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Fuzzy crossed product algebras

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Abstract. We introduce fuzzy groupoid graded rings and, as a particular case, fuzzy crossed product algebras. We show that there is a bijection between the set of fuzzy graded isomorphism equivalence classes of fuzzy crossed product algebras and the associated second cohomology group. This generalizes a classical result for crossed product algebras to the fuzzy situation. Thereby, we quantize the difference of richness between the fuzzy and the crisp case. We give several examples showing that in the fuzzy case the associated second cohomology group is much finer than in the classical situation. In particular, we show that the cohomology group may by *infinite* in the fuzzy case even though it is *trivial* in the crisp case.

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1. Introduction

About fifty years ago, Zadeh [25] introduced the notion of a fuzzy set as a function from the given set to the unit interval. Two years later, Goguen [7] replaced the unit interval by an arbitrary complete ordered lattice. Six years later, Rosenfeld [20] extended the concept of fuzzy sets to algebra by defining fuzzy subgroups of a group. Since then a lot of work has been devoted to proposing different versions of what fuzzy algebras of distinct types may be, e.g. fuzzy semigroups, fuzzy groups, fuzzy rings, fuzzy ideals, fuzzy semirings, fuzzy near-rings and fuzzy categories, see e.g. [1], [2], [3], [4], [5], [6], [8], [9], [10], [11], [12], [13], [14], [16], [17], [21], [22], [18], [19], [26] and [27]. The fuzzy algebraical structures are richer and more complicated than in the classical case. The purpose of this article is to quantize this difference in richness for the class of fuzzy crossed product algebras. To this end, we show that there is a bijection between the set of fuzzy graded isomorphism equivalence classes of fuzzy crossed product algebras and the associated second cohomology group (see Theorem 1.2) and that this cohomology group, in many cases, is much finer than in the classical case.

Recall that the classical crossed product algebras are defined in the following way. Suppose that A is a ring. We always assume that A is associative and that it has a multiplicative identity 1. We let $U(A)$ denote the group of units of A and we let $Z(A)$ denote the center of A i.e. the set of $a \in A$ satisfying $ab = ba$ for all $b \in A$. Furthermore, we let $Endom(A)$ (or $Aut(A)$) denote the set of ring endomorphisms (or automorphisms) of A. Let G be a group written multiplicatively with identity element e. If $\sigma : G \to Aut(A)$ and $\alpha : G \times G \to U(A)$ are maps satisfying the following three relations for all $g, h, p \in G$ and all $a \in A$

$$\sigma_g(\sigma_h(a))\alpha_{g,h} = \alpha_{g,h}\sigma_{gh}(a) \tag{1.1}$$

$$\alpha_{g,h}\alpha_{gh,p} = \sigma_g(\alpha_{h,p})\alpha_{g,hp} \tag{1.2}$$

$$\alpha_{g,e} = \alpha_{e,g} = 1 \tag{1.3}$$

then the crossed product algebra defined by σ and α, denoted by $A \rtimes_\alpha^\sigma G$, is the collection of formal sums $\sum_{g\in G} a_g g$, for $a_g \in A$, $g \in G$, where $a_g = 0$ for all but finitely many $g \in G$. The addition on $A \rtimes_\alpha^\sigma G$ is defined pointwise and the multiplication on $A \rtimes_\alpha^\sigma G$ is defined by the biadditive extension of the relation

$$(ag)(bh) = (a\sigma_g(b)\alpha_{g,h})(gh), \tag{1.4}$$

for $a, b \in A$ and $g, h \in G$. Note that (1.1) and (1.2) imply that the multiplication on $A \rtimes_\alpha^\sigma G$ is associative and by (1.3) we get that the element $1e$ is a multiplicative identity for $A \rtimes_\alpha^\sigma G$. It is clear that $R = A \rtimes_\alpha^\sigma G$ is a G-graded ring i.e. that $R = \oplus_{g\in G} R_g$ as additive groups and $R_g R_h \subseteq R_{gh}$, for $g, h \in G$, where $R_g = Ag$, for $g \in G$. Recall that if S is another G-graded ring, then a ring homomorphism $f : R \to S$ is called graded if $f(R_g) \subseteq S_g$, for $g \in G$. In the classical theory, two crossed product algebras $A \rtimes_\alpha^\sigma G$ and $A \rtimes_{\alpha'}^\sigma G$ are said to be equivalent if there is a graded ring isomorphism between them that is simultaneously an A-bimodule isomorphism. The set of such equivalence classes can be described by group cohomology in the following elegant way.

