Special timelike Smrandache curves in Minkowski 3-space

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Abstract. In this paper, we acquaint a special timelike Smarandache curves $\mathcal{Z}$ reference the Darboux frame of a timelike curve χ in Minkowski 3-space $\mathfrak{R}_1^3$. We investigate the Frenet invariants of $\mathcal{Z}$ and also give some properties when the curve χ is a geodesic, an asymptotic and a principal curve. Finally, we give an example to illustrate these curves.

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1 Introduction

In Smarandache geometry, a regular non-null curve in Minkowski 3-space, whose position vector is collected by the Frenet frame vectors of other regular non-null curve, is said to be Smarandache curve [1]. Recently in Euclidean and Minkowski space-times, special Smarandache curves according to different types of frames have been studied by some authors [2, 3, 8, 10].

In this paper, we introduce a special timelike Smarandache curves recording to the Darboux frame of a curve χ on timelike surface M in Minkowski 3-space $\mathfrak{R}_1^3$. In Section 2, we give the basic concepts of Minkowski 3-space and Darboux frame that will be used throughout the paper. Section 3 is devoted to the study of special four timelike Smarandache curves $T\mathsf{n}$, $T\mathsf{g}$, gn, and $T\mathsf{gn}$-Smarandache curves by considering the relationship with invariants $\kappa_n(\sigma)$, $\kappa_g(\sigma)$ and $\tau_g(\sigma)$ of χ in Minkowski 3-space $\mathfrak{R}_1^3$. From that point, we give some properties of these curves when χ is a geodesic, an asymptotic, or a principal curve. Finally, we illustrate these curves with an example.

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2 Preliminaries

The Minkowski 3-space is three-dimensional Euclidean space provided with the Lorentzian inner product

$$\mathfrak{F} = -(du_1)^2 + (du_2)^2 + (du_3)^2,$$

where $u = (u_1, u_2, u_3)$ is a rectangular coordinate system of $\mathfrak{R}^3_1$. Any vector υ in $\mathfrak{R}^3_1$ can be characterized as follows: the vector υ is called spacelike, lightlike or timelike if $\mathfrak{F}(\upsilon, \upsilon) > 0$ and $\upsilon = 0$, $\mathfrak{F}(\upsilon, \upsilon) = 0$ and $\upsilon \neq 0$ or $\mathfrak{F}(\upsilon, \upsilon) < 0$ respectively. The norm of a vector $\upsilon \in \mathfrak{R}^3_1$ is given by $\|\upsilon\| = \sqrt{|\mathfrak{F}(\upsilon, \upsilon)|}$. Similarly, any arbitrary curve $\chi = \chi(\sigma) : I \to \mathfrak{R}^3_1$ where σ is pseudo-arclength parameter, is called a spacelike curve if $\mathfrak{F}(\chi'(\sigma), \chi'(\sigma)) > 0$, lightlike if $\mathfrak{F}(\chi'(\sigma), \chi'(\sigma)) = 0$ and $\chi'(s) \neq 0$ and timelike if $\mathfrak{F}(\chi'(\sigma), \chi'(\sigma)) < 0$ and for all $\sigma \in I$.

For any unit speed timelike curve χ with Frenet-Serret frame $\{T, N, B\}$, Frenet-Serret formulas of the curve χ can be given as [4, 5, 6]:

$$\begin{pmatrix} T'(\sigma) \\ N'(\sigma) \\ B'(\sigma) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(\sigma) & 0 \\ \kappa(\sigma) & 0 & \tau(\sigma) \\ 0 & -\tau(\sigma) & 0 \end{pmatrix} \begin{pmatrix} T(\sigma) \\ N(\sigma) \\ B(\sigma) \end{pmatrix}, \tag{1}$$

where $-\mathfrak{F}(T, T) = \mathfrak{F}(N, N) = \mathfrak{F}(B, B) = 1$ and $\mathfrak{F}(T, N) = \mathfrak{F}(T, B) = \mathfrak{F}(N, B) = 0$.

Definition 2.1. A spacelike (timelike) surface in the Minkowski 3-space is a surface M in $\mathfrak{R}^3_1$ whose the induced metric is a positive definite Riemannian metric (Lorentz metric). In other words, the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [6].

Let $\Psi : \mathcal{V} \subset \mathfrak{R}^2 \to \mathfrak{R}^3_1$, $\Psi(\mathcal{V}) = M$ and $\beta : I \subset \mathfrak{R} \to \mathcal{V}$ be a timelike embedding and a regular curve, respectively. Then we define a curve $\chi(\sigma) = \Psi(\beta(\sigma))$ on the surface M, and since Ψ is a timelike embedding, we have a unit spacelike normal vector field n along the surface M defined by [7]

$$\mathsf{n} = \frac{\Psi_x \times \Psi_y}{\|\Psi_x \times \Psi_y\|}. \tag{2}$$

Hence we have a pseudo-orthonormal frame $\{T, \mathsf{g}, \mathsf{n}\}$ which is called the Darboux frame along the curve χ where $\mathsf{g}(\sigma) = T(\sigma) \times \mathsf{n}(\sigma)$ is a unit vector. The corresponding Frenet-Serret formulae of χ read

$$\begin{pmatrix} T'(\sigma) \\ \mathsf{g}'(\sigma) \\ \mathsf{n}'(\sigma) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g(\sigma) & \kappa_n(\sigma) \\ \kappa_g(\sigma) & 0 & -\tau_g(\sigma) \\ \kappa_n(\sigma) & \tau_g(\sigma) & 0 \end{pmatrix} \begin{pmatrix} T(\sigma) \\ \mathsf{g}(\sigma) \\ \mathsf{n}(\sigma) \end{pmatrix}, \tag{3}$$

where $\kappa_g(\sigma) = \mathfrak{F}(T'(\sigma), \mathsf{g}(\sigma))$, $\kappa_n(\sigma) = \mathfrak{F}(T'(\sigma), \mathsf{n}(\sigma))$ and $\tau_g(\sigma) = \mathfrak{F}(\mathsf{g}'(\sigma), \mathsf{n}(\sigma))$ are the geodesic curvature, the asymptotic curvature, and the principal curvature of χ on the surface M in $\mathfrak{R}_1^3$, respectively, and σ is arc-length parameter of χ.

The pseudosphere with center at the origin and of radius $r = 1$ in the Minkowski 3-space $\mathfrak{R}_1^3$ is a quadric defined by

$$S_1^2 = \{u \in \mathfrak{R}_1^3 : \mathfrak{F}(u, u) = 1\}.$$

3 Special timelike Smrandache curves in $\mathfrak{R}_1^3$

In this section, we define a special timelike Smarandache curves reference to the Darboux frame in Minkowski 3-space $\mathfrak{R}_1^3$. Additionally, we obtain the Frenet invariants of these curves and give some properties when the curve χ is a geodesic curve or an asymptotic curve or a principal curve.

