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Special equiform Smarandache curves in Minkowski space-time

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ABSTRACT

In this paper, we introduce special equiform Smarandache curves reference to the equiform Frenet frame of a curve ζ on a spacelike surface M in Minkowski 3-space E_1^3. Also, we study the equiform Frenet invariants of the spacial equiform Smarandache curves in E_1^3. Moreover, we give some properties to these curves when the curve ζ has constant curvature or it is a circular helix. Finally, we give an example to illustrate these curves.

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1. Introduction

A regular non-null curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [1]. Recently special Smarandache curves have been studied by some authors [2–5].

In this work, we study special equiform Smarandache curves with reference to the equiform Frenet frame of a curve ζ on a spacelike surface M in Minkowski 3-space E_1^3. In Section 2, we clarify the basic conceptions of Minkowski 3-space E_1^3 and give of equiform Fremet frame that will be used during this work. Section 3 is delicate to the study of the special four equiform Smarandache curves, $T\eta$, $T\xi$, $\eta\xi$ and $T\eta\xi$-equiform Smarandache curves by being the connection with the first and second equiform curvature $k_1(\theta)$, and $k_2(\theta)$ of the equiform spacelike curve ζ in E_1^3. Furthermore, we present some properties on the curves when the curve ζ has constant curvature or it is a circular helix. Finally, we give an example to clarify these curves. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

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2. Preliminaries

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the metric

$$\mathcal{G} = -dz_1^2 + dz_2^2 + dz_3^2,$$

where (z_1, z_2, z_3) is a rectangular coordinate system of E_1^3. Any arbitrary vector $\nu \in E_1^3$ can have one of three Lorentzian clause depicts; it can be timelike if $\mathcal{G}(\nu, \nu) < 0$, spacelike if $\mathcal{G}(\nu, \nu) > 0$ or $\nu = 0$, and lightlike if $\mathcal{G}(\nu, \nu) = 0$ and $\nu \neq 0$. Similarly, any arbitrary curve $\zeta = \zeta(s)$ can be timelike, spacelike or lightlike if all of its velocity vectors $\zeta'(s)$ are timelike, spacelike or lightlike, respectively.

Let $\zeta = \zeta(s)$ be a regular non-null curve parametrized by arc-length in E_1^3 and $\{t, n, b, \kappa, \tau\}$ be its Frenet invariants where $\{t, n, b\}$, κ and τ are the moving Frenet frame and the natural curvature functions respectively. If ζ is a spacelike curve with spacelike principal normal vector, then the Frenet formulas of the curve ζ can be given as [6–8]:

$$\begin{pmatrix} \dot{t}(s) \\ \dot{n}(s) \\ \dot{b}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}, \tag{1}$$

where $\left(\cdot = \dfrac{d}{ds}\right)$, $\mathcal{G}(t, t) = \mathcal{G}(n, n) = -\mathcal{G}(b, b) = 1$, and $\mathcal{G}(t, n) = \mathcal{G}(t, b) = \mathcal{G}(n, b) = 0$.

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Definition 2.1. A surface M in the Minkowski 3-space E_1^3 is said to be timelike, spacelike surface if, respectively the induced metric on the surface is a Lorentz metrica, positive definite Riemannian metric. In other words, the normal vector on the timelike(spacelike) surface is a spacelike(timelike) vector [8].

Let $\zeta : I \to E_1^3$ be a spacelike curve in Minkowski space E_1^3. We define the equiform parameter of ζ by $\theta = \int \kappa ds$. Then, we have $\rho = \frac{ds}{d\theta}$, where $\rho = \frac{1}{\kappa}$ is the radius of curvature of the curve ζ. Let $\mathcal{F}$ be a homothety with the center in the origin and the coefficient μ. If we put $\bar{\zeta} = \mathcal{F}(\zeta)$, then it follows

$\bar{s} = \mu s$ and $\bar{\rho} = \mu \rho$,

where $\bar{s}$ is the arc-length parameter of $\bar{\zeta}$ and $\bar{\rho}$ the radius of curvature of this curve. Therefore, θ is an equiform invariant parameter of ζ [9]. From that point, we recall $\{T, \eta, \xi,\}$ be the moving equiform Frenet frame where $T(\theta) = \rho\, t(s)$, $\eta(\theta) = \rho\, n(s)$ and $\xi(\theta) = \rho\, b(s)$ are the equiform tangent vector, equiform principal normal vector and equiform binormal vector respectively. Additionally, the first and second equiform curvature of the curve $\zeta = \zeta(\theta)$ are defined by $k_1(\theta) = \dot{\rho} = \frac{d\rho}{ds}$ and $k_2(\theta) = \frac{\tau}{\kappa}$. So, the moving equiform Frenet frame of $\zeta = \zeta(\theta)$ is given as [10]:

$$\begin{pmatrix} T'(\theta) \\ \eta'(\theta) \\ \xi'(\theta) \end{pmatrix} = \begin{pmatrix} k_1(\theta) & 1 & 0 \\ -1 & k_1(\theta) & k_2(\theta) \\ 0 & k_2(\theta) & k_1(\theta) \end{pmatrix} \begin{pmatrix} T(\theta) \\ \eta(\theta) \\ \xi(\theta) \end{pmatrix}, \tag{2}$$

where $\left(' = \frac{d}{d\theta} \right)$, $\mathcal{G}(T, T) = \mathcal{G}(\eta, \eta) = -\mathcal{G}(\xi, \xi) = \rho^2$, and $\mathcal{G}(T, \eta) = \mathcal{G}(T, \xi) = \mathcal{G}(\eta, \xi) = 0$.

