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Comparative review of some properties of fuzzy and anti fuzzy subgroups

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Abstract. This paper is to comparatively review some works in fuzzy and anti fuzzy group theory. The aim is to provide anti fuzzy versions of some existing theorems in fuzzy group theory and see how much similar they are to their fuzzy versions. The research therefore focuses on the properties of fuzzy subgroup, fuzzy cosets, fuzzy conjugacy and fuzzy normal subgroups of a group which are mimicked in anti fuzzy group theory.

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1. Introduction

[15] started the research in fuzzy set and he worked on the properties of fuzzy sets in which every element $x \in X$ is assigned a membership value between 0 and 1 depending on to what degree does x belong to X. This was given a group structure by [12]. Furthermore, [2] developed the concept of level subgroups.

In 1984, [6] also introduced the concept of fuzzy left and right cosets. The concept of fuzzy normal subgroup was improved on by the work of [14] in 1997, which studied the concepts such as fuzzy middle cosets, pseudo fuzzy cosets and pseudo fuzzy double coset. The link between fuzzy subgroup and anti fuzzy subgroup was made by [1] in 1990.

2. Preliminaries

Definition 2.1 ([13])**.** Let X be a non-empty set. A fuzzy subset μ of the set X is a function $\mu : X \to [0, 1]$.

Definition 2.2 ([12, 13])**.** Let G be a group and μ a fuzzy subset of G. Then μ is called a fuzzy subgroup of G if
(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\mu(x^{-1}) = \mu(x)$
(iii) μ is called a fuzzy normal subgroup if $\mu(xy) = \mu(yx)$ for all x and y in G.

Definition 2.3 ([13])**.** Let μ and λ be any two fuzzy subsets of a set X. Then
(i) λ and μ are equal if $\mu(x) = \lambda(x)$ for every x in X
(ii) λ and μ are disjoint if $\mu(x) \neq \lambda(x)$ for every x in X
(iii) $\lambda \subseteq \mu$ if $\mu(x) \geq \lambda(x)$

Definition 2.4 ([13, 14])**.** Let μ be a fuzzy subset (subgroup) of X. Then, for some $t \in [0, 1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a t-level subset (subgroup) of the fuzzy subset (subgroup) μ.

Remark 2.5. If μ is a fuzzy subgroup of G and μ_t is a subgroup of G, the set μ_t can be represented as G^t_μ.

Definition 2.6 ([14])**.** Let μ be a fuzzy subgroup of a group G. The set $H = \{x \in G : \mu(x) = \mu(e)\}$ is such that $o(\mu) = o(H)$ and the order of μ is $\mid H \mid$.

Theorem 2.7 ([12])**.** *Let G be a group and μ a fuzzy subset of G. Then μ is called a fuzzy subgroup of G if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$.*

Definition 2.8 ([6, 13])**.** Let μ be a fuzzy (or an anti fuzzy) subgroup of a group G. For a in G, the fuzzy (or anti fuzzy) coset $a\mu$ of G determined by a and μ is defined by $(a\mu)(x) = \mu(a^{-1}x)$ for all x in G.

Definition 2.9 ([1])**.** Let G be a group. A fuzzy subset μ is an anti fuzzy subgroup of G if $\forall x, y \in G, \mu(xy^{-1}) \leq max\{\mu(x), \mu(y)\}$.

Definition 2.10 ([9])**.** Let μ be an anti fuzzy subgroup of G. For $a, b \in G$, anti fuzzy middle coset $(a\mu b)$ is defined by $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$.

Definition 2.11 ([9])**.** Let μ be an anti fuzzy subgroup of G. Then it is anti fuzzy normal if $\mu(xy) = \mu(yx)\ \forall x, y \in G$ or $\mu(x) = \mu(yxy^{-1})\ \forall x, y \in G$.

Definition 2.12 ([6])**.** Let μ be a fuzzy subgroup of G. Then it is fuzzy normal if $\mu(xy) = \mu(yx)\ \forall x, y \in G$ or $\mu(x) = \mu(yxy^{-1})\ \forall x, y \in G$.

Theorem 2.13 ([9])**.** *Let $a^{-1}\mu a$ be a fuzzy middle coset of G for some $a \in G$. Then all such a form the normaliser $N(\mu)$ of fuzzy subgroup μ of G if and only if μ is fuzzy normal.*

Definition 2.14 ([9])**.** Let μ and λ be any two anti fuzzy subgroup of G for any x and some $g \in G$. Then, they are said to be anti fuzzy conjugate if $\lambda(x) = \mu(g^{-1}xg)$

3. ANTI FUZZY COSETS AND CONJUGATES

Many of the results here have been treated in fuzzy version by [9]

Theorem 3.1. *Let $a^{-1}\mu a$ be an anti fuzzy middle coset of G for some $a \in G$. Then all such a form the normalizer $N(\mu)$ of the anti fuzzy subgroup μ of G if and only if μ is anti fuzzy normal.*

Proof. We define the normaliser of the anti fuzzy subgroup μ by $N(\mu) = \{a \in G : \mu(axa^{-1}) = \mu(x)\}$ [8]. Then, $\mu(axa^{-1}) = \mu(x) \Leftrightarrow \mu$ is anti fuzzy normal so that $\mu(axa^{-1}a) = \mu(xa) \Leftrightarrow \mu(ax) = \mu(xa)$.
Conversely, let μ be anti fuzzy normal and $a^{-1}\mu a$ an anti fuzzy middle coset in G. Then, for all $x \in G$ and some $a \in G$,

$$(a^{-1}\mu a)(x) = \mu(axa^{-1}) = \mu(aa^{-1}x) = \mu(x).$$

This implies that $\mu(axa^{-1}) = \mu(x)$. Hence, $\{a\} = N(\mu)$. □

It is important to remark that by 2.13 and 3.1, a fuzzy subgroup and its complement has the same normalizer.