Theorem 1.1. *If A is a ring and σ is a map from G to $Aut(A)$, then there is a bijection between $H^2(G, U(Z(A)))$ and the set of equivalence classes of crossed product algebras $A \rtimes_\alpha^\sigma G$ defined by maps α from $G \times G$ to $U(A)$ satisfying (1.1), (1.2) and (1.3).*

For proof of Theorem 1.1 and more details concerning graded rings, see e.g. [15]. The purpose of this article is to state and prove a fuzzy version of Theorem 1.1 (see Theorem 1.2). To this end, suppose that $\mu : A \to L$ is a fuzzy subring of A where L is a complete distributive lattice. We let $Endom_{\mathrm{fz}}(A)$ (or $Aut_{\mathrm{fz}}(A)$) denote the set of fuzzy ring endomorphisms (or automorphisms) of A. Furthermore, let A_μ denote the subring of A consisting of elements $a \in A$ that satisfy $\mu(ab) \geq \mu(b)$ for all $b \in A$. If $\sigma : G \to Aut_{\mathrm{fz}}(A)$ and $\alpha : G \times G \to U(A_\mu)$ are maps satisfying (1.1), (1.2) and (1.3), then there is an extension of μ to a fuzzy subring of $A \rtimes_\alpha^\sigma G$ defined by $\mu(x) = \bigwedge_{g\in G} \mu(a_g)$ for all $x = \sum_{g\in G} a_g g$ in $A \rtimes_\alpha^\sigma G$ (see Proposition 2.8). We call such fuzzy subrings *fuzzy crossed product algebras*. We say that two fuzzy crossed product algebras $A \rtimes_\alpha^\sigma G$ and $A \rtimes_{\alpha'}^\sigma G$ are *fuzzy equivalent* if there is a graded fuzzy ring isomorphism between them that is simultaneously a fuzzy

A-bimodule isomorphism. In Section 3, we show that the set of such equivalence classes can be described by group cohomology in the following way.

Theorem 1.2. *If μ is a fuzzy subring of a ring A and σ is a map from G to $Aut_{\rm fz}(A)$, then there is a bijection between $H^2(G, U(Z(A_\mu)))$ and the set of fuzzy equivalence classes of fuzzy crossed product algebras $A \rtimes_\alpha^\sigma G$ defined by maps α from $G \times G$ to $U(A_\mu)$ satisfying (1.1), (1.2) and (1.3).*

The article is organized as follows. In Section 2, we recall the definition of the sum $\sum_{i \in I} \mu_i$ of a collection fuzzy subsemigroups μ_i, for $i \in I$, of a commutative semigroup, and, in particular, the direct sum $\oplus_{i \in I} \mu_i$ of such fuzzy semigroups. In the same section, we use this to define fuzzy groupoid graded rings. In Section 3, we introduce an equivalence relation on maps $\alpha : G \times G \to U(A_\mu)$ and we show that equivalence of such maps holds precisely when the corresponding crossed product algebras are equivalent. In the same section, we recall the theory of group cohomology of abelian groups and we apply this theory on the G-modules $U(Z(A_\mu))$ to show Theorem 1.2. In the end of this section, we give several examples of calculations of $H^2(G, U(Z(A_\mu)))$ many of which show that this cohomology group is finer than the classical cohomology group $H^2(G, U(Z(A)))$. In particular, we show that $H^2(G, U(Z(A_\mu)))$ may even be an *infinite* group though $H^2(G, U(Z(A)))$ is *trivial*.

2. Fuzzy Graded Rings

In this section, we fix the notation concerning fuzzy sets and algebras. We also recall the definition of the sum $\sum_{i \in I} \mu_i$ of a collection fuzzy subsemigroups μ_i, for $i \in I$, of a commutative semigroup, and, in particular, the direct sum $\oplus_{i \in I} \mu_i$ of such fuzzy semigroups (see Definition 2.4 and Definition 2.6). We use this to define fuzzy groupoid graded rings (see Definition 2.7). In the end of this section, we show that crossed product algebras defined by groupoids are fuzzy groupoid graded rings (see Proposition 2.8).

For the rest of the article, we let L denote a complete distributive lattice with meet and join denoted by $\vee$ and $\wedge$ respectively. We let the least and greatest elements of L be denoted by 0 and 1 respectively. Let G be a groupoid. By this we mean a set G equipped with a binary operation $G \times G \ni (g, h) \mapsto gh \in G$. Let $\mu : G \to L$ be a fuzzy subset of G. Recall that μ is a fuzzy subgroupoid of G if $\mu(gh) \geq \mu(g) \wedge \mu(h)$ for all $g, h \in G$. If G is a monoid with identity element e, then μ is a fuzzy submonoid of G if it is a fuzzy subgroupoid of G satisfying $\mu(e) = 1$. If G is a group, then μ is a fuzzy subgroup of G if it is fuzzy as submonoid of G satisfying $\mu(g^{-1}) = \mu(g)$ for all $g \in G$. If G is a ring, then μ is called a fuzzy subring of G if it is both a fuzzy subgroup of G with respect to addition and a fuzzy subgroupoid of G with respect to multiplication. Note that if G is a commutative monoid (or abelian group), then we often let the identity element (or the inverse of $g \in G$) be denoted by 0 (or $-g$).

Proposition 2.1. *If G is a semigroup (abelian group, ring) and μ is a fuzzy subsemigroup (subgroup, subring) of G, then the set $G_\mu = \{g \in G \mid \mu(gh) \geq \mu(h),$ for all $h \in H\}$ is a subsemigroup (subgroup, subring) of G.*

