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Article

Fuzzy Adaptive Parameter in the Dai-Liao Optimization Method Based on Neutrosophy

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Abstract: The influence of neutrosophy in the previous period is constantly growing in many areas of science and technology. Moreover, various applications of the neutrosophic approach have become more common in recent years. Our goal in this research is to utilize the neutrosophy to improve the performance of the Dai-Liao conjugate gradient (CG) method. Specifically, in this research, we propose and investigate a new neutrosophic logic system to calculate the key parameter t involved in the Dai–Liao CG iterations. Theoretical analysis and numerical experience indicate that the efficiency and robustness of the new rule for determining t. Combining the neutrosophy and the Dai-Liao conjugate gradient method, we propose and explore a new Dai-Liao CG iterations for solving large-scale unconstrained optimization models. The global convergence is established under common assumptions and the backtracking line search. Finally, by conducting numerical experiments, computational evidence demonstrates that the new fuzzy neutrosophic Dai-Liao conjugate gradient method is computationally effective and robust.

Keywords: Neutrosophic logic systems; Dai-Liao conjugate gradient method; Backtracking line search; Convergence; Unconstrained optimization.

MSC: 90C70; 90C30; 65K05

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1. Introduction and background results

Numerous iterative methods have been developed to solve the large-scale unconstrained optimization problem

$$\text{minimize } f(\mathbf{x}), \ \ \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

in which $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function and bounded below. Continuing well established notation, $\mathbf{g}_k = \mathbf{g}(\mathbf{x}_k) = \nabla f(\mathbf{x}_k)$ stands for the gradient vector of f at the actual iterative point $\mathbf{x}_k$, and further $\mathbf{y}_{k-1} = \mathbf{g}_k - \mathbf{g}_{k-1}$ and $\mathbf{s}_{k-1} = \mathbf{x}_k - x_{k-1}$. Utilizing the extended conjugacy condition

$$\mathbf{d}_k^{\mathrm{T}} \mathbf{y}_{k-1} = -t\mathbf{g}_k^{\mathrm{T}} \mathbf{s}_{k-1}, \ \ t > 0, \tag{2}$$

Dai and Liao in [1] suggested the conjugate gradient (CG) iterations

$$x_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k, \tag{3}$$

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where the $\mathbf{x}_k$ is the last calculated iteration, $\mathbf{x}_{k+1}$ is a new iterative point, α_k is a positive step size parameter defined as the output of an inexact line search, and $\mathbf{d}_k$ is a descent direction. The search directions $\{\mathbf{d}_k,\ k \geq 0\}$ are created by the recurrent regulation

$$\mathbf{d}_k = \begin{cases} -\mathbf{g}_0, & k = 0, \\ -\mathbf{g}_k + \beta_k^{\mathrm{DL}}\, \mathbf{d}_{k-1}, & k \geq 1, \end{cases} \tag{4}$$

where β_k^{DL} is the CG coefficient which describes the type of CG method by the general rule

$$\beta_k^{\mathrm{DL}} = \frac{\mathbf{g}_k^{\mathrm{T}} \mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}} \mathbf{y}_{k-1}} - t \frac{\mathbf{g}_k^{\mathrm{T}} \mathbf{s}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}} \mathbf{y}_{k-1}}, \tag{5}$$

wherein $t > 0$ is an appropriate scalar. The Dai–Liao (DL) method guarantees global convergence for uniformly convex objective functions. These results have attracted a lot of attention, leading to the creation of several methods based on various patterns for defining β_k. Most of these methods were developed by modifying the conjugate gradient parameter β_k^{DL} [2–9]. For more details, see the survey on the DL family of nonlinear CG methods in [10]. One of rules for defining β_k is denoted as β_k^{MHSDL} and defined in [7] by

$$\beta_k^{\mathrm{MHSDL}} = \frac{\mathbf{g}_k^{\mathrm{T}} \widehat{\mathbf{y}_{k-1}}}{\mathbf{d}_{k-1}^{\mathrm{T}} \mathbf{y}_{k-1}} - t \frac{\mathbf{g}_k^{\mathrm{T}} \mathbf{s}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}} \mathbf{y}_{k-1}}, \tag{6}$$

such that $\widehat{\mathbf{y}_{k-1}} = \mathbf{g}_k - \frac{\|\mathbf{g}_k\|}{\|\mathbf{g}_{k-1}\|} \mathbf{g}_{k-1}$ and $t > 0$ is as in (5).

Due to the large influence of the size t on numerical results generated by the DL class of CG methods [11], one of the most common issues is the determination of an appropriate value t. We can distinguish two research directions based on the previous results in determining proper values t in the DL established CG iterations. The first direction of research consists of a group of DL methods that aim to find a suitable constant value for t [1,2,6–8] during iterations, while the second direction consists of a group of DL methods that propose a suitable control in recalculating t in each iteration. In this research, we prioritize the second research stream: find values t that changes appropriately accros iterations. The quantity t determined in kth iterative step will be denoted by $t_k^{(i)}$, where i is a variant of the algorithm for defining t.

Some of the most important adaptive choices for the DL parameter t_k will be presented in the rest of this section. Hager and Zhang in [12,13] suggested the CG-DESCENT method, which is classified into the group of the DL CG methods (5) defined by $t \equiv t_k^{(1)}$ and

$$t_k^{(1)} = 2 \frac{\|\mathbf{y}_{k-1}\|^2}{\mathbf{y}_{k-1}^{\mathrm{T}} \mathbf{s}_{k-1}}. \tag{7}$$

Dai and Kou suggested DK method in [14] where the CG coefficient β_k^{DK} is of the form

$$\beta_k^{\mathrm{DK}} = \frac{\mathbf{g}_k^{\mathrm{T}} \mathbf{y}_{k-1}}{\mathbf{y}_{k-1}^{\mathrm{T}} \mathbf{d}_{k-1}} - \left(\tau_k + \frac{\|\mathbf{y}_{k-1}\|^2}{\mathbf{y}_{k-1}^{\mathrm{T}} \mathbf{s}_{k-1}} - \frac{\mathbf{y}_{k-1}^{\mathrm{T}} \mathbf{s}_{k-1}}{\|\mathbf{s}_{k-1}\|^2} \right) \frac{\mathbf{g}_k^{\mathrm{T}} \mathbf{s}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}} \mathbf{y}_{k-1}}. \tag{8}$$

In the equality (8), the parameter τ_k is defined utilizing the self-scaling memoryless BFGS method. It is also obvious from (8) that the DK method is involved into the DL CG class of methods where $t \equiv t_k^{(2)}$ is defined by

$$t_k^{(2)} = \tau_k + \frac{\|\mathbf{y}_{k-1}\|^2}{\mathbf{y}_{k-1}^{\mathrm{T}} \mathbf{s}_{k-1}} - \frac{\mathbf{y}_{k-1}^{\mathrm{T}} \mathbf{s}_{k-1}}{\|\mathbf{s}_{k-1}\|^2}. \tag{9}$$

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Babaie-Kafaki and Ghanbari in [15] proposed the subsequent two rules for calculating t in (5):

$$t_k^{(3)} = \frac{\mathbf{s}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}{\|\mathbf{s}_{k-1}\|^2} + \frac{\|\mathbf{y}_{k-1}\|}{\|\mathbf{s}_{k-1}\|} \tag{10}$$

and

$$t_k^{(4)} = \frac{\|\mathbf{y}_{k-1}\|}{\|\mathbf{s}_{k-1}\|}. \tag{11}$$

Andrei in [16] originated a new DL class, denoted DLE, where $t \equiv t_k^{(5)}$ is defined by

$$t_k^{(5)} = \frac{\mathbf{s}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}{\|\mathbf{s}_{k-1}\|^2}. \tag{12}$$

A special place in the DL iterations is occupied by the DL method with

$$t_k^{(6)} = \upsilon\frac{\|\mathbf{y}_{k-1}\|^2}{\mathbf{s}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}, \tag{13}$$

where $\upsilon > \frac{1}{4}$ is a constant, has been defined according to the sufficient descent condition

$$\mathbf{g}_k^{\mathrm{T}}\,\mathbf{d}_k \leq -c\,\|\mathbf{g}_k\|^2, \ \ \forall\, k \geq 0, \tag{14}$$

such that $c > 0$ is a constant independent of the cost function convexity and the line search rules (for more details see [17]).

