On A New Smarandache Type Function

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Let $C_n^k = \binom{n}{k}$ denote a binomial coefficient, i.e.

$$C_n^k = \frac{n(n-1)\ldots(n-k+1)}{1*2*\ldots*k} = \frac{n!}{k!(n-k)!} \quad \text{for } 1 \leq k \leq n.$$

Clearly, $n \mid C_n^1$ and $n \mid C_n^{n-1} = C_n^1$. Let us define the following arithmetic function:

$$C(n) = \max \{ k: 1 \leq k < n-1, n \mid C_n^k \} \qquad (1)$$

Clearly, this function is well-defined and $C(n) \geq 1$. We have supposed $k < n - 1$, otherwise on the basis of

$C_n^{n-1} = C_n^1 = n$, clearly we would have $C(n) = n-1$.

By a well-known result on primes, $p \mid C_p^k$ for all primes p and $1 \leq k \leq p-1$.

Thus we get:

$$C(p) = p-2 \text{ for primes } p \geq 3. \qquad (2)$$

Obviously, $C(2) = 1$ and $C(1) = 1$. We note that the above result on primes is usually used in the inductive proof of Fermat's "little" theorem.

This result can be extended as follows:

Lemma: For $(k,n) = 1$, one has $n \mid C_n^k$.

Proof: Let us remark that

$$C_n^k = \frac{n}{k} * \frac{(n-1)\ldots(n-k+1)}{(k-1)!} = \frac{n}{k} * C_{n-1}^{k-1} \qquad (3)$$

thus, the following identity is valid:

$$k * C_n^k = n * C_{n-1}^{k-1} \qquad (3)$$

This gives $n \mid k*C_n^k$, and as $(n,k) = 1$, the result follows.

Theorem: C(n) is the greatest totient of n which is less than or equal to n – 2.

Proof: A totient of n is a number k such that (k,n) = 1. From the lemma and the definition of C(n), the result follows.

Remarks 1) Since (n-2,n) = (2,n) = 1 for odd n, the theorem implies that C(n) = n-2 for $n \geq 3$ and odd. Thus the real difficulty in calculating C(n) is for n an even number.
2) The above lemma and Newton's binomial theorem give an extension of Fermat's divisibility theorem $p \mid (a^p - a)$ for primes p.

References

1. F. Smarandache, *A Function in the Number Theory*. Anal. Univ. Timisoara, vol. XVIII, 1980.
2. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1979.