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On the Pseudo-Smarandache Function and Iteration Problems

Part II: The Sum of Divisors Function

Henry Ibstedt

Abstract: This study is an extension of work done by Charles Ashbacher. Iteration results have been re-defined in terms of invariants and loops. Further empirical studies and analysis of results have helped throw light on a few intriguing questions.

I. Summary of a study by Charles Ashbacher [1]

The following definition forms the basis of Ashbacher's study: For $n>1$, the Z-sigma sequence is the alternating iteration of the sigma, sum of divisors, function followed by the Pseudo-Smarandache function.

The Z-sigma sequence originated by n creates a cycle. Ashbacher identified four 2 cycles and one 12 cycle. These are listed in table 1.

Table 1. Iteration cycles C_1 - C_5.

n	C_k	Cycle
2	C_1	$3 \leftrightarrow 2$
$3 \le n \le 15$	C_2	$24 \leftrightarrow 15$
n=16	C_3	$31 \rightarrow 32 \rightarrow 63 \rightarrow 104 \rightarrow 64 \rightarrow 127 \rightarrow 126 \rightarrow 312 \rightarrow 143 \rightarrow 168 \rightarrow 48 \rightarrow 124$
$17 \le n \le 19$	C_2	$24 \leftrightarrow 15$
n=20	C_3	$42 \leftrightarrow 20$
n=21	C_3	$31 \rightarrow 32 \rightarrow 63 \rightarrow 104 \rightarrow 64 \rightarrow 127 \rightarrow 126 \rightarrow 312 \rightarrow 143 \rightarrow 168 \rightarrow 48 \rightarrow 124$
$22 \le n \le 24$	C_2	$24 \leftrightarrow 15$
n=25	C_3	$31 \rightarrow 32 \rightarrow 63 \rightarrow 104 \rightarrow 64 \rightarrow 127 \rightarrow 126 \rightarrow 312 \rightarrow 143 \rightarrow 168 \rightarrow 48 \rightarrow 124$
n=26	C_3	$42 \leftrightarrow 20$
...		
n=381	C_5	$1023 \leftrightarrow 1536$

The search for new cycles was continued up to n=552,000. No new ones were found. This lead Ashbacher to pose the following questions

1) Is there another cycle generated by the $Z\sigma$ sequence?
2) Is there an infinite number of numbers n that generate the two cycle $42 \leftrightarrow 20$?
3) Are there any other numbers n that generate the two cycle $2 \leftrightarrow 3$?
4) Is there a pattern to the first appearance of a new cycle?

Ashbacher concludes his article by stating that these problems have only been touched upon and encourages others to further explore these problems.

II. An extended study of the Zσ iteration

It is amazing that hundred thousands of integers subject to a fairly simple iteration process all end up with final results that can be described by a few small integers. This merits a closer analysis. In an earlier study of iterations [2] the author classified iteration results in terms of invariants, loops and divergents. Applying the iteration to a member of a loop produces another member of the same loop. The cycles described in the previous section are not loops. The members of a cycle are not generated by the same process, half of them are generated by Z(σ(Z(...σ(n)...))) while the other half is generated by (σ(Z(...σ(n)...)), i.e. we are considering two different operators. This leads to a situation were the iteration process applied to a member of a cycle may generate a member of another cycle as described in table 2.

Table 2. A Zσ iteration applied to an element belonging to one cycle may generate an element belonging to another cycle .

	C_1		C_2		C_3		C_4												C_5	
n	2	3	15	24	20	42	31	32	63	104	64	127	126	312	143	168	48	124	1023	1536
σ(n)	3	4	24	60	42	96	32	63	104	210	127	128	312	840	168	480	124	224	1536	4092
Z(σ(n))	2	7	15	15	20	63	63	27	64	20	126	255	143	224	48	255	31	63	1023	495
σ(Z(σ(n)))		8						40				...		504		...				936
Z(σ(Z(σ(n))))		15						15				15		63		15				143
...																				
Generates	C_1	C_2	C_2	C_2	C_3	C_4	C_4	C_2	C_4	C_3	C_4	C_2	C_4	C_4	C_4	C_2	C_4	C_4	C_5	C_4
*=Shift to other cycle		*				*		*		*		*				*				*

This situation makes it impossible to establish a one-to-one correspondence between a number n to which the sequence of iterations is applied and the cycle that it will generate. Henceforth the iteration function will be Z(σ(n)) which will be denoted Zσ(n) while results included in the above cycles originating from σ(Z(...σ(n)...)) will be considered as intermediate elements. This leads to an unambiguous situation which is shown in table 3.

