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On the hybrid mean value of the Smarandache kn digital sequence with $SL(n)$ function and divisor function $d(n)$[1]

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Abstract The main purpose of this paper is using the elementary method to study the hybrid mean value properties of the Smarandache kn digital sequence with $SL(n)$ function and divisor function $d(n)$, then give two interesting asymptotic formulae for it.

Keywords Smarandache kn digital sequence, $SL(n)$ function, divisor function, hybrid mean value, asymptotic formula.

§1. Introduction and results

For any positive integer k, the famous Smarandache kn-digital sequence $a(k,n)$ is defined as all positive integers which can be partitioned into two groups such that the second part is k times bigger than the first. For example, Smarandache $3n$ digital sequences $a(3,n)$ is defined as $\{a(3,n)\} = \{13, 26, 39, 412, 515, 618, 721, 824, \cdots\}$, for example, $a(3,15) = 1545$. In the reference [1], Professor F. Smarandache asked us to study the properties of $a(k,n)$, about this problem, many people have studied and obtained many meaningful results. In [2], Lu Xiaoping studied the mean value of this sequence and gave the following theorem:

$$\sum_{n\leq N} \frac{n}{a(5,n)} = \frac{9}{50\ln 10} \cdot \ln N + O(1).$$

In [3], Gou Su studied the hybrid mean value of Smarandache kn sequence and divisor function $\sigma(n)$, and gave the following theorem:

$$\sum_{n\leq x} \frac{\sigma(n)}{a(k,n)} = \frac{3\pi^2}{k\cdot 20\cdot \ln 10} \cdot \ln x + O(1),$$

where $1 \leq k \leq 9$.

Inspired by the above conclusions, in this paper, we study the hybrid mean value properties of the Smarandache kn-digital sequence with $SL(n)$ function and divisor function $d(n)$, where

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$SL(n)$ is defined as the smallest positive integer k such that $n|[1,2,\ldots,k]$, that is $SL(n) = \min\{k : k \in N, n|[1,2,\ldots,k]\}$. And obtained the following results:

Theorem 1.1. Let $1 \le k \le 9$, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} \frac{SL(n)}{a(k,n)} = \frac{3\pi^2}{k\cdot 20}\cdot \ln\ln x + O(1).$$

Theorem 1.2. Let $1 \le k \le 9$, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} \frac{d(n)\cdot SL(n)}{a(k,n)} = \frac{\pi^4}{k\cdot 20}\cdot \ln\ln x + O(1).$$

§2. Lemmas

Lemma 2.1. For any real number $x > 1$, we have

$$\sum_{n\le x} \frac{SL(n)}{n} = \frac{\pi^2}{6}\cdot\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

Proof. For any real number $x > 1$, by reference [4] we have the asymptotic formula

$$\sum_{n\le x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Using Abel formula (see [6]) we get

$$\begin{aligned}\sum_{1<n\le x}\frac{SL(n)}{n} &= \frac{1}{x}\left(\frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right)\right) + \int_1^x \frac{1}{t^2}\left(\frac{\pi^2}{12}\cdot\frac{t^2}{\ln t} + O\left(\frac{t^2}{\ln^2 t}\right)\right)\, dt \\ &= \frac{\pi^2}{12}\cdot\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right) + \frac{\pi^2}{12}\int_1^x \frac{1}{\ln t}\, dt + O\left(\int_1^x \frac{1}{\ln^2 t}\, dt\right) \\ &= \frac{\pi^2}{6}\cdot\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).\end{aligned}$$

This proves Lemma 2.1.

Lemma 2.2. For any real number $x > 1$, we have

$$\sum_{n\le x} \frac{d(n)\cdot SL(n)}{n} = \frac{\pi^4}{18}\cdot\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

Proof. For any real number $x > 1$, by reference [5] we have the asymptotic formula

$$\sum_{n\le x} d(n)\cdot SL(n) = \frac{\pi^4}{36}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Using Abel formula (see [6]) we get

$$\begin{aligned}\sum_{1<n\le x}\frac{d(n)\cdot SL(n)}{n} &= \frac{1}{x}\left(\frac{\pi^4}{36}\cdot\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right)\right)+\int_1^x\frac{1}{t^2}\left(\frac{\pi^4}{36}\cdot\frac{t^2}{\ln t}+O\left(\frac{t^2}{\ln^2 t}\right)\right)\,dt\\ &= \frac{\pi^4}{36}\cdot\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right)+\frac{\pi^4}{36}\int_1^x\frac{1}{\ln t}\,dt+O\left(\int_1^x\frac{1}{\ln^2 t}\,dt\right)\\ &= \frac{\pi^4}{18}\cdot\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right).\end{aligned}$$

This proves Lemma 2.2.

