International J.Math. Combin. Vol.1(2015), 24-34

Antidegree Equitable Sets in a Graph

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Abstract: Let $G = (V, E)$ be a graph. A subset S of V is called a Smarandachely antidegree equitable k-set for any integer k, $0 \le k \le \Delta(G)$, if $|deg(u) - deg(v)| \ne k$, for all $u, v \in S$. A Smarandachely antidegree equitable 1-set is usually called an antidegree equitable set. The antidegree equitable number $AD_e(G)$, the lower antidegree equitable number $ad_e(G)$, the independent antidegree equitablenumber $AD_{ie}(G)$ and lower independent antidegree equitable number $ad_{ie}(G)$ are defined as follows:

$AD_e(G) = max\{|S| : \textit{ S is a maximal antidegree equitable set in } G\}$,
$ad_e(G) = min\{|S| : \textit{ S is a maximal antidegree equitable set in } G\}$,
$AD_{ie}(G) = max\{|S| : \textit{ S is a maximal independent and antidegree equitable set in } G\}$,
$ad_{ie}(G) = min\{|S| : \textit{ S is a maximal independent and antidegree equitable set in } G\}$.

In this paper, we study these four parameters on Smarandachely antidegree equitable 1-sets.

Key Words: Smarandachely antidegree equitable k-set, antidegree equitable set, antidegree equitable number, lower antidegree equitable number, independent antidegree equitable number, lower independent antidegree equitable number.

AMS(2010): 05C69

§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The number of vertices in a graph G is called the order of G and number of edges in G is called the size of G. For standard definitions and terminologies on graphs we refer to the books [2] and [3].

In this paper we introduce four graph theoretic parameters which just depend on the basic concept of vertex degrees. We need the following definitions and theorems, which can be found in [2] or [3].

Definition 1.1 *A graph G_1 is isomorphic to a graph G_2, if there exists a bijection ϕ from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if, and only if, $\phi(u)\phi(v) \in E(G_2)$.*

If G_1 is isomorphic to G_2, we write $G_1 \cong G_2$ or sometimes $G_1 = G_2$.

[1]Received June 16, 2014, Accepted February 18, 2015.

Definition 1.2 *The degree of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $deg(v)$ or $deg_G(v)$.*

The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$ respectively.

Theorem 1.3 *In any graph G, the number of odd vertices is even.*

Theorem 1.4 *The sum of the degrees of vertices of a graph G is twice the number of edges.*

Definition 1.5 *The corona of two graphs G_1 and G_2 is defined to be the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.*

Theorem 1.6 *Let G be a simple graph i.e, a undirected graph without loops and multiple edges, with $n \geq 2$. Then G has atleast two vertices of the same degree.*

Definition 1.7 *Any connected graph G having a unique cycle is called a unicyclic graph.*

Definition 1.8 *A graph is called a caterpillar if the deletion of all its pendent vertices produces a path graph.*

Definition 1.9 *A subset S of the vertex set V in a graph G is said to be independent if no two vertices in S are adjacent in G.*

The maximum number of vertices in an independent set of G is called the *independence number* and is denoted by $\beta_0(G)$.

Theorem 1.10 *Let G be a graph and $S \subset V$. S is an independent set of G if, and only if, $V - S$ is a covering of G.*

Definition 1.11 *A clique of a graph is a maximal complete subgraph.*

Definition 1.12 *A clique is said to be maximal if no super set of it is a clique.*

Definition 1.13 *The vertex degrees of a graph G arranged in non-increasing order is called degree sequence of the graph G.*

Definition 1.14 *For any graph G, the set $D(G)$ of all distinct degrees of the vertices of G is called the degree set of G.*

Definition 1.15 *A sequence of non-negative integers is said to be graphical if it is the degree sequence of some simple graph.*

Theorem 1.16([1]) *Let G be any graph. The number of edges in G^{de} the degree equitable graph of G, is given by*

$$\sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2} - \sum_{i=\delta+1}^{\Delta} \binom{|S_i{}'|}{2},$$

where, $S_i = \{v|v \in V,\ deg(v) = i \text{ or } i+1\}$ and $S_i{}' = \{v|v \in V,\ deg(v) = i\}$.

