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COMPLEX SYSTEM WITH FLOWS AND SYNCHRONIZATION

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Abstract : A complex system $\mathscr{S}$ consists of m components, maybe inconsistence with $m \geq 2$ such as those of self-adaptive systems, particularly the biological systems and usually, a system with contradictions, i.e., there are no a classical mathematical subfield applicable. Then, how can we hold its global and local behaviors or true face? All of us know that there always exists universal connections between things in the world, i.e., a topological graph $\overrightarrow{G}$ underlying parts in $\mathscr{S}$. We can thereby establish mathematics over a graph family $\left\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots\right\}$ for characterizing the dynamic behaviors of system $\mathscr{S}$ on the time t, i.e., complex flows. Formally, a complex flow $\overrightarrow{G}^L$ is a topological graph $\overrightarrow{G}$ associated with a mapping $L : (v,u) \to L(v,u)$, 2 end-operators $A^+_{vu} : L(v,u) \to L^{A^+_{vu}}(v,u)$ and $A^+_{uv} : L(u,v) \to L^{A^+_{uv}}(u,v)$ on a Banach space $\mathscr{B}$ over a field $\mathscr{F}$ with $L(v,u) = -L(u,v)$ and $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$ holding with continuity equations

$$\frac{dx_v}{dt} = \sum_{u \in N_G(v)} L^{A^+_{vu}}(v,u), \qquad \forall v \in V\left(\overrightarrow{G}\right),$$

where x_v is a variable on vertex v for $\forall v \in E\left(\overrightarrow{G}\right)$. Particularly, if $dx_v/dt = \mathbf{0}$ for $\forall v \in V\left(\overrightarrow{G}\right)$, such a complex flow $\overrightarrow{G}^L$ is nothing else but an action flow or conservation flow. The main purpose of this lecture is to clarify the complex system with that of contradictory system and its importance to the reality of a thing T by extending Banach or Hilbert spaces to Banach or Hilbert continuity flow spaces over topological graphs $\left\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots\right\}$ and establishing the global differential theory on complex flows, characterize the global dynamic behaviors of complex systems, particularly, complex networks independent on graphs, for instance the synchronization of complex systems by applying global differential on the complex flows $\overrightarrow{G}^L$.

§**1. Introduction.** Is our mathematical theory can already be used for understanding the reality of all things in the world? This is a simple but essential question on the developing direction of mathematics, and it's answer is not positive. All of us live in a world full of colors, encountering various phenomena such as those of gorgeous guppy or peacock shown in Fig. 1 each day and can't help ourselves : Why are they looks like this, not that, i.e., the reality of things in the macro and micro world

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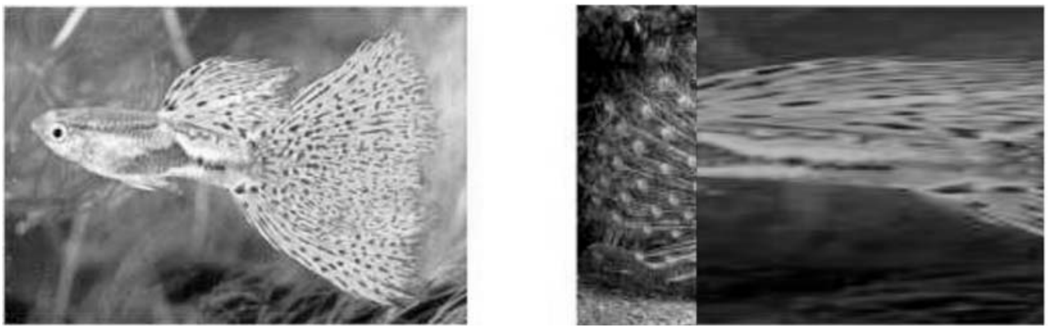

Fig.1

For example, we all known or heard that birds flying in the sky and fishes swimming in the ocean from disorderly to orderly in the macro world, and also words that the superposition, i.e., a quantum particle is both in two or more possible states of being in the micro world. All of these show a beauty of the scenery in one's eyes. It should be noted that the superposition not only appeared in the micro world but also in the macro world. For example, there is a most reluctant to answer question for a Chinese man. That is, if one's wife and his mother fell simultaneously in a river, who will he save first, his mother or his wife? This Chinese question is also equivalent to the famous thought model of Schrödinger's cat, which assumed that a cat, a flask of poison and a radioactive source are placed in a sealed box. If an internal monitor detects radioactivity, the flask is shattered, releasing the poison which kills the cat. Yet, when one looks in the box, one sees the cat either alive or dead, but not both alive and dead. Then, Schrödinger asked: Is the cat alive or dead? Certainly, the two questions both show that the superposition can be also happen in the macro world.

Then, what is the reality of a thing and where do the complex systems come from? The word reality is the state of things as they actually exist, including everything that is and has been, whether or not it is observable or comprehensible. Can one really hold on the reality of things? Usually, a thing T is multilateral or complex, and so to hold on its reality is difficult for human beings, where the world complex implies the cognitive system on a thing T is complex, i.e., a system composes of many components which maybe interact with each other. A typical example for explaining the complex of cognitive system on a thing is the well-known fable "the blind men with an elephant".

In this fable, a group of blind men heard that a strange animal, called an elephant had been brought to the town but none of them were aware of its shape and form. "We must inspect and know it by touch of which we are capable". The first person hand landed on the trunk, said: "this being is like a thick snake". The 2nd one whose hand reached its ear, claimed it like a kind of fan. The 3rd person hand was upon its leg, said the elephant is a pillar like a tree-trunk. The 4th man hand upon its side said: "the elephant is a wall". The 5th felt its tail, described it as a rope and the last felt its tusk, stated the elephant is that being hard, smooth and like a spear. They then entered into an endless argument! "All of you are right"! A wise man explained to them: "why are you telling it differently is because each one of you touched the different part of the elephant".

Fig. 2

What is the philosophical meaning of this fable to human beings? It lies in that the situation of human beings hold on the reality of things is analogous to these blind men, i.e., a complex system. Usually, the reality of thing T identified with known characters on it at one time. For example, let $\nu_i, i \geq 1$ be unknown and $\mu_1, \mu_2, \cdots, \mu_n$ known characters at time t. Then, the reality of thing T should be understood by

$$T = \left(\bigcup_{i=1}^{n}\{\mu_i\}\right) \bigcup \left(\bigcup_{k\geq 1}\{\nu_k\}\right), \tag{1.1}$$

i.e., a Smarandache multispace in logic with an approximation $T^\circ = \bigcup\limits_{i=1}^{n}\{\mu_i\}$ at time t (Smarandache, 1997), which also implies that the cognition on the reality of a thing T is only an approximation, and also the complex, i.e., the reality of a thing T is nothing else but a complex system.

Einstein once said the reality of things with that of mathematics: "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality". Why did Einstein say these words? Because we have no a mathematical subfield applicable to complex system, i.e., the reality of thing T, and generally, we get a contradictory system in mathematics.

The main purpose of this lecture is applying the contradictory universality and the existence of universal connections between things T in the world, i.e., a topological graph $\overrightarrow{G}$ underlying its parts in philosophy to establish a global mathematics over a graph family $\left\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots\right\}$ for characterizing dynamic behaviors of a system on the time t, i.e., complex flows such as those of to extend Banach or Hilbert spaces to Banach or Hilbert continuity flow spaces over topological graphs $\left\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots\right\}$ to establish the global differential theory on complex flows and how to characterize the global dynamic behaviors of complex systems by the global differential theory over graphs. These results can be also applied to complex networks and analyze their dynamic behaviors particularly, for instance the synchronization of complex networks independent on graphs by applying global differential on the complex flows $\overrightarrow{G}^L$.

