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GROUPOID FACTORIZATIONS IN THE SEMIGROUP OF BINARY SYSTEMS

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Abstract. Let $(X, \bullet)$ be a groupoid (binary algebra) and $Bin(\dot{X})$ denote the collection of all groupoids defined on X. We introduce two methods of factorization for this binary system under the binary groupoid product "$\diamond$" in the semigroup $(Bin\,(X)\,, \diamond)$. We conclude that a strong non-idempotent groupoid can be represented as a product of its *similar*- and *signature*- derived factors. Moreover, we show that a groupoid with the orientation property is a product of its *orient*- and *skew*- factors. These unique factorizations can be useful for various applications in other areas of study. Application to algebras such as $B/BCH/BCI/BCK/BH/BI/d$-algebra are widely given throughout this paper.

1. Introduction

Algebraic structures play a vital role in mathematical applications such as information science, network engineering, computer science, cell biology, etc. This encourages sufficient motivation to study abstract algebraic concepts and review previously obtained results. One such concept of interest to many mathematicians over the past two decades or so is that of a simple yet very interesting notion of a single set with one binary operation, historically known as magma and more recently referred to as groupoid. Bruck [8] published the book,"A Survey of Binary Systems" in which the theory of groupoids, loops, quasigroups, and several algebraic structures were discussed. Borůvka in [7] explained the foundations for the theory of groupoids, set decompositions and their application to binary systems.

Given a binary operation "$\bullet$" on a non-empty set X, the groupoid $(X, \bullet)$ is a generalization of the very well-known structure of a group. H. S. Kim and J. Neggers in [33] investigated the structure $(Bin\,(X)\,, \diamond)$ where $Bin\,(X)$ is the collection of all binary systems (groupoids or algebras) defined on a non-empty set X along with an associative binary product $(X, *) \diamond (X, \circ) = (X, \bullet)$ such that $x \bullet y = (x * y) \circ (y * x)$ for all $x, y \in X$. They recognized that the left-zero-semigroup serves as the identity of this semigroup. The present author in [11] introduced the notion of the center $ZBin(X)$ in the semigroup $(Bin\,(X)\,, \diamond)$, and proved that $(X, \bullet) \in ZBin(X)$, if and only if $(X, \bullet)$ is locally-zero. Han and Kim in [13] introduced the notion of hypergroupoids $HBin(X)$, and showed that $(HBin(X), \diamond)$ is a supersemigroup of

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the semigroup $(Bin(X), \diamond)$ via the identification $x \longleftrightarrow \{x\}$. They proved that $(HBin^{*}(X), \ominus, [\emptyset])$ is a BCK-algebra.

In this paper, we investigate the following problem:

Main Problem: Consider the semigroup $(Bin\,(X)\,, \diamond)$. Let the left-zero-semigroup be denoted as $id_{Bin(X)}$. Given a groupoid (binary system) $(X, \bullet) \in Bin(X)$, is it possible to find two groupoid-factors $(X, *)$ and $(X, \circ)$ such that

$$(X, \bullet) = (X, *) \diamond (X, \circ)?$$

If so,

Problem 1 (Uniqueness)**.** *Are the corresponding groupoid-factors:*

(1) *Distinct, i.e.,* $(X, *) \neq (X, \circ)$*?*
(2) *Unique, i.e., if* $(X, \bullet) = (X, *) \diamond (X, \circ)$*, is it possible for* $(X, \bullet) = (X, \lhd) \diamond (X, \rhd)$ *such that* $(X, *) \neq (X, \lhd)$ *and* $(X, \circ) \neq (X, \rhd)$*?*
(3) *Different from* $(X, \bullet)$*, i.e.,* $(X, *) \neq (X, \bullet)$ *and* $(X, \circ) \neq (X, \bullet)$*?*
(4) *Different from the left-zero-semigroup, i.e.,* $(X, *) \neq id_{Bin(X)}$ *and* $(X, \circ) \neq id_{Bin(X)}$*?*

Problem 2 (Derivation)**.** *How do we find the groupoid-factors? Are they:*

(1) *Derived (related to, based off of, dependent on) from: the parent groupoid* $(X, \bullet)$*?*
(2) *Derived from the identity* $id_{Bin(X)}$*?*

Problem 3 (Factorization)**.** *If we use a certain method to find the two groupoid-factors, what is the nature of this factorization?*

(1) *Is it unique?*
(2) *When is it commutative?*

We begin answering these questions by introducing two methods for factoring a random groupoid in $Bin(X)$ using the product "$\diamond$". We will show that both methods result in unique factorizations (Problem 3.1) of a given groupoid and hence we answer Problem 1.2 with a definite yes! Section two provides some definitions and preliminary ideas which are necessary in this context. We also present a summarized table of "logic" algebras for a clear view. Section three describes AU- and UA-factorizations, which comprises the first method (method-1) of factoring. In fact, method-1 factors a groupoid $(X, \bullet)$ by obtaining two *derived* factors from it (Problem 2.1) and from the left-zero-semigroup (Problem 2.2), the *signature*- and *similar*-factors, respectively. We prove that a strong groupoid has a commutative method-1 factorization (Problem 3.2). The possibility of this first method is shown to be feasible and produces non-trivial decompositions (Problem 1.4), however, it is restricted to non-idempotent groupoids only. Hence, section four introduces an OJ- and a JO-factorization, which constitutes our second method (method-2). We will demonstrate that the latter method is sufficient for idempotent as well as non-idempotent groupoids. In addition, an interesting outcome of method-2 is that one of the factors is not *derived* from the parent groupoid (Problems 2.1 and 2.2) while the other factor is; we name them *orient*- and *skew*-factors, respectively. We show that a given groupoid $(X, \bullet)$ with $x \bullet y \in \{x, y\}$, for all x, y in X, has a commutative method-2 factorization (Problem 3.2). Section five briefly applies our two methods to some of the algebras listed in section two; and discusses a promising relationship to graph theory.

Finally, in our last section we generalize and summarize our findings that certain groupoids/algebras decompose into distinct groupoids via (1) an operation on the parent groupoid and the left-zero-semigroup simultaneously, which is a generalization of our first method; or (2) an operation which acts on the parent-groupoid and the left-zero-semigroup separately, hence resulting in a generalization of our second method.

Notions of "method"-*composite,* "method"-*normal*, "factor"-*prime* and "partially"-*left/right-prime* are used to classify and analyze various groupoids as well as other familiar algebras. For simplicity, the left-zero-semigroup will be denoted as $id_{Bin(X)}$.

2. Preliminaries

A *groupoid* [8] $(X, \bullet)$ consists of a non-empty set X together with a binary operation $\bullet : X \times X \to X$ where $x \bullet y \in X$ for all $x, y \in X$.

A groupoid $(X, \bullet)$ is *strong* [33] if and only if for all $x, y \in X$,

$$x \bullet y = y \bullet x \text{ implies } x = y. \tag{2.1}$$

A groupoid $(X, \bullet)$ is *idempotent* if $x \bullet x = x$ for all $x \in X$.

Example 2.1 [12] Let $X = [0, \infty)$ and let $x \bullet y = \max\{0, x - y\}$ for any $x, y \in X$. Then $(X, \bullet)$ is a strong groupoid. To visualize this, let's consider the associated Cayley product table for "$\bullet$". For simplicity, its partial table is displayed below which shows that $x \bullet y = 0$ for all $x \leq y$ and $x \bullet y \neq 0$ for all $x > y$:

$\bullet$	0	1	2	3	4	$\cdots$
0	0	0	0	0	0	$\cdots$
1	1	0	0	0	0	$\cdots$
2	2	1	0	0	0	$\cdots$
3	3	2	1	0	0	$\cdots$
4	4	3	2	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Hence, the strong or anti-commutative property holds for all $x, y \in X$.

Example 2.2 [12] Let $X = \mathbb{R}$ be the set of all real numbers and let $x, y, e \in \mathbb{R}$. If we define a binary operation "$\bullet$" on X by $x \bullet y = (x - y)(x - e) + e$,
then the groupoid $(X, \bullet, e)$ is not strong, since $x = e + \alpha$, $y = e - \alpha$, $\alpha \neq \pm e$ implies $x \bullet y = y \bullet x$, but $x \neq y$.

A groupoid $(X, \bullet)$ is a *left-zero-semigroup* if $x \bullet y = x$ for all $x, y \in X$. Similarly, $(X, \bullet)$ is a *right-zero-semigroup* if $x \bullet y = y$ for all $x, y \in X$. For the theory of semigroups, we refer to [10, 30].

A groupoid $(X, \bullet)$ is *locally-zero* [11] if
(i) $x \bullet x = x$ for all $x \in X$; and
(ii) for any $x \neq y$ in X, $(\{x, y\}, \bullet)$ is either a left-zero-semigroup or a right-zero-semigroup.

Example 2.3 Given a set $X = \{0, 1, 2\}$, let the binary operation "$\bullet$" be defined by the following Cayley product table:

$\bullet$	0	1	2
0	0	0	2
1	1	1	1
2	0	2	2

Then the binary system $(X, \bullet)$ is locally-zero and has the following subtables:

$\bullet$	0	1
0	0	0
1	1	1

$\bullet$	1	2
1	1	1
2	2	2

$\bullet$	0	2
0	0	2
2	0	2

where $(\{0,1\}, \bullet)$ is a left-zero-semigroup; $(\{1,2\}, \bullet)$ is also a left-zero-semigroup; and $(\{0,2\}, \bullet)$ is a right-zero-semigroup.

The notion of the semigroup $(Bin(X), \diamond)$ was introduced by J. Neggers and H.S. Kim in [33]. Given a non-empty set X, let $Bin(X)$ denote the collection of all groupoids $(X, \bullet)$, where $\bullet : X \times X \to X$ is a map. Given elements $(X, *)$ and $(X, \circ)$ of $Bin\,(X)$, define a binary product "$\diamond$" on these groupoids as follows:

$$(X,*) \diamond (X,\circ) = (X,\bullet) \tag{2.2}$$

where

$$x \bullet y = (x * y) \circ (y * x) \tag{2.3}$$

for all $x,\, y \in X$. This turns $(Bin(X), \diamond)$ into a semigroup with identity, the left-zero-semigroup, and an analog of negative one in the right-zero-semigroup.

The present author [11] showed that a groupoid $(X, \bullet)$ commutes, relative to the product "$\diamond$", if and only if any 2-element subset of $(X, \bullet)$ is a subgroupoid that is either a left-zero-semigroup or a right-zero-semigroup. Thus, $(X, \bullet)$ is an element of the *center* $ZBin(X)$ of the semigroup $(Bin(X), \diamond)$, defined as follows:

$$ZBin\,(X) = \{(X,\bullet) \in Bin\,(X) \mid (X,\bullet) \diamond (X,*) = (X,*) \diamond (X,\bullet),\ \forall (X,*) \in Bin(X)\}.$$

In turn, several properties were obtained.

Theorem 2.4 [33] *The collection $(Bin(X), \diamond)$ of all binary systems (groupoids or algebras) defined on X is a semigroup, i.e., the operation "$\diamond$" as defined in general is associative. Furthermore, the left-zero-semigroup is an identity for this operation.*

Proposition 2.5 [33] *Let $(X, \bullet)$ be the right-zero-sermigroup on X. Then $(X, \bullet) \in Str(X)$, the collection of all strong groupoids on X.*

Proposition 2.6 [11] *The left-zero semigroup and right-zero semigroup on X are both in $ZBin(X)$.*

Corollary 2.7. [11] *The collection of all locally-zero groupoids on X forms a subsemigroup of $(Bin(X), \diamond)$.*

Proposition 2.8 [11] *Let $(X, \bullet)$ be a locally-zero groupoid. Then $(X, \bullet) \diamond (X, \bullet) = id_{Bin(X)}$, the left-zero-semigroup on X.*

Let $(X, \bullet)$ be an element of the semigroup $(Bin(X), \diamond)$, we say that $(X, \bullet)$ is a *unit* if and only if there exists an element $(X, *) \in Bin\,(X)$ such that

$$(X,\bullet) \diamond (X,*) = id_{Bin(X)} = (X,*) \diamond (X,\bullet). \tag{2.4}$$

Subsequently, by Proposition 2.8, a locally-zero-groupoid is a unit in $Bin\,(X)$.