Lotfi and Hosseini in [18] suggested the subsequent rule

$$t_k^{(7)} = \max\left\{t_k^{(7^*)},\, t_k^{(6)}\right\}, \tag{15}$$

where

$$t_k^{(7^*)} = \frac{\left(1 - \hbar_k\|g_{k-1\|^r}\right)\mathbf{s}_{k-1}^{\mathrm{T}}\mathbf{g}_k + \frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1}}{\mathbf{y}_{k-1}^{\mathrm{T}}\mathbf{s}_{k-1}}\hbar_k\|\mathbf{g}_{k-1}\|^r\|\mathbf{s}_{k-1}\|^2}{\mathbf{g}_k^{\mathrm{T}}\mathbf{s}_{k-1} + \frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{s}_{k-1}}{\mathbf{s}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\hbar_k\|\mathbf{g}_{k-1}\|^r\|\mathbf{s}_{k-1}\|^2}, \tag{16}$$

$$\hbar_k = C + \max\left\{-\frac{\mathbf{s}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}{\|\mathbf{s}_{k-1}\|^2}, 0\right\}\|\mathbf{g}_{k-1}\|^{-r}, \tag{17}$$

and C, r are positive constants.

Ivanov *et al.* in [19] proposed a variant of the Dai-Liao CG method (6), known as the Effective Dai-Liao (EDL) CG method, where $t \equiv t_k^{(8)}$ is determined as

$$t_k^{(8)} = \frac{\|\mathbf{g}_k\|^2}{\max\left\{1, \mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{g}_k\right\} + \left(\max\left\{0, \frac{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{g}_k}{\|\mathbf{g}_k\|^2}\right\} + 1\right)\|\mathbf{g}_k\|^2}. \tag{18}$$

The experiments performed in [19] verify that the EDL iterations outperform many existing CG variants.

The basics of neutrosophy. Neutrosophic logic was applied in [20] in regulating proper step sizes for a class of accelerated gradient-descent optimization methods. The approach in [20] assumes an additional fuzzy parameter which stabilizes the behavior of an important class of gradient-descent family. Motivated by that approach, in this research we apply neutrosophy in order to enhance performances of DL methods. Based on the review and analysis of the class of DL methods, we propose a new method for determining t_k. The proposed method defines t_k as the output produced by an appropriate neutrosophic logic controller (NLC). Our idea is to replace the classical parameter t_k by an adaptive

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neutrosophic logic parameter ν_k, determined as the output of the NLC. Since $0 \le t_k \le 1$ our decision is to define t_k as the value of ν_k, without additional parameters.

A fuzzy set theory utilizes a membership function (MF) $\mathrm{T}_\Psi(\lambda) \in [0,1], \lambda \in \Lambda$ in the universe Λ that defines the degree of membership of λ in $\Psi \subset \Lambda$ [21]. The intuitionistic fuzzy set (IFS) Ψ is established using both degrees of membership and non-membership function $T_\Psi(\lambda), F_\Psi(\lambda) \in [0,1], \lambda \in \Lambda$ [22], which are mutually corelated by $\mathrm{T}_\Psi(\lambda), \mathrm{F}_\Psi(\lambda) : \Lambda \to [0,1]$ and $0 \le \mathrm{T}_\Psi(\lambda) + \mathrm{F}_\Psi(\lambda) \le 1$. The IFS theory was originally generalized in [23,24] by the the neutrosophic theory. The background of the neutrosophic logic is the utilization of the indeterminacy $\mathrm{I}(\lambda)$. In that direction, entries of a neutrosophic set are determined by three independent MFs [23,24]: the truth-MF $\mathrm{T}(\lambda)$, the indeterminacy-MF $\mathrm{I}(\lambda)$, and the falsity-MF $\mathrm{F}(\lambda)$. Due to the indeterminacy-MF, the neutrosophic logic is based on the symmetry involved in the ordered triple of MFs $(\mathrm{T}, \mathrm{I}, \mathrm{F})$ and the inequality $0 \le \mathrm{T} + \mathrm{I} + \mathrm{F} \le 3$ if all three MFs are independent. Clearly, T is the symmetric pole to its opposite pole F with respect to I, which represents an axis of the symmetry between T and F [25]. The same observation is valid for refined neutrosophic set that assumes the refined indeterminacies I_1 and I_2 between T and F [26]. The MFs of a neutrosophic set Ψ satisfy $\mathrm{T}_\Psi(\lambda), \mathrm{I}_\Psi(\lambda), \mathrm{F}_\Psi(\lambda) : \Lambda \to [0,1]$, which which based on their independence implies $0 \le \mathrm{T}_\Psi(\lambda) + \mathrm{I}_\Psi(\lambda) + \mathrm{F}_\Psi(\lambda) \le 3$, and enables a symmetry between them. In [27], the authors originated a neutrosophic-based multiple criteria decision-making procedure based on previously introduced symmetry measure.

The benefits of the NL approach over the FL and IFL are discussed in [20].

Motivation and highlights of main results. Our task in this paper is to improve the behavior of DL class for solving unconstrained nonlinear optimization problems with the support of an appropriate neutrosophic logic system. The principal results obtained in this paper are presented as follows.

(1) We examine the application of NL in determining the parameter t in the Dai-Liao CG method (5).
(2) A theoretical analysis is accomplished to confirm the global convergence of the proposed method.
(3) A numerical comparison is given between the proposed FDL algorithm and other known DL algorithms.

The sections of the paper are arranged as follows. Introduction, motivation and a brief review of obtained results are presented in Section 1. A neutrosophic-based control for defining appropriate changeable values t_k is proposed in Section 2. Moreover, we present details of the FDL method. The global convergence behavior of the FDL method is examined in Section 3. Numerical comparison of the FDL method with main standard DL methods is presented in Section 4, and a comparison with some known variations of the DL class of methods, is also given. Final conclusions are presented in the concluding section.