Table3. The Zσ iteration process described in terms of invariants, loops and intermediate elements.

	I_1	I_2	I_3	Loop						I_4
n	2	15	20	31	63	64	126	143	48	1023
Z(σ(n))	2	15	20	63	64	126	143	48	31	1023
Intermediate element	3	24	42	32	104	127	312	168	124	1536

We have four invariants I_1, I_2, I_3 and I_4 and one loop L with six elements. No other invariants or loops exist for $n \leq 10^6$. Each number $n \leq 10^6$ corresponds to one of the invariants or the loop. The distribution of results of the Zσ iteration has been examined by intervals of size 50000 as shown in table 4. The stability of this distribution is amazing. It deserves a closer look and will help bringing us closer to answers to the four questions posed by Ashbacher.

Question number 3: Are there any other numbers n that generate the two cycle $2 \leftrightarrow 3$? In the framework set for this study this question will reformulated to: Are there any other numbers than n=2 that belongs to the invariant 2?

Theorem: n=2 is the only element for which $Z(\sigma(n))=2$.

Proof:
$Z(x)=2$ has only one solution which is $x=3$. $Z(\sigma(n))=2$ can therefore only occur when $\sigma(n)=3$ which has the unique solution n=2.
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Table 4. $Z\sigma$ iteration iteration results.

Interval	I_2	I_3	Loop	I_4
3-50000	18824	236	29757	1181
50001-100000	18255	57	30219	1469
100001-150000	17985	49	30307	1659
150001-200000	18129	27	30090	1754
200001-150000	18109	38	30102	1751
250001-300000	18319	33	29730	1918
300001-350000	18207	24	29834	1935
350001-400000	18378	18	29622	1982
400001-450000	18279	21	29645	2055
450001-550000	18182	24	29716	2078
500001-550000	18593	18	29227	2162
550001-600000	18159	19	29651	2171
600001-650000	18596	25	29216	2163
650001-700000	18424	26	29396	2154
700001-750000	18401	20	29409	2170
750001-800000	18391	31	29423	2155
800001-850000	18348	22	29419	2211
850001-900000	18326	15	29338	2321
900001-950000	18271	24	29444	2261
950001-1000000	18517	31	29257	2195
Average	18335	38	29640	1987

Question number 2: Is there an infinite number of numbers n that generate the two cycle $42 \leftrightarrow 20$?

Conjecture: There are infinitely many numbers n which generate the invariant 20 (I_3).

Support:
Although the statistics shown in table 4 only skims the surface of the "ocean of numbers" the number of numbers generating this invariant is as stable as for the other invariants and the loop. To this is added the fact that any number $>10^6$ will either generate a new invariant or loop (highly unlikely) or "catch on to" one of the already existing end results where I_4 will get its share as the iteration "filters through" from 10^6 until it gets locked onto one of the established invariants or the loop.
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Question number 1: Is there another cycle generated by the $Z\sigma$ sequence?

Discussion:
The search up to $n=10^6$ revealed no new invariants or loops. If another invariant or loop exists it must be initiated by $n>10^6$.

Let N be the value of n up to which the search has been completed. For $n=N+1$ there are three possibilities:

Possibility 1.
$Z(\sigma(n))\leq N$. In this case continued iteration repeats iterations which have already been done in the complete search up to $n=N$. No new loops or invariants will be found.