Proof. Suppose that G is a semigroup and take $g, h \in G_\mu$ and $p \in G$. Then $\mu((gh)p) = \mu(g(hp)) \geq \mu(hp) \geq \mu(p)$. Therefore $gh \in G_\mu$. If G is an abelian group and μ is a fuzzy subgroup of G, then $\mu(g^{-1}p) = \mu((g^{-1}p)^{-1}) = \mu(p^{-1}g) = \mu(gp^{-1}) \geq \mu(p^{-1}) = \mu(p)$. Hence $g^{-1} \in G_\mu$. Also $\mu(ep) = \mu(p)$ so $e \in G_\mu$. Suppose now that G is a ring and that μ is a fuzzy subring of G. By the above, we get that with respect to multiplication μ is a fuzzy subsemigroup of G. Also $\mu((g \pm h)p) = \mu(gp \pm hp) \geq \mu(gp) \wedge \mu(hp) \geq \mu(p) \wedge \mu(p) = \mu(p)$ so A_μ is closed under addition and subtraction. Since $\mu(0p) = \mu(0) = 1 \geq \mu(p)$ and $\mu(1p) = \mu(p)$ we get that $0, 1 \in G_\mu$. $\square$

Proposition 2.2. *Let A be a ring and A' a subring of A such that A and A' have the same multiplicative identity. If μ is the fuzzy subset of A defined by $\mu(x) = 1$ if $x \in A'$ and $\mu(x) = 0$ if $x \in A \setminus A'$, then μ is a fuzzy subring of A and $A_\mu = A'$.*

Proof. Take $a, b \in A$. First we show that $\mu(ab) \geq \mu(a) \wedge \mu(b)$. It is enough to show this inequality in the case when $\mu(ab) = 0$. But in that case $ab \in A \setminus A'$ and hence $a \in A \setminus A'$ or $b \in A \setminus A'$. Therefore $\mu(a) = 0$ or $\mu(b) = 0$ and hence $\mu(a) \wedge \mu(a) = 0$. In a completely analogous fashion it is shown that $\mu(a \pm b) \geq \mu(a) \wedge \mu(b)$. So μ is a fuzzy subring of A.

Now we show that $A' = A_\mu$. First we show the inclusion $A' \subseteq A_\mu$. Take $a \in A$ and $a' \in A'$. Then $\mu(a'a) \geq \mu(a) \wedge \mu(a') = \mu(a) \wedge 1 = \mu(a)$. Therefore $a' \in A_\mu$. Now we show the inclusion $A_\mu \subseteq A'$. Take $a'' \in A_\mu$. Then, since $1 \in A_\mu$, we get that $\mu(a'') = \mu(a'' \cdot 1) \geq \mu(1) = 1$. Therefore $a'' \in A'$. $\square$

Proposition 2.3. *The collection of fuzzy subgroupoids of G is a complete ordered lattice with respect to the ordering defined by putting $\mu \leq \mu'$ whenever $\mu(g) \leq \mu'(g)$, for $g \in G$.*

Proof. Define $\nu := \wedge_{i \in I} \mu_i$ of a family μ_i, $i \in I$, of fuzzy subgroupoids of G, by $(\wedge_{i \in I} \mu_i)(g) = \wedge_{i \in I} \mu_i(g)$, for $g \in G$. If $g, h \in G$, then $\nu(gh) = \wedge_{i \in I} \mu_i(gh) \geq \wedge_{i \in I}(\mu_i(g) \wedge \mu_i(h)) = (\wedge_{i \in I} \mu_i(g)) \wedge (\wedge_{i \in I} \mu_i(h)) = \nu(g) \wedge \nu(h)$. $\square$

Definition 2.4. Suppose that G is a groupoid and we are given a non-empty set I and a collection of fuzzy subsets μ_i of G, for $i \in I$. Then we let $\sum_{i \in I} \mu_i$ be the fuzzy subset of G defined in the following way. Take $g \in G$. Let $P(g)$ denote the set of all possible ways of representing g as a product, using parentheses, of some $g_i \in G$, for $i \in J$, where J ranges over all finite subsets of I. Now let $(\sum_{i \in I} \mu_i)(g) = \bigvee_{P(g)} \bigwedge_{i \in J} \mu_i(g_i)$.

Proposition 2.5. *If G is a commutative monoid (group) and μ_i, for $i \in I$, are fuzzy submonoids (subgroups) of G, then $\sum_{i \in I} \mu_i$ is a fuzzy submonoid (subgroup) of G and $\mu_i \leq \sum_{i \in I} \mu_i$ for all $i \in I$.*

Proof. This follows from Theorem 1.5.5 in [13]. For the convenience of the reader, we include the details of the proof for the particular case that we need. Put $\nu = \sum_{i \in I} \mu_i$ and take $g \in G$. From the definition of ν it follows that $\mu_i \leq \nu$ for all $i \in I$. Furthermore, it is clear that $\nu(0) = 1$ and that if G is a group, then $\nu(-g) = \nu(g)$. If $g, h \in G$ and we put $a = gh$, then

$$\nu(a) = \bigvee_{(a_i)_{i \in J} \in P(a)} \bigwedge_{i \in J} \mu_i(a_i) \geq \bigvee_{(g_i)_{i \in J} \in P(g)} \bigvee_{(h_i)_{i \in J} \in P(h)} \bigwedge_{i \in J} \mu_i(g_i h_i) \geq$$

$$\geq \bigvee_{(g_i)_{i\in J}\in P(g)} \bigvee_{(h_i)_{i\in J}\in P(h)} \bigwedge_{i\in J} (\mu_i(g_i) \wedge \mu_i(h_i))) =$$

$$= \bigvee_{(g_i)_{i\in J}\in P(g)} \bigvee_{(h_i)_{i\in J}\in P(h)} (\bigwedge_{i\in J} \mu_i(g_i)) \bigwedge (\bigwedge_{i\in J} \mu_i(h_i))) =$$

$$= \left(\bigvee_{(g_i)_{i\in J}\in P(g)} \bigwedge_{i\in J} \mu_i(g_i) \right) \bigwedge \left(\bigvee_{(h_i)_{i\in J}\in P(h)} \bigwedge_{i\in J} \mu_i(h_i) \right) = \nu(g) \wedge \nu(h).$$