2. Fuzzy neutrosophic Dai-Liao conjugate gradient method

The fuzzy neutrosophic Dai-Liao CG method is defined as a modification of the Dai-Liao CG method (3), where the search directions $\{\mathbf{d}_k\}$ are calculated by the recurrence rule

$$\mathbf{d}_k = \begin{cases} -\mathbf{g}_0, & k=0, \\ -\mathbf{g}_k + \beta_k^{\mathrm{FDL}} \mathbf{d}_{k-1}, & k \ge 1, \end{cases} \tag{19}$$

where the CG coefficient β_k^{FDL} is defined by

$$\beta_k^{\mathrm{FDL}} = \frac{\mathbf{g}_k^{\mathrm{T}} \mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}} \mathbf{y}_{k-1}} - \nu_k \frac{\mathbf{g}_k^{\mathrm{T}} \mathbf{s}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}} \mathbf{y}_{k-1}}, \tag{20}$$

such that $\nu_k := \nu_k$ is a proper fuzzy neutrosophic parameter. Our intention is to define ν_k as a function of $\Delta_k := f(\mathbf{x}_k) - f(\mathbf{x}_{k+1})$ i.e., $\nu_k := \nu_k(\Delta_k)$. More precisely, $\nu_k(\Delta_k)$ is defined subject to the following constraint

$$0 \leq \nu_k(\Delta_k) \leq 1. \tag{21}$$

It is know that $\nu_k(\Delta_k) = 0$ reduces (2) into

$$\mathbf{d}_k^{\mathrm{T}}\, \mathbf{y}_{k-1} = 0. \tag{22}$$

Hence, the equation (22) can be considered as a reflection of the conjugacy condition, which in conjuction with (4) determines the HS parameter [28]

$$\beta_k^{\mathrm{HS}} = \frac{\mathbf{g}_k^{\mathrm{T}} \mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}} \mathbf{y}_{k-1}}. \tag{23}$$

Alternatively, for $\nu_k(\Delta_k) = 1$, the equation (2) is considered as a conjugacy condition that implicitly satisfies the quasi-Newton characteristics. For more details on these cases, see [1,10].

The idea for defining a new parameter tk in the Dai-Liao CG method (5) comes from the neutrosophic logic. According to this decision, our intention is to define $t_k = \nu_k(\Delta_k)$ inside the interval $[0,1]$ according to neutrosophic principles.

The generic layout of the fuzzy neutrosophic Dai-Liao CG method is given in the diagram in Figure 1.

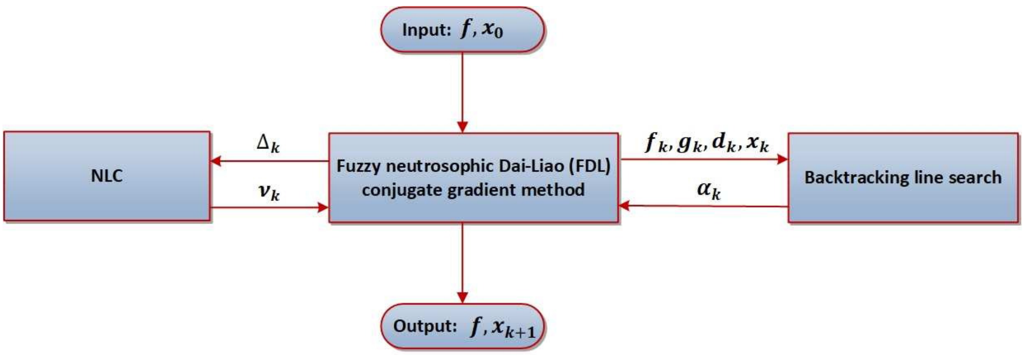

Figure 1. The global structure of the fuzzy neutrosophic Dai-Liao CG method.

The input of the NLC presented in Figure 1 is $\Delta_k := f(\mathbf{x}_k) - f(\mathbf{x}_{k+1})$ and the output is the desired step size ν_k. This means that our basic idea is to define ν_k based on two consecutive values of the objective function f. On the other hand, the backtracking line search is reponsible for appropriate step lengths α_k in (3) and then the descent direction $\mathbf{d}_k$ by (19). Using ν_k it is possible to compute β_k^{FDL} in (20). Finally, (3) generates new iterative point x_{k+1}.

To develop the FDL method, it is necessary to plan three global steps: neutrosophistication, neutrosophic inference engine, and de-neutrosophistication (score function).

(1) *Neutrosophication* maps the input $\Delta_k := f(\mathbf{x}_k) - f(\mathbf{x}_{k+1})$ into neutrosophic ordered triplets $(\mathrm{T}(\Delta_k), \mathrm{I}(\Delta_k), \mathrm{F}(\Delta_k))$. The MFs are defined with the aim to improve the CG iterative rule exploiting numerical experience. The sigmoid function with the slope defined by ς_1 at the crossover point $\Delta = \varsigma_2$ is a proper choice for T:

$$\mathrm{T}(\Delta) = 1/\left(1 + \mathrm{e}^{-\varsigma_1(\Delta - \varsigma_2)}\right). \tag{24}$$

A proper choice for F is the following sigmoid function:

$$\mathrm{F}(\Delta) = 1/(1 + \mathrm{e}^{\varsigma_1(\Delta - \varsigma_2)}). \tag{25}$$

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The Gaussian function with the standard deviation ς_1 and the mean ς_2 defines the indeterminacy I:

$$\mathrm{I}(\Delta) = \mathrm{e}^{-\frac{(\Delta-\varsigma_2)^2}{2\varsigma_1^2}}. \tag{26}$$

Then the neutrosophication of $\Delta \in \mathbb{R}$ is defined as the transition $\Delta \to \langle \mathrm{T}(\Delta), \mathrm{I}(\Delta), \mathrm{F}(\Delta)\rangle$, where the MFs are determined in (24)–(26).

(2) *Neutrosophic inference* between an input fuzzy set $\mathfrak{I}$ and an output fuzzy set is based on the subsequent "IF–THEN" regulations:

$$\Re_1 : \text{If } \mathfrak{I} = S_P \Longrightarrow \mathfrak{O} = \{\mathrm{T}, \mathrm{I}, \mathrm{F}\}$$
$$\Re_2 : \text{If } \mathfrak{I} = S_N \Longrightarrow \mathfrak{O} = \{\mathrm{T}, \mathrm{I}, \mathrm{F}\}.$$

Fuzzy sets S_P and S_N point, respectively, to positive or negative errors. Applying the unification $\Re = \Re_1 \cup \Re_2$, we define $\mathfrak{O}_i = \mathfrak{I} \circ \Re_i$, $i = 1, 2$, where $\circ$ denotes the fuzzy transformation. In addition, for a fuzzy vector $\zeta = \{\mathrm{T}(\Delta_k), \mathrm{I}(\Delta_k), \mathrm{F}(\Delta_k)\}$, it follows $\kappa_{\mathfrak{I}\circ\Re}(\zeta) = \kappa_{\mathfrak{I}\circ\Re_1} \bigvee \kappa_{\mathfrak{I}\circ\Re_2} = \sup(\kappa_{\mathfrak{I}} \bigwedge \kappa_{\mathfrak{O}_i})$, $i = 1, 2$, where $\bigwedge$ and $\bigvee$ denote the (min, max, max) and (max, min, min) operator, respectively. In this research, the centroid defuzzification method is utilized to generate a vector of crisp outputs $\zeta^* \in \mathbb{R}^3$:

$$\zeta^* = \frac{\int_{\mathfrak{O}} \zeta\, \kappa_{\mathfrak{I}\circ\Re}(\zeta) \mathrm{d}\zeta}{\int_{\mathfrak{O}} \kappa_{\mathfrak{I}\circ\Re}(\zeta) \mathrm{d}\zeta}.$$

(3) *De-neutrosophication* is based on the transformation $\langle \mathrm{T}(\Delta_k), \mathrm{I}(\Delta_k), \mathrm{F}(\Delta_k)\rangle \to \nu_k \in \mathbb{R}$ resulting in a crisp value ν_k and suggested as:

$$\nu_k = 2 - (\mathrm{T}(\Delta_k) + \mathrm{I}(\Delta_k) + \mathrm{F}(\Delta_k)). \tag{27}$$

The diagram of Figure 2 presents the NLC based on the neutrosophic rules.