The logic-based BCK/BCI-algebras were introduced by Iséki and Imai in [15] as propositional calculus, but later in [16] developed into the present notion of BCK/BCI which have since then been investigated thoroughly by numerous researchers. J. Neggers and H. S. Kim generalized a BCK-algebra [26] by introducing the notion of a d-algebra in [32]. They also introduced B-algebras in [2]. C. B. Kim and H. S. Kim generalized a B-algebra by defining a BG-algebra in [21].

An algebra $(X, \bullet, 0)$ of type $(2, 0)$ is a *B-algebra* [2] if for all $x, y, z \in X$, it satisfies the following axioms:

B1: $x \bullet x = 0$,
B2: $x \bullet 0 = x$, and
B: $(x \bullet y) \bullet z = x \bullet [z \bullet (0 \bullet y)]$.

An algebra $(X, \bullet, 0)$ of type $(2, 0)$ is a *BG-algebra* [21] if for all $x, y, z \in X$, it satisfies (B1), (B2), and

BG: $x = (x \bullet y) \bullet (0 \bullet y)$.

An algebra $(X, \bullet, 0)$ of type $(2, 0)$ is a *BCI-algebra* [36] if for all $x, y, z \in X$, it satisfies (B2) and:

I: $((x \bullet y) \bullet (x \bullet z)) \bullet (z \bullet y) = 0$,
BH: $x \bullet y = 0$ and $y \bullet x = 0$ implies $x = y$.

Example 2.9 [36] Let $X = \{0, 1, a, b\}$. Define a binary operation "$\bullet$" on X by the following product table:

$\bullet$	0	1	a	b
0	0	0	a	a
1	1	0	a	a
a	a	a	0	0
b	b	a	1	0

Then $(X, \bullet, 0)$ is a BCI-algebra.

A BCI-algebra $(X, \bullet, 0)$ is a *BCK-algebra* [26] if it satisfies the next additional axiom:

K: $0 \bullet x = 0$ for all $x \in X$.

An algebra $(X, \bullet, 0)$ of type $(2, 0)$ is a *d-algebra* provided that for all x, $y \in X$, it satisfies (B1), (K) and (BH).

A d-algebra is *strong* if for all x, $y \in X$:

d-3′: $x \bullet y = y \bullet x$ implies $x = y$.

Otherwise we consider the d-algebra to be *exceptional*. For more information on d-algebras we refer to [5, 6, 32, 31].

Example 2.10 [32] Let $(X, \bullet) = (\mathbb{Z}_5, \bullet)$ where "$\bullet$" is defined by the following Cayley table:

$\bullet$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	3	0
3	3	3	2	0	3
4	4	4	1	1	0

Then $(\mathbb{Z}_5, \bullet, 0)$ is a d-algebra which is not a BCK-algebra. For details on BCK-algebras, see [14, 26, 36].

Y. B. Jun, E. H. Roh and H. S. Kim in [18] introduced the notion of a BH-algebra which is a generalization of $BCK/BCI/BCH$-algebras. There are many other generalizations of similar algebras. We summarize several properties which are used as axioms to define each algebraic structure . Let $(X, \bullet, 0)$ be an algebra of type $(2, 0)$, for any x, y, $z \in X$:

B1: $x \bullet x = 0$,
B2: $x \bullet 0 = x$,
B: $(x \bullet y) \bullet z = x \bullet (z \bullet (0 \bullet y))$,
BG: $x = (x \bullet y) \bullet (0 \bullet y)$,
BM: $(z \bullet x) \bullet (z \bullet y) = y \bullet x$,
BH: $x \bullet y = 0$ and $y \bullet x = 0 \Rightarrow x = y$,
BF: $0 \bullet (x \bullet y) = y \bullet x$,
BN: $(x \bullet y) \bullet z = (0 \bullet z) \bullet (y \bullet x)$,
BO: $x \bullet (y \bullet z) = (x \bullet y) \bullet (0 \bullet z)$,
BP1: $x \bullet (x \bullet y) = y$,
BP2: $(x \bullet z) \bullet (y \bullet z) = x \bullet y$,
Q: $(x \bullet y) \bullet z = (x \bullet z) \bullet y$,
CO: $(x \bullet y) \bullet z = x \bullet (y \bullet z)$,
BZ: $((x \bullet z) \bullet (y \bullet z)) \bullet (x \bullet y) = 0$,
K: $0 \bullet x = 0$,
I: $((x \bullet y) \bullet (x \bullet z)) \bullet (z \bullet y) = 0$,
BI: $x \bullet (y \bullet x) = x$.

An algebra $(X, \bullet, 0)$ of type $(2, 0)$ is classified according to a combination of the above axioms as noted in "**Figure 1**" below. For instance, $(X, \bullet, 0)$ is a BI-algebra [34] if satisfies in (B1) and (BI). For detailed information on each, please see [2-6, 14-26, 31, 32, 34, 36].

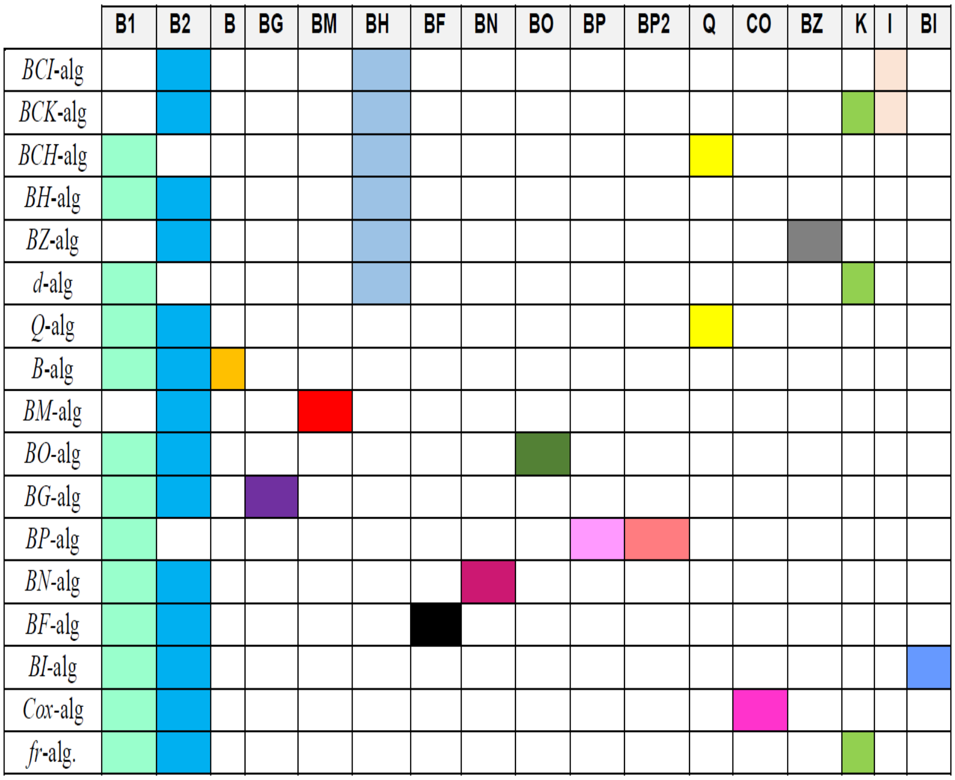


Figure 1. Comparison of Algebras

3. Similar-Signature Factorization

In this section, we present a unique factorization of a given groupoid by "deriving" two factors from it and from the left-zero-semigroup simultaneously.

Let $(X, \bullet)$ be a groupoid of finite order, i.e., $|X| = n$. Then $d^\bullet$ is the *diagonal function* of $(X, \bullet)$ such that $d^\bullet : \mathbb{N} \longrightarrow X$ where $d^\bullet(i) = x_i \bullet x_i$, $i = 1, 2, ..., n$ for all $x_i \in X$.

Example 3.1 Let $(X, \bullet, 0)$ and $(X, *)$ be a d-algebra and an idempotent algebra, respectively. Then $x \bullet x = 0$ and $x * x = x$; or $d^\bullet = 0$ and $d^* = x$ for all $x \in X$.

Two binary systems $(X, *)$ and $(X, \bullet)$ are said to be *similar* if they have the same diagonal function, that is, $d^* = d^\bullet$.

Two binary systems $(X, *)$ and $(X, \bullet)$ are said to be *signature* if
(i) $x * y = x \bullet y$ when $x \neq y$; and
(ii) $x * x \neq x \bullet x$ for all $x \in X$.

Let $(X, \bullet)$ be a groupoid. *Derive* groupoids $(X, *)$ and $(X, \circ)$ from $(X, \bullet)$ and $id_{Bin(X)}$, simultaneously, such that for all $x, y \in X$,

$$x * y = \begin{cases} x & \text{if } x = y, \\ x \bullet y & \text{otherwise.} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} x \bullet x & \text{if } x = y, \\ x & \text{otherwise.} \end{cases} \tag{3.1}$$

The groupoids $(X,*)$ and $(X,\circ)$ are said to be the *signature*- and the *similar-factors* of $(X,\bullet)$, respectively, denoted by $U(X,\bullet)$ and $A(X,\bullet)$. The product "$\diamond$" is associative but not commutative. Hence, for $(X,\bullet)\in Bin(X)$, we may have a *UA-factorization* such that

$$(X,\bullet)=U(X,\bullet)\diamond A(X,\bullet) \tag{3.2}$$

or an *AU-factorization* such that

$$(X,\bullet)=A(X,\bullet)\diamond U(X,\bullet). \tag{3.3}$$

By the equations in 3.1, it follows that for any given groupoid $(X,\bullet)$,

(1) $U(X,\bullet)$ is similar to $id_{Bin(X)}$ while $A(X,\bullet)$ is similar to $(X,\bullet)$; and
(2) $U(X,\bullet)$ is signature with $(X,\bullet)$ while $A(X,\bullet)$ is signature with $id_{Bin(X)}$.

Proposition 3.2 *The similar-factor of a groupoid is strong.*
Proof. Given $(X,\bullet)\in Bin(X)$, let $(X,\circ)=A(X,\bullet)$.

(i) If $x=y$, then $x\circ y=x\circ x=x\bullet x=y\bullet y=y\circ y=y\circ x$.
(ii) If $x\neq y$ and $x\circ y=y\circ x$ for any $x,y\in X$. Then $x\circ y=x$ and $y\circ x=y$. Thus, $x=y$, a contradiction.

Therefore, $(X,\circ)$ is strong.

