ON THE INTEGER PART OF A POSITIVE INTEGER'S K-TH ROOT

Yang Hai
Research Center for Basic Science, Xi'an Jiaotong University, Xi'an, Shaanxi, P.R.China

Fu Ruiqin
School of Science, Xi'an Shiyou University, Xi'an, Shaanxi, P.R.China

Abstract The main purpose of this paper is using the elementary method and analytic method to study the asymptotic properties of the integer part of the k-th root positive integer, and give two interesting asymptotic formulae.

Keywords: k-th root; Integer part; Asymptotic formula.

§1. Introduction And Results

For any positive integer n, let $s(n)$ denote the integer part of k-th root of n. For example, $s(1) = 1$, $s(2) = 1$, $s(3) = 1$, $s(4) = 1, \cdots$, $s(2^k) = 1$, $s(2^k + 1) = 1, \cdots$, $s(3^k) = 1, \cdots$. In problem 80 of [1], Professor F.Smarandache asked us to study the properties of the sequence $s(n)$. About this problem, it seems that none had studied it, at least we have not seen related paper before. In this paper, we use the elementary method and analytic method to study the asymptotic properties of this sequence, and obtain two interesting asymptotic formulae. That is, we shall prove the following:

Theorem 1. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\le x} \Omega(s(n)) = x \ln\ln x + (A - \log k)\, x + O\left(\frac{x}{\ln x}\right),$$

where $\Omega(n)$ denotes the total number of prime divisors of n, A is a constant.

Theorem 2. *Let m be a fixed positive integer and $\varphi(n)$ be the Euler totient function, then for any real number $x \ge 1$, we have the asymptotic formula*

$$\sum_{n\le x} \varphi((s(n), m)) = h(m)x + (k+1)h(m) + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

where $(s(n), m)$ denotes the greatest common divisor of $s(n)$ and m, $h(m) = \frac{\varphi(m)}{m} \prod\limits_{p^\alpha \| m} (1 + \alpha - \frac{\alpha-1}{p})$, and ε is any positive number.

§2. Some Lemmas

To complete the proof of the theorems, we need the following two simple lemmas.

Lemma 1. *For any real number $x > 1$, then we have*

$$\sum_{n\leq x}\Omega(n) = x\log\log x + Ax + O\left(\frac{x}{\log x}\right),$$

where $A = \gamma + \sum_p(\log(1-\frac{1}{p}) + \frac{1}{p}) + \sum_p \frac{1}{p(p-1)}$, *$\gamma$ is the Euler constant.*

Proof. See reference[2].

Lemma 2. *Let m be a fixed positive integer and $\varphi(n)$ be the Euler totient function, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x}\varphi((m,n)) = x\cdot h(m) + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where (m,n) denotes the greatest common divisor of m and n, $h(m) = \frac{\varphi(m)}{m}\prod_{p^\alpha\|m}(1+\alpha-\frac{\alpha-1}{p})$, *and ε is any positive number.*

Proof. Let

$$F(s) = \sum_{n=1}^{\infty}\frac{\varphi((m,n))}{n^s},$$

then from the Euler Product formula [3] and the multiplicative property of $\varphi(m,n)$, we may get

$$\begin{aligned} F(s) &= \prod_p\left(1+\frac{\varphi((m,p))}{p^s}+\frac{\varphi((m,p^2))}{p^{2s}}+\cdots\right)\\ &= \prod_{p^\alpha\|m}\left(1+\frac{\varphi((m,p))}{p^s}+\cdots+\frac{\varphi((m,p^{\alpha-1}))}{p^{(\alpha-1)s}}+\frac{\varphi((m,p^\alpha))}{p^{\alpha s}}(\frac{1}{1-\frac{1}{p^s}})\right)\\ &\quad\times\prod_{p\dagger m}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right)\\ &= \zeta(s)\prod_{p^\alpha\|m}\left(\left(1+\frac{\varphi((m,p))}{p^s}+\cdots+\frac{\varphi((m,p^{\alpha-1}))}{p^{(\alpha-1)s}}\right)\left(1-\frac{1}{p^s}\right)+\frac{\varphi((m,p^\alpha))}{p^{\alpha s}}\right),\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

