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On the Pseudo-Smarandache-Squarefree function and Smarandache function

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Abstract For any positive integer n, Pseudo-Smarandache- Squarefree function $Z_w(n)$ is defined as $Z_w(n) = \min\{m : n|m^n, m \in N\}$. Smarandache function $S(n)$ is defined as $S(n) = \min\{m : n|m!, m \in N\}$. The main purpose of this paper is using the elementary methods to study the mean value properties of the Pseudo-Smarandache-Squarefree function and Smarandache function, and give two sharper asymptotic formulas for it.

Keywords Pseudo-Smarandache-Squarefree function$Z_w(n)$, Smarandache function $S(n)$, me an value, asymptotic formula.

§1. Introduction and result

For any positive integer n, the famous Smarandache function $S(n)$ is defined as $S(n) = \min\{m : n|m!, m \in N\}$, Pseudo-Smarandache-Squarefree function $Z_w(n)$ is defined as the smallest positive integer m such that $n \mid m^n$. That is,

$$Z_w(n) = \min\{m : n|m^n, m \in N\}.$$

For example $Z_w(1) = 1$, $Z_w(2) = 2$, $Z_w(3) = 3$, $Z_w(4) = 2$, $Z_w(5) = 5$, $Z_w(6) = 6$, $Z_w(7) = 7$, $Z_w(8) = 2$, $Z_w(9) = 3$, $Z_w(10) = 10$, $\cdots$. About the elementary properties of $Z_w(n)$, some authors had studied it, and obtained some interesting results. For example, Felice Russo [1] obtained some elementary properties of $Z_w(n)$ as follows:

Property 1. For any positive integer $k > 1$ and prime p, we have $Z_w(p^k) = p$.

Property 2. For any positive integer n, we have $Z_w(n) \leq n$.

Property 3. The function $Z_w(n)$ is multiplicative. That is, if $GCD(m, n) = 1$, then $Z_w(m \cdot n) = Z_w(m) \cdot Z_w(n)$.

The main purpose of this paper is using the elementary methods to study the mean value properties of $Z_w(S(n))$ and $S(n) \cdot Z_w(n)$, and give two sharper asymptotic formulas for it. That is, we shall prove the following conclusions:

Theorem 1. Let $k \geq 2$ be any fixed positive integer. Then for any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n\leq x} Z_w(S(n)) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\Big(\frac{x^2}{\ln^{k+1} x}\Big),$$

where c_i $(i = 2, 3\cdots k)$ are computable constants.

Theorem 2. Let $k \geq 2$ be any fixed positive integer. Then for any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n\leq x} Z_w(n)\cdot S(n) = \frac{\zeta(2)\cdot\zeta(3)}{3\zeta(4)}\prod_p\Big(1-\frac{1}{p+p^3}\Big)\cdot\frac{x^3}{\ln x} + \sum_{i=2}^{k}\frac{e_i\cdot x^3}{\ln^i x} + O\Big(\frac{x^3}{\ln^{k+1} x}\Big),$$

where $\zeta(n)$ is the Riemann zeta-function, $\prod_p$ denotes the product over all primes, e_i $(i = 2, 3\cdots k)$ are computable constants.

§2. A simple lemma

To complete the proof of the theorem, we need the following :

Lemma. For any real number $x \geq 2$ and $s \geq 2$, we have the asymptotic formula

$$\sum_{n\leq\sqrt{x}}\frac{Z_w(n)}{n^s} = \frac{\zeta(s)\cdot\zeta(s-1)}{\zeta(2(s-1))}\cdot\prod_p\Big(1-\frac{1}{p+p^s}\Big) + O\Big(x^{1-\frac{s}{2}}\Big).$$

