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# THE PSEUDO-SMARANDACHE FUNCTION

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**Abstract:**

The Pseudo-Smarandache Function is part of number theory. The function comes from the Smarandache Function. The Pseudo-Smarandache Function is represented by Z(n) where n represents any natural number. The value for a given Z(n) is the smallest integer such that 1+2+3+ . . . + Z(n) is divisible by n. Within the Pseudo-Smarandache Function, there are several formulas which make it easier to find the Z(n) values.

Formulas have been developed for most numbers including:

a) p, where p equals a prime number greater than two;
b) b, where p equals a prime number, x equals a natural number, and $b=p^x$;
c) x, where x equals a natural number, if x/2 equals an odd number greater than two;
d) x, where x equals a natural number, if x/3 equals a prime number greater than three.

Therefore, formulas exist in the Pseudo-Smarandache Function for all values of b except for the following:

a) x, where x = a natural number, if x/3 = a nonprime number whose factorization is not $3^x$;
b) multiples of four that are not powers of two.

All of these formulas are proven, and their use greatly reduces the effort needed to find Z(n) values.

**Keywords:**
Smarandache Function, Pseudo-Smarandache Function, Number Theory, Z(n), g(d), g[Z(n)].

**Introduction.**

The Smarandache (sma-ran-da-ke) Functions, Sequences, Numbers, Series, Constants, Factors, Continued Fractions, Infinite Products are a branch of number theory. There are very interesting patterns within these functions, many worth studying sequences. The name "Pseudo-Smarandache Function" comes from the Smarandache function. [2] The Smarandache Function was named after a Romanian mathematician and poet, Florentin Smarandache. [1] The Smarandache Function is represented as S(n) where n is any natural number. S(n) is defined as the smallest m, where m represents any natural number, such that m! is divisible by n.
To be put simply, the Smarandache Function differs from the Pseudo-Smarandache Function in that in the Smarandache Function, multiplication is used in the form of factorials; in the Pseudo -Smarandache Function, addition is used in the place of the multiplication. The Pseudo-Smarandache Function is represented by Z(n) where n represents all natural numbers. The value for a given Z(n) is the smallest integer such that 1+2+3+ . . . + Z(n) is divisible by n.

| d | g(d) |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 6 |
| 4 | 10 |
| 5 | 15 |
| 6 | 21 |
| 7 | 28 |
| 8 | 36 |
| 9 | 45 |
| 10 | 55 |

**Background**

As previously stated, the value for a given Z(n) is the smallest integer such that 1+2+3+ . . . + Z(n) is divisible by n. Because consecutive numbers are being added, the sum of 1+2+3+ . . . + Z(n) is a triangle number. Triangle numbers are numbers that can be written in the form [d(d+1)]/2 where d equals any natural number. When written in this form, two consecutive numbers must be present in the numerator. In order to better explain the Z(n) function, the g(d) function has been introduced where g(d)=[d(d+1)]/2.

Figure 1: The first ten g(d) values.

| n | Z(n) | g[Z(n)] |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 3 | 6 |
| 3 | 2 | 3 |
| 4 | 7 | 28 |
| 5 | 4 | 10 |
| 6 | 3 | 6 |
| 7 | 6 | 21 |
| 8 | 15 | 120 |
| 9 | 8 | 36 |
| 10 | 4 | 10 |
| 11 | 10 | 55 |
| 12 | 8 | 36 |
| 13 | 12 | 78 |
| 14 | 7 | 28 |
| 15 | 5 | 15 |
| 16 | 31 | 496 |
| 17 | 16 | 136 |
| 18 | 8 | 36 |
| 19 | 18 | 171 |
| 20 | 15 | 120 |

Figure 2: The first 20 Z(n) and g[Z(n)] values.

