SOME SPECIAL CURVES BELONGING TO MANNHEIM CURVES PAIR

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In this paper, we investigate special Smarandache curves with regard to Sabban frame for Mannheim partner curve spherical indicatrix. We created Sabban frame belonging to this curves. It was explained Smarandache curves position vector is consisted by Sabban vectors belonging to this curves. Then, we calculated geodesic curvatures of this Smarandache curves. Found results were expressed depending on the Mannheim curve.

Key Words: Mannheim curve pair, Smarandache curve, Sabban frame, Geodesic curvature**.**

1. Introduction

In differential geometry, special curves have an important role. One of these curves Mannheim curves. Mannheim curve was firstly defined by A. Mannheim in 1878. Any curve can be a Mannheim curve if and only if $\lambda = \dfrac{\kappa}{\kappa^2+\tau^2}$, λ is a nonzere constant, where curvature of curve is κ and curvature of torsion is τ . After a time, Mannheim curve was redefined by Liu and Wang. According to this new definition, when first curves principal normal vector and second curves binormal vector are linearly dependent, first curve was named as Mannheim curve, and second curve was named as Mannheim partner curve [8]. We can found many studies in literature related to Mannheim curves [4, 9]. A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors regular curve is called a Smarandache curve [15]. Special Smarandache curves have been studied by some authors [1, 2, 5, 10, 11]. K. Taşköprü, M. Tosun studied special Smarandache curves according to Sabban frame on S^2 [14]. S. Şenyurt and A. Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves [3]. We investigated special Smarandache curves belonging to Sabban frame drawn on the surface of the sphere by Darboux vector of involute and Bertrand partner curves [12, 13]. Let $\alpha : I \to E^3$ be a unit speed curve, we defined the quantities of the Frenet frame and Frenet formulae, respectively [7],

$$T(s)=\alpha'(s),\, N(s)=\frac{\alpha''(s)}{\|\alpha''(s)\|},\, B(s)=T(s)\wedge N(s), \tag{1.1}$$

$$T'(s)=\kappa(s)N(s),\, N'(s)=-\kappa(s)T(s)+\tau(s)B(s),\, B'(s)=-\tau(s)N(s). \tag{1.2}$$

Let α,α^* be Mannheim pair curve and Frenet apparatus $\{T(s),N(s),B(s),\kappa(s),\tau(s)\}$ and $\{T^*(s),N^*(s),B^*(s),\kappa^*(s),\tau^*(s)\}$ are respectively. The relation between the Frenet apparatus are as follows, [9],

$$T^*=\cos\theta T-\sin\theta B,\;\; N^*=\sin\theta T+\cos\theta B,\, B^*=N,\;\; \kappa^*=\frac{\kappa\theta'}{\lambda\tau\|W\|},\;\; \tau^*=\frac{\kappa}{\lambda\tau}\cdot \tag{1.3}$$

Let $\gamma : I \to S^2$ be a unit speed spherical curve. We denote s as the arc-length parameter of γ. Let us denote by

$$\gamma(s) = \gamma(s), \quad t(s) = \gamma'(s), \quad d(s) = \gamma(s) \wedge t(s) \tag{1.4}$$

$\{\gamma(s), t(s), d(s)\}$ frame is called the Sabban frame of γ on S^2. Then we have the following spherical Frenet formulae of γ

$$\gamma'(s) = t(s), \quad t'(s) = -\gamma(s) + \kappa_g(s)d(s), \quad d'(s) = -\kappa_g(s)t(s) \tag{1.5}$$

where κ_g is called the geodesic curvature of the curve γ on S^2 which is

$$\kappa_g(s) = \langle t'(s), d(s) \rangle. \tag{1.6}$$

2. Main Results

In this section, we investigated special Smarandache curves such as created by Sabban frame, $\{T^*, T_{T^*}, T^* \wedge T_{T^*}\}$, $\{N^*, T_{N^*}, N^* \wedge T_{N^*}\}$ and $\{B^*, T_{B^*}, B^* \wedge T_{B^*}\}$. We will find some results. These results will be expressed depending on the Mannheim curve. Let's find results on this Smarandache curves. $\alpha_{T^*}(s_{T^*}) = T^*(s^*)$, $\alpha_{N^*}(s_{N^*}) = N^*(s^*)$ and $\alpha_{B^*}(s_{B^*}) = B^*(s^*)$ be a regular spherical curves on S^2The Sabban frames of spherical indicatrix belonging to Mannheim partner curve are as follows:

$$T^* = T^*, T_{T^*} = N^*, T^* \wedge T_{T^*} = B^*, \tag{2.1}$$

$$N^* = N^*, \quad T_{N^*} = -\cos\phi^* T^* + \sin\phi^* B^*, \quad N^* \wedge T_{N^*} = \sin\phi^* T^* + \cos\phi^* B^*, \tag{2.2}$$

$$B^* = B^*, \quad T_{B^*} = -N^*, \quad B^* \wedge T_{B^*} = T^*. \tag{2.3}$$

From the equation (1.5), the spherical Frenet formulae of (T^*), (N^*) and (B^*) are, respectively,