The pseudo-Riemannian sphere with center at the origin and of radius r = 1 in the Minkowski 3-space E_1^3 is a quadric defined by

$S_1^2 = \{\vec{u} \in E_1^3 : -u_1^2 + u_2^2 + u_3^2 = 1.\}$

Let $\zeta = \zeta(\theta)$ be a regular non-null curve parametrized by arc-length in Minkowski 3-space E_1^3 with its moving equiform Frenet frame $\{T, \eta, \xi,\}$. Then $T\eta$, $T\xi$, $\eta\xi$ and $T\eta\xi$-equiform Smarandache curves of ζ are defined, respectively as follows [11]:

$$\Im = \Im(\theta^*) = \frac{1}{\sqrt{2}}(T(\theta) + \eta(\theta)),$$

$$\Im = \Im(\theta^*) = \frac{1}{\sqrt{2}}(T(\theta) + \xi(\theta)),$$

$$\Im = \Im(\theta^*) = \frac{1}{\sqrt{2}}(\eta(\theta) + \xi(\theta)),$$

$$\Im = \Im(\theta^*) = \frac{1}{\sqrt{3}}(T(\theta) + \eta(\theta) + \xi(\theta)).$$

3. Special equiform Smarandache curves in E_1^3

In this section, we define the special equiform Smarandache curves reference to the equiform Frenet frame of a curve ζ in Minkowski 3-space E_1^3. Furthermore, we obtain the natural equiform curvature functions of the equiform Smarandache curves lying completely on pesdo-sphere S_1^2 and give some properties on the curves when the curve ζ has constant curvature or it is a circular helix

Definition 3.1. A curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another curve, is called a Smarandache curve.

As consequence with the above definition, we introduce a special form of the equiform Smarandache curves in E_1^3 in the following subsection

3.1. $T\eta$-equiform Smarandache curves in E_1^3

Definition 3.2. Let $\zeta = \zeta(\theta)$ be a regular equiform spacelike curve lying completely on a spacelike surface M in E_1^3 with moving equiform Frenet frame $\{T, \eta, \xi\}$. Then $T\eta$-equiform Smarandache curves are defined by

$$\Im = \Im(\theta^*) = \frac{1}{\sqrt{2}}(T(\theta) + \eta(\theta)). \tag{3}$$

Theorem 3.1. *Let $\zeta = \zeta(s)$ be a spacelike curve with spacelike principal normal vector in E_1^3. If ζ is a circular helix with $\kappa > 0$, then $T\eta$-equiform Smarandache curve is also circular helix and its the natural curvature functions are satisfied the following equation,*

$$\kappa_\Im(\theta^*) = \frac{\sqrt{2}}{\rho(k_2^2 - 2)} : \quad k_2 \neq \pm\sqrt{2},$$
$$\tau_\Im(\theta^*) = \frac{\sqrt{2}}{\rho^2} \frac{k_2(2k_2 + 1) - (k_2^2 - 1)\left[k_2 + k_2^2(k_2 + 2)\right]}{2(k_2^3 + 2)} :$$
$$k_2 \neq -\sqrt[3]{2}. \tag{4}$$

Proof. Let $\Im = \Im(\theta^*)$ be a $T\eta$-equiform Smarandache curves reference to the equiform spacelike curve $\zeta = \zeta(\theta)$. From Eq. (3) and using Eq. (2), we get

$$\Im'(\theta^*) = \frac{d\Im}{d\theta^*}\frac{d\theta^*}{d\theta} = \frac{1}{\sqrt{2}}((k_1 - 1)T(\theta) + (k_1 + 1)\eta(\theta) + k_2\xi(\theta)), \tag{5}$$

hence

$$T_\Im(\theta^*) = \frac{1}{\rho\sqrt{2k_1^2 - k_2^2 - 2}}((k_1 - 1)T(\theta) + (k_1 + 1)\eta(\theta) + k_2\xi(\theta)), \tag{6}$$

where

$$\frac{d\theta^*}{d\theta} = \frac{\rho\sqrt{2k_1^2 - k_2^2 - 2}}{\sqrt{2}}. \tag{7}$$

Now

$$\frac{dT_\Im}{d\theta^*} = \frac{\sqrt{2}}{\rho^2\left[2k_1^2 - k_2^2 - 2\right]^2}(\lambda_1 T(\theta) + \lambda_2\eta(\theta) + \lambda_3\xi(\theta)),$$

where

$$\begin{cases} \lambda_1 = (k_1 - 1)(2k_1k_1' - k_2k_2') + (2k_1^2 - k_2^2 - 2)(k_1' + k_1^2 - 3k_1), \\ \lambda_2 = (k_1 + 1)(2k_1k_1' - k_2k_2') + (2k_1^2 - k_2^2 - 2)(k_1' + k_1^2 + k_2^2 + k_1 - 2), \\ \lambda_3 = k_2(2k_1k_1' - k_2k_2') + (2k_1^2 - k_2^2 - 2)(k_2' + 2k_1k_2). \end{cases}$$

Then

$$\kappa_\Im(\theta^*) = \left\| \frac{dT_\Im}{d\theta^*} \right\| = \frac{\sqrt{2\left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2\right)}}{\rho\left[2k_1^2 - k_2^2 - 2\right]^2}, \tag{8}$$

and

$$N_\Im(\theta^*) = \frac{\lambda_1 T(\theta) + \lambda_2\eta(\theta) + \lambda_3\xi(\theta)}{\rho\sqrt{\lambda_1^2 + \lambda_2^2 - \lambda_3^2}}.$$

Also

$$B_\Im(\theta^*) = \frac{1}{p_1}\{m_1 T(\theta) + m_2\eta(\theta) + m_3\xi(\theta)\},$$

where

$$m_1 = \lambda_2 k_2 - \lambda_3(k_1+1),$$
$$m_2 = \lambda_2 k_2 - \lambda_3(k_1-1),$$
$$m_3 = \lambda_2(k_1-1) - \lambda_1(k_1+1)$$

and $p_1 = \rho\sqrt{2k_1^2 - k_2^2 - 2}\sqrt{\lambda_1^2+\lambda_2^2-\lambda_3^2}$.

