DUAL-COMPLEX NUMBERS AND THEIR HOLOMORPHIC FUNCTIONS

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Abstract. The purpose of this paper is to contribute to development a general theory of dual-complex numbers. We start by define the notion of dual-complex and their algebraic properties. In addition, we develop a simple mathematical method based on matrices, simplifying manipulation of dual-complex numbers. Inspired from complex analysis, we generalize the concept of holomorphicity to dual-complex functions. Moreover, a general representation of holomorphic dual-complex functions has been obtained. Finally and as concrete examples, some usual complex functions have been generalized to the algebra of dual-complex numbers.

1. Introduction

Alternative definitions of the imaginary unit i other than $i^2 = -1$ can give rise to interesting and useful complex number systems. The 16th-century Italian mathematicians G. Cardan (1501–1576) and R. Bombelli (1526–1572) are thought to be among the first to utilize the complex numbers we know today by calculating with a quantity whose square is -1. Since then, various people have modified the original definition of the product of complex numbers. The English geometer W. Clifford (1845–1879) and the German geometer E. Study (1862–1930) added still another variant to the complex products, see [3, 12, 16]. The "dual" numbers arose from the convention that $\varepsilon^2 = 0$.

The ordinary, dual number is particular member of a two-parameter family of complex number systems often called binary number or generalized complex number. Which is two-component number of the form

$$z = x + y\varepsilon, \tag{1.1}$$

where $(x, y) \in \mathbb{R}^2$ and ε is an nilpotent number i.e. $\varepsilon^2 = 0$ and $\varepsilon \neq 0$.

Thus, the dual numbers are elements of the $2-$dimensional real algebra

$$\mathbb{D} = \mathbb{R}[\varepsilon] = \left\{ z = x + y\varepsilon \mid (x, y) \in \mathbb{R}^2,\ \varepsilon^2 = 0 \text{ and } \varepsilon \neq 0 \right\}. \tag{1.2}$$

This nice concept has lots of applications in many fields of fundamental sciences; such, algebraic geometry, Riemannian geometry, quantum mechanics and astrophysics, we refer the reader to [2, 5, 14, 8, 15].

An important point to emphasize is that, what happens if components of dual numbers becom complex numbers. This idea will make the main object of this work.

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The purpose of this work is to contribute to the development of the concept of dual-complex numbers and their holomorphic functions.

In the study of dual-complex functions, natural question arises whether it is possible to extend the concept of holomorphy to dual-complex functions and how can one extend regularly holomorphic complex functions to dual-complex variables.

We Begin by introducing dual-complex numbers and we give some of their basic properties. We define on the algebra of dual-comlex numbers $\mathbb{DC}$ some caracterisctic like conjugations and their associated moduli as well as matrix representation. Also a structure of pseudo-topology is given.

We generalize the notion of holomorphicity to dual-complex functions. To do this, as in complex analysis. We start by study the differentiability of dual-complex functions. The notion of holomorphicity has been introduced and a general representation of holomorphic dual-complex functions was shown. It is proved here that many important properties of holomorphic functions of one complex variable may be extended in the framework of dual-complex analysis.

Further, we also focus on the continuation of complex functions to the algebra $\mathbb{DC}$. We provide the basic assumptions that allow us to extend analytically holomorphic complex functions to the wider dual-complex algebra and we ensure that such an extension is meaningful. As concrete examples, we generalize some usual complex functions to dual-complex variables.

2. Dual-Complex Numbers

We introduce the concept of dual-complex numbers as follows.