Theorem 1.17 *The maximum number of edges in G with radius $r \geq 3$ is given by*

$$\frac{n^2 - 4nr + 5n + 4r^2 - 6r}{2}.$$

Definition 1.18 *A vertex cover in a graph G is such a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha(G)$ of G.*

Recently A. Anitha, S. Arumugam and E. Sampathkumar [1] have introduced degree equitable sets in a graph and studied them. "The characterization of degree equitable graphs" is still an open problem. In this paper we give some necessary conditions for a graph to be degree equitable. For this purpose, we introduce another concept "Antidegree equitable sets" in a graph and we study them.

§2. Antidegree Equitable Sets

Definition 2.1 *Let $G = (V, E)$ be a graph. A non-empty subset S of V is called an antidegree equitable set if $|deg(u) - deg(v)| \neq 1$ for all $u, v \in S$.*

Definition 2.2 *An antidegree equitable set is called a maximal antidegree equitable set if for every $v \in V - S$, there exists at least one element $u \in S$ such that $|deg(u) - deg(v)| = 1$.*

Definition 2.3 *The antidegree equitable number $AD_e(G)$ of a graph G is defined as $AD_e(G) = max\{|S| : S$ is a maximal antidegree equitable set$\}$.*

Definition 2.4 *The lower antidegree equitable number $ad_e(G)$ of a graph G is defined as $ad_e(G) = min\{|S| : S$ is a maximal antidegree equitable set$\}$.*

A few $AD_e(G)$ and $ad_e(G)$ of some graphs are listed in the following:

(i) For the complete bipartite graph $K_{m,n}$, we have

$$AD_e(K_{m,n}) = \begin{cases} m+n & \text{if } |m-n| \neq 1, \\ max\{m,n\} & \text{if } |m-n| = 1 \end{cases}$$

and

$$ad_e(K_{m,n}) = \begin{cases} m+n & \text{if } |m-n| \neq 1, \\ min\{m,n\} & \text{if } |m-n| = 1. \end{cases}$$

(ii) For the wheel W_n on n-vertices, we have

$$AD_e(W_n) = \begin{cases} n & \text{if } n \neq 5, \\ 4 & \text{if } n = 5 \end{cases}$$

and

$$ad_e(W_n) = \begin{cases} n & \text{if } n \neq 5, \\ 1 & \text{if } n = 5. \end{cases}$$

(iii) For the complete graph K_n, we have $AD_e(K_n) = ad_e(K_n) = n - 1$.

Now we study some important basic properties of antidegree equitable sets and independent antidegree equitable sets in a graph.

Theorem 2.5 *Let G be a simple graph on n-vertices. Then*

(i) $1 \leq ad_e(G) \leq AD_e(G) \leq n$;

(ii) $AD_e(G) = 1$ *if, and only if,* $G = K_1$;

(iii) $ad_e(G) = ad_e(\overline{G})$, $AD_e(G) = AD_e(\overline{G})$.

(iv) $ad_e(G) = 1$ *if, and only if, there exists a vertex* $u \in V(G)$ *such that* $|deg(u) - deg(v)| = 1$ *for all* $v \in V - \{u\}$;

(v) *If G is a non-trivial connected graph and $ad_e(G) = 1$, then $AD_e(G) = n - 1$ and n must be odd.*

Proof (i) follows from the definition.

(ii) Suppose $AD_e(G) = 1$ and $G \neq K_1$. Then G is a non-trivial graph and from Theorem 1.6 there exists at least two vertices of same degree and they form an antidegree equitable set in G. So $AD_e(G) \geq 2$ which is a contradiction. The converse is obvious.

(iii) Since $deg_{\overline{G}}(u) = (n-1) - deg_G(u)$, it follows that an antidegree equitable set in G is also an antidegree equitable set in $\overline{G}$.

