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Spacelike Smarandache Curves of Timelike Curves in Anti de Sitter 3-Space

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Abstract: In this paper, we investigate special spacelike Smarandache curves of timelike curves according to Sabban frame in Anti de Sitter 3-Space. Moreover, we give the relationship between the base curve and its Smarandache curve associated with theirs Sabban Frames. However, we obtain some geometric results with respect to special cases of the base curve. Finally, we give some examples of such curves and draw theirs images under stereographic projections from Anti de Sitter 3-space to Minkowski 3-space.

Key Words: Anti de Sitter space, Minkowski space, Semi Euclidean space, Smarandache curve.

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§1. Introduction

It is well known that there are three kinds of Lorentzian space. Minkowski space is a flat Lorentzian space and de Sitter space is a Lorentzian space with positive constant curvature. Lorentzian space with negative constant curvature is called Anti de Sitter space which is quite different from those of Minkowski space and de Sitter space according to causality. The Anti de Sitter space is a vacuum solution of the Einstein's field equation with an attractive cosmological constant in the theory of relativity. The Anti de Sitter space is also important in the string theory and the brane world scenario. Due to this situation, it is a very significant space from the viewpoint of the astrophysics and geometry (Bousso and Randall, 2002; Maldacena, 1998; Witten, 1998).

Smarandache geometry is a geometry which has at least one Smarandachely denied axiom. An axiom is said to be Smarandachely denied, if it behaves in at least two different ways within the same space (Ashbacher, 1997). Smarandache curves are the objects of Smarandache geometry. A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve (Turgut and Yılmaz, 2008). Special Smarandache curves are studied in different ambient spaces by some authors (Bektaş and Yüce, 2013; Koc Ozturk et al., 2013; Taşköprü and Tosun, 2014; Turgut and Yımaz, 2008; Yakut et al., 2014).

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This paper is organized as follows. In section 2, we give local diferential geometry of non-dejenerate regular curves in Anti de Sitter 3-space which is denoted by $\mathbb{H}^3_1$. We call that a curve is *AdS curve* in $\mathbb{H}^3_1$ if the curve is immersed unit speed non-dejenerate curve in $\mathbb{H}^3_1$. In section 3, we consider that any spacelike AdS curve $\boldsymbol{\beta}$ whose position vector is composed by Frenet frame vectors on another timelike AdS curve $\boldsymbol{\alpha}$ in $\mathbb{H}^3_1$. The AdS curve $\boldsymbol{\beta}$ is called *AdS Smarandache curve* of $\boldsymbol{\alpha}$ in $\mathbb{H}^3_1$. We define eleven different types of AdS Smarandache curve $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ according to Sabban frame in $\mathbb{H}^3_1$. Also, we give some relations between Sabban apparatus of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ for all of possible cases. Moreover, we obtain some corollaries for the spacelike AdS Smarandache curve $\boldsymbol{\beta}$ of AdS timelike curve $\boldsymbol{\alpha}$ which is a planar curve, horocycle or helix, respectively. In subsection 3.1, we define *AdS stereographic projection*, that is, the stereographic projection from $\mathbb{H}^3_1$ to $\mathbb{R}^3_1$. Then, we give an example for base AdS curve and its AdS Smarandache curve, which are helices in $\mathbb{H}^3_1$. Finally, we draw the pictures of some AdS curves by using AdS stereographic projection in Minkowski 3-space.

§2. Preliminary

In this section, we give the basic theory of local differential geometry of non-degenerate curves in Anti de Sitter 3-space which is denoted by $\mathbb{H}^3_1$. For more detail and background about Anti de Sitter space, see (Chen et al., 2014; O'Neill, 1983)..

Let $\mathbb{R}^4_2$ denote the four-dimensional semi Euclidean space with index two, that is, the real vector space $\mathbb{R}^4$ endowed with the scalar product

$$\langle \boldsymbol{x}\,, \boldsymbol{y}\rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$$

for all $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ be pseudo-orthonormal basis for $\mathbb{R}^4_2$. Then δ_{ij} is Kronecker-delta function such that $\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \delta_{ij}\varepsilon_j$ for $\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon_3 = \varepsilon_4 = 1$.

A vector $\boldsymbol{x} \in \mathbb{R}^4_2$ is called *spacelike*, *timelike* and *lightlike (null)* if $\langle \boldsymbol{x}\,, \boldsymbol{x}\,\rangle > 0$ (or $\boldsymbol{x} = 0$), $\langle \boldsymbol{x}\,, \boldsymbol{x}\,\rangle < 0$ and $\langle \boldsymbol{x}\,, \boldsymbol{x}\,\rangle = 0$, respectively. The *norm* of a vector $\boldsymbol{x} \in \mathbb{R}^4_2$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}\,, \boldsymbol{x}\,\rangle|}$. The *signature* of a vector $\boldsymbol{x}$ is denoted by

$$sign(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \text{ is spacelike} \\ 0, & \boldsymbol{x} \text{ is null} \\ -1, & \boldsymbol{x} \text{ is timelike} \end{cases}$$

The sets

$$\begin{aligned} \mathbb{S}^3_2 &= \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}\,, \boldsymbol{x}\,\rangle = 1\,\} \\ \mathbb{H}^3_1 &= \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}\,, \boldsymbol{x}\,\rangle = -1\,\} \end{aligned}$$

are called *de Sitter 3-space with index 2* (unit pseudosphere with dimension 3 and index 2 in $\mathbb{R}^4_2$) and *Anti de Sitter 3-space* (unit pseudohyperbolic space with dimension 3 and index 2 in

$\mathbb{R}^4_2$), respectively.

The pseudo vector product of vectors $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3$ is given by

$$\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \boldsymbol{x}^3 = \begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x^1_1 & x^1_2 & x^1_3 & x^1_4 \\ x^2_1 & x^2_2 & x^2_3 & x^2_4 \\ x^3_1 & x^3_2 & x^3_3 & x^3_4 \end{vmatrix} \tag{1}$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ is the canonical basis of $\mathbb{R}^4_2$ and $\boldsymbol{x}^i = (x^i_1, x^i_2, x^i_3, x^i_4)$, $i = 1, 2, 3$. Also, it is clear that

$$\langle \boldsymbol{x}, \boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \boldsymbol{x}^3 \rangle = \det(\boldsymbol{x}, \boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3)$$

for any $\boldsymbol{x} \in \mathbb{R}^4_2$. Therefore, $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \boldsymbol{x}^3$ is pseudo-orthogonal to any $\boldsymbol{x}^i$, $i = 1, 2, 3$.

We give the basic theory of non-degenerate curves in $\mathbb{H}^3_1$. Let $\boldsymbol{\alpha} : I \to \mathbb{H}^3_1$ be regular curve (i.e., an immersed curve) for open subset $I \subset \mathbb{R}$. The regular curve $\boldsymbol{\alpha}$ is said to be spacelike or timelike if $\dot{\boldsymbol{\alpha}}$ is a spacelike or timelike vector at any $t \in I$ where $\dot{\boldsymbol{\alpha}}(t) = d\boldsymbol{\alpha}/dt$. The such curves are called *non-degenerate curve*. Since $\boldsymbol{\alpha}$ is a non-degenerate curve, it admits an arc length parametrization $s = s(t)$. Thus, we can assume that $\boldsymbol{\alpha}(s)$ is a unit speed curve. Then the unit tangent vector of $\boldsymbol{\alpha}$ is given by $\boldsymbol{t}(s) = \boldsymbol{\alpha}'(s)$. Since $\langle \boldsymbol{\alpha}(s), \boldsymbol{\alpha}(s) \rangle = -1$, we have $\langle \boldsymbol{\alpha}(s), \boldsymbol{t}'(s) \rangle = -\delta_1$ where $\delta_1 = sign(\boldsymbol{t}(s))$. The vector $\boldsymbol{t}'(s) - \delta_1 \boldsymbol{\alpha}(s)$ is pseudo-orthogonal to $\boldsymbol{\alpha}(s)$ and $\boldsymbol{t}(s)$. In the case when $\langle \boldsymbol{\alpha}''(s), \boldsymbol{\alpha}''(s) \rangle \neq -1$ and $\boldsymbol{t}'(s) - \delta_1 \boldsymbol{\alpha}(s) \neq 0$, the pirinciple normal vector and the binormal vector of $\boldsymbol{\alpha}$ is given by $\boldsymbol{n}(s) = \frac{\boldsymbol{t}'(s) - \delta_1 \boldsymbol{\alpha}(s)}{\|\boldsymbol{t}'(s) - \delta_1 \boldsymbol{\alpha}(s)\|}$ and $\boldsymbol{b}(s) = \boldsymbol{\alpha}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$, respectively. Also, geodesic curvature of $\boldsymbol{\alpha}$ are defined by $\kappa_g(s) = \|\boldsymbol{t}'(s) - \delta_1 \boldsymbol{\alpha}(s)\|$. Hence, we have pseudo-orthonormal frame field $\{\boldsymbol{\alpha}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ of $\mathbb{R}^4_2$ along $\boldsymbol{\alpha}$. The frame is also called the *Sabban frame* of non-dejenerate curve $\boldsymbol{\alpha}$ on $\mathbb{H}^3_1$ such that