Now, from Eq. (5)

$$\Im''(\theta^*) = \frac{1}{\sqrt{2}}\Big\{[k_1' + k_1^2 - 2k_1 - 1]T(\theta) + [k_1' + k_1^2 + k_2^2 + 2k_1 - 1]\eta(\theta) + [k_2' + 2k_1k_2 + k_2]\xi(\theta)\Big\},$$

and thus

$$\Im'''(\theta^*) = \frac{1}{\sqrt{2}}(\beta_1 T(\theta) + \beta_2\eta(\theta) + \beta_3\xi(\theta)),$$

where

$$\begin{cases} \beta_1 = k_1'' + 3k_1'(k_1-1) + k_1^2(k_1-3) - k_2(k_2+2), \\ \beta_2 = k_1'' + k_1' + 3(k_1k_1' + k_2k_2') + 3k_1(k_1-1) + k_1^2(3k_1+1) - 1, \\ \beta_3 = k_2'' + k_2' + 3(k_1k_1' + k_2k_2') + 3k_1k_2(k_1+1) - k_2. \end{cases}$$

Hence, we have

$$\tau_\Im(\theta^*) = \frac{\sqrt{2}}{\rho^2}\left\{\frac{w_1+w_2+w_3}{\ell_1^2+\ell_2^2-\ell_3^2}\right\}, \tag{9}$$

where

$$w_1 = (k_1' + k_1^2 + k_2^2 + 2k_1 - 1)[\beta_3(k_1-1) - \beta_1k_2],$$
$$w_2 = (k_2' + 2k_1k_2 + k_2)[\beta_1(k_1+1) - \beta_2(k_1-1)],$$
$$w_3 = (k_1' + k_1^2 - 2k_1 - 1)[\beta_2k_2 - \beta_3(k_1+1)],$$
$$\ell_1 = k_1'k_2 - k_2'(k_1+1) + k_2(k_2 - k_1^2 - k_1 - 2),$$
$$\ell_2 = k_1'k_2 - k_2'(k_1-1) - k_1k_2(k_1+1),$$
$$\ell_3 = k_1(2k_1+1) - 2k_1' + k_2^2(k_1+1) + 2.$$

Now, if κ and τ are non-zero constants, then the natural curvature functions $\kappa_\Im$, $\tau_\Im$ are also non-zero constants and satisfying Eq. (4) which means that the $T\eta$-equiform Smarandache curve is circular heilx. □

3.2. $T\xi$-equiform Smarandache curves in E_1^3

Definition 3.3. Let $\zeta = \zeta(\theta)$ be a regular equiform spacelike curve lying completely on a spacelike surface M in E_1^3 with moving equiform Frenet frame $\{T, \eta, \xi\}$. Then $T\xi$-equiform Smarandache curves are defined by

$$\Im = \Im(\theta^*) = \frac{1}{\sqrt{2}}(T(\theta) + \xi(\theta)). \tag{10}$$

Theorem 3.2. *Let $\zeta = \zeta(s)$ be a spacelike curve with spacelike principal normal vector in E_1^3. If ζ is a circular helix with $\kappa > 0$, then $T\xi$-equiform Smarandache curve is contained in a palne and its curvature is satisfied the following equation,*

$$\kappa_\Im(\theta^*) = \frac{\sqrt{2}\sqrt{(1-k_2^2)(k_2+1)^2 - k_2^2(3k_2^2-2)}}{\rho(k_2+1)^2} : \quad k_2 \neq -1. \tag{11}$$

Proof. Let $\Im = \Im(\theta^*)$ be a $T\xi$-equiform Smarandache curves of $\zeta = \zeta(\theta)$. Then from Eq. (10), we have

$$\Im'(\theta^*) = \frac{1}{\sqrt{2}}(k_1T(\theta) + (k_2+1)\eta(\theta) + k_1\xi(\theta)). \tag{12}$$

$$T_\Im(\theta^*) = \frac{1}{\rho(k_2+1)}(k_1T(\theta) + (k_2+1)\eta(\theta) + k_1\xi(\theta)), \tag{13}$$

where

$$\frac{d\theta^*}{d\theta} = \frac{\rho(k_2+1)}{\sqrt{2}}. \tag{14}$$

Now

$$\frac{dT_\Im}{d\theta^*} = \frac{\sqrt{2}}{\rho^2(k_2+1)^3}(\varepsilon_1T(\theta) + \varepsilon_2\eta(\theta) + \varepsilon_3\xi(\theta)),$$

where

$$\begin{cases} \varepsilon_1 = (k_2+1)(k_1' - k_2 - 1) - k_1k_2, \\ \varepsilon_2 = k_1(k_2+1)^2, \\ \varepsilon_3 = (k_2+1)[k_1' + k_2(k_2+1) + k_2^2 - k_1^2] - k_1k_2'. \end{cases}$$

Then

$$\kappa_\Im(\theta^*) = \frac{\sqrt{2}\sqrt{\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2}}{\rho(k_2+1)^3}, \tag{15}$$

and

$$N_\Im(\theta^*) = \frac{\varepsilon_1T(\theta) + \varepsilon_2\eta(\theta) + \varepsilon_3\xi(\theta)}{\rho\sqrt{\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2}}.$$

Also

$$B_\Im(\theta^*) = \frac{1}{p_2}\Big\{[\varepsilon_2k_1 - \varepsilon_3(k_2+1)]T(\theta) + k_1(\varepsilon_1-\varepsilon_3)\eta(\theta) + [\varepsilon_2k_1 - \varepsilon_1(k_2+1)]\xi(\theta)\Big\},$$

where $p_2 = \rho(k_2+1)\sqrt{\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2}$.