$$T^{*\prime} = T_{T^*}, \quad T_{T^*}' = -T^* + \frac{\tau^*}{\kappa^*} T^* \wedge T_{T^*}, \quad (T^* \wedge T_{T^*})' = -\frac{\tau^*}{\kappa^*} T_{T^*}, \tag{2.4}$$

$$N^{*\prime} = T_{N^*}, T_{N^*}' = -N^* + \frac{\phi^{*\prime}}{\|W^*\|} N^* \wedge T_{N^*}, (N^* \wedge T_{N^*})' = -\frac{\phi^{*\prime}}{\|W^*\|} T_{N^*}, \tag{2.5}$$

$$B^{*\prime} = T_{B^*}, T_{B^*}' = -B^* + \frac{\kappa^*}{\tau^*} B^* \wedge T_{B^*}, (B^* \wedge T_{B^*})' = -\frac{\kappa^*}{\tau^*} T_{B^*}. \tag{2.6}$$

Using the equation (1.6) the geodesic curvatures of (T^*), (N^*) and (B^*) are,

$$\kappa_g^{T^*} = \frac{\tau^*}{\kappa^*}, \kappa_g^{N^*} = \frac{\phi^{*\prime}}{\|W^*\|} \text{ and } \kappa_g^{B^*} = \frac{\kappa^*}{\tau^*}. \tag{2.7}$$

Definition 2.1. Let (T^*) curve be of α^*, T^* and T_{T^*} be unit vector of (T^*). In this case β_1-Smarandache curve can be defined by

$$\beta_1(s) = \frac{1}{\sqrt{2}}(T^* + T_{T^*}). \tag{2.8}$$

Theorem 2.1. The $\kappa_g^{\beta_1}$ geodesic curvature belonging to β_1-Smarandache curve of the Mannheim curve is $\kappa_g^{\beta_1} = \dfrac{1}{(2 + (\frac{\|W\|}{\theta'})^2)^{\frac{5}{2}}} (\frac{\|W\|}{\theta'}\overline{\lambda}_1 + \frac{\|W\|}{\theta'}\overline{\lambda}_2 + 2\overline{\lambda}_3)$, where

$$\overline{\lambda}_1=-2-(\frac{\|W\|}{\theta'})^2+(\frac{\|W\|}{\theta'})'(\frac{\|W\|}{\theta'}),\quad \overline{\lambda}_2=-2-3(\frac{\|W\|}{\theta'})^2-(\frac{\|W\|}{\theta'})^4-(\frac{\|W\|}{\theta'})'(\frac{\|W\|}{\theta'})$$

$$\overline{\lambda}_3=2(\frac{\|W\|}{\theta'})+(\frac{\|W\|}{\theta'})^3+(\frac{\|W\|}{\theta'})'.$$

Proof:. $\beta_1(s_{T^*})=\frac{1}{\sqrt{2}}(T^*+T_{T^*})$ and from the equation (2.1), we can write

$$\beta_1(s^*)=\frac{1}{\sqrt{2}}(T^*+N^*). \tag{2.9}$$

Differentiating (2.9), $T_{\beta_1}(s^*)$ is

$$T_{\beta_1}(s^*)=\frac{1}{\sqrt{2+(\frac{\tau^*}{\kappa^*})^2}}(-T^*+N^*+\frac{\tau^*}{\kappa^*}B^*). \tag{2.10}$$

Considering the equations (2.9) and (2.10), with ease seen that

$$(\beta_1\wedge T_{\beta_1})(s^*)=\frac{1}{\sqrt{4+2(\frac{\tau^*}{\kappa^*})^2}}(\frac{\tau^*}{\kappa^*}T^*-\frac{\tau^*}{\kappa^*}N^*+2B^*). \tag{2.11}$$

Differentiating (2.10) equation, T_{β_1}' vector is

$$T_{\beta_1}'(s^*)=\frac{\sqrt{2}}{(2+(\frac{\tau^*}{\kappa^*})^2)^2}(\lambda_1T^*+\lambda_2N^*+\lambda_3B^*), \tag{2.12}$$

where coefficients are

$$\begin{cases}\lambda_1=-2-(\frac{\tau^*}{\kappa^*})^2+(\frac{\tau^*}{\kappa^*})'(\frac{\tau^*}{\kappa^*}),\\ \lambda_2=-2-3(\frac{\tau^*}{\kappa^*})^2-(\frac{\tau^*}{\kappa^*})^4-(\frac{\tau^*}{\kappa^*})'(\frac{\tau^*}{\kappa^*}),\\ \lambda_3=2(\frac{\tau^*}{\kappa^*})+(\frac{\tau^*}{\kappa^*})^3+(\frac{\tau^*}{\kappa^*})'.\end{cases} \tag{2.13}$$

From the equation (1.6), (2.11) and (2.12), $\kappa_g^{\beta_1}$ geodesic curvature of $\beta_1(s^*)$ is

$$\kappa_g^{\beta_1}=\frac{1}{(2+(\frac{\tau^*}{\kappa^*})^2)^{\frac{5}{2}}}(\frac{\tau^*}{\kappa^*}\lambda_1+\frac{\tau^*}{\kappa^*}\lambda_2+2\lambda_3). \tag{2.14}$$