For terminologies and notations not mentioned here, we follow references (Abraham and Marsden, 1978) for mechanics, (Chen, Wang and Li, 2015) for complex network, (Conway, 1990) for functional analysis, (Mao, 2011) for combinatorial geometry, (Murray, 2002) for biological mathematics, (Ho-Kim and Yam, 1998) for elementary particles, (Mao, 2011) and (Smarandache, 1997) for Smarandache systems and multispaces, and all phenomenons discussed in this paper are assumed to be true in the nature.

§**2. Contradictory Systems.** The formula (1.1) implies that one's recognition on a thing T is usually non-completed, which is the origination of contradiction. In classical logic, a contradictory system consists of a logical incompatibility between two or more propositions, which is abandoned without discussion in classical mathematics because a mathematical system should be compatibility in logic. However, different things are contradictory in the eyes of human beings. This is the reason why classical mathematics can not provides a complete recognition on things T.

Fig. 3

Usually, a physical phenomenon of a thing T is characterized by differential equations. If there is only one cell or one bird flying in the sky such as the flying bird, its dynamic behavior can be characterized easily by a orbit in the space, i.e., a differential equation

$$\dot{\overline{x}} = F(t, \overline{x}), \tag{2.1}$$

where, $\dot{\overline{x}} = d\overline{x}/dt$, t is the time parameter and $\overline{x}$ is the position of bird in $\mathbb{R}^3$. But how can one characterize the behavior of a complex system of m cells with $m \geq 2$ in Fig. 3? For example, a water molecule H_2O consists of 2 hydrogen atoms and 1 oxygen atom, and we have known the behavior of a particle is characterized by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi \tag{2.2}$$

in quantum mechanics (Ho-and Yem, 1998)).

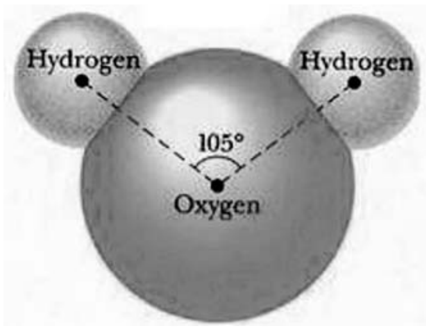

Fig. 4

Can we conclude this equation is absolutely right for atoms H and O in water molecule H_2O? Certainly not because equation (2.2) is always established with an additional assumption, i.e., the geometry on a particle P is a point in classical mechanics or a field in quantum mechanics.

In this case, if equation (2.2) is true for toms H and O, we get three differential equations following

$$\begin{cases} -i\hbar\dfrac{\partial\psi_O}{\partial t} = \dfrac{\hbar^2}{2m_O}\nabla^2\psi_O - V(x)\psi_O \\ -i\hbar\dfrac{\partial\psi_{H_1}}{\partial t} = \dfrac{\hbar^2}{2m_{H_1}}\nabla^2\psi_{H_1} - V(x)\psi_{H_1} \\ -i\hbar\dfrac{\partial\psi_{H_2}}{\partial t} = \dfrac{\hbar^2}{2m_{H_2}}\nabla^2\psi_{H_2} - V(x)\psi_{H_2} \end{cases} \tag{2.3}$$

on atoms H and O. Which is the right model on H_2O, the (2.2) or (2.3) dynamic equations? The answer is not so easy because the equation model (2.2) can only characterizes those of coherent behavior of atoms H and O in H_2O. Although equation (2.3) characterize the different behaviors of atoms H and O but it is non-solvable in mathematics (Mao, 2015). Generally, when one wish to hold on the reality of a thing, i.e., a complex system, he usually get a contradictory system, which also implies that the mathematical known is not absolutely equal to the reality of a thing T. Thus, establish mathematics on non-mathematics, i.e., an envelope theory on mathematics for reality is needed (Mao, 2014).

Now, are these contradictory systems meaningless for human beings? The answer is not! For example, let T_1, T_2, T_3, T_4 and T_1', T_2', T_3', T_4' be respectively two groups of horses running constraint with

$$(LES_4^N)\ \begin{cases} x+y= & 2 \\ x+y= & -2 \\ x-y= & -2 \\ x-y= & 2 \end{cases} \qquad\qquad (LES_4^S)\ \begin{cases} x=y \\ x+y=4 \\ x=2 \\ y=2 \end{cases}$$

on the earth. It is clear that (LES_4^N) is non-solvable because $x+y=-2$ is contradictious to $x+y=2$, and so that for equations $x-y=-2$ and $x-y=2$. But system (LES_4^S) is solvable with $x=2$ and $y=2$. Can we conclude that things T_1', T_2', T_3', T_4' are $x=2, y=2$ and T_1, T_2, T_3, T_4 are nothing? Certainly not because all of them are horses running on the earth,

Fig. 5.

and their solvability only implies the orbits intersection in $\mathbb{R}^2$ such as those shown in Fig. 6.

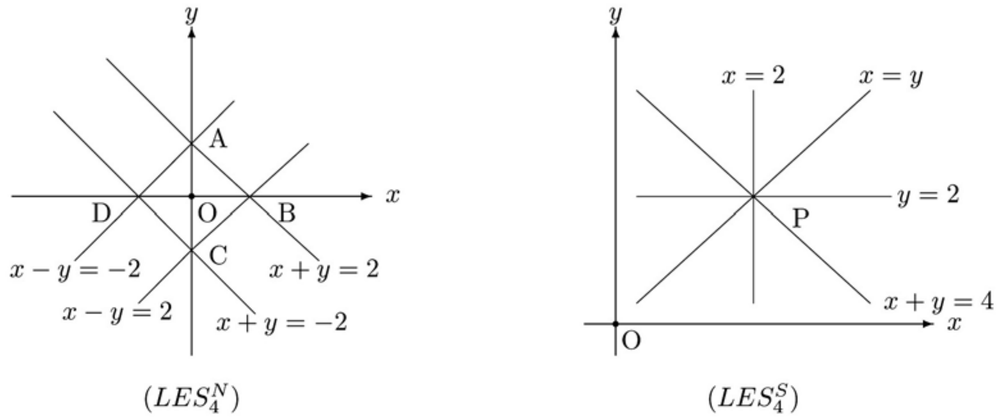

Fig. 6

Denoted by $L_{a,b,c} = \{(x,y)|ax+by=c, ab \neq 0\}$ be points in $\mathbb{R}^2$. We are easily know the behaviors of horses T_1, T_2, T_3, T_4 and T_1', T_2', T_3', T_4' are nothings else but the unions $L_{1,-1,0} \bigcup L_{1,1,4} \bigcup L_{1,0,2} \bigcup L_{0,1,2}$ and $L_{1,1,2} \bigcup L_{1,1,-2} \bigcup L_{1,-1,-2} \bigcup L_{1,-1,2}$, i.e., Smarandache multispaces, respectively.

Generally, let $\mathscr{F}_1, \mathscr{F}_2, \cdots, \mathscr{F}_m$ be m mappings holding in conditions of the implicit mapping theorem and let $S_{\mathscr{F}_i} \subset \mathbb{R}^n$ be a manifold such that $\mathscr{F}_i : S_{\mathscr{F}_i} \to 0$ for integers $1 \leq i \leq m$. Consider the equations

$$\begin{cases} \mathscr{F}_1(x_1, x_2, \cdots, x_n) = 0 \\ \mathscr{F}_2(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathscr{F}_m(x_1, x_2, \cdots, x_n) = 0 \end{cases} \tag{2.4}$$

in Euclidean space $\mathbb{R}^n, n \geq 1$. Geometrically, the system (2.4) is non-solvable or not dependent on $\bigcap\limits_{i=1}^{m} S_{\mathscr{F}_i} = \emptyset$ or $\neq= \emptyset$.