■

Example 3.3 Let $(X,\bullet,0)$ be the BCI-algebra defined in Example 2.9. In accordance with equation 3.1, derive its *signature*- and *similar*- factors $U(X,\bullet,0)$ and $A(X,\bullet,0)$, respectively. Let groupoids $(X,*,0):=U(X,\bullet,0)$ and $(X,\circ,0):=A(X,\bullet,0)$ be given. We obtain:

$*$	0	1	a	b
0	0	0	a	a
1	1	1	a	a
a	a	a	a	0
b	b	a	1	b

and

$\circ$	0	1	a	b
0	0	0	0	0
1	1	0	1	1
a	a	a	0	a
b	b	b	b	0

It remains to verify that $(X,\bullet,0)=(X,*,0)\diamond(X,\circ,0)$ and/or $(X,\bullet,0)=(X,\circ,0)\diamond(X,*,0)$. This will be discussed in more detail in the next section. However, there is a very interesting fact in this example: the two factors are distinct from each other, their parent groupoid, and the left-zero-semigroup. In summary:

(1) $(X,*,0)\neq(X,\circ,0)$; (Problem 1.1)
(2) $(X,*,0)\neq(X,\bullet,0)\neq(X,\circ,0)$; (Problem 1.3)
(3) $(X,*,0)\neq id_{Bin(X)}\neq(X,\circ,0)$. (Problem 1.4)

This is important since it is not always the case that all three distinctions hold as the following example demonstrates.

Example 3.4 Let $(X,\bullet)=(\mathbb{Z}_3,\bullet)$ where "$\bullet$" is defined by the following Cayley table:

$\bullet$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then $(X,\bullet,0)$ is a BI-algebra. Derive its *signature*- and *similar*-factors $U(X,\bullet,0)$ and $A(X,\bullet,0)$, respectively, in accordance to the equations in 3.1. Let $(X,*,0):=$

$U(X,\bullet,0)$ and $(X,\circ,0) := A(X,\bullet,0)$, hence:

$$\begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{array} \quad \text{and} \quad \begin{array}{c|ccc} \circ & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 0 \end{array}$$

Here we observe immediately that the *similar*-factor $(X,\circ,0)$ is equal to $(X,\bullet,0)$ and the *signature*-factor $(X,*,0)$ is equal to $id_{Bin(X)}$. Thus this decomposition is basically a trivial factorization, i.e.,

$$(X,\bullet,0) = (X,*,0) \diamond (X,\circ,0) = id_{Bin(X)} \diamond (X,\bullet,0)$$

and

$$(X,\bullet,0) = (X,\circ,0) \diamond (X,*,0) = (X,\bullet,0) \diamond id_{Bin(X)}.$$

3.1. UA**-Factorization.** In this subsection, we explore a UA-factorization of a given groupoid $(X,\bullet)$ in $Bin(X)$. In the next subsection, a AU-factorization is considered, where the order of the product of the two factors is "reversed". We emphasize that such factorization is unique and not necessarily reversible. Then, we classify a given groupoid as UA- and/or AU-*composite*, *u-composite* or *u-normal;* and as *signature-* or *similar-prime.*

Example 3.1.1 Let $X = \mathbb{Z}$ be the set of all integers and let "$-$" be the usual subtraction on $\mathbb{Z}$. Then $(\mathbb{Z},-)$ is a BH-algebra since it satisfies axioms **B1**, **B2** and **BH** as seen from its partial table below:

$$\begin{array}{c|cccccccccc} - & \cdots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \\ \hline \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ -2 & 1 & 0 & -1 & -2 & -3 & -4 & -5 & -6 & \cdots \\ -1 & 2 & 1 & 0 & -1 & -2 & -3 & -4 & -5 & \cdots \\ 0 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 & \cdots \\ 1 & 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & \cdots \\ 2 & 5 & 4 & 3 & 2 & 1 & 0 & -1 & -2 & \cdots \\ 3 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & -1 & \cdots \\ 4 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Define two binary operations "$*$" and "$\circ$" on $\mathbb{Z}$ such that for all $x, y \in \mathbb{Z}$,

$$x * y = \begin{cases} x & \text{if } x = y, \\ x - y & \text{otherwise.} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} 0 & \text{if } x = y, \\ x & \text{otherwise.} \end{cases}$$

Then it is easy to check that $(\mathbb{Z},-) = (\mathbb{Z},*) \diamond (\mathbb{Z},\circ)$ and $(\mathbb{Z},*) = U(\mathbb{Z},-)$ and $(\mathbb{Z},\circ) = A(\mathbb{Z},-)$. Thus we have a UA-factorization of $(\mathbb{Z},-)$.

A groupoid $(X,\bullet)$ is said to be *signature-prime* if $U(X,\bullet) = id_{Bin(X)}$, and is said to be *similar-prime* if $A(X,\bullet) = id_{Bin(X)}$. Alternatively, if $(X,\bullet)$ is neither *signature-* nor *similar*-prime, then $(X,\bullet)$ is said to be

(1) *UA-composite* if $(X,\bullet) = U(X,\bullet) \diamond A(X,\bullet)$;
(2) *AU-composite* if $(X,\bullet) = A(X,\bullet) \diamond U(X,\bullet)$.

Consequently, $(X,\bullet)$ is said to be *u-composite* if both (1) and (2) hold.

Example 3.1.2 Let $(X, \bullet) = (\mathbb{Z}_5, \bullet)$ where the product "$\bullet$" is defined by the following Cayley table:

$\bullet$	0	1	2	3	4
0	3	2	2	1	1
1	1	3	3	2	3
2	3	3	0	3	0
3	1	0	1	1	2
4	1	1	2	4	2

If we derive its *signature*- and *similar*- factors $(\mathbb{Z}_5, *) = U(\mathbb{Z}_5, \bullet)$ and $A(\mathbb{Z}_5, \bullet) = (\mathbb{Z}_5, \circ)$ as in (3.1), then we have their $\diamond$ product as follows:

$*$	0	1	2	3	4
0	0	2	2	1	1
1	1	1	3	2	3
2	3	3	2	3	0
3	1	0	1	3	2
4	1	1	2	4	4

$\diamond$

$\circ$	0	1	2	3	4
0	3	0	0	0	0
1	1	3	1	1	1
2	2	2	0	2	2
3	3	3	3	1	3
4	4	4	4	4	2

$=$

∇	0	1	2	3	4
0	3	2	2	3	3
1	1	3	1	2	3
2	3	1	0	3	0
3	3	0	1	1	2
4	3	1	2	4	2

We can clearly conclude that $U(\mathbb{Z}_5, \bullet) \diamond A(\mathbb{Z}_5, \bullet) \neq (\mathbb{Z}_5, \bullet)$ since $(\mathbb{Z}_5, \bullet) \neq (\mathbb{Z}_5, \nabla)$ and hence such a groupoid does not have a UA-factorization. Moreover, $(\mathbb{Z}_5, \bullet)$ is not a strong groupoid since $0 \bullet 4 = 4 \bullet 0$. In turn, we have the next theorem.

Theorem 3.1.3 *A strong groupoid has a UA-factorization.*

Proof. Let $(X, \bullet) \in Str\,(X)$, the collection of all strong groupoids defined on X, and let $(X, \odot) = (X, *) \diamond (X, \circ)$ where $(X, *) = U(X, \bullet)$ and $(X, \circ) = A(X, \bullet)$. Then $x \odot y = (x * y) \circ (y * x)$ for all $x, y \in X$. It follows that $x * x = x$, $x * y = x \bullet y$ when $x \neq y$; and $x \circ x = x \bullet x$, $x \circ y = x$ when $x \neq y$.
Next, we show that $(X, \bullet) = (X, \odot)$. Given $x, y \in X$, if $x = y$, then $x \odot x = (x * x) \circ (x * x) = x \circ x = x \bullet x$. Assume $x \neq y$, we claim that $x * y = y * x$ is not possible:

(i) If $x * y = y * x$, then $x \bullet y = x * y = y * x = y \bullet x$. Since $(X, \bullet)$ is strong, we obtain $x = y$, a contradiction.

(ii) If $x * y \neq y * x$, then $x * y = x \bullet y$, $y * x = y \bullet x$, since $x \neq y$.

Therefore $x \odot y = (x * y) \circ (y * x) = (x \bullet y) \circ (y \bullet x) = x \bullet y$, since $x \bullet y \neq y \bullet x$. This proves that $(X, \odot) = (X, \bullet)$.

■

Corollary 3.1.4 *The factorization in Theorem 3.1.3 is unique.*

Proof. Let $(X, \bullet)$ be a strong groupoid with a UA-factorization such that $(X, \bullet) = (X, *) \diamond (X, \circ)$ where $(X, *) = U(X, \bullet)$ and $(X, \circ) = A(X, \bullet)$. Let $(X, \bullet) = (X, \bigtriangledown) \diamond (X, \bigtriangleup)$ where $(X, \bigtriangledown) = U(X, \bullet)$ and $(X, \bigtriangleup) = A(X, \bullet)$. For any $x \in X$, we have $x * x = x = x \bigtriangledown x$, and $x * y = x \bigtriangledown y$ when $x \neq y$. Hence $(X, *) = (X, \bigtriangledown)$. Similarly, if $x \in X$, then $x \circ x = x \bullet x = x \bigtriangleup x$. When $x \neq y$, we have $x \circ y = x = x \bigtriangleup y$, proving that $(X, \circ) = (X, \bigtriangleup)$.

■

Example 3.1.5 [32] Consider the d-algebra $(X, \bullet, 0)$ from Example 2.10. Observe that $(X, \bullet, 0)$ is a strong d-algebra. Let $(X, *, 0) := U(X, \bullet, 0)$ and $(X, \circ, 0) :=$

$A(X,\bullet,0)$, such that $U(X,\bullet,0)$ and $A(X,\bullet,0)$ are its derived *signature*- and *similar*-factors, respectively, as in (3.1). Next, verify that $(X,*,0)\diamond(X,\circ,0)=(X,\bullet,0)$:

$$\begin{array}{c|ccccc} * & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 & 3 & 0 \\ 3 & 3 & 3 & 2 & 3 & 3 \\ 4 & 4 & 4 & 1 & 1 & 4 \end{array} \quad \diamond \quad \begin{array}{c|ccccc} \circ & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 2 & 2 \\ 3 & 3 & 3 & 3 & 0 & 3 \\ 4 & 4 & 4 & 4 & 4 & 0 \end{array} \quad = \quad \begin{array}{c|ccccc} \bullet & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 2 & 2 & 0 & 3 & 0 \\ 3 & 3 & 3 & 2 & 0 & 3 \\ 4 & 4 & 4 & 1 & 1 & 0 \end{array}$$

Indeed we can see that $x\bullet y=(x*y)\circ(y*x)$ for any $x,y\in X$. For instance:

$$\begin{aligned} (1*0)\circ(0*1) &= 1\circ 0=1=1\bullet 0, \\ (3*4)\circ(4*3) &= 3\circ 4=3=3\bullet 4. \end{aligned}$$

Moreover, since $U(X,\bullet,0)\neq id_{Bin(X)}$ and $A(X,\bullet,0)\neq id_{Bin(X)}$, then $(X,\bullet,0)$ is UA-composite.

3.2. AU**-Factorization.** In this subsection we reverse the order of the *signature*- and *similar*-factors of any groupoid $(X,\bullet)$ in $Bin(X)$. We conclude that an arbitrary groupoid $(X,\bullet)$ will always have an AU-factorization. However, this factorization might be trivial and hence the groupoid is either noted as *signature*- or *similar*-prime. Otherwise, if the decomposition is not trivial, we say the groupoid is AU-composite.