Definition 3.1. [9] Let $\chi = \chi(\sigma)$ be a timelike curve lying completely on the timelike surface M in $\mathfrak{R}_1^3$ with the moving Darboux frame $\{T, \mathsf{g}, \mathsf{n}\}$. Then $T\mathsf{g}$-timelike Smarandache curves of χ is defined by

$$\mathcal{Z}\big(\vartheta(\sigma)\big) = \frac{1}{\sqrt{2}}\Big(T(\sigma) + \mathsf{g}(\sigma)\Big). \tag{4}$$

Theorem 3.1. Let $\chi = \chi(\sigma)$ be a timelike curve lying completely on the timelike surface M in $\mathfrak{R}_1^3$ with the moving Darboux frame $\{T, \mathsf{g}, \mathsf{n}\}$. If $\chi(\sigma)$ is a geodesic curve with $\tau_g > \kappa_n$, then the natural curvature functions of $T\mathsf{g}$- timelike Smarandache curves satisfied the following equations,

$$\begin{aligned} \kappa_{\mathcal{Z}}(\vartheta(\sigma)) &= \frac{\sqrt{2(\tau_g^2-\kappa_n^2)}}{(\kappa_n-\tau_g)}, \\ \tau_{\mathcal{Z}}(\vartheta(\sigma)) &= \frac{\sqrt{2}\,\kappa_n(\kappa_n-\tau_g)(\tau_g'-\kappa_n'\tau_g)-\kappa_n\tau_g(\kappa_n'-\tau_g')}{(\kappa_n^2+\tau_g^2)(\kappa_n-\tau_g)^4}. \end{aligned} \tag{5}$$

Proof. Let $\mathcal{Z}=\mathcal{Z}(\vartheta)$ be a $T\mathsf{g}$- timelike Smarandache curves reference to the timelike curve $\chi=\chi(\sigma)$ in Minkowski 3-space $\mathfrak{R}_1^3$. From Eqns. (3) and (5), we get

$$\mathcal{Z}'(\vartheta)=\frac{d\mathcal{Z}}{d\vartheta}\frac{d\vartheta}{d\sigma}=\frac{1}{\sqrt{2}}\Big(\kappa_g\,T(\sigma)+\kappa_g\,\mathsf{g}(\sigma)+(\kappa_n-\tau_g)\mathsf{n}(\sigma)\Big), \tag{6}$$

and

$$T_{\mathcal{Z}}(\vartheta)=\frac{1}{(\kappa_n-\tau_g)}\Big(\kappa_g\,T(\sigma)+\kappa_g\,\mathsf{g}(\sigma)+(\kappa_n-\tau_g)\mathsf{n}(\sigma)\Big), \tag{7}$$

where

$$\frac{d\vartheta}{d\sigma}=\frac{\kappa_n-\tau_g}{\sqrt{2}}. \tag{8}$$

Now

$$\frac{dT_{\mathcal{Z}}}{d\vartheta}=\frac{\sqrt{2}}{\big(\kappa_n-\tau_g\big)^3}\Big(\varepsilon_1(\sigma)T(\sigma)+\varepsilon_2(\sigma)\mathsf{g}(\sigma)+\varepsilon_3(\sigma)\mathsf{n}(\sigma)\Big), \tag{9}$$

where

$$\begin{cases} \varepsilon_1(\sigma)=(\kappa_n-\tau_g)\big[\kappa_g^2+\kappa_g'+\kappa_n(\kappa_n-\tau_g)\big]-\kappa_g(\kappa_n'-\tau_g'), \\ \varepsilon_2(\sigma)=(\kappa_n-\tau_g)\big[\kappa_g^2+\kappa_g'+\tau_g(\kappa_n-\tau_g)\big]-\kappa_g(\kappa_n'-\tau_g'), \\ \varepsilon_3(\sigma)=\kappa_g(\kappa_n-\tau_g)^2. \end{cases}$$

Then

$$\kappa_{\mathcal{Z}}(\vartheta)=\frac{\sqrt{2\big(\varepsilon_2^2+\varepsilon_3^2-\varepsilon_1^2\big)}}{\big(\kappa_n-\tau_g\big)^3},$$

and

$$N_{\mathcal{Z}}(\vartheta)=\frac{1}{\sqrt{\varepsilon_2^2+\varepsilon_3^2-\varepsilon_1^2}}\Big(\varepsilon_1(\sigma)T(\sigma)+\varepsilon_2(\sigma)\mathsf{g}(\sigma)+\varepsilon_3(\sigma)\mathsf{n}(\sigma)\Big).$$

Then, we have

$$B_{\mathcal{Z}}(\vartheta)=\frac{1}{(\kappa_n-\tau_g)\sqrt{\varepsilon_2^2+\varepsilon_3^2-\varepsilon_1^2}}\Big(\ell_1(\sigma)T(\sigma)+\ell_2(\sigma)\mathsf{g}(\sigma)+\ell_3(\sigma)\mathsf{n}(\sigma)\Big),$$

where

$$\begin{cases} \ell_1(\sigma) = \varepsilon_2(\kappa_n - \tau_g) - \varepsilon_3\kappa_g, \\ \ell_2(\sigma) = \varepsilon_1(\kappa_n - \tau_g) - \varepsilon_3\kappa_g, \\ \ell_3(\sigma) = \kappa_g(\varepsilon_2 - \varepsilon_1). \end{cases}$$

From Eqn. (6), we get

$$\begin{aligned} \mathcal{Z}''(\vartheta) = & \frac{1}{\sqrt{2}}\Big[\big[\kappa_g^2 + \kappa_g' + \kappa_n(\kappa_n - \tau_g)\big]T(\sigma) + \big[\kappa_g^2 + \kappa_g' + \tau_g(\kappa_n - \tau_g)\big]\mathsf{g}(\sigma) \\ & + \big[\kappa_n' - \tau_g' + \kappa_g(\kappa_n - \tau_g)\big]\mathsf{n}(\sigma)\Big], \end{aligned}$$

and

$$\mathcal{Z}'''(\vartheta) = \frac{1}{\sqrt{2}}\Big[\mu_1(\sigma)T(\sigma) + \mu_2(\sigma)\mathsf{g}(\sigma) + \mu_3(\sigma)\mathsf{n}(\sigma)\Big],$$

where

$$\begin{cases} \mu_1(\sigma) = \kappa_g^3 + \kappa_g'' + 3\kappa_g\kappa_g' + (\kappa_n - \tau_g)(\kappa_n' + 2\kappa_g\tau_g + \kappa_n\kappa_g) + 3\kappa_n(\kappa_n' - \tau_g'), \\ \mu_2(\sigma) = \kappa_g^3 + \kappa_g'' + 3\kappa_g\kappa_g' + (\kappa_n - \tau_g)(\tau_g' + \kappa_n\kappa_g + \kappa_g\tau_g) + 2\tau_g(\kappa_n' - \tau_g'), \\ \mu_3(\sigma) = \kappa_n'' - \tau_g'' + (\kappa_n - \tau_g)(\kappa_n^2 - \tau_g^2 + \kappa_g + 2\kappa_g') + \kappa_g(\kappa_n' - \tau_g'). \end{cases}$$

