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An Application According to Spatial Quaternionic Smarandache Curve

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Abstract

In this paper, we found the Darboux vector of the spatial quaternionic curve according to the Frenet frame. Then, the curvature and torsion of the spatial quaternionic smarandache curve formed by the unit Darboux vector with the normal vector was calculated. Finally; these values are expressed depending upon the spatial quaternionic curve.

Mathematics Subject Classification: 11R52, 53A04

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1 Introduction

The quaternion was introduced by Hamilton. His initial attempt to generalis the complex numbers by introducing a 3-dimensional object failed in the sense that the algebra he constructed for these 3-dimensional objects did not have

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the desired properties. In 1987, Bharathi and Nagaraj defined the quaternionic curves in $\mathbb{E}^3$, $\mathbb{E}^4$ and studied the differential geometry of space curves and introduced Frenet frames and formulae by using quaternions, [2]. Following, quaternionic inclined curves have been defined and harmonic curvatures studied by Karadağ and Sivridağ, [7]. In, Tuna and Çöken have studied quaternion valued functions and quaternionic inclined curves in the semi-Euclidean space $\mathbb{E}^4_2$, [9]. They have given the Serret-Frenet formulae for the quaternionic curve in the semi-Euclidean space.Then they have defined quaternionic inclined curves and harmonic curvatures for the quaternionic curves in the semi-Euclidean space. Quaternionic rectifying curves have been studied by Güngör and Tosun, [5]. In [1], Ali has introduced some special Smarandache curves in the Euclidean space. He has studied Frenet-Serret invariants of a special case. In [4], Erişir and Güngör have obtained some characterizations of semi-real spatial quaternionic rectifying curves in $\mathbb{IR}^3_1$. Moreover, by the aid of these characterizations, they have investigated semi real quaternionic rectifying curves in semi quaternionic space.

2 Preliminary Notes

In this section, we give the basic elements of the theory of quaternions and quaternionic curves. A more complete elementary treatment of quaternions and quaternionic curves can be found in [2] and [6], respectively. A real quaternion q is an expression of the form

$$q = d + ae_1 + be_2 + ce_3 \tag{2.1}$$

where $a, b, c \in \mathbb{IR}$ and $e_i, 1 \leq i \leq 3$ are quaternionic units which satisfy the non-commutative multiplication rules

$$\begin{cases} {e_1}^2 = {e_2}^2 = {e_3}^2 = e_1 \times e_2 \times e_3 = -1, e_1, e_2, e_3 \in \mathbb{IR}^3 \\ e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_2 \times e_3 = e_1. \end{cases} \tag{2.2}$$

The algebra of the quaternions is denoted Q by and its natural basis is given by $\{e_1, e_2, e_3\}$. A real quaternion can be given by the form

$$q = S_q + V_q \tag{2.3}$$

where $S_q = d$ is scalar part and $V_q = ae_1 + be_2 + ce_3$ is vector part of q. The Hamilton conjugate of $q = S_q + V_q$ is defined by $\bar{q} = S_q - V_q$. Summation of two quaternions $q_1 = S_{q_1} + V_{q_1}$ and $q_2 = S_{q_2} + V_{q_2}$ is defined as $q_1 \oplus q_2 = (S_{q_1} + S_{q_2}) + (V_{q_1} + V_{q_2})$. Multiplication of a quaternion $q = S_q + V_q$ with a scalar $\lambda \in \mathbb{R}$ is identified as $\lambda \odot q = \lambda S_q + \lambda V_q$. These expression the symmetric real-valued, non-degenerate, bilinear form as follows:

$$\langle , \rangle_{|Q} : Q \times Q \to \mathbb{R} \ , \langle q_1, q_2 \rangle|_Q = \frac{1}{2}(q_1 \times \bar{q}_2 + q_2 \times \bar{q}_1) \tag{2.4}$$

which is called the quaternion inner product. Then the norm of q is

$$N(q) = \sqrt{q \times \bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}. \tag{2.5}$$

