

Enumerating AG-Groups with a Study of Smaradache AG-Groups

Muhammad Shah¹

Department of Mathematics
Quaid-i-Azam University, Pakistan
shahmaths_problem@hotmail.com

Charles Gretton² and Volker Sorge

Department of Computer Science
University of Birmingham, UK
charles.gretton@gmail.com, V.Sorge@cs.bham.ac.uk

Abstract

AG-groups are a generalisation of Abelian groups. They correspond to groupoids with a left identity, unique inverses, and satisfy the identity $(xy)z = (zy)x$. We present the first enumeration result for AG-groups up to order 11 and give a lower bound for order 12. The counting is performed with the finite domain enumerator FINDER using bespoke symmetry breaking techniques. We have also developed a function in the GAP computer algebra system to check the generated Cayley tables. This note discusses a few observations obtained from our results, some of which inspired us to examine and discuss Smaradache AG-group structures.

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1 Introduction

In the study of small algebraic structures more general than groups, many interesting questions, such as open existence, classification, and counting problems, have been solved by software tools that enable efficient enumeration of structures. Typically this task involves identifying and exploiting symmetries in the problem at hand. Loops with inverse property (IP-loops) up to order 13 have been counted with model generators using hand

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crafted symmetry breaking constraints and *post-hoc* processing [1]. Monoids up to order 10 and semigroups up to order 9 have been enumerated [3] with off-the-shelf constraint satisfaction software by employing lexicographic symmetry breaking constraints computed using a GAP implementation of the methods described in [5]. A similar approach was more recently used for counting AG-groupoids — groupoids that are left invertive, in the sense $(ab)c = (cb)a$ — up to order 6 [2]. Finally, also related is the enumeration of quasigroups and loops up to size 11 using a mixture of combinatorial considerations and bespoke exhaustive generation software [7].

In this paper we present the enumeration of AG-groups. An AG-group, also called an LA-group, is an AG-groupoid (G, \cdot) with left identity, and unique inverses. With a structure admitting non-associativity, AG-groups lie between quasigroups and Abelian groups. AG-groups were conceived by M.S. Kamran for his PhD thesis [6], first appeared in [8] and then more recently have been studied in [12]. AG-groups generalise Abelian groups. An AG-group satisfies the Lagrange theorem [8]. Moreover, it can easily be verified that AG-groups satisfy both the medial, i.e. $(ab)(cd) = (ac)(bd)$, and paramedial, i.e. $(ab)(cd) = (db)(ca)$, laws (see [12, Lemma 1]).

In Sec. 2 we first count the number of non-isomorphic AG-groups of order up to 11 and give a lower bound for order 12. We then discuss some of the observations we have made when examining the results (Sec. 3) and in particular develop and study a new type of AG-group structure, Smaradache AG-groups (Sec. 4).

2 Counting AG-groups up to Isomorphism

We counted AG-groups by exhaustive enumeration using the FINDER system. Our starting point is the approach developed for counting IP-loops up to isomorphism in [11, 1]. Here, as in that work, FINDER generates a set of candidate tables which contains one table for each minimal element given by a lex(icographic) order over each isomorphism class. A post-processing step — *post-hoc* processing — is used to reject tables that are not minimal in their isomorphism class.³ In order to prevent FINDER from generating an impractical number of candidate tables, further symmetry breaking constraints are posed. Moreover, the validity of *post-hoc* processing is dependent on these constraints being satisfied. In our work we have had to modify those constraints. A summary of the constraints from [11] in their reduced form for AG-groups is given in the following definition.

Definition 2.1. (*Symmetry Breaking Constraints*) *Let N be the order of the AG-group, with elements $x, y \in \{0 \dots, N - 1\}$ and left identity e . Let $f(x)$ abbreviate $(e + 1)(e + x)$ and let $FLAG$ be a boolean variable that is set if the first six elements of the AG-group are self-inverse and $(e + 1)(e + 2)$ is not self-inverse. We then define the following 10 constraints:*

³We consider it an important item for future work to develop *lex-leader* constraints that capture the symmetry breaking that is carried out during this *post-hoc* step.

- (i) $e \leq x$,
- (ii) $x^{-1} < (x + 2)$,
- (iii) $(x^{-1} = x \wedge x < y) \Rightarrow y^{-1} = y$.

For odd values of N :

- (iv) $f(1) < (e + 4)$,
- (v) $(x > 1 \wedge 2x < N) \Rightarrow f(x) < (e + 2x)$.