Then

$$\tau_{\mathcal{Z}}(\vartheta) = \frac{\sqrt{2}\left\{(\mu_1 - \mu_2)\big[\kappa_g(\kappa_n' - \tau_g') - \kappa_g'(\kappa_n - \tau_g)\big] + (\kappa_n - \tau_g)^2(\mu_2\kappa_n - \mu_1\tau_g\mu_3\kappa_g)\right\}}{\Big[\kappa_g(\kappa_n' - \tau_g') - (\kappa_n - \tau_g)\big(\kappa_g' + \tau_g(\kappa_n - \tau_g)\big)^2\Big]^2 - \Big[(\kappa_n - \tau_g)\big(\kappa_g' + \kappa_n(\kappa_n - \tau_g)\big)^2 - \kappa_g(\kappa_n' - \tau_g')\Big]^2 + \kappa_g^2(\kappa_n - \tau_g)^4}$$

So if $\chi(\sigma)$ is a geodesic curve ($\kappa_g = 0$), then Eqn. (5) holds and the proof is complete. □

Definition 3.2. [9] Let $\chi = \chi(\sigma)$ be a timelike curve lying completely on the timelike surface M in $\mathfrak{R}_1^3$ with the moving Darboux frame $\{T, \mathsf{g}, \mathsf{n}\}$. Then $T\mathsf{n}$-timelike Smarandache curves of χ is defined by

$$\mathcal{Z}\big(\vartheta(\sigma)\big) = \frac{1}{\sqrt{2}}\Big(T(\sigma) + \mathsf{n}(\sigma)\Big). \tag{10}$$

Theorem 3.2. Let $\chi = \chi(\sigma)$ be a timelike curve lying completely on the timelike surface M in $\mathfrak{R}_1^3$ with the moving Darboux frame $\{T, \mathsf{g}, \mathsf{n}\}$. If $\chi(\sigma)$ is a an asymptotic line with $\tau_g \geq \kappa_g$, then the natural curvature functions of $T\mathsf{n}$- timelike Smarandache curves satisfied the following equation

$$\begin{aligned} \kappa_{\mathcal{Z}}(\vartheta) &= \frac{\sqrt{2(\tau_g^2-\kappa_g^2)}}{(\kappa_g+\tau_g)}, \\ \tau_{\mathcal{Z}}(\vartheta) &= \frac{\sqrt{2}(\kappa_g\tau_g'-\kappa_g'\tau_g)}{(\kappa_g+\tau_g)(\kappa_g^2+\tau_g^2)}. \end{aligned} \tag{11}$$

Proof. Let $\mathcal{Z}=\mathcal{Z}(\vartheta)$ be a $T\mathsf{n}$- timelike Smarandache curves reference to the timelike curve $\chi=\chi(\sigma)$ in Minkowski 3-space $\mathfrak{R}_1^3$. Then from Eqn. (10), we get

$$\mathcal{Z}'(\vartheta)=\frac{1}{\sqrt{2}}\Big(\kappa_n\,T(\sigma)+(\kappa_g+\tau_g)\,\mathsf{g}(\sigma)+\kappa_n\mathsf{n}(\sigma)\Big), \tag{12}$$

and

$$T_{\mathcal{Z}}(\vartheta)=\frac{1}{\kappa_g+\tau_g}\Big(\kappa_n\,T(\sigma)+(\kappa_g+\tau_g)\,\mathsf{g}(\sigma)+\kappa_n\mathsf{n}(\sigma)\Big), \tag{13}$$

where

$$\frac{d\vartheta}{d\sigma}=\frac{\kappa_g+\tau_g}{\sqrt{2}}\,. \tag{14}$$

Also, one can see that

$$\frac{dT_{\mathcal{Z}}}{d\vartheta}=\frac{\sqrt{2}}{\big(\kappa_g+\tau_g\big)^3}\Big(\gamma_1(\sigma)T(\sigma)+\gamma_2(\sigma)\mathsf{g}(\sigma)+\gamma_3(\sigma)\mathsf{n}(\sigma)\Big),$$

where

$$\begin{aligned} \gamma_1(\sigma) &= (\kappa_g+\tau_g)\big[\kappa_n^2+\kappa_n'+\kappa_g(\kappa_g+\tau_g)\big]-\kappa_n(\kappa_g'+\tau_g'), \\ \gamma_2(\sigma) &= \kappa_n(\kappa_g+\tau_g)^2, \\ \gamma_3(\sigma) &= (\kappa_g+\tau_g)\big[\kappa_n^2+\kappa_n'-\tau_g(\kappa_g+\tau_g)\big]-\kappa_n(\kappa_g'+\tau_g'). \end{aligned}$$

Then,

$$\kappa_{\mathcal{Z}}(\vartheta)=\frac{\sqrt{2\big(\gamma_2^2+\gamma_3^2-\gamma_1^2\big)}}{\big(\kappa_g+\tau_g\big)^3}\,,$$

and

$$N_{\mathcal{Z}}(\vartheta)=\frac{1}{\sqrt{\gamma_2^2+\gamma_3^2-\gamma_1^2}}\Big(\gamma_1(\sigma)T(\sigma)+\gamma_2(\sigma)\mathsf{g}(\sigma)+\gamma_3(\sigma)\mathsf{n}(\sigma)\Big).$$

So

$$B_{\mathcal{Z}}(\vartheta)=\frac{1}{(\kappa_g+\tau_g)\sqrt{\gamma_2^2+\gamma_3^2-\gamma_1^2}}\Big(\delta_1(\sigma)T(\sigma)+\delta_2(\sigma)\mathsf{g}(\sigma)+\delta_3(\sigma)\mathsf{n}(\sigma)\Big),$$

where

$$\begin{cases}\delta_1(\sigma)=\gamma_2\kappa_n-\gamma_3(\kappa_g+\tau_g),\\ \delta_2(\sigma)=(\gamma_1-\gamma_3)\kappa_n,\\ \delta_3(\sigma)=\gamma_2\kappa_n-\gamma_1(\kappa_g+\tau_g).\end{cases}$$

