ON THE 83-TH PROBLEM OF F. SMARANDACHE

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Abstract For any positive integer n, let $m_q(n)$ denote the integer part of k-th root of n. That is, $m_q(n) = \left[n^{\frac{1}{k}}\right]$. In this paper, we study the properties of the sequences $\{m_q(n)\}$, and give an interesting asymptotic formula.

Keywords: Integer part; Mean value; Asymptotic formula.

§1. Introduction

For any positive integer n, let $m_q(n)$ denote the integer part of k-th root of n. That is, $m_q(n) = \left[n^{\frac{1}{k}}\right]$. For example, $m_q(1) = 1$, $m_q(2) = 1$, $m_q(3) = 1$, $m_q(4) = 1$, $\cdots$, $m_q(2^k) = 2$, $m_q(2^k+1) = 2$, $\cdots$, $m_q(3^k) = 3$, $\cdots$. In problem 83 of [1], Professor F. Smarandache asked us to study the properties of the sequence $\{m_q(n)\}$. About this problem, it seems that none had studied it, at least we have not seen related paper before. In this paper, we use the elementary methods to study the properties of this sequence, and give an interesting asymptotic formula. That is, we shall prove the following:

Theorem. *m is any fixed positive integer, α is a real number . For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\le x}\sigma_\alpha((m_q(n),m)) = \frac{(2k-1)\sigma_{1-\alpha}(m)}{m^{1-\alpha}}x + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, ε is any fixed positive number.

When $\alpha = 0, 1$, we have

Corollary. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\le x} d((m_q(n),m)) = \frac{(2k-1)\sigma(m)}{m}x + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

$$\sum_{n\le x} \sigma((m_q(n),m)) = (2k-1)d(m)x + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

where $d(n)$ is divisor function, $\sigma(n)$ is divisor sum function.

§2. One lemma

To prove the theorem, we need the following lemma.

Lemma. *m is any fixed positive integer, α is a real number. For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\le x}\sigma_\alpha((n,m)) = \frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}}x + O\left(x^{\frac{1}{2k}+\varepsilon}\right),$$

where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, ε is any fixed positive number.

Proof. Let

$$g(s) = \sum_{n=1}^{\infty}\frac{\sigma_\alpha((m,n))}{n^s},$$

For m is a fixed number, $f(n) = (m,n)$ is a multiplicative function.we can proof that $\sigma_\alpha((m,n))$ is a multiplicative function too.

From the Euler product formula, we have

$$\begin{aligned}
g(s) &= \prod_p\left(1+\frac{\sigma_\alpha(f(p))}{p^s}+\frac{\sigma_\alpha(f(p^2))}{p^{2s}}+\cdots\right)\\
&= \prod_{p\dagger m}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right)\\
&\quad\times\prod_{p^\beta\|m}\left(1+\frac{1+p^\alpha}{p^s}+\cdots+\frac{\sum\limits_{i=0}^{\beta}(p^i)^\alpha}{p^{\beta s}}+\frac{\sum\limits_{i=0}^{\beta}(p^i)^\alpha}{p^{(\beta+1)s}}+\cdots\right)\\
&= \prod_{p\dagger m}\frac{1}{1-\frac{1}{p^s}}\prod_{p^\beta\|m}\left(1+\frac{1+p^\alpha}{p^s}+\cdots+\frac{\sum\limits_{i=0}^{\beta-1}(p^i)^\alpha}{p^{\beta s}}+\frac{\sum\limits_{i=0}^{\beta}(p^i)^\alpha}{p^{\beta s}}\frac{1}{1-\frac{1}{p^s}}\right)\\
&= \zeta(s)\prod_{p^\beta\|m}\left(1+\frac{1}{p^{s-\alpha}}+\frac{1}{p^{2(s-\alpha)}}+\cdots+\frac{1}{p^{\beta(s-\alpha)}}\right).
\end{aligned}$$

And for

$$|\sigma_\alpha((m,n))| < K = H(x), \qquad \left|\sum_{n=1}^{\infty}\frac{\sigma_\alpha((m,n))}{n^\sigma}\right| < \frac{K}{\sigma-1} = B(\sigma)$$

where K is a constant only about m and α, $\alpha > 1$ is real part of s. So we let $s_0 = 0$, $b = 2$, $T = x^{3/2}$. When x is a half odd, we let $N = x - 1/2$, $\|x\| = |x - N|$. By Perron formula, we have

$$\sum_{n\le x}\sigma_\alpha(f(n))=\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\zeta(s)R(s)\frac{x^s}{s}ds+O(x^{1/2+\varepsilon}).$$

Where

$$R(s)=\prod_{p^\beta\|m}\left(1+\frac{1}{p^{s-\alpha}}+\frac{1}{p^{2(s-\alpha)}}+\cdots+\frac{1}{p^{\beta(s-\alpha)}}\right)$$