Now, from Eq. (12) we have

$$\Im''(\theta^*) = \frac{1}{\sqrt{2}}\Big\{[k_1' + k_1^2 - k_2 + 1]T(\theta) + [k_2' + 2k_1(k_2+2)]\eta(\theta) + [k_1' + k_2(2k_2+1)]\xi(\theta)\Big\},$$

and

$$\Im'''(\theta^*) = \frac{1}{\sqrt{2}}(\delta_1T(\theta) + \delta_2\eta(\theta) + \delta_3\xi(\theta)),$$

where

$$\begin{cases} \delta_1 = k_1'' - k_2' + 3k_1(k_1' - k_2) + k_1(k_1^2 - 1), \\ \delta_2 = k_2'' + k_1' + 3(k_1'k_2 + k_1k_2') + k_1^2(2k_2+3) \\ \qquad + k_2(2k_2^2 + k_2 - 1) + 1, \\ \delta_3 = k_1'' + k_2' + k_1(k_1' + 3k_2) + k_2(5k_2' + 4k_1k_2). \end{cases}$$

Hence, we have

$$\tau_\Im(\theta^*) = \frac{\sqrt{2}}{\rho^2}\left\{\frac{\begin{array}{l}(k_1' + 2k_2^2 + k_2)\big[\delta_1(k_2+1) - \delta_2k_1\big] + k_1(\delta_3-\delta_1) \\ (k_2' + 2k_1k_2 + 2k_1) + (k_1' + k_1^2 - k_2 + 1)\big[\delta_2k_1 - \delta_3(k_2+1)\big]\end{array}}{\begin{array}{l}\big\{k_1k_2' + (k_2+1)(2k_1^2 - k_1') - k_2(2k_2^2 + 3k_2 + 1)\big\}^2 \\ +\big\{k_1[k_1^2 - 2k_2(k_2+1) + 1]\big\}^2 \\ -\big\{k_1k_2' + k_2^2 - k_1^2 - k_1'(k_2+1) + k_2^2(k_2+1) - 1\big\}^2\end{array}}\right\}, \tag{16}$$

So, if κ and τ are non-zero constants, then $\kappa_\Im$ is non-zero constant and satisfying Eq. (11), also $\tau_\Im = 0$ which means that the $T\xi$-equiform Smarandache curve is contained in a plane. □

3.3. $\eta\xi$-equiform Smarandache curves in E_1^3

Definition 3.4. Let $\zeta = \zeta(\theta)$ be a regular equiform spacelike curve lying completely on a spacelike surface M in E_1^3 with moving equiform Frenet frame $\{T, \eta, \xi\}$. Then $\eta\xi$-equiform Smarandache curves are defined by

$$\Im = \Im(\theta^*) = \frac{1}{\sqrt{2}}\Big(\eta(\theta) + \xi(\theta)\Big). \tag{17}$$

Theorem 3.3. *Let $\zeta = \zeta(s)$ be a spacelike curve with spacelike principal normal vector in E_1^3. If ζ is a circular helix with $\kappa > 0$, then* *ηξ-equiform Smarandache curve is also circular helix and its the natural curvature functions are satisfied the following equation,*

$$\kappa_{\Im}(\theta^*) = \frac{\sqrt{2}(k_2^2 - 1)}{\rho}, \quad \tau_{\Im}(\theta^*) = \frac{\sqrt{2}}{\rho^2 k_2} : \quad k_2 \neq 0. \tag{18}$$

Proof. Let $\Im = \Im(\theta^*)$ be a $\eta\xi$-equiform Smarandache curves of the curve $\zeta = \zeta(\theta)$. From Eq. (17), we get

$$\Im'(\theta^*) = \frac{1}{\sqrt{2}}(-T(\theta) + (k_1 + k_2)\eta(\theta) + (k_1 + k_2)\xi(\theta)), \tag{19}$$

hence

$$T_{\Im}(\theta^*) = \frac{1}{\rho}(-T(\theta) + (k_1 + k_2)\eta(\theta) + (k_1 + k_2)\xi(\theta)), \tag{20}$$

where

$$\frac{d\theta^*}{d\theta} = \frac{\rho}{\sqrt{2}}. \tag{21}$$

Now

$$\frac{dT_{\Im}}{d\theta^*} = \frac{\sqrt{2}}{\rho^2}(\gamma_1 T(\theta) + \gamma_2 \eta(\theta) + \gamma_3 \xi(\theta)),$$

where

$$\gamma_1 = -(k_1 + k_2),$$
$$\gamma_2 = k_1' + k_2' + k_2(k_1 + k_2) - 1,$$
$$\gamma_3 = k_1' + k_2' + k_2(k_1 + k_2).$$

Then

$$\kappa_{\Im}(\theta^*) = \frac{\sqrt{2\left(\gamma_1^2 + \gamma_2^2 - \gamma_3^2\right)}}{\rho}, \tag{22}$$

and

$$N_{\Im}(\theta^*) = \frac{\gamma_1 T(\theta) + \gamma_2 \eta(\theta) + \gamma_3 \xi(\theta)}{\rho\sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}}.$$

Also

$$B_{\Im}(\theta^*) = \frac{1}{\rho\sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}}\{[(\gamma_2 - \gamma_3)(k_1 + k_2)]T(\theta) + [\gamma_3 + \gamma_1(k_1 + k_2)]\eta(\theta) - [\gamma_2 + \gamma_1(k_1 + k_2)]\xi(\theta)\}.$$