Now, from Eqn. (12) we get

$$\begin{aligned}\mathcal{Z}''(\vartheta)=&\frac{1}{\sqrt{2}}\Big[\big[\kappa_n^2+\kappa_n'+\kappa_g(\kappa_g+\tau_g)\big]T(\sigma)+\big[\kappa_g'+\tau_g'+\kappa_n(\kappa_g+\tau_g)\big]\mathsf{g}(\sigma)\\&+\big[\kappa_n^2-\tau_g(\kappa_g+\tau_g)\big]\mathsf{n}(\sigma)\Big],\end{aligned}$$

and

$$\mathcal{Z}'''(\vartheta)=\frac{1}{\sqrt{2}}\Big[\nu_1(\sigma)T(\sigma)+\nu_2(\sigma)\mathsf{g}(\sigma)+\nu_3(\sigma)\mathsf{n}(\sigma)\Big],$$

where

$$\begin{cases}\nu_1(\sigma)=\kappa_n^3+\kappa_n''+2\kappa_n\kappa_n'+\kappa_g'(\kappa_g+\tau_g)+2\kappa_g(\kappa_g'+\tau_g'),\\ \nu_2(\sigma)=\kappa_n^3\tau_g+\kappa_g''+\tau_g''+(\kappa_g+\tau_g)(\kappa_n'+\kappa_g^2-\tau_g^2)+\kappa_g(\kappa_n^2+\kappa_n'),\\ \nu_3(\sigma)=\kappa_n^3+3\kappa_n\kappa_n'-\tau_g'(\kappa_g+\tau_g)-2\tau_g(\kappa_g'+\tau_g').\end{cases}$$

Then

$$\tau_{\mathcal{Z}}(\vartheta)=\sqrt{2}\left\{\frac{\begin{array}{c}\nu_2\kappa_n\kappa_n'+(\kappa_g+\tau_g)(2\nu_2\kappa_n\kappa_g-\nu_3\kappa_n')+(\nu_1+\nu_3)\kappa_n(\kappa_g'+\tau_g')\\-(\nu_3\kappa_g+\nu_1\tau_g)(\kappa_g+\tau_g)^2\end{array}}{\begin{array}{l}\big[\tau_g(\kappa_g+\tau_g)^2+\kappa_n(\kappa_g'+\tau_g')\big]^2+\big[\kappa_n\kappa_n'+2\kappa_n\kappa_g(\kappa_g+\tau_g)\big]^2+\big[\kappa_n(\kappa_g'+\tau_g')\\-(\kappa_g+\tau_g)\big(\kappa_n'+\kappa_g(\kappa_g+\tau_g)\big)^2\big]^2\end{array}}\right\}$$

So, if $\chi(\sigma)$ is a an asymptotic line ($\kappa_n=0$), then Eqn. (11) holds and the proof is complete. □

Definition 3.3. [9] Let $\chi=\chi(\sigma)$ be a timelike curve lying completely on the timelike surface M in $\mathfrak{R}_1^3$ with the moving frame $\{T,\mathsf{g},\mathsf{n}\}$. Then gn-Smarandache curves of χ is defined by

$$\mathcal{Z}\big(\vartheta(\sigma)\big)=\frac{1}{\sqrt{2}}\Big(\mathsf{g}(\sigma)+\mathsf{n}(\sigma)\Big). \tag{15}$$

Theorem 3.3. Let $\chi = \chi(\sigma)$ be a timelike curve lying completely on the timelike surface M in $\mathfrak{R}^3_1$ with the moving frame $\{T, \mathsf{g}, \mathsf{n}\}$. If $\chi(\sigma)$ is a principal line with $\kappa_n + \kappa_g \neq 0$, then curvature and torsion of gn-Smarandache curves satisfied the following equations,

$$\begin{aligned} \kappa_{\mathcal{Z}}(\vartheta) &= \frac{\sqrt{2(\kappa_n^2+\kappa_g^2)}}{\kappa_n+\kappa_g}, \\ \tau_{\mathcal{Z}}(\vartheta) &= \sqrt{2}\left\{\frac{(3\kappa_g-\kappa_n)(\kappa_n'-\kappa_g')(\kappa_n^2-\kappa_g^2)+3\kappa_n\kappa_g(\kappa_n+\kappa_g)(\kappa_n'+\kappa_g')}{(\kappa_n+\kappa_g)\left[\kappa_n^2+\kappa_g^2(\kappa_n+\kappa_g)^2\right]}\right\}. \end{aligned} \tag{16}$$

Proof. Let $\mathcal{Z} = \mathcal{Z}(\vartheta)$ be a gn- timelike Smarandache curves reference to the timelike curve $\chi = \chi(\sigma)$ in Minkowski 3-space $\mathfrak{R}^3_1$. Then from Eqn. (15), we get

$$\mathcal{Z}'(\vartheta) = \frac{d\mathcal{Z}}{d\vartheta}\frac{d\vartheta}{ds} = \frac{1}{\sqrt{2}}\Big((\kappa_n+\kappa_g)\,T(\sigma)+\tau_g\,\mathsf{g}(\sigma)-\tau_g\,\mathsf{n}(\sigma)\Big), \tag{17}$$

and

$$T_{\mathcal{Z}}(\vartheta) = \frac{1}{\sqrt{2\tau_g^2-(\kappa_n+\kappa_g)^2}}\Big((\kappa_n+\kappa_g)\,T(\sigma)+\tau_g\,\mathsf{g}(\sigma)-\tau_g\,\mathsf{n}(\sigma)\Big), \tag{18}$$

where

$$\frac{d\vartheta}{ds} = \frac{\sqrt{2\tau_g^2-(\kappa_n+\kappa_g)^2}}{\sqrt{2}}. \tag{19}$$

Then

$$\frac{dT_{\mathcal{Z}}}{d\vartheta} = \frac{\sqrt{2}}{\big(2\tau_g^2-(\kappa_n+\kappa_g)\big)^2}\Big(\eta_1(\sigma)T(\sigma)+\eta_2(\sigma)\mathsf{g}(\sigma)+\eta_3(\sigma)\mathsf{n}(\sigma)\Big), \tag{20}$$

where

$$\begin{aligned} \eta_1(\sigma) &= \left[2\tau_g^2-(\kappa_n+\kappa_g)^2\right]\left[\kappa_n'+\kappa_g'-\tau_g(\kappa_n-\kappa_g)\right]-(\kappa_n+\kappa_g)\left[2\tau_g\tau_g'-(\kappa_n+\kappa_g)(\kappa_n'+\kappa_g')\right], \\ \eta_2(\sigma) &= \left[2\tau_g^2-(\kappa_n+\kappa_g)^2\right]\left[\tau_g'-\tau_g^2+\kappa_g(\kappa_n+\kappa_g)\right]-\tau_g\left[2\tau_g\tau_g'-(\kappa_n+\kappa_g)(\kappa_n'+\kappa_g')\right], \\ \eta_3(\sigma) &= \left[2\tau_g^2-(\kappa_n+\kappa_g)^2\right]\left[\kappa_g(\kappa_n+\kappa_g)-\tau_g'-\tau_g^2\right]+\tau_g\left[2\tau_g\tau_g'-(\kappa_n+\kappa_g)(\kappa_n'+\kappa_g')\right]. \end{aligned}$$