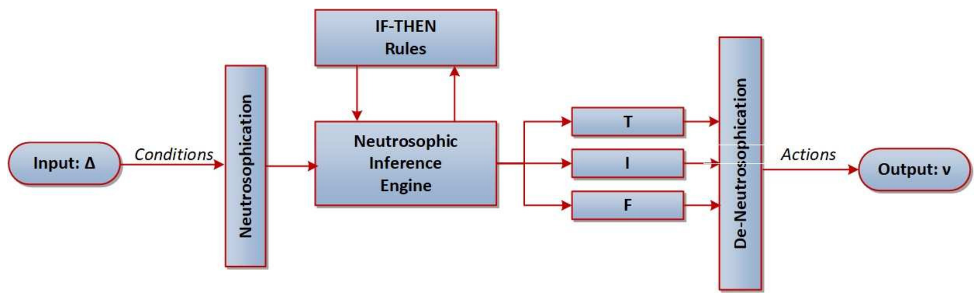

Figure 2. The NLC design based on the neutrosophy.

The settings in the NLC employed in numerical testing are arranged in Table 1.

Table 1: Recommended parameters in NLC.

Set	Membership Function	ς_1	ς_2	Weight
Input	Sigmoid function (24)	1	3	1
	Sigmoid function (25)	1	3	1
	Gaussian function (26)	120	0	1
Output	Score function (27)	-	-	1

Our imperative requirement is $0 \le \nu_k(\Delta_k) \le 1$, requested in (21). This statement is verified in Lemma 1.

Lemma 1. *The inequality* (21) *holds for the given choice of the score function* (27) *and the parameters given in the Table* 1.

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Proof. In order to prove (21), we need to replace the MFs (24), (25), and (26) in (27). After applying the parameters from Table 1, we get

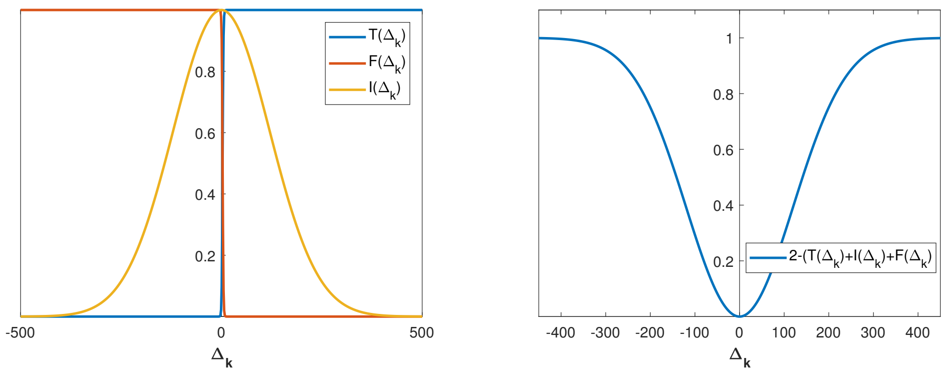
$$\mathrm{T}(\Delta_k) = 1/\left(1 + \mathrm{e}^{-(\Delta-3)}\right), \ \mathrm{F}(\Delta_k) = 1/\left(1 + \mathrm{e}^{\Delta-3}\right), \ \mathrm{I}(\Delta_k) = \mathrm{e}^{-\frac{\Delta_k^2}{2*120^2}}$$

Elementary calculation gives

$$\nu_k(\Delta_k) = 2 - (\mathrm{T}(\Delta_k) + \mathrm{I}(\Delta_k) + \mathrm{F}(\Delta_k)) = 1 - \mathrm{e}^{-\frac{\Delta_k^2}{28800}}. \tag{28}$$

A careful analysis of the function (28) inside the interval $\Delta_k \in (-\infty, +\infty)$ discovers $\min \nu_k(\Delta_k) = 0$ and $\max \nu_k(\Delta_k) = 1$, which proves the inequality (21). □

Graphs of $\mathrm{T}(\Delta_k), \mathrm{I}(\Delta_k), \mathrm{F}(\Delta_k)$ are displayed in Figure 3(**a**). The fulfilment of the requirements (21) in the NLC output ν_k generated throughout the described de-neutrosophication is illustrated in Figure 3(**b**).



(**a**) Neutrosophication.

(**b**) De-neutrosophication.

Figure 3. Neutrosophication (24)–(26) and de-neutrosophication (27) guided by the parameters in Table 1.

Remark 1. *The objective function decreases with the flow of iterations and tends to the minimal value, which means* $\lim_{k\to\infty} \Delta_k = 0$, *i.e.,* $\lim_{k\to\infty} \nu_k(\Delta_k) = 0$. *Such behavior leads to* $\nu_k \to 0$ *as the minimum of* f *approaches, so the impact of the proposed neutrosophic strategy decreases and disappears, which agrees with our goal.*

Remark 2. *Obviously, larger values of* Δ_k *lead to increasing values* $\nu_k(\Delta_k)$ *approaching to* 1, *which will be denoted as* $\nu_k(\Delta_k) \nearrow 1$. *In addition, based on the limit* $\Delta_k \to 0$, *we anticipate smaller values* $\nu_k(\Delta_k)$ *approaching* 0, *i.e.,* $\nu_k(\Delta_k) \searrow 0$ *in final iterations. As a result,* $\nu_k(\Delta_k)$ *is suitable as an adjustable regulator for the quantity* t *in the Dai–Liao CG method.*

The backtracking line search from [29] begins from $\alpha = 1$ and generates further step sizes which ensure decrease of the goal function in each iteration. Algorithm 1, restated from [30], is used to define the primary step size α_k.

Algorithm 1 The backtracking line search.

Input: Objective function $f(\mathbf{x})$, foregoing point $\mathbf{x}_k$, the search direction $\mathbf{d}_k$, a real positive constants $0 < \varphi < 1$, and $0 < \omega < 0.5$.
1: $\ell = 1$.
2: While $f(\mathbf{x}_k + \ell\mathbf{d}_k) > f(\mathbf{x}_k) + \omega\ell\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_k$, do $\ell := \ell\varphi$.
3: Output: $\alpha_k = \ell$.

Algorithm 2 of the FDL method is described as follows:

Algorithm 2 Fuzzy neutrosophic Dai-Liao (FDL) conjugate gradient method.

Input: A starting point x_0 and $0 < \epsilon, \delta < 1$.

1: Assign $k = 0$ and $\mathbf{d}_0 = -\mathbf{g}_0$.
2: If
$$\|\mathbf{g}_k\| \leq \epsilon \quad \text{and} \quad \frac{|f(\mathbf{x}_{k+1}) - f(\mathbf{x}_k)|}{1 + |f(\mathbf{x}_k)|} \leq \delta,$$
STOP;
else go to Step 3.
3: (Backtracking line search) Regulate $\alpha_k \in (0, 1]$ utilizing Algorithm 1.
4: Calculate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$.
5: Calculate $\mathbf{g}_{k+1}$, $\mathbf{y}_k = \mathbf{g}_{k+1} - \mathbf{g}_k$, $\mathbf{s}_k = \mathbf{x}_{k+1} - \mathbf{x}_k$.
6: Calculate $\Delta_k := f_k - f_{k+1}$.
7: Calculate $\mathrm{T}(\Delta_k), \mathrm{I}(\Delta_k)$, and $\mathrm{F}(\Delta_k)$ as in (24)–(26).
8: Calculate $\nu_k := \nu_k(\Delta_k)$ using (27).
9: Calculate $\beta_{k+1}^{\mathrm{FDL}}$ by (20).
10: Generate $\mathbf{d}_{k+1} = -\mathbf{g}_{k+1} + \beta_{k+1}^{\mathrm{FDL}} \mathbf{d}_k$.
11: Set $k := k + 1$, and go to Step 2.

3. Convergence Examination

The subsequent assumptions are necessary during the theoretical examination of the FDL algorithm.