□

Definition 2.6. Let μ and μ_i, for $i \in I$, be fuzzy subsets of a commutative monoid G. We say that μ is the fuzzy direct sum of the μ_i, for $i \in I$, if $\mu = \sum_{i\in I} \mu_i$ and $\mu_j \bigwedge \sum_{i\neq j} \mu_i = E$, for $j \in I$, where E is the fuzzy subset of G defined by $E(0) = 1$ and $E(x) = 0$ for non-zero $x \in G$. In that case we write $\mu = \oplus_{i\in I}\mu_i$. In that case we say that μ is fuzzy graded by the μ_i, for $i \in I$.

Definition 2.7. If ν and ν' are fuzzy additive subgroups of a ring A, then we define a fuzzy subset $\nu\nu'$ of A in the following way. Take $a \in A$. Let $Q(a)$ denote the set of all possible ways of representing $a = \sum_{i=1}^n b_i c_i$, for $b_i, c_i \in A$, where n a positive integer. Following Definition 3.1.3 in [13] we let

$$(\nu\nu')(a) = \bigvee_{Q(a)} \bigwedge_{1\leq i\leq n} \left(\nu(b_i) \bigwedge \nu'(c_i)\right).$$

Suppose that G is a multiplicatively written groupoid. We say that a fuzzy subring μ of A is graded of type G, or A is fuzzy G-graded, if there is a family $\{\mu_g\}_{g\in G}$ of fuzzy additive subgroups of A such that $\mu = \oplus_{g\in G}\mu_g$ and $\mu_g\mu_h \leq \mu_{gh}$ for $g, h \in G$. Suppose that μ is a fuzzy G-graded subring of A. If G has an identity element e, then $\mu_e\mu_e \leq \mu_e$. Hence, in that case, we can define a fuzzy G-grading of μ by $\mu_e = \mu$ and $\mu_g = E$ for nonidentity $g \in G$; we will refer to this fuzzy G-grading as the trivial fuzzy G-grading.

Proposition 2.8. *Suppose that A is a ring, G is a groupoid, μ is a fuzzy subring of A, $\sigma : G \to Endom_{\rm fz}(A)$ and $\alpha : G \times G \to A_\mu$ are maps satisfying (1.1), (1.2) and (1.3). If we define multiplication on $A \rtimes_\alpha^\sigma G$ by (1.4) and we define an extension of μ to $A \rtimes_\alpha^\sigma G$ by $\mu(x) = \wedge_{g\in G}\mu(a_g)$ for all $x = \sum_{g\in G} a_g g$ in $A \rtimes_\alpha^\sigma G$, then μ is a fuzzy G-graded subring of A with $\mu = \oplus_{g\in G}\mu_g$ where we for any $x = \sum_{g\in G} a_g g \in A \rtimes_\alpha^\sigma G$ put $\mu_g(x) = \mu(a_g)$, if $a_h = 0$, for $h \in G \setminus \{g\}$, and $\mu_g(x) = 0$, if $a_h \neq 0$, for some $h \in G \setminus \{g\}$.*

Proof. First we show that μ is a fuzzy subring of $A \rtimes_\alpha^\sigma G$. If $x = \sum_{g\in G} a_g g$ and $y = \sum_{h\in G} b_h h$ in $A \rtimes_\alpha^\sigma G$, where $a_g = b_h = 0$ for all but finitely many $g, h \in G$, then

$$\mu(x-y) = \mu\left(\sum_{g\in G}(a_g - b_g)g\right) = \wedge_{g\in G}\mu(a_g - b_g) \geq \wedge_{g\in G}(\mu(a_g)\wedge\mu(b_g)) = \mu(x)\wedge\mu(y)$$

and

$$\mu(xy) = \bigwedge_{p\in G} \mu\left(\sum_{gh=p} a_g\sigma_g(b_h)\alpha_{g,h}\right) \geq \bigwedge_{p\in G}\bigwedge_{gh=p} \mu(a_g\sigma_g(b_h)\alpha_{g,h}) \geq$$

$$\geq \bigwedge_{p\in G} \bigwedge_{gh=p} \mu(a_g\sigma_g(b_h)) \geq \bigwedge_{p\in G} \bigwedge_{gh=p} (\mu(a_g) \wedge \sigma_g(\mu(b_h))) \geq \bigwedge_{p\in G} \bigwedge_{gh=p} (\mu(a_g) \wedge \mu(b_h)) =$$