Proof. Note that Property 1 and 3, by the Euler product formula (See Theorem 11.7 of [2]), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{Z_w(n)}{n^s} &= \prod_p\Big(1+\frac{Z_w(p)}{p^s}+\frac{Z_w(p^2)}{p^{2s}}+\cdots\Big)\\ &= \prod_p\Big(1+\frac{p}{p^s}+\frac{p}{p^{2s}}+\cdots\Big)\\ &= \prod_p\Big(1+\frac{1}{p^{s-1}}\cdot\frac{1}{1-p^{-s}}\Big)\\ &= \frac{\zeta(s)\zeta(s-1)}{\zeta(2(s-1))}\prod_p\Big(1-\frac{1}{p+p^s}\Big).\end{aligned}\tag{1}$$

From (1) we have

$$\sum_{n\leq\sqrt{x}}\frac{Z_w(n)}{n^s} = \sum_{n=1}^{\infty}\frac{Z_w(n)}{n^s} - \sum_{n>\sqrt{x}}\frac{Z_w(n)}{n^s} = \frac{\zeta(s)\zeta(s-1)}{\zeta(2(s-1))}\prod_p\Big(1-\frac{1}{p+p^s}\Big) + O\Big(x^{1-\frac{s}{2}}\Big).$$

Specially, if $s = 3$, then we have the asymptotic formula

$$\sum_{n\leq\sqrt{x}}\frac{Z_w(n)}{n^s} = \frac{\zeta(3)\zeta(2)}{\zeta(4)}\prod_p\Big(1-\frac{1}{p+p^3}\Big) + O\Big(\frac{1}{\sqrt{x}}\Big).$$

This completes the proof of Lemma.

§3. Proof of the theorems

In this section, we shall use the elementary methods to complete the proof of the theorems.

First we prove Theorem 1. We separate all integer n in the interval $[1, x]$ into two subsets A and B as follows:

A: $p \mid n$ and $p > \sqrt{n}$, where p is a prime. B: other positive integer n such that $n \in [1, x] \setminus A$.

From the definition of the subsets A and B we have

$$\sum_{n \le x} Z_w(S(n)) = \sum_{n \in A} Z_w(S(n)) + \sum_{n \in B} Z_w(S(n)). \tag{2}$$

From Property 1 and the definition of the function $S(n)$ and the subset A we know that if $n \in A$, then we have

$$\sum_{n \in A} Z_w(S(n)) = \sum_{\substack{pn \le x \\ p > n}} Z_w(S(pn)) = \sum_{\substack{pn \le x \\ p > n}} Z_w(p) = \sum_{\substack{pn \le x \\ p > n}} p = \sum_{n \le \sqrt{x}} \sum_{n < p \le \frac{x}{n}} p. \tag{3}$$

By the Abel's summation formula (see Theorem 4.2 of [2]) and the Prime Theorem (see Theorem 3.2 of [3]):

$$\pi(x) = \sum_{p \le x} 1 = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\Big(\frac{x}{\ln^{k+1} x}\Big),$$

where a_i $(i = 1, 2, 3 \cdots k)$ are computable constants and $a_1 = 1$, we have

$$\begin{aligned} \sum_{n < p \le \frac{x}{n}} p &= \frac{x}{n} \pi\Big(\frac{x}{n}\Big) - n\pi(n) - \int_n^{\frac{x}{n}} \pi(t) \mathrm{d}t \\ &= \frac{x^2}{2n^2 \cdot \ln x} + \sum_{i=2}^{k} \frac{b_i \cdot x^2 \cdot \ln^i n}{n^2 \cdot \ln^i x} + O\Big(\frac{x^2}{n^2 \cdot \ln^{k+1} x}\Big), \end{aligned} \tag{4}$$

where b_i $(i = 2, 3 \cdots k)$ are computable constants.

Note that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi 2}{6}$ and $\sum\limits_{n=1}^{\infty} \frac{\ln^i n}{n^2}$ is convergent for all $i = 1, 2, \cdots, k$. Combining (3) and (4) we have

$$\begin{aligned} \sum_{n \in A} Z_w(S(n)) &= \sum_{n \le \sqrt{x}} \left(\frac{x^2}{2n^2 \cdot \ln x} + \sum_{i=2}^{k} \frac{b_i \cdot x^2 \cdot \ln^i n}{n^2 \cdot \ln^i x} + O\left(\frac{x^2}{n^2 \ln^{k+1} x} \right) \right) \\ &= \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\Big(\frac{x^2}{\ln^{k+1} x}\Big), \end{aligned} \tag{5}$$

where c_i $(i = 2, 3 \cdots k)$ are computable constants.