Assumption 1. (1) *The level set* $\mho = \{\mathbf{x} \in \mathbb{R}^n |\, f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$ *of the iterative process* (3) *is bounded.*

(2) *The objective* f *is continuously differentiable in a neighborhood* $\mathcal{P}$ *of* $\mho$ *with the Lipschitz continuous gradient* $\mathbf{g}$. *Such assumption initiates the existence of a constant* $L > 0$ *such that*

$$\|\mathbf{g}(\mathbf{u}) - \mathbf{g}(\mathbf{v})\| \leq L\|\mathbf{u} - \mathbf{v}\|, \ \forall\ \mathbf{u}, \mathbf{v} \in \mathcal{P}. \tag{29}$$

The Assumption 1 provides the existence of quantities Υ and γ such that

$$\|\mathbf{u} - \mathbf{v}\| \leq \Upsilon, \ \forall\ \mathbf{u}, \mathbf{v} \in \mathcal{P} \tag{30}$$

and

$$\|\mathbf{g}(\mathbf{u})\| \leq \gamma, \ \forall\ \mathbf{u} \in \mathcal{P}. \tag{31}$$

If Assumption 1 holds, in view of the uniform convexity of f, there exists $\theta > 0$ satisfying

$$(\mathbf{g}(\mathbf{u}) - \mathbf{g}(\mathbf{v}))^{\mathrm{T}}(\mathbf{u} - \mathbf{v}) \geq \theta\|\mathbf{u} - \mathbf{v}\|^2, \ \ \forall\ \mathbf{u}, \mathbf{v} \in \mho, \tag{32}$$

or equivalently,

$$f(\mathbf{u}) \geq f(\mathbf{v}) + g(\mathbf{v})^{\mathrm{T}}(\mathbf{u} - \mathbf{v}) + \frac{\theta}{2}\|\mathbf{u} - \mathbf{v}\|^2, \ \ \forall\ \mathbf{u}, \mathbf{v} \in \mho. \tag{33}$$

It follows from (32) and (33) that

$$\mathbf{s}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1} \geq \theta\|\mathbf{s}_{k-1}\|^2 \tag{34}$$

and

$$f(\mathbf{x}_{k-1}) - f(\mathbf{x}_k) \geq -\mathbf{g}(\mathbf{x}_k)^{\mathrm{T}}\mathbf{s}_{k-1} + \frac{\theta}{2}\|\mathbf{s}_{k-1}\|^2. \tag{35}$$

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By (29) and (34), one concludes

$$\theta\|\mathbf{s}_{k-1}\|^2 \leq \mathbf{s}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1} \leq L\|\mathbf{s}_{k-1}\|^2, \tag{36}$$

and further $\theta \leq L$.

The inequality (36) implies

$$\mathbf{s}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1} = \alpha_{k-1}\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1} > 0. \tag{37}$$

Taking into account $\alpha_{k-1} > 0$ and the last inequality, we conclude

$$\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1} > 0. \tag{38}$$

Lemma 2. [31,32] *Let the constraints in Assumption 1 hold and the points $\{\mathbf{x}_k\}$ be produced by the iterations (3)-(4). Then the following inequality is satisfied:*

$$\sum_{k=0}^{\infty}\frac{\|\mathbf{g}_k\|^4}{\|\mathbf{d}_k\|^2} < +\infty. \tag{39}$$

Lemma 3. *Observe the suggested Fuzzy neutrosophic Dai-Liao CG method defined by (3), (19), (20). If the search procedure guarantees (38), for all $k \geq 0$, then*

$$\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_k \leq -c\|\mathbf{g}_k\|^2 \tag{40}$$

for some $c \geq 0$.

Proof. In the initial stage it follows $\mathbf{g}_0^{\mathrm{T}}\mathbf{d}_0 = -\|\mathbf{g}_0\|^2$. Since $c = 1$, obviously (40) is satisfied in the initial stage $k = 0$. Assume (40) for some $k \geq 1$. Applying the inner product of both the left and right hand side in (4) with $\mathbf{g}_k^{\mathrm{T}}$, it is concluded

$$\begin{aligned}
\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_k &= -\|\mathbf{g}_k\|^2 + \beta_k^{\mathrm{FDL}}\,\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} \\
&= -\|\mathbf{g}_k\|^2 + \left(\frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}} - \nu_k\frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{s}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\right)\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} \\
&= -\|\mathbf{g}_k\|^2 + \frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} - \nu_k\frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{s}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} \\
&= -\|\mathbf{g}_k\|^2 + \frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} - \nu_k\frac{\alpha_{k-1}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} \\
&= -\|\mathbf{g}_k\|^2 + \frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} - \nu_k\frac{\alpha_{k-1}(\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1})^2}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}.
\end{aligned} \tag{41}$$

Using (21) in conjunction with (38) and $\alpha_{k-1} > 0$, we conclude

$$\nu_k\frac{\alpha_{k-1}(\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1})^2}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}} \geq 0. \tag{42}$$

Now from (41) and (42) it follows

$$\begin{aligned}
\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_k &\leq -\|\mathbf{g}_k\|^2 + \frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} \\
&= \frac{-\|\mathbf{g}_k\|^2(\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1})^2 + (\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1})(\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1})(\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1})}{(\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1})^2}.
\end{aligned} \tag{43}$$

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Applying the inequality $P^TQ \leq \frac{1}{2}(\|P\|^2 + \|Q\|^2)$ to the equality (43) with $P = \frac{1}{\sqrt{2}}(\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})\mathbf{g}_k$ and $Q = \sqrt{2}(\mathbf{g}_k^T\mathbf{d}_{k-1})\mathbf{y}_{k-1}$, it is obtained