$$= (\wedge_{g\in G} a_g) \wedge (\wedge_{h\in G} b_h) = \mu(x) \wedge \mu(y).$$

Next we show that $\mu = \oplus_{g\in G}\mu_g$. Take $g \in G$. First we show that μ_g is a fuzzy additive subgroup of $A \rtimes_\alpha^\sigma G$. First of all $\mu_g(0) = \mu(0) = 1$. Now take $x = \sum_{g\in G} a_g g$ and $y = \sum_{h\in G} b_h h$ in $A \rtimes_\alpha^\sigma G$. Now we show that $\mu_g(x - y) \geq \mu_g(x) \wedge \mu_g(y)$. We only need to consider the case when $\mu_g(x) > 0$ and $\mu_g(y) > 0$. But then $x = a_g g$ and $y = b_g g$ and hence $\mu_g(x-y) = \mu_g(a_g g - b_g g) = \mu_g((a_g - b_g)g) = \mu(a_g - b_g) \geq \mu(a_g) \wedge \mu(b_g) = \mu_g(x) \wedge \mu_g(y)$. Next we show that $\mu = \sum_{g\in G} \mu_g$. The left hand side of this equality is $\mu(x) = \wedge_{g\in G}\mu(a_g)$. Now we calculate the right hand side of the equality. By the definition of $\sum_{g\in G} \mu_g$ we only need to consider the subsets J of $A \rtimes_\alpha^\sigma G$ with $\mu_g(x_g) > 0$ for all $g \in G$. But then we must have $x_g = a_g g$ for all $g \in G$. Hence $(\sum_{g\in G} \mu_g)(x) = \wedge_{g\in G}\mu_g(a_g g) = \wedge_{g\in G}\mu(a_g)$. Hence $\mu = \sum_{g\in G} \mu_g$. Now we show that $\mu = \oplus_{g\in G}\mu_g$. To show that $\mu_g \wedge \sum_{h\in G\setminus\{g\}} \mu_h = E$ we only need to check the equality for $x = a_g g \neq 0$ (otherwise $\mu_g(x) = 0$). But then $x_h = 0$ for all $h \in G \setminus \{g\}$ which implies that $J = \varnothing$ and hence $(\sum_{h\in G\setminus\{g\}} \mu_h)(x) = 0$. Finally we show that $\mu_g\mu_h \leq \mu_{gh}$. First of all, by the definition of $\mu_g\mu_h(x)$ we get that it is enough to consider $n = 1$ and $b_i = a_i g$ and $c_i = a_i' h$ for some $a_i, a_i' \in A$. But then $x = a_{gh} gh$ for some $a_{gh} \in A$. Thus $(\mu_g\mu_h)(x) = \wedge_{aa'=a_{gh}}(\mu(a) \wedge \mu(a')) \leq \mu(a_{gh}) = \mu_{gh}(x)$. □

3. Group Cohomology

In this section, we first recall the definition of group cohomology (see Definition 3.1). Then we use this formalism to show Theorem 1.2. In the end of this section, we give several examples (see Example 3.8, Example 3.9 and Example 3.10) of calculations of $H^2(G, U(Z(A_\mu)))$ many of which show that this cohomology group is finer than the classical cohomology group $H^2(G, U(Z(A)))$. In particular, we show that in some cases $H^2(G, U(Z(A_\mu)))$ is infinite even though $H^2(G, U(Z(A)))$ is trivial (see Example 3.10).

Definition 3.1. Let G be a group and M a G-module. By this we mean that M is an abelian group equipped with a group homomorphism $\sigma : G \to Aut(M)$. Let $Z^2(G, M)$ denote the abelian group of all maps $G \times G \ni (g, h) \mapsto \alpha_{g,h} \in M$ satisfying (1.2) and (1.3). Note that (1.1) is automatically satisfied. Let $B^2(G, M)$ denote the set of all maps $G \times G \ni (g, h) \mapsto \alpha_{g,h} \in M$ such that there are $u_g \in M$, for $g \in G$, satisfying $u_e = 1$ and

$$\alpha_{g,h} = u_g \sigma_g(u_h) u_{gh}^{-1} \tag{3.1}$$

for all $g, h \in G$. Then $B^2(G, M)$ is a subgroup of $Z^2(G, M)$. For the convenience of the reader we include an argument for this claim here. Suppose that α satisfies (3.1) for some $u_g \in M$, for $g \in G$. It is clear that α satisfies (1.3). Now we show (1.2). Take $g, h, p \in G$. Then

$$\alpha_{g,h}\alpha_{gh,p} = u_g\sigma_g(u_h)u_{gh}^{-1}u_{gh}\sigma_{gh}(u_p)u_{ghp}^{-1} = \sigma_g(u_h)\sigma_{gh}(u_p)\sigma_g(u_{hp}^{-1})u_g\sigma_g(u_{hp})u_{ghp}^{-1} =$$

$$= \sigma_g(u_h\sigma_h(u_p)u_{hp}^{-1})u_g\sigma_g(u_{hp})u_{ghp}^{-1} = \sigma_g(\alpha_{h,p})\alpha_{g,hp}.$$

We define $H^2(G,M)$ as the quotient group $Z^2(G,M)/B^2(G,M)$. In the sequel, we will use this formalism in the case when M equals the G-module $U(Z(A_\mu))$ for a given fuzzy subring μ of A and a group homomorphism $\sigma : G \to Aut_{\rm fz}(A)$.