$$\begin{aligned} \boldsymbol{t}(s) \wedge \boldsymbol{n}(s) \wedge \boldsymbol{b}(s) &= \delta_3\, \boldsymbol{\alpha}(s), \quad \boldsymbol{n}(s) \wedge \boldsymbol{b}(s) \wedge \boldsymbol{\alpha}(s) = \delta_1\, \delta_3\, \boldsymbol{t}(s) \\ \boldsymbol{b}(s) \wedge \boldsymbol{\alpha}(s) \wedge \boldsymbol{t}(s) &= -\delta_2\, \delta_3\, \boldsymbol{n}(s), \quad \boldsymbol{\alpha}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s) = \boldsymbol{b}(s). \end{aligned}$$

where $sign(\boldsymbol{t}(s)) = \delta_1$, $sign(\boldsymbol{n}(s)) = \delta_2$, $sign(\boldsymbol{b}(s)) = \delta_3$ and $\det(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}) = -\delta_3$.

Now, if the assumption is $< \boldsymbol{\alpha}''(s), \boldsymbol{\alpha}''(s) > \neq -1$, we can give two different Frenet-Serret formulas of $\boldsymbol{\alpha}$ according to the causal character. It means that if $\delta_1 = 1$ ($\delta_1 = -1$), then $\boldsymbol{\alpha}$ is spacelike (timelike) curve in $\mathbb{H}^3_1$. In that case, the Frenet-Serret formulas are

$$\begin{bmatrix} \boldsymbol{\alpha}'(s) \\ \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \\ \boldsymbol{b}'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \delta_1 & 0 & \kappa_g(s) & 0 \\ 0 & -\delta_1\delta_2\kappa_g(s) & 0 & -\delta_1\delta_3\tau_g(s) \\ 0 & 0 & \delta_1\delta_2\tau_g(s) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}(s) \\ \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{bmatrix} \tag{2}$$

where the geodesic torsion of $\boldsymbol{\alpha}$ is given by $\tau_g(s) = \frac{\delta_1\ \det(\boldsymbol{\alpha}(s), \boldsymbol{\alpha}'(s), \boldsymbol{\alpha}''(s), \boldsymbol{\alpha}'''(s))}{(\kappa_g(s))^2}$.

Remark 2.1 The condition $< \boldsymbol{\alpha}''(s), \boldsymbol{\alpha}''(s) > \neq -1$ is equivalent to $\kappa_g(s) \neq 0$. Moreover, we can show that $\kappa_g(s) = 0$ and $\boldsymbol{t}'(s) - \delta_1 \boldsymbol{\alpha}(s) = 0$ if and only if the non-degenerate curve $\boldsymbol{\alpha}$ is a geodesic in $\mathbb{H}_1^3$.

We can give the following definitions by (Barros et al., 2001; Chen et al., 2014).

Definition 2.2 *Let $\boldsymbol{\alpha} : I \subset \mathbb{R} \to \mathbb{H}_1^3$ is an immersed spacelike (timelike) curve according to the Sabban frame $\{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ with geodesic curvature κ_g and geodesic torsion τ_g. Then,*

(1) If $\tau_g \equiv 0$, $\boldsymbol{\alpha}$ is called a planar curve in $\mathbb{H}_1^3$;

(2) If $\kappa_g \equiv 1$ and $\tau_g \equiv 0$, $\boldsymbol{\alpha}$ is called a horocycle in $\mathbb{H}_1^3$;

(3) If τ_g and κ_g are both non-zero constant, $\boldsymbol{\alpha}$ is called a helix in $\mathbb{H}_1^3$.

Remark 2.3 From now on, we call that $\boldsymbol{\alpha}$ is a spacelike (timelike) *AdS curve* if $\boldsymbol{\alpha} : I \subset \mathbb{R} \to \mathbb{H}_1^3$ is an immersed spacelike (timelike) unit speed curve in $\mathbb{H}_1^3$.

§3. Spacelike Smarandache Curves of Timelike Curves in $\mathbb{H}_1^3$

In this section, we consider any timelike AdS curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ and define its spacelike AdS Smarandache curve $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ according to the Sabban frame $\{\boldsymbol{\alpha}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ of $\boldsymbol{\alpha}$ in $\mathbb{H}_1^3$ where s and $s^\star$ is arc length parameter of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively.

Definition 3.1 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\varphi = \{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. Then the spacelike v_iv_j-Smarandache AdS curve $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ of $\boldsymbol{\alpha}$ is defined by*

$$\boldsymbol{\beta}(s^\star(s)) = \frac{1}{\sqrt{2}}(av_i(s) + bv_j(s)), \tag{3}$$

where $v_i, v_j \in \varphi$ for $i \neq j$ and $a, b \in \mathbb{R}$ such that

v_iv_j	Condition
$\boldsymbol{\alpha t}$	$a^2 + b^2 = 2$
$\boldsymbol{\alpha n}$	$a^2 - b^2 = 2$
$\boldsymbol{\alpha b}$	$a^2 - b^2 = 2$
$\boldsymbol{tn}$	$a^2 - b^2 = 2$
$\boldsymbol{tb}$	$a^2 - b^2 = 2$
$\boldsymbol{nb}$	$a^2 + b^2 = -2$ (Undefined)

. (4)

Theorem 3.2 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\varphi = \{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. If $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ is spacelike v_iv_j-Smarandache AdS curve with Sabban frame $\{\boldsymbol{\beta}, \boldsymbol{t_\beta}, \boldsymbol{n_\beta}, \boldsymbol{b_\beta}\}$ and geodesic curvature $\widetilde{\kappa_g}$, geodesic torsion $\widetilde{\tau}_g$ where $v_i, v_j \in \varphi$ for $i \neq j$, then the Sabban apparatus of $\boldsymbol{\beta}$ can be constructed by the Sabban apparatus* *of* $\boldsymbol{\alpha}$ *such that*

$$\begin{array}{|c|c|}\hline v_iv_j & \text{Condition} \\ \hline \boldsymbol{\alpha t} & b^2\kappa_g(s)^2-2>0 \\ \hline \boldsymbol{\alpha n} & b^2\tau_g(s)^2-(b\kappa_g(s)+a)^2>0 \\ \hline \boldsymbol{\alpha b} & b^2\tau_g(s)^2-a^2>0 \\ \hline \boldsymbol{tn} & 2(\kappa_g(s)^2-1)+b^2\left(\tau_g(s)^2-1\right)>0 \\ \hline \boldsymbol{tb} & (a\kappa_g(s)-b\tau_g(s))^2-a^2>0 \\ \hline \boldsymbol{nb} & \text{(Undefined)} \\ \hline \end{array}\ . \tag{5}$$