$$
\begin{aligned}
\mathbf{g}_k^T\mathbf{d}_k \leq & \frac{-\|\mathbf{g}_k\|^2(\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})^2 + \frac{1}{2}\left(\|\frac{1}{\sqrt{2}}(\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})\mathbf{g}_k\|^2 + \|\sqrt{2}(\mathbf{g}_k^T\mathbf{d}_{k-1})\mathbf{y}_{k-1}\|^2\right)}{(\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})^2} \\
= & \frac{-\|\mathbf{g}_k\|^2(\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})^2 + \frac{1}{2}\left(\frac{1}{2}(\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})^2\|\mathbf{g}_k\|^2 + 2(\mathbf{g}_k^T\mathbf{d}_{k-1})^2\|\mathbf{y}_{k-1}\|^2\right)}{(\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})^2} \\
= & \frac{-\|\mathbf{g}_k\|^2(\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})^2 + \frac{1}{4}(\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})^2\|\mathbf{g}_k\|^2 + (\mathbf{g}_k^T\mathbf{d}_{k-1})^2\|\mathbf{y}_{k-1}\|^2}{(\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})^2} \\
= & -\|\mathbf{g}_k\|^2 + \frac{1}{4}\|\mathbf{g}_k\|^2 + \frac{(\mathbf{g}_k^T\mathbf{d}_{k-1})^2\|\mathbf{y}_{k-1}\|^2}{(\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})^2} \\
\leq & -\|\mathbf{g}_k\|^2 + \frac{1}{4}\|\mathbf{g}_k\|^2 + \frac{\alpha_{k-1}^2(|\mathbf{g}_k^T\mathbf{d}_{k-1}|)^2\|\mathbf{y}_{k-1}\|^2}{(\alpha_{k-1}\mathbf{d}_{k-1}^T\mathbf{y}_{k-1})^2} \\
\leq & -\|\mathbf{g}_k\|^2 + \frac{1}{4}\|\mathbf{g}_k\|^2 + \frac{\alpha_{k-1}^2\|\mathbf{g}_k\|^2\|\mathbf{d}_{k-1}\|^2\|\mathbf{y}_{k-1}\|^2}{(\mathbf{s}_{k-1}^T\mathbf{y}_{k-1})^2} \\
\leq & -\|\mathbf{g}_k\|^2 + \frac{1}{4}\|\mathbf{g}_k\|^2 + \frac{\|\mathbf{g}_k\|^2\|\alpha_{k-1}\mathbf{d}_{k-1}\|^2\|\mathbf{y}_{k-1}\|^2}{\theta^2\|\mathbf{s}_{k-1}\|^4} \\
\leq & -\|\mathbf{g}_k\|^2 + \frac{1}{4}\|\mathbf{g}_k\|^2 + \frac{\|\mathbf{g}_k\|^2\|\mathbf{s}_{k-1}\|^2\|\mathbf{y}_{k-1}\|^2}{\theta^2\|\mathbf{s}_{k-1}\|^4} \\
\leq & -\|\mathbf{g}_k\|^2 + \frac{1}{4}\|\mathbf{g}_k\|^2 + \frac{\|\mathbf{g}_k\|^2\|\mathbf{y}_{k-1}\|^2}{\theta^2\|\mathbf{s}_{k-1}\|^2} \\
\leq & -\|\mathbf{g}_k\|^2 + \frac{1}{4}\|\mathbf{g}_k\|^2 + \frac{L^2\|\mathbf{g}_k\|^2\|\mathbf{s}_{k-1}\|^2}{\theta^2\|\mathbf{s}_{k-1}\|^2} \\
= & -\|\mathbf{g}_k\|^2 + \frac{1}{4}\|\mathbf{g}_k\|^2 + \frac{L^2}{\theta^2}\|\mathbf{g}_k\|^2 \\
= & -\left(1 - \frac{1}{4} - \frac{L^2}{\theta^2}\right)\|\mathbf{g}_k\|^2 = -\left(\frac{3}{4} - \frac{L^2}{\theta^2}\right)\|\mathbf{g}_k\|^2.
\end{aligned}
\tag{44}
$$

The requirement (40) is satisfied for $c = \left(\frac{3}{4} - \frac{L^2}{\theta^2}\right)$ in (44) and an arbitrary $k \geq 0$. □

Theorem 1 confirms the global convergence of the FDL flow.

Theorem 1. *Let the constraints in Assumption 1 be valid and f be uniformly convex. Then the series $\{\mathbf{x}_k\}$ generated on the basis of (3), (19) and (20) fulfils the limit relation*

$$\liminf_{k\to\infty} \|\mathbf{g}_k\| = 0. \tag{45}$$

Proof. Suppose the opposite. Since (45) is not valid, we conclude the existence of $c_1 > 0$ satisfying

$$\|\mathbf{g}_k\| \geq c_1, \quad \text{for all } k. \tag{46}$$

Squaring both sides of (19) one derives

$$\|\mathbf{d}_k\|^2 = \|\mathbf{g}_k\|^2 - 2\beta_k^{FDL}\mathbf{g}_k^T\mathbf{d}_{k-1} + (\beta_k^{FDL})^2\|\mathbf{d}_{k-1}\|^2. \tag{47}$$

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Taking into account (20), we obtain

$$
\begin{aligned}
-2\beta_k^{\mathrm{FDL}}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} &= -2\left(\frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}} - \nu_k\frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{s}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\right)\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} \\
&= -2\left(\frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} - \nu_k\frac{\alpha_{k-1}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1}\right) \\
&= -2\left(\frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} - \nu_k\frac{\alpha_{k-1}(\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1})^2}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\right).
\end{aligned}
\tag{48}
$$

Now from (42), it follows

$$
\begin{aligned}
-2\beta_k^{\mathrm{EDL}}\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1} &\le 2\left|\frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\right||\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_{k-1}| \le 2\frac{\|\mathbf{g}_k\|\|\mathbf{y}_{k-1}\|}{|\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}|}\|\mathbf{g}_k\|\|\mathbf{d}_{k-1}\| \\
&= 2\frac{\alpha_{k-1}\|\mathbf{g}_k\|^2\|\mathbf{y}_{k-1}\|\|\mathbf{d}_{k-1}\|}{\alpha_{k-1}\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}} \\
&= 2\frac{\|\mathbf{g}_k\|^2\|\mathbf{y}_{k-1}\|\|\mathbf{s}_{k-1}\|}{\mathbf{s}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}} \\
&\le 2\frac{\|\mathbf{g}_k\|^2 L\|\mathbf{s}_{k-1}\|\|\mathbf{s}_{k-1}\|}{\theta\|\mathbf{s}_{k-1}\|^2} \\
&= \frac{2L}{\theta}\|\mathbf{g}_k\|^2.
\end{aligned}
\tag{49}
$$

Now, an application of (20) initiates

$$
\begin{aligned}
\beta_k^{\mathrm{FDL}} &= \frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}} - \nu_k\frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{s}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}} = \frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1} - \nu_k\mathbf{g}_k^{\mathrm{T}}\mathbf{s}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}} \\
&\le \left|\frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1} - \nu_k\mathbf{g}_k^{\mathrm{T}}\mathbf{s}_{k-1}}{\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}}\right| = \alpha_{k-1}\frac{|\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1} - \nu_k\mathbf{g}_k^{\mathrm{T}}\mathbf{s}_{k-1}|}{\alpha_{k-1}\mathbf{d}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}} \\
&= \alpha_{k-1}\frac{|\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1} - \nu_k\mathbf{g}_k^{\mathrm{T}}\mathbf{s}_{k-1}|}{\mathbf{s}_{k-1}^{\mathrm{T}}\mathbf{y}_{k-1}} \\
&\le \alpha_{k-1}\frac{|\mathbf{g}_k^{\mathrm{T}}\mathbf{y}_{k-1} - \nu_k\mathbf{g}_k^{\mathrm{T}}\mathbf{s}_{k-1}|}{\theta\|\mathbf{s}_{k-1}\|^2} = \alpha_{k-1}\frac{|\mathbf{g}_k^{\mathrm{T}}(\mathbf{y}_{k-1} - \nu_k\mathbf{s}_{k-1})|}{\theta\alpha_{k-1}^2\|\mathbf{d}_{k-1}\|^2} \\
&\le \frac{\|\mathbf{g}_k\|(\|\mathbf{y}_{k-1}\| + \nu_k\|\mathbf{s}_{k-1}\|)}{\theta\alpha_{k-1}\|\mathbf{d}_{k-1}\|^2} \\
&\le \frac{\|\mathbf{g}_k\|(L\|\mathbf{s}_{k-1}\| + \nu_k)\|\mathbf{s}_{k-1}\|)}{\theta\alpha_{k-1}\|\mathbf{d}_{k-1}\|^2} = \frac{\|\mathbf{g}_k\|(L + \nu_k)\|\mathbf{s}_{k-1}\|}{\theta\alpha_{k-1}\|\mathbf{d}_{k-1}\|^2} \\
&= \frac{\|\mathbf{g}_k\|(L + \nu_k)\alpha_{k-1}\|\mathbf{d}_{k-1}\|}{\theta\alpha_{k-1}\|\mathbf{d}_{k-1}\|^2} \\
&= \frac{(L + \nu_k)\|\mathbf{g}_k\|}{\theta\|\mathbf{d}_{k-1}\|}.
\end{aligned}
\tag{50}
$$

Further (21) and (50) lead to

$$
\beta_k^{\mathrm{FDL}} \le \frac{(L+1)\|\mathbf{g}_k\|}{\theta\|\mathbf{d}_{k-1}\|}.
\tag{51}
$$

