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A note on the near pseudo Smarandache function

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Abstract Vyawahare and Purohit [1] introduced the near pseudo Smarandache function, $K(n)$. In this paper, we derive some more recurrence formulas satisfied by $K(n)$. We also derive some new series, and give an expression for the sum of the first n terms of the sequence $\{K(n)\}$.

Keywords The near pseudo Smarandache function, recurrence formulas, the sum of the first n terms.

§1. Introduction

Vyawahare and Purohit [1] introduced a new function, called the near pseudo Smarandache function, and denoted by $K(n)$, is defined as follows.

Definition 1.1. The near pseudo Smarandache function, $K : N \longrightarrow N$, is

$$K(n) = \sum_{i=1}^{n} i + k(n),$$

where $k(n) = \min\left\{k : k \in N, n | \sum\limits_{i=1}^{n} i + k\right\}$.

The following theorem, due to Vyawahare and Purohit [1], gives explicit expressions for $k(n)$ and $K(n)$.

Theorem 1.1. For any $n \in N$,

$$k(n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ \dfrac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

with

$$K(n) = \begin{cases} \dfrac{n(n+3)}{2}, & \text{if } n \text{ is odd,} \\ \dfrac{n(n+2)}{2}, & \text{if } n \text{ is even.} \end{cases}$$

In [1], Vyawahare and Purohit give a wide range of results related to the near pseudo Smarandache function. Some of them are given in the following lemmas.

Lemma 1.1. $K(2n + 1) - K(2n) = 3n + 2$, for any integer $n \in N$.

Lemma 1.2. $K(2n+1) - K(2n-1) = 4n+3$, for any integer $n \in N$.

This paper gives some recurrence relations satisfied by the near pseudo Smarandache function $K(n)$. These are given in Section 2. In Section 3, we give some series involving the functions $K(n)$ and $k(n)$. We also give an explicit expression for the sum of the first n terms of the sequences $\{K(n)\}_{n=1}^{\infty}$ and$\{k(n)\}_{n=1}^{\infty}$. We conclude this paper with some remarks in the final Section 4.

§2. More recurrence relations

In this section, we derive some new recurrence relations that are satisfied by the near pseudo Smarandache function $K(n)$.

Lemma 2.1. For any integer $n \in N$,

$$K(2n) - K(2n-1) = n+1$$

Proof. Writing $K(2n) - K(2n-1)$ in the following form, and then using Lemma 1.2 and Lemma 1.1 (in this order), we get

$$\begin{aligned} K(2n) - K(2n-1) &= [K(2n+1) - K(2n-1)] - [K(2n+1) - K(2n)] \\ &= (4n+3) - (3n+2), \end{aligned}$$

which now gives the desired result.

We now have the following result.

Corollary 2.1. $K(n)$ is strictly increasing in n.

Proof. From Theorem 1.1, we see that both the subsequences $\{K(2n-1)\}_{n=1}^{\infty}$ and $\{K(2n)\}_{n=1}^{\infty}$ are strictly increasing. This, together with Lemma 1.1 and Lemma 2.1, shows that$\{K(n)\}_{n=1}^{\infty}$ is strictly increasing.

Lemma 2.2. For any integer $n \in N$,

$$K(2n+2) - K(2n) = 4(n+1).$$

Proof. Using Theorem 1.1, we get

$$K(2n+2) - K(2n) = \frac{(2n+2)(2n+4)}{2} - \frac{(2n)(2n+2)}{2} = 4(n+1),$$

after some algebraic simplifications.

Lemma 1.2 shows that the subsequence $\{K(2n-1)\}_{n=1}^{\infty}$ is strictly convex in the sense that

$$K(2n+3) - K(2n+1) = 4n+7 > 4n+3 = K(2n+1) - K(2n-1).$$

From Lemma 2.2, we see that the subsequence $\{K(2n)\}_{n=1}^{\infty}$ is also strictly convex, since

$$K(2n+4) - K(2n+2) = 4(n+2) > 4(n+1) = K(2n+2) - K(2n).$$

However, the sequence $\{K(n)\}_{n=1}^{\infty}$ is not convex, since

$$K(2n+2) - K(2n+1) = n+2 < 3n+2 = K(2n+1) - K(2n).$$

Note that

$$K(2n+1) - K(2n) = 3n+2 > n+1 = K(2n) - K(2n-1).$$

Corollary 2.2. $K(n+1) = K(n)$ has no solution.