For even values of N

- (vi) $f(1) = e$,
- (vii) $(-FLAG \wedge 0 < x < \frac{N}{2}) \Rightarrow f(x) < (e + 2x + 1)$,
- (viii) $(FLAG) \Rightarrow (e + 5)^{-1} = (e + 5)$,
- (ix) $(FLAG \wedge x > 1 \wedge (e + x)^{-1} = (e + x)) \Rightarrow (f(x)^{-1}) \neq f(x)$,
- (x) $(FLAG \wedge 1 < x < y \wedge (e + y)^{-1} = (e + y)) \Rightarrow f(x) < f(y)$. □

The constraints imply that the Cayley table of the AG-group will be filled in an ascending order, where e is always the lexicographical minimal element (i.e., 0). They also have that elements which are self-inverse are ordered first, and otherwise that an element is adjacent to its inverse in the ordering. We have omitted the constraint $x^{-1} = x \Leftrightarrow x = e$ from [1], because this is an invalid symmetry breaker for AG-group counting.

Our enumeration yields all Cayley tables explicitly. We can therefore validate the generated AG-groups using the GAP [4] computer algebra system. Because there was no functionality present in the GAP *loops* package [9] to test whether a Cayley table is an AG-group or not, we have implemented our own function that performs this test. The results of our enumeration were validated using our GAP function.⁴ The algorithm is a straightforward implementation, testing that (1) the Cayley table is a Latin square; i.e., all elements occur exactly once in every row and every column, (2) the identity $(xy)z = (zy)x$ holds, and (3) there exists a left identity.

Our validated results are given in Table 1. We report the number of non-isomorphic AG-groups having order up to 11, and give a lower bound for order 12. In Table 1, for each order we give the total number of AG-groups up to isomorphism, which is further broken down into associative and non-associative AG-groups. Note, associative AG-groups are Abelian groups. For each order we also give the total number of CPU-seconds required to enumerate all groups and the number of tables generated by FINDER that were tested for lex-minimality in post-hoc processing. All counting was carried out on an Intel quad core CPU Q9650 with 4GB of memory. It should be noted that our counting procedure uses negligible computer memory resources.

In the remainder of the paper we discuss some of the observations we made using our enumeration results and, in particular, propose a new interesting class of AG-groups.

3 AG-group of Smallest Order

The smallest AG-group which is not a group is of order 3. The Cayley table for that is given in Example 3.1.

Example 3.1. *Smallest AG-group of order 3 :*

⁴Please contact the authors by email for either the GAP source code, or a copy of the enumerated tables.

Order	AG-groups	Assoc	Non-Assoc	CPU-Time	post-hoc tests
1	1	1	0	< .01	0
2	1	1	0	< .01	1
3	2	1	1	< .01	2
4	4	2	2	< .01	6
5	2	1	1	< .01	7
6	2	1	1	< .01	46
7	2	1	1	.47	97
8	10	3	7	8.44	796
9	5	2	3	102.37	3599
10	2	1	1	1,735.25	16144
11	2	1	1	15,206.26	86406
12	≥ 7	≥ 2	≥ 5	NA	NA

Table 1: Results of AG-group enumeration.

·	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Clearly every Abelian group is an AG-group, however the converse is certainly not always true. We now note a number of contrasts between groups and AG-groups. In particular, we establish the existence of non-associative AG-groups — i.e. non-Abelian groups — of order p and p^2 where p is a prime. In detail, our observations are:

1. Every group of order p is Abelian. We find that non-associative AG-groups of order p exist. The AG-group of order 3 in Example 3.1 is not an Abelian group.
2. Every group of order p^2 is Abelian. We also have that non-associative AG-groups of order p^2 exist. The AG-group in Example 4.3 is not an Abelian group.
3. Every group which satisfies the squaring property $(ab)^2 = a^2b^2$ is Abelian. Although every AG-group clearly satisfies the squaring property, an AG-group is not necessarily Abelian.

4 Smaradache AG-groups

In [10] Padilla Raul introduced the notion of a Smarandache semigroup, here written S -semigroup. An S -semigroup is a semigroup A such that a proper subset of A is a group with respect to the same induced operation [15]. Similarly a Smarandache ring, written S -ring, is defined to be a ring A , such that a proper subset of A is a field with respect to the operations induced. Many other Smarandache structures have also appeared in the

literature. The general concept of Smarandache structures is that, if a special structure happens to be a substructure of a general structure, then that general structure is called Smarandache. In that spirit we propose Smarandache AG-groups here and study them with the help of examples generated during the enumeration given in Sec. 2.

Definition 4.1. *Let G be an AG-group. G is said to be a Smarandache AG-group (S-AG-group) if G has a proper subset P such that P is an Abelian group under the the operation of G .*

The AG-groups G in Examples 4.3 and 4.4 are S-AG-groups, whereas the AG-group G in Examples 3.1 and 4.6 are not.

The following theorem guarantees that an AG-group having a unique nontrivial element of order 2 is always an S-AG-group.