Example 3.2.1 Let $(X,\bullet,0)$ be the strong d-algebra defined in Examples 2.10 and 3.1.5 in which we determined that $(X,\bullet,0)$ is UA-composite. Similarly, we can take the product of $A(X,\bullet,0)$ and $U(X,\bullet,0)$ as follows:

$$\begin{array}{c|ccccc} \circ & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 2 & 2 \\ 3 & 3 & 3 & 3 & 0 & 3 \\ 4 & 4 & 4 & 4 & 4 & 0 \end{array} \quad \diamond \quad \begin{array}{c|ccccc} * & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 & 3 & 0 \\ 3 & 3 & 3 & 2 & 3 & 3 \\ 4 & 4 & 4 & 1 & 1 & 4 \end{array} \quad = \quad \begin{array}{c|ccccc} \bullet & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 2 & 2 & 0 & 3 & 0 \\ 3 & 3 & 3 & 2 & 0 & 3 \\ 4 & 4 & 4 & 1 & 1 & 0 \end{array}$$

By routine checking of $(x\circ y)*(y\circ x)=x\bullet y$ for any $x,y\in X$, we conclude that $(X,\bullet,0)$ has an AU-factorization. Moreover, we can see that this particular groupoid has both, a non-trivial UA- and AU-factorization. Therefore, $(X,\bullet,0)$ is u-composite.

Remark 3.2.2 Note that $A(X,\bullet)\diamond U(X,\bullet)=U(X,\bullet)\diamond A(X,\bullet)$ does not imply that $(X,\bullet)$ is u-composite. It simply implies that the factors of $(X,\bullet)$ commute. This motivates the next definition.

A groupoid $(X,\bullet)$ is said to be *u-normal* if it admits a UA- and an AU-factorization, i.e., if

(i) $(X,\bullet)=U(X,\bullet)\diamond A(X,\bullet)$, and
(ii) $(X,\bullet)=A(X,\bullet)\diamond U(X,\bullet)$.

Theorem 3.2.3 *Any given groupoid has an AU-factorization, i.e., if $(X,\bullet)\in Bin(X)$, then*

$$(X,\bullet)=A(X,\bullet)\diamond U(X,\bullet).$$

Proof. Let $(X,\bullet)\in Bin(X)$ and let $(X,\odot)=(X,\circ)\diamond(X,*)$ where $(X,*)=U(X,\bullet)$

and $(X, \circ) = A(X, \bullet)$. Then $x \odot y = (x \circ y) * (y \circ x)$ for all $x, y \in X$. It follows that $x * x = x$, $x * y = x \bullet y$ when $x \neq y$, and $x \circ x = x \bullet x$, $x \circ y = x$ when $x \neq y$. Given $x, y \in X$, if $x = y$, then $x \odot x = (x \circ x) * (x \circ x) = (x \bullet x) * (x \bullet x) = x \bullet x$. Assume $x \neq y$, then $x \odot y = (x \circ y) * (y \circ x) = x * y = x \bullet y$. This proves that $(X, \odot) = (X, \bullet)$.

■

Corollary 3.2.4 *The factorization in Theorem 3.2.3 is unique.*

Proof. The proof is similar to that of Corollary 3.1.4.

■

Corollary 3.2.5 *A strong groupoid is u-normal.*

Proof. The proof follows directly from Theorems 3.1.3, 3.2.3 and the definition.

■

Example 3.2.6 Let $(X, \bullet) = (\{0, 1, 2\}, +)$ be the cyclic group of order 3. Observe that $(\{0, 1, 2\}, +)$ has an AU-factorization but fails to have a UA-factorization. Take $(\{0, 1, 2\}, *) = U(\{0, 1, 2\}, +)$ and $(\{0, 1, 2\}, \circ) = A(\{0, 1, 2\}, +)$ such that:

$$x \circ y = \begin{cases} (x + x) \bmod 3 & \text{if } x = y, \\ x & \text{otherwise.} \end{cases} \quad \text{and} \quad x * y = \begin{cases} x; & \text{if } x = y, \\ (x + y) \bmod 3 & \text{otherwise.} \end{cases}$$

Routine checking of the product $A(\{0, 1, 2\}, +) \diamond U(\{0, 1, 2\}, +)$ gives $(\{0, 1, 2\}, +)$:

$$\begin{array}{c|ccc} \circ & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \end{array} \quad \diamond \quad \begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 2 \end{array} \quad = \quad \begin{array}{c|ccc} + & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{array}$$

But, the product $U(\{0, 1, 2\}, +) \diamond A(\{0, 1, 2\}, +)$ does not give $(\{0, 1, 2\}, +)$:

$$\begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 2 \end{array} \quad \diamond \quad \begin{array}{c|ccc} \circ & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \end{array} \quad = \quad \begin{array}{c|ccc} \nabla & 0 & 1 & 2 \\ \hline 0 & 0 & 2 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 1 \end{array}$$

Therefore, $(\{0, 1, 2\}, +)$ is not u-normal, it is simply AU-composite.

Proposition 3.2.7 *Any signature- or similar-prime groupoid is u-normal.*

Proof. The proof is straightforward and we omit it.

■

Proposition 3.2.8 *The right-zero-semigroup on X is similar-prime.*

Proof. Let $(X, \bullet)$ be the right-zero-semigroup on X. Then $x \bullet y = y$ for all $x, y \in X$. Let $(X, *) = U(X, \bullet)$ and $(X, \circ) = A(X, \bullet)$, thus

$$x * y = \begin{cases} x & \text{if } x = y, \\ x \bullet y = y & \text{otherwise} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} x \bullet x = x & \text{if } x = y \\ x \circ y = x & \text{otherwise} \end{cases}$$

Hence for all $x, y \in X$, $(X, \bullet) = (X, \bullet) \diamond id_{Bin(X)}$.

■

Example 3.2.9 Let $(X, \bullet) = (\{a, b, c\}, \bullet)$ be the right-zero-semigroup on $\{a, b, c\}$. Its Cayley table together with its associated *signature-similar*-product tables, respectively, are:

$$\begin{array}{c|ccc} \bullet & a & b & c \\ \hline a & a & b & c \\ b & a & b & c \\ c & a & b & c \end{array}$$

$$\begin{array}{c|ccc} * & a & b & c \\ \hline a & a & b & c \\ b & a & b & c \\ c & a & b & c \end{array} \quad \diamond \quad \begin{array}{c|ccc} \circ & a & b & c \\ \hline a & a & a & a \\ b & b & b & b \\ c & c & c & c \end{array} \quad = \quad \begin{array}{c|ccc} \bullet & a & b & c \\ \hline a & a & b & c \\ b & a & b & c \\ c & a & b & c \end{array}$$

Therefore, the right-zero-semigroup of order 3 is *similar*-prime since its *similar*-factor $A(\{a, b, c\}, \bullet)$ is $id_{Bin(X)}$, i.e., the left-zero-semigroup for $\{a, b, c\}$.

Proposition 3.2.10 *A non-locally-zero strong groupoid is u-composite.*

Proof. Let $(X, \bullet) \in Bin(X) - ZBin\,(X)$, then $x \bullet y \neq \{x, y\}$ for any $x, y \in X$. Meaning, $(X, \bullet)$ cannot be the left- nor the right-zero-semigroup on X. By Proposition 3.2.5, $(X, \bullet)$ is u-normal. Let $(X, *) = U\,(X, \bullet)$ and $(X, \circ) = A\,(X, \bullet)$, then

$$x * y = \begin{cases} x & \text{if } x = y, \\ x \bullet y & \text{otherwise} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} x \bullet x & \text{if } x = y \\ x \circ y = x & \text{otherwise} \end{cases}$$

Hence, for all $x, y \in X$, $(X, *) \neq (X, \bullet) \neq (X, \circ)$ and $(X, *) \neq id_{Bin(X)} \neq (X, \circ)$. Therefore, $(X, \bullet)$ is u-composite.

3.3. **Factoring U $(X, \bullet)$ and A $(X, \bullet)$.** Let $Str\,(X)$ be the collection of all strong groupoids on a non-empty set X. Consider a groupoid $(X, \bullet) \in Str\,(X)$, we classify the *signature*- and *similar*-factors of $(X, \bullet)$ as UA-composite, *signature*- or *similar*-prime. We conclude that $U\,(X, \bullet)$ and $A\,(X, \bullet)$ are *similar*- and *signature*-prime, respectively.

Theorem 3.3.1 *The signature-factor of a strong groupoid is similar-prime, and the similar-factor is signature-prime.*

Proof. Let $(X, \bullet) \in Str\,(X)$. Suppose that $(X, *) = U\,(X, \bullet)$ and $(X, \circ) = A\,(X, \bullet)$. Let $(X, \circledast) = U\,(X, *)$ and $(X, \odot) = A\,(X, *)$, then "$\circledast$" and "$\odot$" are defined as:

$$x \circledast y = \begin{cases} x; & \text{if } x = y, \\ x * y = x \bullet y & \text{otherwise} \end{cases} \quad \text{and} \quad x \odot y = \begin{cases} x * x = x & \text{if } x = y, \\ x; & \text{otherwise.} \end{cases}$$

Hence $A\,(X, *) = id_{Bin(X)}$, and therefore $U\,(X, \bullet)$ is *similar*-prime. Similarly, if we let $(X, \boxtimes) = U\,(X, \circ)$ and $(X, \boxdot) = A\,(X, \circ)$, then "$\boxtimes$" and "$\boxdot$" are defined as:

$$x \boxtimes y = \begin{cases} x & \text{if } x = y, \\ x \circ y = x & \text{otherwise} \end{cases} \quad \text{and} \quad x \boxdot y = \begin{cases} x \circ x = x \bullet x & \text{if } x = y, \\ x; & \text{otherwise.} \end{cases}$$

Therefore, $U\,(X, \circ) = id_{Bin(X)}$, and hence $A\,(X, \bullet)$ is *signature*-prime.

■

Corollary 3.3.2. *Let* $(X, \bullet)$ *be any groupoid and let* $(X, *) = U(X, \bullet)$ *and* $(X, \circ) = A(X, \bullet)$*. If* $(X, \bullet)$ *has a* UA*-factorization, i.e., if* $(X, \bullet) = (X, *) \diamond (X, \circ)$*, then*

$$(X, \bullet) = U(X, *) \diamond A(X, \circ).$$

Proof. This follows immediately from the previous theorem. In fact, suppose $(X, \bullet)$ has a UA-factorization, then

$$\begin{aligned}(X, \bullet) &= (X, *) \diamond (X, \circ) \\ &= (U(X, *) \diamond A(X, *)) \diamond (U(X, \circ) \diamond A(X, \circ)) \\ &= \left(U(X, *) \diamond id_{Bin(X)}\right) \diamond \left(id_{Bin(X)} \diamond A(X, \circ)\right) \\ &= U(X, *) \diamond A(X, \circ).\end{aligned}$$

■

Corollary 3.3.3. *Let* $(X, \bullet)$ *be a groupoid and let* $(X, *) = U(X, \bullet)$ *and* $(X, \circ) = A(X, \bullet)$*. If* $(X, \bullet)$ *has a* AU*-factorization then*

$$(X, \bullet) = A(X, \circ) \diamond U(X, *).$$

Proof. The proof is very similar to that of the previous Corollary.

■

Corollary 3.3.4. *Let* $(X, \bullet)$ *be a strong groupoid and let* $(X, *) = U(X, \bullet)$ *and* $(X, \circ) = A(X, \bullet)$*, then*

$$(X, \bullet) = A(X, \circ) \diamond U(X, *) = U(X, *) \diamond A(X, \circ).$$

Proof. This is a direct result of Theorem 3.1.3 and the previous two Corollaries.

■

As a final observation, a groupoid is *similar*-prime if it is similar to the left-zero-semigroup or a locally-zero-groupoid, in other words, if it is idempotent. Hence, we need another method of factorization for idempotent groupoids.