g[Z(n)] values are defined as g(d) values where d equals Z(n). Because of this, it is important to note that all g[Z(n)] values are g(d) values but special ones because they correspond to a particular n value. Since g(d)=[d(d+1)]/2, g[Z(n)]=[Z(n)[Z(n)+1]/2]. Because g(d) is evenly divisible by n, and all g[Z(n)] are also g(d) values, g[Z(n)] is evenly divisible by n. Therefore, the expression [Z(n)[Z(n)+1]/2] can be shortened to n*k (where k is any natural number). If k=x/2 (where x is any natural number) then [Z(n)[Z(n)+1]/2]=(n*x)/2, and the "general form" for a g[Z(n)] value is (n*x)/2. Again, since (n*x)/2 represents a g(d) value, it must contain all of the characteristics of g(d) values. As said before, all g(d) values, when written in the form [d(d+1)]/2, must be able to have two consecutive numbers in their numerator. Therefore, in the expression (n*x)/2, n and x must be consecutive, or they must be able to be factored and rearranged to yield two consecutive numbers. For some values of n, g[Z(n)]=(n*x)/2 where x is much less than n (and they aren't consecutive). This is possible because for certain number combinations n and x can be factored and rearranged in a way that makes them consecutive. For example, Z(n=12) is 8, and g[Z(12)] is 36. This works because the original equation was (12*6)/2=36, but after factoring and rearranging 12 and 6, the equation can be rewritten as (8*9)/2=36.

The Pseudo-Smarandache Function specifies that only positive numbers are used. However, what if both d and n were less than zero? g(d) would then represent the sum of the numbers from d to –1. Under these circumstances, Z(n) values are the same as the Z(n) values in the "regular" system (where all numbers are greater than one) except they are negated. This means that Z(-n)=-[Z(n)]. This occurs because between the positive system and the negative system, the g(d) values are also the same, just negated. For example, g(4)=4+3+2+1=10 and g(-4)= -4+ -3+ -2+ -1=-10. Therefore, the first g(d) value which is evenly divisible by a given value of n won't change between the positive system and the negative system.

## Theorem 1

If ‘p’ is a prime number greater than two, then Z(p)=p-1

*Example:*

| p | Z(p) |
|---|---|
| 3 | 2 |
| 5 | 4 |
| 7 | 6 |
| 11 | 10 |
| 13 | 12 |
| 17 | 16 |
| 19 | 18 |
| 23 | 22 |
| 27 | 26 |
| 29 | 28 |

Figure 3: The first 10 Z(p) values.

*Proof:*

Since we are dealing with specific p values, rather than saying g[Z(n)]=(n*x)/2, we can now say g[j(p)]=(p*x)/2. Therefore, all that must be found is the lowest value of x that is consecutive to p, or the lowest value of x that can be factored and rearranged to be consecutive to p. Since p is prime, it has no natural factors other than one and itself. Therefore, the lowest value of x that is consecutive to p is p-1. Therefore Z(p)=p-1.

## Theorem 2

If x equals any natural number, p equals a prime number greater than two, and b equals $p^x$, then Z(b)=b-1

*Example:*

| b | Z(b) |
|---|---|
| 3 | 2 |
| 9 | 8 |
| 27 | 26 |
| 81 | 80 |
| 243 | 242 |
| 729 | 728 |

| b | Z(b) |
|---|---|
| 5 | 4 |
| 25 | 24 |
| 125 | 124 |
| 625 | 624 |
| 3125 | 3124 |
| 15625 | 15624 |

| b | Z(b) |
|---|---|
| 7 | 6 |
| 49 | 48 |
| 343 | 342 |
| 2401 | 2400 |
| 16807 | 16806 |
| 117649 | 117648 |

Figure 4: the first Z(b) values for different primes.

*Proof:*

The proof for this theorem is similar to the proof of theorem 2. Again, the g(d) function is made up of the product of two consecutive numbers divided by two. Since b's roots are the same, it is impossible for something other than one less than b itself to produce to consecutive natural numbers (even when factored and rearranged). For example, g[Z(25)]=(25*x)/2. When trying to find numbers less than 24 which can be rearranged to make two consecutive natural numbers this becomes g[Z(25)]=(5*5*x)/2. There is no possible value of x (that is less than 24) that can be factored and multiplied into 5*5 to make two consecutive natural numbers. This is because 5 and 5 are prime and equal. They can't be factored as is because the have no divisors. Also, there is no value of x that can be multiplied and rearranged into 5*5, again, because they are prime and equal.