§3. Proof of the theorems

In this section, we shall use the elementary and combinational methods to complete the proof of our theorems. We just prove the case of $k=3$ and $k=5$, for other positive integers we can use the similar methods.

First we prove theorem 1.1. Let $k=3$, for any positive integer $x>3$, there exists a positive integer M such that

$$\underbrace{33\cdots 33}_{M}<x\le\underbrace{33\cdots 33}_{M+1}.$$

So

$$10^M-1<3x\le 10^{M+1}-1,$$

Then we have

$$\frac{\ln 3x}{\ln 10}-1-O\left(\frac{1}{10^M}\right)\le M<\frac{\ln 3x}{\ln 10}-O\left(\frac{1}{10^M}\right). \tag{1}$$

By the definition of $a(3,n)$ we have

$$\begin{aligned}\sum_{1\le n\le x}\frac{SL(n)}{a(3,n)} &= \sum_{n=1}^{3}\frac{SL(n)}{a(3,n)}+\sum_{n=4}^{33}\frac{SL(n)}{a(3,n)}+\sum_{n=34}^{333}\frac{SL(n)}{a(3,n)}+\cdots+\sum_{n=\frac{1}{3}\cdot 10^{M-1}}^{\frac{1}{3}\cdot 10^{M}-1}\frac{SL(n)}{a(3,n)}\\ &\quad+\sum_{\frac{1}{3}\cdot 10^M\le n\le x}\frac{SL(n)}{a(3,n)}\\ &= \sum_{n=1}^{3}\frac{SL(n)}{n(10+3)}+\sum_{n=4}^{33}\frac{SL(n)}{n(10^2+3)}+\sum_{n=34}^{333}\frac{SL(n)}{n(10^3+3)}+\cdots\\ &\quad+\sum_{n=\frac{1}{3}\cdot 10^{M-1}}^{\frac{1}{3}\cdot 10^{M}-1}\frac{SL(n)}{n(10^{M+1}+3)}+\sum_{\frac{1}{3}\cdot 10^M\le n\le x}\frac{SL(n)}{n(10^{M+2}+3)}.\end{aligned} \tag{2}$$

Form (1), (2) and lemma 2.1 we get

$$\begin{aligned}\sum_{n=\frac{1}{3}\cdot 10^{k-1}}^{\frac{1}{3}\cdot 10^{k}-1} \frac{SL(n)}{n\cdot(10^k+3)} &= \sum_{\frac{1}{3}\cdot 10^{k}-1} \frac{SL(n)}{n\cdot(10^k+3)} - \sum_{n=\frac{1}{3}\cdot 10^{k-1}} \frac{SL(n)}{n\cdot(10^k+3)} \\ &= \frac{\pi^2}{6}\cdot\frac{\frac{1}{3}\cdot 10^k-\frac{1}{3}\cdot 10^{k-1}}{10^k+3}\cdot\frac{1}{\ln(\frac{1}{3}\cdot 10^k)}+O\left(\frac{1}{k^2}\right) \\ &= \frac{3\pi^2}{3\cdot 20}\cdot\frac{1}{k}+O\left(\frac{1}{k^2}\right). \qquad (3)\end{aligned}$$

Note that the identity $\sum_{n=1}^{\infty}\frac{1}{n^2}=\pi^2/6$ and the asymptotic formula

$$\sum_{1\le k\le M}\frac{1}{k}=\ln M+\gamma+O\left(\frac{1}{M}\right),$$

where γ is Euler's constant.
Form (1), (2) and (3) we get

$$\begin{aligned}\sum_{1\le n\le x}\frac{SL(n)}{a(3,n)} &= \sum_{n=1}^{3}\frac{SL(n)}{a(3,n)}+\sum_{n=4}^{33}\frac{SL(n)}{a(3,n)}+\sum_{n=34}^{333}\frac{SL(n)}{a(4,n)}+\cdots+\sum_{n=\frac{1}{3}\cdot 10^{M-1}}^{\frac{1}{3}\cdot 10^{M}-1}\frac{SL(n)}{a(3,n)} \\ &\quad+\sum_{\frac{1}{3}\cdot 10^{M}\le n\le x}\frac{SL(n)}{a(3,n)} \\ &= \sum_{k=1}^{M}\frac{3\pi^2}{3\cdot 20}\cdot\frac{1}{k}+O\left(\sum_{k=1}^{M}\frac{1}{k^2}\right) \\ &= \frac{3\pi^2}{3\cdot 20}\ln\ln x+O(1).\end{aligned}$$

