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On the pseudo Smarandache function

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Abstract For any positive integer n, the famous pseudo Smarandache function $Z(n)$ is defined as the smallest positive integer m such that n evenly divides $\sum_{k=1}^{m} k$. That is, $Z(n) = \min\left\{m:\ n\Big|\frac{m(m+1)}{2},\ m\in N\right\}$, where N denotes the set of all positive integers. The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of $\ln Z(n)$, and give an interesting asymptotic formula for it.

Keywords Pseudo Smarandache function, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous pseudo Smarandache function $Z(n)$ is defined as the smallest positive integer m such that n evenly divides $\sum_{k=1}^{m} k$. That is,

$$Z(n) = \min\left\{m:\ n\Big|\frac{m(m+1)}{2},\ m\in N\right\},$$

where N denotes the set of all positive integers. For example, the first few values of $Z(n)$ are:

$$Z(1)=1,\ Z(2)=3,\ Z(3)=2,\ Z(4)=7,\ Z(5)=4,\ Z(6)=3,\ Z(7)=6,\ Z(8)=15,$$
$$Z(9)=8, Z(10)=4,\ Z(11)=10,\ Z(12)=8,\ Z(13)=12,\ Z(14)=7,\ Z(15)=5,$$
$$Z(16)=31,\ Z(17)=16,\ Z(18)=8,\ Z(19)=18,\ Z(20)=15,\ \cdots\cdots.$$

This function was introduced by David Gorski in reference [1], where he studied the elementary properties of $Z(n)$, and obtained a series interesting results. Some of them are as follows:

If $p \geq 2$ be a prime, then $Z(p) = p - 1$;

If $n = 2^k$, then $Z(n) = 2^{k+1} - 1$.

Let p be an odd prime, then $Z(2p) = p$, if $p \equiv 3(\mathrm{mod}4)$; $Z(2p) = p - 1$, if $p \equiv 1(\mathrm{mod}4)$.

For any odd prime p with $p \mid n$ and $n \neq p$, $Z(n) \geq p - 1$.

The other contents related to the pseudo Smarandache function can also be found in references [2], [3] and [4]. In this paper, we consider the mean value properties of $\ln Z(n)$. About this problem, it seems that none had studied it yet, at least we have not seen any related results before. The main purpose of this paper is using the elementary and analytic methods study this problem, and give an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} \ln Z(n) = x \ln x + O(x).$$

Whether there exists an asymptotic formula for the mean value

$$\sum_{n \leq x} Z(n) \quad \text{or} \quad \sum_{n \leq x} \frac{1}{Z(n)}$$

are two open problems.

§2. Proof of the theorem

In this section, we shall complete the proof of Theorem. First we need the following simple conclusion:

Lemma. For all real number $x > 1$, we have the asymptotic formula

$$\sum_{p \leq x} \frac{\ln p}{p} = \ln x + O(1),$$

where $\sum\limits_{p \leq x}$ denotes the summation over all prime p with $2 \leq p \leq x$.

Proof. See Theorem 4.10 of reference [5].

Using this Lemma we can prove our Theorem easily. In fact for any positive integer $n > 1$, note that $n \mid \frac{2n(2n-1)}{2}$, from the definition of $Z(n)$ we know that $Z(n) \leq 2n - 1$. So by the Euler's summation formula we can get

$$\sum_{n \leq x} \ln Z(n) \leq \sum_{n \leq x} \ln(2n-1) \leq x \ln x + O(x). \tag{1}$$

Now let A denotes the set of all square-full numbers n (That is, if $p \mid n$, then $p^2 \mid n$) in the interval $[1,\ x]$. Then we have

$$\sum_{n \leq x} \ln Z(n) = \sum_{\substack{n \leq x \\ n \in A}} \ln Z(n) + \sum_{\substack{n \leq x \\ n \notin A}} \ln Z(n). \tag{2}$$

From reference [6] we know that

$$\sum_{\substack{n \leq x \\ n \in A}} 1 = \frac{\zeta(\frac{3}{2})}{\zeta(3)} x^{\frac{1}{2}} + \frac{\zeta(\frac{2}{3})}{\zeta(2)} x^{\frac{1}{3}} + O\left(x^{\frac{1}{6}} \exp\left(-C \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}\right)\right),$$

where $C > 0$ is a constant. By this estimate and note that $\ln Z(n) \leq \ln(2n)$, we may get

$$\sum_{\substack{n \leq x \\ n \in A}} \ln Z(n) \ll \sqrt{x} \cdot \ln x. \tag{3}$$

If $n \notin A$, then $n = 1$ or there exists at least one prime p with $p \mid n$ and $p^2 \dagger n$. So from Lemma we have

$$\begin{aligned}
\sum_{\substack{n\le x\\ n\notin A}} \ln Z(n) &= \sum_{\substack{np\le x\\ (n,\ p)=1}} \ln Z(np) \ge \sum_{p\le x} \sum_{\substack{n\le \frac{x}{p}\\ (n,\ p)=1}} \ln(p-1)\\
&= \sum_{p\le x} \left[\frac{x}{p} - \frac{x}{p^2} + O(1)\right] \cdot \ln(p-1)\\
&= x \cdot \sum_{p\le x} \frac{\ln p}{p} - x \cdot \sum_{p\le x} \frac{\ln p}{p^2} + O(x)\\
&= x \cdot \ln x + O(x). \qquad (4)
\end{aligned}$$

Combining (1), (2), (3) and (4) we may immediately get the asymptotic formula

$$\sum_{n\le x} \ln Z(n) = x \ln x + O(x).$$

This completes the proof of Theorem.

References

[1] David Gorski, The pseudo Smarandache function, Smarandache Notions Journal, **13** (2002), 140-149.

[2] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[3] Kenichiro Kashihara, Comments and topics on Smarandache notions and problems, Erhus University Press, USA, 1996.

[4] Liu Yanni, On the Smarandache Pseudo Number Sequence, Chinese Quarterly Journal of Mathematics, **10**(2006), No. 4, 42-59.

[5] Tom M. Apostol, Introduction to analytical number theory, Spring-Verlag, New York, 1976.

[6] Aledsandar Ivić, The Riemann zeta-function theory, New York, 1985, 407-413.

[7] Zhang Wenpeng, The elementary number theory, Shaanxi Normal University Press, Xi'an, 2007.

[8] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.