4. Orient-Skew Factorization

We say a groupoid $(X, *)$ has the *orientation property OP* [33] if $x * y \in \{x, y\}$ for all $x, y \in X$. Moreover, $(X, *)$ has the *twisted orientation property TOP* if $x * y = x$ implies $y * x = x$ for all $x, y \in X$. In this section, we introduce a unique factorization which can be applied to groupoids with *OP*. This type of groupoids has proven to be useful in graph theory, where in a directed graph $x * y = x$ can mean there is a path from vertex x to vertex y, i.e. $x \to y$; while $x * y = y$ can mean there is no path from x to y, i.e. $x \nrightarrow y$. In fact, if $\Gamma_{(X,*)}$ is the directed graph on vertex set X and $(X, *) \in TOP(X)$, then $\Gamma_{(X,*)}$ is a simple graph [1]. For more details on groupoids associated with directed and simple graphs we refer to [1, 35].

Example 4.1 Let $X = \{0, 1\}$ and $(X, \leq)$ be a linearly ordered set. Define a binary operation "$\bullet$" on X such that:

$$x \bullet y = \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{otherwise.} \end{cases}$$

Then the binary system $(X, \bullet)$ has the orientation property.

Example 4.2 Let $X = \{a, b, c\}$. Define a binary operation "$\bullet$" on X by the following table:

$\bullet$	a	b	c
a	a	b	c
b	b	b	c
c	c	b	c

Then $(X, \bullet)$ has the twisted orientation property.

We consider three functions to represent operations on the main diagonal and on the anti-diagonal of the associated Cayley table of a binary operation on a finite set.

Let $(X, *)$ be a groupoid of finite order n and binary operation "$*$", i.e., $|X| = n$ and $* : X^2 \longrightarrow X$. Then for all $x_i,\, x_j \in X,\ i, j = 1, 2, ..., n$, and $i + j = n + 1$, we call:

- **diag-1:** $\overline{d^*}$ the *anti-diagonal function* of $(X, *)$ such that $\overline{d^*} : \mathbb{N} \longrightarrow X$, defined by $\overline{d^*}\ (i) = x_i * x_j$.
- **diag-2:** $\widehat{d^*}$ the *reverse-diagonal function* of $(X, *)$ such that $\widehat{d^*} : \mathbb{N} \longrightarrow X$, defined by $\widehat{d^*}\ (i) = x_j * x_j$.
- **diag-3:** $\widetilde{d^*}$ the *skew-diagonal function* of $(X, *)$ such that $\widetilde{d^*} : \mathbb{N} \longrightarrow X$, defined by $\widetilde{d^*}\ (i) = \widehat{\overline{d^*}}(i) = x_j * x_i$.

Example 4.3 Consider the groupoid $(\{0, 1, 2, 3\}, *)$ where "$*$" is given by the following table:

$*$	0	1	2	3
0	0	1	0	3
1	1	1	1	0
2	2	2	2	3
3	0	3	2	3

Observe that $n = 4$ and the main diagonal $d^* = \{0, 1, 2, 3\}$. For instance, $d^*(2) = 2 * 2 = 2$. Also, the anti-diagonal $\overline{d^*} = \{3, 1, 2, 0\}$. For example, $\overline{d^*}\ (1) = x_1 * x_4 = 0 * 3 = 3$. Moreover, the reverse of the diagonal is $\widehat{d^*} = \{3, 2, 1, 0\}$. For instance, $\widehat{d^*}$ $(4) = x_1 * x_1 = 0 * 0 = 0$. So the skew-diagonal defined here is the reverse of the anti-diagonal, hence, $\widetilde{d^*} = \{0, 2, 1, 3\}$. For example, $\widetilde{d^*}\ (3) = \widehat{\overline{d^*}}(3) = x_2 * x_3 = 1 * 2 = 1$.

Given these definitions, we can *derive* the *orient*-factor of a groupoid from $id_{Bin(X)}$, such that all its elements are the same as those of the left-zero-semigroup except elements belonging to the anti-diagonal, which we construct from the skew-diagonal of $id_{Bin(X)}$. Similarly, the *skew*-factor is *derived* from the parent groupoid by letting its anti-diagonal be that of the skew-diagonal of the parent groupoid, otherwise all other elements are kept the same as the parent groupoid.

Let $(X, \bullet)$ be a groupoid. Let $D^\diamond$ denote the main diagonal of $id_{Bin(X)}$. *Derive* groupoids $(X, *)$ and $(X, \circ)$ from $id_{Bin(X)}$ and $(X, \bullet)$, respectively, as follows:
For all $x, y \in X$,

$$\text{(4.1)} \qquad \begin{array}{l} \text{(i) } \overline{d^*} = \widetilde{D^\diamond}, \\ \text{(ii) } x * y = x; \text{ otherwise.} \end{array} \qquad \text{and} \qquad \begin{array}{l} \text{(i) } \overline{d^\circ} = \widetilde{d^\bullet}, \\ \text{(ii) } x \circ y = x \bullet y; \text{ otherwise.} \end{array}$$

Groupoids $(X,*)$ and $(X,\circ)$ are said to be the *orient-* and *skew-factor* of $(X,\bullet)$, respectively, denoted by $O(X,\bullet)$ and $J(X,\bullet)$. As previously mentioned, the product "$\diamond$" is not commutative. Hence, for $(X,\bullet)\in Bin(X)$, we may have an *OJ-factorization* such that

$$(X,\bullet) = O(X,\bullet)\diamond J(X,\bullet) \tag{4.2}$$

or a *JO-factorization* such that

$$(X,\bullet) = J(X,\bullet)\diamond O(X,\bullet). \tag{4.3}$$

Proposition 4.4 *The orient-factor of a given groupoid is locally-zero.*

Proof. Given $(X,\bullet)\in Bin(X)$, let $(X,*) = O(X,\bullet)$. Then, $d^* = D^\diamond$, i.e. $x*x = x$, and $x*y = x$ for all $x,\, y\in X$ except when $x,\, y\in \overline{d^*}$. In fact, for any $x\neq y$ in X, $(\{x,y\},\bullet)$ is either a left-zero-semigroup or a right-zero-semigroup. Moreover, $x\bullet x = x$ for all $x\in X$ which implies that $O(X,\bullet)$ is locally-zero.

■

Corollary 4.5 *The orient-factor of a given groupoid is a unit in $Bin(X)$.*

Proof. This follows immediately from Propositions 2.8 and 4.4.

■

Example 4.6 Let $X = \{e,a,b,c\}$. Define a binary operation "$\bullet$" by the following table:

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

Then, clearly $(X,\bullet,e)$ is a group. Derive its *orient*-factor $(X,*,e) = U(X,\bullet,e)$ as in 4.1 to obtain:

$*$	e	a	b	c
e	e	e	e	c
a	a	a	b	a
b	b	a	b	b
c	e	c	c	c

Hence, $(X,*,e)$ is locally-zero.

4.1. OJ-Factorization.

In this subsection, we explore an OJ-factorization of any groupoid $(X,\bullet)$ in $Bin(X)$, i.e., into its *orient-* and *skew*-factors, respectively. The next subsection discusses a JO-factorization where the product of the two factors is "reversed". Then, we classify $(X,\bullet)$ as OJ- and/or *JO-composite*, *j-composite* or *j-normal;* and as *orient-* or *skew-prime.*

A groupoid $(X,\bullet)$ is *bi-diagonal* if its anti-diagonal is symmetric, meaning if $\overline{d^\bullet} = \widetilde{d^\bullet}$.

Example 4.1.1. Let $(\mathbb{Z},<)$ be a linearly ordered set. Consider groupoid $(\mathbb{Z},\bullet)$ where $x\bullet y = \max\{x,y\}$ for all $x,y\in\mathbb{Z}$. Define two binary operations on $\mathbb{Z}$ such that:

$$x*y = \begin{cases} x & \text{if } x<y,\\ y & \text{otherwise.}\end{cases} \quad \text{and} \quad x\circ y = \begin{cases} x & \text{if } x\leq y,\\ y & \text{otherwise.}\end{cases}$$

Then clearly $(X,*)\diamond(X,\circ)$ is an OJ-factorization of $(X,\bullet)$, where $(X,*)=O(X,\bullet)$ and $(X,\circ)=J(X,\bullet)$. Moreover, $(\mathbb{Z},\ \bullet)$ is bi-diagonal.

A groupoid $(X,\bullet)$ is said to be *orient-prime* if $O\,(X,\bullet)=id_{Bin(X)}$, and is said to be *skew-prime* if $J\,(X,\bullet)=id_{Bin(X)}$. Alternatively, if $(X,\bullet)$ is neither *orient*- nor *skew*-prime, then $(X,\bullet)$ is said to be

(1) *OJ-composite* if $(X,\bullet)=O\,(X,\bullet)\diamond J\,(X,\bullet)$;

(2) *JO-composite* if $(X,\bullet)=J\,(X,\bullet)\diamond O\,(X,\bullet)$.

Consequently, $(X,\bullet)$ is said to be *j-composite* if both (1) and (2) hold.

Just as with UA-factorization, not every groupoid will have a JO-factorization. But it is possible to *derive* an OJ-factorization of any given groupoid.

Theorem 4.1.2 *Any given groupoid has an OJ-factorization, i.e., if $(X,\bullet)\in Bin(X)$, then*

$$(X,\bullet)=O(X,\bullet)\diamond J(X,\bullet).$$

Proof. Let $(X,\bullet)\in Bin\,(X)$ such that $O(X,\bullet)$ and $J(X,\bullet)$ are defined as in 4.1. Let $(X,\odot)=(X,*)\ \diamond(X,\circ)$ where $(X,*)=O(X,\bullet)$ and $(X,\circ)=J(X,\bullet)$. Then $x\odot y=(x*y)\circ(y*x)$ for all $x,y\in X$. It follows that

(i) If $x=y$, $x*x=x$ and $x\circ x=x\bullet x$.

(ii) If $x\neq y$, then if $x*y\in\overline{d^*}$, $x*y\in\widetilde{D^\diamond}$, and for $x\circ y\in\overline{d^\circ}$, then $x\circ y\in\widetilde{d^\bullet}$. Otherwise, $x*y=x$, and $x\circ y=x\bullet y$.

Next, we show that $(X,\bullet)=(X,\odot)$. Given $x,y\in X$,

(i) If $x=y$, $x\odot x=(x\circ x)*(x\circ x)=x\circ x=x\bullet x$.

(ii) If $x\neq y$, then if $x*y=y*x$, then $x\odot y=(x*y)\circ(y*x)=x\circ y=x\bullet y$ and $y\odot x=(y*x)\circ(x*y)=y\circ x=x\bullet y$. If $x*y\neq y*x$, then $x\odot y=(x*y)\circ(y*x)=(x*y)\bullet(y*x)\in\{x\bullet y,\ y\bullet x\}$.

Thus, $x\odot y=x\bullet y$ for all $x,y\in X$. This proves that $(X,\odot)=(X,\bullet)$.

■

Corollary 4.1.3 *The factorization in Theorem 4.1.2 is unique.*

Proof. Let $(X,\bullet)\in Bin\,(X)$ with an OJ-factorization such that $(X,\bullet)=(X,*)\ \diamond(X,\circ)$ where $(X,*)=O(X,\bullet)$ and $(X,\circ)=J(X,\bullet)$. Let $(X,\bullet)=(X,\bigtriangledown)\ \diamond(X,\bigtriangleup)$ where $(X,\bigtriangledown)=O(X,\bullet)$ and $(X,\bigtriangleup)=J(X,\bullet)$. For any $x\in X$, we have $x*x=x=x\bigtriangledown x$, and $x*y=x\bigtriangledown y$ when $x\neq y$. Hence $(X,*)=(X,\bigtriangledown)$. Similarly, if $x\in X$, then $x\circ x=x\bullet x=x\bigtriangleup x$. When $x\neq y$, we have $x\circ y=x\bullet y=x\bigtriangleup y$, proving that $(X,\circ)=(X,\bigtriangleup)$.

