**Scientia Magna**
**Vol. 3 (2007), No. 1, 98-101**

# On the near pseudo Smarandache function

Yongfeng Zhang

Department of Mathematics, Xianyang Normal College,
Xianyang, Shaanxi, P.R.China

Received June 13, 2006

**Abstract** For any positive integer $n$, the near pseudo Smarandache function $K(n)$ is defined as $K(n) = m = \frac{n(n+1)}{2} + k$, where $k$ is the smallest positive integer such that $n$ divides $m$. The main purpose of this paper is using the elementary method to study the calculating problem of an infinite series involving the near pseudo Smarandache function $K(n)$, and give an exact calculating formula.

**Keywords** Near pseudo Smarandache function, infinite series, exact calculating formula.

## §1. Introduction and results

For any positive integer $n$, the near pseudo Smarandache function $K(n)$ is defined as follows:

$$K(n) = m,$$

where $m = \frac{n(n+1)}{2} + k$, and $k$ is the smallest positive integer such that $n$ divides $m$.

The first few values of $K(n)$ are $K(1) = 2$, $K(2) = 4$, $K(3) = 9$, $K(4) = 12$, $K(5) = 20$, $K(6) = 24$, $K(7) = 35$, $K(8) = 40$, $K(9) = 54$, $K(10) = 50$, $K(11) = 77$, $K(12) = 84$, $K(13) = 104$, $K(14) = 112$, $K(15) = 135$, $\cdots$. This function was introduced by A.W.Vyawahare and K.M.Purohit in [1], where they studied the elementary properties of $K(n)$, and obtained a series interesting results. For example, they proved that 2 and 3 are the only solutions of $K(n) = n^2$; If $a$, $b > 5$, then $K(a \cdot b) > K(a) \cdot K(b)$; If $a > 5$, then for all positive integer $n$, $K(a^n) > n \cdot K(a)$; The Fibonacci numbers and the Lucas numbers do not exist in the sequence $\{K(n)\}$; Let $C$ be the continued fraction of the sequence $\{K(n)\}$, then $C$ is convergent and $2 < C < 3$; $K(2^n - 1) + 1$ is a triangular number; The series $\sum\limits_{n=1}^{\infty} \frac{1}{K(n)}$ is convergent. The other contents related to the near pseudo Smarandache function can also be found in references [2], [3] and [4].

In this paper, we use the elementary method to study the calculating problem of the series

$$\sum_{n=1}^{\infty} \frac{1}{K^s(n)}, \tag{1}$$

and give an exact calculating formula for (1). That is, we shall prove the following conclusion:

**Theorem.** For any real number $s > \frac{1}{2}$, the series (1) is convergent, and

(a) $$\sum_{n=1}^{\infty}\frac{1}{K(n)}=\frac{2}{3}\ln 2+\frac{5}{6};$$

(b) $$\sum_{n=1}^{\infty}\frac{1}{K^2(n)}=\frac{11}{108}\cdot\pi^2-\frac{22+2\ln 2}{27}.$$

In fact for any positive integer $s$, using our method we can give an exact calculating formula for (1), but the calculation is very complicate if $s$ is large enough.

## §2. Proof of the theorem

In this section, we shall prove our theorem directly. In fact for any positive integer $n$, it is easily to deduce that $K(n)=\frac{n(n+3)}{2}$ if $n$ is odd and $K(n)=\frac{n(n+2)}{2}$ if $n$ is even. So from this properties we may immediately get

$$\frac{n^2}{2}<K(n)<\frac{(n+3)^2}{2},$$

or

$$\frac{1}{(n+3)^{2s}}\ll\frac{1}{K^s(n)}\ll\frac{1}{n^{2s}}.$$

So the series (1) is convergent if $s>\frac{1}{2}$.