Substituting the equations (1.3) into equation (2.9), (2.10), (2.11) and (2.12), Sabban apparatus of the β_1-Smarandache curve for Mannheim curve are

$$\beta_1(s)=\frac{1}{\sqrt{2}}((\sin\theta+\cos\theta)T+(\cos\theta-\sin\theta)B),$$

$$T_{\beta_1}(s)=\frac{\theta'(\sin\theta-\cos\theta)}{\sqrt{\|W\|^2+2\theta'^2}}T+\frac{\|W\|}{\sqrt{\|W\|^2+2\theta'^2}}N+\frac{\theta'(\cos\theta+\sin\theta)}{\sqrt{\|W\|^2+2\theta'^2}}B,$$

$$(\beta_1\wedge T_{\beta_1})(s)=\frac{\|W\|(\cos\theta+\sin\theta)}{\sqrt{2\|W\|^2+4\theta'^2}}T+\frac{2\theta'}{\sqrt{2\|W\|^2+4\theta'^2}}N+\frac{\|W\|(\cos\theta-\sin\theta)}{\sqrt{2\|W\|^2+4\theta'^2}}B,$$

$$T_{\beta_1}'(s)=\frac{(\theta')^4\sqrt{2}(\bar{\lambda}_1\cos\theta+\bar{\lambda}_2\sin\theta)}{(\|W\|^2+2\theta'^2)^2}T+\frac{(\theta')^4\sqrt{2}\bar{\lambda}_3}{(\|W\|^2+2\theta'^2)^2}N+\frac{(\theta')^4\sqrt{2}(\bar{\lambda}_2\cos\theta-\bar{\lambda}_1\sin\theta)}{(\|W\|^2+2\theta'^2)^2}B$$

and $\kappa_g^{\beta_1}$ geodesic curvature is

$$\kappa_g^{\beta_1}=\frac{1}{(2+(\frac{\|W\|}{\theta'})^2)^{\frac{5}{2}}}(\frac{\|W\|}{\theta'}\bar{\lambda}_1+\frac{\|W\|}{\theta'}\bar{\lambda}_2+2\bar{\lambda}_3).$$

Definition 2.2. *Let* (T^*) *curve be of* α^*, T_{T^*} *and* $T^*\wedge T_{T^*}$ *be unit vector of* (T^*). *In this case* β_2*-Smarandache curve can be defined by*

$$\beta_2(s)=\frac{1}{\sqrt{2}}(T_{T^*}+T^*\wedge T_{T^*}). \tag{2.15}$$

Theorem 2.2. *The* $\kappa_g^{\beta_2}$ *geodesic curvature belonging to* β_2*-Smarandache curve of the Mannheim curve is* $\kappa_g^{\beta_2}=\frac{1}{(1+2(\frac{\|W\|}{\theta'})^2)^{\frac{5}{2}}}(2\frac{\|W\|}{\theta'}\bar{\varepsilon}_1-\bar{\varepsilon}_2+\bar{\varepsilon}_3)$, where

$$\bar{\varepsilon}_1=(\frac{\|W\|}{\theta'})+2(\frac{\|W\|}{\theta'})^3+2(\frac{\|W\|}{\theta'})'(\frac{\|W\|}{\theta'}),\quad \bar{\varepsilon}_2=-1-3(\frac{\|W\|}{\theta'})^2-2(\frac{\|W\|}{\theta'})^4-(\frac{\|W\|}{\theta'})'$$

$$\bar{\varepsilon}_3=-(\frac{\|W\|}{\theta'})^2-2(\frac{\|W\|}{\theta'})^4+(\frac{\|W\|}{\theta'})'.$$

Proof: From the equations (1.3), (2.1) and (2.15), we can write

$$\beta_2(s)=\frac{1}{\sqrt{2}}(\sin\theta T+N+\cos\theta B).$$

Herein, if Sabban apparatus calculate

$$T_{\beta_2}(s)=\frac{-\theta'\cos\theta-\|W\|\sin\theta}{\sqrt{2\|W\|^2+\theta'^2}}T+\frac{\|W\|}{\sqrt{2\|W\|^2+\theta'^2}}N+\frac{\theta'\sin\theta-\|W\|\cos\theta}{\sqrt{2\|W\|^2+\theta'^2}}B,$$

$$T_{\beta_2}'(s)=\frac{(\theta')^4\sqrt{2}(\bar{\varepsilon}_2\sin\theta+\bar{\varepsilon}_1\cos\theta)}{2(\|W\|^2+\theta'^2)^2}T+\frac{(\theta')^4\bar{\varepsilon}_3\sqrt{2}}{(2\|W\|^2+\theta'^2)^2}N+\frac{(\theta')^4\sqrt{2}(\bar{\varepsilon}_2\cos\theta-\bar{\varepsilon}_1\sin\theta)}{(2\|W\|^2+\theta'^2)^2}B,$$

$$(\beta_2\wedge T_{\beta_2})(s)=\frac{2\|W\|\cos\theta-\theta'\sin\theta}{\sqrt{4\|W\|^2+2\theta'^2}}T+\frac{\theta'}{\sqrt{4\|W\|^2+2\theta'^2}}N-\frac{2\|W\|\sin\theta+\theta'\cos\theta}{\sqrt{4\|W\|^2+2\theta'^2}}B,$$