■

Example 4.1.4 [32] Consider the groupoid $(X,\bullet)=(\{1,2,3,4\},\bullet)$ where "$\bullet$" is defined by the following Cayley table:

$\bullet$	1	2	3	4
1	1	1	3	1
2	2	2	3	2
3	1	2	3	4
4	4	4	3	4

By *deriving* its *orient*- and *skew*-factors $O\,(X,\bullet)$ and $J\,(X,\bullet)$, respectively, and by letting $(X,*)=O\,(X,\bullet)$ and $(X,\circ)=J\,(X,\bullet)$ shows that $(X,*)\diamond(X,\circ)=(X,\bullet)$. Indeed, $(X,\bullet)$ has an OJ-factorization:

$*$	1	2	3	4
1	1	1	1	4
2	2	2	3	2
3	3	2	3	3
4	1	4	4	4

$\diamond$

$\circ$	1	2	3	4
1	1	1	3	4
2	2	2	2	2
3	1	3	3	4
4	1	4	3	4

$=$

$\bullet$	1	2	3	4
1	1	1	3	1
2	2	2	3	2
3	1	2	3	4
4	4	4	3	4

Also, since $O(X,\bullet) \neq id_{Bin(X)} \neq J(X,\bullet)$, then $(X,\bullet)$ is OJ-composite.

4.2. JO**-Factorization.** In this subsection, we reverse the product of the *orient*- and *skew*-factors of a given groupoid $(X,\bullet) \in Bin(X)$. We find that an arbitrary groupoid admits a JO-factorization if it has the orientation property.

Example 4.2.1 Consider the groupoid $(X,\bullet) = (\{1,2,3,4\},\bullet)$ defined as in Example 4.1.4:

$\bullet$	1	2	3	4
1	1	1	3	1
2	2	2	3	2
3	1	2	3	4
4	4	4	3	4

Through routine calculations, we find that $(X,\bullet)$ admits a JO-factorization since $J(X,\bullet) \diamond O(X,\bullet) = (X,\circ) \diamond (X,*) = (X,\bullet)$. In addition, $(X,\bullet) \in OP(X)$.

A groupoid $(X,\bullet)$ is said to be j*-normal* if it admits an OJ- and a JO-factorization, i.e., if

(i) $(X,\bullet) = O(X,\bullet) \diamond J(X,\bullet)$ and
(ii) $(X,\bullet) = J(X,\bullet) \diamond O(X,\bullet)$.

Theorem 4.2.3 *A groupoid $(X,\bullet)$ with the orientation property has a JO-factorization.*

Proof. Let $(X,\bullet) \in OP(X)$. Define $(X,\odot) = (X,*) \diamond (X,\circ)$ where $(X,*) = O(X,\bullet)$ and $(X,\circ) = J(X,\bullet)$. Then $x \odot y = (x*y) \circ (y*x)$ for all $x,y \in X$. It follows that

(i) If $x = y$, then $x * x = x$ and $x \circ x = x \bullet x$.
(ii) If $x \neq y$, the two cases arise: if $x*y \in \overline{d^*}$ and $x \circ y \in \overline{d^\circ}$, then $x * y \in \widetilde{D^\diamond}$ and $x \circ y \in \widetilde{d^\bullet}$ which also $\in \{x,\ y\}$. Otherwise, $x*y = x$,and $x \circ y = x \bullet y$.

Next, we show that $(X,\bullet) = (X,\odot)$. Given $x,y \in X$,

(i) If $x = y$, then $x \odot x = (x \circ x) * (x \circ x) = x * x = x \bullet x$.
(ii) If $x \neq y$, then $x \odot y = (x \circ y) * (y \circ x)$. If $x \circ y = y \circ x$, then $x \odot y = (x \circ y) * (x \circ y) = x \circ y = x \bullet y$. If $x \circ y \neq y \circ x$, then $x \odot y = (x*y) \bullet (y*x) \in \{x \bullet y,\ y \bullet x\}$.

Thus $x \odot y = x \bullet y$ for all $x,y \in X$. This proves that $(X,\odot) = (X,\bullet)$. ■

Corollary 4.2.4 *The factorization in Theorem 4.2.3 is unique.*

Proof. The proof is very similar to that of Corollary 4.1.3 so we omit it. ■

Proposition 4.2.5 *A groupoid with the orientation property is j-normal.*

Proof. The result follows from Theorems 4.1.2, 4.2.3 and the definition. ■

Example 4.2.6 Let $(X, \bullet)$ be defined as in Example 4.2.1 where we determined that $(X, \bullet)$ admits an OJ-factorization. It can be verified that $J(X, \bullet) \diamond O(X, \bullet) = (X, \bullet)$, which shows that $(X, \bullet)$ admits a JO-factorization as well. Therefore, $(X, \bullet)$ is j-normal in $(Bin(X), \diamond)$. Additionally, $J(X, \bullet) \neq id_{Bin(X)} \neq O(X, \bullet)$ implies that $(X, \bullet)$ is j-composite.

4.3. **Factoring $\mathbf{O}(X, \bullet)$ and $\mathbf{J}(X, \bullet)$.** In this subsection, the *orient*- and *skew*-factors of $(X, \bullet) \in OP(X)$ are factored to deduce that $O(X, \bullet)$ is *skew*-prime while $J(X, \bullet)$ is *binary*-equivalent to $(X, \bullet)$.

Let $(X, \bullet)$ and $(X, \circ)$ be groupoids in $Bin(X)$. We say that $(X, \circ)$ is *binary-equivalent* to $(X, \bullet)$ if there exists $(X, *) \in Bin(X)$ such that
(i) $(X, \bullet) = (X, *) \diamond (X, \circ)$; and
(ii) $(X, \circ) = (X, *) \diamond (X, \bullet)$.

Theorem 4.3.1 *Given a groupoid $(X, \bullet)$ with the orientation property. Its orient-factor is skew-prime, and its skew-factor is binary-equivalent to $(X, \bullet)$.*

Proof. Let $(X, \bullet) \in OP(X)$. Suppose that $(X, *) = O(X, \bullet)$ and $(X, \circ) = J(X, \bullet)$. Then by Theorem 4.1.2 $(X, \bullet) = O(X, \bullet) \diamond J(X, \bullet) = (X, *) \diamond (X, \circ)$. Let $(X, \circledast) = O(X, *)$ and $(X, \odot) = J(X, *)$, then for $\circledast$: (i) $\overline{d^{\circledast}} = \widetilde{D^{\diamond}}$, (ii) $x \circledast y = x$, otherwise; and for $\odot$: (i) $\overline{d^{\odot}} = \widetilde{d^{*}} = D^{\diamond}$, (ii) $x \odot y = x * y = x$, otherwise. Hence,

$$(X, *) = (X, *) \diamond id_{Bin(X)}$$

and $O(X, \bullet)$ is *skew*-prime. Similarly, if we let $(X, \boxtimes) = O(X, \circ)$ and $(X, \boxdot) = J(X, \circ)$, then for $\boxtimes$: (i) $\overline{d^{\boxtimes}} = \widetilde{D^{\diamond}}$, (ii) $x \boxtimes y = x$, otherwise; and for $\boxdot$: (i) $\overline{d^{\boxdot}} = \widetilde{d^{\circ}} = \overline{d^{\bullet}}$, (ii) $x \boxdot y = x \circ y = x \bullet y$, otherwise. Thus,

$$(X, \circ) = (X, *) \diamond (X, \bullet)$$

and the final result follows.

■

Example 4.3.2 Consider the locally-zero groupoid $(X, \bullet) = (\{0, 1, 2, 3, 4, 5\}, \bullet)$ where "$\bullet$" is defined by the following Cayley table:

$\bullet$	0	1	2	3	4	5
0	0	1	0	0	4	0
1	0	1	2	3	1	5
2	2	1	2	3	4	2
3	3	1	2	3	3	3
4	0	4	2	4	4	4
5	5	1	5	5	5	5

Since $(X, \bullet)$ has the orientation property, then $(X, \bullet)$ is j-normal by Proposition 4.2.5.
Factoring its *orient*- and *skew*-factors $(X, *) = O(X, \bullet)$ and $(X, \circ) = J(X, \bullet)$ into their respective *orient*- and *skew*-factors, $O(X, *)$, $J(X, *)$ and $O(X, \circ)$, $J(X, \circ)$,

is observed through their respective product tables:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	1	1	1	4	1
2	2	2	2	3	2	2
3	3	3	2	3	3	3
4	4	1	4	4	4	4
5	0	5	5	5	5	5

$=$

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	1	1	1	4	1
2	2	2	2	3	2	2
3	3	3	2	3	3	3
4	4	1	4	4	4	4
5	0	5	5	5	5	5

$\diamond$

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5

$\circ$	0	1	2	3	4	5
0	0	1	0	0	4	5
1	0	1	2	3	4	5
2	2	1	2	2	4	2
3	3	1	3	3	3	3
4	0	1	2	4	4	4
5	0	1	5	5	5	5

$=$

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	1	1	1	4	1
2	2	2	2	3	2	2
3	3	3	2	3	3	3
4	4	1	4	4	4	4
5	0	5	5	5	5	5

$\diamond$

$\bullet$	0	1	2	3	4	5
0	0	1	0	0	4	0
1	0	1	2	3	1	5
2	2	1	2	3	4	2
3	3	1	2	3	3	3
4	0	4	2	4	4	4
5	5	1	5	5	5	5

Indeed, $(X,*) = O(X,*) \diamond J(X,*) = (X,*) \diamond id_{Bin(X)}$ and $(X,\circ) = O(X,\circ) \diamond J(X,\circ) = (X,*) \diamond (X,\bullet)$. This clearly shows the results of Theorem 4.3.1.

Theorem 4.3.3 *The right-zero-semigroup on X is j-composite.*

Proof. Let $(X,\bullet)$ be the right-zero-semigroup on X. Suppose that $(X,*) = O(X,\bullet)$ and $(X,\circ) = J(X,\bullet)$. By applying Proposition 4.2.5, $(X,\bullet)$ is j-normal. Thus, $(X,\bullet) = (X,*)\diamond(X,\circ) = (X,\circ)\diamond(X,*)$. Consider $(X,*)$: (i) $\overline{d^*} = \widetilde{D^\diamond}$, (ii) $x*y = x$, otherwise; and for $(X,\circ)$: (i) $\overline{d^\circ} = \widetilde{d^\bullet}$, (ii) $x \circ y = x \bullet y = y$, otherwise. Since neither one of the factors is the left-zero-semigroup for $Bin(X)$, $(X,\bullet)$ is j-composite.

■

Example 4.3.4 Let $(X,\bullet)$ be the right-zero-semigroup as in Example 3.2.9 where $X = \{a,b,c\}$. Let $(X,*) = O(X,\bullet)$ and $(X,\circ) = J(X,\bullet)$, we can check that $(X,\bullet)$ is in fact OJ- and JO-composite. Hence, $(X,\bullet)$ is j-composite:

$*$	a	b	c
a	a	a	c
b	b	b	b
c	a	c	c

$\diamond$

$\circ$	a	b	c
a	a	b	a
b	a	b	c
c	c	b	c

$=$

$\bullet$	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

Moreover, its *orient*-factor $(X,*)$ has the following subtables:

$*$	a	b
a	a	a
b	b	b

$*$	a	c
a	a	c
c	a	c

$*$	b	c
b	b	b
c	c	c

which implies that $(X,*)$ is locally-zero.