Remark 3.2. Suppose that G is a group and M is a G-module. Let M^G denote the set of elements $m \in M$ satisfying $\sigma_g(m) = m$ for all $g \in G$. Now suppose that G is a finite cyclic group of order p with generator g. Let $n(M)$ denote the set of elements in M of the form $\prod_{i=0}^{p-1} \sigma_g^i(m)$ for $m \in M$. Let $M^{(p)}$ denote the set of elements in M of the form m^p for $m \in M$. Then $H^2(G,M)$ equals the quotient group $M^G/n(M)$ (see any standard book on group cohomology, e.g. Proposition 3.2.1 in [24]). Hence, if the action of G on M is trivial, i.e. if $\sigma_g = id_M$, then $H^2(G,M)$ equals $M/M^{(p)}$.

Definition 3.3. Let A be a ring, μ a fuzzy subring of A and G a group. Suppose that $\sigma : G \to Endom_{\rm fz}(A)$ and $\alpha, \alpha' : G \times G \to A_\mu$ are maps that satisfy (1.1), (1.2) and (1.3), then we say that α and α' are equivalent, denoted $\alpha \sim \alpha'$, if there are $u_g \in U(A_\mu)$, for $g \in G$, such that $u_e = 1$, $\sigma_g(a)u_g = u_g\sigma_g(a)$ and $\alpha'_{g,h}u_{gh} = u_g\sigma_g(u_h)\alpha_{g,h}$ for all $a \in A$ and all $g,h \in G$.

Proposition 3.4. *The relation $\sim$ is an equivalence relation.*

Proof. Suppose that $\alpha, \alpha', \alpha'' : G \times G \to A_\mu$ are maps such all three maps α, α' and α'' satisfy (1.1), (1.2) and (1.3). It is clear that $\sim$ is reflexive, with $u_g = 1$, for $g \in G$. Now we show that $\sim$ is symmetric. Suppose that $\alpha \sim \alpha'$. Then there are $u_g \in U(A_\mu)$, for $g \in G$, such that $\alpha'_{g,h}u_{gh} = u_g\sigma_g(u_h)\alpha_{g,h}$ for $g,h \in G$, and $u_e = 1$. Then $\sigma_g(u_h)^{-1}u_g^{-1}\alpha'_{g,h} = \alpha_{g,h}u_{gh}^{-1}$ and hence we get that $\alpha_{g,h}v_{gh} = v_g\sigma_g(v_h)\alpha'_{g,h}$ with $v_g = u_g^{-1}$, for $g \in G$. Now we show that $\sim$ is transitive. Suppose that also $\alpha' \sim \alpha''$. Then there are $v_g \in U(A_\mu)$, for $g \in G$, such that $\alpha''_{g,h}v_{gh} = v_g\sigma_g(v_h)\alpha'_{g,h}$ for $g,h \in G$, and $v_e = 1$. If we put $w_g = v_gu_g$, for $g \in G$, then we get that $\alpha''_{g,h}w_{gh} = \alpha''_{g,h}v_{gh}u_{gh} = v_g\sigma_g(v_h)\alpha'_{g,h}u_{gh} = v_g\sigma_g(v_h)u_g\sigma_g(u_h)\alpha_{g,h} = v_gu_g\sigma_g(v_h)\sigma_g(u_h)\alpha_{g,h} = v_gu_g\sigma_g(v_hu_h)\alpha_{g,h} = w_g\sigma_g(w_h)\alpha_{g,h}$. □

Proposition 3.5. *If A is a ring, G is a groupoid, μ is a fuzzy subring of A, $\sigma : G \to Endom_{\rm fz}(A)$ and $\alpha, \alpha' : G \times G \to A_\mu$ are maps satisfying (1.1), (1.2) and (1.3), then the fuzzy crossed products $A \rtimes_\sigma^\alpha G$ and $A \rtimes_\sigma^{\alpha'} G$ are fuzzy equivalent if and only if α and α' are equivalent.*

Proof. Suppose that $f : A \rtimes_\sigma^\alpha G \to A \rtimes_{\sigma'}^{\alpha'} G$ is a graded homomorphism of fuzzy crossed products with $f(a) = a$, for $a \in A$. Since $f(a) = a$ for all $a \in A$, we get that $f(ag) = au_gg$ for some $u_g \in A$, $g \in G$. Furthermore, since $\sigma_g(a)u_gg = \sigma_g(a)f(g) = f(\sigma_g(a)g) = f(ga) = f(g)a = u_gga = u_g\sigma_g(a)g$, we get that $\sigma_g(a)u_g = u_g\sigma_g(a)$. Since f is a homomorphism of fuzzy rings, we get that $\mu(au_g) \geq \mu(a)$ for all $a \in A$. Therefore $u_g \in A_\mu$. Since f is an isomorphism, we get that f^{-1} is also a graded homomorphism of fuzzy crossed products. We put $f^{-1}(g) = v_gg$ for some $v_g \in A$, for $g \in G$. Since $f \circ f^{-1} = id$ and $f^{-1} \circ f = id$, we get that $u_gv_g = v_gu_g = 1$, for $g \in G$. Therefore $u_g \in U(A_\mu)$, for $g \in G$. Since f respects multiplication, we get that $f(gh) = f(g)f(h)$, for $g,h \in G$. This is equivalent to $f(\alpha_{g,h}gh) = (u_gg)(u_hh)$ that is $\alpha_{g,h}u_{gh} = u_g\sigma_g(u_h)\alpha'_{g,h}$, for $g,h \in G$. This is means that α and α' are equivalent. On the other hand it is easy to see that if α and α' are equivalent, then there is a graded isomorphism of fuzzy crossed products $f : A \rtimes_\sigma^\alpha G \to A \rtimes_{\sigma'}^{\alpha'} G$ defined by $f(g) = u_g g$, for $g \in G$, and this map is also an A-bimodule isomorphism. □