**Theorem 3**

If x equals two to any natural power, then Z(x)=2x-1.

*Example:*

| X | Z(x) |
|---|---|
| 2 | 3 |
| 4 | 7 |
| 8 | 15 |
| 16 | 31 |
| 32 | 63 |
| 64 | 127 |
| 128 | 255 |
| 256 | 511 |
| 512 | 1023 |
| 1024 | 2047 |
| 2048 | 4095 |
| 4096 | 8191 |
| 8192 | 16383 |
| 16384 | 32767 |
| 32768 | 65535 |

Figure 5: The first six Z(x) values.

*Proof:*

According to past logic, it may seem like Z(x) would equal x-1. However, the logic changes when dealing with even numbers. The reason Z(x)≠x-1 is because (x-1)/2 can not be an integral value because x-1 is odd (any odd number divided by two yields a number with a decimal). Therefore, [x(x-1)]/2 is not an even multiple of x. In order to solve this problem, the numerator has to be multiplied by two. In a sense, an extra two is multiplied into the equation so that when the whole equation is divided by two, the two that was multiplied in is the two that is divided out. That way, it won't effect the "important" part of the equation, the numerator, containing the factor of x. Therefore, the new equation becomes 2[x(x-1)]/2, or [2x(x-1)]/2. The only numbers consecutive to 2x are 2x-1 and 2x+1. Therefore, the smallest two consecutive numbers are 2x-1 and 2x.
Therefore, Z(x)=2x-1.

## Theorem 4

If 'j' is any natural number where j/2 equals an odd number greater than two then

$$Z(j) = \begin{cases} \frac{j}{2}-1, & \frac{j}{2}-1 \text{ is evenly divisble by } 4 \\ \frac{j}{2}, & \frac{j}{2}-1 \text{ is not evenly divisble by } 4 \end{cases}$$

*Example:*

| j | Z(j) | j/2 | (j/2)-1 |
|---|---|---|---|
| 6 | 3 | 3 | 2 |
| 10 | 4 | 5 | 4 |
| 14 | 7 | 7 | 6 |
| 18 | 8 | 9 | 8 |
| 22 | 11 | 11 | 10 |
| 26 | 12 | 13 | 12 |
| 30 | 15 | 15 | 14 |
| 34 | 16 | 17 | 16 |
| 38 | 19 | 19 | 18 |
| 42 | 20 | 21 | 20 |
| 46 | 23 | 23 | 22 |
| 50 | 24 | 25 | 24 |
| 54 | 27 | 27 | 26 |
| 58 | 28 | 29 | 28 |
| 62 | 31 | 31 | 30 |
| 66 | 32 | 33 | 32 |

Figure 6: The first twenty j(z) values.

*Proof:*

When finding the smallest two consecutive numbers that can be made from a j value, start by writing the general form but instead of writing n substitute j in its place. That means g[Z(j)]=(j*x)/2. The next step is to factor j as far as possible making it easier to see what x must be. This means that g[Z(j)]=(2*j/2*x)/2. Since the equation is divided by two, if left alone as g[Z(j)]=(**2***j/2*x)/2, the boldface 2 would get divided out. This falsely indicates that j/2*x (what is remaining after the boldface 2 is divided out) is evenly divisible by j for every natural number value of x. However, j/2*x isn't always evenly divisible by j for every natural number value of x. The two that was just divided out must be kept in the equation so that one of the factors of the g(d) value being made is j. In order to fix this the whole equation must be multiplied by two so that every value of x is evenly divisible by j. In a sense, an extra two is multiplied into the equation so that so that when the whole equation is divided by two, the two that was multiplied in is the two that gets divided out. That way, it won't effect the "important" part of the equation containing the factor of two. Therefore it becomes g[Z(j)]=(2*2*j/2*f)/2 where f represents any natural number. This is done so that even when divided by two there is still one factor of j. At this point, it looks as though the lowest consecutive integers that can be made from g[Z(j)]=(2*2*j/2*f) are (j/2) and(j/2)-1. However, this is only sometimes the case. This is where the formula changes for every other value of j. If (j/2)-1 is evenly divisible by the '2*2' (4), then Z(j)=(j/2)-1. However, if (j/2)-1 is not evenly divisible by 4, then the next lowest integer consecutive to j/2 is (j/2)+1. (Note: If (j/2)-1 is not evenly divisible by 4, then the next lowest integer consecutive to j/2 is (j/2)+1. (Note: If (j/2)-1 is not evenly divisible by four, then (j/2)+1 must be evenly divisible by 4 because 4 is evenly divisible by every other multiple of two.) Therefore, if (j/2)-1 is not evenly divisible by 4 then g[Z(j)]=[(j/2)[(j/2)+1]]/2 or Z(j)=j/2.