DEFINITION 2.1 *A G-solution of system* (2.4) *is a labeling graph G^L defined by*

$$V(G) = \left\{ S_{\mathscr{F}_i},\ 1 \leq i \leq n \right\};$$

$$E(G) = \left\{ (S_{\mathscr{F}_i}, S_{\mathscr{F}_j}) \ \ if\ \ S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_j} \neq \emptyset \ \ for\ integers\ \ 1 \leq i,j \leq n \right\} \ \ with\ a\ labeling$$

$$L : S_{\mathscr{F}_i} \to S_{\mathscr{F}_i}, \qquad (S_{\mathscr{F}_i}, S_{\mathscr{F}_j}) \to S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_j}$$

For example, the G-solutions of (LES_4^N) and (LES_4^S) are respectively labeled graphs C_4^L and K_4^L shown in Fig. 7.

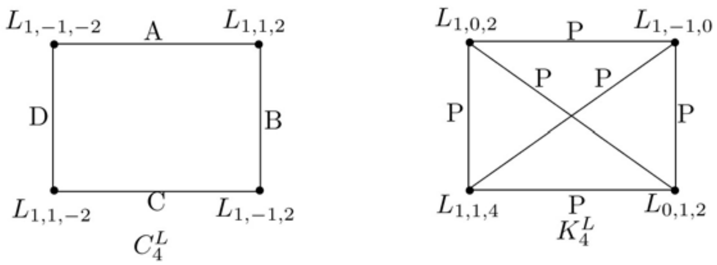

Fig. 7

EXAMPLE 2.2 *Let $(LDES_6^1)$ be a system of linear homogeneous differential equations*

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

where $\ddot{x} = \dfrac{d^2x}{dt^2}$ *and* $\dot{x} = \dfrac{dx}{dt}$. *Clearly,* $(LDES_6^1)$ *is a non-solvable system with solution bases* $\{e^t, e^{2t}\}$, $\{e^{2t}, e^{3t}\}$, $\{e^{3t}, e^{4t}\}$, $\{e^{4t}, e^{5t}\}$, $\{e^{5t}, e^{6t}\}$, $\{e^{6t}, e^t\}$ *respectively on equations* $(1)-(6)$ *and G-solution shown in Fig.* 8,

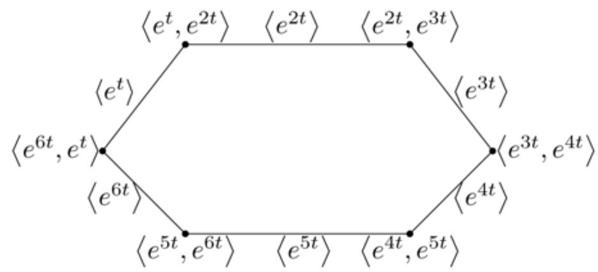

Fig. 8

where $\langle\Delta\rangle$ *denotes the linear space generalized by elements in* Δ.

A more interesting application of the G-solution is it can be applied to characterizing the global stability of differential equations (2.4), even it is non-solvable. See (Mao, 2013, 2014, 2015) for details.

§3. **Complex Flows**

3.1 Complex Flows. Holding on the reality of things, i.e., complex systems enables us to present an element called complex flow in mathematics on oriental philosophy, i.e., there always exist the universal contradiction and connection between things in the world. Then, what is a complex flow? what is its role in understanding things in the world?

DEFINITION 3.1 *A continuity flow* $\left(\overrightarrow{G}; L, A\right)$ *is an oriented embedded graph* $\overrightarrow{G}$ *in a topological space* $\mathscr{S}$ *associated with a mapping* $L : v \to L(v)$, $(v,u) \to L(v,u)$, 2 *end-operators* $A^+_{vu} : L(v,u) \to L^{A^+_{vu}}(v,u)$ *and* $A^+_{uv} : L(u,v) \to L^{A^+_{uv}}(u,v)$ *on a Banach space* $\mathscr{B}$ *over a field* $\mathscr{F}$

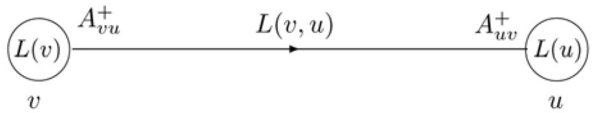

Fig. 9

with $L(v,u) = -L(u,v)$ *and* $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ *holding with continuity equation*

$$\sum_{u \in N_G(v)} L^{A^+_{vu}}(v,u) = L(v) \quad for \quad \forall v \in V\left(\overrightarrow{G}\right)$$

such as those shown for vertex v *in Fig.* 10 *following*

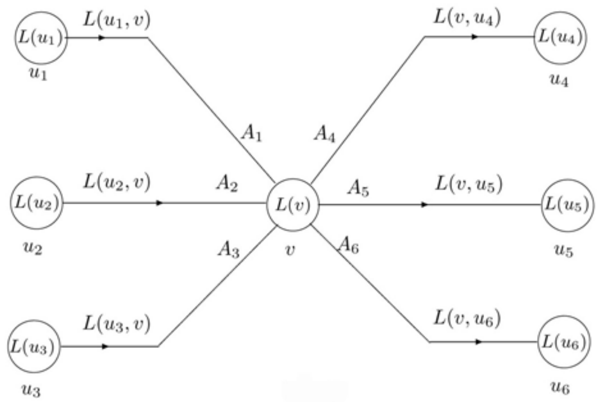

Fig. 10

with a continuity equation

$$L^{A_1}(v,u_1)+L^{A_2}(v,u_2)+L^{A_3}(v,u_3)-L^{A_4}(v,u_4)-L^{A_5}(v,u_5)-L^{A_6}(v,u_6)=L(v),$$

where $L(v)$ is the surplus flow on vertex v.

Particularly, we have known continuity flows following:

(1) $L(v)=\dot{x}_v$, $v\in V\left(\overrightarrow{G}\right)$. In this case, $\left(\overrightarrow{G};L,A\right)$ is said to be a *complex flow*, discussed in this lecture;

(2) For $v\in V\left(\overrightarrow{G}\right)$, x_v is a constant $\boldsymbol{v}_v$ dependent on v. In this case, $\left(\overrightarrow{G};L,A\right)$ is said to be an *action flow*, which was discussed extensively in (Mao, 2016) with applications (Mao, 2015, 2016) to elementary physics and biological systems;

(3) $L(v)$ =constant independent on v, $v\in V\left(\overrightarrow{G}\right)$, which is a special case of action flow called A_0 $\overrightarrow{G}$*-flow* and shown can be applied to synchronization of system in this lecture;

(4) If $A=\mathbf{1}_{\approx}$, $\left(\overrightarrow{G};L,A\right)$ is said to be a $\overrightarrow{G}$-flow, which was discussed in (Mao, 2015);

(5) If $A=\mathbf{1}_{\mathscr{V}}$ and $\mathscr{V}$ is a number field $\mathbb{Z}$ or $\mathbb{R}$, $\left(\overrightarrow{G};L,A\right)$ is said to be a complex network, which was already discussed extensively in publications, for examples, (Albert and Barabaśi, 2000, Barabaśi and Albert, 1999, Chen, Wang and Li, 2015 and Pecora and Carrol, 1998).

For example, let the $L:(v,u)\to L(v,u)\in\mathbb{R}^n\times\mathbb{R}^+$ with end-operators $A^+_{vu}=a_{vu}\dfrac{\partial}{\partial t}$ and $a_{vu}:\mathbb{R}^n\to\mathbb{R}$ for any edge $(v,u)\in E\left(\overrightarrow{G}\right)$ in Fig. 11 following.

