha} + \lambda_0 K_{2_\alpha}M_{2_\alpha} + \lambda_0 K_{3_\alpha}M_{3_\alpha}]}{\sqrt{2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2}} \tag{3.11}$$

$T_\beta \times n_\beta \times \beta''' = C_1 M_{1_\alpha} + C_2 M_{2_\alpha} + C_3 M_{3_\alpha}$ where

$$C_1 = \frac{\sqrt{2}[\lambda_0\lambda_3 K_{1_\alpha}K_{2_\alpha} - \lambda_0\lambda_2 K_{1_\alpha}K_{3_\alpha} - (\lambda_1 K_{1_\alpha} + \lambda_2 K_{2_\alpha} + \lambda_3 K_{3_\alpha})(\lambda_3 K_{2_\alpha} - \lambda_2 K_{3_\alpha})]}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}(2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2)}$$

$$C_2 = \frac{\sqrt{2}[\lambda_0\lambda_3 K_{1_\alpha}^2 - \lambda_0\lambda_1 K_{1_\alpha}K_{3_\alpha} - (\lambda_1 K_{1_\alpha} + \lambda_2 K_{2_\alpha} + \lambda_3 K_{3_\alpha})(\lambda_3 K_{1_\alpha} - \lambda_1 K_{3_\alpha})]}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}(2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2)}$$

$$C_3 = \frac{\sqrt{2}[\lambda_0\lambda_2 K_{1_\alpha}^2 - \lambda_0\lambda_1 K_{1_\alpha}K_{2_\alpha} - (\lambda_1 K_{1_\alpha} + \lambda_2 K_{2_\alpha} + \lambda_3 K_{3_\alpha})(\lambda_1 K_{2_\alpha} - \lambda_2 K_{2_\alpha})]}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}(2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2)}$$

The second binormal of the curve β is given by the following formula

$$b_{2\beta} = \frac{C_1 M_{1_\alpha} + C_2 M_{2_\alpha} + C_3 M_{3_\alpha}}{\sqrt{C_1^2 + C_2^2 + C_3^2}} \tag{3.12}$$

The first binormal of the curve β is given by the following formula

$$\begin{aligned} b_{1\beta} &= b_{2\beta} \times T_\beta \times n_\beta \\ &= \gamma_0 T_\alpha + \gamma_1 M_{1_\alpha} + \gamma_2 M_{2_\alpha} + \gamma_3 M_{3_\alpha} \end{aligned} \tag{3.13}$$

where the constants are given by

$$\gamma_0 = \frac{C_1\lambda_3 K_{2_\alpha} - C_1\lambda_2 K_{3_\alpha} + C_2\lambda_1 K_{3_\alpha} - C_2\lambda_3 K_{1_\alpha} + C_3\lambda_2 K_{1_\alpha} - C_3\lambda_1 K_{2_\alpha}}{\sqrt{C_1^2 + C_2^2 + C_3^2}\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\sqrt{2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2}}$$

$$\gamma_1 = \frac{C_2\lambda_3 K_{1_\alpha} + C_2\lambda_0 K_{3_\alpha} - C_3\lambda_2 K_{1_\alpha} - C_3\lambda_0 K_{1_\alpha}}{\sqrt{C_1^2 + C_2^2 + C_3^2}\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\sqrt{2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2}}$$

$$\gamma_2 = \frac{C_1\lambda_3 K_{1_\alpha} + C_1\lambda_0 K_{3_\alpha} - C_3\lambda_1 K_{1_\alpha} - C_3\lambda_0 K_{1_\alpha}}{\sqrt{C_1^2 + C_2^2 + C_3^2}\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\sqrt{2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2}}$$

$$\gamma_3 = \frac{C_1\lambda_2 K_{1_\alpha} + C_1\lambda_0 K_{2_\alpha} - C_2\lambda_1 K_{1_\alpha} - C_2\lambda_0 K_{1_\alpha}}{\sqrt{C_1^2 + C_2^2 + C_3^2}\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\sqrt{2K_{1_\alpha}^2 + K_{2_\alpha}^2 + K_{3_\alpha}^2}}$$

The parallel transport frame for the curve β has the form

$$\begin{aligned} M_{1_\beta} = {} & \left(\frac{\lambda_0 \cos\theta_\beta \cos\psi_\beta}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}} + \gamma_0 \cos\theta_\beta \sin\psi_\beta\right)T_\alpha \\ & + \left(\frac{\lambda_1 \cos\theta_\beta \cos\psi_\beta}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}} + \gamma_1 \cos\theta_\beta \sin\psi_\beta \right. \\ & \left. - \frac{C_1 \sin\theta_\beta}{\sqrt{C_1^2 + C_2^2 + C_3^2}}\right)M_{1\alpha} + \left(\frac{\lambda_2 \cos\theta_\beta \cos\psi_\beta}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \right. \\ & \left. + \gamma_2 \cos\theta_\beta \sin\psi_\beta - \frac{C_2 \sin\theta_\beta}{\sqrt{C_1^2 + C_2^2 + C_3^2}}\right)M_{2\alpha} \\ & + \left(\frac{\lambda_3 \cos\theta_\beta \cos\psi_\beta}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}} + \gamma_3 \cos\theta_\beta \sin\psi_\beta \right. \\ & \left. - \frac{C_3 \sin\theta_\beta}{\sqrt{C_1^2 + C_2^2 + C_3^2}}\right)M_{3\alpha} \end{aligned} \tag{3.14}$$