Proof. If n is odd, say, $n = 2m-1$ for some integer $m > 1$, then from Lemma 2.1,

$$K(2m) - K(2m-1) = m+1 > 0,$$

and if n is even, say, $n = 2m$ for some integer $m > 1$, then from Lemma 1.1,

$$K(2m+1) - K(2m) = 3m+2 > 0.$$

These two inequalities establish the result.

Corollary 2.3. $K(n+2) = K(n)$ has no solution.

Proof. If n is odd, say, $n = 2m-1$ for some integer $m > 1$, then from Lemma 1.2,

$$K(2m+1) - K(2m-1) = 4m+3 > 0,$$

and if n is even, say, $n = 2m$ for some integer $m > 1$, then from Lemma 2.2,

$$K(2m+2) - K(2m) = 4(m+1) > 0.$$

Thus, the result is established.

Lemma 2.3. For any integers $m, n \in N$ with $m > n$,

(1) $K(2m-1) - K(2n-1) = (m-n)(2m+2n+1)$,

(2) $K(2m) - K(2n) = 2(m-n)(m+n+1)$.

Proof. For any integers m and n with $m \geq n \geq 1$, from Theorem 1.1,

(1)$K(2m-1) - K(2n-1) = (2m-1)(m+1) - (2n-1)(n+2)$

$= 2(m^2 - n^2) + (m-n)$

$= (m-n)(2m+2n+1)$,

(2) $K(2m) - K(2n) = 2m(m+1) - 2n(n+2)$

$= 2(m^2 - n^2) + 2(m-n)$

$= 2(m-n)(m+n+1)$,

which we intended to prove.

Corollary 2.4. For any integers $m, n \in N$ with $m > n$,

(1) $K(2m-1) - K(2n-1) = \dfrac{m-n}{m+n+2} K(2m+2n+1)$,

(2) $K(2m) - K(2n) = \dfrac{m-n}{m+n} K(2m+2n)$.

Proof. Let n and m be any two integers with $n \geq m \geq 1$.

(1) Since

$$K(2m+2n+1) = (m+n+2)(2m+2n+1),$$

we get the result by virtue of part (1) of Lemma 2.3 above.

(2) Note that

$$K(2m+2n)=2(m+n)(m+n+1).$$

This, together with part (2) of Lemma 2.3, gives the desired result.

It may be mentioned here that, Lemma 1.2 is a particular case of part (1) of Corollary 2.4(when $m=n+1$) and Lemma 2.2 is a particular case of part (2) of Corollary 2.4 (when $m=n+1$).

§3. Series involving the functions $K(n)$ and $k(n)$

In this section, we derive some results in series involving $K(n)$ and $k(n)$. We also give explicit expressions of the $n-th$ partial sums in both the cases.

Lemma 3.1. For any integer $n \in N$,

(1) $\sum\limits_{m=1}^{n}[K(2m)-K(2m-1)]=\frac{n(n+3)}{2}$,

(2) $\sum\limits_{m=1}^{n}[K(2m+1)-K(2m)]=3\frac{n(n+3)}{2}-n=\frac{n(3n+7)}{2}$.

Proof. (1) From Lemma 2.1,

$K(2m)-K(2m-1)=m+1$ for any integer $m \geq 1$.

Now, summing over m from 1 to n, we get

$$\sum_{m=1}^{n}[K(2m)-K(2m-1)]=\sum_{m=1}^{n}(m+1)=\frac{n(n+1)}{2}+n=\frac{n(n+3)}{2},$$

which is the result desired.

(2) From Lemma 1.1,

$K(2m+1)-K(2m)+1=3(m+1)$ for any integer $m \geq 1$.

Therefore, summing over m from 1 to n, we get

$$\sum_{m=1}^{n}[K(2m+1)-K(2m)+1]=3\sum_{m=1}^{n}(m+1)=3\left\{\frac{n(n+1)}{2}+n\right\}=3\frac{n(n+3)}{2},$$

that is,

$$\sum_{m=1}^{n}[K(2m+1)-K(2m)]+n=3\frac{n(n+3)}{2}$$

from which the desired result follows immediately.

Corollary 3.1. If n is an odd integer, then

(1) $\sum\limits_{m=1}^{n}[K(2m)-K(2m-1)]=K(n)$,

(2) $\sum\limits_{m=1}^{n}[K(2m+1)-K(2m)]=3K(n)-n$.

Proof. Both the results follow immediately by virtue of Theorem 1.1 and Lemma 3.1.