(iv) If $ad_e(G) = 1$ and there is no such vertex u in G, then $\{u\}$ is not a maximal antidegree equitable set for any $u \in V(G)$ and hence $ad_e(G) \geq 2$ which is a contradiction. The converse is obvious.

(v) Suppose G is a non-trivial connected graph with $ad_e(G) = 1$. Then there exists a vertex $u \in V$ such that $|deg(u) - deg(v)| = 1$, $\forall\ v \in V - \{u\}$. Clearly, $|deg(v) - deg(w)| = 0$ *or* 2, $\forall\ v,\ w \in V - \{u\}$. Hence, $AD_e(G) = |V - \{u\}| = n - 1$. It follows from Theorem 1.4 that $(n-1)$ is even and thus n is odd. □

Theorem 2.6 *Let G be a non-trivial connected graph on n-vertices. Then $2 \leq AD_e(G) \leq n$ and $AD_e(G) = 2$ if, and only if, $G \cong K_2$ or P_2 or P_3 or $L(H)$ or $L^2(H)$ where H is the caterpillar T_5 with spine $P = (v_1 v_2)$.*

Proof By Theorem 2.5, for a non-trivial connected graph G on n-vertices, we have $2 \leq AD_e(G) \leq n$. Suppose $AD_e(G) = 2$. Then for each antidegree equitable set S in G, we have $|S| \leq 2$. Let $D(G) = \{d_1, d_2, \ldots, d_k\}$, where $d_1 < d_2 < d_3 < \cdots < d_k$. As there are at least two vertices with same degree, we have $k \leq n - 1$. Since $AD_e(G) = 2$, more than two vertices cannot have the same degree. Let $d_i \in D(G)$ be such that exactly two vertices of G have degree d_i. Since the cardinality of each antidegree equitable set S cannot exceed two, it follows that

$\cdots, d_i-3, d_i-2, d_i+2, d_i+3, d_i+4, \cdots$ do not belong to $D(G)$. Thus $D(G) \subset \{d_i-1, d_i, d_i+1\}$.

Case 1. If $d_i - 1, d_i + 1$ do not belong to $D(G)$ then $D(G) = \{d_i\}$ and the degree sequence $\{d_i, d_i\}$ is clearly graphical. Thus $n = 2$ and $d_i = 1$ which implies $G = K_2$.

Case 2. If $d_i - 1, d_i + 1 \in D(G)$, then the degree sequence $\{d_i - 1, d_i, d_i, d_i + 1\}$ is graphical. Thus $n = 4$ and $d_i = 2$ which implies $G \cong L(H)$, where H is the caterpillar T_5 with spine $P = (v_1v_2)$.

Case 3. If $d_i-1 \in D(G)$ and d_i+1 does not belong to $D(G)$, then d_i-1 may or may not repeat twice in degree sequence. Thus degree sequence is given by $\{d_i-1, d_i, d_i\}$ or $\{d_i-1, d_i-1, d_i, d_i\}$. The first sequence is not graphical but the second sequence is graphical. Thus $n = 4$ and $d_i = 2$ which implies $G \cong P_4$.

Case 4. If $d_i - 1$ does not belong to $D(G)$ and $d_i + 1 \in D(G)$, then the degree sequence is given by $\{d_i, d_i, d_i + 1\}$ or $\{d_i, d_i, d_i + 1, d_i + 1\}$. Both sequences are graphical. In the first case $n = 3$, $d_i = 1$ which implies $G \cong P_2$, and in the second case $n = 4$, $d_i = 1$ or 2 which implies $G \cong P_3$ or $G \cong L^2(H)$ respectively.