A dual-complex number w is an ordered pair of complex numbers (z,t) associated with the complex unit 1 and dual unit ε, where ε is an nilpotent number i.e. $\varepsilon^2 = 0$ and $\varepsilon \neq 0$. A dual-complex number is usually denoted in the form

$$w = z + t\varepsilon. \tag{2.1}$$

We denote by $\mathbb{DC}$ the set of dual-complex numbers defined as

$$\mathbb{DC} = \left\{ w = z + t\varepsilon \mid z, t \in \mathbb{C} \text{ where } \varepsilon^2 = 0, \varepsilon \neq 0 \text{ and } \varepsilon^0 = 1 \right\} \tag{2.2}$$

If $z = x_1 + ix_2$ and $t = x_3 + ix_4$, where $x_1, x_2, x_3, x_4 \in \mathbb{R}$, then w can be explicitly written

$$w = x_1 + x_2 i + x_3 \varepsilon + x_4 \varepsilon i. \tag{2.3}$$

We will denote by $\text{real}\,(w)$ the real part of w given by

$$\text{real}\,(w) = x_1. \tag{2.4}$$

z and t are called the complex and dual parts, respectively, of the dual-complex number w.

There are many ways to choose the dual unit number ε, see for more details and examples the book of W. B. V. Kandasamy and F. Smarandache [9]. As simple example, we can take the real matrix

$$\varepsilon = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{2.5}$$

Addition and multiplication of the dual-complex numbers are defined by

$$(z_1 + t_1\varepsilon) + (z_2 + t_2\varepsilon) = (z_1 + z_2) + (t + t_2)\,\varepsilon, \tag{2.6}$$

$$(z_1 + t_1\varepsilon) . (z_2 + t_2\varepsilon) = (z_1 z_2) + (z_1 t_2 + z_2 t_2)\,\varepsilon. \tag{2.7}$$

One can verify, using (2.7), that the power of w is

$$w^n = (z + \varepsilon t)^n = z^n + nz^{n-1}t\varepsilon, \quad n \geq 1. \tag{2.8}$$

The division of two complex-dual numbers can be computed as

$$\begin{aligned} \frac{w_1}{w_2} &= \frac{z_1 + t_1\varepsilon}{z_2 + t_2\varepsilon} \\ \frac{(z_1 + t_1\varepsilon)(z_2 - t_2\varepsilon)}{(z_2 + t_2\varepsilon)(z_2 - t_2\varepsilon)} &= \frac{z_1}{z_2} + \frac{z_2 t_1 - z_1 t_2}{z_2^2}\varepsilon. \end{aligned} \tag{2.9}$$

The division $\frac{w_1}{w_2}$ is possible and unambiguous if $\mathrm{Re}\,(w_2) \neq 0$.

Thus, dual-complex numbers form a commutative ring with characteristic 0. Moreover the inherited multiplication gives the dual-comlex numbers the structure of $2-$dimensional comlex Clifford Algebra and $4-$dimensional real Clifford Algebra.

In abstract algebra terms, the dual numbers can be described as the quotient of the polynomial ring $\mathbb{C}[X]$ by the ideal generated by the polynomial X^2, i.e.

$$\mathbb{DC} \approx \mathbb{C}[X]/X^2. \tag{2.10}$$

The algebra $\mathbb{DC}$ is not a division algebra or field since the elements of the form $0 + t\varepsilon$ are not invertible. All elements of this form are zero divisors. Hence, we can define the set $\mathcal{A}$ of zero divisors of $\mathbb{DC}$, which can be called the null-plane, by

$$\mathcal{A} = \{t\varepsilon \mid t \in \mathbb{C}\}.$$

Thus, $\mathbb{DC} - \mathcal{A}$ is a multiplicative group.

Complex and dual conjugations play an important role both for algebraic and geometric properties of $\mathbb{C}$ and $\mathbb{D}$. For dual-complex numbers, there are five possible conjugations. Let $w = z + t\varepsilon$ a dual-complex number. Then we define the five conjugations as

1. Complex conjugation.

$$w^{\dagger_1} = \overline{z} + \overline{t}\varepsilon, \tag{2.11}$$

where $\overline{z}$ represent the standard complex conjugate of the complex number z.