Obviously, we have inequality

$$|\varphi((m,n))| < K, \qquad \left|\sum_{n=1}^{\infty}\frac{\varphi((m,n))}{n^\sigma}\right| < \frac{K}{\sigma-1},$$

where $\sigma > 1$ is the real part of s. So by Perron formula [4], taking $b = 2, T = x^{\frac{1}{2}}, H(x) = K, B(\sigma) = \frac{K}{\sigma-1}$, then we have

$$\sum_{n\leq x}\varphi((m,n)) = \frac{1}{2\pi i}\int_{2-iT}^{2+iT}\zeta(s)R(s)\frac{x^s}{s}ds + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where

$$R(s)=\prod_{p^\alpha\|m}\left(\left(1+\frac{\varphi((m,p))}{p^s}+\cdots+\frac{\varphi((m,p^{\alpha-1}))}{p^{(\alpha-1)s}}\right)\left(1-\frac{1}{p^s}\right)+\frac{\varphi((m,p^\alpha))}{p^{\alpha s}}\right).$$

To estimate the main term

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\zeta(s)R(s)\frac{x^s}{s}ds$$

we move the integral line from $s=2\pm iT$ to $s=1/2\pm iT$. This time, we have a simple pole point at $s=1$ with residue $R(1)x$. That is

$$\frac{1}{2\pi i}\left(\int_{2-iT}^{2+iT}+\int_{2+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}+\int_{\frac{1}{2}-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^s}{s}ds=R(1)x.$$

Taking $T=x^{3/2}$, we can easily get the estimate

$$\begin{aligned}&\left|\frac{1}{2\pi i}\left(\int_{2+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^s}{s}ds\right|\\&\ll\int_{\frac{1}{2}}^{2}\left|\zeta(\sigma+iT)R(s)\frac{x^2}{T}\right|d\sigma\ll\frac{x^2}{T}=x^{\frac{1}{2}},\end{aligned}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\zeta(s)R(s)\frac{x^s}{s}ds\right|\ll\int_0^T\left|\zeta(\frac{1}{2}+it)R(s)\frac{x^{\frac{1}{2}}}{t}\right|dt\ll x^{\frac{1}{2}+\varepsilon}.$$

Noting that

$$\begin{aligned}R(1)&=\prod_{p^\alpha\|m}\left(\left(1+\frac{\varphi((m,p))}{p}+\cdots+\frac{\varphi((m,p^{\alpha-1}))}{p^{(\alpha-1)}}\right)\left(1-\frac{1}{p}\right)+\frac{\varphi((m,p^\alpha))}{p^\alpha}\right)\\&=\frac{\varphi(m)}{m}\prod_{p^\alpha\|m}(1+\alpha-\frac{\alpha-1}{p}).\end{aligned}$$

So we have the asympotic formula

$$\sum_{n\le x}\varphi((m,n))=x\cdot h(m)+O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $h(m)=\frac{\varphi(m)}{m}\prod\limits_{p^\alpha\|m}(1+\alpha-\frac{\alpha-1}{p})$. This completes the proof of Lemma 2.

§3. Proof of Theorems

In this section, we will complete the proof of Theorems. First we prove Theorem 1. For any real number $x > 1$, let M be a fixed positive integer with $M^k \le x \le (M+1)^k$, from the definition of $s(n)$ we have

$$\begin{aligned}\sum_{n\le x}\Omega(s(n)) &= \sum_{t=1}^{M}\sum_{(t-1)^k\le n<t^k}\Omega(s(n)) + \sum_{M^k\le n<x}\Omega(s(n))\\ &= \sum_{t=1}^{M-1}\sum_{t^k\le n<(t+1)^k}\Omega(s(n)) + \sum_{M^k\le n\le x}\Omega(M)\\ &= \sum_{t=1}^{M-1}[(t+1)^k - t^k]\Omega(t) + O\left(\sum_{M^k\le n<(M+1)^k}\Omega(M)\right)\\ &= k\sum_{t=1}^{M} t^{k-1}\Omega(t) + O\left(M^{k-1}\log M\right),\end{aligned}$$

where we have used the estimate $\Omega(M) \ll \log n$.