Let $\{S_n\}$ be the sequence of $n-th$ partial sums of the sequence $\{K(n)\}_{n=1}^{\infty}$, so that

$$S_n=\sum_{m=1}^{n}K(m), n \geqslant 1$$

and likewise, let $\{s_n\}$ be the sequence of $n-th$ partial sums of $\{k(n)\}_{n=1}^{\infty}$.
Then, we have the following result.

Lemma 3.2. For any integer $n \geq 1$,

(1) $S_{2n} = \frac{n}{6}(8n^2 + 21n + 7)$,

(2) $S_{2n+1} = \frac{1}{6}(8n^3 + 33n^2 + 37n + 12)$.

Proof. From Theorem 1.1, for any integer $m \geq 1$,

$$K(2m-1) + K(2m) = (2m-1)(m+1) + 2m(m+1) = 4m^2 + 3m - 1.$$

(1) Since S_{2n} can be written as

$$S_{2n} = K(1) + K(2) + \cdots + K(2n) = \sum_{m=1}^{n} [K(2m-1) + K(2m)],$$

we get,

$$\begin{aligned} S_{2n} &= \sum_{m=1}^{n} (4m^2 + 3m - 1) = 4\sum_{m=1}^{n} m^2 + 3\sum_{m=1}^{n} m - n \\ &= 4\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 3\left\{\frac{n(n+1)}{2}\right\} - n \end{aligned}$$

which now gives the desired result after some algebraic simplifications.

(2) Since $S_{2n+1} = S_{2n} + K(2n+1)$
from part (1) above, together with Theorem 1.1, we get

$$S_{2n+1} = \frac{n}{6}(8n^2 + 21n + 7) + (2n+1)(2n+3),$$

which gives the desired expression for S_{2n+1} after algebraic manipulations.

From Definition 1.1, we see that

$$k(2n-1) = 2n - 1 = k(2(2n-1)) \text{ for any integer } n \geq 1.$$

It then follows that the $n-th$ term of the subsequence $\{k(2n-1)\}_{n=1}^{\infty}$ is $2n-1$, while the $n-th$ term of the subsequence $\{k(2n)\}_{n=1}^{\infty}$ is n.

Lemma 3.3. For any integer $n \geq 1$,

(1) $s_{2n} = \frac{n}{2}(3n+1)$,

(2) $s_{2n+1} = \frac{1}{2}(3n^2 + 5n + 2)$.

Proof. We first note that, for any integer $m \geq 1$,

$$k(2m-1) + k(2m) = 2m - 1 + m = 3m - 1.$$

(1) We get the result from the following expression for s_{2n} :

$$s_{2n} = k(1) + k(2) + \cdots + k(2n) = \sum_{m=1}^{n} [k(2m-1) + k(2m)] = \sum_{m=1}^{n} (3m-1).$$

(2) Using part (1) above, we get

$$s_{2n+1} = s_{2n} + k(2n+1) = \frac{n}{2}(3n+1) + (2n+1),$$

which gives the result after some algebraic simplifications.

Lemma 3.4. For any integer $n \geq 0$,

$$\sum_{m=0}^{n} K(a^m) = \begin{cases} \frac{1}{2}\,\frac{a^{n+1}-1}{a^2-1}(a^{n+1}+3a+4), & \text{if } a \text{ is odd} \\ \frac{1}{2}\left[\frac{a^{n+1}-1}{a^2-1}(a^{n+1}+2a+3)+1\right], & \text{if } a \text{ is even} \end{cases}$$

Proof. We consider the two cases separately.

(1) When a is odd.

In this case,

$$\sum_{m=0}^{n} K(a^m) = \sum_{m=0}^{n} \frac{a^m(a^m+3)}{2} = \frac{1}{2}\left(\sum_{m=0}^{n} a^{2m} + 3\sum_{m=0}^{n} a^m\right).$$

Now, the first series on the right is a geometric series with common ratio a^2, while the second one is geometric with common ratio a. Therefore,

$$\sum_{m=0}^{n} K(a^m) = \frac{1}{2}\left(\frac{a^{2(n+1)}-1}{a^2-1} + 3\frac{a^{n+1}-1}{a-1}\right),$$

which gives the desired result after some algebraic simplifications.

(2) When a is even.

In this case,

$$\sum_{m=0}^{n} K(a^m) = 2 + \sum_{m=1}^{n} \frac{a^m(a^m+2)}{2} = \frac{1}{2}\left(1+\sum_{m=0}^{n} a^{2m}\right) + \sum_{m=0}^{n} a^m$$
$$= \frac{1}{2}\left(1+\frac{a^{2(n+1)}-1}{a^2-1}\right) + \frac{a^{n+1}-1}{a-1}.$$

Now, simplifying the above, we get the result desired.