2. Dual conjugation.

$$w^{\dagger_2} = z - t\varepsilon. \tag{2.12}$$

3. Coulped conjugation.

$$w^{\dagger_3} = \overline{z} - \overline{t}\varepsilon. \tag{2.13}$$

4. Dual-complex conjugation. Suppose that $w \in \mathbb{DC} - \mathcal{A}$. The dual-complex conjuguate of w is caraterized by the relations

$$\begin{cases} ww^{\dagger_4} \in \mathbb{R}, \\ \mathrm{real}\,(w) = \mathrm{real}\left(w^{\dagger_4}\right). \end{cases} \tag{2.14}$$

We can easily verify that

$$w^{\dagger_4} = \overline{z}\left(1 - \frac{t}{z}\varepsilon\right). \tag{2.15}$$

5. Anti-dual conjugation.

$$w^{\dagger_5} = t - z\varepsilon. \tag{2.16}$$

The below Lemma follows.

Proposition 1. *Let $w = z + \varepsilon t \in \mathbb{DC}$ be a dual-complex number. Then, the following assertions hold*

$$w + w^{\dagger_1} = 2\,\mathrm{real}\,(z) + 2\,\mathrm{real}\,(t)\,\varepsilon \in \mathbb{D}, \tag{2.17}$$

$$ww^{\dagger_1} = |z|^2 + 2\,\mathrm{Re}\left(z\overline{t}\right)\varepsilon \in \mathbb{D}, \tag{2.18}$$

$$w + w^{\dagger_2} = 2z \in \mathbb{C}, \tag{2.19}$$

$$ww^{\dagger_2} = z^2 \in \mathbb{C}, \tag{2.20}$$

$$w + w^{\dagger_3} = 2\,\mathrm{real}\,(z) + 2\,\mathrm{Im}\,(t)\,\varepsilon i, \tag{2.21}$$

$$ww^{\dagger_3} = |z|^2 - 2\,\mathrm{Im}\left(z\overline{t}\right)\varepsilon i, \tag{2.22}$$

$$zw^{\dagger_4} = \overline{z}w^{\dagger_2}, \quad (w \in \mathbb{DC} - \mathcal{A}), \tag{2.23}$$

$$ww^{\dagger_4} = |z|^2 \in \mathbb{R}, \quad (w \in \mathbb{DC} - \mathcal{A}), \tag{2.24}$$

$$\begin{cases} z = w - w^{\dagger_5}\varepsilon, \\ t = w^{\dagger_5} + w\varepsilon. \end{cases} \tag{2.25}$$

where $|z|$ represents the usual complex modulus of the complex number z.

The five kinds of conjugation all have some of the standard properties of conjugations, such as:

Proposition 2. *Let w, w_1, $w_2 \in \mathbb{DC}$ are dual-complex numbers. Then*

$$\begin{cases} \left(w^{\dagger_i}\right)^{\dagger_i} = w, \ i = 1, ..., 4 \text{ where } z \neq 0 \text{ for } \dagger_4, \\ \left(w^{\dagger_5}\right)^{\dagger_5} = -w. \end{cases} \tag{2.26}$$

$$(w_1 + w_2)^{\dagger_i} = w_1^{\dagger_i} + w_2^{\dagger_i}, \ i = 1, 2, 3 \text{ and } 5, \tag{2.27}$$

$$(w_1 w_2)^{\dagger_i} = w_1^{\dagger_i} w_2^{\dagger_i}, \ i = 1, ..., 4 \text{ where } z \neq 0 \text{ for } \dagger_4. \tag{2.28}$$

$$\overline{\left(\frac{1}{w}\right)}^{\dagger_i} = \frac{1}{w^{\dagger_i}}, \ (z \neq 0), \ i = 1, ..., 4. \tag{2.29}$$

Denoting now by $\dagger_0$ the trivial conjugation.

$$w^{\dagger_0} = w \ \ \forall w \in \mathbb{DC}. \tag{2.30}$$

The following result holds.