Proof We suppose that $v_iv_j=\boldsymbol{\alpha t}$. Now, let $\boldsymbol{\beta}=\boldsymbol{\beta}(s^\star)$ be spacelike $\boldsymbol{\alpha t}-$Smarandache AdS curve of timelike AdS curve $\boldsymbol{\alpha}=\boldsymbol{\alpha}(s)$. Then, $\boldsymbol{\beta}$ is defined by

$$\boldsymbol{\beta}(s^\star(s))=\frac{1}{\sqrt{2}}(a\boldsymbol{\alpha}(s)+b\boldsymbol{t}(s)) \tag{6}$$

such that $a^2+b^2=2,\ a,b\in\mathbb{R}$ from the Definition 3.1. Differentiating both sides of (6) with respect to s, we get

$$\boldsymbol{\beta}'(s^\star(s))=\frac{d\boldsymbol{\beta}}{ds^\star}\frac{ds^\star}{ds}=\frac{1}{\sqrt{2}}\left(a\boldsymbol{\alpha}'(s)+b\boldsymbol{t}'(s)\right)$$

and by using (2),

$$\boldsymbol{t}_{\boldsymbol{\beta}}(s^\star(s))\frac{ds^\star}{ds}=\frac{1}{\sqrt{2}}\left(a\,\boldsymbol{t}(s)+b\left(-\boldsymbol{\alpha}(s)+\kappa_g(s)\boldsymbol{n}(s)\right)\right),$$

where

$$\frac{ds^\star}{ds}=\sqrt{\frac{b^2\kappa_g(s)^2-2}{2}} \tag{7}$$

with condition $b^2\kappa_g(s)^2-2>0$.

(From now on, unless otherwise stated, we won't use the parameters "s" and "$s^\star$" in the following calculations for the sake of brevity).

Hence, the tangent vector of spacelike $\boldsymbol{\alpha t}-$Smarandache AdS curve $\boldsymbol{\beta}$ is to be

$$\boldsymbol{t}_{\boldsymbol{\beta}}=\frac{1}{\sqrt{\sigma}}\left(-b\boldsymbol{\alpha}+a\boldsymbol{t}+b\kappa_g\boldsymbol{n}\right), \tag{8}$$

where $\sigma=b^2{\kappa_g}^2-2$.

Differentiating both sides of (8) with respect to s, we have

$${\boldsymbol{t}_{\boldsymbol{\beta}}}'=\frac{\sqrt{2}}{\sigma^2}\left(\lambda_1\boldsymbol{\alpha}+\lambda_2\boldsymbol{t}+\lambda_3\boldsymbol{n}+\lambda_4\boldsymbol{b}\right) \tag{9}$$

by using again (2) and (7), where

$$\begin{aligned} \lambda_1 &= b^3\kappa_g{\kappa_g}' - a\sigma \\ \lambda_2 &= -ab^2\kappa_g{\kappa_g}' + b\left({\kappa_g}^2 - 1\right)\sigma \\ \lambda_3 &= -2b{\kappa_g}' + a\kappa_g\sigma \\ \lambda_4 &= b\kappa_g\tau_g\sigma\ . \end{aligned} \tag{10}$$

Now, we can compute

$$\boldsymbol{t_\beta}' - \boldsymbol{\beta} = \frac{1}{\sqrt{2}\sigma^2}\left(\left(2\lambda_1 - a\sigma^2\right)\boldsymbol{\alpha} + \left(2\lambda_2 - b\sigma^2\right)\boldsymbol{t} + 2\lambda_3\boldsymbol{n} + 2\lambda_4\boldsymbol{b}\right) \tag{11}$$

and

$$\left\|\boldsymbol{t_\beta}' - \boldsymbol{\beta}\right\| = \frac{1}{\sigma^2}\sqrt{-\sigma^4 + 2\left(a\lambda_1 + b\lambda_2\right)\sigma^2 + 2\left(-{\lambda_1}^2 - {\lambda_2}^2 + {\lambda_3}^2 + {\lambda_4}^2\right)}. \tag{12}$$

From the equations (11) and (12), the principal normal vector of $\boldsymbol{\beta}$ is

$$\boldsymbol{n_\beta} = \frac{1}{\sqrt{2\mu}}\left(\left(2\lambda_1 - a\sigma^2\right)\boldsymbol{\alpha} + \left(2\lambda_2 - b\sigma^2\right)\boldsymbol{t} + 2\lambda_3\boldsymbol{n} + 2\lambda_4\boldsymbol{b}\right) \tag{13}$$

and the geodesic curvature of $\boldsymbol{\beta}$ is

$$\widetilde{\kappa_g} = \frac{\sqrt{\mu}}{\sigma^2}, \tag{14}$$

where

$$\mu = -\sigma^4 + 2\left(a\lambda_1 + b\lambda_2\right)\sigma^2 + 2\left(-{\lambda_1}^2 - {\lambda_2}^2 + {\lambda_3}^2 + {\lambda_4}^2\right). \tag{15}$$

Also, from the equations (6), (8) and (13), the binormal vector of $\boldsymbol{\beta}$ as pseudo vector product of $\boldsymbol{\beta}, \boldsymbol{t_\beta}$ and $\boldsymbol{n_\beta}$ is given by

$$\boldsymbol{b_\beta} = \frac{1}{\sqrt{\sigma\mu}}\left(\left(-b^2\kappa_g\lambda_4\right)\boldsymbol{\alpha} + \left(ab\kappa_g\lambda_4\right)\boldsymbol{t} + 2\lambda_4\boldsymbol{n} + \left(-b^2\kappa_g\lambda_1 + ab\kappa_g\lambda_2 - 2\lambda_3\right)\boldsymbol{b}\right). \tag{16}$$

Finally, differentiating both sides of (9) with respect to s, we get

$$\boldsymbol{t_\beta}'' = \frac{-2}{\sigma^{7/2}}\left(\begin{array}{l} \left(2\lambda_1\sigma' - ({\lambda_1}' - \lambda_2)\sigma\right)\boldsymbol{\alpha} + \left(2\lambda_2\sigma' - (\lambda_1 + {\lambda_2}' + \kappa_g\lambda_3)\sigma\right)\boldsymbol{t} \\ + \left(2\lambda_3\sigma' - (\kappa_g\lambda_2 + {\lambda_3}' - \tau_g\lambda_4)\sigma\right)\boldsymbol{n} + \left(2\lambda_4\sigma' - (\tau_g\lambda_3 + {\lambda_4}')\sigma\right)\boldsymbol{b} \end{array}\right) \tag{17}$$

by using again (2) and (7). Hence, from the equations (6), (8), (9), (14) and (17), the geodesic torsion of $\boldsymbol{\beta}$ is

$$\widetilde{\tau_g} = \frac{2}{\sigma\mu}\left(\begin{array}{l} \kappa_g(b\lambda_1 - a\lambda_2)(b\tau_g\lambda_3 + a\lambda_4 + b{\lambda_4}') - b\kappa_g(b{\lambda_1}' - a{\lambda_2}')\lambda_4 \\ +2\tau_g({\lambda_3}^2 + {\lambda_4}^2) + ab{\kappa_g}^2\lambda_3\lambda_4 - 2({\lambda_3}'\lambda_4 - \lambda_3{\lambda_4}') \end{array}\right) \tag{18}$$

under the condition $a^2 + b^2 = 2$. Thus, we obtain the Sabban aparatus of $\boldsymbol{\beta}$ for the choice $v_iv_j = \boldsymbol{\alpha t}$.