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Using (49) and (51) in (47), we get

$$\begin{aligned} \|\mathbf{d}_k\|^2 &\le \|\mathbf{g}_k\|^2 + \frac{2L}{\theta}\|\mathbf{g}_k\|^2 + \frac{(L+1)^2\|\mathbf{g}_k\|^2}{\theta^2\|\mathbf{d}_{k-1}\|^2}\|\mathbf{d}_{k-1}\|^2 \\ &= \|\mathbf{g}_k\|^2 + \frac{2L}{\theta}\|\mathbf{g}_k\|^2 + \frac{(L+1)^2\|\mathbf{g}_k\|^2}{\theta^2} \\ &= \left(1 + \frac{2L}{\theta} + \frac{(L+1)^2}{\theta^2}\right)\|\mathbf{g}_k\|^2 \\ &= \left(\frac{\theta+2L}{\theta} + \frac{(L+1)^2}{\theta^2}\right)\|\mathbf{g}_k\|^2 \\ &= \frac{(\theta+2L)\theta + (L+1)^2}{\theta^2}\|\mathbf{g}_k\|^2. \end{aligned} \tag{52}$$

Dividing both sides of (52) by $\|\mathbf{g}_k\|^4$ and utilizing (46), it can be decided

$$\begin{aligned} \frac{\|\mathbf{d}_k\|^2}{\|\mathbf{g}_k\|^4} &\le \frac{(\theta+2L)\theta + (L+1)^2}{\theta^2} \cdot \frac{1}{c_1^2}, \\ \frac{\|\mathbf{g}_k\|^4}{\|\mathbf{d}_k\|^2} &\ge \frac{\theta^2 \cdot c_1^2}{(\theta+2L)\theta + (L+1)^2}. \end{aligned} \tag{53}$$

The inequalities in (53) imply

$$\sum_{k=0}^{\infty} \frac{\|\mathbf{g}_k\|^4}{\|\mathbf{d}_k\|^2} \ge \sum_{k=0}^{\infty} \frac{\theta^2 \cdot c_1^2}{(\theta+2L)\theta + (L+1)^2} = \infty. \tag{54}$$

Therefore, $\|\mathbf{g}_k\| \ge c_1$ causes a contradiction with Lemma 2. □

4. Numerical Results

In this section, numerical results obtained by the FDL method are analyzed and compared with the numerical results generated by the EDL [19] and DL [1] methods.

All the algorithms were written in Matlab R2017a and ran on a 64-bit Lenovo laptop (Intel Core i3 2.0 GHz, RAM 8 GB) with the Windows 10 operating system. The implementation of the FDL method is based on Algorithm 2, while EDL and DL implementation are based on algorithms given in [19] and [1], respectively.

The numerical testing is performed on 50 test functions collected in [33,34], with dimensions from the range $[100, 20000]$. All three tested methods used start from the same initial point x_0 for each test function. Each case in testing is evaluated 10 times with gradually increased dimensions $n = 10^2, 5\times10^2, 10^3, 3\times10^3, \times10^3, 7\times10^3, 8\times10^3, 10^4$, 1.5×10^4 and 2×10^4.

The uniform terminating criteria for observed DL, EDL, and FDL algorithms are

$$\|\mathbf{g}_k\| \le \epsilon \quad \text{and} \quad \frac{|f(\mathbf{x}_{k+1}) - f(\mathbf{x}_k)|}{1 + |f(\mathbf{x}_k)|} \le \delta, \;\; \epsilon = 10^{-6},\, \delta = 10^{-16}.$$

We are going to evaluate the efficiency of the FDL method and compare it with the EDL and DL methods under the backtracking search based on the parameters $\omega = 0.0001$ and $\varphi = 0.8$.

Summary numerical results for DL, FDL, and EDL methods, performed on 50 test functions, are shown in Table 2, where 'Test function', 'Nitr', 'Nfe', and 'Tcpu' represent the name of the tested function, the total number of iterations, the total number of function evaluations, and the running time, respectively.

To visually compare the performance of tested methods, we used the performance profiles technique [35] on numerical results corresponding to Nitr, Nfe, and Tcpu criteria

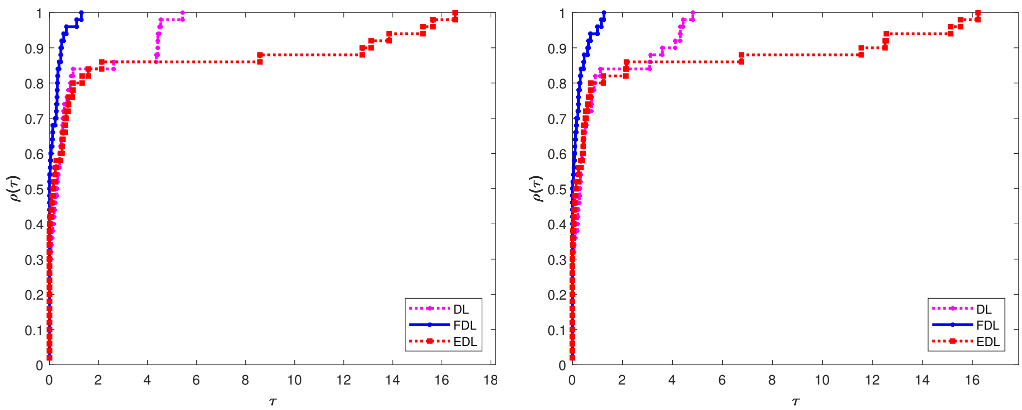
Table 2: Summary numerical results on unconstrained problems of DL, FDL, and EDL methods for the Nitr, Nfe, and Tcpu.