If $q = 1$, then q is called unit quaternion. Let $q_1 = S_{q_1} + V_{q_1} = d_1 + a_1e_1 + b_1e_2 + c_1e_3$ and $q_2 = S_{q_2} + V_{q_2} = d_2 + a_2e_1 + b_2e_2 + c_2e_3$ be two quaternions in, Q, then the quaternion product of q_1 and q_2 is given by

$$\begin{aligned} q_1 \times q_2 &= d_1d_2 - (a_1a_2 + b_1b_2 + c_1c_2) + (d_1a_2 + a_1d_2 + b_1c_2 - c_1b_2)e_1 \\ &+ (d_1b_2 + b_1d_2 + b_1a_2 - a_1b_2)e_2 + +(d_1c_2 + c_1d_2 + a_1b_2 - b_1a_2)e_3 \end{aligned}$$

or

$$q_1 \times q_2 = S_{q_1}S_{q_2} - \langle V_{q_1}, V_{q_2} \rangle + S_{q_1}V_{q_2} + S_{q_2}V_{q_1} + V_{q_1} \wedge V_{q_1} \tag{2.6}$$

where $\langle , \rangle$ and $\wedge$ denote the inner product and vector product in Euclidean 3-space. q is called a spatial quaternion whenever $q + \bar{q} = 0$ and called a temporal quaternion whenever $q - \bar{q} = 0$. A general quaternion q can be given as $q = \frac{1}{2}(q + \bar{q}) + \frac{1}{2}(q - \bar{q})$. The three-dimensional Euclidean space is identified with the space of spatial quaternions, [2].
$Q_H = \{q \in Q \mid q + \bar{q} = 0\}$ in an obvious manner. Let $I = [0, 1]$ be an interval in the real line $\mathbb{IR}$ and $s \in I$ be the arc-length parameter along the smooth curve

$$\gamma : [0, 1] \to Q_H, \ \ \gamma(s) = \sum_{i=1}^{3} \gamma_i(s)e_i. \tag{2.7}$$

The tangent vector $\gamma'(s) = t(s)$ has unit length $\|t(s)\| = 1$ for all s. It follows

$$t' \times \bar{t} + t + (\bar{t})' = 0$$

which implies t' is orthogonal to t and $t' \times \bar{t}$ is a spatial quaternion. Let $\gamma : [0, 1] \to Q_H$ be a differentiable spatial quaternions curve with arc-length parameter s and $\{t(s), n_1(s), n_2(s)\}$ be the Frenet frame of γ at the point $\gamma(s)$, where

$$\begin{cases} t(s) = \gamma'(s) \\ n_1(s) = \dfrac{\gamma''(s)}{N(\gamma''(s))} \\ n_2(s) = t(s) \times n_1(s), \end{cases} \tag{2.8}$$

and the curve $\gamma(s)$ is non unit speed curve then we say that

$$\begin{cases} t(s) = \dfrac{\gamma'(s)}{\nu(s)}, \ \nu(s) = N(\gamma'(s)) \\ n_1(s) = n_2(s) \times t(s) \\ n_2(s) = \dfrac{\gamma'(s) \times \gamma''(s) + \nu(s)\nu'(s)}{N\big(\gamma'(s) \times \gamma''(s) + \nu(s)\nu'(s)\big)}. \end{cases} \tag{2.9}$$

Let $\{t(s), n_1(s), n_2(s)\}$ be the Frenet frame of $\gamma(s)$. Then Frenet formula, curvature and the torsion are given by

$$\begin{cases} t'(s) = k(s)n_1(s) \\ {n_1}'(s) = -k(s)t(s) + r(s)n_2(s) \\ {n_2}'(s) = -r(s)n_1(s), \end{cases} \tag{2.10}$$

and

$$\begin{cases} k(s) = \dfrac{N\big(\gamma'(s) \times \gamma''(s) + \nu(s)\nu'(s)\big)}{\nu(s)^3} \\ r(s) = \dfrac{\langle \gamma'(s) \times \gamma''(s), \gamma'''(s)\rangle_{|Q}}{\Big[N\big(\gamma'(s) \times \gamma''(s) + \nu(s)\nu'(s)\big)\Big]^2}, \end{cases} \tag{2.11}$$

where $t(s), n_1(s), n_2(s)$ are the unit tangent, the unit principal normal andthe unit binormal vector of a quaternionic curve, respectively ([2], [8]). The functions k, r are called the principal curvature and the torsion, respectively. These are

$$\begin{aligned} t(s) \times t(s) &= n_1(s) \times n_1(s) = n_2(s) \times n_2(s) = -1 \\ t(s) \times n_1(s) &= -n_1(s) \times t(s) = n_2(s) \\ n_1(s) \times n_2(s) &= -n_2(s) \times n_1(s) = t(s) \\ n_2(s) \times t(s) &= -t(s) \times n_2(s) = n_1(s). \end{aligned}$$