Proposition 3. *The composition of the conjugates $\dagger_0, \dagger_1, \dagger_2$ and $\dagger_3$ gives the four dimensional abelian Klein group:*

$\circ$	$\dagger_0$	$\dagger_1$	$\dagger_2$	$\dagger_3$
$\dagger_0$	$\dagger_0$	$\dagger_1$	$\dagger_2$	$\dagger_3$
$\dagger_1$	$\dagger_1$	$\dagger_0$	$\dagger_3$	$\dagger_2$
$\dagger_2$	$\dagger_2$	$\dagger_3$	$\dagger_0$	$\dagger_1$
$\dagger_3$	$\dagger_3$	$\dagger_2$	$\dagger_1$	$\dagger_0$

(2.21)

Elsewhere, we know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in $\mathbb{R}^2$. The analogs of this, for dual-complex numbers, are the following. Let $w = z + \varepsilon t$ a dual-complex number, then we find

$$|w|_{\dagger_1}^2 = ww^{\dagger_1} = |z|^2 + 2\operatorname{Re}\left(z\bar{t}\right)\varepsilon \in \mathbb{D}, \tag{2.32}$$

$$|w|_{\dagger_2}^2 = ww^{\dagger_2} = z^2 \in \mathbb{C}, \tag{2.33}$$

$$|w|_{\dagger_3}^2 = ww^{\dagger_3} = |z|^2 - 2\operatorname{Im}\left(z\bar{t}\right)\varepsilon i, \tag{2.34}$$

$$\text{if } w \in \mathbb{DC} - \mathcal{A} \text{ then } |w|_{\dagger_4}^2 = ww^{\dagger_4} = |z|^2 \in \mathbb{R}. \tag{2.35}$$

Remarks here that if $z = 0$ then $|w|_{\dagger_i} = 0, \ \ i = 1, 2, 3$. We also admit that if $z = 0$ then $|w|_{\dagger_4} = 0$.

We can then evaluate the inverse of any dual-complex number $w \in \mathbb{DC} - \mathcal{A}$ as follows

$$\frac{1}{w} = \frac{w^{\dagger_2}}{|w|_{\dagger_2}^2} = \frac{w^{\dagger_4}}{|w|_{\dagger_4}^2}. \tag{2.36}$$

It is also important to know that every dual-complex number has another representation, using matrices.

Introducing the unit dual-complex vector $\mathcal{E}$ defined by

$$\mathcal{E} = \begin{bmatrix} 1 \\ i \\ \varepsilon \\ \varepsilon i \end{bmatrix}. \tag{2.37}$$

Denoting by $\mathcal{G}$ the subset of $\mathcal{M}_4(\mathbb{R})$ given by

$$\mathcal{G} = \left\{ A \in \mathcal{M}_4(\mathbb{R}) \mid A = \begin{bmatrix} x_1 & x_2 & 0 & 0 \\ x_2 & -x_1 & 0 & 0 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & -x_3 & x_2 & -x_1 \end{bmatrix} \right\}. \tag{2.38}$$

One can easily verify that $\mathcal{G}$ is a subring of $\mathcal{M}_4(\mathbb{R})$ which forms a 4$-$dimensional real associative and commutative Algebra.

Under the additional condition $x_1^2 + x_2^2 \neq 0$, $\mathcal{G}$ becomes a subgroup of $GL(4)$.

Let us now define the map

$$\left\{ \begin{array}{c} \mathcal{N} : \mathbb{DC} \longrightarrow \mathcal{G}, \\ \mathcal{N}\left(x_1 + x_2 i + x_3\varepsilon + x_4\varepsilon i\right) = \begin{bmatrix} x_1 & x_2 & 0 & 0 \\ x_2 & -x_1 & 0 & 0 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & -x_3 & x_2 & -x_1 \end{bmatrix} \end{array} \right. \tag{2.39}$$

The following results give us a correspondence between the two algebra $\mathbb{DC}$ and $\mathcal{G}$ via the map $\mathcal{N}$.