Theorem 4.2. *If there is a unique nontrivial element a of order 2 in an AG-group G then $\{e, a\}$ is an Abelian subgroup of G .*

Proof. Take $a \in G$ satisfying $a^2 = e$. Now we have to identify an element for the '?' cell in the following table:

\cdot	e	a
e	e	a
a	?	e

Taking $y = ae$ and using the paramedial law we have $y^2 = (ae)^2 = e^2a^2 = a^2 = e$. Thus, y has order 2. Because G has a single element of order 2, we have $y = ae = a$. Thus, a is the required element, and our table can now be completed:

\cdot	e	a
e	e	a
a	a	e

Here $\{e, a\}$ is an AG-subgroup of G of order 2, and is therefore an Abelian group, hence G is an S-AG-group. □

We illustrate Theorem 4.2, by considering the following example.

Example 4.3. *An AG-group of order 8:*

\cdot	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	6	7	4	5
2	2	3	1	0	5	6	7	4
3	3	2	0	1	7	4	5	6
4	6	4	7	5	3	0	2	1
5	7	5	4	6	0	2	1	3
6	4	6	5	7	2	1	3	0
7	5	7	6	4	1	3	0	2

Here 1 has order 2, and 1 is the unique such element of G . Hence $\{0, 1\}$ is an Abelian subgroup of G .

The converse of Theorem 4.2 is not true. That is, if an AG-group G has an Abelian subgroup, then it is not necessary that G will have a unique non-trivial element of order 2. This can be observed with the following example:

Example 4.4. *An AG-group of order 9:*

·	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	2	0	1	5	6	7	8	3	4
2	1	2	0	7	8	3	4	5	6
3	5	3	7	8	0	4	2	6	1
4	6	4	8	0	7	2	3	1	5
5	3	7	5	6	1	8	0	4	2
6	4	8	6	1	5	0	7	2	3
7	7	5	3	4	2	6	1	8	0
8	8	6	4	2	3	1	5	0	7

The AG-group in Example 4.4 has $\{0, 7, 8\}$ as Abelian subgroup and hence is a Smarandache AG-group. However, the element of order 2 is not unique. In fact, there are two nontrivial elements of order 2, namely $\{1, 2\}$.

Remark 4.5. Because AG-groups satisfy Lagrange's Theorem, the unique element of order 2 can exist in AG-groups of even order only.

However, if G has more than one element of order 2, then it is not necessary that G will have an Abelian subgroup. In Example 4.6 all non-trivial elements of G are of order 2, however we also see that G has no Abelian subgroup. Note that G has four proper AG-subgroups, namely $\{0, 1, 2\}$, $\{0, 3, 7\}$, $\{0, 4, 6\}$, and $\{0, 5, 8\}$. None of those is commutative.

Example 4.6. *An AG-group order of 9:*

·	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	2	0	1	4	5	3	7	8	6
2	1	2	0	5	3	4	8	6	7
3	7	6	8	0	2	1	5	3	4
4	6	8	7	1	0	2	4	5	3
5	8	7	6	2	1	0	3	4	5
6	4	3	5	8	6	7	0	2	1
7	3	5	4	7	8	6	1	0	2
8	5	4	3	6	7	8	2	1	0

AG-groups satisfy Lagrange's Theorem, so AG-groups of prime order cannot have a proper AG-subgroup, hence cannot have a proper Abelian subgroup. We record this fact as the following theorem.

Theorem 4.7. *An AG-group G of prime order cannot be an S-AG-group.* □

The notion of S-AG-group can be generalised to S-AG-groupoid as follows.

Definition 4.8. *Let S be an AG-groupoid. S is said to be an Smaradache AG-groupoid (S-AG-groupoid) if S has a proper subset P such that P is a commutative semigroup under the operation of S .*

The examples given in the case of AG-groups can also be considered for S-AG-groupoids. We now provide two further examples to show that this notion holds generally.

Example 4.9. *An AG-groupoid order 4.*

·	0	1	2	3
0	0	0	2	3
1	0	1	2	3
2	3	3	0	2
3	2	2	3	0

Example 4.10. *An AG-groupoid order 4.*

·	0	1	2	3
0	1	2	0	3
1	3	0	2	1
2	2	1	3	0
3	0	3	1	2

The AG-groupoid S in Example 4.9 has a proper subset $\{0, 1\}$ which is a commutative semigroup, and therefore S is an S-AG-groupoid. Although somewhat tedious, one can check manually that the AG-groupoid S in Example 4.10 has no proper subset having the desired property, and therefore we have S is not a S-AG-groupoid.

5 Future Work

Due to the obvious limitations of enumerative counting, we have only been able to study AG-groups of small order here. An important future direction is to pursue counting algebraically, especially where such work can inform a constructive procedure for larger AG-groups. We have also begun characterising S-AG-groups here, however that notion requires significant further study.

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