$\kappa_g^{\beta_2}$ geodesic curvature is

$$\kappa_g^{\beta_2}=\frac{1}{(1+2(\frac{\|W\|}{\theta'})^2)^{\frac{5}{2}}}(2\frac{\|W\|}{\theta'}\bar{\varepsilon}_1-\bar{\varepsilon}_2+\bar{\varepsilon}_3).$$

Definition 2.3. *Let* (T^*) *curve be of* α^*, T^*, T_{T^*} *and* $T^*\wedge T_{T^*}$ *be unit vector of* (T^*). *In this case* β_3*-Smarandache curve can be defined by*

$$\beta_3(s)=\frac{1}{\sqrt{3}}(T^*+T_{T^*}+T^*\wedge T_{T^*}). \tag{2.16}$$

Theorem 2.3 *The* $\kappa_g^{\beta_3}$ *geodesic curvature belonging to* β_3*-Smarandache curve of the Mannheim curve is* $\kappa_g^{\beta_3}=\dfrac{1}{4\sqrt{2}(1+\frac{\phi'}{\|W\|}+(\frac{\phi'}{\|W\|})^2)^{\frac{5}{2}}}((2\frac{\phi'}{\|W\|}-1)\overline{\varphi}_1-(1+\frac{\phi'}{\|W\|})\overline{\varphi}_2+(2-\frac{\phi'}{\|W\|})\overline{\varphi}_3),$ where

$$\begin{cases}\overline{\varphi}_1=-2+4(\frac{\|W\|}{\theta'})-4(\frac{\|W\|}{\theta'})^2+2(\frac{\|W\|}{\theta'})^3+(\frac{\|W\|}{\theta'})'(2\frac{\|W\|}{\theta'}-1)\\ \overline{\varphi}_2=-2+2(\frac{\|W\|}{\theta'})-4(\frac{\|W\|}{\theta'})^2+(\frac{\|W\|}{\theta'})^3-2(\frac{\|W\|}{\theta'})^4-(\frac{\|W\|}{\theta'})'(1+\frac{\|W\|}{\theta'})\\ \overline{\varphi}_3=2(\frac{\|W\|}{\theta'})-4(\frac{\|W\|}{\theta'})^2+4(\frac{\|W\|}{\theta'})^3-2(\frac{\|W\|}{\theta'})^4+(\frac{\|W\|}{\theta'})'(2-\frac{\|W\|}{\theta'}).\end{cases}$$

Proof: From the equations (1.3),(2.1) and (2.16), we can write

$$\beta_3(s)=\frac{1}{\sqrt{3}}((\sin\theta+\cos\theta)T+N+(\cos\theta-\sin\theta)B).$$

Then if Sabban apparatus calculate

$$T_{\beta_3}(s)=\frac{\theta'\cos\theta-(\|W\|-\theta')\sin\theta}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}T+\frac{\|W\|}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}N-\frac{\theta'\sin\theta+(\|W\|-\theta')\cos\theta}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}B,$$

$$T_{\beta_3}'(s)=\frac{(\theta')^4\sqrt{3}(\overline{\varphi}_2\sin\theta+\overline{\varphi}_1\cos\theta)}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}T+\frac{(\theta')^4\sqrt{3}\overline{\varphi}_3}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}N$$
$$+\frac{(\theta')^4\sqrt{3}(\overline{\varphi}_2\cos\theta-\overline{\varphi}_1\sin\theta)}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}B,$$

$$(\beta_3\wedge T_{\beta_3})(s)=\frac{(2\|W\|-\theta')\cos\theta-(\|W\|+\theta')\sin\theta}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}T+\frac{2\theta'-\|W\|}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}N$$
$$-\frac{(2\|W\|-\theta')\sin\theta+(\|W\|+\theta')\cos\theta}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}B,$$

$\kappa_g^{\beta_3}$ geodesic curvature is

$$\kappa_g^{\beta_3}=\frac{1}{4\sqrt{2}(1+\frac{\|W\|}{\theta'}+(\frac{\|W\|}{\theta'})^2)^{\frac{5}{2}}}((2\frac{\|W\|}{\theta'}-1)\overline{\varphi}_1-(1+\frac{\|W\|}{\theta'})\overline{\varphi}_2+(2-\frac{\|W\|}{\theta'})\overline{\varphi}_3).$$