Let $B(y) = \sum\limits_{n\le y}\Omega(n)$, then by Able's identity and Lemma 1, we can easily deduce that

$$\begin{aligned}&\sum_{t=1}^{M} t^{k-1}\Omega(t) = M^{k-1}B(M) - (k-1)\int_2^M y^{k-2}B(y)dy\\ =\ & M^{k-1}(M\log\log M + AM) - (k-1)\int_2^M (y^{k-1}\log\log y + Ay^{k-1})dy\\ &+O\left(\frac{M^k}{\log M}\right)\\ =\ & M^k\log\log M + AM^k - \frac{k-1}{k}(M^k\log\log M + AM^k) + O\left(\frac{M^k}{\log M}\right)\\ =\ & \frac{1}{k}M^k\log\log M + \frac{1}{k}AM + O\left(\frac{M^k}{\log M}\right).\end{aligned}$$

Therefore, we can obtain the asymptotic formula

$$\sum_{n\le x}\Omega(s(n)) = M^k\log\log M + AM + O\left(\frac{M^k}{\log M}\right).$$

On the other hand, we also have the estimate

$$0 \le x - M^k < (M+1)^k - M^k \ll x^{\frac{k-1}{k}}.$$

Now combining the above, we may immediately obtain the asymptotic formula

$$\sum_{n\leq x}\Omega(s(n)) = x\log\log x + (A-\log k)\,x + O\left(\frac{x}{\log x}\right).$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. For any fixed positive integer m, we have

$$\begin{aligned}
&\sum_{n\leq x}\varphi((s(n),m)) = \sum_{n\leq x}\varphi(([n^{\frac{1}{k}}],m))\\
=&\sum_{1\leq i<2^k}\varphi(([i^{\frac{1}{k}}],m)) + \cdots + \sum_{N\leq i<(N+1)^k}\varphi(([i^{\frac{1}{k}}],m)) + O\left(N^{\varepsilon}\right)\\
=&\sum_{j\leq N}[(j+1)^k - j^k]\varphi((j,m)) + O\left(N^{\varepsilon}\right).
\end{aligned}$$

From Lemma 2, we can let

$$\begin{aligned}
A(N) &= \sum_{j\leq N}\varphi((j,m)) = N\cdot h(m) + O\left(N^{\frac{1}{2}+\varepsilon}\right),\\
f(j) &= [(j+1)^k - j^k],
\end{aligned}$$

Then by Able's identity, we can easily obtain

$$\begin{aligned}
&\sum_{j\leq N}[(j+1)^k - j^k]\varphi((j,m))\\
=& A(N)f(N) - A(1)f(1) - \int_1^N A(t)f'(t)dt\\
=& [N\cdot h(m) + O\left(N^{\frac{1}{2}+\varepsilon}\right)][(N+1)^k - N^k]\\
&-\int_1^N [t\cdot h(m) + O\left(t^{\frac{1}{2}+\varepsilon}\right)]\cdot k[(t+1)^{k-1} - t^{k-1}]dt\\
=& k\cdot N^k h(m) + O\left(k\cdot N^{k-\frac{1}{2}+\varepsilon}\right) - (k-1)h(m)(n^k - 1)\\
=& (k + N^k + 1)h(m) + O\left(k\cdot N^{k-\frac{1}{2}+\varepsilon}\right)\\
=& h(m)x + (k+1)h(m) + +O\left(x^{1-\frac{1}{2k}+\varepsilon}\right).
\end{aligned}$$

This completes the proof of Theorem 2.

References

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