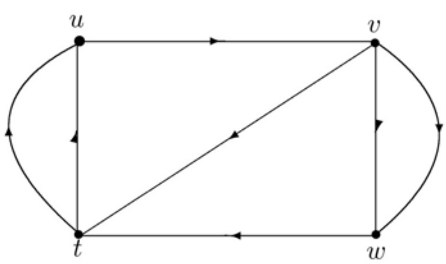

Fig. 11

Then the conservation laws are partial differential equations

$$\begin{cases} a_{tu^1}\dfrac{\partial L(t,u)^1}{\partial t}+a_{tu^2}\dfrac{\partial L(t,u)^2}{\partial t}=a_{uv}\dfrac{\partial L(u,v)}{\partial t} \\ a_{uv}\dfrac{\partial L(u,v)}{\partial t}=a_{vw^1}\dfrac{\partial L(v,w)^1}{\partial t}+a_{vw^2}\dfrac{\partial L(v,w)^2}{\partial t}+a_{vt}\dfrac{\partial L(v,t)}{\partial t} \\ a_{vw^1}\dfrac{\partial L(v,w)^1}{\partial t}+a_{vw^2}\dfrac{\partial L(v,w)^2}{\partial t}=a_{wt}\dfrac{\partial L(w,t)}{\partial t} \\ a_{wt}\dfrac{\partial L(w,t)}{\partial t}+a_{vt}\dfrac{\partial L(v,t)}{\partial t}=a_{tu^1}\dfrac{\partial L(t,u)^1}{\partial t}+a_{tu^2}\dfrac{\partial L(t,u)^2}{\partial t} \end{cases}$$

which maybe solvable or not but characterize the behavior of things.

3.2 Extended Linear Space. Let $\overrightarrow{\mathscr{G}}, \overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ be oriented graphs embedded in $\mathscr{S}$ with $\overrightarrow{\mathscr{G}} = \bigcup\limits_{i=1}^{n} \overrightarrow{G}_i$, i.e., each $\overrightarrow{G}_i$ be a subgraph of $\overrightarrow{\mathscr{G}}$ for integers $1 \le i \le n$. In this case, these is naturally an embedding $\iota : \overrightarrow{G}_i \to \overrightarrow{\mathscr{G}}$. *Can we construct linear space by reviewing continuity flows* $\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}$ *not only labeling graphs but mathematical elements*? The answer is yes!

Let $\mathscr{V}$ be a linear space over a field $\mathscr{F}$. A vector labeling $L : \overrightarrow{G} \to \mathscr{V}$ is a mapping with $L(v), L(e) \in \mathscr{V}$ for $\forall v \in V(\overrightarrow{G}), e \in E(\overrightarrow{G})$. Define

$$\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2} = \left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)^{L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_1+L_2} \bigcup \left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)^{L_2} \tag{3.1}$$

and

$$\lambda \cdot \overrightarrow{G}^{L} = \overrightarrow{G}^{\lambda \cdot L} \tag{3.2}$$

for $\forall \lambda \in \mathscr{F}$. Clearly, if $\overrightarrow{G}^{L}, \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}$ are continuity flows with linear end-operators A_{vu}^{+} and A_{uv}^{+} for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, $\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}$ and $\lambda \cdot \overrightarrow{G}^{L}$ are continuity flows also. If we consider each continuity flow $\overrightarrow{G}_i^{L}$ a continuity subflow of $\overrightarrow{\mathscr{G}}^{\widehat{L}}$, where $\widehat{L} : \overrightarrow{G}_i = L(\overrightarrow{G}_i)$ but $\widehat{L} : \overrightarrow{\mathscr{G}} \setminus \overrightarrow{G}_i \to \mathbf{0}$ for integers $1 \le i \le n$, and define $\boldsymbol{O} : \overrightarrow{\mathscr{G}} \to \mathbf{0}$, then we get the following result.

THEOREM 3.1 (Mao, 2017) *If* A_{vu}^{+} *and* A_{uv}^{+} *are linear end-operators for* $\forall (v,u) \in E\left(\overrightarrow{\mathscr{G}}\right)$, *all continuity flows on oriented graphs* $\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ *naturally form a linear space, denoted by* $\left(\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}; +, \cdot\right)$ *over a field* $\mathscr{F}$ *under operations* (3.1) *and* (3.2).

Particularly, for action flows, we get the following result.

THEOREM 3.2 (Mao, 2015) *Let* $\mathscr{G}$ *be all action flows* $\left(\overrightarrow{G}; L, A\right)$ *with linear end-operators* $A \in \boldsymbol{O}(\mathscr{V})$. *Then*

$$\dim \mathscr{G} = \left(\dim \boldsymbol{O}(\mathscr{V}) \times \dim \mathscr{V}\right)^{\beta(\overrightarrow{G})}$$

if both $\mathscr{V}$ *and* $\boldsymbol{O}(\mathscr{V})$ *are finite. Otherwise,* $\dim \mathscr{G}$ *is infinite.*

Particularly, if operators $A \in \mathscr{V}^*$, *the dual space of* $\mathscr{V}$ *on graph* $\overrightarrow{G}$, *then*

$$\dim \mathscr{G} = \left(\dim \mathscr{V}\right)^{2\beta(\overrightarrow{G})},$$

where $\beta\left(\overrightarrow{G}\right) = \varepsilon\left(\overrightarrow{G}\right) - \left|\overrightarrow{G}\right| + 1$ *is the Betti number of* $\overrightarrow{G}$.

Notice that $\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2} \neq \overrightarrow{G}_1^{L_1}$ or $\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2} \neq \overrightarrow{G}_2^{L_2}$ if and only if $\overrightarrow{G}_1 \not\preceq \overrightarrow{G}_2$ with $L_1 : \overrightarrow{G}_1 \setminus \overrightarrow{G}_2 \not\to \mathbf{0}$ or if $\overrightarrow{G}_2 \not\preceq \overrightarrow{G}_1$ with $L_2 : \overrightarrow{G}_2 \setminus \overrightarrow{G}_1 \not\to \mathbf{0}$, and generally, we say a continuity flow family $\left\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\right\}$ is linear irreducible if for any integer i,

$$\overrightarrow{G}_i \not\preceq \bigcup_{l\neq i} \overrightarrow{G}_l \quad \text{with} \quad L_i : \overrightarrow{G}_i \setminus \bigcup_{l\neq i} \overrightarrow{G}_l \not\to \mathbf{0},$$

where $1 \leq i \leq n$. We know the following result on linear generated sets.