Proposition 3.6. *Suppose that A is a ring, G is a group, μ is a fuzzy subring of A, $\sigma : G \to Aut_{\rm fz}(A)$ and $\alpha : G \times G \to U(A_\mu)$ are maps satisfying (1.1), (1.2) and (1.3). If $\beta : G \times G \to U(A_\mu)$, then $\beta\alpha$ satisfies (1.1), (1.2) and (1.3) if and only if $\beta \in Z^2(G, U(Z(A_\mu)))$. In that case, the fuzzy crossed products $A \rtimes_\sigma^\alpha G$ and $A \rtimes_\sigma^{\beta\alpha} G$ are fuzzy equivalent if and only if $\beta \in B^2(G, U(Z(A_\mu)))$.*

Proof. First we show the "if" statement. To show (1.1), we take $a \in A$ and $g, h \in G$. Then $\sigma_g(\sigma_h(a))\alpha_{g,h}\beta_{g,h} = \alpha_{g,h}\sigma_{gh}(a)\beta_{g,h} = \alpha_{g,h}\beta_{g,h}\sigma_{gh}(a)$. Since both α and β satisfy (1.2) and (1.3) it follows that $\beta\alpha$ also satisfies (1.2) and (1.3).

Now we show the "only if" statement. Put $\alpha' = \beta\alpha$. Then $\beta = \alpha'\alpha^{-1}$. We claim that β takes it's values in $Z(A_\mu)$. If we assume that the claim holds, then, since $U(A_\mu) \cap Z(A_\mu) = U(Z(A_\mu))$, the map β is well defined. Now we show the claim. Take $g, h, p \in G$ and $a \in A$. Since σ_g and σ_h are surjective, we only need to show that $\beta_{g,h}\sigma_g(\sigma_h(a)) = \sigma_g(\sigma_h(a))\beta_{g,h}$. Now

$$\beta_{g,h}\sigma_g((\sigma_h(a)) = \alpha'_{g,h}\alpha_{g,h}^{-1}\sigma_g((\sigma_h(a))\alpha_{g,h}\alpha_{g,h}^{-1} = \alpha'_{g,h}\alpha_{g,h}^{-1}\alpha_{g,h}\sigma_{gh}(a)\alpha_{g,h}^{-1} =$$
$$= \alpha'_{g,h}\sigma_{gh}(a)\alpha_{g,h}^{-1} = \sigma_g(\sigma_h(a))\alpha'_{g,h}\alpha_{g,h}^{-1} = \sigma_g(\sigma_h(a))\beta_{g,h}.$$

Now we show that β satisfies (1.2).

$$\beta_{g,h}\beta_{gh,p} = \alpha'_{g,h}\alpha_{g,h}^{-1}\alpha'_{gh,p}\alpha_{gh,p}^{-1} = \alpha'_{g,h}\alpha'_{gh,p}\alpha_{gh,p}^{-1}\alpha_{g,h}^{-1} = \alpha'_{g,h}\alpha'_{gh,p}(\alpha_{g,h}\alpha_{gh,p})^{-1} =$$
$$= \sigma_g(\alpha'_{h,p})\alpha'_{g,hp}(\sigma_g(\alpha_{h,p})\alpha_{g,hp})^{-1} = \sigma_g(\alpha'_{h,p})\alpha'_{g,hp}\alpha_{g,hp}^{-1}\sigma_g(\alpha_{h,p})^{-1} =$$
$$= \sigma_g(\alpha'_{h,p})\beta_{g,hp}\sigma_g(\alpha_{h,p})^{-1} = \sigma_g(\alpha'_{h,p})\sigma_g(\alpha_{h,p})^{-1}\beta_{g,hp} = \sigma_g(\beta_{h,p})\beta_{g,hp}.$$

It is clear that β satisfies (1.3). The last part of the proof follows from the first part and Proposition 3.5. □

Proof of Theorem 1.2. This follows immediately from Proposition 3.5 and Proposition 3.6. □

Proposition 3.7. *Let G be a group and A a ring. Suppose that A' is a subring of A equipped with the same multiplicative identity as A and μ is the fuzzy subring of A defined by $\mu(x) = 1$ if $x \in A'$ and $\mu(x) = 0$ if $x \in A \setminus A'$. If there is a group action $\sigma : G \to Aut_{\rm fz}(A)$ which restricts to a group action $\sigma : G \to Aut_{\rm fz}(A')$, then $H^2(G, U(Z(A_\mu))) = H^2(G, U(Z(A')))$.*

Proof. This follows from Proposition 2.2. □

Example 3.8. Suppose that $A = \mathbb{C}$ and $G = \{e, g\}$ is the group with two elements. Let G act on A trivially. Then $U(A)$ is a G-module and by Remark 3.2 we get that $H^2(G, U(A)) = (U(A))^G/n(U(A)) = U(A)/U(A)^{(2)} = 1$, since all complex units are complex squares. This means that

- there is only one equivalence class of crossed product algebras $A \rtimes_\alpha^\sigma G$ defined by maps α from $G \times G$ to $U(A)$ satisfying (1.1), (1.2) and (1.3).