## Theorem 5

If 'p' is any natural number where p/3 equals a prime number greater than 3 then

$$Z(p) = \begin{cases} \frac{p}{3}-1, & \frac{p}{3}-1 \text{ is evenly divisible by } 3 \end{cases}$$

*Example:*

| p | Z(p) | p/3 | (p/3)-1 |
|---|---|---|---|
| 15 | 5 | 5 | 4 |
| 21 | 6 | 7 | 6 |
| 33 | 11 | 11 | 10 |
| 39 | 12 | 13 | 12 |
| 51 | 17 | 17 | 16 |
| 57 | 18 | 19 | 18 |
| 69 | 23 | 23 | 22 |
| 87 | 28 | 29 | 28 |
| 93 | 31 | 31 | 30 |
| 111 | 36 | 37 | 36 |

Figure 7: The first ten Z(p) values.

*Proof:*

The proof for this theorem is very similar to the proof for theorem 4. Since p values are being dealt with, p must be substituted into the general form. Therefore, g[Z(p)]=(p*x)/2. Since what made p is already known, p can be factored further so that g[Z(p)]=(3*p/3*x)/2. At this point it looks like the consecutive numbers that will be made out of (the numerator) 3*p/3*x are p/3 and (p/3)-1 (this is because the greatest value already in the numerator is p/3). However, this is only sometimes the case. When p/3-1 is divisible by 3, the consecutive integers in the numerator are p/3 and (p/3)-1. This means that Z(p)=p/3-1 if p/3-1 is evenly divisible by 3. However, if p/3-1 is not divisible by three, the next smallest number that is consecutive to p/3 is (p/3)+1. If (p/3)-1 is not divisible by 3 then (p/3)+1 must be divisible by 3 (see *1 for proof of this statement). Therefore, the consecutive numbers in the numerator are p/3 and (p/3)+1. This means that Z(p) = p/3 if (p/3)-1 is not evenly divisible by three.

Note: Although there is a similar formula for some multiples of the first two primes, this formula does not exist for the next prime number, 5.

| |
|---|
| 3 |
| **4** |
| 5 |
| **6** |
| 7 |
| **8** |
| 9 |
| **10** |
| 11 |
| **12** |
| 13 |

Figure 10

*1 – "If (p/3)-1 is not divisible by 3, then (p/3)+1 must be divisible by 3."

In the table to the left, the underlined values are those that are divisible by three. The bold numbers are those that are divisible by two (even). Since p/3 is prime it cannot be divisible by three. Therefore, the p/3 values must fall somewhere between the underlined numbers. This leaves numbers like 4, 5, 7, 8, 10, 11, etc. Out of these numbers, the only numbers where the number before (or (p/3)-1) is not divisible by three are the numbers that precede the multiples of three. This means that the p/3 values must be the numbers like 5, 8, 11, etc. Since all of these p/3 values precede multiples of 3, (p/3)+1 must be divisible by 3 if (p/3)-1 is not divisible by 3.

**Theorem 6**

If 'n' equals any natural number, Z(n)≠n.

*Proof:*

**Theorem 6: Part A**

If r is any natural odd number, Z(r)≤1

*Proof:*

When r is substituted into the general form, g[Z(r)=[r*(r-1)]/2. Since r is odd r-1 is even. Therefore, when r-1 is divided by two, an integral value is produced. Therefore, (r*r-1)/2 is an even multiple of r and it is also a g(d) value. Because of this, Z(r)≤1. Since Z(r)≤1, Z(r)≠r.