It can be easily seen that the other types of v_iv_j-Smarandache curves $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ by using same method as the above. The proof is complete. □

Corollary 3.3 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve and $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ be spacelike $v_i v_j-$Smarandache AdS curve of $\boldsymbol{\alpha}$, then the following table holds for the special cases of $\boldsymbol{\alpha}$ under the conditions (4) and (5):*

$v_i v_j$	$\boldsymbol{\alpha}$ *is planar curve*	$\boldsymbol{\alpha}$ *is horocycle*	$\boldsymbol{\alpha}$ *is helix*
$\boldsymbol{\alpha t}$	*planar curve*	*undefined*	*helix*
$\boldsymbol{\alpha n}$	*undefined*	*undefined*	*helix*
$\boldsymbol{\alpha b}$	*undefined*	*undefined*	*helix*
$\boldsymbol{tn}$	*planar curve*	*undefined*	*helix*
$\boldsymbol{tb}$	*planar curve*	*undefined*	*helix*

Definition 3.4 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\varphi = \{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. Then the spacelike $v_i v_j v_k-$Smarandache AdS curve $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ of $\boldsymbol{\alpha}$ is defined by*

$$\boldsymbol{\beta}(s^\star(s)) = \frac{1}{\sqrt{3}}(a v_i(s) + b v_j(s) + c v_k(s)), \tag{19}$$

where $v_i, v_j, v_k \in \varphi$ for $i \neq j \neq k$ and $a, b, c \in \mathbb{R}$ such that

$v_i v_j v_k$	*Condition*
$\boldsymbol{\alpha t n}$	$a^2 + b^2 - c^2 = 3$
$\boldsymbol{\alpha t b}$	$a^2 + b^2 - c^2 = 3$
$\boldsymbol{\alpha n b}$	$a^2 - b^2 - c^2 = 3$
$\boldsymbol{t n b}$	$a^2 - b^2 - c^2 = 3$

. (20)

Theorem 3.5 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\varphi = \{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. If $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ is spacelike $v_i v_j v_k-$Smarandache AdS curve with Sabban frame $\{\boldsymbol{\beta}, \boldsymbol{t_\beta}, \boldsymbol{n_\beta}, \boldsymbol{b_\beta}\}$ and geodesic curvature $\widetilde{\kappa_g}$, geodesic torsion $\widetilde{\tau_g}$ where $v_i, v_j, v_k \in \varphi$ for $i \neq j \neq k$, then the Sabban apparatus of $\boldsymbol{\beta}$ can be constructed by the*

Sabban apparatus of $\boldsymbol{\alpha}$ such that

$v_i v_j v_k$	*Condition*
$\boldsymbol{\alpha t n}$	$(b^2-c^2)\kappa_g(s)^2 - 2ac\kappa_g(s) + c^2(\tau_g(s)^2-1) - 3 > 0$
$\boldsymbol{\alpha t b}$	$(b\kappa_g(s) - c\tau_g(s))^2 - \left(c^2+3\right) > 0$
$\boldsymbol{\alpha n b}$	$\left(b^2+c^2\right)\tau_g(s)^2 - (a+b\kappa_g(s))^2 > 0$
$\boldsymbol{t n b}$	$(a\kappa_g(s) - c\tau_g(s))^2 + b^2\left(\tau_g(s)^2 - \kappa_g(s)^2\right) - a^2 > 0$

. (21)

Proof We suppose that $v_i v_j v_k = \boldsymbol{\alpha t b}$. Now, let $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ be spacelike $\boldsymbol{\alpha t b}-$Smarandache AdS curve of timelike AdS curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$. Then, $\boldsymbol{\beta}$ is defined by

$$\boldsymbol{\beta}(s^\star(s)) = \frac{1}{\sqrt{3}}(a\boldsymbol{\alpha}(s) + b\boldsymbol{t}(s) + c\boldsymbol{b}(s)) \tag{22}$$

such that $a^2+b^2-c^2=3,\ a,b,c\in\mathbb{R}$ from the Definition 3.4. Differentiating both sides of (22) with respect to s, we get

$$\boldsymbol{\beta'}(s^\star(s)) = \frac{d\boldsymbol{\beta}}{ds^\star}\frac{ds^\star}{ds} = \frac{1}{\sqrt{3}}\left(a\boldsymbol{\alpha'}(s) + b\boldsymbol{t'}(s) + c\boldsymbol{b'}(s)\right)$$

and by using (2),

$$\boldsymbol{t_\beta}(s^\star(s))\frac{ds^\star}{ds} = \frac{1}{\sqrt{3}}\left(a\boldsymbol{t}(s) + b\left(-\boldsymbol{\alpha}(s) + \kappa_g(s)\boldsymbol{n}(s)\right) + c\left(-\tau_g(s)\boldsymbol{n}(s)\right)\right)$$

where

$$\frac{ds^\star}{ds} = \sqrt{\frac{(b\,\kappa_g(s) - c\,\tau_g(s))^2 - (c^2+3)}{3}} \tag{23}$$

with the condition $(b\kappa_g(s) - c\tau_g(s))^2 - \left(c^2+3\right) > 0$.

(From now on, unless otherwise stated, we won't use the parameters "s" and "$s^\star$" in the following calculations for the sake of brevity).

Hence, the tangent vector of spacelike $\boldsymbol{\alpha t b}-$Smarandache AdS curve $\boldsymbol{\beta}$ is to be

$$\boldsymbol{t_\beta} = \frac{1}{\sqrt{\sigma}}\left(-b\boldsymbol{\alpha} + a\boldsymbol{t} + (b\kappa_g - c\tau_g)\,\boldsymbol{n}\right), \tag{24}$$

where $\sigma = (b\,\kappa_g - c\,\tau_g)^2 - \left(c^2+3\right)$.

Differentiating both sides of (24) with respect to s, we have

$$\boldsymbol{t_\beta}' = \frac{\sqrt{3}}{\sigma^2}\left(\lambda_1\boldsymbol{\alpha} + \lambda_2\boldsymbol{t} + \lambda_3\boldsymbol{n} + \lambda_4\boldsymbol{b}\right) \tag{25}$$

by using again (2) and (23), where

$$\begin{cases} \lambda_1 &= b\left(b\kappa_g - c\tau_g\right)\left(b\kappa_g{}' - c\tau_g{}'\right) - a\sigma \\ \lambda_2 &= -a\left(b\kappa_g - c\tau_g\right)\left(b\kappa_g{}' - c\tau_g{}'\right) + \left(b\left(-1+\kappa_g{}^2\right) - c\kappa_g\tau_g\right)\sigma \\ \lambda_3 &= -\left(3+c^2\right)\left(b\kappa_g{}' - c\tau_g{}'\right) + a\kappa_g\sigma \\ \lambda_4 &= \tau_g\left(b\kappa_g - c\tau_g\right)\sigma \end{cases} \tag{26}$$

Now, we can compute

$$\boldsymbol{t_\beta}' - \boldsymbol{\beta} = \frac{1}{\sqrt{3}\sigma^2}\left(\left(3\lambda_1 - a\sigma^2\right)\boldsymbol{\alpha} + \left(3\lambda_2 - b\sigma^2\right)\boldsymbol{t} + 3\lambda_3\boldsymbol{n} + \left(3\lambda_4 - c\sigma^2\right)\boldsymbol{b}\right) \tag{27}$$

and

$$\|\boldsymbol{t_\beta}' - \boldsymbol{\beta}\| = \frac{1}{\sigma^2}\sqrt{-\sigma^4 + 2\left(a\lambda_1 + b\lambda_2 - c\lambda_4\right)\sigma^2 + 3\left(-\lambda_1{}^2 - \lambda_2{}^2 + \lambda_3{}^2 + \lambda_4{}^2\right)}. \tag{28}$$

From the equations (27) and (28), the principal normal vector of $\boldsymbol{\beta}$ is

$$\boldsymbol{n_\beta} = \frac{1}{\sqrt{3\mu}}\left(\left(3\lambda_1 - a\sigma^2\right)\boldsymbol{\alpha} + \left(3\lambda_2 - b\sigma^2\right)\boldsymbol{t} + 3\lambda_3\boldsymbol{n} + \left(3\lambda_4 - c\sigma^2\right)\boldsymbol{b}\right) \tag{29}$$

and the geodesic curvature of $\boldsymbol{\beta}$ is

$$\widetilde{\kappa_g} = \frac{\sqrt{\mu}}{\sigma^2}, \tag{30}$$

where

$$\mu = -\sigma^4 + 2\left(a\lambda_1 + b\lambda_2 - c\lambda_4\right)\sigma^2 + 3\left(-\lambda_1{}^2 - \lambda_2{}^2 + \lambda_3{}^2 + \lambda_4{}^2\right). \tag{31}$$