Then

$$\kappa_{\mathcal{Z}}(\vartheta) = \frac{\sqrt{2\big(\eta_2^2+\eta_3^2-\eta_1^2\big)}}{\big(2\tau_g^2-(\kappa_n+\kappa_g)\big)^2},$$

and

$$N_{\mathcal{Z}}(\vartheta) = \frac{1}{\sqrt{\eta_2^2+\eta_3^2-\eta_1^2}}\Big(\eta_1\,T+\eta_2\,\mathsf{g}+\eta_3\,\mathsf{n}\Big).$$

So

$$B_{\mathcal{Z}}(\vartheta) = \frac{\upsilon_1(\sigma)T(\sigma) + \upsilon_2(\sigma)\mathsf{g}(\sigma) + \upsilon_3(\sigma)\mathsf{n}(\sigma)}{\sqrt{2\tau_g^2 - (\kappa_n + \kappa_g)^2}\sqrt{\eta_2^2 + \eta_3^2 - \eta_1^2}},$$

where

$$\begin{cases} \upsilon_1(\sigma) = -(\eta_2 + \eta_3)\tau_g, \\ \upsilon_2(\sigma) = -\eta_1\tau_g - \eta_3(\kappa_n + \kappa_g), \\ \upsilon_3(\sigma) = \eta_2(\kappa_n + \kappa_g) - \eta_1\tau_g. \end{cases}$$

From Eqn. (17), we have

$$\begin{aligned} \mathcal{Z}''(\vartheta) =& \frac{1}{\sqrt{2}}\Big[\big[(\kappa_n' + \kappa_g') + \tau_g(\kappa_n - \kappa_g)\big]T(\sigma) + \big[\kappa_g(\kappa_n + \kappa_g) + \tau_g' - \tau_g^2\big]\mathsf{g}(\sigma) \\ &+ \big[\kappa_n(\kappa_n + \kappa_g) - \tau_g' - \tau_g^2\big]\mathsf{n}(\sigma)\Big], \end{aligned}$$

and

$$\mathcal{Z}'''(\vartheta) = \frac{1}{\sqrt{2}}\Big[\omega_1(\sigma)T(\sigma) + \omega_2(\sigma)\mathsf{g}(\sigma) + \omega_3(\sigma)\mathsf{n}(\sigma)\Big],$$

where

$$\begin{cases} \omega_1(\sigma) = \kappa_n'' + \kappa_g'' + (\kappa_n + \kappa_g)\big[\kappa_n^2 + \kappa_g^2 + \tau_g^2\big] - (\kappa_n - \kappa_g)(2\tau_g' + \tau_g), \\ \omega_2(\sigma) = \tau_g'' - \tau_g^3 - 3\tau_g'\tau_g + (\kappa_n + \kappa_g)(\kappa_g' + \kappa_n\tau_g) - \kappa_g\tau_g(\kappa_n - \kappa_g) + 2\kappa_g(\kappa_n' - \kappa_g'), \\ \omega_3(\sigma) = (\kappa_n + \kappa_g)(\kappa_n' - \kappa_g\tau_g) - \kappa_n\tau_g(\kappa_n - \kappa_g) + 2\kappa_n(\kappa_n' - \kappa_g') - \tau_g'' - \tau_g^3. \end{cases}$$

Then

$$\tau_{\mathcal{Z}}(\vartheta) = \sqrt{2}\left\{ \frac{\begin{array}{l} (\kappa_n + \kappa_g)\big[(\omega_2 - \omega_3)(\kappa_g^2 + \tau_g^2 - \kappa_n^2) + (\omega_2 + \omega_3)(\tau_g' + \kappa_n\kappa_g)\big] \\ -(\omega_2 + \omega_3)\tau_g(\kappa_n' + \kappa_g') + (\omega_2 - \omega_3)\tau_g^2(\kappa_n - \kappa_g) + \omega_1\big[\tau_g(\kappa_n + \kappa_g)^2 - 2\tau_g^3\big] \end{array}}{\begin{array}{l} \big[\tau_g(\kappa_n + \kappa_g)^2 - 2\tau_g^3\big]^2 + \big[(\kappa_n + \kappa_g)(\tau_g^2 + \tau_g' - \kappa_n^2 - \kappa_n\kappa_g) - \tau_g(\kappa_n' + \kappa_g') \\ -\tau_g^2(\kappa_n - \kappa_g)\big]^2 + \big[(\kappa_n + \kappa_g)(\kappa_g^2 + \kappa_n\kappa_g - \tau_g' - \tau_g^2) \\ -\tau_g(\kappa_n' + \kappa_g') - \tau_g^2(\kappa_n - \kappa_g)\big]^2 \end{array}} \right\}$$

So, if $\chi(\sigma)$ is a principal line ($\tau_g = 0$), then Eqn. (16) holds and the proof is complete. □

Definition 3.4. [9] Let $\chi = \chi(\sigma)$ be a timelike curve lying completely on the timelike surface M in $\mathfrak{R}_1^3$ with the moving frame $\{T, \mathsf{g}, \mathsf{n}\}$. Then $T\mathsf{gn}$-Smarandache curves of χ is defined by

$$\mathcal{Z}\big(\vartheta(\sigma)\big) = \frac{1}{\sqrt{3}}\Big(T(\sigma) + \mathsf{g}(\sigma) + \mathsf{n}(\sigma)\Big). \tag{21}$$

Now, we can investigate the Frenet invariants of the $T\mathsf{gn}$- timelike Smarandache curves reference to the timelike curve $\chi = \chi(\sigma)$ in Minkowski 3-space $\mathfrak{R}_1^3$. From Eqn. (22), we get

$$\mathcal{Z}'(\vartheta) = \frac{d\mathcal{Z}}{d\vartheta}\frac{d\vartheta}{ds} = \frac{1}{\sqrt{3}}\Big((\kappa_n + \kappa_g)\, T(\sigma) + (\kappa_g + \tau_g)\, \mathsf{g}(\sigma) + (\kappa_n - \tau_g)\, \mathsf{n}(\sigma)\Big), \tag{22}$$

and

$$T_{\mathcal{Z}}(\vartheta) = \frac{(\kappa_n + \kappa_g)\, T(\sigma) + (\kappa_g + \tau_g)\, \mathsf{g}(\sigma) + (\kappa_n - \tau_g)\, \mathsf{n}(\sigma)}{\sqrt{(\kappa_g + \tau_g)^2 - (\kappa_n + \kappa_g)^2 + (\kappa_n - \tau_g)^2}}\,, \tag{23}$$

where

$$\frac{d\vartheta}{ds} = \sqrt{\frac{(\kappa_g + \tau_g)^2 - (\kappa_n + \kappa_g)^2 + (\kappa_n - \tau_g)^2}{3}}\,. \tag{24}$$