Define a fuzzy subring μ of A by $\mu(x) = 1$ when $x \in \mathbb{R}$ and $\mu(x) = 0$ otherwise. By Remark 3.2 and Proposition 3.7, we get that $H^2(G, U(A_\mu)) = H^2(G, U(\mathbb{R})) = U(\mathbb{R})/U(\mathbb{R})^{(2)} = \mathbb{Z}_2$. This means that

- there are two fuzzy equivalence classes of fuzzy crossed product algebras $A \rtimes_\alpha^\sigma G$ defined by maps α from $G \times G$ to $U(A_\mu)$ satisfying (1.1), (1.2) and (1.3).

If we define a fuzzy subring μ of A by $\mu(x) = 1$ when $x \in \mathbb{Q}$ and $\mu(x) = 0$ otherwise, then we get that $H^2(G, U(A_\mu)) = H^2(G, U(\mathbb{Q})) = U(\mathbb{Q})/U(\mathbb{Q})^{(2)}$. The latter quotient group equals the direct sum of infinite countably many copies of $\mathbb{Z}_2$. Hence, we get that

- there are infinitely many fuzzy equivalence classes of fuzzy crossed product algebras $A \rtimes_\alpha^\sigma G$ defined by maps α from $G \times G$ to $U(A_\mu)$ satisfying (1.1), (1.2) and (1.3).

Example 3.9. Suppose that $A = \mathbb{C}$ and $G = \{e, g\}$ is the group with two elements. Define an action of G on A by $\sigma_e = id_A$ and $\sigma_g =$ complex conjugation. Then $U(A)$ is a G-module and by Remark 3.2, we get that $H^2(G, U(A)) = (U(A))^G/n(U(A)) = U(\mathbb{R})/U(\mathbb{R})^{(2)} = \mathbb{Z}_2$. This means that

- there are two equivalence class of crossed product algebras $A \rtimes_\alpha^\sigma G$ defined by maps α from $G \times G$ to $U(A)$ satisfying (1.1), (1.2) and (1.3).

Now put $A' = \mathbb{Z}[\sqrt{2}, i]$ and let μ be the fuzzy subring of A defined by $\mu(x) = 1$, for $x \in A'$ and $\mu(x) = 0$, for $x \in A \backslash A'$. The group of units in A' is $\langle \pm 1, (1+\sqrt{2})^n \mid n \in \mathbb{Z} \rangle$ (see any standard book on number theory e.g. Section 12.1 in [23]). Therefore, by Proposition 3.7, we get that

$$H^2(G, U(A_\mu)) = H^2(G, U(A')) =$$

$$= \langle \pm 1, (1-\sqrt{2})^n \mid n \in \mathbb{Z} \rangle / \langle (1-\sqrt{2})^{2n} \mid n \in \mathbb{Z} \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

This means that

- there are four equivalence classes of fuzzy crossed product algebras $A \rtimes_\alpha^\sigma G$ defined by maps α from $G \times G$ to $U(A_\mu)$ satisfying (1.1), (1.2) and (1.3).

Example 3.10. Suppose that G is a finite cyclic group with p elements. Let $A = \mathbb{C}$ and suppose that G acts trivially on A. Then $U(A)$ is a G-module and by Remark 3.2 we get that $H^2(G, U(A)) = (U(A))^G/n(U(A)) = U(A)/U(A)^{(p)} = 1$, since all complex units are complex pth roots. This means that

- there is only one equivalence class of crossed product algebras $A \rtimes_\alpha^\sigma G$ defined by maps α from $G \times G$ to $U(A)$ satisfying (1.1), (1.2) and (1.3).

Now put $A' = \mathbb{Z}[\sqrt{2}]$ and let μ be the fuzzy subring of A defined by $\mu(x) = 1$, for $x \in A'$ and $\mu(x) = 0$, for $x \in A \setminus A'$. By Proposition 3.7, we get that

$$H^2(G, U(A_\mu)) = H^2(G, U(A')) = \langle \pm 1, (1+\sqrt{2})^n \mid n \in \mathbb{Z} \rangle / \langle (-1)^p, (1+\sqrt{2})^{pn} \mid n \in \mathbb{Z} \rangle.$$

Hence, we get that

$$H^2(G, U(A_\mu)) = \mathbb{Z}_2 \times \mathbb{Z}_p$$

when p is even and

$$H^2(G, U(A_\mu)) = \mathbb{Z}_p$$

when p is odd. In particular, this implies that

- if p is even (or odd), then there are $2p$ (or p) equivalence classes of fuzzy crossed product algebras $A \rtimes_\alpha^\sigma G$ defined by maps α from $G \times G$ to $U(A_\mu)$ satisfying (1.1), (1.2) and (1.3).

If we use the fuzzy subring μ from the end of Example 3.8 and p is odd (even), then we get that $H^2(G, U(A_\mu))$ equals a direct sum of ($\mathbb{Z}_2$ and) infinitely countable many copies of $\mathbb{Z}_p$. Thus there are infinitely many equivalence classes of fuzzy crossed product algebras $A \rtimes_\alpha^\sigma G$ defined by maps α from $G \times G$ to $U(A_\mu)$ satisfying (1.1), (1.2) and (1.3).

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