Also, from the equations (22), (24) and (29), the binormal vector of $\boldsymbol{\beta}$ as pseudo vector product of $\boldsymbol{\beta}, \boldsymbol{t_\beta}$ and $\boldsymbol{n_\beta}$ is given by

$$\boldsymbol{b_\beta} = \frac{1}{\sqrt{\sigma\mu}}\begin{pmatrix} (c(b\kappa_g - c\tau_g)\lambda_2 - (ac)\lambda_3 - b(b\kappa_g - c\tau_g)\lambda_4)\,\boldsymbol{\alpha} \\ -\left(c(b\kappa_g - c\tau_g)\lambda_1 + (bc)\lambda_3 - a(b\kappa_g - c\tau_g)\lambda_4\right)\boldsymbol{t} \\ -\left((ac)\lambda_1 + (bc)\lambda_2 - (c^2+3)\lambda_4\right)\boldsymbol{n} \\ -\left((b\kappa_g - c\tau_g)(b\lambda_1 - a\lambda_2) + (c^2+3)\lambda_3\right)\boldsymbol{b} \end{pmatrix}. \tag{32}$$

Finally, differentiating both sides of (25) with respect to s, we get

$$\boldsymbol{t_\beta}'' = \frac{-3}{\sigma^{7/2}}\begin{pmatrix} \left(2\lambda_1\sigma' - (\lambda_1{}' - \lambda_2)\sigma\right)\boldsymbol{\alpha} + \left(2\lambda_2\sigma' - (\lambda_1 + \lambda_2{}' + \kappa_g\lambda_3)\sigma\right)\boldsymbol{t} \\ +\left(2\lambda_3\sigma' - (\kappa_g\lambda_2 + \lambda_3{}' - \tau_g\lambda_4)\sigma\right)\boldsymbol{n} + \left(2\lambda_4\sigma' - (\tau_g\lambda_3 + \lambda_4{}')\sigma\right)\boldsymbol{b} \end{pmatrix} \tag{33}$$

by using again (2) and (23). Hence, from the equations (22), (24), (25), (30) and (33), the geodesic torsion of $\boldsymbol{\beta}$ is

$$\widetilde{\tau_g} = \frac{3}{\sigma\mu}\begin{pmatrix} c\left(a\lambda_3 - \lambda_2\left(b\kappa_g - c\tau_g\right)\right)\left(\lambda_2 - \lambda_1{}'\right) - c\left(b\lambda_3 + \lambda_1\left(b\kappa_g - c\tau_g\right)\right)\left(\lambda_1 + \kappa_g\lambda_3 + \lambda_2{}'\right) \\ +\lambda_4\left(b\kappa_g - c\tau_g\right)\left(b\left(\lambda_2 - \lambda_1{}'\right) + a\left(\lambda_1 + \kappa_g\lambda_3 + \lambda_2{}'\right)\right) + c\left(a\lambda_1 + b\lambda_2\right)\left(\kappa_g\lambda_2 - \tau_g\lambda_4 + \lambda_3{}'\right) \\ -\left(3+c^2\right)\lambda_4\left(\kappa_g\lambda_2 - \tau_g\lambda_4 + \lambda_3{}'\right) + \left(\left(3+c^2\right)\lambda_3 + \left(b\lambda_1 - a\lambda_2\right)\left(b\kappa_g - c\tau_g\right)\right)\left(\tau_g\lambda_3 + \lambda_4{}'\right) \end{pmatrix} \tag{34}$$

under the condition $a^2 + b^2 - c^2 = 3$. Thus, we obtain the Sabban aparatus of $\boldsymbol{\beta}$ for the choice

$v_i v_j v_k = \boldsymbol{\alpha t b}$.

It can be easily seen that the other types of $v_i v_j v_k-$Smarandache curves $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ by using same method as the above. The proof is complete. $\square$

Corollary 3.6 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve and $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ be spacelike $v_i v_j v_k-$Smarandache AdS curve of $\boldsymbol{\alpha}$, then the following table holds for the special cases of $\boldsymbol{\alpha}$ under the conditions (20) and (21):*

$v_i v_j v_k$	$\boldsymbol{\alpha}$ *is planar curve*	$\boldsymbol{\alpha}$ *is horocycle*	$\boldsymbol{\alpha}$ *is helix*
$\boldsymbol{\alpha t n}$	*planar curve*	*undefined*	*helix*
$\boldsymbol{\alpha t b}$	*planar curve*	*undefined*	*helix*
$\boldsymbol{\alpha n b}$	*undefined*	*undefined*	*helix*
$\boldsymbol{t n b}$	*planar*	*undefined*	*helix*

Definition 3.7 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. Then the spacelike $\boldsymbol{\alpha t n b}-$Smarandache AdS curve $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ of $\boldsymbol{\alpha}$ is defined by*

$$\boldsymbol{\beta}(s^\star(s)) = \frac{1}{\sqrt{4}}(a_0 \boldsymbol{\alpha}(s) + b_0 \boldsymbol{t}(s) + c_0 \boldsymbol{n}(s) + d_0 \boldsymbol{b}(s)), \tag{35}$$

where $a_0, b_0, c_0, d_0 \in \mathbb{R}$ such that

$$a_0{}^2 + b_0^2 - c_0{}^2 - d_0{}^2 = 4. \tag{36}$$

Theorem 3.8 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve with Sabban frame $\{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ and geodesic curvature κ_g and geodesic torsion τ_g. If $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ is spacelike $\boldsymbol{\alpha t n b}-$Smarandache AdS curve with Sabban frame $\{\boldsymbol{\beta}, \boldsymbol{t_\beta}, \boldsymbol{n_\beta}, \boldsymbol{b_\beta}\}$ and geodesic curvature $\widetilde{\kappa_g}$, geodesic torsion $\widetilde{\tau_g}$, then the Sabban apparatus of $\boldsymbol{\beta}$ can be constructed by the Sabban apparatus of $\boldsymbol{\alpha}$ under the condition*

$$(b_0 \kappa_g(s) - d_0 \tau_g(s))^2 - (a_0 + c_0 \kappa_g(s))^2 + c_0{}^2 \tau_g(s)^2 - b_0{}^2 > 0. \tag{37}$$

Proof Let $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ be spacelike $\boldsymbol{\alpha t n b}-$Smarandache AdS curve of timelike AdS curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$. Then, $\boldsymbol{\beta}$ is defined by

$$\boldsymbol{\beta}(s^\star(s)) = \frac{1}{\sqrt{4}}(a_0 \boldsymbol{\alpha}(s) + b_0 \boldsymbol{t}(s) + c_0 \boldsymbol{n}(s) + d_0 \boldsymbol{b}(s)) \tag{38}$$

such that $a_0{}^2 + b_0{}^2 - c_0{}^2 - d_0{}^2 = 4,\ a_0, b_0, c_0, d_0 \in \mathbb{R}$ from the Definition 3.7. Differentiating both sides of (38) with respect to s, we get

$$\boldsymbol{\beta}'(s^\star(s)) = \frac{d\boldsymbol{\beta}}{ds^\star}\frac{ds^\star}{ds} = \frac{1}{\sqrt{4}}\left(a_0\boldsymbol{\alpha}'(s) + b_0\boldsymbol{t}'(s) + c_0\boldsymbol{n}' + d_0\boldsymbol{b}'(s)\right)$$

and by using (2),

$$\boldsymbol{t}_{\boldsymbol{\beta}}(s^\star(s))\frac{ds^\star}{ds} = \frac{1}{\sqrt{4}}\left(a_0\boldsymbol{t}(s) + b_0\left(-\boldsymbol{\alpha}(s) + \kappa_g(s)\boldsymbol{n}(s)\right) + c_0\left(\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{b}(s)\right) + d_0\left(-\tau_g(s)\boldsymbol{n}(s)\right)\right)$$

where

$$\frac{ds^\star}{ds} = \sqrt{\frac{\left(b_0\kappa_g(s) - d_0\tau_g(s)\right)^2 - \left(a_0 + c_0\kappa_g(s)\right)^2 + {c_0}^2\tau_g(s)^2 - {b_0}^2}{4}} \tag{39}$$

with the condition $\left(b_0\kappa_g(s) - d_0\tau_g(s)\right)^2 - \left(a_0 + c_0\kappa_g(s)\right)^2 + {c_0}^2\tau_g(s)^2 - {b_0}^2 > 0$.