Then

$$\frac{dT_{\mathcal{Z}}}{d\vartheta} = \frac{\sqrt{3}\Big(\lambda_1(\sigma)T(\sigma) + \lambda_2(\sigma)\mathsf{g}(\sigma) + \lambda_3(\sigma)\mathsf{n}(\sigma)\Big)}{\Big((\kappa_g + \tau_g)^2 - (\kappa_n + \kappa_g)^2 + (\kappa_n - \tau_g)^2\Big)^2}\,, \tag{25}$$

where

$$\begin{aligned}
\lambda_1(\sigma) = {} & (\kappa_n' + \kappa_g')\big[(\kappa_g + \tau_g)^2 + (\kappa_n - \tau_g)^2\big] + (\kappa_g + \tau_g)\Big\{\kappa_g\big[(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 \\
& + (\kappa_n - \tau_g)^2\big] - (\kappa_n + \kappa_g)(\kappa_n' + \kappa_g')\Big\} + (\kappa_n - \tau_g)\Big\{\kappa_n\big[(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 \\
& + (\kappa_n - \tau_g)^2\big] - (\kappa_n + \kappa_g)(\kappa_n' - \tau_g')\Big\},
\end{aligned}$$

$$\begin{aligned}
\lambda_2(\sigma) = {} & (\kappa_g' + \tau_g')\big[(\kappa_g + \tau_g)^2 + (\kappa_n - \tau_g)^2\big] + (\kappa_n + \kappa_g)\Big\{\kappa_g\big[(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 \\
& + (\kappa_n - \tau_g)^2\big] - (\kappa_g + \tau_g)(\kappa_n' + \kappa_g')\Big\} + (\kappa_n - \tau_g)\Big\{\tau_g\big[(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 \\
& + (\kappa_n - \tau_g)^2\big] - (\kappa_g + \tau_g)(\kappa_n' - \tau_g')\Big\}, \\
\lambda_3(\sigma) = {} & (\kappa_n' + \tau_g')\big[(\kappa_g + \tau_g)^2 + (\kappa_n + \kappa_g)^2\big] + (\kappa_n + \tau_g)\Big\{\kappa_n\big[(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 \\
& + (\kappa_n - \tau_g)^2\big] - (\kappa_n - \tau_g)(\kappa_n' + \kappa_g')\Big\} - (\kappa_g + \tau_g)\Big\{\tau_g\big[(\kappa_n + \kappa_g)^2 + (\kappa_g + \tau_g)^2 \\
& + (\kappa_n - \tau_g)^2\big] + (\kappa_n - \tau_g)(\kappa_g' + \tau_g')\Big\}.
\end{aligned}$$

Then

$$\kappa_{\mathcal{Z}}(\vartheta)=\frac{\sqrt{3\big(\lambda_2^2+\lambda_3^2-\lambda_1^2\big)}}{\Big((\kappa_g+\tau_g)^2-(\kappa_n+\kappa_g)^2+(\kappa_n-\tau_g)^2\Big)^2},$$

and

$$N_{\mathcal{Z}}(\vartheta)=\frac{1}{\sqrt{\lambda_2^2+\lambda_3^2-\lambda_1^2}}\Big(\lambda_1(\sigma)T(\sigma)+\lambda_2(\sigma)\mathsf{g}(\sigma)+\lambda_3(\sigma)\mathsf{n}(\sigma)\Big).$$

So

$$B_{\mathcal{Z}}(\vartheta)=\frac{\rho_1(\sigma)T(\sigma)+\rho_2(\sigma)\mathsf{g}(\sigma)+\rho_3(\sigma)\mathsf{n}(\sigma)}{\sqrt{(\kappa_g+\tau_g)^2-(\kappa_n+\kappa_g)^2+(\kappa_n-\tau_g)^2}\sqrt{\lambda_2^2+\lambda_3^2-\lambda_1^2}}$$

where

$$\begin{cases}\rho_1(\sigma)=\lambda_2(\kappa_n-\tau_g)-\lambda_3(\kappa_g+\tau_g),\\ \rho_2(\sigma)=\lambda_1(\kappa_n-\tau_g)-\lambda_3(\kappa_n+\kappa_g),\\ \rho_3(\sigma)=\lambda_2(\kappa_n+\kappa_g)-\lambda_1(\kappa_g+\tau_g).\end{cases}$$

Then from Eqn. (22), we get

$$\begin{aligned}\mathcal{Z}''(\vartheta)=&\frac{1}{\sqrt{3}}\Big[\big[\kappa_n'+\kappa_g'+\kappa_g(\kappa_g+\tau_g)+\kappa_n(\kappa_n-\tau_g)\big]T(\sigma)+\big[\kappa_g'+\tau_g'+\kappa_g(\kappa_n+\kappa_g)\\ &+\tau_g(\kappa_n-\tau_g)\big]\mathsf{g}(\sigma)+\big[\kappa_n'-\tau_g'+\kappa_n(\kappa_n+\kappa_g)-\tau_g(\kappa_g+\tau_g)\big]\mathsf{n}(\sigma)\Big],\end{aligned}$$

and

$$\mathcal{Z}'''(\vartheta)=\frac{1}{\sqrt{3}}\Big[\xi_1(\sigma)T(\sigma)+\xi_2(\sigma)\mathsf{g}(\sigma)+\xi_3(\sigma)\mathsf{n}(\sigma)\Big],$$

where

$$\begin{cases}\xi_1(\sigma)=\kappa_n''+\kappa_g''+(\kappa_n+\kappa_g)(\kappa_n^2+\kappa_g^2)+(\kappa_g+\tau_g)(\kappa_g'-\kappa_n\tau_g)+(\kappa_n-\tau_g)(\kappa_n'+\kappa_g\tau_g)\\ \qquad +2\kappa_n(\kappa_n'-\tau_g')+2\kappa_g(\kappa_g'+\tau_g'),\\ \xi_2(\sigma)=\kappa_g''+\tau_g''+(\kappa_g+\tau_g)(\kappa_g^2-\tau_g^2)+(\kappa_n-\tau_g)(\tau_g'+\kappa_n\kappa_g)+(\kappa_n+\kappa_g)(\kappa_g'+\kappa_n\tau_g)\\ \qquad +2\kappa_g(\kappa_n'+\kappa_g')+2\tau_g(\kappa_n'+\tau_g'),\\ \xi_3(\sigma)=\kappa_n''-\tau_g''+(\kappa_n-\tau_g)(\kappa_n^2-\tau_g^2)+(\kappa_n+\kappa_g)(\kappa_n'-\kappa_g\tau_g)+(\kappa_g+\tau_g)(\kappa_n\kappa_g-\tau_g')\\ \qquad +2\kappa_n(\kappa_n'+\kappa_g')-2\tau_g(\kappa_g'+\tau_g').\end{cases}$$

