Scientia Magna
Vol. 4 (2008), No. 2, 1-3

Two problems related to the Smarandache function

Wenpeng Zhang† and Ling Li† ‡

† Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China
‡ Basic Department, Shaanxi Polytechnic Institute, Xianyang, Shaanxi, P.R.China

Abstract For any positive integer n, the famous pseudo Smarandache function $Z(n)$ is defined as the smallest positive integer m such that $n \mid \frac{m(m+1)}{2}$. That is, $Z(n) = \min\left\{m:\ n \mid \frac{m(m+1)}{2},\ n \in N\right\}$. The Smarandache reciprocal function $S_c(n)$ is defined as $S_c(n) = \max\{m:\ y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } m+1 \dagger n!\}$. That is, $S_c(n)$ is the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$. The main purpose of this paper is to study the solvability of some equations involving the pseudo Smarandache function $Z(n)$ and the Smarandache reciprocal function $S_c(n)$, and propose some interesting conjectures.

Keywords The pseudo Smarandache function, the Smarandache reciprocal function, the d-ual function, equation, positive integer solutions, conjecture.

§1. Introduction and results

For any positive integer n, the famous pseudo Smarandache function $Z(n)$ is defined as the smallest positive integer m such that $n \mid \frac{m(m+1)}{2}$. That is,

$$Z(n) = \min\left\{m:\ n \mid \frac{m(m+1)}{2},\ n \in N\right\}.$$

Its dual function $Z^*(n)$ is defined as $Z^*(n) = \max\left\{m: \frac{m(m+1)}{2} \mid n,\ m \in N\right\}$, where N denotes the set of all positive integers. From the definition of $Z(n)$ we can find that the first few values of $Z(n)$ are: $Z(1) = 1$, $Z(2) = 3$, $Z(3) = 2$, $Z(4) = 7$, $Z(5) = 4$, $Z(6) = 3$, $Z(7) = 6$, $Z(8) = 15$, $Z(9) = 8$, $Z(10) = 4$, $Z(11) = 10$, $Z(12) = 8$, $Z(13) = 12$, $Z(14) = 7$, $Z(15) = 5$, $Z(16) = 31$, $\cdots\cdots$. About the elementary properties of $Z(n)$, many authors had studied it, and obtained some interesting results, see references [1], [2], [3], [4], [5] and [6]. For example, the first author [6] studied the solvability of the equations:

$$Z(n) = S(n) \quad \text{and} \quad Z(n) + 1 = S(n),$$

and obtained their all positive integer solutions, where $S(n)$ is the Smarandache function.

On the other hand, in reference [7], A.Murthy introduced another function $S_c(n)$, which called the Smarandache reciprocal function. It is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$. That is, $S_c(n) = \max\{m : \ y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } \ m+1 \dagger n!\}$. For example, the first few values of $S_c(n)$ are:

$$S_c(1) = 1,\ S_c(2) = 2,\ S_c(3) = 3,\ S_c(4) = 4,\ S_(5) = 6,\ S_c(6) = 6,\ S_c(7) = 10,$$
$$S_c(8) = 10,\ S_c(9) = 10,\ S_c(10) = 10,\ S_c(11) = 12,\ S_c(12) = 12,\ S_c(13) = 16,$$
$$S_c(14) = 16,\ S_5(15) = 16,\ S_c(16) = 16,\ S_c(17) = 18,\ S_c(18) = 18,\ \cdots\cdots .$$

A.Murthy [7], Ding Liping [8] and Ren Zhibin [9] also studied the elementary properties of $S_c(n)$, and obtained some interesting conclusions, one of them is that if $S_c(n) = x$ and $n \neq 3$, then $x+1$ is the smallest prime greater than n.