(From now on, unless otherwise stated, we won't use the parameters "s" and "$s^\star$" in the following calculations for the sake of brevity).

Hence, the tangent vector of spacelike $\boldsymbol{\alpha t n b}$−Smarandache AdS curve $\boldsymbol{\beta}$ is to be

$$\boldsymbol{t}_{\boldsymbol{\beta}} = \frac{1}{\sqrt{\sigma}}\left(-b_0\boldsymbol{\alpha} + \left(a_0 + c_0\kappa_g\right)\boldsymbol{t} + \left(b_0\kappa_g - d_0\tau_g\right)\boldsymbol{n} + c_0\tau_g\boldsymbol{b}\right), \tag{40}$$

where $\sigma = \left(b_0\kappa_g - d_0\tau_g\right)^2 - \left(a_0 + c_0\kappa_g\right)^2 + {c_0}^2{\tau_g}^2 - {b_0}^2$.

Differentiating both sides of (40) with respect to s, we have

$$\boldsymbol{t}_{\boldsymbol{\beta}}' = \frac{2}{\sigma^2}\left(\lambda_1\boldsymbol{\alpha} + \lambda_2\boldsymbol{t} + \lambda_3\boldsymbol{n} + \lambda_4\boldsymbol{b}\right) \tag{41}$$

by using again (2) and (39) where

$$\begin{aligned}
\lambda_1 &= -b_0\left(a_0c_0 + c_0^2\kappa_g - b_0\left(b_0\kappa_g - d_0\tau_g\right)\right){\kappa_g}' + b_0\left(c_0^2\tau_g - d_0\left(b_0\kappa_g - d_0\tau_g\right)\right){\tau_g}' - \left(a_0 + c_0\kappa_g\right)\sigma \\
\lambda_2 &= \left(-b_0^2\left(c_0 + a_0\kappa_g\right) + b_0d_0\left(a_0 - c_0\kappa_g\right)\tau_g + c_0\left({c_0}^2 + {d_0}^2\right)\tau_g^2\right){\kappa_g}' \\
&\quad + \left(b_0d_0\kappa_g\left(a_0 + c_0\kappa_g\right) - \left(c_0^2 + d_0^2\right)\left(a_0 + c_0\kappa_g\right)\tau_g\right){\tau_g}' + \left(b_0\left(\kappa_g^2 - 1\right) - d_0\kappa_g\tau_g\right)\sigma \\
\lambda_3 &= -\left(a_0c_0\left(b_0\kappa_g + d_0\tau_g\right) + c_0^2\left(d_0\kappa_g\tau_g - b_0\left(\tau_g^2 - 1\right)\right) + b_0\left(4 + d_0^2\right)\right){\kappa_g}' \\
&\quad + \left(2a_0c_0d_0\kappa_g + c_0^2\left(d_0\left(1 + \kappa_g^2\right) - b_0\kappa_g\tau_g\right) + d_0\left(4 + d_0^2\right)\right){\tau_g}' + \left(a_0\kappa_g + c_0\left(\kappa_g^2 - \tau_g^2\right)\right)\sigma \\
\lambda_4 &= c_0\left(c_0\left(a_0 + c_0\kappa_g\right) - b_0\left(b_0\kappa_g - d_0\tau_g\right)\right)\tau_g{\kappa_g}' + \\
&\quad c_0\left(\tau_g\left(b_0d_0\kappa_g - \left(c_0^2 + d_0^2\right)\tau_g\right) + \sigma\right){\tau_g}' + \left(b_0\kappa_g - d_0\tau_g\right)\tau_g\sigma
\end{aligned} \tag{42}$$

Now, we can compute

$$\boldsymbol{t}_{\boldsymbol{\beta}}' - \boldsymbol{\beta} = \frac{1}{2\sigma^2}\left(\left(4\lambda_1 - a_0\sigma^2\right)\boldsymbol{\alpha} + \left(4\lambda_2 - b_0\sigma^2\right)\boldsymbol{t} + \left(4\lambda_3 - c_0\sigma^2\right)\boldsymbol{n} + \left(4\lambda_4 - d_0\sigma^2\right)\boldsymbol{b}\right) \tag{43}$$

and

$$\left\|\boldsymbol{t}_{\boldsymbol{\beta}}' - \boldsymbol{\beta}\right\| = \frac{1}{\sigma^2}\sqrt{-\sigma^4 + 2\left(a_0\lambda_1 + b_0\lambda_2 - c_0\lambda_3 - d_0\lambda_4\right)\sigma^2 + 4\left(-{\lambda_1}^2 - {\lambda_2}^2 + {\lambda_3}^2 + {\lambda_4}^2\right)}. \tag{44}$$

From the equations (43) and (44), the principal normal vector of $\boldsymbol{\beta}$ is

$$\boldsymbol{n}_{\boldsymbol{\beta}} = \frac{1}{2\sqrt{\mu}} \left(\left(4\lambda_1 - a_0\sigma^2\right) \boldsymbol{\alpha} + \left(4\lambda_2 - b_0\sigma^2\right) \boldsymbol{t} + \left(4\lambda_3 - c_0\sigma^2\right) \boldsymbol{n} + \left(4\lambda_4 - d_0\sigma^2\right) \boldsymbol{b} \right) \tag{45}$$

and the geodesic curvature of $\boldsymbol{\beta}$ is

$$\widetilde{\kappa_g} = \frac{\sqrt{\mu}}{\sigma^2}, \tag{46}$$

where

$$\mu = -\sigma^4 + 2\left(a_0\lambda_1 + b_0\lambda_2 - c_0\lambda_3 - d_0\lambda_4\right)\sigma^2 + 4\left(-{\lambda_1}^2 - {\lambda_2}^2 + {\lambda_3}^2 + {\lambda_4}^2\right) \tag{47}$$

Also, from the equations (38),(40) and (45), the binormal vector of $\boldsymbol{\beta}$ as pseudo vector product of $\boldsymbol{\beta}, \boldsymbol{t}_{\boldsymbol{\beta}}$ and $\boldsymbol{n}_{\boldsymbol{\beta}}$ is given by

$$\begin{aligned} \boldsymbol{b}_{\boldsymbol{\beta}} \;=\; & \frac{1}{\sqrt{\mu}\sigma} \big((-b_0^2\kappa_g\lambda_4 + c_0(-d_0\kappa_g\lambda_3 + a_0\lambda_4) - c_0^2(\tau_g\lambda_2 - \kappa_g\lambda_4) - d_0(d_0\tau_g\lambda_2 + a_0\lambda_3) \\ & + b_0(c_0\tau_g\lambda_3 + d_0(\kappa_g\lambda_2 + \tau_g\lambda_4)))\boldsymbol{\alpha} + (b_0(-d_0(\kappa_g\lambda_1 + \lambda_3) + (c_0 + a_0\kappa_g)\lambda_4) \\ & + (c_0^2\lambda_1 - a_0c_0\lambda_3 + d_0(d_0\lambda_1 - a_0\lambda_4))\tau_g)\boldsymbol{t} + (a_0^2\lambda_4 - b_0(d_0\lambda_2 - b_0\lambda_4) \\ & - c_0\lambda_1(d_0\kappa_g - b_0\tau_g) - a_0(d_0\lambda_1 + c_0(\tau_g\lambda_2 - \kappa_g\lambda_4)))\boldsymbol{n} + (c_0^2\kappa_g\lambda_1 - a_0^2\lambda_3 \\ & - b_0^2(\kappa_g\lambda_1 + \lambda_3) + b_0(c_0\lambda_2 + d_0\tau_g\lambda_1) + a_0(c_0(\lambda_1 - \kappa_g\lambda_3) + (b_0\kappa_g - d_0\tau_g)\lambda_2 t))\boldsymbol{b} \big) \end{aligned} \tag{48}$$