Let $\gamma : [0,1] \to Q_H$ be a unit speed regular curve and $\{t(s), n_1(s), n_2(s)\}$ be its moving Serret-Frenet frame. In this case tn_1, n_1n_2, tn_1n_2- Quaternionic Smarandache curves can be defined by

$$\begin{aligned}\beta_{tn_1} &= \frac{1}{\sqrt{2}}(t(s)+n_1(s))\\ \beta_{n_1n_2} &= \frac{1}{\sqrt{2}}(n_1(s)+n_2(s))\\ \beta_{tn_1n_2} &= \frac{1}{\sqrt{3}}(t(s)+n_1(s)+n_2(s)), (see[1],[3],[8]).\end{aligned}$$

3 An Application According to Spatial Quaternionic Smarandache Curve

$\gamma : [0,1] \to Q_H$ spatial quaternions curve, $\{t(s), n_1(s), n_2(s)\}$ moving frame moves with a certain angular velocity around each axis s instantly. This axis is called instantaneous rotation axis of the spatial quaternionic curve. If Darboux axis vector in the direction indicated by D

$$D = xt + yn_1 + zn_2.$$

From Darboux equations, $t', {n_1}', {n_2}' \in Q_H$ derivative vectors

$$\begin{aligned} t' &= D\times t = zn_1 - yn_2\\ {n_1}' &= D\times n_1 = -zt + xn_2\\ {n_2}' &= D\times n_2 = yt - xn_1\end{aligned}$$

and from (2.10) $z = k, y = 0, x = r$. If we write these values

$$D = rt + kn_2. \tag{3.1}$$

The norm is

$$N(D) = \sqrt{D\times\overline{D}} = \sqrt{k^2+r^2}. \tag{3.2}$$

Let D is instantaneos pfaff vector of γ curve. If the angle between D and n_2 is φ, from Fig.1, it is obtained that

$$\cos\varphi = \frac{\langle D, kn_2\rangle_{|Q}}{N(D)N(kn_2)} = \frac{D\times\overline{kn_2} + kn_2\times\overline{D}}{2k\sqrt{k^2+r^2}},$$

$$\sin\varphi = \frac{\langle D, rt\rangle_{|Q}}{N(D)N(rt)} = \frac{D\times\overline{rt} + rt\times\overline{D}}{2r\sqrt{k^2+r^2}}$$

and

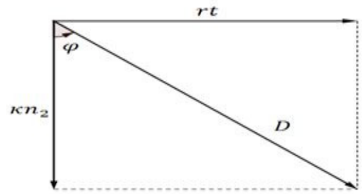

Figure 1: *Quaternionic Darboux Vector*

$$\cos\varphi = \frac{k}{\sqrt{k^2+r^2}}, \quad \sin\varphi = \frac{r}{\sqrt{k^2+r^2}} \tag{3.3}$$

If the unit vector of quaternionic darboux vector indicated by w

$$w = \frac{D}{N(D)} = \sin\varphi t + \cos\varphi n_2.$$

Conclusion 3.1 *Let $\gamma : [0,1] \to Q_H$ be a unit speed regular curve and $\{t(s), n_1(s), n_2(s)\}$ be its moving Serret-Frenet frame. For an arbitrary curve γ, with curvature and torsion, $k(s)$ and $r(s)$ respectively. Darboux vector in the direction of the axis of the quaternionic curve D*

$$D = rt + kn_2$$

and $\cos\varphi = \frac{k}{N(D)}$, $\sin\varphi = \frac{r}{N(D)}$ including, unit Darboux vector is

$$w = \sin\varphi t + \cos\varphi n_2.$$

Let $\gamma : [0,1] \to Q_H$ be a unit speed regular curve and $\{t(s), n_1(s), n_2(s)\}$ be its moving Serret-Frenet frame. Quaternionic n_1w- Smarandache curves can be defined by

$$\beta(s) = \frac{1}{\sqrt{2}}(n_1(s) + w(s)). \tag{3.4}$$

Now, we can investigate Serret-Frenet invariants of quaternionic n_1w- Smarandache curves according to $\gamma = \gamma(s)$. Differentiating (3.4) with respect to s_β, we get

$$\beta' = t_\beta(s)\frac{ds_\beta}{ds} = \frac{1}{\sqrt{2}}\big[(\varphi'\cos\varphi - k)t + (r - \varphi'\sin\varphi)n_2\big] \tag{3.5}$$

where

$$\frac{ds_\beta}{ds} = \frac{\sqrt{(\varphi')^2 + N(D)^2 - 2\varphi' N(D)}}{\sqrt{2}}.$$

