ON THE 82-TH SMARANDACHE'S PROBLEM

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Abstract The main purpose of this paper is using the elementary method to study the asymptotic properties of the integer part of the k-th root positive integer, and give two interesting asymptotic formulae.

Keywords: k-th root; Integer part; Asymptotic formula.

§1. Introduction And Results

For any positive integer n, let $s_k(n)$ denote the integer part of k-th root of n. For example, $s_k(1) = 1$, $s_k(2) = 1$, $s_k(3) = 1$, $s_k(4) = 1, \cdots$, $s_k(2^k) = 2$, $s_k(2^k + 1) = 2, \cdots$, $s_k(3^k) = 3, \cdots$. In problem 82 of [1], Professor F.Smarandache asked us to study the properties of the sequence $s_k(n)$. About this problem, some authors had studied it, and obtained some interesting results. For instance, the authors [5] used the elementary method to study the mean value properties of $S(s_k(n))$, where Smarandache function $S(n)$ is defined as following:

$$S(n) = \min\{m : m \in N, n \mid m!\}.$$

In this paper, we use elementary method to study the asymptotic properties of this sequence in the following form: $\sum\limits_{n\le x} \frac{\varphi(s_k(n))}{s_k(n)}$ and $\sum\limits_{n\le x} \frac{1}{\varphi(s_k(n))}$, where $x \ge 1$ be a real number, $\varphi(n)$ be the Euler totient function, and give two interesting asymptotic formulae. That is, we shall prove the following:

Theorem 1. *For any real number $x > 1$ and any fixed positive integer $k > 1$, we have the asymptotic formula*

$$\sum_{n\le x} \frac{\varphi(s_k(n))}{s_k(n)} = \frac{6}{\pi^2}x + O\left(x^{1-\frac{1}{k}-\varepsilon}\right),$$

where ε is any real number.

Theorem 2. *For any real number $x > 1$ and any fixed positive integer $k > 1$, we have the asymptotic formula*

$$\sum_{n\le x}\frac{1}{\varphi(s_k(n))}=\frac{k\zeta(2)\zeta(3)}{(k-1)\zeta(6)}x^{1-\frac{1}{k}}+A+O\left(x^{1-\frac{2}{k}}\log x\right),$$

where $A=\gamma\sum\limits_{n=1}^{\infty}\frac{\mu^2(n)}{n\varphi(n)}-\sum\limits_{n=1}^{\infty}\frac{\mu^2(n)\log n}{n\varphi(n)}$.

§2. Proof of Theorems

In this section, we will complete the proof of Theorems. First we come to prove Theorem 1. For any real number $x > 1$, let M be a fixed positive integer with $M^k \le x \le (M+1)^k$, from the definition of $s_k(n)$ we have

$$\begin{aligned}\sum_{n\le x}\frac{\varphi(s_k(n))}{s_k(n)} &= \sum_{t=1}^{M}\sum_{(t-1)^k\le n<t^k}\frac{\varphi(s_k(n))}{s_k(n)}+\sum_{M^k\le n<x}\frac{\varphi(s_k(n))}{s_k(n)}\\ &= \sum_{t=1}^{M-1}\sum_{t^k\le n<(t+1)^k}\frac{\varphi(s_k(n))}{s_k(n)}+\sum_{M^k\le n\le x}\frac{\varphi(M)}{M}\\ &= \sum_{t=1}^{M-1}[(t+1)^k-t^k]\frac{\varphi(t)}{t}+O\left(\sum_{M^k\le n<(M+1)^k}\frac{\varphi(M)}{M}\right)\\ &= k\sum_{t=1}^{M}t^{k-1}\frac{\varphi(t)}{t}+O\left(M^{k-1-\varepsilon}\right), \qquad (1)\end{aligned}$$

where we have used the estimate $\frac{\varphi(n)}{n}\ll n^{-\varepsilon}$.

Note that(see reference [3])

$$\sum_{n\le x}\frac{\varphi(n)}{n}=\frac{6}{\pi^2}x+O\left((\log x)^{\frac{2}{3}}(\log\log x)^{\frac{4}{3}}\right). \qquad (2)$$

Let $B(y)=\sum\limits_{t\le y}\frac{\varphi(t)}{t}$, then by Abel's identity (see Theorem 4.2 of [2]) and (2), we can easily deduce that

$$\begin{aligned}\sum_{t=1}^{M}t^{k-1}\frac{\varphi(t)}{t} &= M^{k-1}B(M)-B(1)-(k-1)\int_1^M y^{k-2}B(y)dy\\ &= M^{k-1}\left(\frac{6}{\pi^2}M+O\left((\log M)^{\frac{2}{3}}(\log\log M)^{\frac{4}{3}}\right)\right)\\ &- (k-1)\int_1^M (y^{k-2}\left(\frac{6}{\pi^2}y+O\left((\log y)^{\frac{2}{3}}(\log\log y)^{\frac{4}{3}}\right)\right)dy\\ &= \frac{6}{k\pi^2}M^k+O\left((\log M)^{\frac{2}{3}}(\log\log M)^{\frac{4}{3}}\right). \qquad (3)\end{aligned}$$

Applying (1) and (3) we can obtain the asymptotic formula

$$\sum_{n\le x}\frac{\varphi(s_k(n))}{s_k(n)}=\frac{6}{\pi^2}M^k+O\left(M^{k-1-\varepsilon}\right). \qquad (4)$$

On the other hand, note that the estimate

$$0\le x-M^k<(M+1)^k-M^k\ll x^{\frac{k-1}{k}} \qquad (5)$$

Now combining (4) and (5) we can immediately obtain the asymptotic formula

$$\sum_{n\le x}\frac{\varphi(s_k(n))}{s_k(n)}=\frac{6}{\pi^2}x+O\left(x^{1-\frac{1}{k}-\varepsilon}\right).$$

This proves Theorem 1.

Similarly, note that(see reference [4])

$$\sum_{n\le x}\frac{1}{\varphi(n)}=\frac{\zeta(2)\zeta(3)}{\zeta(6)}\log x+A+O\left(\frac{\log x}{x}\right),$$

where $A=\gamma\sum\limits_{n=1}^{\infty}\frac{\mu^2(n)}{n\varphi(n)}-\sum\limits_{n=1}^{\infty}\frac{\mu^2(n)\log n}{n\varphi(n)}$. We can use the same method to obtain the result of Theorem 2.

References

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