Theorem 4. *$\mathcal{N}$ is an isomorphism of rings.*

From now on we denote by $\mathcal{P}$ the map

$$\left\{ \begin{array}{c} \mathcal{P} : \mathbb{DC} \longrightarrow \mathbb{R}_+, \\ \mathcal{P}(w) = |w|_{\dagger_4} = |z|. \end{array} \right. \tag{2.40}$$

It is easy to verify that

$$\left\{ \begin{array}{c} \mathcal{P}(w_1 + w_2) \leq \mathcal{P}(w_1) + \mathcal{P}(w_2) \quad \forall w_1, w_2 \in \mathbb{DC}, \\ \mathcal{P}(w_1 w_2) = \mathcal{P}(w_1)\mathcal{P}(w_2) \quad \forall w_1, w_2 \in \mathbb{DC}, \\ \mathcal{P}(w) \geq 0 \text{ with } \mathcal{P}(w) = 0 \text{ iff } w \in \mathcal{A}. \end{array} \right. \tag{2.41}$$

This implies in particular

$$\left\{\begin{array}{c}\mathcal{P}\left(\lambda w_2\right)=\left|\lambda\right|\mathcal{P}\left(w_2\right)\quad\forall w_1,w_2\in\mathbb{DC},\quad\forall\lambda\in\mathbb{C},\\\mathcal{P}\left(\alpha w_2\right)=\left|\mathrm{Re}\left(\alpha\right)\right|\mathcal{P}\left(w_2\right)\quad\forall w_1,w_2\in\mathbb{DC},\quad\forall\alpha\in\mathbb{D},\end{array}\right. \tag{2.42}$$

So, $\mathcal{P}$ defines a pseudo-modulus on $\mathbb{DC}$. It inducess a structure of pseudo-topology over the algebra $\mathbb{DC}$.

The following result holds.

Proposition 5. *Let $w\in\mathbb{DC}$ be a dual-complex number and $n\in\mathbb{N}$. Then*

$$\mathcal{P}\left(w\right)=\left|\det\left(\mathcal{N}\left(w\right)\right)\right|^{\frac{1}{2}}, \tag{2.43}$$

$$\mathcal{P}\left(w^{\dagger_i}\right)=\mathcal{P}\left(w\right),\quad i=1,...,4. \tag{2.44}$$

$$\mathcal{P}\left(w^n\right)=\mathcal{P}\left(w\right)^n, \tag{2.45}$$

$$\mathcal{P}\left(\frac{1}{w}\right)=\frac{1}{\mathcal{P}\left(w\right)},\quad\left(w\in\mathbb{DC}-\mathcal{A}\right). \tag{2.46}$$

Thus, we can construct the dual-complex disk and dual-complex sphere of centre $w_0=z_0+t_0\varepsilon\in\mathbb{DC}$ and radius $r>0$, respectively, as follows

$$D\left(w_0,r\right)=\left\{w=z+t\varepsilon\in\mathbb{DC}\mid p\left(w-w_0\right)<r\right\}\approx D_c\left(z_0,r\right)\times\mathbb{C}, \tag{2.47}$$

$$S\left(w_0,r\right)=\left\{w=z+t\varepsilon\in\mathbb{DC}\mid p\left(w-w_0\right)=r\right\}\approx S_c\left(z_0,r\right)\times\mathbb{C}. \tag{2.48}$$

where $D_c\left(z_0,r\right)$ and $S_c\left(z_0,r\right)$ are, repectively, the complex disk and complex sphere of centre z_0 and radius $r>0$.

$S\left(w_0,r\right)$ can be also called the complex Galilean sphere.

Definition 1. *1. We say that Ω is a dual-complex subset of $\mathbb{DC}$ if there exists a subset $O\subset\mathbb{C}$ such that*

$$\Omega=O+\mathbb{C}\varepsilon\approx O\times\mathbb{C}. \tag{2.49}$$

O is called the generator of Ω.

2. We say that Ω is an open dual-complex subset of $\mathbb{DC}$ if the generator of Ω is an open subset of $\mathbb{C}$.

3. Ω is said to be a closed dual-complex subset of $\mathbb{DC}$ if his complement is an open subset of $\mathbb{DC}$.

Note that the algebra $\mathbb{DC}$ equipped with the previous pseudo-topology is not Hausdorff space.

We discuss now some properties of dual-complex functions. We investigate the continuity of dual-complex functions and the differentiability in the dual-complex sense, which can be also called holomorphicity, as in complex case. In the following definitions, we suppose that $\mathbb{DC}$ is equipped with the usual topology of $\mathbb{C}^2$.