The converse is obvious. □

Theorem 2.7 *If a and b are positive integers with $a \leq b$, then there exists a connected simple graph G with $ad_e(G) = a$ and $AD_e(G) = b$ except when $a = 1$ and $b = 2m + 1$, $m \in N$.*

Proof If $a = b$ then for any regular graph of order a, we have $ad_e(G) = AD_e(G) = a$. If $b = a + 1$, then for the complete bipartite graph $G = k_{a,a+1}$ we have $ad_e(G) = a$ and $AD_e(G) = a + 1 = b$. If $b \geq a + 2$, $a \geq 2$, and $b > 4$, then for the graph G consisting of the wheel W_{b-1} and the path $P_a = (v_1v_2v_3 \ldots v_a)$ with an edge joining a pendant vertex of P_a to the center of the wheel W_{b-1}, we have $ad_e(G) = a$, $AD_e(G) = b$. If $a = 1$ and $b = 2m$, $m \in N$, then the graph consisting of two cycles C_m and C_{m+1} along with edges joining i^{th} vertex of C_m to i^{th} vertex of C_{m+1}, we have $ad_e(G) = 1 = a$ and $AD_e(G) = 2m = b$.

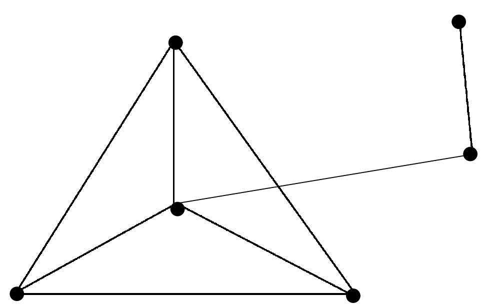

Figure 1

For $a = 2$ and $b = 4$ we consider graph G in Figure 1, for which $ad_e(G) = 2$ and $AD_e(G) = 4$. Also, it follows from Theorem 2.5 that there is no graph G with $ad_e(G) = 1$ and $AD_e(G) = 2m + 1$. □

Theorem 2.8 *Let G be a non-trivial connected graph on n vertices and let S^* be a subset of V such that $|deg(u) - deg(v)| \geq 2$ for all $u, v \in S^*$. Then $1 \leq |S^*| \leq \left[\frac{\Delta - \delta}{2}\right] + 1$ and also, if S^* is a maximal subset of V such that $|deg(u) - deg(v)| \geq 2$ for all u, $v \in S^*$, then $S = \bigcup_{v \in S^*} S_{deg(v)}$ is a maximal antidegree equitable set in G, where $S_{deg(v)} = \{u \in V : deg(u) = deg(v)\}$.*

Proof For any two vertices $u, v \in S^*$, $d(u)$ and $d(v)$ cannot be two successive members of $A = \{\delta, \delta+1, \delta+2, \ldots, \delta+k = \Delta\}$ and $D(G) \subset A$. Hence

$$|S^*| \leq \left[\frac{|D(G)| + 1}{2}\right] \leq \left[\frac{|A| + 1}{2}\right] = \left[\frac{\Delta - \delta}{2}\right] + 1.$$

If $a, b \in S = \bigcup_{v \in S^*} S_{deg(v)}$, then it is clear that either $|deg(a) - deg(b)| = 0$ or $|deg(a) - deg(b)| \geq 2$ and hence S is an antidegree equitable set. Suppose $u \in V - S$. Then $deg(u) \neq deg(v)$ for any $v \in S^*$. So, u do not belong to S^* and hence $|deg(u) - deg(v)| = 1$ for all $v \in S$. This implies that S is a maximal antidegree equitable set. □

Theorem 2.9 *Given a positive integer k, there exists graphs G_1 and G_2 such that $ad_e(G_1) - ad_e(G_1 - e) = k$ and $ad_e(G_2 - e) - ad_e(G_2) = k$.*

Proof Let $G_1 = K_{k+2}$. Then $ad_e(G_1) = k + 2$ and $ad_e(G_1 - e) = 2$, where $e \in E(G_1)$. Hence $ad_e(G_1) - ad_e(G_1 - e) = k$. Let G_2 be the graph obtained from C_{k+1} by attaching one leaf e at $(k+1)^{th}$ vertex of C_{k+1}. Then $ad_e(G_2 - e) - ad_e(G_2) = k$. □

Theorem 2.10 *Given two positive integers n and k with $k \leq n$. Then there exists a graph G of order n with $ad_e(G) = k$.*

Proof If $k < \frac{n}{2}$, then we take G to be the graph obtained from the path $P_k = (v_1 v_2 v_3 \ldots v_k)$ and the complete graph K_{n-k} by joining v_1 and a vertex of K_{n-k} by an edge. Clearly, $ad_e(G) = k$. If $k \geq \frac{n}{2}$, then we take G to be the graph obtained from the cycle C_k by attaching exactly one leaf at $(n - k)$ vertices of C_k. Clearly, $ad_e(G) = k$. □

§3. Independent Antidegree Equitable Sets

In this section, we introduce the concepts of independent antidegree equitable number and lower independent antidegree equitable number and establish important results on these parameters.