Now we prove the case of $k=5$, for any positive integer $x>1$, there exists a positive integer M such that

$$\underbrace{200\cdots 00}_{M}<x\le\underbrace{199\cdots 99}_{M+1}.$$

So

$$10^M<5x\le 10^{M+1}-5,$$

Then we have

$$\frac{\ln 5x}{\ln 10}-1-O\left(\frac{1}{10^M}\right)\le M<\frac{\ln 5x}{\ln 10}. \qquad (4)$$

By the definition of $a(5,n)$ we have

$$\begin{aligned}\sum_{1\le n\le x}\frac{SL(n)}{a(5,n)} &= \sum_{n=1}\frac{SL(n)}{a(5,n)}+\sum_{n=2}^{19}\frac{SL(n)}{a(5,n)}+\sum_{n=20}^{199}\frac{SL(n)}{a(5,n)}+\cdots+\sum_{n=\frac{1}{5}\cdot 10^{M-1}}^{\frac{1}{5}\cdot 10^{M}-1}\frac{SL(n)}{a(5,n)}\\ &\quad +\sum_{\frac{1}{5}\cdot 10^{M}\le n\le x}\frac{SL(n)}{a(5,n)}\\ &= \sum_{n=1}\frac{SL(n)}{n(10+5)}+\sum_{n=2}^{19}\frac{SL(n)}{n(10^2+5)}+\sum_{n=20}^{199}\frac{SL(n)}{n(10^3+5)}+\cdots\\ &\quad +\sum_{n=\frac{1}{5}\cdot 10^{M-1}}^{\frac{1}{5}\cdot 10^{M}-1}\frac{SL(n)}{n(10^{M+1}+5)}+\sum_{\frac{1}{5}\cdot 10^{M}\le n\le x}\frac{SL(n)}{n(10^{M+2}+5)}. \end{aligned} \tag{5}$$

Form (4), (5) and lemma 2.1 we get

$$\begin{aligned}\sum_{n=\frac{1}{5}\cdot 10^{k-1}}^{\frac{1}{5}\cdot 10^{k}-1}\frac{SL(n)}{n\cdot(10^k+5)} &= \sum_{\frac{1}{5}\cdot 10^{k}-1}\frac{SL(n)}{n\cdot(10^k+5)}-\sum_{\frac{1}{5}\cdot 10^{k-1}}\frac{SL(n)}{n\cdot(10^k+5)}\\ &= \frac{\pi^2}{6}\cdot\frac{\frac{1}{5}\cdot 10^k-\frac{1}{5}\cdot 10^{k-1}}{10^k+5}\cdot\frac{1}{\ln(\frac{1}{5}\cdot 10^k)}+O\left(\frac{1}{k^2}\right)\\ &= \frac{3\pi^2}{5\cdot 20}\cdot\frac{1}{k}+O\left(\frac{1}{k^2}\right).\end{aligned}$$

Similar to the proof $k=3$, we get

$$\begin{aligned}\sum_{1\le n\le x}\frac{SL(n)}{a(5,n)} &= \sum_{n=1}\frac{SL(n)}{a(5,n)}+\sum_{n=2}^{19}\frac{SL(n)}{a(5,n)}+\sum_{n=20}^{199}\frac{SL(n)}{a(5,n)}+\cdots+\sum_{n=\frac{1}{5}\cdot 10^{M-1}}^{\frac{1}{5}\cdot 10^{M}-1}\frac{SL(n)}{a(5,n)}\\ &\quad +\sum_{\frac{1}{5}\cdot 10^{M}\le n\le x}\frac{SL(n)}{a(5,n)}\\ &= \sum_{k=1}^{M}\frac{3\pi^2}{5\cdot 20}\cdot\frac{1}{k}+O\left(\sum_{k=1}^{M}\frac{1}{k^2}\right)\\ &= \frac{3\pi^2}{5\cdot 20}\ln\ln x+O(1).\end{aligned}$$

By using the same methods, we can also prove that the theorem holds for all integers $1\le k\le 9$. This completes the proof of theorem 1.1.

Similar to the proof of theorem 1.1, we can immediately prove theorem 1.2, we don't repeated here. As the promotion of this article, we can consider the hybrid mean value of Smarandache kn sequence with other functions such as $SL^*(n), Sdf(n), \sigma(S(n)), \Omega(S^*(n))$, and obtain the corresponding asymptotic formula.

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