**Theorem 6: Part B**

If v is an natural even number, Z(v)≠v.

*Proof:*

If Z(v) = v, the general form would appear as the following: g[Z(v)]=[v(v+1)]/2. This is not possible because if v is even then v+1 is odd. When v+1 is divided by two, a non-integral value is produced. Therefore, (v*v+1)/2 is not an integral multiple of v. Therefore, Z(v)≠v.

**Theorem 7**

If w is any natural number except for numbers whose prime factorization equals 2 to any power, Z(w)<w.

*Proof*

As in several other proofs, this proof can be broken down into two separate parts, a part for r values (r is any natural odd number) and one for v values (v is any natural even number). As proven in Theorem 6: Part A, Z(r)≤1. This proves that Z(r) is less than r.

For v values, v must be substituted into the general form in order to be able to see patterns. Therefore, g[Z(v)]=(v*x)/2. Since v is even it must be divisible by two. Therefore, v can be factored making g[Z(v)]=[2*(v/2)*x]/2. Since the numerator is being divided by two, when done with the division, one whole factor of v will not always be left. Therefore, an extra two must be multiplied into the equation so that even when divided by two, there is still one whole factor of v left. Therefore, g[Z(v)]≤[4*(v/2)*x]/2. At this point, the equation can be simplified to g[Z(v)]≤v*x. Therefore, x=v-1, and Z(v)<v-1. Z(v) is less than v-1 rather than less than or equal to v-1 because as proven in theorem 4, Z(v)≠v-1.

## Conclusion

| n | Z(n) |
|---|---|
| 12 | 8 |
| 20 | 15 |
| 24 | 15 |
| 28 | 7 |
| 36 | 8 |
| 40 | 15 |
| 44 | 32 |
| 48 | 32 |
| 52 | 39 |
| 56 | 48 |

Figure 8

| n/3 | n | Z(n) |
|---|---|---|
| 9 | 27 | 8 |
| 15 | 45 | 9 |
| 21 | 63 | 27 |
| 25 | 75 | 24 |
| 33 | 99 | 44 |
| 35 | 105 | 14 |
| 45 | 135 | 54 |
| 49 | 147 | 48 |
| 55 | 165 | 44 |
| 65 | 195 | 39 |

Figure 9

Through researching the relationships between different groups of natural numbers, patterns and formulas have been developed to find Z(n) values for most numbers. Formulas have been developed for most numbers including:

a) p, where p equals a prime number greater than two
b) b, where p equals a prime number, x equals a natural number, and $b=p^x$
c) x, where x equals a natural number, if x/2 equals an odd number greater than two
d) x, where x equals a natural number, if x/3 equals a prime number greater than three

In fact there are only two remaining groups of numbers for which there are no formulas or shortcuts. Formulas exist in the Pseudo-Smarandache Function for all values of b except for the following:

a) multiples of four that that are not powers of two (figure 8)
b) x, where x = a natural number, if x/3 = a nonprime number whose factorization is not $3^x$ (figure9)

If p equals a prime number greater than two then Z(p)=p-1. If p equals a prime number greater than two, x equals a natural number, and $b=p^x$ then Z(b)=b-1. However, if p=2 then Z(b)=2b-1. If x equals a natural number, and x/2 equals an odd number greater than two then if (x/2)-1 is evenly divisible by four then Z(x)=(x/2)-1. Otherwise, if x/2-1 is not evenly divisible by four then Z(x)=z/2. If x equals a natural number, and x/3 equals a prime number greater than three then if (x/3)-1 is evenly divisible by three then Z(x)=(x/3)-1. Otherwise, if x/3-1 is not evenly divisible by three then Z(x)=x/3. All of these formulas are proven, and their use greatly reduces the effort needed to find Z(n) values.

**References:**

[1] Ashbacher, Charles, "Introduction to the Smarandache Function", J. Recreational Mathematics (1996-1997): 249-251.

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