Given two distinct groupoids $(X,\triangleright)$ and $(X,\triangleleft)$ in $Bin(X)$. Suppose that $(X,\triangleright) \neq id_{Bin(X)}$ and $(X,\triangleleft) \neq id_{Bin(X)}$. Let $(X,\bullet)$ be a groupoid such that $(X,\triangleright) \neq (X,\bullet) \neq (X,\triangleleft)$. Then $(X,\bullet)$ is said to be:

(i) *partially-right-prime,* ∂_r-prime, if $(X,\bullet) = (X,\bullet) \diamond (X,\triangleright)$;

(ii) *partially-left-prime,* ∂_l-prime, if $(X,\bullet) = (X,\triangleleft) \diamond (X,\bullet)$.

Whence $(X,\triangleright)$ and $(X,\triangleleft)$ behave like *right*- and *left*-identities respectively. Here, $(X,\triangleright)$ and $(X,\triangleleft)$ could be either $O(X,\bullet)$, $J(X,\bullet)$, $U(X,\bullet)$, $A(X,\bullet)$ or any other factor of $(X,\bullet)$. The next proposition demonstrates one such case.

Proposition 4.3.5 *A bi-diagonal groupoid is partially-left-prime.*

Proof. Given a bi-diagonal groupoid $(X, \bullet)$, then its *skew*-factor $J(X, \bullet) = (X, \bullet)$ since $\overline{d^{\circ}} = \widetilde{d^{\bullet}} = \overline{d^{\bullet}}$ and $x \circ y = x \bullet y$ otherwise. Meanwhile, its *orient*-factor $O(X, \bullet)$ is not affected by the bi-diagonal property. By Theorem 4.1.2, $(X, \bullet)$ has an OJ-factorization,

$$\begin{aligned} (X, \bullet) &= O(X, \bullet) \diamond J(X, \bullet) \\ &= O(X, \bullet) \diamond (X, \bullet). \end{aligned}$$

Therefore, $O(X, \bullet)$ is a left-identity in $(Bin\,(X)\,, \diamond)$ and the result follows.

■

Example 4.3.6 Consider the group $(X, \bullet, e)$ as defined in Example 4.5. Then clearly $(X, \bullet, e)$ is bi-diagonal. Recall its *orient*-factor $(X, *, e) = O(X, \bullet, e)$ and derive its *skew*-factor $(X, \circ, e) = J(X, \bullet, e)$ to obtain:

$*$	e	a	b	c
e	e	e	e	c
a	a	a	b	a
b	b	a	b	b
c	e	c	c	c

and

$\circ$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

Then $(X, \bullet, e) = O(X, \bullet, e) \diamond (X, \bullet, e)$ and therefore the group $(X, \bullet, e)$ is ∂_l-prime.

5. Application

Recall some of the algebras described in "**Figure 1**" of Section 2.

We shall say an algebra $(X, \bullet, 0)$ of type $(2, 0)$ is a *strong* $B1$-*algebra* if it satisfies (B1) and equation 2.1. Meaning, if for all $x, y \in X$,

(i) $x \bullet x = 0$,
(ii) $x \bullet y = y \bullet x$ implies $x = y$.

A groupoid $(X, \bullet, 0)$ is *semi-neutral* if for all $x, y \in X$,

(i) $x \bullet x = 0$,
(ii) $x \bullet y = x$.

A $B1$-algebra $(X, \bullet, 0)$ is *semi-neutral* if for $x \neq y$, $x \bullet y = x$ for all $x, y \in X$.

A normal/composite groupoid is *semi-normal (resp., semi-composite)* if only one of its factors is semi-neutral.

Proposition 5.1 *A semi-neutral groupoid is signature-prime and* OJ*-composite.*

Proof. Let $(X, \bullet, 0)$ be the semi-neutral groupoid on X. Then $x \bullet y = x$ for all $x, y \in X$ and $x \bullet x = 0$. Let $(X, *, 0) = U\,(X, \bullet, 0)$ and $(X, \circ, 0) = A\,(X, \bullet, 0)$, its *signature*- and *similar*-factors, respectively. Deriving them according to 3.1 gives:

$$x * y = \begin{cases} x & \text{if } x = y, \\ x \bullet y = x & \text{otherwise.} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} x \bullet x = 0 & \text{if } x = y, \\ x & \text{otherwise.} \end{cases}$$

Hence for all $x, y \in X$, $(X, \bullet, 0) = id_{Bin(X)} \diamond (X, \bullet, 0)$.
By Theorem 4.1.2, $(X, \bullet, 0)$ has an OJ-factorization. Let $(X, \circledast, 0) = O\,(X, \bullet, 0)$ and $(X, \odot, 0) = J\,(X, \bullet, 0)$, its *orient*- and *skew*-factors, respectively. Deriving them according to 4.1 gives: for $\circledast$: (i) $\overline{d^{\circledast}} = \widetilde{D^{\diamond}}$, (ii) $x \circledast y = x$, otherwise; and for $\odot$: (i)

$\overline{d^{\odot}} = \widetilde{d^{\bullet}} \neq D^{\diamond}$, (ii) $x \odot y = x \bullet y$, otherwise. Thus, $(X,*,0) \neq id_{Bin(X)} \neq (X,\circ,0)$ and $(X,*,0) \neq (X,\bullet,0) \neq (X,\circ,0)$.

■

Corollary 5.2 *A semi-neutral groupoid is semi-normal.*

Proof. This is a direct result of Proposition 5.1 and the definition of a semi-normal groupoid.

■

Proposition 5.3 *The product of semi-neutral groupoids is semi-neutral.*

Proof. Consider semi-netural groupoids $(X,*,0)$ and $(X,\circ,0)$. Let $(X,*,0) \diamond (X,\circ,0) = (X,\bullet,0)$ such that $x \bullet y = (x * y) \circ (y * x)$. Then, $x \bullet x = (x * x) \circ (x * x) = 0$. If $x \neq y$,.$x \bullet y = x \circ y$. It follows that $(X,\bullet,0) = (X,\circ,0)$ and therefore is semi-neutral.

■

Proposition 5.4 *The similar-factor of a $B1$-algebra is semi-neutral.*

Proof. Let $(X,\bullet,0)$ be a $B1$-algebra. Consider the AU-factorization $(X,\bullet,0) = A(X,\bullet,0) \diamond U(X,\bullet,0)$. Let $(X,*,0) := U(X,\bullet,0)$ and $(X,\circ,0) := A(X,\bullet,0)$, its *signature*- and *similar*-factors, respectively. Deriving them according to 3.1 gives:

$$x * y = \begin{cases} x & \text{if } x = y, \\ x \bullet y & \text{otherwise.} \end{cases} \quad \text{and} \quad x \circ y = \begin{cases} x \bullet x = 0 & \text{if } x = y, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $(X,\circ,0)$ is semi-neutral.

■

Corollary 5.5 *A strong $B1$-algebra is semi-normal.*

Proof. This is a direct result of Corollary 3.2.5, Proposition 5.4 and the definition of a semi-normal algebra.

■

Corollary 5.6 *A strong $B1$-algebra $(X,\bullet,0)$ is semi-composite if it is not semi-neutral, i.e., if $x \bullet y \neq x$* for all x, $y \in X$.

Proof. Let $(X,\bullet,0)$ be a strong $B1$-algebra. Let $(X,*,0) := U(X,\bullet,0)$ and $(X,\circ,0) := A(X,\bullet,0)$, its *signature*- and *similar*-factors respectively. Deriving them according to 3.1. Assume that $x \bullet y = x$. Then $x * y = x$ for all $x, y \in X$. Thus, $(X,\bullet,0) = id_{Bin(X)} \diamond (X,\bullet,0)$ which makes it signature-prime and not *u*-composite.

■

Example 5.7 Let $(X,\bullet,0) = (\{0,1,2\},\bullet)$ be a strong BCK-algebra of order 3 where "$\bullet$" is defined by the following Cayley table:

$\bullet$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Let $(\{0,1,2\},*) = U(\{0,1,2\},\bullet)$ and $(\{0,1,2\},\circ) = A(\{0,1,2\},\bullet)$. Its UA.-factorization is:

$$\begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{array} \quad \diamond \quad \begin{array}{c|ccc} \circ & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 0 \end{array} \quad = \quad \begin{array}{c|ccc} \bullet & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 0 \end{array}$$

Therefore, $(\{0,1,2\},\bullet)$ is *signature*-prime and u-normal. Moreover, $(\{0,1,2\},\bullet)$ as defined is semi-neutral. Next, *derive* its *orient*- and *skew*-factors $O(X,\bullet,0)$ and $J(X,\bullet,o)$, respectively. Let $(X,\circledast,0) = O(X,\bullet,0)$ and $(X,\odot,0) = J(X,\bullet,0)$. We have the following product:

$$\begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 2 \end{array} \quad \diamond \quad \begin{array}{c|ccc} \circ & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{array} \quad = \quad \begin{array}{c|ccc} \bullet & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 0 \end{array}$$

Hence, $O(X,\bullet,0) \neq id_{Bin(X)} \neq J(X,\bullet,0)$ implies that $(X,\bullet,0)$ is OJ-composite.

Example 5.8 Let $(X,\bullet,0) = \{(0,1,2),\bullet\}$ be a strong Q-algebra of order 3 where "$\bullet$" is given by the following Cayley table:

$$\begin{array}{c|ccc} \bullet & 0 & 1 & 2 \\ \hline 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 1 & 0 \end{array}$$

Let $(\{0,1,2\},*) = U(\{0,1,2\},\bullet)$ and $(\{0,1,2\},\circ) = A(\{0,1,2\},\bullet)$. Its UA.-factorization is:

$$\begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 2 \end{array} \quad \diamond \quad \begin{array}{c|ccc} \circ & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 0 \end{array} \quad = \quad \begin{array}{c|ccc} \bullet & 0 & 1 & 2 \\ \hline 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 1 & 0 \end{array}$$

Since $(\{0,1,2\},*,0) \neq Id_{Bin(X)}$ and $(\{0,1,2\},\circ,0)$ is semi-neutral, we can conclude that $(\{0,1,2\},\bullet,0)$ is semi-composite.

P.J. Allen, H.S. Kim and Neggers in [4] introduced the notion of Smarandache disjointness in algebras. Two groupoids (algebras) $(X,\bullet)$ and $(X,*)$ are said to be Smarandache disjoint if we add some axioms of an algebra $(X,\bullet)$ to an algebra $(X,*)$, then the algebra $(X,*)$ becomes a trivial algebra, i.e., $|X| = 1$.

Proposition 5.9 *The class of abelian groupoids and the class of u-normal groupoids are Smarandache disjoint.*

Proof. Let $(X,\bullet) \in Ab(X)$, the collection of all abelian groupoids defined on X. Suppose that $(X,\circ) = A(X,\bullet)$ and $(X,*) = U\ (X,\bullet)$. By Theorem 3.2.3, $(X\bullet)$ admits an AU-factorization. Consider $(X,*) \diamond (X,\circ)$, then for $x = y$,

$$\begin{aligned} x \diamond x &= (x * x) \circ (x * x) \\ &= x \circ x \\ &= x \bullet x. \end{aligned}$$

If $x \neq y$,

$$\begin{aligned} x \diamond y &= (x * y) \circ (y * x) \\ &= (x \bullet y) \circ (y \bullet x) \\ &= (x \bullet y) \bullet (x \bullet y) \,. \end{aligned}$$

Hence, $(X\bullet)$ admits a UA-factorization only if $(x \bullet y) \bullet (x \bullet y) = x \bullet y$. This means that $(X, \bullet)$ is u-normal if it is either the left- or right-zero-semigroup. Since both such groupoids are not abelian, then X must only have one element and the conclusion follows.