It can be shown that (see Yongfeng Zhang [2]) the series $\sum\limits_{n=1}^{\infty} \frac{1}{[K(n)]^s}$ is convergent for any real number $s > \frac{1}{2}$, and $\sum\limits_{n=1}^{\infty} \frac{1}{[K(n)]^s}$ is convergent for any real number $s > 1$ (Yu Wang [3]) with

$$\sum_{n=1}^{n} \frac{1}{[k(n)]^s} = \zeta(s)(2-\frac{1}{2^s}),$$

where $\zeta(s)$ is the Riemann zeta function.

§4. Some remarks

Since

$$\frac{n(n+3)}{2} > n,\ \frac{n(n+2)}{2} > n \text{ for any integer } n \geqslant 1,$$

it follows that $K(n) > n$ for any integer $n \geq 1$. A consequence of this is that, the equation $K(n) = n$ has no solution.

Let T_m be the $m-th$ triangular number, that is

$$T_m = \frac{m(m+1)}{2}, m \geq 1.$$

Then, T_m satisfies the following recurrence relation :

$$T_{m+1} = T_m + m + 1, m \geq 1.$$

Now, by Definition 1.1,

$$K(m) - k(m) = T_m, m \geq 1,$$

so that

$$K(m) - [k(m) - m] + 1 = T_m + (m+1) = T_{m+1}.$$

Now, if m is odd, say, $m = 2n - 1$ (for some integer $n \geq 1$), then $k(m) - m = 0$, so that

$$K(2n-1) + 1 = T_{2n}$$

is a triangular number.

Again, since

$$K(m) - \left[k(m) - \frac{m}{2}\right] + \frac{m}{2} + 1 = T_{m+1},$$

it follows that, when m is even, say, $m = 2n$ (for some integer $n \geq 1$), then

$$K(2n-1) + n + 1 = T_{2n+1}$$

is a triangular number.

In a recent paper, [2] introduced a new function, which may be called the near pseudo Smarandache function of order t, where $t \geq 1$ is a fixed integer, and is defined as

$$K_t(n) = \sum_{i=1}^{n} i^t + k_t(n), \text{for any } n \in N$$

where

$$k_t(n) = \min\left\{k : k \in N, n | \sum_{i=1}^{n} i^t + k\right\}.$$

Then, the function introduced by Vyawahare and Purohit [1] is the near pseudo Smarandache function of order 1, that is,

$$K(n) = K_1(n) \quad \text{with } k(n) = k_1(n), n \in N.$$

In [3], Yu Wang has given the explicit expressions for $k_2(n)$ and $k_3(n)$, including the convergence of two infinite series involving these two functions. But the properties of $K_t(n)$ and $k_t(n)$ still remain to be investigated. The following two lemmas give the expressions for $k_2(n)$ and $k_3(n)$, due to Yu Wang [3].

Lemma 4.1. For any integer $n \geq 1$,

$$k_2(n) = \begin{cases} \dfrac{5n}{6}, & \text{if } n = 6m \\ \dfrac{n}{3}, & \text{if } n = 3(2m+1) \\ n, & \text{if } n = 6m+1 \text{ or } n = 6m+5 \\ \dfrac{n}{2}, & \text{if } n = 2(3m+1) \text{ or } n = 2(3m+2) \end{cases}$$

Lemma 4.2. For any integer $n \geq 1$,

$$k_3(n) = \begin{cases} \dfrac{n}{2}, \text{if } n = 2(2m+1) \\ n, \text{ otherwise} \end{cases}$$

Using the above two lemmas, we get the expressions for $K_2(n)$ and $K_3(n)$, given below.

Lemma 4.3. For any integer $n \geq 1$,

$$K_2(n) = \begin{cases} \dfrac{n}{6}(n^2+3n+6), & \text{if } n = 6m \\ \dfrac{n}{6}(n^2+3n+3), & \text{if } n = 3(2m+1) \\ \dfrac{n}{6}(n^2+3n+7), & \text{if } n = 6m+1 \text{ or } n = 6m+5 \\ \dfrac{n}{6}(n^2+3n+4), & \text{if } n = 2(3m+1) \text{ or } n = 2(3m+2) \end{cases}$$

Lemma 4.4. For any integer $n \geq 1$,

$$K_3(n) = \begin{cases} \dfrac{1}{4}n(n+2)(n^2+1), & \text{if } n = 2(2m+1) \\ \dfrac{1}{4}n(n^3+2n^2+n+4), & \text{otherwise} \end{cases}$$

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