Now we estimate the error terms in set B. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of n into prime powers. If $n \in B$, then we have

$$S(n) = \max_{1 \le i \le s} \Big(S(p_i^{\alpha_i})\Big) \le \max_{1 \le i \le s} (\alpha_i p_i) \le \sqrt{n} \ln n \ll n^{\frac{5}{6}}. \tag{6}$$

From (6) and Property 2 we have

$$\sum_{n\in B} Z_w(S(n)) \ll \sum_{n\leq x} n^{\frac{5}{6}} \ll x^{\frac{11}{6}}. \tag{7}$$

Combining (2), (5) and (7) we have

$$\sum_{n\leq x} Z_w(S(n)) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\Big(\frac{x^2}{\ln^{k+1} x}\Big).$$

This proves Theorem 1.

Now we prove Theorem 2. From Property 1 and Property 3 we have

$$\sum_{n\in A} S(n)\cdot Z_w(n) = \sum_{\substack{pn\leq x\\ n<p}} S(pn)\cdot Z_w(pn) = \sum_{n\leq\sqrt{x}}\sum_{n<p\leq\frac{x}{n}} p^2\cdot Z_w(n). \tag{8}$$

$$\begin{aligned}\sum_{n<p\leq\frac{x}{n}} p^2 &= \frac{x^2}{n^2}\pi\Big(\frac{x}{n}\Big) - n^2\pi(n) - 2\int_n^{\frac{x}{n}} t\cdot\pi(t)\mathrm{d}t\\ &= \frac{x^3}{3n^3\cdot\ln x} + \sum_{i=2}^{k}\frac{d_i\cdot x^3\cdot\ln^i n}{n^3\cdot\ln^i x} + O\Big(\frac{x^3}{n^3\cdot\ln^{k+1} x}\Big),\end{aligned} \tag{9}$$

where d_i $(i=2,3\cdots k)$ are computable constants.

Note that the lemma and $\sum_{n=1}^{\infty}\frac{\ln^i n\cdot Z_w(n)}{n^3}$ is convergent for all $i=1,2,\cdots,k$. Combining (8) and (9) we have

$$\begin{aligned}\sum_{n\in A} S(n)\cdot Z_w(n) &= \sum_{n\leq\sqrt{x}}\left(\frac{x^3}{3n^3\cdot\ln x} + \sum_{i=2}^{k}\frac{d_i\cdot x^3\cdot\ln^i n}{n^3\cdot\ln^i x} + O\left(\frac{x^3}{n^3\cdot\ln^{k+1} x}\right)\right)\cdot Z_w(n)\\ &= \frac{\zeta(2)\zeta(3)}{3\zeta(4)}\prod_p\Big(1-\frac{1}{p+p^3}\Big)\cdot\frac{x^3}{\ln x} + \sum_{i=2}^{k}\frac{e_i\cdot x^3}{\ln^i x} + O\Big(\frac{x^3}{\ln^{k+1} x}\Big),\end{aligned} \tag{10}$$

where e_i $(i=2,3\cdots k)$ are computable constants.

If $n\in B$, then we have

$$\sum_{n\in B} S(n)\cdot Z_w(n) \ll \sum_{n\leq x}\sqrt{n}\ln n\cdot n \ll x^{\frac{5}{2}}\ln x. \tag{11}$$

Combining (10) and (11) we have

$$\sum_{n\leq x} Z_w(n)\cdot S(n) = \frac{\zeta(2)\cdot\zeta(3)}{3\zeta(4)}\prod_p\Big(1-\frac{1}{p+p^3}\Big)\cdot\frac{x^3}{\ln x} + \sum_{i=2}^{k}\frac{e_i\cdot x^3}{\ln^i x} + O\Big(\frac{x^3}{\ln^{k+1} x}\Big).$$

This proves Theorem 2.

References

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