Definition 2. *A dual-complex function is a mapping from a subset $\Omega\subset\mathbb{DC}$ to $\mathbb{DC}$.*

Let Ω be an open subset of $\mathbb{DC}$, $w_0=z_0+\varepsilon t_0\in\Omega$ and $f:\Omega\longrightarrow\mathbb{DC}$ a dual-complex function.

Definition 3. *We say that the dual-complex function f is continuous at $w_0=z_0+t_0\varepsilon$ if*

$$\lim_{w\to w_0}f\left(w\right)=f\left(w_0\right). \tag{2.50}$$

where the limit is calculated coordinate by coordinate, this means that

$$\lim_{w\to w_0}f\left(w\right)=\lim_{z\to z_0,\ t\to t_0}f\left(w\right)=f\left(w_0\right). \tag{2.51}$$

Definition 4. *The function f is continuous in $\Omega \subset \mathbb{DC}$ if it is continuous at every point of Ω.*

Definition 5. *The dual-complex function f is said to be differentiable in the dual-complex sense at $w_0 = z_0 + t_0\varepsilon$ if the following limit exists*

$$\frac{df}{dw}(w_0) = \lim_{z \to z_0,\ t \to t_0} \frac{f(w) - f(w_0)}{w - w_0}, \tag{2.52}$$

$\frac{df}{dw}(w_0)$ is called the derivative of f at the point w_0.

If f is differentiable for all points in a neighbourhood of the point w then f is called holomorphic at w.

Definition 6. *The function f is holomorphic in $\Omega \subset \mathbb{DC}$ if it is holomorphic at every point of Ω.*

In the following results we generalize the Cauchy-Riemann formulas to dual-complex functions.

Theorem 6. *Let f be a dual-complex function in $\Omega \subset \mathbb{DC}$, which can be written in terms of its complex and dual parts as*

$$f = p + q\varepsilon. \tag{2.53}$$

Then, f is holomorphic in $\Omega \subset \mathbb{DC}$ if and only if the derivative of f satisfies

$$\frac{df}{dw} = \frac{\partial f}{\partial z} = \frac{\partial p}{\partial z} + \frac{\partial q}{\partial z}\varepsilon. \tag{2.54}$$

Corollary 7. *Let f be a dual-complex function in $\Omega \subset \mathbb{DC}$, which can be written in terms of its complex and dual parts as $f = p + q\varepsilon$ and suppose that the partial derivatives of f exist. Then,*

1. f is holomorphic in $\Omega \subset \mathbb{DC}$ if and only it satisfies

$$\mathcal{D}(f) = 0, \tag{2.56}$$

where $\mathcal{D}$ is the differential operator

$$\mathcal{D}(f) = -\varepsilon\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t}. \tag{2.57}$$

2. f is holomorphic in $\Omega \subset \mathbb{DC}$ if and only if its complex and dual parts satisfy the following generalized Cauchy-Riemann equations,

$$\begin{cases} \frac{\partial p}{\partial z} = \frac{\partial q}{\partial t}, \\ \\ \frac{\partial p}{\partial t} = 0. \end{cases} \tag{2.58}$$

Furthermore, as in complex analysis, the Cauchy-Riemann equations can be also reformulated using the partial derivative with respect to the anti-dual conjugate. For this, we can write using the formula (2.25)

$$\begin{cases} dz = dw - dw^{\dagger_5}\varepsilon, \\ dt = dw^{\dagger_5} + dw\varepsilon. \end{cases} \tag{2.59}$$

Replacing in the total differential of f, we find

$$df = \left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t}\varepsilon\right)dw + \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial z}\varepsilon\right)dw^{\dagger_5}. \tag{2.60}$$

This allows us to properly introduce the differential operators $\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial w^{\dagger_5}}$ as

$$\begin{cases} \frac{\partial}{\partial w} = \frac{\partial}{\partial z} + \frac{\partial}{\partial t}\varepsilon, \\ \frac{\partial}{\partial w^{\dagger_5}} = \frac{\partial}{\partial t} - \frac{\partial}{\partial z}\varepsilon. \end{cases} \tag{2.61}$$