The main purpose of this paper is to study the solvability of some equations related to the Smarandache function, and propose some interesting problems. That is, we have the following:

Unsolved problem 1. Whether there exist infinite positive integers n such that the equation

$$S_c(n) + Z(n) = 2n. \tag{1}$$

Unsolved problem 2. Find all positive integer solutions of the equation

$$S_c(n) = Z^*(n) + n. \tag{2}$$

§2. Some results on these unsolved problems

In this section, we shall give some new progress on these unsolved problems. First for the problem 1, it is clear that $n = 1$ satisfy the equation (1). $n = 3$ does not satisfy the equation (1). If $p \geq 5$ and $p^\alpha + 2$ are two odd primes, then $n = p^\alpha$ satisfy the equation (1). In fact this time, we have $Z\left(p^\alpha\right) = p^\alpha - 1$, $S_c\left(p^\alpha\right) = p^\alpha + 1$. Therefore, $S_c\left(p^\alpha\right) + Z\left(p^\alpha\right) = p^\alpha + 1 + p^\alpha - 1 = 2 \cdot p^\alpha$. So $n = p^\alpha$ satisfy the equation (1). For example, $n = 1$, 5, 11, 17, 29 and 41 are six solutions of the equation (1). We think that the equation (1) has infinite positive integer solutions. Even more, we have the following:

Conjecture 1. For any positive integer n, the equation

$$S_c(n) + Z(n) = 2n$$

holds if and only if $n = 1$, 3^α and $p^{2\beta+1}$, where $\alpha \geq 2$ be any integer such that $3^\alpha + 2$ be a prime, $p \geq 5$ be any prime, $\beta \geq 0$ be any integer such that $p^{2\beta+1} + 2$ be a prime.

For the problem 2, it is clear that $n = 3$ does not satisfy the equation (2). If $p \geq 5$ be a prime, $n = p^{2\alpha+1}$ such that $n+2$ be a prime, then $S_c(n) = n+1$, $Z^*(n) = 1$, so $S_c(n) = Z^*(n) + n$. Therefore, $n = p^{2\alpha+1}$ satisfy the equation (2). Besides these, whether there exist any other positive integer n satisfying the equation (2) is an open problem. We believe that the following conjecture is true.

Conjecture 2. For any positive integer n, the equation

$$S_c(n) = Z^*(n) + n$$

In our conjectures, if prime $p \geq 5$, then $p^{2\beta} + 2$ can be divided by 3. So if $p^{\alpha} + 2$ be a prime, then α must be an odd number.

From our conjectures we also know that there exists close relationship between the solutions of the equations (1), (2) and the twin primes. So we think that the above unsolved problems are very interesting and important.

References

[1] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] Kenichiro Kashihara, Comments and topics on Smarandache notions and problems, Erhus University Press, USA, 1996.

[3] David Gorski, The pseudo Smarandache function, Smarandache Notions Journal, **13**(2002), No.1-2-3, 140-149.

[4] Lou Yuanbing, On the pseudo Smarandache function, Scientia Magna, **3**(2007), No.4, 48-50.

[5] Zheng Yani, On the Pseudo Smarandache function and its two conjectures, Scientia Magna, **3**(2007), No.4, 74-76.

[6] Zhang Wenpeng, On two problems of the Smarandache function, Journal of Northwest University, **38** (2008), No.2, 173-176.

[7] A. Murthy, Smarandache reciprocal function and an elementary inequality, Smarandache Notions Journal, **11**(2000), No.1-2-3, 312-315.

[8] Liping Ding, On the Smarandache reciprocal function and its mean value, Scientia Magna. **4**(2008), No.1, 120-123.

[9] Zhibin Ren, On an equation involving the Smarandache reciprocal function and its positive integer solutions, Scientia Magna, **4** (2008), No.1, 23-25.

[10] Zhang Wenpeng, The elementary number theory (in Chinese), Shaanxi Normal University Press, Xi'an, 2007.

[11] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[12] I. Balacenoiu and V. Seleacu, History of the Smarandache function, Smarandache Notions Journal, **10**(1999), No.1-2-3, 192-201.