THEOREM 3.3 (Mao, 2017) *Let* $\mathscr{V}$ *be a linear space over a field* $\mathscr{F}$ *and let* $\left\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\right\}$ *be an linear irreducible family,* $L_i : \overrightarrow{G}_i \to \mathscr{V}$ *for integers* $1 \leq i \leq n$ *with linear operators* A_{vu}^+, A_{uv}^+ *for* $\forall (v, u) \in E\left(\overrightarrow{G}\right)$. *Then,* $\left\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\right\}$ *is an independent generated set of* $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{V}}$, *called basis, i.e.,*

$$\dim \left\langle \overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{V}} = n.$$

3.3 Extended Commutative Rings. Furthermore, if $\mathscr{V}$ is a commutative ring $(\mathbb{R}; +, \cdot)$, *can we extend it over oriented graph family* $\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n\}$ *by introducing operation* "$+$" *with* (3.1) *and operation* "$\cdot$" *following:*

$$\overrightarrow{G}_1^{L_1} \cdot \overrightarrow{G}_2^{L_2} = \left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)^{L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_1 \cdot L_2} \bigcup \left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)^{L_2},$$

where $L_1 \cdot L_2 : x \to L_1(x) \cdot L_2(x)$? The answer is yes! We get the following result:

THEOREM 3.4 (Mao, 2017) *Let* $(\mathbb{R}; +, \cdot)$ *be a commutative ring and let* $\left\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\right\}$ *be a linear irreducible family,* $L_i : \overrightarrow{G}_i \to \mathbb{R}$ *for integers* $1 \leq i \leq n$ *with linear operators* A_{vu}^+, A_{uv}^+ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. *Then,* $\left(\left\langle \overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathbb{R}}; +, \cdot\right)$ *is a commutative ring.*

3.4 Banach or Hilbert Space. We have shown that $\overrightarrow{G}^{\mathscr{V}}$ is a Banach space, and furthermore, Hilbert space if $\mathscr{V}$ is a Banach or Hilbert space for an oriented graph $\overrightarrow{G}$ embedded in topological space $\mathscr{S}$ in (Mao, 2015). Generally, let $\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\}$ be a basis of $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{V}}$, where $\mathscr{V}$ is a Banach space with a norm $\| \cdot \|$. *Can we extend Banach space* V *over* $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n\right\rangle$*? And similarly, can we extend Hilbert space* V *over* $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n\right\rangle$? The answer is yes!

For $\forall \overrightarrow{G}^L \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{V}}$, define

$$\left\|\overrightarrow{G}^L\right\| = \sum_{e \in E\left(\overrightarrow{G}\right)} \|L(e)\|$$

or

$$\left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \right\rangle = \sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)} \langle L_1(e), L_1(e)\rangle + \sum_{e \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)} \langle L_1(e), L_2(e)\rangle + \sum_{e \in E\left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)} \langle L_2(e), L_2(e)\rangle . \quad (2.10)$$

Then we are easily know also that

THEOREM 3.5 (Mao, 2017) *Let* $\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ *be oriented graphs embedded in a space* $\mathscr{S}$ *and* $\mathscr{V}$ *a Banach space over a field* $\mathscr{F}$. *Then* $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{V}}$ *with linear operators* A_{vu}^+, A_{uv}^+ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ *is a Banach space, and furthermore, if* $\mathscr{V}$ *is a Hilbert space,* $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{V}}$ *is a Hilbert space too.*

Therefore, we can consider calculus and differentials on Hilbert space $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{V}}$.

Now, if L is kth differentiable to t on a domain $\mathscr{D}\subset\mathbb{R}$, where $k\geq 1$ and we define

$$\frac{d\overrightarrow{G}^L}{dt}=\overrightarrow{G}^{\frac{dL}{dt}}\quad\text{and}\quad\int_0^t\overrightarrow{G}^L dt=\overrightarrow{G}^{\int_0^t Ldt}.$$

Then, *what will happens*? We can generalize Taylor formula on differentiable functions in $\left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathscr{V}}$ following.

THEOREM 3.6 (Taylor)(Mao, 2017) *Let $\overrightarrow{G}^L\in\left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathbb{R}\times\mathbb{R}^n}$ and there exist kth order derivative of L to t on a domain $\mathscr{D}\subset\mathbb{R}$, where $k\geq 1$. If A_{vu}^+, A_{uv}^+ are linear for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, then*

$$\overrightarrow{G}^L=\overrightarrow{G}^{L(t_0)}+\frac{t-t_0}{1!}\overrightarrow{G}^{L'(t_0)}+\cdots+\frac{(t-t_0)^k}{k!}\overrightarrow{G}^{L^{(k)}(t_0)}+o\left((t-t_0)^{-k}\,\overrightarrow{G}\right),$$

for $\forall t_0\in\mathscr{D}$, where $o\left((t-t_0)^{-k}\,\overrightarrow{G}\right)$ denotes such an infinitesimal term $\widehat{L}$ of L that

$$\lim_{t\to t_0}\frac{\widehat{L}(v,u)}{(t-t_0)^k}=0\quad for\quad\forall(v,u)\in E\left(\overrightarrow{G}\right).$$

Particularly, if $L(v,u)=f(t)c_{vu}$, where c_{vu} is a constant, denoted by $f(t)\overrightarrow{G}^{L_C}$ with $L_C:(v,u)\to c_{vu}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ and

$$f(t)=f(t_0)+\frac{(t-t_0)}{1!}f'(t_0)+\frac{(t-t_0)^2}{2!}f''(t_0)+\cdots+\frac{(t-t_0)^k}{k!}f^{(k)}(t_0)+o\left((t-t_0)^k\right),$$

then

$$f(t)\overrightarrow{G}^{L_C}=f(t)\cdot\overrightarrow{G}^{L_C}.$$

This formula for continuity flow $\overrightarrow{G}^L$ enables one to find interesting results and formulas on $\overrightarrow{G}^L$ by $f\left(t\overrightarrow{G}\right)$ such as those of the following.

COROLLARY 3.7 *Let $f(t)$ be a k differentiable function to t on a domain $\mathscr{D}\subset\mathbb{R}$ with $0\in\mathscr{D}$ and $f(0\overrightarrow{G})=f(0)\overrightarrow{G}$. If A_{vu}^+, A_{uv}^+ are linear for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, then*

$$f(t)\overrightarrow{G}=f\left(t\overrightarrow{G}\right).$$

For examples,

$$e^{t\overrightarrow{G}}=e^t\overrightarrow{G}=\overrightarrow{G}+\frac{t}{1!}\overrightarrow{G}+\frac{t^2}{2!}\overrightarrow{G}+\cdots+\frac{t^k}{k!}\overrightarrow{G}+\cdots$$

and for a real number α if $|t| < 1$,

$$\left(\overrightarrow{G} + t\overrightarrow{G}\right)^{\alpha} = \overrightarrow{G} + \frac{\alpha t}{1!}\overrightarrow{G} + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)t^n}{n!}\overrightarrow{G} + \cdots.$$

§4. **Synchronization Independent on Graphs.** How can we characterize the behavior of a self-adaptive system with cells $m \geq 2$, for instance a flock of m birds? A natural way for characterizing the behavior of m birds is to collect all dynamic equations of cells, i.e.,

$$\begin{cases} \dot{\overline{x}}_1 = F_1(t, \overline{x}_1) \\ \dot{\overline{x}}_2 = F_2(t, \overline{x}_2) \\ \cdots\cdots\cdots\cdots\cdots \\ \dot{\overline{x}}_m = F_m(t, \overline{x}_m) \end{cases} \qquad (4.1)$$

to characterize the global behavior of the system.

However, birds or generally, cells in a self-adaptive system are interacted each other. The system (1.2) only is a collection of equation of each cell, not a global characterizing of the biological system in space. Including the interaction of cells enables one to apply m geometrical points in $\mathbb{R}^3$ and characterizing the system by a system of differential equations following

$$\begin{cases} \dot{\overline{x}}_1 = F_1(t, \overline{x}_1) + \sum\limits_{j\neq 1} H_j(\overline{x}_j \to \overline{x}_1) \\ \dot{\overline{x}}_2 = F_2(t, \overline{x}_2) + \sum\limits_{j\neq 2} H_j(\overline{x}_j \to \overline{x}_2) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{\overline{x}}_m = F_m(t, \overline{x}_m) + \sum\limits_{j\neq m} H_j(\overline{x}_j \to \overline{x}_m) \end{cases}, \qquad (4.2)$$

where $F_i : \mathbb{R}^3 \to \mathbb{R}^3$ is generally a nonlinear function characterizing the external appearance of ith cell and $H_j(x_j \to x_i)$ is the action strength of the jth cell to the ith cell in this system for integers $1 \leq i, j \leq m$.