Hence, Cauchy-Riemann formulas have the particular compact form

$$\frac{\partial f}{\partial w^{\dagger_5}} = 0. \tag{2.62}$$

Theorem 8. *The function f is holomorphic in the open subset $\Omega \subset \mathbb{DC}$, (with respect to the topology of $\mathbb{C}^2$), if and only if there exists a pair of complex functions p and r, such that $p \in \mathrm{C}^2(P_z(\Omega))$ and $r \in \mathrm{C}^1(P_z(\Omega))$, where P_z is the projection with respect to the first complex variable z, so that the function f has the explicite expression*

$$f(w) = p(z) + \left(\frac{dp}{dz}t + r(z)\right)\varepsilon \quad \forall w \in \Omega. \tag{2.63}$$

Remark 1. *The formula (2.63) gives, taking into account the fact that $\frac{df}{dw} = \frac{\partial f}{\partial z}$,*

$$\frac{df}{dw} = \frac{dp}{dz} + \left(\frac{d^2p}{dz^2}t + \frac{dr}{dz}\right)\varepsilon. \tag{2.64}$$

In particular, since p and r are holomorphic from Cauchy's integral formula f is analytic in Ω and we have

$$\frac{d^m f}{dw^m} = \frac{d^m p}{dz^m} + \left(\frac{d^{m+1}p}{dz^{m+1}}t + \frac{d^m r}{dz^m}\right)\varepsilon \quad \forall m \geq 1. \tag{2.65}$$

In the following theorem we give two basic results concerning the continuation of holomorphic dual-complex functions and that of holomorphic complex functions to dual-complex numbers.

Theorem 9. ***1.*** *Let f be an holomorphic dual-complex function in an open subset $\Omega \subset \mathbb{DC}$. Then, f can be holomorphically extended to the open dual-complex subset $P_z(\Omega) + \mathbb{C}\varepsilon$.*

2. *Let f be an holomorphic complex function in an open subset $O \subset \mathbb{C}$. Then, there exists a unique holomorphic dual-complex function F defined in the open dual-complex subset $O + \mathbb{C}\varepsilon$ such that*

$$F(z) = f(z) \quad \forall z \in O. \tag{2.66}$$

and we have

$$F(z + t\varepsilon) = f(z) + \frac{df}{dz}t\varepsilon \quad \forall z + t\varepsilon \in O + \mathbb{C}\varepsilon. \tag{2.67}$$

The proof follows directly from the previous theorem.

3. Usual dual functions

We can think of applying the statement of theorem 10, which asserts that any holomorphic complex function can be holomorphically extended to dual-complex numbers, to build dual-complex functions similar to the usual complex functions, obtained as their extensions.

3.1. **The dual-complex Exponential function.** The complex exponential function e^z defined for all $z \in \mathbb{C}$ can be extended to the algebra $\mathbb{DC}$ as follows

$$\exp(w) = e^w = e^z + e^z t\varepsilon = e^z(1 + t\varepsilon). \tag{3.1}$$

The derivative of e^w is

$$\frac{de^w}{dz} = \frac{de^z}{dx} + \frac{de^z}{dx} t\varepsilon = e^w \quad \forall z \in \mathbb{DC}. \tag{3.2}$$

By recurrence, we find

$$\frac{d^n e^w}{dz^w} = e^w \quad \forall z \in \mathbb{DC}, \quad \forall n \in \mathbb{N}. \tag{3.3}$$

Thus, any dual number $w = z + t\varepsilon \in \mathbb{DC} - \mathcal{A}$ has the exponential representation

$$w = ze^{\frac{t}{z}\varepsilon}. \tag{3.4}$$

Denoting by $\arg_d$ the complex number, called the dual argument of the dual-complex number w,

$$\arg_d w = \frac{t}{z}, \quad w \in \mathbb{DC} - \mathcal{A}. \tag{3.5}$$

Some properties are collected in the followings.

Proposition 10. *1.* $e^{w_1 + w_2} = e^{w_1} e^{w_2}$.
2. $e^{-w} = \frac{1}{ew}$.
3. $e^w \neq 0 \quad \forall w \in \mathbb{DC}$.