Definition 3.1 *The independent antidegree equitable number $AD_{ie}(G) = max\{|S| : S \subset V, S$ is a maximal independent and antidegree equitable set in $G\}$.*

Definition 3.2 *The lower independent antidegree equitable number $ad_{ie}(G) = min\{|S| :$*

S is a maximal independent and antidegree equitable set in $G\}$.

A few AD_{ie} and ad_{ie} of graphs are listed in the following.

(i) For the star graph $K_{1,n}$ we have, $AD_{ie}(K_{1,n}) = n$ and $ad_{ie}(K_{1,n}) = 1$.

(ii) For the complete bipartite graph $K_{m,n}$ we have $AD_{ie}(K_{m,n}) = max\{m,n\}$ and $ad_{ie}(K_{m,n}) = min\{m,n\}$.

(iii) For any regular graph G we have, $AD_{ie}(G) = ad_{ie}(G) = \beta_o(G)$.

The following theorem shows that on removal of an edge in G, $AD_{ie}(G)$ can decrease by at most one and increase by at most 2.

Theorem 3.3 *Let G be a connected graph, $e = uv \in E(G)$. Then*

$$AD_{ie}(G) - 1 \leq AD_{ie}(G - e) \leq AD_{ie}(G) + 2.$$

Proof Let S be an independent antidegree equitable set in G with $|S| = AD_{ie}(G)$. After removing an edge $e = uv$ from the graph G, we shall give an upper and a lower bound for $AD_{ie}(G - e)$.

Case 1. If u, v does not belong to S, then S is a maximal independent antidegree equitable set in $G - e$ as well as in G. Hence, $AD_{ie}(G - e) = AD_{ie}(G)$.

Case 2. If $u \in S$ and v does not belong to S, then $S - \{u\}$ is an independent antidegree equitable set in $G - e$. Hence, $AD_{ie}(G - e) \geq |S - \{u\}| = AD_{ie}(G) - 1$. Thus, $AD_{ie}(G) - 1 \leq AD_{ie}(G - e)$.

Now, Let S be an independent antidegree equitable set in $G - e$ with $|S| = AD_{ie}(G - e)$.

Case 3. If $u, v \in S$, then $S - \{u, v\}$ is an independent antidegree equitable set in G. Hence, by definition $AD_{ie}(G) \geq |S - \{u, v\}| = AD_{ie}(G - e) - 2$.

Case 4. If $u \in S$ and v does not belong to S, then $S - \{u\}$ is an independent antidegree equitable set in G. Hence, by definition $AD_{ie}(G) \geq |S - \{u\}| = AD_{ie}(G - e) - 1$.

Case 5. If u, v do not belong to S, then S is an independent antidegree equitable set in G. Hence, by definition $AD_{ie}(G) \geq |S| = AD_{ie}(G-e)$. It follows that $AD_{ie}(G) \geq AD_{ie}(G-e)-2$. Hence,

$$AD_{ie}(G) - 1 \leq AD_{ie}(G - e) \leq AD_{ie}(G) + 2. \qquad \square$$

Theorem 3.4 *Let G be a connected graph. $AD_{ie}(G) = 1$ if, and only if, $G \cong K_n$ or for any two non-adjacent vertices $u, v \in V$, $|deg(u) - deg(v)| = 1$.*

Proof Suppose $AD_{ie}(G) = 1$.

Case 1. If $G \cong K_n$, then there is nothing to prove.