Then, what is the synchronization of a self-adaptive system? The synchronization characterizes the behavior of a self-adaptive system from disorderly to orderly, such as those of birds flock or fishes shoal. By system (4.2) of differential equations, the synchronization nature is formally defined following.

DEFINITION 4.1 (Chen, Wang and Li, 2015) *The system* (4.2) *is said to be complete synchronization if*

$$\lim_{t\to\infty} \|\overline{x}_i(t) - \overline{x}_j(t)\| = 0$$

for all integers $i, j = 1, 2\cdots, m$, *where* $\|\cdot\|$ *is the Euclidean norm.*

In the past decades, many researchers discussed the synchronization of (4.2) in case of $F_1 = F_2 = \cdots = F_m$ and $H_j = H$, i.e., a network of m identical nodes with a constant coupling c and action strength $H(x_j)$ of node x_j to x_i for $i, j = 1, 2, \cdots, m$ such as those shown in the following model (Albert and Barabaśi, 2000, Barabaśi and Albert, 1999, Chen, Wang and Li, 2015 and Pecora and Carrol, 1998)

$$\begin{cases} \dot{\overline{x}}_1 = f(\overline{x}_1) + c\sum\limits_{j=1}^{m} a_{1j}H(\overline{x}_j) \\ \dot{\overline{x}}_2 = f(\overline{x}_2) + c\sum\limits_{j=1}^{m} a_{2j}H(\overline{x}_j) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{\overline{x}}_m = f(\overline{x}_m) + c\sum\limits_{j=1}^{m} a_{mj}H(\overline{x}_j) \end{cases} \tag{4.3}$$

where $\overline{x}_i = \left(x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(n)}\right)^T \in \mathbb{R}^n$ is the state vector, f is generally a nonlinear function satisfying a Lipschitz condition, $H : \mathbb{R}^n \to \mathbb{R}^n$ is the inner action and $A = [a_{ij}]_{m\times m}$ is the outer coupling matrix defined by $a_{ij} = a_{ji} = 1$ if there is a connection between nodes i and j, $i \neq j$, otherwise, $a_{ij} = a_{ji} = 0$ and the diagonal elements with $i = j$ are defined by

$$a_{ii} = -\sum_{j=1, j\neq i} a_{ij} = -\sum_{j=1, j\neq i} a_{ji} = -k_i, \quad i = 1, 2, \cdots, m$$

where k_i is the degree of node i. Hence, the matrix A is actually the negative Laplacian matrix.

Today, we have known the synchronization of (4.3) is so dependent on eigenvalues λ_2 and λ_m of matrix A (Albert and Barabaśi, 2000, Barabaśi and Albert, 1999, Chen, Wang and Li, 2015 and Pecora and Carrol, 1998) , which classified the regions leading to the synchronization of (4.3), called synchronized region into 4 cases following by a master stability function (Pecora and Carrol, 1998) :

Type I. Synchronized region is (α_1, ∞). In this type, the synchronization of (4.3) is determined by λ_2, i.e., if $c\lambda_2 > \alpha_1$, the system (4.3) is synchronized.

Type II. Synchronized region is $(\alpha_2, \alpha_3) \subset (0, \infty)$. In this type, the synchronization of (4.3) is determined by λ_2 and λ_m, i.e., if $\dfrac{\alpha_2}{\lambda_2} < c < \dfrac{\alpha_3}{\lambda_m}$, the system (4.3) is synchronized.

Type III. Synchronized region is the union of several intervals of $(0, \infty)$, for instance $(\alpha_2, \alpha_3)\bigcup(\alpha_4, \alpha_5)(\alpha_6, \infty)$.

Type IV. Synchronized region does not exist.

But, the criterions I-IV were so strange that the synchronization is a global behavior of individuals in a self-adaptive system and can not be completely dependent on its underlying graph or in other words, the eigenvalues of matrix A. However, they appears because of one's assumptions on system (4.3), i.e., the synchronization of a self-adaptive system should be independent on the underlying structure of individuals in general. Can we view a self-adaptive system as a mathematical element and characterize the synchronization of system? The answer is positive, i.e., by complex flows $\overrightarrow{G}^L$!

Notice that the synchronization state of a complex flow $\overrightarrow{G}^L$ is nothing else but a non-zero A_0 flows, i.e., $L(v) = \boldsymbol{v} \neq \mathbf{0}$ for $\forall v \in V\left(\overrightarrow{G}\right)$.

EXAMPLE 4.2 *Let $\overrightarrow{G} = \overrightarrow{C}_n$ or $\overrightarrow{P}_n$ for an integer $n \geq 1$. If there is an A_0 $\overrightarrow{G}$-flow on $\overrightarrow{C}_n$ such as those shown in Fig.* 12.

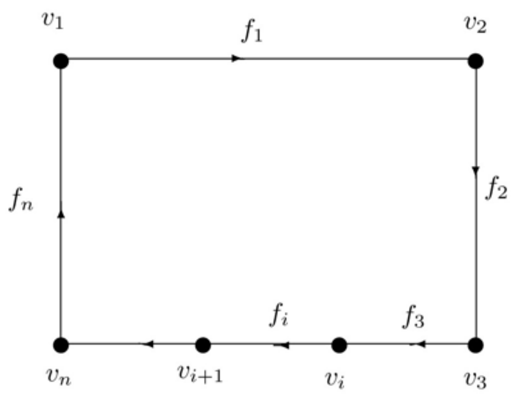

Fig. 12

We are easily know that

$$f_1 - f_n = f_2 - f_1 = f_3 - f_2 = \cdots = f_{i+1} - f_i = \cdots = f_n - f_{n-1}$$

by the definition of A_0-flow, which only have solutions $f_1 = f_2 = \cdots = f_n$. Thus, it is a zero A_0 flows.

Similarly, if there is an A_0 $\overrightarrow{G}$-flow on $\overrightarrow{P}_n$ such as those shown in Fig.13.

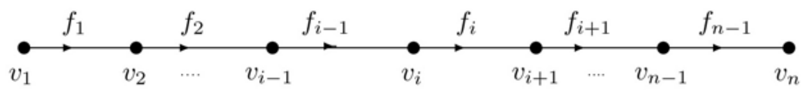

Fig. 13

We are easily know that

$$-f_1 = f_2 - f_1 = \cdots = f_i - f_{i-1} = \cdots = f_{n-1} - f_{n-2} = f_{n-1}$$

by the definition of A_0 flow, which only have solutions $f_1 = f_2 = \cdots = f_n = \mathbf{0}$. Thus, it is a zero A_0 flows also.

A complex A_0 flow $\overrightarrow{G}^L$ exists in $\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathbb{R}\times\mathbb{R}^n}$ if and only if $L(v) = F(t, \overline{x})$ for $\forall v \in V\left(\overrightarrow{G}\right)$, where $F(t, \overline{x})$ is independent on the interaction in $\overrightarrow{G}^L$. i.e., the system of continuity equations

$$\sum_{u \in N_G(v)} L^{A_{vu}^+}(v, u) = F(t, \overline{x}), \quad \forall v \in V\left(\overrightarrow{G}\right) \tag{4.4}$$

with the same solvable differential equation

$$\frac{dx_v}{dt} = F(t, \overline{x})$$

characterizing the behavior of variables on $v \in V\left(\overrightarrow{G}\right)$, which is homogenous. Thus, we know the following result by definition.