From Eq. (19), we have

$$\Im''(\theta^*) = \frac{1}{\sqrt{2}}\{-[2k_1 + k_2]T(\theta) + [k_1' + k_2' + (k_1 + k_2)^2 - 1]\eta(\theta) + [k_1' + k_2' + (k_1 + k_2)^2]\xi(\theta)\},$$

and

$$\Im'''(\theta^*) = \frac{1}{\sqrt{2}}(\omega_1 T(\theta) + \omega_2 \eta(\theta) + \omega_3 \xi(\theta)),$$

where

$$\begin{cases} \omega_1 = -\left[3k_1' + 2k_2' + k_1(2k_1 + k_2) + (k_1 + k_2)^2 - 1\right], \\ \omega_2 = k_1'' + k_2'' - 3k_1 - k_2 + 3(k_1 + k_2)(k_1' + k_2') + (k_1 + k_2)^3, \\ \omega_3 = k_1'' + k_2'' + 3(k_1 + k_2)(k_1' + k_2') + (k_1 + k_2)^3. \end{cases}$$

Hence, we have

$$\tau_{\Im}(\theta^*) = \frac{\sqrt{2}\begin{Bmatrix} [k_1' + k_2' + (k_1 + k_2)^2 - 1][\omega_3 + \omega_1(k_1 + k_2)] \\ -[k_1' + k_2' + (k_1 + k_2)^2][\omega_2 + \omega_1(k_1 + k_2)] \\ -(\omega_3 - \omega_2)(k_1 + k_2)(2k_1 + k_2) \end{Bmatrix}}{\rho^2\left\{k_1^2 - k_2^2 + 2(k_1' + k_2')\right\}}. \tag{23}$$

Then, if κ and τ are non-zero constants, then the natural curvature functions $\kappa_{\Im}$, $\tau_{\Im}$ are also non-zero constants and satisfying Eq. (18) which means that the $\eta\xi$-equiform Smarandache curve is circular heilx. □

3.4. *Tηξ-equiform Smarandache curves in* E_1^3

Definition 3.5. Let $\zeta = \zeta(\theta)$ be a regular equiform spacelike curve lying completely on a spacelike surface M in E_1^3 with moving equiform Frenet frame $\{T, \eta, \xi\}$. Then $T\eta\xi$-equiform Smarandache curves are defined by

$$\Im = \Im(\theta^*) = \frac{1}{\sqrt{3}}(T(\theta) + \eta(\theta) + \xi(\theta)). \tag{24}$$

Theorem 3.4. *Let $\zeta = \zeta(s)$ be a spacelike curve with spacelike principal normal vector in E_1^3. If ζ is a circular helix with $\kappa > 0$, then $T\eta\xi$-equiform Smarandache curve is also circular helix and its the natural curvature functions are satisfied the following equation,*

$$\kappa_{\Im}(\theta^*) = \frac{\sqrt{3}\sqrt{2(1 - k_2)}}{2\rho} : \quad |k_1| < 1, \quad \tau_{\Im}(\theta^*) = \frac{\sqrt{3}}{3\rho^2}\frac{k_2(k_2^2 + 1)}{(k_2 - 3)(k_2 + 1)^2} : \quad k_2 \neq -1, 3. \tag{25}$$

Proof. Let $\Im = \Im(\theta^*)$ be a $T\eta\xi$-equiform Smarandache curves of the curve $\zeta = \zeta(\theta)$. Then from Eq. (24), we get

$$\Im'(\theta^*) = \frac{1}{\sqrt{3}}((k_1 - 1)T(\theta) + (k_1 + k_2 + 1)\eta(\theta) + (k_1 + k_2)\xi(\theta)). \tag{26}$$

$$T_{\Im}(\theta^*) = \frac{1}{\rho\sqrt{k_1^2 + 2k_2 + 2}}((k_1 - 1)T(\theta) + (k_1 + k_2 + 1)\eta(\theta) + (k_1 + k_2)\xi(\theta)), \tag{27}$$

where

$$\frac{d\theta^*}{d\theta} = \frac{\rho\sqrt{k_1^2 + 2k_2 + 2}}{\sqrt{3}}. \tag{28}$$

Now

$$\frac{dT_{\Im}}{d\theta^*} = \frac{\sqrt{3}}{\rho^2\left[k_1^2 + 2k_2 + 2\right]^2}(\chi_1 T(\theta) + \chi_2 \eta(\theta) + \chi_3 \xi(\theta)),$$

where

$$\begin{cases} \chi_1 = (k_1 - 1)(k_1 k_1' + k_2') - (k_2 + 1)(k_1^2 + 2k_2 + 2), \\ \chi_2 = (k_1^2 + 2k_2 + 2)[k_1' + k_2' + k_1 + k_2(k_1 + k_2) - 1] \\ \qquad + (k_1 k_1' + k_2')(k_1 + k_2 + 1), \\ \chi_3 = (k_1^2 + 2k_2 + 2)[k_1' + k_2' + k_2(k_1 + k_2 + 1)] \\ \qquad + (k_1 + k_2)(k_1 k_1' + k_2'). \end{cases}$$

Then

$$\kappa_{\Im}(\theta^*) = \frac{\sqrt{3\left(\chi_1^2 + \chi_2^2 - \chi_3^2\right)}}{\rho\left[k_1^2 + 2k_2 + 2\right]^2}, \tag{29}$$

and

$$N_{\Im}(\theta^*) = \frac{\chi_1 T(\theta) + \chi_2 \eta(\theta) + \chi_3 \xi(\theta)}{\rho\sqrt{\chi_1^2 + \chi_2^2 - \chi_3^2}}.$$

Also

$$B_{\Im}(\theta^*) = \frac{1}{\rho\sqrt{k_1^2 + 2k_2 + 2}\sqrt{\chi_1^2 + \chi_2^2 - \chi_3^2}} \times \Big\{[-\chi_3 + (\chi_2 - \chi_3)(k_1 + k_2)]T(\theta) + [\chi_1(k_1 + k_2) - \chi_1(k_1 - 1)]\eta(\theta) + [\chi_2(k_2 - 1) - \chi_1(k_1 + k_2 + 1)]\xi(\theta)\Big\},$$