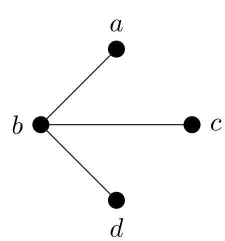
■

Suppose that in $Bin(X)$ we consider all those groupoids $(X, *)$ with the orientation property. Thus, $x * x = x$ as a consequence. If $(X, *)$ and $(X, \circ)$ both have the orientation property, then for $x \diamond\ y = (x * y) \circ (y * x)$ we have the possibilities: $x * x = x$, $y * y = y$, $x * y \in \{x, y\}$ and $y * x \in \{x, y\}$, so that $x \diamond y \in \{x, y\}$. It follows that if $OP(X)$ denotes this collection of groupoids, then $(OP(X), \diamond)$ is a subsemigroup [33] of $(Bin(X), \diamond)$.

In a sequence of papers Nebeský ([27], [28], [29]) associated with graphs (V, E) groupoids $(V, *)$ with various properties and conversely. He defined a *travel groupoid* $(X, *)$ as a groupoid satisfying the axioms: $(u*v)*u = u$ and $(u*v)*v = u$ implies $u = v$. If one adds these two laws to the orientation property, then $(X, *)$ is an OP-travel-groupoid. In this case $u * v = v$ implies $v * u = u$, i.e., $uv \in E$ implies $vu \in E$, i.e., the digraph (X, E) is a (simple) graph if $uu \notin E$, with $u*u = u$. Also, if $u \neq v$, then $u*v = u$ implies $(u*v)*v = u*v = u$ is impossible, whence $u*v = v$ and $uv \in E$, so that (X, E) is a complete (simple) graph.

In a recent paper, Ahn, Kim and Neggers [1] related graphs with binary systems in the center of $Bin\,(X)$. Given an element of $ZBin(X)$, say $(X, \bullet)$, they constructed a graph, Γ_X by letting $V(\Gamma_X) = X$ and $(x, y) \in E(\Gamma_X)$, the edge set of Γ_X, such that $x \neq y$, $y \bullet x = y$ and $x \bullet y = x$. Thus, if $(x, y) \in E(\Gamma_X)$, then $(y, x) \in E(\Gamma_X)$ as well and they identify $(x, y) = (y, x)$ as an undirected edge of Γ_X. Then they concluded that if $(X, \bullet)$ is the left-zero-semigroup, then Γ_X is the complete graph on X. Also, if $(X, \bullet)$ is the right-zero-semigroup, then Γ_X is the null graph on X, since $E(\Gamma_X) = \varnothing$.

Example 5.10 Let $X = \{a, b, c, d\}$ and consider the simple graph on X:



Then the associated groupoid table with binary operation "$\bullet$" is:

$\bullet$	a	b	c	d
a	a	a	c	d
b	b	b	b	b
c	a	c	c	d
d	a	d	c	d

By applying Proposition 4.2.5 to $(X, \bullet)$, we have the product of $O(X, \bullet)$ and $J(X, \bullet)$ given by their respective tables:

$*$	a	b	c	d
a	a	a	a	d
b	b	b	c	b
c	c	b	c	c
d	a	d	d	d

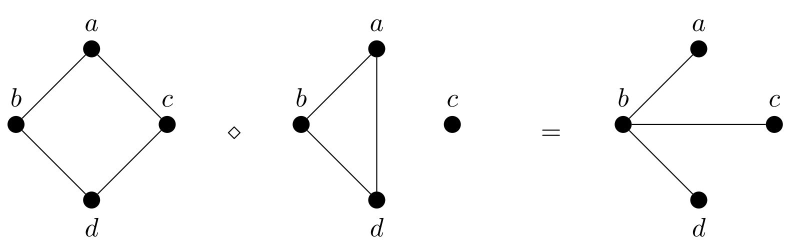
$\diamond$

$\circ$	a	b	c	d
a	a	a	c	a
b	b	b	c	b
c	a	b	c	d
d	d	d	c	d

$=$

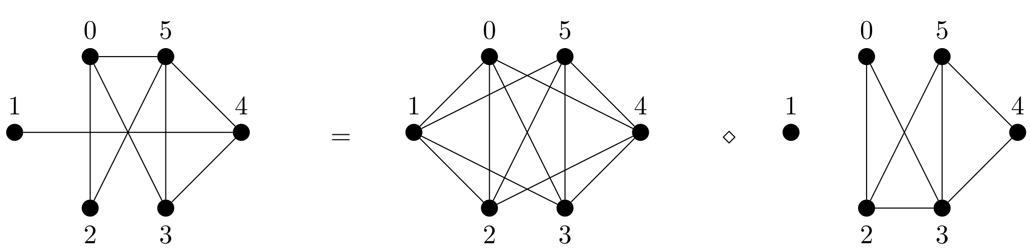
$\bullet$	a	b	c	d
a	a	a	c	d
b	b	b	b	b
c	a	c	c	d
d	a	d	c	d

We can visualize this product with the associated graphs of groupoids $(X, *)$ and $(X, \circ)$:



Thus, any simple graph constructed in this manner can be decomposed into two or more other factors with the binary product "$\diamond$". This fact is further illustrated in the next example.

Example 5.11 Let $(X, \bullet) = (\{0, 1, 2, 3, 4, 5\}, \bullet)$ be the locally-zero groupoid defined as in Example 4.3.2. Then its associated graph decomposes into its factors $(X, *)$ and $(X, \circ)$:



6. Generalization and Summary

In this final note, we discuss two generalizations which can serve as grounds for future exploration of groupoid factorizations or algebra decompositions via the groupoid product "$\diamond$".

6.1. ***Ψ-type-Factorization.*** Let Ψ be a groupoid operation that interchanges elements of any two given groupoids and produces two other (possibly identical) groupoids. Given groupoid $(X, \bullet) \in Bin\,(X)$ and the left-zero-semigroup as $id_{Bin(X)}$, define $\Psi : Bin\,(X) \times Bin\,(X) \rightarrow Bin\,(X) \times Bin\,(X)$. A Ψ-type-factorization of $(X, \bullet)$ gives a pair of groupoid factors as follows:

$$\Psi((X, \bullet)\,, id_{Bin(X)}) = ((X, \bullet)_L\,, (X, \bullet)_R)$$

where $(X, \bullet)_L = \Psi_\alpha((X, \bullet)\,, id_{Bin(X)})$ and $(X, \bullet)_R = \Psi_\alpha(id_{Bin(X)}, (X, \bullet))$, the *left-* and *right-Ψ-factors* of $(X, \bullet)$, respectively, such that the maps Ψ_α and α are defined as $\Psi_\alpha : Bin\,(X) \times Bin\,(X) \rightarrow Bin\,(X)$ and $\alpha : Bin\,(X) \rightarrow Bin\,(X)$.

Let $(X, *) := (X, \bullet)_L$ and $(X, \circ) := (X, \bullet)_R$, then $(X, \bullet)$ can be represented as a product of the groupoid pair, i.e.,

$$\begin{aligned} (X, \bullet) &= (X, *) \diamond (X, \circ) \text{ and/or} \\ (X, \bullet) &= (X, \circ) \diamond (X, *) \end{aligned}$$

thus rendering $(X, \bullet)$ as:

(i) Ψ-*prime*, if $(X, \bullet)_L = id_{Bin(X)}$ or $(X, \bullet)_R = id_{Bin(X)}$; or
(ii) Ψ-*normal* if $(X, *) \diamond (X, \circ) = (X, \circ) \diamond (X, *)$; or
(iii) Ψ-*composite* if $(X, \bullet)$ is Ψ-normal but not Ψ-prime.

An example of this Ψ-type-factorization is our first method of *similar-signature-*factorization where

$$\Psi_d((X, \bullet)\,, id_{Bin(X)}) = \{(X, \bullet)\,|d\,(X, \bullet) = d\left(id_{Bin(X)}\right)\}$$

and

$$\Psi_d(id_{Bin(X)}, (X, \bullet)) = \{id_{Bin(X)}|d\left(id_{Bin(X)}\right) = d\,(X, \bullet)\}$$

The Ψ in that case switched the diagonal d of the parent groupoid $(X, \bullet)$ with that of the left-zero-semigroup, $id_{Bin(X)}$, to obtain the *signature-* and *similar-*factors $(X, \circ)$ and $(X, *)$, respectively. Hence, the *signature-* and *similar-*factors of a groupoid are *Ψ-type-factors*.

6.2. ***τ-type-Factorization.*** Let τ be a groupoid operation that manipulates elements of any given pair of groupoid in the same fashion. Given groupoid $(X, \bullet) \in Bin\,(X)$ and the left-zero-semigroup as $id_{Bin(X)}$, define $\tau : Bin\,(X) \times Bin\,(X) \rightarrow Bin\,(X) \times Bin\,(X)$. A *$\tau$-type-factorization* of $(X, \bullet)$ is given as follows:

$$\tau\left((X, \bullet)\,, id_{Bin(X)}\right) = ((X, \bullet)_L\,, (X, \bullet)_R)$$

where $(X, \bullet)_L = \theta(id_{Bin(X)})$ and and $(X, \bullet)_R = \theta\,(X, \bullet)$ such that the map $\theta : Bin\,(X) \rightarrow Bin\,(X)$, the *left-* and *right-τ-factors* of $(X, \bullet)$, respectively. Let $(X, *) := (X, \bullet)_L$ and $(X, \circ) := (X, \bullet)_R$, then $(X, \bullet)$ could factor into a product of the groupoid pair, i.e.,

$$\begin{aligned} (X, \bullet) &= (X, *) \diamond (X, \circ) \text{ and/or} \\ (X, \bullet) &= (X, \circ) \diamond (X, *)\,. \end{aligned}$$

Once again rendering $(X, \bullet)$ as:

(i) τ-*prime*, if $(X,\bullet)_L = id_{Bin(X)}$ or $(X,\bullet)_R = id_{Bin(X)}$; or
(ii) τ-*normal* if $(X,*)\diamond(X,\circ) = (X,\circ)\diamond(X,*)$; or
(iii) τ-*composite* if $(X,\bullet)$ is τ-normal but not τ-prime.

An example of this τ-type-factorization is our second method of *orient-skew*-factorization where $O(X,\bullet) := (X,\bullet)_L$ and $J(X,\bullet) := (X,\bullet)_R$. The τ (indeed, θ) in that scenario reversed the anti-diagonal of a given groupoid. Hence, applying τ to the left-zero-semigroup $id_{Bin(X)}$ and to the parent groupoid $(X,\bullet)$ results in the *orient*- and *skew*-factors $(X,*)$ and $(X,\circ)$, respectively. In conclusion, the *orient*- and *skew*-factors of a groupoid are τ-*type-factors*.

6.3. **Summary.** The goal of this paper was to gain more insight about the dynamics of binary systems, namely groupoids or algebras equipped with a single binary operation. We have shown that a strong groupoid can be represented as a "composite" groupoid of its *similar*- and *signature*- derived factors. Moreover, we concluded that an idempotent groupoid with the orientation property, can be decomposed into a product of its *orient*- and *skew*- factors. An application into the fields of logic-algebras and graph theory were briefly introduced. We found that a *semi*-neutral $B1$-algebra is *signature*-prime, OJ-composite and *semi*-normal. Meanwhile, a strong $B1$-algebra is then *semi*-composite if it is not *semi*-neutral. We finished our note with generalizations of our two methods in hopes that other factorizations can be discovered in the near future. It may be interesting to find other conditions for a groupoid to have such decompositions. As a final reminder, factorization can be useful in various applications such as algebraic cryptography and DNA code theory. We intend to extend our investigation in the future to hypergroupoid, semigroups as well as determine other factorizations and explore their applications.

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