The tangent vector of curve β can be written as follow

$$t_\beta(s) = \frac{(\varphi'\cos\varphi - k)t + (r - \varphi'\sin\varphi)n_2}{\sqrt{(\varphi')^2 + N(D)^2 - 2\varphi' N(D)}}, \tag{3.6}$$

differentiating (3.6) with respect to s, we obtain

$$t_\beta{}'(s) = \frac{\sqrt{2}(\lambda_1 t + \lambda_2 n_1 + \lambda_3 n_2)}{\big((\varphi')^2 + N(D)^2 - 2\varphi' N(D)\big)^2}, \tag{3.7}$$

where

$$\begin{cases}
\lambda_1 = r^2\varphi''\cos\varphi - k\varphi'\varphi''\cos^2\varphi - r\varphi'\varphi''\sin\varphi\cos\varphi - (\varphi')^4\sin\varphi - k^2(\varphi')^2\sin\varphi \\
\quad -r^2(\varphi')^2\sin\varphi + 2k(\varphi')^3\sin\varphi\cos\varphi + 2r(\varphi')^3\sin^2\varphi - k'(\varphi')^2 - r^2k' \\
\quad -2k'k\varphi'\cos\varphi - 2k'r\varphi'\sin\varphi - rr'\varphi'\cos\varphi + k^2(\varphi')^2\cos^2\varphi + r(\varphi')^2\sin\varphi\cos\varphi \\
\quad +k\varphi'\varphi'' + krr' - kr\varphi''\sin\varphi - \varphi'r'k\sin\varphi \\
\lambda_2 = k(\varphi')^3\cos\varphi + 3k^3\varphi'\cos\varphi + 3r^2k\varphi'\cos\varphi - 2k^2(\varphi')^2\cos^2\varphi - 4kr(\varphi')^2\sin\varphi\cos\varphi \\
\quad -k^2(\varphi')^2 - k^4 - 2k^2r^2 + 3k^2r\varphi'\sin\varphi - r^2(\varphi')^2 + 3r^3\varphi'\sin\varphi + r(\varphi')^3\sin\varphi \\
\quad -2r^2\varphi'^2\sin^2\varphi \\
\lambda_3 = r'(\varphi')^2 + k^2r' - 2kr'\varphi'\cos\varphi - k^2\varphi''\sin\varphi + k\varphi'\varphi''\sin\varphi\cos\varphi + r\varphi'\varphi''\sin^2\varphi \\
\quad -(\varphi')^4\cos\varphi - k^2(\varphi')^2\cos\varphi - k^2(\varphi')^2\cos\varphi + 2k(\varphi')^3\cos^2\varphi + 2r(\varphi')^3\sin\varphi\cos\varphi \\
\quad -r\varphi'\varphi'' - rkk' + rk\varphi''\cos\varphi + rk'\varphi'\cos\varphi + kk'\varphi'\sin\varphi - k'(\varphi')^2\sin\varphi\cos\varphi \\
\quad -r'(\varphi')^2\sin^2\varphi
\end{cases}$$

The principal curvature and principal normal vector field of curve β are respectively,

$$\kappa_\beta = \frac{\sqrt{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}}{\big((\varphi')^2 + N(D)^2 - 2\varphi' N(D)\big)^2} \tag{3.8}$$

and

$$n_\beta = \frac{\lambda_1 t + \lambda_2 n_1 + \lambda_3 n_2}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}. \tag{3.9}$$

On the other hand, we express $b_\beta = t_\beta \times n_\beta$. So, the binormal vector of curve

$$\begin{aligned} b_\beta &= \frac{\lambda_2(\varphi' \sin\varphi - r)t + \big(\lambda_1(r - \varphi' \sin\varphi) - \lambda_3(\varphi' \cos\varphi - k)\big)n_1}{\sqrt{\big((\varphi')^2 + N(D)^2 - 2\varphi' N(D)\big)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}} \\ &\quad + \frac{\lambda_2(\varphi' \cos\varphi - k)n_2}{\sqrt{\big((\varphi')^2 + N(D)^2 - 2\varphi' N(D)\big)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}} \end{aligned} \tag{3.10}$$