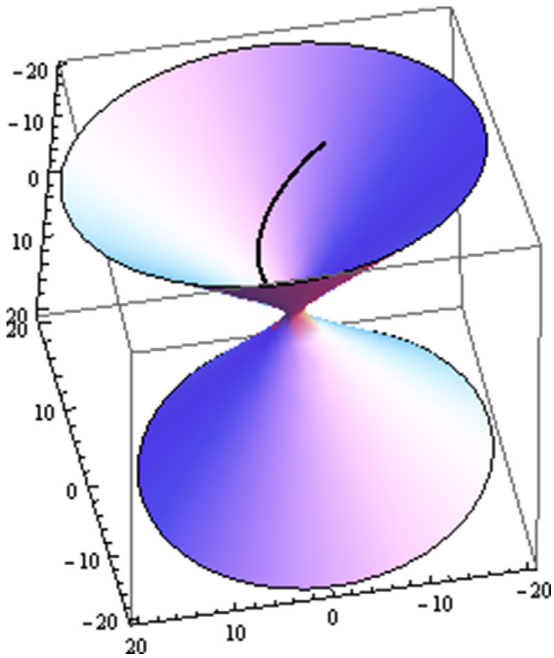

Fig. 1. Spacelike curve $\zeta = \zeta(s)$ on S_1^2.

From Eq. (26), we have

$$\Im''(\theta^*) = \frac{1}{\sqrt{3}}\{[k_1^2 - k_1 - k_2 - 1]T(\theta) + [k_1' + k_2' + 2k_1 + (k_1+k_2)^2 - 1]\eta(\theta) + [k_1' + k_2' + k_2 + (k_1+k_2)^2]\xi(\theta)\},$$

and thus

$$\Im'''(\theta^*) = \frac{1}{\sqrt{3}}(\phi_1 T(\theta) + \phi_2 \eta(\theta) + \phi_3 \xi(\theta)),$$

where

$$\begin{cases} \phi_1 = 2k_1'(k_1+1) - 2k_2' + k_1(k_1^2 - k_1 - k_2 - 2) + 1, \\ \phi_2 = k_1'' + k_2'' + 2k_1' + k_2(k_2 - 1) + 3(k_1+k_2)(k_1' + k_2') \\ \quad + k_1(2k_1^2 + k_1 - 1) + (k_1+k_2)^3 - 1, \\ \phi_3 = k_1'' + k_2'' + +k_2' + k_2(3k_1 - 1) + 3(k_1+k_2)(k_1' + k_2') \\ \quad + (k_1+k_2)^3. \end{cases}$$

Hence, we have

$$\tau_{\Im}(\theta^*) = \frac{\sqrt{3}}{\rho^2}\left\{\frac{\upsilon_1 + \upsilon_2 + \upsilon_3}{q^2 + q_2^2 - q_3^2}\right\}, \tag{30}$$

where

$$\begin{aligned} \upsilon_1 &= (k_1^2 - k_1 - k_2 - 1)[\phi_2(k_1+k_2) - \phi_3(k_1+k_2-1)], \\ \upsilon_2 &= [k_1' + k_2' + 2k_1 + (k_1+k_2)^2 - 1][\phi_3(k_1-1) - \phi_1(k_1+k_2)], \\ \upsilon_3 &= [k_1' + k_2' + k_2 + (k_1+k_2)^2][\phi_1(k_1+k_2-1) - \phi_2(k_1-1)], \\ q_1 &= (k_1+k_2)(2k_1 - k_2 - 1) - [k_1' + k_2' + 2k_1 + (k_1+k_2)^2], \\ q_2 &= -(k_1+k_2)(k_1k_2+1) - (k_1-1)(k_1' + k_2' + k_2), \\ q_3 &= (k_1-1)[k_1' + k_2' + 2k_1 + (k_1+k_2)^2] + k_1(2k_1 - 3) \\ &\quad + (k+1+k_2+1)[2(k_2+1) - k_1(k_1+1)] + 1. \end{aligned}$$

Now, if κ and τ are non-zero constants, then the natural curvature functions $\kappa_{\Im}$, $\tau_{\Im}$ are also non-zero constants and satisfying Eq. (25) which means that the $T\eta$-equiform Smarandache curve is circular heilx. □

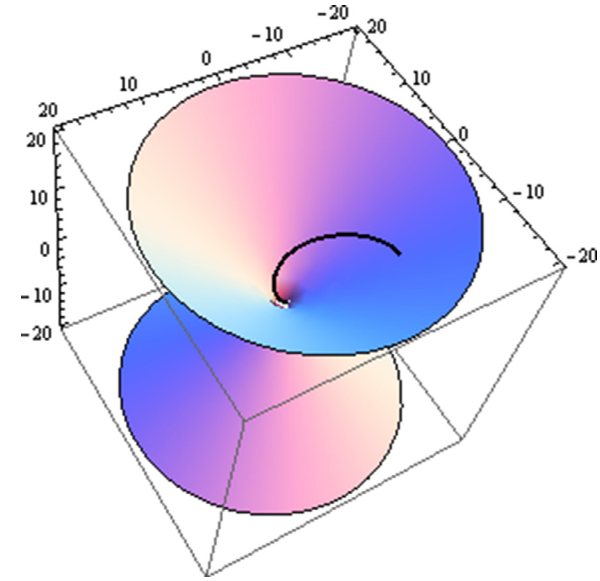

Fig. 2. Equiform spacelike curve $\zeta = \zeta(\theta)$ on S_1^2.