Proposition 11. *1. The map* $\arg_d : (\mathbb{DC} - \mathcal{A}, .) \longrightarrow (\mathbb{C}, +)$ *is a morphism of groups where the kernel is given by*

$$\ker(\arg_d) = \mathbb{C}^*. \tag{3.6}$$

2. $\begin{cases} w^{\dagger_1} = \overline{z} e^{\overline{\arg_d(w)}\varepsilon}, \\ w^{\dagger_2} = z e^{-(\arg_d w)\varepsilon}, \\ w^{\dagger_3} = \overline{z} e^{-\overline{\arg_d(w)}\varepsilon}, \\ w^{\dagger_4} = \overline{z} e^{-(\arg_d w)\varepsilon}. \end{cases}$

3.2. **The dual-complex Trigonometric functions.** The trigonometric functions: sine, cos, etc, have their dual-complex analogues. In fact, we can define them by the formulas

$$\sin w = \sin z + (\cos z)\, t\varepsilon \quad \forall z \in \mathbb{DC}, \tag{3.7}$$

$$\cos w = \cos z - (\sin z)\, t\varepsilon \ \forall z \in \mathbb{DC}, \tag{3.8}$$

The below properties can be mostly deduced from the previous definition.

Proposition 12. *1.* $\sin$ *and* $\cos$ *are* $2\pi-$*periodic functions.*
2. $\sin(-w) = -\sin w, \quad \cos(-w) = \cos w$.
3. $\sin(w_1 + w_2) = \sin w_1 \cos w_2 + \cos w_1 \sin w_2$.
4. $\cos(w_1 + w_2) = \cos w_1 \cos w_2 - \sin w_1 \sin w_2$.
5. $\sin^2 w + \cos^2 w = 1$.
6. $\frac{d \sin w}{dw} = \cos w$.
7. $\frac{d \cos w}{dw} = -\sin w$.
8. $\sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$.
9. $\cos w = \dfrac{e^{iw} + e^{-iw}}{2}$.

3.3. **The dual-complex Hyperbolic functions.** The dual-complex hyperbolic functions are defined by

$$\sinh w = \sinh z + (\cosh z)\, t\varepsilon \quad \forall w \in \mathbb{DC}, \tag{3.9}$$

$$\cosh w = \cosh z + (\sinh z)\, t\varepsilon \quad \forall w \in \mathbb{DC}, \tag{3.10}$$

These are equivalent, as in the complex case, to

$$\sinh w = \frac{e^{w} - e^{-w}}{2} \quad \forall w \in \mathbb{DC}, \tag{3.11}$$

$$\cosh w = \frac{e^{w} + e^{-w}}{2} \quad \forall w \in \mathbb{DC}, \tag{3.12}$$

The following collects some basic properties.

Proposition 13. *1.* $\sinh(-w) = -\sinh w, \;\; \cosh(-w) = \cosh w.$

2. $\cosh^2 w - \sinh^2 w = 1.$

3. $\frac{d \sinh w}{dw} = \cosh w.$

4. $\frac{d \cosh w}{dw} = \sinh w.$

5. $\sinh(iw) = i \sin w$ *and* $\cosh(iw) = \cos w.$

3.4. **The dual Logarithmic function.** We define the dual Logarithmic function by the formula

$$\log w = \log z + \frac{t}{z}\varepsilon = \log z + (\arg_d w)\,\varepsilon \quad \forall w \in \mathbb{DC} - \mathcal{A}. \tag{3.13}$$

It is straightforward to verify that dual Logarithmic function, satisfies some properties, given by

Proposition 14. *1.* $\log\left(\frac{1}{w}\right) = -\log w.$

2. $\log(w_1 w_2) = \log(w_1) + \log(w_2).$

3. If $\arg(z) \in \left]-\pi, \pi\right],$ *(principal representation), then* $e^{\log w} = \log(e^{w}) = w.$

5. $\frac{d \log w}{dz} = \frac{1}{w}.$

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