Case 2. Let $G \neq K_n$, and u, v be any two non-adjacent vertices in G. Since $AD_{ie}(G) = 1$, $\{u, v\}$ is not an antidegree equitable set and hence $|deg(u) - deg(v)| = 1$. The converse is obvious. □

Theorem 3.5 *Let G be a connected graph. $ad_{ie}(G) = 1$ if, and only if, either $\Delta = n-1$ or for any two non-adjacent vertices $u, v \in V$, $|deg(u) - deg(v)| = 1$.*

Proof Suppose $ad_{ie}(G) = 1$, then for any two non-adjacent vertices u and v, $\{u, v\}$ is not an antidegree equitable set.

Case 1. If $\Delta = n-1$, then there is nothing to prove.

Case 2. Let $\Delta < n-1$, and u, v be any two non-adjacent vertices in G. Then $\{u, v\}$ is not an antidegree equitable set and hence, $|deg(u) - deg(v)| = 1$.

The converse is obvious. □

Remark 3.6 Theorems 3.4 and 3.5 are equivalent.

§4. Degree Equitable and Antidegree Equitable Graphs

After studying the basic properties of antidegree equitable and independent antidegree equitable sets in a graph, in this section we give some conditions for a graph to be degree equitable. We recall the definition of degree equitable graph given by A. Anitha, S. Arumugam, and E. Sampathkumar [1].

Definition 4.1 *Let $G = (V, E)$ be a graph. The degree equitable graph of G, denoted by G^{de} is defined as follows:$V(G^{de}) = V(G)$ and two vertices u and v are adjacent vertices in G^{de} if, and only if, $|deg(u) - deg(v)| \leq 1$.*

Example 4.2 For any regular graph G on n vertices, we have $G^{de} = K_n$.

Definition 4.3 *A graph H is called degree equitable graph if there exists a graph G such that $H \cong G^{de}$.*

Example 4.4 Any complete graph K_n is a degree equitable graph because $K_n = G^{de}$ for any regular graph G on n-vertices.

Theorem 4.5 *Let $G = (V, E)$ be any graph on n vertices with radius $r \geq 3$. Then*

(i) $1 \leq \beta_0(G^{de}) \leq \sqrt{n^2 - 4nr + 5n + 4r^2 - 6r}$.
(ii) $\beta_0(G^{de}) \leq \left[\frac{\Delta-\delta}{2}\right] + 1$, *where* $\Delta = \Delta(G)$ *and* $\delta = \delta(G)$.

Proof (*i*) Let A be an independent set of G^{de} such that $|A| = \beta_0(G^{de})$. Then A is an antidegree equitable set in G and hence

$$\sum_{v \in V} deg_G(v) \geq \sum_{v \in A} deg_G(v) = \sum_{\ell=1}^{\beta_0(G^{de})} 2\ell - 1 = {\beta_0}^2(G^{de}).$$

By Theorem 1.17 it follows that

$$2\left(\frac{n^2-4nr+5n+4r^2-6r}{2}\right) \geq {\beta_0}^2(G^{de}).$$

Therefore,

$$1 \leq \beta_0(G^{de}) \leq \sqrt{n^2-4nr+5n+4r^2-6r}.$$

(ii) We know that every independent set A in G^{de} is an antidegree equitable set in G and hence by Theorem 2.8,

$$|A| \leq \left[\frac{\Delta(G)-\delta(G)}{2}\right]+1.$$

Therefore,

$$\beta_0(G^{de}) \leq \left[\frac{\Delta(G)-\delta(G)}{2}\right]+1.$$

This completes the proof. □

Theorem 4.6 *Let H be any degree equitable graph on n vertices and $H = G^{de}$ for some graph G. Then*

$$\sqrt{\sum_{v\in A} deg_G(v)} \leq \left[\frac{\Delta(G)-\delta(G)}{2}\right]+1$$

where A is an independent set in G^{de} such that $|A| = \beta_0(G^{de})$.