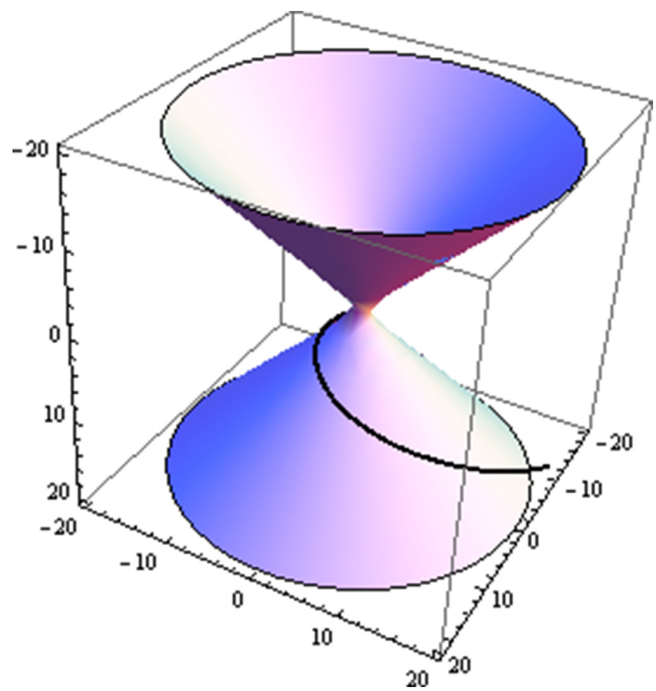

Fig. 3. The $T\eta$-equiform Smarandache curve $\Im(\theta^*)$ on S_1^2.

4. Example

Let $\zeta(s) = \left(\sqrt{3}s, s\sin(\sqrt{3}\ln s), s\cos(\sqrt{3}\ln s)\right)$ be a unit speed spacelike curve parametrized by arc-length s with spacelike principal normal vector in E_1^3 (see Fig. 1). Then it is easy to show that

$$\begin{cases} t(s) = (\sqrt{3}, \sin(\sqrt{3}\ln s) + \sqrt{3}\cos(\sqrt{3}\ln s), \\ \quad \cos(\sqrt{3}\ln s) - \sqrt{3}\sin(\sqrt{3}\ln s)), \\ n(s) = \frac{1}{\sqrt{2}}(0, \cos(\sqrt{3}\ln s) - \sqrt{3}\sin(\sqrt{3}\ln s), \\ \quad -\sin(\sqrt{3}\ln s) - \sqrt{3}\cos(\sqrt{3}\ln s)), \\ \kappa = \frac{2\sqrt{3}}{s}, \qquad \rho = \frac{s}{2\sqrt{3}}, \qquad k_1 = \frac{1}{2\sqrt{3}}, \\ b(s) = (2, \frac{\sqrt{3}}{2}\sin(\sqrt{3}\ln s) + \frac{3}{2}\cos(\sqrt{3}\ln s), \\ \quad \frac{\sqrt{3}}{2}\cos(\sqrt{3}\ln s) - \frac{3}{2}\sin(\sqrt{3}\ln s)), \\ \tau = \frac{3}{s}, \qquad k_2 = \frac{\sqrt{3}}{2}. \end{cases}$$

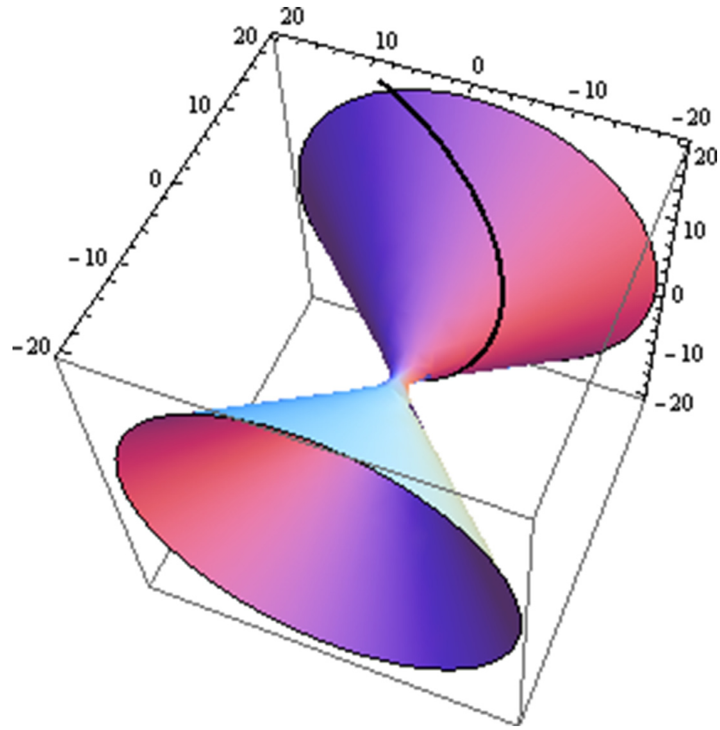

Fig. 4. The $T\xi$-equiform Smarandache curve $\Im(\theta^*)$ on S_1^2.

Hence, the equiform parameter is $\theta = \int \kappa\, ds = 2\sqrt{3}\,s + c$. Here we take $c = 0$, then we have $s = e^{\theta/2\sqrt{3}}$ and $\rho = \frac{e^{\theta/2\sqrt{3}}}{2\sqrt{3}}$. So the equiform spacelike curve ζ is define as (see Fig. 2)

$$\zeta(\theta) = \left(\sqrt{3}\,e^{\theta/2\sqrt{3}}, e^{\theta/2\sqrt{3}} \sin\left(\frac{\theta}{2}\right), e^{\theta/2\sqrt{3}} \cos\left(\frac{\theta}{2}\right)\right).$$

It easy to show that

$$T(\theta) = \frac{e^{\theta/2\sqrt{3}}}{2}\left(1, \frac{1}{\sqrt{3}} \sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right), \frac{1}{\sqrt{3}} \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right).$$

It is clear that T is an equiform spacelike vector. Also

$$\eta(\theta) = \frac{e^{\theta/2\sqrt{3}}}{4}\left(0, \frac{1}{\sqrt{3}} \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right), \frac{-1}{\sqrt{3}} \sin\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\right),$$

and

$$\xi(\theta) = \frac{e^{\theta/2\sqrt{3}}}{4}\left(\frac{4}{\sqrt{3}}, \sin\left(\frac{\theta}{2}\right) + \sqrt{3}\cos\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right) - \sqrt{3}\sin\left(\frac{\theta}{2}\right)\right).$$

Then η is an equiform spacelike vector and ξ is an equiform timelike vector.