Possibility 2.
$Z(\sigma(n))=n$. If this happens then $n=N+1$ is a new invariant. A necessary condition for an invariant is therefore that

$$\frac{n(n+1)}{2\sigma(n)} = q \text{, where q is positive integer.} \qquad (1)$$

If in addition **no** $m<n$ exists so that

$$\frac{m(m+1)}{2\sigma(n)} = q_1 \text{, } q_1 \text{ integer, then n is invariant.} \qquad (2)$$

There are 111 potential invariant candidates for n up to $3\cdot 10^8$ satisfying the necessary condition (1). Only four of them n = 2, 15, 20 and 1023 satisfied condition (2). It seems that for a given solution to (1) there is always, for $n>N>1023$, a solution to (2) with $m<n$. This is plausible since we know [4] that $\sigma(n)=O(n^{1+\delta})$ for every positive δ which means that $\sigma(n)$ is small compared to $n(n+1)\approx n^2$ for large n.

Example: The largest $n<3\cdot 10^8$ for which (1) is satisfied is n=292,409,999 with $\sigma(292{,}409{,}999)=341145000$ and $292409999 \cdot 292410000/(2\cdot 341145000)=125318571$. But $m=61370000<n$ exists for which $61370000\cdot 61370001/(2\cdot 341145000)=5520053$, an integer, which means that n is not invariant.

Possibility 3.
$Z(\sigma(n))>N$. This could lead to a new loop or invariant. Let's suppose that a new loop of length $k\geq 2$ is created. All elements of this loop must be greater than N otherwise the iteration sequence will fall below N and end up on a previously known invariant or loop. A necessary condition for a loop is therefore that

$$Z(\sigma(n))>n \text{ and } Z(\sigma(Z(\sigma(n))))\geq n. \qquad (3)$$

Denoting the k^th^ iteration $(Z\sigma)_k(n)$ we must finally have

$$(Z\sigma)_k(n)= (Z\sigma)_j(n) \text{ for some } k\neq j \text{, interpreting } (Z\sigma)_0(n)=n \qquad (4)$$

There isn't much hope for all this to happen since, for large n, already $Z(\sigma(n))>n$ is a scarce event and becomes scarcer as we increase n. A study of the number of incidents where $(Z\sigma)_3(n)>n$ for $n<800{,}000$ was made. There are only 86 of them, of these 65 occurred for $n<100{,}000$. From n=510,322 to n=800,000 there was not a single incident.

Question number 4: No particular patterns were found.

Epilog:

In empirical studies of numbers the search for patterns and general behaviors is an interesting and important part. In this iteration study it is amazing that all these numbers, where not even the sky is the limit[1], after a few iterations filter down to end up on one of three invariants or a single loop. The other amazing thing is the relative stability of distribution between the three invariants and the loop with increasing n (see table 4) . When $(Z\sigma)_k(n)$ drops below n it catches on to an integer which has already been iterated and which has therefore already been classified to belong to one of the four terminal events. This in my mind explains the relative stability. In general the end result is obtained after only a few iterations. It is interesting to see that $\sigma(n)$ often assumes the same value for values of n which are fairly close together. Here is an example: $\sigma(n)=3024$ for n=1020, 1056, 1120, 1230, 1284, 1326, 1420, 1430, 1484, 1504, 1506, 1564, 1670, 1724, 1826, 1846, 1886, 2067, 2091, 2255, 2431, 2515, 2761, 2839, 2911, 3023. I may not have brought this subject much further but I hope to have contributed some light reading in the area of recreational mathematics.

References

1. H. Ibstedt, *Surfing on the Ocean of Numbers*, Erhus University Press, 1997.
2. Charles Ashbacher, *Pluckings From the Tree of Smarandache Sequences and Functions*, American Research Press, 1998.
3. Charles Ashbacher, On Iterations That Alternate the Pseudo-Smarandache and Classic Functions of Number Theory, *Smarandache Notions Journal, Vol. 11, No 1-2-3.*
4. G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*. Oxford University Press, 1938.

7, Rue du Sergent Blandan
92130 Issy les Moulineaux, France

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[1] "Not even the sky is the limit" expresses the same dilemma as the title of the authors book "Surfing on the ocean of numbers". Even with for ever faster computers and better software for handling large numbers empirical studies remain very limited.