Then

$$\tau_{\mathcal{Z}}(\vartheta)=\sqrt{3}\left\{\frac{\begin{aligned}&\xi_1\big[(\kappa_n-\tau_g)(\kappa_n'+\kappa_g')-(\kappa_n+\kappa_g)(\kappa_n'-\tau_g')\big]+(\kappa_n+\kappa_g)(\kappa_n-\tau_g)(\xi_3\tau_g-\xi_1\kappa_g)\\&+(\kappa_g'+\tau_g')\big[\xi_3(\kappa_n+\kappa_g)-\xi_1(\kappa_n-\tau_g)\big]+(\kappa_g+\tau_g)\big[\xi_1(\kappa_n'-\tau_g')-\xi_3(\kappa_n'+\kappa_g')\big]\\&+(\kappa_g+\tau_g)\big[\kappa_g(\kappa_n-\tau_g)+\kappa_n\big(\xi_1(\kappa_n+\kappa_g)-\xi_3(\kappa_n-\tau_g)\big)\big]+(\kappa_n+\kappa_g)^2(\xi_3\kappa_g\\&-\xi_2\kappa_n)+(\kappa_n-\tau_g)^2(\xi_2\kappa_n-\xi_1\tau_g)-(\kappa_g+\tau_g)^2(\xi_1\tau_g+\xi_3\kappa_g)\end{aligned}}{\begin{aligned}&\Big[(\kappa_n+\kappa_g)(\kappa_n'-\tau_g')-(\kappa_g'+\tau_g')(\kappa_n-\tau_g)+(\kappa_n+\kappa_g)\big[\kappa_n(\kappa_n+\kappa_g)-\kappa_g(\kappa_n-\tau_g)\big]\\&-\tau_g\big[(\kappa_n+\kappa_g)^2+(\kappa_n-\tau_g)^2\big]\Big]^2-\Big[(\kappa_n-\tau_g)(\kappa_n'+\kappa_g')\\&-(\kappa_n'-\tau_g')(\kappa_n+\kappa_g)+(\kappa_n+\kappa_g)\big[\kappa_g(\kappa_n-\tau_g)+\tau_g(\kappa_n+\kappa_g)\big]\\&+\kappa_n\big[(\kappa_n-\tau_g)^2-(\kappa_n+\kappa_g)^2\big]\Big]^2+\Big[(\kappa_n+\kappa_g)(\kappa_g'+\tau_g')-(\kappa_n'+\kappa_g')(\kappa_g+\tau_g)\\&+(\kappa_n-\tau_g)\big[\tau_g(\kappa_n+\kappa_g)-\kappa_n(\kappa_g+\tau_g)\big]+\kappa_g\big[(\kappa_n+\kappa_g)^2-(\kappa_g+\tau_g)^2\big]\Big]^2\end{aligned}}\right\}$$

Example 3.1. Let the timelike surface (see Figure 1) in the Minkowski 3-space $\mathfrak{R}_1^3$ is defined as

$$\begin{aligned}&\Psi:\mathcal{V}\subset\mathfrak{R}^2\to\mathfrak{R}_1^3,\\&(\sigma,t)\to\Psi(\sigma,t)=\chi(\sigma)+t\,e(\sigma),\\&\Psi(\sigma,t)=(\sigma,t\sin\sigma,t\cos\sigma),\end{aligned}\tag{26}$$

where $t\in(-1,1)$. Then we get the moving Darboux frame $\{T,\mathsf{g},\mathsf{n}\}$ along the curve χ as follows:

$$\begin{aligned}&T(\sigma)=(1,0,0),\\&\mathsf{g}(\sigma)=\big(0,-\sin\sigma,-\cos\sigma\big),\\&\mathsf{n}(\sigma)=\frac{1}{\sqrt{1-t^2}}\Big(-t,-\cos\sigma,\sin\sigma\Big),\end{aligned}\tag{27}$$

where $\mathsf{g}(\sigma)$ and $\mathsf{n}(\sigma)$ are a unit spacelike vectors. Moreover, the geodesic curvature $\kappa_g(\sigma)$, the asymptotic curvature $\kappa_n(\sigma)$, and the principal curvature $\tau_g(\sigma)$ of the curve χ have the form

$$\begin{aligned}&\kappa_n(\sigma)=\kappa_g(\sigma)=0,\\&\tau_g(\sigma)=\frac{1}{\sqrt{1-t^2}}\,.\end{aligned}\tag{28}$$

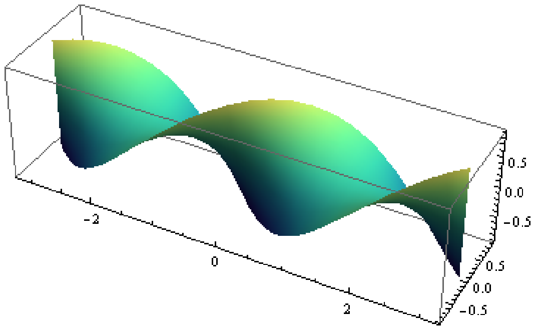

Figure 1: The timelike surface $\Psi(\sigma, t)$.

Then, the Tg-timelike Smarandache curves $\mathcal{Z}$ of the curve χ is given by (see Figure 2)

$$\mathcal{Z}\big(\vartheta(\sigma)\big) = \frac{1}{\sqrt{2}}\Big(1, -\sin\sigma, -\cos\sigma\Big). \tag{29}$$

The Tn-timelike Smarandache curves $\mathcal{Z}$ of the curve χ is given by (see Figure 3)

$$\mathcal{Z}\big(\vartheta(\sigma)\big) = \frac{1}{\sqrt{2}}\Big(1 - \frac{t}{\sqrt{1-t^2}}, -\frac{\cos\sigma}{\sqrt{1-t^2}}, \frac{\sin\sigma}{\sqrt{1-t^2}}\Big). \tag{30}$$

The gn-timelike Smarandache curves $\mathcal{Z}$ of the curve χ is given by (see Figure 4)

$$\mathcal{Z}\big(\vartheta(\sigma)\big) = \frac{1}{\sqrt{2}}\Big(- \frac{t}{\sqrt{1-t^2}}, -\frac{\cos\sigma}{\sqrt{1-t^2}} - \sin\sigma, \frac{\sin\sigma}{\sqrt{1-t^2}} - \cos\sigma\Big). \tag{31}$$

The Tgn-timelike Smarandache curves $\mathcal{Z}$ of the curve χ is given by (see Figure 5)

$$\mathcal{Z}\big(\vartheta(\sigma)\big) = \frac{1}{\sqrt{3}}\Big(1 - \frac{t}{\sqrt{1-t^2}}, -\frac{\cos\sigma}{\sqrt{1-t^2}} - \sin\sigma, \frac{\sin\sigma}{\sqrt{1-t^2}} - \cos\sigma\Big) \tag{32}$$

References

[1] C. Ashbacher, Smarandache geometries, Smarandache Notions Journal, vol. 8 (1-3) (1997) , 212–215.