The $T\eta$-equiform Smarandache curve $\Im(\theta^*)$ of the curve $\zeta(\theta)$ is given by (see Fig. 3)

$$\Im(\theta^*) = \frac{\sqrt{6}\,e^{\theta/2\sqrt{3}}}{24}\left(2\sqrt{3}, (2\sqrt{3}+1)\cos\left(\frac{\theta}{2}\right) + (2-\sqrt{3})\sin\left(\frac{\theta}{2}\right), (2-\sqrt{3})\cos\left(\frac{\theta}{2}\right) - (2\sqrt{3}+1)\sin\left(\frac{\theta}{2}\right)\right).$$

The $T\xi$-equiform Smarandache curve $\Im(\theta^*)$ of the curve $\zeta(\theta)$ is given by (see Fig. 4)

$$\Im(\theta^*) = \frac{\sqrt{6}\,e^{\theta/2\sqrt{3}}}{24}\left(2(2+\sqrt{3}), (2+\sqrt{3})\sin\left(\frac{\theta}{2}\right) + (2\sqrt{3}+3)\cos\left(\frac{\theta}{2}\right), (2+\sqrt{3})\cos\left(\frac{\theta}{2}\right) - (2\sqrt{3}+3)\sin\left(\frac{\theta}{2}\right)\right).$$

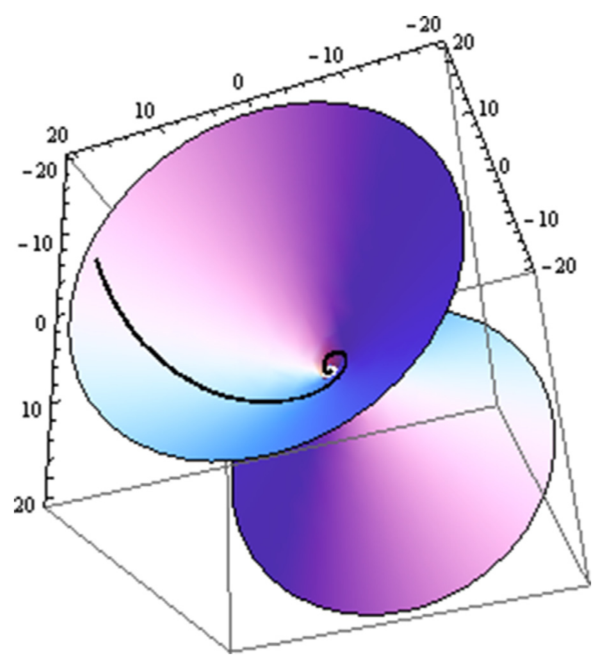

Fig. 5. The $\eta\xi$-equiform Smarandache curve $\Im(\theta^*)$ on S_1^2.

The $\eta\xi$-equiform Smarandache curve $\Im(\theta^*)$ of the curve $\zeta(\theta)$ is given by (see Fig. 5)

$$\Im(\theta^*) = \frac{\sqrt{6}\,e^{\theta/2\sqrt{3}}}{6}\left(1, \cos\left(\frac{\theta}{2}\right), -\sin\left(\frac{\theta}{2}\right)\right).$$

The $T\eta\xi$-equiform Smarandache curve $\Im(\theta^*)$ of the curve $\zeta(\theta)$ is given by (see Fig. 6)

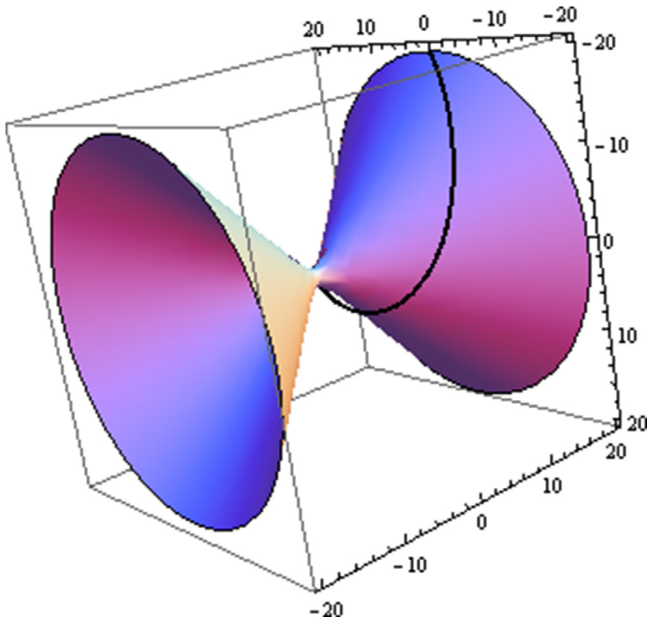

Fig. 6. The $T\eta\xi$-equiform Smarandache curve $\Im(\theta^*)$ on S_1^2.

$$\Im(\theta^*) = \frac{\sqrt{9}\,e^{\theta/2\sqrt{3}}}{18}\left(2+\sqrt{3}, \sin\left(\frac{\theta}{2}\right)+(2+\sqrt{3})\cos\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right) -(2+\sqrt{3})\sin\left(\frac{\theta}{2}\right)\right).$$

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