THEOREM 4.3 *A complex A_0 flow $\overrightarrow{G}^L$ exists in Hilbert space $\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathbb{R}\times\mathbb{R}^n}$ if and only if the ordinary differential equation*

$$\frac{dx}{dt} = F(t, x) \tag{4.5}$$

is solvable in Hilbert space $\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathbb{R}\times\mathbb{R}^n}$.

Such a solution is usually called a multispace solution of (4.5).

DEFINITION 4.4 *Let $\overrightarrow{G}^L, \overrightarrow{G}_1^{L_1} \in \left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}$ with L, L_1 dependent on a variable $t \in [a, b] \subset (-\infty, +\infty)$ and linear continuous end-operators A_{vu}^+ for $\forall (v, u) \in E\left(\overrightarrow{G}\right)$. For $t_0 \in [a, b]$ and any number $\varepsilon > 0$, if there is always a number $\delta(\varepsilon)$ such that if $|t - t_0| \le \delta(\varepsilon)$ then $\left\| \overrightarrow{G}_1^{L_1} - \overrightarrow{G}^L \right\| < \varepsilon$, then, $\overrightarrow{G}_1^{L_1}$ is said to be converged to $\overrightarrow{G}^L$ as $t \to t_0$, denoted by $\lim\limits_{t \to t_0} \overrightarrow{G}_1^{L_1} = \overrightarrow{G}^L$. Particularly, if $\overrightarrow{G}^L$ is a continuity flow with a constant $L(v)$ for $\forall v \in V\left(\overrightarrow{G}\right)$ and $t_0 = +\infty$, $\overrightarrow{G}_1^{L_1}$ is said to be $\overrightarrow{G}$-synchronized.*

These is a well-known result on liner operators following, which is useful to determining the synchronization of systems.

THEOREM 4.5 (Conway, 1990) *Let $\mathscr{B}_1$, $\mathscr{B}_2$ be Banach spaces over a field $\mathscr{F}$ with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then, a linear operator $\boldsymbol{T}:\mathscr{B}_1\to\mathscr{B}_2$ is continuous if and only if it is bounded, or equivalently,*

$$\|\boldsymbol{T}\| := \sup_{\mathbf{0}\neq\boldsymbol{v}\in\mathscr{B}_1}\frac{\|\boldsymbol{T}(\boldsymbol{v})\|_2}{\|\boldsymbol{v}\|_1} < +\infty.$$

According to Theorem 4.5, if A^+_{vu} is liner continuous operator there must be a constant c_{vu} such that $\|A^+_{vu}\|\leq c_{vu}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. Let

$$c^{\max}_{G_1G} = \left\{\max_{(v,u)\in E(G_1)} c^+_{vu}, \max_{(v,u)\in E(G_1)} c^+_{vu}\right\}.$$

Then, we get an equivalent condition for $\lim\limits_{t\to t_0}\overrightarrow{G}_1^{L_1} = \overrightarrow{G}^L$ following.

THEOREM 4.6 $\lim\limits_{t\to t_0}\overrightarrow{G}_1^{L_1} = \overrightarrow{G}^L$ *if and only if for any number $\varepsilon>0$ there is always a number $\delta(\varepsilon)$ such that if $|t-t_0|\leq\delta(\varepsilon)$ then $\|L_1(v,u)\|<\varepsilon$ for $(v,u)\in E\left(\overrightarrow{G}_1\setminus\overrightarrow{G}\right)$, $\|(L_1-L)(v,u)\|<\varepsilon$ for $(v,u)\in E\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}\right)$ and $\|-L(v,u)\|<\varepsilon$ for $(v,u)\in E\left(\overrightarrow{G}\setminus\overrightarrow{G}_1\right)$, i.e., $\overrightarrow{G}_1^{L_1}-\overrightarrow{G}^L$ is an infinitesimal or $\lim\limits_{t\to t_0}\left(\overrightarrow{G}_1^{L_1}-\overrightarrow{G}^L\right)=\boldsymbol{O}$.*

Proof: Clearly,

$$\begin{aligned}
\left\|\overrightarrow{G}_1^{L_1}-\overrightarrow{G}^L\right\| &= \left\|\left(\overrightarrow{G}_1\setminus\overrightarrow{G}\right)^{L_1}\right\| + \left\|\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}\right)^{L_1-L}\right\| + \left\|\left(\overrightarrow{G}\setminus\overrightarrow{G}_1\right)^{-L}\right\| \\
&= \sum_{u\in N_{G_1\setminus G}(v)}\left\|L_1^{A'^+_{vu}}(v,u)\right\| + \sum_{u\in N_{G_1\bigcap G}(v)}\left\|\left(L_1^{A'^+_{vu}}-L_{vu}^{A^+_{vu}}\right)(v,u)\right\| \\
&\quad + \sum_{u\in N_{G\setminus G_1}(v)}\left\|-L^{A^+_{vu}}(v,u)\right\| \\
&\leq \sum_{u\in N_{G_1\setminus G}(v)} c^+_{vu}\|L_1(v,u)\| + \sum_{u\in N_{G_1\bigcap G}(v)} c^+_{vu}\|\left(L_1-L\right)(v,u)\| \\
&\quad + \sum_{u\in N_{G\setminus G_1}(v)} c^+_{vu}\|-L(v,u)\|
\end{aligned}$$

and $\|L(v,u)\| \geq 0$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ and $E\left(\overrightarrow{G}_1\right)$. If $\left\|\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^L\right\| < \varepsilon$, we are easily knowing that $\|L_1(v,u)\| < c_{G_1G}^{\max}\varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)$, $\|(L_1 - L)(v,u)\| < c_{G_1G}^{\max}\varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)$ and $\|-L(v,u)\| < c_{G_1G}^{\max}\varepsilon$ for $(v,u) \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)$.

Conversely, if $\|L_1(v,u)\| < \varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)$, $\|(L_1 - L)(v,u)\| < \varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)$ and $\|-L(v,u)\| < \varepsilon$ for $(v,u) \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)$, we easily find that

$$\begin{aligned}
\left\|\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^L\right\| &= \sum_{u \in N_{G_1 \setminus G}(v)} \left\|L_1^{A'^{+}_{vu}}(v,u)\right\| + \sum_{u \in N_{G_1 \bigcap G}(v)} \left\|\left(L_1^{A'^{+}_{vu}} - L_{vu}^{A^{+}_{vu}}\right)(v,u)\right\| \\
&\quad + \sum_{u \in N_{G \setminus G_1}(v)} \left\|-L^{A^{+}_{vu}}(v,u)\right\| \\
&\leq \sum_{u \in N_{G_1 \setminus G}(v)} c_{vu}^{+}\|L_1(v,u)\| + \sum_{u \in N_{G_1 \bigcap G}(v)} c_{vu}^{+}\|\left(L_1 - L\right)(v,u)\| \\
&\quad + \sum_{u \in N_{G \setminus G_1}(v)} c_{vu}^{+}\|-L(v,u)\| \\
&< \left|\overrightarrow{G}_1 \setminus \overrightarrow{G}\right| c_{G_1G}^{\max}\varepsilon + \left|\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right| c_{G_1G}^{\max}\varepsilon + \left|\overrightarrow{G} \setminus \overrightarrow{G}_1\right| c_{G_1G}^{\max}\varepsilon \\
&= \left|\overrightarrow{G}_1 \bigcup \overrightarrow{G}\right| c_{G_1G}^{\max}\varepsilon.
\end{aligned}$$

This completes the proof.

An application of Theorem 4.6 enables us to get a result on synchronization of complex flows following, which is independent on the underlying structure of cells of a self-adaptive system.