To estimate the main term

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\zeta(s)R(s)\frac{x^s}{s}ds,$$

we move the integral line from $2\pm iT$ to $1/2\pm iT$. This time, the function

$$\zeta(s)R(s)\frac{x^s}{s}$$

have a simple pole point at $s=1$, so we have

$$\frac{1}{2\pi i}\left(\int_{2-iT}^{2+iT}+\int_{2+iT}^{1/2+iT}+\int_{1/2+iT}^{1/2-iT}+\int_{1/2-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^s}{s}ds=R(1)x.$$

Taking $T=x^{\frac{3}{2}}$, we have

$$\begin{aligned}&\left|\frac{1}{2\pi i}\left(\int_{2+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^s}{s}ds\right|\\ \ll\ &\int_{\frac{1}{2}}^{2}\left|\zeta(\sigma+iT)R(s)\frac{x^2}{T}\right|d\sigma\\ \ll\ &\frac{x^2}{T}=x^{\frac{1}{2}};\end{aligned}$$

And we can easy get the estimate

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\zeta(s)R(s)\frac{x^s}{s}ds\right|\ll\int_0^T\left|\zeta(\frac{1}{2}+it)R(s)\frac{x^{\frac{1}{2}}}{t}\right|dt\ll x^{\frac{1}{2}+\varepsilon};$$

For

$$R(1)=\prod_{p^\beta\|m}\left(1+\frac{1}{p^{1-\alpha}}+\frac{1}{p^{2(1-\alpha)}}+\cdots+\frac{1}{p^{\beta(1-\alpha)}}\right)=\frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}}$$

We can have

$$\sum_{n\le x}\sigma_\alpha(f(n))=\frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}}x+O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

This completes the proof of Lemma.

§3. Proof of the theorem

In this section, we shall complete the proof of Theorem. For any real number $x \geq 1$, let N be a fixed positive integer such that

$$N^k \leq x < (N+1)^k.$$

from the definition of $m_q(n)$ we have

$$\begin{aligned}
\sum_{n\leq x} \sigma_\alpha((m_q(n), m)) &= \sum_{n\leq x} \sigma_\alpha(([n^{\frac{1}{k}}], m)) \\
&= \sum_{1^k \leq i < 2^k} \sigma_\alpha(([i^{\frac{1}{k}}], m)) + \sum_{2^k \leq i < 3^k} \sigma_\alpha(([i^{\frac{1}{k}}], m)) \\
&+ \cdots + \sum_{N^k \leq i \leq x < (N+1)^k} \sigma_\alpha(([i^{\frac{1}{k}}], m)) + O(N^\varepsilon)t) \\
&= (2^k - 1)\sigma_\alpha((1, m)) + (3^k - 2^k)\sigma_\alpha((2, m)) \\
&+ \cdots + [(N+1)^k - N^k]\sigma_\alpha((N, m)) + O(N^\varepsilon) \\
&= \sum_{j\leq N} [(j+1)^k - j^k]\sigma_\alpha((j, m)) + O(N^\varepsilon),
\end{aligned}$$

where ε is any fixed positive number.
Let $A(N) = \sum_{j\leq N} \sigma_\alpha((j, m))$. From Lemma, we have

$$A(N) = \sum_{j\leq N} \sigma_\alpha((j, m)) = \frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}} N + O\left(N^{\frac{1}{2}+\varepsilon}\right),$$

And letting $f(j) = [(j+1)^k - j^k]$. By Abel's identity, we have

$$\begin{aligned}
&\sum_{j\leq N} [(j+1)^k - j^k]\sigma_\alpha((j, m)) \\
= \ & A(N)f(N) - A(1)f(1) - \int_1^N A(t)f'(t)dt \\
= \ & [\frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}} N + O\left(N^{\frac{1}{2}+\varepsilon}\right)][(N+1)^k - N^k \\
& -A(1)f(1) - \int_1^N [\frac{\sigma_{1-\alpha}(m)}{m^{1-\alpha}} t + O\left(t^{\frac{1}{2}+\varepsilon}\right)] \\
& [k(t+1)^{k-1} - kt^{k-1}]dt
\end{aligned}$$

From the binomial theorem, we have

$$\sum_{j\leq N} [(j+1)^k - j^k]\sigma_\alpha((j, m)) = \frac{(2k-1)\sigma_{1-\alpha}(m)}{m^{1-\alpha}} N^k + O\left(N^{k-\frac{1}{2}+\varepsilon}\right)$$

So

$$\begin{aligned}\sum_{n\le x}\sigma_\alpha((m_q(n),m)) &= \sum_{n\le x}\sigma_\alpha(([n^{\frac{1}{k}}],m))\\ &= \sum_{j\le N}[(j+1)^k-j^k]\sigma_\alpha((j,m))+O(N^\varepsilon)\\ &= \frac{(2k-1)\sigma_{1-\alpha}(m)}{m^{1-\alpha}}x+O\left(x^{1-\frac{1}{2k}+\varepsilon}\right).\end{aligned}$$

This completes the proof of Theorem.

References

[1] F. Smarandache, Only Problems, Not Solutions, Xiquan Publishing House, Chicago, 1993.

[2] Pan Chengdong and Pan Chengbiao, Elements of the Analytic Number Theory, Beijing, Science Press, 1991, pp. 98.