Definition 2.4. *Let* (N^*) *curve be of* α^*, N^* *and* T_{N^*} *be unit vector of* (N^*). *In this case* β_4*-Smarandache curve can be defined by*

$$\beta_4(s)=\frac{1}{\sqrt{4}}(N^*+T_{N^*}). \tag{2.17}$$

Theorem 2.4. *The* $\kappa_g^{\beta_4}$ *geodesic curvature belonging to* β_4*-Smarandache curve of the Mannheim curve is* $\kappa_g^{\beta_4}=\dfrac{1}{(2+\eta^2)^{\frac{5}{2}}}(\eta\overline{\chi}_1-\eta\overline{\chi}_2+2\overline{\chi}_3),$ where $\eta=(\dfrac{\|W\|}{\sqrt{(\theta')^2+\|W\|^2}})'\dfrac{\lambda\tau}{\theta'}\csc\theta$ and

$\overline{\chi}_1=-2-\eta^2+\eta'\eta,\ \overline{\chi}_2=-2-3\eta^2-\eta^4-\eta'\eta,\ \overline{\chi}_3=2\eta+\eta^3+\eta'.$

Proof: From the equations (1.3), (2.2) and (2.17), we can say

$$\beta_4(s)=\frac{\sqrt{\theta'^2+\|W\|^2}\sin\theta-\theta'\cos\theta}{\sqrt{2\theta'^2+2\|W\|^2}}T+\frac{\|W\|}{\sqrt{2\theta'^2+2\|W\|^2}}N+\frac{\theta'\sin\theta+\sqrt{\theta'^2+\|W\|^2}\cos\theta}{\sqrt{2\theta'^2+2\|W\|^2}}B.$$

If Sabban apparatus calculate

$$T_{\beta_4}(s)=\frac{(\eta\|W\|-\theta')\cos\theta-\sqrt{\|W\|^2+\theta'^2}\sin\theta}{\sqrt{\|W\|^2+\theta'^2}\sqrt{2+\eta^2}}T+\frac{\eta\theta'+\|W\|}{\sqrt{\|W\|^2+\theta'^2}\sqrt{2+\eta^2}}N$$
$$-\frac{(\eta\|W\|+\theta')\sin\theta+\sqrt{\|W\|^2+\theta'^2}\cos\theta}{\sqrt{\|W\|^2+\theta'^2}\sqrt{2+\eta^2}}B,$$

$$T_{\beta_4}'(s)=\frac{(\overline{\chi}_3\|W\|-\overline{\chi}_2\theta')\sqrt{2}\cos\theta+\sqrt{2\|W\|^2+2\theta'^2}\,\overline{\chi}_1\sin\theta}{(2+\eta^2)^2\sqrt{\|W\|^2+\theta'^2}}T+\frac{(\overline{\chi}_3\theta'+\overline{\chi}_2\|W\|)\sqrt{2}}{(2+\eta^2)^2\sqrt{\|W\|^2+\theta'^2}}N$$
$$+\frac{(\overline{\chi}_2\theta'-\overline{\chi}_3\|W\|)\sqrt{2}\sin\theta+\sqrt{2\|W\|^2+2\theta'^2}\,\overline{\chi}_1\cos\theta}{(2+\eta^2)^2\sqrt{\|W\|^2+\theta'^2}}B,$$

$$(\beta_4\wedge T_{\beta_4})(s)=\frac{(2\|W\|-\eta\theta')\cos\theta-\eta\sqrt{\|W\|^2+\theta'^2}\sin\theta}{\sqrt{4+2\eta^2}\sqrt{\|W\|^2+\theta'^2}}T+\frac{2\theta'-\eta\|W\|}{\sqrt{4+2\eta^2}\sqrt{\|W\|^2+\theta'^2}}N$$
$$+\frac{(\eta\theta'-2\|W\|)\sin\theta-\eta\sqrt{\|W\|^2+\theta'^2}\cos\theta}{\sqrt{4+2\eta^2}\sqrt{\|W\|^2+\theta'^2}}B,$$

$\kappa_g^{\beta_4}$ geodesic curvature is

$$\kappa_g^{\beta_4}=\frac{1}{(2+\eta^2)^{\frac{5}{2}}}(\eta\overline{\chi}_1-\eta\overline{\chi}_2+2\overline{\chi}_3).$$

Definition 2.5. *Let* (N^*) *curve be of* α^*, T_{N^*} *and* $N^*\wedge T_{N^*}$ *be unit vector of* (N^*). *In this case* β_5 *-Smarandache curve can be defined by*

$$\beta_5(s)=\frac{1}{\sqrt{2}}(T_{N^*}+N^*\wedge T_{N^*}). \tag{2.18}$$

Theorem 2.5. *The* $\kappa_g^{\beta_5}$ *geodesic curvature belonging to* β_5 *-Smarandache curve of the Mannheim curve is* $\kappa_g^{\beta_5}=\dfrac{1}{(2+\eta^2)^{\frac{5}{2}}}(2\eta\overline{\delta}_1-\overline{\delta}_2+\overline{\delta}_3)$, where

$$\overline{\delta}_1=\eta+2\eta^3+2\eta'\eta,\quad \overline{\delta}_2=-1-3\eta^2-2\eta^4-\eta',\quad \overline{\delta}_3=-\eta^2-2\eta^4+\eta'$$