$$M_{2_\beta} = \left[\frac{\lambda_0(-\cos\phi_\beta \sin\psi_\beta + \sin\phi_\beta \sin\theta_\beta \cos\psi_\beta)}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}} + \gamma_0(\cos\phi_\beta \cos\psi_\beta + \sin\phi_\beta \sin\theta_\beta \sin\psi_\beta)\right]T_\alpha + \left[\frac{\lambda_1(-\cos\phi_\beta \sin\psi_\beta + \sin\phi_\beta \sin\theta_\beta \cos\psi_\beta)}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}} + \gamma_1(\cos\phi_\beta \cos\psi_\beta + \sin\phi_\beta \sin\theta_\beta \sin\psi_\beta) + \left(\frac{C_1 \sin\phi_\beta \cos\theta_\beta}{\sqrt{C_1^2+C_2^2+C_3^2}}\right)\right]M_{1_\alpha} + \left[\frac{\lambda_2(-\cos\phi_\beta \sin\psi_\beta + \sin\phi_\beta \sin\theta_\beta \cos\psi_\beta)}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}} + \gamma_2(\cos\phi_\beta \cos\psi_\beta + \sin\phi_\beta \sin\theta_\beta \sin\psi_\beta) + \left(\frac{C_2 \sin\phi_\beta \cos\theta_\beta}{\sqrt{C_1^2+C_2^2+C_3^2}}\right)\right]M_{2_\alpha} + \left[\frac{\lambda_3(-\cos\phi_\beta \sin\psi_\beta + \sin\phi_\beta \sin\theta_\beta \cos\psi_\beta)}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}} + \gamma_3(\cos\phi_\beta \cos\psi_\beta + \sin\phi_\beta \sin\theta_\beta \sin\psi_\beta) + \left(\frac{C_3 \sin\phi_\beta \cos\theta_\beta}{\sqrt{C_1^2+C_2^2+C_3^2}}\right)\right]M_{3_\alpha} \tag{3.15}$$

$$M_{3_\beta} = \left[\frac{\lambda_0(\sin\phi_\beta \sin\psi_\beta + \cos\phi_\beta \sin\theta_\beta \sin\psi_\beta)}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}} + \gamma_0(-\sin\phi_\beta \cos\psi_\beta + \cos\phi_\beta \sin\theta_\beta \sin\psi_\beta)\right]T_\alpha + \left[\frac{\lambda_1(\sin\phi_\beta \sin\psi_\beta + \cos\phi_\beta \sin\theta_\beta \sin\psi_\beta)}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}} + \gamma_1(-\sin\phi_\beta \cos\psi_\beta + \cos\phi_\beta \sin\theta_\beta \sin\psi_\beta) + \frac{C_1 \cos\phi_\beta \cos\theta_\beta}{\sqrt{C_1^2+C_2^2+C_3^2}}\right]M_{1\alpha} + \left[\frac{\lambda_2(\sin\phi_\beta \sin\psi_\beta + \cos\phi_\beta \sin\theta_\beta \sin\psi_\beta)}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}} + \gamma_2(-\sin\phi_\beta \cos\psi_\beta + \cos\phi_\beta \sin\theta_\beta \sin\psi_\beta) + \frac{C_2 \cos\phi_\beta \cos\theta_\beta}{\sqrt{C_1^2+C_2^2+C_3^2}}\right]M_{2\alpha} + \left[\frac{\lambda_3(\sin\phi_\beta \sin\psi_\beta + \cos\phi_\beta \sin\theta_\beta \sin\psi_\beta)}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}} + \gamma_3(-\sin\phi_\beta \cos\psi_\beta + \cos\phi_\beta \sin\theta_\beta \sin\psi_\beta) + \frac{C_3 \cos\phi_\beta \cos\theta_\beta}{\sqrt{C_1^2+C_2^2+C_3^2}}\right]M_{3\alpha} \tag{3.16}$$

Note that

$$\theta_\beta = \int \frac{k_{3_\beta}}{\sqrt{k_{1_\beta}^2 + k_{2_\beta}^2}} ds_\beta, \quad \psi_\beta = -\int \left[k_{2_\beta} + k_{3_\beta}\frac{\sqrt{k_{3_\beta}^2 - \theta_\beta'^2}}{\sqrt{k_{1_\beta}^2 + k_{2_\beta}^2}}\right] ds_\beta,$$

and $\phi_\beta = -\int \frac{\sqrt{k_{3_\beta}^2 - \theta_\beta'^2}}{\cos\theta_\beta} ds_\beta$

The second curvature of the curve β according to Frenet frame is given by

$$k_{2_\beta} = \sqrt{\frac{C_1^2+C_2^2+C_3^2}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}}} \tag{3.17}$$

The third curvature of the curve β according to Frenet frame is given by

$$k_{3_\beta} = \frac{-(C_1K_{1_\alpha} + C_2K_{2_\alpha} + C_3K_{3_\alpha})(\lambda_1K_{1_\alpha} + \lambda_2K_{2_\alpha} + \lambda_3K_{3_\alpha})}{\sqrt{C_1^2+C_2^2+C_3^2}} \tag{3.18}$$

The first curvature of the curve β according to parallel transport frame reads

$$K_{1_\beta} = \sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\cos\theta_\beta \cos\psi_\beta \tag{3.19}$$

The second curvature of the curve β according to parallel transport frame reads

$$K_{2_\beta} = \sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}[-\cos\phi_\beta \sin\psi_\beta + \sin\phi_\beta \sin\theta_\beta \cos\psi_\beta] \tag{3.20}$$

The third curvature of the curve β according to parallel transport frame reads

$$K_{3_\beta} = \sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}[\sin\phi_\beta \sin\psi_\beta + \cos\phi_\beta \sin\theta_\beta \cos\psi_\beta] \tag{3.21}$$

The proof is complete. □

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