Now from the properties of $K(n)$ we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{K(n)} &= \sum_{n=1}^{\infty}\frac{1}{K(2n-1)}+\sum_{n=1}^{\infty}\frac{1}{K(2n)}\\ &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)(n+1)}+\sum_{n=1}^{\infty}\frac{1}{2n(n+1)}\\ &= \frac{2}{3}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n+2}\right)+\frac{1}{2}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \frac{2}{3}\cdot\lim_{N\to\infty}\left(\sum_{n\le N}\frac{1}{2n-1}-\sum_{n\le N}\frac{1}{2n+2}\right)+\frac{1}{2}\\ &= \frac{2}{3}\cdot\lim_{N\to\infty}\left(\sum_{n\le 2N}\frac{1}{n}-\frac{1}{2N+2}+\frac{1}{2}-\sum_{n\le N}\frac{1}{n}\right)+\frac{1}{2}. \qquad (2)\end{aligned}$$

Note that for any $N>1$, we have the asymptotic formula (See Theorem 3.2 of [5])

$$\sum_{n\le N}\frac{1}{n}=\ln N+\gamma+O\left(\frac{1}{N}\right), \qquad (3)$$

where $\gamma$ is the Euler constant.

Combining (2) and (3) we may immediately obtain

$$\sum_{n=1}^{\infty}\frac{1}{K(n)}=\frac{2}{3}\cdot\lim_{N\to\infty}\left[\ln(2N)+\gamma+\frac{1}{2}-\ln N-\gamma+O\left(\frac{1}{N}\right)\right]+\frac{1}{2}=\frac{2}{3}\ln 2+\frac{5}{6}.$$

This completes the proof of (a) in Theorem.

Now we prove (b) in Theorem. From the definition and properties of $K(n)$ we also have

$$\sum_{n=1}^{\infty}\frac{1}{K^2(n)} = \sum_{n=1}^{\infty}\frac{1}{K^2(2n-1)}+\sum_{n=1}^{\infty}\frac{1}{K^2(2n)} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2(n+1)^2}+\sum_{n=1}^{\infty}\frac{1}{(2n)^2(n+1)^2}. \quad (4)$$

Note that the identities

$$\frac{1}{(2n-1)^2(n+1)^2}=\frac{2}{27}\left(\frac{1}{2n+2}-\frac{1}{2n-1}\right)+\frac{1}{9}\frac{1}{(2n-1)^2}+\frac{1}{9}\frac{1}{(2n+2)^2}, \quad (5)$$

$$\frac{1}{n^2(n+1)^2}=2\left(\frac{1}{n+1}-\frac{1}{n}\right)+\frac{1}{n^2}+\frac{1}{(n+1)^2}, \quad (6)$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}=\frac{\pi^2}{6}-1. \quad (7)$$

From (3), (4), (5), (6) and (7) we may deduce that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{K^2(n)} &= \frac{2}{27}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{2n+2}-\frac{1}{2n-1}\right)+\frac{1}{9}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^2}+\frac{1}{(2n+2)^2}\right)\\ &+\frac{1}{2}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n}\right)+\frac{1}{4}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}\right)\\ &= \frac{2}{27}\cdot\lim_{N\to\infty}\left[\sum_{n\le N}\frac{1}{2n+2}-\sum_{n\le N}\frac{1}{2n-1}\right]+\frac{\pi^2}{72}+\frac{\pi^2}{216}-\frac{1}{36}\\ &+\frac{1}{2}\cdot\lim_{N\to\infty}\left[\sum_{n\le N}\frac{1}{n+1}-\sum_{n\le N}\frac{1}{n}\right]+\frac{\pi^2}{24}+\frac{\pi^2}{24}-\frac{1}{4}\\ &= \frac{2}{27}\cdot\lim_{N\to\infty}\left[-\frac{1}{2}+\ln N-\ln(2N)+O\left(\frac{1}{N}\right)\right]+\frac{\pi^2}{54}-\frac{1}{36}\\ &-\frac{1}{2}+\frac{\pi^2}{12}-\frac{1}{4}\\ &= \frac{11}{108}\cdot\pi^2-\frac{22+2\ln 2}{27}.\end{aligned}$$

This completes the proof of (b) in Theorem.

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