We differentiate (3.5) with respect to s in order to calculate the torsion

$$\begin{aligned} \beta'' &= \frac{(\varphi'' \cos\varphi - (\varphi')^2 \sin\varphi - k')t + (k\varphi' \cos\varphi + r\varphi' \sin\varphi - k^2 - r^2)n_1}{\sqrt{2}} \\ &\quad + \frac{(r' - \varphi'' \sin\varphi - (\varphi')^2 \cos\varphi)n_2}{\sqrt{2}} \end{aligned} \tag{3.11}$$

and similarly

$$\beta''' = \frac{\omega_1 t + \omega_2 n_1 + \omega_3 n_2}{\sqrt{2}}$$

where

$$\begin{cases} \omega_1 = \varphi''' \cos\varphi - 3\varphi'\varphi'' \sin\varphi - (\varphi')^3 \cos\varphi - k'' - k^2\varphi' \cos\varphi - kr\varphi' \sin\varphi + k^3 + kr^2 \\ \omega_2 = 2k\varphi'' \cos\varphi - 2k(\varphi')^2 \sin\varphi - 3kk' + k\varphi'' \cos\varphi + r'\varphi' \sin\varphi + 2r\varphi'' \sin\varphi \\ \qquad +2(\varphi')^2 \cos\varphi - 3rr' \\ \omega_3 = (kr\varphi' - 3\varphi'\varphi'') \cos\varphi + (r^2\varphi' - \varphi''' + (\varphi')^3) \sin\varphi - k^2 r - r^3 + r'' \end{cases}$$

The torsion of curve β is

$$\tau_\beta = \frac{\sqrt{2}(\varpi_1\omega_1 + \varpi_2\omega_2 + \varpi_3\omega_3)}{\varpi_1^2 + \varpi_2^2 + \varpi_3^2} \tag{3.12}$$

where

$$\begin{cases} \varpi_1 = (k(\varphi')^2 \sin\varphi)\cos\varphi + (r(\varphi')^2 \sin\varphi - k^2\varphi' - 2r^2\varphi')\sin\varphi + k^2 r + r^3 \\ \\ \varpi_2 = (r\varphi'' - \tau'\varphi' - k(\varphi')^2)\cos\varphi + (k'\varphi' - r(\varphi')^2 - k\varphi'')\sin\varphi - rk' + (\varphi')^3 + kr' \\ \\ \varpi_3 = (k(\varphi')^2\cos\varphi + r(\varphi')^2\sin\varphi - 2k^2\varphi' - r^2\varphi')\cos\varphi - kr\varphi'\sin\varphi + k^3 + kr^2. \\ \end{cases}$$

Example: Let be spatial quaternionic curve

$$\gamma(s) = \Big(\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{5}} + \frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{5}}\Big)e_1 - \frac{2s}{\sqrt{5}}e_2 + \Big(-\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{5}} + \frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{5}}\Big)e_3.$$

In terms of definition, we obtain special $n_1 w-$ smarandache curve according to Frenet frame of spatial quaternionic curve, (see Figure 3).

$$\beta(s) = \Big(-\frac{1}{2}\cos\frac{s}{\sqrt{5}} - \frac{1}{2}\sin\frac{s}{\sqrt{5}}\Big)e_1 - \frac{1}{\sqrt{2}}e_2 + \Big(\frac{1}{2}\cos\frac{s}{\sqrt{5}} - \frac{1}{2}\sin\frac{s}{\sqrt{5}}\Big)e_3.$$

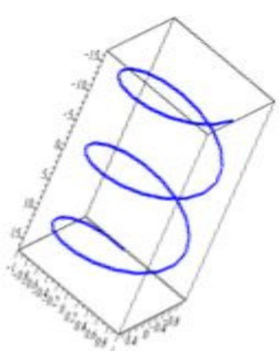

Figure 2: γ Spatial Quaternionic Curve

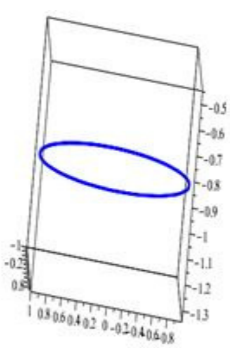

Figure 3: $\beta-$ Smarandache Curve

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