Proof: From the equations (1.3),(2.2) and (2.18), we can write

$$\beta_5(s)=\frac{(\|W\|-\theta')\cos\theta}{\sqrt{2\theta'^2+2\|W\|^2}}T+\frac{\theta'+\|W\|}{\sqrt{2\theta'^2+2\|W\|^2}}N+\frac{(\theta'-\|W\|)\sin\theta}{\sqrt{2\theta'^2+2\|W\|^2}}B.$$

Here, if Sabban apparatus calculate

$$T_{\beta_5}(s)=\frac{\eta(\|W\|+\theta')\cos\theta-\sqrt{\|W\|^2+\theta'^2}\sin\theta}{\sqrt{1+2\eta^2}\sqrt{\|W\|^2+\theta'^2}}T+\frac{\eta(\theta'-\|W\|)}{\sqrt{1+2\eta^2}\sqrt{\|W\|^2+\theta'^2}}N$$
$$-\frac{\eta(\|W\|+\theta')\sin\theta+\sqrt{\|W\|^2+\theta'^2}\cos\theta}{\sqrt{1+2\eta^2}\sqrt{\|W\|^2+\theta'^2}}B,$$

$$T_{\beta_5}'(s)=\frac{(\overline{\delta}_3\|W\|-\overline{\delta}_2\theta')\sqrt{2}\cos\theta+\overline{\delta}_1\sqrt{2\|W\|^2+2\theta'^2}\sin\theta}{(1+2\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}T+\frac{(\overline{\delta}_3\theta'+\overline{\delta}_2\|W\|)\sqrt{2}}{(1+2\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}N$$
$$+\frac{(\overline{\delta}_2\theta'-\overline{\delta}_3\|W\|)\sqrt{2}\sin\theta+\overline{\delta}_1\sqrt{2\|W\|^2+2\theta'^2}\cos\theta}{(1+2\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}B,$$

$$(\beta_5\wedge T_{\beta_5})(s)=\frac{(\|W\|+\theta')\cos\theta+2\eta\sqrt{\|W\|^2+\theta'^2}\sin\theta}{\sqrt{2+4\eta^2}\sqrt{\|W\|^2+\theta'^2}}T+\frac{\theta'-\|W\|}{\sqrt{2+4\eta^2}\sqrt{\|W\|^2+\theta'^2}}N$$
$$+\frac{2\eta\sqrt{\|W\|^2+\theta'^2}\cos\theta-(\|W\|+\theta')\sin\theta}{\sqrt{2+4\eta^2}\sqrt{\|W\|^2+\theta'^2}}B,$$

$\kappa_g^{\beta_5}$ geodesic curvature is

$$\kappa_g^{\beta_5}=\frac{1}{(2+\eta^2)^{\frac{5}{2}}}(2\eta\overline{\delta}_1-\overline{\delta}_2+\overline{\delta}_3).$$

Definition 2.6 *Let* (N^*) *curve be of* α^*, N^*, T_{N^*} *and* $N^*\wedge T_{N^*}$ *be unit vector of* (N^*). *In this case* β_6*-Smarandache curve can be defined by*

$$\beta_6(s)=\frac{1}{\sqrt{3}}(N^*+T_{N^*}+N^*\wedge T_{N^*}). \tag{2.19}$$

Theorem 2.6 *The* $\kappa_g^{\beta_6}$ *geodesic curvature belonging to* β_6*-Smarandache curve of the Mannheim curve is* $\kappa_g^{\beta_6}=\dfrac{(2\eta-1)\overline{\rho}_1-(1+\eta)\overline{\rho}_2+(2-\eta)\overline{\rho}_3}{4\sqrt{2}(1-\eta+\eta^2)^{\frac{5}{2}}}$, where

$$\overline{\rho}_1=-2+4\eta-4\eta^2+2\eta^3+2\eta'(2\eta-1),\quad \overline{\rho}_2=-2+2\eta-4\eta^2+2\eta^3-2\eta^4-\eta'(1+\eta),$$
$$\overline{\rho}_3=2\eta-4\eta^2+4\eta^3-2\eta^4+\eta'(2-\eta).$$

Proof: From the equations (1.3), (2.2) and (2.19), we can write

$$\beta_6(s)=\frac{(\|W\|-\theta')\cos\theta+\sqrt{\theta'^2+\|W\|^2}\sin\theta}{\sqrt{3\theta'^2+3\|W\|^2}}T+\frac{\theta'+\|W\|}{\sqrt{3\theta'^2+3\|W\|^2}}N$$
$$+\frac{(\theta'-\|W\|)\sin\theta+\sqrt{\theta'^2+\|W\|^2}\cos\theta}{\sqrt{3\theta'^2+3\|W\|^2}}B.$$