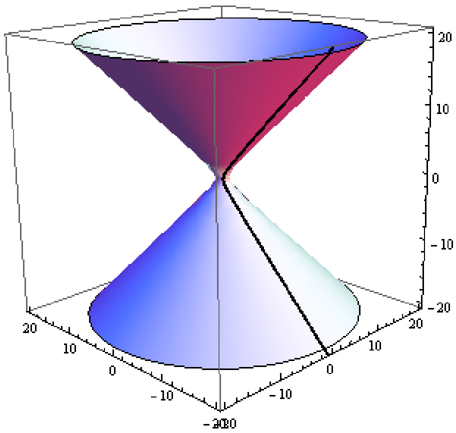

Figure 2: The $T\mathsf{g}$-timelike Smarandache curves $\mathcal{Z}$ on S_1^2.

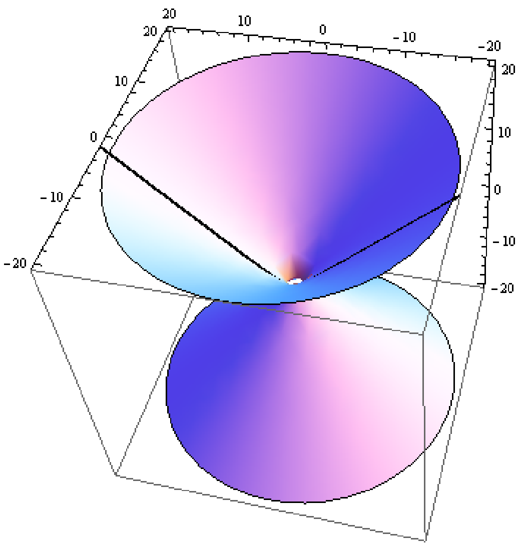

Figure 3: The $T\mathsf{n}$-timelike Smarandache curves $\mathcal{Z}$ on S_1^2.

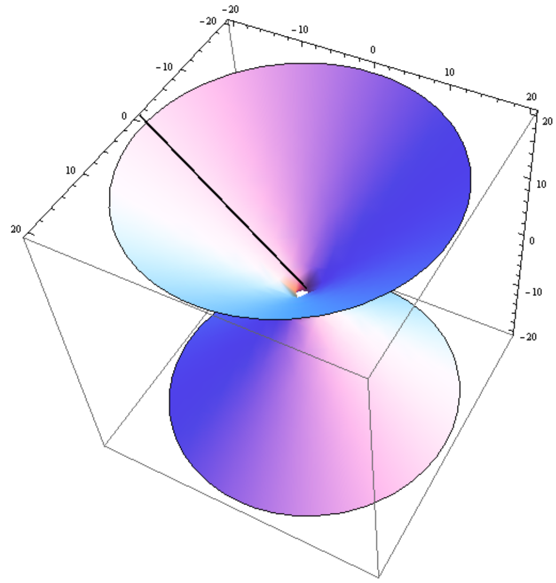

Figure 4: The gn-timelike Smarandache curves $\mathcal{Z}$ on S_1^2.

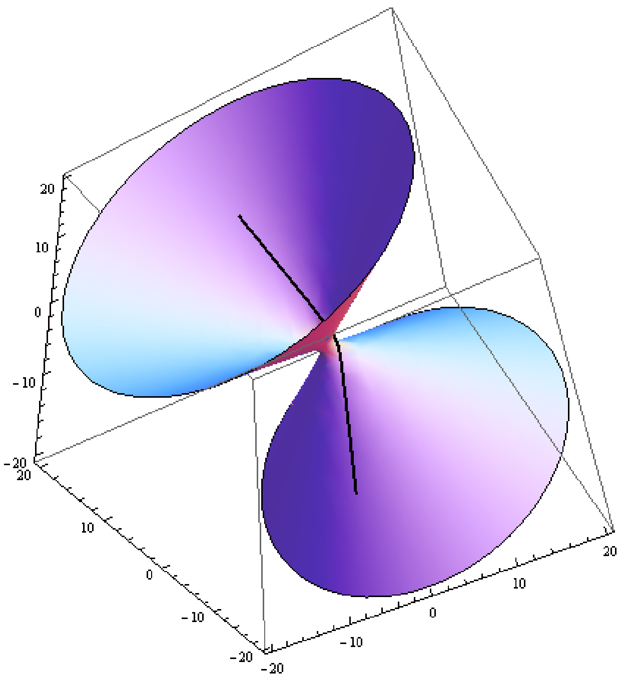

Figure 5: The Tgn-timelike Smarandache curves $\mathcal{Z}$ on S_1^2.

[2] Ö. Bektaş, S. Yce, Special Smarandache curves according to Darboux Fframe in Euclidean 3- Space, Romanian Journal of Mathematics and Computer Science, 3 (2013), 48–59.

[3] M. Çetin, Y. Tunçer, M. K. Karacan, Smarandache curves according to Bishop frame in Euclidean 3- space, General Mathematics Notes, 20 (2) (2014), 50–66

[4] M.P. Do Carmo, Differential geometry of curves and surfaces, Prentice Hall, Englewood Cliffs, NJ, 1976.

[5] R. López, Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space, International Electronic Journal of Geometry, 7 (1) (2014), 44–107.

[6] B. O'Neill, Semi-Riemannian geometry with applications to relativity, Academic press, New York, 1983.

[7] U. Ozturk and E. B. Koc Ozturk, Smarandache Curves according to Curves on a Spacelike Surface in Minkowski 3-Space R_1^3, Journal of Discrete Mathematics, Volume 2014, Article ID 829581, 10 pages, http://dx.doi.org/10.1155/2014/829581.

[8] E. M. Solouma, Investigation of special Smarandache curves according to Bishop frame in Minkowski space-time, Nonlinear Engenering-Modeling and Application (NLENG), accepted (To appear).

[9] M. Turgut, S. Yılmaz, Smarandache curves in Minkowski space-time, International Journal of Mathematical Combinatorics, 3 (2008), 51–55.

[10] K. Taşköprü and M. Tosun, Smarandache curves according to Sabban frame on, Boletim da Sociedade Paraneanse de Matematica, 32 (1) (2014), 51–59.