Proposition 3.2. *Let μ be an anti fuzzy normal subgroup of G. Then every anti fuzzy middle coset $a\mu b$ by a and b coincides with some left and right anti fuzy cosets $c\mu$ and μc respectively, where c is the product $b^{-1}a^{-1}$.*

Proof. By associativity in G and 2.11, we have that

$$(a\mu b)(x) = \mu((a^{-1}x)b^{-1}) = \mu(b^{-1}(a^{-1}x))$$
$$\mu(b^{-1}a^{-1}x) = \mu(cx) = \mu(xc) \text{ still by 2.11.}$$

Therefore, $(a\mu b)(x) = (c\mu)(x) = (\mu c)(x)$
Thus, $(a\mu b) = c\mu = \mu c$. □

Theorem 3.3. *Let G be a group of order 2 and μ an anti fuzzy normal subgroup of G. Then, for some $a \in G$ and $\forall x \in G$, the anti fuzzy middle coset $a\mu a$ coincides with the anti fuzzy subgroup μ*

Proof. In the anti fuzzy middle coset $a\mu b$, take $a = b$. By associativity in G, we have

$$(a\mu a)(x) = \mu((a^{-1}x)a^{-1})$$

. By 3.2,

$$\mu((a^{-1}x)a^{-1}) = \mu(a^{-2}x).$$

Since $a^{-1} \in G$ and G is of order 2,

$$\mu(a^{-2}x) = \mu((a^{-1})^2x) = \mu(ex) = \mu(x).$$

Therefore,

$$a\mu a = \mu.$$

□

Remark 3.4. [9] has define that any two elements $a, b \in G$ are fuzzy μ-commutative if μ is a fuzzy subgroup when $a\mu b = b\mu a$. Similarly, we extend that this is also the definition when μ is an anti fuzzy subgroup of G.

Theorem 3.5. *Let μ be an anti fuzzy normal subgroup of G. Then any two elements a and b in G are anti fuzzy μ-commutative.*

Proof. $(a\mu b)(x) = \mu(a^{-1}xb^{-1})$. Then, 2.10, 2.11 and associativity of G,

$$\mu(a^{-1}xb^{-1}) = \mu(b^{-1}xa^{-1}) = (b\mu a)(x)$$

. Thus,

$$a\mu b = b\mu a$$

. □

Theorem 3.6. *Every anti fuzzy middle coset $a\mu b$ of a group G is an anti fuzzy subgroup if μ is anti fuzzy conjugate to some anti fuzzy subgroup λ of G.*

Proof. Let $b = a^{-1}$ for some $a, b \in G$ and μ and λ be anti fuzzy conjugate subgroups of G.

$$(a\mu b)(xy^{-1}) = (a\mu a^{-1})(xy^{-1}) = \mu(a^{-1}xy^{-1}a) = \lambda(xy^{-1}) \leq max\{\lambda(x), \lambda(y)\}$$

This implies that

$$max\{\lambda(x), \lambda(y)\} = max\{\mu(a^{-1}xa), \mu(a^{-1}ya)\} = max\{(a\mu a^{-1})(x), (a\mu a^{-1})(y)\}.$$

Hence,

$$(a\mu b)(xy^{-1}) \leq max\{(a\mu b)(x), (a\mu b)(y)\}.$$

□

CONCLUSION

Combining the results of [3], [4], [5], [7] [9], [10], [13] and this work, it can be seen that many algebraic properties of the underlying set(s) of fuzzy and anti fuzzy subgroups are very similar. It may then be concluded that many algebraic properties of a fuzzy subgroup of a group will hold similarly for the complement of the fuzzy subgroup, which is an anti fuzzy subgroup of the group. Variations may only occur when level subsets are involved as can be seen in [10] and [11]. This is because, if μ is a fuzzy subgroup and $t_1, t_2 \in [0, 1]$ such that $t_1 \leq t_2$, then $\mu_{t_2} \subseteq \mu_{t_1}$. But if μ is an anti fuzzy subgroup for such t_1 and t_2, $\mu_{t_1} \subseteq \mu_{t_2}$.

Hence, if results are established in fuzzy group theory, it may not worth much rigour to work on its anti fuzzy version. As a matter of fact, the anti fuzzy version can be established easily by just taking the complement of the fuzzy subgroup in the following sense: If μ is a fuzzy subgroup of a group, then, $1 - \mu$, its complement, is an anti fuzzy subgroup.

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