Herein, if Sabban apparatus calculate

$$T_{\beta_6}(s)=\frac{(\eta\|W\|-(1-\eta)\theta')\cos\theta-\sqrt{\|W\|^2+\theta'^2}\sin\theta}{\sqrt{2(1-\eta+\eta^2)}\sqrt{\|W\|^2+\theta'^2}}T+\frac{\eta\theta'+(1-\eta)\|W\|}{\sqrt{2(1-\eta+\eta^2)}\sqrt{\|W\|^2+\theta'^2}}N$$
$$+\frac{((1-\eta)\theta'-\eta\|W\|)\sin\theta-\sqrt{\|W\|^2+\theta'^2}\cos\theta}{\sqrt{2(1-\eta+\eta^2)}\sqrt{\|W\|^2+\theta'^2}}B,$$

$$T_{\beta_6}'(s)=\frac{(\overline{\rho}_3\theta'-\overline{\rho}_2\|W\|)\sqrt{3}\cos\theta+\overline{\rho}_1\sqrt{3\|W\|^2+3\theta'^2}\sin\theta}{4(1-\eta+\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}T+\frac{(\overline{\rho}_3\|W\|+\overline{\rho}_2\theta')\sqrt{3}}{4(1-\eta+\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}N$$
$$+\frac{(\overline{\rho}_2\|W\|-\overline{\rho}_3\theta')\sqrt{3}\sin\theta+\overline{\rho}_1\sqrt{3\|W\|^2+3\theta'^2}\cos\theta}{4(1-\eta+\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}B,$$

$$(\beta_6\wedge T_{\beta_6})(s)=\frac{((2-\eta)\|W\|+(1+\eta)\theta')\cos\theta+(2\eta-1)\sqrt{\|W\|^2+\theta'^2}\sin\theta}{\sqrt{6-6\eta+6\eta^2}\sqrt{\|W\|^2+\theta'^2}}T$$
$$+\frac{(2-\eta)\theta'-(1+\eta)\|W\|}{\sqrt{6-6\eta+6\eta^2}\sqrt{\|W\|^2+\theta'^2}}N$$
$$+\frac{(2\eta-1)\sqrt{\|W\|^2+\theta'^2}\cos\theta-((2-\eta)\|W\|-(1+\eta)\theta')\sin\theta}{\sqrt{6-6\eta+6\eta^2}\sqrt{\|W\|^2+\theta'^2}}B,$$

$\kappa_g^{\beta_6}$ geodesic curvature is

$$\kappa_g^{\beta_6}=\frac{(2\eta-1)\overline{\rho}_1-(1+\eta)\overline{\rho}_2+(2-\eta)\overline{\rho}_3}{4\sqrt{2}(1-\eta+\eta^2)^{\frac{5}{2}}}.$$

Definition 2.7. *Let (B^*) curve be of α^*, T_{B^*} and $B^*\wedge T_{B^*}$ be unit vector of (B^*). In this case β_7-Smarandache curve can be defined by*

$$\beta_7(s)=\frac{1}{\sqrt{2}}(T_{B^*}+T^*\wedge T_{B^*}). \qquad (2.20)$$

Theorem 2.7. *The $\kappa_g^{\beta_7}$ geodesic curvature belonging to β_7-Smarandache curve of the Mannheim curve is* $\kappa_g^{\beta_7}=\dfrac{1}{(1+2(\frac{\theta'}{\|W\|})^2)^{\frac{5}{2}}}(2\frac{\theta'}{\|W\|}\overline{\psi}_1-\overline{\psi}_2+\overline{\psi}_3),$ where

$$\overline{\psi}_1=(\frac{\theta'}{\|W\|})+(\frac{\theta'}{\|W\|})^3+2(\frac{\theta'}{\|W\|})'(\frac{\theta'}{\|W\|}),\qquad \overline{\psi}_2=-1-3(\frac{\theta'}{\|W\|})^2-2(\frac{\theta'}{\|W\|})^4-(\frac{\theta'}{\|W\|})'$$
$$\overline{\psi}_3=-(\frac{\theta'}{\|W\|})^2-2(\frac{\theta'}{\|W\|})^4+(\frac{\theta'}{\|W\|})'.$$

Proof: From the equations (1.3), (2.3) and (2.20), we can write

$$\beta_7(s)=\frac{1}{\sqrt{2}}(-\sin\theta T+N-\cos\theta B).$$

If Sabban apparatus calculate

$$T_{\beta_7}(s)=\frac{\theta'\cos\theta-\|W\|\sin\theta}{\sqrt{2\|W\|^2+\theta'^2}}T-\frac{\|W\|}{\sqrt{2\|W\|^2+\theta'^2}}N-\frac{\theta'\sin\theta+\|W\|\cos\theta}{\sqrt{2\|W\|^2+\theta'^2}}B,$$

$$T_{\beta_7}'(s)=\frac{\|W\|^4\sqrt{2}(\overline{\omega}_3\cos\theta-\overline{\omega}_2\sin\theta)}{(2\|W\|^2+\theta'^2)^2}T+\frac{\|W\|^4\sqrt{2}\overline{\omega}_1}{(2\|W\|^2+\theta'^2)^2}N-\frac{\|W\|^4\sqrt{2}(\overline{\omega}_2\cos\theta+\overline{\omega}_3\sin\theta)}{(2\|W\|^2+\theta'^2)^2}B,$$

$$(\beta_7\wedge T_{\beta_7})(s)=\frac{2\|W\|\cos\theta+\theta'\sin\theta}{\sqrt{4\|W\|^2+2\theta'^2}}T+\frac{\theta'}{\sqrt{4\|W\|^2+2\theta'^2}}N+\frac{\theta'\cos\theta-2\|W\|\sin\theta}{\sqrt{4\|W\|^2+2\theta'^2}}B,$$