Test function	DL	FDL	EDL
	Nitr/Nfe/Tcpu	Nitr/Nfe/Tcpu	Nitr/Nfe/Tcpu
Extended Penalty	1905/77578/32.438	1610/62534/24.266	2304/82602/39.344
Perturbed Quadratic	14555/606750/379.359	10800/440213/206.5	10012/408474/248.5
Raydan 1	4337/114595/98.984	5497/122843/76.813	4194/109164/96.938
Raydan 2	1427/2864/3.188	67/144/0.281	2572540/5145090/894.453
Diagonal 1	5809/223750/245.578	5488/212491/227.781	4673/178295/219.109
Diagonal 3	5247/196745/423.766	4531/168162/307.594	4596/171636/366.203
Hager	1742/31516/103.672	1242/22799/47.063	1940/33206/98.766
Generalized Tridiagonal 1	2058/32313/49.5	2160/32033/27.5	2161/33285/44.703
Extended Tridiagonal 1	310/2932/8.391	182/2501/6.297	308/4129/12.766
Extended TET	1140/9840/11.031	619/5808/5.484	749/6362/5.969
Diagonal 5	1394/2798/6.938	60/130/0.609	3053907/6107824/3124.875
Extended Himmelblau	50/2431/1.016	51/2602/0.813	50/2413/0.938
Perturbed quadratic diagonal	1837/69156/18.453	1261/36785/13.875	2157/86977/34.797
Quadratic QF1	13895/526995/187.313	21989/846402/376.156	10199/379554/122.844
Extended quadratic penalty QP1	1080/17440/9.922	1524/23840/8.25	1157/18043/9.109
Extended quadratic penalty QP2	218/9479/11.047	112/5513/4.953	218/9194/8.906
Quadratic QF2	19211/847031/348.781	18861/816310/225.891	15555/689736/250.891
Extended quadratic exponential EP1	1254/3443/3.172	56/404/0.516	21431/43829/7.531
Extended Tridiagonal 2	22468/998473/549.484	3668/114169/87.438	10989/510713/93.609
TRIDIA (CUTE)	33278/1647913/967.234	40156/1977068/950.547	29133/1428866/675.422
ARWHEAD (CUTE)	1624/81625/44.875	1529/72379/31.594	1219/57140/28.672
Almost Perturbed Quadratic	14904/621925/259.797	19675/829784/357.359	13201/543372/188.047
LIARWHD (CUTE)	30/2705/1.281	30/2732/1.25	30/2739/1.438
POWER (CUTE)	532442/44419504/16742.672	580790/48609979/17435.609	629342/52431424/23630.781
ENGVAL1 (CUTE)	2489/33103/13.781	2400/32299/10.719	1975/27260/12.922
INDEF (CUTE)	21/1924/2.125	26/2238/2.5	30/2610/4.266
Diagonal 6	1583/3197/4.531	74/185/0.359	7052401/14105032/5037.219
DIXON3DQ (CUTE)	320921/1775846/1083.281	229757/1368033/727.172	257451/1517252/1045.328
COSINE (CUTE)	20/1600/1.891	20/1697/1.891	20/1700/2
BIGGSB1 (CUTE)	249919/1400798/832.375	259475/1549293/810.766	236612/1389720/945.672
Generalized Quartic	866/11273/3.984	1099/8951/4.063	959/10662/3.125
Diagonal 7	1453/4564/6.875	68/162/0.469	469477/940686/140.172
Diagonal 8	1371/3962/5.359	67/199/0.422	594522/1193760/195.094
Full Hessian FH3	2237/6202/7.125	52/513/0.688	767988/1537759/188.469
Diagonal 9	3312/138545/225.719	5344/217150/224.906	4520/189307/260.453
HIMMELH (CUTE)	20/1690/4.797	20/1758/4.531	20/1760/4.891
FLETCHCR (CUTE)	303212/10189775/5073.688	300227/10011849/4704.125	289670/9702961/4411.453
Extended BD1 (Block Diagonal)	1597/16783/7.625	1227/15639/5.875	1200/12605/6.625
Extended Maratos	72/3366/1.188	50/2069/0.719	40/1975/0.75
Extended Cliff	234/2992/2.078	217/6000/4.891	950/13187/6.188
Extended Hiebert	70/7215/1.938	70/7220/1.828	70/7228/1.859
NONDIA (CUTE)	33/3066/1.375	30/2829/1.266	32/3031/1.625
NONDQUAR (CUTE)	58/4652/18.047	45/3666/17.219	86/4989/19.016
DQDRTIC (CUTE)	3456/87105/26.453	2327/59047/16.406	3637/92315/34.953
Extended Freudenstein and Roth	1376/46597/10.734	3390/111830/28.516	2018/66654/16.172
Generalized Rosenbrock	282948/8410218/4125.516	280440/8335396/4088.547	281792/8373946/4055.172
Extended White and Holst	76/5794/9.219	50/3171/7.281	59/4022/11.563
Extended Beale	118/6791/14.047	72/3118/5.906	181/4748/6.75
EG2 (CUTE)	507/29388/47.547	697/48512/119.875	811/39769/122.469
EDENSCH (CUTE)	1694/23160/89.453	2089/27821/83.266	1684/22731/116.844

generated by DL, FDL, and EDL methods. An upper graph in a performance profile corresponds to the method that shows better performance. The vertical axis of each performance profile in figures undicates the percentage of test functions for which the considered method is the winner between compared methods, whereby the right-hand side corresponds to the percentage of successfully solved test functions.

Figures 4 and 5 plot the performance profiles for the data in Table 2. Graphs in Figure 4 illustrate the performance profiles Nitr and Nfe for DL, FDL, and EDL iterations based on the data from Table 2. In Figure 4(**a**), it is noticeable that DL, FDL, and EDL methods are able to solve all tested functions, wherein the FDL method produces the best results in 54.0% (27 out of 50) of test functions compared with DL (26.0% (13 out of 50)) and EDL (38.0% (19 out of 50)). From Figure 4(**a**), it is observed that the FDL graph reaches the top first, so FDL is the best relative to other examined methods with respect to the Nitr criterion.

Figure 4(**b**) indicates that the FDL graph is the most efficient and successfully solves all test cases. In addition, the obtained numerical results confirm that FDL performs well in most cases. Most specifically, FDL is the fastest because it solves about 48.0% (24 out of 50) of tested functions with the least Nfe compared to the DL and EDL methods. Meanwhile, the DL and EDL are superior for solving 22.0% (11 out of 50) and 30.0% (15 out of 50) test functions, respectively. Hence, the numerical behavior of FDL is superior compared to the DL and EDL methods for the given test functions.



(**a**) Nitr performance profile. (**b**) Nfe performance profile.

Figure 4. Performance profile for DL, FDL, and EDL methods.

Figure 5 shows Tcpu performance profiles graphs of DL, FDL, and EDL methods. It is observable that DL, FDL, and EDL are able to solve all tested functions. Further examination leads to the conclusion that the FDL method is the best in 74.0% (37 out of 50) of the test cases compared with DL (12.0% (6 out of 50)) and EDL (16.0% (8 out of 50)). Analysing the graphs in Figure 5, a clear conclusion is that the FDL graph comes to the top first, which confirms its dominance in terms of Tcpu.

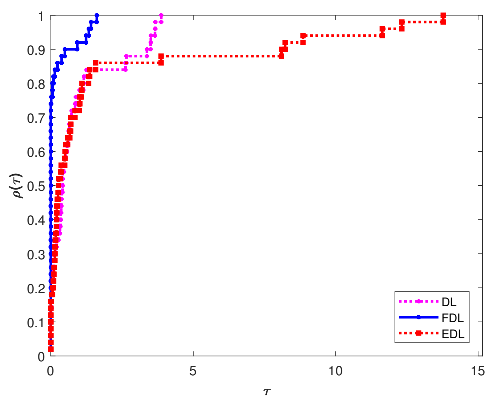


Figure 5. Tcpu performance profile for DL, FDL, and EDL methods.

By analyzing the results shown in Figures 4, 5 and in Table 2, we conclude that the FDL method achieved better results. This observation leads us to the final conclusion that the proposed FDL method is effective for solving unconstrained optimization problems in terms of all three criteria (iterations, function evaluations, and processor time).

5. Conclusion

In current research, we propose a novel approach to determining the parameter t in the Dai-Liao CG iterations. New approach is based on finding suitable values for a non-negative parameter t in the DL method using neutrosophic logic. Utilizing $t = \nu_k(\Delta_k)$ in (5), an original strategy in defining the Dai-Liao CG parameter β_k^{FDL} is proposed and a novel fuzzy neutrosophic Dai-Liao (FDL) CG method is presented.

Numerical experiments and comparisons with some well-known CG methods and the theoretical convergence analysis show effectiveness of the method. The numerical testing and initiated comparison are based on standard performance profiles, such as the total number of iterations (Nitr), the total number of function evaluations (Nfe), and the running time (Tcpu) performances for each function and each method. Analysis of the obtained numerical results revealed that the FDL method is the most efficient.

We are convinced that the obtained results will be a motivation for further research in defining improved DL methods strengthened by the neutrosophic logic.

Future scientific research in this direction can be continued in in several directions. Previous research has shown the effectiveness of the neutrosophic principle in scaled gradient descent methods and DL class of CG methods. The challenge is to apply such a principle to other non-linear optimization methods. On the other hand, there is a wide variety of different possibilities for defining the principles of neutrosophication and de-neutrosophication, which can be considered in future research. Finally, there is a great opportunity in improving the Neutrosophic inference engine used in this research.

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