Finally, differentiating both sides of (41) with respect to s, we get

$$\boldsymbol{t}_{\boldsymbol{\beta}}'' = \frac{-4}{\sigma^{7/2}} \begin{pmatrix} \left(2\lambda_1\sigma' - \left({\lambda_1}' - \lambda_2\right)\sigma\right)\boldsymbol{\alpha} + \left(2\lambda_2\sigma' - (\lambda_1 + {\lambda_2}' + \kappa_g\lambda_3)\sigma\right)\boldsymbol{t} \\ + \left(2\lambda_3\sigma' - (\kappa_g\lambda_2 + {\lambda_3}' - \tau_g\lambda_4)\sigma\right)\boldsymbol{n} + \left(2\lambda_4\sigma' - (\tau_g\lambda_3 + {\lambda_4}')\sigma\right)\boldsymbol{b} \end{pmatrix} \tag{49}$$

by using again (2) and (49). Hence, from the equations (38), (40), (41), (46) and (49), the geodesic torsion of $\boldsymbol{\beta}$ is

$$\begin{aligned} \widetilde{\tau_g} \;=\; & \frac{4}{\mu\sigma} \big((b_0^2\kappa_g\lambda_4 + (a_0 + c_0\kappa_g)(d_0\lambda_3 - c_0\lambda_4) + (c_0^2 + d_0^2)\tau_g\lambda_2 \\ & - b_0(c_0\tau_g\lambda_3 + d_0(\kappa_g\lambda_2 + \tau_g\lambda_4)))(\lambda_2 - \lambda_1') \\ & + (b_0(-d_0(\kappa_g\lambda_1 + \lambda_3) + (c_0 + a_0\kappa_g)\lambda_4) + (c_0^2\lambda_1 - a_0c_0\lambda_3 \\ & + d_0\tau_g(d_0\lambda_1 - a_0\lambda_4)))(\lambda_1 + \kappa_g\lambda_3 + \lambda_2') + (d_0((a_0 + c_0\kappa_g)\lambda_1 \\ & + b_0\lambda_2) - (a_0(a_0 + c_0\kappa_g) + b_0^2)\lambda_4 - c_0\tau_g(b_0\lambda_1 - a_0\lambda_2))(\kappa_g\lambda_2 - \lambda_4\tau_g + \lambda_3') \\ & + (-c_0^2\kappa_g\lambda_1 + a_0^2\lambda_3 + b_0^2(\kappa_g\lambda_1 + \lambda_3) - b_0(c_0\lambda_2 + d_0\lambda_1\tau_g) + a_0(c_0(-\lambda_1 + \kappa_g\lambda_3) \\ & + \lambda_2(-b_0\kappa_g + d_0\tau_g)))(\lambda_3\tau_g + \lambda_4') \big) \end{aligned} \tag{50}$$

under the condition (36). The proof is complete. □

Corollary 3.9 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a timelike AdS curve and $\boldsymbol{\beta} = \boldsymbol{\beta}(s^\star)$ be spacelike $\boldsymbol{\alpha tnb}-$Smarandache AdS curve of $\boldsymbol{\alpha}$, then the following table holds for the special cases of $\boldsymbol{\alpha}$ under the conditions (36) and (37):*

	$\boldsymbol{\alpha}$ *is planar curve*	$\boldsymbol{\alpha}$ *is horocycle*	$\boldsymbol{\alpha}$ *is helix*
$\boldsymbol{\alpha t n b}$	*planar curve*	*undefined*	*helix*

Consequently, we can give the following corollaries by Corollary 3.3, Corollary 3.6, Corollary 3.9.

Corollary 3.10 *Let $\boldsymbol{\alpha}$ be a timelike horocycle in $\mathbb{H}_1^3$. Then, there exist no spacelike Smarandache AdS curve of $\boldsymbol{\alpha}$ in $\mathbb{H}_1^3$.*

Corollary 3.11 *Let $\boldsymbol{\alpha}$ be a timelike AdS curve and $\boldsymbol{\beta}$ be any spacelike Smarandache AdS curve of $\boldsymbol{\alpha}$. Then, $\boldsymbol{\alpha}$ is helix if and only if $\boldsymbol{\beta}$ is helix.*

§4. Examples and AdS Stereographic Projection

Let $\mathbb{R}_1^3$ denote Minkowski 3-space (three-dimensional semi Euclidean space with index one), that is, the real vector space $\mathbb{R}^3$ endowed with the scalar product

$$\langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}} \rangle_\star = -\overline{x_1}\,\overline{y_1} + \overline{x_2}\,\overline{y_2} + \overline{x_3}\,\overline{y_3}$$

for all $\overline{\boldsymbol{x}} = (\overline{x_1}, \overline{x_2}, \overline{x_3})$, $\overline{\boldsymbol{y}} = (\overline{y_1}, \overline{y_2}, \overline{y_3}) \in \mathbb{R}^3$. The set

$$\mathbb{S}_1^2 = \left\{ \overline{\boldsymbol{x}} \in \mathbb{R}_1^3 \mid \langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star = 1 \right\}$$

is called *de Sitter plane* (unit pseudosphere with dimension 2 and index 1 in $\mathbb{R}_1^3$). Then, the stereographic projection Φ from $\mathbb{H}_1^3$ to $\mathbb{R}_1^3$ and its inverse is given by

$$\Phi : \mathbb{H}_1^3 \backslash \Gamma \to \mathbb{R}_1^3 \backslash \mathbb{S}_1^2, \ \Phi(\boldsymbol{x}) = \left(\frac{x_2}{1+x_1}, \frac{x_3}{1+x_1}, \frac{x_4}{1+x_1} \right)$$

and

$$\Phi^{-1} : \mathbb{R}_1^3 \backslash \mathbb{S}_1^2 \to \mathbb{H}_1^3 \backslash \Gamma, \ \Phi^{-1}(\overline{\boldsymbol{x}}) = \left(\frac{1 + \langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star}{1 - \langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star}, \frac{2\overline{x_1}}{1 - \langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star}, \frac{2\overline{x_2}}{1 - \langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star}, \frac{2\overline{x_3}}{1 - \langle \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \rangle_\star} \right)$$

according to set $\Gamma = \left\{ \boldsymbol{x} \in \mathbb{H}_1^3 \mid x_1 = -1 \right\}$, respectively. It is easily seen that Φ is conformal map.

Hence, the stereographic projection Φ of $\mathbb{H}_1^3$ is called *AdS stereographic projection*. Now, we can give the following important proposition about projection regions of any AdS curve.

Proposition 4.1 *Let Φ be AdS stereographic projection. Then the following statements are satisfied for all $\boldsymbol{x} \in \mathbb{H}_1^3$:*

(a) $x_1 > -1 \Leftrightarrow \langle \Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}) \rangle_\star < 1$;

(b) $x_1 < -1 \Leftrightarrow \langle \Phi(\boldsymbol{x}), \Phi(\boldsymbol{x}) \rangle_\star > 1$.

Now, we give an example for timelike AdS curve as helix and some spacelike Smarandache AdS curves of the base curve. Besides, we draw pictures of these curves by using Mathematica.