$\kappa_g^{\beta_7}$ geodesic curvature is

$$\kappa_g^{\beta_7}=\frac{1}{(2+(\frac{\theta'}{\|W\|})^2)^{\frac{5}{2}}}(\frac{\theta'}{\|W\|}\overline{\omega}_1-\frac{\theta'}{\|W\|}\overline{\omega}_2+2\overline{\omega}_3).$$

Definition 2.8. *Let (B^*) curve be of α^*, B^*, T_{B^*} and $B^*\wedge T_{B^*}$ be unit vector of (B^*). In this case β_8-Smarandache curve can be defined by*

$$\beta_8(s)=\frac{1}{\sqrt{3}}(B^*+T_{B^*}+B^*\wedge T_{B^*}). \tag{2.21}$$

Theorem 2.8. *The $\kappa_g^{\beta_8}$ geodesic curvature belonging to β_8-Smarandache curve of the Mannheim curve is* $\kappa_g^{\beta_8}=\frac{1}{4\sqrt{2}(1+\frac{\theta'}{\|W\|}+(\frac{\theta'}{\|W\|})^2)^{\frac{5}{2}}}((2\frac{\theta'}{\|W\|}-1)\overline{\zeta}_1-(1+\frac{\theta'}{\|W\|})\overline{\zeta}_2+(2-\frac{\theta'}{\|W\|})\overline{\zeta}_3),$ where

$$\begin{cases}\overline{\zeta}_1=-2+4\frac{\theta'}{\|W\|}+4(\frac{\theta'}{\|W\|})-(\frac{\theta'}{\|W\|})^2+2(\frac{\theta'}{\|W\|})^3+(\frac{\theta'}{\|W\|})'(2\frac{\theta'}{\|W\|}-1)\\ \overline{\zeta}_2=-2+2\frac{\theta'}{\|W\|}-4(\frac{\theta'}{\|W\|})^2+(\frac{\theta'}{\|W\|})^3-2(\frac{\theta'}{\|W\|})^4-(\frac{\theta'}{\|W\|})'(1+\frac{\theta'}{\|W\|})\\ \overline{\zeta}_3=2\frac{\theta'}{\|W\|}-4(\frac{\theta'}{\|W\|})^2+4(\frac{\theta'}{\|W\|})^3-2(\frac{\theta'}{\|W\|})^4+(\frac{\theta'}{\|W\|})'(2-\frac{\theta'}{\|W\|}).\end{cases}$$

Proof: From the equations (1.3), (2.3) and (2.21), we can write

$$\beta_8(s)=\frac{1}{\sqrt{3}}((\cos\theta-\sin\theta)T+N-(\cos\theta+\sin\theta)B).$$

Here if Sabban apparatus calculate

$$T_{\beta_8}(s)=\frac{\theta'\cos\theta+(\theta'-\|W\|)\sin\theta}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}T-\frac{\|W\|}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}N-\frac{\theta'\sin\theta-(\theta'-\|W\|)\cos\theta}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}B,$$

$$T_{\beta_8}'(s)=\frac{\|W\|^4\sqrt{3}(\overline{\zeta}_3\cos\theta-\overline{\zeta}_2\sin\theta)}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}T+\frac{\|W\|^4\sqrt{3}\overline{\zeta}_1}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}N-\frac{\|W\|^4\sqrt{3}(\overline{\zeta}_2\cos\theta+\overline{\zeta}_3\sin\theta)}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}B,$$

$$(\beta_8\wedge T_{\beta_8})(s)=\frac{(2\|W\|-\theta')\cos\theta+(\|W\|+\theta')\sin\theta}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}T+\frac{2\theta'-\|W\|}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}N$$
$$+\frac{(\|W\|+\theta')\cos\theta-(2\|W\|-\theta')\sin\theta}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}B,$$

$\kappa_g^{\beta_8}$ geodesic curvature is

$$\kappa_g^{\beta_8} = \frac{1}{4\sqrt{2}(1+\frac{\theta'}{\|W\|}+(\frac{\theta'}{\|W\|})^2)^{\frac{5}{2}}}((2\frac{\theta'}{\|W\|}-1)\overline{\zeta}_1-(1+\frac{\theta'}{\|W\|})\overline{\zeta}_2+(2-\frac{\theta'}{\|W\|})\overline{\zeta}_3).$$

3. Conclusion

In this study, we studied Mannheim Partner Curves and Smarandache curves, which are well known in Differential Geometry. The Sabban frames of the differential curves drawn by the Frenet vectors of the Mannheim Partner Curve on the unit sphere surface were calculated. Then Smarandache curves were defined with the help of these frames. Finally, geodesic curvatures of these curves were calculated according to Sabban frames.

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