THEOREM 4.7 *A complex flow $\overrightarrow{G}^L$ with linear continuous end-operator A_{vu}^{+} for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$ is $\overrightarrow{G}$-synchronized if and only if for any number $\varepsilon > 0$ if $t \geq N(\varepsilon)$ then $\|L(v) - L(u)\| < \varepsilon$ for $\forall v,u \in V\left(\overrightarrow{G}\right)$, i.e., flows on vertex are synchronized.*

Proof: By definition, if $\overrightarrow{G}^L$ is synchronized, there must be a non-zero A_0 flow $\overrightarrow{G}_0^{L_0}$ and a number $N(\varepsilon)$ such that $\left\|\overrightarrow{G}^L - \overrightarrow{G}_0^{L_0}\right\| < \varepsilon$ if $t \geq N(\varepsilon)$, which implies that $\|L(v,u)\| < \varepsilon$ for $u \in V\left(\overrightarrow{G} \setminus \overrightarrow{G}_0\right)$, $\|(L - L_0)(v,u)\| < \varepsilon$ for $u \in V\left(\overrightarrow{G} \bigcap \overrightarrow{G}_0\right)$ and $\|-L_0(v,u)\| < \varepsilon$ for $u \in V\left(\overrightarrow{G}_0 \setminus \overrightarrow{G}\right)$ by Theorem 4.6.

Therefore,

$$\|L(v)\| = \sum_{u\in N_G(v)} \left\|L^{A^+_{vu}}(v,u)\right\| \le \sum_{u\in N_G(v)} c^+_{vu}\|L(v,u)\| \le |N_G(v)| c^{\max}_{GG_0}\varepsilon$$

for $\forall v \in V\left(\overrightarrow{G}\right)$ by applying Theorem 4.6. Thus,

$$\|L(v)-L(u)\| \le \|L(v)\| + \|L(u)\| \le (|N_G(v)|+|N_G(u)|)c^{\max}_{GG_0}\varepsilon < \varepsilon$$

for $\forall v,u \in V\left(\overrightarrow{G}\right)$ if $t \ge N\left(\frac{\varepsilon}{c^{\max}_{GG_0}\left(|N_G(v)|+|N_G(u)|\right)}\right)$.

Conversely, if there is a number $\varepsilon > 0$ such that $\|L(v)-L(u)\| < \varepsilon$ if $t \ge N(\varepsilon)$ for $\forall v,u \in V\left(\overrightarrow{G}\right)$, we are easily know that $\lim\limits_{t\to\infty} L(v) = \lim\limits_{t\to\infty} L(u) = \boldsymbol{v}$ for $\forall v,u \in V\left(\overrightarrow{G}\right)$. Let $\lim\limits_{t\to\infty} \overrightarrow{G}^L = \overrightarrow{G}^{L_0}$. Then, $\overrightarrow{G}^{L_0}$ is a non-zero A_0 flow by definition and follows that

$$\begin{aligned}
\left\|\overrightarrow{G}^L - \overrightarrow{G}^{L_0}\right\| &= \sum_{u\in N_G(v)} \left\|L^{A^+_{vu}}(v,u)\right\| \\
&\le \sum_{(v,u)\in E\left(\overrightarrow{G}\right)} c^{\max}_G\|(L-L_0)(v,u)\| \\
&= \frac{1}{2}\sum_{v\in V\left(\overrightarrow{G}\right)} c^{\max}_G\left\|\left(L-L_0\right)(v)\right\| \le \left|\overrightarrow{G}\right| c^{\max}_G\varepsilon < \varepsilon
\end{aligned}$$

if $t \ge N\left(\frac{\varepsilon}{c^{\max}_G\left|\overrightarrow{G}\right|}\right)$, i.e., an infinitesimal which completes the proof.

Denoted by $\overrightarrow{\triangle} = \overrightarrow{G}_1^{L_1} - \overrightarrow{G}^L$ in Definition 4.1. Then, $\overrightarrow{\triangle}$ is an infinitesimal by Theorem 4.5, denoted by $o\left(t\overrightarrow{G}\right)$. We therefore know a conclusion following by Theorem 4.7, which completely changed the notion that synchronization dependent on the structure of $\overrightarrow{G}$.

THEOREM 4.8 *A continuity flow $\overrightarrow{G}^L$ with liner continuous end-operator A^+_{vu} for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$ is $\overrightarrow{G}$-synchronized if and only if there is a non-zero A_0 flow $\overrightarrow{G}_0^{L_0}$ such that*

$$\overrightarrow{G}^L = \overrightarrow{G}_0^{L_0} + o\left(t^{-1}\overrightarrow{G}\right),$$

independent on the structure of $\overrightarrow{G}$.

Notice that

$$\frac{d}{dt}\left(\sum_{u\in N_G(v)} L^{A_{vu}^+}(v,u)\right) = \sum_{u\in N_G(v)} \frac{d}{dt} L^{A_{vu}^+}(v,u)$$

for $\forall v \in V\left(\overrightarrow{G}\right)$ and $\frac{d\ln|t|}{dt} = t^{-1}$. We get the following result on a synchronized complex flow.

THEOREM 4.9 *A complex flow $\overrightarrow{G}^L$ with liner continuous end-operators A_{vu}^+ for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$ is $\overrightarrow{G}$-synchronized if and only if there is a non-zero A_0 flow $\overrightarrow{G}^{L_0}$ such that*

$$\overrightarrow{G}^L = \overrightarrow{G}^{L_0} + o\left(\ln|t|\overrightarrow{G}\right).$$

Particularly, if each A_{vu}^+ is a constant for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$, we get the conclusion following.

COROLLARY 4.10 *A complex flow $\overrightarrow{G}^L$ with $A_{vu}^+ = c_{vu}$, a constant for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$ is synchronized if and only if there is a non-zero A_0 flow $\overrightarrow{G}^{L_0}$ such that*

$$\overrightarrow{G}^L = \overrightarrow{G}^{L_0} + o\left(\ln|t|\overrightarrow{G}\right).$$

For example, let $A_{v_iv_{i+1}}^+ = 1$, $A_{v_iv_{i-1}}^+ = 2$ and

$$f_i = \frac{f_1 + \left(2^{i-1} - 1\right) F(t,\overline{x})}{2^{i-1}}$$

for integers $1 \le i \le n$ in Fig.12. We have known $\overrightarrow{C}_n^f$ with $f : (v_i, v_{i+1}) \to f_i$ is a non-zero A_0 flow. Construct a complex flow $\overrightarrow{C}_n^L$ by letting

$$L : (v_i, v_{i+1}) \to \frac{f_1 + \left(2^{i-1} - 1\right) F(t,\overline{x})}{2^{i-1}} + \frac{n!}{t^i}$$

and

$$L_\Delta : (v_i, v_{i+1}) \to \frac{n!}{t^i}.$$

Notice that $\overrightarrow{C}_n^{L_\Delta} = o\left(t^{-1}\overrightarrow{C}_n\right)$. We therefore known that the complex flow $\overrightarrow{C}_n^L$ is $\overrightarrow{G}$-synchronized by Corollary 4.10. However, by the master stability functions in [Pecora and Carrol, 1998] we can only conclude that it is difficult to attain the synchronization for $\overrightarrow{C}_n, n \ge 3$.

§**5 Conclusion.** The reality of a thing T is essentially a complex system, even a contradictory system in the eyes of human beings, and there are no a mathematical subfield applicable until today. Thus, a new mathematical theory should be established for holding on the reality of things in the world. For this objective, the *mathematical combinatorics*, i.e., mathematics over graphs and particularly, the mathematics on complex flows $\overrightarrow{G}^L$ is a candidate because every thing T is not isolated but connected with other things in the world, and a complex system or a contradictory system in classical is nothing else but a mathematics over a graph $\overrightarrow{G}$.

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