On a Generalized Bisector Theorem

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In the book [1] by Smarandache (see also [2]) appears the following generalization of the well-known bisector theorem.

Let AM be a cevian of the triangle which forms the angles u and v with the sides AB and AC, respectively. Then

$$\frac{AB}{AC} = \frac{MB}{MC} \cdot \frac{\sin v}{\sin u}. \tag{76}$$

We wish to mention here that relation (1) **also** appeared in my book [3] on page 112, where it is used for a generalization of Steiner's theorem. Namely, the following result holds true (see Theorem 25 in page 112):

Let AD and AE be two cevians ($D, E \in (BC)$) forming angles α, β with the sides AB, AC, respectively. If $\widehat{A} \leq 90°$ and $\alpha \leq \beta$, then

$$\frac{BD \cdot BE}{CD \cdot CE} \leq \frac{AB^2}{AC^2}. \tag{77}$$

Indeed, by applying the area resp. trigonometrical formulas of the area of a triangle, we get

$$\frac{BD}{CD} = \frac{A(ABD)}{A(ACD)} = \frac{AB \sin \alpha}{AC \sin(A - \alpha)}$$

(i.e. relation (1) with $u = \alpha$, $v = \beta - \alpha$). Similarly one has

$$\frac{BE}{CE} = \frac{AB \sin(A - \beta)}{AC \sin \beta}.$$

Therefore

$$\frac{BD \cdot BE}{CD \cdot CE} = \left(\frac{AB}{AC}\right)^2 \frac{\sin \alpha}{\sin \beta} \cdot \frac{\sin(A - \beta)}{\sin(A - \alpha)}. \tag{78}$$

Now, identity (3), by $0 < \alpha \leq \beta < 90°$ and $0 < A - \beta \leq A - \alpha < 90°$ gives immediately relation (2). This solution appears in [3]. For $\alpha = \beta$ one has

$$\frac{BD \cdot BE}{CD \cdot CE} = \left(\frac{AB}{AC}\right)^2 \tag{79}$$

which is the classical Steiner theorem. When $D \equiv E$, this gives the well known bisector theorem.

References

[1] F. Smarandache, *Problèmes avec et sans... problèmes!*, Somipress, Fes, Marocco, 1983.

[2] M.L. Perez, htpp/www.gallup.unm.edu/~smarandache/

[3] J. Sándor, *Geometric Inequalities* (Hungarian), Ed. Dacia, 1988.