Proof We know that if A is an independent set in H then it is an antidegree equitable set in G. Hence,

$$\sum_{v\in A} deg_G(v) \leq \sum_{\ell=1}^{\beta_0(H)} 2\ell-1 = {\beta_0}^2(H).$$

By Theorem 4.5

$$\sum_{v\in A} deg_G(v) \leq \left(\left[\frac{\Delta(G)-\delta(G)}{2}\right]+1\right)^2.$$

Therefore,

$$\sqrt{\sum_{v\in A} deg_G(v)} \leq \left[\frac{\Delta(G)-\delta(G)}{2}\right]+1.$$ □

We introduce a new concept antidegree equitable graph and present some basic results.

Definition 4.7 *Let $G = (V, E)$ be a graph. The antidegree equitable graph of G, denoted by G^{ade} defined as follows: $V(G^{ade}) = V(G)$ and two vertices u and v are adjacent in G^{ade} if, and only if, $|deg(u) - deg(v)| \neq 1$.*

Example 4.8 For a complete bipartite graph $K_{m,n}$, we have

$$G^{ade} = \begin{cases} K_{m+n} & \text{if } |m-n| \geq 2, \text{or} = 0 \\ K_m \cup K_n & \text{if } |m-n| = 1. \end{cases}$$

Definition 4.9 *A graph H is called an antidegree equitable graph if there exists a graph G such that $H \cong G^{ade}$.*

Example 4.10 Any complete graph K_n is an antidegree equitable graph because $K_n = G^{ade}$ for any regular graph G on n-vertices.

Theorem 4.11 *Let G be any graph on n vertices. Then the number of edges in G^{ade} is given by*

$$\binom{n}{2} - \sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2} + \binom{|S_\delta{}'|}{2} + 2\sum_{i=\delta+1}^{\Delta} \binom{|S_i{}'|}{2},$$

where $S_i = \{v |\ v \in V\ deg_G(v) = i \text{ or } i+1\}$, $S_i{}' = \{v |\ v \in V\ deg_G(v) = i\}$, $\Delta = \Delta(G)$ and $\delta = \delta(G)$.

Proof By Theorem 1.16, we have the number of edges in G^{ade} with end vertices having the difference degree greater than two in G is

$$\binom{n}{2} - \sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2} + \sum_{i=\delta+1}^{\Delta} \binom{|S_i{}'|}{2}.$$

and also, the number of edges in G^{ade} with end vertices having the same degree is

$$\sum_{i=\delta}^{\Delta} \binom{|S_i{}'|}{2}.$$

Hence, the total number of edges in G^{ade} is

$$\binom{n}{2} - \sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2} + \sum_{i=\delta+1}^{\Delta} \binom{|S_i{}'|}{2} + \sum_{i=\delta}^{\Delta} \binom{|S_i{}'|}{2}$$

$$= \binom{n}{2} - \sum_{i=\delta}^{\Delta-1} \binom{|S_i|}{2} + \binom{|S_\delta{}'|}{2} + 2\sum_{i=\delta+1}^{\Delta} \binom{|S_i{}'|}{2}.$$ □

Theorem 4.12 *Let G be any graph on n vertices. Then*

(i) $\alpha(G^{ade}) \le \sqrt{n(n-1)}$;
(ii) $\alpha(G^{ade}) \le \left[\frac{\Delta-\delta}{2}\right] + 1$, *where $\Delta = \Delta(G)$ and $\delta = \delta(G)$.*

Proof Let $A \subset V$ be the set of vertices that covers all edges of G^{ade}. Then A is an antidegree equitable set in G. Hence,

$$\sum_{v \in A} deg_G(v) \ge \sum_{\ell=1}^{\alpha(G^{ade})} 2\ell - 1 = \alpha^2(G^{ade}).$$

Therefore,

$$2\left(\frac{n(n-1)}{2}\right) \ge \alpha^2(G^{ade}),$$

$$\alpha(G^{ade}) \leq \sqrt{n(n-1)}.$$

Since, the set A is an antidegree equitable set in G, by Theorem 2.8, we have

$$|A| \leq \left[\frac{\Delta - \delta}{2}\right] + 1.$$

This implies

$$\alpha(G^{ade}) \leq \left[\frac{\Delta - \delta}{2}\right] + 1. \qquad \square$$

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