Example 4.2 Let AdS curve $\boldsymbol{\alpha}$ be

$$\begin{aligned}\boldsymbol{\alpha}(s) &= \Big(\sqrt{2}\cosh(\sqrt{2}s), 2^{1/4}\cosh(\sqrt{5}s)+\sqrt{1+\sqrt{2}}\sinh(\sqrt{5}s),\\ &\qquad \sqrt{2}\sinh(\sqrt{2}s), \sqrt{1+\sqrt{2}}\cosh(\sqrt{5}s)+2^{1/4}\sinh\sqrt{5}s\Big).\end{aligned}$$

Then the tangent vector of $\boldsymbol{\alpha}$ is given by

$$\begin{aligned}\boldsymbol{t}(s) &= \Big(2\sinh(\sqrt{2}s), \sqrt{5\left(1+\sqrt{2}\right)}\cosh\sqrt{5}s+2^{1/4}\sqrt{5}\sinh(\sqrt{5}s),\\ &\qquad 2\cosh(\sqrt{2}s), 2^{1/4}\sqrt{5}\cosh(\sqrt{5}s)+\sqrt{5\left(1+\sqrt{2}\right)}\sinh(\sqrt{5}s)\Big),\end{aligned}$$

and since

$$\langle \boldsymbol{t}(s)\,,\,\boldsymbol{t}(s)\rangle = -1,$$

$\boldsymbol{\alpha}$ is timelike AdS curve. By direct calculations, we get easily the following rest of Sabban frame's elements of $\boldsymbol{\alpha}$:

$$\begin{aligned}\boldsymbol{n}(s) &= \Big(\cosh(\sqrt{2}s), 2^{3/4}\cosh(\sqrt{5}s)+\sqrt{2\left(1+\sqrt{2}\right)}\sinh(\sqrt{5}s),\\ &\qquad \sinh(\sqrt{2}s), \sqrt{2\left(1+\sqrt{2}\right)}\cosh(\sqrt{5}s)+2^{3/4}\sinh(\sqrt{5}s)\Big),\\ \boldsymbol{b}(s) &= \Big(\sqrt{5}\sinh(\sqrt{2}s), 2\sqrt{1+\sqrt{2}}\cosh(\sqrt{5}s)+2^{5/4}\sinh(\sqrt{5}s),\\ &\qquad \sqrt{5}\cosh(\sqrt{2}s), 2^{5/4}\cosh(\sqrt{5}s)+2\sqrt{1+\sqrt{2}}\sinh(\sqrt{5}s)\Big).\end{aligned}$$

and the geodesic curvatures of $\boldsymbol{\alpha}$ are obtained by

$$\kappa_g = 3\sqrt{2}\,,\ \tau_g = -\sqrt{10}.$$

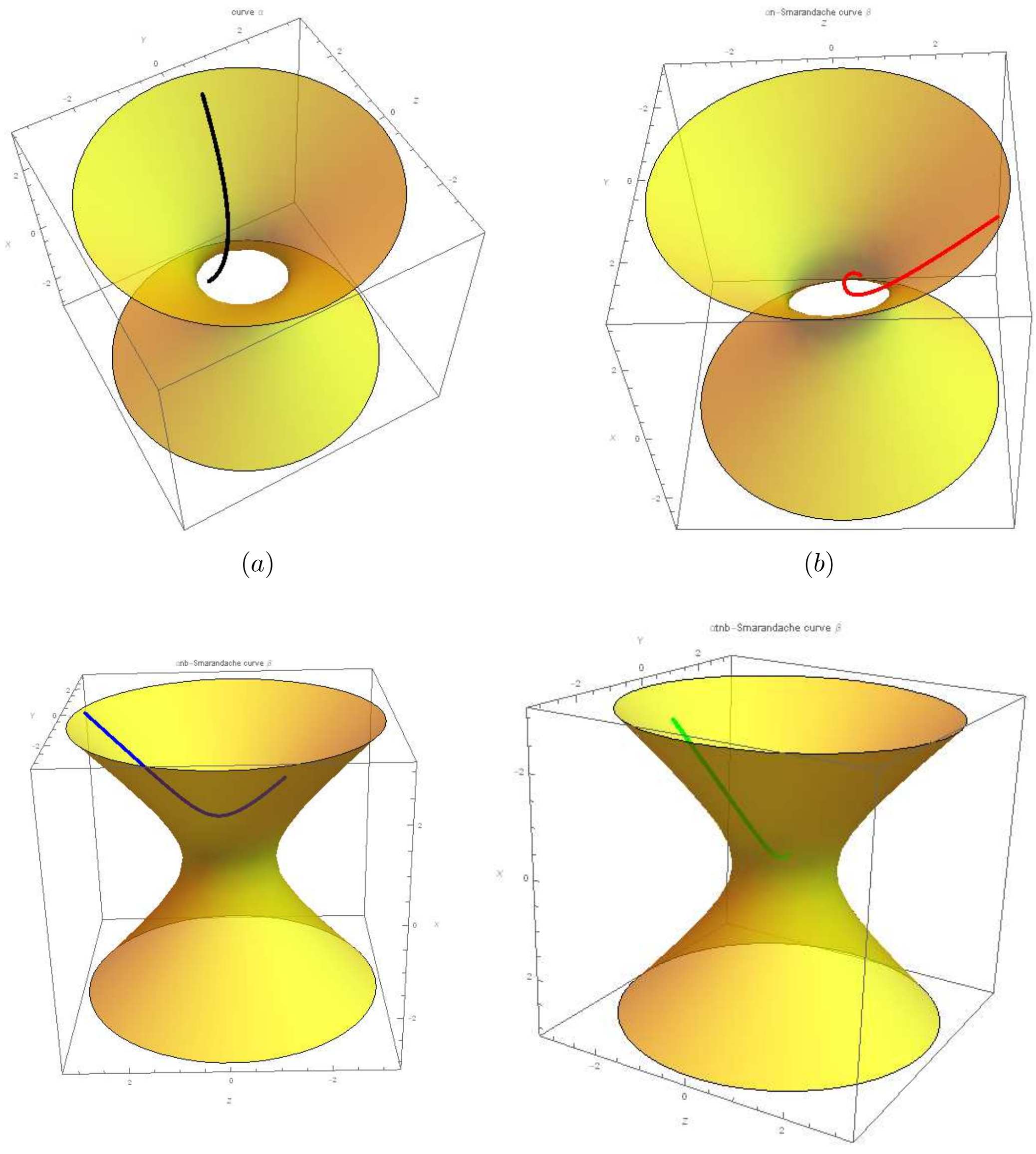
Thus, $\boldsymbol{\alpha}$ is a helix in $\mathbb{H}^3_1$. Now, we can define some spacelike Smarandache AdS curves of $\boldsymbol{\alpha}$ as the following:

$$\begin{aligned}{}_{\boldsymbol{\alpha n}}\boldsymbol{\beta}(s^\star(s)) &= \tfrac{1}{\sqrt{2}}\left(\sqrt{3}\,\boldsymbol{\alpha}(s)-\boldsymbol{n}(s)\right)\\ {}_{\boldsymbol{\alpha nb}}\boldsymbol{\beta}(s^\star(s)) &= \tfrac{1}{\sqrt{3}}\left(\sqrt{6}\,\boldsymbol{\alpha}(s)-\sqrt{2}\,\boldsymbol{n}(s)+\boldsymbol{b}(s)\right)\\ {}_{\boldsymbol{\alpha tnb}}\boldsymbol{\beta}(s^\star(s)) &= \tfrac{1}{2}\left(\tfrac{\sqrt{71}}{6}\boldsymbol{\alpha}(s)-\tfrac{3}{2}\boldsymbol{t}(s)+\tfrac{1}{3}\boldsymbol{n}(s)+\tfrac{1}{3}\boldsymbol{b}(s)\right)\end{aligned}$$

and theirs geodesic curvatures are obtained by

$$_{\boldsymbol{\alpha n}}\kappa_g = 1.9647\,,\ \ _{\boldsymbol{\alpha n}}\tau_g = -0.0619$$
$$_{\boldsymbol{\alpha nb}}\kappa_g = 1.9773\,,\ \ _{\boldsymbol{\alpha nb}}\tau_g = -0.0126$$
$$_{\boldsymbol{\alpha tnb}}\kappa_g = 2.0067\,,\ \ _{\boldsymbol{\alpha tnb}}\tau_g = -0.0044$$

in numeric form, respectively. Hence, the above spacelike Smarandache AdS curves of $\boldsymbol{\alpha}$ are also helix in $\mathbb{H}_1^3$, seeing Figure 1.



(a) (b)

(c) (d)

Figure 1

where, (a) is the timelike AdS helix $\boldsymbol{\alpha}$, (b) the spacelike $\boldsymbol{\alpha n}$-Smarandache AdS helix of $\boldsymbol{\alpha}$, (c) the spacelike $\boldsymbol{\alpha nb}$-Smarandache AdS helix of $\boldsymbol{\alpha}$ and (d) the spacelike $\boldsymbol{\alpha tnb}$-Smarandache AdS helix of $\boldsymbol{\alpha}$.

§5. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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